Advanced Quantum Field Theory

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Lectures given at the Florence University during the academic year 1998/99.

Contents

Chapter 1

Notations and Conventions

1.1 Units

In quantum relativistic theories the two fundamental constants c e $\hbar$, the light velocity and the Planck constant respectively, appear everywhere. Therefore it is convenient to choose a unit system where their numerical value is given by

$$c = \hbar = 1 \tag{1.1}$$

For the electromagnetism we will use the Heaviside-Lorentz system, where we take also

$$\epsilon_0 = 1 \tag{1.2}$$

From the relation $\epsilon_0 \mu_0 = 1/c^2$ it follows

$$\mu_0 = 1 \tag{1.3}$$

In these units the Coulomb force is given by

$$|\vec{F}| = \frac{e_1 e_2}{4\pi} \frac{1}{|\vec{x}_1 - \vec{x}_2|^2} \tag{1.4}$$

and the Maxwell equations appear without any visible constant. For instance the gauss law is

$$\vec{\nabla} \cdot \vec{E} = \rho \tag{1.5}$$

dove ρ is the charge density. The dimensionless fine structure constant

$$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} \tag{1.6}$$

is given by

$$\alpha = \frac{e^2}{4\pi} \tag{1.7}$$

Any physical quantity can be expressed equivalently by using as fundamental unit energy, mass, lenght or time in an equivalent fashion. In fact from our choice the following equivalence relations follow

$$
\begin{aligned}
ct &\approx \ell \Longrightarrow \text{time} \approx \text{lenght} \\
E &\approx mc^2 \Longrightarrow \text{energy} \approx \text{mass} \\
E &\approx pv \Longrightarrow \text{energy} \approx \text{momentum} \\
Et &\approx \hbar \Longrightarrow \text{energy} \approx (\text{time})^{-1} \approx (\text{lenght})^{-1}
\end{aligned} \tag{1.8}
$$

In practice, it is enough to notice that the product $c\hbar$ has dimensions $[E \cdot \ell]$. Therefore

$$
c\hbar = 3 \cdot 10^8 \text{ mt} \cdot \text{sec}^{-1} \cdot 1.05 \cdot 10^{-34} \text{ J} \cdot \text{sec} = 3.15 \cdot 10^{-26} \text{ J} \cdot \text{mt} \tag{1.9}
$$

Recalling that

$$
1 \text{ eV} = e \cdot 1 = 1.602 \cdot 10^{-19} \text{ J} \tag{1.10}
$$

it follows

$$
c\hbar = \frac{3.15 \cdot 10^{-26}}{1.6 \cdot 10^{-13}} \text{ MeV} \cdot \text{mt} = 197 \text{ MeV} \cdot \text{fermi} \tag{1.11}
$$

From which

$$
1 \text{ MeV}^{-1} = 197 \text{ fermi} \tag{1.12}
$$

Using this relation we can easily convert a quantity given in MeV (the typical unit used in elementary particle physics) in fermi. For instance, using the fact that also the elementary particle masses are usually given in MeV, the wave lenght of an electron is given by

$$
\lambda_{\text{Compton}}^e = \frac{1}{m_e} \approx \frac{1}{0.5 \; MeV} \approx \frac{200 \; MeV \cdot \text{fermi}}{0.5 \; MeV} \approx 400 \text{ fermi} \tag{1.13}
$$

Therefore the approximate relation to keep in mind is $1 = 200 \; MeV \cdot$ fermi. Furthermore, using

$$
c = 3 \cdot 10^{23} \text{ fermi} \cdot \text{sec}^{-1} \tag{1.14}
$$

we get

$$
1 \text{ fermi} = 3.3 \cdot 10^{-24} \text{ sec} \tag{1.15}
$$

and

$$
1 \text{ MeV}^{-1} = 6.58 \cdot 10^{-22} \text{ sec} \tag{1.16}
$$

Also, using

$$
1 \text{ barn} = 10^{-24} \text{ cm}^2 \tag{1.17}
$$

it follows from (1.12)

$$
1 \text{ GeV}^{-2} = 0.389 \text{ mbarn} \tag{1.18}
$$

1.2 Relativity and Tensors

Our conventions are as follows: the metric tensor $g_{\mu\nu}$ is diagonal with eigenvalues $(+1, -1, -1, -1)$. The position and momentum four-vectors are given by

$$x^\mu = (t, \vec{x}), \qquad p^\mu = (E, \vec{p}), \quad \mu = 0, 1, 2, 3 \qquad (1.19)$$

where $\vec{x}$ and $\vec{p}$ are the three-dimensional position and momentum. The scalar product between two four-vectors is given by

$$a \cdot b = a^\mu b^\nu g_{\mu\nu} = a_\mu b^\mu = a^\mu b_\mu = a^0 b^0 - \vec{a} \cdot \vec{b} \qquad (1.20)$$

where the indices have been lowered by

$$a_\mu = g_{\mu\nu} a^\nu \qquad (1.21)$$

and can be raised by using the inverse metric tensor $g^{\mu\nu} \equiv g_{\mu\nu}$. The four-gradient is defined as

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \vec{x}} \right) = \left(\partial_t, \vec{\nabla} \right) \qquad (1.22)$$

The four-momentum operator in position space is

$$p^\mu \to i \frac{\partial}{\partial x_\mu} = (i\partial_t, -i\vec{\nabla}) \qquad (1.23)$$

The following relations will be useful

$$p^2 = p_\mu p^\mu \to -\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu} = -\Box \qquad (1.24)$$

$$x \cdot p = Et - \vec{p} \cdot \vec{x} \qquad (1.25)$$

1.3 The Noether's theorem for relativistic fields

We will now review the Noether's theorem. This allows to relate symmetries of the action with conserved quantities. More precisely, given a transformation involving both the fields and the coordinates, if it happens that the action is invariant under this transformation, then a conservation law follows. When the transformations are limited to the fields one speaks about **internal transformations**. When both types of transformations are involved the total variation of a local quantity $F(x)$ (that is a function of the space-time point) is given by

$$\begin{aligned} \Delta F(x) \quad &= \tilde{F}(x') - F(x) = \tilde{F}(x + \delta x) - F(x) \\ &\cong \tilde{F}(x) - F(x) + \delta x^\mu \frac{\partial F(x)}{\partial x^\mu} \end{aligned} \qquad (1.26)$$

6

The total variation Δ keeps into account both the variation of the reference frame and the form variation of F. It is then convenient to define a **local** variation δF, depending only on the form variation

$$\delta F(x) = \tilde{F}(x) - F(x) \tag{1.27}$$

Then we get

$$\Delta F(x) = \delta F(x) + \delta x^\mu \frac{\partial F(x)}{\partial x^\mu} \tag{1.28}$$

Let us now start form a generic four-dimensional action

$$S = \int_V d^4x \, \mathcal{L}(\phi^i, x), \qquad i = 1, \dots, N \tag{1.29}$$

and let us consider a generic variation of the fields and of the coordinates, $x'^\mu = x^\mu + \delta x^\mu$

$$\Delta \phi^i(x) = \tilde{\phi}^i(x') - \phi^i(x) \approx \delta \phi^i(x) + \delta x^\mu \frac{\partial \phi}{\partial x^\mu} \tag{1.30}$$

If the action is invariant under the transformation, then

$$\tilde{S}_{V'} = S_V \tag{1.31}$$

This gives rise to the conservation equation

$$\partial_\mu \left[\mathcal{L}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} \Delta \phi^i - \delta x^\nu \frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} \phi^i_{,\nu} \right] = 0 \tag{1.32}$$

This is the general result expressing the local conservation of the quantity in parenthesis. According to the choice one does for the variations δx^μ and $\Delta \phi^i$, and of the corresponding symmetries of the action, one gets different kind of conserved quantities.

Let us start with an action invariant under space and time translations. In the case we take $\delta x^\mu = a^\mu$ with a^μ independent on x e $\Delta \phi^i = 0$. From the general result in eq. (1.32) we get the following local conservation law

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} \phi^i_{,\nu} - \mathcal{L} g^\mu_\nu, \quad \partial_\mu T^\mu_\nu = 0 \tag{1.33}$$

$T_{\mu\nu}$ is called the energy-momentum tensor of the system. From its local conservation we get four constant of motion

$$P_\mu = \int d^3x T^0_\mu \tag{1.34}$$

P_μ is the four-momentum of the system. In the case of internal symmetries we take $\delta x^\mu = 0$. The conserved current will be

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} \Delta \phi^i = \frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} \delta \phi^i, \qquad \partial_\mu J^\mu = 0 \tag{1.35}$$

7

with an associated constant of motion given by

$$Q = \int d^3x J^0 \tag{1.36}$$

In general, if the system has more that one internal symmetry, we may have more that one conserved charge Q, that is we have a conserved charge for any Δ.

The last case we will consider is the invariance with respect to Lorentz transformations. Let us recall that they are defined as the transformations leaving invariant the norm of a four-vector

$$x^2 = x'^2 \tag{1.37}$$

For an infinitesimal transformation

$$x'_\mu = \Lambda_{\mu\nu}x^\nu \approx x_\mu + \epsilon_{\mu\nu}x^\nu \tag{1.38}$$

with

$$\epsilon_{\mu\nu} = -\epsilon_{\nu\mu} \tag{1.39}$$

We see that the number of independent parameters characterizing a Lorentz transformation is six. As well known, three of them correspond to spatial rotations, whereas the remaining three correspond to Lorentz boosts. In general, the relativistic fields are chosen to belong to a representation of the Lorentz group (for instance the Klein-Gordon field belongs to the scalar representation). This means that under a Lorentz transformation the components of the field mix together, as, for instance, a vector field does under rotations. Therefore, the transformation law of the fields ϕ^i under an infinitesimal Lorentz transformation can be written as

$$\Delta\phi^i = -\frac{1}{2}\Sigma^{ij}_{\mu\nu}\epsilon^{\mu\nu}\phi^j \tag{1.40}$$

where we have required that the transformation of the fields is of first order in the Lorentz parameters $\epsilon_{\mu\nu}$. The coefficients $\Sigma_{\mu\nu}$ (antisymmetric in the indices (μ, ν)) define a matrix in the indices (i, j) which can be shown to be the representative of the infinitesimal generators of the Lorentz group in the field representation. Using this equation and the expression for δx_μ we get the local conservation law

$$
\begin{aligned}
0 &= \partial_\mu \left[\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\phi^i_{,\nu} - \mathcal{L}g^\mu_\nu \right) \epsilon^{\nu\rho}x_\rho + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\Sigma^{ij}_{\nu\rho}\epsilon^{\nu\rho}\phi^j \right] \\
&= \frac{1}{2}\epsilon^{\nu\rho}\partial_\mu \left[\left(T^\mu_\nu x_\rho - T^\mu_\rho x_\nu \right) + \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\Sigma^{ij}_{\nu\rho}\phi^j \right]
\end{aligned}
\tag{1.41}
$$

and defining (watch at the change of sign)

$$\mathcal{M}^\mu_{\rho\nu} = x_\rho T^\mu_\nu - x_\nu T^\mu_\rho - \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\Sigma^{ij}_{\rho\nu}\phi^j \tag{1.42}$$

8

it follows the existence of six locally conserved currents (one for each Lorentz transformation)

$$\partial_\mu \mathcal{M}^\mu_{\nu\rho} = 0 \tag{1.43}$$

and consequently six constants of motion (notice that the lower indices are antisymmetric)

$$M_{\nu\rho} = \int d^3x \mathcal{M}^0_{\nu\rho} \tag{1.44}$$

Three of these constants (the ones with ν and ρ assuming spatial values) are nothing but the components of the angular momentum of the field.

1.4 Field Quantization

The quantization procedure for a field goes through the following steps:

- construction of the lagrangian density and determination of the canonical momentum density

$$\Pi_\alpha(x) = \frac{\partial \mathcal{L}}{\partial \phi_\alpha(x)} \tag{1.45}$$

where the index α describes both the spin and internal degrees of freedom

- quantization through the requirement of the equal time canonical commutation relations for a spin integer field or through canonical anticommutation relations for half-integer fields

$$[\phi_\alpha(\vec{x}, t), \Pi_\beta(\vec{y}, t)]_\pm = i\delta_{\alpha\beta}\delta^3(\vec{x} - \vec{y}) \tag{1.46}$$

$$[\phi_\alpha(\vec{x}, t), \phi_\beta(\vec{y}, t)]_\pm = 0, \quad [\Pi_\alpha(\vec{x}, t), \Pi_\beta(\vec{y}, t)]_\pm = 0 \tag{1.47}$$

In the case of a free field, or for an interacting field in the interaction picture (and therefore evolving with the free hamiltonian), we can add the following steps

- expansion of $\phi_\alpha(x)$ in terms of a complete set of solutions of the Klein-Gordon equation, allowing the definition of creation and annihilation operators;

- construction of the Fock space through the creation and annihilation operators.

In the following we will give the main results about the quantization of the scalar, the Dirac and the electromagnetic field.

1.4.1 The Real Scalar Field

The relativistic real scalar free field $\phi(x)$ obeys the Klein-Gordon equation

$$(\Box + m^2)\phi(x) = 0 \tag{1.48}$$

which can be derived from the variation of the action

$$S = \int d^4x \mathcal{L} \tag{1.49}$$

where $\mathcal{L}$ is the lagrangian density

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu \phi\, \partial^\mu \phi - m^2 \phi^2\right] \tag{1.50}$$

giving rise to the canonical momentum

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} \tag{1.51}$$

From this we get the equal time canonical commutation relations

$$[\phi(\vec{x},t),\dot{\phi}(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y}), \quad [\phi(\vec{x},t),\phi(\vec{y},t)] = [\dot{\phi}(\vec{x},t),\dot{\phi}(\vec{y},t)] = 0 \tag{1.52}$$

By using the box normalization, a complete set of solutions of the Klein-Gordon equation is given by

$$f_{\vec{k}}(x) = \frac{1}{\sqrt{V}}\frac{1}{\sqrt{2\omega_k}}e^{-ikx}, \quad f_{\vec{k}}^*(x) = \frac{1}{\sqrt{V}}\frac{1}{\sqrt{2\omega_k}}e^{ikx} \tag{1.53}$$

corresponding respectively to positive and negative energies, with

$$k_0 = \omega_k = \sqrt{|\vec{k}|^2 + m^2} \tag{1.54}$$

and

$$\vec{k} = \frac{2\pi}{L}\vec{n} \tag{1.55}$$

with

$$\vec{n} = n_1\vec{i}_1 + n_2\vec{i}_2 + n_3\vec{i}_3, \quad n_i \in \mathbb{Z} \tag{1.56}$$

and L being the side of the quantization box ($V = L^3$). For any two solutions f and g of the Klein-Gordon equation, the following quantity is a constant of motion and defines a scalar product (not positive definite)

$$\langle f|g\rangle = i\int d^3\vec{x}\, f^*\, \partial_t^{(-)} g \tag{1.57}$$

where

$$f^* \partial_t^{(-)} g = f^*\dot{g} - \dot{f}^* g \tag{1.58}$$

The solutions of eq. (1.53) are orthogonal with respect to this scalar product, for instance

$$\langle f_{\vec{k}} | f_{\vec{k}'} \rangle = \delta_{\vec{k}, \vec{k}'} \equiv \prod_{i=1}^{3} \delta_{n_i, n_i'} \tag{1.59}$$

We can go to the <u>continuum</u> normalization by the substitution

$$\frac{1}{\sqrt{V}} \to \frac{1}{\sqrt{(2\pi)^3}} \tag{1.60}$$

and sending the Krönecker delta function into the Dirac delta function. The expansion of the field in terms of these solutions is then (in the continuum)

$$\phi(x) = \int d^3\vec{k} [f_{\vec{k}}(x) a(\vec{k}) + f_{\vec{k}}^*(x) a^\dagger(\vec{k})] \tag{1.61}$$

or, more explicitly

$$\phi(x) = \int d^3\vec{k} \frac{1}{\sqrt{(2\pi)^3 2\omega_k}} [a(\vec{k}) e^{-ikx} + a^\dagger(\vec{k}) e^{ikx}] \tag{1.62}$$

using the orthogonality properties of the solutions one can invert this expression

$$a(\vec{k}) = i \int d^3\vec{x} f_{\vec{k}}^*(x) \partial_t^{(-)} \phi(x), \quad a^\dagger(\vec{k}) = i \int d^3\vec{x} \phi(x) \partial_t^{(-)} f_{\vec{k}}(x) \tag{1.63}$$

and obtain the commutation relations

$$[a(\vec{k}), a^\dagger(\vec{k}')] = \delta^3(\vec{k} - \vec{k}') \tag{1.64}$$

$$[a(\vec{k}), a(\vec{k}')] = [a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0 \tag{1.65}$$

From these equations one constructs the Fock space starting from the vacuum state which is defined by

$$a(\vec{k})|0\rangle = 0 \tag{1.66}$$

for any $\vec{k}$. The states

$$a^\dagger(\vec{k}_1) \cdots a^\dagger(\vec{k}_n)|0\rangle \tag{1.67}$$

describe n Bose particles of equal mass $m^2 = k_i^2$, and of total four-momentum $k = k_1 + \cdots k_n$. This follows by recalling the expression (1.33) for the energy-momentum tensor, that for the Klein-Gordon field is

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \left(\partial_\rho \phi \partial^\rho \phi - m^2 \phi^2 \right) g_{\mu\nu} \tag{1.68}$$

from which

$$P^0 = H = \int d^3x T^{00} = \frac{1}{2} \int d^3x \left(\Pi^2 + |\vec{\nabla}\phi|^2 + m^2\phi^2 \right) \tag{1.69}$$

11

$$P^i = \int d^3x T^{0i} = -\int d^3x \Pi \frac{\partial \phi}{\partial x^i}, \quad (\vec{P} = -\int d^3x \Pi \vec{\nabla} \phi) \tag{1.70}$$

and for the normal ordered operator P_μ we get (the normal ordering is defined by putting all the creation operators to the left of the annihilation operators)

$$: P_\mu := \int d^3\vec{k} \, k_\mu a^\dagger(\vec{k}) a(\vec{k}) \tag{1.71}$$

implying

$$: P_\mu : a^\dagger(\vec{k})|0\rangle = k_\mu a^\dagger(\vec{k})|0\rangle \tag{1.72}$$

The propagator for the scalar field is given by

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = -i\Delta_F(x-y) \tag{1.73}$$

where

$$\Delta_F(x) = -\int \frac{d^4k}{(2\pi)^4} e^{-ikx} \Delta_F(k) \tag{1.74}$$

and

$$\Delta_F(k) = \frac{1}{k^2 - m^2 + i\epsilon} \tag{1.75}$$

1.4.2   The Charged Scalar Field

The charged scalar field can be described in terms of two real scalar fields of equal mass

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{2} \left[(\partial_\mu \phi_i)(\partial^\mu \phi_i) - m^2 \phi_i^2 \right] \tag{1.76}$$

and we can write immediately the canonical commutation relations

$$[\phi_i(\vec{x},t), \dot{\phi}_j(\vec{y},t)] = i\delta_{ij}\delta^3(\vec{x}-\vec{y}) \tag{1.77}$$

$$[\phi_i(\vec{x},t), \phi_j(\vec{y},t)] = [\dot{\phi}_i(\vec{x},t), \dot{\phi}_j(\vec{y},t)] = 0 \tag{1.78}$$

The charged field is better understood in a complex basis

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \qquad \phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \tag{1.79}$$

The lagrangian density becomes

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \tag{1.80}$$

The theory is invariant under the phase transformation $\phi \to e^{i\alpha}\phi$, and therefore it admits the conserved current (see eq. (1.35))

$$j_\mu = i\left[(\partial_\mu \phi)\phi^\dagger - (\partial_\mu \phi^\dagger)\phi\right] \tag{1.81}$$

with the corresponding constant of motion

$$Q = i \int d^3x \; \phi^\dagger \partial_t^{(-)} \phi \qquad (1.82)$$

The commutation relations in the new basis are given by

$$[\phi(\vec{x}, t), \dot{\phi}^\dagger(\vec{y}, t)] = i\delta^3(\vec{x} - \vec{y}) \qquad (1.83)$$

and

$$[\phi(\vec{x}, t), \phi(\vec{y}, t)] = [\phi^\dagger(\vec{x}, t), \phi^\dagger(\vec{y}, t)] = 0$$
$$[\dot{\phi}(\vec{x}, t), \dot{\phi}(\vec{y}, t)] = [\dot{\phi}^\dagger(\vec{x}, t), \dot{\phi}^\dagger(\vec{y}, t)] = 0 \qquad (1.84)$$

Let us notice that these commutation relations could have also been obtained directly from the lagrangian (1.80), since

$$\Pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^\dagger, \qquad \Pi_{\phi^\dagger} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^\dagger} = \dot{\phi} \qquad (1.85)$$

Using the expansion for the real fields

$$\phi_i(x) = \int d^3k \left[f_{\vec{k}}(x) a_i(\vec{k}) + f_{\vec{k}}^*(x) a^\dagger_i(\vec{k}) \right] \qquad (1.86)$$

we get

$$\phi(x) = \int d^3k \left[f_{\vec{k}}(x) \frac{1}{\sqrt{2}} (a_1(\vec{k}) + ia_2(\vec{k})) + f_{\vec{k}}^*(x) \frac{1}{\sqrt{2}} (a^\dagger_1(\vec{k}) + ia^\dagger_2(\vec{k})) \right] \qquad (1.87)$$

Introducing the combinations

$$a(\vec{k}) = \frac{1}{\sqrt{2}} (a_1(\vec{k}) + ia_2(\vec{k})), \qquad b(\vec{k}) = \frac{1}{\sqrt{2}} (a_1(\vec{k}) - ia_2(\vec{k})) \qquad (1.88)$$

it follows

$$\phi(x) = \int d^3k \left[f_{\vec{k}}(x) a(\vec{k}) + f_{\vec{k}}^*(x) b^\dagger(\vec{k}) \right]$$
$$\phi^\dagger(x) = \int d^3k \left[f_{\vec{k}}(x) b(\vec{k}) + f_{\vec{k}}^*(x) a^\dagger(\vec{k}) \right] \qquad (1.89)$$

from which we can evaluate the commutation relations for the creation and annihilation operators in the complex basis

$$[a(\vec{k}), a^\dagger(\vec{k}')] = [b(\vec{k}), b^\dagger(\vec{k}')] = \delta^3(\vec{k} - \vec{k}') \qquad (1.90)$$

$$[a(\vec{k}), b(\vec{k}')] = [a(\vec{k}), b^\dagger(\vec{k}')] = 0 \qquad (1.91)$$

13

We get also

$$: P_\mu := \int d^3k\, k_\mu \sum_{i=1}^{2} a^\dagger_{\ i}(\vec{k}) a_i(\vec{k}) = \int d^3k\, k_\mu \left[a^\dagger(\vec{k}) a(\vec{k}) + b^\dagger(\vec{k}) b(\vec{k}) \right] \qquad (1.92)$$

Therefore the operators $a^\dagger(\vec{k})$ e $b^\dagger(\vec{k})$ both create particles states with momentum k, as the original operators $a^\dagger_{\ i}$. For the normal ordered charge operator we get

$$: Q := \int d^3k \left[a^\dagger(\vec{k}) a(\vec{k}) - b^\dagger(\vec{k}) b(\vec{k}) \right] \qquad (1.93)$$

showing explicitly that $a^\dagger$ and $b^\dagger$ create particles of charge $+1$ and -1 respectively.

The only non vanishing propagator for the scalar field is given by

$$\langle 0 | T(\phi(x)\phi^\dagger(y)) | 0 \rangle = -i\Delta_F(x - y) \qquad (1.94)$$

1.4.3 The Dirac Field

The Dirac field is described by the action

$$S = \int_V d^4x\ \bar{\psi}(i\hat{\partial} - m)\psi \qquad (1.95)$$

where we have used the following notations for four-vectors contracted with the γ matrices

$$\hat{v} \equiv v_\mu \gamma^\mu \qquad (1.96)$$

The canonical momenta result to be

$$\Pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^\dagger, \qquad \Pi_{\psi^\dagger} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger} = 0 \qquad (1.97)$$

The canonical momenta do not depend on the velocities. In principle, this creates a problem in going to the hamiltonian formalism. In fact a rigorous treatment requires an extension of the classical hamiltonian approach which was performed by Dirac himself. In this particular case, the result one gets is the same as proceeding in a naive way. For this reason we will avoid to describe this extension, and we will proceed as in the standard case. From the general expression for the energy momentum tensor (see eq. 1.33) we get

$$T^\mu_\nu = i\bar{\psi}\gamma^\mu \psi_{,\nu} - g^\mu_\nu (\bar{\psi}(i\hat{\partial} - m)\psi) \qquad (1.98)$$

and using the Dirac equation

$$T^\mu_\nu = i\bar{\psi}\gamma^\mu \psi_{,\nu} \qquad (1.99)$$

The momentum of the field is given by

$$P^k = \int d^3x\ T^{0k} \implies \vec{P} = -i \int d^3x\ \psi^\dagger \vec{\nabla} \psi \qquad (1.100)$$

14

and the hamiltonian

$$H = \int d^3x \, T^{00} \Longrightarrow H = i \int d^3x \, \psi^\dagger \partial_t \psi \qquad (1.101)$$

For a Lorentz transformation, the field variation is defined by

$$\Delta \phi^i = -\frac{1}{2} \Sigma_{\mu\nu}^{ij} \epsilon^{\mu\nu} \phi^j \qquad (1.102)$$

In the Dirac case one has

$$\Delta \psi(x) = \psi'(x') - \psi(x) = -\frac{i}{4} \sigma_{\mu\nu} \epsilon^{\mu\nu} \psi(x) \qquad (1.103)$$

from which

$$\Sigma_{\mu\nu} = \frac{i}{2} \sigma_{\mu\nu} = -\frac{1}{4} [\gamma_\mu, \gamma_\nu] \qquad (1.104)$$

Therefore the angular momentum density (see eq. (1.42)) is

$$\mathcal{M}_{\rho\nu}^\mu = i\bar{\psi}\gamma^\mu \left(x_\rho \partial_\nu - x_\nu \partial_\rho - \frac{i}{2}\sigma_{\rho\nu} \right) \psi = i\bar{\psi}\gamma^\mu \left(x_\rho \partial_\nu - x_\nu \partial_\rho + \frac{1}{4}[\gamma_\rho, \gamma_\nu] \right) \psi$$
$$(1.105)$$

By taking the spatial components we obtain

$$\vec{J} = (M^{23}, M^{31}, M^{12}) = \int d^3x \, \psi^\dagger \left(-i\vec{x} \wedge \vec{\nabla} + \frac{1}{2}\vec{\sigma} \otimes 1_2 \right) \psi \qquad (1.106)$$

where 1_2 is the identity matrix in 2 dimensions, and we have defined

$$\vec{\sigma} \otimes 1_2 = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \qquad (1.107)$$

The expression of $\vec{J}$ shows the decomposition of the total angular momentum in the orbital and in the spin part. The theory has a further conserved quantity, the current $\bar{\psi}\gamma^\mu\psi$ resulting from the phase invariance $\psi \to e^{i\alpha}\psi$.

The decomposition of the Dirac field in plane waves is obtained by using the spinors $u(p, \pm n)$ e $v(p, \pm n)$ solutions of the Dirac equation in momentum space for positive and negative energies respectively. We write

$$\psi(x) = \sum_{\pm n} \int \frac{d^3 p}{\sqrt{(2\pi)^3}} \sqrt{\frac{m}{E_p}} \left[b(p, n)u(p, n)e^{-ipx} + d^\dagger(p, n)v(p, n)e^{ipx} \right] \quad (1.108)$$

$$\psi^\dagger(x) = \sum_{\pm n} \int \frac{d^3 p}{\sqrt{(2\pi)^3}} \sqrt{\frac{m}{E_p}} \left[d(p, n)\bar{v}(p, n)e^{-ipx} + b^\dagger(p, n)\bar{u}(p, n)e^{ipx} \right] \gamma_0$$
$$(1.109)$$

where $E_p = \sqrt{|\vec{p}|^2 + m^2}$. We will collect here the various properties of the spinors. The four-vector n_μ is the one that identifies the direction of the spin quantization in the rest frame. For instance, by quantizing the spin along the third axis one takes n_μ in the rest frame as $n_R^\mu = (0, 0, 0, 1)$. Then n^μ is obtained by boosting from the rest frame to the frame where the particle has four-momentum p_μ.

15

- Dirac equation

$$
\begin{aligned}
(\hat{p} - m)u(p,n) &= \bar{u}(p,n)(\hat{p} - m) = 0 \\
(\hat{p} + m)v(p,n) &= \bar{v}(p,n)(\hat{p} + m) = 0
\end{aligned}
\tag{1.110}
$$

- Orthogonality

$$
\begin{aligned}
\bar{u}(p,n)u(p,n') &= -\bar{v}(p,n)v(p,n') = \delta_{nn'} \\
u^{\dagger}(p,n)u(p,n') &= v^{\dagger}(p,n)v(p,n') = \frac{E_p}{m}\delta_{nn'} \\
\bar{v}(p,n)u(p,n') &= v^{\dagger}(p,n)u(\tilde{p},n') = 0
\end{aligned}
\tag{1.111}
$$

where, if $p^{\mu} = (E_p, \vec{p})$, then $\tilde{p}^{\mu} = (E_p, -\vec{p})$.

- Completeness

$$
\begin{aligned}
\sum_{\pm n} u(p,n)\bar{u}(p,n) &= \frac{\hat{p} + m}{2m} \\
\sum_{\pm n} v(p,n)\bar{v}(p,n) &= \frac{\hat{p} - m}{2m}
\end{aligned}
\tag{1.112}
$$

The Dirac field is quantized by canonical anticommutation relations in order to satisfy the Pauli principle

$$
\left[\psi(\vec{x},t), \psi(\vec{y},t)\right]_{+} = \left[\psi^{\dagger}(\vec{x},t), \psi^{\dagger}(\vec{y},t)\right]_{+} = 0
\tag{1.113}
$$

and

$$
\left[\Pi_{\psi}(\vec{x},t), \psi(\vec{y},t)\right]_{+} = i\delta^{3}(\vec{x} - \vec{y})
\tag{1.114}
$$

or

$$
\left[\psi(\vec{x},t), \psi^{\dagger}(\vec{y},t)\right]_{+} = \delta^{3}(\vec{x} - \vec{y})
\tag{1.115}
$$

The normal ordered four-momentum is given by

$$
: P_{\mu} := \sum_{\pm n} \int d^{3}p \, p_{\mu}[b^{\dagger}(p,n)b(p,n) + d^{\dagger}(p,n)d(p,n)]
\tag{1.116}
$$

If we couple the Dirac field to the electromagnetism through the minimal substitution we find that the free action (1.95) becomes

$$
S = \int_{V} d^{4}x\bar{\psi}(i\hat{\partial} - e\hat{A} - m)\psi
\tag{1.117}
$$

Therefore the electromagnetic field is coupled to the conserved current

$$
j^{\mu} = e\bar{\psi}\gamma^{\mu}\psi
\tag{1.118}
$$

This forces us to say that the integral of the fourth component of the current should be the charge operator. In fact, we find

$$
\begin{aligned}
: Q : &= e \int d^3x \ \psi^\dagger \psi \\
&= \sum_{\pm n} \int d^3p \ e \left[b^\dagger(p,n)b(p,n) - d^\dagger(p,n)d(p,n) \right]
\end{aligned}
\tag{1.119}
$$

The expressions for the momentum angular momentum and charge operators show that the operators $b^\dagger(\vec{p})$ and $d^\dagger(\vec{p})$ create out of the vacuum particles of spin $1/2$, four-momentum p_μ and charge e and $-e$ respectively.

The propagator for the Dirac field is given by

$$
\langle 0 | T(\psi_\alpha(x)\bar{\psi}_\beta(y)) | 0 \rangle = i S_F(x-y)_{\alpha\beta}
\tag{1.120}
$$

where

$$
S_F(x) = -(i\hat{\partial} + m)\Delta_F(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} S_F(k)
\tag{1.121}
$$

and

$$
S_F(k) = \frac{\hat{k}+m}{k^2-m^2+i\epsilon} = \frac{1}{\hat{k}-m+i\epsilon}
\tag{1.122}
$$

1.4.4 The Electromagnetic Field

The lagrangian density for the free electromagnetic field, expressed in terms of the four-vector potential is

$$
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}
\tag{1.123}
$$

where

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu
\tag{1.124}
$$

and

$$
\vec{E} = -\vec{\nabla}A^0 - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \wedge \vec{A}
\tag{1.125}
$$

from which

$$
\vec{E} = (F^{10}, F^{20}, F^{30}), \quad \vec{B} = (-F^{23}, -F^{31}, -F^{12})
\tag{1.126}
$$

The resulting equations of motion are

$$
\Box A_\mu - \partial_\mu(\partial^\nu A_\nu) = 0
\tag{1.127}
$$

The potentials are defined up to a gauge transformation

$$
A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x)
\tag{1.128}
$$

In fact, A_μ and A'_μ satisfy the same equations of motion and give rise to the same electromagnetic field. It is possible to use the gauge invariance to require some

17

particular condition on the field A_μ. For instance, we can perform a gauge transformation in such a way that the transformed field satisfies

$$\partial_\mu A^\mu = 0 \tag{1.129}$$

This is called the Lorentz gauge. Or we can choose a gauge such that

$$A_0 = \vec{\nabla} \cdot \vec{A} = 0 \tag{1.130}$$

This is called the Coulomb gauge. Since in this gauge all the gauge freedom is completely fixed (in contrast with the Lorentz gauge), it is easy to count the independent degrees of freedom of the electromagnetic field. From the equations (1.130) we see that A_μ has only two degrees of freedom. Another way of showing that there are only two independent degrees of freedom is through the equations of motion. Let us consider the four dimensional Fourier transform of $A_\mu(x)$

$$A_\mu(x) = \int d^4k \; e^{ikx} A_\mu(k) \tag{1.131}$$

Substituting this expression in the equations of motion we get

$$-k^2 A_\mu(k) + k_\mu(k^\nu A_\nu(k)) = 0 \tag{1.132}$$

Let us now decompose $A_\mu(k)$ in terms of four independent four vectors, which can be chosen as $k^\mu = (E, \vec{k})$, $\tilde{k}^\mu = (E, -\vec{k})$, and two further four vectors $e_\mu^\lambda(k)$, $\lambda = 1, 2$, orthogonal to k^μ

$$k^\mu e_\mu^\lambda = 0, \qquad \lambda = 1, 2 \tag{1.133}$$

The decomposition of $A_\mu(k)$ reads

$$A_\mu(k) = a_\lambda(k)e_\mu^\lambda + b(k)k_\mu + c(k)\tilde{k}_\mu \tag{1.134}$$

From the equations of motion we get

$$-k^2(a_\lambda e_\mu^\lambda + bk_\mu + c\tilde{k}_\mu) + k_\mu(bk^2 + c(k \cdot \tilde{k})) = 0 \tag{1.135}$$

The term in $b(k)$ cancels, therefore it is left undetermined by the equations of motion. For the other quantities we have

$$k^2 a_\lambda(k) = c(k) = 0 \tag{1.136}$$

The arbitrariness of $b(k)$ is a consequence of the gauge invariance. In fact if we gauge transform $A_\mu(x)$

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x) \tag{1.137}$$

then

$$A_\mu(k) \to A_\mu(k) + ik_\mu \Lambda(k) \tag{1.138}$$

where

$$\Lambda(x) = \int d^4k \; e^{ikx} \Lambda(k) \tag{1.139}$$

Since the gauge transformation amounts to a translation in $b(k)$ by an arbitrary function of k, we can always to choose it equal to zero. Therefore we are left with the two degrees of freedom described by the amplitudes $a_\lambda(k)$, $\lambda = 1, 2$. Furthermore these amplitudes are different from zero only if the dispersion relation $k^2 = 0$ is satisfied. This shows that the corresponding quanta will have zero mass. With the choice $b(k) = 0$, the field $A_\mu(k)$ becomes

$$A_\mu(k) = a_\lambda(k)e_\mu^\lambda(k) \tag{1.140}$$

showing that $k^\mu A_\mu(k) = 0$. Therefore the choice $b(k) = 0$ is equivalent to the choice of the Lorentz gauge.

Let us consider now the quantization of this theory. Due to the lack of manifest covariance of the Coulomb gauge we will discuss here the quantization in the Lorentz gauge, where however we will encounter other problems. If we want to maintain the explicit covariance of the theory we have to require non trivial commutation relations for all the component of the field. That is

$$[A_\mu(\vec{x}, t), \Pi^\nu(\vec{y}, t)] = ig_\mu^\nu \delta^3(\vec{x} - \vec{y}) \tag{1.141}$$

$$[A_\mu(\vec{x}, t), A_\nu(\vec{y}, t)] = [\Pi^\mu(\vec{x}, t), \Pi^\nu(\vec{y}, t)] = 0 \tag{1.142}$$

with

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} \tag{1.143}$$

To evaluate the conjugated momenta is better to write the lagrangian density (see eq. (1.123) in the following form

$$\mathcal{L} = -\frac{1}{4}[A_{\mu,\nu} - A_{\nu,\mu}][A^{\mu,\nu} - A^{\nu,\mu}] = -\frac{1}{2}A_{\mu,\nu}A^{\mu,\nu} + \frac{1}{2}A_{\mu,\nu}A^{\nu,\mu} \tag{1.144}$$

Therefore

$$\frac{\partial \mathcal{L}}{\partial A_{\mu,\nu}} = -A^{\mu,\nu} + A^{\nu,\mu} = F^{\mu\nu} \tag{1.145}$$

implying

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0} \tag{1.146}$$

It follows

$$\Pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0 \tag{1.147}$$

We see that it is impossible to satisfy the condition

$$[A_0(\vec{x}, t), \Pi^0(\vec{y}, t)] = i\delta^3(\vec{x} - \vec{y}) \tag{1.148}$$

19

We can try to find a solution to this problem modifying the lagrangian density in such a way that $\Pi^0 \neq 0$. But doing so we will not recover the Maxwell equation. However we can take advantage of the gauge symmetry, modifying the lagrangian density in such a way to recover the equations of motion in a particular gauge. For instance, in the Lorentz gauge we have

$$\Box A_\mu(x) = 0 \tag{1.149}$$

and this equation can be obtained by the lagrangian density

$$\mathcal{L} = -\frac{1}{2} A_{\mu,\nu} A^{\mu,\nu} \tag{1.150}$$

(just think to the Klein-Gordon case). The minus sign is necessary to recover a positive hamiltonian density. We now express this lagrangian density in terms of the gauge invariant one, given in eq. (1.123). To this end we observe that the difference between the two lagrangian densities is nothing but the second term of eq. (1.144)

$$
\begin{aligned}
\frac{1}{2} A_{\mu,\nu} A^{\nu,\mu} &= \partial^\mu \left[\frac{1}{2} A_{\mu,\nu} A^\nu \right] - \frac{1}{2} (\partial^\mu A_{\mu,\nu}) A^\nu \\
&= \partial^\mu \left[\frac{1}{2} A_{\mu,\nu} A^\nu \right] - \partial^\nu \left[\frac{1}{2} (\partial^\mu A_\mu) A_\nu \right] + \frac{1}{2} (\partial^\mu A_\mu)^2 \tag{1.151}
\end{aligned}
$$

Then, up to a four divergence, we can write the new lagrangian density in the form

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial^\mu A_\mu)^2 \tag{1.152}$$

One can check that this form gives the correct equations of motion. In fact from

$$\frac{\partial \mathcal{L}}{\partial A_{\mu,\nu}} = -A^{\mu,\nu} + A^{\nu,\mu} - g^{\mu\nu} (\partial^\lambda A_\lambda), \qquad \frac{\partial \mathcal{L}}{\partial A_\mu} = 0 \tag{1.153}$$

we get

$$0 = -\Box A^\mu + \partial^\mu (\partial^\nu A_\nu) - \partial^\mu (\partial^\lambda A_\lambda) = -\Box A^\mu \tag{1.154}$$

the term

$$-\frac{1}{2} (\partial^\mu A_\mu)^2 \tag{1.155}$$

which is not gauge invariant, is called the **gauge fixing term**. More generally, we could add to the original lagrangian density a term of the form

$$-\frac{\lambda}{2} (\partial^\mu A_\mu)^2 \tag{1.156}$$

The corresponding equations of motion would be

$$\Box A_\mu - (1 - \lambda) \partial^\mu (\partial^\lambda A_\lambda) = 0 \tag{1.157}$$

In the following we will use $\lambda = 1$. From eq. (1.153) we see that

$$\Pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = -\partial^\mu A_\mu \qquad (1.158)$$

In the Lorentz gauge we find again $\Pi^0 = 0$. To avoid the corresponding problem we can ask that $\partial^\mu A_\mu = 0$ does not hold as an operator condition, but rather as a condition upon the physical states

$$\langle \text{phys}|\partial^\mu A_\mu|\text{phys}\rangle = 0 \qquad (1.159)$$

The price to pay to quantize the theory in a covariant way is to work in a Hilbert space much bigger than the physical one. The physical states span a subspace which is defined by the previous relation. A further bonus is that in this way one has to do with local commutation relations. On the contrary, in the Coulomb gauge, one needs to introduce non local commutation relations for the canonical variables. We will come back later to the condition (1.159).

Since we don't have to worry any more about the operator condition $\Pi^0 = 0$, we can proceed with our program of canonical quantization. The canonical momentum densities are

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0} - g^{\mu 0}(\partial^\lambda A_\lambda) \qquad (1.160)$$

or, explicitly

$$\begin{aligned} \Pi^0 &= -\partial^\lambda A_\lambda = -\dot{A}_0 - \vec{\nabla} \cdot \vec{A} \\ \Pi^i &= \partial^i A_0 - \partial^0 A^i = -\dot{A}^i + \partial^i A^0 \end{aligned} \qquad (1.161)$$

Since the spatial gradient of the field commutes with the field itself at equal time, the canonical commutator (1.141) gives rise to

$$[A_\mu(\vec{x}, t), \dot{A}_\nu(\vec{y}, t)] = -i g_{\mu\nu} \delta^3(\vec{x} - \vec{y}) \qquad (1.162)$$

To get the quanta of the field we look for plane wave solutions of the wave equation. We need four independent four vectors in order to expand the solutions in the momentum space. In a given frame, let us consider the unit four vector which defines the time axis. This must be a time-like vector, $n^2 = 1$, and we will choose $n^0 > 0$. For instance, $n^\mu = (1, 0, 0, 0)$. Then we take two four vectors $\epsilon_\mu^{(\lambda)}$, $\lambda = 1, 2$,in the plane orthogonal to n^μ and k^μ. Notice that now $k^2 = 0$,since we are considering solutions of the wave equation. Therefore

$$k^\mu \epsilon_\mu^{(\lambda)} = n^\mu \epsilon_\mu^{(\lambda)} = 0, \qquad \lambda = 1, 2 \qquad (1.163)$$

The four vectors $\epsilon_\mu^{(\lambda)}$, being orthogonal to n^μ are space-like, then they will be chosen orthogonal and normalized in the following way

$$\epsilon_\mu^{(\lambda)} \epsilon^{(\lambda')\mu} = -\delta_{\lambda\lambda'} \qquad (1.164)$$

21

Next, we define a unit space-like four vector, orthogonal to n^μ and lying in the plane (k, n)

$$n^\mu \epsilon^{(3)}{}_\mu = 0 \qquad (1.165)$$

with

$$\epsilon^{(3)}_\mu \epsilon^{(3)\,\mu} = -1 \qquad (1.166)$$

By construction $\epsilon^{(3)}_\mu$ is orthogonal to $\epsilon^{(\lambda)}_\mu$. This four vector is completely fixed by the previous conditions, and we get

$$\epsilon^{(3)}_\mu = \frac{k_\mu - (n \cdot k) n_\mu}{(n \cdot k)} \qquad (1.167)$$

As a last unit four vector we choose n^μ

$$\epsilon^{(0)}_\mu = n_\mu \qquad (1.168)$$

These four vectors are orthonormal

$$\epsilon^{(\lambda)}_\mu \epsilon^{(\lambda')\,\mu} = g^{\lambda\lambda'} \qquad (1.169)$$

and linearly independent. Then they satisfy the completeness relation

$$\epsilon^{(\lambda)}_\mu \epsilon^{(\lambda')}_\nu g_{\lambda\lambda'} = g_{\mu\nu} \qquad (1.170)$$

In the frame where $n^\mu = (1, \vec{0})$ and $k^\mu = (k, 0, 0, k)$, we have

$$\epsilon^{(1)\,\mu} = (0, 1, 0, 0), \qquad \epsilon^{(2)\,\mu} = (0, 0, 1, 0), \qquad \epsilon^{(3)\,\mu} = (0, 0, 0, 1) \qquad (1.171)$$

The plane wave expansion of A_μ is

$$A_\mu(x) = \int \frac{d^3 k}{\sqrt{2\omega_k (2\pi)^3}} \sum_{\lambda=0}^{3} \epsilon^{(\lambda)}_\mu(k) \left[a_\lambda(k) e^{-ikx} + a^\dagger_\lambda(k) e^{ikx} \right] \qquad (1.172)$$

where we have included the hermiticity condition for $A_\mu(x)$. For any fixed μ, this expansion is the same as the one that we wrote for the Klein-Gordon field, with the substitution $\epsilon^{(\lambda)}_\mu a_\lambda(k) \to a(k)$. Then, from eq. (1.63)

$$\epsilon^{(\lambda)}_\mu(k) a_\lambda(k) = i \int d^3 x \; f^*_{\vec{k}}(x) \partial^{(-)}_t A_\mu(x) \qquad (1.173)$$

with the functions $f_{\vec{k}}(x)$ defined as in eq. (1.53). Using the orthogonality of the $\epsilon^{(\lambda')\,\mu}$'s we find

$$a_\lambda(k) = i g_{\lambda\lambda'} \int d^3 x \; \epsilon^{(\lambda')\,\mu}(k) f^*_{\vec{k}}(x) \partial^{(-)}_t A_\mu(x) \qquad (1.174)$$

and analogously

$$a^\dagger_\lambda(k) = i g_{\lambda\lambda'} \int d^3 x \; \epsilon^{(\lambda')\,\mu}(k) A_\mu(x) \partial^{(-)}_t f_{\vec{k}}(x) \qquad (1.175)$$

From these expressions we can evaluate the commutator

$$[a_\lambda(k), a_{\lambda'}^\dagger(k')] =$$

$$= \int d^3x \, d^3y \Big[- f_{\vec{k}}^*(x) \dot{f}_{\vec{k}'}(y) \Big(i g_{\mu\nu} g_{\lambda\lambda''} \epsilon^{(\lambda'')\mu}(k) \epsilon^{(\lambda''')\nu}(k') g_{\lambda'\lambda'''} \delta^3(\vec{x} - \vec{y}) \Big)$$

$$- \dot{f}_{\vec{k}}^*(x) f_{\vec{k}'}(y) \Big(-i g_{\mu\nu} g_{\lambda\lambda''} \epsilon^{(\lambda'')\mu}(k) \epsilon^{(\lambda''')\nu}(k') g_{\lambda'\lambda'''} \delta^3(\vec{x} - \vec{y}) \Big) \Big]$$

$$= - \int d^3x \, f_{\vec{k}}^*(x) i\partial_t^{(-)} f_{\vec{k}'}(x) g_{\lambda\lambda'} \tag{1.176}$$

and using the orthogonality relations for the functions $f)\vec{k}$

$$[a_\lambda(k), a_{\lambda'}^\dagger(k')] = -g_{\lambda\lambda'} \delta^3(\vec{k} - \vec{k}') \tag{1.177}$$

Analogously

$$[a_\lambda(k), a_{\lambda'}(k')] = [a_\lambda^\dagger(k), a_{\lambda'}^\dagger(k')] = 0 \tag{1.178}$$

The commutation rules we have derived for the operators $a_\lambda(k)$ create some problem. Let us consider a one-particle state

$$|1, \lambda\rangle = \int d^3k \, f(k) a_\lambda^\dagger(k)|0\rangle \tag{1.179}$$

its norm is given by

$$\begin{aligned}
\langle 1, \lambda|1, \lambda\rangle &= \int d^3k \, d^3k' \, f^\star(k) f(k') \langle 0|a_\lambda(k) a_\lambda^\dagger(k')|0\rangle \\
&= \int d^3k \, d^3k' \, f^\star(k) f(k') \langle 0|[a_\lambda(k), a_\lambda^\dagger(k')|0\rangle \\
&= -g_{\lambda\lambda} \int d^3k \, |f(k)|^2 \tag{1.180}
\end{aligned}$$

Therefore the states with $\lambda = 0$ have negative norm. This problem does not come out completely unexpected. In fact, our expectation is that only the transverse states ($\lambda = 1, 2$), are physical states. For the moment being we have ignored the gauge fixing condition $\langle \text{phys}|\partial^\mu A_\mu|\text{phys}\rangle = 0$, but its meaning is that only part of the total Hilbert space is physical. Therefore the relevant thing is to show that the states satisfying the Lorentz condition have positive norm. To discuss the gauge fixing condition, let us notice that formulated in the way we did, being bilinear in the states, it could destroy the linearity of the Hilbert space. So we will try to modify the condition in a linear one

$$\partial^\mu A_\mu|\text{phys.}\rangle = 0 \tag{1.181}$$

But this would be a too strong requirement. Not even the vacuum state satisfies it. However, if we consider the positive and negative frequency parts of the field

$$A_\mu^{(+)}(x) = \int \frac{d^3k}{\sqrt{2\omega_k(2\pi)^3}} \sum_{\lambda=0}^3 \epsilon_\mu^{(\lambda)}(k) a_\lambda(k) e^{-ikx}, \qquad A_\mu^{(-)}(x) = (A^{(+)}(x))^\dagger \tag{1.182}$$

23

it is possible to weaken the condition, and require

$$\partial^\mu A_\mu^{(+)}(x)|\text{phys.}\rangle = 0 \qquad (1.183)$$

This allows us to satisfy automatically the original requirement

$$\langle\text{phys.}|(\partial^\mu A_\mu^{(+)} + \partial^\mu A_\mu^{(-)})|\text{phys.}\rangle = 0 \qquad (1.184)$$

To make this condition more explicit let us evaluate the four divergence of $A_\mu^{(+)}$

$$i\partial^\mu A_\mu^{(+)}(x) = \int \frac{d^3k}{\sqrt{2\omega_k(2\pi)^3}} e^{-ikx} \sum_{\lambda=0,3} k^\mu \epsilon_\mu^{(\lambda)}(k) a_\lambda(k) \qquad (1.185)$$

Using eq. (1.167), we get

$$k^\mu \epsilon_\mu^{(3)} = -(n \cdot k), \qquad k^\mu \epsilon_\mu^{(0)} = (n \cdot k) \qquad (1.186)$$

from which

$$[a_0(k) - a_3(k)]|\text{phys.}\rangle = 0 \qquad (1.187)$$

Notice that

$$[a_0(k) - a_3(k), a_0^\dagger(k') - a_3^\dagger(k')] = -\delta^3(\vec{k} - \vec{k'}) + \delta^3(\vec{k} - \vec{k'}) = 0 \qquad (1.188)$$

Let us denote by $\Phi_{\vec{k}}(n_0, n_3)$ the state with n_0 scalar photons (that is with polarization $\epsilon_\mu^{(0)}(k)$), and with n_3 longitudinal photons (that is with polarization $\epsilon_\mu^{(3)}(k)$). Then the following states satisfy the condition (1.183)

$$\Phi_{\vec{k}}^{(m)} = \frac{1}{m!}(a_0^\dagger(k) - a_3^\dagger(k))^m \Phi_{\vec{k}}(0,0) \qquad (1.189)$$

These states have vanishing norm

$$||\Phi_{\vec{k}}^{(m)}||^2 = 0 \qquad (1.190)$$

More generally we can make the following observation. Let us consider the number operator for scalar and longitudinal photons

$$N = \int d^3k \ (a_3^\dagger(k)a_3(k) - a_0^\dagger(k)a_0(k)) \qquad (1.191)$$

Notice the minus sign that is a consequence of the commutation relations, and it ensures that N has positive eigenvalues. For instance

$$Na_0^\dagger(k)|0\rangle = -\int d^3k' \ a_0^\dagger(k')[a_0(k'), a_0^\dagger(k)]|0\rangle = a_0^\dagger(k)|0\rangle \qquad (1.192)$$

24

Let us consider a physical state with a total number n of scalar and longitudinal photons. Then

$$\langle \varphi_n | N | \varphi_n \rangle = 0 \tag{1.193}$$

since a_0 and a_3 act in the same way on a physical state (see eq. (1.187)). It follows

$$n \langle \varphi_n | \varphi_n \rangle = 0 \tag{1.194}$$

Therefore all the physical states with a total definite number of scalar and longitudinal photons have zero norm, except for the vacuum state ($n = 0$). Then

$$\langle \varphi_n | \varphi_n \rangle = \delta_{n,0} \tag{1.195}$$

A generic physical state with zero transverse photons is a linear superposition of the previous states

$$|\varphi\rangle = c_0 |\varphi_0\rangle + \sum_{i \neq 0} c_i |\varphi_i\rangle \tag{1.196}$$

This state has a positive definite norm

$$\langle \varphi | \varphi \rangle = |c_0|^2 \geq 0 \tag{1.197}$$

The proof that a physical state has a positive norm can be extended to the case in which also transverse photons are present. Of course, the coefficients c_i, appearing in the expression of a physical state, are completely arbitrary, but this is not going to modify the values of the observables. For instance, consider the hamiltonian, we have

$$
\begin{aligned}
H &= \int d^3x \; : [\Pi^\mu \dot{A}_\mu - \mathcal{L}] : \\
&= \int d^3x \; : \left[F^{\mu 0} \dot{A}_\mu - (\partial^\lambda A_\lambda) \dot{A}_0 + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2}(\partial^\lambda A_\lambda)^2 \right] : \quad (1.198)
\end{aligned}
$$

One can easily show the hamiltonian is given by the sum of all the degrees of freedom appearing in A_μ

$$
\begin{aligned}
H &= \frac{1}{2} \int d^3x \; : \left[\sum_{i=1}^{3} \left(\dot{A}_i^2 + (\vec{\nabla} A_i)^2 \right) - \dot{A}_0^2 - \vec{\nabla} A_0^2 \right] : \\
&= \int d^3k \; \omega_k : \left[\sum_{\lambda=1}^{3} a_\lambda^\dagger(k) a_\lambda(k) - a_0^\dagger(k) a_0(k) \right] : \quad (1.199)
\end{aligned}
$$

Since on the physical states a_0 and a_3 act in the same way, we get

$$\langle \text{phys.} | H | \text{phys.} \rangle = \langle \text{phys.} | \int d^3k \; \omega_k \sum_{\lambda=1}^{2} a_\lambda^\dagger(k) a_\lambda(k) | \text{phys.} \rangle \tag{1.200}$$

25

The generic physical state is of the form $|\varphi_T\rangle \otimes |\varphi\rangle$. with $|\varphi\rangle$ defined as in eq. (1.196). Since only $|\varphi_T\rangle$, contributes to the evaluation of an observable quantity , we can always choose $|\varphi\rangle$ proportional to $|\varphi_0\rangle$. However, this does not mean that we are always working in the restricted physical space, because in a sum over the intermediate states we need to include all the degrees of freedom. This is crucial for the explicit covariance and locality of the theory.

The arbitrariness in defining the state $|\varphi\rangle$ has, in fact, a very simple interpretation. It corresponds to add to A_μ a four gradient, that is it corresponds to perform a gauge transformation. Consider the following matrix element

$$\langle\varphi|A_\mu(x)|\varphi\rangle = \sum_{n,m} c_n^\star c_m \langle\varphi_n|A_\mu(x)|\varphi_m\rangle \tag{1.201}$$

Since A_μ change the occupation number by one unit and all the states $|\varphi_n\rangle$ have zero norm (except for the state with $n = 0$), the only non vanishing contributions come from $n = 0$, $m = 1$ and $n = 1$, $m = 0$

$$\langle\varphi|A_\mu(x)|\varphi\rangle = c_0^\star c_1 \langle 0| \int \frac{d^3k}{\sqrt{2\omega_k(2\pi)^3}} \ e^{-ikx} [\epsilon_\mu^{(3)}(k)a_3(k) + \epsilon_\mu^{(0)}(k)a_0(k)]|\varphi_1\rangle + \text{c.c.} \tag{1.202}$$

In order to satisfy the gauge condition the state $|\varphi_1\rangle$ must be of the form

$$|\varphi_1\rangle = \int \ d^3q \ f(\vec{q})[a_3^\dagger(q) - a_0^\dagger(q)]|0\rangle \tag{1.203}$$

and therefore

$$\langle\varphi|A_\mu(x)|\varphi\rangle = \int \frac{d^3k}{\sqrt{2\omega_k(2\pi)^3}} [\epsilon_\mu^{(3)}(k) + \epsilon_\mu^{(0)}(k)][c_0^\star c_1 e^{-ikx} f(\vec{k}) + \text{c.c.}] \tag{1.204}$$

From eqs. (1.167) and (1.168) we have

$$\epsilon_\mu^{(3)} + \epsilon_\mu^{(0)} = \frac{k_\mu}{(k \cdot n)} \tag{1.205}$$

from which

$$\langle\varphi|A_\mu(x)|\varphi\rangle = \partial_\mu\Lambda(x) \tag{1.206}$$

with

$$\Lambda(x) = \int \frac{d^3k}{\sqrt{2\omega_k(2\pi)^3}} \frac{1}{n \cdot k} (ic_0^\star c_1 e^{-ikx} f(\vec{k}) + \text{c.c.}) \tag{1.207}$$

It is important to notice that this gauge transformation leaves A_μ in the Lorentz gauge, since

$$\Box\Lambda = 0 \tag{1.208}$$

26

because the momentum k inside the integral satisfies $k^2 = 0$. From the expansion in eq. (1.172) one gets immediately the expression for the photon propagator

$$\langle 0|T(A_\mu(x)A_\nu(y)|0\rangle = -ig_{\mu\nu}\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{k^2+i\epsilon} \tag{1.209}$$

By defining

$$D(x) = \int \frac{d^4k}{(2\pi)^4}e^{-ikx}D(k) \tag{1.210}$$

and

$$D(k) = -\frac{1}{k^2+i\epsilon} \tag{1.211}$$

we get

$$\langle 0|T(A_\mu(x)A_\nu(y)|0\rangle = ig_{\mu\nu}D(x-y) \tag{1.212}$$

1.5 Perturbation Theory

1.5.1 The Scattering Matrix

In order to describe a scattering process we will assign to the vector of state a condition at $t = -\infty$

$$|\Phi(-\infty)\rangle \equiv |\Phi_i\rangle \tag{1.213}$$

where the state Φ_i will be specified by assigning the set of incoming free particles in terms of eigenstates of momentum, spin and so on. For instance, in QED we will have to specify how many electrons, positrons and photons are in the initial state and we will have to specify their momenta, the spin projection of fermions and the polarization of the photons. The equations of motion will tell us how this state evolves with time and it will be possible to evaluate the state at $t = +\infty$, where, ideally, we will detect the final states. In practice the preparation and the detection processes are made at some finite times. It follows that our ideal description will be correct only if these times are much bigger than the typical interaction time of the scattering process. Once we know $\Phi(+\infty)$, we are interested to evaluate the probability amplitude of detecting at $t = +\infty$ a given set of free particles specified by a vector state Φ_f. The amplitude is

$$S_{fi} = \langle \Phi_f|\Phi(+\infty)\rangle \tag{1.214}$$

We will define the S matrix as the operator that give us $|\Phi(+\infty)\rangle$ once we know $|\Phi(-\infty)\rangle$

$$|\Phi(+\infty)\rangle = S|\Phi(-\infty)\rangle \tag{1.215}$$

The amplitude S_{fi} is then

$$S_{fi} = \langle \Phi_f|S|\Phi_i\rangle \tag{1.216}$$

Therefore S_{fi} is the S matrix element between free states ($|\Phi_i\rangle$ and $\langle\Phi_f|$ are called in and out states respectively). In the interaction representation defined by

$$|\Phi(t)\rangle = e^{iH_S^0 t}|\Phi_S(t)\rangle, \quad O(t) = e^{iH_S^0 t}O_S(t)e^{-iH_S^0 t} \tag{1.217}$$

where the index S identifies the Schrödinger representation and H_S^0 is the free hamiltonian, the state vectors satisfy the Schrödinger equation with the interaction hamiltonian (evaluated in the interaction picture)

$$i\frac{\partial}{\partial t}|\Phi(t)\rangle = H_I|\Phi(t)\rangle \tag{1.218}$$

whereas the operators evolve with the free hamiltonian. To evaluate the S matrix we first transform the Schrödinger equation in the interaction representation in an integral equation

$$|\Phi(t)\rangle = |\Phi(-\infty)\rangle - i\int_{-\infty}^{t} dt_1\ H^I(t_1)|\Phi(t_1)\rangle \tag{1.219}$$

One can verify that this is indeed a solution, and that it satisfies explicitly the boundary condition at $t = -\infty$. The perturbative expansion consists in evaluating $|\Phi(t)\rangle$ by iterating this integral equation. The result in terms of ordered T-products is

$$S = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} dt_1 \cdots \int_{-\infty}^{+\infty} dt_n\ T\left(H^I(t_1)\cdots H^I(t_n)\right) \tag{1.220}$$

The T-product of n terms means that the factors have to be written from left to right with decreasing times. For instance, if $t_1 \geq t_2 \geq \cdots \geq t_n$, then

$$T\left(H^I(t_1)\cdots H^I(t_n)\right) = H^I(t_1)\cdots H^I(t_n) \tag{1.221}$$

This result can be written in a more compact form by introducing the T-ordered exponential

$$S = T\left(e^{-i\int_{-\infty}^{+\infty} dt\ H^I(t)}\right) \tag{1.222}$$

This expression is a symbolic one and it is really defined by its series expansion. The motivation for introducing the T-ordered exponential is that it satisfies the following factorization property

$$T\left(e^{\int_{t_1}^{t_3} O(t)dt}\right) = T\left(e^{\int_{t_2}^{t_3} O(t)dt}\right)T\left(e^{\int_{t_1}^{t_2} O(t)dt}\right) \tag{1.223}$$

If there are no derivative interactions we have

$$S = T\left(e^{-i\int_{-\infty}^{+\infty} dt\ H^I(t)}\right) = T\left(e^{+i\int d^4x\ \mathcal{L}_{\text{int}}}\right) \tag{1.224}$$

It follows that if the theory is Lorentz invariant, also the S matrix enjoys the same property.

1.5.2 Wick's theorem

The matrix elements of the S matrix between free particle states can be expressed as vacuum expectation values (VEV's) of T-products. These $VEV's$ satisfy an important theorem due to Wick that states that the T-products of an arbitrary number of free fields (the ones we have to do in the interaction representation) can be expressed as combinations of T-products among two fields, that is in terms of Feynman propagators. The Wick's theorem is summarized in the following equation

$$T\Big(e^{-i\int d^4x\ j(x)\phi(x)}\Big)$$
$$=: e^{-i\int d^4x\ j(x)\phi(x)}: e^{-\frac{1}{2}\int d^4x\ d^4y\ j(x)j(y)\langle 0|T(\phi(x)\phi(y))|0\rangle} \tag{1.225}$$

where $\phi(x)$ is a free real scalar field and $j(x)$ an ordinary real function. The previous formula can be easily extended to charged scalar, fermionic and photon fields. The Wick's theorem is then obtained by expanding both sides of this equation in powers of $j(x)$ and taking the VEV of both sides. Let us now expand both sides of eq. (1.225) in a series of $j(x)$ and compare term by term. We will use the simplified notation $\phi_i \equiv \phi(x_i)$. We get

$$T(\phi) = :\phi: \tag{1.226}$$

$$T(\phi_1\phi_2) = :\phi_1\phi_2: +\langle 0|T(\phi_1\phi_2)|0\rangle \tag{1.227}$$

$$T(\phi_1\phi_2\phi_3) = :\phi_1\phi_2\phi_3: + \sum_{i\neq j\neq k=1}^{3} :\phi_i: \langle 0|T(\phi_j\phi_k)|0\rangle \tag{1.228}$$

$$T(\phi_1\phi_2\phi_3\phi_4) = :\phi_1\phi_2\phi_3\phi_4: + \sum_{i\neq j\neq k\neq l=1}^{4}\Big[:\phi_i\phi_j: \langle 0|T(\phi_k\phi_l)|0\rangle$$
$$+ \langle 0|T(\phi_i\phi_j)|0\rangle\langle 0|T(\phi_k\phi_l)|0\rangle\Big] \tag{1.229}$$

and so on. By taking the VEV of these expression, and recalling that the VEV of a normal product is zero, we get the Wick's theorem. The T-product of two field operators is sometimes called the **contraction** of the two operators. Therefore to evaluate the VEV of a T-product of an arbitrary number of free fields, it is enough to consider all the possible contractions of the fields appearing in the T-product. For instance, from the last of the previous relations we get

$$\langle 0|T(\phi_1\phi_2\phi_3\phi_4)|0\rangle = \sum_{i\neq j\neq k\neq l=1}^{4} \langle 0|T(\phi_i\phi_j)|0\rangle\langle 0|T(\phi_k\phi_l)|0\rangle \tag{1.230}$$

An analogous theorem holds for the photon field. For the fermions one has to remember that the T-product is defined in a slightly different way. This gives a minus sign any time we have a permutation of the fermion fields which is odd with respect to the original ordering. As an illustration the previous formula becomes

$$\langle 0|T(\psi_1\psi_2\psi_3\psi_4)|0\rangle = \sum_{i\neq j\neq k\neq l=1}^{4} \sigma_P \langle 0|T(\psi_i\psi_j)|0\rangle\langle 0|T(\psi_k\psi_l)|0\rangle \qquad (1.231)$$

where $\sigma_P = \pm 1$ is the sign of the permutation (i,j,k,l) with respect to the fundamental one $(1,2,3,4)$ appearing on the right hand side. More explicitly

$$
\begin{aligned}
\langle 0|T(\psi_1\psi_2\psi_3\psi_4)|0\rangle &= \langle 0|T(\psi_1\psi_2)|0\rangle\langle 0|T(\psi_3\psi_4)|0\rangle \\
&- \langle 0|T(\psi_1\psi_3)|0\rangle\langle 0|T(\psi_2\psi_4)|0\rangle \\
&+ \langle 0|T(\psi_1\psi_4)|0\rangle\langle 0|T(\psi_2\psi_3)|0\rangle
\end{aligned}
\qquad (1.232)
$$

1.5.3 Feynman diagrams in momentum space

We will discuss here the Feynman rules for the scalar and spinor QED. Let us recall that the electromagnetic interaction is introduced via the minimal substitution

$$\partial_\mu \to \partial_\mu + ieA_\mu \qquad (1.233)$$

For instance, for the charged scalar field we get

$$
\begin{aligned}
\mathcal{L}_{\text{free}} &= \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \to \\
&\to [(\partial_\mu + ieA_\mu)\phi)]^\dagger [(\partial^\mu + ieA^\mu)\,\phi] - m^2\phi^\dagger\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}
\end{aligned}
\qquad (1.234)
$$

The interaction part is given by

$$\mathcal{L}_{\text{int.}} = -ie\left[\phi^\dagger\partial_\mu\phi - (\partial_\mu\phi^\dagger)\phi\right]A^\mu + e^2A^2\phi^\dagger\phi \qquad (1.235)$$

Therefore the electromagnetic field is coupled to the current

$$j_\mu = ie\left[\phi^\dagger\partial_\mu\phi - (\partial_\mu\phi^\dagger)\phi\right] \qquad (1.236)$$

but another interaction term appears (the seagull term). In the spinor case we have a simpler situation

$$\mathcal{L}_{\text{free}} = \bar\psi(i\hat\partial - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \to \bar\psi(i\hat\partial - e\hat A - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (1.237)$$

giving the interaction term

$$\mathcal{L}_{\text{int}} = -e\bar\psi\gamma_\mu\psi A^\mu \qquad (1.238)$$

In a typical experiment in particle physics we prepare beams of particles with definite momentum, polarization, etc. At the same time we measure momenta and polarizations of the final states. For this reason it is convenient to work in a momentum representation. The result for a generic matrix element of the S matrix can be expressed in the following form

$$\langle f|S|i\rangle = (2\pi)^4\delta^4(\sum_{\text{in}} p_{\text{in}} - \sum_{\text{out}} p_{\text{out}}) \prod_{\text{ext. bosons}} \sqrt{\frac{1}{2EV}} \prod_{\text{ext. fermions}} \sqrt{\frac{m}{VE}}\mathcal{M} \quad (1.239)$$

Here $f \neq i$, and we are using the box normalization. The Feynman amplitude $\mathcal{M}$ (often one defines a matrix T wich differs from $\mathcal{M}$ by a factor i) can be obtained by drawing at a given perturbative order all the connected and topologically inequivalent Feynman's diagrams. The diagrams are obtained by combining together the graphics elements given in Figures 1.5.1 and 1.5.2. The amplitude $\mathcal{M}$ is given by the sum of the amplitudes associated to these diagrams, after extraction of the factor $(2\pi)^4\delta^4(\sum p)$ in (1.239).

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Fermion line
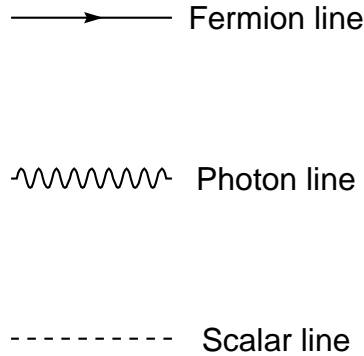
Photon line

Scalar line

Fig. 1.5.1 - *Graphical representation for both internal and external particle lines.*

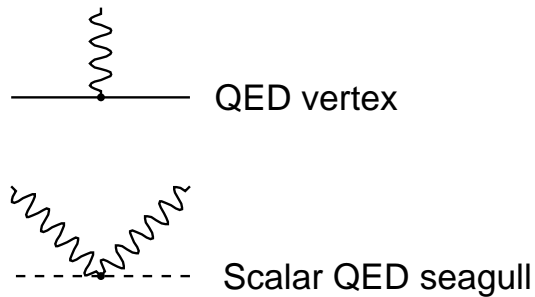.

QED vertex

Scalar QED seagull

Fig. 1.5.2 - *Graphical representation for the vertices in scalar and spinor QED.*

31

At each diagram we associate an amplitude obtained by multiplying together the following factors

- for each ingoing and/or outgoing external photon line a factor ϵ_μ^λ (or its complex conjugate for outgoing photons if we consider complex polarizations as the circular one)

- for each ingoing and/or outgoing external fermionic line a factor $u(p)$ and/or $\bar{u}(p)$

- for each ingoing and/or outgoing external antifermionic line a factor $\bar{v}(p)$ and/or $v(p)$

- for each internal scalar line a factor $i\Delta_F(p)$ (propagator)

- for each internal photon line a factor $ig^{\mu\nu}D(p)$ (propagator)

- for each internal fermion line a factor $iS_F(p)$ (propagator)

- for each vertex in scalar QED a factor $-ie(p+p')(2\pi)^4\delta^4(\sum_{\text{in}} p_i - \sum_{\text{out}} p_f)$, where p and p' are the momenta of the scalar particle (taking p ingoing and p' outgoing)

- For each seagull term in scalar QED a factor $2ie^2 g_{\mu\nu}(2\pi)^4\delta^4(\sum_{\text{in}} p_i - \sum_{\text{out}} p_f)$

- for each vertex in spinor QED a factor $-ie\gamma_\mu(2\pi)^4\delta^4(\sum_{\text{ingoing}} p_i - \sum_{\text{outgoing}} p_f)$

- Integrate over all the internal momenta with measure $d^4p/(2\pi)^4$. This gives rise automatically to the conservation of the total four-momentum.

- In a graph containing closed lines (loops), a factor (-1) for each fermionic loop (see Fig. 1.5.3)
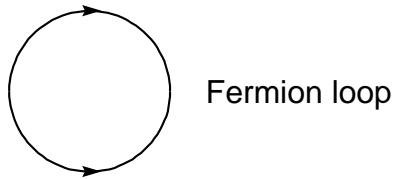
.

Fermion loop

Fig. 1.5.3 - *A fermion loop can appear inside any Feynman diagram (attaching the rest of the diagram by photonic lines).*

In the following we will use need also the rules for the interaction with a static external e.m. field. The corresponding graphical element is given in Fig. 1.5.4. The

rule is to treat it as a normal photon, except that one needs to substitute the wave function of the photon

$$\left(\frac{1}{2EV}\right)^{1/2} \epsilon_\mu^{(\lambda)}(q) \tag{1.240}$$

with the Fourier transform of the external field

$$A_\mu^{\text{ext}}(\vec{q}) = \int d^3q e^{-i\vec{q}\cdot\vec{x}} A_\mu^{\text{ext}}(\vec{x}) \tag{1.241}$$

where $\vec{q}$ is the momentum in transfer at the vertex where the external field line is attached to. Also, since the static external field breaks translational invariance, we loose the factor $(2\pi)^3\delta(\sum_{\text{in}}\vec{p}_{\text{in}} - \sum_{\text{out}}\vec{p}_{\text{out}})$.

.

Fig. 1.5.4 - *Graphical representation for an external electromagnetic field. The upper end of the line can be attached to any fermion line.*

1.5.4 The cross-section

Let us consider a scattering process with a set of initial particles with four momenta $p_i = (E_i, \vec{p}_i)$ which collide and produce a set of final particles with four momenta $p_f = (E_f, \vec{p}_f)$. From the rules of the previous Chapter we know that each external photon line contributes with a factor $(1/2VE)^{1/2}$, whereas each external fermionic line contributes with $(m/VE)^{1/2}$. Furthermore the conservation of the total four momentum gives a term $(2\pi)^4\delta^4(\sum_i p_i - \sum_f p_f)$. If we exclude the uninteresting case $f = i$ we can write

$$S_{fi} = (2\pi)^4\delta^4\left(\sum_i p_i - \sum_f p_f\right) \prod_{\text{fermioni}} \left(\frac{m}{VE}\right)^{1/2} \prod_{\text{bosoni}} \left(\frac{1}{2VE}\right)^{1/2} \mathcal{M} \tag{1.242}$$

where $\mathcal{M}$ is the Feynman amplitude which can be evaluated by using the rules of the previous Section.

Let us consider the typical case of a two particle collision giving rise to an N particles final state. Therefore, the probability per unit time of the transition is given by $w = |S_{fi}|^2$ with

$$w = V(2\pi)^4\delta^4(P_f - P_i) \prod_i \frac{1}{2VE_i} \prod_f \frac{1}{2VE_f} \prod_{\text{fermioni}} (2m)|\mathcal{M}|^2 \tag{1.243}$$

where, for reasons of convenience, we have written

$$\frac{m}{VE} = (2m)\frac{1}{2VE} \tag{1.244}$$

w gives the probability per unit time of a transition toward a state with well defined quantum numbers, but we are rather interested to the final states having momenta between $\vec{p}_f$ and $\vec{p}_f + d\vec{p}_f$. Since in the volume V the momentum is given by $\vec{p} = 2\pi\vec{n}/L$, the number of final states is given by

$$\left(\frac{L}{2\pi}\right)^3 d^3p \tag{1.245}$$

The cross-section is defined as the probability per unit time divided by the flux of the ingoing particles, and has the dimensions of a length to the square

$$[t^{-1}\ell^2 t] = [\ell^2] \tag{1.246}$$

In fact the flux is defined as $v_{\rm rel}\rho = v_{\rm rel}/V$, since in our normalization we have a particle in the volume V, and $v_{\rm rel}$ is the relative velocity of the ingoing particles. For the bosons this follows from the normalization condition of the functions $f_{\vec{k}}(x)$. For the fermions recall that in the box normalization the wave function is

$$\sqrt{\frac{m}{EV}}u(p)e^{-ipx} \tag{1.247}$$

from which

$$\int_V d^3x\,\rho(x) = \int_V d^3x\,\frac{m}{VE}u^\dagger(p)u(p) = 1 \tag{1.248}$$

Then the cross-section for getting the final states with momenta between $\vec{p}_f$ e $\vec{p}_f + d\vec{p}_f$ is given by

$$d\sigma = w\frac{V}{v_{\rm rel}}dN_F = w\frac{V}{v_{\rm rel}}\prod_f\frac{Vd^3p_f}{(2\pi)^3} \tag{1.249}$$

We obtain

$$\begin{aligned}
d\sigma &= \frac{V}{v_{\rm rel}}\prod_f\frac{Vd^3p_f}{(2\pi)^3}V(2\pi)^4\delta^4(P_f - P_i)\frac{1}{4V^2E_1E_2}\prod_f\frac{1}{2VE_f}\prod_{\rm fermioni}(2m)|\mathcal{M}|^2 \\
&= (2\pi)^4\delta^4(P_f - P_i)\frac{1}{4E_1E_2v_{\rm rel}}\prod_{\rm fermioni}(2m)\prod_f\frac{d^3p_f}{(2\pi)^32E_f}|\mathcal{M}|^2 \tag{1.250}
\end{aligned}$$

Notice that the dependence on the quantization volume V disappears, as it should be, in the final equation. Furthermore, the total cross-section, which is obtained by integrating the previous expression over all the final momenta, is Lorentz invariant.

In fact, as it follows from the Feynman rules, $\mathcal{M}$ is invariant, as the factors $d^3p/2E$. Furthermore

$$\vec{v}_{\text{rel}} = \vec{v}_1 - \vec{v}_2 = \frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2} \tag{1.251}$$

but in the frame where the particle 2 is at rest (laboratory frame) we have $p_2 = (m_2, \vec{0})$ and $\vec{v}_{\text{rel}} = \vec{v}_1$, from which

$$
\begin{aligned}
E_1 E_2 |\vec{v}_{\text{rel}}| &= E_1 m_2 \frac{|\vec{p}_1|}{E_1} = m_2 |\vec{p}_1| = m_2 \sqrt{E_1^2 - m_1^2} \\
&= \sqrt{m_2^2 E_1^2 - m_1^2 m_2^2} = \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}
\end{aligned}
\tag{1.252}
$$

We see that also this factor is Lorentz invariant.

Chapter 2

One-loop renormalization

2.1 Divergences of the Feynman integrals

Let us consider the Coulomb scattering. If we expand the S matrix up to the third order in the electric charge, and we assume that the external field is weak enough such to take only the first order, we can easily see that the relevant Feynman diagrams are the ones of Fig. 2.1.1

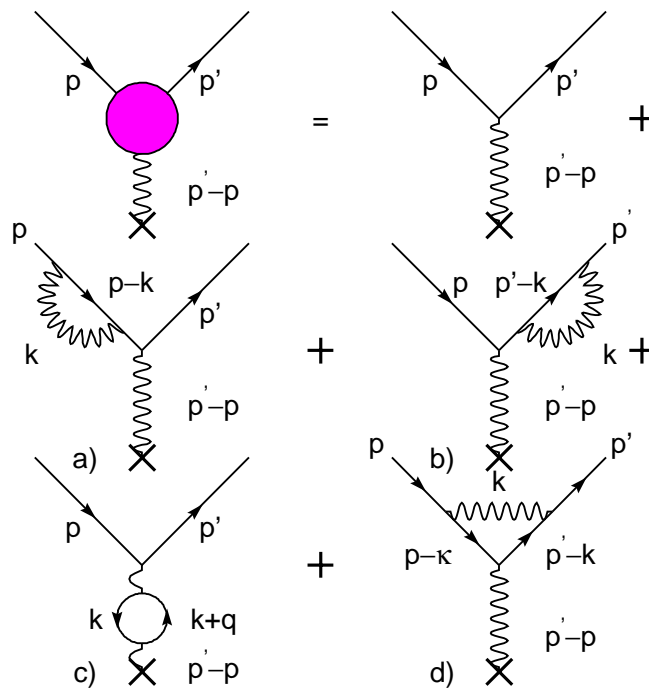.   .

Fig. 2.1.1 - *The Feynman diagram for the Coulomb scattering at the third order in the electric charge and at the first order in the external field.*

The Coulomb scattering can be used, in principle, to define the physical electric charge of the electron. This is done assuming that the amplitude is linear in e_{phys}, from which we get an expansion of the type

$$e_{\text{phys}} = e + a_2 e^3 + \cdots = e(1 + a_2 e^2 + \cdots) \tag{2.1}$$

in terms of the parameter e which appears in the original lagrangian. The first problem we encounter is that we would like to have the results of our calculation in terms of measured quantities as e_{phys}. This could be done by inverting the previous expansion, but, and here comes the second problem, the coefficient of the expansion are divergent quantities. To show this, consider, for instance the self-energy contribution to one of the external photons (as the one in Fig. 2.1.1a). We have

$$\mathcal{M}_a = \bar{u}(p')(-ie\gamma^\mu)A_\mu^{\text{ext}}(\vec{p}\,' - \vec{p})\frac{i}{\hat{p} - m + i\epsilon}\left[ie^2\Sigma(p)\right]u(p) \tag{2.2}$$

where

$$
\begin{aligned}
ie^2\Sigma(p) &= \int \frac{d^4k}{(2\pi)^4}\,(-ie\gamma_\mu)\frac{-ig^{\mu\nu}}{k^2 + i\epsilon}\frac{i}{\hat{p} - \hat{k} - m + i\epsilon}(-ie\gamma_\nu) \\
&= -e^2\int \frac{d^4k}{(2\pi)^4}\,\gamma_\mu\frac{1}{k^2 + i\epsilon}\frac{1}{\hat{p} - \hat{k} - m + i\epsilon}\gamma^\mu
\end{aligned}
\tag{2.3}
$$

or

$$\Sigma(p) = i\int \frac{d^4k}{(2\pi)^4}\,\gamma_\mu\frac{\hat{p} - \hat{k} + m}{(p-k)^2 - m^2 + i\epsilon}\gamma^\mu\frac{1}{k^2 + i\epsilon} \tag{2.4}$$

For large momentum, k, the integrand behaves as $1/k^3$ and the integral diverges linearly. Analogously one can check that all the other third order contributions diverge. Let us write explicitly the amplitudes for the other diagrams

$$\mathcal{M}_b = \bar{u}(p')ie^2\Sigma(p')\frac{i}{\hat{p}\,' - m + i\epsilon}(-ie\gamma^\mu)A_\mu^{\text{ext}}(\vec{p}\,' - \vec{p})u(p) \tag{2.5}$$

$$\mathcal{M}_c = \bar{u}(p')(-ie\gamma^\mu)\frac{-ig_{\mu\nu}}{q^2 + i\epsilon}ie^2\Pi^{\nu\rho}(q)A_\rho^{\text{ext}}(\vec{p}\,' - \vec{p})u(p), \quad q = p' - p \tag{2.6}$$

where

$$ie^2\Pi^{\mu\nu}(q) = (-1)\int \frac{d^4k}{(2\pi)^4}\,Tr\left[\frac{i}{\hat{k} + \hat{q} - m + i\epsilon}(-ie\gamma^\mu)\frac{i}{\hat{k} - m + i\epsilon}(-ie\gamma^\nu)\right] \tag{2.7}$$

(the minus sign originates from the fermion loop) and therefore

$$\Pi^{\mu\nu}(q) = i\int \frac{d^4k}{(2\pi)^4}\,Tr\left[\frac{1}{\hat{k} + \hat{q} - m}\gamma^\mu\frac{1}{\hat{k} - m + i\epsilon}\gamma^\nu\right] \tag{2.8}$$

The last contribution is

$$\mathcal{M}_d = \bar{u}(p')(-ie)e^2\Lambda^\mu(p',p)u(p)A_\mu^{\text{ext}}(\vec{p}\,' - \vec{p}) \tag{2.9}$$

37

where

$$e^2 \Lambda^\mu(p', p) = \int \frac{d^4 k}{(2\pi)^4} \; (-ie\gamma^\alpha) \frac{i}{\hat{p}\,' - \hat{k} - m + i\epsilon} \gamma^\mu \frac{i}{\hat{p} - \hat{k} - m + i\epsilon} (-ie\gamma^\beta) \frac{-ig_{\alpha\beta}}{k^2 + i\epsilon}$$
(2.10)

or

$$\Lambda^\mu(p', p) = -i \int \frac{d^4 k}{(2\pi)^4} \; \gamma^\alpha \frac{1}{\hat{p}\,' - \hat{k} - m + i\epsilon} \gamma^\mu \frac{1}{\hat{p} - \hat{k} - m + i\epsilon} \gamma_\alpha \frac{1}{k^2 + i\epsilon}$$
(2.11)

The problem of the divergences is a serious one and in order to give some sense to the theory we have to define a way to define our integrals. This is what is called the regularization procedure of the Feynman integrals. That is we give a prescription in order to make the integrals finite. This can be done in various ways, as introducing an ultraviolet cut-off, or, as we shall see later by the more convenient means of dimensional regularization. However, we want that the theory does not depend on the way in which we define the integrals, otherwise we would have to look for some physical meaning of the regularization procedure we choose. This bring us to the other problem, the inversion of eq. (2.1). Since now the coefficients are finite we can indeed perform the inversion and obtaining e as a function of e_{phys} and obtain all the observables in terms of the physical electric charge (that is the one measured in the Coulomb scattering). By doing so, a priori we will introduce in the observables a dependence on the renormalization procedure. We will say that the theory is renormalizable when this dependence cancels out. Thinking to the regularization in terms of a cut-off this means that considering the observable quantities in terms of e_{phys}, and removing the cut-off (that is by taking the limit for the cut-off going to the infinity), the result should be finite. Of course, this cancellation is not obvious at all, and in fact in most of the theories this does not happen. However there is a restrict class of renormalizable theories, as for instance QED. We will not discuss the renormalization at all order and neither we will prove which criteria a theory should satisfy in order to be renormalizable. We will give these criteria without a proof but we will try only to justify them On a physical basis. As far QED is concerned we will study in detail the renormalization at one-loop.

