Supersymmetry

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Contents

Part 1

Fermions in Quantum Field Theory and the Standard Model

1

Two-component formalism for Spin-1/2 Fermions

In this chapter, we examine the incorporation of spin-1/2 fermions into quantum field theory. Underlying the relativistic theory of quantized fields is special relativity and the invariance of the Lagrangian under the Poincaré group, which consists of the Lorentz group plus space-time translations. Representations of the Poincaré group correspond to particle states of definite mass and spin.

1.1 The Lorentz group and its Lie algebra

Under Lorentz transformations, $\Lambda^\mu{}_\nu$, the spacetime coordinates, x^μ, transform as $x'^\mu = \Lambda^\mu{}_\nu x^\nu$.[1] The condition that $g_{\mu\nu}x^\mu x^\nu$ is invariant under Lorentz transformations implies that

$$\Lambda^\mu{}_\nu g_{\mu\rho}\Lambda^\rho{}_\lambda = g_{\lambda\nu}\,. \tag{1.1}$$

We can also define $\Lambda_\mu{}^\nu$ which governs the transformation of a covariant four-vector: $x'_\mu = \Lambda_\mu{}^\nu x_\nu$. One then easily derives: $\Lambda_\mu{}^\nu = g_{\mu\alpha}g^{\nu\beta}\Lambda^\alpha{}_\beta$. Using eq. (1.1), it follows that $\Lambda_\mu{}^\nu\Lambda^\mu{}_\beta = g^\nu{}_\beta = \delta^\nu_\beta$. The set of all possible Λ forms a Lie group O(3,1).

Eq. (1.1) implies that Λ possesses the following two properties: (i) $\det\Lambda = \pm 1$ and (ii) $|\Lambda^0{}_0| \geq 1$. Thus, Lorentz transformations fall into four disconnected classes, which can be denoted by a pair of signs: $\big(\mathrm{sgn}(\det\Lambda)\,,\,\mathrm{sgn}(\Lambda^0{}_0)\big)$. The proper orthochronous Lorentz transformations correspond to $(+,+)$ and are continuously connected to the identity. The group of such transformations forms a subgroup of O(3,1), which we shall denote by $\mathrm{SO}_+(3{,}1)$.[2] If $\Lambda \in \mathrm{SO}_+(3,1)$, we can

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[1] Our conventions for spacetime notation can be found in Appendix A.

[2] In this notation, the S (which stands for "special") corresponds to the condition $\det\Lambda = 1$ and the subscript + corresponds to $\Lambda^0{}_0 \geq +1$.

generate all elements of the other classes of Lorentz transformations by introducing the space-inversion matrix, $\Lambda_P = \text{diag}(1, -1, -1, -1)$, the time-inversion matrix $\Lambda_T = \text{diag}(-1, 1, 1, 1)$ and the spacetime inversion matrix $\Lambda_P \Lambda_T = -I_4$. Then, $\{\Lambda\,,\, \Lambda_P\Lambda\,,\, \Lambda_T\Lambda\,,\, \Lambda_P\Lambda_T\Lambda\}$ spans the full Lorentz group.

Infinitesimal Lorentz transformations must be proper and orthochronous since these are continuously connected to the identity. These transformations are best studied by exploring the properties of the SO(3,1) Lie algebra. Later (see section 1.7), we shall examine the implications of the improper Lorentz transformations—space-inversion and time-inversion.

The most general proper orthochronous Lorentz transformation, corresponding to a rotation by an angle θ about an axis $\widehat{\boldsymbol{n}}$ $[\vec{\boldsymbol{\theta}} \equiv \theta\widehat{\boldsymbol{n}}]$ and a boost vector $\vec{\boldsymbol{\zeta}} \equiv \hat{\boldsymbol{v}} \tanh^{-1}\beta$ [where $\hat{\boldsymbol{v}} \equiv \vec{\boldsymbol{v}}/|\vec{\boldsymbol{v}}|$ and $\beta \equiv |\vec{\boldsymbol{v}}|/c$],[3] is a 4×4 matrix given by:

$$\Lambda = \exp\left(-i\,\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\,\vec{\boldsymbol{s}} - i\,\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\,\vec{\boldsymbol{k}}\right) = \exp\left(-\tfrac{i}{2}\theta^{\alpha\beta}s_{\alpha\beta}\right)\,, \tag{1.2}$$

where $\theta^i \equiv \frac{1}{2}\epsilon^{ijk}\theta_{jk}$, $\zeta^i \equiv \theta^{i0} = -\theta^{0i}$, $s^i \equiv \frac{1}{2}\epsilon^{ijk}s_{jk}$, $k^i \equiv s^{0i} = -s^{i0}$ and

$$(s^{\alpha\beta})^\mu{}_\nu = i(g^{\alpha\mu}\,g^\beta{}_\nu - g^{\beta\mu}\,g^\alpha{}_\nu)\,. \tag{1.3}$$

Here, the indices $i, j, k = 1, 2, 3$ and $\epsilon^{123} = +1$.

Note that $s^{\mu\nu}$ is an antisymmetric 4×4 matrix, *i.e.*, $s^{\mu\nu} = -s^{\nu\mu}$, and satisfies the commutation relations of the SO(3,1) $\simeq$ SL(2,C) Lie algebra,

$$[s^{\alpha\beta}, s^{\rho\sigma}] = i(g^{\beta\rho}\,s^{\alpha\sigma} - g^{\alpha\rho}\,s^{\beta\sigma} - g^{\beta\sigma}\,s^{\alpha\rho} + g^{\alpha\sigma}\,s^{\beta\rho}). \tag{1.4}$$

The fields of spin-zero and spin-one transform under a general Lorentz transformation as

$$\phi'(x') = \phi(x)\,, \qquad \text{spin } 0\,, \tag{1.5}$$

$$A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x)\,, \qquad \text{spin } 1\,. \tag{1.6}$$

For a field of spin-s, the general transformation law reads

$$\psi'_\alpha(x') = \exp\left(-\tfrac{i}{2}\theta^{\mu\nu}S_{\mu\nu}\right)_\alpha{}^\beta\;\psi_\beta(x)\,, \tag{1.7}$$

where the $S_{\mu\nu}$ are (finite-dimensional) irreducible matrix representations of the Lie algebra of the Lorentz group, and α and β label the components of the matrix representation space. The dimension of this space is related to the spin of the particle. In particular, $S_{\mu\nu}$ is an antisymmetric tensor, $S_{\mu\nu} = -S_{\nu\mu}$, that satisfies the commutation relations of the SL(2,C) Lie

[3] Henceforth, we work in units where $c = 1$.

algebra [eq. (1.4)]. Different irreducible finite dimensional representations of SL(2,C) correspond to particles of different spin. It is convenient to identify the following pieces of $S^{\mu\nu}$

$$S^i \equiv \tfrac{1}{2}\epsilon^{ijk} S_{jk}\,, \qquad\qquad K^i \equiv S^{0i}\,, \tag{1.8}$$

where $i, j, k = 1, 2, 3$. One can show that the S^i generate three-dimensional rotations in space and the K^i generate the Lorentz-boosts [2]. Using eq. (1.4), it follows that S^i and N^i satisfy the commutation relations

$$[S^i, S^j] = i\epsilon^{ijk} S^k\,, \tag{1.9}$$
$$[S^i, K^j] = i\epsilon^{ijk} K^k\,, \tag{1.10}$$
$$[K^i, K^j] = -i\epsilon^{ijk} S^k\,. \tag{1.11}$$

It is convenient to define the following linear combinations of the generators

$$\vec{\boldsymbol{S}}_+ \equiv \tfrac{1}{2}(\vec{\boldsymbol{S}} + i\vec{\boldsymbol{K}})\,, \tag{1.12}$$
$$\vec{\boldsymbol{S}}_- \equiv \tfrac{1}{2}(\vec{\boldsymbol{S}} - i\vec{\boldsymbol{K}})\,. \tag{1.13}$$

Then, the commutation relations [eqs. (1.9)–(1.11)] decouple into two independent SU(2) algebras

$$[S^i_+, S^j_+] = i\epsilon^{ijk} S^k_+\,, \tag{1.14}$$
$$[S^i_-, S^j_-] = i\epsilon^{ijk} S^k_-\,, \tag{1.15}$$
$$[S^i_+, S^j_-] = 0\,. \tag{1.16}$$

The finite-dimensional irreducible representations of SU(2) are well known: these are the $(2s + 1) \times (2s + 1)$ representation matrices corresponding to spin-s, where $s = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2} \ldots$ (whose matrix elements appear in most textbooks of quantum mechanics). Hence, the irreducible representations of the Lorentz group can be characterized by two numbers (s_+, s_-), where $s_\pm$ is non-negative and either an integer or half-integer. The eigenvalues of $\vec{\boldsymbol{S}}^{\,2}_\pm$ are given by $s_\pm(s_\pm + 1)$. The dimension of the representation corresponding to (s_+, s_-) is $(2s_+ + 1)(2s_- + 1)$.

Using eqs. (1.7) and (1.8), one can write an infinitesimal Lorentz transformation as

$$M \equiv \exp\left(-\tfrac{i}{2}\theta_{\mu\nu} S^{\mu\nu}\right) \simeq I - i\,\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{S}} - i\,\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{K}}\,, \tag{1.17}$$

where θ^i and ζ^i are defined below eq. (1.2) and I is the identity. The simplest (trivial) representation is the one-dimensional $(0, 0)$ representation which corresponds to a spin-zero scalar field. In this representation,

$\vec{\boldsymbol{S}} = \vec{\boldsymbol{K}} = 0$ and we recover from eq. (1.7) the transformation law given in eq. (1.5). The spin-one transformation of eq. (1.6) corresponds to the four-dimensional $(\frac{1}{2}, \frac{1}{2})$ representation. However, in a quantum field theory of spin-one fields, only three of the four degrees of freedom are physical. Moreover, in gauge theories of massless spin-one fields, gauge invariance introduces an additional constraint and only two degrees of freedom are physical. This is described in detail in ref. [1] to which we refer the reader.

1.2 The Poincaré group and its Lie algebra

The Poincaré group consists of Lorentz transformations and spacetime translations. That is, under a Poincaré transformation, the spacetime co-ordinates transform as $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$, where Λ is given by eq. (1.2) and a^μ is a constant four-vector. For a field of spin s, the transformation law given by eq. (1.7) still holds under Poincaré transformations. It is convenient to rewrite this transformation law as follows:

$$\psi'_\alpha(x) = \exp\left(-\tfrac{i}{2}\theta^{\mu\nu} S_{\mu\nu}\right)_\alpha{}^\beta \; \psi_\beta\left(\Lambda^{-1}(x-a)\right) , \tag{1.18}$$

where we have used $x = \Lambda^{-1}(x'-a)$ and redefined the dummy variable x' by removing the prime. The Poincaré algebra is obtained by considering an infinitesimal Poincaré transformation. Expanding in a Taylor series about $\Lambda = I_4$ and $a = 0$, we may rewrite eq. (1.18) as:[4]

$$\psi'_\alpha(x) \simeq \left[I + ia^\mu P_\mu - \tfrac{i}{2}\theta^{\mu\nu}(L_{\mu\nu} + S_{\mu\nu})\right]_\alpha{}^\beta \; \psi_\beta(x) \,, \tag{1.19}$$

where P_μ and $L_{\mu\nu}$ are the differential operators

$$P_\mu \equiv i\partial_\mu \,, \qquad\qquad L_{\mu\nu} \equiv i(x_\mu\partial_\nu - x_\nu\partial_\mu) \,. \tag{1.20}$$

We further define $J_{\mu\nu} \equiv L_{\mu\nu} + S_{\mu\nu}$. Thus, the Poincaré algebra consists of the ten generators P^μ and $J^{\mu\nu}$ which obey the following commutation relations:[5]

$$[P^\mu, P^\nu] = 0 \,, \tag{1.21}$$

$$\left[J^{\mu\nu}, P^\lambda\right] = i(g^{\nu\lambda}P^\mu - g^{\mu\lambda}P^\nu) \,, \tag{1.22}$$

$$\left[J^{\alpha\beta}, J^{\rho\sigma}\right] = i(g^{\beta\rho}\, J^{\alpha\sigma} - g^{\alpha\rho}\, J^{\beta\sigma} - g^{\beta\sigma}\, J^{\alpha\rho} + g^{\alpha\sigma}\, J^{\beta\rho}) \,. \tag{1.23}$$

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[4] The operators I, P^μ and $L^{\mu\nu}$ include an implicit factor of $\delta_\alpha{}^\beta$, whereas the spin operator $S^{\mu\nu}$ depends non-trivially on α and β (except for the case of spin zero, when $S = 0$).

[5] It is easy to show that eq. (1.22) follows from the transformation law of the four-vector P^μ under Lorentz transformations.

The Poincaré algebra possesses two independent Casimir operators (these are polynomial functions of the generators that commute with the generators P^μ and $J^{\mu\nu}$). The first one is $P^2 \equiv P_\mu P^\mu$. To construct the second one, we introduce the Pauli-Lubanski vector w^μ:

$$w^\mu \equiv -\tfrac{1}{2}\epsilon^{\mu\nu\rho\lambda} J_{\nu\rho} P_\lambda \,, \tag{1.24}$$

where $\epsilon^{0123} = 1$. Explicitly,

$$w^\mu = (\vec{\boldsymbol{J}} \cdot \vec{\boldsymbol{P}} \,;\; P^0 \vec{\boldsymbol{J}} + \vec{\boldsymbol{N}} \times \vec{\boldsymbol{P}}) \,, \tag{1.25}$$

where $J^i \equiv \frac{1}{2}\epsilon^{ijk} J_{jk}$ and $N^i \equiv J^{0i}$. From its definition, it follows that $w_\mu P^\mu = 0$. The second Casimir operator is $w^2 \equiv w_\mu w^\mu$. The representations of the Poincaré algebra can be labelled by the eigenvalues of P^2 and w^2 when acting on the physical states.[6] The eigenvalue of P^2 is m^2, where m is the mass. To see the physical interpretation of w^2, let us evaluate the eigenvalue of w^2, consider first the case of $m \neq 0$. In this case, we are free to evaluate w^2 in the particle rest frame (since w^2 is a Lorentz scalar). In this frame, $w^\mu = (0\,;\, m\vec{\boldsymbol{S}})$, where S^i is defined in eq. (1.8). Hence, $w^2 = -m^2 \vec{\boldsymbol{S}}^2$, with eigenvalues $-m^2 s(s+1)$, $s = 0, \frac{1}{2}, 1, \ldots$. We conclude that massive (positive energy) states can be labelled by (m, s), where m is the mass and s is the spin of the state.

If $m = 0$, the previous analysis is not valid, since we cannot evaluate w^2 in the rest frame. Nevertheless, if we take the $m \to 0$ limit, it follows from the results above that either $w^2 = 0$ or the corresponding states have infinite spin. We reject the second possibility (which does not appear to be realized in nature) and assume that $w^2 = 0$. Thus, we must solve the equations $w^2 = P^2 = w_\mu P^\mu = 0$. It is simplest to choose a frame in which $P = P^0(1; 0, 0, 1)$ where $P^0 > 0$. In this frame, it is easy to show that $w = w^0(1; 0, 0, 1)$. That is, in any Lorentz frame,

$$w^\mu = h P^\mu \,, \tag{1.26}$$

where h is called the helicity operator. From eq. (1.26), we derive:[7]

$$h = \frac{w^0}{P^0} = \frac{\vec{\boldsymbol{J}} \cdot \vec{\boldsymbol{P}}}{P^0} = \frac{\vec{\boldsymbol{S}} \cdot \vec{\boldsymbol{P}}}{P^0} \,. \tag{1.27}$$

Eigenvalues of h are called the helicity (and are denoted by λ). One can check that h commutes with all the generators of the Poincaré algebra (as long as $P^0 = |\vec{\boldsymbol{P}}| \neq 0$). Thus, λ can be used to label states. In

[6] Here, we restrict our consideration to states of non-negative energy P^0. This is possible since the sign of P^0 is also invariant under proper Poincaré transformations.

[7] We define the differential operator $L^i \equiv \frac{1}{2}\epsilon^{ijk} L_{jk}$. Then, noting that $\vec{\boldsymbol{L}} = \vec{\boldsymbol{x}} \times \vec{\boldsymbol{P}}$, it follows that $\vec{\boldsymbol{J}} \cdot \vec{\boldsymbol{P}} = \vec{\boldsymbol{S}} \cdot \vec{\boldsymbol{P}}$.

particular, noting that for massless states, the eigenvalue of $\vec{\boldsymbol{S}}\cdot\vec{\boldsymbol{P}}/P^0$ is equal to that of $\vec{\boldsymbol{S}}\cdot\hat{\boldsymbol{P}}$ (where $\hat{\boldsymbol{P}}\equiv\vec{\boldsymbol{P}}/|\vec{\boldsymbol{P}}|$) and corresponds to the projection of the the spin along its direction of motion. The spectrum of $\vec{\boldsymbol{S}}\cdot\hat{\boldsymbol{P}}$ consists of $\lambda=0,\pm\frac{1}{2},\pm1,\ldots$. Under a CPT transformation, $\lambda\to-\lambda$. Thus, massless (positive energy) particles are labeled by $|\lambda|$ (colloquially called spin-$|\lambda|$), where both $\pm|\lambda|$ helicity states appear in the theory.

1.3 Spin-1/2 representation of the Lorentz group

In this chapter, we focus on the simplest non-trivial irreducible representations of the Lorentz algebra. These are the two-dimensional representations: $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$. Explicitly, the corresponding two-dimensional representations of the Lorentz generators are given by

$$(\tfrac{1}{2},0):\quad \vec{\boldsymbol{S}}_+=\tfrac{1}{2}(\vec{\boldsymbol{S}}+i\vec{\boldsymbol{K}})=\tfrac{1}{2}\vec{\boldsymbol{\sigma}}\,,\qquad \vec{\boldsymbol{S}}_-=\tfrac{1}{2}(\vec{\boldsymbol{S}}-i\vec{\boldsymbol{K}})=0\,,\tag{1.28}$$

which corresponds to $\vec{\boldsymbol{S}}=\vec{\boldsymbol{\sigma}}/2$ and $\vec{\boldsymbol{K}}=-i\vec{\boldsymbol{\sigma}}/2$, and

$$(0,\tfrac{1}{2}):\quad \vec{\boldsymbol{S}}_+=\tfrac{1}{2}(\vec{\boldsymbol{S}}+i\vec{\boldsymbol{K}})=0\,,\qquad \vec{\boldsymbol{S}}_-=\tfrac{1}{2}(\vec{\boldsymbol{S}}-i\vec{\boldsymbol{K}})=\tfrac{1}{2}\vec{\boldsymbol{\sigma}}\,,\tag{1.29}$$

which corresponds to $\vec{\boldsymbol{S}}=\vec{\boldsymbol{\sigma}}/2$ and $\vec{\boldsymbol{K}}=i\vec{\boldsymbol{\sigma}}/2$. Here, $\vec{\boldsymbol{\sigma}}=(\sigma^1,\sigma^2,\sigma^3)$ are the usual Pauli spin matrices

$$\sigma^1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \sigma^2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\qquad \sigma^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.\tag{1.30}$$

We shall define a fourth matrix, $\sigma^0\equiv I_2$, which is the 2×2 identity matrix.

Consider the infinitesimal Lorentz transformation in the $(\frac{1}{2},0)$ representation. Inserting the $(\frac{1}{2},0)$ generators [eq. (1.28)] into eq. (1.17) yields:

$$M\simeq I_2-\tfrac{i}{2}\,\vec{\boldsymbol{\theta}}\cdot\vec{\boldsymbol{\sigma}}-\tfrac{1}{2}\,\vec{\boldsymbol{\zeta}}\cdot\vec{\boldsymbol{\sigma}}\,.\tag{1.31}$$

A two-component $(\frac{1}{2},0)$ spinor field is denoted by $\xi_\alpha(x)$, and transforms as

$$\xi_\alpha(x)\to\xi'_\alpha(x')=M_\alpha{}^\beta\xi_\beta(x),\qquad \alpha,\beta=1,2.\tag{1.32}$$

If M is a matrix representation of SL(n,C), then, $M^*,(M^{-1})^T$, and $(M^{-1})^\dagger$ are also matrix representations of the same dimension. For $n>2$, all four representations are inequivalent. For SL(2,C), there are at most two distinct matrix representations corresponding to a given dimension:

(j_1, j_2) and (j_2, j_1). Using eq. (1.31), and the following property of Pauli matrices:

$$\sigma^2 \vec{\boldsymbol{\sigma}} (\sigma^2)^T = \vec{\boldsymbol{\sigma}}^T \,, \tag{1.33}$$

where the transpose of the σ-matrices are: $\vec{\boldsymbol{\sigma}}^T = (\sigma^1, -\sigma^2, \sigma^3)$. It is a simple matter to check that M and $(M^{-1})^T$ are related by

$$(M^{-1})^T = i\sigma^2 M (i\sigma^2)^T \,. \tag{1.34}$$

Since $(i\sigma^2)^T = (i\sigma^2)^{-1}$, the matrices M and $(M^{-1})^T$ are related by a similarity transformation corresponding to a unitary change in basis. Hence, M and $(M^{-1})^T$ are equivalent representations.[8] It is convenient to introduce the two-component spinor field $\xi^\alpha(x)$, $(\alpha = 1, 2)$, which transforms under the contragradient representation $(M^{-1})^T$

$$\begin{aligned} \xi^\alpha(x) \to \xi'^\alpha(x') &= (M^{-1})^{T\alpha}{}_\beta \, \xi^\beta(x) \,, \\ &= [i\sigma^2 M (i\sigma^2)^T]^\alpha{}_\beta \, \xi^\beta(x) \,. \end{aligned} \tag{1.35}$$

This motivates the definitions

$$\epsilon^{\alpha\beta} \equiv i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \qquad \epsilon_{\alpha\beta} \equiv (i\sigma^2)^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \tag{1.36}$$

i.e. $\epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12} = 1$. The ϵ-tensor satisfies:

$$\epsilon_{\alpha\beta}\epsilon^{\rho\tau} = -\delta_\alpha{}^\rho \delta_\beta{}^\tau + \delta_\alpha{}^\tau \delta_\beta{}^\rho \,, \tag{1.37}$$

from which it follows that:[9]

$$\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha{}^\gamma \,, \qquad \epsilon^{\gamma\beta}\epsilon_{\beta\alpha} = \delta^\gamma{}_\alpha \,. \tag{1.38}$$

Then, eqs. (1.32) and (1.35) imply that

$$\xi^\alpha = \epsilon^{\alpha\beta}\xi_\beta \,, \qquad \xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta \,. \tag{1.39}$$

Since M and $(M^{-1})^T$ are equivalent representations, either ξ_α or ξ^α are equally good candidates to describe the $(\frac{1}{2}, 0)$ representation.

Consider next the infinitesimal Lorentz transformation in the $(0, \frac{1}{2})$ representation [eqs. (1.17) and (1.29)],

$$(M^{-1})^\dagger \simeq I_2 - \tfrac{i}{2}\, \vec{\boldsymbol{\theta}} \boldsymbol{\cdot} \vec{\boldsymbol{\sigma}} + \tfrac{1}{2}\, \vec{\boldsymbol{\zeta}} \boldsymbol{\cdot} \vec{\boldsymbol{\sigma}} \,, \tag{1.40}$$

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[8] This corresponds to the well known result that the **2** and **2*** representations of SU(2) are equivalent.

[9] Of course, $\delta_\alpha{}^\gamma = \delta^\gamma{}_\alpha$, so the distinction is somewhat pedantic. Nevertheless, it is useful to keep track of this distinction when reinterpreting such equations in terms of matrix multiplication.

where we have used eq. (1.31) to identify the right-hand side above as $(M^{-1})^\dagger$.

A two-component $(0,\frac{1}{2})$ spinor field is denoted by $\bar\eta^{\dot\alpha}(x)$, and transforms as

$$\bar\eta^{\dot\alpha}(x) \to \bar\eta'^{\dot\alpha}(x') = (M^{-1})^{\dagger\,\dot\alpha}{}_{\dot\beta}\,\bar\eta^{\dot\beta}(x)\,, \qquad \dot\alpha,\dot\beta = \dot 1, \dot 2. \tag{1.41}$$

The so-called "dotted" indices have been introduced to distinguish the $(0,\frac{1}{2})$ representation from the $(\frac{1}{2},0)$ representation. The "bar" over the spinor is a convention that serves as an extra (admittedly redundant) reminder that this is the $(0,\frac{1}{2})$-representation. The equivalent description of this representation is obtained via the conjugate representation M^*. That is, M^* is related by a similarity transformation to $(M^{-1})^\dagger$. Under Lorentz transformations

$$\bar\eta_{\dot\alpha}(x) \to \bar\eta'_{\dot\alpha}(x') = (M^*)_{\dot\alpha}{}^{\dot\beta}\bar\eta_{\dot\beta}(x), \tag{1.42}$$

where

$$\bar\eta_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\eta^{\dot\beta}\,, \qquad\qquad \bar\eta^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\eta_{\dot\beta}, \tag{1.43}$$

and

$$\epsilon^{\dot\alpha\dot\beta} = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \qquad \epsilon_{\dot\alpha\dot\beta} = (i\sigma^2)^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,. \tag{1.44}$$

Hence, $\epsilon^{\dot\alpha\dot\beta}$ and $\epsilon^{\alpha\beta}$ are numerically equal, but the dotted and undotted indices transform differently under Lorentz transformations [eqs. (1.32) and (1.41)] and must always be kept distinct.

The dotted epsilon tensor satisfies:

$$\epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\rho\dot\tau} = -\delta_{\dot\alpha}{}^{\dot\rho}\delta_{\dot\beta}{}^{\dot\tau} + \delta_{\dot\alpha}{}^{\dot\tau}\delta_{\dot\beta}{}^{\dot\rho}\,, \tag{1.45}$$

from which it follows that[10]

$$\epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma} = \delta_{\dot\alpha}{}^{\dot\gamma}\,, \qquad\qquad \epsilon^{\dot\gamma\dot\beta}\epsilon_{\dot\beta\dot\alpha} = \delta^{\dot\gamma}{}_{\dot\alpha}\,. \tag{1.46}$$

Note that $\bar\eta_{\dot\alpha}$ and $(\eta^\dagger)_\alpha$ have the same transformation law, as do $\bar\eta^{\dot\alpha}$ and $(\eta^\dagger)^\alpha$. Hence, we may equate them:[11]

$$\bar\eta_{\dot\alpha} = (\eta^\dagger)_\alpha\,, \qquad\qquad \bar\eta^{\dot\alpha} = (\eta^\dagger)^\alpha\,. \tag{1.47}$$

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[10] Of course, $\delta_{\dot\alpha}{}^{\dot\gamma} = \delta^{\dot\gamma}{}_{\dot\alpha}$ See footnote 9.

[11] One could also consider the complex-conjugated two-component spinor fields. However, in quantum field theory, the two-component spinor fields are also operators, and the use of the dagger (†) indicates hermitian conjugation with respect to the quantum Hilbert space. The hermitian conjugated fields rather than the complex-conjugated fields are the ones that appear in the Lagrangian of the theory.