2.2   Higher order corrections

In this section we will illustrate, in a simplified context, the previous considerations. Consider again the Coulomb scattering. At the lowest order the Feynman amplitude in presence of an external static electromagnetic field is simply given by the following expression (see Fig. 2.1.1), $(f \neq i)$

$$S_{fi} = (2\pi)\delta(E_f - E_i) \frac{m}{VE} \mathcal{M}^{(1)}$$
(2.12)

where

$$\mathcal{M}^{(1)} = \bar{u}(p')(-ie\gamma^\mu)u(p)A_\mu^{\text{ext}}(\vec{p}\,' - \vec{p})$$
(2.13)

and

$$A_\mu^{\text{ext}}(\vec{q}) = \int d^3x \, A_\mu^{\text{ext}}(\vec{x})e^{-i\vec{q}\cdot\vec{x}} \tag{2.14}$$

is the Fourier transform of the static e.m. field. In the case of a point-like source, of charge Ze we have

$$A_\mu^{\text{ext}}(\vec{x}) = (-\frac{Ze}{4\pi|\vec{x}|}, \vec{0}) \tag{2.15}$$

For the following it is more convenient to re-express the previous amplitude in terms of the external current generating the external e.m. field. Since we have

$$\Box A_\mu = j_\mu \tag{2.16}$$

we can invert this equation in momentum space, by getting

$$A_\mu(q) = -\frac{g_{\mu\nu}}{q^2 + i\epsilon}j^\nu(q) \tag{2.17}$$

from which

$$\mathcal{M}^{(1)} = \bar{u}(p')(-ie\gamma_\mu)u(p)\frac{-ig^{\mu\nu}}{q^2 + i\epsilon}(-ij_\nu^{\text{ext}}(\vec{p}' - \vec{p})) \tag{2.18}$$

with

$$j_\mu^{\text{ext}}(\vec{x}) = (Ze\delta^3(\vec{x}), \vec{0}) \tag{2.19}$$

From these expressions one can get easily the differential cross-section $d\sigma/d\Omega$. This quantity could be measured and then we could compare the result with the theoretical result obtained from the previous expression. In this way we get a measure of the electric charge. However the result is tied to a theoretical equation evaluated at the lowest order in e. In order to get a better determination one has to evaluate the higher order corrections, and again extract the value of e from the data. The lowest order result is at the order e^2 in the electric charge. the next-to-leading contributions ar at the order e^4. For instance, the diagram in Fig. 2.1.1c gives one of these contribution. This is usually referred to as the vacuum polarization contribution and in this Section we will consider only this correction, just to exemplify how the renormalization procedure works. This contribution is given in eqs. (2.6) and (2.7). The total result is

$$\mathcal{M}^{(1)} + \mathcal{M}_c = \bar{u}(p')(-ie\gamma_\mu)u(p)\left[\frac{-ig^{\mu\nu}}{q^2 + i\epsilon} + \frac{-ig^{\mu\rho}}{q^2 + i\epsilon}(ie^2\Pi_{\rho\lambda})\frac{-ig^{\lambda\nu}}{q^2 + i\epsilon}\right](-ij_\nu^{\text{xt}}(\vec{q})) \tag{2.20}$$

The sum gives a result very similar the to one obtained from $\mathcal{M}^{(1)}$ alone, but with the photon propagator modified in the following way

$$\frac{-ig^{\mu\nu}}{q^2} \to \frac{-ig^{\mu\nu}}{q^2} + \frac{-ig^{\mu\rho}}{q^2}(ie^2\Pi_{\rho\lambda}(q^2))\frac{-ig^{\lambda\nu}}{q^2} = \frac{-ig^{\mu\nu}}{q^2} + \frac{-ie^2\Pi^{\mu\nu}}{q^4} \tag{2.21}$$

As discussed in the previous Section $\Pi_{\mu\nu}$ is a divergent quantity. In fact, at high values of the momentum it behaves as

$$\int \frac{d^4p}{p^2} \tag{2.22}$$

and therefore it diverges in a quadratic way. This follows by evaluating the integral by cutting the upper limit by a cut-off Λ. As we shall see later on, the real divergence is weaker due to the gauge invariance. We will see that the correct behaviour is given by

$$\int \frac{d^4p}{p^4} \tag{2.23}$$

The corresponding divergence is only logarithmic, but in any case the integral is divergent. By evaluating it with a cut-off we get

$$\Pi_{\mu\nu}(q^2) = -g_{\mu\nu}q^2 I(q^2) + \cdots \tag{2.24}$$

The terms omitted are proportional to $q^\mu q^\nu$, and since the photon is always attached to a conserved current, they do not contribute to the final result. The functions $I(q^2)$ can be evaluated by various tricks (we will evaluate it explicitly in the following) and the result is

$$I(q^2) = \frac{1}{12\pi^2} \log \frac{\Lambda^2}{m^2} - \frac{1}{2\pi^2} \int_0^1 dz \ z(1-z) \log \left[1 - \frac{q^2 z(1-z)}{m^2}\right] \tag{2.25}$$

For small values of q^2 ($-q^2 << m^2$) one can expand the logarithm obtaining

$$I(q^2) \approx \frac{1}{12\pi^2} \log \frac{\Lambda^2}{m^2} + \frac{1}{60\pi^2} \frac{q^2}{m^2} \tag{2.26}$$

Summarizing the propagator undergoes the following modification

$$\frac{-ig^{\lambda\nu}}{q^2} \to \frac{-ig^{\mu\nu}}{q^2}[1 - e^2 I(q^2)] \tag{2.27}$$

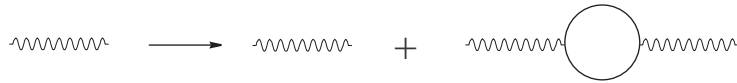This relation is shown in graphical terms in Fig. 2.2.1.

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Fig. 2.2.1 - *The vacuum polarization correction to the propagator.*

The total result is then

$$\mathcal{M}^{(1)} + \mathcal{M}_c = \bar{u}(p')(-ie\gamma_0)u(p)\frac{-i}{q^2}\left(1 - \frac{e^2}{12\pi^2}\log\frac{\Lambda^2}{m^2} - \frac{e^2}{60\pi^2}\frac{q^2}{m^2}\right)(-iZe) \tag{2.28}$$

in the limit $q \to 0$ the result looks exactly as at the lowest order, except that we need to introduce a renormalized charge e_R given by

$$e_R = e \left[1 - \frac{e^2}{12\pi^2} \log \frac{\Lambda^2}{m^2} \right]^{1/2} \tag{2.29}$$

In term of e_R and taking into account that we are working at the order e^4 we can write

$$\mathcal{M}^{(1)} + \mathcal{M}_c = \bar{u}_f(ie_R\gamma_0)u_i\frac{-i}{q^2}\left[1 - \frac{e_R^2}{60\pi^2}\frac{q^2}{m^2}\right](-iZe_R) \tag{2.30}$$

Now we compare this result with the experiment where we measure $d\sigma/d\Omega$ in the limit $q \to 0$. We can extract again from the data the value of the electric charge, but this time we will measure e_R. Notice that in this expression there are no more divergent quantities, since they have been eliminated by the definition of the renormalized charge. It must be stressed that the quantity that is fixed by the experiments is not the "parameter" e appearing in the original lagrangian, but rather e_R. This means that the quantity e is not really defined unless we specify how to relate it to measured quantities. This relation is usually referred to as the renormalization condition. As we have noticed the final expression we got does not contain any explicit divergent quantity. Furthermore, also e_R, which is obtained from the experimental data, is a finite quantity. It follows that the parameter e, must diverge for $\Lambda \to \infty$. In other words, the divergence which we have obtained in the amplitude, is compensated by the divergence implicit in e. Therefore the procedure outlined here has meaning only if we start with an infinite charge. Said that, the renormalization procedure is a way of reorganizing the terms of the perturbative expansion in such a way that the parameter controlling it is the renormalized charge e_R. If this is possible, the various terms in the expansion turn out to be finite and the theory is said to be renormalizable. As already stressed this is not automatically satisfied. In fact renormalizability is not a generic feature of the relativistic quantum field theories. Rather the class of renormalizable theories (that is the theories for which the previous procedure is meaningful) is quite restricted.

We have seen that the measured electric charge is different from the quantity appearing in the expression of the vertex. It will be convenient in the following to call this quantity e_B (tha bare charge). On the contrary, the renormalized charge will be called e, rather than e_R as done previously. For the moment being we have included only the vacuum polarization corrections, however if we restrict our problem to the charge renormalization, the divergent part coming from the self-energy and vertex corrections cancel out (see later). Let us consider now a process like $e^-\mu^- \to e^-\mu^-$. Neglecting other corrections the total amplitude is given by Fig. 2.2.2.

We will define the charge by choosing an arbitrary value of the square of the momentum in transfer q_μ, such that $Q^2 = -q^2 = \mu^2$. In this way the renormalized charge depends on the scale μ. In the previous example we were using $q \to 0$. The graphs in Fig. 2.2.2 sum up to give (omitting the contribution of the fermionic

41

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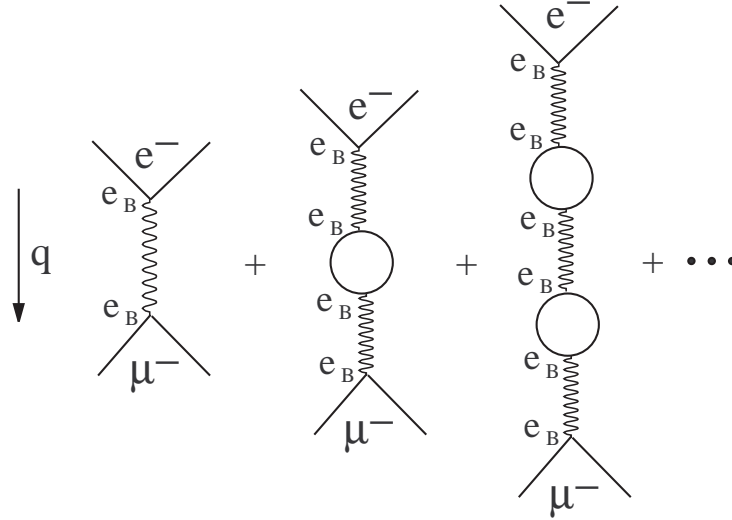
Fig. 2.2.2 - *The vacuum polarization contributions to the $e^- \mu^- \to e^- \mu^-$ scattering.*

lines,)

$$e_B^2 \left[\frac{-ig^{\mu\nu}}{q^2} + \frac{-ig^{\mu\rho}}{q^2} (ie_B^2 \Pi_{\rho\lambda}) \frac{-ig^{\lambda\nu}}{q^2} + \frac{-ig^{\mu\rho}}{q^2} (ie_B^2 \Pi_{\rho\lambda}) \frac{-ig^{\lambda\sigma}}{q^2} (ie_B^2 \Pi_{\sigma\tau}) \frac{-ig^{\tau\nu}}{q^2} + \cdots \right]_{Q^2 = \mu^2} \tag{2.31}$$

By considering only the $g_{\mu\nu}$ contribution to $\Pi_{\mu\nu}$, we get

$$e_B^2 \frac{-ig_{\mu\nu}}{q^2} \left[1 - e_B^2 I(q^2) + e_B^4 I^2(q^2) + \cdots \right]_{Q^2 = \mu^2} \tag{2.32}$$

This expression looks as the lowest order contribution to the propagator be defining the renormalized electric charge as

$$e^2 = e_B^2 [1 - e_B^2 I(\mu^2) + e_B^4 I^2(\mu^2) + \cdots] \tag{2.33}$$

Since the fermionic contribution is the same for all the diagrams, by calling it $F(Q^2)$, we get the amplitude in the form

$$\mathcal{M}(e_B) = e_B^2 F(Q^2)[1 - e_B^2 I(Q^2) + e_B^4 I^2(Q^2) \cdots] \tag{2.34}$$

where $e_B^2 F(Q^2)$ is the amplitude at the lowest order in e_B^2. As we know $I(Q^2)$ is a divergent quantity and therefore the amplitude is not defined. However, following the discussion of the previous Section we can try to re-express everything in terms of the renormalized charge (2.33). We can easily invert by series the eq. (2.33) obtaining at the order e^4

$$e_B^2 = e^2 [1 + e^2 I(\mu^2) + \cdots] \tag{2.35}$$

42

By substituting this expression in $\mathcal{M}(e_0)$ we get

$$
\begin{aligned}
\mathcal{M}(e_B) & = & e^2[1 + e^2 I(\mu^2) + \cdots]F(Q^2)[1 - e^2 I(Q^2) + \cdots] = \\
& = & e^2 F(Q^2)[1 - e^2(I(Q^2) - I(\mu^2)) + \cdots] \equiv \mathcal{M}_R(e) \quad (2.36)
\end{aligned}
$$

From the expression (2.25) for $I(Q^2)$ we get

$$
I(Q^2) = \frac{1}{12\pi^2} \log \frac{\Lambda^2}{m^2} - \frac{1}{2\pi^2} \int_0^1 dz\, z(1-z) \log\left[1 + \frac{Q^2 z(1-z)}{m^2}\right] \quad (2.37)
$$

Since the divergent part is momentum independent it cancels out in the expression for $\mathcal{M}_R(e)$. Furthermore, the renormalized charge e will be measured through the cross-section, and therefore, by definition, it will be finite. Therefore, the charge renormalization makes the amplitude finite. Evaluating $I(Q^2) - I(\mu^2)$ per large Q^2 (and μ^2) we get

$$
\begin{aligned}
e^2(I(Q^2) - I(\mu^2)) & = & -\frac{2\alpha}{\pi} \int_0^1 dz\, z(1-z) \log\left[\frac{m^2 + Q^2 z(1-z)}{m^2 + \mu^2 z(1-z)}\right] \rightarrow \\
& \rightarrow & -\frac{2\alpha}{\pi} \int_0^1 dz\, z(1-z) \log \frac{Q^2}{\mu^2} = -\frac{\alpha}{3\pi} \log \frac{Q^2}{\mu^2} \quad (2.38)
\end{aligned}
$$

and in this limit

$$
\mathcal{M}_R(e) = e^2 F(Q^2) \left[1 - \frac{\alpha}{3\pi} \log \frac{Q^2}{\mu^2} + \cdots\right] \quad (2.39)
$$

At first sight $\mathcal{M}_R(e)$ depends on μ, but actually this is not the case. In fact, e is defined through the eq. (2.33) and therefore it depends also on μ. What is going on is that the implicit dependecnce of e on μ cancels out the explicit dependence of $\mathcal{M}_R(e)$. This follows from $\mathcal{M}_R(e) = \mathcal{M}(e_B)$ and on the observation that $\mathcal{M}(e_B)$ does not depend on μ. This can be formalized in a way that amounts to introduce the concept of renormalization group that we will discuss in much more detail later on. In fact, let us be more explicit by writing

$$
\mathcal{M}(e_B) = \mathcal{M}_R(e(\mu), \mu) \quad (2.40)
$$

By differentiating this expression with respect to μ we get

$$
\frac{d\mathcal{M}(e_B)}{d\mu} = \frac{\partial \mathcal{M}_R}{\partial \mu} + \frac{\partial \mathcal{M}_R}{\partial e} \frac{\partial e}{\partial \mu} = 0 \quad (2.41)
$$

This equation say formally that the amplitude expressed as a function of the renormalized charge is indeed independent on μ. This equation is the prototype of the renormalization group equations, and tells us how we have to change the coupling, when we change the definition scale μ.

Going back to the original expression for $\mathcal{M}(e_0)$, we see that this is given by the tree amplitude $e_B^2 F(Q^2)$ times a geometric series arising from the iteration of

the vacuum polarization diagram. Therefore, it comes natural to define a coupling running with the momentum of the photon.

$$e^2(Q^2) = e_B^2[1 - e_B^2 I(Q^2) + e_B^4 I^2(Q^2) + \cdots] = \frac{e_B^2}{1 + e_B^2 I(Q^2)} \tag{2.42}$$

Also this expression is cooked up in terms of the divergent quantities e_B and $I(Q^2)$, but again $e^2(Q^2)$ is finite. This can be shown explicitly by evaluating the renormalized charge, $e(\mu^2)$, in terms of the bare one

$$e^2(\mu^2) = \frac{e_B^2}{1 + e_B^2 I(\mu^2)} \tag{2.43}$$

We can put the last two equations in the form

$$\frac{1}{e^2(Q^2)} = \frac{1}{e_B^2} + I(Q^2), \quad \frac{1}{e^2(\mu^2)} = \frac{1}{e_B^2} + I(\mu^2) \tag{2.44}$$

and subtracting them

$$\frac{1}{e^2(Q^2)} - \frac{1}{e^2(\mu^2)} = I(Q^2) - I(\mu^2) \tag{2.45}$$

we get

$$e^2(Q^2) = \frac{e^2(\mu^2)}{1 + e^2(\mu^2)[I(Q^2) - I(\mu^2)]} \tag{2.46}$$

By using this result we can write $\mathcal{M}_R(e)$ as

$$\mathcal{M}_R(e) = e^2(Q^2)F(Q^2) \tag{2.47}$$

It follows that the true expansion parameter in the perturbative series is the running coupling $e(Q^2)$. By using eq. (2.38) we obtain

$$\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{3\pi} \log \frac{Q^2}{\mu^2}} \tag{2.48}$$

We see that $\alpha(Q^2)$ increases with Q^2, therefore the perturbative approach looses validity for those values Q^2 of such that $\alpha(Q^2) \approx 1$, that is when

$$1 - \frac{\alpha(\mu^2)}{3\pi} \log \frac{Q^2}{\mu^2} = \alpha(\mu^2) \tag{2.49}$$

or

$$\frac{Q^2}{\mu^2} = e^{3\pi \left(\frac{1}{\alpha(\mu^2)} - 1 \right)} \tag{2.50}$$

44

For $\alpha(\mu^2) \approx 1/137$ we get

$$\frac{Q^2}{\mu^2} \approx e^{1281} \approx 10^{556} \tag{2.51}$$

Actually $\alpha(Q^2)$ is divergent for

$$\frac{Q^2}{\mu^2} = e^{\dfrac{3\pi}{\alpha(\mu^2)}} \approx e^{1291} \approx 10^{560} \tag{2.52}$$

This value of Q^2 is referred to as the Landau pole. Therefore in the case of QED the perturbative approach maintains its validity up to gigantic values of the momenta. This discussion shows clearly that the behaviour of the running coupling constant is determined by $I(Q^2)$, that is from the vaccum polarization contribution. In particular, $\alpha(Q^2)$ increases with Q^2 since in the expression $e^2(\mu^2)(I(Q^2) - I(\mu^2))$ the coefficient of the logarithm is negative (equal to $-\alpha(\mu^2)/(3\pi)$). We will see that in the so called non-abelian gauge theories the running is opposite, that is the theory becomes more and more perturbative, by increasing the energy.

2.3   The analysis by counterterms

The previous way of defining a renormalizable theory amounts to say that the original parameters in the lagrangian, as e, should be infinite and that their divergences should compensate the divergences of the Feynman diagrams. Then one can try to separate the infinite from the finite part of the parameters (this separation is ambiguous, see later). The infinite contributions are called counterterms, and by definition they have the same operator structure of the original terms in the lagrangian. On the other hand, the procedure of regularization can be performed by adding to the original lagrangian counterterms cooked in such a way that their contribution kills the divergent part of the Feynman integrals. This means that the coefficients of these counterterms have to be infinite. However they can also be regularized in the same way as the other integrals. We see that the theory will be renormalizable if the counterterms we add to make the theory finite have the same structure of the original terms in the lagrangian, in fact, if this is the case, they can be absorbed in the original parameters, which however are arbitrary, because they have to be fixed by the experiments (renormalization conditions). We will show now that at one-loop the divergent contributions come only from the three diagrams discussed before. In fact, let us define D as the superficial degree of divergence of a one-loop diagram the difference between 4 (the number of integration over the mimentum) and the number of powers of momentum coming from the propagators (we assume here that no powers of momenta are coming from the vertices, which is not true in scalar QED). We get

$$D = 4 - I_F - 2I_B \tag{2.53}$$

45

where I_F and I_B are the number of internal fermionic and bosonic lines. Going through the loop one has to have a number of vertices, V equal to the number of internal lines

$$V = I_F + I_B \tag{2.54}$$

Also, if at each vertex there are N_i lines of the particle of type i, then we have

$$N_i V = 2I_i + E_i \tag{2.55}$$

from which

$$2V = 2I_F + E_F, \quad V = 2I_B + E_B \tag{2.56}$$

From these equations we can get V, I_F and I_B in terms of E_F and E_B

$$I_F = \frac{1}{2}E_F + E_B, \quad I_B = \frac{1}{2}E_F, \quad V = E_F + E_B \tag{2.57}$$

from which

$$D = 4 - \frac{3}{2}E_F - E_B \tag{2.58}$$

It follows that the only one-loop superficially divergent diagrams in spinor QED are the ones in Fig. 2.3.1. In fact, all the diagrams with an odd number of external photons vanish. This is trivial at one-loop, since the trace of an odd number of γ matrices is zero. In general (Furry's theorem) it follows from the conservation of charge conjugation, since the photon is an eigenstate of C with eigenvalue equal to -1.

Furthermore, the effective degree of divergence can be less than the superficial degree. In fact, gauge invariance often lowers the divergence. For instance the diagram of Fig. 2.3.1 with four external photons (light-light scattering) has $D = 0$ (logarithmic divergence), but gauge invariance makes it finite. Also the vacuum polarization diagram has an effective degree of divergence equal to 2. Therefore, there are only three divergent one-loop diagrams and all the divergences can be brought back to the three functions $\Sigma(p)$, $\Pi_{\mu\nu}(q)$ e $\Lambda^\mu(p', p)$. This does not mean that an arbitrary diagram is not divergence, but it can be made finite if the previous functions are such. In such a case one has only to show that eliminating these three divergences (primitive divergences) the theory is automatically finite. In particular we will show that the divergent part of $\Sigma(p)$ can be absorbed into the definition of the mass of the electron and a redefinition of the electron field (wave function renormalization). The divergence in $\Pi_{\mu\nu}$, the photon self-energy, can be absorbed in the wave function renormalization of the photon (the mass of the photon is not renormalized due to the gauge invariance). And finally the divergence of $\Lambda_\mu(p', p)$ goes into the definition of the parameter e. To realize this program we divide up the lagrangian density in two parts, one written in terms of the physical parameters, the other will contain the counterterms. We will call also the original parameters and fields of the theory the bare parameters and the bare fields and we will use an index B in order to distinguish them from the physical quantities. Therefore the
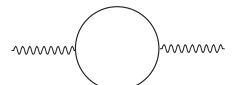
.
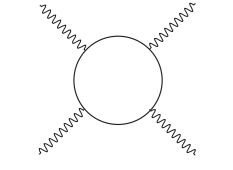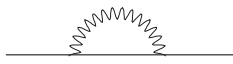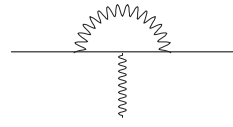
E_F	E_B	D
0	2	2
0	4	0
2	0	1
2	1	0

Fig. 2.3.1 - *The superficially divergent one-loop diagrams in spinor QED*

two pieces of the lagrangian should look like as follows: the piece in terms of the physical parameters
L_p

$$\mathcal{L}_p = \bar{\psi}(i\hat{\partial} - m)\psi - e\bar{\psi}\gamma_\mu\psi A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 \qquad (2.59)$$

and the counter terms piece $\mathcal{L}_{\text{c.t.}}$

$$\mathcal{L}_{\text{c.t.}} = iB\bar{\psi}\hat{\partial}\psi - A\bar{\psi}\psi - \frac{C}{4}F_{\mu\nu} - \frac{E}{2}(\partial_\mu A^\mu)^2 - eD\bar{\psi}\hat{A}\psi \qquad (2.60)$$

and we have to require that la sum of these two contributions should coincide with the original lagrangian written in terms of the bare quantities. Adding together $\mathcal{L}_p$ and $\mathcal{L}_{\text{c.t.}}$ we get

$$\mathcal{L} = (1+B)i\bar{\psi}\hat{\partial}\psi - (m+A)\bar{\psi}\psi - e(1+D)\bar{\psi}\gamma_\mu\psi A^\mu - \frac{1+C}{4}F_{\mu\nu}F^{\mu\nu} + \text{gauge} - \text{fixing} \qquad (2.61)$$

where, for sake of simplicity, we have omitted the gauge fixing term. Defining the renormalization constant of the fields

$$Z_1 = (1+D), \qquad Z_2 = (1+B), \qquad Z_3 = (1+C) \qquad (2.62)$$

we write the bare fields as

$$\psi_B = Z_2^{1/2}\psi, \qquad A_B^\mu = Z_3^{1/2}A^\mu \qquad (2.63)$$

47

we obtain

$$\mathcal{L} = i\bar{\psi}_B \hat{\partial}\psi_B - \frac{m+A}{Z_2}\bar{\psi}_B\psi_B - \frac{eZ_1}{Z_2 Z_3^{1/2}}\bar{\psi}_B\gamma_\mu\psi_B A_B^\mu - \frac{1}{4}F_{B,\mu\nu}F_B^{\mu\nu} + \text{gauge} - \text{fixing}$$

(2.64)

and putting

$$m_B = \frac{m+A}{Z_2}, \qquad e_B = \frac{eZ_1}{Z_2 Z_3^{1/2}}$$

(2.65)

we get

$$\mathcal{L} = i\bar{\psi}_B \hat{\partial}\psi_B - m_B\bar{\psi}_B\psi_B - e_B\bar{\psi}_B\gamma_\mu\psi_B A_B^\mu - \frac{1}{4}F_{B,\mu\nu}F_B^{\mu\nu} + \text{gauge} - \text{fixing}$$ (2.66)

So we have succeeded in the wanted separation. Notice that the division of the parameters in physical and counter term part is well defined, because the finite piece is fixed to be an observable quantity. This requirement gives the renormalization conditions. The counter terms A, B, ... are determined recursively at each perturbative order in such a way to eliminate the divergent parts and to respect the renormalization conditions. We will see later how this works in practice at one-loop level. Another observation is that Z_1 and Z_2 have to do with the self-energy of the electron, and as such they depend on the electron mass. Therefore if we consider the theory for a different particle, as the muon, which has the same interactions as the electron and differs only for the value of the mass ($m_\mu \approx 200 m_e$), one would get a different bare electric charge for the two particles. Or, phrased in a different way, one would have to tune the bare electric charge at different values in order to get the same physical charge. This looks very unnatural, but the gauge invariance of the theory implies that at all the perturbative orders $Z_1 = Z_2$. As a consequence $e_B = e/Z_3^{1/2}$, and since Z_3 comes from the photon self-energy, the relation between the bare and the physical electric charge is universal (that is it does not depend on the kind of charged particle under consideration).

Summarizing, one starts dividing the bare lagrangian in two pieces. Then we regularize the theory giving some prescription to get finite Feynman integrals. The part containing the counter terms is determined, order by order, by requiring that the divergences of the Feynman integrals, which come about when removing the regularization, are cancelled out by the counter term contributions. Since the separation of an infinite quantity into an infinite plus a finite term is not well defined, we use the renormalization conditions, to fix the finite part. After evaluating a physical quantity we remove the regularization. Notice that although the counter terms are divergent quantity when we remove the cut off, we will order them according to the power of the coupling in which we are doing the perturbative calculation. That is we have a double limit, one in the coupling and the other in some parameter (regulator) which defines the regularization. The order of the limit is first to work at some order in the coupling, at fixed regulator, and then remove the regularization.

Before going into the calculations for QED we want to illustrate some general results about the renormalization. If one considers only theory involving scalar,

fermion and massless spin 1 (as the photon) fields, it is not difficult to construct an algorithm which allows to evaluate the ultraviolet (that is for large momenta) divergence of any Feynman diagram. In the case of the electron self-energy (see Figs. 2.1.1a and 2.1.1b) one has an integration over the four momentum p and a behaviour of the integrand, coming from the propagators, as $1/p^3$, giving a linear divergence (it turns out that the divergence is only logarithmic). From this counting one can see that only the lagrangian densities containing monomials in the fields with mass dimension smaller or equal to the number of space-time dimensions have a finite number of divergent diagrams. It turns out also that these are renormalizable theories (a part some small technicalities). The mass dimensions of the fields can be easily evaluated from the observation that the action is dimensionless in our units ($\hbar = 1$). Therefore, in n space-time dimensions, the lagrangian density, defined as

$$\int d^n x \, \mathcal{L} \tag{2.67}$$

has a mass dimension n. Looking at the kinetic terms of the bosonic fields (two derivatives) and of the fermionic fields (one derivative), we see that

$$\dim[\phi] = \dim[A_\mu] = \frac{n}{2} - 1, \quad \dim[\psi] = \frac{n-1}{2} \tag{2.68}$$

In particular, in 4 dimensions the bosonic fields have dimension 1 and the fermionic 3/2. Then, we see that QED is renormalizable, since all the terms in the lagrangian density have dimensions smaller or equal to 4

$$\dim[\bar{\psi}\psi] = 3, \quad \dim[\bar{\psi}\gamma_\mu\psi A^\mu] = 4, \quad \dim[(\partial^\mu A_\mu)^2] = 4 \tag{2.69}$$

The condition on the dimensions of the operators appearing in the lagrangian can be translated into a condition over the coupling constants. In fact each monomial $\mathcal{O}_i$ will appear multiplied by a coupling g_i

$$\mathcal{L} = \sum_i g_i \mathcal{O}_i \tag{2.70}$$

therefore

$$\dim[g_i] = 4 - \dim[\mathcal{O}_i] \tag{2.71}$$

The renormalizability requires

$$\dim[\mathcal{O}_i] \leq 4 \tag{2.72}$$

from which

$$\dim[g_i] \geq 0 \tag{2.73}$$

that is the couplings must have positive dimension in mass are to be dimensionless. In QED the only couplings are the mass of the electron and the electric charge

49

which is dimensionless. As a further example consider a single scalar field. The most general renormalizable lagrangian density is characterized by two parameters

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \rho\phi^3 - \lambda\phi^4 \tag{2.74}$$

Here ρ has dimension 1 and λ is dimensionless. We see that the linear σ-models are renormalizable theories.

Giving these facts let us try to understand what makes renormalizable and non renormalizable theories different. In the renormalizable case, if we have written the most general lagrangian, the only divergent diagrams which appear are the ones corresponding to the processes described by the operators appearing in the lagrangian. Therefore adding to $\mathcal{L}$ the counter term

$$\mathcal{L}_{\text{c.t.}} = \sum_i \delta g_i \mathcal{O}_i \tag{2.75}$$

we can choose the δg_i in such a way to cancel, order by order, the divergences. The theory depends on a finite number of arbitrary parameters equal to the number of parameters g_i. Therefore the theory is a predictive one. In the non renormalizable case, the number of divergent diagrams increase with the perturbative order. At each order we have to introduce new counter terms having an operator structure different from the original one. At the end the theory will depend on an infinite number of arbitrary parameters. As an example consider a fermionic theory with an interaction of the type $(\bar{\psi}\psi)^2$. Since this term has dimension 6, the relative coupling has dimension -2

$$\mathcal{L}_{\text{int}} = -g_2(\bar{\psi}\psi)^2 \tag{2.76}$$

At one loop the theory gives rise to the divergent diagrams of Fig. 2.3.2. The divergence of the first diagram can be absorbed into a counter term of the original type

$$-\delta g_2(\bar{\psi}\psi)^2 \tag{2.77}$$

The other two need counter terms of the type

$$\delta g_3(\bar{\psi}\psi)^3 + \delta g_4(\bar{\psi}\psi)^4 \tag{2.78}$$

These counter terms originate new one-loop divergent diagrams, as for instance the ones in Fig. 2.3.3. The first diagram modifies the already introduced counter term $(\bar{\psi}\psi)^4$, but the second one needs a new counter term

$$\delta g_5(\bar{\psi}\psi)^5 \tag{2.79}$$

This process never ends.

The renormalization requirement restricts in a fantastic way the possible field theories. However one could think that this requirement is too technical and one could imagine other ways of giving a meaning to lagrangians which do not satisfy this

.
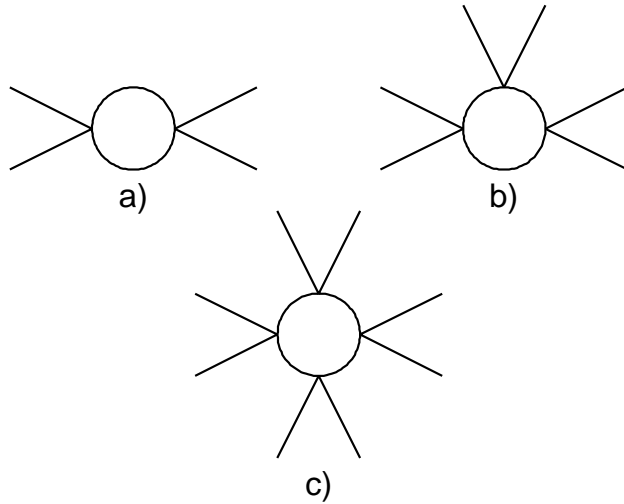
Fig. 2.3.2 - *Divergent diagrams coming from the interaction $(\bar{\psi}\psi)^2$.*

condition. But suppose we try to give a meaning to a non renormalizable lagrangian, simply requiring that it gives rise to a consistent theory at any energy. We will show that this does not happen. Consider again a theory with a four-fermion interaction. Since dim $g_2 = -2$, if we consider the scattering $\psi + \psi \to \psi + \psi$ in the high energy limit (where we can neglect all the masses), on dimensional ground we get that the total cross-section behaves like

$$\sigma \approx g_2^2 E^2 \tag{2.80}$$

Analogously, in any non renormalizable theory, being there couplings with negative dimensions, the cross-section will increase with the energy. But the cross-section has to do with the S matrix which is unitary. Since a unitary matrix has eigenvalues of
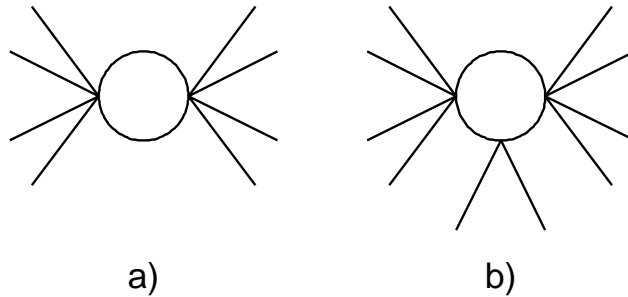
.

Fig. 2.3.3 - *Divergent diagrams coming from the interactions $(\bar{\psi}\psi)^2$ and $(\bar{\psi}\psi)^4$.*

modulus 1, it follows that its matrix elements must be bounded. Translating this argument in the cross-section one gets the bound

$$\sigma \leq \frac{c^2}{E^2} \tag{2.81}$$

where c is some constant. For the previous example we get

$$g_2 E^2 \leq c \tag{2.82}$$

This implies a violation of the S matrix unitarity at energies such that

$$E \geq \sqrt{\frac{c}{g_2}} \tag{2.83}$$

It follows that we can give a meaning also to non renormalizable theories, but only for a limited range of values of the energy. This range is fixed by the value of the non renormalizable coupling. It is not difficulty to realize that non renormalizability and bad behaviour of the amplitudes at high energies are strictly connected.

2.4 Dimensional regularization of the Feynman integrals

As we have discussed in the previous Section we need a procedure to give sense at the otherwise divergent Feynman diagrams. The simplest of these procedures is just to introduce a cut-off Λ and define our integrals as $\int_0^\infty \to \lim_{\Lambda \to \infty} \int_0^\Lambda$). Of course, in the spirit of renormalization we have first to perform the perturbative expansion and then take the limit over the cut-off. Although this procedure is very simple it results to be inadequate in situations like gauge theories. In fact one can show that the cut-off regularization breaks the translational invariance and creates problems in gauge theories. One kind of regularization which nowadays is very much used is the dimensional regularization. This consists in considering the integration in an arbitrary number of space-time dimensions and taking the limit of four dimensions at the end. This way of regularizing is very convenient because it respects all the symmetries. In fact, except for very few cases the symmetries do not depend on the number of space-time dimensions. Let us what is dimensional regularization about. We want to evaluate integrals of the type

$$I_4(k) = \int d^4p \ F(p, k) \tag{2.84}$$

with $F(p, k) \approx p^{-2}$ or p^{-4}. The idea is that integrating on a lower number of dimensions the integral improves the convergence properties in the ultraviolet. For

instance, if $F(p, k) \approx p^{-4}$, the integral is convergent in 2 and in 3 dimensions. Therefore, we would like to introduce a quantity

$$I(\omega, k) = \int d^{2\omega}p \ F(p, k) \tag{2.85}$$

to be regarded as a function of the complex variable ω. Then, if we can define a complex function $I'(\omega, k)$ on the entire complex plane, with definite singularity, and as such that it coincides with I on some common domain, then by analytic continuation I and I' define the same analytic function. A simple example of this procedure is given by the Euler's Γ. This complex function is defined for $Re \ z > 0$ by the integral representation

$$\Gamma(z) = \int_0^\infty dt \ e^{-t} t^{z-1} \tag{2.86}$$

If Re $z \le 0$, the integral diverges as

$$\frac{dt}{t^{1+|\text{Re } z|}} \tag{2.87}$$

in the limit $t \to 0$. However it is easy to get a representation valid also for $Re \ z \le 0$. Let us divide the integration region in two parts defined by a parameter α

$$\Gamma(z) = \int_0^\alpha dt \ e^{-t} t^{z-1} + \int_\alpha^\infty dt \ e^{-t} t^{z-1} \tag{2.88}$$

Expanding the exponential in the first integral and integrating term by term we get

$$
\begin{aligned}
\Gamma(z) &= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_0^\alpha dt \ t^{n+z-1} + \int_\alpha^\infty dt \ e^{-t} t^{z-1} \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{\alpha^{n+z}}{n+z} + \int_\alpha^\infty dt \ e^{-t} t^{z-1}
\end{aligned} \tag{2.89}
$$

The second integral converges for any z since $\alpha > 0$. This expression coincides with the representation for the Γ function for $Re \ z > 0$, but it is defined also for $Re \ z < 0$ where it has simple poles located at $z = -n$. Therefore it is a meaningful expression on all the complex plane z. Notice that in order to isolate the divergences we need to introduce an arbitrary parameter α. However the result does not depend on the particular value of this parameter. This the Weierstrass representation of the Euler $\Gamma(z)$. From this example we see that we need the following three steps

- Find a domain where $I(\omega, k)$ is convergent. Typically this will be for Re $\omega < 2$.

- Construct an analytic function identical to $I(\omega, k)$ in the domain of convergence, but defined on a larger domain including the point $\omega = 2$.

- At the end of the calculation take the limit $\omega \to 2$.

2.5 Integration in arbitrary dimensions

Let us consider the integral

$$I_N = \int d^N p F(p^2) \tag{2.90}$$

with N an integer number and p a vector in an Euclidean N-dimensional space. Since the integrand is invariant under rotations of the N-dimensional vector p, we can perform the angular integration by means of

$$d^N p = d\Omega_N p^{N-1} dp \tag{2.91}$$

where $d\Omega_N$ is the solid angle element in N-dimensions. Therefore $\int d\Omega_N = S_N$, with S_N the surface of the unit sphere in N-dimensions. Then

$$I_N = S_N \int_0^\infty p^{N-1} F(p^2) dp \tag{2.92}$$

The value of the sphere surface can be evaluated by the following trick. Consider

$$I = \int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi} \tag{2.93}$$

By taking N of these factors we get

$$I^N = \int dx_1 \cdots dx_N e^{-(x_1^2 + \cdots + x_N^2)} = \pi^{N/2} \tag{2.94}$$

The same integral can be evaluated in polar coordinates

$$\pi^{N/2} = S_N \int_0^\infty \rho^{N-1} e^{-\rho^2} d\rho \tag{2.95}$$

By putting $x = \rho^2$ we have

$$\pi^{N/2} = \frac{1}{2} S_N \int_0^\infty x^{N/2-1} e^{-x} dx = \frac{1}{2} S_N \Gamma\left(\frac{N}{2}\right) \tag{2.96}$$

where we have used the representation of the Euler Γ function given in the previous Section. Therefore

$$S_N = \frac{2\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)} \tag{2.97}$$

and

$$I_N = \frac{\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)} \int_0^\infty x^{N/2-1} F(x) dx \tag{2.98}$$

with $x = p^2$.

The integrals we will be interested in are of the type

$$I_N^{(M)} = \int \frac{d^N p}{(p^2 - a^2 + i\epsilon)^A} \tag{2.99}$$

with p a vector in a Ndimensional Minkowski space. We can perform an anti-clockwise rotation of 90^0 (Wick's rotation) in the complex plane of p_0 without hitting any singularity . Then we do a change of variables $p_0 \to ip_0$, and we use the definition $p^2 \to -p^2 = \sum_{i=1}^N p_i^2$, obtaining

$$I_N^{(M)} = i \int \frac{d^N p}{(-p^2 - a^2)^A} = i(-1)^A I_N \tag{2.100}$$

where I_N is an integral of the kind discussed at the beginning of this Section with $F(x)$ given by

$$F(x) = (x + a^2)^{-A} \tag{2.101}$$

It follows

$$I_N = \frac{\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)} \int_0^\infty \frac{x^{N/2-1}}{(x + a^2)^A} dx \tag{2.102}$$

By putting $x = a^2 y$ we get

$$I_N = (a^2)^{N/2-A} \frac{\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)} \int_0^\infty y^{N/2-1}(1 + y)^{-A} dx \tag{2.103}$$

and recalling the integral representation for the Euler $B(x, y)$ function (valid for $Re x, y > 0$)

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} = \int_0^\infty t^{x-1}(1 + t)^{-(x+y)} dt \tag{2.104}$$

it follows

$$I_N = \pi^{N/2} \frac{\Gamma(A - N/2)}{\Gamma(A)} \frac{1}{(a^2)^{A-N/2}} \tag{2.105}$$

We have obtained this representation for $N/2 > 0$ and $Re(A - N/2) > 0$. But we know how to extend the Euler gamma-function to the entire complex plane, and therefore we can extend this formula to complex dimensions $N = 2\omega$

$$I_{2\omega} = \pi^\omega \frac{\Gamma(A - \omega)}{\Gamma(A)} \frac{1}{(a^2)^{A-\omega}} \tag{2.106}$$

This shows that $I_{2\omega}$ has simple poles located at

$$\omega = A, A + 1, \cdots \tag{2.107}$$

Therefore our integral will be perfectly defined at all ω such that $\omega \neq A, A + 1, \cdots$. At the end we will have to consider the limit $\omega \to 2$. The original integral in Minkowski space is then

$$\int \frac{d^{2\omega} p}{(p^2 - a^2)^A} = i\pi^\omega (-1)^A \frac{\Gamma(A - \omega)}{\Gamma(A)} \frac{1}{(a^2)^{A-\omega}} \tag{2.108}$$

For the following it will be useful to derive another formula. Let us put in the previous equation $p = p' + k$ and $b^2 = -a^2 + k^2$, then

$$\int \frac{d^{2\omega}p'}{(p'^2 + 2p' \cdot k + k^2 - a^2)^A} = i\pi^\omega (-1)^A \frac{\Gamma(A - \omega)}{\Gamma(A)} \frac{1}{(a^2)^{A-\omega}} \tag{2.109}$$

from which

$$\int \frac{d^{2\omega}p}{(p^2 + 2p \cdot k + b^2)^A} = i\pi^\omega (-1)^A \frac{\Gamma(A - \omega)}{\Gamma(A)} \frac{1}{(k^2 - b^2)^{A-\omega}} \tag{2.110}$$

Differentiating with respect to k_μ we get various useful relations as

$$\int d^{2\omega}p \frac{p_\mu}{(p^2 + 2p \cdot k + b^2)^A} = i\pi^\omega (-1)^A \frac{\Gamma(A - \omega)}{\Gamma(A)} \frac{-k_\mu}{(k^2 - b^2)^{A-\omega}} \tag{2.111}$$

and

$$\int d^{2\omega}p \frac{p_\mu p_\nu}{(p^2 + 2p \cdot k + b^2)^A} = i \frac{\pi^\omega (-1)^A}{\Gamma(A)(k^2 - b^2)^{A-\omega}}$$
$$\times \left[\Gamma(A - \omega) k_\mu k_\nu - \frac{1}{2} g_{\mu\nu}(k^2 - b^2)\Gamma(A - \omega - 1) \right] \tag{2.112}$$

Since at the end of our calculation we will have to take the limit $\omega \to 2$, it will be useful to recall the expansion of the Gamma function around its poles

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + \mathcal{O}(\epsilon) \tag{2.113}$$

where

$$\gamma = 0.5772... \tag{2.114}$$

is the Euler-Mascheroni constant, and for $(n \geq 1)$:

$$\Gamma(-n + \epsilon) = \frac{(-1)^n}{n!} \left[\frac{1}{\epsilon} + \psi(n + 1) + \mathcal{O}(\epsilon) \right] \tag{2.115}$$

where

$$\psi(n + 1) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma \tag{2.116}$$

2.6 One loop regularization of QED

In this Section we will regularize, by using dimensional regularization, the relevant divergent quantities in QED, that is $\Sigma(p)$, $\Pi_{\mu\nu}(q)$ e $\Lambda^\mu(p, p')$. Furthermore, in order to define the counter terms we will determine the expressions which become divergent in the limit of $\omega \to 2$. It will be also convenient to introduce a parameter μ such that these quantities have the same dimensions in the space with $d = 2\omega$ as in $d = 4$.

The algebra of the Dirac matrices is also easily extended to arbitrary dimensions d. For instance, starting from

$$[\gamma_\mu, \gamma_\nu]_+ = 2g_{\mu\nu} \tag{2.117}$$

we get

$$\gamma^\mu \gamma_\mu = d \tag{2.118}$$

and

$$\gamma_\mu \gamma_\nu \gamma^\mu = (2 - d)\gamma_\nu \tag{2.119}$$

Other relations can be obtained by starting from the algebraic properties of the γ-matrices. Let us start with the electron self-energy which we will require to have dimension 1 as in $d = 4$. From eq. (2.4) we have

$$\Sigma(p) = i\mu^{4-2\omega} \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \gamma_\mu \frac{\hat{p} - \hat{k} + m}{(p - k)^2 - m^2 + i\epsilon} \gamma^\mu \frac{1}{k^2 + i\epsilon} \tag{2.120}$$

In order to use the equations of the previous Section it is convenient to combine together the denominators of this expression into a single one. This is done by using a formula due to Feynman

$$\frac{1}{ab} = \int_0^1 \frac{dz}{[az + b(1 - z)]^2} \tag{2.121}$$

which is proven using

$$\frac{1}{ab} = \frac{1}{b - a}\left[\frac{1}{a} - \frac{1}{b}\right] = \frac{1}{b - a} \int_a^b \frac{dx}{x^2} \tag{2.122}$$

and doing the change of variables

$$x = az + b(1 - z) \tag{2.123}$$

We get

$$\Sigma(p) = i\mu^{4-2\omega} \int_0^1 dz \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \gamma_\mu \frac{\hat{p} - \hat{k} + m}{[(p - k)^2 z - m^2 z + k^2(1 - z)]^2} \gamma^\mu \tag{2.124}$$

The denominator can be written in the following way

$$[...] = p^2 z - m^2 z + k^2 - 2p \cdot kz \tag{2.125}$$

and the term $p \cdot k$ can be eliminated through the following change of variables $k = k' + pz$. We find

$$[...] = (p^2 - m^2)z + (k' + pz)^2 - 2p \cdot (k' + pz)z = k'^2 - m^2 z + p^2 z(1 - z) \tag{2.126}$$

It follows (we put $k' = k$)

$$\Sigma(p) = i\mu^{4-2\omega} \int_0^1 dz \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \gamma_\mu \frac{\hat{p}(1-z) - \hat{k} + m}{[k^2 - m^2 z + p^2 z(1-z)]^2} \gamma^\mu \qquad (2.127)$$

The linear term in k has vanishing integral, therefore

$$\Sigma(p) = i\mu^{4-2\omega} \int_0^1 dz \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \gamma_\mu \frac{\hat{p}(1-z) + m}{[k^2 - m^2 z + p^2 z(1-z)]^2} \gamma^\mu \qquad (2.128)$$

and integrating over k

$$\Sigma(p) = i\mu^{4-2\omega} \int_0^1 dz \frac{1}{(2\pi)^{2\omega}} i\pi^\omega \frac{\Gamma(2-\omega)}{\Gamma(2)} \gamma_\mu \frac{\hat{p}(1-z) + m}{[m^2 z - p^2 z(1-z)]^{2-\omega}} \gamma^\mu \qquad (2.129)$$

By defining $\epsilon = 4 - 2\omega$ we get

$$\Sigma(p) = -\mu^\epsilon \int_0^1 dz \frac{1}{(2\pi)^{4-\epsilon}} \pi^{(4-\epsilon)/2} \Gamma(\epsilon/2) \gamma_\mu \frac{\hat{p}(1-z) + m}{[m^2 z - p^2 z(1-z)]^{\epsilon/2}} \gamma^\mu \qquad (2.130)$$

Contracting the γ matrices

$$\Sigma(p) = -\frac{1}{16\pi^2} \Gamma(\epsilon/2) \int_0^1 dz (4\pi\mu^2)^{\epsilon/2} \frac{(\epsilon-2)\hat{p}(1-z) + (4-\epsilon)m}{[m^2 z - p^2 z(1-z)]^{\epsilon/2}} \qquad (2.131)$$

we obtain

$$\Sigma(p) = \frac{1}{16\pi^2} \Gamma(\epsilon/2)$$
$$\times \int_0^1 dz [2\hat{p}(1-z) - 4m - \epsilon(\hat{p}(1-z) - m)] \left[\frac{m^2 z - p^2 z(1-z)}{4\pi\mu^2} \right]^{-\epsilon/2} \qquad (2.132)$$

Defining

$$A = 2\hat{p}(1-z) - 4m, \quad B = -\hat{p}(1-z) + m, \quad C = \frac{m^2 z - p^2 z(1-z)}{4\pi\mu^2} \qquad (2.133)$$

and expanding for $\epsilon \to 0$

$$\begin{aligned}
\Sigma(p) &= \frac{1}{16\pi^2} \int_0^1 dz \left[\frac{2A}{\epsilon} + 2B - \gamma A \right] \left[1 - \frac{\epsilon}{2} \log C \right] \\
&= \frac{1}{16\pi^2} \int_0^1 dz \left[\frac{2A}{\epsilon} - A\log C + 2B - \gamma A \right] \\
&= \frac{1}{8\pi^2\epsilon}(\hat{p} - 4m) - \frac{1}{16\pi^2}[\hat{p} - 2m + \gamma(\hat{p} - 4m)] \\
&\quad - \frac{1}{8\pi^2} \int_0^1 dz [\hat{p}(1-z) - 2m] \log \frac{m^2 z - p^2 z(1-z)}{4\pi\mu^2} \\
&= \frac{1}{8\pi^2\epsilon}(\hat{p} - 4m) + \text{finite terms} \qquad (2.134)
\end{aligned}$$

Consider now the vacuum polarization (see eq. (2.8))

$$
\begin{aligned}
\Pi^{\mu\nu}(q) &= i\mu^{4-2\omega} \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} Tr\left[\frac{1}{\hat{k}+\hat{q}-m}\gamma^\mu \frac{1}{\hat{k}-m}\gamma^\nu\right] \\
&= i\mu^{4-2\omega} \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{Tr[\gamma_\mu(\hat{k}+m)\gamma_\nu(\hat{k}+\hat{q}+m)]}{(k^2-m^2)((k+q)^2-m^2)}
\end{aligned}
\tag{2.135}
$$

Using again the Feynman representation for the denominators

$$
\Pi^{\mu\nu}(q) = i\mu^{4-2\omega} \int_0^1 dz \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{Tr[\gamma_\mu(\hat{k}+m)\gamma_\nu(\hat{k}+\hat{q}+m)]}{[(k^2-m^2)(1-z)+((k+q)^2-m^2)z]^2}
\tag{2.136}
$$

We write the denominator as

$$
[...] = k^2 + q^2 z - m^2 + 2k\cdot qz
\tag{2.137}
$$

through the change of variable $k = k' - qz$ we cancel the mixed term obtaining

$$
[...] = (k'-qz)^2 + 2(k'-qz)\cdot qz + q^2 z - m^2 = k'^2 + q^2 z(1-z) - m^2
\tag{2.138}
$$

and therefore (we put again $k' = k$)

$$
\Pi^{\mu\nu}(q) = i\mu^{4-2\omega} \int_0^1 dz \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{Tr[\gamma_\mu(\hat{k}-\hat{q}z+m)\gamma_\nu(\hat{k}+\hat{q}(1-z)+m)]}{[k^2+q^2z(1-z)-m^2]^2}
\tag{2.139}
$$

Since the integral of the odd terms in k is zero, it is enough to evaluate the contribution of the even term to the trace

$$
\begin{aligned}
Tr[...]_{\text{even}} &= Tr[\gamma_\mu(\hat{k}-\hat{q}z+m)\gamma_\nu(\hat{k}+\hat{q}(1-z)]_{\text{pari}} + m^2 Tr[\gamma_\mu\gamma_\nu] \\
&= Tr[\gamma_\mu\hat{k}\gamma_\nu\hat{k}] - Tr[\gamma_\mu\hat{q}\gamma_\nu\hat{q}]z(1-z) + m^2 Tr[\gamma_\mu\gamma_\nu]
\end{aligned}
\tag{2.140}
$$

If we define the γ as matrices of dimension $2^\omega \times 2^\omega$ we can repeat the standard calculation by obtaining a factor 2^ω instead of 4. Therefore

$$
\begin{aligned}
Tr[...]_{\text{even}} &= 2^\omega[2k_\mu k_\nu - g_{\mu\nu}k^2 - (2q_\mu q_\nu - g_{\mu\nu}q^2)z(1-z) + m^2 g_{\mu\nu}] \\
&= 2^\omega[2k_\mu k_\nu - 2z(1-z)(q_\mu q_\nu - g_{\mu\nu}q^2) - g_{\mu\nu}(k^2 - m^2 + q^2 z(1-z)]
\end{aligned}
\tag{2.141}
$$

and we find

$$
\begin{aligned}
\Pi^{\mu\nu}(q) &= i\mu^{4-2\omega}2^\omega \int_0^1 dz \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}\left[\frac{2k_\mu k_\nu}{[k^2+q^2z(1-z)-m^2]^2}\right. \\
&\left. - \frac{2z(1-z)(q_\mu q_\nu - g_{\mu\nu}q^2)}{[k^2+q^2z(1-z)-m^2]^2} - \frac{g_{\mu\nu}}{[k^2+q^2z(1-z)-m^2]}\right]
\end{aligned}
\tag{2.142}
$$

From the relations of the previous Section we get

$$
\int d^{2\omega}p \frac{2p_\mu p_\nu}{[p^2-a^2]^2} = -ig_{\mu\nu}\pi^\omega \frac{\Gamma(1-\omega)}{(a^2)^{1-\omega}}
\tag{2.143}
$$

59

whereas

$$\int d^{2\omega}p \frac{1}{[p^2 - a^2]} = -i\pi^\omega \frac{\Gamma(1 - \omega)}{(a^2)^{1-\omega}} \tag{2.144}$$

Therefore the first and the third contribution to the vacuum polarization cancel out and we are left with

$$\Pi^{\mu\nu}(q) = -i\mu^{4-2\omega}2^\omega(q_\mu q_\nu - g_{\mu\nu}q^2) \int_0^1 dz\ 2z(1-z) \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{[k^2 + q^2z(1-z) - m^2]^2} \tag{2.145}$$

Notice that the original integral was quadratically divergent, but due to the previous cancellation the divergence is only logarithmic. The reason is again gauge invariance. In fact it is possible to show that this implies $q^\mu\Pi_{\mu\nu}(q) = 0$. Performing the momentum integration we have

$$\Pi^{\mu\nu}(q) = -i2\mu^{4-2\omega}2^\omega(q_\mu q_\nu - g_{\mu\nu}q^2) \int_0^1 dz\ z(1-z) \frac{i\pi^\omega}{(2\pi)^{2\omega}} \frac{\Gamma(2-\omega)}{[m^2 - q^2z(1-z)]^{2-\omega}} \tag{2.146}$$

By putting again $\epsilon = 4 - 2\omega$ and expanding the previous expression

$$
\begin{aligned}
\Pi^{\mu\nu}(q) &= 2\mu^\epsilon 2^{2-\epsilon/2}(q_\mu q_\nu - g_{\mu\nu}q^2) \int_0^1 dz\ z(1-z) \frac{\pi^{2-\epsilon/2}}{(2\pi)^{4-\epsilon}} \frac{\Gamma(\epsilon/2)}{[m^2 - q^2z(1-z)]^{\epsilon/2}} \\
&= \frac{2}{16\pi^2}2^{2-\epsilon/2}(q_\mu q_\nu - g_{\mu\nu}q^2) \int_0^1 dz\ z(1-z)\Gamma(\epsilon/2) \left[\frac{m^2 - q^2z(1-z)}{4\pi\mu^2}\right]^{-\epsilon/2} \\
&= \frac{2}{16\pi^2}(4 - 2\epsilon\log 2)(q_\mu q_\nu - g_{\mu\nu}q^2) \\
&\times \int_0^1 dz\ z(1-z)(\frac{2}{\epsilon} - \gamma)\left[1 - \frac{\epsilon}{2}\log C\right] \tag{2.147}
\end{aligned}
$$

where C is definite in eq. (2.133). Finally

$$
\begin{aligned}
\Pi_{\mu\nu}(q) &= \frac{1}{8\pi^2}(q_\mu q_\nu - g_{\mu\nu}q^2) \int dz\ z(1-z)\left[\frac{8}{\epsilon} - 4\gamma - 4\log 2\right]\left[1 - \frac{\epsilon}{2}\log C\right] \\
&= \frac{1}{2\pi^2}(q_\mu q_\nu - g_{\mu\nu}q^2) \\
&\times \left[\frac{1}{3\epsilon} - \frac{\gamma}{6} - \int dz\ z(1-z)\log\left[\frac{m^2 - q^2z(1-z)}{2\pi\mu^2}\right]\right] \tag{2.148}
\end{aligned}
$$

or, in abbreviated way

$$\Pi_{\mu\nu}(q) = \frac{1}{6\pi^2}(q_\mu q_\nu - g_{\mu\nu}q^2)\frac{1}{\epsilon} + \text{finite terms} \tag{2.149}$$

We have now to evaluate $\Lambda_\mu(p', p)$. From eq. (2.11) we have

$$\Lambda^\mu(p', p) = -i\mu^{4-2\omega} \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}\gamma^\alpha \frac{\hat{p}\,' - \hat{k} + m}{(p' - k)^2 - m^2}\gamma^\mu \frac{\hat{p} - \hat{k} + m}{(p - k)^2 - m^2}\gamma_\alpha\frac{1}{k^2} \tag{2.150}$$

the general formula to reduce n denominators to a single one is

$$\prod_{i=1}^{n}\frac{1}{a_i} = (n-1)!\int_0^1 \prod_{i=1}^{n} d\beta_i \frac{\delta(1-\sum_{i=1}^{n}\beta_i)}{[\sum_{i=1}^{n}\beta_i a_i]^n} \tag{2.151}$$

To show this equation notice that

$$\prod_{i=1}^{n}\frac{1}{a_i} = \int_0^\infty \prod_{i=1}^{n} d\alpha_i e^{-\sum_{i=1}^{n}\alpha_i a_i} \tag{2.152}$$

Introducing the identity

$$1 = \int_0^\infty d\lambda\,\delta(\lambda - \sum_{i=1}^{n}\alpha_i) \tag{2.153}$$

and changing variables $\alpha_i = \lambda\beta_i$

$$\prod_{i=1}^{n}\frac{1}{a_i} = \int_0^\infty \prod_{i=1}^{n} d\alpha_i d\lambda\,\delta(\lambda - \sum_{i=1}^{n}\alpha_i)e^{-\sum_{i=1}^{n}\alpha_i a_i}$$

$$= \int_0^\infty \prod_{i=1}^{n} d\beta_i \lambda^n \frac{d\lambda}{\lambda} e^{-\lambda\sum_{i=1}^{n}\beta_i a_i} \delta(1 - \sum_{i=1}^{n}\beta_i) \tag{2.154}$$

The integration over β_i can be restricted to the interval $[0,1]$ due to the delta function, and furthermore

$$\int_0^\infty d\lambda\,\lambda^{n-1}e^{-\rho\lambda} = \frac{(n-1)!}{\rho^n} \tag{2.155}$$

In our case we get

$$\Lambda^\mu(p',p) = -2i\mu^{4-2\omega}\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}\int_0^1 dx \int_0^{1-x} dy$$

$$\times\frac{\gamma^\alpha(\hat{p}\,'-\hat{k}+m)\gamma^\mu(\hat{p}-\hat{k}+m\gamma_\alpha}{[k^2(1-x-y)+(p-k)^2x - m^2x + (p'-k)^2y - m^2y]^3} \tag{2.156}$$

The denominator is

$$[...] = k^2 - m^2(x+y) + p^2x + p'^2y - 2k\cdot(px + p'y) \tag{2.157}$$

Changing variable, $k = k' + px + py$

$$[...] = (k'+px+p'y)^2 - m^2(x+y) + p^2x + p'^2y - 2(k'+px+p'y)\cdot(px+p'y)$$

$$= k'^{\,2} - m^2(x+y) + p^2x(1-x) + p'^2y(1-y) - 2p\cdot p'xy \tag{2.158}$$

Letting again $k' \to k$

$$\Lambda^\mu(p',p) = -2i\mu^{4-2\omega} \int_0^1 dx \int_0^{1-x} dy \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}$$

$$\times \frac{\gamma^\alpha(\hat{p}\,'(1-y) - \hat{p}x - \hat{k} + m)\gamma^\mu(\hat{p}(1-x) - \hat{p}'y - \hat{k} + m)\gamma_\alpha}{[k^2 - m^2(x+y) + p^2x(1-x) + p'^2y(1-y) - 2p\cdot p'xy]^3} \quad (2.159)$$

The odd term in k is zero after integration, the term in k^2 is logarithmically divergent, whereas the remaining part is convergent. Separating the divergent piece, $\Lambda_\mu^{(1)}$, from the convergent one, $\Lambda_\mu^{(3)}$,

$$\Lambda_\mu = \Lambda_\mu^{(1)} + \Lambda_\mu^{(2)} \quad (2.160)$$

we get for the first term

$$
\begin{aligned}
\Lambda_\mu^{(1)}(p',p) &= -2i\mu^{4-2\omega} \int_0^1 dx \int_0^{1-x} dy \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \\
&\qquad \gamma^\alpha\gamma_\lambda\gamma_\mu\gamma_\sigma\gamma_\alpha \frac{k^\lambda k^\sigma}{[k^2 - m^2(x+y) + p^2x(1-x) + p'^2y(1-y) - 2p\cdot p'xy]^3} \\
&= -2i\mu^{4-2\omega} \int_0^1 dx \int_0^{1-x} dy \frac{i\pi^\omega(-1)^3}{\Gamma(3)}\left(-\frac{1}{2}\right)\Gamma(2-\omega) \\
&\qquad \frac{\gamma^\alpha\gamma_\lambda\gamma^\mu\gamma^\lambda\gamma_\alpha}{[m^2(x+y) - p^2x(1-x) - p'^2y(1-y) + 2p\cdot p'xy]^{2-\omega}} \\
&= \frac{1}{2}\mu^{4-2\omega}\left(\frac{1}{4\pi}\right)^\omega \Gamma(2-\omega) \int_0^1 dx \int_0^{1-x} dy \\
&\qquad \frac{\gamma^\alpha\gamma_\lambda\gamma^\mu\gamma^\lambda\gamma_\alpha}{[m^2(x+y) - p^2x(1-x) - p'^2y(1-y) + 2p\cdot p'xy]^{2-\omega}} \quad (2.161)
\end{aligned}
$$

Using the relation

$$\gamma^\alpha\gamma_\lambda\gamma^\mu\gamma^\lambda\gamma_\alpha = (2-d)^2\gamma^\mu \quad (2.162)$$

we obtain ($\epsilon = 4 - 2\omega$)

$$
\begin{aligned}
\Lambda_\mu^{(1)}(p',p) &= \frac{1}{2}\mu^\epsilon\left(\frac{1}{4\pi}\right)^{2-\epsilon/2}\Gamma(\epsilon/2)(\epsilon-2)^2\gamma_\mu \int_0^1 dx \int_0^{1-x} dy \\
&\qquad \frac{1}{[m^2(x+y) - p^2x(1-x) - p'^2y(1-y) + 2p\cdot p'xy]^{\epsilon/2}} \\
&= \frac{1}{32\pi^2}\left[\frac{2}{\epsilon} - \gamma\right][4 - 4\epsilon]\gamma_\mu \int_0^1 dx \int_0^{1-x} dy \\
&\qquad \left[\frac{m^2(x+y) - p^2x(1-x) - p'^2y(1-y) + 2p\cdot p'xy}{4\pi\mu^2}\right]^{-\epsilon/2} \\
&= \frac{1}{8\pi^2\epsilon}\gamma_\mu - \frac{1}{16\pi^2}(\gamma+2)\gamma_\mu - \frac{1}{8\pi^2}\gamma_\mu \int_0^1 dx \int_0^{1-x} dy
\end{aligned}
$$

62

$$\times \log \left[\frac{m^2(x+y) - p^2 x(1-x) - p'^2 y(1-y) + 2p \cdot p' xy}{4\pi\mu^2} \right] \quad (2.163)$$

and finally

$$\Lambda_\mu^{(1)}(p', p) = \frac{1}{8\pi^2 \epsilon} \gamma_\mu + \text{finite terms} \quad (2.164)$$

In the convergent part we can put directly $\omega = 2$

$$
\begin{aligned}
\Lambda_\mu^{(2)}(p', p) &= -\frac{i}{8\pi^4} \int_0^1 dx \int_0^{1-x} dy \frac{i\pi^2(-1)^3}{\Gamma(3)} \\
&\quad \frac{\gamma^\alpha(\hat{p}'(1-y) - \hat{p}x + m)\gamma^\mu(\hat{p}(1-x) - \hat{p}'y + m)\gamma_\alpha}{m^2(x+y) - p^2 x(1-x) - p'^2 y(1-y) + 2p \cdot p' xy} \\
&= -\frac{1}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \\
&\quad \frac{\gamma^\alpha(\hat{p}'(1-y) - \hat{p}x + m)\gamma^\mu(\hat{p}(1-x) - \hat{p}'y + m)\gamma_\alpha}{m^2(x+y) - p^2 x(1-x) - p'^2 y(1-y) + 2p \cdot p' xy} \quad (2.165)
\end{aligned}
$$

2.7 One loop renormalization

We summarize here the results of the previous Section

$$\Sigma(p) = \frac{1}{8\pi^2 \epsilon}(\hat{p} - 4m) + \Sigma^f(p) \quad (2.166)$$

$$\Pi_{\mu\nu}(q) = (q_\mu q_\nu - g_{\mu\nu}q^2) \left[\frac{1}{6\pi^2 \epsilon} + \Pi^f(q) \right] \equiv (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(q^2) \quad (2.167)$$

$$\Lambda_\mu(p', p) = \frac{1}{8\pi^2 \epsilon}\gamma_\mu + \Lambda_\mu^f(p', p) \quad (2.168)$$

where the functions with the superscript f represent the finite contributions. Let us start discussing the electron self-energy. As shown in eq. (2.2), the effect of $\Sigma(p)$ is to correct the electron propagator. In fact we have (see Fig. 2.7.1):

$$S_F(p) = \frac{i}{\hat{p} - m} + \frac{i}{\hat{p} - m}ie^2\Sigma(p)\frac{i}{\hat{p} - m} + \cdots \quad (2.169)$$
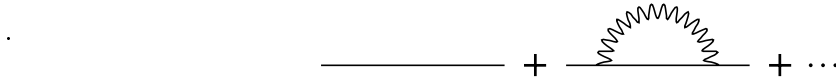
.

Fig. 2.7.1 - *The loop expansion for the electron propagator.*

from which, at the same order in the perturbative expansion

$$S_F(p) = \frac{i}{\hat{p} - m} \left(1 + \frac{e^2\Sigma(p)}{\hat{p} - m} \right)^{-1} = \frac{i}{\hat{p} - m + e^2\Sigma(p)} \quad (2.170)$$

Therefore the effect of the divergent terms is to modify the coefficients of $\hat{p}$ and m:

$$iS_F^{-1}(p) = \hat{p} - m + e^2\Sigma(p) = \hat{p}\left(1 + \frac{e^2}{8\pi^2\epsilon}\right) - m\left(1 + \frac{e^2}{2\pi^2\epsilon}\right) + \text{finite terms} \quad (2.171)$$

This allows us to define the counter terms to be added to the lagrangian expressed in terms of the physical parameters in such a way to cancel these divergences

$$(\mathcal{L}_1)_{ct} = iB\bar{\psi}\hat{\partial}\psi - A\bar{\psi}\psi \quad (2.172)$$

$(\mathcal{L}_1)_{ct}$ modifies the Feynman rules adding two particles interacting terms. These can be easily evaluated noticing that the expression of the propagator, taking into account $(\mathcal{L}_1)_{ct}$, is

$$\frac{i}{(1+B)\hat{p} - (m+A)} \approx \frac{i}{\hat{p}-m}\left(1 - \frac{B\hat{p}-A}{\hat{p}-m}\right)$$
$$\approx \frac{i}{\hat{p}-m} + \frac{i}{\hat{p}-m}(iB\hat{p} - iA)\frac{i}{\hat{p}-m} \quad (2.173)$$

where, consistently with our expansion we have taken only the first order terms in A and B. We can associate to these two terms the diagrams of Fig. 2.7.2, with contributions $-iA$ to the mass term, and $iB\hat{p}$ to $\hat{p}$.

.

$$-iA \qquad\qquad iB\not{p}$$
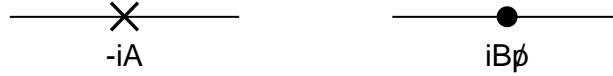
Fig. 2.7.2 - *The counter terms for the self-energy ($\not{p}$ in the figure should be read as $\hat{p}$).*

The propagator at the second order in the coupling constant is then given by adding the diagrams of Fig. 2.7.3.
At this order we get

$$S_F(p) = \frac{i}{\hat{p}-m} + \frac{i}{\hat{p}-m}\left(ie^2\Sigma(p) + iB\hat{p} - iA\right)\frac{i}{\hat{p}-m} \quad (2.174)$$

and the correction to the free propagator is given by

$$e^2\Sigma(p) + B\hat{p} - A = \left(\frac{e^2}{8\pi^2\epsilon} + B\right)\hat{p} - \left(\frac{e^2}{2\pi^2\epsilon}m + A\right) + \text{finite terms} \quad (2.175)$$

We can now fix the counter terms by choosing

$$B = -\frac{e^2}{8\pi^2}\left(\frac{1}{\epsilon} + F_2\left(\frac{m}{\mu}\right)\right) \quad (2.176)$$
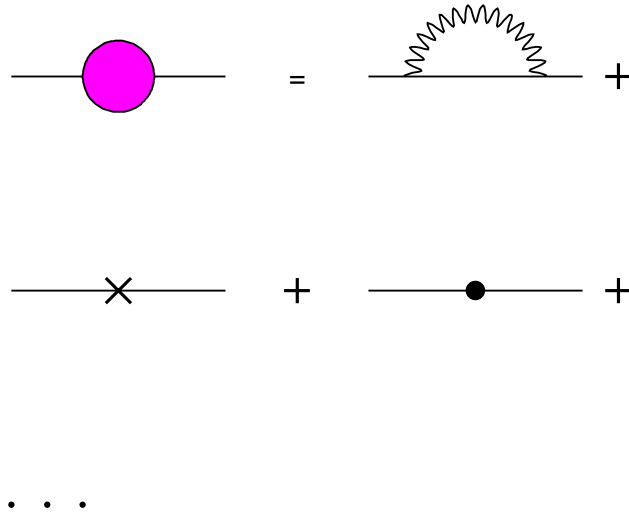
Fig. 2.7.3 - *The second order contributions at the electron propagator.*

$$A = -\frac{me^2}{2\pi^2}\left(\frac{1}{\epsilon} + F_m\left(\frac{m}{\mu}\right)\right) \tag{2.177}$$

with F_2 and F_m finite for $\epsilon \to 0$. Notice also that these two functions are dimensionless and arbitrary up to this moment. However they can be determined by the renormalization conditions, that is by fixing the arbitrary constants appearing in the lagrangian. In fact, given

$$iS_F^{-1}(p) \equiv \Gamma^{(2)}(p) = \hat{p} - m + B\hat{p} - A + e^2\Sigma(p) \tag{2.178}$$

we can require that at the physical pole, $\hat{p} = m$, the propagator coincides with the free propagator

$$S_F(p) \approx \frac{i}{\hat{p} - m}, \quad \text{for} \ \ \hat{p} = m \tag{2.179}$$

From here we get two conditions. The first one is

$$0 = \Gamma^{(2)}(\hat{p} = m) = \frac{e^2}{8\pi^2\epsilon}(\hat{p} - 4m) + e^2\Sigma^f(\hat{p} = m) - \frac{e^2}{8\pi^2}\left(\frac{1}{\epsilon} + F_2\right)\hat{p} + \frac{me^2}{2\pi^2\epsilon}\left(\frac{1}{\epsilon} + F_m\right) \tag{2.180}$$

from which

$$e^2\Sigma^f(\hat{p} = m) - \frac{me^2}{8\pi^2}F_2 + \frac{me^2}{2\pi^2}F_m = 0 \tag{2.181}$$

The second condition is

$$\left.\frac{\partial\Gamma^{(2)}(p)}{\partial\hat{p}}\right|_{\hat{p}=m} = 1 \tag{2.182}$$

65

which gives

$$e^2 \frac{\partial \Sigma^f(p)}{\partial \hat{p}}\bigg|_{\hat{p}=m} - \frac{e^2}{8\pi^2} F_2 = 0 \tag{2.183}$$

One should be careful because these particular conditions of renormalization gives some problems related to the zero mass of the photon. In fact one finds some ill-defined integral in the infrared region. However these are harmless divergences, because giving a small to the photon and letting it to zero at the end of the calculations gives rise to finite results. Notice that these conditions of renormalization have the advantage of being expressed directly in terms of the measured parameters, as the electron mass. However, one could renormalize at an arbitrary mass scale, M. In this case the parameters comparing in $\mathcal{L}_p$ are not the directly measured parameters, but they can be correlated to the actual parameters by evaluating some observable quantity. From this point of view one could avoid the problems mentioned above by choosing a different point of renormalization.

As far as the vacuum polarization is concerned, $\Pi_{\mu\nu}$ gives rise to the following correction of the photon propagator (illustrated in Fig. 2.7.4)

$$D'_{\mu\nu}(q) = \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\lambda}}{q^2} ie^2 \Pi^{\lambda\rho}(q) \frac{-ig_{\rho\nu}}{q^2} + \cdots \tag{2.184}$$

.
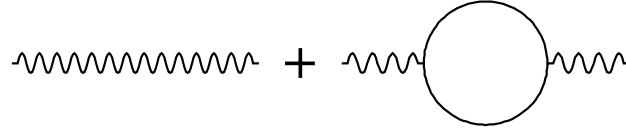
Fig. 2.7.4 - *The loop expansion for the photon propagator.*

from which

$$\begin{aligned}
D'_{\mu\nu}(q) &= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\lambda}}{q^2} \left[(ie^2)(q^\lambda q^\rho - g^{\lambda\rho}q^2)\Pi(q^2) \right] \frac{-ig_{\rho\nu}}{q^2} + \cdots \\
&= \frac{-ig_{\mu\nu}}{q^2} \left[1 - e^2\Pi(q^2) \right] - i\frac{q_\mu q_\nu}{q^4} e^2 \Pi(q^2) + \cdots
\end{aligned} \tag{2.185}$$

We see that the one loop propagator has a divergent part in $g_{\mu\nu}$, and also divergent and finite pieces in the term proportional to the momenta. Therefore the propagator is not any more in the Feynman gauge. It follows that we should add to $\mathcal{L}_p$

$$\mathcal{L}_2 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 = \frac{1}{2} A^\mu g_{\mu\nu} \Box A^\nu \tag{2.186}$$

66

the two counterterms

$$(\mathcal{L}_2)_{\text{ct}} = -\frac{C}{4}F_{\mu\nu}F^{\mu\nu} - \frac{E}{2}(\partial_\mu A^\mu)^2 = -C\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu A^\mu)^2\right) - \frac{E-C}{2}(\partial_\mu A^\mu)^2 \tag{2.187}$$

As for the electron propagator, we can look at these two contributions as perturbations to the free lagrangian, and evaluate the corresponding Feynman rules, or evaluate the effect on the propagator, and then expand in the counter terms. The effect of these two terms to the equation which defines the propagator in momentum space is

$$[(1+C)q^2 g_{\mu\nu} - (C-E)q_\mu q_\nu]D^{\nu\lambda}(q) = -ig_\mu^\lambda \tag{2.188}$$

We solve this equation by putting

$$D_{\mu\nu}(q) = \alpha(q^2)g_{\mu\nu} + \beta(q^2)q_\mu q_\nu \tag{2.189}$$

Substituting in the previous equation we determine α e β. The result is

$$\alpha = -\frac{i}{q^2(1+C)}, \qquad \beta = -\frac{i}{q^4}\frac{C-E}{(1+C)(1+E)} \tag{2.190}$$

The free propagator, including the corrections at the first order in C and E is

$$D_{\mu\nu}(q) = \frac{-ig_{\mu\nu}}{q^2}(1-C) - i\frac{q_\mu q_\nu}{q^4}(C-E) \tag{2.191}$$

and the total propagator

$$D'_{\mu\nu}(q) = \frac{-ig_{\mu\nu}}{q^2}[1 - e^2\Pi(q^2) - C] - i\frac{q_\mu q_\nu}{q^4}[e^2\Pi(q^2) + C - E] \tag{2.192}$$

We can now choose $E = 0$ and cancel the divergence by the choice

$$C = -\frac{e^2}{6\pi^2\epsilon} + F_3\left(\frac{m}{\mu}\right) \tag{2.193}$$

In fact we are free to choose the finite term of the gauge fixing, since this choice does not change the physics. This is because the terms proportional to $q_\mu q_\nu$, as it follows from the gauge invariance, and the conservation of the electromagnetic current. For instance, if we have a vertex with a virtual photon (that is the vertex is connected to an internal photon line) and two external electrons, the term proportional to $q_\mu q_\nu$ is saturated with

$$\bar{u}(p')\gamma_\mu u(p), \qquad q = p' - p \tag{2.194}$$

The result is zero, bay taking into account the Dirac equation. Let us see what the full propagator says about the mass of the photon. We have

$$D'_{\mu\nu}(q) = \frac{-ig_{\mu\nu}}{q^2}\left[1 - F_3 - e^2\Pi^f\right] - i\frac{q_\mu q_\nu}{q^4}\left[F_3 + e^2\Pi^f\right] \equiv \frac{-ig_{\mu\nu}}{q^2}[1 - \tilde{\Pi}] - i\frac{q_\mu q_\nu}{q^4}\tilde{\Pi} \tag{2.195}$$

where
$$\tilde{\Pi} = e^2 \Pi^f + F_3 \qquad (2.196)$$
At the second order in the electric charge we can write the propagator in the form

$$D'_{\mu\nu}(q) = \frac{-i}{q^2(1 + \tilde{\Pi}(q))} \left[g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \tilde{\Pi}(q) \right] \qquad (2.197)$$

and we see that the propagator has a pole at $q^2 = 0$, since $\tilde{\Pi}(q^2)$ is finite for $q^2 \to 0$. Therefore the photon remains massless after renormalization. This is part of a rather general aspect of renormalization which says that if the regularization procedure respects the symmetries of the original lagrangian, the symmetries are preserved at any perturbative order. An exception is the case of the anomalous symmetries, which are symmetries at the classical level, but are broken by the quantum corrections. All the anomalous symmetries can be easily identified in a given theory.

Also in this case we will require that at the physical pole, $q^2 \to 0$, the propagator has the free form, that is
$$\tilde{\Pi}(0) = 0 \qquad (2.198)$$
from which
$$F_3 = -e^2 \Pi^f(0) \qquad (2.199)$$
For small momenta we can write

$$\tilde{\Pi}(q) \approx e^2 q^2 \frac{d\Pi^f(q)}{dq^2} \bigg|_{q^2=0} \qquad (2.200)$$

since F_3 is a constant. Using the expression for Π^f from the previous Section, we get

$$\Pi^f(q) \approx -\frac{1}{2\pi^2} \left(\frac{\gamma}{6} + \frac{1}{6} \log \left(\frac{m^2}{2\pi\mu^2} \right) \right) + \frac{1}{2\pi^2} \int_0^1 dz \; z^2 (1-z)^2 \frac{q^2}{m^2} + \cdots \quad (2.201)$$

and
$$\tilde{\Pi}(q) \approx \frac{e^2 q^2}{60\pi^2 m^2} \qquad (2.202)$$
from which
$$D'_{\mu\nu} = -i \frac{g_{\mu\nu}}{q^2} \left[1 - \frac{e^2 q^2}{60\pi^2 m^2} \right] + \text{gauge terms} \qquad (2.203)$$

The first term, $1/q^2$, gives rise to the Coulomb potential, $e^2/4\pi r$. Therefore this is modified by a constant term in momentum space, or by a term proportional to the delta function in the real space

$$\Delta_{12} = e^2 \int dt \int \frac{d^4 q}{(2\pi)^4} e^{-iq(x_1 - x_2)} \left[\frac{1}{q^2} - \frac{e^2}{60\pi^2 m^2} \right] = -\frac{e^2}{4\pi r} - \frac{e^4}{60\pi^2 m^2} \delta^3(\vec{r}) \qquad (2.204)$$

68

This modification of the Coulomb potential modifies the energy levels of the hydrogen atom, and it is one of the contributions to the Lamb shift, which produces a splitting of the levels $2S_{1/2}$ and $2P_{1/2}$. The total Lamb shift is the sum of all the self-energy and vertex corrections, and turns out to be about $1057.9\ MHz$. The contribution we have just calculated is only $-27.1\ MHz$, but it is important since the agreement between experiment and theory is of the order of $0.1\ MHz$.

We have now to discuss the vertex corrections. We have seen that the divergent contribution is $\Lambda_\mu^{(1)}(p',p)$, and this is proportional to γ_μ. The counter term to add to interacting part of $\mathcal{L}_p$

$$\mathcal{L}_p^{\text{int}} = -e\bar{\psi}\gamma_\mu\psi A^\mu \tag{2.205}$$

is

$$(\mathcal{L}_3)_{\text{ct}} = -eD\bar{\psi}\gamma_\mu\psi A^\mu \tag{2.206}$$

The complete vertex is given by (see Fig. 2.7.5)

$$-ie\left[\gamma_\mu + e^2\Lambda^\mu + D\gamma_\mu\right] \tag{2.207}$$

.
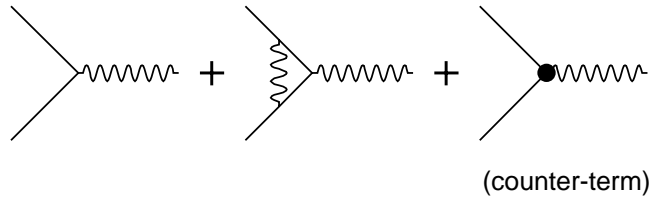
(counter-term)

Fig. 2.7.5 - *The one loop vertex corrections.*

We fix the counter term by

$$D = -\frac{e^2}{8\pi^2}\left[\frac{1}{\epsilon} + F_2\right] \tag{2.208}$$

with F_2 the same as in eq. (2.7). The reason is that with this choice we get the equality of the wave function renormalization factors $Z_1 = Z_2$. This equality follows from the conservation of the current, and therefore we can use the arbitrariness in the finite part of the vertex counter term to guarantee it. The one-loop vertex is then

$$(\Lambda^\mu)' = \left[\gamma_\mu + e^2(\Lambda_\mu^f - \frac{F_2}{8\pi^2}\gamma_\mu)\right] \tag{2.209}$$

One can see that this choice of F_2 is such that for spinors on shell the vertex is the free one

$$\bar{u}(p)(\Lambda^\mu)'u(p)|_{\hat{p}=m} = \bar{u}(p)\gamma^\mu u(p) \tag{2.210}$$

69

We will evaluate now the radiative corrections to the $g - 2$ of the electron. Here g is the gyromagnetic ratio, which the Dirac equation predicts to be equal to be two. To this end we first need to prove the Gordon identity for the current of a Dirac particle

$$\bar{u}(p')\gamma_\mu u(p) = \bar{u}(p') \left[\frac{p^\mu + p^{\mu\prime}}{2m} + \frac{i}{2m}\sigma_{\mu\nu}q^\nu \right] u(p) \qquad (2.211)$$

con $q = p' - p$. For the proof we start from

$$\hat{p}\gamma_\mu u(p) = (-m\gamma_\mu + 2p_\mu)u(p) \qquad (2.212)$$

and

$$\gamma_\mu \hat{p} u(p) = m\gamma_\mu \qquad (2.213)$$

Subtracting these two expressions we obtain

$$\gamma_\mu u(p) = \left(\frac{p_\mu}{m} - \frac{i}{m}\sigma_{\mu\nu}p^\nu \right) u(p) \qquad (2.214)$$

An analogous operation on the barred spinor leads to the result. We observe also that the Gordon identity shows immediately that $g = 2$, because it implies that the coupling with the electromagnetic field is just

$$\frac{e}{2m}\sigma_{\mu\nu}F^{\mu\nu}(q) \qquad (2.215)$$

To evaluate the correction to this term from the one loop diagrams, it is enough to evaluate the matrix element

$$e^2 \bar{u}(p')\Lambda_\mu^{(2)}(p', p)u(p) \qquad (2.216)$$

in the limit $p' \to p$ and for on-shell momenta. In fact $\Lambda_\mu^{(1)}$ contributes only to the terms in γ_μ, and in the previous limit they have to build up the free vertex, as implied by the renormalization condition. Therefore we will ignore all the terms proportional to γ_μ and we will take the first order in the momentum q. For momenta on shell, the denominator of $\Lambda_\mu^{(2)}$ is given by

$$[...] = m^2(x + y) - m^2 x(1 - x) - m^2 y(1 - y) + 2m^2 xy = m^2(x + y)^2 \qquad (2.217)$$

In order to evaluate the numerator, let us define

$$V_\mu = \bar{u}(p')\gamma^\alpha \left[\hat{p}'(1 - y) - \hat{p}x + m \right] \gamma^\mu \left[\hat{p}(1 - x) - \hat{p}'y + m \right] \gamma_\alpha u(p) \qquad (2.218)$$

Using $\hat{p}\gamma_\alpha = -\gamma_\alpha \hat{p} + 2p_\alpha$, and an analogous equation for $\hat{p}'$, we can bring $\hat{p}$ to act on the spinor at the right of the expression, and $\hat{p}'$ on the spinor at the left, obtaining

$$V_\mu = \bar{u}(p') \left[my\gamma^\alpha + 2(1 - y)p'^\alpha - \gamma^\alpha \hat{p}x \right] \gamma^\mu \left[mx\gamma_\alpha + 2(1 - x)p_\alpha - \hat{p}'y\gamma_\alpha \right] u(p) \qquad (2.219)$$

Making use of

$$\gamma_\alpha \gamma^\mu \gamma_\alpha = -2\gamma^\mu \tag{2.220}$$

$$\gamma_\alpha \gamma_\mu \gamma_\nu \gamma^\alpha = 4g_{\mu\nu} \tag{2.221}$$

$$\gamma_\alpha \hat{p} \gamma^\mu \hat{p}' \gamma^\alpha = -2\hat{p}' \gamma^\mu \hat{p} \tag{2.222}$$

we get

$$
\begin{aligned}
V_\mu &= \bar{u}(p') \Big[-2m^2 xy\gamma^\mu + 2my(1-x)(-m\gamma^\mu + 2p^\mu) \\
&- 4my^2 p'^\mu + 2mx(1-y)(-m\gamma^\mu + 2p'^\mu) + 4(1-x)(1-y)m^2\gamma^\mu \\
&- 2y(1-y)m^2\gamma^\mu - 4mx^2 p^\mu - 2m^2 x(1-x)\gamma^\mu - 2xym^2\gamma^\mu \Big] u(p) \tag{2.223}
\end{aligned}
$$

and for the piece which does not contain γ_μ

$$V_\mu = 4m\bar{u}(p') \left[p^\mu(y - xy - x^2) + p'^\mu(x - xy - y^2) \right] u(p) \tag{2.224}$$

Therefore the relevant part of the vertex contribution is

$$
\begin{aligned}
e^2 \bar{u}(p') \Lambda_\mu^{(2)}(p', p)u(p) \quad &\rightarrow \quad -\frac{e^2}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \, \frac{4}{m} \bar{u}(p') \Big[p^\mu \frac{(y - xy - x^2)}{(x+y)^2} \\
&+ \quad p'^\mu \frac{(x - xy - y^2)}{(x+y)^2} \Big] u(p) \tag{2.225}
\end{aligned}
$$

By changing variable $z = x + y$ we have

$$
\begin{aligned}
\int_0^1 dx \int_0^{1-x} dy \frac{y - xy - x^2}{(x+y)^2} &= \int_0^1 dx \int_x^1 dz \left[\frac{1-x}{z} - \frac{x}{z^2} \right] \\
&= \int_0^1 \left[-(1-x)\log x + x - 1 \right] \\
&= \left[-x\log x + x + \frac{x^2}{2}\log x - \frac{x^2}{4} + \frac{x^2}{2} - x \right]_0^1 = \frac{1}{4} \tag{2.226}
\end{aligned}
$$

from which

$$e^2 \bar{u}(p') \Lambda_\mu^{(2)}(p', p)u(p) \rightarrow -\frac{e^2}{16\pi^2 m} \bar{u}(p')[p_\mu + p'_\mu]u(p) \tag{2.227}$$

Using the Gordon in this expression, and eliminating the further contribution in γ_μ we obtain

$$e^2 \bar{u}(p') \Lambda_\mu^{(2)}(p', p)u(p)|_{\text{mom. magn.}} \rightarrow \frac{ie^2}{16\pi^2 m} \bar{u}(p')\sigma_{\mu\nu}q^\nu u(p) \tag{2.228}$$

Finally we have to add this correction to the vertex part taken at $p' = p$, which coincides with the free vertex

$$\bar{u}(p')[\gamma_\mu + \frac{ie^2}{16\pi^2 m}\sigma_{\mu\nu}q^\nu]u(p) = \bar{u}(p')[\frac{p_\mu + p'_\mu}{2m} + \frac{i}{2m}\left(1 + \frac{e^2}{8\pi^2}\right)\sigma_{\mu\nu}q^\nu]u(p) \tag{2.229}$$

71

Therefore the correction is

$$\frac{e}{2m} \rightarrow \frac{e}{2m}\left(1 + \frac{\alpha}{2\pi}\right) \tag{2.230}$$

Recalling that g is the ratio between $\vec{S} \cdot \vec{B}$ and $e/2m$, we get

$$\frac{g}{2} = 1 + \frac{\alpha}{2\pi} + \mathcal{O}(\alpha^2) \tag{2.231}$$

This correction was evaluated by Schwinger in 1948. Actually we know the first three terms of the expansion

$$a_{\mathrm{th}} = \frac{1}{2}(g-2) = \frac{1}{2}\frac{\alpha}{\pi} - 0.32848\left(\frac{\alpha}{\pi}\right)^2 + 1.49(\frac{\alpha}{\pi})^3 + \cdots = (1159652.4 \pm 0.4) \times 10^{-9} \tag{2.232}$$

to be compared with the experimental value

$$a_{\mathrm{exp}} = (1159652.4 \pm 0.2) \times 10^{-9} \tag{2.233}$$

Chapter 3

Vacuum expectation values and the S matrix

3.1 In- and Out-states

For the future we will need a formulation of field theory showing that any S-matrix element can be reduced to the evaluation of the vacuum expectation value of a T-product of field operators. Such a formulation is called LSZ (from the authors l (Lehaman, Symanzik and Zimmermann). This formalism is based on a series of very general properties as displacement and Lorentz invariance. It is interesting to see how far one can go in determining the features of a field theory by using only invariance arguments. In the following we will work in the Heisenberg picture, that is the operators will be time-dependent and evolving with the full hamiltonian. Formally this representation can be obtained from eq. (1.217) by sending the free hamiltonian into the full hamiltonian. It should be also kept in mind that since no exact solution of an interacting field theory is known, we are not sure that the assumptions we are going to make are consistent. Said that, our assumptions are:

1. The eigenvalues of the four-momentum lie within the forward light cone

$$p^2 = p^\mu p_\mu \geq 0, \qquad p^0 \geq 0 \tag{3.1}$$

2. There exists a nondegenerate Lorentz invariant states of minimum energy. This is called the **vacuum state**

$$\Phi_0 \equiv |0\rangle \tag{3.2}$$

   By convention, we assume that the corresponding energy is zero

$$\mathbf{P}_0|0\rangle = 0 \tag{3.3}$$

   From eq. (3.1) we get also

$$\vec{\mathbf{P}}|0\rangle = 0 \rightarrow \mathbf{P}_\mu|0\rangle = 0 \tag{3.4}$$

Since $p_\mu = 0$ is a Lorentz invariant condition, it follows that Φ_0 appears as the vacuum states in all the Lorentz frames.

3. For each stable particle of mass m, there exists a stable stae of single particle

$$\Phi_1 \equiv |p\rangle \qquad (3.5)$$

where Φ_1 is a momentum eigenstate with eigenvalue p_μ such that $p^2 = m^2$.

4. Neglecting the complications from the zero mass states we can add a further requirement: the vacuum and the single particle states give a discrete spectrum in p_μ. For instance, the case of the π mesons is given in Fig. 3.1.
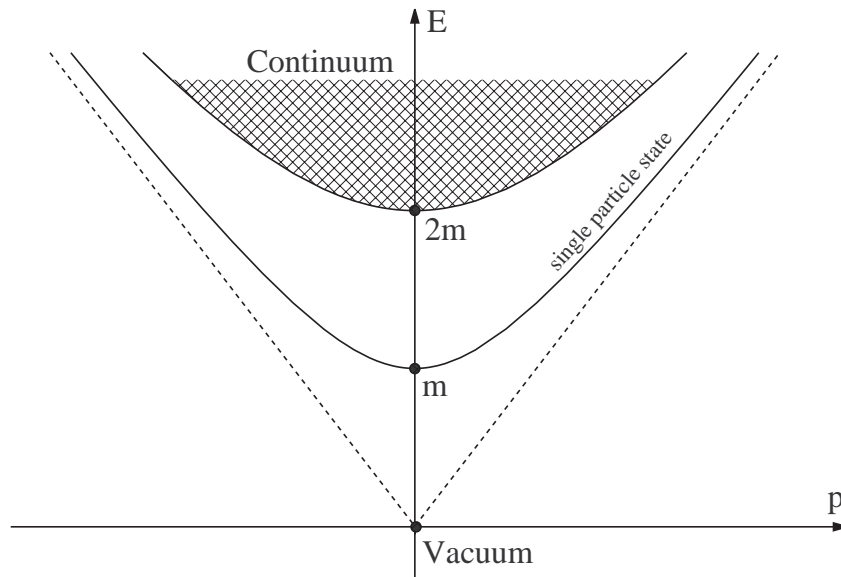
Fig. 3.1 - *Energy-momentum spectrum for particles of mass m.*

Remember that the pion is not a stable particle and decays into $\mu + \bar{\nu}_\mu$ with a lifetime of the order of $\approx 2 \cdot 10^{-8}$ sec., but this number is very large with respect to the natural decay frequency $1/m_\pi \approx 5 \cdot 10^{-24}$ sec. $(m_\pi \approx 140\ MeV)$. Therefore, by neglecting the weak interactions (responsible for the decay) we can assume that the pion is stable. The same argument can be applied to quarks and leptons that will be the main actors in the following. However we mentioned also the pions, to say that the arguments presented here can be applied (in the appropriate range of energy) also to composite particles. Notice that this assumption is very much in the spirit of perturbation theory. That is we introduce a field for each particle, assuming that the interactions do not modify the spectrum in a violent way.

The main tools of investigation in the field of elementary particle physics are the scattering experiments. Ideally these experiments are realized by preparing at

$t = -\infty$ a system of free particles. That is assuming that they do not interact each other (this requires that the particles are well separated in space). These particles will interact with a target at some finite time. After that scattering process we imagine that at $t = +\infty$ the emerging particles will behave again as free particles. In real life the times $t = \pm\infty$ correspond to laboratory time. That is we prepare the beam at some finite time $-T$, the scattering process will occur around the time $t = 0$ (assuming this as the time scale) and then we will measure the outgoing particles at a time T. The ideal description considered before can be considered as correct if the interaction time Δt around $t = 0$ is much smaller that T. For instance, imagine an electron prepared in an eigenstate of momentum p, impinging over an atom. Here there is a further simplification that we are doing since in a finite time or a finite volume we cannot prepare a state of definite momentum. This is one of the reasons why analyzing a scattering process it is always convenient to quantize in a box of the dimension of the experiment itself (that is about the distance between the source of the beam and the detector). The same consideration holds for the energy, that is one has always to remember that the experiments is taking place between to finite times $-T$ and T. Having prepared the electron in the initial state at the time $-T$, the particle will travel since it will reach the target where it will be subject to the interaction with the atoms of target itself. The interaction time will be of the order of the time necessary to cross the atomic dimensions for the case of a single scattering, or, at maximum, of the order of the thickness of the target for a multiple scattering. After that we will let again the electron to move freely since it will reach the detector where we will measure its properties as momentum and polarization. In conclusion we can think to describe the particles at $t = \pm\infty$ in terms of free fields, except that they will be subject to the self-interaction which, as we know, cannot be neglected. The free fields that we are going to use will be denoted by $\phi_{\text{in}}(x)$ for $t \to -\infty$ and by $\phi_{\text{out}}(x)$ for $t \to +\infty$. The interacting field $\phi(x)$) can be thought of as constructed in terms of these free fields operators and such that at $t \to \pm\infty$ it reduces to $\phi_{\text{out}}(x)$ or $\phi_{\text{in}}(x)$. In order that the in- and out-fields describe correctly the incoming and outgoing free particles we have to require some properties. In particular we will require

1. $\phi_{\text{in}}(x)$ must transform as the corresponding $\phi(x)$ with respect to the symmetries of the theory. In particular, with respect to translation it must satisfy

$$[\mathbf{P}_\mu,] = -i\frac{\partial\phi_{\text{in}}(x)}{\partial x^\mu} \tag{3.6}$$

equivalent to require

$$\phi_{\text{in}}(x + a) = e^{iP \cdot a}\phi_{\text{in}}(x)e^{-iP \cdot a} \tag{3.7}$$

2. $\phi_{\text{in}}(x)$ must satisfy the free Klein-Gordon equation (for a scalar field, or the Dirac free equation for a spinor field, etc.) corresponding to the **physical mass** m

$$(\Box + m^2)\phi_{\text{in}}(x) = 0 \tag{3.8}$$

75

Notice that this requirement implies that we have taken into account the self-interactions of the field which are responsible for the mass renormalization (we will mention the wave function renormalization later).

The same properties are required for the out-fields. By using these two properties it follows at once that the in-field creates from the vacuum the physical one particle state from the vacuum. In fact, let $|p\rangle$ an eigenstate of $\mathbf{P}_\mu$ with eigenvalue p_μ. We have

$$-i\frac{\partial}{\partial x^\mu}\langle p|\phi_{\text{in}}(x)|0\rangle = \langle p|[\mathbf{P}_\mu, \phi_{\text{in}}(x)]|0\rangle = p_\mu\langle p|\phi_{\text{in}}(x)|0\rangle \tag{3.9}$$

By applying $-i\partial/\partial x_\mu$ once more, we find

$$-\Box\langle p|\phi_{\text{in}}(x)|0\rangle = p^2\langle p|\phi_{\text{in}}(x)|0\rangle \tag{3.10}$$

and using the Klein-Gordon equation

$$(p^2 - m^2)\langle p|\phi_{\text{in}}(x)|0\rangle = 0 \tag{3.11}$$

We see that the only state that $\phi_{\text{in}}(x)$ can create out of the vacuum is the one with p_μ such that $p^2 = m^2$. From our starting hypotheses we have that this is the state of single particle with mass m. Since the in- and out-fields are free fields we can apply all the corresponding formalism. In particular they can be Fourier expanded

$$\phi_{\text{in}}(x) = \int d^3\vec{k}\left[a_{\text{in}}(\vec{k})f_{\vec{k}}(x) + a_{\text{in}}^\dagger(\vec{k})f_{\vec{k}}^*(x)\right] \tag{3.12}$$

with

$$f_{\vec{k}}(x) = \frac{1}{\sqrt{(2\pi)^3 2\omega_{\vec{k}}}}e^{-ikx}, \qquad \omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2} \tag{3.13}$$

Inverting this relation, we find

$$a_{\text{in}}(\vec{k}) = i\int d^3\vec{x}\left[f_{\vec{k}}^*(x)\partial_0\phi_{\text{in}}(x) - \partial_0 f_{\vec{k}}^*(x)\phi_{\text{in}}(x)\right] \tag{3.14}$$

from which

$$\begin{aligned}
[\mathbf{P}_i, a_{\text{in}}(\vec{k})] &= \int d^3\vec{x}\left[f_{\vec{k}}^*(x)\left(\frac{\partial}{\partial x^0}\frac{\partial\phi_{\text{in}}(x)}{\partial x^i}\right) - \left(\frac{\partial}{\partial x^0}f_{\vec{k}}^*(x)\right)\frac{\partial\phi_{\text{in}}(x)}{\partial x^i}\right] \\
&= -\int d^3\vec{x}\left[\frac{\partial f_{\vec{k}}^*(x)}{\partial x_i}\left(\frac{\partial}{\partial x^0}\phi_{\text{in}}(x)\right) - \left(\frac{\partial}{\partial x^0}\frac{\partial f_{\vec{k}}^*(x)}{\partial x_i}\right)\phi_{\text{in}}(x)\right] \\
&= -k_i a_{\text{in}}(\vec{k}) \tag{3.15}
\end{aligned}$$

Since $f_{\vec{k}}^*(x)$ and $\phi_{\text{in}}(x)$ both satisfy the free Klein-Gordon equation, we see that the operators defined in eq. (3.14) are time independent. This follows immediately, by

recalling that if $\phi_1(x)$ and $\phi_2(x)$ are both solutions of the Klein-Gordon equation, then the four-vector

$$j_\mu = \phi_1 \frac{\partial \phi_2}{\partial x^\mu} - \frac{\partial \phi_1}{\partial x^\mu} \phi_2 \tag{3.16}$$

is a conserved current, and therefore the charge

$$Q = \int d^3 \vec{x} j_0 \tag{3.17}$$

is a constant of motion. Using this, it follows

$$
\begin{aligned}
[\mathbf{P}_0, a_{\text{in}}(\vec{k})] &= \int d^3 \vec{x} \left[f_{\vec{k}}^*(x) \left(\frac{\partial}{\partial x^0} \frac{\partial \phi_{\text{in}}(x)}{\partial x^0} \right) - \left(\frac{\partial}{\partial x^0} f_{\vec{k}}^*(x) \right) \frac{\partial \phi_{\text{in}}(x)}{\partial x^0} \right] \\
&= -\int d^3 \vec{x} \left[\frac{\partial f_{\vec{k}}^*(x)}{\partial x^0} \left(\frac{\partial}{\partial x^0} \phi_{\text{in}}(x) \right) - \left(\frac{\partial}{\partial x^0} \frac{\partial f_{\vec{k}}^*(x)}{\partial x^0} \right) \phi_{\text{in}}(x) \right] \\
&= -k_0 a_{\text{in}}(\vec{k}) \tag{3.18}
\end{aligned}
$$

In conclusion

$$[\mathbf{P}_\mu, a_{\text{in}}(\vec{k})] = -k_\mu a_{\text{in}}(\vec{k}) \tag{3.19}$$

Also, by hermitian conjugation we get

$$[\mathbf{P}_\mu, a_{\text{in}}^\dagger(\vec{k})] = k_\mu a_{\text{in}}^\dagger(\vec{k}) \tag{3.20}$$

It follows that $a_{\text{in}}^\dagger(\vec{k})$ creates states of four-momentum k_μ whereas $a_{\text{in}}(\vec{k})$ annihilates states with momentum k_μ. For instance,

$$\mathbf{P}_\mu a_{\text{in}}^\dagger(\vec{k}_1) a_{\text{in}}^\dagger(\vec{k}_2)|0\rangle = [\mathbf{P}_\mu, a_{\text{in}}^\dagger(\vec{k}_1) a_{\text{in}}^\dagger(\vec{k}_2)]|0\rangle = (k_1 + k_2)_\mu a_{\text{in}}^\dagger(\vec{k}_1) a_{\text{in}}^\dagger(\vec{k}_2)|0\rangle \tag{3.21}$$

$$\mathbf{P}_\mu a_{\text{in}}(\vec{k}_1) a_{\text{in}}^\dagger(\vec{k}_2)|0\rangle = [\mathbf{P}_\mu, a_{\text{in}}(\vec{k}_1) a_{\text{in}}^\dagger(\vec{k}_2)]|0\rangle = (k_2 - k_1)_\mu a_{\text{in}}(\vec{k}_1) a_{\text{in}}^\dagger(\vec{k}_2)|0\rangle \tag{3.22}$$

By using the condition on the spectrum of the momentum operator we get also

$$a_{\text{in}}(\vec{k})|0\rangle = 0 \tag{3.23}$$

since, otherwise, this state would have a negative eigenvalue of the energy. We get also

$$\langle p_1, \cdots, p_M; \text{in}|k_1, \cdots, k_N; \text{in}\rangle = 0 \tag{3.24}$$

unless $M = N$ and the set $(p_1, \cdots, p_M)$ is identical to the set $(k_1, \cdots, k_N)$.

Let us now look for a relation between the in- and the ϕ-field. The field $\phi(x)$ will satisfy an equation of the type

$$(\Box + m^2)\phi(x) = j(x) \tag{3.25}$$

where $j(x)$ (the scalar current) contains the self-interactions of the fields and possible interactions with other fields in the theory. We need to solve this equation with the

boundary conditions at $t = \pm\infty$ relating $\phi(x)$ with the in- and out-fields. Notice that in writing the previous equation we have included the effects of mass renormalization inside the current $j(x)$, but we know that at the same time we have also a wave function renormalization. Therefore asymptotically the field $\phi(x)$ cannot be the same as the in- and out-fields but it will differ by a factor $\sqrt{Z}$, where $\sqrt{Z}$ is the wave function renormalization. Therefore, the required asymptotic conditions are

$$\lim_{t \to -\infty} \phi(x) = \lim_{t \to -\infty} \sqrt{Z}\phi_{\text{in}}(x), \quad \lim_{t \to +\infty} \phi(x) = \lim_{t \to +\infty} \sqrt{Z}\phi_{\text{out}}(x) \tag{3.26}$$

The solution for $\phi(x)$ is then

$$\phi(x) = \sqrt{Z}\phi_{\text{in}}(x) + \int d^4y \Delta_{\text{ret}}(x - y; m^2)j(y) \tag{3.27}$$

We will see in a moment the meaning of Z in this context. The function $\Delta_{\text{ret}}(x; m^2)$ is the retarded Green's function, defined by

$$(\Box + m^2)\Delta_{\text{ret}}(x; m^2) = \delta^4(x) \tag{3.28}$$

such that

$$\Delta_{\text{ret}}(x; m^2) = 0, \quad \text{for} \quad x_0 < 0 \tag{3.29}$$

The need of Z in this context, is because we want the the in-field properly normalized in order to produce single particle states. In fact, both $\langle 1|\phi_{\text{in}}(x)|0\rangle$ e $\langle 1|\phi(x)|0\rangle$ have the same x-dependence (they must be both proportional to $\exp(ipx)$, see eq. (3.6)), but $\phi(x)$ may create out of the vacuum states other than the single particle state, and therefore the normalization of the two previous matrix elements cannot be the same.

It can be shown that the asymptotic condition (3.26), cannot hold as an operator statement. This is because it is not possible to isolate the operator $\phi(x)$ from the scalar current $j(x)$ at $t = -\infty$, since this includes also the self-interactions. However the statement can be given in a weak form for the matrix elements. The correct way of doing it is first to smear out the field over a space-like region

$$\phi^f(t) = i \int d^3x \left(f^*(\vec{x}, t)\partial_0 \phi(\vec{x}, t) - \partial_0 f^*(\vec{x}, t)\phi(\vec{x}, t) \right) \tag{3.30}$$

with $f(\vec{x}, t)$ an arbitrary normalizable solution of the Klein-Gordon equation

$$(\Box + m^2)f(x) = 0 \tag{3.31}$$

Then the asymptotic condition is formulated as

$$\lim_{t \to -\infty} \langle \alpha|\phi^f(t)|\beta\rangle = \sqrt{Z} \lim_{t \to -\infty} \langle \alpha|\phi^f(t)_{\text{in}}|\beta\rangle \tag{3.32}$$

where $|\alpha\rangle$ and $|\beta\rangle$ are any two normalizable states.

Similar considerations can be made for the out-field. In particular we can find a relation between the in- and out-fields

$$\phi(x) = \sqrt{Z}\phi_{\text{out}}(x) + \int d^4y \Delta_{\text{adv}}(x - y; m^2)j(y) \tag{3.33}$$

where

$$\Delta_{\text{adv}}(x; m^2) = 0, \quad \text{for} \quad x_0 > 0 \tag{3.34}$$

$$(\Box + m^2)\Delta_{\text{adv}}(x; m^2) = \delta^4(x) \tag{3.35}$$

is the advanced Green's function. By comparing this expression with eq. (3.27) we find

$$\sqrt{Z}\phi_{\text{in}}(x) = \sqrt{Z}\phi_{\text{out}}(x) + \int d^4y \Delta(x - y; m^2)j(y) \tag{3.36}$$

with

$$\Delta(x - y; m^2) = \Delta_{\text{adv}}(x - y; m^2) - \Delta_{\text{ret}}(x - y; m^2) \tag{3.37}$$

The construction of the out-operators of creation and annihilation goes along the same lines discussed for the in-operators.

3.2 The S matrix

We have shown in the previous Section that an initial state of n non-interacting particles can be described by

$$|p_1, \cdots, p_n; \text{in}\rangle = a_{\text{in}}^\dagger(p_1) \cdots a_{\text{in}}^\dagger(p_n)|0; \text{in}\rangle \equiv |\alpha; \text{in}\rangle \tag{3.38}$$

where α stands for the all collection of the quantum numbers of the initial state. After the scattering process we are interested to evaluate the probability amplitude for the final state to be a state of non-interacting particles described by the out-operators

$$|p_1', \cdots, p_n'; \text{out}\rangle = a_{\text{out}}^\dagger(p_1') \cdots a_{\text{out}}^\dagger(p_n')|0; \text{out}\rangle \equiv |\beta; \text{out}\rangle \tag{3.39}$$

The probability amplitude is given by the S matrix elements

$$S_{\beta\alpha} = \langle\beta; \text{out}|\alpha; \text{in}\rangle \tag{3.40}$$

We can also define an operator S transforming the in-states into the out-states

$$\langle\beta; \text{in}|S = \langle\beta; \text{out}| \tag{3.41}$$

such that

$$S_{\beta\alpha} = \langle\beta; \text{in}|S|\alpha; \text{in}\rangle \tag{3.42}$$

From the definition we can derive some important properties for the operator S

1. Since the vacuum is stable and not degenerate, it follows $|S_{00}| = 1$, and therefore
$$\langle 0\text{in}|S = \langle 0\text{out}| = e^{i\phi}\langle 0\text{in}| \tag{3.43}$$
By choosing the phase equal to zero we get
$$|0; \text{out}\rangle = |0; \text{in}\rangle \equiv |0\rangle \tag{3.44}$$

2. In an analogous way the stability of the one particle state requires
$$|p; \text{in}\rangle = |p; \text{out}\rangle \tag{3.45}$$

3. The operator S maps the in- to the out-fields
$$\phi_{\text{in}}(x) = S\phi_{\text{out}}(x)S^{-1} \tag{3.46}$$
In fact, let us consider the following matrix element
$$\langle \beta; \text{out}|\phi_{\text{out}}|\alpha; \text{in}\rangle = \langle \beta; \text{in}|S\phi_{\text{out}}|\alpha; \text{in}\rangle \tag{3.47}$$
But $\langle \beta; \text{out}|\phi_{\text{out}}$ is an out-state and therefore
$$\langle \beta; \text{out}|\phi_{\text{out}} = \langle \beta; \text{in}|\phi_{\text{in}}S \tag{3.48}$$
Then
$$\langle \beta; \text{out}|\phi_{\text{out}}(x)|\alpha; \text{in}\rangle = \langle \beta; \text{in}|S\phi_{\text{out}}(x)|\alpha; \text{in}\rangle = \langle \beta; \text{in}|\phi_{\text{in}}(x)S|\alpha; \text{in}\rangle \tag{3.49}$$
which shows the relation (3.46).

4. The S is unitary. This follows from the very definition (3.41) implying
$$S^{\dagger}|\alpha; \text{in}\rangle = |\alpha; \text{out}\rangle \tag{3.50}$$
from which
$$\langle \beta; \text{in}|SS^{\dagger}|\alpha; \text{in}\rangle = \langle \beta; \text{out}|\alpha; \text{out}\rangle = \delta_{\alpha\beta} \tag{3.51}$$
showing that
$$SS^{\dagger} = 1 \tag{3.52}$$
From eq. (3.50) and the unitarity of S we get
$$|\alpha; \text{in}\rangle = S^{\dagger^{-1}}|\alpha; \text{out}\rangle = S|\alpha; \text{out}\rangle \tag{3.53}$$

5. S is Lorentz invariant, as it follows from the fact that transform in- into out-fields having the same Lorentz properties. For the same reason the S matrix is invariant under any symmetry of the theory.

3.3 The reduction formalism

In this Section we will describe a systematic way of evaluating the S matrix elements in terms of the vacuum expectation values of T-products of field operators. Let us start considering the following S matrix element

$$S_{\beta,\alpha p} = \langle \beta; \text{out}|\alpha, p; \text{in}\rangle \tag{3.54}$$

where $|\alpha, p; \text{in}\rangle$ is the state of a set of particles α plus an additional incoming particle of momentum p_μ. Here we will consider the case of scalar particles, but the formalism can be easily extended to other cases. We want to show that, by using the asymptotic condition, it is possible to **extract** the particle of momentum p_μ from the state by introducing the corresponding field operator. Let us consider the following identities

$$
\begin{aligned}
\langle \beta; \text{out}|\alpha, p; \text{in}\rangle &= \langle \beta; \text{out}|a_{\text{in}}^\dagger(p)|\alpha; \text{in}\rangle \\
&= \langle \beta; \text{out}|a_{\text{out}}^\dagger(p)|\alpha; \text{in}\rangle + \langle \beta; \text{out}|a_{\text{in}}^\dagger(p) - a_{\text{out}}^\dagger(p)|\alpha; \text{in}\rangle \\
&= \langle \beta - p; \text{out}|\alpha; \text{in}\rangle \\
&\quad - i \int d^3\vec{x} \Big[f_{\vec{p}}(x) \left(\frac{\partial}{\partial x^0} \langle \beta; \text{out}|\phi_{\text{in}}(x) - \phi_{\text{out}}(x)|\alpha; \text{in}\rangle \right) \\
&\quad - \left(\frac{\partial}{\partial x^0} f_{\vec{p}}(x) \right) \langle \beta; \text{out}|\phi_{\text{in}}(x) - \phi_{\text{out}}(x)|\alpha; \text{in}\rangle \Big]
\end{aligned}
\tag{3.55}
$$

where we have used the adjoint of eq. (3.14). We have denoted $|\beta - p; \text{out}\rangle$ the out-state obtained by removing the particle with momentum p_μ, if present in the state, otherwise it must be taken as the null vector. As we know the integral appearing in the previous expression is time-independent since $f_{\vec{p}}(x)$, $\phi_{\text{in}}(x)$ and $\phi_{\text{out}}(x)$ are solutions of the Klein-Gordon equation. Therefore we are allowed to make the following substitutions (in the weak sense, since we are taking matrix elements)

$$
\begin{aligned}
\lim_{x^0 \to -\infty} \phi_{\text{in}}(x) &= \lim_{x^0 \to -\infty} \frac{1}{\sqrt{Z}} \phi(x) \\
\lim_{x^0 \to +\infty} \phi_{\text{out}}(x) &= \lim_{x^0 \to +\infty} \frac{1}{\sqrt{Z}} \phi(x)
\end{aligned}
\tag{3.56}
$$

obtaining

$$
\begin{aligned}
\langle \beta; \text{out}|\alpha, p; \text{in}\rangle &= \langle \beta - p; \text{out}|\alpha; \text{in}\rangle \\
&\quad + \frac{i}{\sqrt{Z}} \left(\lim_{x^0 \to +\infty} - \lim_{x^0 \to -\infty} \right) \int d^3\vec{x} \Big[f_{\vec{p}}(x) \left(\frac{\partial}{\partial x^0} \langle \beta; \text{out}|\phi(x)|\alpha; \text{in}\rangle \right) \\
&\quad - \left(\frac{\partial}{\partial x^0} f_{\vec{p}}(x) \right) \langle \beta; \text{out}|\phi(x)|\alpha; \text{in}\rangle \Big]
\end{aligned}
\tag{3.57}
$$

By using the identity

$$
\int_{-\infty}^{+\infty} d^4x \frac{\partial}{\partial x^0} \left[g_1(x) \left(\frac{\partial}{\partial x^0} g_2(x) \right) - \left(\frac{\partial}{\partial x^0} g_1(x) \right) g_2(x) \right] =
$$

$$
\begin{aligned}
= & \left(\lim_{x^0 \to +\infty} - \lim_{x^0 \to -\infty} \right) \int d^3\vec{x} \left[g_1(x) \left(\frac{\partial}{\partial x^0} g_2(x) \right) - \left(\frac{\partial}{\partial x^0} g_1(x) \right) g_2(x) \right] \\
= & \int_{-\infty}^{+\infty} d^4x \left[g_1(x) \ddot{g}_2(x) - \ddot{g}_1(x) g_2(x) \right] \quad\quad\quad\quad\quad\quad\quad\quad (3.58)
\end{aligned}
$$

we can write

$$
\begin{aligned}
\langle \beta; \text{out} | \alpha, p; \text{in} \rangle = & \ \langle \beta - p; \text{out} | \alpha; \text{in} \rangle \\
& + \ \frac{i}{\sqrt{Z}} \int d^4x \langle \beta; \text{out} | \left[f_{\vec{p}}(x) \ddot{\phi}(x) - \ddot{f}_{\vec{p}}(x) \phi(x) \right] | \alpha; \text{in} \rangle \quad (3.59)
\end{aligned}
$$

By using the Klein-Gordon equation for $f_{\vec{p}}(x)$ we get

$$
\begin{aligned}
\langle \beta; \text{out} | \alpha, p; \text{in} \rangle = & \ \langle \beta - p; \text{out} | \alpha; \text{in} \rangle \\
+ \ \frac{i}{\sqrt{Z}} \int d^4x \langle \beta; \text{out} | & \left[f_{\vec{p}}(x) \ddot{\phi}(x) - \left((\nabla^2 - m^2) f_{\vec{p}}(x) \right) \phi(x) \right] | \alpha; \text{in} \rangle \quad (3.60)
\end{aligned}
$$

Integrating by parts

$$
\begin{aligned}
\langle \beta; \text{out} | \alpha, p; \text{in} \rangle = & \ \langle \beta - p; \text{out} | \alpha; \text{in} \rangle \\
& + \ \frac{i}{\sqrt{Z}} \int d^4x f_{\vec{p}}(x) (\Box + m^2) \langle \beta; \text{out} | \phi(x) | \alpha; \text{in} \rangle \quad (3.61)
\end{aligned}
$$

We can iterate this procedure in order to extract all the in- and out-particles from the states. Let us see also the case in which we want to extract from the previous matrix element an out-particle. This exercise is interesting because it shows how the T-product comes about. Let us suppose that the out state is of the type $\beta = (\gamma, p')$, with p' the momentum of a single particle. We want to extract this particle from the state

$$
\begin{aligned}
\langle \beta; \text{out} | \phi(x) | \alpha; \text{in} \rangle = & \ \langle \gamma, p'; \text{out} | \phi(x) | \alpha; \text{in} \rangle = \langle \gamma; \text{out} | a_{\text{out}}(p') \phi(x) | \alpha; \text{in} \rangle \\
= & \ \langle \gamma; \text{out} | [a_{\text{out}}(p') \phi(x) - \phi(x) a_{\text{in}}(p')] | \alpha; \text{in} \rangle + \langle \gamma; \text{out} | \phi(x) a_{\text{in}}(p') | \alpha; \text{in} \rangle \\
= & \ \langle \gamma; \text{out} | \phi(x) | \alpha - p'; \text{in} \rangle \\
+ \ i \int d^3 \vec{y} & \left[f_{\vec{p}'*}(y) \left(\frac{\partial}{\partial y^0} \langle \gamma; \text{out} | (\phi_{\text{out}}(y) \phi(x) - \phi(x) \phi_{\text{in}}(y)) | \alpha; \text{in} \rangle \right) \right. \\
- & \left. \left(\frac{\partial}{\partial y^0} f_{\vec{p}'}^*(y) \right) \langle \gamma; \text{out} | (\phi_{\text{out}}(y) \phi(x) - \phi(x) \phi_{\text{in}}(y)) | \alpha; \text{in} \rangle \right] \quad (3.62)
\end{aligned}
$$

Using again the time independence of the integral we have

$$
\begin{aligned}
& \langle \gamma; \text{out} | [\phi_{\text{out}}(y) \phi(x) - \phi(x) \phi_{\text{in}}(y)] | \alpha; \text{in} \rangle \\
= & \ \frac{1}{\sqrt{Z}} \left(\lim_{y^0 \to +\infty} - \lim_{y^0 \to -\infty} \right) \langle \gamma; \text{out} | T(\phi(y) \phi(x)) | \alpha; \text{in} \rangle \quad (3.63)
\end{aligned}
$$

and eq. (3.58)

$$\langle\beta;\text{out}|\phi(x)|\alpha;\text{in}\rangle = \langle\gamma;\text{out}|\phi(x)|\alpha-p';\text{in}\rangle$$

$$+ \frac{i}{\sqrt{Z}}\left(\lim_{y^0\to+\infty} - \lim_{y^0\to-\infty}\right)\int d^3\vec{y}\left[f^*_{\vec{p}'}(y)\left(\frac{\partial}{\partial y^0}\langle\gamma;\text{out}|T(\phi(y)\phi(x))|\alpha;\text{in}\rangle\right)\right.$$

$$- \left.\left(\frac{\partial}{\partial y^0}f^*_{\vec{p}'}(y)\right)\langle\gamma;\text{out}|T(\phi(y)\phi(x))|\alpha;\text{in}\rangle\right]$$

$$= \langle\gamma;\text{out}|\phi(x)|\alpha-p';\text{in}\rangle$$

$$+ \frac{i}{\sqrt{Z}}\int d^4y\langle\gamma;\text{out}|\left[f^*_{\vec{p}'}(y)\frac{\partial^2}{\partial y_0^2}T(\phi(y)\phi(x)) - \ddot{f}^*_{\vec{p}'}(y)T(\phi(y)\phi(x))\right]|\alpha;\text{in}\rangle$$

$$= \langle\gamma;\text{out}|\phi(x)|\alpha-p';\text{in}\rangle$$

$$+ \frac{i}{\sqrt{Z}}\int d^4y f^*_{\vec{p}'}(y)(\Box+m^2)_y\langle\gamma;\text{out}|T(\phi(y)\phi(x))|\alpha;\text{in}\rangle \qquad (3.64)$$

Substituting inside the (3.61)

$$\langle\beta;\text{out}|\alpha,p;\text{in}\rangle$$

$$= \langle\beta-p;\text{out}|\alpha;\text{in}\rangle$$

$$+ \frac{i}{\sqrt{Z}}\int d^4x f_{\vec{p}}(x)(\Box+m^2)_x\langle\gamma;\text{out}|\phi(x)|\alpha-p';\text{in}\rangle$$

$$+ \left(\frac{i}{\sqrt{Z}}\right)^2\int d^4x d^4y$$

$$f_{\vec{p}}(x)f^*_{\vec{p}'}(y)(\Box+m^2)_x(\Box+m^2)_y\langle\gamma;\text{out}|T(\phi(y)\phi(x))|\alpha;\text{in}\rangle \qquad (3.65)$$

More generally if we want to remove all the particles in $(q_1,\cdots,q_m)$ and in $(p_1,\cdots,p_n)$ with the condition $p_i\neq q_j$ we get

$$\langle p_1,\cdots,p_m;\text{out}|q_1,\cdots,q_n;\text{in}\rangle$$

$$= \left(\frac{i}{\sqrt{Z}}\right)^{m+n}\int\prod_{i,j=1}^{m,n}d^4x_i d^4y_j f_{\vec{q}_i}(x_i)f^*_{\vec{p}_j}(y_j)(\Box+m^2)_{x_i}(\Box+m^2)_{y_j}\cdot$$

$$\langle 0|T(\phi(y_1)\cdots\phi(y_n)\phi(x_1)\cdots\phi(x_m))|0\rangle \qquad (3.66)$$

If some of the q_i's are equal to some of the p_j's, we can reduce further the matrix elements of the type $\langle\beta;\text{out}|\alpha-p;\text{in}\rangle$. Therefore we have shown that it is possible to evaluate all the S matrix elements once one knows the vacuum expectation values of the T-products (called also the n-point Green's function). The result obtained here can be easily extended to generic fields simply by substituting to the wave functions $f_{\vec{p}}(x)$ the corresponding solutions of the wave equation (for instance, for spinor fields, spinors times plane waves) and to the Klein-Gordon operator the corresponding wave operators (for the spinor case the Dirac operator $(i\slashed{\partial}-m)$). In the Dirac case one halso to change $i/\sqrt{Z}$ to $-i/\sqrt{Z_2}$ for fermions and to $+i/\sqrt{Z_2}$ for antifermions, where Z_2 is the wave function renormalization for the spinor field.

We will show in the following that a simple technique to evaluate perturbatively the vacuum expectation value of n fields, a n-point Green's function, can be obtained in the context of the path-integral formulation of quantum mechanics.

Chapter 4

Path integral formulation of quantum mechanics

4.1 Feynman's formulation of quantum mechanics

In 1948 Feynman gave a formulation of quantum mechanics quite different from the standard one. This formulation did not make use of operators and of Schrödinger equation, but rather it was giving an explicit expression for the quantum-mechanical amplitude, although using a mathematical formalism which was far from being well defined, which was obtained directly from the physical idea that the probability amplitude to go from one state to another can be obtained by summing together all the possible ways with an appropriate weight. This was as apply the ordinary composition laws in probability theory, to the **probability amplitudes** (remember that the probability is the square of the amplitude). The key observation to get the appropriate weight was taken by the Dirac's book where it was shown that, for one degree of freedom, the probability amplitude

$$\langle q', t + \Delta t | q, t \rangle \tag{4.1}$$

for Δt infinitesimal, is given by the e^{iS} where S is the classical action to go from q at the time t to q' at the tome $t + \Delta t$. In fact we will show that the Feynman's formulation can be derived from the usual formulation of quantum mechanics. Although the path integral formulation (as the Feynman's approach is usually called) is just a different way of formulating quantum mechanics, at the beginning of the 70's it turned out to be a crucial tool for the quantization of gauge theories. Also it is one the few methods which can be used for studying situations outside the perturbative approach. In order to derive the Feynman's formulation we will start studying a quantum system described by one degree of freedom, and we will see later how this can be extended also to infinite degrees of freedom as needed in quantum field theory. First of all we notice that the quantum mechanical problem is to evaluate

the time evolution, which is given by the propagator. This can be evaluated in any basis, for instance in configuration space, its matrix elements are given by

$$\langle q', t' | q, t \rangle, \quad t' \geq t \tag{4.2}$$

It will be convenient to work in the Heisenberg representation, that is using time-dependent operators $\mathbf{q}(t)$. The states of the previous equation are defined as eigenstates of the operator $\mathbf{q}(t)$ at the time t, that is

$$\mathbf{q}(t) | q, t \rangle = | q, t \rangle q \tag{4.3}$$

It follows immediately that the relation between these states at different times is given by

$$| q, t \rangle = e^{iHt} | q, 0 \rangle \tag{4.4}$$

for a time independent hamiltonian. The matrix element (4.2) can be evaluated by slicing the time interval $t' - t$ in infinitesimal pieces and using the completeness relation. we get

$$
\begin{aligned}
\langle q', t' | q, t \rangle &= \lim_{n \to \infty} \int dq_1 \cdots dq_{n-1} \langle q', t' | q_{n-1}, t_{n-1} \rangle \langle q_{n-1}, t_{n-1} | q_{n-2}, t_{n-2} \rangle \cdot \\
&\qquad \cdots \langle q_1, t_1 | q, t \rangle
\end{aligned}
\tag{4.5}
$$

where

$$t_k = t + k\epsilon, \quad t_0 \equiv t, \quad t_n \equiv t' \tag{4.6}$$

Therefore

$$\langle q', t' | q, t \rangle = \lim_{n \to \infty} \int \left(\prod_{k=1}^{n-1} dq_k \right) \left(\prod_{k=0}^{n-1} \langle q_{k+1}, t_{k+1} | q_k, t_k \rangle \right) \tag{4.7}$$

We have still to evaluate the matrix element (4.2), but now for an infinitesimal time interval ϵ. let us use again the completeness in momentum space with states taken at a time $\tilde{t}$ in between t_k e t_{k+1}:

$$\langle q_{k+1}, t_k + \epsilon | q_k, t_k \rangle = \int dp \langle q_{k+1}, t_k + \epsilon | p, \tilde{t} \rangle \langle p, \tilde{t} | q_k, t_k \rangle \tag{4.8}$$

The mixed matrix elements can be written as

$$\langle q_{k+1}, t_k + \epsilon | p, \tilde{t} \rangle = \langle q_{k+1}, 0 | e^{-iH(\mathbf{p}, \mathbf{q})(t_k + \epsilon - \tilde{t})} | p, 0 \rangle \tag{4.9}$$

(remember $\hbar = 1$). making use of the canonical commutation relations for $\mathbf{p}$ and $\mathbf{q}$, the hamiltonian can be written in such a way that all the operators $\mathbf{q}$ are at the left of all the operators $\mathbf{p}$. The result will be denoted by the symbol $H_+(\mathbf{q}, \mathbf{p})$. By doing so we get

$$
\begin{aligned}
\langle q_{k+1}, t_k + \epsilon | p, \tilde{t} \rangle &\simeq \langle q_{k+1}, 0 | p, 0 \rangle \left(1 - iH_+(q_{k+1}, p)(t_k + \epsilon - \tilde{t}) \right) \\
&\simeq \langle q_{k+1}, 0 | p, 0 \rangle e^{-iH_+(q_{k+1}, p)(t_k + \epsilon - \tilde{t})}
\end{aligned}
\tag{4.10}
$$

The function $H_+(q,p)$ is obtained by substituting in $H_+(\mathbf{q},\mathbf{p})$ the operators $\mathbf{q}$ and $\mathbf{p}$ with their eigenvalues. Notice that although $H(\mathbf{q},\mathbf{p})$ and $H_+(\mathbf{q},\mathbf{p})$ are the same operator, $H(q,p)$ e $H_+(q,p)$ are, in general, different functions. In this way we get

$$\langle q_{k+1}, t_k + \epsilon | p, \tilde{t} \rangle \simeq \frac{1}{\sqrt{2\pi}} e^{i(pq_{k+1} - H_+(q_{k+1},p)(t_{k+1} - \tilde{t}))} \tag{4.11}$$

and

$$\langle p, \tilde{t} | q_k, t_k \rangle \simeq \frac{1}{\sqrt{2\pi}} e^{i(-pq_k - H_-(q_k,p)(\tilde{t} - t_k))} \tag{4.12}$$

where H_- is the eigenvalue of the operator $H_-(\mathbf{p},\mathbf{q})$ defined by putting the operators $\mathbf{q}$ to the right of $\mathbf{p}$ by using the canonical commutation relations. By choosing $\tilde{t} = (t_k + t_{k+1})/2$ and the previous two equations, we get

$$\langle q_{k+1}, t_k + \epsilon | q_k, t \rangle \simeq \int \frac{dp}{2\pi} e^{i\epsilon(p(q_{k+1} - q_k)/\epsilon - H_c(q_{k+1},q_k,p))} \tag{4.13}$$

where

$$H_c(q_{k+1}, q_k, p) = \frac{1}{2}(H_+(q_{k+1},p) + H_-(q_k,p)) \tag{4.14}$$

Since in the limit $\epsilon \to 0$ we have

$$\langle q_{k+1}, t_k + \epsilon | q_k, t_k \rangle \to \delta(q_{k+1} - q_k) \tag{4.15}$$

it is reasonable to define a variable **velocity** as

$$\dot{q}_k = \frac{q_{k+1} - q_k}{\epsilon} \tag{4.16}$$

and write

$$H_c(q_{k+1}, q_k, p) \equiv H(q_k, p) \tag{4.17}$$

In the same limit $H(q_k, p)$ is nothing but the hamiltonian of the system evaluated in the phase space. Therefore

$$\langle q_{k+1}, t_k + \epsilon | q_k, t_k \rangle \simeq \int \frac{dp}{2\pi} e^{i\epsilon L(q_k,p)} \tag{4.18}$$

where

$$L(q_k, p) = p\dot{q}_k - H(q_k, p) \tag{4.19}$$

is the lagrangian in phase space. Coming back to (4.7) and using (4.18) we obtain

$$\langle q', t' | q, t \rangle = \lim_{n \to \infty} \int \left(\prod_{k=1}^{n-1} dq_k \right) \left(\prod_{k=0}^{n-1} \frac{dp_k}{2\pi} \right) e^{i \sum_{k=0}^{n-1} \epsilon L(q_k, p_k)} \tag{4.20}$$

In the limit, the argument of the exponential gives the integral between t and t' of the lagrangian in the phase space, therefore we get the canonical action

$$S = \int_t^{t'} dt\, L(q, p) \tag{4.21}$$

In the limit $n \to \infty$, the multiple integral in eq. (4.20) is called the **functional integral** and we will denote it symbolically as

$$\langle q', t'|q, t\rangle = \int_{q,t}^{q',t'} d\mu(q(t))d\mu(p(t))e^{i\int_t^{t'} dtL(q,p)} \tag{4.22}$$

where

$$d\mu(q(t)) = \prod_{t<t''<t'} dq(t''), \quad d\mu(p(t)) = \prod_{t\leq t''\leq t'} \frac{dp(t'')}{2\pi} \tag{4.23}$$

The asymmetry between the integration in $q(t)$ and $p(t)$ implies that the integral is not invariant under a generic canonical transformation. Notice that the integration is made over all the functions $q(t'')$ such that $q(t) = q$ e $q(t') = q'$. On the contrary there are no limitations on $p(t'')$, as it required by the uncertainty principle.

4.2   Path integral in configuration space

The path integral in eq. (4.20) can be easily written as a functional integral in configuration space when the hamiltonian of the system is of the type

$$H(\mathbf{q}, \mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{q}) \tag{4.24}$$

In such a case

$$H_+(q, p) = H_-(q, p) = H(q, p) = \frac{p^2}{2m} + V(q) \tag{4.25}$$

The momentum integration can be done naively

$$
\begin{aligned}
\langle q', t'|q, t\rangle &= \lim_{n\to\infty} \int \left(\prod_{k=1}^{n-1} dq_k\right) \\
&\quad \cdot \prod_{k=0}^{n-1} \frac{dp_k}{2\pi} e^{i[p_k(q_{k+1}-q_k) - \epsilon p_k^2/2m - \epsilon(V(q_{k+1})+V(q_k))/2]} = \\
&= \lim_{n\to\infty} \int \left(\prod_{k=1}^{n-1} dq_k\right) \left(\frac{1}{2\pi}\sqrt{\frac{2m\pi}{i\epsilon}}\right)^n \prod_{k=0}^{n-1} e^{(-2m/i\epsilon)(q_{k+1}-q_k)^2/4} \\
&\quad \cdot e^{-i\epsilon(V(q_{k+1})+V(q_k))/2} =
\end{aligned}
$$

$$
= \lim_{n \to \infty} \int \left(\sqrt{\frac{m}{2i\pi\epsilon}} \right)^n \left(\prod_{k=1}^{n-1} dq_k \right)
$$

$$
\cdot \prod_{k=0}^{n-1} e^{i\epsilon[m((q_{k+1} - q_k)/\epsilon)^2/2 - (V(q_{k+1}) + V(q_k))/2]} \tag{4.26}
$$

with the result

$$
\langle q', t' | q, t \rangle = \lim_{n \to \infty} \int \left(\sqrt{\frac{m}{2i\pi\epsilon}} \right)^n \left(\prod_{k=1}^{n-1} dq_k \right)
$$

$$
\cdot \; e^{\displaystyle i\epsilon \sum_{k=0}^{n-1} [m((q_{k+1} - q_k)/\epsilon)^2/2 - (V(q_{k+1}) + V(q_k))/2]} \tag{4.27}
$$

We have also

$$
\lim_{n \to \infty} \sum_{k=0}^{n-1} \epsilon \left[\frac{m}{2} \left(\frac{q_{k+1} - q_k}{\epsilon} \right)^2 - \frac{V(q_{k+1}) + V(q_k)}{2} \right] = \int_t^{t'} dt L(q, \dot{q}) \tag{4.28}
$$

where $L(q, \dot{q})$ is the lagrangian in configuration space. Therefore we will write

$$
\langle q', t' | q, t \rangle = \int_{q,t}^{q',t'} \mathcal{D}(q(t)) e^{\displaystyle i \int_t^{t'} dt L(q, \dot{q})} \tag{4.29}
$$

with the functional measure given by

$$
\mathcal{D}(q(t)) = \lim_{n \to \infty} \left(\sqrt{\frac{m}{2i\pi\epsilon}} \right)^n \prod_{k=1}^{n-1} dq_k \tag{4.30}
$$

It is the expression (4.29) (really defined by (4.27)) which was originally formulated by Feynman. In fact it has a simple physical. But before discussing the interpretation let us notice that for N degrees of freedom and with hamiltonians of the type

$$
H(\mathbf{q}_i, \mathbf{p}_i) = \sum_{i=1}^N \frac{\mathbf{p}_i^2}{2m} + V(\mathbf{q}_i) \quad i = 1, \dots, N \tag{4.31}
$$

the expression (4.29) can be easily generalized. However for arbitrary system one has to start with the formulation in the phase space of the previous Section, which also can be easily generalized to an arbitrary number of degrees of freedom.

For discussing the interpretation of the expression (4.29), let us recall the boundary conditions

$$
q(t) = q, \qquad q(t') = q' \tag{4.32}
$$
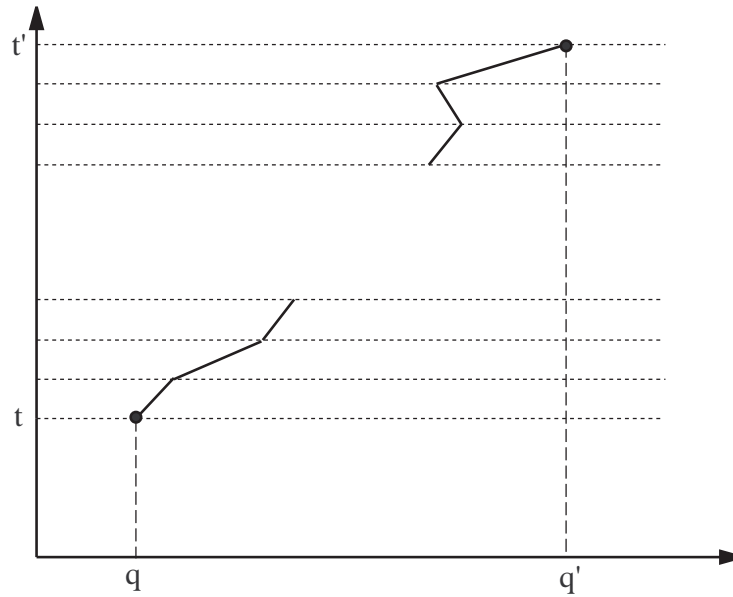
Fig. 4.2.1 - *The Feynman definition for the path integral.*

The expression in the exponent of (4.29) is nothing but the action evaluated for a continuous path starting from q at time t and ending at q' at time t'. This path is approximated in eq. (4.27) by many infinitesimal straight paths as shown in Fig. 4.2.1.

Therefore the interpretation is that in order to evaluate the amplitude to go from (q, t) to (q', t') is obtained by multiplying together the amplitudes corresponding to the infinitesimal straight paths. Each of these amplitudes (neglecting the normalization factor $\sqrt{m/2i\pi\epsilon}$) is given by the exponential of i times the action evaluated along the path. Furthermore we need to sum (or integrate functionally) over all the possible paths joining the two given end points. In a more concise way one says that the amplitude to from one point to another is obtained by summing over all the possible paths joining the two points with a weight proportional to the exponential of iS, where S is the classical action evaluated along the path.

4.3 The physical interpretation of the path integral

Feynman arrived to the path integral formulation in configuration space on a more physical basis. First of all he knew that the amplitude for infinitesimal time differences is proportional to the exponential of iS, with S the classical action evaluated along the infinitesimal path. From that one can derive directly the final formula in the following way. Let us consider the classical double slit experiment, as illustrated in Fig. 4.3.1, where S is the electron source. The electrons come out of the slit A and arrive to the screen C, where they are detected by D, going through one of the two silts A or B. By counting the electrons with the detectpr D we can determine, as a function of x, how many electrons arrive on C, and therefore the probability distribution of the electrons on the screen C. Under the conditions of Fig. 4.3.1, that is with both slits 1 and 2 open, one gets the distribution given in Fig. 4.3.2. In order to explain this distribution it would seem reasonable to make the following assumptions:
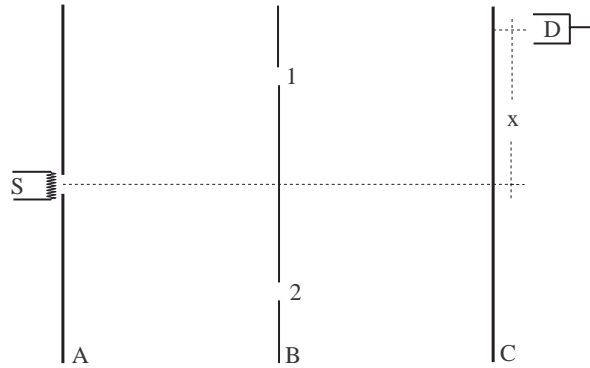
.

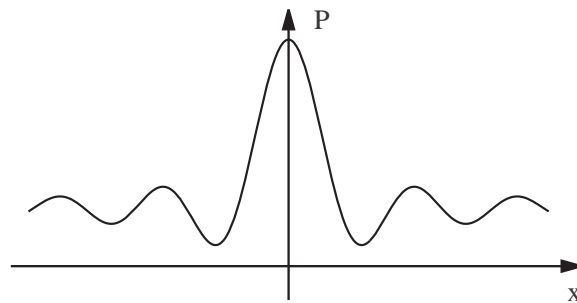**Fig. 4.3.1** - *The double slit experiment.*

.

Fig. 4.3.2 - *The experimental probability distribution, P, with both slits open.*

- Each electron must go through the slit 1 or through the slit 2.

- As a consequence the probability for the electron to reach the screen at the point x must be given by the sum of the probability to reach x through 1 with the probability to go through 2.

These hypotheses can be easily checked by closing one of the two slits and doing again the experiment. The results obtained in the two cases are given in Fig. 4.3.3.
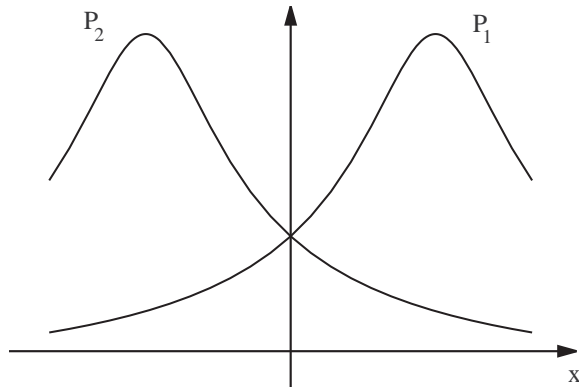.

Fig. 4.3.3 - *The experimental probability distributions, P_1 and P_2 obtained keeping open the sli1 1 or the slit 2 respectively.*

We see that $P \neq P_1 + P_2$, therefore one of our hypotheses must be wrong. In fact, quantum mechanics tells us that we can say that an electron went through one of the two slits with absolute certainty, only if we detect its passage through the slit. Only in this case one would get $P = P_1 + P_2$. In fact, this is the result that one obtains when doing the experiment with a further detector telling us which of the holes have been crossed by the electron. This means that with this experimental setting the interference gets destroyed. On the contrary the experiment tells us that when we do not detect the hole crossed by the electron, the total amplitude for the process is given by the sum of the amplitudes corresponding to the two different possibilities, that is

$$P = |A|^2 = |A_1 + A_2|^2 = |A_1|^2 + |A_2|^2 + A_1^\star A_2 + A_1 A_2^\star \neq |A_1|^2 + |A_2|^2 = P_1 + P_2 \quad (4.33)$$

The two expressions differ for a term which is not positive definite and which is the responsible for the interference pattern. Therefore, quantum mechanics tells us that the usual rules for combining together probabilities are correct, but the result that we get follows from our ignorance about the path followed by the particle, unless we measure it. That is it does not make sense, under such circumstances, define a concept as the path followed by a particle. The Feynnman's interpretation is a little bit different although at the end the theories turn out to be equivalent. He says that it is still possible to speak about trajectories of the electron if one gives up to the usual rules of combining probabilities. This point of view requires taking into account different type of alternatives according to different experimental settings. In particular one defines the following alternatives

1. **Exclusive alternative** - We will speak about exclusive alternatives when we use an apparatus to determine the alternative chosen by the system. For instance one we use a detector to determine the slit crossed by the electrons. In this case the total probability is given by the sum of the probabilities corresponding to the different alternatives,

$$P = \sum_i P_i. \tag{4.34}$$

2. **Interfering alternatives** - This is the situation where the alternative chosen by the system is not detected by an experimental apparatus. In this case we associate to each alternative an amplitude A_i and assume that the total amplitude is given by the sum of all the amplitudes A_i

$$A = \sum_i A_i \tag{4.35}$$

The total probability for the process is evaluated as

$$P = |A|^2 = |\sum_i A_i|^2 \tag{4.36}$$

Therefore the probability rules are strictly related to the experimental setting. There are situations in which one may have both type of alternatives. For instance, if we want to know the total probability of having an electron in C within the interval (a, b), having a detector in C and not looking at the slit crossed by the electrons. In this case the total probability will be given by

$$P = \int_a^b dx |A_1(x) + A_2(x)|^2 \tag{4.37}$$

Since the alternatives of getting the electron at different points in C are exclusive ones.

From this point of view it is well conceivable to speak about trajectories of the electron, but we have to associate to each possible path an amplitude and then sum all over the possible paths. For instance, suppose that we want to evaluate the amplitude to go from S to the screen C when the screen B is eliminated. This can be done by increasing the number of holes in the screen B. Then, if we denote by $A(x, y)$ the amplitude for the electron to reach x going through the hole placed at a distance y from the center of B, the total amplitude will be given by

$$A(x) = \int dy A(x, y) \tag{4.38}$$

We can continue by using more and more screen of type B with more and more holes. That is we can construct a lattice of points between A and C and construct

all the possible paths from A to the point x on C joining the sites of the lattice, as shown in Fig. 4.3.4. By denoting with $\mathcal{C}$ the generic path we get that the total amplitude is given by

$$A(x) = \sum_{\mathcal{C}} A(x, \mathcal{C}) \tag{4.39}$$

.
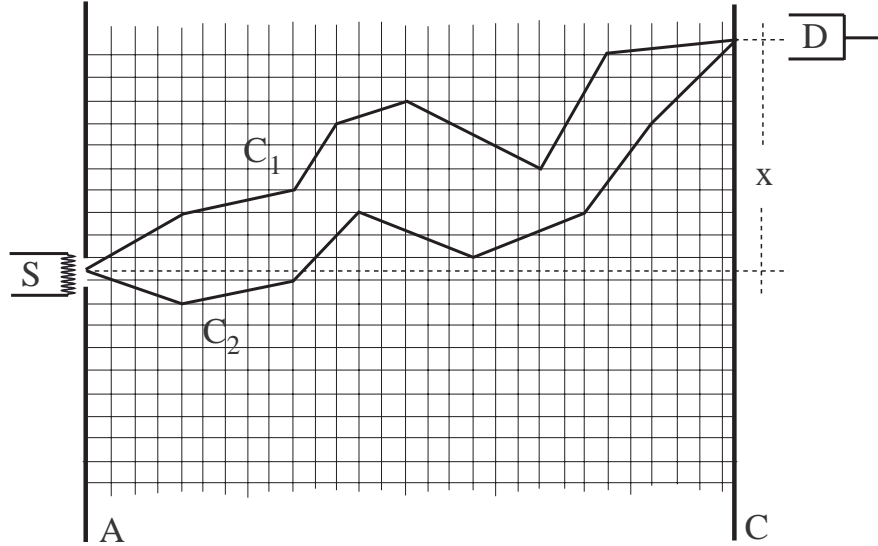
Fig. 4.3.4 - *The lattice for the evaluation of equation (4.39).*

From this point it is easy to get the path integral expression for the total amplitude, since each path is obtained by joining together many infinitesimal paths, and for each of these paths the amplitude is proportional to $\exp(iS)$ where S is the action for the infinitesimal path. Therefore, for the single path $\mathcal{C}$ the amplitude will be given by $\exp(iS(\mathcal{C}))$. This is the final result, except for a proportionality factor. This is easy to get if one starts from the phase space formulation. This is important because in many cases the simple result outlined above is not the correct one. For instance, consider the following lagrangian

$$L = \frac{m}{2}\dot{q}^2 f(q) - V(q) \tag{4.40}$$

where $f(q)$ is an arbitrary function of q. The canonical momentum is given by

$$p = m\dot{q}f(q) \tag{4.41}$$

and the corresponding hamiltonian is

$$H = \frac{1}{2mf(q)}p^2 + V(q) \tag{4.42}$$

Notice that for $q_k \approx q_{k+1}$ we have $f(\bar{q}_k) \approx (f(q_k) + f(q_{k+1}))/2$, where $\bar{q}_k = (q_k + q_{k+1})/2$. Therefore, form eq. (4.20) we get

$$\langle q',t'|q,t\rangle = \lim_{n\to\infty} \int \left(\prod_{k=1}^{n-1} dq_k \right)$$

$$\cdot \prod_{k=0}^{n-1} \frac{dp_k}{2\pi} e^{i[p_k(q_{k+1} - q_k) - \epsilon p_k^2/(2mf(\bar{q}_k)) - \epsilon(V(q_{k+1}) + V(q_k))/2]} \quad (4.43)$$

This expression is formally the same as the one in eq. (4.26) after sending $m \to mf(\bar{q}_k)$. Therefore we get

$$\langle q',t'|q,t\rangle = \lim_{n\to\infty} \int \left(\sqrt{\frac{mf(\bar{q}_k)}{2i\pi\epsilon}} \right)^n \left(\prod_{k=1}^{n-1} dq_k \right)$$

$$\cdot e^{i\epsilon \sum_{k=0}^{n-1} [mf(\bar{q}_k)((q_{k+1} - q_k)/\epsilon)^2/2 - (V(q_{k+1}) + V(q_k))/2]} \quad (4.44)$$

We can rewrite this expression in a way similar to the one used in eq. (4.29)

$$\langle q',t'|q,t\rangle = \int_{q,t}^{q',t'} \mathcal{D}(q(t)) e^{i \int_t^{t'} dt L(q,\dot{q})} \quad (4.45)$$

with $L(q,\dot{q})$ given in (4.40), but the functional measure is now

$$\mathcal{D}(q(t)) = \lim_{n\to\infty} \left(\sqrt{\frac{mf(\bar{q}_k)}{2i\pi\epsilon}} \right)^n \prod_{k=1}^{n-1} dq_k \quad (4.46)$$

The formal expression for the path integral is always the same, but the integration measure must be determined by using the original phase space formulation. In the continuum limit we have

$$\lim_{n\to\infty} \prod_{k=0}^{n-1} \sqrt{f(\bar{q}_k)} = \lim_{n\to\infty} e^{\frac{1}{2} \sum_{k=0}^{n-1} \log f(\bar{q}_k)} =$$

$$= \lim_{n\to\infty} e^{\frac{1}{2\epsilon} \sum_{k=0}^{n-1} \epsilon \log f(\bar{q}_k)} = e^{\frac{1}{2}\delta(0) \int_t^{t'} dt \log f(q)} \quad (4.47)$$

where we have used

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \epsilon = \int_t^{t'} dt \ , \qquad \lim_{n\to\infty} \frac{\delta_{hk}}{\epsilon} = \delta(t_h - t_k) \quad (4.48)$$

95

Therefore the result can be also interpreted by saying that the classical lagrangian gets modified by quantum effects as follows

$$L \rightarrow L - \frac{i}{2}\delta(0)\log f(q) \tag{4.49}$$

The reason why we say that this is a quantum modification is because the second term must be proportional to $\hbar$, as it follows from dimensional arguments.

An interesting question is how one gets the classical limit from this formulation. Let us recall that the amplitude for a single path is $exp(iS(\mathcal{C})/\hbar)$ (where we have put back the factor $\hbar$). To find the total amplitude we must sum over all the possible paths $\mathcal{C}$. Since the amplitudes are phase factors the will cancel except for the paths which are very close to a stationary points of S. Around this point the phases will add coherently, whereas they will cancel going away from this point due to the fast variation of the phase. Roughly we can expect that the coherence effect will take place for those paths such that their phase differs from the stationary value, $S(\mathcal{C}_{class})/\hbar$, of the order of one. For a macroscopical particle ($m \approx 1$ gr and times of about 1 sec) the typical action is about $10^{27}\hbar$, and the amplitude gets contribution only from the classical path. For instance, take a free particle starting from the origin at $t = 0$ and arriving at x_0 at the time t_0. The classical path (the one making S stationary) is

$$x_{class}(t) = x_0\frac{t}{t_0} \tag{4.50}$$

The corresponding value for S is

$$S(x_{class}) = \frac{1}{2}m\int_0^{t_0}\dot{q}^2 = \frac{1}{2}m\frac{x_0^2}{t_0} \tag{4.51}$$

Let us now consider another path joining the origin with x_0 as, for instance,

$$x(t) = x_0\frac{t^2}{t_0^2} \tag{4.52}$$

The corresponding action is

$$S(x) = \frac{2}{3}m\frac{x_0^2}{t_0} \tag{4.53}$$

and we get

$$\Delta S = S(x) - S(x_{class}) = \frac{1}{6}m\frac{x_0^2}{t_0} \tag{4.54}$$

Let us take $t_0 = 1$ sec and $x_0 = 1$ cm. We get $\Delta S = m/6$. For a macroscopic particle (say $m = 1$ gr) and recalling that $\hbar \approx 10^{-27}$ erg·sec, we get $\Delta S = 1.6 \cdot 10^{26}\hbar$ and this non-classical path can be safely ignored. However, for an electron $m \approx 10^{-27}$ gr, and $\Delta S = \hbar/6$. It follows that in the microscopic world, many paths may contribute to the amplitude.

4.4   The free particle

We shall now consider the case of the one-dimensional free motion. In this case we can easily make the integrations going back to the original definition (eq. (4.27). The expression we wish to evaluate is

$$\langle q',t'|q,t\rangle = \lim_{n\to\infty}\int\left(\sqrt{\frac{m}{2i\pi\epsilon}}\right)^n\left(\prod_{k=1}^{n-1}dq_k\right)e^{i\epsilon\sum_{k=0}^{n-1}m((q_{k+1}-q_k)/\epsilon)^2/2} \tag{4.55}$$

since the lagrangian is

$$L = \frac{1}{2}m\dot{q}^2 \tag{4.56}$$

The integrations can be done immediately starting from the gaussian integral

$$\int_{-\infty}^{\infty}dx e^{ax^2+bx} = \sqrt{-\frac{\pi}{a}}e^{-b^2/4a}, \quad \mathrm{Re}\,a\le0 \tag{4.57}$$

which allows us to evaluate the expression

$$\int dx e^{a(x_1-x)^2+b(x_2-x)^2} = \sqrt{-\frac{\pi}{a+b}}e^{ab(x_1-x_2)^2/(a+b)} \tag{4.58}$$

let us start by integrating q_1

$$\int dq_1 e^{im[(q_2-q_1)^2+(q_1-q_0)^2]/(2\epsilon)} = \sqrt{\frac{i\pi\epsilon}{m}}e^{im(q_2-q_0)^2/(2(2\epsilon))}$$

$$= \frac{2\pi i\epsilon}{m}\left(\frac{m}{2\pi i(2\epsilon)}\right)^{\frac{1}{2}}e^{im(q_2-q_0)^2/(2(2\epsilon))} \tag{4.59}$$

Multiplying by the exponential depending on q_2 and integrating over this variable we get

$$\frac{2\pi i\epsilon}{m}\left(\frac{m}{2\pi i(2\epsilon)}\right)^{\frac{1}{2}}\int dq_2 e^{im[(q_3-q_2)^2+(q_2-q_0)^2/2]/(2\epsilon)}$$

$$= \frac{2\pi i\epsilon}{m}\left(\frac{m}{2\pi i(2\epsilon)}\right)^{\frac{1}{2}}\left(\frac{2\pi i(2\epsilon)}{3m}\right)^{\frac{1}{2}}e^{im(q_3-q_0)^2/(2(3\epsilon))}$$

$$= \left(\frac{2\pi i\epsilon}{m}\right)^{\frac{3}{2}}\left(\frac{m}{2\pi i(3\epsilon)}\right)^{\frac{1}{2}}e^{im(q_3-q_0)^2/(2(3\epsilon))} \tag{4.60}$$

By repeating this procedure, after $n-1$ steps we find

$$\int\left(\prod_{k=1}^{n-1}dq_k\right)e^{i\epsilon\sum_{k=0}^{n-1}m((q_{k+1}-q_k)/\epsilon)^2/2}$$

$$= \left(\frac{2\pi i\epsilon}{m}\right)^{\frac{n}{2}}\left(\frac{m}{2\pi i(n\epsilon)}\right)^{\frac{1}{2}}e^{im(q_n-q_0)^2/(2(n\epsilon))} \tag{4.61}$$

97

The first factor in the right hand side of this equation cancels out an analogous factor in the integration measure (see eq. (4.55))

$$\langle q', t' | q, t \rangle = \left[\frac{m}{2\pi i (t' - t)} \right]^{\frac{1}{2}} e^{im(q' - q)^2 / (2(t' - t))} \tag{4.62}$$

This expression can be generalized to a free system with N degrees of freedom

$$\langle q_i', t' | q_i, t \rangle = \left[\frac{m}{2\pi i (t' - t)} \right]^{\frac{N}{2}} e^{im \sum_{i=1}^{N} (q_i' - q_i)^2 / (2(t' - t))} \quad , \quad i = 1, \cdots, N \tag{4.63}$$

Notice also that

$$\lim_{t' \to t} \langle q', t' | q, t \rangle = \delta(q' - q) \tag{4.64}$$

The classical path connecting (q, t) to (q', t') for the free particle is given by

$$q_{\mathrm{cl}}(\tau) = q + \frac{\tau - t}{t' - t}(q' - q) \tag{4.65}$$

The corresponding classical action is

$$S_{\mathrm{cl}} = \int_t^{t'} \frac{m}{2} \dot{q}_{\mathrm{cl}}^2 d\tau = \frac{m}{2} \frac{(q' - q)^2}{(t' - t)^2} \int_t^{t'} d\tau = \frac{m}{2} \frac{(q' - q)^2}{(t' - t)} \tag{4.66}$$

Therefore our result can be written as

$$\langle q', t' | q, t \rangle = \left[\frac{m}{2\pi i (t' - t)} \right]^{\frac{1}{2}} e^{iS_{\mathrm{cl}}} \tag{4.67}$$

We shall see that for all the lagrangians quadratic in the coordinates an in the momenta this result, that is that the amplitude is proportional to the exponential of iS with S the action evaluated along the classical path, is generally true.

Of course, one could formulate quantum mechanics directly in terms of the path integral. It is not difficult to show that it is possible to recover all the results of the ordinary formulation and prove that one has complete equivalence. However we will not do it here, but we will limit ourselves to notice that the formulations agree in the free case. we take a fixed initial point of propagation, for instance $(q, t) = (0, 0)$. Then we define the wave function as

$$\psi(q', t') = \langle q', t' | 0, 0 \rangle = \int_{0,0}^{q', t'} \mathcal{D}(q(t)) e^{iS} \tag{4.68}$$

For the free particle we get

$$\psi(q, t) = \left[\frac{m}{2\pi i t} \right]^{\frac{1}{2}} e^{i \frac{mq^2}{2t}} \tag{4.69}$$

This is indeed the wave function for a free particle in configuration space, as it can be checked by going in the momentum space

$$\psi(p,t) = \left[\frac{m}{2\pi it}\right]^{\frac{1}{2}} \int dq\, e^{-ipq} e^{i\frac{mq^2}{2t}} = \left[\frac{m}{2\pi it}\right]^{\frac{1}{2}} \left[\frac{2\pi it}{m}\right]^{\frac{1}{2}} e^{-i\frac{p^2}{2m}t} = e^{-iEt} \quad (4.70)$$

where $E = p^2/2m$.