The Lorentz transformation property of $\bar{\eta}_{\dot{\alpha}}$ can then be written in the form $(\eta^\dagger)_\alpha \to (\eta^\dagger)_\beta (M^\dagger)^\beta{}_\alpha$, where $(M^\dagger)^\beta{}_\alpha = (M^*)_\alpha{}^\beta$ simply corresponds to the well known definition of the hermitian adjoint matrix as the complex conjugate transpose of the matrix.

So far, we have employed a notation in which unbarred two-component fermion fields correspond to the $(\frac{1}{2}, 0)$ representation of the Lorentz group, while barred fields correspond to the $(0, \frac{1}{2})$ representation. Of course, one could always consider a $(0, \frac{1}{2})$ as a fundamental field and assign it a symbol without a bar. However, it is useful to establish a convention in which *all* unbarred fields correspond to the $(\frac{1}{2}, 0)$ representation. Henceforth, we shall always work in this convention. Since $(\eta^\dagger)^\dagger = \eta$, this convention can be adopted with no loss of generality.

We will encounter two types of spinor quantities in this book. First, we shall examine spinor fields that are used to construct Lagrangians for field theoretic models. These spinors are anti-commuting objects. We shall then consider free-field plane wave expansions of these spinor fields. That is, we expand the spinor fields in terms of sums of plane waves multiplied by spinor wave-functions times the appropriate creation or annihilation operator. These wave-functions are *commuting* objects that satisfy the free-field Dirac equation, whereas the creation and annihilation operators satisfy the usual canonical anti-commutation relations.

1.4 Bilinear covariants of two-component spinors

It is well known that one can construct Lorentz scalar, vector and tensor quantities that are bilinear in the fermion fields. Subsequently, these vector and tensor quantities can be further combined to make new Lorentz scalars. Such scalars can be utilized in the construction of Lagrangians for theories of fermion fields. The Lagrangian of a relativistic quantum field theory is Lorentz invariant, *i.e.*, it must transform as a Lorentz scalar.

The simplest quantum field theory of fermions consists of a single $(\frac{1}{2}, 0)$ fermion field. More generally, one can consider a theory of a multiplet of fermions fields. Since the corresponding Lagrangian is bilinear in the fermion fields, it is sufficient to consider two $(\frac{1}{2}, 0)$ fermion fields and their Lorentz transformation properties:

$$\xi_\alpha \to M_\alpha{}^\beta \xi_\beta \,, \qquad \chi_\alpha \to M_\alpha{}^\beta \chi_\beta \,. \tag{1.48}$$

In order to construct a Lorentz invariant (*i.e.*, scalar) bilinear of the two-component fermion fields, we first note a basic property of the Lorentz transformation matrix M. From the definition of the determinant,

$$\epsilon^{\alpha\beta} M_\alpha{}^\rho M_\beta{}^\sigma = \epsilon^{\rho\sigma} \det M = \epsilon^{\rho\sigma}, \tag{1.49}$$

where we have used $\det M = 1$ at the last step. It then follows that the bilinear combination

$$\chi\xi \equiv \chi^\alpha \xi_\alpha = \epsilon^{\alpha\beta}\chi_\beta\xi_\alpha \tag{1.50}$$

is invariant under Lorentz transformations. Similarly, given two $(0, \frac{1}{2})$ fermion fields, the bilinear combination

$$\bar{\chi}\bar{\xi} \equiv \bar{\chi}_{\dot{\alpha}}\bar{\xi}^{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\chi}^{\dot{\beta}}\xi^{\dot{\alpha}} \tag{1.51}$$

is also Lorentz invariant. Note carefully the placement of the indices. We have adopted a convention that the heights of indices must be consistent in the sense that lowered indices must always be contracted with raised indices. Moreover, whenever possible undotted and dotted indices are contracted with the placement of indices as follows:

$${}^{\alpha}{}_{\alpha} \qquad \text{and} \qquad {}_{\dot{\alpha}}{}^{\dot{\alpha}} \,. \tag{1.52}$$

In this case, the spinor indices can be omitted without ambiguity as indicated in eqs. (1.50) and (1.51).

We consider next the behavior of eqs. (1.50) and (1.51) under hermitian conjugation. Since hermitian conjugation of a spinor product reverses the order of the spinors, it follows that

$$(\chi\xi)^\dagger = (\chi^\alpha\xi_\alpha)^\dagger = \bar{\xi}_{\dot{\alpha}}\chi^{\dot{\alpha}} = \bar{\xi}\bar{\chi}\,. \tag{1.53}$$

Finally, we note that for *anti-commuting* two-component fermion fields,

$$\chi\xi = \epsilon^{\alpha\beta}\chi_\beta\xi_\alpha = -\epsilon^{\alpha\beta}\xi_\alpha\chi_\beta = \epsilon^{\beta\alpha}\xi_\alpha\chi_\beta = \xi\chi\,, \tag{1.54}$$

where we have used $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ and relabeled the dummy indices. Likewise,

$$\bar{\chi}\bar{\xi} = \bar{\xi}\bar{\chi} \tag{1.55}$$

$$(\chi\xi)^\dagger = \bar{\chi}\bar{\xi}\,. \tag{1.56}$$

In order to construct fermion bilinears that transform as Lorentz vectors and tensors, we must introduce the sigma-matrices, $\sigma^\mu_{\alpha\dot{\beta}}$ and $\overline{\sigma}^{\mu\dot{\alpha}\beta}$, defined by

$$\sigma^\mu \equiv (I_2\,;\, \vec{\boldsymbol{\sigma}})\,, \qquad \overline{\sigma}^\mu \equiv (I_2\,;\, -\vec{\boldsymbol{\sigma}})\,, \tag{1.57}$$

where $\mu = 0, 1, 2, 3$. These quantities possess a specific spinor index structure. To determine it, first note that

$$p_\mu\sigma^\mu = p_0 I_2 - \vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{\sigma}} = \begin{pmatrix} p_0 - p_3 & -p_1 + ip_2 \\ -p_1 - ip_2 & p_0 + p_3 \end{pmatrix} \tag{1.58}$$

is an hermitian 2×2 matrix. Moreover, any hermitian 2×2 matrix can be written in the form of eq. (1.58). Since $Mp_\mu\sigma^\mu M^\dagger$ is also hermitian, there exists a p'_μ such that

$$p'_\mu \sigma^\mu = M p_\mu \sigma^\mu M^\dagger \,. \tag{1.59}$$

Using $\det(p_\mu\sigma^\mu) = p_0^2 - |\vec{p}\,|^2$ and $\det M = 1$, it follows that

$$p_0'^2 - |\vec{\boldsymbol{p}}'|^2 = p_0^2 - |\vec{\boldsymbol{p}}|^2 \,, \tag{1.60}$$

and hence $p_\mu \to p'_\mu$ is a Lorentz transformation. Moreover, one can check that for $M \simeq I_2 - \frac{i}{2}\,\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\,\vec{\boldsymbol{\sigma}} - \frac{1}{2}\,\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\,\vec{\boldsymbol{\sigma}}$, one obtains $p'^\mu = \Lambda^\mu{}_\nu p^\nu$ where Λ is the 4×4 Lorentz transformation matrix [eq. (1.2)] parameterized by $\vec{\boldsymbol{\theta}}$ and $\vec{\boldsymbol{\beta}}$. In order for eq. (1.59) to be Lorentz-covariant, both sides of the equation must possess the same index structure. Using the known index structure of the Lorentz transformation matrices M [eq. (1.32)] and M^* [eq. (1.42)], it follows that the spinor index structure of eq. (1.59) is given by

$$p'_\mu \sigma^\mu_{\alpha\dot{\alpha}} = M_\alpha{}^\beta p_\mu \sigma^\mu_{\beta\dot{\beta}} (M^*)_{\dot{\alpha}}{}^{\dot{\beta}} \,, \tag{1.61}$$

where we have used $(M^\dagger)^{\dot{\beta}}{}_{\dot{\alpha}} = (M^*)_{\dot{\alpha}}{}^{\dot{\beta}}$. Likewise, for the same four-vectors p and p' as above,

$$p'_\mu \overline{\sigma}^{\mu\dot{\alpha}\alpha} = [(M^\dagger)^{-1}]^{\dot{\alpha}}{}_{\dot{\beta}} p_\mu \overline{\sigma}^{\mu\dot{\beta}\beta} (M^{-1})_\beta{}^\alpha \,. \tag{1.62}$$

We thus obtain the index structure:

$$\sigma^\mu_{\alpha\dot{\alpha}} \quad \text{and} \quad \overline{\sigma}^{\mu\dot{\alpha}\alpha} \,. \tag{1.63}$$

Both σ^μ and $\overline{\sigma}^\mu$ are hermitian matrices, which is expressed by the following relations:

$$(\sigma^{\mu\,*})_{\alpha\dot{\beta}} = \sigma^\mu_{\beta\dot{\alpha}} \,, \qquad\qquad (\overline{\sigma}^{\mu\,*})^{\dot{\alpha}\beta} = \overline{\sigma}^{\mu\dot{\beta}\alpha} \,. \tag{1.64}$$

Some useful relations between σ^μ and $\overline{\sigma}^\mu$ are

$$\sigma^\mu_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\overline{\sigma}^{\mu\,\dot{\beta}\beta} \,, \qquad\qquad \overline{\sigma}^{\mu\,\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\beta\dot{\beta}} \,, \tag{1.65}$$

$$\epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\overline{\sigma}^{\mu\dot{\beta}\alpha} \,, \qquad\qquad \epsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\alpha\dot{\beta}} = \epsilon_{\alpha\beta}\overline{\sigma}^{\mu\dot{\alpha}\beta} \,. \tag{1.66}$$

Finally, we note the following identities:

$$[\sigma^\mu\overline{\sigma}^\nu + \sigma^\nu\overline{\sigma}^\mu]_\alpha{}^\beta = 2g^{\mu\nu}\delta_\alpha{}^\beta \,, \tag{1.67}$$

$$[\overline{\sigma}^\mu\sigma^\nu + \overline{\sigma}^\nu\sigma^\mu]^{\dot{\alpha}}{}_{\dot{\beta}} = 2g^{\mu\nu}\delta^{\dot{\alpha}}{}_{\dot{\beta}} \,. \tag{1.68}$$

We may now construct the bilinear covariant that transforms as a Lorentz vector:

$$\chi\sigma^\mu\bar{\xi} \equiv \chi^\alpha \sigma^\mu_{\alpha\dot{\beta}} \bar{\xi}^{\dot{\beta}} \,. \tag{1.69}$$

Note that the convention of eq. (1.52) is respected and allows us to suppress the spinor indices. Moreover, the structure of the spinor and Lorentz indices guarantees that $\chi\sigma^\mu\bar{\xi}$ transforms as a Lorentz vector. Nevertheless, an explicit proof is instructive. If $\chi\sigma^\mu\bar{\xi}$ and A_μ are Lorentz vectors, then $\chi\sigma^\mu\bar{\xi}A_\mu$ is a Lorentz scalar. Under a Lorentz transformation, A_μ, χ^α and $\bar{\xi}^{\dot{\beta}}$ transform according to eqs. (1.6), (1.35) and (1.41), respectively. Then, $\chi\sigma^\mu\bar{\xi}A_\mu$ is a Lorentz scalar if the following condition is satisfied:

$$(M^{-1})_\alpha{}^\beta \sigma^\mu_{\beta\dot{\beta}} (M^{-1})^{\dagger\dot{\beta}}{}_{\dot{\alpha}} = \Lambda^\mu{}_\nu \sigma^\nu_{\alpha\dot{\alpha}} \,. \tag{1.70}$$

To verify eq. (1.70), we need an explicit expression for M. This will be given after the introduction of the second-rank tensor bilinear covariants below, so we postpone the rest of the proof until then.

Likewise, we may define:

$$\bar{\xi}\overline{\sigma}^\mu\chi \equiv \bar{\xi}_{\dot{\alpha}} \overline{\sigma}^{\mu\dot{\alpha}\alpha} \chi_\alpha \,. \tag{1.71}$$

One may rewrite this Lorentz covariant bilinear in terms of σ^μ by using the identity:

$$\bar{\xi}\bar{\sigma}^\mu\chi = -\chi\sigma^\mu\bar{\xi} \,, \tag{1.72}$$

which is valid for anti-commuting two-component spinors. Under hermitian conjugation,

$$(\chi\sigma^\mu\bar{\xi})^\dagger = \xi\sigma^\mu\bar{\chi} \,, \tag{1.73}$$

a result that holds true for both commuting and anti-commuting spinors. Combining eq. (1.72) and eq. (1.73) yields

$$(\chi\sigma^\mu\bar{\xi})^\dagger = -\bar{\chi}\overline{\sigma}^\mu\xi \tag{1.74}$$

Using the fact that the quantity $\bar{\xi}\overline{\sigma}^\mu\chi A_\mu$ is a Lorentz scalar, we obtain a second consistency conditions following similar arguments given above:

$$M^{\dagger\dot{\alpha}}{}_{\dot{\beta}} \, \overline{\sigma}^{\mu\dot{\beta}\beta} M_\beta{}^\alpha = \Lambda^\mu{}_\nu \, \overline{\sigma}^{\nu\dot{\alpha}\alpha} \,. \tag{1.75}$$

Next we proceed to construct bilinear covariants that transform as a Lorentz second-rank tensor. Using the matrices σ^μ and $\overline{\sigma}^\mu$ we first define

$$(\sigma^{\mu\nu})_\alpha{}^\beta \equiv \frac{i}{4}\left(\sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^{\nu\dot{\alpha}\beta} - \sigma^\nu_{\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta}\right) , \tag{1.76}$$

$$(\overline{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \equiv \frac{i}{4}\left(\bar{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^\nu_{\alpha\dot{\beta}} - \bar{\sigma}^{\nu\dot{\alpha}\alpha}\sigma^\mu_{\alpha\dot{\beta}}\right) . \tag{1.77}$$

These matrices satisfy self-duality relations:

$$\sigma^{\mu\nu} = -\tfrac{1}{2} i\epsilon^{\mu\nu\rho\kappa}\sigma_{\rho\kappa}\,, \qquad \overline{\sigma}^{\mu\nu} = \tfrac{1}{2} i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_{\rho\kappa}\,. \tag{1.78}$$

The components of $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$ are easily evaluated:

$$\sigma^{ij} = \bar{\sigma}^{ij} = \tfrac{1}{2}\epsilon^{ijk}\sigma^k\,, \tag{1.79}$$

$$\sigma^{i0} = -\sigma^{0i} = -\bar{\sigma}^{i0} = \bar{\sigma}^{0i} = \tfrac{i}{2}\sigma^i\,. \tag{1.80}$$

The hermiticity of $\vec{\boldsymbol{\sigma}}$ implies that

$$(\sigma^{\mu\nu\dagger})_\alpha{}^\beta = \overline{\sigma}^{\mu\nu\dot{\beta}}{}_{\dot{\alpha}}\,. \tag{1.81}$$

Previously [see eqs. (1.17), (1.28) and (1.29)], we noted that the Lorentz transformation matrices for the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations, respectively are given by

$$\exp\left(-\frac{i}{2}\theta_{\mu\nu}S^{\mu\nu}\right) \simeq \begin{cases} I_2 - \frac{i}{2}\,\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\,\vec{\boldsymbol{\sigma}} - \frac{1}{2}\,\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\,\vec{\boldsymbol{\sigma}}\,, & \text{for } (\frac{1}{2},0)\,, \\ I_2 - \frac{i}{2}\,\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\,\vec{\boldsymbol{\sigma}} + \frac{1}{2}\,\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\,\vec{\boldsymbol{\sigma}}\,, & \text{for } (0,\frac{1}{2})\,. \end{cases} \tag{1.82}$$

From this we deduce that

$$S^{\mu\nu} = \sigma^{\mu\nu}\,, \quad \text{for the } (\tfrac{1}{2},0) \text{ representation}\,, \tag{1.83}$$

$$S^{\mu\nu} = \overline{\sigma}^{\mu\nu}\,, \quad \text{for the } (0,\tfrac{1}{2}) \text{ representation}\,. \tag{1.84}$$

Indeed, $S^i = \frac{1}{2}\epsilon^{ijk}S_{jk} = \frac{1}{2}\sigma^i$ for both $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations, and thus $w^2 = -m^2\vec{\boldsymbol{S}}^2$ with eigenvalues $-\frac{3}{4}m^2$ as expected for a spin-1/2 fermion.

We may now construct two bilinear covariants that transform as second-rank Lorentz tensors:

$$\chi\sigma^{\mu\nu}\xi \equiv \chi^\alpha(\sigma^{\mu\nu})_\alpha{}^\beta\xi_\beta\,, \tag{1.85}$$

$$\bar{\chi}\overline{\sigma}^{\mu\nu}\bar{\xi} \equiv \bar{\chi}_{\dot{\alpha}}\,(\overline{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\,\bar{\xi}^{\dot{\beta}}\,. \tag{1.86}$$

Again, we note that the convention of eq. (1.52) is respected and allows us to suppress the spinor indices. For anti-commuting two-component spinors, one easily verifies that

$$\chi\sigma^{\mu\nu}\xi = -\xi\sigma^{\mu\nu}\chi\,, \qquad \bar{\chi}\overline{\sigma}^{\mu\nu}\bar{\xi} = -\bar{\xi}\overline{\sigma}^{\mu\nu}\bar{\chi}\,. \tag{1.87}$$

Under hermitian conjugation, eq. (1.81) yields

$$(\chi\sigma^{\mu\nu}\xi)^\dagger = \bar{\xi}\overline{\sigma}^{\mu\nu}\bar{\chi}\,, \tag{1.88}$$

a result that holds true for both commuting and anti-commuting spinors.

We now return to complete the proof of eqs. (1.70) and (1.75). It is sufficient to use the infinitesimal forms[12] for M, $M^\dagger$ and Λ

$$M \simeq I_2 - \tfrac{i}{2}\theta_{\mu\nu}\sigma^{\mu\nu}\,, \tag{1.89}$$
$$M^\dagger \simeq I_2 + \tfrac{i}{2}\theta_{\mu\nu}\overline{\sigma}^{\mu\nu}\,, \tag{1.90}$$
$$\Lambda_\mu{}^\nu \simeq \delta_\mu{}^\nu + \tfrac{1}{2}\left(\theta^{\alpha\nu}g_{\alpha\mu} - \theta^{\nu\beta}g_{\beta\mu}\right)\,. \tag{1.91}$$

Inserting these results in eqs. (1.70) and (1.75), it is straightforward to verify that these results are equivalent to the identities:

$$\sigma^{\mu\nu}\sigma^\rho - \sigma^\rho\overline{\sigma}^{\mu\nu} = i(g^{\nu\rho}\sigma^\mu - g^{\mu\rho}\sigma^\nu)\,, \tag{1.92}$$
$$\overline{\sigma}^{\mu\nu}\overline{\sigma}^\rho - \overline{\sigma}^\rho\sigma^{\mu\nu} = i(g^{\nu\rho}\overline{\sigma}^\mu - g^{\mu\rho}\overline{\sigma}^\nu)\,, \tag{1.93}$$

which are easily verified. This completes the proof of eqs.(1.70) and (1.75). Many more useful identities involving σ^μ, $\overline{\sigma}^\mu$, $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$ can be found in the Appendix.

The complete list of independent bilinear covariants is given in Table 1.1. To show that the list is complete, we note that an arbitrary 2×2 complex matrix can be written as a complex linear combination of four linearly independent 2×2 matrices, which may be chosen, *e.g.*, to be $\sigma^\mu_{\alpha\dot\beta}$ $(\mu = 0,1,2,3)$. However, in constructing the bilinear covariants, both undotted and dotted two-component spinors are employed. Thus, the relevant space is the sixteen-dimensional product space of two independent spaces of 2×2 matrices, spanned by the sixteen matrices[13]

$$\{\delta_\alpha{}^\beta\,,\quad \delta^{\dot\alpha}{}_{\dot\beta}\,,\quad \sigma^\mu_{\alpha\dot\beta}\,,\quad \overline{\sigma}^{\mu\dot\alpha\beta}\,,\quad \sigma^{\mu\nu}{}_\alpha{}^\beta\,,\quad \overline{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta}\}\,. \tag{1.94}$$

Here, it is important to note that $\sigma^{\mu\nu}{}_\alpha{}^\beta$ and $\overline{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta}$ each possess only three independent components due to the self-duality relations of eq. (1.78). Hence, the total number of independent components of the bilinear covariants is indeed equal to 16, as indicated in Table 1.1. One consequence of these considerations is that products of three or more σ or $\overline{\sigma}$ matrices can always be expressed as a linear combination of the matrices given in eq. (1.94) . Examples can be found in the Appendix.

In a theory of a single two-component spinor, there are only six independent components of the non-vanishing bilinear covariants. To obtain this result, let us set $\chi = \xi$ in the bilinear covariants listed in Table 1.1, and note that $\bar\xi\overline{\sigma}^\mu\xi = -\xi\sigma^\mu\bar\xi$ and $\chi\sigma^{\mu\nu}\chi = \bar\chi\overline{\sigma}^{\mu\nu}\bar\chi = 0$. The

[12] The inverses of these quantities are obtained (to first order in θ) by replacing $\theta \to -\theta$.

[13] For example, inside the sixteen dimensional product space there exist four-dimensional subspaces spanned by $\{\delta_\alpha{}^\beta\,,\ \sigma^{\mu\nu}{}_\alpha{}^\beta\}$ and $\{\delta^{\dot\alpha}{}_{\dot\beta}\,,\ \overline{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta}\}$ respectively.

Table 1.1. Independent bilinear covariants involving a pair of two-component spinor fields.

bilinear covariants	number of components
$\chi\xi$	1
$\bar{\chi}\bar{\xi}$	1
$\chi\sigma^\mu\bar{\xi}$	4
$\bar{\chi}\overline{\sigma}^\mu\xi$	4
$\chi\sigma^{\mu\nu}\xi$	3
$\bar{\chi}\bar{\sigma}^{\mu\nu}\bar{\xi}$	3

latter result is derived as follows. Using eqs. (1.38), (1.54) and (1.85),

$$\begin{aligned}\chi\sigma^{\mu\nu}\chi &= \epsilon_{\beta\gamma}(\sigma^{\mu\nu})_\alpha{}^\beta\,\chi^\alpha\chi^\gamma \\ &= -\tfrac{1}{2}\epsilon_{\beta\gamma}\epsilon^{\alpha\gamma}(\sigma^{\mu\nu})_\alpha{}^\beta\,\chi\chi \\ &= \tfrac{1}{2}\chi\chi\,\mathrm{Tr}\;\sigma^{\mu\nu} = 0\,. \end{aligned}\tag{1.95}$$

The derivation of $\bar{\chi}\overline{\sigma}^{\mu\nu}\bar{\chi} = 0$ is similar. One consequence of this result is that the magnetic moment of a real Majorana fermion must vanish.

1.5 Lagrangians for free spin-1/2 fermions

We now have the ingredients to construct a Lorentz invariant hermitian Lagrangian for free fermions. First, consider a theory of a single two-component fermion, χ. From the previous section, it is clear that the following quantity is a Lorentz scalar:

$$i\overline{\chi}_{\dot\alpha}\,\overline{\sigma}^{\mu\dot\alpha\beta}\partial_\mu\chi_\beta \equiv i\overline{\chi}\,\overline{\sigma}^\mu\partial_\mu\chi. \tag{1.96}$$

The factor of i is inserted since the hermitian quantity

$$\tfrac{i}{2}\overline{\chi}\,\overline{\sigma}^\mu \overleftrightarrow{\partial_\mu}\,\chi \equiv \tfrac{i}{2}\left[\overline{\chi}\,\overline{\sigma}^\mu(\partial_\mu\chi) - (\partial_\mu\overline{\chi})\overline{\sigma}^\mu\chi\right]\,, \tag{1.97}$$

differs from $i\overline{\chi}\,\overline{\sigma}^\mu\partial_\mu\chi$ by a total divergence. Hence, eq. (1.96) is a candidate for a kinetic energy term in the Lagrangian. A second hermitian Lorentz scalar,

$$m\chi\chi + m^*\bar{\chi}\bar{\chi}\,, \tag{1.98}$$

is a candidate for a mass term in the Lagrangian for a theory of fermions.

Thus, the Lagrangian for a free two-component fermion is given by

$$\mathcal{L} = i\overline{\chi}\,\overline{\sigma}^\mu\partial_\mu\chi - \tfrac{1}{2}(m\chi\chi + m^*\overline{\chi}\,\overline{\chi})\,. \tag{1.99}$$

Without loss of generality, we can absorb the phase of the parameter m into the definition of the field χ. Henceforth, we shall always work in a convention in which m is real and non-negative. The corresponding field equations are:

$$i\overline{\sigma}^\mu \partial_\mu \chi - m\overline{\chi} = 0\,, \tag{1.100}$$

$$i\sigma^\mu \partial_\mu \overline{\chi} - m\chi = 0\,, \tag{1.101}$$

where eq. (1.101) is the conjugate of eq. (1.100). This is a theory of a (neutral) Majorana spin-1/2 fermion.

Next, consider a theory of two free two-component fermion fields, χ_1 and χ_2. The corresponding Lagrangian can be written in the following form:[14]

$$\mathcal{L} = i\overline{\chi}_1\overline{\sigma}^\mu\partial_\mu\chi_1 + i\overline{\chi}_2\overline{\sigma}^\mu\partial_\mu\chi_2 - \tfrac{1}{2}m_1(\chi_1\chi_1 + \overline{\chi}_1\overline{\chi}_1) - \tfrac{1}{2}m_2(\chi_2\chi_2 + \overline{\chi}_2\overline{\chi}_2)\,, \tag{1.102}$$

where m_1 and m_2 are real and non-negative mass parameters. A special case arises when $m_1 = m_2 = m$. In this case one can perform complex field redefinitions

$$\xi = \frac{1}{\sqrt{2}}(\chi_1 + i\chi_2)\,, \tag{1.103}$$

$$\eta = \frac{1}{\sqrt{2}}(\chi_1 - i\chi_2)\,, \tag{1.104}$$

to obtain

$$\mathcal{L} = i\overline{\xi}\,\overline{\sigma}^\mu\partial_\mu\xi + i\overline{\eta}\,\overline{\sigma}^\mu\partial_\mu\eta - m(\xi\eta + \overline{\xi}\,\overline{\eta})\,. \tag{1.105}$$

The corresponding field equations for ξ and η are:

$$i\overline{\sigma}^\mu\partial_\mu\xi - m\overline{\eta} = 0\,, \qquad i\overline{\sigma}^\mu\partial_\mu\eta - m\overline{\xi} = 0\,, \tag{1.106}$$

$$i\sigma^\mu\partial_\mu\overline{\eta} - m\xi = 0\,, \qquad i\sigma^\mu\partial_\mu\overline{\xi} - m\eta = 0\,, \tag{1.107}$$

The theory of two free two-component fermion fields of equal mass [eq. (1.102) with $m_1 = m_2$] possesses a global internal O(2) symmetry,[15] $\chi_i \to O_{ij}\chi_j$, where $O^T O = I_2$. Corresponding to this symmetry is an hermitian conserved Noether current:

$$J^\mu = i(\overline{\chi}_1\overline{\sigma}^\mu\chi_2 - \overline{\chi}_2\overline{\sigma}^\mu\chi_1)\,, \tag{1.108}$$

[14] The theory specified by eq. (1.99) is in a form in which the mass matrix is diagonal. In Section 1.6, we will generalize to the case of n two-component fields ($n \geq 2$) and discuss the general procedure of fermion mass matrix diagonalization.

[15] The kinetic energy term alone is invariant under a larger global U(2) symmetry. However, when $m \neq 0$, the global symmetry of the free fermion field theory is O(2).

with a corresponding conserved charge, $Q = \int J^0 d^3x$. In the χ_i-basis, the Noether current is off-diagonal. However, after converting to the ξ–η basis via eqs. (1.103) and (1.104), the current is diagonal:

$$J^\mu = \bar{\xi}\overline{\sigma}^\mu \xi - \overline{\eta}\overline{\sigma}^\mu \eta\,, \tag{1.109}$$

which provides the motivation for this basis choice. In particular, we can now identify ξ and η as fields of definite and opposite charge Q. Together, ξ and η constitute a single Dirac spin-1/2 fermion.

Finally, we introduce the concept of *off-shell* degrees of freedom and contrast this with *on-shell* or physical degrees of freedom. Here, off-shell [on-shell] is shorthand for off- [on-] mass shell and distinguishes between virtual and real particle propagation, respectively. Given a particle with four-momentum p, real particle propagation must satisfy $p^2 = m^2$ (the mass-shell condition), whereas for virtual particles (*e.g.*, particle exchange inside Feynman diagrams), the four-momentum p and the mass m are independent. The field operator corresponding to real particle propagation satisfies the field equations, whereas the field operator of a virtual particle is not constrained by the field equations.

For scalar (spin-0) fields, there is no distinction between on-shell and off-shell degrees of freedom. The on-shell scalar field satisfies the Klein-Gordon equation, but this does not alter the fact that a real (complex) scalar field consists of one (two) degree(s) of freedom. For a spin-s field ($s > 0$), the number of on-shell degrees of freedom is less than the number of off-shell degrees of freedom. Although all on-shell spin-s fields satisfy the Klein-Gordon equation, the spin-s field equations provide additional constraints that reduce the number of degrees of freedom originally present. We illustrate this in the case of free fermion field theory.

A theory of a neutral self-conjugate fermion is described in terms of a single complex two-component fermion field $\chi_\alpha(x)$. This corresponds initially to four off-shell degrees of freedom, since χ_α and $\overline{\chi}_{\dot\alpha} \equiv (\chi^\dagger)_\alpha$ are independent degrees of freedom. But, when the field equations are imposed [see eq. (1.100)], $\overline{\chi}(x)$ is determined in terms of $\chi(x)$. Thus, the number of physical ("on-shell") degrees of freedom for a neutral self-conjugate fermion is equal to two.

A theory of a charged fermion is described in terms of a pair of mass-degenerate two-component fermions. We shall work in a basis, where the charged fermion is represented by a pair of two-component fermion fields ξ and η. The number of off-shell degrees of freedom for a charged fermion is eight. Applying the field equations determines $\overline{\xi}$ in terms of η and $\bar{\eta}$ in terms of ξ [see eqs. (1.106) and (1.107)]. Thus, the number of physical on-shell degrees of freedom for a charged fermion is equal to four.

Although we have illustrated the counting of degrees of freedom for free fermion field theory, the same counting applies in an interacting theory.

1.6 The fermion mass-matrix and its diagonalization

We now generalize the discussion of Section 1.5 and consider a collection of n free anti-commuting two-component spin-1/2 fields, $\zeta_{\alpha i}(x)$, which transform as $(\frac{1}{2}, 0)$ fields under the Lorentz group. Here, α is the spinor index, and i is a flavor index ($i = 1, 2, \ldots, n$) that labels the distinct fields of the collection. The free-field Lagrangian is given by

$$\mathcal{L} = i\bar{\hat{\chi}}^i \overline{\sigma}^\mu \partial_\mu \hat{\chi}_i - \tfrac{1}{2} M^{ij} \hat{\chi}_i \hat{\chi}_j - \tfrac{1}{2} M_{ij} \bar{\hat{\chi}}^i \bar{\hat{\chi}}^j \,, \tag{1.110}$$

where we are following the flavor index conventions of Appendix A. In particular, note that $M_{ij} \equiv (M^{ij})^*$. Moreover, M^{ij} is a complex symmetric matrix, since the product of anticommuting two-component fields satisfies $\hat{\chi}_i \hat{\chi}_j = \hat{\chi}_j \hat{\chi}_i$ [with the spinor contraction rule according to eq. (1.52)].

In general, the matrix M is not diagonal, so we cannot directly identify $\hat{\chi}_i(x)$ as a field of definite mass. If the Lagrangian were also to include interaction terms, we would call $\hat{\chi}_i(x)$ an *interaction*-eigenstate field.[16] One can diagonalize the mass matrix and rewrite the Lagrangian in terms of mass eigenstates $\chi_{\alpha i}$ that exhibit real non-negative masses m_i, respectively. To do this, we introduce the *mass*-eigenstates:

$$\hat{\chi}_i = \Omega_i{}^j \chi_j \,, \qquad\qquad \bar{\hat{\chi}}^i = \Omega^i{}_j \bar{\chi}^j \,, \tag{1.111}$$

where Ω is a unitary $n \times n$ matrix (to be determined by eq. (1.113) below), and $\Omega^i{}_j \equiv (\Omega_i{}^j)^*$. Under this transformation, the kinetic energy term is unchanged: $\bar{\chi}^i \overline{\sigma}^\mu \partial_\mu \chi_i = \bar{\hat{\chi}}^i \overline{\sigma}^\mu \partial_\mu \hat{\chi}_i$, but the mass term is transformed:

$$M^{ij} \hat{\chi}_i \hat{\chi}_j = M^{ij} \Omega_i{}^k \Omega_j{}^\ell \chi_k \chi_\ell = \sum_k m_k \chi_k \chi_k \,, \tag{1.112}$$

where the m_k are real and non-negative, and $M^{ij} \Omega_i{}^k \Omega_j{}^\ell = m_k \delta^{k\ell}$, (no sum over k). Equivalently, in matrix notation with suppressed indices,

$$\Omega^T M \, \Omega = M_D \equiv \mathrm{diag}(m_1, m_2, \ldots). \tag{1.113}$$

One can prove that a unitary matrix Ω satisfying eq. (1.113) always exists. This result corresponds to the Takagi factorization of a general complex symmetric matrix [3, 4]. That is, for any complex symmetric matrix

[16] In general, we shall use "hatted" fields to represent interaction eigenstate fields and "un-hatted" fields to describe states of definite mass

M there exists a unitary matrix Ω such that $\Omega^T M \Omega$ is diagonal with non-negative entries given by the positive square roots of the eigenvalues of $MM^\dagger$ (or $M^\dagger M$). To compute the values of the (non-negative real) diagonal elements of M_D, note that eq. (1.113) implies that

$$\Omega^T M M^\dagger \Omega^* = M_D M_D^\dagger = M_D^2 \,. \tag{1.114}$$

However, we know that the hermitian matrix $MM^\dagger$ can be diagonalized by a unitry matrix.[17] Hence, the diagonal elements of M_D^2 are the eigenvlaues of $MM^\dagger$, which implies that the diagonal elements of M_D are the non-negative square roots of the corresponding eigenvalues of $MM^\dagger$, as asserted above. If the eigenvalues of $MM^\dagger$ are non-degenerate, then Ω is unique up to permuations of the rows (or columns) of Ω and a possible overall multiplication of any row (or column) of Ω by -1. If there are degenerate eigenvalues, then if Ω satisfies eq. (1.113) and so does ΩK, where K is real and orthogonal within the degenerate subspace.

Thus, in terms of the mass eigenstates, the general form for the free-field Lagrangian of n two-component spinors is given by:

$$\mathcal{L} = i\bar{\chi}^i \overline{\sigma}^\mu \partial_\mu \chi_i - \tfrac{1}{2} \sum_i m_i (\chi_i \chi_i + \bar{\chi}^i \bar{\chi}^i) \,. \tag{1.115}$$

If all the masses are non-degenerate, then this Lagrangian does not exhibit any nontrivial global (internal) symmetry. However, if some of the masses are degenerate, global internal symmetries will exist (some of which may still be respected once interactions are included). For example, if all masses are degenerate, then the model exhibits an $\mathrm{O}(n)$ global symmetry corresponding to $\chi_i \to O_{ij}\chi_j$ with $O^T O = I_n$.

In the case where some of the spin-1/2 fermion fields are massive Dirac fermions carrying a conserved charge, it is often convenient to modify the above procedure. If ξ_α is a charged massive field, then there must be an associated independent two-component spinor field η_α of equal mass with the opposite charge. Although the preceding mass diagonalization procedure will always work, it is often more convenient to employ mass eigenstates that are also eigenstates of the charge operator. In the case of two fields of opposite charge, this means writing the corresponding Lagrangian in the form given by eq. (1.105) [whereas the form given by eq. (1.102) results from the diagonalization procedure described above].

A general theory of charged fermions deserves special attention. Consider a collection of such free anti-commuting charged massive

[17] Note however that one cannot in general use eq. (1.114) to determine the matrix Ω. For example, it is possible to have a non-diagonal matrix M such that $MM^\dagger$ is proportional to the identity matrix. In this case, Ω drops out completely from eq. (1.114), and must be determined directly from eq. (1.113).

spin-1/2 fields, which can be represented by pairs of two-component (interaction-eiegenstate) fields $\hat{\xi}_{\alpha i}(x)$, $\hat{\eta}_{\alpha i}(x)$, where $\hat{\eta}_{\alpha i}(x)$ transforms in a (possibly reducible) representation of the unbroken symmetry group that is the complex conjugate of the representation of $\hat{\xi}_{\alpha i}(x)$. The most general free-field Lagrangian is given by

$$\mathcal{L} = i\bar{\hat{\xi}}^i \overline{\sigma}^\mu \partial_\mu \hat{\xi}_i + i\bar{\hat{\eta}}^i \overline{\sigma}^\mu \partial_\mu \hat{\eta}_i - M^{ij} \hat{\xi}_i \hat{\eta}_j - M_{ij} \bar{\hat{\xi}}^i \bar{\hat{\eta}}^j \,, \qquad (1.116)$$

where M^{ij} is an arbitrary complex matrix, and $M_{ij} \equiv (M^{ij})^*$ as before. We diagonalize the mass matrix by introducing the mass-eigenstate fields χ_i and η_i and unitary matrices L and R (to be determined below),

$$\hat{\chi}_i = L_i{}^k \chi_k \,, \qquad \hat{\eta}_i = R_i{}^k \eta_k \,, \qquad (1.117)$$

and demand that $M^{ij} L_i{}^k R_j{}^\ell = m_k \delta^{k\ell}$ (no sum over k). In matrix form, this is written as:

$$L^T M R = M_D = \mathrm{diag}(m_1, m_2, \ldots), \qquad (1.118)$$

with the m_i real and non-negative. One can prove that the unitary matrices L and R satisfying eq. (1.118) always exist. This result corresponds to the singular value decomposition of a general complex matrix [5]. That is, for any complex matrix M there exist unitary matrices L and R such that $L^T M R$ is diagonal with non-negative entries given by the positive square roots of the eigenvalues of $MM^\dagger$ (or $M^\dagger M$). In particular, from eq. (1.118) it follows that

$$L^T (MM^\dagger) L^* = M_D M_D^\dagger = M_D^2, \qquad (1.119)$$

$$R^\dagger (M^\dagger M) R = M_D^\dagger M_D = M_D^2. \qquad (1.120)$$

This illustrates the fact that since $MM^\dagger$ and $M^\dagger M$ are both hermitian (these two matrices are not equal in general, although they possess the same real non-negative eigenvalues), they can be diagonalized by unitary matrices. The diagonal elements of M_D are therefore the non-negative square roots of the corresponding eigenvalues of $MM^\dagger$ (or $M^\dagger M$), as asserted above.

Thus, in terms of the mass eigenstates,

$$\mathcal{L} = i\bar{\chi}^i \overline{\sigma}^\mu \partial_\mu \chi_i + i\bar{\eta}^i \overline{\sigma}^\mu \partial_\mu \eta_i - \sum_i m_i (\chi_i \eta_i + \bar{\chi}^i \bar{\eta}^i) \,. \qquad (1.121)$$

The mass matrix now consists of 2×2 blocks $\begin{pmatrix} 0 & m_i \\ m_i & 0 \end{pmatrix}$ along the diagonal. More importantly, the mass matrix is diagonal with doubly-degenerate entries m_i^2 and describes a collection of Dirac fermions.[18]

In the most general theory of spin-1/2 fields, the Lagrangian can be written in terms of two-component $(\frac{1}{2}, 0)$ fermion fields $\hat{\psi}_i(x)$, which in general may consist of neutral fermions $\hat{\chi}_i$, and charged fermion pairs $\hat{\xi}_i$ and $\hat{\eta}_i$. The mass eigenstate basis is achieved by a unitary rotation $U_i{}^j$ on the flavor indices. In matrix form:

$$\hat{\psi} \equiv \begin{pmatrix} \hat{\chi} \\ \hat{\xi} \\ \hat{\eta} \end{pmatrix} = U\psi \equiv \begin{pmatrix} \Omega & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & R \end{pmatrix} \begin{pmatrix} \chi \\ \xi \\ \eta \end{pmatrix} , \tag{1.122}$$

where Ω, L, and R are constructed as described above. The result of the mass diagonalization procedure in a general theory therefore always consists of a collection of Majorana fermions as in eq. (1.115), plus a collection of Dirac fermions as in eq. (1.121).

1.7 Discrete spacetime and internal symmetries

The Lorentz bilinear covariant quantities given in Table 1.1 transform as a scalar, vector or second-rank tensor under the proper Lorentz transformations (which are continuously connected to the identity). But, it also proves useful to study the behavior of the bilinear covariants under improper Lorentz transformations. To accomplish this, it is sufficient to study the behavior of the bilinear covariants under time reversal (T) and under parity (P). In addition, for bilinear covariants that depend on a pair of two-component spinor fields, a discrete transformation that can be interpreted as charge conjugation (C) is also relevant. These symmetries act on the Hilbert space of states via the anti-unitarity operator $\mathcal{T}$ and the unitary operators $\mathcal{P}$ and $\mathcal{C}$, respectively. The second-quantized fermion fields, which are operators that act on on the Hilbert space, transform under T, P and C via similarity transformations: $\chi \to \mathcal{T}\chi\mathcal{T}^{-1}$, $\chi \to \mathcal{P}\chi\mathcal{P}^{-1}$ and $\chi \to \mathcal{C}\chi\mathcal{C}^{-1}$.

The transformation laws for the two-component spin-1/2 fermions under P and T cannot be determined *a priori* from the transformation laws under proper Lorentz transformations. This can be understood

[18] Of course, one could always choose instead to treat the Dirac fermions in a basis with a fully diagonalized mass matrix, as in equation (1.115), by defining $\chi_{2i-1} = (\xi_i + \eta_i)/\sqrt{2}$ and $\chi_{2i} = -i(\xi_i - \eta_i)/\sqrt{2}$. These fermion fields do not carry well-defined charges, and are analogous to writing a charged scalar field ϕ and its oppositely-charged conjugate ϕ^* in terms of their real and imaginary parts. However, it is rarely, if ever, convenient to do so; practical calculations only require that the mass matrix is diagonal, and it is of course more pleasant to use fields that carry well-defined charges.

by noting that the two-component spinors transform under SL(2,C) which is the universal covering group of $SO_+(3,1)$, the group of proper orthochronous Lorentz transformations.[19] In particular, SL(2,C) does not contain space-inversion or time-inversion. As a result, there is some freedom in defining the action of P and T on the two-component fermion fields. We shall fix this freedom by demanding that the free-field fermionic Lagrangians respect the P and T discrete symmetries. That is, $\mathcal{L}^P(\vec{\boldsymbol{x}},t) = \mathcal{L}(-\vec{\boldsymbol{x}},t)$ and $\mathcal{L}^T(\vec{\boldsymbol{x}},t) = \mathcal{L}(\vec{\boldsymbol{x}},-t)$, which ensures that the corresponding free-field action is invariant under P and T, respectively.

We expect that under a parity transformation a $(\frac{1}{2},0)$ fermion is transformed into a $(0,\frac{1}{2})$ fermion and vice versa. Thus, it is convenient to define an hermitian matrix P such that

$$\mathcal{P}\chi_\alpha(x)\mathcal{P}^{-1} = i\eta_P P_{\alpha\dot{\beta}}\overline{\chi}^{\dot{\beta}}(x_P)\,, \quad \mathcal{P}\overline{\chi}_{\dot{\alpha}}(x)\mathcal{P}^{-1} = -i\eta_P^* P_{\beta\dot{\alpha}}\chi^\beta(x_P)\,, \tag{1.123}$$

where $x_P \equiv (t\,;\,-\vec{\boldsymbol{x}})$ and η_P is initially an arbitrary complex phase. Note that the hermiticity of P implies that $(P^*)_{\alpha\dot{\beta}} = P_{\beta\dot{\alpha}}$. By imposing $\mathcal{L}^P(\vec{\boldsymbol{x}},t) = \mathcal{L}(-\vec{\boldsymbol{x}},t)$, where $\mathcal{L}$ is the free fermion Lagrangian [eq. (1.99)], the matrix P is determined and the possible values of η_P are restricted. Here, we follow the standard convention where the mass term is of the form $\mathcal{L}_m = -\frac{1}{2}(m\chi\chi + m^*\overline{\chi}\,\overline{\chi})$ with m real and non-negative.[20] Invariance of the kinetic energy term [eq. (1.96)] determines $P_{\alpha\dot{\beta}} = \sigma^0_{\alpha\dot{\beta}}$ (up to an overall sign that can be absorbed into the definition of η_P). To derive this result, we note that any hermitian 2×2 matrix can be written as $P = a_\mu\sigma^\mu$. Invariance of the kinetic energy term then implies that $a_0 = \pm 1$ and $a_i = 0$. It is convenient to separate out an explicit factor of i in the definition of η_P, as we have done in eq. (1.123). Then, the invariance of the mass term requires $\eta_P = \eta_P^*$ or $\eta_P = \pm 1$, as shown in Section 1.8.

Time-reversal transforms a $(\frac{1}{2},0)$ fermion into itself [and likewise for the $(0,\frac{1}{2})$ fermion]. Thus, it is convenient to define an hermitian matrix T such that

$$\mathcal{T}\chi_\alpha(x)\mathcal{T}^{-1} = \eta_T T_{\beta\dot{\alpha}}\chi^\beta(x_T)\,, \qquad \mathcal{T}\overline{\chi}_{\dot{\alpha}}(x)\mathcal{T}^{-1} = \eta_T^* T_{\alpha\dot{\beta}}\overline{\chi}^{\dot{\beta}}(x_T)\,, \tag{1.124}$$

where $x_T \equiv (-t\,;\,\vec{\boldsymbol{x}})$ and η_T is initially an arbitrary complex phase. Here, it should be noted that $\mathcal{T}\chi_\alpha(\vec{\boldsymbol{x}},t)\mathcal{T}^{-1}$ transforms as $\overline{\chi}_{\dot{\alpha}}$ and $\mathcal{T}\overline{\chi}_{\dot{\alpha}}(\vec{\boldsymbol{x}},t)\mathcal{T}^{-1}$

[19] As Lie groups, $SO_+(3,1) \cong SL(2,C)/Z_2$, corresponding to a double covering of $SO_+(3,1)$ by SL(2,C). For example, the SL(2,C) matrices I_2 and $-I_2$ correspond to the identity element $I_4 \in SO_+(3,1)$. For fermions, this is significant, since the fermion wave function changes by an overall minus sign under a 360° rotation.

[20] Note that if m is initially complex, one is always free to absorb its phase in a field redefinition of χ.

transforms as χ_α, which explains the spinor index structure of eq. (1.124). This behavior is a consequence of the fact that $\mathcal{T}$ is an anti-unitary operator that satisfies $\mathcal{T} z \mathcal{T}^{-1} = z^*$ for any complex number z. By imposing $\mathcal{L}^T(\vec{\boldsymbol{x}}, t) = \mathcal{L}(\vec{\boldsymbol{x}}, -t)$, where $\mathcal{L}$ is the free fermion Lagrangian [eq. (1.99)], the matrix T is determined and the possible values of η_T are restricted. Invariance of the kinetic energy term [eq. (1.96)] determines $T_{\alpha\dot{\beta}} = \sigma^0_{\alpha\dot{\beta}}$ (up to an overall sign that can be absorbed into the definition of η_T). Invariance of the mass term requires $\eta_T = \eta_T^*$ or $\eta_T = \pm 1$, as shown in Section 1.9.

Finally, we examine charge conjugation. In a free-fermion theory, charge conjugation simply interchanges particles and antiparticles, so $\mathcal{C}^2 = 1$. In a theory of a single neutral two-component Majorana fermion (which is its own antiparticle), $\mathcal{C}\chi_\alpha\mathcal{C}^{-1} = \eta_C \chi_\alpha$. Then, $\mathcal{C}^2 = 1$ implies that $\eta_C^2 = 1$, and we conclude that $\eta_C = \pm 1$. Clearly, in the free fermion theory charge conjugation is trivial (and we are free to choose $\eta_C = 1$). However, when interactions are included, in some cases charge conjugation symmetry is maintained only if $\eta_C = -1$. Charge conjugation is less trivial in a theory of charged fermions. Consider a theory of a pair of two-component fermion fields of equal (non-zero) mass. The corresponding free-fermion Lagrangian is given by eq. (1.102) with $m_1 = m_2$. As before, we assume that the phases of the fields have been chosen such that m_1 and m_2 are real and positive. In this case, the Lagrangian exhibits a global O(2) symmetry, $\chi_i \to C_{ij}\chi_j$, where $C \in$O(2). Corresponding to this symmetry is an hermitian conserved Noether current [eq. (1.108)] with a corresponding conserved charge Q. Under the action of charge conjugation, the eigenvalues of Q change sign (*i.e.*, $\mathcal{C} Q \mathcal{C}^{-1} = -Q$). Thus, the charge conjugation operator satisfies:

$$\mathcal{C} J^\mu \mathcal{C}^{-1} = -J^\mu \,. \tag{1.125}$$

We can use this result to fix the form of the charge conjugation matrix C. To accomplish this, consider the transformation law of the fields under charge conjugation:

$$\mathcal{C}\chi_i(x)\mathcal{C}^{-1} = C_{ij}\chi_j(x) \,, \qquad \mathcal{C}\overline{\chi}_i(x)\mathcal{C}^{-1} = C_{ij}\overline{\chi}_j(x) \,, \tag{1.126}$$

where C is a real orthogonal 2×2 matrix. Using eq. (1.125) and the explicit form of J^μ [eq. (1.108)], it follows that det $C = -1$. The most general transformation of this type is given by

$$\begin{pmatrix} \chi_1^C \\ \chi_2^C \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} . \tag{1.127}$$

However, one can always redefine the fermion fields to absorb the angle θ (while leaving the free-field Lagrangian unchanged). Thus, without loss of generality one can choose $C=\eta_C\sigma^3$, where $\eta_C=\pm 1$. Explicitly,

$$\mathcal{C}\chi_1(x)\mathcal{C}^{-1}=\eta_C\chi_1(x)\,,\qquad\qquad \mathcal{C}\chi_2(x)\mathcal{C}^{-1}=-\eta_C\chi_2(x)\,. \tag{1.128}$$

It is often more convenient to employ two-component fermion fields of definite (and opposite) charge, denoted by ξ and η. More precisely, for a fermion of charge $+1$ (in arbitrary units), the fields ξ and $\bar{\eta}$ correspond to charge $+1$ fields and the fields $\bar{\xi}$ and η correspond to charge -1 fields. The fields ξ and η can be expressed in terms of χ_1 and χ_2 using eqs. (1.103) and (1.104). Acting on ξ and η, the charge conjugation transformation [eq. (1.128)] is given by

$$\mathcal{C}\xi(x)\mathcal{C}^{-1}=\eta_C\eta(x)\,,\qquad\qquad \mathcal{C}\eta(x)\mathcal{C}^{-1}=\eta_C\xi(x)\,. \tag{1.129}$$

Free field theories are always separately invariant under P, C and T. When interactions are included, this may no longer be the case. To test whether an interacting field theory is invariant under one or more of the discrete symmetries, one must exhibit some choice of the phases (η_P, η_C and η_T) of the fields for which the Lagrangian of the interacting theory is invariant. However, even if none of the discrete symmetries are separately conserved, a deep theorem of quantum field theory asserts that the combined discrete symmetry of CPT must be conserved by all relativistic local (free or interacting) quantum field theories. The CPT-invariance of quantum field theory imposes a condition on the overall phase $\eta_{CPT}=\eta_C\eta_P\eta_T$. For *all* spin-$s$ bosonic fields, $\eta_{CPT}=(-1)^s$. For fermion fields, η_{CPT} must also be independent of particle species (although its sign is not determined since any term in a Lagrangian must contain an even number of fermion fields). By convention, we shall assume that $\eta_{CPT}=+1$ for all species of spin-1/2 fermion fields. Then, for a charged fermion, represented by a pair of two-component fermion fields of definite and opposite charge, ξ and η,

$$\mathcal{CPT}\xi(x)(\mathcal{CPT})^{-1}=-i\bar{\xi}(-x)\,,\quad \mathcal{CPT}\eta(x)(\mathcal{CPT})^{-1}=-i\bar{\eta}(-x)\,. \tag{1.130}$$

We now examine P, T and C transformations in more detail.

1.8 Parity transformation of two-component spinors

Under the parity (or more precisely, the space-inversion) transformation, $x_P^\mu=(\Lambda_P)^\mu{}_\nu x^\nu=(t\,;\,-\vec{\boldsymbol{x}})$, where

$$(\Lambda_P)^\mu{}_\nu=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\,. \tag{1.131}$$

Consider a theory of a single two-component fermion field $\chi(x)$. Under parity the two-component spinor transforms as

$$\mathcal{P}\chi_\alpha(x)\mathcal{P}^{-1} \equiv \chi^P_\alpha(x) = \eta_P i\sigma^0_{\alpha\dot\beta}\overline{\chi}^{\dot\beta}(x_P)\,, \tag{1.132}$$

$$\mathcal{P}\chi^\alpha(x)\mathcal{P}^{-1} \equiv \chi^{P\alpha}(x) = -\eta_P i\overline{\sigma}^{0\dot\beta\alpha}\overline{\chi}_{\dot\beta}(x_P)\,, \tag{1.133}$$

$$\mathcal{P}\overline{\chi}_{\dot\alpha}(x)\mathcal{P}^{-1} \equiv \overline{\chi}^P_{\dot\alpha}(x) = -\eta^*_P i\sigma^0_{\beta\dot\alpha}\chi^\beta(x_P)\,, \tag{1.134}$$

$$\mathcal{P}\overline{\chi}^{\dot\alpha}(x)\mathcal{P}^{-1} \equiv \overline{\chi}^{P\dot\alpha}(x) = \eta^*_P i\overline{\sigma}^{0\dot\alpha\beta}\chi_\beta(x_P)\,. \tag{1.135}$$

where $|\eta_P| = 1$. Eq. (1.133) is obtained from the first by raising the indices [see eq. (1.39)] and using eq. (1.66). Eqs. (1.134) and (1.135) are obtained from eqs. (1.132) and (1.133) by hermitian conjugation, respectively. Note that when the parity transformation is applied twice, we obtain:

$$(\chi^P)^P = -\chi\,, \qquad (\bar\chi^P)^P = -\bar\chi\,. \tag{1.136}$$

This is consistent with the more general result that for any neutral self-conjugate spin-s field, $P^2 = (-1)^{2s}$.