4.5   The case of a quadratic action

In this Section we will consider the case of a quadratic lagrangian for a single degree of freedom (it can be easily extended to many degrees of freedom)

$$L = \frac{1}{2}m\dot{q}^2 + b(\tau)\dot{q}q - \frac{1}{2}c(\tau)q^2 + d(\tau)\dot{q} - e(\tau)q - f(\tau) \quad (4.71)$$

To evaluate the integral (4.29) let us start by the Euler-Lagrange equations associated to eq. (4.71)

$$\frac{d}{d\tau}\left[m\dot{q} + b(\tau)q + d(\tau)\right] - \left[b(\tau)\dot{q} - c(\tau)q - e(\tau)\right] = 0 \quad (4.72)$$

that is

$$m\ddot{q} + \dot{b}q + \dot{d} + cq + e = 0 \quad (4.73)$$

Let us denote by $q_{\text{cl}}(\tau)$ the solution of the classical equations of motion satisfying the boundary conditions

$$q_{\text{cl}}(t) = q, \qquad\qquad q_{\text{cl}}(t') = q' \quad (4.74)$$

We then perform the change of variables

$$\eta(\tau) = q(\tau) - q_{\text{cl}}(\tau) \quad (4.75)$$

which in terms of our original definition (4.29) means

$$\eta_k = q_k - q_{\text{cl}\,k}, \qquad\qquad k = 1, \cdots, n-1 \quad (4.76)$$

Since this is a translation the functional measure remains invariant. The new variable $\eta(\tau)$ satisfies the boundary condition

$$\eta(t) = \eta(t') = 0 \quad (4.77)$$

The lagrangian in terms of the new variables is given by

$$\begin{aligned}
L(q) &= L(q_{\text{cl}} + \eta) = L(q_{\text{cl}}) + \frac{1}{2}m\dot{\eta}^2 + b\dot{\eta}\eta - \frac{1}{2}c\eta^2 \\
&+ d\dot{\eta} - e\eta + m\dot{q}_{\text{cl}}\dot{\eta} + b\dot{q}_{\text{cl}}\eta + b\dot{\eta}q_{\text{cl}} - cq_{\text{cl}}\eta
\end{aligned} \quad (4.78)$$

Inside the action we can integrate by parts the terms containing both $q_{\rm cl}$ and η. By using the boundary conditions for η we get

$$L(q) = L(q_{\rm cl}) + \frac{1}{2}m\dot{\eta}^2 + b\dot{\eta}\eta - \frac{1}{2}c\eta^2 + d\dot{\eta} - e\eta - \left[m\ddot{q}_{\rm cl} + \dot{b}q_{\rm cl} + cq_{\rm cl}\right]\eta \qquad (4.79)$$

Recalling the equations of motion for $q_{\rm cl}$ and performing a further integration by parts we find

$$\begin{aligned}
L(q) &= L(q_{\rm cl}) + \frac{1}{2}m\dot{\eta}^2 + b\dot{\eta}\eta - \frac{1}{2}c\eta^2 + d\dot{\eta} - e\eta - \left[-\dot{d} - e\right]\eta \\
&= L(q_{\rm cl}) + \frac{1}{2}m\dot{\eta}^2 + b\dot{\eta}\eta - \frac{1}{2}c\eta^2
\end{aligned} \qquad (4.80)$$

Therefore we can write the path integral as follows

$$\langle q',t'|q,t\rangle = e^{iS_{\rm cl}(q,q')} \int_{0,t}^{0,t'} \mathcal{D}(\eta(\tau)) e^{i\int_t^{t'} (\frac{1}{2}m\dot{\eta}^2 + b\eta\dot{\eta} - \frac{1}{2}c\eta^2)d\tau} \qquad (4.81)$$

Notice that here all the dependence on the variables q and q' is in the classical action. In fact the path integral which remains to be evaluated depends only on t and t' due to the boundary conditions satisfied by η. For the future purposes of extension of this approach to field theory, this term turns out to be irrelevant. However it is important for systems with a finite number of degrees of freedom and therefore we will give here an idea how one can evaluate it in general. First, let us notice that the term $b\eta\dot{\eta}$ is a trivial one, since we can absorb it into a re-definition of $c(\tau)$, through an integration by part

$$b\eta\dot{\eta} = \frac{d}{d\tau}\left(\frac{1}{2}b\eta^2\right) - \frac{1}{2}\dot{b}\eta^2 \qquad (4.82)$$

and defining

$$\tilde{c} = c + \dot{b} \qquad (4.83)$$

In this way we get

$$\begin{aligned}
i\int_t^{t'}\left(\frac{1}{2}m\dot{\eta}^2 - \frac{1}{2}\tilde{c}\eta^2\right)d\tau &= \lim_{n\to\infty} i\sum_{k=0}^{n-1}\left(\frac{m}{2\epsilon}(\eta_{k+1}-\eta_k)^2 - \frac{1}{2}\epsilon\tilde{c}_k\eta_k^2\right) \\
&= \lim_{n\to\infty}\sum_{k=1}^{n-1}\left(\frac{im}{2\epsilon}\left(2\eta_k^2 - 2\eta_k\eta_{k+1}\right) - \frac{i}{2}\epsilon\tilde{c}_k\eta_k^2\right) \\
&= \lim_{n\to\infty} i\eta^T\sigma\eta
\end{aligned} \qquad (4.84)$$

where we have used the boundary conditions on η ($\eta_0 = \eta_n = 0$) and we have introduced the vector

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \cdot \\ \cdot \\ \eta_{n-1} \end{bmatrix} \qquad (4.85)$$

100

and the matrix

$$
\sigma = \frac{m}{2\epsilon}
\begin{bmatrix}
2 & -1 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 2 & -1 & \cdots & 0 & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 & 0 \\
. & . & . & \cdots & . & . & . \\
0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
0 & 0 & 0 & \cdots & 0 & -1 & 2
\end{bmatrix}
- \frac{\epsilon}{2}
\begin{bmatrix}
\tilde{c}_1 & 0 & \cdots & 0 & 0 \\
0 & \tilde{c}_2 & \cdots & 0 & 0 \\
. & . & \cdots & . & . \\
. & . & \cdots & . & . \\
0 & 0 & \cdots & \tilde{c}_{n-2} & 0 \\
0 & 0 & \cdots & 0 & \tilde{c}_{n-1}
\end{bmatrix}
\tag{4.86}
$$

Therefore

$$
\begin{aligned}
G(t',t) &\equiv \int_{0,t}^{0,t'} \mathcal{D}(\eta(\tau)) e^{i\int_t^{t'}(\frac{1}{2}m\dot{\eta}^2 + b\eta\dot{\eta} - \frac{1}{2}c\eta^2)d\tau} \\
&= \lim_{n\to\infty} \left(\frac{m}{2\pi i\epsilon}\right)^{\frac{n}{2}} \int \left(\prod_{k=1}^{n-1} d\eta_k\right) e^{i\eta^T\sigma\eta}
\end{aligned}
\tag{4.87}
$$

Since σ is a symmetric matrix, it can be diagonalized by an orthogonal transformation

$$
\sigma = U^T \sigma_D U
\tag{4.88}
$$

The eigenvectors of σ, $\zeta = U\eta$, are real and the Jacobian of the transformation is one, since $det|U| = 1$. We get

$$
\int \left(\prod_{k=1}^{n-1} d\eta_k\right) e^{i\eta^T\sigma\eta} = \int \left(\prod_{k=1}^{n-1} d\zeta_k\right) e^{i\zeta^T\sigma_D\zeta} = \prod_{k=1}^{n-1} \sqrt{\frac{\pi}{(-i\sigma_k)}} = \frac{\pi^{\frac{n-1}{2}}}{\sqrt{det(-i\sigma)}}
\tag{4.89}
$$

We get

$$
G(t',t) = \lim_{n\to\infty} \left[\left(\frac{m}{2\pi i\epsilon}\right)^n \frac{\pi^{n-1}}{det(-i\sigma)}\right]^{\frac{1}{2}} = \lim_{n\to\infty} \left[\frac{m}{2\pi i}\frac{1}{\epsilon}\frac{1}{(\frac{2i\epsilon}{m})^{n-1}det(-i\sigma)}\right]^{\frac{1}{2}}
\tag{4.90}
$$

Let us define

$$
f(t',t) = \lim_{n\to\infty} \left[\epsilon\left(\frac{2i\epsilon}{m}\right)^{n-1} det(-i\sigma)\right]
\tag{4.91}
$$

then

$$
G(t',t) = \sqrt{\frac{m}{2\pi i f(t',t)}}
\tag{4.92}
$$

In order to evaluate $f(t',t)$, let us define (for $j = 1, \cdots, n-1$)

$$
P_j =
\begin{bmatrix}
2 & -1 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 2 & -1 & \cdots & 0 & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 & 0 \\
. & . & . & \cdots & . & . & . \\
0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
0 & 0 & 0 & \cdots & 0 & -1 & 2
\end{bmatrix}
- \frac{\epsilon^2}{m}
\begin{bmatrix}
\tilde{c}_1 & 0 & 0 & \cdots & 0 & 0 \\
0 & \tilde{c}_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & \tilde{c}_3 & \cdots & 0 & 0 \\
. & . & . & \cdots & . & . \\
0 & 0 & 0 & \cdots & \tilde{c}_{j-1} & 0 \\
0 & 0 & 0 & \cdots & 0 & \tilde{c}_j
\end{bmatrix}
\tag{4.93}
$$

and

$$p_j = det|P_j| \tag{4.94}$$

Clearly

$$\left(\frac{2i\epsilon}{m}\right)^{n-1} det(-i\sigma) = p_{n-1} \tag{4.95}$$

For the first values of j, p_j is given by

$$p_1 = 2 - \frac{\epsilon^2}{m}\tilde{c}_1$$

$$p_2 = \left(2 - \frac{\epsilon^2}{m}\tilde{c}_2\right)p_1 - 1$$

$$p_3 = \left(2 - \frac{\epsilon^2}{m}\tilde{c}_3\right)p_2 - p_1 \tag{4.96}$$

Therefore we find the recurrency relation

$$p_{j+1} = \left(2 - \frac{\epsilon^2}{m}\tilde{c}_{j+1}\right)p_j - p_{j-1}, \quad j = 1, \cdots, n-1 \tag{4.97}$$

where we define $p_0 = 1$. It is convenient to put this expression in the form

$$\frac{p_{j+1} - 2p_j + p_{j-1}}{\epsilon^2} = -\frac{\tilde{c}_{j+1}p_j}{m} \tag{4.98}$$

It is also convenient to introduce the finite difference operator

$$\Delta p_j = \frac{p_{j+1} - p_j}{\epsilon} \tag{4.99}$$

a time $\tau = t + j\epsilon$, and a function $\phi(\tau)$ such that

$$\lim_{\epsilon \to 0} \epsilon p_j = \phi(\tau) \tag{4.100}$$

in the limit in which we keep τ fixed. It follows

$$\lim_{\epsilon \to 0} \Delta \epsilon p_j = \frac{d\phi(\tau)}{d\tau} \tag{4.101}$$

The equation (4.98) can be written as

$$\Delta^2 p_{j-1} = -\frac{\tilde{c}_{j+1}p_j}{m} \tag{4.102}$$

therefore we get the following differential equation for $\phi(\tau)$

$$\frac{d^2\phi(\tau)}{d\tau^2} = -\frac{\tilde{c}(\tau)}{m}\phi(\tau) \tag{4.103}$$

102

Furthermore, $\phi(\tau)$ satisfies the following boundary conditions

$$\phi(t) = \lim_{\epsilon \to 0} \epsilon p_0 = 0$$

$$\frac{d\phi(\tau)}{d\tau}\Big|_{\tau=t} = \lim_{\epsilon \to 0} \epsilon \frac{p_1 - p_0}{\epsilon} = \lim_{\epsilon \to 0} 2 - \frac{\epsilon^2}{m}\tilde{c}_1 - 1 = 1 \qquad (4.104)$$

Recalling eq. (4.91) we have

$$f(t', t) = \lim_{\epsilon \to 0} \epsilon p_{n-1} = \phi(t') \qquad (4.105)$$

Therefore, for quadratic lagrangian we get the general result

$$\langle q', t' | q, t \rangle = \sqrt{\frac{m}{2\pi i f(t', t)}} e^{i S_{\mathrm{cl}}} \qquad (4.106)$$

with $f(t', t) = \phi(t')$.

Examples:

The free particle.

We need only to evaluate $f(t', t)$. The function $\phi(\tau)$ satisfies

$$\frac{d^2\phi(\tau)}{d\tau^2} = 0, \quad \phi(t) = 0, \quad \frac{d\phi(\tau)}{d\tau}\Big|_{\tau=t} = 1 \qquad (4.107)$$

with solution

$$\phi(\tau) = \tau - t \qquad (4.108)$$

from which

$$f(t', t) = t' - t \qquad (4.109)$$

in agreement with the direct calculation.

The harmonic oscillator.

In this case $b = 0$ and $c = m\omega^2$. Therefore

$$\frac{d^2\phi(\tau)}{d\tau^2} = -\omega^2\phi(\tau), \quad \phi(t) = 0, \quad \frac{d\phi(\tau)}{d\tau}\Big|_{\tau=t} = 1 \qquad (4.110)$$

from which

$$\phi(\tau) = \frac{1}{\omega}\sin\omega(\tau - t) \qquad (4.111)$$

by putting $T = t' - t$:

$$f(t', t) = \frac{1}{\omega}\sin\omega T \qquad (4.112)$$

To evaluate the classical action we need to solve the classical equations of motion

$$m\ddot{q} + m\omega^2 q = 0 \tag{4.113}$$

with boundary conditions

$$q(t) = q, \quad q(t') = q' \tag{4.114}$$

The general solution is given by

$$q(\tau) = A \sin \omega(\tau - t) + B \cos \omega(\tau - t) \tag{4.115}$$

From the boundary conditions

$$A = \frac{q' - q \cos \omega T}{\sin \omega T}, \quad B = q \tag{4.116}$$

and substituting inside the lagrangian

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 = \frac{1}{2}m\omega^2 \left[(A^2 - B^2) \cos 2\omega(\tau - t) - 2AB \sin 2\omega(\tau - t)\right] \tag{4.117}$$

Recalling the following integrals

$$\int_t^{t'} d\tau \cos 2\omega(\tau - t) = \frac{\sin 2\omega T}{2\omega}, \quad \int_t^{t'} d\tau \sin 2\omega(\tau - t) = \frac{1 - \cos 2\omega T}{2\omega} \tag{4.118}$$

we get

$$S_{\text{cl}} = \frac{1}{2}\frac{m\omega}{\sin \omega T} \left[(q^2 + q'^2) \cos \omega T - 2qq'\right] \tag{4.119}$$

and

$$\langle q', t'|q, t\rangle = \sqrt{\frac{m\omega}{2\pi i \sin \omega T}} e^{\frac{i}{2}\frac{m\omega}{\sin \omega T}\left[(q^2 + q'^2)\cos \omega T - 2qq'\right]} \tag{4.120}$$

In the limit $\omega \to 0$ we get back correctly the free case.

4.6   Functional formalism

The most natural mathematical setting to deal with the path integral is the theory of functionals. The concept of functional is nothing but an extension of the concept of function. Recall that a function is simply a mapping from a manifold $\mathcal{M}$ to R^1, that a law which associates a number to each point of the manifold. A real functional is defined as an application from a space of functions (and therefore an ∞-dimensional space) to R^1. There are many spaces of functions. For instance, the space of the square integrable real functions of a real variable. Therefore a real

functional associate a real number to a real function. If we denote the space of functions by $\mathcal{L}$, a real functional is the mapping

$$F: \quad \mathcal{L} \to R^1 \tag{4.121}$$

We will make use of the notation

$$F[\eta], \quad F, \quad F[\cdot], \qquad \eta \in \mathcal{L} \tag{4.122}$$

to denote the functional of defined in eq. (4.121). Examples of functionals are

- $\mathcal{L}=L^1[-\infty,+\infty]=$ the space of the integrable functions in R^1 with respect to the measure $w(x)dx$:

$$F_1[\eta] = \int_{-\infty}^{+\infty} dx w(x)\eta(x) \tag{4.123}$$

- $\mathcal{L}=L^2[-\infty,+\infty]=$ space of the square integrable functions in R^1:

$$F_2[\eta] = e^{-\dfrac{1}{2}\int dx \eta^2(x)} \tag{4.124}$$

- $\mathcal{L}=\mathcal{C}=$ space of the continuous functions:

$$F_3[\eta] = \eta(x_0) \tag{4.125}$$

F_3 is the functional that associates to each continuous function its value at x_0.

The next step is to define the derivative of a functional. We need to understand how a functional varies when we vary the function. A simple way of understand this point is to consider some path on the definition manifold of the functions. For instance consider functions from R^1 to R^1, and take an interval (t,t') of R^1 made of many infinitesimal pieces, as we did for defining the path integral. Then we can approximate a function $\eta(t)$ with a convenient limit of its discrete approximation $(\eta_1,\eta_2,\cdots,\eta_n)$ obtained by evaluating the functions at the discrete points in which the interval has been divided. In this approximation we can think of the functional F as an application from $R^n \to R^1$. That is

$$F[\eta] \to F(\eta_1,\cdots,\eta_n) \equiv F(\eta_i), \quad i = 1,\cdots,n \tag{4.126}$$

Then we can consider the variation of $F(\eta_i)$ when we vary the quantities η_i

$$F(\eta_i + \delta\eta_i) - F(\eta_i) = \sum_{j=1}^{n} \frac{\partial F(\eta_i)}{\partial \eta_j}\delta\eta_j \tag{4.127}$$

Let us put

$$K_i = \frac{1}{\epsilon}\frac{\partial F}{\partial \eta_i} \tag{4.128}$$

105

where ϵ is the infinitesimal interval where we evaluate the discrete apoproximation to the function. Then, for $\epsilon \to 0$

$$F(\eta_i + \delta\eta_i) - F(\eta_i) = \sum_{j=1}^{n} K_j \delta\eta_j \epsilon \to \int K \delta\eta dx \tag{4.129}$$

that is

$$\delta F[\eta] = \int K(x) \delta\eta(x) dx \tag{4.130}$$

Since $\delta\eta(x)$ is the variation of the function evaluated at x, we will define the *functional derivative* as

$$\delta F[\eta] = \int \frac{\delta F}{\delta\eta(x)} \delta\eta(x) dx \tag{4.131}$$

In other words. to evaluate the functional derivative of a functional we first evaluate its infinitesimal variation and then we can get the functional derivative by comparison with the previous formula. Looking at the eq. (4.128) we get

$$\frac{\delta F[\eta]}{\delta\eta(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \frac{\partial F(\eta_i)}{\partial \eta_i}$$

$$\eta_i = \eta(x_i), \qquad x_i = x_1 + (i-1)\epsilon, \qquad \epsilon = \frac{x_n - x_1}{n-1}, \qquad i = 1, \cdots, n \tag{4.132}$$

let us now evaluate the functional derivatives of the previous functionals:

$$\delta F_1[\eta] = \int w(x) \delta\eta(x) dx \tag{4.133}$$

giving

$$\frac{\delta F_1[\eta]}{\delta\eta(x)} = w(x) \tag{4.134}$$

Then

$$\delta F_2[\eta] = e^{-\frac{1}{2} \int dx \eta^2(x)} \left(- \int \eta(x) \delta\eta(x) \right) \tag{4.135}$$

and therefore

$$\frac{\delta F_2[\eta]}{\delta\eta(x)} = -\eta(x) F_2[\eta] \tag{4.136}$$

In order to evaluate the derivative of F_3 let us write

$$F_3[\eta] = \int \eta(x) \delta(x - x_0) dx \tag{4.137}$$

In this way F_3 looks formally as a functional of type F_1, and

$$\frac{\delta F_3[\eta]}{\delta\eta(x)} = \delta(x - x_0) \tag{4.138}$$

106

Let us consider also

$$F_4[\eta] = e^{-\dfrac{1}{2}\int dx dy K(x,y)\eta(x)\eta(y)} \tag{4.139}$$

then

$$\delta F_4[\eta] = e^{-\dfrac{1}{2}\int dx dy K(x,y)\eta(x)\eta(y)} \left(-\int dx dy K(x,y)\delta\eta(x)\eta(y)\right) \tag{4.140}$$

where we have used the symmetry of the integral with respect to $x \leftrightarrow y$. We obtain

$$\frac{\delta F_4[\eta]}{\delta\eta(x)} = -\int dy K(x,y)\eta(y)F_4[\eta] \tag{4.141}$$

A further example is the action functional. In fact the action is a functional of the path

$$S[q] = \int_t^{t'} L(q,\dot{q})dt \tag{4.142}$$

By varying it

$$\delta S[q] = \int_t^{t'}\left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}\right)dt = \int_t^{t'}\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\delta q\, dt \tag{4.143}$$

Since we consider variations with respect to functions with fixed boundary conditions, we have $\delta q(t) = \delta q(t') = 0$. We get

$$\frac{\delta S}{\delta q(t)} = \frac{\partial L}{\partial q(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}(t)} \tag{4.144}$$

Starting from the functional derivative one can generalize the Taylor expansion to the functional case. This generalization is called the Volterra expansion of a functional

$$F[\eta_0 + \eta_1] = \sum_{k=0}^{\infty}\frac{1}{k!}\int dx_1\cdots dx_k \left.\frac{\delta^k F[\eta]}{\delta\eta(x_1)\cdots\delta\eta(x_k)}\right|_{\eta=\eta_0}\eta_1(x_1)\cdots\eta_1(x_k) \tag{4.145}$$

All these definitions are easily generalized to the case of η a function from $R^m \to R^n$, or from a manifold M to a manifold N.

4.7 General properties of the path integral

In this Section we would like to stress some important properties of the path integral, namely

- **Invariance of the functional measure under translations** - It follows immediately from the definition that

$$
\begin{aligned}
q(\tau) &\to q(\tau) + \eta(\tau), & \eta(t) &= \eta(t') = 0 \\
p(\tau) &\to p(\tau) + \pi(\tau), & &\text{for any } \pi(t),\ \pi(t')
\end{aligned}
\tag{4.146}
$$

- **Factorization of the path integral**. From the very definition of the path it follows that in order to evaluate the amplitude for going from q at time t to q' at time t' we can first go to q" to a time $t \le t$" $\le t'$, sum up over all the paths from$q$ a q" and from q" to q', and eventually integrating over all the possible intermediate points q" (see Fig. 4.7.1)).

.
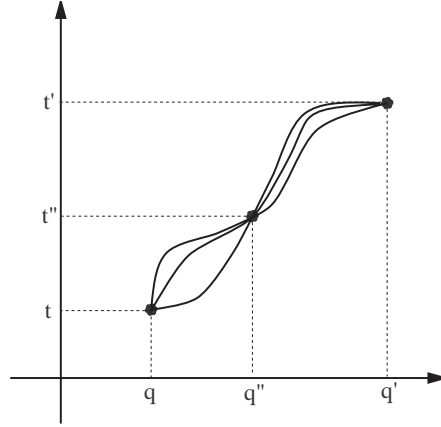
Fig. 4.7.1 - *The factorization of the path integral.*

In equations we get

$$
\begin{aligned}
\langle q', t' | q, t \rangle &= \int_{q,t}^{q't'} d\mu(q(\tau))d\mu(p(\tau))e^{\displaystyle i\int_t^{t"} Ld\tau + i\int_{t"}^{t'} Ld\tau} \\[2mm]
&= \int dq" \int_{q,t}^{q",t"} d\mu(q(\tau))d\mu(p(\tau))e^{\displaystyle i\int_t^{t"} Ld\tau} \\[2mm]
&\quad \times \int_{q",t"}^{q',t'} d\mu(q(\tau))d\mu(p(\tau))e^{\displaystyle i\int_{t"}^{t'} Ld\tau} \\[2mm]
&= \int dq" \langle q', t' | q", t" \rangle \langle q", t" | q, t \rangle
\end{aligned}
\tag{4.147}
$$

as it follows from the factorization properties of the measure

$$
d\mu(q(\tau)) = \prod_{t < \tau < t'} dq(\tau) = dq" \prod_{t < \tau < t"} dq(\tau) \prod_{t" < \tau < t'} dq(\tau)
$$

$$d\mu(p(\tau)) = \prod_{t \le \tau \le t"} dp(\tau) = \prod_{t \le \tau \le t"} dp(\tau) \prod_{t" \le \tau \le t'} dp(\tau) \qquad (4.148)$$

It is clear that the factorization property is equivalent to the completeness of the states.

Using these properties it is possible to show that the wave function, as defined in (4.68) satisfies the Schrödinger equation.

The next important property of the path integral is that it gives an easy way of evaluating expectation values. To this end we will consider expressions of the type (which will be called the average value of F)

$$\int_{q,t}^{q',t'} d\mu(q(\tau))d\mu(p(\tau))F[q(\tau),p(\tau)]e^{iS} \qquad (4.149)$$

where F is an arbitrary functional of q, p and of their time derivatives. Let us start with

$$\int_{q,t}^{q',t'} d\mu(q(\tau))d\mu(p(\tau))e^{iS}q(t"), \qquad t \le t" \le t' \qquad (4.150)$$

We begin by the observation that

$$\int_{q,t}^{q',t'} d\mu(q(\tau))d\mu(p(\tau))e^{iS}q(t') = q' \int_{q,t}^{q',t'} d\mu(q(\tau))d\mu(p(\tau))e^{iS} = q'\langle q',t'|q,t\rangle \qquad (4.151)$$

since $q(t)$ satisfies the boundary conditions $q(t) = q$, $q(t') = q'$. By using the factorization we get

$$
\begin{aligned}
\int_{q,t}^{q',t'} d\mu(q(\tau))d\mu(p(\tau))e^{iS}q(t") &= \int dq" \int_{q,t}^{q",t"} d\mu(q(\tau))d\mu(p(\tau))e^{iS}q(t") \\
&\quad \cdot \int_{q",t"}^{q',t'} d\mu(q(\tau))d\mu(p(\tau))e^{iS} \\
&= \int dq" \langle q',t'|q",t"\rangle q" \langle q",t"|q,t\rangle \qquad (4.152)
\end{aligned}
$$

Since $|q,t\rangle$ is an eigenstate of $\mathbf{q}(t)$ it follows

$$
\begin{aligned}
\int_{q,t}^{q',t'} d\mu(q(\tau))d\mu(p(\tau))e^{iS}q(t") &= \int dq" \langle q',t'|\mathbf{q}(t")|q",t"\rangle\langle q",t"|q,t\rangle \\
&= \langle q',t'|\mathbf{q}(t")|q,t\rangle \qquad (4.153)
\end{aligned}
$$

Therefore the average value of $q(t")$ evaluated with $\exp(iS)$ coincides with the expectation value of the operator $\mathbf{q}(t")$.

Now let us study the average value of $q(t_1)q(t_2)$ con $t \le t_1, t_2 \le t'$. By proceeding as before we will get the expectation value of $\mathbf{q}(t_1)\mathbf{q}(t_2)$ if $t_1 \ge t_2$, or $\mathbf{q}(t_2)\mathbf{q}(t_1)$ if

$t_2 \geq t_1$. In terms of the T-product we get

$$\int_{q,t}^{q',t'} d\mu(q(\tau))d\mu(p(\tau))e^{iS}q(t_1)q(t_2) = \langle q',t'|T(\mathbf{q}(t_1)\mathbf{q}(t_2)|q,t\rangle \qquad (4.154)$$

Analogously

$$\int_{q,t}^{q',t'} d\mu(q(\tau))d\mu(p(\tau))e^{iS}q(t_1)\cdots q(t_n) = \langle q',t'|T(\mathbf{q}(t_1)\cdots\mathbf{q}(t_n))|q,t\rangle \qquad (4.155)$$

Let us now consider a functional of $q(t)$such to admit an expansion in a Volterra's series

$$\int_{q,t}^{q',t'} d\mu(q(\tau))d\mu(p(\tau))F[q]e^{iS}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \int dt_1 \cdots dt_k \left.\frac{\delta^k F[q]}{\delta q(t_1)\cdots\delta q(t_k)}\right|_{q=0}$$

$$\cdot \int_{q,t}^{q',t'} d\mu(q(\tau))d\mu(p(\tau))e^{iS}q(t_1)\cdots q(t_k)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \int dt_1 \cdots dt_k \left.\frac{\delta^k F[q]}{\delta q(t_1)\cdots\delta q(t_k)}\right|_{q=0} \langle q',t'|T(\mathbf{q}(t_1)\cdots\mathbf{q}(t_k))|q,t\rangle$$

$$\equiv \langle q',t'|T(F[\mathbf{q}])|q,t\rangle \qquad (4.156)$$

where

$$T(F[\mathbf{q}]) \equiv \sum_{k=0}^{\infty} \frac{1}{k!} \int dt_1 \cdots dt_k \left.\frac{\delta^k F[q]}{\delta q(t_1)\cdots\delta q(t_k)}\right|_{q=0} T(\mathbf{q}(t_1)\cdots\mathbf{q}(t_k)) \qquad (4.157)$$

If the functional depends also on the time derivatives a certain caution is necessary. For instance, let us study the average value of $q(t_1)\dot{q}(t_2)$ ($t \leq t_1, t_2 \leq t'$):

$$\int_{q,t}^{q',t'} d\mu(q(\tau))d\mu(p(\tau))e^{iS}q(t_1)\dot{q}(t_2)$$

$$= \lim_{\epsilon\to 0} \int_{q,t}^{q',t'} d\mu(q(\tau))d\mu(p(\tau))e^{iS}q(t_1)\frac{q(t_2+\epsilon)-q(t_2)}{\epsilon}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\langle q',t'|T\Big(\mathbf{q}(t_1)\mathbf{q}(t_2+\epsilon)\Big) - T\Big(\mathbf{q}(t_1)\mathbf{q}(t_2)\Big)|q,t\rangle$$

$$= \frac{d}{dt_2}\langle q',t'|T\Big(\mathbf{q}(t_1)\mathbf{q}(t_2)\Big)|q,t\rangle \qquad (4.158)$$

Let us define an ordered T*-product (or covariant T-product) as

$$T^\star\Big(\mathbf{q}(t_1)\dot{\mathbf{q}}(t_2)\Big) = \frac{d}{dt_2}T\Big(\mathbf{q}(t_1)\mathbf{q}(t_2)\Big) \qquad (4.159)$$

110

Then it is not difficult to show that for an arbitrary functional of $q(t)$ e $\dot{q}(t)$ one has

$$\int_{q,t}^{q',t'} d\mu(q(\tau))d\mu(p(\tau))F[q,\dot{q}]e^{iS} = \langle q',t'|T^{\star}(F[\mathbf{q},\dot{\mathbf{q}}])|q,t\rangle \tag{4.160}$$

The proof can be done by expanding $F[q,\dot{q}]$ in a Volterra's series of q and $\dot{q}$, integrating by parts and using (4.156). At the end one has to integrate again by parts. The T* product is obtained as in (4.158),

The power of this formulation of quantum mechanics comes about when we want to get the fundamental properties of quantum mechanics as the equations of motion for the operators, the commutation relations and so on so forth. The fundamental relation follows from the following property of the path integral

$$\int_{q,t}^{q',t'} d\mu(q(\tau))d\mu(p(\tau))\frac{\delta}{\delta q(\tau)}\left[F[q(\tau),p(\tau)]e^{iS}\right] = 0 \tag{4.161}$$

This is a simple consequence of the definition of the path integral and of the functional derivative

$$\lim_{n\to\infty}\int\left(\prod_{k=1}^{n-1}dq_k\right)\left(\prod_{k=0}^{n-1}\frac{dp_k}{2\pi}\right)\frac{1}{\epsilon}\frac{\partial}{\partial q_k}\left[F(q_1,\cdots,q_{n-1})e^{i\sum_{k=0}^{n-1}\epsilon L(q_k,p_k)}\right] = 0 \tag{4.162}$$

(of course we are assuming that the integrand goes to zero for large values of the q_i's. To this end one has to define the integral in an appropriate way as it will be shown later on). On the other hand the previous equation follows from the invariance under translation of the functional measure. In fact, let us introduce the following notation

$$d\mu_{q,p} = d\mu(q(\tau))d\mu(p(\tau)) \tag{4.163}$$

It follows for an infinitesimal function η such that $\eta(t) = \eta(t') = 0$

$$\int_{q,t}^{q',t'} d\mu_{q,p}\, A[q] = \int_{q,t}^{q',t'} d\mu_{q,p}\, A[q+\eta] = \int_{q,t}^{q',t'} d\mu_{q,p}\, A[q] + \int_{q,t}^{q',t'} d\mu_{q,p}\int \eta(t)\frac{\delta A[q]}{\delta q(t)}dt \tag{4.164}$$

Proving the validity of eq. (4.161). By performing explicitly the functional derivative in eq. (4.161)we get a relation which is the basis of the Schwinger's formulation of quantum mechanics, that is

$$i\langle q',t'|T^{\star}\left(F[\mathbf{q}]\frac{\delta S}{\delta \mathbf{q}(\tau)}\right)|q,t\rangle = -\langle q',t'|T^{\star}\left(\frac{\delta F[\mathbf{q}]}{\delta \mathbf{q}(\tau)}\right)|q,t\rangle \tag{4.165}$$

The importance of this relation follows from the fact that we can use this relation to get all the properties of quantum mechanics by selecting the functional F in a

111

convenient way. For instance, by choosing $F[q] = 1$ we get

$$\langle q', t'|T^\star \left(\frac{\delta S}{\delta \mathbf{q}(t_1)} \right) |q, t\rangle = 0 \qquad (4.166)$$

that is the quantum equations of motion. With the choice $F[q] = q$ we get

$$\langle q', t'|T^\star \left(\mathbf{q}(t_1)\frac{\delta S}{\delta \mathbf{q}(t_2)} \right) |q, t\rangle = i\langle q', t'|q, t\rangle \delta(t_1 - t_2) \qquad (4.167)$$

and since this relation holds for arbitrary states

$$T^\star \left(\mathbf{q}(t_1)\frac{\delta S}{\delta \mathbf{q}(t_2)} \right) = i\delta(t_1 - t_2) \qquad (4.168)$$

For simplicity, let us consider the following action

$$S = \int_t^{t'} \left(\frac{1}{2}m\dot{q}^2 - V(q) \right) dt \qquad (4.169)$$

then

$$\frac{\delta S}{\delta q(t_2)} = - \left(m\ddot{q}(t_2) + \frac{\partial V(q(t_2))}{\partial q(t_2)} \right) \qquad (4.170)$$

from which

$$T^\star \left[\mathbf{q}(t_1) \left(m\ddot{\mathbf{q}}(t_2) + \frac{\partial V(\mathbf{q})}{\partial \mathbf{q}(t_2)} \right) \right] = -i\delta(t_1 - t_2) \qquad (4.171)$$

By using the definition of the $T^\star$-product we get

$$
\begin{aligned}
T^\star(\mathbf{q}(t_1)\ddot{\mathbf{q}}(t_2) &= \frac{d^2}{dt_2^2}T(\mathbf{q}(t_1)\mathbf{q}(t_2)) \\
&= \frac{d^2}{dt_2^2}[\theta(t_1 - t_2)\mathbf{q}(t_1)\mathbf{q}(t_2) + \theta(t_2 - t_1)\mathbf{q}(t_2)\mathbf{q}(t_1)] \\
&= \frac{d}{dt_2}\Big[-\delta(t_1 - t_2)\mathbf{q}(t_1)\mathbf{q}(t_2) + \delta(t_1 - t_2)\mathbf{q}(t_2)\mathbf{q}(t_1) \\
&\quad + theta(t_1 - t_2)\mathbf{q}(t_1)\dot{\mathbf{q}}(t_2) + \theta(t_2 - t_1)\dot{\mathbf{q}}(t_2)\mathbf{q}(t_1)\Big] \\
&= \delta(t_1 - t_2)\mathbf{q}(t_1)\dot{\mathbf{q}}(t_2) + \delta(t_1 - t_2)\dot{\mathbf{q}}(t_2)\mathbf{q}(t_1) + T(\mathbf{q}(t_1)\ddot{\mathbf{q}}(t_2)) \\
&= \delta(t_1 - t_2)[\mathbf{q}(t_1), \dot{\mathbf{q}}(t_1)] + T(\mathbf{q}(t_1)\ddot{\mathbf{q}}(t_2)) \qquad (4.172)
\end{aligned}
$$

Inserting this inside the eq. (4.171

$$-m\delta(t_1 - t_2)[\mathbf{q}(t_1), \dot{\mathbf{q}}(t_2)] + T\left[\mathbf{q}(t_1) \left(m\ddot{\mathbf{q}}(t_2) + \frac{\partial V}{\partial \mathbf{q}(t_2)} \right) \right] = -i\delta(t_1 - t_2) \quad (4.173)$$

and using the equations of motion, we get

$$[\mathbf{q}(t), m\dot{\mathbf{q}}(t)] = i \qquad (4.174)$$

Therefore we get the canonical commutator

$$[\mathbf{q}(t), \mathbf{p}(t)] = i \qquad (4.175)$$

112

4.8 The generating functional of the Green's functions

In field theory the relevant amplitudes are the ones evaluated between $t = -\infty$ and $t = +\infty$. We have seen that using the reduction formulas we are lead to evaluate the vacuum expectation value of T-products of local operators. Therefore we will consider matrix elements of the type (limiting ourselves, for the moment being, to a single degree of freedom)

$$\langle 0|T(\mathbf{q}(t_1)\cdots\mathbf{q}(t_n))|0\rangle \tag{4.176}$$

Here $|0\rangle$ is the ground state of system. Notice that more generally one can consider the matrix element

$$\langle q', t'|T(\mathbf{q}(t_1)\cdots\mathbf{q}(t_n))|q, t\rangle = \int_{q,t}^{q',t'} d\mu(q(\tau))d\mu(p(\tau))e^{iS}q(t_1)\cdots q(t_n) \tag{4.177}$$

These matrix elements can be derived in a more compact form, by introducing the generating functional

$$\langle q', t'|q, t\rangle_J = \int_{q,t}^{q',t'} d\mu(q(\tau))d\mu(p(\tau))e^{iS_J} \tag{4.178}$$

where

$$S_J = \int_t^{t'} d\tau \left[p\dot{q} - H + Jq\right] \tag{4.179}$$

and $J(\tau)$ is an arbitrary external source. By differentiating this expression with respect to the external source and evaluating the derivatives at $J = 0$ we get

$$\langle q', t'|T(\mathbf{q}(t_1)\cdots\mathbf{q}(t_n))|q, t\rangle = \left(\frac{1}{i}\right)^n \frac{\delta^n}{\delta J(t_1)\cdots\delta J(t_n)}\langle q', t'|q, t\rangle_J\bigg|_{J=0} \tag{4.180}$$

Now we want to see how it is possible to bring back the evaluation of the matrix element in (4.176) to the knowledge of the generating functional. Let us introduce the wave function of the ground state

$$\Phi_0(q, t) = e^{-iE_0 t}\langle q|0\rangle = \langle q|e^{-iHt}|0\rangle = \langle q, t|0\rangle \tag{4.181}$$

then

$$\langle 0|T(\mathbf{q}(t_1)\cdots\mathbf{q}(t_n))|0\rangle = \int dq'dq\langle 0|q', t'\rangle\langle q', t'|T(\mathbf{q}(t_1)\cdots\mathbf{q}(t_n))|q, t\rangle\langle q, t|0\rangle$$

$$= \int dq'dq\Phi_0^\star(q', t')\langle q', t'|T(\mathbf{q}(t_1)\cdots\mathbf{q}(t_n))|q, t\rangle\Phi_0(q, t) \tag{4.182}$$

Next we define the functional $Z[J]$, the generating functional of the vacuum amplitudes, that is

$$Z[J] = \int dq' dq \, \Phi_0^\star(q', t') \langle q', t' | q, t \rangle_J \Phi_0(q, t) = \langle 0 | 0 \rangle_J \qquad (4.183)$$

It follows

$$\langle 0 | T(\mathbf{q}(t_1) \cdots \mathbf{q}(t_n)) | 0 \rangle = \left(\frac{1}{i} \right)^n \frac{\delta^n}{\delta J(t_1) \cdots \delta J(t_n)} Z[J] \Big|_{J=0} \qquad (4.184)$$

We want to show that it is possible evaluate $Z[J]$ as a path integral with arbitrary boundary conditions on q and q'. That is we want to prove the following relation

$$Z[J] = \lim_{t \to +i\infty,\ t' \to -i\infty} \frac{e^{iE_0(t' - t)}}{\Phi_0^\star(q) \Phi_0(q')} \langle q', t' | q, t \rangle_J \qquad (4.185)$$

wit q and q' arbitrary and with the ground state wave functions taken at $t = 0$. To this end let us consider an external source $J(t)$ vanishing outside the interval (t'', t''') with $t' > t''' > t'' > t$. Then, we can write

$$\langle q', t' | q, t \rangle_J = \int dq'' dq''' \langle q', t' | q''', t''' \rangle \langle q''', t''' | q'', t'' \rangle_J \langle q'', t'' | q, t \rangle \qquad (4.186)$$

with

$$\langle q', t' | q''', t''' \rangle = \langle q' | e^{-iH(t' - t''')} | q''' \rangle = \sum_n \Phi_n(q') \Phi_n^\star(q''') e^{-iE_n(t' - t''')} \qquad (4.187)$$

Since E_0 is the lowest eigenvalue we get

$$\lim_{t' \to -i\infty} e^{iE_0 t'} \langle q', t' | q''', t''' \rangle = \lim_{t' \to -i\infty} \sum_n \Phi_n(q') \Phi_n^\star(q''') e^{iE_n t'''} e^{-i(E_n - E_0)t'} =$$

$$= \Phi_0(q') \Phi_0^\star(q''') e^{iE_0 t'''} = \Phi_0^\star(q''', t''') \Phi_0(q') \qquad (4.188)$$

Analogously

$$\langle q'', t'' | q, t \rangle = \sum_n \Phi_n(q'') \Phi_n^\star(q) e^{-iE_n(t'' - t)} \qquad (4.189)$$

and

$$\lim_{t \to +i\infty} e^{-iE_0 t} \langle q'', t'' | q, t \rangle = \lim_{t \to +i\infty} \sum_n \Phi_n(q'') \Phi_n^\star(q) e^{-iE_n t''} e^{i(E_n - E_0)t} =$$

$$= \Phi_0(q'') \Phi_0^\star(q) e^{-iE_0 t''} = \Phi_0(q'', t'') \Phi_0^\star(q) \qquad (4.190)$$

And finally we get the result

$$im_{t \to +i\infty, t' \to -\infty} e^{iE_0(t' - t)} \langle q', t' | q, t \rangle_J$$

$$= \int dq'' dq''' \Phi_0^\star(q) \Phi_0(q') \Phi_0^\star(q''', t''') \langle q''', t''' | q'', t'' \rangle_J \Phi_0(q'', t'') =$$

$$= \Phi_0^\star(q) \Phi_0(q') Z[J] \qquad (4.191)$$

114

Let us notice again that the values q and q' are completely arbitrary, and furthermore that the coefficient in front of the matrix element is J-independent. Therefore it is physically irrelevant. In fact, the relevant physical quantities are the ratios

$$\frac{\langle 0|T(\mathbf{q}(t_1)\cdots\mathbf{q}(t_n))|0\rangle}{\langle 0|0\rangle} = \left(\frac{1}{i}\right)^n \frac{1}{Z[0]} \frac{\delta^n}{\delta J(t_1)\cdots\delta J(t_n)} Z[J]\Big|_{J=0} \tag{4.192}$$

where the J-independent factor cancels out. Then we can write

$$Z[J] = \lim_{t\to+i\infty, t'\to-i\infty} N \int_{q,t}^{q',t'} \mathcal{D}(q(\tau)) e^{i\int_t^{t'}(L+Jq)d\tau} \tag{4.193}$$

where N is a J-independent normalization factor. This expression suggests the definition of an **euclidean** generating functional $Z_E[J]$. This is obtained by introducing an euclidean time $\tau = it$ (Wick's rotation)

$$Z_E[J] = \lim_{\tau'\to+\infty, \tau\to-\infty} N \int_{q,\tau}^{q',\tau'} \mathcal{D}(q(\tau)) e^{\int_\tau^{\tau'}(L(q, i\frac{dq}{d\tau})+Jq)d\tau} \tag{4.194}$$

Consider for instance the case

$$L = \frac{1}{2}m\left(\frac{dq}{dt}\right)^2 - V(q) \tag{4.195}$$

from which

$$L(q, i\frac{dq}{d\tau}) = -\frac{1}{2}m\left(\frac{dq}{d\tau}\right)^2 - V(q) \tag{4.196}$$

It is then convenient to define the euclidean lagrangian and euclidean action

$$L_E = \frac{1}{2}m\left(\frac{dq}{d\tau}\right)^2 + V(q) \tag{4.197}$$

$$S_E = \int_\tau^{\tau'} L_E d\tau \tag{4.198}$$

Notice that the euclidean lagrangian coincides formally with the hamiltonian. More generally

$$L_E = -L(q, i\frac{dq}{d\tau}) \tag{4.199}$$

Therefore

$$Z_E[J] = \lim_{\tau'\to+\infty, \tau\to-\infty} N \int_{q,\tau}^{q',\tau'} \mathcal{D}(q(\tau)) e^{-S_E + \int_\tau^{\tau'} Jq\, d\tau} \tag{4.200}$$

Since e^{-S_E} is a positive definite functional, the path integral turns out to be well defined and convergent if S_E is positive definite.the vacuum expectation values of the operators $\mathbf{q}(t)$ can be evaluated by using Z_E and continuing the result for real times

$$\frac{1}{Z[0]}\frac{\delta^n Z[J]}{\delta J(t_1)\cdots \delta J(t_n)}\bigg|_{J=0} = (i)^n \frac{1}{Z_E[0]}\frac{\delta^n Z_E[J]}{\delta J(\tau_1)\cdots \delta J(\tau_n)}\bigg|_{J=0,\tau_i=it_i} \tag{4.201}$$

Since the values of q and q' are arbitrary a standard choice is to set them at zero.

4.9 The Green's functions for the harmonic oscillator

It is interesting to evaluate the generating functional for the harmonic oscillator starting from both the equations (4.185) and (4.200). Let us start from eq. (4.183), where we do not need to specify the boundary conditions

$$Z[J] = \int \mathcal{D}(q(\tau))\Phi_0^\star(q',t')e^{iS_J}\Phi_0(q,t)dqdq' \tag{4.202}$$

with

$$\Phi_0(q,t) = \left(\frac{m\omega}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}m\omega q^2} e^{-\frac{i}{2}\omega t} \tag{4.203}$$

$$\Phi_0^\star(q',t') = \left(\frac{m\omega}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}m\omega q'^2} e^{\frac{i}{2}\omega t'} \tag{4.204}$$

and

$$S_J = \int_t^{t'} \left[\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 + Jq\right]d\tau \tag{4.205}$$

In this calculation we are not interested in the normalization factors but only on the J dependence. Then, let us consider the argument of the exponential in eq. (4.9 and, for the moment being, let us neglect the time dependent terms from the oscillator wave functions

$$\arg = -\frac{1}{2}m\omega(q^2 + q'^2) + i\int_t^{t'} \left[\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 + Jq\right]d\tau \tag{4.206}$$

let us perform the change of variable

$$q(\tau) = x(\tau) + x_0(\tau) \tag{4.207}$$

where we will try to arrange $x_0(\tau)$ in such a way to extract all the dependence from the source J out of the integral. We have

$$\arg = -\frac{1}{2}m\omega[x(t)^2 + x^2(t')] - \frac{1}{2}m\omega[x_0(t)^2 + x_0^2(t')] - m\omega[x(t)x_0(t) + x(t')x_0(t')]$$

$$+ \quad i \int_t^{t'} \left[\frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 \right] d\tau + i \int_t^{t'} \left[\frac{1}{2} m \dot{x}_0^2 - \frac{1}{2} m \omega^2 x_0^2 + J x_0 \right] d\tau$$

$$+ \quad i \int_t^{t'} \left[m \dot{x} \dot{x}_0 - m \omega^2 x x_0 + J x \right] d\tau \tag{4.208}$$

Integrating by parts the last integral can be written as

$$im[x \dot{x}_0]_t^{t'} - i \int_t^{t'} [m \ddot{x}_0 + m \omega^2 x_0 - J] x \, d\tau \tag{4.209}$$

and choosing x_0 as a solution of the wave equation in presence of the source J

$$\ddot{x}_0 + \omega^2 x_0 = \frac{1}{m} J \tag{4.210}$$

we get

$$im[x(t') \dot{x}_0(t') - x(t) \dot{x}_0(t)] \tag{4.211}$$

Let us take the following boundary conditions on x_0:

$$i \dot{x}_0(t') = \omega x_0(t'), \quad i \dot{x}_0(t) = -\omega x_0(t) \tag{4.212}$$

In this way, the term (4.211) cancels the mixed terms

$$\begin{aligned} \text{arg} \quad = \quad & -\frac{1}{2} m \omega [x^2(t) + x^2(t')] - \frac{1}{2} m \omega [x_0^2(t) + x_0^2(t')] \\ & + \quad i \int_t^{t'} [\frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2] d\tau \\ & + \quad i \int_t^{t'} [\frac{1}{2} m \dot{x}_0^2 - \frac{1}{2} m \omega^2 x_0^2 + J x_0] d\tau \end{aligned} \tag{4.213}$$

Integrating by parts the last term we get

$$\begin{aligned} i \int_t^{t'} [\frac{1}{2} m \dot{x}_0^2 - \frac{1}{2} m \omega^2 x_0^2 + J x_0] d\tau = \\ = \quad \int_t^{t'} [-\frac{1}{2} m x_0 (\ddot{x}_0 + \omega^2 x_0 - \frac{1}{m} J) + \frac{1}{2} J x_0] d\tau + \frac{i}{2} m [x_0 \dot{x}_0]_t^{t'} = \\ = \quad \frac{1}{2} m \omega [x_0^2(t') + x_0^2(t)] + \frac{i}{2} \int_t^{t'} J x_0 d\tau \end{aligned} \tag{4.214}$$

Keeping in mind the boundary conditions (4.212), the term coming from the integration by parts cancels the second term in eq. (4.213), and therefore

$$\text{arg} = -\frac{1}{2} m \omega [x^2(t) + x^2(t')] + i \int_t^{t'} [\frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2] d\tau + \frac{i}{2} \int_t^{t'} J x_0 d\tau \tag{4.215}$$

The J dependence is now in the last term, and we can extract it from the integral. The first two term give rise to the functional at $J = 0$. We obtain

$$Z[J] = e^{\frac{i}{2} \int_t^{t'} J(\tau)x_0(\tau)d\tau} Z[0] \tag{4.216}$$

with $x_0(\tau)$ solution of the equations of motion (4.210) with boundary conditions (4.212). For $t < \tau < t'$ this solution can be written as

$$x_0(\tau) = i \int_t^{t'} \Delta(\tau - s)J(s)ds \tag{4.217}$$

with

$$\left[\frac{d^2}{d\tau^2} + \omega^2 \right] \Delta(\tau - s) = -\frac{i}{m}\delta(\tau - s) \tag{4.218}$$

and

$$i\dot{\Delta}(t' - s) = \omega\Delta(t' - s), \quad i\dot{\Delta}(t - s) = -\omega\Delta(t - s) \tag{4.219}$$

These equations are easily solved fort $t' > s$ and $t < s$

$$\Delta(\tau) = Ae^{-i\omega\tau}, \quad \tau > 0$$
$$\Delta(\tau) = Be^{+i\omega\tau}, \quad \tau < 0 \tag{4.220}$$

Therefore we have

$$\Delta(\tau) = A\theta(\tau)e^{-i\omega\tau} + B\theta(-\tau)e^{+i\omega\tau} \tag{4.221}$$

The constants A e B are fixed by the requirement that $\Delta(\tau)$ satisfies eq. (4.218). We have

$$\dot{\Delta}(\tau) = (A - B)\delta(\tau) - iA\omega\theta(\tau)e^{-i\omega\tau} + iB\omega\theta(-\tau)e^{+i\omega\tau} \tag{4.222}$$

and

$$\ddot{\Delta}(\tau) = (A - B)\dot{\delta}(\tau) - i\omega(A + B)\delta(\tau) - \omega^2\Delta(\tau) \tag{4.223}$$

from which

$$A = B, \quad A = \frac{1}{2\omega m} \tag{4.224}$$

and finally

$$\Delta(\tau) = \frac{1}{2\omega m} \left[\theta(\tau)e^{-i\omega\tau} + \theta(-\tau)e^{+i\omega\tau} \right] \tag{4.225}$$

The functional we were looking for is

$$Z[J] = e^{-\frac{1}{2} \int_t^{t'} dsds' J(s)\Delta(s - s')J(s')} Z[0] \tag{4.226}$$

118

We see that $\Delta(s - s')$ is the vacuum expectation value of the T-product of the position operators at the times s and s':

$$\Delta(s - s') = -\frac{1}{Z[0]} \frac{\delta^2 Z[J]}{\delta J(s) \delta J(s')} \bigg|_{J=0} = \frac{\langle 0 | T(\mathbf{q}(s) \mathbf{q}(s')) | 0 \rangle}{\langle 0 | 0 \rangle} \tag{4.227}$$

The analogue calculation in the euclidean version is simpler since the path integral is of the type already considered (quadratic action)

$$Z_E[J] \approx e^{-S_E^{(cl)}} \tag{4.228}$$

Here $S_E^{(cl)}$ is th euclidean classical action. We have already noticed that here the boundary conditions are arbitrary and therefore we choose $q(\tau) \to 0$ for $\tau \to \pm\infty$. we have to solve the equations of motion in presence of the external source

$$\frac{\delta}{\delta q(\tau)} \left[S_E - \int_{-\infty}^{+\infty} Jq d\tau \right] = -m\ddot{q}(\tau) + m\omega^2 q(\tau) - J = 0 \tag{4.229}$$

that is

$$\ddot{q} - \omega^2 q = -\frac{1}{m} J \tag{4.230}$$

Let us now define the euclidean Green's function

$$\left[\frac{d^2}{d\tau^2} - \omega^2 \right] D_E(\tau) = -\frac{1}{m} \delta(\tau) \tag{4.231}$$

with the boundary condition

$$\lim_{\tau \to \pm\infty} D_E(\tau) = 0 \tag{4.232}$$

then, the solution of the eq. (4.230) is

$$q(\tau) = \int_{-\infty}^{+\infty} D_E(\tau - s) J(s) ds \tag{4.233}$$

We can evaluate $D_E(\tau)$ starting from the Fourier transform

$$D_E(\tau) = \int d\nu e^{-i\nu\tau} D_E(\nu) \tag{4.234}$$

Substituting inside (4.231) we get

$$-\int d\nu e^{-i\nu\tau} [\nu^2 + \omega^2] D_E(\nu) = -\frac{1}{m} \delta(\tau) \tag{4.235}$$

from which

$$D_E(\nu) = \frac{1}{2\pi m} \frac{1}{\nu^2 + \omega^2} \tag{4.236}$$

119

and

$$D_E(\tau) = \frac{1}{2\pi m} \int_{-\infty}^{+\infty} d\nu \frac{1}{\nu^2 + \omega^2} e^{-i\nu\tau} \tag{4.237}$$

To evaluate this integral we close the integral with the circle at the ∞ in the lower ν-plane for $\tau > 0$ or in the upper ν-plane for $\tau < 0$. The result is

$$D_E(\tau) = \frac{1}{2\pi m} \frac{e^{-\omega\tau}}{(-2i\omega)}(-2\pi i) = \frac{1}{2m\omega} e^{-\omega\tau} \tag{4.238}$$

By doing the same calculation for $\tau < 0$ we finally get

$$D_E(\tau) = \frac{1}{2m\omega} e^{-\omega|\tau|} \tag{4.239}$$

The classical action is given by

$$S_E - \int_{-\infty}^{+\infty} Jq\, d\tau = \int_{-\infty}^{+\infty} \left[-\frac{1}{2} m \left(\ddot{q} - \omega^2 q + \frac{1}{m} J \right) q - \frac{1}{2} Jq \right] d\tau = -\frac{1}{2} \int_{-\infty}^{+\infty} Jq\, d\tau \tag{4.240}$$

and therefore

$$Z_E[J] = e^{\frac{1}{2} \int_{-\infty}^{+\infty} Jq\, d\tau} Z[0] = e^{\frac{1}{2} \int_{-\infty}^{+\infty} J(s) D_E(s - s') J(s') ds\, ds'} Z[0] \tag{4.241}$$

It follows

$$\frac{1}{Z_E[0]} \frac{\delta^2 Z_E[J]}{\delta J(s) \delta J(s')} \bigg|_{J=0} = D_E(s - s') \tag{4.242}$$

and using the eq. (4.201)

$$\frac{1}{Z[0]} \frac{\delta^2 Z[J]}{\delta J(t_1) \delta J(t_2)} \bigg|_{J=0} = -\frac{1}{Z_E[0]} \frac{\delta^2 Z_E[J]}{\delta J(\tau_1) \delta J(\tau_2)} \bigg|_{J=0, \tau_i = it_i} \tag{4.243}$$

we get

$$\Delta(t) = D_E(it) \tag{4.244}$$

To compare these two expressions let us introduce the Fourier transform of $\Delta(t)$. Since it satisfies eq. (4.218) we obtain

$$\Delta(t) = \int d\nu e^{-i\nu t} \Delta(\nu) \tag{4.245}$$

with

$$(-\nu^2 + \omega^2) \Delta(\nu) = -\frac{i}{2\pi m} \tag{4.246}$$

that is

$$\Delta(\nu) = \frac{i}{2\pi m} \frac{1}{\nu^2 - \omega^2} \tag{4.247}$$

120

Whereas eq. (4.237) does not present any problem ($\nu^2 + \omega^2 > 0$), in eq. (4.246) we need to define the singularities present in the denominator. In fact, this expression has two poles at $\nu = \pm \omega$, and therefore we need to specify how the integral is defined. In order to reproduce the euclidean result, it is necessary to specify the integration path as in Fig. 4.9.1. For instance, for $t > 0$ by closing the integral in the lower plane we get

$$\frac{i}{2\pi m}(-2\pi i)\frac{1}{2\omega}e^{-i\omega t} = \frac{1}{2m\omega}e^{-i\omega t} \tag{4.248}$$

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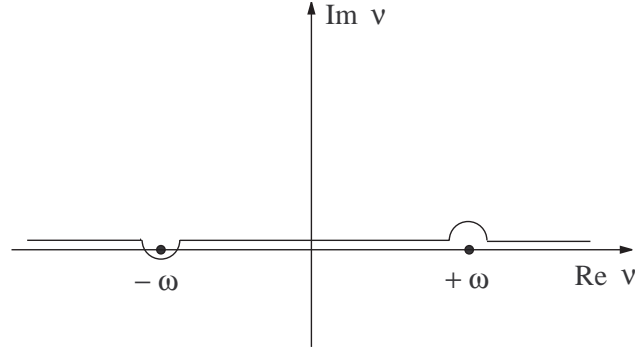
Fig. 4.9.1 - *The integration path for equation (4.245).*

We would get the same result by integrating along the real axis but translating the denominator of $+i\epsilon$ ($\epsilon > 0$) in such a way to move the pole at ω in the lower plane and the one at $-\omega$ in the upper plane. We get

$$\Delta(t) = \lim_{\epsilon \to 0^+} \frac{i}{2\pi m} \int_{-\infty}^{+\infty} \frac{e^{-i\nu t}}{\nu^2 - \omega^2 + i\epsilon} d\nu \tag{4.249}$$

With this prescription we can safely rotate the integration path anti-clockwise by 90^0, since we do not encounter any singularity (see Fig. 4.9.2)

$$\Delta(t) = \frac{i}{2\pi m} \int_{-i\infty}^{+i\infty} \frac{e^{-i\nu t}}{\nu^2 - \omega^2} d\nu \tag{4.250}$$

We have omitted here the limit since the expression is now regular. Performing the change of variable $\nu = i\nu'$

$$\Delta(t) = -\frac{1}{2\pi m} \int_{-\infty}^{+\infty} \frac{e^{+\nu' t}}{-\nu'^2 - \omega^2} d\nu' = D_E(it) \tag{4.251}$$

Therefore we have shown how it works the analytic continuation implicit in eq. (4.244). By doing so we have also seen that the Feynman prescription (the $+i\epsilon$ term) is equivalent to the euclidean formulation. One could arrive to this connection also

121

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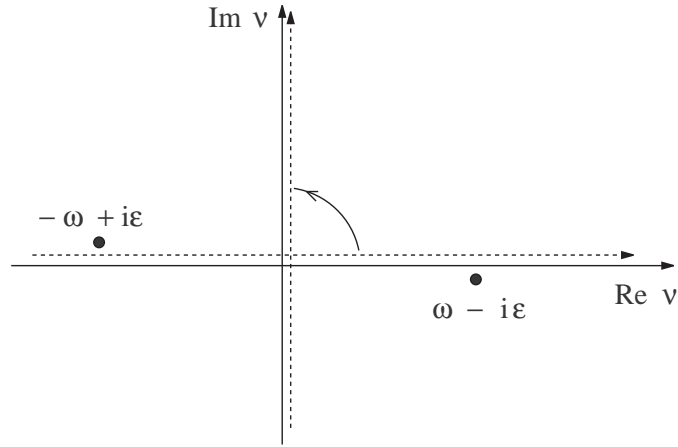
Fig. 4.9.2 - *The Wick rotation in the ν-plane.*

by noticing that the Feynman prescription is equivalent to substitute in the path integral the term

$$e^{-\dfrac{i}{2}m\omega^2 \int q^2(t)dt}$$
(4.252)

with

$$e^{-\dfrac{i}{2}m(\omega^2 - i\epsilon) \int q^2(t)dt}$$
(4.253)

Therefore the Feynman prescription is there to make the integral convergent.

$$e^{-\dfrac{1}{2}m\epsilon \int q^2(t)dt}$$
(4.254)

The same convergence is ensured by the euclidean formulation. Therefore we have shown that, in the case of quadratic actions, the path integral can be formulated as an euclidean integral, making it a well defined mathematical expression. This fact is enough to guarantee that all the manipulations that we make with the path integral in the perturbative expansion are well defined, since we will be expanding around the quadratic case.

Chapter 5

The path integral in field theory

5.1 The path integral for a free scalar field

We will extend now the previous approach to the case of a scalar field theory. We will show later how to extend the formulation to the case of Fermi fields. Let us start with a free scalar field. We have denoted the quantum operator by $\phi(x)$. Here we will denote the classical field by $\varphi(x)$. For a neutral particle this is a real function, and its dynamics is described by the lagrangian

$$S = \int_V d^4x \frac{1}{2} \left[\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2 \right] \equiv \int_t^{t'} dt \int d^3\vec{x} \frac{1}{2} \left[\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2 \right] \tag{5.1}$$

A free field theory can be seen as a continuous collection of non-interacting harmonic oscillators. As it is well known this can be seen by looking for the normal modes. To this end, let us consider the Fourier transform of the field

$$\varphi(\vec{x}, t) = \frac{1}{(2\pi)^3} \int d^3\vec{k} e^{i\vec{k} \cdot \vec{x}} q(\vec{k}, t) \tag{5.2}$$

By substituting inside eq. (5.1) we get

$$
\begin{aligned}
S &= \frac{1}{2} \int_V d^4x \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{d^3\vec{k}'}{(2\pi)^3} [\dot{q}(\vec{k}, t)\dot{q}(\vec{k}', t) \\
&+ (\vec{k} \cdot \vec{k}' - m^2)q(\vec{k}, t)q(\vec{k}', t)] e^{i\vec{k} \cdot \vec{x} + i\vec{k}' \cdot \vec{x}} \\
&= \frac{1}{2} \int_t^{t'} dt \int \frac{d^3\vec{k}}{(2\pi)^3} [\dot{q}(\vec{k}, t)\dot{q}(-\vec{k}, t) - (|\vec{k}|^2 + m^2)q(\vec{k}, t)q(-\vec{k}, t)] \tag{5.3}
\end{aligned}
$$

Since $\varphi(\vec{x}, t)$ is a real field

$$q^*(\vec{k}, t) = q(-\vec{k}, t) \tag{5.4}$$

and

$$S = \int_t^{t'} dt \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2} \left[|\dot{q}(\vec{k}, t)|^2 - \omega_k^2 |q(\vec{k}, t)|^2 \right], \quad \omega_k^2 = |\vec{k}|^2 + m^2 \tag{5.5}$$

Therefore the system is equivalent to a continuous set of complex oscillators $q(\vec{k}, t)$ (or a pair of real oscillators)). The equations of motion are

$$\ddot{q}(\vec{k}, t) + \omega_{\vec{k}}^2 q(\vec{k}, t) = 0 \tag{5.6}$$

We are now in the position of evaluating the generating functional $Z[J]$ by adding an external source term to the action

$$S = \int_V d^4x \left\{ \frac{1}{2} \left[\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2 \right] + J\varphi \right\} \tag{5.7}$$

From the Fourier decomposition we get

$$S = \int_t^{t'} dt \int \frac{d^3\vec{k}}{(2\pi)^3} \left\{ \frac{1}{2} \left[|\dot{q}(\vec{k}, t)|^2 - \omega_{\vec{k}}^2 |q(\vec{k}, t)|^2 \right] + J(-\vec{k}, t) q(\vec{k}, t) \right\} \tag{5.8}$$

where we have also expanded $J(x)$ in terms of its Fourier components. Let us separate $q(\vec{k}, t)$ and $J(\vec{k}, t)$ into their real and imaginary parts

$$q(\vec{k}, t) = x(\vec{k}, t) + iy(\vec{k}, t)$$
$$J(\vec{k}, t) = l(\vec{k}, t) + im(\vec{k}, t) \tag{5.9}$$

It follows from the reality conditions that the real parts are even functions of $\vec{k}$ whereas the imaginary parts are odd functions

$$x(\vec{k}, t) = x(-\vec{k}, t), \qquad l(\vec{k}, t) = l(-\vec{k}, t)$$
$$y(\vec{k}, t) = -y(-\vec{k}, t), \qquad m(\vec{k}, t) = -m(-\vec{k}, t) \tag{5.10}$$

Now we can use the result found in the previous Section for a single oscillator. it will be enough to take the mass equal to one and sum over all the degrees of freedom. In particular we get for the exponent in eq. (4.226)

$$-\frac{1}{2} \int_t^{t'} ds ds' J(s) \Delta(s - s') J(s') \rightarrow$$

$$\rightarrow -\frac{1}{2} \int_t^{t'} ds ds' \int \frac{d^3\vec{k}}{(2\pi)^3} \Big[l(\vec{k}, s) \Delta(s - s'; \omega_{\vec{k}}) l(\vec{k}, s')$$

$$+ \quad m(\vec{k}, s) \Delta(s - s'; \omega_{\vec{k}}) m(\vec{k}, s') \Big]$$

$$= \quad -\frac{1}{2} \int_t^{t'} ds ds' \int \frac{d^3\vec{k}}{(2\pi)^3} J(-\vec{k}, s) \Delta(s - s'; \omega_{\vec{k}}) J(\vec{k}, s') \tag{5.11}$$

where we have made use of eq. (5.9). Going back to $J(\vec{x}, t)$, we get

$$-\frac{1}{2} \int d^4x d^4y J(x) \int \frac{d^3\vec{k}}{(2\pi)^3} \Delta(s - s'; \omega_{\vec{k}}) e^{i\vec{k} \cdot (\vec{x} - \vec{y})} J(y)$$

$$\equiv -\frac{i}{2} \int d^4x d^4y J(x) \Delta_F(x - y; m^2) J(y) \tag{5.12}$$

where we have defined $x^\mu = (\vec{x}, s)$, $y^\mu = (\vec{y}, s')$ and

$$i\Delta_F(x, m^2) = \int \frac{d^3\vec{k}}{(2\pi)^3} \Delta(s; \omega_{\vec{k}}) e^{i\vec{k}\cdot\vec{x}} = \lim_{\epsilon\to 0^+} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ikx} \quad (5.13)$$

Again we have put $k^\mu = (\nu, \vec{k})$. Therefore, the generating functional is

$$Z[J] = e^{-\frac{i}{2} \int d^4x d^4y J(x)\Delta_F(x - y; m^2)J(y)} Z[0] \quad (5.14)$$

In particular we get

$$\frac{1}{\langle 0|0\rangle}\langle 0|T(\phi(x_1)\phi(x_2))|0\rangle = -\frac{1}{Z[0]}\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} = i\Delta_F(x_1 - x_2; m^2) \quad (5.15)$$

We could repeat the exercise in the euclidean formulation, with the result

$$Z_E[J] = e^{\frac{1}{2} \int d^4x d^4y J(x) G_0(x - y; m^2)J(y)} Z_E[0] \quad (5.16)$$

where

$$G_0(x; m^2) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2} e^{-ikx} \quad (5.17)$$

We go from Minkowskian to the Euclidean description through the substitution

$$-i\Delta_F \to G_0 \quad (5.18)$$

For the future it will be convenient to use the following notation for the N-points Green's functions

$$G^{(N)}(x_1, \cdots, x_N) = \frac{\langle 0|T(\phi(x_1)\cdots\phi(x_N))|0\rangle}{\langle 0|0\rangle} \quad (5.19)$$

In the free case we can get evaluate easily the generic $G^{(N)}$ In fact $G_0^{(N)}$ (the index recalls that we are in the free case) is obtained by developing the generating functional

$$\frac{Z[J]}{Z[0]} = \sum_{n=0}^\infty \int d^4x_1 \cdots d^4x_n J(x_1)\cdots J(x_n)\frac{(i)^n}{n!}\frac{\langle 0|T(\phi(x_1)\cdots\phi(x_n))|0\rangle}{\langle 0|0\rangle} \equiv$$

$$\equiv \sum_{n=0}^\infty \int d^4x_1 \cdots d^4x_n J(x_1)\cdots J(x_n)\frac{(i)^n}{n!}G_0^{(n)}(x_1, \cdots, x_n) \quad (5.20)$$

By comparison with the expansion of eq. (5.14)

$$\begin{aligned}
\frac{Z[J]}{Z[0]} &= 1 - \frac{i}{2}\int d^4x_1 d^4x_2 J(x_1)J(x_2)\Delta_F(x_1 - x_2) \\
&\quad - \frac{1}{8}\int d^4x_1 d^4x_2 J(x_1)J(x_2)\Delta_F(x_1 - x_2) \\
&\quad \cdot \int d^4x_3 d^4x_4 J(x_3)J(x_4)\Delta_F(x_3 - x_4) + \cdots
\end{aligned} \quad (5.21)$$

Since $Z[J]$ is even in J we have

$$G_0^{(2k+1)}(x_1, \cdots, x_{2k+1}) = 0 \tag{5.22}$$

For $G_0^{(2)}$ we get back eq. (5.13). To evaluate $G_0^{(4)}$ we have first to notice that all the $G_0^{(N)}$'s are symmetric in their arguments. therefore in order to extract the coefficients we need to symmetrize. Therefore we write

$$\int d^4x_1 \cdots d^4x_4 J(x_1) \cdots J(x_4) \Delta_F(x_1 - x_2) \Delta_F(x_3 - x_4)$$

$$= \frac{1}{3} \int d^4x_1 \cdots d^4x_4 J(x_1) \cdots J(x_4) \Big[\Delta_F(x_1 - x_2) \Delta_F(x_3 - x_4)$$

$$+ \Delta_F(x_1 - x_3) \Delta_F(x_2 - x_4) + \Delta_F(x_1 - x_4) \Delta_F(x_2 - x_3) \Big] \tag{5.23}$$

It follows

$$G_0^{(4)}(x_1, \cdots, x_4) = -\Big[\Delta_F(x_1 - x_2) \Delta_F(x_3 - x_4) + \Delta_F(x_1 - x_3) \Delta_F(x_2 - x_4)$$

$$+ \Delta_F(x_1 - x_4) \Delta_F(x_2 - x_3) \Big] \tag{5.24}$$

If we represent $G_0^{(2)}(x, y)$ by a line as in Fig. 5.1.1, $G_0^{(4)}$ will be given by Fig. 5.1.2 since it is obtained in terms of products of $G_0^{(2)}$. Therefore $G_0^{(2n)}$ is given by a combination of products of n two point functions. We will call disconnected a diagram in which we can isolate two subdiagrams with no connecting lines. We see that in the free case all the non vanishing Green's functions are disconnected except for the two point function. Therefore it is natural to introduce a functional generating only the connected Green's functions. In the free case we can put

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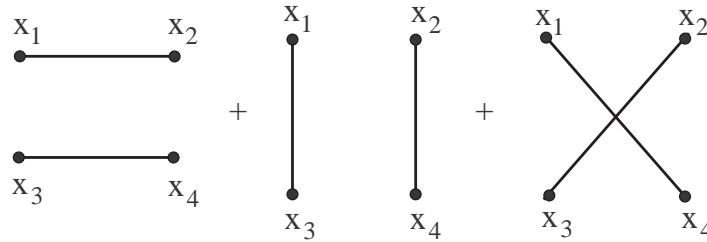**Fig. 5.1.1** - *The two point Green's function $G_0^{(2)}(x, y)$.*

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Fig. 5.5.2 - *The expansion of the four point Green's function $G_0^{(4)}(x_1, x_2, x_3, x_4)$.*

$$Z[J] = e^{iW[J]} \tag{5.25}$$

126

with

$$W[J] = -\frac{1}{2} \int d^4x_1 d^4x_2 J(x_1) J(x_2) \Delta_F(x_1 - x_2) + W[0] \qquad (5.26)$$

and we have

$$\frac{1}{Z[0]} \frac{\delta^2 Z[J]}{\delta J(x_1) \delta J(x_2)} \bigg|_{J=0} = i \frac{\delta^2 W[J]}{\delta J(x_1) \delta J(x_2)} \bigg|_{J=0} = -i\Delta_F(x_1 - x_2) \qquad (5.27)$$

Therefore the derivatives of $W[J]$ give the connected diagrams with the proper normalization(that is divided by $1/Z[0]$). In the next Section we will see that this property generalizes to the interacting case. That is $W[J]$ is defined through eq. (5.25), and its Volterra expansion gives rise to the connected Green's functions

$$iW[J] = \sum_{n=0}^{\infty} \frac{(i)^n}{n!} \int d^4x_1 \cdots d^4x_n J(x_1) \cdots J(x_n) G_c^{(n)}(x_1, \cdots, x_n) \qquad (5.28)$$

where

$$G_c^{(n)}(x_1, \cdots, x_n) = \frac{1}{\langle 0|0 \rangle} \langle 0|T(\phi(x_1) \cdots \phi(x_n))|0 \rangle_{\text{conn}} \qquad (5.29)$$

The index "conn" denotes the connected Green's functions. Notice that once we find the connected Green's functions the theory is completely solved, since the generating functional is recovered by exponentiation of $W[J]$.

5.2 The generating functional of the connected Green's functions

Let us start by defining recursively the connected Green's functions.

$$G^{(1)}(x_1) = G_c^{(1)}(x_1) \equiv G_c^{(1)};$$
$$G^{(2)}(x_1, x_2) = G_c^{(2)}(x_1, x_2) + G_c^{(1)}(x_1) G_c^{(1)}(x_2) \equiv G_c^{(2)} + G_c^{(1)^2};$$
$$G^{(3)}(x_1, x_2, x_3) = G_c^{(3)}(x_1, x_2, x_3) + G_c^{(1)}(x_1) G_c^{(2)}(x_2, x_3)$$
$$+ G_c^{(1)}(x_2) G_c^{(2)}(x_1, x_3) + G_c^{(1)}(x_3) G_c^{(2)}(x_1, x_2) + G_c^{(1)}(x_1) G_c^{(1)}(x_2) G_c^{(1)}(x_3) \equiv$$
$$\equiv G_c^{(3)} + 3G_c^{(1)} G_c^{(2)} + G_c^{(1)^3} \qquad (5.30)$$

On the right hand side we have used a symbolic notation by adding together the topological equivalent configurations. To define $G_c^{(n)}(x_1, \cdots, x_n)$ we decompose the set $(x_1, \cdots, x_n)$ in all the possible subsets, and then we write

$$G^{(n)}(x_1, \cdots, x_n) = \sum G_c^{(n_1)}(x_{p_1}, \cdots, x_{p_{n_1}}) G_c^{(n_2)}(x_{q_1}, \cdots, x_{q_{n_2}}) \cdots G_c^{(n_k)}(x_{r_1}, \cdots, x_{r_{n_k}}) \qquad (5.31)$$

where we sum over all the indices n_i such that $n_1 + \cdots + n_k = n$ and over all the permutations from 1 to n of the indices in the set $p_1, \cdots, p_{n_1}$, etc. A graphical

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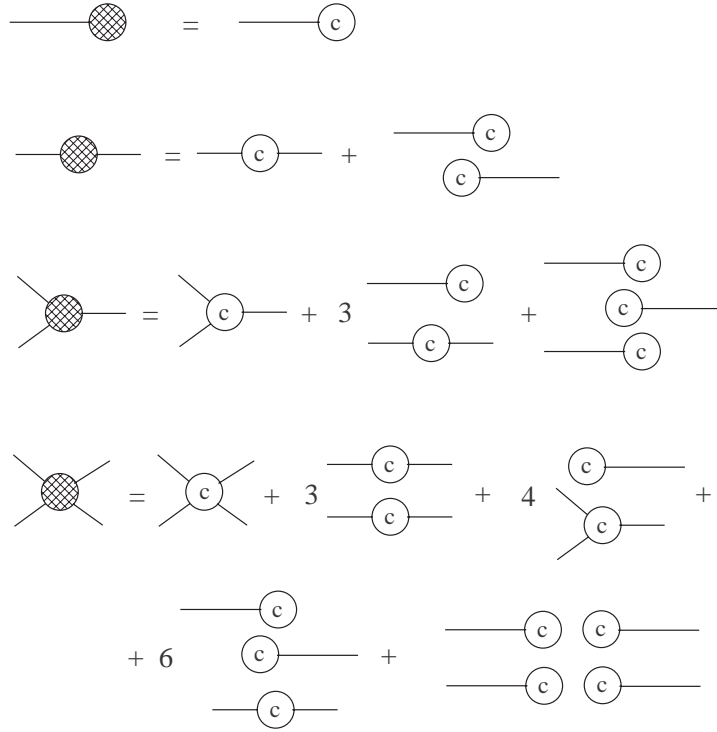
Fig. 5.2.1 - *The connected Green's functions.*

description is given in Fig. 5.2.1. By using these definitions it is not difficult to check that $W[J]$ is the generating functional of the connected Green's functions. It is enough to compare the expansion of $Z[J]$ in series of $G_c^{(n)}$ with the expansion from the $G^{(n)}$. This can be done by induction. We will limit ourselves to an explicit check up to the 4th order. By defining $G^{(0)} = 1$ and using the same symbolic notation as before, we get

$$
\begin{aligned}
\frac{Z[J]}{Z[0]} &= \sum_{n=0}^{\infty} \frac{(i)^n}{n!} J^n G^{(n)} = e^{i(W[J] - W[0])} \\
&= \exp\left[iG_c^{(1)} J + \frac{i^2}{2!}G_c^{(2)} J^2 + \frac{i^3}{3!}G_c^{(3)} J^3 + \frac{i^4}{4!}G_c^{(4)} J^4 + \cdots \right] \\
&\approx 1 + iG_c^{(1)} J + \frac{i^2}{2!}G_c^{(2)} J^2 + \frac{i^3}{3!}G_c^{(3)} J^3 + \frac{i^4}{4!}G_c^{(4)} J^4 \\
&+ \frac{1}{2!}\left[i^2 G_c^{(1)^2} J^2 + \frac{i^4}{4!}G_c^{(2)^2} J^4 + i^3 G_c^{(1)} G_c^{(2)} J^3 + \frac{i^4}{3}G_c^{(1)} G_c^{(3)} J^4 \right] \\
&+ \frac{1}{3!}\left[i^3 G_c^{(1)^3} J^3 + \frac{3}{2}i^4 G_c^{(1)^2} G_c^{(2)} J^4 \right] \\
&+ \frac{1}{4!}\left[i^4 G_c^{(1)^4} J^4 \right] + \cdots
\end{aligned}
$$

128

$$\begin{aligned}
= \quad & 1 + iJG_c^{(1)} + \frac{i^2}{2!}\left[G_c^{(2)} + G_c^{(1)\,2}\right] \\
+ \quad & \frac{i^3}{3!}J^3\left[G_c^{(3)} + 3G_c^{(1)}G_c^{(2)} + G_c^{(1)\,3}\right] \\
+ \quad & \frac{i^4}{4!}J^4\left[G_c^{(4)} + 3G_c^{(2)\,2} + 4G_c^{(1)}G_c^{(3)} + 6G_c^{(1)\,2}G_c^{(2)} + G_c^{(1)\,4}\right] + \cdots \quad (5.32)
\end{aligned}$$

5.3 The perturbative expansion for the theory $\lambda\varphi^4$

Let us consider an interacting scalar field. The action will be of the type

$$S = \int_V d^4x \left(\frac{1}{2}\left[\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2\right] - V(\varphi)\right) \qquad (5.33)$$

where $V(\varphi)$ is the interaction potential. A typical example is the theory $\lambda\varphi^4$ with the potential given by

$$V(\varphi) = \frac{\lambda}{4!}\varphi^4 \qquad (5.34)$$

The derivation of the perturbative expansion is very simple in this formalism. We take advantage of the following identity valid for any functional F

$$\int \mathcal{D}(\varphi(x))F[\varphi]e^{iS + i\int d^4xJ(x)\varphi(x)}$$
$$= \quad F\left[\frac{1}{i}\frac{\delta}{\delta J}\right]\int \mathcal{D}(\varphi(x))e^{iS + i\int d^4xJ(x)\varphi(x)} \qquad (5.35)$$

We separate in eq. (5.33) the quadratic part of the action, S_0, from the interacting part and then we use the previous identity

$$\begin{aligned}
Z[J] \quad = \quad & N\int \mathcal{D}(\varphi(x))e^{iS + i\int d^4xJ(x)\varphi(x)} \\
= \quad & N\int \mathcal{D}(\varphi(x))e^{-i\int d^4xV(\varphi)}e^{iS_0 + i\int d^4xJ(x)\varphi(x)} \\
= \quad & e^{-i\int d^4xV\left(\frac{1}{i}\frac{\delta}{\delta J}\right)}Z_0[J] \qquad (5.36)
\end{aligned}$$

Notice that we have suppressed the temporal limits in the expression for the generating functional. $Z_0[J]$ is the generating functional for the free case, evaluated in eq. (5.14). Therefore we get

$$Z[J] = e^{iW[J]} = Z[0]e^{-i\int d^4xV\left(\frac{1}{i}\frac{\delta}{\delta J}\right)}e^{-\frac{i}{2}\int d^4xd^4yJ(x)\Delta_F(x-y)J(y)}$$
$$(5.37)$$

In this expression we have suppressed the mass in the argument of Δ_F. For the following calculation it will be useful to introduce the following shorthand notation

$$\int d^4x_1 \cdots d^4x_n F(x_1, \cdots, x_n) \equiv \langle F(x_1, \cdots, x_n) \rangle \tag{5.38}$$

Then, eq. (5.37) can be rewritten in the following way

$$
\begin{aligned}
Z[J] = e^{iW[J]} &= e^{-i\langle V\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\rangle} e^{-\frac{i}{2}\langle J(1)\Delta_F(1,2)J(2)\rangle} Z[0] \\
&= e^{-i\langle V\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\rangle} e^{iW_0[J]}
\end{aligned}
\tag{5.39}
$$

This gives rise to the following expression for the generating functional of the connected Green's functions

$$
\begin{aligned}
W[J] &= -i\log\left[e^{iW_0[J]} + \left(e^{-i\langle V\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\rangle} - 1 \right) e^{iW_0[J]} \right] \\
&= W_0[J] - i\log\left[1 + e^{-iW_0[J]} \left(e^{-i\langle V\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\rangle} - 1 \right) e^{iW_0[J]} \right]
\end{aligned}
\tag{5.40}
$$

This expression can be easily expanded in V. In fact, by defining

$$\delta[J] = e^{-iW_0[J]} \left(e^{-i\langle V\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\rangle} - 1 \right) e^{iW_0[J]} \tag{5.41}$$

we see that we can expand $W[J]$ in a series of δ. Notice also that the expansion is in term of the interaction lagrangian. At the second order in δ we get

$$W[J] = W_0[J] - i\left(\delta - \frac{1}{2}\delta^2 \right) + \cdots \tag{5.42}$$

Let us now consider the case of $\lambda\varphi^4$. the functional $\delta[J]$ can be in turn expanded in a series of the dimensionless coupling λ

$$\delta = \lambda\delta_1 + \lambda^2\delta_2 + \cdots \tag{5.43}$$

It follows

$$W[J] = W_0[J] - i\lambda\delta_1 - i\lambda^2 \left(\delta_2 - \frac{1}{2}\delta_1^2 \right) + \cdots \tag{5.44}$$

130

Then, from eq. (5.43) we get

$$\delta_1 = -\frac{i}{4!}e^{-iW_0[J]}\langle\left(\frac{1}{i}\frac{\delta}{\delta J}\right)^4\rangle e^{iW_0[J]}$$

$$\delta_2 = -\frac{1}{2(4!)^2}e^{-iW_0[J]}\langle\left(\frac{1}{i}\frac{\delta}{\delta J}\right)^4\rangle\langle\left(\frac{1}{i}\frac{\delta}{\delta J}\right)^4\rangle e^{iW_0[J]} \tag{5.45}$$

By using the following list of functional derivatives

$$\frac{\delta}{\delta J(x)}e^{iW_0[J]} = -i\Delta(x,1)J(1)e^{iW_0[J]}$$

$$\frac{\delta^2}{\delta J(x)^2}e^{iW_0[J]} = \left(-i\Delta(x,x) - \Delta(x,1)\Delta(x,2)J(1)J(2)\right)e^{iW_0[J]}$$

$$\frac{\delta^3}{\delta J(x)^3}e^{iW_0[J]} = \Big(-3\Delta(x,x)\Delta(x,1)J(1)$$

$$+ \ i\Delta(x,1)\Delta(x,2)\Delta(x,3)J(1)J(2)J(3)\Big)e^{iW_0[J]}$$

$$\frac{\delta^4}{\delta J(x)^4}e^{iW_0[J]} = (-3\Delta(x,x)^2 + 6i\Delta(x,x)\Delta(x,1)\Delta(x,2)J(1)J(2)$$

$$+\Delta(x,1)\Delta(x,2)\Delta(x,3)\Delta(x,4)J(1)J(2)J(3)J(4))e^{iW_0[J]} \tag{5.46}$$

where the integration on the variables $1,2,3,4$ is understood, we obtain

$$\delta_1 \ = \ -\frac{i}{4!}\Big[\langle\Delta(y,1)\Delta(y,2)\Delta(y,3)\Delta(y,4)J(1)J(2)J(3)J(4)\rangle$$

$$+ \ 6i\langle\Delta(y,y)\Delta(y,1)\Delta(y,2)J(1)J(2)\rangle - 3\langle\Delta^2(y,y)\rangle\Big] \tag{5.47}$$

The quantity δ_2 can be made in terms of δ_1 by writing

$$\delta_2 \ = \ -\frac{1}{2(4!)^2}e^{-iW_0[J]}\langle\left(\frac{1}{i}\frac{\delta}{\delta J}\right)^4\rangle\left(e^{iW_0[J]}e^{-iW_0[J]}\langle\left(\frac{1}{i}\frac{\delta}{\delta J}\right)^4\rangle e^{iW_0[J]}\right)$$

$$= \ -\frac{i}{2(4!)}e^{-iW_0[J]}\langle\left(\frac{1}{i}\frac{\delta}{\delta J}\right)^4\rangle\left(e^{iW_0[J]}\delta_1\right) \tag{5.48}$$

Then we have

$$\langle\left(\frac{1}{i}\frac{\delta}{\delta J}\right)^4\rangle\left(e^{iW_0[J]}\delta_1\right) \ = \ \langle\frac{\delta^4 e^{iW_0[J]}}{\delta J(x)^4}\rangle\delta_1 + 4\langle\frac{\delta^3 e^{iW_0[J]}}{\delta J(x)^3}\frac{\delta\delta_1}{\delta J(x)}\rangle$$

$$+ \ 6\langle\frac{\delta^2 e^{iW_0[J]}}{\delta J(x)^2}\frac{\delta^2\delta_1}{\delta J(x)^2}\rangle + 4\langle\frac{\delta e^{iW_0[J]}}{\delta J(x)}\frac{\delta^3\delta_1}{\delta J(x)^3}\rangle$$

$$+ \ e^{iW_0[J]}\langle\frac{\delta^4\delta_1}{\delta J(x)^4}\rangle \tag{5.49}$$

131

and therefore

$$
\begin{aligned}
\delta_2 \;=\;& \frac{1}{2}\delta_1^2 - \frac{i}{2\cdot 4!}e^{-iW_0[J]}\Big[4\langle\frac{\delta^3 e^{iW_0[J]}}{\delta J(x)^3}\frac{\delta\delta_1}{\delta J(x)}\rangle \\
& +\; 6\langle\frac{\delta^2 e^{iW_0[J]}}{\delta J(x)^2}\frac{\delta^2\delta_1}{\delta J(x)^2}\rangle + 4\langle\frac{\delta e^{iW_0[J]}}{\delta J(x)}\frac{\delta^3\delta_1}{\delta J(x)^3}\rangle \\
& +\; e^{iW_0[J]}\langle\frac{\delta^4\delta_1}{\delta J(x)^4}\rangle\Big]
\end{aligned}
\tag{5.50}
$$

The δ_1^2 gives a disconnected contribution since there are no propagators connecting the two terms. In fact the previous expression shows that δ_2 contains a term which cancels precisely the term δ_1^2 in eq. (5.44). By using the expression for δ_1 and eqs. (5.46) we get

$$
\begin{aligned}
& \delta_2 - \frac{1}{2}\delta_1^2 = \\
=\;& \left(-\frac{i}{2\cdot 4!}\right)\left(-\frac{i}{4!}\right)\langle 4\Big(-3\Delta(x,x)\Delta(x,1)J(1) \\
& +\; i\Delta(x,1)\Delta(x,2)\Delta(x,3)J(1)J(2)J(3)\Big) \\
& \cdot\; \Big(4\Delta(y,x)\Delta(y,4)\Delta(y,5)\Delta(y,6)J(4)J(5)J(6) + 12i\Delta(y,y)\Delta(y,x)\Delta(y,4)J(4)\Big) \\
& +\; 6\Big(-i\Delta(x,x) - \Delta(x,1)\Delta(x,2)J(1)J(2)\Big)\Big(12\Delta^2(y,x)\Delta(y,3)\Delta(y,4)J(3)J(4) \\
& +\; 12i\Delta(y,y)\Delta^2(y,x)\Big) \\
& +\; 4\Big(-i\Delta(x,1)J(1)\Big)\Big(24\Delta^3(y,x)\Delta(y,2)J(2)\Big) + 24\Delta^4(y,x)\rangle
\end{aligned}
\tag{5.51}
$$

By omitting the terms independent on the external source we have

$$
\begin{aligned}
& \delta_2 - \frac{1}{2}\delta_1^2 = \\
=\;& \frac{i}{2}\langle J(1)\Delta(1,x)\Big(\frac{1}{6}\Delta^3(x,y) + \frac{1}{4}\Delta(x,x)\Delta(x,y)\Delta(y,y)\Big)\Delta(y,2)J(2)\rangle \\
& +\; \frac{i}{8}\langle J(1)\Delta(1,x)\Delta^2(x,y)\Delta(y,y)\Delta(x,2)J(2)\rangle \\
& +\; \frac{1}{12}\langle J(1)\Delta(1,x)\Delta(x,x)\Delta(x,y)\Delta(y,2)\Delta(y,3)\Delta(y,4)J(2)J(3)J(4)\rangle \\
& +\; \frac{1}{16}\langle J(1)J(2)\Delta(1,x)\Delta(2,x)\Delta^2(x,y)\Delta(y,3)\Delta(y,4)J(3)J(4)\rangle \\
& -\; \frac{i}{72}\langle J(1)J(2)J(3)\Delta(1,x)\Delta(2,x)\Delta(3,x)\Delta(x,y)\Delta(y,4) \\
& \cdot\; \Delta(y,5)\Delta(y,6)J(4)J(5)J(6)\rangle
\end{aligned}
\tag{5.52}
$$

Now, by differentiating functionally eq. (5.44) and putting $J = 0$ we can evaluate the connected Green's functions. For instance, recalling that

$$iG_c^{(2)}(x_1, x_2) = \frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)}\Big|_{J=0} \tag{5.53}$$

we get

$$G_c^{(2)}(x_1, x_2) = i\Delta_F(x_1 - x_2) - \frac{\lambda}{2}\int d^4x \Delta_F(x_1 - x)\Delta_F(x - x)\Delta_F(x - x_2)$$

$$- i\lambda^2 \int d^4x d^4y \Delta_F(x_1 - x)\Big[\frac{1}{6}\Delta_F^3(x - y)$$

$$+ \frac{1}{4}\Delta_F(x - y)\Delta_F(x - x)\Delta_F(y - y)\Big]\Delta_F(y - x_2)$$

$$- i\frac{\lambda^2}{4}\int d^4x d^4y \Delta_F(x_1 - x)\Delta_F^2(x - y)\Delta_F(y - y)\Delta_F(x - x_2) \tag{5.54}$$

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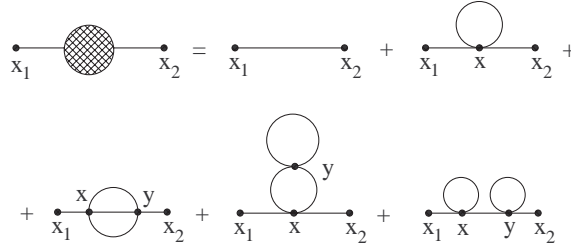
Fig. 5.3.1 - *The graphical expansion for the two point connected Green's function $G_c^{(2)}(x, y)$.*

In Fig. 5.3.1) we give a graphical form to this expression. as we see the result has an easy interpretation. The diagrams of Fig. 5.3.1) are given in terms of two elements, the propagator $\Delta_F(x - y)$, corresponding to the lines and the vertex $\lambda\varphi^4$ corresponding to a point joining four lines. In order to get $G_c^{(n)}$ we have to draw all the possible connected diagrams containing a number of interaction vertices corresponding to the fixed perturbative order. The analytic expression is obtained associating a factor $i\Delta_F(x - y)$ to each line connecting the point x with the point y, and a factor $(-i\lambda/4!)$ for each interaction vertex. The numerical factors appearing in eq. (5.54) can be obtained by counting the possible ways to draw a given diagram. For instance let us consider the diagram of Fig. 5.3.2. This diagram contains the two external propagators, the internal one, and the vertex at x. To get the numerical factor we count the number of ways to attach the external propagators to the vertex, after that we have only a possibility to attach the vertex at the internal propagator. There are four ways to attach the the vertex at the first propagator and three for the second. Therefore

$$\left(\frac{1}{4!}\right) \cdot 4 \cdot 3 = \frac{1}{2} \tag{5.55}$$

133
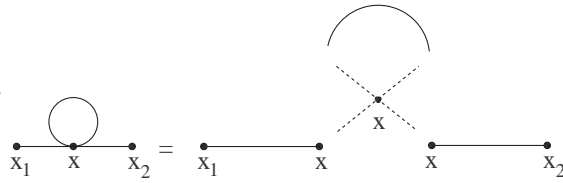
Fig. 5.3.2 - *How to get the numerical factor for the first order contributions to* $G_c^{(2)}(x_1, x_2)$.

where the factor $1/4!$ comes from the vertex. We proceed in the same way for the diagram of Fig. 5.3.3.
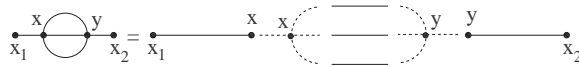
Fig. 5.3.3 - *How to get the numerical factor for one of the second order contributions to* $G_c^{(2)}(x_1, x_2)$.
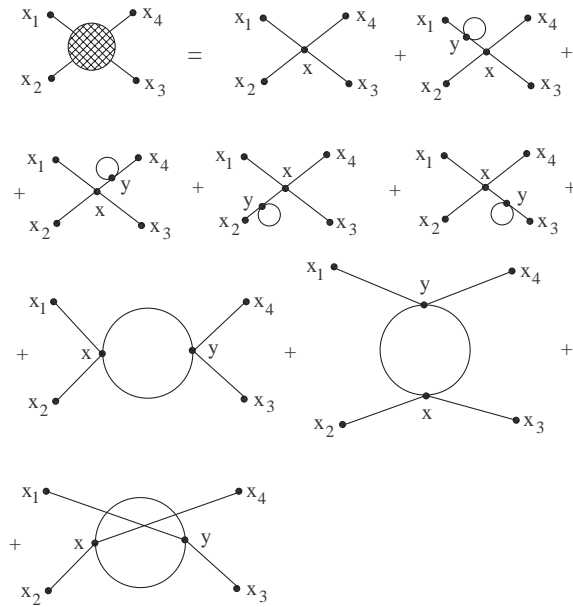
Fig. 5.3.4 - *The graphical expansion up to the second order of* $G_c^{(4)}(x_1, x_2, x_3, x_4)$.

There are four ways to attach the first vertex to the first propagator and four for attaching the second vertex to the second propagator. Three ways for attaching the second leg of the first vertex to the second vertex and two ways of attaching the the third leg of the first vertex to the second one. Therefore we get

$$\left(\frac{1}{4!}\right) \cdot 4 \cdot 4 \cdot 3 \cdot 2 = \frac{1}{6} \tag{5.56}$$

134

The four point function can be evaluated in analogous way and the result is given in Fig. 5.3.4. Of course we can get it by functional differentiation of $W[J]$. The first order contribution is

$$G_c^{(4)}(x_1, x_2, x_3, x_4) = -i\lambda \int d^4x \Delta_F(x_1-x)\Delta_F(x_2-x)\Delta_F(x_3-x)\Delta_F(x_4-x) \quad (5.57)$$

where the numerical coefficient is one, since there are $4 \cdot 3 \cdot 2$ of attaching the vertex to the four external lines.

5.4 The Feynman's rules in momentum space

The rules for the perturbative expansion of the Green's functions in space-time that we got in the previous Section can be easily extended to the momentum space. We will consider again the two point function given in eq. (5.54) where we will take separately the various contributions in λ that we will denote by $G_c^{(2)(i)}$, where i stands for the power of λ. let us also define the Fourier transform of $G_c^{(2)}(x_1, x_2)$ as

$$G_c^{(2)}(p_1, p_2) = \int d^4x_1 d^4x_2 G_c^{(2)}(x_1, x_2)e^{ip_1x_1 + ip_2x_2} \quad (5.58)$$

Then we get

$$G_c^{(2)(0)}(p_1, p_2) = (2\pi)^4\delta^4(p_1+p_2)\frac{i}{p_1^2 - m^2 + i\epsilon} \quad (5.59)$$

and

$$G_c^{(2)(1)}(p_1, p_2) =$$

$$= -i\frac{\lambda}{2}\int d^4x_1 d^4x_2 d^4x e^{ip_1x_1 + ip_2x_2}\int \frac{d^4p}{(2\pi)^4}e^{-ip(x_1 - x)}\frac{i}{p^2 - m^2 + i\epsilon}$$

$$\cdot \int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2 - m^2 + i\epsilon}\int \frac{d^4k}{(2\pi)^4}e^{-ik(x - x_2)}\frac{i}{k^2 - m^2 + i\epsilon}$$

$$= -i\frac{\lambda}{2}\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}(2\pi)^4\delta^4(p - k)(2\pi)^4\delta^4(p_1 - p)(2\pi)^4\delta^4(p_2 + k)$$

$$\cdot \frac{i}{p^2 - m^2 + i\epsilon}\frac{i}{q^2 - m^2 + i\epsilon}\frac{i}{k^2 - m^2 + i\epsilon} \quad (5.60)$$

The product of the δ^4 functions may be rewritten in such a way to pull out the overall four-momentum conservation

$$(2\pi)^4\delta^4(p_1 + p_2)(2\pi)^4\delta^4(p_1 - p)(2\pi)^4\delta^4(p - k) \quad (5.61)$$

By integrating over p and k we get

$$G_c^{(2)(1)}(p_1, p_2) = -i\frac{\lambda}{2}(2\pi)^4\delta^4(p_1+p_2)\left(\frac{i}{p_1^2 - m^2 + i\epsilon}\right)^2\int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2 - m^2 + i\epsilon} \quad (5.62)$$
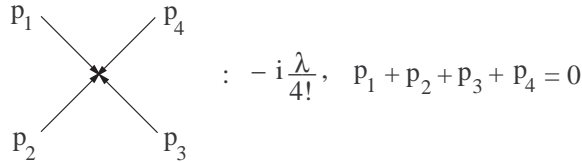
Also in momentum space we can make use of a diagrammatic expansion with the following rules:

- For each propagator draw a line with associated momentum p (see Fig. 5.4.1).

.

$$\text{———————} \quad : \quad \frac{i}{p^2 - m^2 + i\varepsilon}$$

Fig. 5.4.1 - *The propagator in momentum space.*

- For each factor $(-i\lambda/4!)$ draw a vertex with the convention that the momentum flux is zero (see Fig. 5.4.2).

.

$$: \quad -i\frac{\lambda}{4!}, \quad p_1 + p_2 + p_3 + p_4 = 0$$

Fig. 5.4.2 - *The vertex in momentum space.*

- To get $G_c^{(n)}$ draw all the topological inequivalent diagrams after having fixed the external legs. The number of ways of drawing a given diagram is its topological weight. The contribution of such a diagram is multiplied by its topological weight.

- After the requirement of the conservation of the four-momentum at each vertex we need to integrate over all the independent internal four-momenta with weight

$$\int \frac{d^4 q}{(2\pi)^4} \tag{5.63}$$

A more systematic way is to associate a factor

$$-\frac{i\lambda}{4!}(2\pi)^4 \delta^4 \left(\sum_{i=1}^{4} p_i \right) \tag{5.64}$$

to each vertex and integrate over all the momenta of the internal lines. This gives automatically a factor

$$(2\pi)^4 \delta^4 \left(\sum_{i=1}^{n} p_i \right), \quad (\text{n} = \text{numero linee esterne}) \tag{5.65}$$

corresponding to the conservation of the total four-momentum of the diagram.

By using these rules we can easily evaluate the 2nd order contribution of Fig. 5.4.3.

136
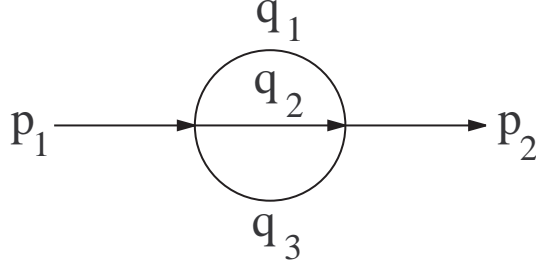
Fig. 5.4.3 - *One of the second order contributions to $G_c^{(2)}$.*

We get

$$4 \times 4 \times 3 \times 2 \times \left(-\frac{i\lambda}{4!}\right)^2 \frac{i}{p_2^2 - m^2 + i\epsilon} \int \left(\prod_{i=1}^3 \frac{d^4 q_i}{(2\pi)^4} \frac{i}{q_i^2 - m^2 + i\epsilon}\right)$$

$$\cdot (2\pi)^4 \delta^4(p_1 - q_1 - q_2 - q_3)(2\pi)^4 \delta^4(q_1 + q_2 + q_3 - p_2) \frac{i}{p_1^2 - m^2 + i\epsilon}$$

$$= \frac{\lambda^2}{6} \frac{1}{(p_1^2 - m^2 + i\epsilon)^2}(2\pi)^4 \delta^4(p_1 + p_2)$$

$$\cdot \int \left(\prod_{i=1}^3 \frac{d^4 q_i}{(2\pi)^4} \frac{i}{q_i^2 - m^2 + i\epsilon}\right)(2\pi)^4 \delta^4(p_1 - q_1 - q_2 - q_3) \qquad (5.66)$$

5.5 Power counting in $\lambda \varphi^4$

We start by going to the euclidean formulation. Let us denote the time by t_R. The relation with the euclidean time is $t_E = i t_R$. Recalling from the Section 4.9 that for the harmonic oscillator of unit mass one has (we have defined $x_R = (t_R, \vec{x})$, $x_E = (t_E, \vec{x})$)

$$i\Delta_F(x_R) = \int \frac{d^3\vec{p}}{(2\pi)^3} \Delta(t_R, \omega_k^2) e^{i\vec{p} \cdot \vec{x}} = \int \frac{d^3\vec{p}}{(2\pi)^3} D_E(t_E, \omega_k^2) e^{i\vec{p} \cdot \vec{x}}$$

$$= \int \frac{d^4 p_E}{(2\pi)^4} \frac{e^{-i p_E x_E}}{p_E^2 + m^2} \qquad (5.67)$$

where $p_E = (E_E, \vec{p})$, and we have used eq. (4.237) for unit mass and with $\nu = E_E$.

From this we get the following modifications to Feynman's rules in the euclidean version

- Associate to each line a propagator

$$\frac{1}{p^2 + m^2} \qquad (5.68)$$

137

- To each vertex associate a factor

$$-\frac{\lambda}{4!} \tag{5.69}$$

This rule follows because the weight in the euclidean path-integral is given by e^{-S_E} instead of e^{iS}.

The other rules of the previous Section are unchanged. The reason to use the euclidean version is because in this way the divergences of the integrals become manifest by simply going in polar coordinates.

We will now in the position to make a general analysis of the divergences. For simplicity we will consider only scalar particles. First of all consider the number of independent momenta in a given diagram, or the number of "loops", L. This is given, by definition, by the number of momenta upon which we are performing the integration, since they are not fixed by the four-momentum conservation at the vertices. Since one of this conservation gives rise to the overall four-momentum conservation we see that the number of loops is given by

$$L = I - (V - 1) = I - V + 1 \tag{5.70}$$

where I is the number of internal lines and V the number of vertices. As in Section 2.3 we will be interested in evaluating the superficial degree of divergence of a diagram. To this end notice that

- The L independent integrations give a factor $\prod_{k=1}^{L} d^d p_k$ where d is the number of space-time dimensions.

- Each internal line gives rise to a propagator containing two inverse powers in momenta

$$\prod_{i=1}^{I} \frac{1}{p_i^2 + m^2} \tag{5.71}$$

Given that, the superficial degree of divergence is defined as

$$D = dL - 2I \tag{5.72}$$

Notice that D has to do with a common scaling of all the integration variables, therefore it may well arise the situation where $D < 0$ (corresponding to a convergent situation), but there are divergent sub-integrations. Therefore we will need a more accurate analysis that we will do at the end of this Section. For the moment being let us stay with this definition. We will try now to get a relation between D and the number of external legs in a diagram. By putting V_N equal to the total number of vertices with N lines we have (see Section 2.3)

$$NV_N = E + 2I \tag{5.73}$$

with E the number of external legs and I the number of internal ones (recall we are dealing with scalar particles only). Eliminating I between this relation and the previous one, we get

$$I = \frac{1}{2}(NV_N - E) \tag{5.74}$$

from which

$$D = d(I - V_N + 1) - 2I = (d-2)I - d(V_N - 1) = \frac{d-2}{2}(NV_N - E) - d(V_N - 1) \tag{5.75}$$

This relation can be rewritten as

$$D = d - \frac{1}{2}(d-2)E + \left(\frac{N-2}{2}d - N\right)V_N \tag{5.76}$$

In particular, for $d = 4$ we have

$$D = 4 - E + (N - 4)V_N \tag{5.77}$$

For the $\lambda\varphi^4$ theory this relation becomes particularly simple since then D depends only on the number of external legs

$$D = 4 - E \tag{5.78}$$

Therefore the only diagrams superficially divergent, $D \geq 0$, are the ones with $E = 2$ and $E = 4$.

As noticed before, this does not prove that a diagram with $E > 4$ or $D < 0$ is convergent. Let us discuss more in detail this point. Let us consider the diagram of Fig. 5.5.1, and let us suppose that the diagrams 1 and 2 have degrees of superficial divergence D_1 and D_2 respectively. Since they are connected by n lines, we have $n - 1$ independent momenta, and therefore

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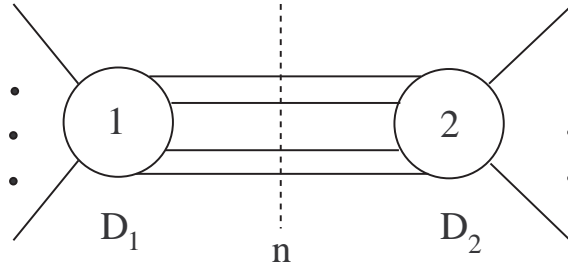
Fig. 5.5.1 - *Two diagrams connected by n-lines.*

$$D = D_1 + D_2 + 4(n - 1) - 2n = D_1 + D_2 + 2n - 4 \tag{5.79}$$

Therefore we may have $D < 0$ with D_1 and D_2 positive, that is both divergent. However for $n > 1$ we have

$$D \geq D_1 + D_2 \tag{5.80}$$

139

and therefore in order to have $D < 0$ with 1 or 2 being divergent diagrams, we need D_1 or D_2 to be negative enough to overcome the other. To exclude the case $n = 1$ means to consider only the one-particle irreducible (1PI) Feynman's diagrams. In Fig. 5.5.2 we give an example of a one-particle reducible diagram.

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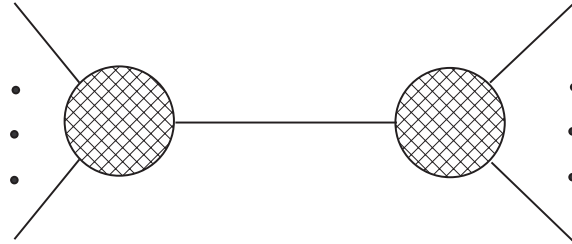
Fig. 5.5.2 - *One-particle reducible Feynman's diagram.*

The reason of excluding these diagrams from our considerations is that the momentum of the internal line connecting the two diagrams is fixed, and therefore it cannot give rise to divergences. Furthermore, the one-particle reducible diagrams can be easily reconstructed from the 1PI's. After that we look for the diagrams having $D < 0$ but yet divergent, in order to have a complete classification of the divergences. The idea is simply to look for the reducibility of the diagram. We will now stay with the $\lambda \varphi^4$ theory. Then, we can consider the 2-, 3-particle reducible diagrams. In fact, for the nature of the vertex it is impossible to have a connected 4-particle irreducible diagram. We begin with the 3-particle reducibility. In this case we look for diagrams of the type of Fig. 5.5.3.

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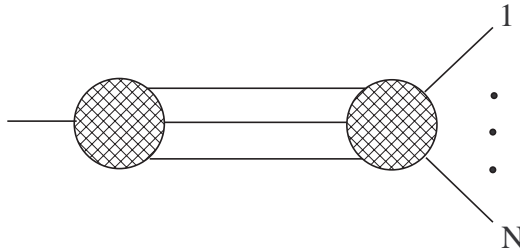
Fig. 5.5.3 - *Three-particle reducible Feynman's diagram.*

In this case we have $D_1 = 0$ and $D_2 = 1 - N$, therefore

$$D = D_1 + D_2 + 2 = 3 - N \qquad (5.81)$$

In order the superficial degree of divergence is negative, we must have $N > 4$. In this diagram we have a primitively divergent four point function and a diagram with $D < 0$ which can be again analyzed in the same way. In analogous way we look for two-particle reducible diagrams with $D < 0$ and we extract contributions of the forms given in Fig 5.5.4.
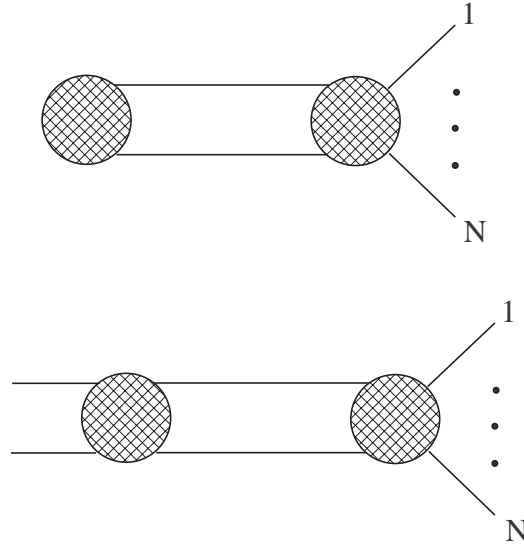
Fig. 5.5.4 - *Two-particle reducible Feynman's diagram.*

In the first case we have $D_1 = 2$ and $D_2 = 2 - N$, therefore

$$D = D_1 + D_2 = 4 - N \tag{5.82}$$

Again this requires $N > 4$. The second diagram has again $D_2 < 0$ and we can iterate the procedure. In the second case we have $D_1 = 0$ and $D_2 = 2 - N$, therefore

$$D = D_1 + D_2 = 2 - N \tag{5.83}$$

requiring $N > 2$. By proceeding in this way we see that truly convergent diagrams do not contain hidden two and four point divergent functions. What we have shown is that for the $\lambda\varphi^4$ theory in 4 dimensions a 1PI Feynman diagram is convergent if $D < 0$ and it cannot be split up into 3- or 2-particle reducible parts containing two-and four-point divergent functions. This turns out to be a general theorem due to Weinberg, valid in any field theory and it says that a Feynman diagram is convergent if its superficial degree of divergence and that of all its subgraphs are negative.

In the case of the $\lambda\phi^4$ theory the Weinberg's theorem tells us that all the divergences rely in the two and four point functions. If it happens that all the divergent diagrams (in the primitive sense specified above) correspond to terms which are present in the original lagrangian, we say that the theory is renormalizable. In general we can look at this question by studying the equation for the superficial divergence that we got before

$$D = d - \frac{1}{2}(d - 2)E + \left(\frac{N - 2}{2} d - N \right) V_N \tag{5.84}$$

141

For $d = 4$ we recall

$$D = 4 - E + (N - 4)V_N \tag{5.85}$$

We see that D increases with V_N for $N > 4$. Therefore we have infinite divergent diagrams and the theory is not renormalizable. We can have a renormalizable scalar theory only for $N \leq 4$, meaning that the dimension N of the vertex operator, φ^N, must be less or equal to 4. An interesting case is the one of the theory $\lambda \varphi^3$ in 6 dimensions. For $d = 6$ we get

$$D = 6 - 2E + (2N - 6)V_N \tag{5.86}$$

and for $N = 3$

$$D = 6 - 2E \tag{5.87}$$

and the only divergent functions are the one- two- and three-point ones. In $d = 6$ we have $dim[\varphi] = 2$ and again in order to have renormalizability we must have that the dimensions of the vertex operator must be less or equal to 6.

5.6 Regularization in $\lambda \varphi^4$

In this Section we will evaluate the one-loop contributions to the two- and four-point Green's functions by using the dimensional regularization. To this end, let us consider the action in dimensions $d = 2\omega$

$$S_{2\omega} = \int d^{2\omega}x \left[\frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi + \frac{1}{2}m^2 \varphi^2 + \frac{\lambda}{4!}\varphi^4 \right] \tag{5.88}$$

In our unit system we have ($\hslash = c = 1$), and therefore the action $S_{2\omega}$ should be dimensionless. As discussed in Section 2.3 this allows to evaluate the dimensions of the fields, and we recall that for the scalar field we have

$$dim[\varphi] = \frac{d}{2} - 1 = \omega - 1 \tag{5.89}$$

From this it follows at once that

$$dim[m] = 1, \quad dim[\lambda] = 4 - 2\omega \tag{5.90}$$

As we did in QED for the electric charge, here too it is convenient to introduce a dimensionful parameter μ in order to be able to define a dimensionless coupling

$$\lambda_{\text{new}} = \lambda_{\text{old}} (\mu^2)^{\omega - 2} \tag{5.91}$$

Then the action will be $(\lambda = \lambda_{\text{new}})$

$$S_{2\omega} = \int d^{2\omega}x \left[\frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi + \frac{1}{2}m^2 \varphi^2 + \frac{\lambda}{4!}(\mu^2)^{2-\omega} \varphi^4 \right] \tag{5.92}$$

The Feynman's rules are slightly modified

- The scalar product among two four-vectors becomes a sum over 2ω components.

- In the loop integrals we will have a factor

$$\int \frac{d^{2\omega}p}{(2\pi)^{2\omega}}$$

(5.93)

and the δ function in 2ω-dimensions will be defined by

$$\int d^{2\omega}p\,\delta^{(2\omega)}\Big(\sum_i p_i\Big) = 1$$

(5.94)

- The coupling λ goes into $\lambda(\mu^2)^{2-\omega}$.

Also, we will do all the calculation in the euclidean version. Remember that in Section 2.5 we gave the relevant integrals both in the euclidean and in the minkowskian version.

We start our calculation from the first order contribution to the two-point function corresponding to the diagram (tadpole) of Fig. 5.6.1. This is given by

$$(2\pi)^{2\omega}\delta^{(2\omega)}(p_1 + p_2)\frac{1}{p_1^2 + m^2}T_2\frac{1}{p_1^2 + m^2}$$

(5.95)

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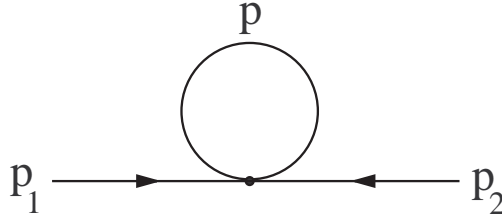
Fig. 5.6.1 - *The one-loop contribution to the two-point function in the $\lambda\varphi^4$ theory.*

with

$$
\begin{aligned}
T_2 &= \frac{1}{2}(-\lambda)(\mu^2)^{2-\omega}\int \frac{d^{2\omega}p}{(2\pi)^{2\omega}}\frac{1}{p^2 + m^2} = -\frac{\lambda}{2}(\mu^2)^{2-\omega}\frac{\pi^\omega}{(2\pi)^{2\omega}}\Gamma(1-\omega)\frac{1}{(m^2)^{1-\omega}} \\
&= -\frac{\lambda}{2}\frac{m^2}{(4\pi)^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{2-\omega}\Gamma(1-\omega)
\end{aligned}
$$

(5.96)

For $\omega \to 2$ we get

$$
\begin{aligned}
T_2 &\approx -\frac{\lambda}{2}\frac{m^2}{(4\pi)^2}\left[1 + (2-\omega)\log\frac{4\pi\mu^2}{m^2} + \cdots\right] \times (-1)\left[\frac{1}{2-\omega} + \psi(2) + \cdots\right] \\
&= \frac{\lambda}{32\pi^2}m^2\left[\frac{1}{2-\omega} + \psi(2) + \log\frac{4\pi\mu^2}{m^2} + \cdots\right]
\end{aligned}
$$

(5.97)

143
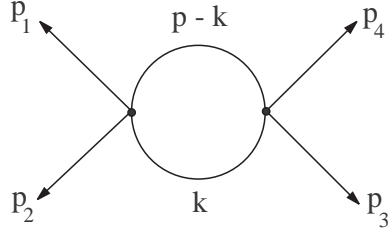
Fig. 5.6.2 - *The one-loop contribution to the four-point function in the $\lambda \varphi^4$ theory.*

This expression has a simple pole at $\omega = 2$ and a finite part depending explicitly on μ^2.

Let us consider now the one-loop contribution to the four-point function. The corresponding diagram is given in Fig. 5.6.2, with an amplitude

$$(2\pi)^{2\omega} \delta^{(2\omega)}(p_1 + p_2 + p_3 + p_4) \prod_{k=1}^{4} \frac{1}{p_k^2 + m^2} T_4 \tag{5.98}$$

with

$$T_4 = \frac{1}{2}(-\lambda)^2 (\mu^2)^{4-2\omega} \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{k^2 + m^2} \frac{1}{(k-p)^2 + m^2}, \qquad p = p_1 + p_2 \tag{5.99}$$

We reduce the two denominator to a single one, through the equation (see Section 2.6)

$$
\begin{aligned}
\frac{1}{k^2 + m^2} \frac{1}{(k-p)^2 + m^2} &= \int_0^1 dx \frac{1}{\left[x(k^2 + m^2) + (1-x)\left((k-p)^2 + m^2\right)\right]^2} \\
&= \int_0^1 dx \frac{1}{\left[x(k^2 + m^2) + (1-x)\left(k^2 + m^2 - 2k \cdot p + p^2\right)\right]^2} \\
&= \int_0^1 dx \frac{1}{\left[(k^2 + m^2) + (1-x)(p^2 - 2k \cdot p)\right]^2} \tag{5.100}
\end{aligned}
$$

the denominator can be written as

$$
\begin{aligned}
(k^2 + m^2) + (1-x)(p^2 - 2k \cdot p) &= \left[k - (1-x)p\right]^2 + m^2 + p^2(1-x) - p^2(1-x)^2 \\
&= \left[k - (1-x)p\right]^2 + m^2 + p^2 x(1-x) \tag{5.101}
\end{aligned}
$$

Since our integral is a convergent one, it is possible to translate k into $k - p(1-x)$ obtaining

$$T_4 = \frac{\lambda^2}{2}(\mu^2)^{4-2\omega} \int_0^1 dx \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{\left[k^2 + m^2 + p^2 x(1-x)\right]^2} \tag{5.102}$$

144

By performing the momentum integration we get

$$T_4 = \frac{\lambda^2}{2}(\mu^2)^{4-2\omega} \int_0^1 \frac{dx}{(4\pi)^\omega} \Gamma(2-\omega) \frac{1}{[m^2 + p^2 x(1-x)]^{2-\omega}} \tag{5.103}$$

In the limit $\omega \to 2$

$$
\begin{aligned}
T_4 &\approx \frac{\lambda^2}{2}(\mu^2)^{2-\omega} \left[1 + (2-\omega)\log\mu^2\right] \int_0^1 \frac{dx}{(4\pi)^2} \left[\frac{1}{2-\omega} + \psi(1)\right] \\
&\times \left[1 - (2-\omega)\log\frac{m^2 + p^2 x(1-x)}{4\pi}\right] \\
&\approx (\mu^2)^{2-\omega} \frac{\lambda^2}{32\pi^2} \left[\frac{1}{2-\omega} + \psi(1) - \int_0^1 dx \log\frac{m^2 + p^2 x(1-x)}{4\pi\mu^2}\right] \tag{5.104}
\end{aligned}
$$

Also the x integration can be done by using the formula

$$\int_0^1 dx \log\left[1 + \frac{4}{a}x(1-x)\right] = -2 + \sqrt{1+a}\log\frac{\sqrt{1+a}+1}{\sqrt{1+a}-1}, \quad a > 0 \tag{5.105}$$

The final result is

$$
\begin{aligned}
T_4 &\approx (\mu^2)^{2-\omega} \frac{\lambda^2}{32\pi^2} \left[\frac{1}{2-\omega} + \psi(1) + 2 + \log\frac{4\pi\mu^2}{m^2}\right. \\
&\left. - \sqrt{1 + \frac{4m^2}{p^2}}\log\frac{\sqrt{1 + \frac{4m^2}{p^2}} + 1}{\sqrt{1 + \frac{4m^2}{p^2}} - 1} + \cdots\right] \tag{5.106}
\end{aligned}
$$

5.7 Renormalization in the theory $\lambda\varphi^4$

In order to renormalize the theory $\lambda\varphi^4$ we proceed as in QED. That is we split the original lagrangian, expressed in terms of bare couplings and bare fields in a part

$$\mathcal{L}_p = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\mu^{4-2\omega}\varphi^4 \tag{5.107}$$

depending on the renormalized parameters m and λ and field φ, and in the counterterms contribution

$$\mathcal{L}_{ct} = \frac{1}{2}\delta Z \partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}\delta m^2\varphi^2 + \frac{\delta\lambda}{4!}\mu^{4-2\omega}\varphi^4 \tag{5.108}$$

The sum of these two expressions

$$\mathcal{L} = \frac{1}{2}(1 + \delta Z)\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}(m^2 + \delta m^2)\varphi^2 + \frac{(\lambda + \delta\lambda)}{4!}\mu^{4-2\omega}\varphi^4 \tag{5.109}$$

gives back the original lagrangian, if we redefine couplings and fields as follows

$$\varphi_B = (1 + \delta Z)^{1/2} \varphi \equiv Z_\varphi^{1/2} \varphi \tag{5.110}$$

$$m_B^2 = \frac{m^2 + \delta m^2}{1 + \delta Z} = (m^2 + \delta m^2) Z_\varphi^{-1} \tag{5.111}$$

$$\lambda_B = \mu^{4-2\omega} \frac{\lambda + \delta\lambda}{(1 + \delta Z)^2} = \mu^{4-2\omega} (\lambda + \delta\lambda) Z_\varphi^{-2} \tag{5.112}$$

In fact

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi_B \partial^\mu \varphi_B + \frac{1}{2} m_B^2 \varphi_B^2 + \frac{\lambda_0}{4!} \varphi_B^4 \tag{5.113}$$

The counter-terms are then evaluated by the requirement of removing the divergences arising in the limit $\omega \to 2$, except for a finite part which is fixed by the renormalization conditions. From the results of the previous Section, by summing up
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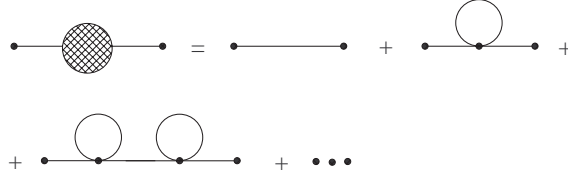
Fig. 5.7.1 - *The contributions to the teo-point connected Green's function.*

all the 1PI contributions we get the following expression for the two-point connected Green's function (see Fig. 5.7.1)

$$\begin{aligned} G_c^{(2)}(p_1, p_2) &= (2\pi)^{2\omega} \delta^{2\omega}(p_1 + p_2) \left[\frac{1}{p_1^2 + m^2} + \frac{1}{p_1^2 + m^2} T_2 \frac{1}{p_1^2 + m^2} + \cdots \right] \\ &\approx (2\pi)^{2\omega} \delta^{2\omega}(p_1 + p_2) \frac{1}{p_1^2 + m^2 - T_2} \end{aligned} \tag{5.114}$$

Therefore, the effect of the first order contribution is to produce a mass correction determined by T_2. The divergent part can be taken in care by the counterterm δm^2 which produces an additional contribution to the one-loop diagram, represented in Fig. 5.7.2. We choose the counterterm as

$$\frac{1}{2} \delta m^2 \varphi^2 = \frac{\lambda}{64\pi^2} m^2 \left[\frac{2}{\epsilon} + F_1 \left(\omega, \frac{m^2}{\mu^2} \right) \right] \varphi^2 \tag{5.115}$$

with $\epsilon = 4 - 2\omega$ and F_1 an arbitrary dimensionless function Adding the contribution of the counterterm to the previous result we get

$$G_c^{(2)}(p_1, p_2) = (2\pi)^{2\omega} \delta^{2\omega}(p_1 + p_2) \left[\frac{1}{p_1^2 + m^2 - T_2} + \frac{1}{p_1^2 + m^2} (-\delta m^2) \frac{1}{p_1^2 + m^2} \right] \tag{5.116}$$

146

$$= -\delta m^2$$

Fig. 5.7.2 - *The Feynman'rule for the counterterm δm^2.*

at the order λ we have

$$G_c^{(2)}(p_1, p_2) = (2\pi)^{2\omega}\delta^{2\omega}(p_1 + p_2)\frac{1}{p_1^2 + m^2 + \delta m^2 - T_2} \tag{5.117}$$

Of course the same result could have been obtained by noticing that the counterterm modifies the propagator by shifting m^2 into $m^2 + \delta m^2$. Now the combination $\delta m^2 - T_2$ is finite and given by

$$\delta m^2 - T_2 = \frac{\lambda}{32\pi^2}m^2\left[F_1 - \psi(2) + \log\frac{m^2}{4\pi\mu^2}\right] \tag{5.118}$$

The finite result at the order λ is then

$$G_c^{(2)}(p_1, p_2) = (2\pi)^{2\omega}\delta^{2\omega}(p_1 + p_2)\frac{1}{p_1^2 + m^2\left(1 + \frac{\lambda}{32\pi^2}\left[F_1 - \psi(2) + \log\frac{m^2}{4\pi\mu^2}\right]\right)} \tag{5.119}$$

This expression has a pole in Minkowski space, and we can define the renormalization condition by requiring that the inverse propagator at the physical mass is

$$p^2 + m_{phys}^2 \tag{5.120}$$

This choice determines uniquely the F_1 term. However we will discuss more in detail the renormalization conditions in the next Chapter when we will speak about the renormalization group. Notice that at one loop there is no wave function renormalization, but this comes about at two loops, as we shall see later.

Consider now the four-point connected Green's function. Its diagrammatic expansion at one loop is given in Fig. 5.7.3 (again we have included only 1PI diagrams).

From the results of the previous Section and introducing the Mandelstam invariants

$$s = (p_1 + p_2)^2, \quad t = (p_1 + p_3)^2, \quad u = (p_1 + p_4)^2 \tag{5.121}$$

we get

$$G_c^{(4)}(p_1, p_2, p_3, p_4) = (2\pi)^{2\omega}\delta^{(2\omega)}(p_1 + p_2 + p_3 + p_4)\prod_{k=1}^{4}\frac{1}{p_k^2 + m^2}$$

$$\times \quad (-\mu^{4-2\omega})\lambda\left[1 - \frac{3\lambda}{32\pi^2}\left(\frac{2}{\epsilon} + \psi(1) + 2 - \log\frac{m^2}{4\pi\mu^2} - \frac{1}{3}A(s, t, u)\right)\right] \tag{5.122}$$

147
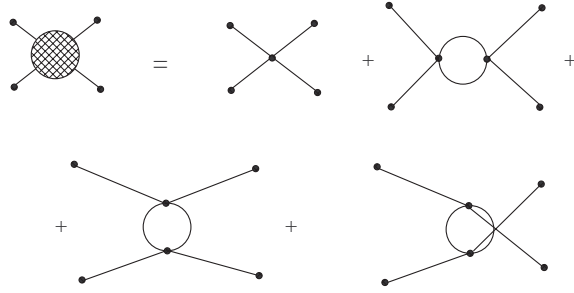
Fig. 5.7.3 - *The one-loop expansion for the four-point Green's function.*

with

$$A(s,t,u) = \sum_{z=s,t,u} \sqrt{1 + \frac{4m^2}{z}} \log \frac{\sqrt{1 + \frac{4m^2}{z}} + 1}{\sqrt{1 + \frac{4m^2}{z}} - 1} \tag{5.123}$$

The divergent part can be disposed off by the counterterm

$$\frac{\delta\lambda}{4!}\mu^{4-2\omega}\varphi^4 = \frac{1}{4!}\mu^{4-2\omega}\frac{3\lambda^2}{32\pi^2}\left[\frac{2}{\epsilon} + G_1\left(\omega, \frac{m^2}{\mu^2}\right)\right]\varphi^4 \tag{5.124}$$

with G_1 an arbitrary dimensionless function. This counterrtem produces an additional contribution with a Feynman rule represented in Fig. 5.7.4.
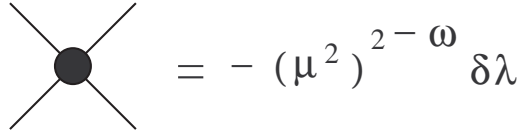
Fig. 5.7.4 - *The Feynman's rule for the couterterm $\delta\lambda$.*

By adding the counterterm we get

$$G_c^{(4)}(p_1, p_2, p_3, p_4) = (2\pi)^{2\omega}\delta^{(2\omega)}(p_1 + p_2 + p_3 + p_4)\prod_{k=1}^{4}\frac{1}{p_k^2 + m^2}(-\mu^{2-4\omega})\lambda$$

$$\times\left[1 - \frac{3\lambda}{32\pi^2}\left(-G_1 + \psi(1) + 2 - \log\frac{m^2}{4\pi\mu^2} - \frac{1}{3}A(s,t,u)\right)\right] \tag{5.125}$$

A possible renormalization condition is to fix the scattering amplitude at some value of the invariants (for instance, $s = 4m_{phys}^2$, $t = u = 0$) to be equal to the lowest order value

$$G_c^{(4)}(p_1, p_2, p_3, p_4) = (2\pi)^{2\omega}\delta^{(2\omega)}(p_1 + p_2 + p_3 + p_4)\prod_{k=1}^{4}\frac{1}{p_k^2 + m^2} \times (-\mu^{2-4\omega})\lambda \tag{5.126}$$

148

In any case G_1 is completely fixed by this condition.

Without entering into the calculation we want now show how the renormalization program works at two loops. We will examine only the two-point function. Its complete expansion up to two loops, including also the counterterm contributions is given in Fig. 5.7.5. By omitting the external propagators and the delta-function of

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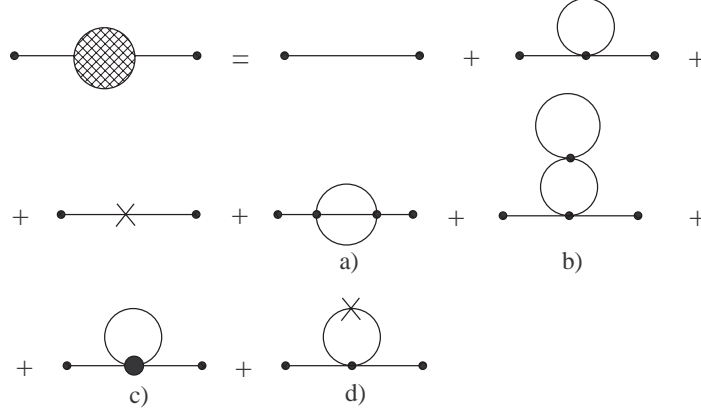
Fig. 5.7.5 - *The two-loop expansion for the two-point Green's function.*

conservation of the four-momentum, the various two-loop contributions are (notice that a counterterm inside a loop gives a second order contribution as a pure two-loops diagram):

- Contribution from the diagram a) of Fig 5.7.5

$$-\frac{\lambda^2}{6(16\pi^2)^2}\left[\frac{1}{2\epsilon}p^2 + \frac{6m^2}{\epsilon^2} + \frac{6m^2}{\epsilon}\left(\frac{3}{2} + \psi(1) - \log\frac{m^2}{4\pi\mu^2}\right)\right] \qquad (5.127)$$

- Contribution from the diagram b) of Fig 5.7.5

$$-\frac{\lambda^2}{4(16\pi^2)^2}\left[\frac{4}{\epsilon^2} + \frac{2}{\epsilon}\left(\psi(2) + \psi(1) - 2\log\frac{m^2}{4\pi\mu^2}\right)\right] \qquad (5.128)$$

- Contribution from the diagram c) of Fig 5.7.5

$$\frac{3\lambda^2}{4(16\pi^2)^2}m^2\left[\frac{4}{\epsilon^2} + \frac{2}{\epsilon}\left(\psi(2) + G_1 - \log\frac{m^2}{4\pi\mu^2}\right)\right] \qquad (5.129)$$

- Contribution from the diagram d) of Fig 5.7.5

$$\frac{\lambda^2}{4(16\pi^2)^2}m^2\left[\frac{4}{\epsilon^2} + \frac{2}{\epsilon}\left(\psi(1) + F_1 - \log\frac{m^2}{4\pi\mu^2}\right)\right] \qquad (5.130)$$

149

Adding together these terms we find

$$-\frac{\lambda^2}{12(16\pi^2)^2}\frac{p^2}{\epsilon} + m^2\frac{2\lambda^2}{(16\pi^2)^2}\frac{1}{\epsilon^2} + m^2\frac{\lambda^2}{2(16\pi^2)^2}\frac{1}{\epsilon}[F_1 + 3G_1 - 1] \qquad (5.131)$$

where we have used $\psi(2) - \psi(1) = 1$. This expression is still divergent but at the order λ^2. Therefore we can make finite the diagram by modifying at the second order the δm^2 counterterm

$$\frac{1}{2}\left(\delta m_1^2 + \delta m_2^2\right)\varphi^2 \qquad (5.132)$$

with δm_1^2 given in eq. (5.115) and

$$\delta m_2^2 = m^2\frac{\lambda^2}{2(16\pi^2)^2}\left[\frac{4}{\epsilon^2} + \frac{1}{\epsilon}(F_1 + 3G_1 - 1) + F_2\left(\omega, \frac{m^2}{\mu^2}\right)\right] \qquad (5.133)$$

and F_2 a new dimensionless function. The further divergence in p^2 can be eliminated through the wave function renormalization counterterm, that is

$$\frac{1}{2}\delta Z\partial_\mu\varphi\partial^\mu\varphi \qquad (5.134)$$

with the choice

$$\delta Z = -\frac{\lambda^2}{24(16\pi^2)^2}\left[\frac{2}{\epsilon} + H_2(\omega, \frac{m^2}{\mu^2})\right] \qquad (5.135)$$

with H_2 dimensionless. All the functions F_1, F_2, G_1 H_2 are finite in the limit $\omega \to 2$. Notice also that we are not introducing more and more freedom, because only particular combinations of these functions appear in the counterterms, and are precisely these combinations which are fixed by the renormalization conditions. Also, the function H_2 is fixed by the renormalization condition on the inverse propagator, which are really two conditions: a) the position of the pole, b) the residuum at the pole.
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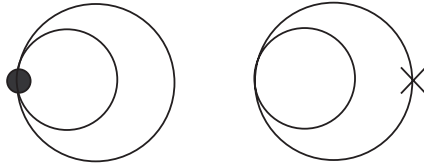
Fig. 5.7.6 - *Insertion of counterterms in multi-loop diagrams.*

The previous procedure can be easily extended to higher and higher orders in the expansion in the coupling constant. What makes the theory renormalizable is the fact that the new divergences that one encounters can all be disposed off by modifying the counterterms δm^2 and λ. This depends from the fact that the divergent terms are constant or quadratic in the external momentum, and therefore they can be reproduced by operators as φ^2, $\partial_\mu\varphi\partial^\mu\varphi$ and φ^4. The non trivial part of

the renormalization program is just the last one. In fact consider diagrams of the type represented in Fig. 5.7.6, where we have taken out all the possible external lines. These diagrams contain $1/\epsilon$ terms coming from the counterterms times $\log p^2$ terms from the loop integration. In the previous two-loop calculation we had no such terms but we got their strict relatives $\log(m^2/4\pi\mu^2)$. If present the $\log p^2$ terms would be a real disaster since they are non-local in configuration space and as such they cannot be absorbed by local operators. However if one makes all the required subtractions, that is one subtracts correctly also the diagrams of the type in Fig. 5.7.6, it is possible to show that the $\log p^2$ terms cancel out. For instance, in the expression for δm_2^2 there are not such terms. In fact they cancel in adding up the four two-loop contributions.

Chapter 6

The renormalization group

6.1 The renormalization group equations

For the following considerations it will be convenient to introduce the one-particle irreducible truncated n-point functions $\Gamma^{(n)}(p_1, p_2, \cdots, p_n)$ defined by removing the propagators on the external legs and omitting the delta-function expressing the conservation of the four-momentum

$$(2\pi)^{2\omega}\delta^{(2\omega)}(\sum_{i=1}^{n} p_i)\Gamma^{(n)}(p_1, p_2, \cdots, p_n) = G_{1PI}^{(n)}(p_1, p_2, \cdots, p_n)\prod_{i=1}^{n} D_F(p_i)^{-1} \qquad (6.1)$$

where

$$D_F(p) = \int d^{2\omega}x e^{ipx}\langle 0|T(\varphi(x)\varphi(0))|0\rangle \qquad (6.2)$$

is the exact euclidean propagator. We have two ways of evaluating these functions, we can start from the original lagrangian and evaluate them in terms of the bare couplings constants (in the case of $\lambda\varphi^4$, λ_B and m_B). On the other hand we can express the bare parameters in terms of the renormalized ones, and, at the same time, renormalizing the operators through the wave function renormalization ($\varphi_B = Z_\varphi^{1/2}\varphi$), obtaining the relation

$$\Gamma_B^{(n)}(p_1, p_2, \cdots, p_n; \lambda_B, m_B, \omega) = Z_\varphi^{-n/2}\Gamma^{(n)}(p_1, p_2, \cdots, p_n; \lambda, m, \mu, \omega) \qquad (6.3)$$

The factor $Z_\varphi^{-n/2}$ originates from a factor $Z_\varphi^{n/2}$ coming from the renormalization factors in $G_c^{(n)}$ and from n factors Z_φ^{-1} from the inverse propagators. Of course, in a renormalizable theory the $\Gamma^{(n)}$'s are finite by construction. In the previous relation we must regard the renormalized parameters as function of the bare ones. In the $\lambda\varphi^4$ case, this means λ and m depending on λ_B and m_B. However the right hand side of the previous equation does not depend on the choice of the parameter μ. This implies that also the functions $\Gamma^{(n)}$ do not depend on μ except for the field rescaling factor $Z_\varphi^{-n/2}$. These considerations can be condensed into a differential

equation obtained by requiring that the total derivative with respect to μ of the right hand side of eq. (6.3) vanishes

$$\left[\mu \frac{\partial}{\partial \mu} + \mu \frac{\partial \lambda}{\partial \mu} \frac{\partial}{\partial \lambda} + \mu \frac{\partial m}{\partial \mu} \frac{\partial}{\partial m} - \frac{n}{2} \mu \frac{\partial \log Z_\varphi}{\partial \mu} \right] \Gamma^{(n)}(p_1, p_2, \cdots, p_n; \lambda, m, \mu, \omega) = 0$$

(6.4)

The content of this relation is that a change of the scale μ can be compensated by a convenient change of the couplings and of the field normalization. Notice that since $\Gamma^{(n)}$ is finite in the limit $\omega \to 2$, it follows that also the derivative of $\log Z_\varphi$ is finite. It is convenient to define the following dimensionless coefficients

$$\mu \frac{\partial \lambda}{\partial \mu} = \beta \left(\lambda, \frac{m}{\mu}, \omega \right)$$
$$\frac{1}{2} \mu \frac{\partial \log Z_\varphi}{\partial \mu} = \gamma_d \left(\lambda, \frac{m}{\mu}, \omega \right)$$
$$\frac{\mu}{m} \frac{\partial m}{\partial \mu} = \gamma_m \left(\lambda, \frac{m}{\mu}, \omega \right)$$

(6.5)

obtaining

$$\left[\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \lambda} + \gamma_m m \frac{\partial}{\partial m} - n \gamma_d \right] \Gamma^{(n)}(p_1, p_2, \cdots, p_n; \lambda, m, \mu, \omega) = 0$$

(6.6)

We can try to eliminate from this equation the term $\partial/\partial \mu$. To this end let us consider the dimensions of $\Gamma^{(n)}$. In dimensions $d = 2\omega$ the delta function has dimensions -2ω and $dim[\varphi] = \omega - 1$, therefore

$$G_c^{(n)}(p_1, \cdots, p_n) = \int \prod_{i=1}^{n} d^{2\omega} x_i e^{i(p_1 x_1 + \cdots + p_n x_n)} \langle 0 | T(\varphi(x_1) \cdots \varphi(x_n) | 0 \rangle_{\text{conn.}}$$

(6.7)

has dimensions $n(\omega - 1) - 2n\omega = -n(\omega + 1)$. The propagator $D_F(p)$ has dimensions $2(\omega - 1) - 2\omega = -2$, and we get

$$dim \left[\Gamma^{(n)} \right] - 2\omega = -n(\omega + 1) + 2n \rightarrow dim \left[\Gamma^{(n)} \right] = n + \omega(2 - n)$$

(6.8)

and putting $\epsilon = 4 - 2\omega$

$$dim \left[\Gamma^{(n)} \right] = 4 - n + \frac{\epsilon}{2}(n - 2)$$

(6.9)

Therefore $\Gamma^{(n)}$ must be a homogeneous function of degree $4 - n + \epsilon(n - 2)/2$ in the dimensionful parameters p_i, m e μ. Introducing a common scale s for the momenta we get (from the Euler theorem)

$$\left[\mu \frac{\partial}{\partial \mu} + s \frac{\partial}{\partial s} + m \frac{\partial}{\partial m} - (4 - n + \frac{\epsilon}{2}(n - 2)) \right] \Gamma^{(n)}(sp_i; \lambda, m, \mu, \epsilon) = 0$$

(6.10)

153

Since everything is finite for $\epsilon \to 0$ we can take the limit and eliminate $\partial/\partial\mu$ between this equation and eq. (6.6)

$$\left[-s\frac{\partial}{\partial s} + \beta\frac{\partial}{\partial\lambda} + (\gamma_m - 1)m\frac{\partial}{\partial m} - n\gamma_d + 4 - n \right] \Gamma^{(n)}(sp_i; \lambda, m, \mu, \epsilon) = 0 \qquad (6.11)$$

This equation tells us how the $\Gamma^{(n)}$ scale with the momenta, and allows us to evaluate them at momenta sp_i once we know them at momenta p_i. To this end one needs to know the coefficients β, γ_m and γ_d. For this purpose one needs to specify the renormalization conditions since these coefficients depend on the arbitrary functions $F_1, G_1, \cdots$.

In order to use the renormalization group equations we need to know more about the structure of the functions β, γ_d and γ_m. We have seen that the counterterms are singular expressions in ϵ, therefore it will be possible to express them as a Laurent series in the renormalized parameters

$$\lambda_B = \mu^\epsilon \left[a_0 + \sum_{k=1}^{\infty} \frac{a_k}{\epsilon^k} \right] \qquad (6.12)$$

$$m_B^2 = m^2 \left[b_0 + \sum_{k=1}^{\infty} \frac{b_k}{\epsilon^k} \right] \qquad (6.13)$$

$$Z_\varphi = \left[c_0 + \sum_{k=1}^{\infty} \frac{c_k}{\epsilon^k} \right] \qquad (6.14)$$

with the coefficients functions of the dimensionless parameters λ and m/μ. Notice that a_0, b_0 and c_0 may depend on ϵ but only in an analytical way. Recalling from the previous Section the expressions for the counterterms at the second order in λ

$$\delta\lambda = \frac{3\lambda^2}{32\pi^2} \left[\frac{2}{\epsilon} + G_1 \right] \qquad (6.15)$$

$$\delta m^2 = \frac{\lambda}{32\pi^2}m^2 \left[\frac{2}{\epsilon} + F_1 \right] + \frac{\lambda^2}{2(16\pi^2)^2}m^2 \left[\frac{4}{\epsilon^2} + \frac{1}{\epsilon}(F_1 + 3G_1 - 1) + F_2 \right] \qquad (6.16)$$

$$\delta Z = -\frac{\lambda^2}{24(16\pi^2)^2} \left[\frac{2}{\epsilon} + H_2 \right] \qquad (6.17)$$

we get for the bare parameters

$$\lambda_B = \mu^\epsilon \left[\lambda + \frac{3\lambda^2}{32\pi^2}\left(\frac{2}{\epsilon} + G_1 \right) \right] + \mathcal{O}(\lambda^3) \qquad (6.18)$$

154

$$
\begin{aligned}
m_B^2 \;=\; & m^2 \Bigg[1 + \frac{\lambda}{32\pi^2}\left(\frac{2}{\epsilon} + F_1\right) \\
& + \frac{\lambda^2}{2(16\pi^2)^2}\left(\frac{4}{\epsilon^2} + \frac{1}{\epsilon}(F_1 + 3G_1 - 1) + F_2\right) \\
& + \frac{\lambda^2}{24(16\pi^2)^2}\left(\frac{2}{\epsilon} + H_2\right) \Bigg] + \mathcal{O}(\lambda^3)
\end{aligned}
\tag{6.19}
$$

$$
Z_\varphi = 1 - \frac{\lambda^2}{24(16\pi^2)^2}\left(\frac{2}{\epsilon} + H_2\right) + \mathcal{O}(\lambda^3)
\tag{6.20}
$$

Therefore the coefficients are given by

$$
a_0 = \lambda + \frac{3\lambda^2}{32\pi^2}G_1 + \mathcal{O}(\lambda^3)
\tag{6.21}
$$

$$
a_1 = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3)
\tag{6.22}
$$

$$
b_0 = 1 + \frac{\lambda}{32\pi^2}F_1 + \frac{\lambda^2}{2(16\pi^2)^2}(F_2 + \frac{1}{12}H_2) + \mathcal{O}(\lambda^3)
\tag{6.23}
$$

$$
b_1 = \frac{\lambda}{16\pi^2} + \frac{\lambda^2}{2(16\pi^2)^2}(F_1 + 3G_1 - 1) + \frac{\lambda^2}{12(16\pi^2)^2} + \mathcal{O}(\lambda^3)
\tag{6.24}
$$

$$
b_2 = \frac{2\lambda^2}{(16\pi^2)^2} + \mathcal{O}(\lambda^3)
\tag{6.25}
$$

$$
c_0 = 1 - \frac{\lambda^2}{24(16\pi^2)^2}H_2 + \mathcal{O}(\lambda^3)
\tag{6.26}
$$

$$
c_1 = -\frac{\lambda^2}{12(16\pi^2)^2} + \mathcal{O}(\lambda^3)
\tag{6.27}
$$

Notice that the dependence on m/μ comes only from the arbitrary functions defining the finite parts of the counterterms. This is easily understood since the divergent terms originate from the large momentum dependence where the masses can be neglected. This is an essential feature, since it is very difficult to solve the group renormalization equations when the coefficients β, γ_m e γ_d depend on both variables λ and m/μ. On the other hand since these coefficients depend on the renormalization conditions, it is possible to define a scheme where they do not depend on m/μ. Another option would be to try to solve the equations in the large momentum limit where the mass dependence can be neglected.

Callan and Szymanzik have considered a different version of the renormalization group equations. They have considered the dependence of $\Gamma^{(n)}$ on the physical mass. In that case one can show that β and γ_d depend only on λ. Furthermore in their equation the terms γ_m and $\mu/\partial\mu$ do not appear. In their place there is a inhomogeneous term in the large momentum limit.

6.2 Renormalization conditions

Let us consider again the renormalization conditions. The renormalization procedure introduces in the theory a certain amount of arbitrariness in terms of the scale μ and of the arbitrary functions $F_1, F_2 \cdots$. Recall that the lagrangian has been decomposed as

$$\mathcal{L} = \mathcal{L}_p + \mathcal{L}_{ct} \tag{6.28}$$

therefore, the finite part of $\mathcal{L}_{ct}$ can be absorbed into a redefinition of the parameters appearing in $\mathcal{L}$ since both have the same operatorial structure. This is another way of saying that the finite terms in $\mathcal{L}_{ct}$ are fixed once we assign λ and m. In turn these are fixed by relating them to some observable quantity as a scattering cross-section. Furthermore we have to fix a scale μ at which we define the parameters λ and m. The renormalization group equations then tell us how to change the couplings when we change μ, in order to leave invariant the physical quantities. As we have said the procedure for fixing m and λ is largely arbitrary and we will exhibit various possibilities. To do that it is convenient to write explicitly the expression for the two- and four-point truncated amplitudes

$$
\begin{aligned}
\Gamma^{(2)}(p) &= \left(1 - \frac{\lambda^2}{24(16\pi^2)^2}H_2\right)p^2 \\
&+ m^2\left(1 + \frac{\lambda}{32\pi^2}\left(F_1 - \psi(2) + \log\frac{m^2}{4\pi\mu^2}\right)\right. \\
&+ \left.\frac{\lambda^2}{2(16\pi^2)^2}F_2 + \cdots\right)
\end{aligned}
\tag{6.29}
$$

$$\Gamma^{(4)}(p_1, p_2, p_3, p_4) = -\mu^\epsilon\lambda\left(1 - \frac{3\lambda}{32\pi^2}\left(\psi(1) + 2 - \log\frac{m^2}{4\pi\mu^2} - \frac{1}{3}A(s,t,u) - G_1\right)\right) \tag{6.30}$$

In the two point function we have omitted the finite terms coming from the second order contributions for which we have given only the divergent part. let us now discuss various possibilities of renormalization conditions (sometimes one speaks of subtraction conditions)

- A first possibility is to renormalize at zero momenta, that is requiring

$$\left.\Gamma^{(2)}(p)\right|_{p^2 \sim 0} = p^2 + m_A^2 \tag{6.31}$$

$$\left.\Gamma^{(4)}(p_1, p_2, p_3, p_4)\right|_{p_i = 0} = -\mu^\epsilon\lambda_A \tag{6.32}$$

where m_A^2 and λ_A are quantities to be fixed by comparing some theoretical cross-section with the experimental data. Then, comparing with eqs. (6.29) and (6.30) and using $A(s,t,u) = 6$ for $p_i = 0$, we get

$$H_2^A = 0, \quad F_1^A = \psi(2) - \log\frac{m_A^2}{4\pi\mu^2}, \quad G_1^A = \psi(1) - \log\frac{m_A^2}{4\pi\mu^2} \tag{6.33}$$

156

Another possibility would be to fix the momenta at some physical value in the Minkowski space. For instance we could require $\Gamma(-p^2 = \bar{m}^2) = 0$ where $\bar{m}$ is the physical mass and then fix $\Gamma^{(4)}$ with all the momenta on shell.

- To choose the subtraction point at zero momenta is generally dangerous for massless particles (in this way one would introduce spurious infrared divergences). As a consequence a popular choice is to subtract at an arbitrary value of the momenta. For instance one requires

$$\Gamma^{(2)}(p) = p^2 + m_B^2, \qquad p^2 = M^2 \tag{6.34}$$

$$\Gamma^{(4)}(p_1, p_2, p_3, p_4) = -\mu^\epsilon \lambda_B, \quad s = t = u = M^2 \tag{6.35}$$

Notice that the point chosen for the four-point function is a non-physical one. In this case we get

$$H_2^B = 0, \quad F_1^B = \psi(2) - \log \frac{m_B^2}{4\pi\mu^2}, \quad G_1^B = \psi(1) + 2 - \log \frac{m_B^2}{4\pi\mu^2} - \frac{1}{3} A(M^2, M^2, M^2) \tag{6.36}$$

This way of renormalize has however the problem of leaving an explicit dependence on the mass in the coefficients of the renormalization group equations.

- The last way of renormalizing we will consider is due to Weinberg and 't Hooft and the idea is simply to put equal to zero all the arbitrary functions $F_1, F_2, \cdots$ at each order in the expansion. In this way the coefficients a_i, b_i and c_i appearing in the Laurent series defining the bare couplings turn out to be mass independent.

The Weinberg and 't Hooft prescription allows to evaluate the renormalization group equations coefficients in a very simple way. For instance, let us start with the bare coupling

$$\lambda_B = \mu^\epsilon \left[\lambda + \sum_{k=1}^\infty \frac{a_k(\lambda)}{\epsilon^k} \right] \tag{6.37}$$

Differentiating with respect to μ and recalling that λ_B does not depend on this variable, we get

$$0 = \epsilon \left[\lambda + \sum_{k=1}^\infty \frac{a_k(\lambda)}{\epsilon^k} \right] + \mu \frac{\partial\lambda}{\partial\mu} \left[1 + \sum_{k=1}^\infty \frac{a_k'(\lambda)}{\epsilon^k} \right] \tag{6.38}$$

where

$$a_k'(\lambda) = \frac{da_k(\lambda)}{d\lambda} \tag{6.39}$$

Since $\beta = \mu\partial\lambda/\partial\mu$ is analytic for $\epsilon \to 0$, for small ϵ we can write $\beta = A + B\epsilon$, obtaining

$$0 = \epsilon \left[\lambda + \sum_{k=1}^\infty \frac{a_k(\lambda)}{\epsilon^k} \right] + (A + B\epsilon) \left[1 + \sum_{k=1}^\infty \frac{a_k'(\lambda)}{\epsilon^k} \right] \tag{6.40}$$

157

and identifying the coefficients of the various powers in ϵ

$$B + \lambda = 0 \tag{6.41}$$

$$a_1 + A + Ba_1' = 0 \tag{6.42}$$

$$a_{k+1} + Aa_k' + Ba_{k+1}' = 0 \tag{6.43}$$

from which

$$B = -\lambda \tag{6.44}$$

$$A = -\left(1 - \lambda \frac{d}{d\lambda}\right) a_1 \tag{6.45}$$

We then get

$$\beta(\lambda) = -\left(1 - \lambda \frac{d}{d\lambda}\right) a_1 - \lambda \epsilon \tag{6.46}$$

and for $\epsilon \to 0$

$$\beta(\lambda) = -\left(1 - \lambda \frac{d}{d\lambda}\right) a_1 \tag{6.47}$$

and

$$\left(1 - \lambda \frac{d}{d\lambda}\right) a_{k+1} = \frac{da_k}{d\lambda} \left(1 - \lambda \frac{d}{d\lambda}\right) a_1 \tag{6.48}$$

From the expression of the previous Section for a_1 at the second order in λ, we get

$$\mu \frac{\partial \lambda}{\partial \mu} = \beta(\lambda) = \frac{3\lambda^2}{16\pi^2} \tag{6.49}$$

This equation can be integrated in order to know how we have to change λ when we change the value of the scale μ. The function $\lambda(\mu)$ is known as the running coupling constant. Its knowledge, together with the knowledge of $m(\mu)$ allows us to integrate easily the renormalization group equations. In fact, let us define

$$t = \log s \tag{6.50}$$

and let us consider the following partial differential equation (we shall see that the renormalization group equations can be brought back to this form)

$$\frac{\partial}{\partial t} F(x_i, t) = \beta_i(x_i) \frac{\partial}{\partial x_i} F(x_i, t) \tag{6.51}$$

The general solution to this equation is obtained by using the method of the "characteristics". That is one considers the integral curves (the characteristic curves) defined by the ordinary differential equations

$$\frac{dx_i(t)}{dt} = \beta_i(x_i(t)) \tag{6.52}$$

with given boundary conditions $x_i(0) = \bar{x}_i$. Then, the general solution is

$$F(x_i, t) = \psi(x_i(t)) \tag{6.53}$$

where

$$\psi(\bar{x}_i) = F(\bar{x}_i, 0) \tag{6.54}$$

is an arbitrary function to be determined by the condition at $t = 0$ required for F. In fact we can check that ψ satisfies the differential equation

$$\frac{\partial}{\partial t}\psi(x_i(t)) = \frac{dx_i(t)}{dt}\frac{\partial \psi}{\partial x_i} = \beta_i \frac{\partial \psi}{\partial x_i} \tag{6.55}$$

In our case the equation contains a non-derivative term

$$\frac{\partial}{\partial t}G(x_i, t) = \beta_i(x_i)\frac{\partial}{\partial x_i}G(x_i, t) + \gamma(x_i)G(x_i, t) \tag{6.56}$$

with $\gamma(x_i)$ a given function. By putting

$$G(x_i, t) = e^{\int_0^t dt' \gamma(x_i(t'))}F(x_i, t) \tag{6.57}$$

we see that $F(x_i, t)$ satisfies eq. (6.51) and therefore the general solution will be

$$G(x_i, t) = e^{\int_0^t dt' \gamma(x_i(t'))}\psi(x_i(t)) \tag{6.58}$$

with $x_i(t)$ satisfying eq. (6.52). For the $\Gamma^{(n)}$ we get ($t = \log s$)

$$\Gamma^{(n)}(sp_i; m, \lambda, \mu) = \Gamma^{(n)}(p_i; m(s), \lambda(s), \mu)s^{4-n}e^{-n\int_1^s \gamma_d(\lambda(s'))\frac{ds'}{s'}} \tag{6.59}$$

with

$$s\frac{\partial \lambda(s)}{\partial s} = \beta(\lambda(s)) \tag{6.60}$$

$$s\frac{\partial m(s)}{\partial s} = m(s)(\gamma_m(\lambda(s)) - 1) \tag{6.61}$$

and $\lambda(s = 1) = \lambda$, $m(s = 1) = m$.

This result tells us that when we perform a re-scaling on the momenta, the amplitudes $\Gamma^{(n)}$ do not scale only with the trivial factor s^{4-n} due to their dimensions, but that they undergo also a non trivial scaling, due to $\gamma_d \neq 0$, which is necessary to compensate the variations of λ and m with the scale.

6.3 Application to QED

In this Section we will apply the previous results to QED. In particular, we want to check that the running coupling obtained via the renormalization group equations is the same that we have defined in Section 2.2. First of all let us start with the QED lagrangian in $d = 2\omega$ dimensions (neglecting the gauge fixing terms)

$$S_{2\omega} = \int d^{2\omega}x \left[\bar{\psi}(i\hat{\partial} - m)\psi - e\bar{\psi}\hat{A}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \right] \tag{6.62}$$

Since

$$dim[\bar{\psi}\hat{A}\psi] = \omega - 1 + 2\left(\omega - \frac{1}{2}\right) = 3\omega - 2 \tag{6.63}$$

it follows

$$dim[e] = 2 - \omega \tag{6.64}$$

Therefore we define

$$e_{new} = e_{old}(\mu)^{\omega-2} = e_{old}\mu^{-\epsilon/2} \tag{6.65}$$

As before we write the lagrangian as $\mathcal{L}_p + \mathcal{L}_{ct}$, with (we ignore here the gauge fixing term)

$$\mathcal{L}_p = \bar{\psi}(i\hat{\partial} - m)\psi - e\mu^{\epsilon/2}\bar{\psi}\hat{A}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{6.66}$$

and

$$\mathcal{L}_{ct} = i\delta Z_2\bar{\psi}\hat{\partial}\psi - \delta m\bar{\psi}\psi - e\delta Z_1\mu^{\epsilon/2}\bar{\psi}\hat{A}\psi - \frac{\delta Z_3}{4}F_{\mu\nu}F^{\mu\nu} \tag{6.67}$$

giving

$$\mathcal{L} = i(1 + \delta Z_2)\bar{\psi}\hat{\partial}\psi - (m + \delta m)\bar{\psi}\psi - e(1 + \delta Z_1)\mu^{\epsilon/2}\bar{\psi}\hat{A}\psi - \frac{1 + \delta Z_3}{4}F_{\mu\nu}F^{\mu\nu} \tag{6.68}$$

After re-scaling of the fields one gets back the original lagrangian via, in particular, the identification

$$e_B = \mu^{\epsilon/2}e\frac{(1 + \delta Z_1)}{(1 + \delta Z_2)(1 + \delta Z_3/2)} = \mu^{\epsilon/2}e(1 - \delta Z_3/2) \tag{6.69}$$

where we have used $\delta Z_1 = \delta Z_2$. From Section 2.7 we recall that

$$\delta Z_3 = C = -\frac{e^2}{6\pi^2\epsilon} \tag{6.70}$$

and therefore

$$e_B = \mu^{\epsilon/2}e\left(1 + \frac{e^2}{12\pi^2\epsilon}\right) \tag{6.71}$$

We proceed as in the previous Section by defining

$$e_B = \mu^{\epsilon/2}\left(e + \sum_k \frac{a_k}{\epsilon^k}\right) \tag{6.72}$$

with

$$a_1 = \frac{e^3}{12\pi^2} \tag{6.73}$$

Requiring that e_B is μ-independent, and in the limit $\epsilon \to 0$ we get

$$\beta(e) = \mu \frac{\partial e}{\partial \mu} = -\frac{1}{2}\left(1 - e\frac{d}{de}\right)a_1 \tag{6.74}$$

from which

$$\beta(e) = \frac{e^3}{12\pi^2} \tag{6.75}$$

The differential equation defining the running coupling constant is

$$\mu\frac{\partial e(\mu)}{\partial \mu} = \beta(e) = \frac{e^3}{12\pi^2} \tag{6.76}$$

Integrating this equation we obtain

$$\frac{1}{e^2(\mu^2)} = -\frac{1}{12\pi^2}\log\mu^2 + K \tag{6.77}$$

where K is an integration constant which can be eliminated by evaluating e at another momentum scale Q^2. We get

$$\frac{1}{\alpha(Q^2)} - \frac{1}{\alpha(\mu^2)} = -\frac{1}{3\pi}\log\frac{Q^2}{\mu^2} \tag{6.78}$$

or

$$\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{3\pi}\log\frac{Q^2}{\mu^2}} \tag{6.79}$$

which agrees with the result obtained in Section 2.2.

6.4 Properties of the renormalization group equations

Integrating eq. (6.49)

$$\mu\frac{d\lambda}{d\mu} = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3) \tag{6.80}$$

we get

$$-\frac{1}{\lambda(\mu)} = \frac{3}{16\pi^2}\log\frac{\mu}{\mu_s} - \frac{1}{\lambda_s} \tag{6.81}$$

where μ_s is a reference scale and $\lambda_s = \lambda(\mu_s)$. Therefore

$$\lambda(\mu) = \lambda_s\frac{1}{1 - \frac{3\lambda_s}{16\pi^2}\log\frac{\mu}{\mu_s}} \tag{6.82}$$

161

From this expression we see that starting from $\mu = \mu_s$, λ increases with μ. Suppose that we start with a small λ in such a way that the perturbative expansion is sensible. However increasing μ we will need to add more and more terms to the expansion due to the increasing of λ. Therefore the $\lambda\varphi^4$ theory allows a perturbative expansion only at small mass scales, or equivalently at large distances. In this theory we can give a definite sense to the asymptotic states. The increasing of λ is related to the sign of the $\beta(\lambda)$ function. For the $\lambda\varphi^4$ theory and for QED the β function is positive, however if it would be negative the theory would become perturbative at large mass scales or at small distances. In this case one could solve the renormalization group equations at large momenta by using the perturbative expansion at a lower order to evaluate the β and the other coefficients. However in this case the coupling is increasing at large distances and this would be a problem for our approach based on th in- and out- states. As we shall see this is the situation in QCD. here the fields that are useful to describe the dynamics at short distances (the quark fields) cannot describe the asymptotic states, which can be rather described by bound states of quarks as mesons and baryons.

Going back to the eq. (6.82) we see that, assuming the validity of the expression for the β function up to large momenta, the coupling constant becomes infinite at the value of μ given by

$$\mu = \mu_s e^{\dfrac{16\pi^2}{3\lambda_s}} \tag{6.83}$$

which is a very big scale if λ_s is small. The pole in eq. (6.82) is called the Landau pole (see Section 2.2 in the case of QED). Of course there are no motivations why eq. (6.82) should be valid for large values of λ.

Let us now discuss several possible scenarios starting from a theory where $\beta = 0$ for $\lambda = 0$.

- If $\beta(\lambda) > 0$ for any λ, than the running coupling is always increasing, and it will become infinite at some value of μ. If this happens for a finite μ we say that we have a Landau pole at that scale.

- Suppose that $\beta(\lambda)$ is positive at small λ and that it becomes negative going through a zero at $\lambda = \lambda_F$ (see Fig. 6.4.1)).

  The point λ_F is called a fixed point since if we start with the initial condition $\lambda = \lambda_F$ than λ remains fixed at that value. To study the behaviour of λ around the fixed point, let us expand the $\beta(\lambda)$ around its zero

  $$\beta(\lambda) \approx (\lambda - \lambda_F)\beta'(\lambda_F) \tag{6.84}$$

  It follows

  $$\mu\frac{d\lambda}{d\mu} = (\lambda - \lambda_F)\beta'(\lambda_F) + \cdots \tag{6.85}$$

162
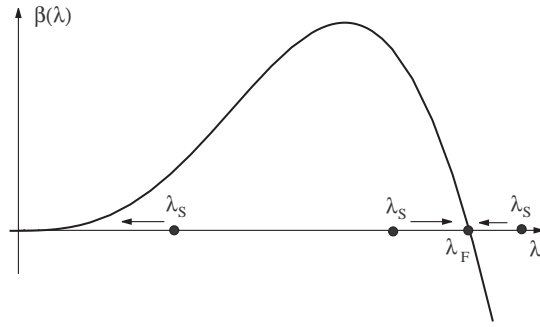
Fig. 6.4.1 - *The behaviour of $\beta(\lambda)$ giving rise to an ultraviolet fixed point.*

from which

$$\frac{d\lambda}{\lambda - \lambda_F} = \beta'(\lambda_F)\frac{d\mu}{\mu} \tag{6.86}$$

Integrating this equation we get

$$\frac{\lambda - \lambda_F}{\lambda_s - \lambda_F} = \left(\frac{\mu}{\mu_s}\right)^{\beta'(\lambda_F)} \tag{6.87}$$

Notice that the sign of $\beta'(\lambda_F)$ plays here a very important role. In the present case the sign is negative since $\beta(\lambda) > 0$ for $\lambda < \lambda_F$ and $\beta(\lambda) < 0$ for $\lambda > \lambda_F$. For large values of μ we have that $\lambda \to \lambda_F$ independently on the initial value λ_s. For this reason λ_F is referred to as a ultraviolet (UV) fixed point (see Fig. 6.4.2)). The large scale (or small distance) behaviour of a field theory of
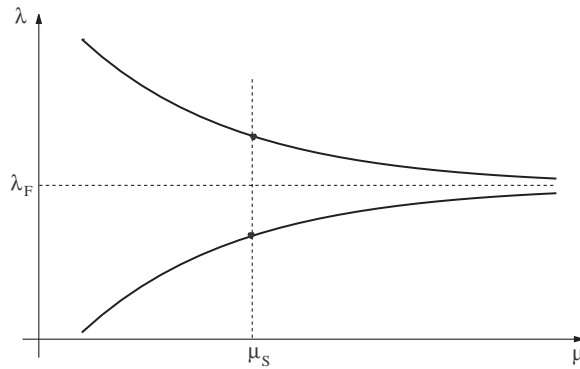
Fig. 6.4.2 - *The behaviour of λ close to a UV fixed point.*

this type would be dictated by the value of λ_F. In particular if $\lambda_F \ll 1$ and $\lambda_s < \lambda_F$ we would be always in the perturbative regime. Otherwise, starting from $\lambda_s > \lambda_F$ the theory would become perturbative at large scales (see Fig. 6.4.1). About this point, notice that $\lambda = 0$ is a fixed point since $\beta(0) = 0$. However this is an infrared (IR) fixed point, since the coupling goes to zero for

163

$\mu \to 0$ independently on the initial value λ_s. We say also that the fixed point is repulsive since λ goes away from the fixed point for increasing μ, whereas a UV fixed point is said to be attractive.