If one redefines $\chi \to e^{i\zeta}\chi$ [and $\bar\chi \to e^{-i\zeta}\bar\chi$], then $\eta_P \to e^{2i\zeta}\eta_P$. If the mass of the fermion is nonzero,[21] we may restrict the choice of the phase η_P by establishing a convention in which the mass parameter m in the Lagrangian of eq. (1.99) is taken to be real and positive. Then, $\eta_P = \pm 1$[22] as a consequence of the parity invariance of $\chi\chi + \overline{\chi}\,\overline{\chi}$, as we now demonstrate.

To show that the Lagrangian of eq. (1.99) is invariant under parity, we investigate the transformation properties of the scalar bilinear covariants. For $\chi_1\chi_2$ and $\overline{\chi}_1\,\overline{\chi}_2$ we obtain

$$\chi^P_1(x)\chi^P_2(x) = \eta_{P1}\eta_{P2}\overline{\chi}_1(x_P)\overline{\chi}_2(x_P)\,, \tag{1.137}$$

$$\overline{\chi}^P_1(x)\overline{\chi}^P_2(x) = (\eta_{P1}\eta_{P2})^*\chi_1(x_P)\chi_2(x_P)\,, \tag{1.138}$$

where we have used $\sigma^0_{\gamma\dot\alpha}\overline{\sigma}^{0\dot\alpha\beta} = \delta^\beta_\gamma$. As promised, $\chi\chi + \overline{\chi}\,\overline{\chi}$, transforms as a scalar for $\eta_P = \pm 1$. In this case, the hermitian linear combination $i(\chi\chi - \overline{\chi}\,\overline{\chi})$ transforms as a pseudoscalar.

The Lorentz vector bilinear covariants transform as

$$\overline{\chi}^P_1(x)\overline{\sigma}^\mu\chi^P_2(x) = \eta^*_{P1}\eta_{P2}(\Lambda_P)^\mu{}_\nu\chi_1(x_P)\sigma^\nu\overline{\chi}_2(x_P)\,, \tag{1.139}$$

$$\chi^P_1(x)\sigma^\mu\bar\chi^P_2(x) = \eta_{P1}\eta^*_{P2}(\Lambda_P)^\mu{}_\nu\overline{\chi}_1(x_P)\overline{\sigma}^\nu\chi_2(x_P)\,, \tag{1.140}$$

---

[21] For massless fermions, there are no additional restriction in the choice of phases that enter the discrete symmetry transformation laws. However, in order to have a continuous $m \to 0$ limit, we will choose these phase factors according to the restrictions (if any) of the massive theory.

[22] The standard phase convention in the literature (and in textbooks that treat such things carefully) absorbs the factor of i into η_P in eqs. (1.132)–(1.135). In this convention, $\eta_P = \pm i$. In this chapter, we have chosen the simpler procedure of making the factor of i explicit so that in our convention $\eta_P = \pm 1$.

where Λ_P is given in eq. (1.131).[23] In deriving these results, we have made use of:[24]

$$\overline{\sigma}^{0\dot{\beta}\alpha}\,\sigma^{\mu}_{\alpha\dot{\gamma}}\,\overline{\sigma}^{0\dot{\gamma}\beta} = (\Lambda_P)^{\mu}{}_{\nu}\,\overline{\sigma}^{\nu\dot{\beta}\beta}\,, \tag{1.141}$$

$$\sigma^{0}_{\beta\dot{\alpha}}\,\overline{\sigma}^{\mu\dot{\alpha}\gamma}\,\sigma^{0}_{\gamma\dot{\beta}} = (\Lambda_P)^{\mu}{}_{\nu}\,\sigma^{\nu}_{\beta\dot{\beta}}\,. \tag{1.142}$$

The kinetic energy term of the action [see eq. (1.96)] is therefore a scalar under parity[25] (independently of the choice of η_P). This is easily deduced from eq. (1.139) by noting that $\partial_\mu = (\Lambda_P)^{\nu}{}_{\mu}\partial^P_\nu$ (where $\partial^P_\nu \equiv \partial/\partial x^\nu_P$ is the parity-transformed derivative).

For the Lorentz tensor bilinear covariants we have

$$\chi_1^P(x)\sigma^{\mu\nu}\chi_2^P(x) = \eta_{P1}\eta_{P2}(\Lambda_P)^{\mu}{}_{\rho}(\Lambda_P)^{\nu}{}_{\tau}\overline{\chi}_1(x_P)\overline{\sigma}^{\rho\tau}\overline{\chi}_2(x_P)\,, \tag{1.143}$$

$$\overline{\chi}_1^P(x)\overline{\sigma}^{\mu\nu}\overline{\chi}_2^P(x) = \eta^*_{P1}\eta^*_{P2}(\Lambda_P)^{\mu}{}_{\rho}(\Lambda_P)^{\nu}{}_{\tau}\chi_1(x_P)\sigma^{\rho\tau}\chi_2(x_P)\,. \tag{1.144}$$

In deriving these results, we have made use of:

$$\overline{\sigma}^{0\dot{\beta}\alpha}\,\sigma^{\mu\nu}{}_{\alpha}{}^{\gamma}\,\sigma^{0}_{\gamma\dot{\alpha}} = (\Lambda_P)^{\mu}{}_{\rho}(\Lambda_P)^{\nu}{}_{\tau}\,\overline{\sigma}^{\rho\tau\dot{\beta}}{}_{\dot{\alpha}}\,, \tag{1.145}$$

$$\sigma^{0}_{\beta\dot{\alpha}}\,\overline{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\gamma}}\,\overline{\sigma}^{0\dot{\gamma}\alpha} = (\Lambda_P)^{\mu}{}_{\rho}(\Lambda_P)^{\nu}{}_{\tau}\,\sigma^{\rho\tau}{}_{\beta}{}^{\alpha}\,. \tag{1.146}$$

In theories with multiple two-component fermion fields with no global symmetries, the parity properties of the kth fermion is given by eqs. (1.132)–(1.135), with $(\eta_P)_k = \pm 1$ (in the convention where all mass parameters are real). When interactions are included, if there is some choice of the $(\eta_P)_k$ such that the action is invariant under parity, then the theory is parity conserving. If there is an internal global symmetry ($\chi_i \to U_{ij}\chi_j$) under which the Lagrangian is invariant, then there are relations among the parity transformations of the fermion fields. Here, we examine the simplest case of two mass-degenerate fermion fields. As shown in section 1.7, a theory with a pair of mass-degenerate two-component fermion fields possesses a conserved charge Q [see eq. (1.108)]. This implies that one can choose linear combinations of the fermions that are eigenstates of Q. In particular, ξ and η [with the properties noted in

[23] Numerically, $(\Lambda_P)^{\mu}{}_{\nu} = g_{\mu\nu}$, and many books therefore employ $g_{\mu\mu}$ (no sum over μ) in the parity transformation of the bilinear covariants. We prefer the more accurate notation above.

[24] Roughly speaking, the factor of $(\Lambda_P)^{\mu}{}_{\nu}$ converts σ^μ into $\overline{\sigma}^\mu$, while the spinor index structure is preserved by the multiplication of the appropriate factors of the identity matrix (either σ^0 or $\overline{\sigma}^0$).

[25] Under parity, the Lagrangian satisfies $\mathcal{P}\mathcal{L}(x)\mathcal{P}^{-1} = \mathcal{L}(x_P)$. Integrating this result to get the action $S \equiv \int \mathcal{L}\, d^4x$ yields $\mathcal{P}S\mathcal{P}^{-1} = S$ as desired.

eq. (1.129)] are states of definite (and opposite) charge. The fields ξ and η transform under parity as:

$$\mathcal{P}\xi_\alpha(x)\mathcal{P}^{-1} \equiv \xi^P_\alpha(x) = \eta_P i\sigma^0_{\alpha\dot\beta}\bar\eta^{\dot\beta}(x)\,, \tag{1.147}$$

$$\mathcal{P}\eta_\alpha(x)\mathcal{P}^{-1} \equiv \eta^P_\alpha(x) = \eta^*_P i\sigma^0_{\alpha\dot\beta}\overline{\xi}^{\dot\beta}(x)\,. \tag{1.148}$$

That is, the phases that appear in the parity transformation laws of two-component fermion fields of opposite charge are complex conjugates of each other. This ensures that the mass term of the free Lagrangian, $m(\chi\eta+\bar\chi+\bar\eta$ with m real, is parity invariant. Notice that if we return to the case of a single uncharged two-component fermion by taking $\xi=\eta$ in the transformation laws above [eqs. (1.147) and (1.148)], then consistency of the ξ and η transformation laws implies that $\eta_P=\eta^*_P$ as previously noted. In the case of the charged fermion pair, the phase η_P is not constrained. However, we can always choose to work in another basis. For example, if we work in terms of the fields χ_1 and χ_2 [eqs. (1.103) and (1.104)], then it follows that:

$$\mathcal{P}\chi_{1\alpha}(x)\mathcal{P}^{-1} = (\mathrm{Re}\ \eta_P)i\sigma^0_{\alpha\dot\beta}\overline{\chi}_1^{\dot\beta}(x_P) - (\mathrm{Im}\ \eta_P)i\sigma^0_{\alpha\dot\beta}\overline{\chi}_2^{\dot\beta}(x_P)\,, \tag{1.149}$$

$$\mathcal{P}\chi_{2\alpha}(x)\mathcal{P}^{-1} = (\mathrm{Im}\ \eta_P)i\sigma^0_{\alpha\dot\beta}\overline{\chi}_1^{\dot\beta}(x_P) + (\mathrm{Re}\ \eta_P)i\sigma^0_{\alpha\dot\beta}\overline{\chi}_2^{\dot\beta}(x_P)\,. \tag{1.150}$$

If we choose $\eta_P = \eta^*_P$, then χ_1 and χ_2 have simple transformation properties under parity:

$$\mathcal{P}\chi_{i\alpha}(x)\mathcal{P}^{-1} = \eta_P i\sigma^0_{\alpha\dot\beta}\overline{\chi}_i^{\dot\beta}(x_P)\,, \qquad (i=1,2)\,. \tag{1.151}$$

That is, the fields χ_1 and χ_2 obey the standard parity transformation laws of a single two-component fermion field [eqs. (1.132)–(1.135)], with the same phase factor η_P in each case. Had one chosen $\eta_P \neq \eta^*_P$ in eqs. (1.147) and (1.148), then the parity transformation laws of χ_1 and χ_2 would have been more complicated [eqs. (1.149) and (1.150)]. But, in this case, one is free to make a further SO(2) rotation to transform χ_1 and χ_2 into new fields that do exhibit the simpler parity transformation laws [eq. (1.151)].

We noted previously that $P^2=-1$ for a neutral self-conjugate spin-1/2 fermion. However, for a charged fermion, eqs. (1.147) and (1.148) imply that:

$$\mathcal{P}^2\xi_\alpha(x)(\mathcal{P}^2)^{-1} = -\eta_P^2\xi_\alpha(x)\,, \tag{1.152}$$

$$\mathcal{P}^2\bar\eta^{\dot\alpha}(x)(\mathcal{P}^2)^{-1} = -\eta_P^2\bar\eta^{\dot\alpha}(x)\,. \tag{1.153}$$

That is, $P^2=-\eta_P^2$ when acting on ξ and $\bar\eta$ (these are fields with the same value of the conserved charge). Note that unless $\eta_P=\pm1$ or $\pm i$,

P^2 is not an element of the Lorentz group! Some authors insist that $P^2 = \pm 1$ when applied to charged fermions, although there is nothing inconsistent with a more general phase factor. In fact, for any choice of η_P, one can always redefine P (by multiplication by an appropriate gauge transformation) to have any desired behavior. Nevertheless, there is some motivation for choosing $\eta_P = \pm 1$ so that $P^2 = -1$ as in the case of the neutral self-conjugate spin-1/2 fermion.

One can now work out the parity properties of bilinear covariants constructed out of the fields of a charged fermion pair. The results are listed in Table B.3.

1.9 Time-reversal transformation of two-component spinors

Under the time-reversal (or more precisely, the time-inversion) transformation, $x^\mu_T = (\Lambda_T)^\mu{}_\nu x^\nu = (-t\,;\,\vec{\boldsymbol{x}})$, where

$$(\Lambda_T)^\mu{}_\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{1.154}$$

Since the time-reversal transformation is anti-unitary it requires a careful consideration. Time-reversal is treated differently in the first-quantized and the second-quantized theories.[26] Here, we shall only consider the second-quantized version of time-reversal, which is governed by an anti-unitary operator $\mathcal{T}$ that acts on the Hilbert space. Consider a theory of a single two-component fermion field $\chi(x)$. Under time-reversal, the two-component spinor transforms as

$$\mathcal{T}\chi_\alpha(x)\mathcal{T}^{-1} \equiv \overline{\chi}^t_{\dot\alpha}(x) = \eta_T \sigma^0_{\beta\dot\alpha}\chi^\beta(x_T)\,, \tag{1.155}$$

$$\mathcal{T}\chi^\alpha(x)\mathcal{T}^{-1} \equiv \overline{\chi}^{t\dot\alpha}(x) = -\eta_T \overline{\sigma}^{0\dot\alpha\beta}\chi_\beta(x_T)\,, \tag{1.156}$$

$$\mathcal{T}\overline{\chi}_{\dot\alpha}(x)\mathcal{T}^{-1} \equiv \chi^t_\alpha(x) = \eta^*_T \sigma^0_{\alpha\dot\beta}\overline{\chi}^{\dot\beta}(x_T)\,, \tag{1.157}$$

$$\mathcal{T}\overline{\chi}^{\dot\alpha}(x)\mathcal{T}^{-1} \equiv \chi^{t\alpha}(x) = -\eta^*_T \overline{\sigma}^{0\dot\beta\alpha}\overline{\chi}_{\dot\beta}(x_T)\,, \tag{1.158}$$

where $|\eta_T| = 1$. The notation requires some explanation. Due to the anti-unitary nature of the operator $\mathcal{T}$, the two-component spinor quantity $\mathcal{T}\chi_\alpha(x)\mathcal{T}^{-1}$ transforms as the complex conjugate of a $(\frac{1}{2}, 0)$ spinor with a lower spinor index. Thus, we shall denote this quantity by $\overline{\chi}^t_{\dot\alpha}(x)$ (where the superscript t is used in place of T in order to avoid confusion with the symbol for the matrix transpose). Eq. (1.156) is obtained from

[26] For parity and charge conjugation the classical and field theoretic descriptions are identical. See ref. [6] for a more detailed discussion.

eq. (1.155) by first raising the indices [eq. (1.43)] and then using eq. (1.66). Eqs. (1.157) and (1.158) are obtained from eqs. (1.155) and (1.156) by hermitian conjugation, respectively. Note that when time-reversal is applied twice in succession, we use $\mathcal{T}\eta_T\mathcal{T}^{-1} = \eta_T^*$ and $|\eta_T| = 1$ to obtain:

$$(\chi^t)^t = -\chi\,, \qquad (\bar\chi^t)^t = -\bar\chi\,. \tag{1.159}$$

This is consistent with the more general result that:

$$T^2 = (-1)^{2s} \tag{1.160}$$

for any (charged or neutral) spin-s quantum field.

If one redefines $\chi \to e^{i\zeta}\chi$ [and $\bar\chi \to e^{-i\zeta}\bar\chi$], then $\eta_T \to e^{-2i\zeta}\eta_T$. As before, we may restrict the choice of the phase η_T by choosing the convention in which the mass parameter m in the Lagrangian of eq. (1.99) is taken to be real and non-negative. Then, as shown below, $\eta_T = \pm 1$ as a consequence of the time-reversal invariance of $\chi\chi + \overline{\chi}\,\overline{\chi}$.

To show that the Lagrangian of eq. (1.99) is invariant under time reversal, we investigate the transformation properties of the scalar bilinear covariants. For $\chi_1\chi_2$ and $\overline{\chi}_1\,\overline{\chi}_2$ we obtain

$$\begin{aligned}
\mathcal{T}\chi_1(x)\chi_2(x)\mathcal{T}^{-1} &= \mathcal{T}\chi_1^\alpha\mathcal{T}^{-1}\mathcal{T}\chi_{2\alpha}\mathcal{T}^{-1} \\
&= \overline{\chi}_1^{t\dot\alpha}(x)\overline{\chi}^t_{2\dot\alpha}(x) \\
&= \eta_{T1}\eta_{T2}\chi_1(x_T)\chi_2(x_T)\,,
\end{aligned} \tag{1.161}$$

and

$$\mathcal{T}\overline{\chi}_1(x)\overline{\chi}_2(x)\mathcal{T}^{-1} = (\eta_{T1}\eta_{T2})^*\overline{\chi}_1(x_T)\overline{\chi}_2(x_T)\,. \tag{1.162}$$

Thus, $\chi\chi + \overline{\chi}\,\overline{\chi}$ is even under time-reversal if $\eta_T = \pm 1$. In this case, the hermitian linear combination $i(\chi\chi - \overline{\chi}\,\overline{\chi})$ is odd under time-reversal (after noting that $\mathcal{T}i\mathcal{T}^{-1} = -i$).

The Lorentz vector bilinear covariants transform as

$$\mathcal{T}\overline{\chi}_{1\dot\alpha}\overline{\sigma}^{\mu\dot\alpha\alpha}\chi_{2\alpha}\mathcal{T}^{-1} = (\mathcal{T}\overline{\chi}_{1\dot\alpha}\mathcal{T}^{-1})(\mathcal{T}\overline{\sigma}^{\mu\dot\alpha\alpha}\mathcal{T}^{-1})(\mathcal{T}\chi_{2\alpha}\mathcal{T}^{-1})\,, \tag{1.163}$$

and an analogous equation involving σ^μ. Since $\mathcal{T}\overline{\chi}_{\dot\alpha}(x)\mathcal{T}^{-1} \equiv \chi^t_\alpha(x)$ and $\mathcal{T}\chi_\alpha(x)\mathcal{T}^{-1} \equiv \overline{\chi}^t_{\dot\alpha}(x)$, in order to have the spinor indices properly matched in eq. (1.163) [and in the analogous equation involving σ^μ], one must make use of the following results:

$$\mathcal{T}\sigma^\mu_{\alpha\dot\beta}\mathcal{T}^{-1} = \sigma^\mu_{\beta\dot\alpha}\,, \qquad \mathcal{T}\overline{\sigma}^{\mu\dot\alpha\beta}\mathcal{T}^{-1} = \overline{\sigma}^{\mu\dot\beta\alpha}\,, \tag{1.164}$$

which follow from $\mathcal{T}\vec{\boldsymbol{\sigma}}\mathcal{T}^{-1} = \vec{\boldsymbol{\sigma}}^* = \vec{\boldsymbol{\sigma}}^T$. We then obtain:

$$\begin{aligned}
\mathcal{T}\overline{\chi}_1(x)\overline{\sigma}^\mu\chi_2(x)\mathcal{T}^{-1} &= \eta^*_{T1}\eta_{T2}\overline{\chi}_1^{\dot{\gamma}}(x_T)\sigma^0_{\alpha\dot{\gamma}}(\overline{\sigma}^{\mu\dot{\beta}\alpha})\sigma^0_{\delta\dot{\beta}}\chi_2^\delta(x_T) \\
&= -\eta^*_{T1}\eta_{T2}\chi_2^\delta(x_T)\sigma^0_{\delta\dot{\beta}}(\overline{\sigma}^{\mu\dot{\beta}\alpha})\sigma^0_{\alpha\dot{\gamma}}\overline{\chi}_1^{\dot{\gamma}}(x_T) \\
&= \quad \eta^*_{T1}\eta_{T2}(\Lambda_T)^\mu{}_\nu\chi_2(x_T)\sigma^\nu\overline{\chi}_1(x_T) \\
&= -\eta^*_{T1}\eta_{T2}(\Lambda_T)^\mu{}_\nu\overline{\chi}_1(x_T)\overline{\sigma}^\nu\chi_2(x_T)\,. \qquad (1.165)
\end{aligned}$$

The minus sign in the second step arises due to anti-commuting spinors. In the third step, we have made use of:

$$\overline{\sigma}^{0\dot{\beta}\alpha}\,\sigma^\mu_{\alpha\dot{\gamma}}\,\overline{\sigma}^{0\dot{\gamma}\beta} = -(\Lambda_T)^\mu{}_\nu\,\overline{\sigma}^{\nu\dot{\beta}\beta}\,, \qquad (1.166)$$

$$\sigma^0_{\beta\dot{\alpha}}\,\overline{\sigma}^{\mu\dot{\alpha}\gamma}\,\sigma^0_{\gamma\dot{\beta}} = -(\Lambda_T)^\mu{}_\nu\,\sigma^\nu_{\beta\dot{\beta}}\,, \qquad (1.167)$$

which follow immediately from eqs. (1.141) and (1.142) after noting that $(\Lambda_T)^\mu{}_\nu = -(\Lambda_P)^\mu{}_\nu$. Finally, applying eq. (1.72) yields the final result given in eq. (1.165). Similarly,

$$\mathcal{T}\chi_1(x)\sigma^\mu\overline{\chi}_2(x)\mathcal{T}^{-1} = -\eta_{T1}\eta^*_{T2}(\Lambda_T)^\mu{}_\nu\chi_1(x_T)\sigma^\nu\overline{\chi}_2(x_T)\,. \quad (1.168)$$

The kinetic energy term of the action [see eq. (1.96)] is therefore a scalar under time reversal (independently of the choice of η_P). This is easily deduced from eq. (1.165) by noting that $\partial_\mu = -(\Lambda_T)^\nu{}_\mu\partial^t_\nu$ (where $\partial^t_\nu = \partial/\partial x^\nu_T$ is the time reversal-transformed derivative) and remembering to complex conjugate the factor of i.

For the Lorentz tensor bilinear covariants, we examine:

$$\mathcal{T}\chi_1^\alpha(\sigma^{\mu\nu})_\alpha{}^\beta\chi_{2\beta}\mathcal{T}^{-1} = (\mathcal{T}\chi_1^\alpha\mathcal{T}^{-1})(\mathcal{T}(\sigma^{\mu\nu})_\alpha{}^\beta\mathcal{T}^{-1})(\mathcal{T}\chi_{2\beta}\mathcal{T}^{-1})\,, \quad (1.169)$$

and an analogous equation involving $\overline{\sigma}^{\mu\nu}$. We then employ results similar to those of eq. (1.164):

$$\mathcal{T}(\sigma^{\mu\nu})_\alpha{}^\beta\mathcal{T}^{-1} = (\overline{\sigma}^{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}\,, \qquad \mathcal{T}(\overline{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\mathcal{T}^{-1} = (\sigma^{\mu\nu})_\beta{}^\alpha\,. \quad (1.170)$$

The rest of the derivation is straightforward, and the end result is:

$$\mathcal{T}\chi_1(x)\sigma^{\mu\nu}\chi_2(x)\mathcal{T}^{-1} = -\eta_{T1}\eta_{T2}(\Lambda_T)^\mu{}_\rho(\Lambda_T)^\nu{}_\tau\chi_1(x_T)\sigma^{\rho\tau}\chi_2(x_T)\,, \quad (1.171)$$

$$\mathcal{T}\overline{\chi}_1(x)\overline{\sigma}^{\mu\nu}\,\overline{\chi}_2(x)\mathcal{T}^{-1} = -\eta^*_{T1}\eta^*_{T2}(\Lambda_T)^\mu{}_\rho(\Lambda_T)^\nu{}_\tau\overline{\chi}_1(x_T)\overline{\sigma}^{\rho\tau}\,\overline{\chi}_2(x_T)\,, \quad (1.172)$$

where we have used eqs. (1.87) and the following results:

$$\overline{\sigma}^{0\dot{\beta}\alpha}\,\sigma^{\mu\nu}{}_\alpha{}^\gamma\,\sigma^0_{\gamma\dot{\alpha}} = (\Lambda_T)^\mu{}_\rho(\Lambda_T)^\nu{}_\tau\,\overline{\sigma}^{\rho\tau\dot{\beta}}{}_{\dot{\alpha}}\,, \qquad (1.173)$$

$$\sigma^0_{\beta\dot{\alpha}}\,\overline{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\gamma}}\,\overline{\sigma}^{0\dot{\gamma}\alpha} = (\Lambda_T)^\mu{}_\rho(\Lambda_T)^\nu{}_\tau\,\sigma^{\rho\tau}{}_\beta{}^\alpha\,. \qquad (1.174)$$

which follow immediately from eqs. (1.145) and (1.146).

In theories with multiple two-component fermion fields with no global symmetries, the time reversal properties of each fermion is given by eqs. (1.155)–(1.158), with $\eta_T = \pm 1$ (in the convention where all mass parameters are real). When interactions are included, if there is some choice of the η_T such that the action is invariant under time reversal, then the theory is time reversal invariant. If there is an internal global symmetry ($\chi_i \to U_{ij}\chi_j$) under which the Lagrangian is invariant, then there are relations among the time reversal transformations of the fermion fields. Here, we again examine the simplest case of two mass-degenerate fermion fields, χ_1 and χ_2. In terms of the linear combinations ξ and η [eqs. (1.103) and (1.104)] corresponding to the fields of definite charge, we define the transformations under time reversal as follows. The fields ξ and η transform under time reversal as:

$$\mathcal{T}\xi_\alpha(x)\mathcal{T}^{-1} \equiv \bar{\xi}^t_{\dot{\alpha}}(x) = \eta_T \sigma^0_{\beta\dot{\alpha}}\xi^\beta(x_T)\,, \tag{1.175}$$

$$\mathcal{T}\eta_\alpha(x)\mathcal{T}^{-1} \equiv \bar{\eta}^t_{\dot{\alpha}}(x) = \eta_T^* \sigma^0_{\beta\dot{\alpha}}\eta^\beta(x_T)\,, \tag{1.176}$$

That is, the phases that appear in the time-reversal transformation laws of two-component fermion fields of opposite charge are complex conjugates of each other. If we apply the time-reversal operator twice,

$$\mathcal{T}^2\xi_\alpha(x)(\mathcal{T}^2)^{-1} = -\xi_\alpha(x)\,, \tag{1.177}$$

$$\mathcal{T}^2\eta_\alpha(x)(\mathcal{T}^2)^{-1} = -\eta_\alpha(x)\,, \tag{1.178}$$

and so $T^2 = -1$ for both neutral and charged spin-1/2 fermions [as noted in eq. (1.160)]. In contrast to the case of P^2 considered in section 1.8, the phase η_T drops out in the computation of T^2 due to the anti-linearity of the time-reversal operator.

Notice that if we return to the case of a single uncharged two-component fermion by taking $\xi = \eta$ in the transformation laws above [eqs. (1.175) and (1.176)], then consistency of the ξ and η transformation laws implies that $\eta_T = \eta_T^*$ as previously noted. In the case of the charged fermion pair, the phase η_T is not constrained. However, we can always choose to work in another basis. For example, if we work in terms of the fields χ_1 and χ_2 [eqs. (1.103) and (1.104)], then it follows that:

$$\mathcal{T}\chi_{1\alpha}(x)\mathcal{T}^{-1} = (\mathrm{Re}\ \eta_T)\sigma^0_{\alpha\dot{\beta}}\chi_1^\beta(x_T) - (\mathrm{Im}\ \eta_T)\sigma^0_{\dot{\alpha}\beta}\chi_2^\beta(x_T)\,, \tag{1.179}$$

$$\mathcal{T}\chi_{2\alpha}(x)\mathcal{T}^{-1} = -(\mathrm{Im}\ \eta_P)\sigma^0_{\alpha\dot{\beta}}\chi_1^\beta(x_T) - (\mathrm{Re}\ \eta_P)\sigma^0_{\dot{\alpha}\beta}\chi_2^\beta(x_T)\,. \tag{1.180}$$

If we choose $\eta_T = \eta_T^*$, then χ_1 and χ_2 have simple transformation properties under time reversal:

$$\mathcal{T}\chi_{1\alpha}(x)\mathcal{T}^{-1} = \eta_T\sigma^0_{\dot{\alpha}\beta}\chi_1^\beta(x_T)\,, \tag{1.181}$$

$$\mathcal{T}\chi_{2\alpha}(x)\mathcal{T}^{-1} = -\eta_T\sigma^0_{\dot{\alpha}\beta}\chi_2^\beta(x_T)\,. \tag{1.182}$$

That is, the fields χ_1 and χ_2 obey the standard time reversal transformation laws of a single two-component fermion field [eqs. (1.155)–(1.158)], but with *opposite* sign phase factors η_T in each case. Had one chosen $\eta_T \neq \eta_T^*$ in eqs. (1.175) and (1.176), then the time reversal transformation laws of χ_1 and χ_2 would have been more complicated [eqs. (1.179) and (1.180)]. But as before, one is free to make a further SO(2) rotation to transform χ_1 and χ_2 into new fields that do exhibit the simpler time reversal transformation laws [eq. (1.151)].

One can now work out the time reversal properties of bilinear covariants constructed out of the fields of a charged fermion pair. The results are listed in Table B.3.

1.10 Charge conjugation of two-component spinors

Charge conjugation was introduced in section 1.7. The charge conjugation operator is a discrete operator that interchanges particles and their C-conjugates. Here, conjugation refers to some conserved charge operator Q, where $\mathcal{C}Q\mathcal{C}^{-1} = -Q$. The conjugate of the conjugate field is the original field, so that $C^2 = 1$. For a single two-component fermion field

$$\mathcal{C}\chi_\alpha(x)\mathcal{C}^{-1} \equiv \chi^C_\alpha(x) = \eta_C\chi_\alpha(x)\,, \tag{1.183}$$

$$\mathcal{C}\overline{\chi}_{\dot\alpha}(x)\mathcal{C}^{-1} \equiv \overline{\chi}^C_{\dot\alpha}(x) = \eta_C^*\overline{\chi}_{\dot\alpha}(x)\,, \tag{1.184}$$

where $|\eta_C| = 1$. In this case, no conserved charge exists, so that charge conjugation is trivial. In particular, as noted in section 1.7, the invariance of the mass term $\chi\chi + \overline{\chi}\,\overline{\chi}$ implies that $\eta_C^* = \eta_C$, and hence $\eta_C = \pm 1$.[27] The behavior of the bilinear covariants under C is simple:

$$\chi_1^C\mathcal{O}\chi_2^C = \eta_{C1}\eta_{C2}\chi_1\mathcal{O}\chi_2\,, \tag{1.185}$$

for any $\mathcal{O} = I_2, \sigma_\mu, \overline{\sigma}_\mu, \sigma_{\mu\nu}$ and $\overline{\sigma}_{\mu\nu}$ (where bars should appear over the appropriate χ_i depending on the choice of $\mathcal{O}$).

In theories with multiple two-component fermion fields with no global symmetries, the charge conjugation properties of each fermion is given by eqs. (1.183) and (1.184), with $\eta_C = \pm 1$. When interactions are included, if there is some choice of the η_C such that the action is invariant under charge conjugation, then the theory is charge conjugation invariant. If there is an internal global symmetry ($\chi_i \to U_{ij}\chi_j$) under which the Lagrangian is invariant, then there are relations among the charge conjugation transformations of the fermion fields. Here, we consider

[27] In a theory with just one fermion field, no physical quantity can depend on the sign of η_C since the fermion field must appear quadratically in the Lagrangian. So, without loss of generality, we can take $\eta_C = 1$.

the simplest case of two mass-degenerate fermion fields, χ_1 and χ_2, and identify the conserved charge as $Q = \int J^0 d^3x$, where the conserved current J^μ is given in eq. (1.108). In terms of the linear combinations ξ and η [eqs. (1.103) and (1.104)] corresponding to the fields of definite charge, we define the charge conjugation transformations as follows.

$$\mathcal{C}\xi_\alpha(x)\mathcal{C}^{-1} \equiv \xi^C_\alpha(x) = \eta_C \eta_\alpha(x)\,, \quad \mathcal{C}\overline{\xi}_{\dot\alpha}(x)\mathcal{C}^{-1} \equiv \overline{\xi}^C_{\dot\alpha}(x) = \eta^*_C \bar\eta_{\dot\alpha}(x)\,, \tag{1.186}$$

$$\mathcal{C}\eta_\alpha(x)\mathcal{C}^{-1} \equiv \eta^C_\alpha(x) = \eta^*_C \xi_\alpha(x)\,, \quad \mathcal{C}\bar\eta_{\dot\alpha}(x)\mathcal{C}^{-1} \equiv \bar\eta^C_{\dot\alpha}(x) = \eta_C \overline{\xi}_{\dot\alpha}(x)\,. \tag{1.187}$$

Notice that if we return to the case of a single uncharged two-component fermion by taking $\xi = \eta$ in the transformation laws above [eqs. (1.186) and (1.187)], then consistency of the ξ and η transformation laws implies that $\eta_C = \eta^*_C$ as previously noted. In the case of the charged fermion pair, the phase η_C is not constrained (*i.e.*, $C^2 = 1$ independent of the value of the phase η_C). However, we can always choose to work in another basis. For example, if we work in terms of the fields χ_1 and χ_2 [eqs. (1.103) and (1.104)], then it follows that:

$$\mathcal{C}\chi_1(x)\mathcal{C}^{-1} = (\mathrm{Re}\ \eta_C)\chi_1(x) + (\mathrm{Im}\ \eta_C)\chi_2(x)\,, \tag{1.188}$$

$$\mathcal{C}\chi_2(x)\mathcal{C}^{-1} = (\mathrm{Im}\ \eta_C)\chi_1(x) - (\mathrm{Re}\ \eta_C)\chi_2(x)\,. \tag{1.189}$$

If we choose $\eta_C = \eta^*_C$, then χ_1 and χ_2 have simple transformation properties under charge conjugation:

$$\mathcal{C}\chi_1(x)\mathcal{C}^{-1} = \eta_C\chi_1(x)\,, \qquad \mathcal{C}\chi_2(x)\mathcal{C}^{-1} = -\eta_C\chi_2(x)\,. \tag{1.190}$$

That is, the fields χ_1 and χ_2 obey the standard charge conjugation transformation laws of a single two-component fermion field [eqs. (1.183) and (1.184)], but with *opposite* sign phase factors η_C in each case. Had one chosen $\eta_C \neq \eta^*_C$ in eqs. (1.186) and (1.187), then the charge conjugation transformation laws of χ_1 and χ_2 would have been more complicated [eqs. (1.188) and (1.189)]. But once again, one is free to make a further SO(2) rotation to transform χ_1 and χ_2 into new fields that do exhibit the simpler charge conjugation transformation laws [eq. (1.190)].

One can now work out the charge conjugation properties of bilinear covariants constructed out of the fields of a charged fermion pair. The results are listed in Table B.3.

1.11 CP and CPT conjugation of two-component spinors

Given the results of the previous three sections, it is a simple matter to work out the effects of CP and CPT transformations on the two-component spinors and the bilinear covariants. Here, we focus on the case of two mass-degenerate fermion fields, χ_1 and χ_2, with the associated conserved charge J^μ given in eq. (1.108). In terms of the linear combinations ξ and η [eqs. (1.103) and (1.104)] corresponding to the fields of definite charge, the CP transformations of the two-component spinors are given by:

$$\mathcal{CP}\xi_\alpha(x)(\mathcal{CP})^{-1} = \eta_{CP} i\sigma^0_{\alpha\dot{\beta}}\overline{\xi}^{\dot{\beta}}(x_P)\,, \tag{1.191}$$

$$\mathcal{CP}\eta_\alpha(x)(\mathcal{CP})^{-1} = \eta^*_{CP} i\sigma^0_{\alpha\dot{\beta}}\bar{\eta}^{\dot{\beta}}(x_P)\,, \tag{1.192}$$

where $\eta_{CP} \equiv \eta_C\eta_P$. Again, observe that if we return to the case of a single uncharged two-component fermion by taking $\xi = \eta$ in the transformation laws above [eqs. (1.191) and (1.192)], then consistency of the ξ and η transformation laws implies that $\eta_{CP} = \eta^*_{CP}$ as expected. In the case of the charged fermion pair, the phase η_{CP} is not constrained. However, if we work in terms of the fields χ_1 and χ_2 [eqs. (1.103) and (1.104)], and choose $\eta_{CP} = \pm 1$ (as required for uncharged two-component fields) then as before it follows that the χ_i have simple transformation properties:

$$\mathcal{CP}\chi_{1\alpha}(x)(\mathcal{CP})^{-1} = \eta_{CP} i\sigma^0_{\alpha\dot{\beta}}\overline{\chi}_1^{\dot{\beta}}(x_P)\,, \tag{1.193}$$

$$\mathcal{CP}\chi_{2\alpha}(x)(\mathcal{CP})^{-1} = -\eta_{CP} i\sigma^0_{\alpha\dot{\beta}}\overline{\chi}_2^{\dot{\beta}}(x_P)\,. \tag{1.194}$$

That is, the fields χ_1 and χ_2 obey the standard CP transformation laws of a single two-component fermion field [eqs. (1.191) and (1.192)], but with *opposite* sign phase factors η_{CP} in each case. This is the origin of the oft-quoted statement in the literature that a theory of two mass-degenerate Majorana fermions of opposite CP quantum numbers can be combined into a single Dirac fermion.

In general, when acting on charged fermion fields $\mathcal{CP} \neq \mathcal{PC}$. A simple computation yields: $PC = (\eta_P^*)^2 CP$ when acting on the ξ and $\bar{\eta}$ fields.[28] However, for self-conjugate spin-1/2 fermion fields, $PC = CP$ since $\eta_P = \pm 1$. This provides another motivation for the choice of $\eta^*_{CP} = \eta_{CP}$ above in the case of charged fermion fields. One can also examine the behavior of the fermion fields under the other possible products of C, P and T. Again, the order of the operators can be significant. For example, $TC = (\eta_C^*\eta_T^*)^2 CT$ and $TP = -(\eta_P^*)^2 PT$ when acting on the ξ and $\bar{\eta}$ fields, due to the anti-linearity of the time-reversal operator. For self-conjugate spin-1/2 fermion fields, $TC = CT$ and $TP = -PT$ since η_C, η_P and η_T are real (and equal to either ± 1). One is also free to employ these phase conventions in the case of charged fermion fields.

[28] When acting on the $\bar{\xi}$ and η fields (whose conserved charge is opposite to that of ξ and $\bar{\eta}$), one must employ the complex conjugate of the phases above.

Finally, we consider the effect of a CPT transformation.[29] The corresponding transformation laws under CPT conjugation take on even simpler forms:

$$\mathcal{CPT}\,\xi_\alpha(x)(\mathcal{CPT})^{-1} = -i\eta_{CPT}\overline{\xi}_{\dot\alpha}(-x)\,, \tag{1.195}$$
$$\mathcal{CPT}\,\eta_\alpha(x)(\mathcal{CPT})^{-1} = -i\eta^*_{CPT}\bar\eta_{\dot\alpha}(-x)\,, \tag{1.196}$$

where $\eta_{CPT} \equiv \eta_C\eta_P\eta_T$ and $-x \equiv (-t\,;\,-\vec{\boldsymbol{x}})$. Again, we observe that when $\xi = \eta$, consistency of the CPT transformation laws implies that for self-conjugate neutral fermion fields, $\eta_{CPT} = \pm 1$. Note that when $\mathcal{CPT}$ is applied twice,[30]

$$(\mathcal{CPT})^2\xi_\alpha(x)[(\mathcal{CPT})^2]^{-1} = -(\eta^*_{CPT})^2\xi_\alpha(x)\,, \tag{1.197}$$
$$(\mathcal{CPT})^2\eta_\alpha(x)[(\mathcal{CPT})^2]^{-1} = -(\eta_{CPT})^2\eta_\alpha(x)\,. \tag{1.198}$$

Thus, for self-conjugate neutral spin-1/2 fermion fields, it follows that $(CPT)^2 = -1$. As before, if we also choose $\eta_{CPT} = \pm 1$ in the case of charged fields ξ and η, then one obtains simple CPT-transformation laws when the charged fermion fields are rewritten in terms of χ_1 and χ_2 [eqs. (1.103) and (1.104)]:

$$\mathcal{CPT}\,\chi_{i\alpha}(x)(\mathcal{CPT})^{-1} = -i\eta_{CPT}\overline{\chi}_{i\dot\alpha}(-x)\,, \qquad i = 1, 2\,. \tag{1.199}$$

However, the choice of the phase η_{CPT} is more than just a convenience. In general, in order to guarantee that $\mathcal{CPT}\mathcal{L}(x)(\mathcal{CPT})^{-1} = \mathcal{L}(-x)$, we demand that η_{CPT} is independent of the particle species. For integral spin s particles, one must choose $\eta_{CPT} = (-1)^s$. For half-integral spin particles, *all* fermions must possess the *same* value of η_{CPT}, which must therefore be equal to one of the two allowed values $\eta_{CPT} = \pm 1$ for neutral self-conjugate fermions. However, in the latter case, the overall sign choice for η_{CPT} is not physically meaningful, since an even number of fermion fields always appears in each term of $\mathcal{L}$. Nevertheless, one is free to choose a phase convention such that for any half-integer spin particle,

$$\eta_{CPT} \equiv \eta_C\eta_P\eta_T = +1\,. \tag{1.200}$$

Consider the bilinear covariants of the form $\chi_i\mathcal{O}\chi_j$ where $\mathcal{O} = I_2, \sigma_\mu, \overline{\sigma}_\mu, \sigma_{\mu\nu}$ or $\overline{\sigma}_{\mu\nu}$ (and bars should appear over the appropriate χ_i depending on the choice of $\mathcal{O}$). These can be used to construct

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[29] In principle, there are six possible orderings of C, P and T; the effect of the corresponding operators differ by at most a phase which is easily determined from the results previously obtained above.

[30] Using the results quoted above, when applied to the spin-1/2 fermions fields ξ and $\bar\eta$ [see footnote 28], $(CPT)^2 = (\eta^*_{CPT})^2C^2(PT)^2 = -(\eta^*_{CPT})^2$, where we have noted that $C^2 = 1$ and $(PT)^2 = -1$ independently of the phase choices.

hermitian bilinear quantities that are either scalar, vectors or second-rank antisymmetric tensors with respect to proper Lorentz transformations. Denote these by B, B^μ and $B^{\mu\nu}$, respectively. Then, starting from eq. (1.199) and using eqs. (1.164) and (1.170) and the anti-commutativity of the fermion fields, it follows that:

$$\mathcal{CPT}\, B(x)\, (\mathcal{CPT})^{-1} = B(-x)\,, \tag{1.201}$$

$$\mathcal{CPT}\, B^\mu(x)\, (\mathcal{CPT})^{-1} = -B^\mu(-x)\,, \tag{1.202}$$

$$\mathcal{CPT}\, B^{\mu\nu}(x)\, (\mathcal{CPT})^{-1} = B^{\mu\nu}(-x)\,, \tag{1.203}$$

provided that the phase η_{CPT} is chosen to be the same for *every* two-component fermion field (as noted above). Thus, we see that all Lorentz tensors of the same (integer) rank behave the same way under a CPT transformation. This ensures that the Lagrangian for any local quantum field theory (which is a hermitian Lorentz scalar) is CPT-invariant.

Problems

1. Consider a proper orthochronous Lorentz transformation that is a pure boost. We follow the notation of eqs. (1.2) and (1.3).

(a) Prove the following two relations:

$$\Lambda = \exp[-i\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{k}}] = I_4 - i\hat{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{k}}\sinh\zeta + (\hat{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{k}})^2[1-\cosh\zeta]\,, \tag{1.204}$$

$$\exp\left(\tfrac{1}{2}\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\right) = \cosh(\zeta/2) + \hat{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sinh(\zeta/2)\,, \tag{1.205}$$

where $\vec{\boldsymbol{\zeta}} = \hat{\boldsymbol{v}}\tanh^{-1}\beta$ and $\zeta \equiv |\vec{\boldsymbol{\zeta}}|$.

(b) Prove the following two identities:

$$\sqrt{p\!\cdot\!\sigma} \equiv \frac{E+m-\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}}{\sqrt{2(E+m)}}\,, \tag{1.206}$$

$$\sqrt{p\!\cdot\!\overline{\sigma}} \equiv \frac{E+m+\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}}{\sqrt{2(E+m)}}\,, \tag{1.207}$$

where $p^\mu \equiv (E\,;\vec{\boldsymbol{p}})$. The matrix square root of $p\!\cdot\!\sigma$ [or $p\!\cdot\!\overline{\sigma}$] as defined here is the unique hermitian matrix with non-negative eigenvalues whose square is equal to $p\!\cdot\!\sigma$ [or $p\!\cdot\!\overline{\sigma}$].

(c) Using the results of parts (a) and (b), and noting that $\cosh\zeta = E/m$, show that for a pure boost (where $\theta_{ij} = 0$ and $\zeta^i = \theta^{i0} = -\theta^{0i}$), the Lorentz transformations for the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations, respectively, are given by:

$$\exp\left(-\frac{i}{2}\theta_{\mu\nu}S^{\mu\nu}\right) = \begin{cases} \exp\left(-\dfrac{1}{2}\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\right) = \sqrt{\dfrac{p\!\cdot\!\sigma}{m}}\,, & \text{for } (\dfrac{1}{2},0)\,, \\ \exp\left(\dfrac{1}{2}\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\right) = \sqrt{\dfrac{p\!\cdot\!\overline{\sigma}}{m}}\,, & \text{for } (0,\dfrac{1}{2})\,. \end{cases} \tag{1.208}$$

2. Starting from the free-field Dirac equations in two-component notation given in eq. (1.106), show that these equations are form invariant under a proper orthochronous Lorentz transformation, $x'^\mu = \Lambda^\mu{}_\nu x^\nu$. That is, by writing $\xi'_\alpha(x') = M_\alpha{}^\beta\xi_\beta(x)$ and $\bar{\eta}'^{\dot\alpha}(x') = (M^{-1})^{\dagger\,\dot\alpha}{}_{\dot\beta}\bar{\eta}^{\dot\beta}(x)$, and noting that $\partial_\mu = \Lambda^\nu{}_\mu\partial'_\nu$, show that ξ'_α and $\bar{\eta}'^{\dot\alpha}$ satisfy the Dirac equation in the transformed reference frame if eq. (1.75) is satisfied.

3. (a) Show that there is no solution for M in eq. (1.75) for $\Lambda = \Lambda_P$, where Λ_P is given by eq. (1.131).

(b) Starting from the free-field Dirac equations in two-component notation given in eqs. (1.106) and (1.107), show that these equations are form invariant under a parity transformation, $x'^{\mu} = (\Lambda_P)^{\mu}{}_{\nu}x^{\nu}$. That is, by writing $\xi'_{\alpha}(x') = iP_{\alpha\dot{\beta}}\bar{\eta}^{\dot{\beta}}(x)$ and $\bar{\eta}'^{\dot{\alpha}} = i\overline{P}^{\dot{\alpha}\beta}\xi_{\beta}(x)$, where $\overline{P}^{\dot{\alpha}\beta} \equiv \epsilon^{\beta\gamma}\epsilon^{\dot{\alpha}\dot{\delta}}(P^{\dagger})_{\gamma\dot{\delta}}$, show that ξ' and $\bar{\eta}'$ satisfy the Dirac equation in the space-reflected reference frame if:

$$\overline{P}^{-1}\overline{\sigma}^{\mu}P = (\Lambda_P)^{\mu}{}_{\nu}\sigma^{\nu}\,. \tag{1.209}$$

Determine P (up to an overall phase factor).

4. (a) Show that there is no solution for M in eq. (1.75) for $\Lambda = \Lambda_T$, where Λ_T is given by eq. (1.154).

(b) Starting from the free-field Dirac equations in two-component notation given in eqs. (1.106) and (1.107), show that these equations are form invariant under a time-reversal transformation, $x'^{\mu} = (\Lambda_T)^{\mu}{}_{\nu}x^{\nu}$. That is, by writing $\xi'_{\alpha}(x') = T_{\alpha}{}^{\dot{\beta}}\overline{\xi}_{\dot{\beta}}(x)$ and $\bar{\eta}'^{\dot{\alpha}} = -\overline{T}^{\dot{\alpha}}{}_{\beta}\eta^{\beta}(x)$, where $\overline{T}^{\dot{\alpha}}{}_{\beta} \equiv \epsilon^{\dot{\alpha}\dot{\delta}}\epsilon_{\beta\gamma}(T^{\dagger})^{\gamma}{}_{\dot{\delta}}$, show that ξ' and $\bar{\eta}'$ satisfy the Dirac equation in the time-reversed reference frame if:

$$\overline{T}^{-1}\overline{\sigma}^{\mu}T = (\Lambda_T)^{\mu}{}_{\nu}\overline{\sigma}^{\nu}\,. \tag{1.210}$$

Determine T (up to an overall phase factor).

5. Consider the following Lagrangian of two free two-component fermion fields, χ_1 and χ_2:

$$\mathcal{L} = i\overline{\chi}_1\overline{\sigma}^{\mu}\partial_{\mu}\chi_1 + i\overline{\chi}_2\overline{\sigma}^{\mu}\partial_{\mu}\chi_2 - m(\chi_1\chi_2 + \overline{\chi}_1\overline{\chi}_2) - \tfrac{1}{2}M(\chi_2\chi_2 + \overline{\chi}_2\overline{\chi}_2)\,. \tag{1.211}$$

(a) Determine the mass-eigenstates and find the corresponding masses of the two fermions.

(b) In the seesaw mechanism for light neutrino masses, one assumes that $m \ll M$. From the results of part (a), find the mass of the lightest fermion (keeping terms of order m^2/M). Determine the numerical value of M, assuming $m = m_{\tau} = 1.777$ GeV and $m_{\nu_{\tau}} \simeq 5 \times 10^{-2}$ eV.

6. Consider the following Lagrangian of n free two-component fermion fields, ζ_i:

$$\mathcal{L} = iK_i{}^j\,\bar{\hat{\zeta}}^i\,\overline{\sigma}^{\mu}\partial_{\mu}\hat{\zeta}_j - \tfrac{1}{2}M^{ij}\hat{\zeta}_i\hat{\zeta}_j - \tfrac{1}{2}M_{ij}\bar{\hat{\zeta}}^i\bar{\hat{\zeta}}^j\,, \tag{1.212}$$

(a) What are the constraints on $K_i{}^j$, assuming that the Lagrangian is hermitian?

(b) Define a new set of fields ζ_i in terms of the $\hat{\zeta}_i$, such that the kinetic energy term is canonical (*i.e.*, $K_i{}^j = \delta_i{}^j$). Re-express the Lagrangian in terms of the new fields ζ_i and show that the resulting expression takes the form given in eq. (1.110).

(c) How does the presence of $K_i{}^j$ affect the mass diagonalization procedure of Section 1.6?

References

[1] S. Weinberg, *The Quantum Theory of Fields, Volume I: Foundations* (Cambridge University Press, Cambridge, UK, 1995).

[2] References to the Lorentz algebra and the interpretations of boosts.

[3] R.A. Horn and C.R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, England, 1990);

[4] R.N. Mohapatra and P.B. Pal, *Massive Neutrinos in Physics and Astrophysics*, 2nd edition (World Scientific, Singapore, 1998).

[5] The singular value decomposition of a complex matrix is discussed in many linear algebra textbooks. See, *e.g.*, ref. [3].

[6] D. Bailin, *Weak Interactions*, second edition (Adam Hilger Ltd., Bristol, England, 1982).

2
Feynman Rules for Fermions

In this chapter, we devise a set of Feynman rules to describe matrix elements of processes involving fermions. The rules are developed for two-component fermions and compared to the usual rules for four-component fermions that can be found in any textbook on quantum field theory.

2.1 Fermion creation and annihilation operators

We begin by describing the properties of a free neutral massive anti-commuting spin-1/2 field, denoted $\chi_\alpha(x)$, which transforms as $(\frac{1}{2}, 0)$ under the Lorentz group. The Lagrangian density is given by eq. (1.99). The corresponding field equations (*i.e.*, the two-component version of the Dirac equation) are given by eqs. (1.100) and (1.101). By virtue of the field equations, the field $\chi_\alpha(x)$ can be expanded in a Fourier series:

$$\chi_\alpha(x) = \sum_\lambda \int \frac{d^3\boldsymbol{p}}{(2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}} \times \left[x_\alpha(\vec{\boldsymbol{p}}, \lambda) a(\vec{\boldsymbol{p}}, \lambda) e^{-ip\cdot x} + y_\alpha(\vec{\boldsymbol{p}}, \lambda) a^\dagger(\vec{\boldsymbol{p}}, \lambda) e^{ip\cdot x}\right], \quad (2.1)$$

where $E_{\boldsymbol{p}} \equiv (|\vec{\boldsymbol{p}}|^2 + m^2)^{1/2}$, and the creation and annihilation operators $a^\dagger$ and a satisfy anticommutation relations:

$$\{a(\vec{\boldsymbol{p}}, \lambda), a^\dagger(\vec{\boldsymbol{p}}', \lambda')\} = \delta^3(\vec{\boldsymbol{p}} - \vec{\boldsymbol{p}}')\delta_{\lambda\lambda'}\,, \quad (2.2)$$

and all other anticommutators vanish. Applying eq. (1.100) to eq. (2.1), we find that the x_α and y_α satisfy momentum space Dirac equations. We shall write these equations out explicitly and study the propereties of these two-component spinor wave functions in Section 2.2.