- Let us consider now the case of $\beta(\lambda) < 0$ for small values of λ and decreasing monotonically. For instance, suppose $\beta(\lambda) = -a\lambda^2$ with $a > 0$. Integrating the equation for the running coupling we get

$$\lambda(\mu) = \lambda_s \frac{1}{1 + a\lambda_s \log \frac{\mu}{\mu_s}} \tag{6.88}$$

In this case λ is a decreasing function of μ and goes to zero for $\mu \to \infty$ (that is $\lambda = 0$ is a UV fixed point). This property is known as "asymptoyic freedom" and it holds for non-abelian gauge theories that we will describe later on in the course. these are the only 4-dimensional theories to share this property. In higher dimensions other theories enjoy this property. For instance in $d = 6$ the theory $\lambda\varphi^3$ is asymptotically free. Notice that also this time the running coupling has a pole, but at a smaller scale than μ_s

$$\frac{\mu}{\mu_s} = e^{-\frac{1}{a\lambda_s}} \tag{6.89}$$

Therefore the coupling increases at large distances.

- Finally let us consider the case of $\beta(\lambda) < 0$ for small values of λ, becomes zero at λ_F and then positive. Then $\beta'(\lambda_F) > 0$. By expanding $\beta(\lambda)$ around λ_F we get (by the same analysis we did before)

$$\frac{\lambda - \lambda_F}{\lambda_s - \lambda_F} = \left(\frac{\mu}{\mu_s}\right)^{\beta'(\lambda_F)} \tag{6.90}$$

and we see that $\lambda \to \lambda_F$ $\mu \to 0$ in a way independent on λ_s, whereas it goes away from λ_s for increasing μ. Therefore λ_F is an IR fixed point (see Fig. 6.4.3)).

Let us now derive the perturbative expressions for γ_m and γ_d using the same method followed for $\beta(\lambda)$. è possibile ricavare. We differentiate the eq. (6.13) with respect to μ (notice that in this renormalization scheme $b_0 = 1$)

$$0 = 2m\frac{\partial m}{\partial \mu}\left[1 + \sum_{k=1}^{\infty} \frac{b_k}{\epsilon^k}\right] + m^2 \frac{\partial \lambda}{\partial \mu}\sum_{k=1}^{\infty} \frac{b_k'}{\epsilon^k} \tag{6.91}$$

from which

$$0 = 2\gamma_m\left[1 + \sum_{k=1}^{\infty} \frac{b_k}{\epsilon^k}\right] + \beta(\lambda)\sum_{k=1}^{\infty} \frac{b_k'}{\epsilon^k} \tag{6.92}$$

164
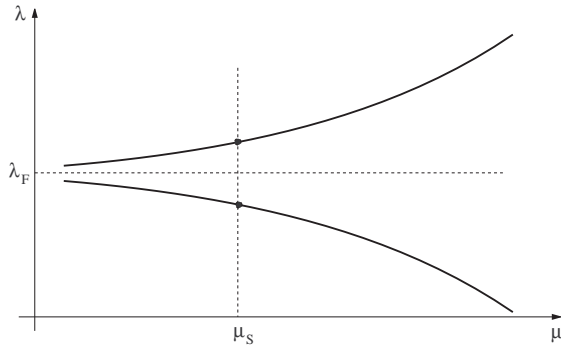
Fig. 6.4.3 - *The behaviour of λ close to an IR fixed point.*

where we have made use of the definition $\gamma_m = (\mu/m)(\partial m/\partial \mu)$. Using eq. (6.46) we find

$$2\gamma_m - \lambda \frac{db_1}{d\lambda} = 0$$

$$2\gamma_m b_k - \frac{db_k}{d\lambda} \left(1 - \lambda \frac{d}{d\lambda}\right) a_1 - \lambda \frac{db_{k+1}}{d\lambda} \tag{6.93}$$

or

$$\gamma_m = \frac{1}{2}\lambda \frac{db_1}{d\lambda} \tag{6.94}$$

and

$$\lambda \frac{db_1}{d\lambda}b_k - \frac{db_k}{d\lambda}\left(1 - \lambda \frac{d}{d\lambda}\right)a_1 - \lambda \frac{db_{k+1}}{d\lambda} = 0 \tag{6.95}$$

Using the eq. (6.24) for b_1 we get for γ_m

$$\gamma_m(\lambda) = \frac{\lambda}{32\pi^2} - \frac{5}{12}\left(\frac{\lambda}{16\pi^2}\right)^2 + \mathcal{O}(\lambda^3) \tag{6.96}$$

Let us now consider Z_φ (recall that although this quantity is infinite in the limit $\epsilon \to 0$, γ_d is finite). By differentiating eq. (6.14) we find ($c_0 = 1$)

$$Z_\varphi \gamma_d = \frac{1}{2}\mu \frac{\partial Z_\varphi}{\partial \mu} = \frac{1}{2}\mu \frac{d\lambda}{d\mu}\sum_{k=1}^{\infty}\frac{c_k'}{\epsilon^k} = \frac{1}{2}\beta(\lambda)\sum_{k=1}^{\infty}\frac{c_k'}{\epsilon^k} \tag{6.97}$$

or

$$\left(1 + \sum_{k=1}^{\infty}\frac{c_k}{\epsilon^k}\right)\gamma_d = \frac{1}{2}(A + B\epsilon)\sum_{k=1}^{\infty}\frac{c_k'}{\epsilon^k} \tag{6.98}$$

Using again the expression for $\beta(\lambda)$ (see eq. (6.46)

$$\gamma_d = -\frac{1}{2}\lambda \frac{dc_1}{d\lambda} \tag{6.99}$$

165

$$\lambda \frac{dc_{k+1}}{d\lambda} = \lambda c_k \frac{dc_1}{d\lambda} - \frac{dc_k}{d\lambda} \left(1 - \lambda \frac{d}{d\lambda} \right) a_1 \tag{6.100}$$

At the lowest order, using eq. (6.27) we get

$$\gamma_d = \frac{1}{12} \left(\frac{\lambda}{16\pi^2} \right)^2 + \mathcal{O}(\lambda^3) \tag{6.101}$$

To conclude this Section, let us study the behaviour of $\Gamma^{(n)}$ when the theory has a UV fixed point at $\lambda = \lambda_F$. Since for large scales $\lambda \to \lambda_F$ we may suppose that γ_m and γ_d go to $\gamma_m(\lambda_F)$ e $\gamma_d(\lambda_F)$ respectively. Then, integrating the following equation (see eq. (6.61))

$$s \frac{dm}{ds} = m(\gamma_m(\lambda_F) - 1) \tag{6.102}$$

we have

$$m(s) = m(1) s^{\gamma_m(\lambda_F) - 1} \tag{6.103}$$

Analogously

$$\int_1^s \frac{ds'}{s'} \gamma_d(\lambda_F) = \gamma_d(\lambda_F) \log s \tag{6.104}$$

Then, for large values of s (see eq. (6.59)

$$\Gamma^{(n)}(sp_i; m, \lambda, \mu) \to s^{(4-n-n\gamma_d(\lambda_F))} \Gamma^{(n)}(p_i; m s^{\gamma_m(\lambda_F)-1}, \lambda_F, \mu) \tag{6.105}$$

Therefore, if $1 - \gamma_m(\lambda_F) > 0$ we can safely neglect the mass on the right hand side of this equation. From this equation we understand also the reason why γ_d is called the anomalous dimension.

In non abelian gauge theories there is a UV fixed point at $\lambda = 0$ where $\gamma_m = 0$ and therefore one can neglect the masses at large momenta.

To justify why we have assumed that in the evaluation of γ_m and γ_d the only important contribution comes from $\lambda = \lambda_F$ we notice that the integral

$$\int_1^s \frac{ds'}{s'} \gamma_m(\lambda(s')) \tag{6.106}$$

can be rewritten (using $ds/s = d\lambda/\beta$)

$$\int_\lambda^{\lambda(s)} d\lambda' \frac{\gamma_m(\lambda')}{\beta(\lambda')} \tag{6.107}$$

Therefore, barring the case in which γ_m and β have simultaneous zeroes, the integral is dominated by the points where there is a zero of the β function.

6.5 The coefficients of the renormalization group equations and the renormalization conditions

We have already noticed that the coefficients of the renormalization group equations depend on the subtraction procedure. In particular, using the first two renormalization conditions considered in Section 6.2 a mass dependence of the coefficients is introduced. To study better this problem, let us consider, for generic renormalization conditions, the expression (6.12) for the bare coupling

$$\lambda_B = \mu^\epsilon \left(a_0 + \sum_{k=1}^\infty \frac{a_k}{\epsilon^k} \right) \tag{6.108}$$

Differentiating with respect to μ we find

$$\epsilon \left(a_0 + \sum_{k=1}^\infty \frac{a_k}{\epsilon^k} \right) + \mu \frac{\partial \lambda}{\partial \mu} \left(a_0' + \sum_{k=1}^\infty \frac{a_k'}{\epsilon^k} \right) + \mu \frac{\partial (m/\mu)}{\partial \mu} \left(\dot{a}_0 + \sum_{k=1}^\infty \frac{\dot{a}_k}{\epsilon^k} \right) = 0 \quad (6.109)$$

where the dot stands for the derivative with respect to m/μ. Notice also that (see eq. (6.5)

$$\mu \frac{\partial}{\partial \mu} \left(\frac{m}{\mu} \right) = \frac{m}{\mu} (\gamma_m - 1) \tag{6.110}$$

With a familiar procedure, we put $\beta = A + B\epsilon$, obtaining

$$a_0 + B a_0' = 0 \tag{6.111}$$

$$a_1 + A a_0' + B a_1' + \frac{m}{\mu} (\gamma_m - 1) \dot{a}_0 = 0 \tag{6.112}$$

At the second order in λ

$$B = -\lambda \left(1 + \frac{3\lambda}{32\pi^2} G_1 \right) \left(1 + \frac{3\lambda}{16\pi^2} G_1 \right)^{-1} \approx -\lambda \left(1 - \frac{3\lambda}{32\pi^2} G_1 \right) \tag{6.113}$$

and

$$A a_0' = -\frac{3\lambda^2}{16\pi^2} + \lambda \left(1 - \frac{3\lambda}{32\pi^2} G_1 \right) \frac{3\lambda}{8\pi^2} - \frac{m}{\mu} (\gamma_m - 1) \frac{3\lambda^2}{32\pi^2} \dot{G}_1 \tag{6.114}$$

Since the right hand side is at least of the second order in λ, neglecting higher order terms we get

$$A = \frac{3\lambda^2}{16\pi^2} + \frac{m}{\mu} \frac{3\lambda^2}{32\pi^2} \dot{G}_1 \tag{6.115}$$

and for $\epsilon \to 0$

$$\beta = \frac{3\lambda^2}{16\pi^2} + \frac{m}{\mu} \frac{3\lambda^2}{32\pi^2} \dot{G}_1 \tag{6.116}$$

We see directly how the β function depends on the subtraction procedure. Similar expression can be obtained for γ_m e γ_d. Therefore, when using subtraction procedure

such that the coefficients of the renormalization group equations depend explicitly on the masses, it is convenient to look for solutions in a region of momenta where the masses can be neglected. Another possibility is to choose the arbitrary scale μ such that m/μ can be neglected.

Of course, changing the renormalization procedure will induce a finite renormalization in the couplings. For instance we get expressions of the type

$$\lambda' = f\left(\lambda, \frac{m}{\mu}\right) = \lambda + \mathcal{O}(\lambda^2) \tag{6.117}$$

Therefore we obtain

$$\beta' = \mu\frac{\partial\lambda'}{\partial\mu} = \mu\frac{\partial\lambda}{\partial\mu}f' + \mu\frac{\partial(m/\mu)}{\partial\mu}\dot{f} = \beta f' + \frac{m}{\mu}(\gamma_m - 1)\dot{f} \tag{6.118}$$

Working in a region where we can neglect the mass we have

$$\beta'(\lambda') = \beta(\lambda)f'(\lambda) \tag{6.119}$$

and we see that a fixed point at $\lambda = \lambda_F$ implies a fixed point in λ'. Therefore the presence of a fixed point does not depend on the scheme used (at least in this approximation). It is possible to show that also the sign of the β function is scheme-independent.

Chapter 7

The path integral for Fermi fields

7.1 Fermionic oscillators and Grassmann algebras

The Feynman's formulation of the path integral can be extended to field theories for fermions. However, since the path integral is strictly related to the classical description of the theory, it could be nice to have such a "classical" description for fermionic systems. In fact it turns out that it is possible to have a "classical" (called pseudoclassical) description in a way such that its quantization leads naturally to anticommutation relations. In this theory one introduces anticommuting variables already at the classical level. This type of mathematical objects form the so called Grassmann algebras. Before proceeding further let us introduce some concepts about these algebras which will turn out useful in the following.

First of all we recall that an algebra V is nothing but a vector space where it is defined a bilinear mapping $V \times V \to V$ (the algebra product). A Grassmann algebra $\mathcal{G}_n$ is a vector space of dimensions 2^n and it can be described in terms of n generators θ_i $(i = 1, \cdots, n)$ satisfying the relations

$$\theta_i \theta_j + \theta_j \theta_i \equiv [\theta_i, \theta_j]_+ = 0, \quad i, j = 1, \cdots, n \tag{7.1}$$

Then, the elements constructed by all the possible products among the generators form a basis of $\mathcal{G}_n$:

$$1, \ \theta_i, \ \theta_i \theta_j, \ \theta_i \theta_j \theta_k, \cdots, \theta_1 \theta_2 \cdots \theta_n \tag{7.2}$$

Notice that there are no other possible elements since

$$\theta_i^2 = 0 \tag{7.3}$$

from eq. (7.1). The subset of elements of $\mathcal{G}_n$ generated by the product of an even number of generators is called the even part of the algebra, $\mathcal{G}_n^{(0)}$, whereas the subset generated by the product of an odd number of generators is the odd part, $\mathcal{G}_n^{(1)}$. It follows

$$\mathcal{G}_n = \mathcal{G}_n^{(0)} \oplus \mathcal{G}_n^{(1)} \tag{7.4}$$

A decomposition of this kind is called a **grading** of the vector space, if it is possible to assign a grade to each element of the subspaces. In this case the grading is as follows

1. If $x \in \mathcal{G}_n^{(0)}$, $\deg(x) = +1$.

2. If $x \in \mathcal{G}_n^{(1)}$, $\deg(x) = -1$.

Notice that this grading is also consistent with the algebra product, since if x, y are homogeneous elements of $\mathcal{G}_n$ (that is to say even or odd), then

$$\deg(xy) = \deg(x)\deg(y) \tag{7.5}$$

Since we have no time to enter into the details of the classical mechanics of the Grassmann variables, we will formulate the path integral starting directly from quantum mechanics. Let us recall that in the Feynman's formulation one introduces states which are eigenstates of a complete set of observables describing the configuration space of the system. In the case of field theories one takes eigenstates of the field operators in the Heisenberg representation

$$\phi(\vec{x}, t)|\varphi(\vec{x}), t\rangle = \varphi(\vec{x})|\varphi(\vec{x}), t\rangle \tag{7.6}$$

this is possible since the fields commute at equal times

$$[\phi(\vec{x}, t), \phi(\vec{y}, t)]_- = 0 \tag{7.7}$$

and as a consequence it is possible to diagonalize simultaneously at fixed time the set of operators $\phi(\vec{x}, t)$. For Fermi fields such a possibility does not arise since they anticommute at equal times. Therefore it is not possible to define simultaneous eigenstates of Fermi fields. To gain a better understanding of this point let us stay for the moment with free fields, and let us recall that this is nothing but a set of harmonic oscillators. This is true also for Fermi fields, except for the fact that these oscillators are of Fermi type, that is they are described by creation and annihilation operators satisfying the anticommutation relations

$$[a_i, a_j]_+ = \left[a_i^\dagger, a_j^\dagger\right]_+ = 0, \quad \left[a_i, a_j^\dagger\right]_+ = \delta_{ij} \tag{7.8}$$

For greater simplicity take the case of a single Fermi oscillator, and let us try to diagonalize the corresponding annihilation operator

$$a|\theta\rangle = \theta|\theta\rangle \tag{7.9}$$

Then, from $a^2 = 0$, as it foillows from eqs. (7.8), we must have $\theta^2 = 0$. Therefore the eigenvalue θ cannot be an ordinary number. However it can be thought as an odd element of a Grassmann algebra. If we have n Fermi oscillators and we require

$$a_i|\theta_1, \cdots, \theta_n\rangle = \theta_i|\theta_1, \cdots, \theta_n\rangle \tag{7.10}$$

170

we get

$$a_i \left(a_j | \theta_1, \cdots, \theta_n \rangle \right) = \theta_j \left(a_i | \theta_1, \cdots, \theta_n \rangle \right) = \theta_j \theta_i | \theta_1, \cdots, \theta_n \rangle \tag{7.11}$$

But $a_i a_j = -a_j a_i$ and therefore

$$[\theta_i, \theta_j]_+ = 0 \tag{7.12}$$

We see that the equations (7.10) are meaningful if we identify the eigenvalues θ_i with the generators of a Grassmann algebra $\mathcal{G}_n$. In this case the equation (7.9) can be solved easily

$$|\theta\rangle = e^{\theta a^\dagger} |0\rangle \tag{7.13}$$

where $|0\rangle$ is the ground state of the oscillator. In fact

$$
\begin{aligned}
a|\theta\rangle &= a(1 + \theta a^\dagger)|0\rangle = \theta a a^\dagger |0\rangle = \theta(1 - a^\dagger a)|0\rangle \\
&= \theta|0\rangle = \theta(1 + \theta a^\dagger |0\rangle = \theta|\theta\rangle
\end{aligned} \tag{7.14}
$$

where we have used

$$e^{\theta a^\dagger} = 1 + \theta a^\dagger \tag{7.15}$$

since $\theta^2 = a^2 = 0$. We have also

$$a^\dagger |\theta\rangle = a^\dagger (1 + \theta a^\dagger)|0\rangle = a^\dagger |0\rangle = \frac{\partial}{\partial\theta}(1 + \theta a^\dagger)|0\rangle = \frac{\partial}{\partial\theta}|\theta\rangle \tag{7.16}$$

Here we have used the left-derivative on the $\mathcal{G}_n$ defined by

$$
\begin{aligned}
\frac{\partial}{\partial\theta_i}\theta_{i_1}\theta_{i_2}\cdots\theta_{i_k} &= \delta_{i_1,i}\theta_{i_2}\theta_{i_3}\cdots\theta_{i_k} - \delta_{i_2,i}\theta_{i_1}\theta_{i_3}\cdots\theta_{i_k} \\
&+ \cdots + (-1)^{k-1}\delta_{i_k,i}\theta_{i_1}\theta_{i_3}\cdots\theta_{i_{k-1}}
\end{aligned} \tag{7.17}
$$

Suppose we want to define a scalar product in the vector space of the eigenstates of a (notice that these are vector spaces with coefficients in a Grassmann algebra. mathematically such an object is called a modulus over the algebra). Then it is convenient to extend the algebra $\mathcal{G}_1$ generated by θ to a comples Grassmann algebra, $\mathcal{G}_2$, generated by θ and θ^*. The operation "$\star$" is defined as the complex conjugation on the ordinary numbers and it is an involution (automorphism of the algebra with square one) with the property

$$(\theta_{i_1}\theta_{i_2}\cdots\theta_{i_k})^* = \theta_{i_k}^* \cdots \theta_{i_2}^* \theta_{i_1}^* \tag{7.18}$$

In the algebra $\mathcal{G}_2$ we can define normalized eigenstates

$$|\theta\rangle = e^{\theta a^\dagger}|0\rangle N(\theta, \theta^*) \tag{7.19}$$

where $N(\theta, \theta^*) \in \mathcal{G}_2^{(0)}$ and the phase is fixed by the convention

$$N^*(\theta, \theta^*) = N(\theta, \theta^*) \tag{7.20}$$

171

The corresponding bra will be

$$\langle\theta| = N(\theta,\theta^*)\langle 0|e^{\theta^* a} \tag{7.21}$$

the normalization factor $N(\theta,\theta^*)$ is determined by the condition

$$
\begin{aligned}
1 &= \langle\theta|\theta\rangle = N^2(\theta,\theta^*)\langle 0|e^{\theta^* a}e^{\theta a^\dagger}|0\rangle \\
&= N^2(\theta,\theta^*)\langle 0|(1+\theta^* a)(1+\theta a^\dagger)|0\rangle \\
&= N^2(\theta,\theta^*)\langle 0|(1+\theta^*\theta aa^\dagger)|0\rangle \\
&= N^2(\theta,\theta^*)(1+\theta^*\theta) = N^2(\theta,\theta^*)e^{-\theta\theta^*}
\end{aligned}
\tag{7.22}
$$

It follows

$$N(\theta,\theta^*) = e^{\theta\theta^*/2} \tag{7.23}$$

Let us recall some simple properties of a Fermi oscillator. the occupation number operator has eigenvalues 1 and 0. In fact

$$(a^\dagger a)^2 = a^\dagger a a^\dagger a = a^\dagger(1-a^\dagger a)a = a^\dagger a \tag{7.24}$$

from which

$$a^\dagger a(a^\dagger a - 1) = 0 \tag{7.25}$$

Starting from the state with zero eigenvalue for $a^\dagger a$, we can build up only the eigenstate with eigenvalue 1, $a^\dagger|0\rangle$, since $a^{\dagger 2}|0\rangle = 0$. Therefore the space of the eigenstates of $a^\dagger a$ is a two-dimensional space. By defining the ground state as

$$|0\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{7.26}$$

one sees easily that

$$a = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad a^\dagger = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{7.27}$$

(one writes down a as the most general 2×2 matrix and then requires $a|0\rangle = 0$, $a^2 = 0$ and $\left[a,a^\dagger\right]_+ = 1$). The state with occupation number equal to 1 is

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{7.28}$$

In this basis the state $|\theta\rangle$ is given by

$$
\begin{aligned}
|\theta\rangle &= e^{\theta\theta^*/2}e^{\theta a^\dagger}|0\rangle = e^{\theta\theta^*/2}(1+\theta a^\dagger)|0\rangle \\
&= e^{\theta\theta^*/2}\left(|0\rangle + \theta|1\rangle\right) = e^{\theta\theta^*/2}\begin{bmatrix} \theta \\ 1 \end{bmatrix}
\end{aligned}
\tag{7.29}
$$

whereas

$$\langle\theta| = e^{\theta\theta^*/2}\begin{bmatrix} \theta^* & 1 \end{bmatrix} \tag{7.30}$$

172

We see that in order to introduce eigenstates of the operator a has forced us to generalize the ordinary notion of vector space on the real or on the complex numbers, to a vector space on a Grassmann algebra (a modulus). In this space the linear combinations of the vectors are taken with coefficients in $\mathcal{G}_2$ (or more generally in $\mathcal{G}_{2n}$).

Let us now come to the definition of the integral over a Grassmann algebra. To this end we recall that the crucial element in order to derive the path integral formalism from quantum mechanics is the completeness relation for the eigenstates of the position operator

$$\int dq |q\rangle\langle q| = 1 \tag{7.31}$$

What we are trying to do here is to think to the operators a as generalized position operators, and therefore we would like its eigenstates to satisfy a completeness relation of the previous type. Therefore we will "define" the integration over the Grassmann algebra $\mathcal{G}_2$ by requiring that the states defined in eqs. (7.29) and (7.30) satisfy

$$\int d\theta^* d\theta |\theta\rangle\langle\theta| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv 1_2 \tag{7.32}$$

where the identity on the right hand side must be the identity on the two-dimensional vector space. Using the explicit representation for the states

$$
\begin{aligned}
\int d\theta^* d\theta |\theta\rangle\langle\theta| &= \int d\theta^* d\theta\, e^{\theta\theta^*} \begin{pmatrix} \theta \\ 1 \end{pmatrix} (\,\theta^* \quad 1\,) \\
&= \int d\theta^* d\theta\, e^{\theta\theta^*} \begin{pmatrix} \theta\theta^* & \theta \\ \theta^* & 1 \end{pmatrix} \\
&= \int d\theta^* d\theta \begin{pmatrix} \theta\theta^* & \theta \\ \theta^* & 1 + \theta\theta^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
\tag{7.33}
$$

Therefore the integration must satisfy

$$\int d\theta^* d\theta \cdot \theta\theta^* = 1 \tag{7.34}$$

$$\int d\theta^* d\theta \cdot \theta = \int d\theta^* d\theta \cdot \theta^* = \int d\theta^* d\theta \cdot 1 = 0 \tag{7.35}$$

These relations determine the integration in a unique way, since they fix the value of the integral over all the basis elements of $\mathcal{G}_2$. The integral over a generic element of the algebra is then defined by linearity. the previous integration rules can be rewritten as integration rules over the algebra $\mathcal{G}_1$ requiring

$$[d\theta, d\theta^*]_+ = [\theta, d\theta^*]_+ = [\theta, d\theta]_+ = [\theta^*, d\theta]_+ = [\theta^*, d\theta^*]_+ = 0 \tag{7.36}$$

Then, we can reproduce the rules (7.34) and (7.35) by requiring

$$\int d\theta \cdot \theta = 1, \qquad \int d\theta \cdot 1 = 0 \tag{7.37}$$

For n Fermi oscillators we get

$$|\theta_1, \cdots, \theta_n\rangle = e^{\sum_{i=1}^{n} \theta_i \theta_i^*/2} \, e^{\sum_{i=1}^{n} \theta_i a_i^\dagger} |0\rangle \tag{7.38}$$

with

$$\int \left(\prod_{i=1}^{n} d\theta_i^* d\theta_i \right) |\theta_1, \cdots, \theta_n\rangle\langle\theta_1, \cdots, \theta_n| = 1 \tag{7.39}$$

and integration rules

$$\int d\theta_i \theta_j = \delta_{ij} \tag{7.40}$$

Assuming also

$$[d\theta_i, \theta_j]_+ = [d\theta_i, d\theta_j]_+ = 0 \tag{7.41}$$

Notice that the measure $d\theta$ is translationally invariant. In fact if $f(\theta) = a + b\theta$ and $\alpha \in \mathcal{G}_1^{(1)}$

$$\int d\theta\, f(\theta) = \int d\theta\, f(\theta + \alpha) \tag{7.42}$$

since'

$$\int d\theta\, b\alpha = 0 \tag{7.43}$$

7.2 Integration over Grassmann variables

Let us consider the Grassmann algebra $\mathcal{G}_n$. Its generic element can be expanded over the basis elements as

$$f(\theta_1, \cdots, \theta_n) = f_0 + \sum_{i_1} f_1(i_1)\theta_{i_1} + \sum_{i_1, 1_2} f_2(1_1.i_2)\theta_{i_1}\theta_{i_2} + \cdots \sum_{i_1, \cdots, i_n} f_n(i_1, \cdots, i_n)\theta_{i_1} \cdots \theta_{i_n} \tag{7.44}$$

Of course the coefficients of the expansion are antisymmetric. If we integrate f over the algebra $\mathcal{G}_n$ only the last element of the expansion gives a non vanishing contribution. Noticing also that for the antisymmetry we have

$$\sum_{i_1, \cdots, i_n} f_n(i_1, \cdots, i_n)\theta_{i_1} \cdots \theta_{i_n} = n! f_n(1, \cdots, n)\theta_1 \cdots \theta_n \tag{7.45}$$

it follows

$$\int d\theta_n \cdots d\theta_1 f(\theta_1, \cdots, \theta_n) = n! f_n(1, \cdots, n) \tag{7.46}$$

An interesting property of the integration rules for Grassmann variables is the integration by parts formula. This can be seen either by the observation that a derivative destroys a factor θ_i in each element of the basis and therefore

$$\int d\theta_n \cdots d\theta_1 \frac{\partial}{\partial \theta_j} f(\theta_1, \cdots, \theta_n) = 0 \tag{7.47}$$

174

either by the property that the integral is invariant under translations (see the previous Section and Section 4.7). Let us now consider the transformation properties of the measure under a change of variable. For simplicity consider a linear transformation

$$\theta_i = \sum_k a_{ik}\theta'_k \tag{7.48}$$

The eq. (7.40) must be required to hold in each basis, since it can be seen easily that the previous transformation leaves invariant the multiplication rules. In fact,

$$\theta'_i\theta'_j = c_{ik}c_{jh}\theta_k\theta_h = -c_{ik}c_{jh}\theta_h\theta_k == -c_{jh}c_{ik}\theta_h\theta_k = -\theta'_j\theta'_i \tag{7.49}$$

where c_{ij} is the inverse of the matrix a_{ij}. Therefore we require

$$\int d\theta_i\theta_j = \int d\theta'_i\theta'_j = \delta_{ij} \tag{7.50}$$

By putting

$$d\theta_i = \sum_k d\theta'_k b_{ki} \tag{7.51}$$

we obtain

$$\int d\theta_i\theta_j = \int \sum_{h,k} d\theta'_k b_{ki}a_{jh}\theta'_h = \sum_{h,k} b_{ki}a_{jh}\delta_{hk} = (ab)_{ij} = \delta_{ij} \tag{7.52}$$

from which

$$b_{ij} = (a^{-1})_{ij} \tag{7.53}$$

The transformation properties of the measure are

$$d\theta_n \cdots d\theta_1 = \det|a^{-1}|d\theta'_n \cdots d\theta'_1 \tag{7.54}$$

where we have used the anticommutation properties of $d\theta_i$ among themselves. We see also that

$$\det|a| = \det\left|\frac{\partial\theta}{\partial\theta'}\right| \tag{7.55}$$

and

$$d\theta'_n \cdots d\theta'_1 = \det^{-1}\left|\frac{\partial\theta'}{\partial\theta}\right| d\theta_n \cdots d\theta_1 \tag{7.56}$$

Therefore the rule for a linear change of variables is

$$\int d\theta'_n \cdots d\theta'_1 f(\theta'_i) = \int d\theta_n \cdots d\theta_1 \det^{-1}\left|\frac{\partial\theta'}{\partial\theta}\right| f(\theta'_i(\theta_i)) \tag{7.57}$$

This rule should be compared with the one for commuting variables

$$\int dx'_1 \cdots dx'_n f(x'_i) = \int dx_1 \cdots dx_n \det\left|\frac{\partial x'}{\partial x}\right| f(x'_i(x_i)) \tag{7.58}$$

We see that the Grassmann measure transforms according to the inverse jacobian rather than with the jacobian itself. As in the commuting case, the more important integral to be evaluated for the path integral approach is the gaussian one

$$I = \int d\theta_n \cdots d\theta_1 e^{\sum_{i,j} A_{ij}\theta_i\theta_j/2} \equiv \int d\theta_n \cdots d\theta_1 e^{\theta^T A\theta/2} \tag{7.59}$$

The matrix A is supposed to be real antisymmetric

$$A^T = -A \tag{7.60}$$

As shown in the Appendix a real and antisymmetric matrix can be reduced to the form

$$A_s = \begin{bmatrix} 0 & \lambda_1 & 0 & 0 & \cdot & \cdot & \cdot \\ -\lambda_1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \lambda_2 & \cdot & \cdot & \cdot \\ 0 & 0 & -\lambda_2 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{7.61}$$

by means of an orthogonal transformation S. Then, let us do the following change of variables

$$\theta_i = \sum_j (S^T)_{ij}\theta'_j \tag{7.62}$$

we get

$$\theta^T A\theta = (S^T\theta')AS^T\theta' = \theta'^T SAS^T\theta' = \theta'^T A_s\theta' \tag{7.63}$$

and using $\det|S| = 1$

$$I = \int d\theta'_n \cdots d\theta'_1 e^{\theta'^T A_s\theta'/2} \tag{7.64}$$

The expression in the exponent is given by

$$\frac{1}{2}\theta'^T A_s\theta' = \lambda_1\theta'_1\theta'_2 + \cdots + \lambda_{\frac{n}{2}}\theta'_{n-1}\theta'_n, \quad \text{for } n \text{ even}$$

$$\frac{1}{2}\theta'^T A_s\theta' = \lambda_1\theta'_1\theta'_2 + \cdots + \lambda_{\frac{n-1}{2}}\theta'_{n-2}\theta'_{n-1}, \quad \text{for } n \text{ odd} \tag{7.65}$$

From eq. (7.46) we get contribution to the integral only from the term in the expansion of the exponential $(\theta'^T A_s\theta')^{\frac{n}{2}}$ for n even. In the case of n odd we get $I = 0$ since $\theta'^T A_s\theta'$ does not contain θ'_n. Therefore, for n even

$$\begin{aligned} I &= \int d\theta'_n \cdots d\theta'_1 e^{\theta'^T A_s\theta'/2} = \int d\theta'_n \cdots d\theta'_1 \frac{1}{(\frac{n}{2})!}(\frac{1}{2}\theta'^T A_s\theta')^{\frac{n}{2}} \\ &= \int d\theta'_n \cdots d\theta'_1 \frac{1}{(\frac{n}{2})!}(\frac{n}{2})!\lambda_1 \cdots \lambda_{\frac{n}{2}}\theta'_1 \cdots \theta'_n \\ &= \lambda_1 \cdots \lambda_{\frac{n}{2}} = \sqrt{\det A} \end{aligned} \tag{7.66}$$

where we have used $\det|A_s| = \det|A|$. Then

$$\int d\theta_n \cdots d\theta_1 e^{\theta^T A\theta/2} = \sqrt{\det A} \tag{7.67}$$

This result is valid also for n odd since in this case $\det|A| = 0$. A similar result holds for commuting variables. In fact, let us define

$$J = \int dx_1 \cdots dx_n e^{-x^T Ax/2} \tag{7.68}$$

with A a real and symmetric matrix. A can be diagonalized by an orthogonal transformation with the result

$$J = \int dx'_1 \cdots dx'_n e^{-x'^T A_d x'/2} = (2\pi)^{\frac{n}{2}} \frac{1}{\sqrt{\det A}} \tag{7.69}$$

Again, the result for fermionic variables is opposite to the result of the commuting case. Another useful integral is

$$I(A;\chi) = \int d\theta_n \cdots d\theta_1 e^{\theta^T A\theta/2 + \chi^T \theta} \tag{7.70}$$

with χ_i Grassmann variables. To evaluate this integral let us put

$$\theta = \theta' + A^{-1}\chi \tag{7.71}$$

We get

$$\begin{aligned}
\frac{1}{2}\theta^T A\theta + \chi^T\theta &= \frac{1}{2}(\theta'^T - \chi^T A^{-1})A(\theta' + A^{-1}\chi) + \chi^T(\theta' + A^{-1}\chi) \\
&= \frac{1}{2}\theta'^T A\theta' + \frac{1}{2}\chi^T A^{-1}\chi
\end{aligned} \tag{7.72}$$

where we have used the antisymmetry of A and $\chi^T\theta' = -\theta'^T\chi$. Since the measure is invariant under translations we find

$$I(A;\chi) = \int d\theta'_n \cdots d\theta'_1 e^{\theta'^T A\theta'/2 + \chi^T A^{-1}\chi/2} = e^{\chi^T A^{-1}\chi/2}\sqrt{\det A} \tag{7.73}$$

again to be compared with the result for commuting variables

$$I(A;J) = \int dx_1 \cdots dx_n e^{-x^T Ax/2 + J^T x} = e^{J^T A^{-1}J/2}\frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}} \tag{7.74}$$

The previous equations can be generalized to complex variables

$$\int d\theta_1 d\theta_1^* \cdots d\theta_n d\theta_n^* e^{\theta^\dagger A\theta} = \det|A| \tag{7.75}$$

177

and

$$\int d\theta_1 d\theta_1^* \cdots d\theta_n d\theta_n^* e^{\theta^\dagger A\theta + \chi^\dagger \theta + \theta^\dagger \chi} = e^{-\chi^\dagger A^{-1}\chi} \det|A| \qquad (7.76)$$

The second of these equations follows from the first one as in the real case. we will prove the first one for the case $n = 1$. Starting from

$$\int d\theta_2 d\theta_1 e^{\theta^T A\theta/2} = A_{12} \qquad (7.77)$$

and putting

$$\theta_1 = \frac{1}{\sqrt{2}}(\eta + \eta^*), \quad \theta_2 = -\frac{i}{\sqrt{2}}(\eta - \eta^*) \qquad (7.78)$$

we get

$$\frac{1}{2}\theta^T A\theta = -i\eta^* A_{12}\eta \qquad (7.79)$$

and recalling that (remember that here one has to use the inverse of the jacobian)

$$d\theta_2 d\theta_1 = id\eta d\eta^* \qquad (7.80)$$

we get

$$\int d\theta_2 d\theta_1 e^{\theta^T A\theta/2} = i \int d\eta d\eta^* e^{-i\eta^* A_{12}\eta} \qquad (7.81)$$

and

$$\int d\eta d\eta^* e^{\eta^*(-iA_{12})\eta} = -iA_{12} \qquad (7.82)$$

7.3 The path integral for the fermionic theories

In this Section we will discuss the path integral formulation for a system described by anticommuting variables. Considering a Fermi oscillator, one could start from the ordinary quantum mechanical description and then by using the completeness in terms of Grassmann variable, it would be possible to derive the corresponding path integral formulation. We will give here only the result of this analysis. It turns out that the amplitude among eigenstates of the Fermi annihilation operator is given by

$$\langle \theta', t'|\theta, t\rangle = \int_{\theta(t)=\theta,t}^{\theta^*(t')=\theta'^*,t'} \mathcal{D}(\theta, \theta^*) e^{\theta'^* \theta(t')/2 + \theta^*(t)\theta/2 + iS} \qquad (7.83)$$

where

$$\mathcal{D}(\theta, \theta^*) = \prod d\theta d\theta^* \qquad (7.84)$$

and

$$S = \int_t^{t'} \left[\frac{i}{2}(\theta^*\dot\theta - \dot\theta^*\theta) - H(\theta, \theta^*) \right] dt \qquad (7.85)$$

where, for the Fermi oscillator $H = \theta^*\theta$. The expression is very similar to the one obtained in the commuting case if we realize that the previous formula is the analogue in the phase space using complex coordinates of the type $q \pm ip$. It should be noticed the particular boundary conditions for the θ variables originating from the fact that the equations of motion are of the first order. Correspondingly we can assign only the coordinates at a particular time. In any case, since at the end we are interested in field theory where only the generating functional is relevant and where the boundary conditions are arbitrary (for instance we can take the fields vanishing at $t = \pm\infty$), and in any case do not affect the dependence of the generating functional on the external source. The extension to the Fermi field is done exactly as we have done in the bosonic case, and correspondingly the generating functional for the free Dirac theory will be

$$Z[\eta, \bar{\eta}] = \int \mathcal{D}(\psi, \bar{\psi}) e^{i \int [\bar{\psi}(i\slashed{\partial} - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta]} \tag{7.86}$$

where η and $\bar{\eta}$ are the external sources (of Grassmann character). Using the integration formula (7.73) we get formally

$$Z[\eta, \bar{\eta}] = Z[0] e^{-(i\bar{\eta})\frac{-i}{i\slashed{\partial} - m}(i\eta)} = Z[0] e^{-i\bar{\eta}\frac{1}{i\slashed{\partial} - m}\eta} \tag{7.87}$$

where the determinant of the Dirac operator has been included in the term at zero source. In this expression we have to specify how to take the inverse operator. We know that this is nothing but the Dirac propagator defined by

$$(i\slashed{\partial} - m)S_F(x - y) = \delta^4(x - y) \tag{7.88}$$

It follows

$$Z[\eta, \bar{\eta}] = Z[0] e^{-i \int d^4x d^4y \bar{\eta}(x) S_F(x - y)\eta(y)} \tag{7.89}$$

with

$$S_F(x) = \lim_{\epsilon \to 0^+} \int \frac{d^4k}{(2\pi)^4} \frac{1}{\slashed{k} - m + i\epsilon} e^{-ikx} = \lim_{\epsilon \to 0^+} \int \frac{d^4k}{(2\pi)^4} \frac{\slashed{k} + m}{k^2 - m^2 + i\epsilon} e^{-ikx} \tag{7.90}$$

The Green's functions are obtained by differentiating the generating functional with respect to the external sources (Grassmann differentiation). In particular, the two point function is given by

$$\frac{1}{Z[0]} \frac{\delta Z}{\delta\bar{\eta}(x)\delta\eta(y)}\bigg|_{\eta=\bar{\eta}=0} = \frac{1}{\langle 0|0\rangle} \langle 0|T(\psi(x)\bar{\psi}(y))|0\rangle = iS_F(x - y) \tag{7.91}$$

The T-product in this expression is the one for the Fermi fields, as it follows from the derivation made in the bosonic case, taking into account that the classical field of this case are anticommuting.

We will not insist here further on this discussion, but all we have done in the bosonic case can be easily generalized for Fermi fields.

179

Chapter 8

The quantization of the gauge fields

8.1   QED as a gauge theory

Many field theories possess global symmetries. These are transformations leaving invariant the action of the system and are characterized by a certain number of parameters which are independent on the space-time point. As a prototype we can consider the free Dirac lagrangian

$$\mathcal{L}_0 = \bar{\psi}(x)[i\partial\!\!\!/ - m]\psi(x) \tag{8.1}$$

which is invariant under the global phase transformation

$$\psi(x) \to \psi'(x) = e^{-iQ\alpha}\psi(x) \tag{8.2}$$

If one has more than one field, Q is a diagonal matrix having as eigenvalues the charges of the different fields measured in unit e. For instance, a term as $\bar{\psi}_2\psi_1\phi$, with ϕ a scalar field, is invariant by choosing $Q(\psi_1) = Q(\phi) = 1$, and $Q(\psi_2) = 2$. This is a so called **abelian symmetry** since

$$e^{-i\alpha Q}e^{-i\beta Q} = e^{-i(\alpha + \beta)Q} = e^{-i\beta Q}e^{-i\alpha Q} \tag{8.3}$$

It is also referred to as a $U(1)$ symmetry. The physical meaning of this invariance lies in the possibility of assigning the phase to the fields in an arbitrary way, without changing the observable quantities. This way of thinking is in some sort of contradiction with causality, since it requires to assign the phase of the fields simultaneously at all space-time points. It looks more physical to require the possibility of assigning the phase in an arbitrary way at each space-time point. This invariance, formulated by Weyl in 1929, was called gauge invariance. The free lagrangian (8.1) cannot be gauge invariant due to the derivative coming from the kinetic term. The idea is

simply to generalize the derivative ∂_μ to a so called covariant derivative D_μ having the property that $D_\mu\psi$ transforms as ψ, that is

$$D_\mu\psi(x) \to [D_\mu\psi(x)]' = e^{-iQ\alpha(x)}D_\mu\psi(x) \tag{8.4}$$

In this case the term

$$\bar{\psi}D_\mu\psi \tag{8.5}$$

will be invariant as the mass term under the local phase transformation. To construct the covariant derivative, we need to enlarge the field content of the theory, by introducing a vector field, the **gauge field** A_μ, in the following way

$$D_\mu = \partial_\mu + ieQA_\mu \tag{8.6}$$

The transformation law of A_μ is obtained from eq. (8.4)

$$
\begin{aligned}
{[(\partial_\mu + ieQA_\mu)\psi]'} &= (\partial_\mu + ieQA'_\mu)\psi'(x) \\
&= (\partial_\mu + ieQA'_\mu)e^{-iQ\alpha(x)}\psi \\
&= e^{-iQ\alpha(x)}\left[\partial_\mu + ieQ(A'_\mu - \frac{1}{e}\partial_\mu\alpha)\right]\psi
\end{aligned} \tag{8.7}
$$

from which

$$A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\alpha \tag{8.8}$$

The lagrangian density

$$\mathcal{L}_\psi = \bar{\psi}[i\slashed{D} - m]\psi = \bar{\psi}[i\gamma^\mu(\partial_\mu + ieQA_\mu) - m)\psi = \mathcal{L}_0 - e\bar{\psi}Q\gamma^\mu\psi A_\mu \tag{8.9}$$

is then invariant under gauge transformations, or under the local group $U(1)$. We see also that by requiring local invariance we reproduce the electromagnetic interaction as obtained through the minimal substitution we discussed before.

In order to determine the kinetic term for the vector field A_μ we notice that eq. (8.4) implies that under a gauge transformation, the covariant derivative undergoes a unitary transformation

$$D_\mu \to D_\mu{}' = e^{-iQ\alpha(x)}D_\mu e^{iQ\alpha(x)} \tag{8.10}$$

Then, also the commutator of two covariant derivatives

$$[D_\mu, D_\nu] = [\partial_\mu + ieQA_\mu, \partial_\nu + ieQA_\nu] = ieQF_{\mu\nu} \tag{8.11}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{8.12}$$

transforms in the same way

$$F_{\mu\nu} \to e^{-iQ\alpha(x)}F_{\mu\nu}e^{iQ\alpha(x)} = F_{\mu\nu} \tag{8.13}$$

The last equality follows from the commutativity of $F_{\mu\nu}$ with the phase factor. The complete lagrangian density is then

$$\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_A = \bar{\psi}[i\gamma^\mu(\partial_\mu + ieQA_\mu) - m]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{8.14}$$

The gauge principle has automatically generated an interaction between the gauge field and the charged field. We notice also that gauge invariance prevents any mass term, $\frac{1}{2}M^2A^\mu A_\mu$. Therefore, the photon field is massless. Also, since the local invariance implies the global ones, by using the Noether's theorem we find the conserved current as

$$j_\mu = \frac{\partial \mathcal{L}}{\partial \psi_{,\mu}}\delta\psi = \bar{\psi}\gamma_\mu(Q\alpha)\psi \tag{8.15}$$

from which, eliminating the infinitesimal parameter α,

$$J_\mu = \bar{\psi}\gamma_\mu Q\psi \tag{8.16}$$

8.2   Non-abelian gauge theories

The approach of the previous section can be easily extended to local non-abelian symmetries. We will consider the case of N Dirac fields. The free lagrangian

$$\mathcal{L}_0 = \sum_{a=1}^{N} \bar{\psi}_a(i\slashed{\partial} - m)\psi_a \tag{8.17}$$

is invariant under the global transformation

$$\Psi(x) \to \Psi'(x) = A\Psi(x) \tag{8.18}$$

where A is a unitary $N \times N$ matrix, and we have denoted by Ψ the column vector with components ψ_a. In a more general situation the actual symmetry could be a subgroup of $U(N)$. For instance, when the masses are not all equal. So we will consider here the gauging of a subgroup G of $U(N)$. The fields $\psi_a(x)$ will belong, in general, to some reducible representation of G. Denoting by U the generic element of G, we will write the corresponding matrix U_{ab} acting upon the fields ψ_a as

$$U = e^{-i\alpha_A T^A}, \qquad U \in G \tag{8.19}$$

where T^A denote the generators of the Lie algebra associated to G, Lie(G), (that is the vector space spanned by the infinitesimal generators of the group) in the fermion representation. The generators T^A satisfy the algebra

$$[T^A, T^B] = if_C^{AB}T^C \tag{8.20}$$

182

where f_C^{AB} are the structure constants of Lie(G). For instance, if $G = SU(2)$, and we take the fermions in the fundamental representation,

$$\Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \tag{8.21}$$

we have

$$T^A = \frac{\sigma^A}{2}, \qquad A = 1, 2, 3 \tag{8.22}$$

where σ^A are the Pauli matrices. In the general case the T^A's are $N \times N$ hermitian matrices that we will choose normalized in such a way that

$$Tr(T^A T^B) = \frac{1}{2}\delta^{AB} \tag{8.23}$$

To make local the transformation (8.19), means to promote the parameters α_A to space-time functions

$$\alpha_A \to \alpha_A(x) \tag{8.24}$$

Notice, that now the group does not need to be abelian, and therefore, in general

$$e^{-i\alpha_A T^A} e^{-i\beta_A T^A} \neq e^{-i\beta_A T^A} e^{-i\alpha_A T^A} \tag{8.25}$$

Let us now proceed to the case of the local symmetry by defining again the concept of covariant derivative

$$D_\mu \Psi(x) \to [D_\mu \Psi(x)]' = U(x)[D_\mu \psi(x)] \tag{8.26}$$

We will put again

$$D_\mu = \partial_\mu + ig B_\mu \tag{8.27}$$

where B_μ is a $N \times N$ matrix acting upon $\Psi(x)$. In components

$$D_{ab}^\mu = \delta_{ab}\partial^\mu + ig(B^\mu)_{ab} \tag{8.28}$$

The eq. (8.26) implies

$$\begin{aligned}
D_\mu \Psi &\to (\partial_\mu + ig B'_\mu)U(x)\Psi \\
&= U(x)\partial_\mu \Psi + U(x)[U^{-1}(x)ig B'_\mu U(x)]\Psi + (\partial_\mu U(x))\Psi \\
&= U(x)[\partial_\mu + U^{-1}(x)ig B'_\mu U(x) + U^{-1}(x)\partial_\mu U(x)]\Psi
\end{aligned} \tag{8.29}$$

therefore

$$U^{-1}(x)ig B'_\mu U(x) + U^{-1}(x)\partial_\mu U(x) = ig B_\mu \tag{8.30}$$

and

$$B'_\mu(x) = U(x)B_\mu(x)U^{-1}(x) + \frac{i}{g}(\partial_\mu U(x))U^{-1}(x) \tag{8.31}$$

For an infinitesimal transformation

$$U(x) \approx 1 - i\alpha_A(x)T^A \tag{8.32}$$

we get

$$\delta B_\mu(x) = -i\alpha_A(x)[T^A, B_\mu(x)] + \frac{1}{g}(\partial_\mu \alpha_A(x))T^A \tag{8.33}$$

Since $B_\mu(x)$ acquires a term proportional to T^A, the transformation law is consistent with a B_μ linear in the generators of the Lie algebra, that is

$$(B^\mu)_{ab} \equiv A^\mu_A (T^A)_{ab} \tag{8.34}$$

The transformation law for A_μ becomes

$$\delta A^\mu_C = f^{AB}_C \alpha_A A^\mu_B + \frac{1}{g} \partial^\mu \alpha_C \tag{8.35}$$

The difference with respect to the abelian case is that the field undergoes also a homogeneous transformation.

The kinetic term for the gauge fields is constructed as in the abelian case. In fact the quantity

$$[D_\mu, D_\nu]\Psi \equiv igF_{\mu\nu}\Psi \tag{8.36}$$

in virtue of the eq. (8.26), transforms as Ψ under gauge transformations, that is

$$\begin{aligned}([D_\mu, D_\nu]\Psi)' &= igF'_{\mu\nu}\Psi' = igF'_{\mu\nu}U(x)\Psi \\ &= U(x)([D_\mu, D_\nu]\Psi) = U(x)\,(igF_{\mu\nu})\,\Psi \end{aligned} \tag{8.37}$$

This time the tensor $F_{\mu\nu}$ is not invariant but transforms homogeneously, since it does not commute with the gauge transformation as in the abelian case

$$F'_{\mu\nu} = U(x)F_{\mu\nu}U^{-1}(x) \tag{8.38}$$

The invariant kinetic term will be assumed as

$$\mathcal{L}_A = -\frac{1}{2}Tr[F_{\mu\nu}F^{\mu\nu}] \tag{8.39}$$

Let us now evaluate $F_{\mu\nu}$

$$\begin{aligned}igF_{\mu\nu} &= [D_\mu, D_\nu] = [\partial_\mu + igB_\mu, \partial_\nu + igB_\nu] \\ &= ig(\partial_\mu B_\nu - \partial_\nu B_\mu) - g^2[B_\mu, B_\nu]\end{aligned} \tag{8.40}$$

or

$$F_{\mu\nu} = (\partial_\mu B_\nu - \partial_\nu B_\mu) + ig[B_\mu, B_\nu] \tag{8.41}$$

in components

$$F^{\mu\nu} = F^{\mu\nu}_C T^C \tag{8.42}$$

184

with

$$F_C^{\mu\nu} = \partial^\mu A_C^\nu - \partial^\nu A_C^\mu - g f_C^{AB} A_A^\mu A_B^\nu \tag{8.43}$$

The main feature of the non-abelian gauge theories is the bilinear term in the previous expression. Such a term comes because $f_C^{AB} \neq 0$, expressing the fact that G is not abelian. The kinetic term for the gauge field, expressed in components, is given by

$$\mathcal{L}_A = -\frac{1}{4} \sum_A F_{\mu\nu A} F_A^{\mu\nu} \tag{8.44}$$

Therefore, whereas in the abelian case $\mathcal{L}_A$ is a free lagrangian (it contains only quadratic terms), now it contains interaction terms cubic and quartic in the fields. The physical motivation lies in the fact that the gauge fields couple to everything which transforms in a non trivial way under the gauge group. Therefore they couple also to themselves (remember the homogeneous piece of transformation).

To derive the equations of motion for the gauge fields, let us consider the total action

$$\int_V d^4 x \left[\bar{\Psi}(i\partial\!\!\!/ - m)\Psi - g\bar{\Psi}\gamma_\mu B^\mu \Psi \right] + S_A \tag{8.45}$$

where

$$S_A = -\frac{1}{2} \int_V d^4 x \, Tr(F_{\mu\nu} F^{\mu\nu}) \tag{8.46}$$

and the variation of S_A

$$\delta S_A = - \int_V d^4 x \, Tr(F_{\mu\nu} \delta F^{\mu\nu}) \tag{8.47}$$

Using the definition (8.41) for the field strength we get

$$\delta F_{\mu\nu} = \partial_\mu \delta B_\nu + ig(\delta B_\mu) B_\nu + ig B_\mu \delta(B_\nu) - (\mu \leftrightarrow \nu) \tag{8.48}$$

from which

$$\delta S_A = -2 \int_V d^4 x \, Tr[F^{\mu\nu}(\partial_\mu \delta B_\nu + ig(\delta B_\mu) B_\nu + ig B_\mu (\delta B_\nu))] \tag{8.49}$$

where we have taken into account the antisymmetry properties of $F_{\mu\nu}$. Integrating by parts we obtain

$$\begin{aligned}
\delta S_A &= -2 \int_V d^4 x \, Tr[-(\partial_\mu F^{\mu\nu}) \delta B_\nu - ig B_\mu F^{\mu\nu} \delta B_\nu + ig F^{\mu\nu} B_\mu \delta B_\nu] \\
&= 2 \int_V d^4 x \, Tr\left[(\partial_\mu F^{\mu\nu} + ig[B_\mu, F^{\mu\nu}]) \delta B_\nu \right] \\
&= \int_V d^4 x \, (\partial_\mu F^{\mu\nu} + ig[B_\mu, F^{\mu\nu}])_A \, \delta A_{\nu A}
\end{aligned} \tag{8.50}$$

where we used the cyclic property of the trace. By taking into account also the free term for the Dirac fields and the interaction we find the equations of motion

$$\partial_\mu F^{\mu\nu A} + ig[B_\mu, F^{\mu\nu}]^A = g\bar{\Psi}\gamma^\nu T^A \Psi$$
$$(i\slashed{\partial} - m)\Psi = g\gamma_\mu B^\mu \Psi \tag{8.51}$$

From the first equation we see that the currents $\bar{\Psi}\gamma_\mu T^A \Psi$ are not conserved. In fact the conserved currents turn out to be

$$J_\nu^A = \bar{\Psi}\gamma_\nu T^A \Psi - i[B^\mu, F_{\mu\nu}]^A \tag{8.52}$$

The reason is that under a global transformation of the symmetry group, the gauge fields are not invariant, said in different words they are <u>charged</u> fields with respect to the gauge fields. In fact we can verify immediately that the previous currents are precisely the Noether's currents. Under a global variation we have

$$\delta A_C^\mu = f_C^{AB}\alpha_A A_B^\mu, \qquad \delta\Psi = -i\alpha_A T^A \Psi \tag{8.53}$$

and we get

$$j^\mu = \frac{\partial\mathcal{L}}{\partial\Psi_{,\mu}}\delta\Psi + \frac{\partial\mathcal{L}}{\partial A_{\nu,\mu C}}\delta A_{\nu C} \tag{8.54}$$

from which

$$j^\mu = \bar{\Psi}\gamma^\mu \alpha_A T^A \Psi - F_C^{\mu\nu} f_C^{AB}\alpha_A A_{\nu B} \tag{8.55}$$

In the case of simple compact Lie groups one can define $f^{ABC} = f_C^{AB}$ with the property $f^{ABC} = f^{BCA}$. It follows

$$F_C^{\mu\nu} f^{ABC} A_{\nu B} T^A = i[B^\nu, F_{\nu\mu}] \equiv i[B^\nu, F_{\nu\mu}]^A T^A \tag{8.56}$$

Therefore

$$j^\mu = \bar{\Psi}\gamma^\mu \alpha_A T^A \Psi - i[B_\nu, F^{\nu\mu}]^A \alpha_A \tag{8.57}$$

After division by α_A we get the Noether's currents (8.52). The contribution of the gauge fields to the currents is also crucial in order they are conserved quantities. In fact, the divergence of the fermionic contribution is given by

$$\partial^\mu(\bar{\Psi}\gamma_\mu T^A \Psi) = -ig\bar{\Psi}\gamma_\mu T^A B^\mu \Psi + ig\bar{\Psi}\gamma_\mu B^\mu T^A \Psi = -ig\bar{\Psi}\gamma_\mu[T^A, B^\mu]\Psi \tag{8.58}$$

which vanishes for abelian gauge fields, whereas it is compensated by the gauge fields contribution in the non abelian case.

8.3 Path integral quantization of the gauge theories

As we have seen at the beginning of this course that the quantization of the electromagnetic field is far from being trivial, the difficulties being related to the gauge

invariance of the theory. In this Section we would like to discuss how these difficulties arise also in the path integral quantization. To this end let us recall the general scheme of field quantization in this approach. First of all we will deal only with the perturbative approach, meaning that the only thing that one needs to evaluate is the generating functional for the free case. The free case is defined by the quadratic part of the action. Therefore the evaluation of the path integral reduces to calculate a Gaussian integral of the type given in eqs. (7.73) and (7.73) for the fermionic and bosonic case respectively. For instance, in the bosonic case one evaluates

$$I(A; J) = \int dx_1 \cdots dx_n e^{-x^T A x/2 + J^T x} = e^{J^T A^{-1} J/2} \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}} \qquad (8.59)$$

In our case the operator A appearing in the gaussian factor is nothing but the wave operator. The result of the integration is meaningful only if the operator A is non-singular (it needs to have an inverse) otherwise the integral is not well defined. This is precisely the point where we run into problems in the case of a gauge theory. Remember from eq. (1.132) that the wave operator for the electromagnetic field in momentum space is given by

$$(-k^2 g^{\mu\nu} + k^\mu k^\nu) A_\nu(k) = 0 \qquad (8.60)$$

We see that the operator has a null eigenvector k_ν, since

$$(-k^2 g^{\mu\nu} + k^\mu k^\nu) k_\nu = 0 \qquad (8.61)$$

and therefore it is a singular operator. This is strictly related to the gauge invariance, which in momentum space reads

$$A_\mu(k) \to A_\mu(k) + k_\mu \Lambda(k) \qquad (8.62)$$

In fact, we see that the equations of motion are invariant under a gauge transformation precisely because the wave operator has k_μ as a null eigenvector. Another way of seeing this problem is the following: in order to get physical results, as to evaluate S matrix elements, it is necessary to integrate over gauge-invariant functionals

$$\int \mathcal{D}(A_\mu) F[A_\mu] e^{iS} \qquad (8.63)$$

with $F[A_\mu] = F[A_\mu^\Omega]$ (here A_μ^Ω is the gauge transformed of A_μ). But being the action and the functional F gauge invariant, it follows that the integrand is invariant along the gauge group orbits. The gauge group orbits are defined as the set of points, in the field space, which can be reached by a given A_μ via a gauge transformation. In other words, given A_μ, its orbit is given by all the fields A_μ^Ω obtained by varying the parameters of the gauge group (that is varying Ω), see Fig. 8.3.1. For instance, in the abelian case

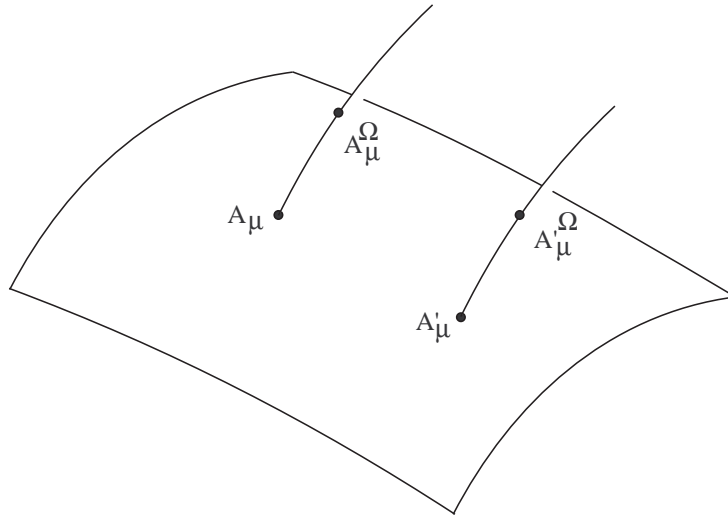$$A_\mu^\Omega = A_\mu + \partial_\mu \Omega \qquad (8.64)$$

187

Fig. 8.3.1 - *The orbits of the gauge group in the field space.*

Since the orbits define equivalence classes (or cosets) we can imagine the field space as the set of all the orbits. Correspondingly we can divide our functional integral in a part parallel and in one perpendicular to the orbits. Then, the reason why our functional integral is not well defined depends from the fact that the integrand along the space parallel to the orbits is infinite (the integrand being gauge invariant does not depend on the corresponding variables). However this observation suggests a simple solution to our problem. We could define the integral by dividing it by the integral along the part parallel to the orbit (notice that is the same as the volume of the gauge group). In order to gain a clear understanding of this problem let us consider the following very simple example in two dimensions

$$
Z = \int_{-\infty}^{+\infty} dx dy e^{-\frac{1}{2}a^2(x-y)^2}
\tag{8.65}
$$

This is as reducing the space-time two two points with x and y the possible values of the field in those points. At the same time the exponent can be seen as an euclidean action with the (gauge) symmetry

$$
x^\Omega = x + \Omega, \quad y^\Omega = y + \Omega
\tag{8.66}
$$

The integral (8.65) is infinite due to this invariance. In fact this implies that the integrand depends only on $x - y$, and therefore the integration over the remaining variable gives rise to an infinite result. In order to proceed along the lines we have described above, let us partition the space of the field, that is the two-dimensional space (x, y) in the gauge group orbits. Given a point (x_0, y_0), the group orbit is given by the set of points $(x^\Omega = x + \Omega, y^\Omega = y + \Omega)$ corresponding to the line $x - y = x_0 - y_0$, as shown in Fig. 8.3.2 We can perform the integral by integrating

188

.
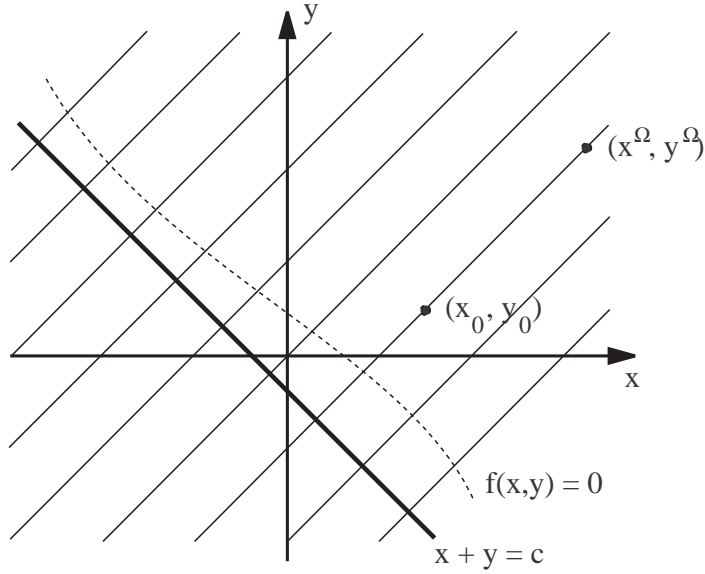
Fig. 8.3.2 - *The field space (x, y) partitioned in the orbits of the gauge group. We show also the line $x + y = c$ and the arbitrary line $f(x, y) = 0$ (dashed).*

along the perpendicular space, the line $x + y = c$ (c is a constant), and then along the parallel space. It is this last integral which gives a divergent result, since we are adding up the same contribution an infinite number of times. therefore the obvious way to do is to introduce as integration variables $x - y$ and $x + y$, integrating only on $x - y$. Or, said in a different way, perform both integrals and dividing up by the integral over $x + y$, which cancels the divergence at the numerator. This can be done in a more systematic way by using the following identity

$$1 = \int_{-\infty}^{+\infty} d\Omega \delta \left(\frac{1}{2}(x + y) + \Omega \right) \tag{8.67}$$

Then we multiply Z per by this factor obtaining

$$Z = \int d\Omega \left[\int dx dy \delta \left(\frac{1}{2}(x + y) + \Omega \right) e^{-\frac{1}{2}a^2(x - y)^2} \right] \tag{8.68}$$

The integral inside the parenthesis is what we are looking for since it is evaluated along the line $x + y = constant$. Of course the result should not depend on the choice of this constant (it corresponds to translate the integration line in a way parallel to the orbits). In fact, by doing the change of variables

$$x \to x + \Omega, \qquad y \to y + \Omega \tag{8.69}$$

189

we get

$$Z = \int d\Omega \left[\int dx dy \delta \left(\frac{1}{2}(x+y) \right) e^{-\frac{1}{2}a^2(x-y)^2} \right] \tag{8.70}$$

As promised the infinite factor is now factorized out and we see also clearly that this nothing but the volume of the gauge group (Ω parameterize the gauge group, which in the present case is isomorphic to R^1). Then, we define our integral as

$$Z' = \frac{Z}{V_G} = \int dx dy \delta \left(\frac{1}{2}(x+y) \right) e^{-\frac{1}{2}a^2(x-y)^2} \tag{8.71}$$

with

$$V_G = \int d\Omega \tag{8.72}$$

In this way the integral is defined by integrating over the particular surface $x+y=0$. Of course we would get the same result integrating over any other line (or surface in the general case) having the property of intersecting each orbit only once. For instance, let us choose the line (see Fig. 8.3.2)

$$f(x,y) = 0 \tag{8.73}$$

Then, proceeding as in eq. (8.67) we consider the following identity, which allows us to define a gauge invariant function $\Delta_f(x,y)$

$$1 = \Delta_f(x,y) \int d\Omega \delta \left[f(x^\Omega, y^\Omega) \right] = \Delta_f(x,y) \int d\Omega \delta \left[f(x+\Omega, y+\Omega) \right] \tag{8.74}$$

In the previous case we had $f(x,y) = (x+y)/2 = 0$ giving rise to $f(x^\Omega, y^\Omega) = (x+y)/2 + \Omega = 0$, from which $\Delta_f(x,y) = 1$. Multiplying Z by the expression (8.74) we get

$$Z = \int d\Omega \left[\int dx dy \Delta_f(x,y) \delta \left[f(x+\Omega, y+\Omega) \right] e^{-\frac{1}{2}a^2(x-y)^2} \right] \tag{8.75}$$

Let us show that $\Delta_f(x,y)$ is gauge invariant. In fact by the transformation $x \to x + \Omega'$, $y \to y + \Omega'$ we obtain

$$1 = \Delta_f(x+\Omega', y+\Omega') \int d\Omega \delta \left[f(x+\Omega+\Omega', y+\Omega+\Omega') \right] \tag{8.76}$$

and changing the integration variable $\Omega \to \Omega + \Omega'$

$$1 = \Delta_f(x+\Omega', y+\Omega') \int d\Omega \delta \left[f(x+\Omega, y+\Omega) \right] \tag{8.77}$$

190

Comparison with eq. (8.74) gives

$$\Delta_f(x, y) = \Delta_f(x + \Omega', y + \Omega') \tag{8.78}$$

The integral inside the parenthesis in eq. (8.75) does not depend on Ω (that is is gauge invariant). This can be shown exactly as we did before in the case $f(x, y) = (x + y)/2$. Therefore we can again factorize the integration over Ω, being the sum of infinite gauge copies

$$Z = \int d\Omega \left[\int dx dy \Delta_f(x, y) \delta\left[f(x, y) \right] e^{-\frac{1}{2}a^2(x - y)^2} \right] \tag{8.79}$$

and define

$$Z' = \frac{Z}{V_G} = \int dx dy \Delta_f(x, y) \delta\left[f(x, y) \right] e^{-\frac{1}{2}a^2(x - y)^2} \tag{8.80}$$

The function $\Delta_f(x, y)$ (called the Faddeev-Popov determinant) can be calculated directly from its definition. To this end notice that in (8.79), $\Delta_f(x, y)$ appears multiplied by $\delta[f(x, y)]$. As a consequence, we need to know Δ_f only at points very close at the surface $f(x, y) = 0$. But in this case, the equation $f(x^\Omega, y^\Omega) = 0$ has the only solution $\Omega = 0$, since by hypothesis the surface $f(x, y)$ crosses the gauge orbits only once. Therefore we can write

$$\delta[f(x^\Omega, y^\Omega)] = \frac{1}{\left[\dfrac{\partial f^\Omega}{\partial \Omega} \right]_{\Omega=0}} \delta[\Omega] \tag{8.81}$$

giving rise to

$$\Delta_f(x, y) = \left. \frac{\partial f^\Omega}{\partial \Omega} \right|_{\Omega=0} \tag{8.82}$$

where $f^\Omega = f(x^\Omega, y^\Omega)$. Clearly Δ_f represents the answer of the gauge fixing $f = 0$ to an infinitesimal gauge transformation. As a last remark, notice that our original integral can be written as

$$Z = \int d^2x e^{-\frac{1}{2}x_i A_{ij} x_j} \tag{8.83}$$

with

$$A = a^2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{8.84}$$

a singular matrix. As we have already noticed the singularity of A is related to the gauge invariance, and the solution we have given here solves also the problem of the singularity of A. In fact, integrating over $x - y$ only, means to integrate only on the eigenvectors of A corresponding to non zero eigenvalues.

Since the solution we have found to our problem is of geometrical type, it is very easy to generalize it to the case of a gauge field theory, abelian or not. We start by choosing a surface in the gauge field space given by

$$f_B(A_B^\mu) = 0, \qquad B = 1, \cdots, n \tag{8.85}$$

where n is the number of parameters of Lie$G$. Also, this surface should intersect the gauge orbits only once. To write down the analogue of equation (8.74) we need also a measure on the gauge group. This can be obtained as follows. We have seen that we are interested in the solutions of

$$f_B(A_\mu^\Omega) = 0 \tag{8.86}$$

around $f_B(A_\mu) = 0$, therefore it is enough to know the measure around the identity of the gauge group. In such a case the transformation Ω can be written as

$$\Omega \approx 1 + i\alpha_A(x)T^A \tag{8.87}$$

Since at the first order in the group parameters

$$\Omega\Omega' \approx 1 + i(\alpha_A + \alpha'_A)T^A \tag{8.88}$$

we see that the invariant measure of the group, around the identity, is given by

$$d\mu(\Omega) = \prod_{A,x} d\alpha_A(x) \tag{8.89}$$

It follows that for infinitesimal transformations

$$d\mu(\Omega\Omega') = d\mu(\Omega'\Omega) = d\mu(\Omega) \tag{8.90}$$

next we define the Faddeev-Popov determinant as in (8.74)

$$1 = \Delta_f[A_\mu] \int d\mu(\Omega)\delta[f_A(A_\mu^\Omega)] \tag{8.91}$$

In this expression we have introduced a functional delta-function, defined by the identity

$$1 = \int d\mu(\Omega)\delta[\Omega] \tag{8.92}$$

having the usual properties extended to the functional case. We can see that Δ_f is a gauge invariant functional, since

$$\Delta_f^{-1}[A_\mu^{\Omega'^{-1}}] = \int d\mu(\Omega)\delta[f_A(A_\mu^{(\Omega'^{-1}\Omega)}] \tag{8.93}$$

and changing variables $\Omega'\Omega = \Omega''$ we get

$$\Delta_f^{-1}(a_\mu^{\Omega'^{-1}}) = \int d\mu(\Omega''\Omega')\delta[f_A(A_\mu^{\Omega''})] = \int d\mu(\Omega)\delta[f_A(A_\mu^\Omega)] \tag{8.94}$$

192

Consider now the naive path integral

$$Z = \int \mathcal{D}(A_\mu) F[A_\mu] e^{i \int \mathcal{L}_A d^4 x} \qquad (8.95)$$

where F is a gauge invariant functional. Since in the following F will not play any particular role we will discuss only the case $F = 1$, but all the following considerations are valid for an arbitrary gauge invariant F. We multiply Z by the identity (8.91)

$$Z = \int d\mu(\Omega) \left[\int \mathcal{D}(A_\mu) e^{i \int \mathcal{L}_A d^4 x} \delta[f_A(A_\mu^\Omega)] \Delta_f[A_\mu] \right] \qquad (8.96)$$

The functional measure is invariant under gauge transformations

$$A_\mu \to \Omega A_\mu \Omega^{-1} + \frac{i}{g} (\partial_\mu \Omega) \Omega^{-1} \qquad (8.97)$$

since the homogeneous part has determinant one, and the inhomogeneous part gives rise to a translation. Then by performing the change of variables $A_\mu \to A_\mu^\Omega$, and using the gauge invariance of $\mathcal{D}(A_\mu)$, $\mathcal{L}_A$ and $\Delta_f[A_\mu]$, we get

$$Z = \int d\mu(\Omega) \left[\int \mathcal{D}(A_\mu) e^{i \int \mathcal{L}_A d^4 x} \delta[f_A(A_\mu)] \Delta_f[A_\mu] \right] \qquad (8.98)$$

Again, we define the functional integral as

$$Z \to \frac{Z}{V_G} = \frac{Z}{\int d\mu(\Omega)} = \int \mathcal{D}(A_\mu) e^{i \int \mathcal{L}_A d^4 x} \delta[f_A(A_\mu)] \Delta_f[A_\mu] \qquad (8.99)$$

The evaluation of Δ_f goes as before. Since in the previous equation it appears multiplied by $\delta[f_A(A_\mu)]$, the only solution of $f_A(A_\mu^\Omega) = 0$ in this neighborhood is $\Omega = 1$, or $\alpha_A = 0$. therefore

$$\delta[f_A(A_\mu^\Omega)] = \det^{-1} \left\| \frac{\delta f_A(A_\mu^\Omega)}{\delta \alpha_B} \bigg|_{\alpha_A = 0} \right\| \delta[\alpha_A] \qquad (8.100)$$

from which

$$\Delta_f[A_\mu] = \det \left\| \frac{\delta f_A(A_\mu^\Omega)}{\delta \alpha_B} \bigg|_{\alpha_A = 0} \right\| \qquad (8.101)$$

Once again, the Faddeev-Popov determinant $\delta_f[A_\mu]$ is the answer of the gauge-fixing to an infinitesimal gauge transformation. The final expression that we get for Z is

$$Z = \int \mathcal{D}(A_\mu) \delta[f_A(A_\mu)] \det \left\| \frac{\delta f_A(A_\mu^\Omega)}{\delta \alpha_B} \bigg|_{\alpha_A = 0} \right\| e^{i \int \mathcal{L}_A d^4 x} \qquad (8.102)$$

193

This expression gives the right measure to be used in the path integral quantization of the gauge fields. In particular, the vacuum expectation value (v.e.v.) of a gauge field functional will be defined as

$$\langle 0|T^*(F[\mathbf{A}])|0\rangle_f = \int \mathcal{D}(A_\mu)\delta[f]\Delta_f e^{iS}F[A] \qquad (8.103)$$

where the index f specifies the gauge in which we evaluate the v.e.v. of F. About this point, we can prove an important result, that is: if $F[A]$ is a gauge invariant functional

$$F[A^\Omega] = F[A] \qquad (8.104)$$

than its v.e.v. does not depend on the gauge fixing $f_A = 0$. We will prove this result by showing the relation between the v.e.v.'s of $F[A]$ in due different gauges. Consider the identity

$$\langle 0|T^*(F[\mathbf{A}])|0\rangle_{f_1} = \int \mathcal{D}(A_\mu)\delta[f_1]\Delta_{f_1}e^{iS}F[A]\int d\mu(\Omega)\delta[f_2^\Omega]\Delta_{f_2} \qquad (8.105)$$

where we have used eq. (8.91)and defined $f_2^\Omega[A] \equiv f_2[A^\Omega]$. By change of variable $A_\mu \to A_\mu^\Omega$, we get

$$\begin{aligned}
\langle 0|T^*(F[\mathbf{A}])|0\rangle_{f_1} &= \int d\mu(\Omega)\int \mathcal{D}(A_\mu)\delta[f_1^{\Omega^{-1}}]\Delta_{f_1}e^{iS}F[A^{\Omega^{-1}}]\delta[f_2]\Delta_{f_2} \\
&= \int d\mu(\Omega)\int \mathcal{D}(A_\mu)e^{iS}\delta[f_2]\Delta_{f_2}\left[\delta[f_1^\Omega]\Delta_{f_1}F[A^\Omega]\right] \quad (8.106)
\end{aligned}$$

where in the second line we have made a further change of variables $\Omega \to \Omega^{-1}$ and used the invariance of $d\mu(\Omega)$. Then

$$\langle 0|T^*(F[\mathbf{A}])|0\rangle_{f_1} = \int d\mu(\Omega)\langle 0|T^*(\delta[f_1^\Omega]\Delta_{f_1}F[\mathbf{A}^\Omega])|0\rangle_{f_2} \qquad (8.107)$$

This is the relation we were looking for. If F is gauge invariant we get at once

$$\langle 0|T^*(F[\mathbf{A}])|0\rangle_{f_1} = \langle 0|T^*(F[\mathbf{A}])|0\rangle_{f_2} \qquad (8.108)$$

For the following we will define a generating functional

$$Z_f[\eta_\mu^A] = \int \mathcal{D}(A_\mu)\delta[f]\Delta_f e^{iS}e^{i\int d^4x\eta_\mu^A A_A^\mu} \qquad (8.109)$$

which is not gauge invariant, but in terms of which we can easily construct the v.e.v.'s of gauge invariant quantities, since it is built up in terms of the correct functional measure.

For the purpose of building up the perturbative theory it is convenient to write the integrand of eq. (8.109) in the form $\exp(iS_{\text{eff}})$, where S_{eff} is an effective action

194

taking into account the gauge fixing $f = 0$ and the Faddeev-Popov determinant. We will discuss later the exponentiation of Δ_f. As far as $\delta[f]$ it is concerned, we can exponentiate it by using its Fourier representation, or, more conveniently by choosing the gauge condition in the form $f_A(A) - B_A(x) = 0$, with $B_A(x)$ a set of arbitrary functions independent on the gauge fields. Then

$$\Delta_{(f-B)}[A] = \det \left\| \frac{\delta\left(f_A[A^\Omega] - B_A\right)}{\delta \alpha_B} \Big|_{\alpha_A=0} \right\| = \det \left\| \frac{\delta f_A[A^\Omega]}{\delta \alpha_B} \Big|_{\alpha_A=0} \right\| = \Delta_f[A] \quad (8.110)$$

and therefore

$$1 = \Delta_f[A] \int d\mu(\Omega) \delta[f_A^\Omega - B_A] \quad (8.111)$$

We can further multiply this identity by the following expression which does not depend on the gauge fields

$$\text{constant} = \int \mathcal{D}(B) e^{-\dfrac{i}{2\beta} \int d^4x \sum_A B_A^2(x)} \quad (8.112)$$

We get

$$
\begin{aligned}
\text{constant} &= \Delta_f[A] \int d\mu(\Omega)\mathcal{D}(B) e^{-\dfrac{i}{2\beta} \int d^4x \sum_A (f_A^\Omega(x))^2} \delta(f_A - B_A) \\
&= \Delta_f[A] \int d\mu(\Omega) e^{-\dfrac{i}{2\beta} \int d^4x \sum_A (f_A^\Omega(x))^2} \quad (8.113)
\end{aligned}
$$

Multiplying this expression by the functional (8.95)we can show as before that the gauge volume factor factorizes, allowing us to define a class of generating functionals

$$Z_\beta[\eta_\mu^A] = \int \mathcal{D}(A_\mu)\Delta_f[A] e^{i \int d^4x (\mathcal{L}_A - \frac{1}{2\beta} \sum_A (f_A)^2)} e^{i \int d^4x \eta_\mu^A A_A^\mu} \quad (8.114)$$

Finally, defining

$$S_f = -\frac{1}{2\beta} \int d^4x \sum_A (f_A(x))^2 \quad (8.115)$$

we find the analogue of eq. (8.107) is

$$\langle 0|T^*(F[\mathbf{A}])|0\rangle_{f_1} = \int d\mu(\Omega)\langle 0|T^*(\Delta_{f_1} e^{iS_{f_1}^\Omega} F[\mathbf{A}^\Omega])|0\rangle_{f_2} \quad (8.116)$$

showing again that the v.e.v. a gauge invariant functional gives rise to a gauge invariant v.e.v. for the corresponding operator.