Similarly,

$$\bar{\chi}_{\dot{\alpha}}(x) \equiv (\chi_\alpha)^\dagger = \sum_\lambda \int \frac{d^3\boldsymbol{p}}{(2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}} \times \left[\bar{x}_{\dot{\alpha}}(\vec{\boldsymbol{p}},\lambda)a^\dagger(\vec{\boldsymbol{p}},\lambda)e^{ip\cdot x} + \bar{y}_{\dot{\alpha}}(\vec{\boldsymbol{p}},\lambda)a(\vec{\boldsymbol{p}},\lambda)e^{-ip\cdot x}\right]. \tag{2.3}$$

We employ covariant normalization of the one particle states, *i.e.*, we act with one creation operator on the vacuum with the following convention

$$|\vec{\boldsymbol{p}},\lambda\rangle \equiv (2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}a^\dagger(\vec{\boldsymbol{p}},\lambda)\,|0\rangle\,, \tag{2.4}$$

so that $\langle\vec{\boldsymbol{p}},\lambda|\vec{\boldsymbol{p}}',\lambda'\rangle = (2\pi)^3(2E_{\boldsymbol{p}})\delta^3(\vec{\boldsymbol{p}}-\vec{\boldsymbol{p}}')\delta_{\lambda\lambda'}$. Therefore,

$$\langle 0|\,\chi_\alpha(x)\,|\vec{\boldsymbol{p}},\lambda\rangle = x_\alpha(\vec{\boldsymbol{p}},\lambda)e^{-ip\cdot x}\,, \qquad \langle 0|\,\bar{\chi}_{\dot{\alpha}}(x)\,|\vec{\boldsymbol{p}},\lambda\rangle = \bar{y}_{\dot{\alpha}}(\vec{\boldsymbol{p}},\lambda)e^{-ip\cdot x}\,, \tag{2.5}$$

$$\langle\vec{\boldsymbol{p}},\lambda|\,\chi_\alpha(x)\,|0\rangle = y_\alpha(\vec{\boldsymbol{p}},\lambda)e^{ip\cdot x}\,, \qquad \langle\vec{\boldsymbol{p}},\lambda|\,\bar{\chi}_{\dot{\alpha}}(x)\,|0\rangle = \bar{x}_{\dot{\alpha}}(\vec{\boldsymbol{p}},\lambda)e^{ip\cdot x}\,. \tag{2.6}$$

It should be emphasized that $\chi_\alpha(x)$ is an anticommuting spinor field, whereas x_α and y_α are *commuting* two-component spinor wave functions. The anticommuting properties of the fields are carried by the creation and annihilation operators $a^\dagger(\vec{\boldsymbol{p}},\lambda)$ and $a(\vec{\boldsymbol{p}},\lambda)$.

2.2 Properties of the two-component spinor wave functions

In this section, we explore in some detail the properties of the two-component spinor wave functions $x_\alpha(\vec{\boldsymbol{p}},\lambda)$ and $y_\alpha(\vec{\boldsymbol{p}},\lambda)$. Applying eq. (1.100) to eq. (2.1), x_α and y_α satisfy momentum space Dirac equations:

$$(p\cdot\overline{\sigma})^{\dot{\alpha}\beta}x_\beta = m\bar{y}^{\dot{\alpha}}\,, \qquad (p\cdot\sigma)_{\alpha\dot{\beta}}\bar{y}^{\dot{\beta}} = mx_\alpha\,, \tag{2.7}$$

$$(p\cdot\sigma)_{\alpha\dot{\beta}}\bar{x}^{\dot{\beta}} = -my_\alpha\,, \qquad (p\cdot\overline{\sigma})^{\dot{\alpha}\beta}y_\beta = -m\bar{x}^{\dot{\alpha}}\,, \tag{2.8}$$

$$x^\alpha(p\cdot\sigma)_{\alpha\dot{\beta}} = -m\bar{y}_{\dot{\beta}}\,, \qquad \bar{y}_{\dot{\alpha}}(p\cdot\overline{\sigma})^{\dot{\alpha}\beta} = -mx^\beta\,, \tag{2.9}$$

$$\bar{x}_{\dot{\alpha}}(p\cdot\overline{\sigma})^{\dot{\alpha}\beta} = my^\beta\,, \qquad y^\alpha(p\cdot\sigma)_{\alpha\dot{\beta}} = m\bar{x}_{\dot{\beta}}\,, \tag{2.10}$$

where $x_\alpha \equiv x_\alpha(\vec{\boldsymbol{p}},\lambda)$, $y_\alpha \equiv y_\alpha(\vec{\boldsymbol{p}},\lambda)$, *etc.* These equations imply that both x_α and y_α must satisfy the mass-shell condition, $p^2 = m^2$ (or equivalently, $p^0 = E_{\boldsymbol{p}}$).

The quantum number λ labels the spin or helicity of the spin-1/2 fermion. In order to construct the spin-1/2 helicity states, consider a basis of two-component spinors ζ_λ that are eigenstates of $\frac{1}{2}\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{p}}$, *i.e.*,

$$\tfrac{1}{2}\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{p}}\;\zeta_\lambda = \lambda\,\zeta_\lambda, \qquad \lambda = \pm\tfrac{1}{2}\,. \tag{2.11}$$

If $\hat{\boldsymbol{p}}$ is a unit vector with polar angle θ and azimuthal angle ϕ with respect to a fixed z-axis, then the two-component spinors are

$$\zeta_{1/2}(\hat{\boldsymbol{p}}) = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}, \qquad \zeta_{-1/2}(\hat{\boldsymbol{p}}) = \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}. \tag{2.12}$$

The two-component helicity spinors satisfy:

$$\zeta_{-\lambda}(\hat{\boldsymbol{p}}) = 2\lambda\epsilon\zeta^*_\lambda(\hat{\boldsymbol{p}})\,, \tag{2.13}$$

$$\zeta_\lambda(-\hat{\boldsymbol{p}}) = -2\lambda e^{2i\lambda\phi}\zeta_{-\lambda}(\hat{\boldsymbol{p}})\,, \tag{2.14}$$

where ϵ is the 2×2 matrix whose matrix elements are $\epsilon_{\alpha\beta}$, and $-\hat{\boldsymbol{p}}$ is a unit vector with polar angle $\pi-\theta$ and azimuthal angle $\phi+\pi$ with respect to the fixed z-axis. Alternatively, we could construct spin states where the spin is quantized in the particle's rest frame along a fixed axis, pointing along the unit three-vector $\hat{\boldsymbol{s}}$. If we denote ζ_s to be an eigenstate of $\frac{1}{2}\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{s}}$, then we may use the above formulas, where the angles θ and ϕ are now the polar and azimuthal angles of $\hat{\boldsymbol{s}}$. In relativistic scattering processes, it is usually more convenient to employ helicity states. Note that for massless particles, there is no rest frame and one must use helicity states.

For massive fermions, it is possible to define the spin four-vector s^μ, which satisfies $s\cdot p = 0$ and $s\cdot s = -1$. In the rest frame of the particle, $s^\mu(\lambda) = 2\lambda(0;\hat{\boldsymbol{s}})$, and $\lambda = \pm 1/2$ corresponds to spin-up and spin-down with respect to the spin quantization axis that points in the direction of the unit three-vector $\hat{\boldsymbol{s}}$. For helicity states, the spin four-vector is defined as[1]

$$s^\mu(\lambda) = \frac{2\lambda}{m}\left(|\vec{\boldsymbol{p}}|\,;\,E\hat{\boldsymbol{p}}\right)\,, \tag{2.15}$$

where $2\lambda = \pm 1$ is twice the spin-1/2 particle helicity. Note that in the rest frame, $s^\mu = 2\lambda\,(0\,;\,\hat{\boldsymbol{p}})$, whereas in the high energy limit (where $E \gg m$), $s^\mu = 2\lambda p^\mu/m + \mathcal{O}(m/E)$. For a massless fermion, the spin four-vector does not exist (there is no rest frame). Nevertheless, one can obtain consistent results by working with massive helicity states and taking the $m \to 0$ limit at the end of the computation. In this case, we can simply use $s^\mu = 2\lambda p^\mu/m + \mathcal{O}(m/E)$; in practical computations the final result will be well-defined in the zero mass limit.

The two-component spinors x and y can now be given explicitly in terms of the ζ_λ defined in eq. (2.12):

$$x_\alpha(\vec{\boldsymbol{p}},\lambda) = \sqrt{p\cdot\sigma}\,\zeta_\lambda\,, \qquad x^\alpha(\vec{\boldsymbol{p}},\lambda) = -2\lambda\zeta^\dagger_{-\lambda}\sqrt{p\cdot\overline{\sigma}}\,, \tag{2.16}$$

$$y_\alpha(\vec{\boldsymbol{p}},\lambda) = 2\lambda\sqrt{p\cdot\sigma}\,\zeta_{-\lambda}\,, \qquad y^\alpha(\vec{\boldsymbol{p}},\lambda) = \zeta^\dagger_\lambda\sqrt{p\cdot\overline{\sigma}}\,, \tag{2.17}$$

[1] The overall sign of s^μ depends on the choice of λ. Thus, we write $s^\mu(\lambda)$ to emphasize this dependence. Later on in the text, we will often write s^μ for $s^\mu(\lambda)$ when there is no specific need to highlight the dependence on λ.

or equivalently

$$\bar{x}^{\dot{\alpha}}(\vec{\boldsymbol{p}},\lambda) = -2\lambda\sqrt{p\cdot\overline{\sigma}}\,\zeta_{-\lambda}\,, \qquad \bar{x}_{\dot{\alpha}}(\vec{\boldsymbol{p}},\lambda) = \zeta^{\dagger}_{\lambda}\sqrt{p\cdot\sigma}\,, \tag{2.18}$$

$$\bar{y}^{\dot{\alpha}}(\vec{\boldsymbol{p}},\lambda) = \sqrt{p\cdot\overline{\sigma}}\,\zeta_{\lambda}\,, \qquad \bar{y}_{\dot{\alpha}}(\vec{\boldsymbol{p}},\lambda) = 2\lambda\zeta^{\dagger}_{-\lambda}\sqrt{p\cdot\sigma}\,. \tag{2.19}$$

In the above equations, $p^0 = E_{\boldsymbol{p}}$ is satisfied, and the (hermitian) matrices $\sqrt{p\cdot\sigma}$ and $\sqrt{p\cdot\overline{\sigma}}$ are given in eqs. (1.206) and (1.207).

The phase choices employed in eqs. (2.16)–(2.19) are conventional and consistent with the phase choices for four-component spinor wave functions [see Section 3.3]. We again emphasize that in eqs. (2.16)–(2.19), one may either choose ζ_λ to be an eigenstate of $\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{p}}$ (which yields the helicity spinor wave functions), or choose ζ_λ to be an eigenstate of $\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{s}}$, where the spin is measured in the rest frame along the quantization axis $\hat{\boldsymbol{s}}$.

The following equations can now be verified by explicit computation:

$$(s\cdot\overline{\sigma})^{\dot{\alpha}\beta}x_\beta = \bar{y}^{\dot{\alpha}}\ , \qquad (s\cdot\sigma)_{\alpha\dot{\beta}}\bar{y}^{\dot{\beta}} = -x_\alpha\ , \tag{2.20}$$

$$(s\cdot\sigma)_{\alpha\dot{\beta}}\bar{x}^{\dot{\beta}} = -y_\alpha\ , \qquad (s\cdot\overline{\sigma})^{\dot{\alpha}\beta}y_\beta = \bar{x}^{\dot{\alpha}}\ , \tag{2.21}$$

$$x^\alpha(s\cdot\sigma)_{\alpha\dot{\beta}} = -\bar{y}_{\dot{\beta}}\ , \qquad \bar{y}_{\dot{\alpha}}(s\cdot\overline{\sigma})^{\dot{\alpha}\beta} = x^\beta\ , \tag{2.22}$$

$$\bar{x}_{\dot{\alpha}}(s\cdot\overline{\sigma})^{\dot{\alpha}\beta} = y^\beta\ , \qquad y^\alpha(s\cdot\sigma)_{\alpha\dot{\beta}} = -\bar{x}_{\dot{\beta}}\ , \tag{2.23}$$

where $s^\mu \equiv s^\mu(\lambda)$, $x_\alpha \equiv x_\alpha(\vec{\boldsymbol{p}},\lambda)$, $y_\alpha \equiv y_\alpha(\vec{\boldsymbol{p}},\lambda)$, *etc.* Eqs. (2.20)–(2.23) are the analogues of the momentum space Dirac equations [eqs. (2.7)–(2.10)]. From these equations, one can check that both x_α and y_α must satisfy $s\cdot s = -1$ and $p\cdot s = 0$.

It is useful to combine the results of eqs. (2.7)–(2.10) and eqs. (2.20)–(2.23) as follows:

$$(p^\mu - ms^\mu)\overline{\sigma}^{\dot{\alpha}\beta}_\mu x_\beta = 0\,, \qquad (p_\mu - ms_\mu)\sigma^\mu_{\alpha\dot{\beta}}\bar{x}^{\dot{\beta}} = 0\,, \tag{2.24}$$

$$(p^\mu + ms^\mu)\overline{\sigma}^{\dot{\alpha}\beta}_\mu y_\beta = 0\,, \qquad (p_\mu + ms_\mu)\sigma^\mu_{\alpha\dot{\beta}}\bar{y}^{\dot{\beta}} = 0\,, \tag{2.25}$$

$$x^\alpha\sigma^\mu_{\alpha\dot{\beta}}(p_\mu - ms_\mu) = 0\,, \qquad \bar{x}_{\dot{\alpha}}\overline{\sigma}^{\dot{\alpha}\beta}_\mu(p^\mu - ms^\mu) = 0\,, \tag{2.26}$$

$$y^\alpha\sigma^\mu_{\alpha\dot{\beta}}(p_\mu + ms_\mu) = 0\,, \qquad \bar{y}_{\dot{\alpha}}\overline{\sigma}^{\dot{\alpha}\beta}_\mu(p^\mu + ms^\mu) = 0\,. \tag{2.27}$$

The above results are applicable only for massive fermions (where the spin four-vector s^μ exists). However, in the case of a massless fermion we can obtain valid results for helicity spinors simply by setting $s^\mu = 2\lambda p^\mu/m$ and then taking the $m \to 0$ limit. In particular, replacing $ms^\mu = 2\lambda p^\mu$ in eqs. (2.24)–(2.27), and using the results of eqs. (2.7)–(2.10) [before taking the $m \to 0$ limit] yields

$$(1+2\lambda)x(\vec{\boldsymbol{p}},\lambda) = 0\,, \qquad (1-2\lambda)y(\vec{\boldsymbol{p}},\lambda) = 0\,, \tag{2.28}$$

where λ is the helicity. Finally, we take the $m \to 0$ limit. The meaning of the end result is clear; for massless fermions, only one helicity component of x and y is non-zero. Applying this result to neutrinos, we find that massless neutrinos are left-handed ($\lambda = -1/2$), while anti-neutrinos are right-handed ($\lambda = +1/2$).

Having defined explicit forms for the two-component spinor wave functions, we can now write down the spin projection operators. These operators can be easily obtained by making use of the following result:

$$\tfrac{1}{2}\left[1 + \frac{1}{m}\sqrt{p\cdot\sigma}\big[s(\lambda)\cdot\overline{\sigma}\big]\sqrt{p\cdot\sigma}\right]\zeta_{\lambda'} = \delta_{\lambda\lambda'}\zeta_\lambda\,. \tag{2.29}$$

We then have for example, with both spinor indices assumed to be in the lowered position,

$$\begin{aligned} x(\vec{\boldsymbol{p}},\lambda)\bar{x}(\vec{\boldsymbol{p}},\lambda) &= \sqrt{p\cdot\sigma}\,\zeta_\lambda\zeta_\lambda^\dagger\,\sqrt{p\cdot\sigma} \\ &= \tfrac{1}{2}\sqrt{p\cdot\sigma}\left[1 + \frac{1}{m}\sqrt{p\cdot\sigma}s\cdot\overline{\sigma}\sqrt{p\cdot\sigma}\right]\sum_{\lambda'}\zeta_{\lambda'}\zeta_{\lambda'}^\dagger\sqrt{p\cdot\sigma} \\ &= \tfrac{1}{2}\left[p\cdot\sigma + \frac{1}{m}p\cdot\sigma s\cdot\overline{\sigma}p\cdot\sigma\right] \\ &= \tfrac{1}{2}\left[p\cdot\sigma - ms\cdot\sigma\right]\,, \end{aligned} \tag{2.30}$$

where $s \equiv s(\lambda)$. In deriving eq. (2.30), we made use of eq. (2.29) and the completeness of the ζ_λ. The product of three dot-products was simplified by noting that $p\cdot s = 0$ implies $s\cdot\overline{\sigma}\ p\cdot\sigma = -p\cdot\overline{\sigma}\ s\cdot\sigma$. The other spin projection formulas can be similarly derived. For massive fermions, they are:

$$x_\alpha(\vec{\boldsymbol{p}},\lambda)\bar{x}_{\dot\beta}(\vec{\boldsymbol{p}},\lambda) = \tfrac{1}{2}(p_\mu - ms_\mu)\sigma^\mu_{\alpha\dot\beta}\,, \tag{2.31}$$

$$\bar{y}^{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)y^\beta(\vec{\boldsymbol{p}},\lambda) = \tfrac{1}{2}(p^\mu + ms^\mu)\overline{\sigma}_\mu^{\dot\alpha\beta}\,, \tag{2.32}$$

$$x_\alpha(\vec{\boldsymbol{p}},\lambda)y^\beta(\vec{\boldsymbol{p}},\lambda) = \tfrac{1}{2}\left(m\delta_\alpha{}^\beta - [s\cdot\sigma\,p\cdot\overline{\sigma}]_\alpha{}^\beta\right)\,, \tag{2.33}$$

$$\bar{y}^{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)\bar{x}_{\dot\beta}(\vec{\boldsymbol{p}},\lambda) = \tfrac{1}{2}\left(m\delta^{\dot\alpha}{}_{\dot\beta} + [s\cdot\overline{\sigma}\,p\cdot\sigma]^{\dot\alpha}{}_{\dot\beta}\right)\,, \tag{2.34}$$

or equivalently,

$$\bar{x}^{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)x^\beta(\vec{\boldsymbol{p}},\lambda) = \tfrac{1}{2}(p^\mu - ms^\mu)\overline{\sigma}_\mu^{\dot\alpha\beta}\,, \tag{2.35}$$

$$y_\alpha(\vec{\boldsymbol{p}},\lambda)\bar{y}_{\dot\beta}(\vec{\boldsymbol{p}},\lambda) = \tfrac{1}{2}(p_\mu + ms_\mu)\sigma^\mu_{\alpha\dot\beta}\,, \tag{2.36}$$

$$y_\alpha(\vec{\boldsymbol{p}},\lambda)x^\beta(\vec{\boldsymbol{p}},\lambda) = -\tfrac{1}{2}\left(m\delta_\alpha{}^\beta + [s\cdot\sigma\,p\cdot\overline{\sigma}]_\alpha{}^\beta\right)\,, \tag{2.37}$$

$$\bar{x}^{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)\bar{y}_{\dot\beta}(\vec{\boldsymbol{p}},\lambda) = -\tfrac{1}{2}\left(m\delta^{\dot\alpha}{}_{\dot\beta} - [s\cdot\overline{\sigma}\,p\cdot\sigma]^{\dot\alpha}{}_{\dot\beta}\right)\,. \tag{2.38}$$

For the case of massless spin-1/2 fermions, we must use helicity spinor wave functions. Setting $s = 2\lambda p/m$ in the above formulas and letting $m \to 0$ yields

$$x_\alpha(\vec{\boldsymbol{p}},\lambda)\bar{x}_{\dot\beta}(\vec{\boldsymbol{p}},\lambda) = (\tfrac{1}{2}-\lambda)p\cdot\sigma_{\alpha\dot\beta}\,, \qquad x_\alpha(\vec{\boldsymbol{p}},\lambda)y^\beta(\vec{\boldsymbol{p}},\lambda) = 0\,, \tag{2.39}$$

$$\bar{y}^{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)y^\beta(\vec{\boldsymbol{p}},\lambda) = (\tfrac{1}{2}+\lambda)p\cdot\overline{\sigma}^{\dot\alpha\beta}\,, \qquad \bar{y}^{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)\bar{x}_{\dot\beta}(\vec{\boldsymbol{p}},\lambda) = 0\,, \tag{2.40}$$

$$\bar{x}^{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)x^\beta(\vec{\boldsymbol{p}},\lambda) = (\tfrac{1}{2}-\lambda)p\cdot\overline{\sigma}^{\dot\alpha\beta}\,, \qquad y_\alpha(\vec{\boldsymbol{p}},\lambda)x^\beta(\vec{\boldsymbol{p}},\lambda) = 0\,, \tag{2.41}$$

$$y_\alpha(\vec{\boldsymbol{p}},\lambda)\bar{y}_{\dot\beta}(\vec{\boldsymbol{p}},\lambda) = (\tfrac{1}{2}+\lambda)p\cdot\sigma_{\alpha\dot\beta}\,, \qquad \bar{x}^{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)\bar{y}_{\dot\beta}(\vec{\boldsymbol{p}},\lambda) = 0\,. \tag{2.42}$$

Having listed the projection operators for definite spin projection or helicity, we may now sum over spins to derive the spin-sum identities. These arise when computing squared matrix elements for unpolarized scattering and decay. There are only four basic identities, but for convenience we list each of them with the two index height permutations that can occur in squared amplitudes by following the rules given in this paper. The results can be derived by inspection of the spin projection operators, since summing over $\lambda = \pm 1/2$ simply removes all terms linear in the spin four-vector (which is proportional to λ).

$$\sum_\lambda x_\alpha(\vec{\boldsymbol{p}},\lambda)\bar{x}_{\dot\beta}(\vec{\boldsymbol{p}},\lambda) = p\cdot\sigma_{\alpha\dot\beta}\,, \qquad \sum_\lambda \bar{x}^{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)x^\beta(\vec{\boldsymbol{p}},\lambda) = p\cdot\overline{\sigma}^{\dot\alpha\beta}\,, \tag{2.43}$$

$$\sum_\lambda \bar{y}^{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)y^\beta(\vec{\boldsymbol{p}},\lambda) = p\cdot\overline{\sigma}^{\dot\alpha\beta}\,, \qquad \sum_\lambda y_\alpha(\vec{\boldsymbol{p}},\lambda)\bar{y}_{\dot\beta}(\vec{\boldsymbol{p}},\lambda) = p\cdot\sigma_{\alpha\dot\beta}\,, \tag{2.44}$$

$$\sum_\lambda x_\alpha(\vec{\boldsymbol{p}},\lambda)y^\beta(\vec{\boldsymbol{p}},\lambda) = m\delta_\alpha{}^\beta\,, \qquad \sum_\lambda y_\alpha(\vec{\boldsymbol{p}},\lambda)x^\beta(\vec{\boldsymbol{p}},\lambda) = -m\delta_\alpha{}^\beta\,, \tag{2.45}$$

$$\sum_\lambda \bar{y}^{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)\bar{x}_{\dot\beta}(\vec{\boldsymbol{p}},\lambda) = m\delta^{\dot\alpha}{}_{\dot\beta}\,, \qquad \sum_\lambda \bar{x}^{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)\bar{y}_{\dot\beta}(\vec{\boldsymbol{p}},\lambda) = -m\delta^{\dot\alpha}{}_{\dot\beta}\,. \tag{2.46}$$

These results hold for both massive and massless spin-1/2 fermions.

2.3 Charged two-component fermion fields

Consider a collection of free anti-commuting two-component spin-1/2 fields, $\chi_{\alpha i}(x)$, which transform as $(\frac{1}{2},0)$ fields under the Lorentz group. As shown in Section 1.6, one can diagonalize the fermion mass matrix. As a result, the free-field Lagrangian in terms of mass-eigenstate fields is given by eq. (1.115). Each $\chi_{\alpha i}$ can now be expanded in a Fourier series, as in Section 2.2:

$$\chi_{\alpha i}(x) = \sum_\lambda \int \frac{d^3p}{(2\pi)^{3/2}(2E_{i\boldsymbol{p}})^{1/2}} \times \left[x_\alpha(\vec{\boldsymbol{p}},\lambda) a_i(\vec{\boldsymbol{p}},\lambda) e^{-ip\cdot x} + y_\alpha(\vec{\boldsymbol{p}},\lambda) a_i^\dagger(\vec{\boldsymbol{p}},\lambda) e^{ip\cdot x} \right], \tag{2.47}$$

where $E_{i\boldsymbol{p}} \equiv (|\vec{\boldsymbol{p}}|^2 + m_i^2)^{1/2}$, and the creation and annihilation operators, $a_i^\dagger$ and a_i satisfy anticommutation relations:

$$\{a_i(\vec{\boldsymbol{p}},\lambda), a_j^\dagger(\vec{\boldsymbol{p}}',\lambda')\} = \delta^3(\vec{\boldsymbol{p}} - \vec{\boldsymbol{p}}')\delta_{\lambda\lambda'}\delta_{ij}\,, \tag{2.48}$$

and all other anticommutators vanish. We employ covariant normalization of the one particle states (separately for each flavor i) as in eq. (2.4).

In the case where some of the fields are massive Dirac fermions carrying a conserved charge, it is more convenient to work in terms of mass-eigenstate fields of definite charge. If ξ_α is a charged massive field, then there must be an associated independent two-component spinor field η_α of equal mass with the opposite charge. The corresponding Lagrangian is given by eq. (1.105). Together, ξ and η constitute a single Dirac spin-1/2 fermion. We can then write:

$$\xi_\alpha(x) = \sum_\lambda \int \frac{d^3p}{(2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}} \times \left[x_\alpha(\vec{\boldsymbol{p}},\lambda) a(\vec{\boldsymbol{p}},\lambda) e^{-ip\cdot x} + y_\alpha(\vec{\boldsymbol{p}},\lambda) b^\dagger(\vec{\boldsymbol{p}},\lambda) e^{ip\cdot x} \right], \tag{2.49}$$

$$\eta_\alpha(x) = \sum_\lambda \int \frac{d^3p}{(2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}} \times \left[x_\alpha(\vec{\boldsymbol{p}},\lambda) b(\vec{\boldsymbol{p}},\lambda) e^{-ip\cdot x} + y_\alpha(\vec{\boldsymbol{p}},\lambda) a^\dagger(\vec{\boldsymbol{p}},\lambda) e^{ip\cdot x} \right], \tag{2.50}$$

where $E_{\boldsymbol{p}} \equiv (|\vec{\boldsymbol{p}}|^2 + m^2)^{1/2}$, and the creation and annihilation operators, $a^\dagger$, $b^\dagger$, a and b satisfy anticommutation relations:

$$\{a(\vec{\boldsymbol{p}},\lambda), a^\dagger(\vec{\boldsymbol{p}}',\lambda')\} = \{b(\vec{\boldsymbol{p}},\lambda), b^\dagger(\vec{\boldsymbol{p}}',\lambda')\} = \delta^3(\vec{\boldsymbol{p}} - \vec{\boldsymbol{p}}')\delta_{\lambda\lambda'}\,, \tag{2.51}$$

and all other anticommutators vanish. We now must distinguish between two types of one particle states, which we can call fermion (F) and anti-fermion ($\overline{F}$):

$$|\vec{\boldsymbol{p}},\lambda;F\rangle \equiv (2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2} b^\dagger(\vec{\boldsymbol{p}},\lambda)\,|0\rangle\,, \tag{2.52}$$

$$|\vec{\boldsymbol{p}},\lambda;\overline{F}\rangle \equiv (2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2} a^\dagger(\vec{\boldsymbol{p}},\lambda)\,|0\rangle\,. \tag{2.53}$$

Note that both $\xi(x)$ and $\bar{\eta}(x)$ can create $|\vec{\boldsymbol{p}}, \lambda; F\rangle$ from the vacuum, while $\bar{\xi}(x)$ and $\eta(x)$ can create $\left|\vec{\boldsymbol{p}}, \lambda; \overline{F}\right\rangle$. The one-particle wave functions are given by:

$$\langle 0|\, \eta_\alpha(x)\, |\vec{\boldsymbol{p}}, \lambda; F\rangle = x_\alpha(\vec{\boldsymbol{p}}, \lambda) e^{-ip\cdot x}\,, \quad \langle 0|\, \bar{\xi}_{\dot{\alpha}}(x)\, |\vec{\boldsymbol{p}}, \lambda; F\rangle = \bar{y}_{\dot{\alpha}}(\vec{\boldsymbol{p}}, \lambda) e^{-ip\cdot x}\,, \tag{2.54}$$

$$\langle F; \vec{\boldsymbol{p}}, \lambda|\, \xi_\alpha(x)\, |0\rangle = y_\alpha(\vec{\boldsymbol{p}}, \lambda) e^{ip\cdot x}\,, \quad \langle F; \vec{\boldsymbol{p}}, \lambda|\, \bar{\eta}_{\dot{\alpha}}(x)\, |0\rangle = \bar{x}_{\dot{\alpha}}(\vec{\boldsymbol{p}}, \lambda) e^{ip\cdot x}\,, \tag{2.55}$$

$$\langle 0|\, \xi_\alpha(x)\, \left|\vec{\boldsymbol{p}}, \lambda; \overline{F}\right\rangle = x_\alpha(\vec{\boldsymbol{p}}, \lambda) e^{-ip\cdot x}\,, \quad \langle 0|\, \bar{\eta}_{\dot{\alpha}}(x)\, \left|\vec{\boldsymbol{p}}, \lambda; \overline{F}\right\rangle = \bar{y}_{\dot{\alpha}}(\vec{\boldsymbol{p}}, \lambda) e^{-ip\cdot x}\,, \tag{2.56}$$

$$\left\langle \overline{F}; \vec{\boldsymbol{p}}, \lambda\right|\, \eta_\alpha(x)\, |0\rangle = y_\alpha(\vec{\boldsymbol{p}}, \lambda) e^{ip\cdot x}\,, \quad \left\langle \overline{F}; \vec{\boldsymbol{p}}, \lambda\right|\, \bar{\xi}_{\dot{\alpha}}(x)\, |0\rangle = \bar{x}_{\dot{\alpha}}(\vec{\boldsymbol{p}}, \lambda) e^{ip\cdot x}\,, \tag{2.57}$$

and the eight other single-particle matrix elements vanish.

Note that the two-component wave functions, $x_\alpha(\vec{\boldsymbol{p}})$ and $y_\alpha(\vec{\boldsymbol{p}})$ that appear in the Fourier expansions of the two-component fermion fields [eqs. (2.1), (2.47), (2.49) and (2.50)] do not depend on the flavor indices (or whether the field is neutral or charged). The properties of these functions have been given in Section 2.2.

2.4 Feynman rules for external two-component fermion lines

Let us consider a general Feynman diagram in which fermions of definite mass (i.e., the so-called fermion mass-eigenstates) can appear as initial and final states. The rules for assigning two-component external state spinors are then as follows.

- For an initial-state left-handed $(\frac{1}{2}, 0)$ fermion of momentum $\vec{\boldsymbol{p}}$ and helicity λ: $x(\vec{\boldsymbol{p}}, \lambda)$.
- For an initial-state right-handed $(0, \frac{1}{2})$ fermion of momentum $\vec{\boldsymbol{p}}$ and helicity λ: $\bar{y}(\vec{\boldsymbol{p}}, \lambda)$.
- For a final-state left-handed $(\frac{1}{2}, 0)$ fermion of momentum $\vec{\boldsymbol{p}}$ and helicity λ: $\bar{x}(\vec{\boldsymbol{p}}, \lambda)$.
- For a final-state right-handed $(0, \frac{1}{2})$ fermion of momentum $\vec{\boldsymbol{p}}$ and helicity λ: $y(\vec{\boldsymbol{p}}, \lambda)$.

Note that the two-component external state fermion wave functions are distinguished by their Lorentz group transformation properties, rather than by their particle or antiparticle status as in four-component Feynman rules. This helps to explain why two-component notation is especially convenient for either theories with Majorana particles, in which there is no fundamental distinction between particles and antiparticles, or chiral theories where the left and right-handed fermions transform under different representations of the gauge group.

These rules are summarized in the following mnemonic diagram:

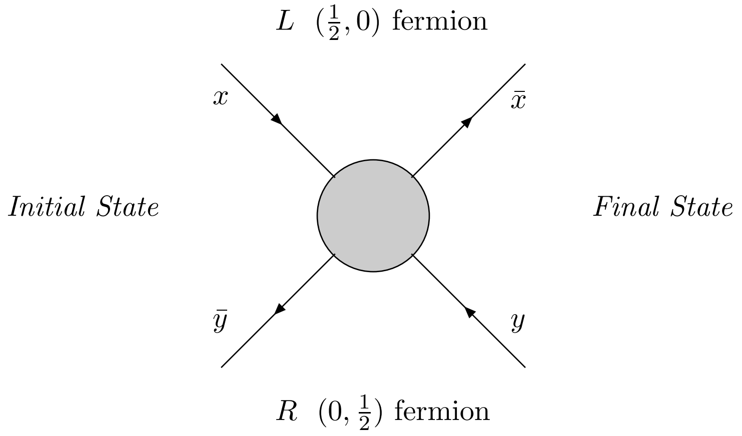


Fig. 2.1. The external wave-function spinors should be assigned as indicated here, for initial-state and final-state left-handed $(\frac{1}{2}, 0)$ and right-handed $(0, \frac{1}{2})$ fermions.

In contrast to four-component Feynman rules, the direction of the arrows do *not* correspond to the flow of charge or fermion number. Nevertheless, the above choice is convenient—the arrows of $(\frac{1}{2}, 0)$ fermions always point in the direction of their momenta while the arrows of $(0, \frac{1}{2})$ fermions always point opposite to their momenta. These rules simply correspond to the formulas for the one-particle wave functions given in eqs. (2.5) and (2.6) [with the convention that $|\vec{\boldsymbol{p}}, \lambda\rangle$ is an initial-state fermion and $\langle\vec{\boldsymbol{p}}, \lambda|$ is a final-state fermion].

The rules above apply to any mass eigenstate two-component fermion external wave functions. In particular, the same rules apply for the two-component fermions governed by the Lagrangians of eq. (1.115) [Majorana] and eq. (1.121) [Dirac].

2.5 Feynman rules for two-component fermion propagators

Next we examine the fermion propagators for two-component fermions. These are the Fourier transforms of the free-field vacuum expectation values of time-ordered products of two fermion fields. They are easily obtained by inserting the free-field expansion of the two-component fermion field and evaluating the spin sums using the formulas given in eqs. (2.43)–(2.46). For the case of a single neutral two-component fermion field ξ of mass m

$$\langle 0|\, T\xi_\alpha(x)\bar{\xi}_{\dot{\beta}}(y)\,|0\rangle_{\rm FT} = \frac{i}{p^2-m^2+i\epsilon}\, p\cdot\sigma_{\alpha\dot{\beta}}\,, \tag{2.58}$$

$$\langle 0|\, T\bar{\xi}^{\dot{\alpha}}(x)\xi^{\beta}(y)\,|0\rangle_{\rm FT} = \frac{i}{p^2-m^2+i\epsilon}\, p\cdot\overline{\sigma}^{\dot{\alpha}\beta}\,, \tag{2.59}$$

$$\langle 0|\, T\xi_\alpha(x)\xi^{\beta}(y)\,|0\rangle_{\rm FT} = \frac{i}{p^2-m^2+i\epsilon}\, m\delta_\alpha{}^\beta\,, \tag{2.60}$$

$$\langle 0|\, T\bar{\xi}^{\dot{\alpha}}(x)\bar{\xi}_{\dot{\beta}}(y)\,|0\rangle_{\rm FT} = \frac{i}{p^2-m^2+i\epsilon}\, m\delta^{\dot{\alpha}}{}_{\dot{\beta}}\,, \tag{2.61}$$

where the subscript FT indicates the Fourier transform from position to momentum space. These results have an obvious diagrammatic representation as shown in Fig. 2.2.

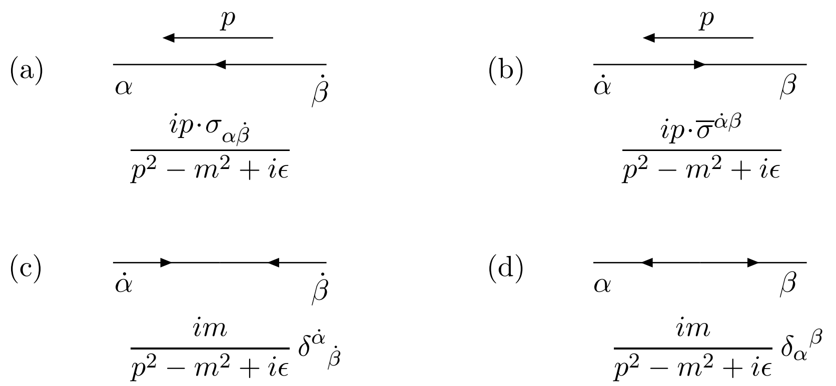
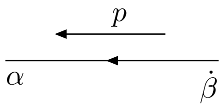


Fig. 2.2. Feynman rules for propagator lines of a neutral two-component spin-1/2 fermion.

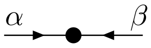
Note that the direction of the momentum flow p^μ here is determined by the creation operator that appears in the evaluation of the free-field propagator. Arrows on fermion lines always run away from dotted indices at a vertex and toward undotted indices at a vertex.

There are two types of fermion propagators. The first type preserves the direction of arrows, so it has one dotted and one undotted index. For this type of propagator, it is convenient to establish a convention where p^μ in the diagram is defined to be the momentum flowing in the direction of the arrow on the fermion propagator. With this convention, the two rules above for propagators of the first type can be summarized by one rule, as shown in Figure 2.3. Here the choice of the σ or the $\overline{\sigma}$ version of the rule is uniquely determined by the height of the indices on the vertex

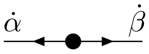


$$\frac{ip\cdot\sigma_{\alpha\dot\beta}}{p^2-m^2+i\epsilon} \quad \underline{\text{or}} \quad \frac{-ip\cdot\overline{\sigma}^{\dot\beta\alpha}}{p^2-m^2+i\epsilon}$$

Fig. 2.3. This one rule summarizes the results of Fig. 2.2(a) and (b).



$$-im\delta_\alpha{}^\beta$$



$$-im\delta^{\dot\alpha}{}_{\dot\beta}$$

Fig. 2.4. Fermion mass insertions can be treated as a type of interaction vertex, using the Feynman rules shown here.

to which the propagator is connected. These heights should always be chosen so that they are contracted as in eq. (1.52). The second type of propagator shown above does not preserve the direction of arrows, and corresponds to an odd number of mass insertions. The indices on $\delta_\alpha{}^\beta$ and $\delta^{\dot\alpha}{}_{\dot\beta}$ are staggered as shown to indicate that α or $\dot\alpha$ are to be contracted with an expression to the left, while β or $\dot\beta$ are to be contracted with an expression to the right, in accord with eq. (1.52).

Mass insertions on fermion lines can instead be handled as interaction vertices, as shown in Figure 2.4. By summing up an infinite chain of such mass insertions between massless fermion propagators, one can easily reproduce the massive fermion propagators of both types.

It is convenient to treat separately the case of charged massive fermions. Consider a charged Dirac fermion of mass m, which is described by two two-component fields ξ and η, whose free-field Lagrangian is given by eq. (1.104). Using the free field expansions given by eqs. (2.49) and (2.50), and the appropriate spin-sums [eqs. (2.43)–(2.46)], the two-component free-field propagators are easily obtained:

$$\langle 0|\, T\xi_\alpha(x)\bar\xi_{\dot\beta}(y)\,|0\rangle_{\rm FT} = \langle 0|\, T\eta_\alpha(x)\bar\eta_{\dot\beta}(y)\,|0\rangle_{\rm FT} = \frac{i}{p^2-m^2}\,p\cdot\sigma_{\alpha\dot\beta}\,, \quad (2.62)$$

$$\langle 0|\, T\bar{\xi}^{\dot{\alpha}}(x)\xi^{\beta}(y)\,|0\rangle_{\rm FT} = \langle 0|\, T\bar{\eta}^{\dot{\alpha}}(x)\eta^{\beta}(y)\,|0\rangle_{\rm FT} = \frac{i}{p^2-m^2}\,p\cdot\overline{\sigma}^{\dot{\alpha}\beta}\,, \quad (2.63)$$

$$\langle 0|\, T\xi_{\alpha}(x)\eta^{\beta}(y)\,|0\rangle_{\rm FT} = \langle 0|\, T\eta_{\alpha}(x)\xi^{\beta}(y)\,|0\rangle_{\rm FT} = \frac{i}{p^2-m^2}\,m\,\delta_{\alpha}{}^{\beta}\,, \quad (2.64)$$

$$\langle 0|\, T\bar{\xi}^{\dot{\alpha}}(x)\bar{\eta}_{\dot{\beta}}(y)\,|0\rangle_{\rm FT} = \langle 0|\, T\bar{\eta}^{\dot{\alpha}}(x)\bar{\xi}_{\dot{\beta}}(y)\,|0\rangle_{\rm FT} = \frac{i}{p^2-m^2}\,m\,\delta^{\dot{\alpha}}{}_{\dot{\beta}}\,. \quad (2.65)$$

For all other combinations of fermion bilinears, the corresponding two-point functions vanish. These results again have a simple diagrammatic representation, as shown in Figure 2.5.[2]

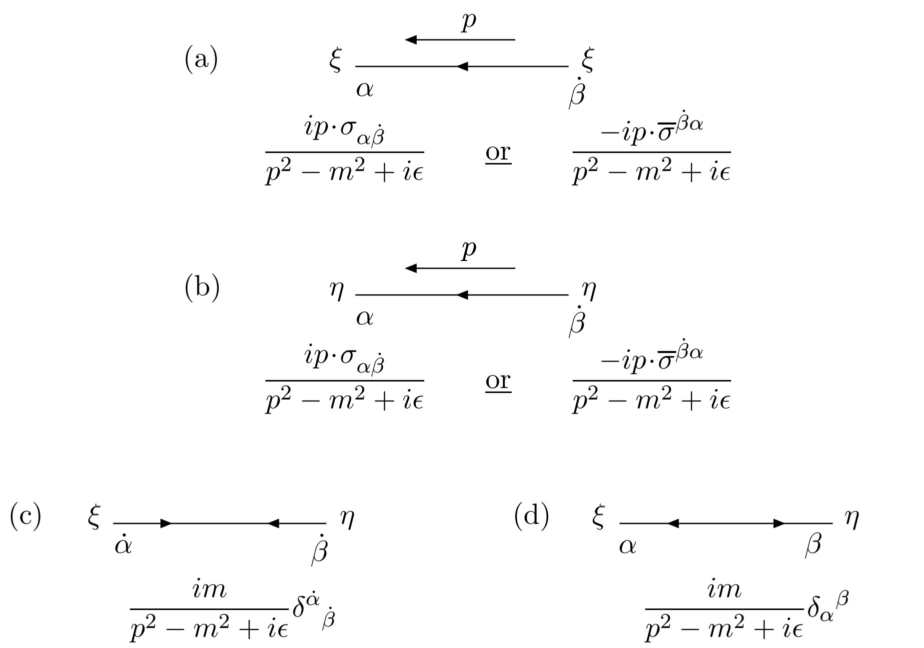


Fig. 2.5. Feynman rules for propagator lines of a charged two-component spin-1/2 fermion.

Note that for Dirac fermions, the propagators with opposing arrows (proportional to a mass) necessarily change the identity (ξ or η) of the two-component fermion, while the single-arrow propagators never do. In processes involving such a charged fermion, one must of course carefully distinguish between the ξ and η fields.

[2] In Fig. 2.5, the diagrams are drawn with the momentum along the arrow direction, as in Fig. 2.3.

2.6 Feynman rules for two-component fermion interactions

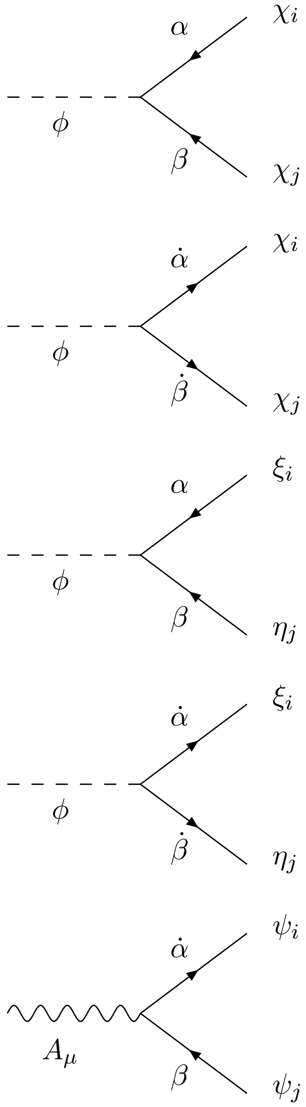
We next examine the possible interaction vertices. Renormalizable Lorentz invariant interactions involving fermions must consist of bilinears in the fermion fields, which transform as a Lorentz scalar or vector, coupled to the appropriate bosonic Lorentz scalar or vector field to make an overall Lorentz scalar quantity. Here, we shall first consider the case of fermion pairs coupled to a real scalar field ϕ and a real vector field A_μ. We assume that the two-component fermion mass matrix has been diagonalized as discussed in Section 1.6, so that the fermions consist of a collection of two-component $(\frac{1}{2}, 0)$ fermion mass-eigenstate fields. These may include both neutral two-component fields χ and/or pairs of oppositely-charged fields ξ and η. The interaction Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\rm int} = &-\tfrac{1}{2}(\lambda^{ij}\chi_i\chi_j + \lambda_{ij}\bar{\chi}^i\bar{\chi}^j)\phi - (\kappa^{ij}\xi_i\eta_j + \kappa_{ij}\bar{\xi}^i\bar{\eta}^j)\phi \\ &-(G_\chi)_i{}^j\,\bar{\chi}^i\overline{\sigma}^\mu\chi_j A_\mu - [(G_\xi)_i{}^j\bar{\xi}^i\overline{\sigma}^\mu\xi_j + (G_\eta)_i{}^j\bar{\eta}^i\overline{\sigma}^\mu\eta_j]A_\mu\,, \quad (2.66)\end{aligned}$$

where λ is a complex symmetric matrix, κ is an arbitrary complex matrix and G_ξ, G_χ and G_η are hermitian matrices. We have suppressed the spinor indices in eq. (2.66); the product of two component spinors is always performed according to the index convention indicated in eq. (1.52). We also have employed the convention concerning the "flavor" labels i and j described in Section A.2. That is, flipping the heights of all flavor indices of an object corresponds to complex conjugation. In this convention, raised indices can only be contracted with lowered indices and vice versa. The Feynman rules for the vertices that arise from this interaction Lagrangian [eq. (2.66)] are shown in Fig. 2.6.

One clarification in the labeling is helpful. Consider a line in Fig. 2.6 labeled ψ_i. This means that the corresponding state is given by $|\psi_i\rangle$ [as in eqs. (2.4), (2.52) or (2.53)], independent of the direction of the arrow. The fact that there are two separate rules corresponding to the same pair of outgoing states (differentiated by the arrow directions) is a consequence of the two terms proportional to $\psi_i\psi_j$ and $\bar{\psi}^i\bar{\psi}^j$ in eq. (2.66).

In Fig. 2.6, two versions are given for each of the boson-fermion-fermion Feynman rules. The correct version to use depends in a unique way on the heights of indices used to connect each fermion line to the rest of the diagram. For example, the way of writing the vector-fermion-fermion interaction rule depends on whether we used $-\psi_j\sigma^\mu\bar{\psi}^i$, or its equivalent form $\bar{\psi}^i\overline{\sigma}^\mu\psi_j$, from eq. (2.66). Note the different heights of the spinor indices $\dot{\alpha}$ and β on σ^μ and $\overline{\sigma}^\mu$. The choice of which rule to use is thus dictated by the height of the indices on the lines that connect to the vertex. These heights should always be chosen so that they are contracted as in eq. (1.52). Similarly, for the scalar-fermion-fermion vertices, one should



$$-i\lambda^{ij}\delta_\alpha{}^\beta \quad \text{or} \quad -i\lambda^{ij}\delta_\beta{}^\alpha$$

$$-i\lambda_{ij}\delta^{\dot\alpha}{}_{\dot\beta} \quad \text{or} \quad -i\lambda_{ij}\delta^{\dot\beta}{}_{\dot\alpha}$$

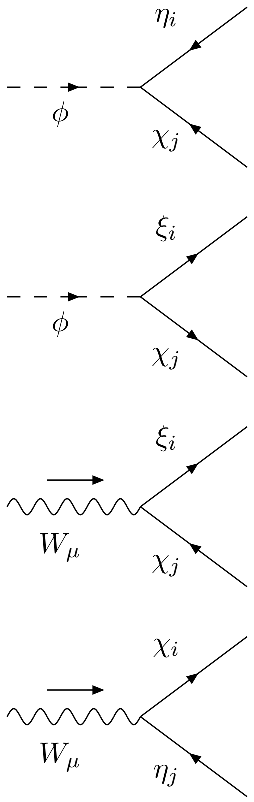
$$-i\kappa^{ij}\delta_\alpha{}^\beta \quad \text{or} \quad -i\kappa^{ij}\delta_\beta{}^\alpha$$

$$-i\kappa_{ij}\delta^{\dot\alpha}{}_{\dot\beta} \quad \text{or} \quad -i\kappa_{ij}\delta^{\dot\beta}{}_{\dot\alpha}$$

$$-i(G_\psi)_i{}^j\overline{\sigma}_\mu^{\dot\alpha\beta} \quad \text{or} \quad i(G_\psi)_i{}^j\sigma_{\mu\beta\dot\alpha}$$

Fig. 2.6. The form of the Feynman rules for two-component fermion interactions with a neutral boson in a general renormalizable field theory. In the diagram with the external vector boson, one may take $\psi = \chi$, ξ or η, respectively.

choose the rule which correctly matches the indices with the rest of the diagram. (However, when all indices are suppressed, the scalar-fermion-fermion rules will have an identical appearance for both cases anyway, since they are just proportional to the identity matrix on the 2×2 spinor



$$-i(\kappa_2)^{ij}$$

$$-i(\kappa_1)_{ij}$$

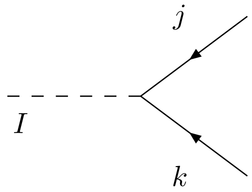
$$-i(G_1)_i{}^j\overline{\sigma}_\mu \quad \text{or} \quad i(G_1)_i{}^j\sigma_\mu$$

$$-i(G_2)_i{}^j\overline{\sigma}_\mu \quad \text{or} \quad i(G_2)_i{}^j\sigma_\mu$$

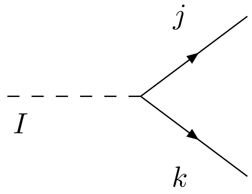
Fig. 2.7. The form of the Feynman rules for two-component fermion interactions with a charged boson in a general renormalizable field theory. For each diagram shown above, there is a charge-conjugated diagram in which all arrows are reversed. The corresponding rules are obtained simply by raising all lowered flavor indices and lowering all raised flavor indices [*c.f.* Section A.2]. Spinor indices are suppressed [*c.f.* Fig. 2.6]. The arrows above on the charged scalar lines and above the vector boson lines indicate the flow of charge. On both charged and neutral fermion lines, the arrows indicate the flow of chirality.

space.) These comments will be clarified by examples below.

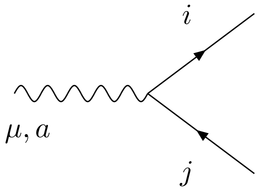
We can also treat the interactions of fermions to complex scalar and vector fields. As in eq. (2.66), we assume that the fermion mass matrices have been diagonalized (see Section 1.6). We denote a set of neutral fermion mass-eigenstates fields by χ_i and a set of charged fermion mass-eigenstate fields by pairs of oppositely charged fields ξ_j and η_j. The charged scalar and vector bosons are complex fields denoted by ϕ and W, respectively. Here, we shall only consider the simplest case where



$$-iU_m{}^j y^{Imn} U_n{}^k$$



$$-iU^m{}_j y_{Imn} U^n{}_k$$



$$-igU_i{}^k(\boldsymbol{T}^a)_k{}^m U_m{}^j\,\overline{\sigma}_\mu$$

or

$$igU_i{}^k(\boldsymbol{T}^a)_k{}^m U_m{}^j\,\sigma_\mu$$

Fig. 2.8. Feynman rules for Yukawa [gauge] couplings of scalars [vector bosons] to two-component fermions. Spinor indices are suppressed [*c.f.* Fig. 2.6]. The y^{Imn} are the Yukawa coupling matrices in the interaction-eigenstate basis, and the matrices U achieve the rotation to the mass eigenstate basis. The flavor index conenvtions of Section A.2 imply that $y_{Ijk} \equiv (y^{Ijk})^*$ and $U^i{}_j \equiv (U_i{}^j)^*$.

the charges of ϕ, W and ξ are assumed to be equal. In this case, the interaction Lagrangian is given by:

$$\begin{aligned}\mathcal{L}_{\rm int} = &-\tfrac{1}{2}\phi^*[\kappa_1^{ij}\xi_i\chi_j + (\kappa_2)_{ij}\bar\eta^i\bar\chi^j] - \tfrac{1}{2}\phi[\kappa_2^{ij}\eta_i\chi_j + (\kappa_1)_{ij}\bar\xi^i\bar\chi^j] \\ &-\tfrac{1}{2}W_\mu[(G_1)_i{}^j\bar\xi^i\overline{\sigma}^\mu\chi_j + (G_2)_i{}^j\bar\chi_i\overline{\sigma}^\mu\eta_j] \\ &-\tfrac{1}{2}W^*_\mu[(G_1)^i{}_j\bar\chi^j\overline{\sigma}^\mu\xi_i + (G_2)^i{}_j\bar\eta^j\overline{\sigma}^\mu\chi_i]\,, \end{aligned} \tag{2.67}$$

where κ_1 and κ_2 are complex symmetric matrices and G_1 and G_2 are hermitian matrices. The corresponding Feynman rules are given in Fig. 2.7.