8.4 Path integral quantization of QED

Let us now discuss the path integral quantization of QED in a covariant gauge. We start from the lagrangian

$$\mathcal{L} = \bar{\psi}\left[i\gamma^{\mu}(\partial_{\mu} + ieA_{\mu}) - m\right]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{8.117}$$

with

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \tag{8.118}$$

invariant under

$$
\begin{aligned}
\psi(x) &\rightarrow \psi'(x) = e^{-ie\alpha(x)}\psi(x) \\
A_{\mu}(x) &\rightarrow A'_{\mu}(x) = A_{\mu}(x) + \frac{1}{e}\partial_{\mu}\alpha(x)
\end{aligned}
\tag{8.119}
$$

The Lorentz gauge is defined by

$$f(A_{\mu}) = \partial^{\mu}A_{\mu}(x) = 0 \tag{8.120}$$

and

$$f(A_{\mu}^{\Omega}) = \partial^{\mu}A_{\mu}(x) + \frac{1}{e}\Box\alpha(x) \tag{8.121}$$

with

$$\left.\frac{\delta f(A_{\mu}^{\Omega})}{\delta\alpha(y)}\right|_{\alpha=0} = \frac{1}{e}\Box_{x}\delta^{4}(x-y) \tag{8.122}$$

Then the Faddeev-Popov determinant turns out to be field-independent and it can be absorbed into the normalization of the generating functional, which is given by

$$Z[\eta,\bar{\eta},\eta_{\mu}] = N\int\mathcal{D}(\psi,\bar{\psi})\mathcal{D}(A_{\mu})e^{i\int d^{4}x\mathcal{L}}\delta[\partial_{\mu}A^{\mu}]e^{i\int[\bar{\eta}\psi + \bar{\psi}\eta + \eta_{\mu}A^{\mu}]d^{4}x} \tag{8.123}$$

We extract the interaction term in the usual way

$$Z[\eta,\bar{\eta},\eta_{\mu}] = e^{-ie\int d^{4}x\left(-\frac{1}{i}\frac{\delta}{\delta\eta}\right)\gamma^{\mu}\left(\frac{1}{i}\frac{\delta}{\delta\eta^{\mu}}\right)\left(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}}\right)}Z_{0}[\eta,\bar{\eta},\eta_{\mu}] \tag{8.124}$$

with

$$Z_{0}[\eta,\bar{\eta},\eta_{\mu}] = N\int\mathcal{D}(\psi,\bar{\psi})\mathcal{D}(A_{\mu})e^{i\int d^{4}x\mathcal{L}_{0}}\delta[\partial_{\mu}A^{\mu}]e^{i\int[\bar{\eta}\psi + \bar{\psi}\eta + \eta_{\mu}A^{\mu}]d^{4}x} \tag{8.125}$$

Writing

$$S_{A} = -\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}d^{4}x = \frac{1}{2}\int d^{4}x\; A_{\mu}(g^{\mu\nu}\Box - \partial^{\mu}\partial^{\nu})A_{\nu} \tag{8.126}$$

using $\delta[\partial_\mu A^\mu]$ and integrating over the Fermi fields we get

$$
\begin{aligned}
Z_0[\eta, \bar{\eta}, \eta_\mu] \ &= \ e^{-i\langle \bar{\eta}(x) S_F(x-y)\eta(y)\rangle} \\
&\cdot N \int \mathcal{D}(A_\mu)\delta[\partial_\mu A^\mu]e^{\frac{i}{2}\langle A_\mu(x)\Box A^\mu(x)\rangle}e^{i\langle \eta_\mu(x) A^\mu(x)\rangle}
\end{aligned}
\tag{8.127}
$$

Then we exponentiate the $\delta[\partial_\mu A^\mu]$ through its Fourier transform

$$
\delta[\partial_\mu A^\mu] = \int \mathcal{D}(C)e^{i\langle C(x)\partial_\mu A^\mu(x)\rangle} = \int \mathcal{D}(C)e^{-i\langle \partial_\mu C(x)A^\mu(x)\rangle}
\tag{8.128}
$$

obtaining the following expression for Z_0

$$
Z_0[\eta, \bar{\eta}, \eta_\mu] = e^{-i\langle \bar{\eta}(x)S_F(x-y)\eta(y)\rangle}
$$
$$
\cdot N \int \mathcal{D}(A_\mu)\mathcal{D}(C)e^{i\langle \frac{1}{2}A_\mu(x)\Box A^\mu(x) + (\eta_\mu(x) - \partial_\mu C(x))A^\mu(x)\rangle}
\tag{8.129}
$$

Doing the gaussian integral over A_μ we get

$$
\begin{aligned}
Z_0[\eta, \bar{\eta}, \eta_\mu] \ &= \ e^{-i\langle \bar{\eta}(x)S_F(x-y)\eta(y)\rangle} \\
&\cdot N \int \mathcal{D}(C)e^{\frac{1}{2}\langle i(\eta_\mu(x) - \partial_\mu C(x))\left(\frac{i}{\Box}\right)_{x,y} i(\eta^\mu(y) - \partial^\mu C(y))\rangle} \\
&= \ e^{-i\langle \bar{\eta}(x)S_F(x-y)\eta(y)\rangle}e^{-\frac{i}{2}\langle \eta_\mu \frac{1}{\Box}\eta^\mu\rangle} \\
&\cdot \int \mathcal{D}(C)e^{i\langle \partial_\mu C \frac{1}{\Box}\eta^\mu\rangle - \frac{i}{2}\langle \partial_\mu C \frac{1}{\Box}\partial^\mu C\rangle}
\end{aligned}
\tag{8.130}
$$

After integrating by parts, we can rewrite the integral over $C(x)$ as

$$
\int \mathcal{D}(C)e^{-i\langle C \frac{1}{\Box}\partial_\mu \eta^\mu\rangle + \frac{i}{2}\langle CC\rangle}
\tag{8.131}
$$

and doing again this gaussian integral we find

$$
e^{\frac{1}{2}\langle \left(-\frac{i}{\Box}\partial_\mu \eta^\mu\right)(+i)\left(-\frac{i}{\Box}\partial_\nu \eta^\nu\right)\rangle} = e^{\frac{i}{2}\langle \eta^\mu \frac{\partial_\mu \partial_\nu}{\Box^2}\eta^\nu\rangle}
\tag{8.132}
$$

The final result is

$$
Z_0[\eta, \bar{\eta}, \eta_\mu] = Z[0]e^{-i\langle \bar{\eta}(x)S_F(x-y)\eta(y)\rangle}e^{-\frac{i}{2}\langle \eta^\mu \left(\frac{g_{\mu\nu}}{\Box} - \frac{\partial_\mu \partial_\nu}{\Box^2}\right)\eta^\nu\rangle}
\tag{8.133}
$$

197

Introducing the function

$$G_{\mu\nu} = \lim_{\epsilon \to 0^+} \int \frac{d^4k}{(2\pi)^4} \left[-\frac{g_{\mu\nu}}{k^2 + i\epsilon} + \frac{k_\mu k_\nu}{(k^2 + i\epsilon)^2} \right] e^{-ikx} \tag{8.134}$$

we write

$$Z_0[\eta, \bar\eta, \eta_\mu] = Z[0] e^{-i\langle \bar\eta(x) S_F(x-y)\eta(y)\rangle} e^{-\frac{i}{2}\langle \eta^\mu(x) G_{\mu\nu}(x-y)\eta^\nu(y)\rangle} \tag{8.135}$$

The photon propagator in this gauge is given by

$$\frac{1}{\langle 0|0\rangle} \langle 0|T(A_\mu(x) A_\nu(x))|0\rangle = i G_{\mu\nu}(x-y) \tag{8.136}$$

Notice that this propagator is transverse, that is it satisfies the Lorentz condition

$$\partial^\mu G_{\mu\nu}(x) = 0 \tag{8.137}$$

The Feynman's rules in this gauge are the same discussed in Section 1.5.3, except for th photon propagator which is

$$-i \left[\frac{g_{\mu\nu}}{k^2 + i\epsilon} - \frac{k_\mu k_\nu}{(k^2 + i\epsilon)^2} \right] \tag{8.138}$$

Of course, the additional term in the propagator proportional to $k_\mu k_\nu$ does not affect the physics since the photon is coupled to a conserved current.

We can perform an analogous calculation but using the gauge fixing in the form $f - B$ discussed at the end of the previous Section, that is using the generating functional (8.114)

$$\begin{aligned}
Z_\beta[\eta, \bar\eta, \eta_\mu] &= N \int \mathcal{D}(\psi, \bar\psi)\mathcal{D}(A_\mu) e^{i\int d^4x \left[\mathcal{L} - \frac{1}{2\beta}(\partial_\mu A^\mu)^2 \right]} \\
&\cdot e^{\int d^4x \left[\bar\eta\psi + \bar\psi\eta + \eta_\mu A^\mu \right]}
\end{aligned} \tag{8.139}$$

In this case the only effect of the gauge fixing is to change the lagrangian $\mathcal{L}$ to $\mathcal{L}' = \mathcal{L} - (\partial_\mu A^\mu)^2/(2\beta)$. Of course, the addition of this term removes the problem of the singular wave operator. In fact we can write

$$\int d^4x \mathcal{L}' = \frac{1}{2} \int d^4x A^\mu \left(g_{\mu\nu}\Box - (1 - \frac{1}{\beta})\partial_\mu\partial_\nu \right) A^\nu \tag{8.140}$$

Therefore we have to evaluate the functional integral

$$Z_0[\eta, \bar\eta, \eta_\mu] = N e^{-i\langle \bar\eta(x) S_F(x-y)\eta(y)\rangle} \int \mathcal{D}(A_\mu) e^{i\int d^4x \left(\frac{1}{2} A^\mu K_{\mu\nu} A^\nu + \eta_\mu A^\mu \right)} \tag{8.141}$$

198

with

$$K_{\mu\nu} = g_{\mu\nu}\Box - \frac{\beta - 1}{\beta}\partial_\mu\partial_\nu \tag{8.142}$$

obtaining

$$Z_\beta[\eta, \bar\eta, \eta_\mu] = Z[0]e^{-i\langle\bar\eta(x)S_F(x-y)\eta(y)\rangle}e^{-\frac{i}{2}\langle\eta_\mu(x)G^{\mu\nu}(x-y)\eta^\nu(y)\rangle} \tag{8.143}$$

where

$$K_{\mu\nu}G^{\nu\rho}(x) = \delta_\mu^\rho\delta^4(x) \tag{8.144}$$

We can see that $K_{\mu\nu}$ is non singular for $\beta \neq \infty$. Going to momentum space, eq. (8.144) gives

$$\int \frac{d^4k}{(2\pi)^4}\left(-g_{\mu\nu}k^2 + \frac{\beta-1}{\beta}k_\mu k_\nu\right)G^{\nu\rho}(k)e^{-ikx} = \delta_\mu^\rho\int\frac{d^4k}{(2\pi)^4}e^{-ikx} \tag{8.145}$$

or

$$\left(-g_{\mu\nu}k^2 + \frac{\beta-1}{\beta}k_\mu k_\nu\right)G^{\nu\rho}(k) = \delta_\nu^\rho \tag{8.146}$$

This can be solved by writing the most general second rank symmetric tensor function of k_μ

$$G_{\mu\nu}(k) = Ag_{\mu\nu} + Bk_\mu k_\nu \tag{8.147}$$

with the coefficients such that

$$-Ak^2 = 1, \qquad \frac{\beta-1}{\beta}A - \frac{1}{\beta}Bk^2 = 0 \tag{8.148}$$

Solving these equations

$$G_{\mu\nu}(k) = -\frac{g_{\mu\nu}}{k^2} + (1-\beta)\frac{k_\mu k_\nu}{k^4} \tag{8.149}$$

The singularities are defined by the usual $i\epsilon$ term, and we get

$$G_{\mu\nu}(x) = \lim_{\epsilon\to 0^+}\int\frac{d^4k}{(2\pi)^4}\left[-\frac{g_{\mu\nu}}{k^2+i\epsilon} + (1-\beta)\frac{k_\mu k_\nu}{(k^2+i\epsilon)^2}\right]e^{-ikx} \tag{8.150}$$

The discussion made at the beginning of this Section corresponds to $\beta = 0$ (the so called Landau gauge). In fact for $\beta \to 0$

$$\lim_{\beta\to 0}e^{-\frac{i}{2\beta}\int d^4x(\partial_\mu A^\mu)^2} \to \delta[\partial_\mu A^\mu] \tag{8.151}$$

The choice corresponding to the quantization made in Section 1.5.3 is $\beta = 1$. In this case

$$G_{\mu\nu}(x) = -g_{\mu\nu}\Delta_F(x; m^2 = 0) \tag{8.152}$$

199

This is called the Feynman's gauge. As we see the parameter β selects different covariant gauges. As already stressed the physics does not depend on β since the photon couples to the fermionic current $\bar{\psi}\gamma^\mu\psi$, which is conserved in the abelian case, and as a consequence the β dependent part of the propagator, being proportional to $k_\mu k_\nu$ does not contribute. The situation is quite different in the non-abelian case where the fermionic current is not conserved since the gauge fields are not neutral. Therefore we need further contributions in order to cancel the β dependent terms. Such contributions arise from the Faddeev-Popov determinant.

8.5 Path integral quantization of the non-abelian gauge theories

In the case of a non-abelian gauge theory there are several differences with respect to QED. The pure gauge lagrangian is not quadratic, it contains three-linear and four-linear interaction terms and furthermore the Faddeev-Popov determinant typically depends on the gauge fields. In this Section we will consider the pure gauge case, since the fermionic coupling is completely analogue to the abelian case. Separating the lagrangian in the quadratic part plus the rest, we get

$$
\begin{aligned}
\mathcal{L}_A &= -\frac{1}{4}\sum_C F_{\mu\nu C}F_C^{\mu\nu} \\
&= -\frac{1}{4}\left(\partial_\mu A_{\nu C} - \partial_\nu A_{\mu C} - gf_C^{AB}A_{\mu A}A_{\nu B}\right)\left(\partial^\mu A_C^\nu - \partial^\nu A_C^\mu - gf_C^{DE}A_D^\mu A_E^\nu\right) \\
&= \mathcal{L}_A^{(2)} + gf_C^{AB}A_{\mu A}A_{\nu B}\partial^\mu A_C^\nu - \frac{1}{4}g^2 f_C^{AB}f_C^{DE}A_A^\mu A_B^\nu A_{\mu D}A_{\nu E} \\
&\equiv \mathcal{L}_A^{(2)} + \mathcal{L}_A^I
\end{aligned}
\tag{8.153}
$$

where $\mathcal{L}_A^{(2)}$ and $\mathcal{L}_A^{(I)}$ are the quadratic and the interacting part respectively. The interaction terms can be extracted by the functional integration by the usual trick

$$
Z[\eta_\mu] = e^{\,iS_A^{(I)}\left[\frac{1}{i}\frac{\delta}{\delta\eta_\mu^A}\right]} Z_0[\eta_\mu^A]
\tag{8.154}
$$

with $S_A^{(I)} = \int d^4x \mathcal{L}_A^{(I)}$ and

$$
Z_0[\eta_\mu] = N\int \mathcal{D}(A_\mu)e^{\,i\int d^4x\left[\mathcal{L}_A^{(2)} - \frac{1}{2\beta}\sum_A(\partial_\mu A_A^\mu)^2\right]} e^{\,i\int d^4x\eta_A^\mu A_\mu^A}\Delta_f[A_\mu]
\tag{8.155}
$$

Consider now $\Delta_f[A_\mu]$. In the Lorentz gauge

$$
f_A[A_\mu] = \partial_\mu A_A^\mu
\tag{8.156}
$$

Under an infinitesimal gauge transformation (for later convenience we have changed the definition of the gauge parameters, sending $\alpha_A \to g\alpha_A$)

$$\delta A_C^\mu = gf_C^{AB}\alpha_A A_B^\mu + \partial^\mu \alpha_C \tag{8.157}$$

we have

$$f_C[A_\mu^\Omega] = \partial_\mu A_C^\mu + gf_C^{AB}\partial_\mu(\alpha_A A_B^\mu) + \Box\alpha_C \tag{8.158}$$

Then

$$\left.\frac{\delta f_C[A_\mu^\Omega(x)]}{\delta\alpha_B(y)}\right|_{\alpha_A=0} = \delta_{BC}\Box_x^2\delta^4(x-y) + gf_C^{BA}\partial_\mu(\delta^4(x-y)A_A^\mu) = M_{CB}(x-y) \tag{8.159}$$

We see that $\Delta_f[A]$ depends explicitly on the gauge fields. In order to get an expression for the generating functional which is convenient fir deriving the Feynman rules, it is useful to give to it an exponential form. This can be done, recalling from eq. (7.76)) that a determinant can be written as a gaussian integral over Grassmann variables.

$$\Delta_f[A_\mu] = \det\left\|\left.\frac{\delta f_A[A_\mu^\Omega(x)]}{\delta\alpha_B(y)}\right|_{\alpha_A=0}\right\| = N\int \mathcal{D}(c,c^*)e^{i\int c_A^*(x)M_{AB}(x-y)c_B(y)d^4x d^4y} \tag{8.160}$$

The Grassmann fields $c_A(x)$ are called the ghost fields. It should be noticed that the ghost carry zero spin and therefore in their particle interpretation lead to a violation of the spin-statistics theorem. However this fact has no consequence, since our generating functional do not generate amplitudes with external ghosts. The action for the ghost fields can be read from the previous equation and we get

$$S_{FPG} = \int d^4x\, c_A^*(x)\left[\delta^{AB}\Box - gf_B^{AC}\partial_\mu A_C^\mu\right]c_B(x) \tag{8.161}$$

The ghosts are zero mass particles interacting with the gauge fields through the interaction term

$$S_{FPG}^{(I)} = -g\int d^4x\, c_A^* f_B^{AC}\partial_\mu A_C^\mu c_B \tag{8.162}$$

In order to deal with this interaction it turns out convenient to define a new generating functional where we introduce also external sources for the ghost fields

$$Z[\eta,\eta*,\eta_\mu] = e^{iS_A^{(I)}\left[\frac{1}{i}\frac{\delta}{\delta\eta_\mu}\right]}e^{iS_{FPG}^{(I)}\left[-\frac{1}{i}\frac{\delta}{\delta\eta},\frac{1}{i}\frac{\delta}{\delta\eta^*}\right]}Z_0[\eta,\eta^*,\eta_\mu] \tag{8.163}$$

with

$$
\begin{aligned}
Z_0[\eta,\eta^*,\eta_\mu] = & \int \mathcal{D}(A_\mu)\mathcal{D}(c,c^*) \\
& \cdot e^{i\int d^4x\left[\mathcal{L}_A^{(2)} - \frac{1}{2\beta}\sum_A(\partial_\mu A_A^\mu)^2 + \eta_\mu^A A_A^\mu\right]} \\
& \cdot e^{i\int d^4x\left[c_A^*\Box c_A + c_A^*\eta^A + \eta^{A^*}c_A\right]}
\end{aligned} \tag{8.164}
$$

The integration over A_A^μ like in the abelian case. For the ghost field we use eq. (7.76) with the result

$$\int \mathcal{D}(c, c^*) e^{i\langle c_A^* \Box c_A + c_A^* \eta^A + \eta^{A^*} c_A\rangle} = e^{-\langle (i\eta^{A^*})\left(-\frac{i}{\Box}\right)(i\eta^A)\rangle} = e^{-i\langle \eta^{A^*} \frac{1}{\Box} \eta^A\rangle}$$

(8.165)

Defining

$$\Delta_{AB}(x) = -\delta_{AB} \lim_{\epsilon \to 0^+} \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-ikx}}{k^2 + i\epsilon}$$

(8.166)

we get

$$Z_0[\eta, \eta^*, \eta_\mu] = e^{-\frac{i}{2}\langle \eta_\mu^A(x) G_{AB}^{\mu\nu}(x-y)\eta_B^\nu(y)\rangle} e^{-i\langle \eta^{A^*}(x)\Delta_{AB}(x-y)\eta^B(y)\rangle}$$

(8.167)

with

$$G_{AB}^{\mu\nu} = \delta_{AB} G^{\mu\nu}(x-y)$$

(8.168)

It is then easy to read the Feynman's rules. For the propagators we get

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Fig. 8.5.1 - *Gauge field propagator*:
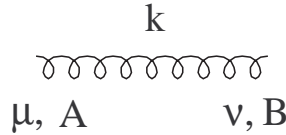
$$-i\delta_{AB}\left(g_{\mu\nu} - (1-\beta)\frac{k_\mu k_\nu}{k^2 + i\epsilon}\right)\frac{1}{k^2 + i\epsilon}$$

(8.169)

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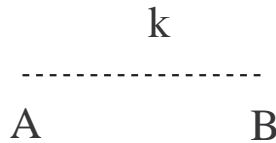
Fig. 8.5.2 - *Ghost field propagator*:

$$i\delta_{AB}\frac{1}{k^2 + i\epsilon}$$

(8.170)

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The Feynman's rules for the vertices are obtained by taking the Fourier transform of the interaction parts (here we have written $f_C^{AB} \equiv f_{ABC}$).

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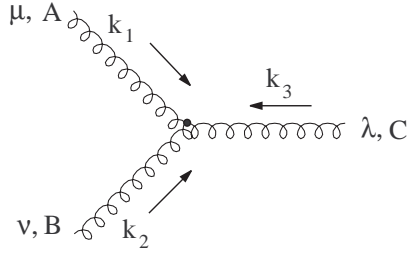
Fig. 8.5.3 - *Three-linear gluon vertex*:

$$g f_{ABC}(2\pi)^4 \delta^4(k_1 + k_2 + k_3) \left[g_{\mu\nu}(k_1 - k_2)_\lambda + g_{\nu\lambda}(k_2 - k_3)_\mu + g_{\lambda\mu}(k_3 - k_1)_\nu \right] \quad (8.171)$$

.
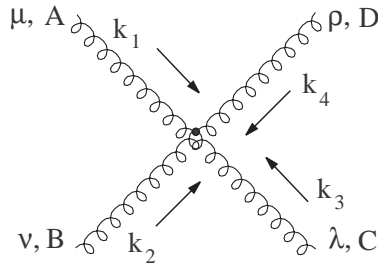
Fig. 8.5.4 - *Four-linear gluon vertex*:

$$-ig^2(2\pi)^4 \delta^4(\sum_{i=1}^{4} k_i) \Big[f_{ABE} f_{CDE}(g_{\mu\lambda} g_{\nu\rho} - g_{\nu\lambda} g_{\mu\rho})$$

$$+ \quad f_{ACE} f_{BDE}(g_{\mu\nu} g_{\lambda\rho} - g_{\lambda\nu} g_{\mu\rho}) + f_{ADE} f_{CBE}(g_{\mu\lambda} g_{\rho\nu} - g_{\rho\lambda} g_{\mu\nu}) \Big] \quad (8.172)$$

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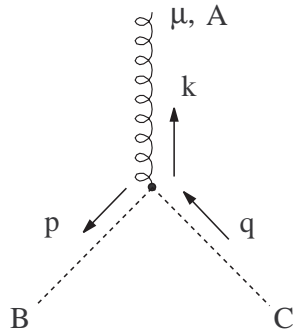
Fig. 8.5.5 - *Ghost-gluon vertex*:

$$-g f_{ABC}(2\pi)^4 \delta^4(k + p - q) p_\mu \quad\quad\quad (8.173)$$

When we have fermions coupled to the gauge fields we need to add the rule for the vertex (the propagator is the usual one). By a simple comparison with the coupling in QED we see that in this case the rule for the vertex is

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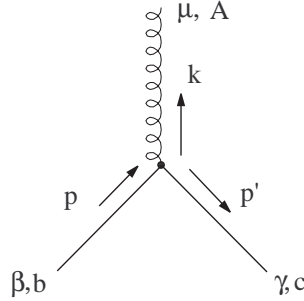
Fig. 8.5.6 - *Fermion-gluon vertex*:

$$-igT^A_{bc}(\gamma_\mu)_{\beta\gamma}(2\pi)^4\delta^4(p - p' - k) \tag{8.174}$$

8.6 The β-function in non-abelian gauge theories

We will skip a careful treatment of the renormalization in non-abelian gauge theories. However, since all the interaction terms have dimensions ≤ 4, the theory is renormalizable by power counting. On the other hand we will evaluate the *beta* function, since in these theories there is the possibility of having $\beta < 0$, giving rise to asymptotic freedom. As it is known, QCD the theory of strong interactions is in fact a gauge theory based on the gauge group $SU(3)$, and it is precisely ths feature of being an asymptotically free theory that has made it very attractive phenomenologically. In non-abelian gauge theories there are several ways to define the coupling, in fact we can look at the fermionic coupling, at the trilinear or at the four-linear couplings. If the theory is gauge invariant (and the renormalization should preserve this feature), all these ways should give the same result. This is in fact ensured by a set of identities among the renormalization constants (Ward identities), which are the analogue of the equality $Z_1 = Z_2$ in QED. Therefore we will study the definition coming from the fermionic coupling. In this case the relation between the bare coupling and the renormalized one is the same as in QED (see Section 6.3))

$$g_B = \mu^{\epsilon/2}g\frac{Z_1}{Z_2Z_3^{1/2}} \tag{8.175}$$

where Z_1 is the vertex renormalization constant, and $Z_@$ and Z_3 are the wave function renormalization constants for the fermion and the gauge particle respectively.

As we shall see, one of the differences with QED is that now $Z_1 \neq Z_2$. As a consequence we will need to evaluate all the three renormalization constants. Let us start with the fermion self-energy. The one-loop contribution is given in Fig. 8.6.1.

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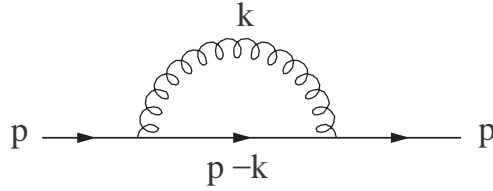
Fig. 8.6.1 - *One-loop fermion self-energy.*

We will work in the Feynman-'t Hooft gauge, corresponding to the choice $\beta = 1$ for the gauge field propagator (see eq. (8.169)). The Feynman rules for this diagram are the same as in QED except for the group factors. In order to avoid group theory complications, we will consider the case of a gauge group $SU(2)$ with fermions in the spinor representation, where all the group factors are easily evaluated. At the same time we will exhibit also the general results for a theory $SU(N)$ with fermions in the fundamental representation (that is the one of dimensions N). Then, recalling the expression for the electron self-energy in QED (see eq. (2.166)), we get

$$\Sigma(p) = \sum_A (T^A T^A) \Sigma_{\text{QED}}(p) = \sum_A (T^A T^A) \left[\frac{1}{8\pi^2\epsilon}(\hat{p} - 4m) + \Sigma_{\text{QED}}^f(p) \right] \quad (8.176)$$

For $SU(2)$ we have

$$\sum_A (T^A T^A)_{ab} = \sum_A \left(\frac{\tau^A}{2} \frac{\tau^A}{2} \right)_{ab} = \frac{3}{4}\delta_{ab} \quad (8.177)$$

In the case of a group $SU(N)$ and fermions in the fundamental representation it is possible to show that

$$\sum_A (T^A T^A)_{ab} = C_2(F)\delta_{ab} \quad (8.178)$$

with

$$C_2(F) = \frac{N^2 - 1}{2N} \quad (8.179)$$

This result is justified by the observation that the operator $\sum_A T^A T^A$ (called the Casimir operator) commutes with all the generators and therefore it must be proportional to the identity matrix. From the previous expression for Σ we find immediately (see Section 2.7)

$$Z_2 = 1 - \frac{g^2}{8\pi^2\epsilon} C_2(F) \quad (8.180)$$
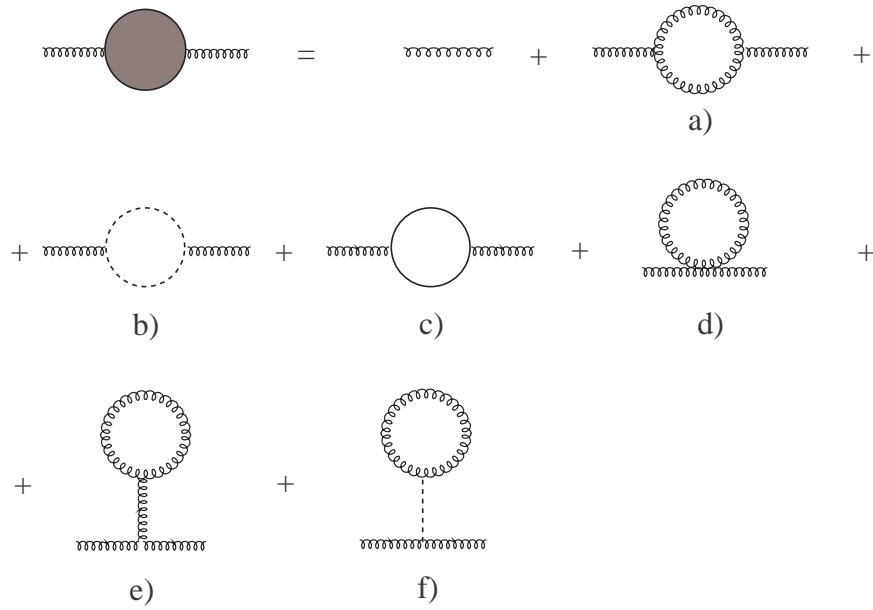
Fig. 8.6.2 - *One-loop expansion of the vacuum polarization diagram in non-abelian gauge theories.*

The next task is the evaluation of Z_3. To this end we need the one-loop expansion of the gauge particle propagator (the vacuum polarization diagram). This is given in Fig. 8.6.2.

Since we are using dimensional regularization we can see immediately that the last three diagrams (diagrams d), e) and f)) of Fig. 8.6.2 are zero (this is not true in other regularizations). In fact all these diagrams involve the integral

$$\int \frac{d^{2\omega}k}{k^2} \tag{8.181}$$

which, in dimensional regularization gives (including a mass term)

$$\int \frac{d^{2\omega}k}{k^2 - a^2} = -i\pi^\omega \Gamma(1-\omega)(a^2)^{\omega-1} \tag{8.182}$$

By taking the limit $a \to 0$, we see that for $\omega > 1$ the integral vanishes.

Let us begin our calculation from the gauge particle contributions to the vacuum polarization. The kinematics is specified in Fig. 8.6.3. We will evaluate the truncated Green's function which differs by a factor ig^2 from the vacuum polarization vacuum $\Pi_{\mu\nu}$ as defined in Section 2.1. We have

$$ig^2\Pi_{\mu\nu,AB}^{(a)} = \frac{1}{2}(-g)^2 f^{ADC} f^{DBC} \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} E_{\mu\nu} \frac{-i}{(p+k)^2} \frac{-i}{k^2} \tag{8.183}$$

The factor $1/2$ is a symmetry factor associated to the diagram and the tensor $E_{\mu\nu}$ comes from the trilinear vertices. It is given by

$$
\begin{aligned}
E_{\mu\nu} &= \left[g^\sigma_\mu (p-k)^\rho + g^{\sigma\rho}(2k+p)_\mu - g^\rho_\mu(2p+k)^\sigma \right] \\
&\times \left[g_{\sigma\nu}(p-k)_\rho - g_{\nu\rho}(2p+k)_\sigma + g_{\rho\sigma}(2k+p)_\nu \right] \\
&= g_{\mu\nu}\left[(p-k)^2 + (2p+k)^2 \right] + 2\omega(2k+p)_\mu(2k+p)_\nu \\
&- \left[(2p+k)_\mu(2k+p)_\nu - (2k+p)_\mu(p-k)_\nu + (p-k)_\mu(2p+k)_\nu \right]
\end{aligned}
\tag{8.184}
$$

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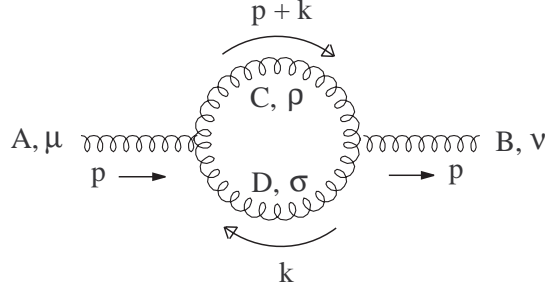
Fig. 8.6.3 - *One-loop gauge particle contribution to the vacuum polarization.*

The group factor is easily evaluated for $SU(2)$, since in this case $f^{ABC} = \epsilon_{ABC}$. Then we get

$$
\epsilon_{ADC}\epsilon_{DBC} = -2\delta_{AB} \tag{8.185}
$$

The general result for $SU(N)$ is (taking into account that f^{ABC} are completely antisymmetric)

$$
f^{ADC}f^{DBC} = -C_2(G)\delta_{AB} \tag{8.186}
$$

with

$$
C_2(G) = N \tag{8.187}
$$

This result can be understood by noticing that $(T^A)_{BC} = if^{ABC}$ gives a representation of the Lie algebra of the group (the adjoint representation), and therefore $C_2(G)$ is nothing but the Casimir of the adjoint. By using the standard trick we reduce the two denominators in the integral to a single one

$$
\frac{1}{k^2}\frac{1}{(k+p)^2} = \int_0^1 dz \frac{1}{(k'^2 + z(1-z)p^2)^2} \tag{8.188}
$$

with

$$
k = k' - pz \tag{8.189}
$$

Operating this substitution into the tensor $E_{\mu\nu}$, taking into account that the integration over the odd terms in k' gives zero, and that for symmetry we can make the substitution

$$
k'_\mu k'_\nu \to \frac{g_{\mu\nu}}{2\omega}k'^2 \tag{8.190}
$$

207

we get (by calling k'^2 with k^2)

$$
\begin{aligned}
E_{\mu\nu} &= g_{\mu\nu}k^2 6\left(1 - \frac{1}{2\omega}\right) + g_{\mu\nu}p^2\left[(1+z)^2 + (2-z)^2\right] \\
&+ p_\mu p_\nu\left[2\omega(1-2z)^2 - 2\left((1+z)^2 + (2-z)(1-2z)\right)\right] \\
&\equiv a_{\mu\nu}k^2 + b_{\mu\nu}
\end{aligned}
\tag{8.191}
$$

Performing the integral over k with the formulas of Section 2.5 and putting everything together, we get

$$
ig^2\Pi^{(a)}_{\mu\nu,AB} = -\frac{1}{2}g^2(-C_2(G))\delta_{AB}\frac{i\pi^\omega}{(2\pi)^{2\omega}}\int_0^1 dz\left[\omega z(1-z)p^2\Gamma(1-\omega)a_{\mu\nu} + \Gamma(2-\omega)b_{\mu\nu}\right]
\tag{8.192}
$$

By taking only the $1/\epsilon$ term ($\epsilon = 4 - 2\omega$) in the expansion and integrating over z we get

$$
ig^2\Pi^{(a)}_{\mu\nu,AB} = i\frac{g^2}{16\pi^2\epsilon}C_2(G)\delta_{AB}\left[\frac{19}{6}g^{\mu\nu}p^2 - \frac{11}{3}p^\mu p^\nu\right]
\tag{8.193}
$$

As we see this contribution is not transverse, that is it is not proportional to $(p_\mu p_\nu - g_{\mu\nu}p^2)$, as it should be. in fact, we will see that adding to this diagram the contribution of the ghost loop we recover the transverse factor. The ghost contribution is the part b) of the expansion of Fig. 8.6.2. The relevant kinematics is given in Fig. 8.6.4.

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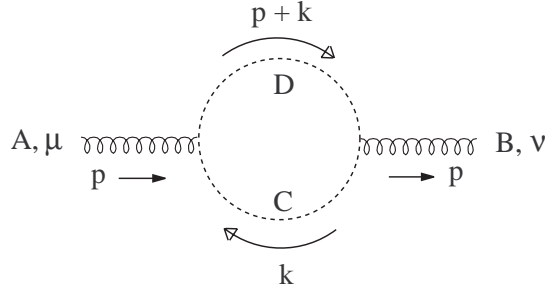
Fig. 8.6.4 - *One-loop ghost contribution to the vacuum polarization.*

We have

$$
ig^2\Pi^{(b)}_{\mu\nu,AB} = (-1)g^2 f^{ACD}f^{BDC}\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{i}{(p+k)^2}\frac{i}{k^2}(p+k)_\mu k_\nu
\tag{8.194}
$$

By the standard procedure we get ($k = k' - pz$))

$$
ig^2\Pi^{(b)}_{\mu\nu,AB} = -g^2 C_2(G)\delta_{AB}\int_0^1 dz\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{k'_\mu k'_\nu - p_\mu p_\nu z(1-z)}{(k'^2 + z(1-z)p^2)^2}
\tag{8.195}
$$

where we have eliminated the terms linear in k'. Then, we perform the integral over k'

$$ig^2\Pi^{(b)}_{\mu\nu,AB} = -ig^2\frac{\pi^\omega}{(2\pi)^{2\omega}}C_2(G)\delta_{AB}\int_0^1 dz\, z(1-z)\left[\frac{1}{2}p^2\Gamma(1-\omega) - p_\mu p_\nu\Gamma(2-\omega)\right]$$
(8.196)

We perform now the integral in z. Then taking the limit $\omega \to 2$, and keeping only the leading term, we get

$$ig^2\Pi^{(b)}_{\mu\nu,AB} = i\frac{g^2}{16\pi^2\epsilon}C_2(G)\delta_{AB}\left[\frac{1}{6}p^2 g_{\mu\nu} + \frac{1}{3}p_\mu p_\nu\right]$$
(8.197)

Putting together this contribution with the former one we get

$$ig^2(\Pi^{(a)}_{\mu\nu,AB} + \Pi^{(b)}_{\mu\nu,AB}) = -i\frac{g^2}{8\pi^2\epsilon}\frac{5}{3}C_2(G)\delta_{AB}\left[p_\mu p_\nu - g_{\mu\nu}p^2\right]$$
(8.198)

The fermionic loop contribution is given in Fig. 8.6.5, and except for the group factor is the same evaluated in Section 2.6.

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$p+k$

A, μ B, ν

$p \longrightarrow$ $\longrightarrow p$

k
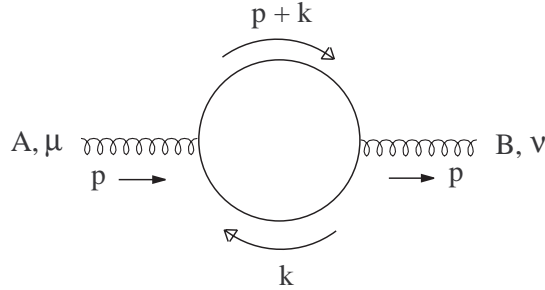
Fig. 8.6.5 - *One-loop fermion contribution to the vacuum polarization.*

The leading contribution is

$$ig^2\Pi^{(c)}_{\mu\nu,AB} = ig^2 Tr(T^A T^B)\Pi^{QED}_{\mu\nu} = i\frac{g^2}{12\pi^2\epsilon}n_F\delta_{AB}\left[p_\mu p_\nu - g_{\mu\nu}p^2\right]$$
(8.199)

where we have used the conventional normalization for the group generators in the fermionic representation

$$Tr(T^A T^B) = \frac{1}{2}n_F\delta_{AB}$$
(8.200)

where n_F is the number of fundamental representations. For instance, in the case of QCD, we have three different kind of quarks corresponding to different flavors. Then, the one-loop leading term contribution to the vacuum polarization is given by

$$\begin{aligned} ig^2\Pi_{\mu\nu,AB} &= ig^2(\Pi^{(a)}_{\mu\nu,AB} + \Pi^{(b)}_{\mu\nu,AB} + \Pi^{(c)}_{\mu\nu,AB}) \\ &= -i\frac{g^2}{8\pi^2\epsilon}\left(\frac{5}{3}C_2(G) - \frac{2}{3}n_F\right)\left[p_\mu p_\nu - g_{\mu\nu}p^2\right] \end{aligned}$$
(8.201)

209

By comparison with Section 2.7 we get

$$Z_3 = 1 + \frac{g^2}{8\pi^2\epsilon}\left(\frac{5}{3}C_2(G) - \frac{2}{3}n_F\right) \tag{8.202}$$

We are now left with the vertex corrections. The fermion contribution is given by the diagram of Fig. 8.6.6.

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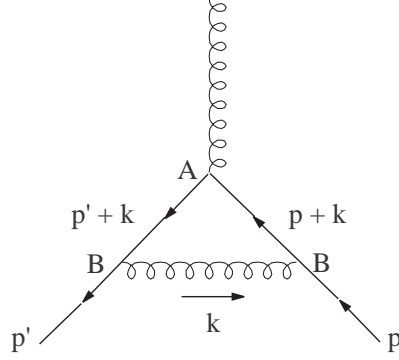
Fig. 8.6.6 - *Fermion one-loop vertex correction.*

Again, this is the same as in QED except for the group factors. the leading contribution is

$$-ig^3\Lambda_f^{\mu,A} = -i\frac{g^3}{8\pi^2\epsilon}(T^B T^A T^B)\gamma^\mu \tag{8.203}$$

The group factor can be elaborated as follows

$$
\begin{aligned}
T^B T^A T^B &= T^B(T^B T^A + if^{ABC}T^C) = C_2(F)T^A + if^{ABC}T^B T^C \\
&= C_2(F)T^A - \frac{1}{2}f^{ABC}f^{BCD}T^D = \left(C_2(F) - \frac{1}{2}C_2(G)\right)T^A
\end{aligned}
\tag{8.204}
$$

Therefore we obtain

$$-ig^3\Lambda_f^{\mu,A} = -i\frac{g^3}{8\pi^2\epsilon}\left(C_2(F) - \frac{1}{2}C_2(G)\right)T^A\gamma^\mu \tag{8.205}$$

The final diagram to be evaluated is the vertex correction from the trilinear gauge coupling given in Fig. 8.6.7.
We get $(q = p' - p)$

$$
\begin{aligned}
-ig^3\Lambda_g^{\mu,A} &= (-ig\gamma_\nu T^B)\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{i\hat{k}}{k^2}\frac{-i}{(p'-k)^2} \\
&\quad\cdot\ (-g)f^{ABC}\left[g^{\mu\nu}(q+p'-k)^\rho - g^{\nu\rho}(p+p'-2k)^\mu + g^{\rho\mu}(p-k-q)_\nu\right] \\
&\quad\cdot\ \frac{-i}{(p-k)^2}(-ig\gamma_\rho T^C)
\end{aligned}
\tag{8.206}
$$

210
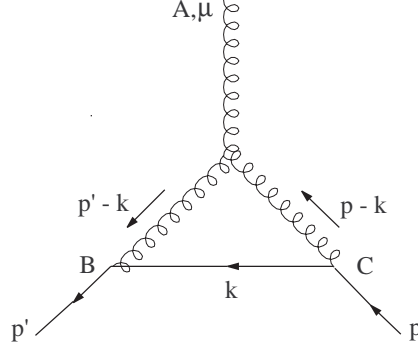
Fig. 8.6.7 - *One-loop vertex correction from the trilinear gauge coupling.*

Since we are interested only in the divergent piece, the calculation can be simplified a lot by the observation that the denominator goes as $(k^2)^3$, whereas the numerator goes at most as k^2. Therefore the divergent piece is obtained by neglecting all the external momenta. Therefore the expression simplifies to

$$-ig^3\Lambda_g^{\mu,A} = ig^3 f^{ABC}T^B T^C \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{\gamma_\nu \hat{k}(g^{\mu\nu}k^\rho - 2g^{\nu\rho}k^\mu + g^{\mu\rho}k^\nu)\gamma_\rho}{(k^2)^3} \qquad (8.207)$$

The group factor gives

$$f^{ABC}T^B T^C = \frac{i}{2}f^{ABC}f^{BCD}T^D = \frac{i}{2}C_2(G)T^A \qquad (8.208)$$

whereas the numerator gives simply $-3k^2\gamma^\mu$. Then we have

$$-ig^3\Lambda_g^{\mu,A} = -\frac{g^3}{(2\pi)^{2\omega}}\frac{3}{2}C_2(G)T^A\gamma^\mu \int \frac{d^{2\omega}k}{(k^2)^2} = -i\frac{g^3}{8\pi^2\epsilon}\frac{3}{2}C_2(G)T^A\gamma^\mu \qquad (8.209)$$

Adding up the two vertex contributions we obtain

$$-ig^3\Lambda^{\mu,A} = -ig^3(\Lambda_f^{\mu,A} + \Lambda_g^{\mu,A}) = -i\frac{g^3}{8\pi^2\epsilon}\left(C_2(F) + C_2(G)\right)T^A\gamma^\mu \qquad (8.210)$$

Again, comparison with Section 2.7 gives

$$Z_1 = 1 - \frac{g^2}{8\pi^2\epsilon}\left(C_2(F) + C_2(G)\right) \qquad (8.211)$$

Using eq. (8.175) we have

$$g_B = \mu^{\epsilon/2}g(1 + \Delta Z_1 - \Delta Z_2 - \frac{1}{2}\Delta Z_3) = \mu^{\epsilon/2}\left(g + \frac{a_1}{\epsilon}\right) \qquad (8.212)$$

and collecting everything together we get

$$a_1 = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}C_2(G) - \frac{2}{3}n_F\right) \qquad (8.213)$$

211

The evaluation of $\beta(g)$ goes as in the QED case (see Section 6.3), that is

$$\beta(g) = \mu \frac{\partial g(\mu)}{\partial \mu} = -\frac{1}{2} \left(1 - g\frac{d}{dg} \right) a_1 = -\frac{1}{2}(a_1 - 3a_1) = a_1 \tag{8.214}$$

that is

$$\beta(g) = -\frac{g^3}{16\pi^2} \left(\frac{11}{3} C_2(G) - \frac{2}{3} n_F \right) \tag{8.215}$$

In the case of QCD where the gauge group is $SU(3)$ we have

$$C_2(G) = 3 \tag{8.216}$$

and therefore

$$\beta_{\text{QCD}}(g) = -\frac{g^3}{16\pi^2} \left(11 - \frac{2}{3} n_F \right) \tag{8.217}$$

We see that QCD is asymptotically free $(\beta(g) < 0)$ for

$$n_F < \frac{33}{2} \tag{8.218}$$

Chapter 9

Spontaneous symmetry breaking

9.1 The linear σ-model

In this Section and in the following we will study, from a classical point of view, some field theory with particular symmetry properties. We will start examining the linear σ-model. This is a model for N scalar fields, with a symmetry $O(N)$. The lagrangian is given by

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{N} \partial_\mu \phi_i \partial^\mu \phi_i - \frac{1}{2} \mu^2 \sum_{i=1}^{N} \phi_i \phi_i - \frac{\lambda}{4} \left(\sum_{i=1}^{N} \phi_i \phi_i \right)^2 \tag{9.1}$$

This lagrangian is invariant under linear transformations acting upon the vector $\vec{\phi} = (\phi_1, \cdots, \phi_N)$ and leaving invariant its norm

$$|\vec{\phi}|^2 = \sum_{i=1}^{N} \phi_i \phi_i \tag{9.2}$$

Consider an infinitesimal transformation (from now on we will omit the index of sum over the indices which are repeated)

$$\delta \phi_i = \epsilon_{ij} \phi_j \tag{9.3}$$

The condition for letting the norm invariant gives

$$|\vec{\phi} + \delta \vec{\phi}|^2 = |\vec{\phi}|^2 \tag{9.4}$$

from which

$$\vec{\phi} \cdot \delta \vec{\phi} = 0 \tag{9.5}$$

or, in components,

$$\phi_i \epsilon_{ij} \phi_j = 0 \tag{9.6}$$

This is satisfied by

$$\epsilon_{ij} = -\epsilon_{ji} \tag{9.7}$$

213

showing that the rotations in N dimensions depend on $N(N-1)/2$ parameters. For a finite transformation we have

$$|\vec{\phi}'|^2 = |\vec{\phi}|^2 \tag{9.8}$$

with

$$\phi'_i = S_{ij}\phi_j \tag{9.9}$$

implying

$$SS^T = 1 \tag{9.10}$$

In fact, by exponentiating the infinitesimal transformation one gets

$$S = e^\epsilon \tag{9.11}$$

with

$$\epsilon^T = -\epsilon \tag{9.12}$$

implying that S is an orthogonal transformation. The matrices S form the rotation group in N dimensions, $O(N)$.

The Noether's theorem implies a conserved current for any symmetry of the theory. In this case we will get $N(N-1)/2$ conserved quantities. It is useful for further generalizations to write the infinitesimal transformation in the form

$$\delta\phi_i = \epsilon_{ij}\phi_j = -\frac{i}{2}\epsilon_{AB}T^{AB}_{ij}\phi_j, \qquad i,j = 1,\cdots,N, \qquad A,B = 1,\cdots,N \tag{9.13}$$

which is similar to what we did in Section 3.3 when we discussed the Lorentz transformations. By comparison we see that the matrices T^{AB} are given by

$$T^{AB}_{ij} = i(\delta^A_i\delta^B_j - \delta^A_j\delta^B_i) \tag{9.14}$$

It is not difficult to show that these matrices satisfy the algebra

$$[T^{AB},T^{CD}] = -i\delta^{AC}T^{BD} + i\delta^{AD}T^{BC} - i\delta^{BD}T^{AC} + i\delta^{BC}T^{AD} \tag{9.15}$$

This is nothing but the Lie algebra of the group $O(N)$, and the T^{AB} are the infinitesimal generators of the group. Applying now the Noether's theorem we find the conserved current

$$j_\mu = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi_i)}\delta\phi_i = -\frac{i}{2}\phi_{i,\mu}\epsilon_{AB}T^{AB}_{ij}\phi_j \tag{9.16}$$

since the $N(N-1)/2$ parameters ϵ_{AB} are linearly independent, we find the $N(N-1)/2$ conserved currents

$$J^{AB}_\mu = -i\phi_{i,\mu}T^{AB}_{ij}\phi_j \tag{9.17}$$

In the case of $N = 2$ the symmetry is the same as for the charged field in a real basis, and the only conserved current is given by

$$J^{12}_\mu = -J^{21}_\mu = \phi_{1,\mu}\phi_2 - \phi_{2,\mu}\phi_1 \tag{9.18}$$

214

One can easily check that the charges associated to the conserved currents close the same Lie algebra as the generators T^{AB}. More generally, if we have conserved currents given by

$$j_\mu^A = -i\phi_{i,\mu} T_{ij}^A \phi_j \tag{9.19}$$

with

$$[T^A, T^B] = i f^{ABC} T^C \tag{9.20}$$

then, using the canonical commutation relations, we get

$$[Q^A, Q^B] = i f^{ABC} Q^C \tag{9.21}$$

with

$$Q^A = \int d^3x \, j_0^A(x) = -i \int d^3x \, \dot{\phi}_i T_{ij}^A \phi_j \tag{9.22}$$

A particular example is the case $N = 4$. We parameterize our fields in the form

$$\vec{\phi} = (\pi_1, \pi_2, \pi_3, \sigma) = (\vec{\pi}, \sigma) \tag{9.23}$$

These fields can be arranged into a 2×2 matrix

$$M = \sigma + i\vec{\tau} \cdot \vec{\pi} \tag{9.24}$$

where $\vec{\tau}$ are the Pauli matrices. Noticing that τ_2 is pure imaginary, τ_1 and τ_3 real, and that τ_2 anticommutes with τ_1 and τ_3, we get

$$M = \tau_2 M^* \tau_2 \tag{9.25}$$

Furthermore we have the relation

$$|\vec{\phi}|^2 = \sigma^2 + |\vec{\pi}|^2 = \frac{1}{2} Tr(M^\dagger M) \tag{9.26}$$

Using this it is easy to write the lagrangian for the σ-model in the form

$$\mathcal{L} = \frac{1}{4} Tr(\partial_\mu M^\dagger \partial^\mu M) - \frac{1}{4}\mu^2 Tr(M^\dagger M) - \frac{1}{16}\lambda \left(Tr(M^\dagger M)\right)^2 \tag{9.27}$$

This lagrangian is invariant under the following transformation of the matrix M

$$M \to LMR^\dagger \tag{9.28}$$

where L and R are two special (that is with determinant equal to 1) unitary matrices, that is $L, R \in SU(2)$. The reason to restrict these matrices to be special is that only in this way the transformed matrix satisfy the condition (9.25). In fact, if A is a 2×2 matrix with $det A = 1$, then

$$\tau_2 A^T \tau_2 = A^{-1} \tag{9.29}$$

Therefore, for $M' = LMR^\dagger$ we get

$$\tau_2 M'^* \tau_2 = \tau_2 L^* M^* R^T \tau_2 = \tau_2 L^* \tau_2 (\tau_2 M^* \tau_2) \tau_2 R^T \tau_2 \tag{9.30}$$

and from (9.29)

$$\begin{aligned}
\tau_2 L^* \tau_2 &= \tau_2 L^{\dagger T} \tau_2 = L^{\dagger^{-1}} = L \\
\tau_2 R^T \tau_2 &= R^{-1} = R^\dagger
\end{aligned} \tag{9.31}$$

since the L and R are independent transformations, the invariance group in this basis is $SU(2)_L \otimes SU(2)_R$. In fact this group and $O(4)$ are related by the following observation: the transformation $M \rightarrow LMR^\dagger$ is a linear transformation on the matrix elements of M, but from the relation (9.26) we see that $M \rightarrow LMR^\dagger$ leaves the norm of the vector $\vec{\phi} = (\vec{\pi}, \sigma)$ invariant and therefore the same must be true for the linear transformation acting upon the matrix elements of M, that is on σ and $\vec{\pi}$. Therefore this transformation must belongs to $O(4)$. This shows that the two groups $SU(2) \otimes SU(2)$ and $O(4)$ are homomorphic (actually there is a 2 to 1 relationship, since $-L$ and $-R$ define the same S as L and R).

We can evaluate the effect of an infinitesimal transformation. To this end we will consider separately left and right transformations. We parameterize the transformations as follows

$$L \approx 1 - \frac{i}{2}\vec{\theta}_L \cdot \vec{\tau}, \quad R \approx 1 - \frac{i}{2}\vec{\theta}_R \cdot \vec{\tau} \tag{9.32}$$

then we get

$$\delta_L M = (-\frac{i}{2}\vec{\theta}_L \cdot \vec{\tau})M = (-\frac{i}{2}\vec{\theta}_L \cdot \vec{\tau})(\sigma + i\vec{\pi} \cdot \vec{\tau}) = \frac{1}{2}\vec{\theta}_L \cdot \vec{\pi} + \frac{i}{2}(\vec{\theta}_L \wedge \vec{\pi} - \vec{\theta}_L \sigma) \cdot \vec{\tau} \tag{9.33}$$

where we have used

$$\tau_i \tau_j = \delta_{ij} + i\epsilon_{ijk}\tau_k \tag{9.34}$$

Since

$$\delta_L M = \delta_L \sigma + i\delta_L \vec{\pi} \cdot \vec{\tau} \tag{9.35}$$

we get

$$\delta_L \sigma = \frac{1}{2}\vec{\theta}_L \cdot \vec{\pi}, \quad \delta_L \vec{\pi} = \frac{1}{2}(\vec{\theta}_L \wedge \vec{\pi} - \vec{\theta}_L \sigma) \tag{9.36}$$

Analogously we obtain

$$\delta_R \sigma = -\frac{1}{2}\vec{\theta}_R \cdot \vec{\pi}, \quad \delta_R \vec{\pi} = \frac{1}{2}(\vec{\theta}_R \wedge \vec{\pi} + \vec{\theta}_R \sigma) \tag{9.37}$$

The combined transformation is given by

$$\delta\sigma = \frac{1}{2}(\vec{\theta}_L - \vec{\theta}_R) \cdot \vec{\pi}, \quad \delta\vec{\pi} = \frac{1}{2}[(\vec{\theta}_L + \vec{\theta}_R) \wedge \vec{\pi} - (\vec{\theta}_L - \vec{\theta}_R)\sigma] \tag{9.38}$$

and we can check immediately that

$$\sigma\delta\sigma + \vec{\pi} \cdot \delta\vec{\pi} = 0 \tag{9.39}$$

as it must be for a transformation leaving the form $\sigma^2 + |\vec{\pi}|^2$ invariant. Of particular interest are the transformations with $\vec{\theta}_L = \vec{\theta}_R \equiv \theta$. In this case we have $L = R$ and

$$M \to LML^\dagger \tag{9.40}$$

These transformations span a subgroup $SU(2)$ of $SU(2)_L \otimes SU(2)_R$ called the diagonal subgroup. In this case we have

$$\delta\sigma = 0, \quad \delta\vec{\pi} = \vec{\theta} \wedge \vec{\pi} \tag{9.41}$$

We see that the transformations corresponding to the diagonal $SU(2)$ are the rotations in the 3-dimensional space spanned by $\vec{\pi}$. These rotations define a subgroup $O(3)$ of the original symmetry group $O(4)$. From the Noether's theorem we get the conserved currents

$$j_\mu^L = \frac{1}{2}\sigma_{,\mu}\vec{\theta}_L \cdot \vec{\pi} + \frac{1}{2}\vec{\pi}_{,\mu} \cdot (\vec{\theta}_L \wedge \vec{\pi} - \vec{\theta}_L\sigma) \tag{9.42}$$

and dividing by $\theta_L/2$

$$\vec{J}_\mu^L = \sigma_{,\mu}\vec{\pi} - \vec{\pi}_{,\mu}\sigma - \vec{\pi}_{,\mu} \wedge \vec{\pi} \tag{9.43}$$

and analogously

$$\vec{J}_\mu^R = -\sigma_{,\mu}\vec{\pi} + \vec{\pi}_{,\mu}\sigma - \vec{\pi}_{,\mu} \wedge \vec{\pi} \tag{9.44}$$

Using the canonical commutation relations one can verify that the corresponding charges satisfy the Lie algebra of $SU(2)_L \otimes SU(2)_R$

$$[Q_i^L, Q_j^L] = i\epsilon_{ijk}Q_k^L, \quad [Q_i^R, Q_j^R] = i\epsilon_{ijk}Q_k^R, \quad [Q_i^L, Q_j^R] = 0 \tag{9.45}$$

By taking the following combinations of the currents

$$\vec{J}_\mu^V = \frac{1}{2}(\vec{J}_\mu^L + \vec{J}_\mu^R), \quad \vec{J}_\mu^A = \frac{1}{2}(\vec{J}_\mu^L - \vec{J}_\mu^R) \tag{9.46}$$

one has

$$\vec{J}_\mu^V = \vec{\pi} \wedge \vec{\pi}_{,\mu} \tag{9.47}$$

and

$$\vec{J}_\mu^A = \sigma_{,\mu}\vec{\pi} - \vec{\pi}_{,\mu}\sigma \tag{9.48}$$

The corresponding algebra of charges is

$$[Q_i^V, Q_j^V] = i\epsilon_{ijk}Q_k^V, \quad [Q_i^V, Q_j^A] = i\epsilon_{ijk}Q_k^A, \quad [Q_i^A, Q_j^A] = i\epsilon_{ijk}Q_k^V \tag{9.49}$$

These equations show that Q_i^V are the infinitesimal generators of a subgroup $SU(2)$ of $SU(2)_L \otimes SU(2)_R$ which is the diagonal subgroup, as it follows from

$$[Q_i^V, \pi_j] = i\epsilon_{ijk}\pi_k, \quad [Q_i^V, \sigma] = 0 \tag{9.50}$$

217

In the following we will be interested in treating the interacting field theories by using the perturbation theory. As in the quantum mechanical case, this is well defined only when we are considering the theory close to a minimum the energy of the system. In fact if we are going to expand around a maximum the oscillation of the system can become very large leading us outside of the domain of perturbation theory. In the case of the linear σ-model the energy is given by

$$H = \int d^3x \, \mathcal{H} = \int d^3x \left[\sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} \dot{\phi}_i - \mathcal{L} \right] = \int d^3x \left[\frac{1}{2} \sum_{i=1}^{N} (\dot{\phi}_i^2 + |\vec{\nabla}\phi_i|^2) + V(|\vec{\phi}|^2) \right]$$
(9.51)

Since in the last member of this relation the first two terms are positive definite, it follows that the absolute minimum is obtained for constant field configurations, such that

$$\frac{\partial V(|\vec{\phi}|^2)}{\partial \phi_i} = 0$$
(9.52)

Let us call by v_i the generic solution to this equation (in general it could happen that the absolute minimum is degenerate). Then the condition for getting a minimum is that the eigenvalues of the matrix of the second derivatives of the potential at the stationary point are definite positive. In this case we define new fields by shifting the original fields by

$$\phi_i \to \phi_i' = \phi_i - v_i$$
(9.53)

The lagrangian density becomes

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_i' \partial^\mu \phi_i' - V(|\vec{\phi}' + \vec{v}|^2)$$
(9.54)

Expanding V in series of ϕ_i' we get

$$V = V(|\vec{v}|^2) + \frac{1}{2} \frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\bigg|_{\vec{\phi}=\vec{v}} \phi_i' \phi_j' + \cdots$$
(9.55)

This equation shows that the particle masses are given by the eigenvalues of the second derivative of the potential at the minimum. In the case of the linear σ-model we have

$$V = \frac{1}{2} \mu^2 |\vec{\phi}|^2 + \frac{\lambda}{4} (|\vec{\phi}|^2)^2$$
(9.56)

Therefore

$$\frac{\partial V}{\partial \phi_i} = \mu^2 \phi_i + \lambda \phi_i |\vec{\phi}|^2$$
(9.57)

In order to have a solution to the stationary condition we must have $\phi_i = 0$, or

$$|\vec{\phi}|^2 = -\frac{\mu^2}{\lambda}$$
(9.58)

218

This equation has real solutions only if $\mu^2/\lambda < 0$. However, in order to have a potential bounded from below one has to require $\lambda > 0$, therefore we may have non zero solutions to the minimum condition only if $\mu^2 < 0$. But, notice that in this case, μ^2 cannot be identified with a physical mass, these are given by the eigenvalues of the matrix of the second derivatives of the potential at the minimum and they are positive definite by definition. We will study this case in the following Sections. In the case of $\mu^2 > 0$ the minimum is given by $\phi_i = 0$ and one can study the theory by taking the term $\lambda(|\vec{\phi}|^2)^2$ as a small perturbation (that is requiring that both λ and the values of ϕ_i, the fluctuations, are small). The free theory is given by the quadratic terms in the lagrangian density, and they describe N particles of common mass m. Furthermore, both the free and the interacting theories are $O(N)$ symmetric.

9.2   Spontaneous symmetry breaking

In this Section we will see that the linear σ-model with $\mu^2 < 0$, is just but an example of a general phenomenon which goes under the name of spontaneous symmetry breaking of symmetry. This phenomenon lies at the basis of the modern description of phase transitions and it has acquired a capital relevance in the last years in all field of physics. The idea is very simple and consists in the observation that a theory with hamiltonian invariant under a symmetry group may not show explicitly the symmetry at the level of the solutions. As we shall see this may happen when the following conditions are realized:

- The theory is invariant under a symmetry group G.

- The fundamental state of the theory is degenerate and transforms in a non trivial way under the symmetry group.

Just as an example consider a scalar field described by a lagrangian invariant under parity

$$P: \quad \phi \to -\phi \tag{9.59}$$

The lagrangian density will be of the type

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi^2) \tag{9.60}$$

If the vacuum state is non degenerate, barring a phase factor, we must have

$$P|0\rangle = |0\rangle \tag{9.61}$$

since P commutes with the hamiltonian. It follows

$$\langle 0|\phi|0\rangle = \langle 0|P^{-1}P\phi P^{-1}P|0\rangle = \langle 0|P\phi P^{-1}|0\rangle = -\langle 0|\phi|0\rangle \tag{9.62}$$

from which

$$\langle 0|\phi|0\rangle = 0 \qquad (9.63)$$

Things change if the fundamental state is degenerate. This would be the case in the example (9.60), if

$$V(\phi^2) = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 \qquad (9.64)$$

with $\mu^2 < 0$. In fact, this potential has two minima located at

$$\phi = \pm v, \qquad v = \sqrt{-\frac{\mu^2}{\lambda}} \qquad (9.65)$$

By denoting with $|R\rangle$ e $|L\rangle$ the two states corresponding to the classical configurations $\phi = \pm v$, we have

$$P|R\rangle = |L\rangle \neq |R\rangle \qquad (9.66)$$

Therefore

$$\langle R|\phi|R\rangle = \langle R|P^{-1}P\phi P^{-1}P|R\rangle = -\langle L|\phi|L\rangle \qquad (9.67)$$

which now does not imply that the expectation value of the field vanishes. In the following we will be rather interested in the case of continuous symmetries. So let us consider two scalar fields, and a lagrangian density with symmetry $O(2)$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - \frac{1}{2}\mu^2\vec{\phi}\cdot\vec{\phi} - \frac{\lambda}{4}(\vec{\phi}\cdot\vec{\phi})^2 \qquad (9.68)$$

where

$$\vec{\phi}\cdot\vec{\phi} = \phi_1^2 + \phi_2^2 \qquad (9.69)$$

For $\mu^2 > 0$ there is a unique fundamental state (minimum of the potential) $\vec{\phi} = 0$, whereas for $\mu^2 < 0$ there are infinite degenerate states given by

$$|\vec{\phi}|^2 = \phi_1^2 + \phi_2^2 = v^2 \qquad (9.70)$$

with v defined as in (9.65). By denoting with $R(\theta)$ the operator rotating the fields in the plane (ϕ_1, ϕ_2), in the non-degenerate case we have

$$R(\theta)|0\rangle = |0\rangle \qquad (9.71)$$

and

$$\langle 0|\phi|0\rangle = \langle 0|R^{-1}R\phi R^{-1}R|0\rangle = \langle 0|\phi^\theta|0\rangle = 0 \qquad (9.72)$$

since $\phi^\theta \neq \phi$. In the case $\mu^2 < 0$ (degenerate case), we have

$$R(\theta)|0\rangle = |\theta\rangle \qquad (9.73)$$

where $|\theta\rangle$ is one of the infinitely many degenerate fundamental states lying on the circle $|\vec{\phi}|^2 = v^2$. Then

$$\langle 0|\phi_i|0\rangle = \langle 0|R^{-1}(\theta)R(\theta)\phi_i R^{-1}(\theta)R(\theta)|0\rangle = \langle \theta|\phi_i^\theta|\theta\rangle \qquad (9.74)$$

with

$$\phi_i^\theta = R(\theta)\phi_i R^{-1}(\theta) \neq \phi_i \tag{9.75}$$

Again, the expectation value of the field (contrarily to the non-degenerate state) does not need to vanish. The situation can be described qualitatively saying that the existence of a degenerate fundamental state forces the system to choose one of these equivalent states, and consequently to break the symmetry. But the breaking is only at the level of the solutions, the lagrangian and the equations of motion preserve the symmetry. One can easily construct classical systems exhibiting spontaneous symmetry breaking. For instance, a classical particle in a double-well potential. This system has parity invariance $x \to -x$, where x is the particle position. The equilibrium positions are around the minima positions, $\pm x_0$. If we put the particle close to x_0, it will perform oscillations around that point and the original symmetry is lost. A further example is given by a ferromagnet which has an hamiltonian invariant under rotations, but below the Curie temperature exhibits spontaneous magnetization, breaking in this way the symmetry. These situations are typical for the so called second order phase transitions. One can describe them through the Landau free-energy, which depends on two different kind of parameters:

- **Control parameters**, as μ^2 for the scalar field, and the temperature for the ferromagnet.

- **Order parameters**, as the expectation value of the scalar field or as the magnetization.

The system goes from one phase to another varying the control parameters, and the phase transition is characterized by the order parameters which assume different values in different phases. In the previous examples, the order parameters were zero in the symmetric phase and different from zero in the broken phase.

The situation is slightly more involved at the quantum level, since spontaneous symmetry breaking cannot happen in finite systems. This follows from the existence of the tunnel effect. Let us consider again a particle in a double-well potential, and recall that we have defined the fundamental states through the correspondence with the classical minima

$$x = x_0 \to |R\rangle$$
$$x = -x_0 \to |L\rangle \tag{9.76}$$

But the tunnel effect gives rise to a transition between these two states and as a consequence it removes the degeneracy. In fact, due to the transition the hamiltonian acquires a non zero matrix element between the states $|R\rangle$ and $|L\rangle$. By denoting with $\underline{H}$ the matrix of the hamiltonian between these two states, we get

$$\underline{H} = \begin{bmatrix} \epsilon_0 & \epsilon_1 \\ \epsilon_1 & \epsilon_0 \end{bmatrix} \tag{9.77}$$

The eigenvalues of $\underline{H}$ are

$$(\epsilon_0 + \epsilon_1, \epsilon_0 - \epsilon_1) \tag{9.78}$$

We have no more degeneracy and the eigenstates are

$$|S\rangle = \frac{1}{\sqrt{2}}(|R\rangle + |L\rangle) \tag{9.79}$$

with eigenvalue $E_S = \epsilon_0 + \epsilon_1$, and

$$|A\rangle = \frac{1}{\sqrt{2}}(|R\rangle - |L\rangle) \tag{9.80}$$

with eigenvalue $E_A = \epsilon_0 - \epsilon_1$. One can show that $\epsilon_1 < 0$ and therefore the fundamental state is the symmetric one, $|S\rangle$. This situation gives rise to the well known effect of quantum oscillations. We can express the states $|R\rangle$ and $|L\rangle$ in terms of the energy eigenstates

$$
\begin{aligned}
|R\rangle &= \tfrac{1}{\sqrt{2}}(|S\rangle + |A\rangle) \\
|L\rangle &= \tfrac{1}{\sqrt{2}}(|S\rangle - |A\rangle)
\end{aligned} \tag{9.81}
$$

Let us now prepare a state, at $t = 0$, by putting the particle in the right minimum. This is not an energy eigenstate and its time evolution is given by

$$|R, t\rangle = \frac{1}{\sqrt{2}}\left(e^{-iE_S t}|S\rangle + e^{-iE_A t}|A\rangle\right) = \frac{1}{\sqrt{2}}e^{-iE_S t}\left(|S\rangle + e^{-it\Delta E}|A\rangle\right) \tag{9.82}$$

with $\Delta E = E_A - E_S$. Therefore, for $t = \pi/\Delta E$ the state $|R\rangle$ transforms into the state $|L\rangle$. The state oscillates with a period given by

$$T = \frac{2\pi}{\Delta E} \tag{9.83}$$

In nature there are finite systems as sugar molecules, which seem to exhibit spontaneous symmetry breaking. In fact one observes right-handed and left-handed sugar molecules. The explanation is simply that the energy difference ΔE is so small that the oscillation period is of the order of $10^4 - 10^6$ years.

The splitting of the fundamental states decreases with the height of the potential between two minima, therefore, for infinite systems, the previous mechanism does not work, and we may have spontaneous symmetry breaking. In fact, coming back to the scalar field example, its expectation value on the vacuum must be a constant, as it follows from the translational invariance of the vacuum

$$\langle 0|\phi(x)|0\rangle = \langle 0|e^{iPx}\phi(0)e^{-iPx}|0\rangle = \langle 0|\phi(0)|0\rangle = v \tag{9.84}$$

and the energy difference between the maximum at $\phi = 0$, and the minimum at $\phi = v$, becomes infinite in the limit of infinite volume

$$H(\phi = 0) - H(\phi = v) = -\int_V d^3x \left[\frac{\mu^2}{2}v^2 + \frac{\lambda}{4}v^4\right] = \frac{\mu^4}{4\lambda}\int_V d^3x = \frac{\mu^4}{4\lambda}V \tag{9.85}$$

222

9.3 The Goldstone theorem

From our point of view, the most interesting consequence of spontaneous symmetry breaking is the Goldstone theorem. This theorem says that for any continuous symmetry spontaneously broken, there exists a massless particle (the Goldstone boson). The theorem holds rigorously in a local field theory, under the following hypotheses

- The spontaneous broken symmetry must be a continuous one.

- The theory must be manifestly covariant.

- The Hilbert space of the theory must have a definite positive norm.

We will limit ourselves to analyze the theorem in the case of a classical scalar field theory. Let us start considering the lagrangian for the linear σ-model with invariance $O(N)$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - \frac{\mu^2}{2}\vec{\phi} \cdot \vec{\phi} - \frac{\lambda}{4}(\vec{\phi} \cdot \vec{\phi})^2 \tag{9.86}$$

The conditions that V must satisfy in order to have a minimum are

$$\frac{\partial V}{\partial \phi_l} = \mu^2 \phi_l + \lambda \phi_l |\vec{\phi}|^2 = 0 \tag{9.87}$$

with solutions

$$\phi_l = 0, \qquad |\vec{\phi}|^2 = v^2, \qquad v = \sqrt{\frac{-\mu^2}{\lambda}} \tag{9.88}$$

The character of the stationary points can be studied by evaluating the second derivatives

$$\frac{\partial^2 V}{\partial \phi_l \partial \phi_m} = \delta_{lm}(\mu^2 + \lambda|\vec{\phi}|^2) + 2\lambda \phi_l \phi_m \tag{9.89}$$

We have two possibilities

- $\mu^2 > 0$, we have only one real solution given by $\vec{\phi} = 0$, which is a minimum, since

$$\frac{\partial^2 V}{\partial \phi_l \partial \phi_m} = \delta_{lm}\mu^2 > 0 \tag{9.90}$$

- $\mu^2 < 0$, there are infinite solutions, among which $\vec{\phi} = 0$ is a maximum. The points of the sphere $|\vec{\phi}|^2 = v^2$ are degenerate minima. In fact, by choosing $\phi_l = v\delta_{lN}$ as a representative point, we get

$$\frac{\partial^2 V}{\partial \phi_l \partial \phi_m} = 2\lambda v^2 \delta_{lN}\delta_{mN} > 0 \tag{9.91}$$

Expanding the potential around this minimum we get

$$V(\vec{\phi}) \approx V\Big|_{\text{minimum}} + \frac{1}{2}\frac{\partial^2 V}{\partial\phi_l\partial\phi_m}\Big|_{\text{minimum}}(\phi_l - v\delta_{lN})(\phi_m - v\delta_{mN}) \tag{9.92}$$

If we are going to make a perturbative expansion, the right fields to be used are $\phi_l - v\delta_{lN}$, and their mass is just given by the coefficient of the quadratic term

$$M_{lm}^2 = \frac{\partial^2 V}{\partial\phi_l\partial\phi_m}\Big|_{\text{minimo}} = -2\mu^2\delta_{lN}\delta_{mN} = \begin{bmatrix} 0 & 0 & \cdot & 0 \\ 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & -2\mu^2 \end{bmatrix} \tag{9.93}$$

Therefore the masses of the fields ϕ_a, $a = 1, \cdots, N-1$, and $\chi = \phi_N - v$, are given by

$$m_{\phi_a}^2 = 0, \qquad m_\chi^2 = -2\mu^2 \tag{9.94}$$

By defining

$$m^2 = -2\mu^2 \tag{9.95}$$

we can write the potential as a function of the new fields

$$V = \frac{m^4}{16\lambda} + \frac{1}{2}m^2\chi^2 + \sqrt{\frac{m^2\lambda}{2}}\chi\left(\sum_{a=1}^{N-1}\phi_a^2 + \chi^2\right) + \frac{\lambda}{4}\left(\sum_{a=1}^{N-1}\phi_a^2 + \chi^2\right)^2 \tag{9.96}$$

In this form the original symmetry $O(N)$ is broken. However a residual symmetry $O(N-1)$ is left. In fact, V depends only on the combination $\sum_{a=1}^{N-1}\phi_a^2$, and it is invariant under rotations around the axis we have chosen as representative for the fundamental state, $(0, \cdots, v)$. It must be stressed that this is not the most general potential invariant under $O(N-1)$. In fact the most general potential (up to the fourth order in the fields) describing N scalar fields with a symmetry $O(N-1)$ would depend on 7 coupling constants, whereas the one we got depends only on the two parameters m and λ. Therefore spontaneous symmetry breaking puts heavy constraints on the dynamics of the system. We have also seen that we have $N-1$ massless scalars. Clearly the rotations along the first $N-1$ directions leave the potential invariant, whereas the $N-1$ rotations on the planes $a-N$ move away from the surface of the minima. This can be seen also in terms of generators. Since the field we have chosen as representative of the ground state is $\phi_i|_{\text{min}} = v\delta_{iN}$, we have

$$T_{ij}^{ab}\phi_j|_{\text{min}} = i(\delta_i^a\delta_j^b - \delta_i^b\delta_j^a)v\delta_{jN} = 0 \tag{9.97}$$

since $a, b = 1, \cdots, N-1$, and

$$T_{ij}^{aN}\phi_j|_{\text{min}} = i(\delta_i^a\delta_j^N - \delta_i^N\delta_j^a)v\delta_{jN} = iv\delta_i^a \neq 0 \tag{9.98}$$

Therefore we have $N-1$ broken symmetries and $N-1$ massless scalars. The generators of $O(N)$ divide up naturally in the generators of the vacuum symmetry

(here $O(N-1)$), and in the so called broken generators, each of them corresponding to a massless Goldstone boson. In general, if the original symmetry group G of the theory is spontaneously broken down to a subgroup H (which is the symmetry of the vacuum), the Goldstone bosons correspond to the generators of G which are left after subtracting the generators of H. Intuitively one can understand the origin of the massless particles noticing that the broken generators allow transitions from a possible vacuum to another. Since these states are degenerate the operation does not cost any energy. From the relativistic dispersion relation this implies that we must have massless particles. One can say that Goldstone bosons correspond to flat directions in the potential.

9.4 The Higgs mechanism

We have seen that the spontaneous symmetry breaking mechanism, in the case of continuous symmetry leads to massless scalar particles, the Goldstone bosons. Also gauge theories lead to massless vector bosons, in fact, as in the electromagnetic case, gauge invariance forbids the presence in the lagrangian of terms quadratic in the fields. Unfortunately in nature the only massless particles we know are the photon and perhaps the neutrinos, which however are fermions. But once one couples spontaneous symmetry breaking to a gauge symmetry, things change. In fact, if we look back at the hypotheses underlying a gauge theory, it turns out that Goldstone theorem does not hold in this context. The reason is that it is impossible to quantize a gauge theory in a way which is at the same time manifestly covariant and has a Hilbert space with positive definite metric. This is well known already for the electromagnetic field, where one has to choose the gauge before quantization. What happens is that, if one chooses a physical gauge, as the Coulomb gauge, in order to have a Hilbert space spanned by only the physical states, than the theory looses the manifest covariance. If one goes to a covariant gauge, as the Lorentz one, the theory is covariant but one has to work with a big Hilbert space, with non-definite positive metric, and where the physical states are extracted through a supplementary condition. The way in which the Goldstone theorem is evaded is that the Goldstone bosons disappear, and, at the same time, the gauge bosons corresponding to the broken symmetries acquire mass. This is the famous **Higgs mechanism**.

Let us start with a scalar theory invariant under $O(2)$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - \frac{\mu^2}{2}\vec{\phi}\cdot\vec{\phi} - \frac{\lambda}{4}(\vec{\phi}\cdot\vec{\phi})^2 \tag{9.99}$$

and let us analyze the spontaneous symmetry breaking mechanism. If $\mu^2 < 0$ the symmetry is broken and we can choose the vacuum as the state

$$\vec{\phi} = (v,0), \qquad v = \sqrt{\frac{-\mu^2}{\lambda}} \tag{9.100}$$

225

After the translation $\phi_1 = \chi + v$, with $\langle 0|\chi|0\rangle = 0$, we get the potential ($m^2 = -2\mu^2$)

$$V = -\frac{m^4}{16\lambda} + \frac{1}{2}m^2\chi^2 + \sqrt{\frac{m^2\lambda}{2}}\chi(\phi_2^2 + \chi^2) - \frac{\lambda}{4}(\phi_2^2 + \chi^2)^2 \qquad (9.101)$$

In this case the group $O(2)$ is completely broken (except for the discrete symmetry $\phi_2 \to -\phi_2$). The Goldstone field is ϕ_2 . This has a peculiar way of transforming under $O(2)$. In fact, the original fields transform as

$$\delta\phi_1 = -\alpha\phi_2, \qquad \delta\phi_2 = \alpha\phi_1 \qquad (9.102)$$

from which

$$\delta\chi = -\alpha\phi_2, \qquad \delta\phi_2 = \alpha\chi + \alpha v \qquad (9.103)$$

We see that the Goldstone field undergoes a rotation plus a translation, αv. This is the main reason for the Goldstone particle to be massless. In fact one can have invariance under translations of the field, only if the potential is flat in the corresponding direction. This is what happens when one moves in a way which is tangent to the surface of the degenerate vacuums (in this case a circle). How do things change if our theory is gauge invariant? In that case we should have invariance under a transformation of the Goldstone field given by

$$\delta\phi_2(x) = \alpha(x)\chi(x) + \alpha(x)v \qquad (9.104)$$

Since $\alpha(x)$ is an arbitrary function of the space-time point, it follows that we can choose it in such a way to make $\phi_2(x)$ vanish. In other words it must be possible to eliminate the Goldstone field from the theory. This is better seen by using polar coordinates for the fields, that is

$$\rho = \sqrt{\phi_1^2 + \phi_2^2}, \qquad \sin\theta = \frac{\phi_2}{\sqrt{\phi_1^2 + \phi_2^2}} \qquad (9.105)$$

Under a finite rotation, the new fields transform as

$$\rho \to \rho, \qquad \theta \to \theta + \alpha \qquad (9.106)$$

It should be also noticed that the two coordinate systems coincide when we are close to the vacuum, as when we are doing perturbation theory. In fact, in that case we can perform the following expansion

$$\rho = \sqrt{\phi_2^2 + \chi^2 + 2\chi v + v^2} \approx v + \chi, \qquad \theta \approx \frac{\phi_2}{v + \chi} \approx \frac{\phi_2}{v} \qquad (9.107)$$

Again, if we make the theory invariant under a local transformation, we will have invariance under

$$\theta(x) \to \theta(x) + \alpha(x) \qquad (9.108)$$

By choosing $\alpha(x) = -\theta(x)$ we can eliminate this last field from the theory. The only remaining degree of freedom in the scalar sector is $\rho(x)$.

Let us study the gauging of this model. It is convenient to introduce complex variables

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \qquad \phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \tag{9.109}$$

The $O(2)$ transformations become phase transformations on ϕ

$$\phi \to e^{i\alpha}\phi \tag{9.110}$$

and the lagrangian (9.99) can be written as

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \tag{9.111}$$

We know that it is possible to promote a global symmetry to a local one by introducing the covariant derivative

$$\partial_\mu\phi \to (\partial_\mu - igA_\mu)\phi \tag{9.112}$$

from which

$$\mathcal{L} = (\partial_\mu + igA_\mu)\phi^\dagger(\partial^\mu - igA^\mu)\phi - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{9.113}$$

In terms of the polar coordinates (ρ, θ) we have

$$\phi = \frac{1}{\sqrt{2}}\rho e^{i\theta}, \qquad \phi^\dagger = \frac{1}{\sqrt{2}}\rho e^{-i\theta} \tag{9.114}$$

By performing the following gauge transformation on the scalars

$$\phi \to \phi' = \phi e^{-i\theta} \tag{9.115}$$

and the corresponding transformation on the gauge fields

$$A_\mu \to A'_\mu = A_\mu - \frac{1}{g}\partial_\mu\theta \tag{9.116}$$

the lagrangian will depend only on the fields ρ and A'_μ (we will put again $A'_\mu = A_\mu$)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu + igA_\mu)\rho(\partial^\mu - igA_\mu)\rho - \frac{\mu^2}{2}\rho^2 - \frac{\lambda}{4}\rho^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{9.117}$$

In this way the Goldstone boson disappears. We have now to translate the field ρ

$$\rho = \chi + v, \qquad \langle 0|\chi|0\rangle = 0 \tag{9.118}$$

and we see that this generates a bilinear term in A_μ, coming from the covariant derivative, given by

$$\frac{1}{2}g^2 v^2 A_\mu A^\mu \tag{9.119}$$

Therefore the gauge field acquires a mass

$$m_A^2 = g^2 v^2 \tag{9.120}$$

It is instructive to count the degrees of freedom before and after the gauge transformation. Before we had 4 degrees of freedom, two from the scalar fields and two from the gauge field. After the gauge transformation we have only one degree of freedom from the scalar sector, but three degrees of freedom from the gauge vector, because now it is a massive vector field. The result looks a little bit strange, but the reason why we may read clearly the number of degrees of freedom only after the gauge transformation is that before the lagrangian contains a mixing term

$$A_\mu \partial^\mu \theta \tag{9.121}$$

between the Goldstone field and the gauge vector which makes complicate to read the mass of the states. The previous gauge transformation realizes the purpose of making that term vanish. The gauge in which such a thing happens is called the **unitary gauge**.

We will consider now the further example of a symmetry $O(N)$. The lagrangian invariant under local transformations is

$$\mathcal{L} = \frac{1}{2}(D_\mu)_{ij}\phi_j (D^\mu)_{ik}\phi_k - \frac{\mu^2}{2}\phi_i\phi_i - \frac{\lambda}{4}(\phi_i\phi_i)^2 \tag{9.122}$$

where

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu + i\frac{g}{2}(T^{AB})_{ij}W_\mu^{AB} \tag{9.123}$$

where $(T^{AB})_{lm} = i(\delta_l^A \delta_m^B - \delta_m^A \delta_l^B)$. In the case of broken symmetry ($\mu^2 < 0$), we choose again the vacuum along the direction N, with v defined as in (9.100)

$$\phi_i = v\delta_{iN} \tag{9.124}$$

Recalling that

$$T_{ij}^{ab}\phi_j\Big|_{\min} = 0, \quad T_{ij}^{aN}\phi_j\Big|_{\min} = iv\delta_i^a, \quad a,b = 1, \cdots, N-1 \tag{9.125}$$

the mass term for the gauge field is given by

$$
\begin{aligned}
&-\frac{1}{8}g^2 T_{ij}^{AB}\phi_j\Big|_{\min}(T^{CD})_{ik}\phi_k\Big|_{\min} W_\mu^{AB}W^{\mu CD} \\
&= -\frac{1}{4}g^2 T_{ij}^{aN}\phi_j\Big|_{\min}(T^{bN})_{ik}\phi_k\Big|_{\min} W_\mu^{aN}W^{\mu bN} \\
&= \frac{1}{4}g^2 v^2 \delta_i^a \delta_i^b W_\mu^{aN}W^{\mu bN} = \frac{1}{4}g^2 v^2 W_\mu^{aN}W^{\mu bN} \tag{9.126}
\end{aligned}
$$

Therefore, the fields W_μ^{aN} associated to the broken directions T^{aN} acquire a mass $g^2 v^2/2$, whereas W_μ^{ab}, associated to the unbroken symmetry $O(N-1)$, remain massless.

In general, if G is the global symmetry group of the lagrangian, H the subgroup of G leaving invariant the vacuum, and G_W the group of local (gauge) symmetries, $G_W \in G$, one can divide up the broken generators in two categories. In the first category fall the broken generators lying in G_W; they have associated massive vector bosons. In the second category fall the other broken generators; they have associated massless Goldstone bosons. Finally the gauge fields associated to generators of G_W lying in H remain massless. From the previous derivation this follows noticing that the generators of H annihilate the minimum of the fields, leaving the corresponding gauge bosons massless, whereas the non zero action of the broken generators generate a mass term for the other gauge fields.

The situation is represented in Fig. 9.4.1.
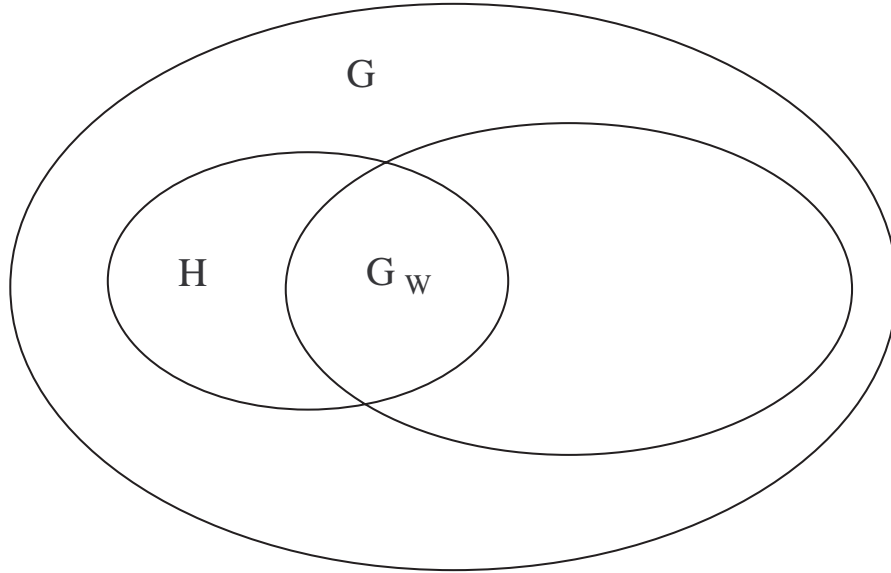
Fig. 6.1 -*This figure shows the various groups, G, the global symmetry of the lagrangian, $H \in G$, the symmetry of the vacuum, and G_W, the group of local symmetries. The broken generators in G_W correspond to massive vector bosons. The broken generators do not belonging to G_W correspond to massless Goldstone bosons. Th unbroken generators in G_W correspond to massless vector bosons.*

We can now show how to eliminate the Goldstone bosons. In fact we can define new fields ξ_a and χ as

$$\phi_i = \left(e^{-iT^{aN}\xi_a} \right)_{iN} (\chi + v) \tag{9.127}$$

where $a = 1, \cdots, N-1$, that is the sum is restricted to the broken directions. The other degree of freedom is in the other factor. The correspondence among the fields

$\vec{\phi}$ and (ξ_a, χ) can be seen easily by expanding around the vacuum

$$\left(e^{-iT^{aN}\xi_a} \right)_{iN} \approx \delta_{iN} - i(T^{aN})_{iN}\xi_a = \delta_{iN} + \delta_i^a \xi_a \tag{9.128}$$

from which

$$\phi_i \approx (\xi_a, \chi + v) \tag{9.129}$$

showing that the ξ_a's are really the Goldstone fields. The unitary gauge is defined through the transformation

$$\phi_i \rightarrow \left(e^{iT^{aN}\xi_a} \right)_{ij} \phi_j = \delta_{iN}(\chi + v) \tag{9.130}$$

$$W_\mu \rightarrow e^{iT^{aN}\xi_a} W_\mu e^{-iT^{aN}\xi_a} - \frac{i}{g}\left(\partial_\mu e^{iT^{aN}\xi_a} \right) e^{-iT^{aN}\xi_a} \tag{9.131}$$

This transformation eliminates the Goldstone degrees of freedom and the resulting lagrangian depends on the field χ, on the massive vector fields W_μ^{aN} and on the massless field W_μ^{ab}. Notice again the counting of the degrees of freedom $N + 2N(N - 1)/2 = N^2$ in a generic gauge, and $1 + 3(N - 1) + 2(N - 1)(N - 2)/2 = N^2$ in the unitary gauge.

9.5 Quantization of a spontaneously broken gauge theory in the R_ξ-gauge

In the previous Section we have seen that in the unitary gauge some of the gauge fields acquire a mass. The corresponding quadratic part of the lagrangian will be (we consider here the abelian case corresponding to a symmetry $U(1)$ or $O(2)$)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}W^2 \tag{9.132}$$

where W_μ is the massive gauge field (called also a Yang-Mills massive field). The free equations of motion are

$$(\Box + m^2)W_\mu - \partial_\mu(\partial^\nu W_\nu) = 0 \tag{9.133}$$

Let us study the solutions of this equation in momentum space. We define

$$W_\mu(x) = \int d^4 \, W_\mu(k)e^{-ikx} \tag{9.134}$$

It is convenient to introduce a set of four independent four-vectors. By putting $k^\mu = (E, \vec{k})$ we introduce

$$\epsilon^{\mu(i)} = (0, \vec{n}_i), \qquad i = 1, 2$$
$$\epsilon^{\mu(3)} = \frac{1}{m}\left(|\vec{k}|, E\frac{\vec{k}}{|\vec{k}|} \right) \tag{9.135}$$

with
$$\vec{k} \cdot \vec{n}_i = 0, \qquad |\vec{n}_i|^2 = 1 \tag{9.136}$$

Then we can decompose $W_\mu(k)$ as follows
$$W_\mu(k) = \epsilon^{\mu(\lambda)} a_\lambda(k) + k_\mu b(k) \tag{9.137}$$

Since $k_\mu \epsilon^{\mu(\lambda)} = 0$, from the equation of motion we get
$$(-k^2 + m^2)(\epsilon^{\mu(\lambda)} a_\lambda(k) + k_\mu b(k)) + k_\mu k^2 b(k) = 0 \tag{9.138}$$

implying
$$(k^2 - m^2) a_\lambda(k) = 0, \qquad m^2 b(k) = 0 \tag{9.139}$$

Therefore, for $m \neq 0$ the field has three degrees of freedom with $k^2 = m^2$, corresponding to transverse and longitudinal polarization. When the momentum is on shell, we can define a fourth unit vector
$$\epsilon^{\mu(0)} = \frac{k^\mu}{m} \tag{9.140}$$

In this way we have four independent unit vectors which satisfy the completeness relation
$$\sum_{\lambda, \lambda'=0}^{3} \epsilon^{\mu(\lambda)} \epsilon^{\nu(\lambda')} g_{\lambda\lambda'} = g^{\mu\nu} \tag{9.141}$$

Therefore the sum over the physical degrees of freedom gives
$$\sum_{\lambda=1}^{3} \epsilon^{\mu(\lambda)} \epsilon^{\nu(\lambda)} = -g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2} \tag{9.142}$$

The propagator is defined by the equation
$$\left[(\Box + m^2) - \partial_\mu \partial_\nu \right] G^{\nu\lambda}(x) = i g_\mu^\lambda \delta^4(x) \tag{9.143}$$

Solving in momentum space we find
$$G^{\mu\nu}(k) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{i}{k^2 - m^2 + i\epsilon} \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2} \right) \tag{9.144}$$

Therefore, the corresponding Feynman's rule for the massive gauge field propagator is
$$-\frac{i}{k^2 - m^2 + i\epsilon} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2} \right) \tag{9.145}$$

Notice that the expression in parenthesis is nothing but the projector over the physical states given in eq. (9.142). This propagator has a very bad high-energy behaviour. In fact it goes to constant for $k \to \infty$. As a consequence the argument

231

for the renormalizability of the theory based on power counting does not hold any more. Although in the unitary gauge a spontaneously broken gauge theory does not seem to be renormalizable, we have to take into account that the gauge invariance allows us to make different choices of the gauge. In fact we will show that there exist an entire class of gauges where the theory is renormalizable by power counting. We will discuss this point only for the abelian case, with gauge group $O(2)$, but the argument cab be easily extended to the non-abelian case. Let us consider the $O(2)$ theory as presented in Section 9.4. Let us recall that under a transformation of $O(2)$ the two real scalar field of the theory transform as

$$\delta\phi_1 = -\alpha\phi_2, \quad \delta\phi_2 = -\alpha\phi_1 \tag{9.146}$$

Therefore, in order to make it a gauge theory we introduce covariant derivatives

$$\begin{aligned} D_\mu\phi_1 &= \partial_\mu\phi_1 + gW_\mu\phi_2 \\ D_\mu\phi_2 &= \partial_\mu\phi_2 - gW_\mu\phi_1 \end{aligned} \tag{9.147}$$

with W_μ transforming under a local transformation as

$$\delta W_\mu(x) = \frac{1}{g}\partial_\mu\alpha(x) \tag{9.148}$$

Assuming that the spontaneous breaking occurs along the direction ϕ_1, we introduce the fields

$$\phi_1 = v + \chi, \quad \phi_2 = \phi \tag{9.149}$$

with

$$v^2 = -\frac{\mu^2}{\lambda} \tag{9.150}$$

Remember that χ is the Higgs field, whereas ϕ is the Goldstone field. In terms of the new variables the lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\chi + gW_\mu\phi)^2 + \frac{1}{2}(\partial_\mu\phi - gW_\mu\chi - gvW_\mu)^2 - V(\chi, phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{9.151}$$

where, the potential up to quadratic terms is given by (see eq. (9.101))

$$V(\chi, \phi) = -\frac{m^4}{16\lambda} + \frac{1}{2}m^2\chi^2 + \cdots \tag{9.152}$$

with $m^2 = -2\mu^2$. We see that the effect of the shift on ϕ_1 is to produce the terms

$$\frac{1}{2}g^2v^2W^2 - gvW_\mu\partial^\mu\phi + g^2vW^2\chi \tag{9.153}$$

The last term is an interaction term and it does not play any role in the following considerations. The first term is the mass term for the W

$$m_W^2 = g^2v^2 \tag{9.154}$$

232

whereas the second term is the mixing between the W and the Goldstone field, that we got rid of by choosing the unitary gauge. However this choice is not unique. In fact let us consider a gauge fixing of the form

$$f - B = 0, \quad f = \partial^\mu W_\mu + gv\xi\phi \tag{9.155}$$

where ξ is an arbitrary parameter and $B(x)$ is an arbitrary function. By proceeding as in Section 8.3 we know that the effect of such a gauge fixing is to add to the lagrangian a term

$$-\frac{1}{2\beta}f^2 = -\frac{1}{2\beta}(\partial^\mu W_\mu + gv\xi\phi)^2 = -\frac{1}{2\beta}(\partial^\mu W_\mu)^2 - \frac{\xi}{\beta}gv\partial^\mu W_\mu\phi - \frac{\xi^2}{2\beta}g^2v^2\phi^2 \tag{9.156}$$

The first term is the usual covariant gauge fixing. The second term, after integration by parts reads

$$+\frac{\xi}{\beta}gvW_\mu\partial^\mu\phi \tag{9.157}$$

and we see that with the choice

$$\beta = \xi \tag{9.158}$$

it cancels the unwanted mixing term. Finally the third contribution

$$-\frac{1}{2}\xi g^2v^2\phi^2 \tag{9.159}$$

is a mass term for the Goldstone field

$$m_\phi^2 = \xi g^2 v^2 = \xi m_W^2 \tag{9.160}$$

However this mass is gauge dependent (it depends on the arbitrary parameter ξ) and this is a signal of the fact that in this gauge the Goldstone field is fictitious. In fact remember that it disappears in the unitary gauge. We will see later, in an example, that evaluating a physical amplitude the contribution of the Goldstone disappears, as it should be. Finally we have to consider the ghost contribution from the Faddeev-Popov determinant. For this purpose we need to know the variation of the gauge fixing f under an infinitesimal gauge transformation

$$\delta f(x) = \frac{1}{g}\Box\alpha(x) + gv\xi\alpha(x)(\chi(x) + v) \tag{9.161}$$

giving

$$\frac{\delta f(x)}{\delta\alpha(y)} = \frac{1}{g}\Box\delta^4(x - y) + gv\xi(\chi(x) + v)\delta^4(x - y) \tag{9.162}$$

Therefore the ghost lagrangian can be written as

$$c^*\Box c + \xi g^2v^2 c^* c + \xi gv\chi c^* c \tag{9.163}$$

Notice that in contrast with QED, the ghosts cannot be ignored, since they are coupled to the Higgs field. We are now in the position of reading the various propagators from the quadratic part of the lagrangian

- Higgs propagator (χ): $\dfrac{i}{k^2 - m^2}$

- Goldstone propagator (ϕ): $\dfrac{i}{k^2 - \xi m_W^2}$

- Ghost propagator (c): $-\dfrac{i}{k^2 - \xi m_W^2}$

The propagator for the vector field is more involved. let us collect the various quadratic terms in W_μ

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_W^2 W^2 - \frac{1}{2\xi} (\partial^\mu W_\mu)^2 \tag{9.164}$$

In momentum space wave operator is

$$(g^{\mu\nu} k^2 - k^\mu k^\nu) - m_W^2 g^{\mu\nu} + \frac{1}{\xi} k^\mu k^\nu \tag{9.165}$$

which can be rewritten as

$$\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right)(k^2 - m_W^2) + \frac{k^\mu k^\nu}{k^2} \frac{1}{\xi} (k^2 - \xi m_W^2) \tag{9.166}$$

The propagator in momentum space is the inverse of the wave operator (except for a factor $-i$). To invert the operator we notice that it is written as a combination of two orthogonal projection operators P_1 and P_2

$$\alpha P_1 + \beta P_2 \tag{9.167}$$

The inverse of such an operator is simply

$$\frac{1}{\alpha} P_1 + \frac{1}{\beta} P_2 \tag{9.168}$$

Therefore the W propagator is given by

$$
\begin{aligned}
&-\frac{i}{k^2 - m_W^2} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) - \frac{i\xi}{k^2 - \xi m_W^2} \frac{k^\mu k^\nu}{k^2} \\
&= -\frac{i}{k^2 - m_W^2} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2 - \xi m_W^2} (1 - \xi) \right)
\end{aligned} \tag{9.169}
$$

We see that for this class of gauges the asymptotic behaviour of the propagator is $1/k^2$ and we can apply the usual power counting. Notice that the unitary gauge is recovered in the limit $\xi \to \infty$. In this limit the W-propagator becomes the one discussed at the beginning of the Section. Of course, the singularity of the limit destroys the asymptotic behaviour that one has for finite values of ξ. Also, for $\xi \to \infty$ the ghost and the Goldstone fields decouple since their mass goes to infinity.

To end this Section we observe that with this approach one is able to prove at the same time that the theory is renormalizable and that it does not contain spurious states. In fact the theory is renormalizable for finite values of ξ and it does not contain spurious states for $\xi \to \infty$. But, since the gauge invariant quantities do not depend on the gauge fixing, they do not depend in particular on ξ, and therefore one gets the wanted result.

9.6 ξ-cancellation in perturbation theory

In this Section we will discuss the cancellation of the ξ dependent terms in the calculation of a simple amplitude at tree level. To this end we consider again the $O(2)$ model of the previous Section, and we add to the theory a fermion field. We will also mimic the standard model by making transform under the gauge group only the left-handed part of the fermion field. That is we decompose ψ as

$$\psi = \psi_L + \psi_R \tag{9.170}$$

where

$$\psi_L = \frac{1 - \gamma_5}{2}\psi, \quad \psi_R = \frac{1 + \gamma_5}{2}\psi \tag{9.171}$$

and we assume the following transformation under the gauge group

$$\psi_L \to e^{i\alpha}\psi_L, \quad \psi_R \to \psi_R \tag{9.172}$$

A fermion mass term would destroy the gauge invariance, since it couples left-handed and right-handed fields. In fact

$$\bar{\psi}_L = \overline{\left(\frac{1 - \gamma_5}{2}\right)\psi} = \psi^\dagger \frac{1 - \gamma_5}{2}\gamma_0 = \bar{\psi}\frac{1 + \gamma_5}{2} \tag{9.173}$$

and therefore

$$\bar{\psi}\psi = \bar{\psi}\left(\frac{1 + \gamma_5}{2} + \frac{1 - \gamma_5}{2}\right)\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L \tag{9.174}$$

Therefore we will introduce a Yukawa coupling between the scalar field and the fermions, in such a way to be gauge invariant and to gives rise to a mass term for the fermion when the symmetry is spontaneously broken. We will write

$$\mathcal{L}_f = \bar{\psi}_L i\hat{D}\psi_L + \bar{\psi}_R i\hat{\partial}\psi_R - \lambda_f(\bar{\psi}_L\psi_R\Phi + \bar{\psi}_R\psi_L\Phi^*) \tag{9.175}$$

with

$$D_\mu = \partial_\mu - igW_\mu \tag{9.176}$$

Here we have used again the complex notation for the scalar field, as in eq. (9.109), and we remind that it transforms as

$$\Phi \to e^{i\alpha}\Phi \tag{9.177}$$

235

Therefore $\mathcal{L}_f$ is gauge invariant. The lagrangian can be rewritten in terms of the field ψ by writing explicitly the chiral projectors $(1 \pm \gamma_5)/2$. We get

$$
\begin{aligned}
\mathcal{L}_f \;=\;\; & \bar{\psi} i \hat{\partial} \frac{1 - \gamma_5}{2} \psi + \bar{\psi} i \hat{\partial} \frac{1 + \gamma_5}{2} \psi + g \bar{\psi} \gamma_\mu \frac{1 - \gamma_5}{2} \psi W^\mu \\
& - \frac{\lambda_f}{\sqrt{2}} \left(\bar{\psi} \frac{1 + \gamma_5}{2} \psi (\phi_1 + i\phi_2) + \bar{\psi} \frac{1 - \gamma_5}{2} \psi (\phi_1 - i\phi_2) \right)
\end{aligned}
\tag{9.178}
$$

from which

$$
\mathcal{L} = \bar{\psi} i \hat{\partial} \psi + g \bar{\psi} \gamma_\mu \frac{1 - \gamma_5}{2} \psi W^\mu - \frac{\lambda_f}{\sqrt{2}} \left(\bar{\psi} \psi (v + \chi) + i \bar{\psi} \gamma_5 \psi \phi \right)
\tag{9.179}
$$

where $\chi + v = \phi_1$ and $\phi = \phi_2$. This expression shows that the fermion field acquires a mass through the Higgs mechanism given by

$$
m_f = \frac{\lambda_f v}{\sqrt{2}}
\tag{9.180}
$$

The previous expression shows also that the Higgs field is a scalar (parity $+1$), whereas the Goldstone field is a pseudoscalar (parity= -1).Consider now the fermion-fermion scattering in this model at tree level. The amplitude gets contribution from the diagrams of Fig. 9.6.1.

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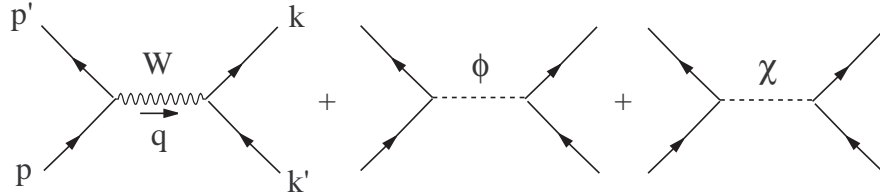
Fig. 9.6.1 - *The fermion-fermion scattering in the $U(1)$ Higgs model.*

We want to show that the ξ contribution cancels out. The ξ dependence arises from the gauge field and the Goldstone field exchange. Therefore we will consider only these two contributions. Let us start with the Goldstone exchange. The Feynman's rule for the vertex is given in Fig. 9.6.2.
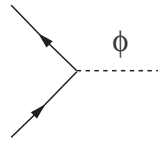
.

Fig. 9.6.2 - *Goldstone-fermion vertex:* $\dfrac{\lambda_f}{\sqrt{2}} \gamma_5$

The corresponding amplitude is

$$\mathcal{M}_f = \left(\frac{\lambda_f}{\sqrt{2}}\right)^2 \bar{u}(p')\gamma_5 u(p)\frac{i}{q^2 - \xi m_W^2}\bar{u}(k')\gamma_5 u(k) \tag{9.181}$$

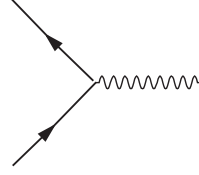where $q = p - p' = k' - k$. The Feynman rule for the W-fermion vertex is given in Fig. 9.6.3.

.

Fig. 9.6.3 - *W-fermion vertex: $ig\gamma_\mu\dfrac{1 - \gamma_5}{2}$*

The corresponding amplitude is

$$
\begin{aligned}
\mathcal{M}_W \;&=\; (ig)^2\bar{u}(p')\gamma_\mu\frac{1 - \gamma_5}{2}u(p) \\
&\times\; \frac{-i}{q^2 - m_W^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2 - \xi m_W^2}(1 - \xi)\right)\bar{u}(k')\gamma_\nu\frac{1 - \gamma_5}{2}u(k)
\end{aligned} \tag{9.182}
$$

It is convenient to isolate the ξ dependence by rewriting the gauge field propagator as

$$
\begin{aligned}
&\frac{-i}{q^2 - m_W^2}\left[g^{\mu\nu} - \frac{q^\mu q^\nu}{m_W^2} + q^\mu q^\nu\left(\frac{1}{m_W^2} - \frac{1 - \xi}{q^2 - \xi m_W^2}\right)\right] \\
=\; =\; &\frac{-i}{q^2 - m_W^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{m_W^2}\right) + \frac{-i}{q^2 - \xi m_W^2}\frac{q^\mu q^\nu}{m_W^2} \tag{9.183}
\end{aligned}
$$

The first term is ξ independent and it is precisely the W propagator in the unitary gauge. The second term can be simplified by using the identity

$$
\begin{aligned}
q^\mu \bar{u}(p')\gamma_\mu\frac{1 - \gamma_5}{2}u(p) &= (p - p')^\mu \bar{u}(p')\gamma_\mu\frac{1 - \gamma_5}{2}u(p) = \\
=\; \bar{u}(p')&\left(\frac{1 + \gamma_5}{2}\hat{p} - \hat{p}'\frac{1 - \gamma_5}{2}\right)u(p) = m_f\bar{u}(p')\gamma_5 u(p) \tag{9.184}
\end{aligned}
$$

By using this relation we have

$$
\begin{aligned}
q^\mu q^\nu \bar{u}(p')&\gamma_\mu\frac{1 - \gamma_5}{2}u(p)\bar{u}(k')\gamma_\mu\frac{1 - \gamma_5}{2}u(k) = \\
=\; &(m_f)\bar{u}(p')\gamma_5 u(p)(-m_f)\bar{u}(k')\gamma_5 u(k) \\
=\; &-\left(\frac{\lambda_f}{\sqrt{2}}\right)^2 v^2\bar{u}(p')\gamma_5 u(p)\bar{u}(k')\gamma_5 u(k) \tag{9.185}
\end{aligned}
$$

237

where we have made use of eq. (9.180). Substituting this result in $\mathcal{M}_W$ we get

$$
\begin{aligned}
\mathcal{M}_W &= (ig)^2 \bar{u}(p') \gamma_\mu \frac{1-\gamma_5}{2} u(p) \frac{-i}{q^2 - m_W^2} \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{m_W^2} \right) \bar{u}(k') \gamma_\mu \frac{1-\gamma_5}{2} u(k) \\
&+ \frac{-i}{q^2 - \xi m_W^2} \frac{g^2 v^2}{m_W^2} \left(\frac{\lambda_f}{\sqrt{2}} \right)^2 \bar{u}(p') \gamma_5 u(p) \bar{u}(k') \gamma_5 u(k)
\end{aligned}
\tag{9.186}
$$

Finally, using $m_W^2 = g^2 v^2$ we see that the last term cancels exactly the $\mathcal{M}_\phi$ amplitude. Therefore the result is the same that we would have obtained in the unitary gauge where there is no Goldstone contribution.

Now let us make some comment about the result. In QED we have used the fact that the $q^\mu q^\nu$ terms in the propagator do not play a physical role, since they are contracted with the fermionic current which is conserved. In the present case this does not happens, but we get a term proportional to the fermion mass. In fact the fermionic current $\bar{\psi} \gamma_\mu (1 - \gamma_5)/2\psi$ it is not conserved because the fermion acquires a mass through the spontaneous breaking of the symmetry. On the other hand, the term that one gets in this way it is just the one necessary to give rise to the cancellation with the Goldstone boson contribution. It is also worth to compare what happens for different choices of ξ. For $\xi = 0$ the gauge propagator is transverse

$$
\frac{-i}{k^2 - m_W^2} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right)
\tag{9.187}
$$

This corresponds to the Lorentz gauge. The propagator has a pole term at $k^2 = 0$, which however it is cancelled by the Goldstone pole which also is at $k^2 = 0$. For the choice $\xi = 1$ the gauge propagator has the very simple form

$$
\frac{-i g^{\mu\nu}}{k^2 - m_W^2}
\tag{9.188}
$$

This is the Feynman-'t Hooft gauge. Here the role of the Goldstone boson, which has a propagator

$$
\frac{i}{k^2 - m_W^2}
\tag{9.189}
$$

it is to cancel the contribution of the time-like component of the gauge field, which is described by the polarization vector $\epsilon_\mu^{(0)}$. In fact the gauge propagator can be rewritten as

$$
\frac{-i g^{\mu\nu}}{k^2 - m_W^2} = \frac{-i}{k^2 - m_W^2} \sum_{\lambda,\lambda'=0}^{3} \epsilon_\mu^{(\lambda)} \epsilon_\nu^{(\lambda')} g_{\lambda\lambda'} = \frac{i}{k^2 - m_W^2} \sum_{\lambda=1}^{3} \epsilon_\mu^{(\lambda)} \epsilon_\nu^{(\lambda)} + \frac{-i}{k^2 - m_W^2} \frac{k^\mu k^\nu}{m_W^2}
\tag{9.190}
$$

The first piece is the propagator in the unitary gauge, whereas the second piece is cancelled by the Goldstone boson contribution.

Chapter 10

The Standard model of the electroweak interactions

10.1 The Standard Model of the electroweak interactions

We will dscribe now the Weinberg Salam model for the electroweak interactions. Let us recall that the Fermi theory of weak interactions is based on a lagrangian involving four Fermi fields. This lagrangian can be written as the product of two currents which are the sum of various pieces.

$$\mathcal{L}_F = \frac{G_F}{\sqrt{2}} J_\mu^{(+)} J^{\mu(-)} \tag{10.1}$$

Limiting for the moment our considerations to the charged current for the electron and its neutrino (here $\nu \equiv \nu_e$)), the current is given by

$$J_\mu^{(+)} = \bar{\psi}_e \gamma_\mu (1 - \gamma_5) \psi_\nu, \qquad J_\mu^{(-)} = J_\mu^{(+)\dagger} = \bar{\psi}_\nu \gamma_\mu (1 - \gamma_5) \psi_e \tag{10.2}$$

To understand the symmetry hidden in this couplings it is convenient to introduce the following spinors with 2×4 components

$$L = \begin{pmatrix} (\psi_\nu)_L \\ (\psi_e)_L \end{pmatrix} = \frac{1 - \gamma_5}{2} \begin{pmatrix} \psi_\nu \\ \psi_e \end{pmatrix} \tag{10.3}$$

where we have introduced left-handed fields. The right-handed spinors, projected out with $(1 + \gamma_5)/2$, don't feel the weak interaction. This is, in fact, the physical meaning of the $V - A$ interaction. By using

$$\bar{L} = \left((\bar{\psi}_\nu)_L, (\bar{\psi}_e)_L \right) = \left(\bar{\psi}_\nu, \bar{\psi}_e \right) \frac{1 + \gamma_5}{2} \tag{10.4}$$

we can write the weak currents in the following way

$$
\begin{aligned}
J_\mu^{(+)} &= 2\bar{\psi}_e \frac{1+\gamma_5}{2} \gamma_\mu \frac{1-\gamma_5}{2} \psi_\nu = 2 \left((\bar{\psi}_\nu)_L, (\bar{\psi}_e)_L \right) \gamma_\mu \begin{pmatrix} 0 \\ (\psi_\nu)_L \end{pmatrix} \\
&= 2 \left((\bar{\psi}_\nu)_L, (\bar{\psi}_e)_L \right) \gamma_\mu \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (\psi_\nu)_L \\ (\psi_e)_L \end{pmatrix} = 2\bar{L}\gamma_\mu \tau_- L
\end{aligned}
\tag{10.5}
$$

where we have defined

$$
\tau_\pm = \frac{\tau_1 \pm i\tau_2}{2}
\tag{10.6}
$$

the τ_i's being the Pauli matrices. Analogously

$$
J_\mu^{(-)} = 2\bar{L}\gamma_\mu \tau_+ L
\tag{10.7}
$$

In the fifties the hypothesis of the intermediate vector bosons was advanced. According to this idea the Fermi theory was to be regarded as the low energy limit of a theory where the currents were coupled to vector bosons, very much as QED. However, due to the short range of the weak interactions these vector bosons had to have mass. The interaction among the vector bosons and the fermions is given by

$$
\mathcal{L}_{\text{int}} = \frac{g}{2\sqrt{2}} \left[J_\mu^{(+)} W^{-\mu} + J_\mu^{(-)} W^{+\mu} \right] = \frac{g}{\sqrt{2}} \bar{L}\gamma_\mu (\tau_- W^{-\mu} + \tau_+ W^{+\mu}) L
\tag{10.8}
$$

The coupling g can be related to the Fermi constant through the relation

$$
\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}
\tag{10.9}
$$

Introducing real fields in place of $W_\pm$

$$
W^\pm = \frac{W_1 \mp iW_2}{\sqrt{2}}
\tag{10.10}
$$

we get

$$
\mathcal{L}_{\text{int}} = \frac{g}{2}\bar{L}\gamma_\mu (\tau_1 W_1^\mu + \tau_2 W_2^\mu) L
\tag{10.11}
$$

Let us now try to write the electromagnetic interaction within the same formalism

$$
\begin{aligned}
\mathcal{L}_{\text{em}} = e\bar{\psi}_e Q \gamma_\mu \psi_e A^\mu &= e\bar{\psi}_e Q \left(\frac{1+\gamma_5}{2}\gamma_\mu \frac{1-\gamma_5}{2} + \frac{1-\gamma_5}{2}\gamma_\mu \frac{1+\gamma_5}{2} \right) \psi_e A^\mu \\
&= e[(\bar{\psi}_e)_L Q \gamma_\mu (\psi_e)_L + (\bar{\psi}_e)_R Q \gamma_\mu (\psi_e)_R] A^\mu
\end{aligned}
\tag{10.12}
$$

where

$$
(\psi_e)_R = \frac{1+\gamma_5}{2}\psi_e, \qquad (\bar{\psi}_e)_R = \bar{\psi}_e \frac{1-\gamma_5}{2}
\tag{10.13}
$$

where $Q = -1$ is the electric charge of the electron, in unit of the electric charge of the proton, e. It is also convenient to denote the right-handed components of the electron as

$$
R = (\psi_e)_R
\tag{10.14}
$$

240

We can write
$$(\bar{\psi}_e)_L \gamma_\mu (\psi_e)_L = \bar{L}\gamma_\mu \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} L = \bar{L}\gamma_\mu \frac{1 - \tau_3}{2} L \qquad (10.15)$$

and
$$(\bar{\psi}_e)_R \gamma_\mu (\psi_e)_R = \bar{R}\gamma_\mu R \qquad (10.16)$$

from which, using $Q = -1$,
$$\mathcal{L}_{\text{em}} = e\left(-\frac{1}{2}\bar{L}\gamma_\mu L - \bar{R}\gamma_\mu R + \bar{L}\gamma_\mu \frac{\tau_3}{2} L\right) A^\mu = e j_\mu^{\text{em}} A^\mu \qquad (10.17)$$

From this equation we can write
$$j_\mu^{\text{em}} = j_\mu^3 + \frac{1}{2}j_\mu^Y \qquad (10.18)$$

with
$$j_\mu^3 = \bar{L}\gamma_\mu \frac{\tau_3}{2} L \qquad (10.19)$$

and
$$j_\mu^Y = -\bar{L}\gamma_\mu L - 2\bar{R}\gamma_\mu R \qquad (10.20)$$

These equations show that the electromagnetic current is a mixture of j_μ^3, which is a partner of the charged weak currents, and of j_μ^Y, which is another neutral current. The following notations unify the charged currents and j_μ^3
$$j_\mu^i = \bar{L}\gamma_\mu \frac{\tau_i}{2} L \qquad (10.21)$$

Using the canonical anticommutators for the Fermi fields, it is not difficukt to prove that the charges associated to the currents
$$Q^i = \int d^3\vec{x}\, j_0^i(x) \qquad (10.22)$$

span the Lie algebra of $SU(2)$,
$$[Q^i, Q^j] = i\epsilon_{ijk}Q^k \qquad (10.23)$$

The charge Q^Y associated to the current j_μ^Y commutes with Q^i, as it can be easily verified noticing that the right-handed and left-handed fields commute with each other. Therefore the algebra of the charges is
$$[Q^i, Q^j] = i\epsilon_{ijk}Q^k, \qquad [Q^i, Q^Y] = 0 \qquad (10.24)$$

This is the Lie algebra of $SU(2) \otimes U(1)$, since the Q^i's generate a group $SU(2)$, whereas Q^Y generates a group $U(1)$, with the two groups commuting among themselves. To build up a gauge theory from these elements we have to start with an initial lagrangian possessing an $SU(2) \otimes U(1)$ global invariance. This will produce

4 gauge vector bosons. Two of these vectors are the charged one already introduce, whereas we would like to identify one of the the other two with the photon. By evaluating the commutators of the charges with the fields we can read immediately the quantum numbers of the various particles. We have

$$[L_c, Q^i] = \int d^3x \, [L_c, L_a^\dagger \left(\frac{\tau_i}{2}\right)_{ab} L_b] = \left(\frac{\tau_i}{2}\right)_{cb} L_b \tag{10.25}$$

and

$$[R, Q^i] = 0 \tag{10.26}$$

$$[L_c, Q^Y] = -L_c, \qquad [R, Q^Y] = -2R \tag{10.27}$$

These relations show that L transforms as an $SU(2)$ spinor, or, as the representation $\underline{2}$, where we have identified the representation with its dimensionality. R belongs to the trivial representation of dimension $\underline{1}$. Putting everything together we have the following behaviour under the representations of $SU(2) \otimes U(1)$

$$L \in (\underline{2}, -1), \qquad R \in (\underline{1}, -2) \tag{10.28}$$

The relation (10.18) gives the following relation among the different charges

$$Q_{\text{em}} = Q^3 + \frac{1}{2} Q^Y \tag{10.29}$$

As we have argued, if $SU(2) \otimes U(1)$ has to be the fundamental symmetry of electroweak interactions, we have to start with a lagrangian exhibiting this global symmetry. This is the free Dirac lagrangian for a massless electron and a left-handed neutrino

$$\mathcal{L}_0 = \bar{L} i \hat{\partial} L + \bar{R} i \hat{\partial} R \tag{10.30}$$

Mass terms, mixing left- and right-handed fields, would destroy the global symmetry. For instance, a typical mass term gives

$$m\bar{\psi}\psi = m\psi^\dagger \left[\frac{1+\gamma_5}{2}\gamma_0\frac{1-\gamma_5}{2} + \frac{1-\gamma_5}{2}\gamma_0\frac{1+\gamma_5}{2}\right]\psi = m(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R) \tag{10.31}$$

We shall see later that the same mechanism giving mass to the vector bosons can be used to generate fermion masses. A lagrangian invariant under the local symmetry $SU(2) \otimes U(1)$ is obtained through the use of covariant derivatives

$$\mathcal{L} = \bar{L} i \gamma^\mu \left(\partial_\mu - ig\frac{\tau_i}{2}W_\mu^i + i\frac{g'}{2}Y_\mu\right) L + \bar{R} i \gamma^\mu \left(\partial_\mu + ig'Y^\mu\right) R \tag{10.32}$$

The interaction term is

$$\mathcal{L}_{\text{int}} = g\bar{L}\gamma_\mu \left(\frac{\vec{\tau}}{2} \cdot \vec{W}^\mu\right) L - \frac{g'}{2}\bar{L}\gamma_\mu L Y^\mu - g'\bar{R}\gamma_\mu R Y^\mu \tag{10.33}$$

242

In this expression we recognize the charged interaction with $W^{\pm}$. We have also, as expected, two neutral vector bosons W^3 and Y. The photon must couple to the electromagnetic current which is a linear combination of j^3 and j^Y. The neutral vector bosons are coupled to these two currents, so we expect the photon field, A_{μ}, to be a linear combination of W^3_{μ} and Y_{μ}. It is convenient to introduce two orthogonal combination of these two fields, Z_{μ}, and A_{μ}. The mixing angle can be identified through the requirement that the current coupled to A_{μ} is exactly the electromagnetic current with coupling constant e. Let us consider the part of $\mathcal{L}_{\text{int}}$ involving the neutral couplings

$$\mathcal{L}_{NC} = g j^3_{\mu} W^{\mu}_3 + \frac{g'}{2} j^Y_{\mu} Y^{\mu} \tag{10.34}$$

By putting (θ is called the Weinberg angle)

$$W_3 = Z \, \cos\theta + A \, \sin\theta$$
$$Y \ = -Z \, \sin\theta + A \, \cos\theta \tag{10.35}$$

we get

$$\mathcal{L}_{NC} = \left[g \sin\theta \, j^3_{\mu} + g' \cos\theta \frac{1}{2} j^Y_{\mu} \right] A^{\mu} + \left[g \cos\theta \, j^3_{\mu} - g' \sin\theta \frac{1}{2} j^Y_{\mu} \right] Z^{\mu} \tag{10.36}$$

The electromagnetic coupling is reproduced by requiring the two conditions

$$g \sin\theta = g' \cos\theta = e \tag{10.37}$$

These two relations allow us to express both the Weinberg angle and the electric charge e in terms of g and g'.

In the low-energy experiments performed before the LEP era (< 1990) the fundamental parameters used in the theory were e (or rather the fine structure constant α) and $\sin^2\theta$. We shall see in the following how the Weinberg angle can be eliminated in favor of the mass of the Z, which is now very well known. By using eq. (10.37), and $j^{\text{em}}_{\mu} = j^3_{\mu} + \frac{1}{2} j^Y_{\mu}$, we can write

$$\mathcal{L}_{NC} = e j^{\text{em}}_{\mu} A^{\mu} + [g \cos\theta j^3_{\mu} - g' \sin\theta (j^{\text{em}}_{\mu} - j^3_{\mu})] Z^{\mu} \tag{10.38}$$

Expressing g' through $g' = g \tan\theta$, the coefficient of Z_{μ} can be put in the form

$$[g \cos\theta + g' \sin\theta] j^3_{\mu} - g' \sin\theta \, j^{\text{em}}_{\mu} = \frac{g}{\cos\theta} j^3_{\mu} - \frac{g}{\cos\theta} \sin^2\theta \, j^{\text{em}}_{\mu} \tag{10.39}$$

from which

$$\mathcal{L}_{NC} = e j^{\text{em}}_{\mu} A^{\mu} + \frac{g}{\cos\theta} [j^3_{\mu} - \sin^2\theta \, j^{\text{em}}_{\mu}] Z^{\mu} \equiv e j^{\text{em}}_{\mu} A^{\mu} + \frac{g}{\cos\theta} j^Z_{\mu} Z^{\mu} \tag{10.40}$$

where we have introduced the neutral current coupled to the Z

$$j^Z_{\mu} = j^3_{\mu} - \sin^2\theta \, j^{\text{em}}_{\mu} \tag{10.41}$$

243

The value of $\sin^2 \theta$ was evaluated initially from processes induced by neutral currents (as the scattering $\nu - e$) at low energy. The approximate value is

$$\sin^2 \theta \approx 0.23 \qquad (10.42)$$

This shows that both g and g' are of the same order of the electric charge. Notice that the eq. (10.37) gives the following relation among the electric charge and the couplings g and g'

$$\frac{1}{e^2} = \frac{1}{g^2} + \frac{1}{g'^2} \qquad (10.43)$$

10.2   The Higgs sector in the Standard Model

According to the discussion made in the previous Section, we need three broken symmetries in order to give mass to $W^\pm$ and Z. Since $SU(2) \otimes U(1)$ has four generators, we will be left with one unbroken symmetry, that we should identify with the group $U(1)$ of the electromagnetism ($U(1)_{\text{em}}$), in such a way to have the corresponding gauge particle (the photon) massless. Therefore, the group $U(1)_{\text{em}}$ must be a symmetry of the vacuum (that is the vacuum should be electrically neutral). To realize this aim we have to introduce a set of scalar fields transforming in a convenient way under $SU(2) \otimes U(1)$. The simplest choice turns out to be a complex representation of $SU(2)$ of dimension 2 (Higgs doublet). As we already noticed the vacuum should be electrically neutral, so one of the components of the doublet must also be neutral. Assume that this component is the lower one, than we will write

$$\Phi = \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix} \qquad (10.44)$$

To determine the weak hypercharge, Q^Y, of Φ, we use the relation

$$Q_{\text{em}} = Q^3 + \frac{1}{2} Q^Y \qquad (10.45)$$

for the lower component of Φ. We get

$$Q^Y(\phi^0) = 2 \times \left[0 - \left(-\frac{1}{2} \right) \right] = 1 \qquad (10.46)$$

Since Q^Y and Q^i commute, both the components of the doublet must have the same value of Q^Y, and using again eq. (10.45), we get

$$Q_{\text{em}}(\phi^+) = \frac{1}{2} + \frac{1}{2} = 1 \qquad (10.47)$$

which justifies the notation ϕ^+ for the upper component of the doublet.

The most general lagrangian for Φ with the global symmetry $SU(2) \otimes U(1)$ and containing terms of dimension lower or equal to four (in order to have a renormalizable theory) is

$$\mathcal{L}_{\text{Higgs}} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2 \tag{10.48}$$

where $\lambda > 0$ and $\mu^2 < 0$. The potential has infinitely many minima on the surface

$$|\Phi|^2_{\text{minimum}} = -\frac{\mu^2}{2\lambda} = \frac{v^2}{2} \tag{10.49}$$

with

$$v^2 = -\frac{\mu^2}{\lambda} \tag{10.50}$$

Let us choose the vacuum as the state

$$\langle 0|\Phi|0 \rangle = \begin{bmatrix} 0 \\ v/\sqrt{2} \end{bmatrix} \tag{10.51}$$

By putting

$$\Phi = \begin{bmatrix} 0 \\ v/\sqrt{2} \end{bmatrix} + \begin{bmatrix} \phi^+ \\ (h + i\eta)/\sqrt{2} \end{bmatrix} \equiv \Phi_0 + \Phi' \tag{10.52}$$

with

$$\langle 0|\Phi'|0 \rangle = 0 \tag{10.53}$$

we get

$$|\Phi|^2 = \frac{1}{2}v^2 + vh + |\phi^+|^2 + \frac{1}{2}(h^2 + \eta^2) \tag{10.54}$$

from which

$$V_{\text{Higgs}} = -\frac{1}{4}\frac{\mu^4}{\lambda} + \lambda v^2 h^2 + 2\lambda v h \left(|\phi^+|^2 + \frac{1}{2}(h^2 + \eta^2)\right) + \lambda \left(|\phi^+|^2 + \frac{1}{2}(h^2 + \eta^2)\right)^2 \tag{10.55}$$

From this expression we read immediately the particle masses $(\phi^- = (\phi^+)^\dagger)$

$$m_\eta^2 = m_{\phi^+}^2 = m_{\phi^-}^2 = 0 \tag{10.56}$$

and

$$m_h^2 = 2\lambda v^2 = -2\mu^2 \tag{10.57}$$

It is also convenient to express the parameters appearing in the original form of the potential in terms of m_h^2 e v ($\mu^2 = -2m_h^2$, $\lambda = 2m_h^2/v^2$). In this way we get

$$V_{\text{Higgs}} = -2m_h^2 + \frac{m_h^2}{2}h^2 + \frac{m_h^2}{v}h\left(|\phi^+|^2 + \frac{1}{2}(h^2 + \eta^2)\right) + \frac{m_h^2}{2v^2}\left(|\phi^+|^2 + \frac{1}{2}(h^2 + \eta^2)\right)^2 \tag{10.58}$$

Summarizing we have three massless Goldstone bosons $\phi^\pm$ and η, and a massive scalar h. This is called the Higgs field.

As usual, by now, we promote the global symmetry to a local one by introducing the covariant derivative. Recalling that $\Phi \in (\underline{2}, 1)$ of $SU(2) \otimes U(1)$, we have

$$D_\mu^H = \partial_\mu - i\frac{g}{2}\vec{\tau} \cdot \vec{W}_\mu - i\frac{g'}{2}Y_\mu \tag{10.59}$$

and

$$\mathcal{L}_{\text{Higgs}} = (D_\mu^H \Phi)^\dagger D^{H\mu}\Phi - \mu^2 \Phi^\dagger \Phi - \lambda(\Phi^\dagger \Phi)^2 \tag{10.60}$$

To study the mass generation it is convenient to write Φ in a way analogous to the one used in eq. (9.127)

$$\Phi = e^{i\vec{\xi} \cdot \vec{\tau}/v} \begin{bmatrix} 0 \\ (v + h)/\sqrt{2} \end{bmatrix} \tag{10.61}$$

According to our rules the exponential should contain the broken generators. In fact there are τ_1, and τ_2 which are broken. Furthermore it should contain the combination of 1 and τ_3 which is broken (remember that $(1 + \tau_3)/2$ is the electric charge of the doublet Φ, and it is conserved). But, at any rate, τ_3 is a broken generator, so the previous expression gives a good representation of Φ around the vacuum. In fact, expanding around this state

$$\begin{aligned}
e^{i\vec{\xi} \cdot \vec{\tau}/v} \begin{bmatrix} 0 \\ (v + h)/\sqrt{2} \end{bmatrix} &\approx \begin{bmatrix} 1 + i\xi_3/v & i(\xi_1 - i\xi_2)/v \\ i(\xi_1 + i\xi_2)/v & 1 - i\xi_3/v \end{bmatrix} \begin{bmatrix} 0 \\ (v + h)/\sqrt{2} \end{bmatrix} \\
&\approx \frac{1}{\sqrt{2}} \begin{bmatrix} i(\xi_1 - i\xi_2) \\ (v + h - i\xi_3) \end{bmatrix}
\end{aligned} \tag{10.62}$$

and introducing real components for Φ ($\phi^+ = (\phi_1 - i\phi_2)/\sqrt{2}$)

$$\Phi = \frac{1}{\sqrt{2}} \begin{bmatrix} \phi_1 - i\phi_2 \\ v + h + i\eta \end{bmatrix} \tag{10.63}$$

we get the relation among the two sets of coordinates

$$(\phi_1, \phi_2, \eta, h) \approx (\xi_2, -\xi_1, -\xi_3, h) \tag{10.64}$$

By performing the gauge transformation

$$\Phi \rightarrow e^{-i\vec{\xi} \cdot \vec{\tau}/v} \Phi \tag{10.65}$$

$$\vec{W}_\mu \cdot \frac{\vec{\tau}}{2} \rightarrow e^{-i\vec{\xi} \cdot \vec{\tau}/v} \vec{W}_\mu \cdot \frac{\vec{\tau}}{2} e^{i\vec{\xi} \cdot \vec{\tau}/v} + \frac{i}{g}\left(\partial_\mu e^{-i\vec{\xi} \cdot \vec{\tau}/v}\right) e^{i\vec{\xi} \cdot \vec{\tau}/v} \tag{10.66}$$

$$Y_\mu \rightarrow Y_\mu \tag{10.67}$$

the Higgs lagrangian remains invariant in form, except for the substitution of Φ with $(0, (v + h)/\sqrt{2})$. We will also denote old and new gauge fields with the same symbol. Defining

$$\Phi_d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \Phi_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{10.68}$$

the mass terms for the scalar fields can be read by substituting, in the kinetic term, Φ with its expectation value

$$\Phi \to \frac{v}{\sqrt{2}} \Phi_d \tag{10.69}$$

We get

$$
\begin{aligned}
-i \left(\frac{g}{2} \vec{\tau} \cdot \vec{W} + \frac{g'}{2} Y \right) \frac{v}{\sqrt{2}} \Phi_d &= -i \left(\frac{g}{\sqrt{2}} (\tau_- W^- + \tau_+ W^+) + \frac{g}{2} \tau_3 W^3 + \frac{g'}{2} Y \right) \frac{v}{\sqrt{2}} \Phi_d \\
&= -i \frac{v}{\sqrt{2}} \left[\frac{g}{\sqrt{2}} W^+ \Phi_u - \frac{1}{2} (g W^3 - g' Y) \Phi_d \right] \tag{10.70}
\end{aligned}
$$

Since Φ_d and Φ_u are orthogonals, the mass term is

$$\frac{v^2}{2} \left[\frac{g^2}{2} |W^+|^2 + \frac{1}{4} (g W^3 - g' Y)^2 \right] \tag{10.71}$$

From eq. (10.37) we have $\tan\theta = g'/g$, and therefore

$$\sin\theta = \frac{g'}{\sqrt{g^2 + g'^2}}, \qquad \cos\theta = \frac{g}{\sqrt{g^2 + g'^2}} \tag{10.72}$$

allowing us to write the mass term in the form

$$\frac{v^2}{2} \left[\frac{g^2}{2} |W^+|^2 + \frac{g^2 + g'^2}{4} (W^3 \cos\theta - Y \sin\theta)^2 \right] \tag{10.73}$$

From eq. (10.35) we see that the neutral fields combination is just the Z field, orthogonal to the photon. In this way we get

$$\frac{g^2 v^2}{4} |W^+|^2 + \frac{1}{2} \frac{g^2 + g'^2}{4} v^2 Z^2 \tag{10.74}$$

Finally the mass of the vector bosons is given by

$$M_W^2 = \frac{1}{4} g^2 v^2, \qquad M_Z^2 = \frac{1}{4} (g^2 + g'^2) v^2 \tag{10.75}$$

Notice that the masses of the $W^\pm$ and of the Z are not independent, since their ratio is determined by the Weinberg angle, or from the gauge coupling constants

$$\frac{M_W^2}{M_Z^2} = \frac{g^2}{g^2 + g'^2} = \cos^2\theta \tag{10.76}$$

Let us now discuss the parameters that we have so far in the theory. The gauge interaction introduces the two gauge couplings g and g', which can also be expressed in terms of the electric charge (or the fine structure constant), and of the Weinberg angle. The Higgs sector brings in two additional parameters, the mass of the Higgs,

m_h, and the expectation value of the field Φ, v. The Higgs particle has not been yet discovered, and at the moment we have only an experimental lower bound on m_h, from LEP, which is given by $m_h > 60 \ GeV$. The parameter v can be expressed in terms of the Fermi coupling constant G_F. In fact, from eq. (10.9)

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8 M_W^2} \tag{10.77}$$

using the expression for M_W^2, we get

$$v^2 = \frac{1}{\sqrt{2} G_F} \approx (246 \ GeV)^2 \tag{10.78}$$

Therefore, the three parameters g, g' and v can be traded for e, $\sin^2 \theta$ and G_F. Another possibility is to use the mass of the Z

$$M_Z^2 = \frac{1}{4} \frac{1}{\sqrt{2} G_F} \frac{e^2}{\sin^2 \theta \cos^2 \theta} = \frac{\pi \alpha}{\sqrt{2} G_F \sin^2 \theta \cos^2 \theta} \tag{10.79}$$

to eliminate $\sin^2 \theta$

$$\sin^2 \theta = \frac{1}{2} \left[1 - \sqrt{1 - \frac{4\pi\alpha}{\sqrt{2} G_F M_Z^2}} \right] \tag{10.80}$$

The first alternative has been the one used before LEP. However, after LEP1, the mass of the Z is very well known

$$M_Z = (91.1863 \pm 0.0020) \ GeV \tag{10.81}$$

Therefore, the parameters that are now used as input in the SM are α, G_F and M_Z.

The last problem we have to solve is how to give mass to the electron, since it is impossible to construct bilinear terms in the electron field which are invariant under the gauge group. The solution is that we can build up trilinear invariant terms in the electron field and in Φ. Once the field Φ acquires a non vanishing expectation value, due to the breaking of the symmetry, the trilinear term generates the electron mass. Recalling the behaviour of the fields under $SU(2) \otimes U(1)$

$$L \in (\underline{2}, -1), \qquad R \in (\underline{1}, -2), \qquad \Phi \in (\underline{2}, +1) \tag{10.82}$$

we see that the following coupling (Yukawian coupling) is invariant

$$\mathcal{L}_Y = g_e \bar{L} \Phi R + \text{h.c.} = g_e \left[(\bar{\psi}_\nu)_L, \quad (\bar{\psi}_e)_L \right] \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix} (\psi_e)_R + \text{h.c.} \tag{10.83}$$

In the unitary gauge, which is obtained by using the transformation (10.65) both for Φ and for the lepton field L, we get

$$\mathcal{L}_Y = g_e \left[(\bar{\psi}_\nu)_L, \quad (\bar{\psi}_e)_L \right] \begin{bmatrix} 0 \\ (v + h)/\sqrt{2} \end{bmatrix} (\psi_e)_R + \text{h.c.} \tag{10.84}$$

from which

$$\mathcal{L}_Y = \frac{g_e v}{\sqrt{2}} (\bar{\psi}_e)_L (\psi_e)_R + \frac{g_e}{\sqrt{2}} (\bar{\psi}_e)_L (\psi_e)_R h + \text{h.c.} \qquad (10.85)$$

Since

$$[(\bar{\psi}_e)_L (\psi_e)_R]^\dagger = (\bar{\psi}_e \frac{1+\gamma_5}{2} \psi_e)^\dagger = \psi_e^\dagger \frac{1+\gamma_5}{2} \gamma_0 \psi_e = \bar{\psi}_e \frac{1-\gamma_5}{2} \psi_e = (\bar{\psi}_e)_R (\psi_e)_L \qquad (10.86)$$

we get

$$\mathcal{L}_Y = \frac{g_e v}{\sqrt{2}} \bar{\psi}_e \psi_e + \frac{g_e}{\sqrt{2}} \bar{\psi}_e \psi_e h \qquad (10.87)$$

Therefore the symmetry breaking generates an electron mass given by

$$m_e = -\frac{g_e v}{\sqrt{2}} \qquad (10.88)$$

In summary, we have been able to reproduce all the phenomenological features of the $V - A$ theory and its extension to neutral currents. This has been done with a theory that, before spontaneous symmetry breaking is renormalizable. In fact, also the Yukawian coupling has only dimension 3. The proof that the renormalizability of the theory holds also in the case of spontaneous symmetry breaking ($\mu^2 < 0$) is absolutely non trivial. In fact the proof was given only at the beginning of the seventies by 't Hooft.

10.3 The electroweak interactions of quarks and the Kobayashi-Maskawa-Cabibbo matrix

So far we have formulated the SM for an electron-neutrino pair, but it can be extended in a trivial way to other lepton pairs. The extension to the quarks is a little bit less obvious and we will follow the necessary steps in this Section. First of all we recall that in the quark model the nucleon is made up of three quarks, with composition, $p \approx uud$, $n \approx udd$. Then, the weak transition $n \to p + e^- + \bar{\nu}_e$ is obtained through

$$d \to u + e^- + \bar{\nu}_e \qquad (10.89)$$

and the assumption is made that quarks are point-like particles as leptons. Therefore, it is natural to write quark current in a form analogous to the leptonic current (see eqs. (10.5) and (10.7))

$$J_d^{\mu(\pm)} = 2 \bar{\Psi}_L \gamma^\mu \tau_\mp \Psi_L \qquad (10.90)$$

with

$$\Psi_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} = \frac{1-\gamma_5}{2} \begin{pmatrix} u \\ d \end{pmatrix} \qquad (10.91)$$

Phenomenologically, from the hyperon decays it was known that it was necessary a further current involving the quark s, given by

$$J_s^{\mu(\pm)} = 2\bar{\Psi}_L' \gamma^\mu \tau_\mp \Psi_L' \tag{10.92}$$

with

$$\Psi_L' = \begin{pmatrix} u_L \\ s_L \end{pmatrix} = \frac{1 - \gamma_5}{2} \begin{pmatrix} u \\ s \end{pmatrix} \tag{10.93}$$

Furthermore the total current contributing to the Fermi interaction had to be of the form

$$J_h^\mu = J_d^\mu \cos\theta_C + J_s^\mu \sin\theta_C \tag{10.94}$$

where θ_C is the Cabibbo angle, and

$$\sin\theta_C = 0.21 \pm 0.03 \tag{10.95}$$

The presence of the angle in the current was necessary in order to explain why the weak decays strangeness violating (that is the ones corresponding to the transition $u \leftrightarrow s$, as $K^+ \to \mu^+ \nu_\mu$) where suppressed with respect to the decays strangeness conserving. Collecting together the two currents we get

$$J_h^{\mu(\pm)} = 2\bar{Q}_L \gamma^\mu \tau_\mp Q_L \tag{10.96}$$

with

$$Q_L = \begin{pmatrix} u_L \\ d_L \cos\theta_C + s_L \sin\theta_C \end{pmatrix} \equiv \begin{pmatrix} u_L \\ d_L^C \end{pmatrix} \tag{10.97}$$

and

$$d_L^C = d_L \cos\theta_C + s_L \sin\theta_C \tag{10.98}$$

This allows us to assign Q_L to the representation $(\underline{2}, 1/3)$ of $SU(2) \otimes U(1)$. In fact, from

$$Q_{\rm em} = Q^3 + \frac{1}{2} Q^Y \tag{10.99}$$

we get

$$Q^Y(u_L) = 2(\frac{2}{3} - \frac{1}{2}) = \frac{1}{3} \tag{10.100}$$

and $(Q_{\rm em}(s) = Q_{\rm em}(d) = -1/3)$

$$Q^Y(d_L^C) = 2(-\frac{1}{3} + \frac{1}{2}) = \frac{1}{3} \tag{10.101}$$

As far as the right-handed components are concerned, we assign them to $SU(2)$ singlets, obtaining

$$u_R \in (\underline{1}, \frac{4}{3}) \qquad d_R^C \in (\underline{1}, -\frac{2}{3}) \tag{10.102}$$

Therefore, the hadronic neutral currents contribution from quarks u, d, s are given by

$$j_\mu^3 = \bar{Q}_L \gamma_\mu \frac{\tau_3}{2} Q_L \tag{10.103}$$

$$j_\mu^Y = \frac{1}{3} \bar{Q}_L \gamma_\mu Q_L + \frac{4}{3} \bar{u}_R \gamma_\mu u_R - \frac{2}{3} \bar{d}_R^C \gamma_\mu d_R^C \tag{10.104}$$

Since the Z is coupled to the combination $j_\mu^3 - \sin^2 \theta j_\mu^Y$, it is easily seen that the coupling contains a bilinear term in the field d^C given by

$$-\frac{\cos^2 \theta}{2} Z^\mu \bar{d}_L^C \gamma_\mu d_L^C + \frac{1}{3} \sin^2 \theta Z^\mu \bar{d}_R^C \gamma_\mu d_R^C \tag{10.105}$$

This gives rise to terms of the type $\bar{d}s$ and $\bar{s}d$ which produce strangeness changing neutral current transitions . However the experimental result is that these transitions are strongly suppressed. In a more general way we can get this result from the simple observation that W_3 is coupled to the current j_μ^3 which has an associated charge Q^3 which can be obtained by commuting the charges Q^1 and Q^2

$$\begin{aligned}
[Q^1, Q^2] &= \int d^3\vec{x} d^3\vec{y} [Q_L^\dagger \tau_1 Q_L, Q_L^\dagger \tau_2 Q_L] = \int d^3\vec{x} Q_L^\dagger [\tau_1, \tau_2] Q_L \\
&= i \int d^3\vec{x} Q_L^\dagger \tau_3 Q_L = i \int d^3\vec{x} (u_L^\dagger u_L - d_L^{C\dagger} d_L^C) \tag{10.106}
\end{aligned}$$

The last term is just the charge associated to a Flavour Changing Neutral Current (FCNC). We can compare the strength of a FCNC transition, which from

$$d^{C\dagger} d^C = d^\dagger d \cos^2 \theta_C + s^\dagger s \sin^2 \theta_C + (d^\dagger s + s^\dagger d) \sin \theta_C \cos \theta_C \tag{10.107}$$

is proportional to $\sin \theta_C \cos \theta_C$, with the strength of a flavour changing charged current transition as $u \to s$, which is proportional to $\sin \theta_C$. From this comparison we expect the two transitions to be of the same order of magnitude. But if we compare $K^0 \to \mu^+ \mu^-$ induced by the neutral current , with $K^+ \to \mu^+ \nu_\mu$ induced by the charged current (see Fig. 2.3), using $BR(K^0 \to \mu^+ \mu^-)$, we find

$$\frac{\Gamma(K^0 \to \mu^+ \mu^-)}{\Gamma(K^+ \to \mu^+ \nu_\mu)} \leq 6 \times 10^{-5} \tag{10.108}$$

This problem was solved in 1970 by the suggestion (Glashow Illiopoulos and Maiani, GIM) that another quark with charge $+2/3$ should exist, the charm. Then, one can form two left-handed doublets

$$Q_L^1 = \begin{pmatrix} u_L \\ d_L^C \end{pmatrix}, \qquad Q_L^2 = \begin{pmatrix} c_L \\ s_L^C \end{pmatrix} \tag{10.109}$$

where

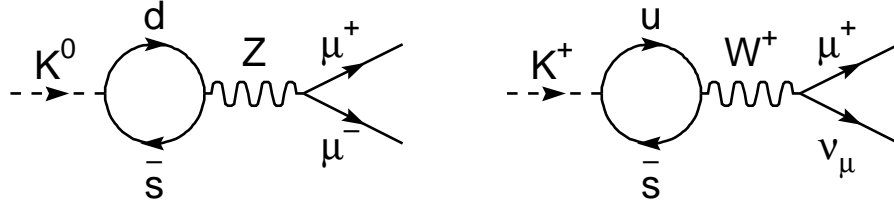$$s_L^C = -d_L \sin \theta_C + s_L \cos \theta_C \tag{10.110}$$

251

Fig. 10.3.1 - *The neutral current flavour changing process $K^0 \to \mu^+\mu^-$ and he charged current flavour changing process $K^+ \to \mu^+\nu_\mu$.*

is the combination orthogonal to d_L^C. As a consequence the expression of Q^3 gets modified

$$Q^3 = \int d^3\vec{x} \left(u_L^\dagger u_L + c_L^\dagger c_L - d_L^{C\dagger} d_L^C - s_L^{C\dagger} s_L^C\right) \tag{10.111}$$

But

$$d_L^{C\dagger} d_L^C + s_L^{C\dagger} s_L^C = d_L^\dagger d_L + s_L^\dagger s_L \tag{10.112}$$

since the transformation matrix

$$\begin{pmatrix} d_L^C \\ s_L^C \end{pmatrix} = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix} \begin{pmatrix} d_L \\ s_L \end{pmatrix} \tag{10.113}$$

is orthogonal. The same considerations apply to Q^Y. This mechanism of cancellation turns out to be effective also at the second order in the interaction, as it should be, since we need a cancellation at least at the order G_F^2 in order to cope with the experimental bounds. In fact, one can see that the two second order graphs contributing to $K^0 \to \mu^+\mu^-$ cancel out except for terms coming from the mass difference between the quarks u and c. One gets

$$A(K^0 \to \mu^+\mu^-) \approx G_F^2(m_c^2 - m_u^2) \tag{10.114}$$

allowing to get a rough estimate of the mass of the charm quark, $m_c \approx 1 \div 3\ GeV$. The charm was discovered in 1974, when it was observed the bound state $\bar{c}c$, with $J^{PC} = 1^{--}$, known as J/ψ. The mass of charm was evaluated to be around $1.5\ GeV$. In 1977 there was the observation of a new vector resonance, the Υ, interpreted as a bound state $\bar{b}b$, where b is a new quark, the bottom or beauty, with $m_b \approx 5\ GeV$. Finally in 1995 the partner of the bottom, the top, t, was discovered at Fermi Lab. The mass of the top is around $175\ GeV$. We see that in association with three

leptonic doublets

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \qquad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, \qquad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L \tag{10.115}$$

there are three quark doublets

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, \qquad \begin{pmatrix} c \\ s \end{pmatrix}_L, \qquad \begin{pmatrix} t \\ b \end{pmatrix}_L \tag{10.116}$$

We will show later that experimental evidences for the quark b to belong to a doublet came out from PEP and PETRA much before the discovery of top. Summarizing we have three generations of quarks and leptons. Each generation includes two left-handed doublets

$$\begin{pmatrix} \nu_A \\ e_A^- \end{pmatrix}_L, \qquad \begin{pmatrix} u_A \\ d_A \end{pmatrix}_L \qquad A = 1, 2, 3 \tag{10.117}$$

and the corresponding right-handed singlets. Inside each generation the total charge is equal to zero. In fact

$$Q_i^{\text{tot}} = \sum_{f \in \text{gen}} Q_f = 0 - 1 + 3 \times \left(\frac{2}{3} - \frac{1}{3} \right) = 0 \tag{10.118}$$

where we have also taken into account that each quark comes in three colors. This is very important because it guarantees the renormalizability of the SM. In fact, in general there are quantum corrections to the divergences of the currents destroying their conservation. However, the conservation of the currents coupled to the Yang-Mills fields is crucial for the renormalization properties. In the case of the SM for the electroweak interactions one can show that the condition for the absence of these corrections (Adler, Bell, Jackiw anomalies) is that the total electric charge of the fields is zero.

We will complete now the formulation of the SM for the quark sector, showing that the mixing of different quarks arises naturally from the fact that the quark mass eigenstates are not necessarily the same fields which couple to the gauge fields. We will denote the last ones by

$$\ell'_{AL} = \begin{pmatrix} \nu'_A \\ e'_A \end{pmatrix}_L, \qquad q'_{AL} = \begin{pmatrix} u'_A \\ d'_A \end{pmatrix}_L, \qquad A = 1, 2, 3 \tag{10.119}$$

and the right-handed singlets by

$$e'_{AR}, \quad u'_{AR}, \quad d'_{AR} \tag{10.120}$$

We assume that the neutrinos are massless and that they do not have any right-handed partner. The gauge interaction is then

$$\begin{aligned} \mathcal{L} &= \sum_{f_L} \bar{f}_{AL} \left(i\partial_\mu + g \frac{\vec{\tau}}{2} \cdot \vec{W}_\mu + g' \frac{Q^Y(f_L)}{2} Y_\mu \right) \gamma^\mu f_{AL} \\ &+ \sum_{f_R} \bar{f}_{AR} \left(i\partial_\mu + g' \frac{Q^Y(f_R)}{2} \right) \gamma^\mu f_{AR} \end{aligned} \tag{10.121}$$

where $f_{AL} = \ell'_{AL}$, q'_{AL}, and $f_{AR} = e'_{AR}$, u'_{AR}, d'_{AR}. We recall also the weak hypercharge assignments

$$Q^Y(\ell'_{AL}) = -1, \quad Q^Y(q'_{AL}) = \frac{1}{3}$$

$$Q^Y(e'_{AR}) = -2, \quad Q^Y(u'_{AR}) = \frac{4}{3}$$

$$Q^Y(d'_{AR}) = -\frac{2}{3} \tag{10.122}$$

Let us now see how to give mass to quarks. We start with up and down quarks, and a Yukawian coupling given by

$$g_d \bar{q}_L \Phi d_R + \text{h.c.} \tag{10.123}$$

where Φ is the Higgs doublet. When this takes its expectation value, $\langle \Phi \rangle = (0, v/\sqrt{2})$, a mass term for the down quark is generated

$$\frac{g_d v}{\sqrt{2}} \left(\bar{d}_L d_R + \bar{d}_R d_L \right) = \frac{g_d v}{\sqrt{2}} \bar{d} d \tag{10.124}$$

corresponding to a mass $m_d = -g_d v/\sqrt{2}$. In order to give mass to the up quark we need to introduce the conjugated Higgs doublet defined as

$$\tilde{\Phi} = i\tau_2 \Phi^\star = \begin{pmatrix} (\phi^0)^\star \\ -(\phi^+)^\star \end{pmatrix} \tag{10.125}$$

Observing that $Q^Y(\tilde{\Phi}) = -1$, another invariant Yukawian coupling is given by

$$g_u \bar{q}_L \tilde{\Phi} u_R + \text{h.c.} \tag{10.126}$$

giving mass to the up quark, $m_u = -g_u v/\sqrt{2}$. The most general Yukawian coupling among quarks and Higgs field is then given by

$$\mathcal{L}_Y = \sum_{AB} \left(g^e_{AB} \bar{\ell}'_{AL} \Phi e'_{BR} + g^u_{AB} \bar{q}'_{AL} \tilde{\Phi} u'_{BR} + g^d_{AB} \bar{q}'_{AL} \Phi d'_{BR} \right) \tag{10.127}$$

where g^i_{AB}, with $i = e$, u, d, are arbitrary 3×3 matrices. When we shift Φ by its vacuum expectation value we get a mass term for the fermions given by

$$\mathcal{L}_{\text{fermion mass}} = \frac{v}{\sqrt{2}} \sum_{AB} \left(g^e_{AB} \bar{\ell}'_{AL} e'_{BR} + g^u_{AB} \bar{u}'_{AL} u'_{BR} + g^d_{AB} \bar{d}'_{AL} d'_{BR} \right) \tag{10.128}$$

Therefore we obtain three 3×3 mass matrices

$$M^i_{AB} = -\frac{v}{\sqrt{2}} g^i_{AB}, \quad i = e, \ u, \ d \tag{10.129}$$

These matrices give mass respectively to the charged leptons and to the up and down of quarks. They can be diagonalized by a biunitary transformation

$$M_d^i = S^\dagger M^i T \tag{10.130}$$

where S and T are unitary matrices (depending on i) and M_d^i are three diagonal matrices. Furthermore one can take the eigenvalues to be positive. This follows from the polar decomposition of an arbitrary matrix

$$M = HU \tag{10.131}$$

where $H^\dagger = H = \sqrt{MM^\dagger}$ is a hermitian definite positive matrix and $U^\dagger = U^{-1}$ is a unitary matrix. Therefore, if the unitary matrix S diagonalizes H we have

$$H_d^i = S^\dagger H S = S^\dagger M U^{-1} S = S^\dagger M T \tag{10.132}$$

with $T = U^{-1}S$ is a unitary matrix. Each of the mass terms has the structure $\bar\psi_L' M \psi_R'$. Then we get

$$\bar\psi_L' M \psi_R' = \bar\psi_L' S (S^\dagger M T) T^\dagger \psi_R' \equiv \bar\psi_L M_d \psi_R \tag{10.133}$$

where we have defined the mass eigenstates

$$\psi_L' = S\psi_L, \qquad \psi_R' = T\psi_R \tag{10.134}$$

We can now express the charged current in terms of the mass eigenstates. We have

$$J_\mu^{+(h)} = 2\bar q_{AL}' \gamma_\mu \tau_- q_{AL}' = 2\bar u_{AL}' \gamma_\mu d_{AL}' = 2\bar u_{AL} \gamma_\mu ((S^u)^{-1} S^d)_{AB} d_{BL} \tag{10.135}$$

Defining

$$V = (S^u)^{-1} S^d, \qquad V^\dagger = V^{-1} \tag{10.136}$$

we get

$$J_\mu^{h(+)} = 2\bar u_{AL} \gamma_\mu d_{AL}^{(V)} \tag{10.137}$$

with

$$d_{AL}^{(V)} = V_{AB} d_{BL} \tag{10.138}$$

The matrix V is called the Cabibbo-Kobayashi-Maskawa (CKM) matrix, and it describes the mixing among the down type of quarks (this is conventional we could have chosen to mix the up quarks as well). We see that its physical origin is that, in general, there are no relations between the mass matrix of the up quarks, M^u, and the mass matrix of the down quarks, M^d. The neutral currents are expressions which are diagonal in the "primed" states, therefore their expression is the same also in the basis of the mass eigenstates, due to the unitarity of the various S matrices. In fact

$$j_\mu^{h(3)} \approx \bar u_L' u_L' - \bar d_L' d_L' = \bar u_L (S^u)^{-1} S^u u_L - \bar d_L (S^d)^{-1} S^d d_L = \bar u_L u_L - \bar d_L d_L \tag{10.139}$$

For the charged leptonic current, since we have assumed massless neutrinos we get

$$J_\mu^{\ell(+)} = 2\bar{\nu}_{AL}\gamma_\mu(S^e)_{AB}e_{BL} \equiv 2\bar{\nu}'_{AL}\gamma_\mu e_{AL} \tag{10.140}$$

where $\nu'_L = (S^e)^\dagger\nu_L$ is again a massless eigenstate. Therefore, there is no mixing in the leptonic sector, among different generations. However this is tied to our assumption of massless neutrinos. By relaxing this assumption we may generate a mixing in the leptonic sector producing a violation of the different leptonic numbers.

It is interesting to discuss in a more detailed way the structure of the CKM matrix. In the case we have n generations of quarks and leptons, the matrix V, being unitary, depends on n^2 parameters. However one is free to choose in an arbitrary way the phase for the $2n$ up and down quark fields. But an overall phase does not change V. Therefore the CKM matrix depends only on

$$n^2 - (2n-1) = (n-1)^2 \tag{10.141}$$

parameters. We can see that, in general, it is impossible to choose the phases in such a way to make V real. In fact a real unitary matrix is nothing but an orthogonal matrix which depends on

$$\frac{n(n-1)}{2} \tag{10.142}$$

real parameters. This means that V will depend on a number of phases given by

$$\text{number of phases} = (n-1)^2 - \frac{n(n-1)}{2} = \frac{(n-1)(n-2)}{2} \tag{10.143}$$

Then, in the case of two generations V depends only on one real parameter (the Cabibbo angle). For three generations V depends on three real parameters and one phase. Since the invariance under the discrete symmetry CP implies that all the couplings in the lagrangian must be real, it follows that the SM, in the case of three generations, implies a CP violation. A violation of CP has been observed experimentally by Christensen et al. in 1964. These authors discovered that the eigenstate of CP, with $CP = -1$

$$|K_2^0\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle - |\bar{K}^0\rangle\right) \tag{10.144}$$

decays into a two pion state with $CP = +1$ with a BR

$$BR(K_2^0 \to \pi^+\pi^-) \approx 2 \times 10^{-3} \tag{10.145}$$

It is not clear yet, if the SM is able to explain the observed CP violation in quantitative terms.

To complete this Section we give the experimental values for some of the matrix elements of V. The elements V_{ud} and V_{us} are determined through nuclear β-, $K-$ and hyperon-decays, using the muon decay as normalization. One gets

$$|V_{ud}| = 0.9736 \pm 0.0010 \tag{10.146}$$

and

$$|V_{us}| = 0.2205 \pm 0.0018 \tag{10.147}$$

The elements V_{cd} and V_{cs} are determined from charm production in deep inelastic scattering $\nu_\mu + N \to \mu + c + X$. The result is

$$|V_{cd}| = 0.224 \pm 0.016 \tag{10.148}$$

and

$$|V_{cs}| = 1.01 \pm 0.18 \tag{10.149}$$

The elements V_{ub} and V_{cb} are determined from the $b \to u$ and $b \to c$ semi-leptonic decays as evaluated in the spectator model, and from the B semi-leptonic exclusive decay $B \to \bar{D}^\star \ell \nu_\ell$. One gets

$$\frac{|V_{ub}|}{|V_{cb}|} = 0.08 \pm 0.02 \tag{10.150}$$

and

$$|V_{cb}| = 0.041 \pm 0.003 \tag{10.151}$$

More recently from the branching ratio $t \to Wb$, CDF (1996) has measured $|V_{tb}|$

$$|V_{tb}| = 0.97 \pm 0.15 \pm 0.07 \tag{10.152}$$

A more recent comprehensive analysis gives the following 90 % C.L. range for the various elements of the CKM matrix

$$V = \begin{pmatrix} 0.9745 - 0.9757 & 0.219 - 0.224 & 0.002 - 0.005 \\ 0.218 - 0.224 & 0.9736 - 0.9750 & 0.036 - 0.046 \\ 0.004 - 0.014 & 0.034 - 0.046 & 0.9989 - 0.9993 \end{pmatrix} \tag{10.153}$$

10.4   The parameters of the SM

It is now the moment to count the parameters of the SM. We start with the gauge sector, where we have the two gauge coupling constants g and g'. Then, the Higgs sector is specified by the parameters μ and the self-coupling λ. We will rather use $v = \sqrt{-\mu^2/\lambda}$ and $m_h^2 = -2\mu^2$. Then we have the Yukawian sector, that is that part of the interaction between fermions and Higgs field giving rise to the quark and lepton masses. If we assume three generations and that the neutrinos are all massless we have three mass parameters for the charged leptons, six mass parameters for the quarks (assuming the color symmetry) and four mixing angles. In order to test the structure of the SM the important parameters are those relative to the gauge and to the Higgs sectors. As a consequence one assumes that the mass matrix for the fermions is known. This was not the case till two years ago, before the top discovery, and its mass was unknown. The masses of the vector bosons can be expressed in terms of the previous parameters. However, the question arises about

the better choice of the input parameters. Before LEP1, the better choice was to use quantities known with great precision related to the parameters (g, g', v). The choice was to use the fine structure constant

$$\alpha = \frac{1}{137.0359895(61)} \tag{10.154}$$

and the Fermi constant

$$G_F = 1.166389(22) \times 10^{-5} \ GeV^{-2} \tag{10.155}$$

as measured from the β-decay of the muon. Then, most of the experimental research of the seventies and eighties was centered about the determination of $\sin^2 \theta$. The other parameters as (m_t, m_h) affect only radiative corrections, which, in this type of experiments can be safely neglected due to the relatively large experimental errors. Therefore, in order to relate the set of parameters $(\alpha, G_F, \sin^2 \theta)$ to (g, g', v) we can use the tree level relations. The situation has changed a lot after the beginning of running of LEP1. In fact, the mass of the Z has been measured with great precision, $\delta M_Z / M_Z \approx 2 \times 10^{-5}$. In this case the most convenient set is (α, G_F, M_Z), with

$$M_Z = (91.1863 \pm 0.0020) \ GeV \tag{10.156}$$

Also the great accuracy of the experimental measurements requires to take into account the radiative corrections. We will discuss this point in more detail at the end of these lectures. However we notice that one gets informations about parameters as (m_t, m_h) just because they affect the radiative corrections. In particular, the radiative corrections are particularly sensitive to the top mass. In fact, LEP1 was able to determine the top mass, in this indirect way, before the top discovery.

Appendix A

A.1 Properties of the real antisymmetric matrices

Given a real and antisymmetric matrix $n \times n$ A, we want to show that it is possible to put it in the form (7.61) by means of an orthogonal transformation. To this end, let us notice that iA is a hermitian matrix and therefore it can be diagonalized through a unitary transformation U:

$$U(iA)U^\dagger = A_d \tag{A.1}$$

where A_d is diagonal and real. The eigenvalues of iA satisfy the equation

$$\det |iA - \lambda \cdot 1| = 0 \tag{A.2}$$

Since $A^T = -A$ it follows

$$\det |iA - \lambda \cdot 1| = \det |(iA - \lambda \cdot 1)^T| = \det |-iA - \lambda \cdot 1| = 0 \tag{A.3}$$

Therefore, if λ is an eigenvalue, the same is for $-\lambda$. lo e'. It follows that A_d can be written in the form

$$A_d = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & -\lambda_1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & \lambda_2 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & -\lambda_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{A.4}$$

Notice that if n is odd, A_d has necessarily a zero eigenvalue, that we will write as $(A_d)_{nn}$. We will assume also to order the matrix in such a way that all the λ_i are positive.Each submatrix of the type

$$\begin{bmatrix} \lambda_i & 0 \\ 0 & -\lambda_i \end{bmatrix} \tag{A.5}$$

can be put in the form

$$\begin{bmatrix} 0 & i\lambda_i \\ -i\lambda_i & 0 \end{bmatrix} \tag{A.6}$$

through the unitary 2×2 matrix

$$V_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix} \tag{A.7}$$

In fact

$$V_2 \begin{bmatrix} \lambda_i & 0 \\ 0 & -\lambda_i \end{bmatrix} V_2^\dagger = \begin{bmatrix} 0 & i\lambda_i \\ -i\lambda_i & 0 \end{bmatrix} \tag{A.8}$$

Defining

$$V = \begin{bmatrix} V_2 & 0 & \cdot & \cdot & \cdot \\ 0 & V_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{A.9}$$

with $V_{nn} = 0$ for n odd, we get

$$V A_d V^\dagger = \begin{bmatrix} 0 & i\lambda_1 & 0 & 0 & \cdot & \cdot & \cdot \\ -i\lambda_1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & i\lambda_2 0 & \cdot & \cdot & \cdot \\ 0 & 0 & -i\lambda_1 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} = iA_s \tag{A.10}$$

with A_s defined in (7.61). Therefore

$$(VU)(iA)(VU)^\dagger = iA_s \tag{A.11}$$

or

$$(VU)(A)(VU)^\dagger = A_s \tag{A.12}$$

But A and A_s are real matrices, and therefore also VU must be real. However being VU real and unitary it must be orthogonal.