If the interaction Lagrangian is given in terms of interaction-eigenstate fields, then the unitary matrices used to rotate to the mass-eigenstate basis will appear in the Feynman rules. Suppoe that the two-component fermion fields that appear in the Lagrangian are written originally in terms of $(\frac{1}{2},0)$-fermion interaction eigenstates, $\hat\psi_i$, which are related to mass eigenstate fields by $\hat{\psi} = U\psi$ [see eq. (1.122)]. In any theory, the most general set of fermion interaction vertices with a (possibly complex) scalar ϕ_I and gauge field A^a_μ can be written as follows:

$$\mathcal{L}_{\text{int}} = -\tfrac{1}{2} y^{Ijk} \phi_I \hat{\psi}_j \hat{\psi}_k - \tfrac{1}{2} y_{Ijk} \phi^{*I} \hat{\bar{\psi}}^j \hat{\bar{\psi}}^k - g A^a_\mu \hat{\bar{\psi}}^i \overline{\sigma}^\mu (\boldsymbol{T}^a)_i{}^j \hat{\psi}_j \,, \quad (2.68)$$

where the complex Yukawa couplings y^{Ijk} are symmetric under interchange of j and k, g is a real gauge coupling and the $\boldsymbol{T}^a$ are the hermitian representation matrices[3] corresponding to the two-component fermions fields. As before, we have employed the flavor index conventions of Section A.2. The mass eigenstate Feynman rules then take the form shown in Fig. 2.8.

2.7 General structure and rules for Feynman graphs

When computing an amplitude for a given process, all possible diagrams should be drawn that conform with the rules given above for external wave functions, propagators, and interactions. Starting from any external wave function spinor, or from any vertex on a fermion loop, factors corresponding to each propagator and vertex should be written down from left to right, following the line until it ends at another external state wave function or at the original point on the fermion loop. If one starts a fermion line at an x or y external state spinor, it should have a raised undotted index in accord with eq. (1.52). Or, if one starts with an $\bar{x}$ or $\bar{y}$, it should have a lowered dotted spinor index. Then, all spinor indices should always be contracted as in eq. (1.52). If one ends with an x or y external state spinor, it will have a lowered undotted index, while if one ends with an $\bar{x}$ or $\bar{y}$ spinor, it will have a raised dotted index. For arrow-preserving fermion propagators and gauge vertices, the preceding determines whether the σ or $\overline{\sigma}$ rule should be used. With only a little experience, one can write down amplitudes immediately with all spinor indices suppressed.

Symmetry factors for identical particles are implemented in the usual way. Fermi-Dirac statistics are implemented by the following rules:

- Each closed fermion loop gets a factor of -1.
- A relative minus sign is imposed between terms contributing to a given amplitude whenever the ordering of external state spinors (written left-to-right) differs by an odd permutation.

[3] For a $U(1)$ gauge group, the $\boldsymbol{T}^a$ are replaced by real numbers corresponding to the $U(1)$ charges of the left-handed $(\frac{1}{2}, 0)$ fermion.

Amplitudes generated according to these rules will contain objects of the form:

$$\mathcal{A} = z_1 \Sigma z_2 \tag{2.69}$$

where z_1 and z_2 are each commuting external spinor wave functions x, $\bar{x}$, y, or $\bar{y}$, and Σ is a sequence of alternating σ and $\overline{\sigma}$ matrices. The complex conjugate of this quantity is given by

$$\mathcal{A}^* = \bar{z}_2 \overleftarrow{\Sigma} \bar{z}_1 \tag{2.70}$$

where $\overleftarrow{\Sigma}$ is obtained from Σ by reversing the order of all the σ and $\overline{\sigma}$ matrices, and using the same rule for suppressed spinor indices. (Notice that this rule for taking complex conjugates has the same form as for anticommuting spinors.) We emphasize that in principle, it does not matter in what direction a diagram is traversed while applying the rules. However, one must associate a sign with each diagram that depends on the ordering of the external fermions. This sign can be fixed by first choosing some canonical ordering of the external fermions. Then for any graph that contributes to the process of interest, the corresponding sign is positive (negative) if the ordering of external fermions is an even (odd) permutation with respect to the canonical ordering. If one chooses a different canonical ordering, then the resulting amplitude changes by an overall sign (is unchanged) if this ordering is an odd (even) permutation of the original canonical ordering.[4] This is consistent with the fact that the amplitude is only defined up to an overall sign, which is not physically observable.

Note that different graphs contributing to the same process will often have different external state wave function spinors, with different arrow directions, for the same external fermion. In particular, there are no arbitrary choices to be made for arrow directions, since one must add together *all* Feynman graphs that obey the rules.

[4] For a process with exactly two external fermions, it is convenient to apply the Feynman rules by starting from the same fermion external state in all diagrams. That way, all terms in the amplitude have the same canonical ordering of fermions and there are no additional minus signs between diagrams. Four a process with four or more external fermions, it may happen that there is no way to choose the same ordering of external state spinors for all graphs when the amplitude is written down. Then the relative signs between different graphs must be chosen according to the relative sign of the permutation of the corresponding external fermion spinors. This guarantees that the total amplitude is antisymmetric under the interchange of any pair of external fermions.

2.8 Simple examples of Feynman diagrams and amplitudes

Some simple examples based on the interaction vertices of Section 2.6 will help clarify the rules of Section 2.7.

2.8.1 Tree-level decays

Let us first consider a theory with a single, uncharged, massive $(\frac{1}{2}, 0)$ fermion χ, and a real scalar ϕ, with interaction

$$\mathcal{L}_{\rm int} = -\tfrac{1}{2}\,(\lambda\chi\chi + \lambda^*\bar{\chi}\bar{\chi})\,\phi. \tag{2.71}$$

Consider the decay $\phi \to \chi(\vec{\boldsymbol{p}}_1, s_1)\chi(\vec{\boldsymbol{p}}_2, s_2)$, where by χ we mean the one particle state given by eq. (2.4). Two diagrams contribute to this process, as shown in Figure 2.9.

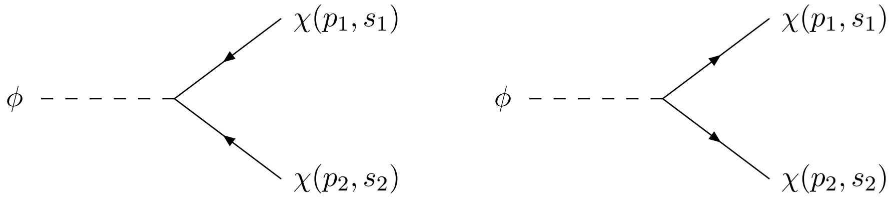


Fig. 2.9. The two tree-level Feynman diagrams contributing to the decay of a scalar into a Majorana fermion pair.

The matrix element is then given by

$$\begin{aligned} i\mathcal{M} &= y(\vec{\boldsymbol{p}}_1, s_1)^\alpha(-i\lambda\delta_\alpha{}^\beta)y(\vec{\boldsymbol{p}}_2, s_2)_\beta + \bar{x}(\vec{\boldsymbol{p}}_1, s_1)_{\dot\alpha}(-i\lambda^*\delta^{\dot\alpha}{}_{\dot\beta})\bar{x}(\vec{\boldsymbol{p}}_2, s_2)^{\dot\beta} \\ &= -i\lambda y(\vec{\boldsymbol{p}}_1, s_1)y(\vec{\boldsymbol{p}}_2, s_2) - i\lambda^*\bar{x}(\vec{\boldsymbol{p}}_1, s_1)\bar{x}(\vec{\boldsymbol{p}}_2, s_2). \end{aligned} \tag{2.72}$$

The second line above could be written down directly by recalling that the sum over suppressed spinor indices is taken according to eq. (1.52). Note that if we reverse the ordering for the external fermions, the overall sign of the amplitude changes sign.[5] Of course, this overall sign is not significant and depends on the order used in constructing the two particle state. One could even make the unconventional (but correct) choice of starting the first diagram from fermion 1, and the second diagram from fermion 2:

$$i\mathcal{M} = -i\lambda y(\vec{\boldsymbol{p}}_1, s_1)y(\vec{\boldsymbol{p}}_2, s_2) - (-1)i\lambda^*\bar{x}(\vec{\boldsymbol{p}}_2, s_2)\bar{x}(\vec{\boldsymbol{p}}_1, s_1)\,. \tag{2.73}$$

[5] This is easily checked, since for the *commuting* spinor wave functions ($\bar{x}$ and y), the spinor products in eq. (2.72) change sign when the order is reversed [see eqs. (B.42) and (B.43)].

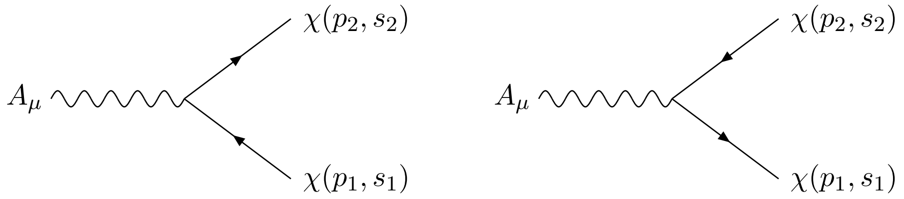


Fig. 2.10. The two tree-level Feynman diagrams contributing to the decay of a massive vector boson A_μ into a pair of Majorana fermions χ.

Here the first term establishes the canonical ordering of fermions (1,2), and the contribution from the second diagram therefore includes the relative minus sign in parentheses. It is easily seen that equations (2.72) and (2.73) are indeed equal. As usual, when a total decay rate is computed, one must multiply the integral over the total phase space by 1/2 to account for the identical particles.

Consider next the decay of a massive neutral vector A_μ into a Majorana fermion pair $A_\mu \to \chi(\vec{\boldsymbol{p}}_1, s_1)\chi(\vec{\boldsymbol{p}}_2, s_2)$, following from the interaction

$$\mathcal{L}_{\text{int}} = -GA^\mu \bar{\chi}\overline{\sigma}_\mu \chi \,, \tag{2.74}$$

where G is a real coupling parameter. The two diagrams shown in Figure 2.10 contribute.

Following the rules of Fig. 2.6, we start from the fermion with momentum p_1 and spin vector s_1, and end at the fermion with momentum p_2 and spin vector s_2. The resulting amplitude for the decay is

$$i\mathcal{M} = \varepsilon^\mu \left[-iG\bar{x}(\vec{\boldsymbol{p}}_1, s_1)\overline{\sigma}_\mu y(\vec{\boldsymbol{p}}_2, s_2) + iGy(\vec{\boldsymbol{p}}_1, s_1)\sigma_\mu \bar{x}(\vec{\boldsymbol{p}}_2, s_2)\right] \tag{2.75}$$

where ε^μ is the vector boson polarization vector. As illustrated in Fig. 2.6, we have used the $\overline{\sigma}$-version of the vector-fermion-fermion rule for the first diagram of Fig. 2.10 and the σ-version for the second diagram of Fig. 2.10, as dictated by the implicit spinor indices, which we have suppressed. However, we could have chosen to evaluate the second diagram of Fig. 2.10 using the $\overline{\sigma}$-version of the vector-fermion-fermion rule by starting from the fermion with momentum p_2. In that case, the term $+iGy(\vec{\boldsymbol{p}}_1, s_1)\sigma_\mu\bar{x}(\vec{\boldsymbol{p}}_2, s_2)$ in eq. (2.75) is replaced by

$$(-1)[-iG\bar{x}(\vec{\boldsymbol{p}}_2, s_2)\overline{\sigma}_\mu y(\vec{\boldsymbol{p}}_1, s_1)] \,. \tag{2.76}$$

In eq. (2.76), the factor of $-iG$ arises from the use of the $\overline{\sigma}$-version of the vector-fermion-fermion rule, whereas the overall factor of -1 is due to the fact that the order of the fermion wave functions has been reversed; *i.e*

(21) is an odd permutation of (12). This is in accord with the ordering rule stated at the end of Section 2.7. Thus, the resulting amplitude for the decay of the vector boson into the pair of Majorana fermions now takes the form:

$$i\mathcal{M} = \varepsilon^\mu \left[-iG\bar{x}(\vec{\boldsymbol{p}}_1, s_1)\overline{\sigma}_\mu y(\vec{\boldsymbol{p}}_2, s_2) + iG\bar{x}(\vec{\boldsymbol{p}}_2, s_2)\overline{\sigma}_\mu y(\vec{\boldsymbol{p}}_1, s_1)\right] . \tag{2.77}$$

By using $y\sigma^\mu\bar{x} = \bar{x}\overline{\sigma}^\mu y$, one trivially shows that eqs. (2.75) and (2.77) are identical. The form given in eq. (2.77) is especially convenient because it explicitly exhibits the fact that the amplitude is antisymmetric under the interchange of the two external identical fermions. Again, the absolute sign of the total amplitude is not significant and depends on the choice of ordering of the outgoing states. When computing the total decay rate, one must again multiply the total integral over phase space by 1/2 to account for identical particles in the final state.

Next, we consider the decay of a neutral vector boson into a charged fermion-antifermion pair. We denote ξ and η as $(\frac{1}{2}, 0)$ fields with charges $Q = 1$ and $Q = -1$, respectively, with couplings to the neutral vector boson as follows:

$$\mathcal{L}_{\rm int} = -A^\mu[G_\xi\bar{\xi}\overline{\sigma}_\mu\xi + G_\eta\bar{\eta}\overline{\sigma}_\mu\eta]. \tag{2.78}$$

There are two contributing graphs to vector boson decay as shown in Figure 2.11. To evaluate the amplitude, we start from the charge $Q = +1$ fermion (with momentum p_1 and spin vector s_1), and end at the charge $Q = -1$ fermion (with momentum p_2 and spin vector s_2). The charge flow follows the direction of the arrow on the fermion line. Note that for the final state fermion lines, the outgoing ξ with arrow pointing outward from the vertex and the outgoing η with arrow pointing *inward* to the vertex both correspond to outgoing $Q = +1$ states. The amplitude for the decay is

$$\begin{aligned} i\mathcal{M} &= \varepsilon^\mu \left[-iG_\xi\bar{x}(\vec{\boldsymbol{p}}_1, s_1)\overline{\sigma}_\mu y(\vec{\boldsymbol{p}}_2, s_2) + iG_\eta y(\vec{\boldsymbol{p}}_1, s_1)\sigma_\mu\bar{x}(\vec{\boldsymbol{p}}_2, s_2)\right] \\ &= \varepsilon^\mu \left[-iG_\xi\bar{x}(\vec{\boldsymbol{p}}_1, s_1)\overline{\sigma}_\mu y(\vec{\boldsymbol{p}}_2, s_2) + iG_\eta\bar{x}(\vec{\boldsymbol{p}}_2, s_2)\overline{\sigma}_\mu y(\vec{\boldsymbol{p}}_1, s_1)\right] . \end{aligned} \tag{2.79}$$

As in the case of the decay to a pair of Majorana fermions, we have exhibited two forms for the amplitude in eq. (2.79) that depend on whether the $\overline{\sigma}$-version or the σ-version of the Feynman rule has been employed. Of course, the resulting amplitude is the same in each method (up to an overall sign of the total amplitude which is not determined).

2.8.2 Tree-level scattering processes

The next level of complexity consists of diagrams that involve fermion propagators. For our first example of this type, consider the tree-level

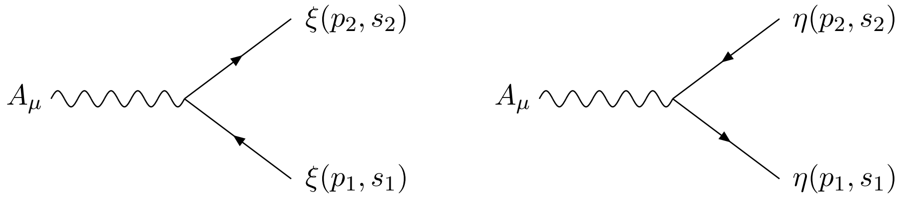


Fig. 2.11. The two tree-level Feynman diagrams contributing to the decay of a massive neutral vector boson A_μ into a Dirac fermion-antifermion pair.

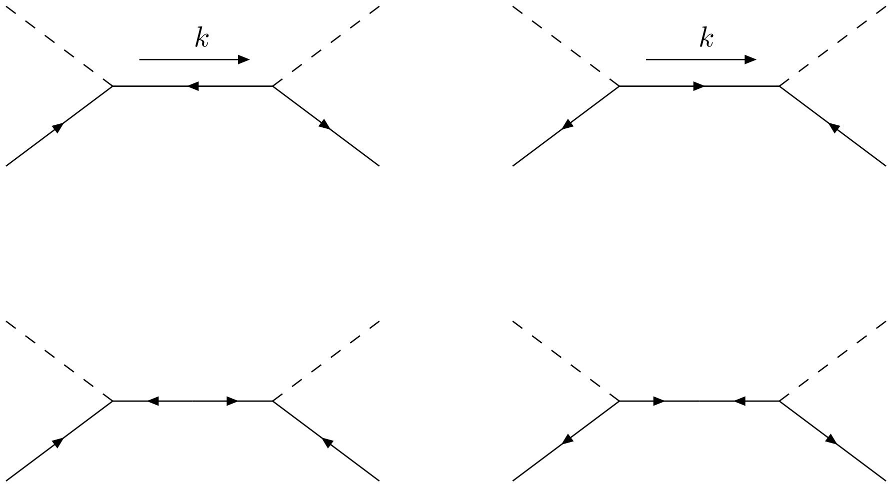


Fig. 2.12. Tree-level Feynman diagrams contributing to the elastic scattering of a neutral scalar and a neutral two-component fermion. There are four more diagrams, obtained from these by crossing the initial and final scalar lines.

matrix element for the scattering of a neutral scalar and a two-component neutral massive fermion ($\phi\xi \to \phi\xi$), with the interaction Lagrangian given by eq. (2.71). Using the corresponding Feynman rules, there are eight contributing diagrams. Four are depicted in Fig. 2.12; there are another four diagrams (not shown) where the initial and final state scalars are crossed (so that the initial [final] state scalar is now attached to the same vertex as the final [initial] state fermion).

We shall write down the amplitudes for these diagrams starting with the final state fermion line and moving toward the initial state. Then,

$$
\begin{aligned}
i\mathcal{M} = \frac{-i}{s-m_\chi^2}\Big\{ & |\lambda|^2 \left[\bar{x}(\vec{\boldsymbol{p}}_{\mathbf{2}}, s_2)\,\overline{\sigma}\!\cdot\! k\, x(\vec{\boldsymbol{p}}_{\mathbf{1}}, s_1) + y(\vec{\boldsymbol{p}}_{\mathbf{2}}, s_2)\,\sigma\!\cdot\! k\, \bar{y}(\vec{\boldsymbol{p}}_{\mathbf{1}}, s_1)\right] \\
& + m_\chi \left[\lambda^2 y(\vec{\boldsymbol{p}}_{\mathbf{2}}, s_2) x(\vec{\boldsymbol{p}}_{\mathbf{1}}, s_1) + (\lambda^*)^2 \bar{x}(\vec{\boldsymbol{p}}_{\mathbf{2}}, s_2)\bar{y}(\vec{\boldsymbol{p}}_{\mathbf{1}}, s_1)\right]\Big\} \\
& + (\text{crossed})\,, \qquad (2.80)
\end{aligned}
$$

where k^μ is the sum of the two incoming (or outgoing) four-momenta, $s = k^2$, (p_1, s_1) are the momentum and spin four-vectors of the incoming fermion, and (p_2, s_2) are those of the outgoing fermion. "Crossed" indicates the same contribution but with the initial and final scalars interchanged. Note that we could have evaluated the diagrams above by starting with the initial vertex and moving toward the final vertex. It is easy to check that the resulting amplitude is the negative[6] of the one obtained in eq. (2.80); the overall sign change simply corresponds to swapping the order of the two fermions and has no physical consequence.

Next, we compute the tree-level matrix element for the scattering of a vector boson and a neutral massive two-component fermion χ with the interaction Lagrangian of eq. (2.74). Again there are eight diagrams: the four diagrams depicted in Fig. 2.13 plus another four (not shown) where the initial and final state vector bosons are crossed. Starting with the final state fermion line and moving toward the initial state, we obtain

$$
\begin{aligned}
i\mathcal{M} = {} & \frac{-iG^2}{s-m_\chi^2} \\
& \times\Big\{ \bar{x}(\vec{\boldsymbol{p}}_{\mathbf{2}}, s_2)\,\overline{\sigma}\!\cdot\!\varepsilon_2^*\,\sigma\!\cdot\! k\,\overline{\sigma}\!\cdot\!\varepsilon_1\, x(\vec{\boldsymbol{p}}_{\mathbf{1}}, s_1) + y(\vec{\boldsymbol{p}}_{\mathbf{2}}, s_2)\,\sigma\!\cdot\!\varepsilon_2^*\,\overline{\sigma}\!\cdot\! k\,\sigma\!\cdot\!\varepsilon_1 \bar{y}\,(\vec{\boldsymbol{p}}_{\mathbf{1}}, s_1) \\
& - m_\xi \left[y(\vec{\boldsymbol{p}}_{\mathbf{2}}, s_2)\,\sigma\!\cdot\!\varepsilon_2^*\,\overline{\sigma}\!\cdot\!\varepsilon_1\, x(\vec{\boldsymbol{p}}_{\mathbf{1}}, s_1) + \bar{x}(\vec{\boldsymbol{p}}_{\mathbf{2}}, s_2)\,\overline{\sigma}\!\cdot\!\varepsilon_2^*\,\sigma\!\cdot\!\varepsilon_1\,\bar{y}(\vec{\boldsymbol{p}}_{\mathbf{1}}, s_1)\right]\Big\} \\
& + (\text{crossed})\,, \qquad (2.81)
\end{aligned}
$$

where ε_1 and ε_2 are the initial and final vector boson polarization four-vectors, respectively. As before, k^μ is the sum of the two incoming (or outgoing) four-momenta and $s = k^2$, and (p_1, s_1) are the momentum and spin four-vectors of the incoming fermion, and (p_2, s_2) are those of the outgoing fermion. "Crossed" indicates the same contribution but with the initial and final vector bosons swapped. If one evaluates the diagrams above by starting with the initial vertex and moving toward the final

[6] The opposite sign is a consequence of eqs. (B.42)–(B.44) and the minus sign difference between the two ways of evaluating the propagator that preserves the arrow direction.

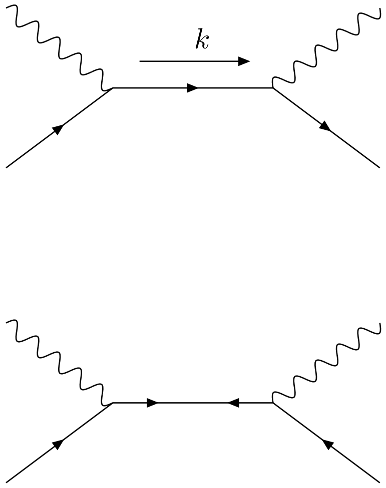


Fig. 2.13. Tree-level Feynman diagrams contributing to the elastic scattering of a neutral vector boson and a neutral two-component fermion. There are four more diagrams, obtained from these by crossing the initial and final scalar lines.

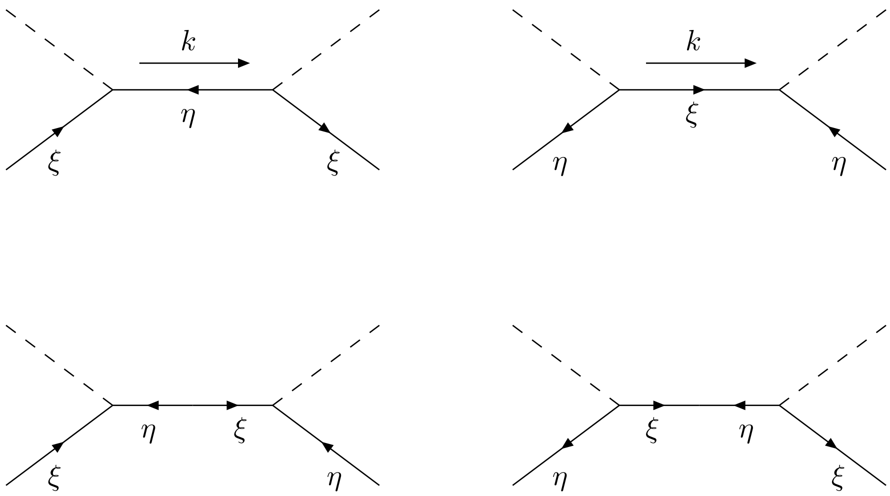


Fig. 2.14. Tree-level Feynman diagrams contributing to the elastic scattering of a neutral scalar and a charged fermion. There are four more diagrams, obtained from these by crossing the initial and final scalar lines.

vertex, the resulting amplitude is the negative of the one obtained in eq. (2.81), as expected.

We now consider the scattering of a charged Dirac fermion with a

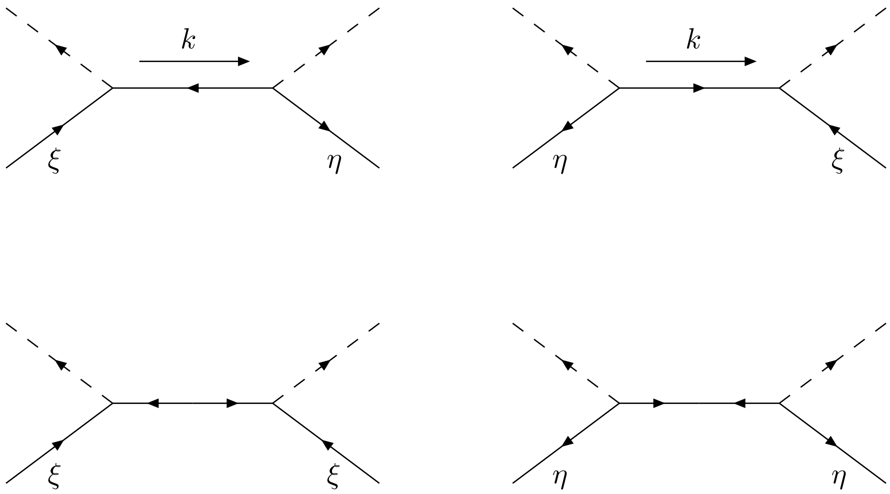


Fig. 2.15. Tree-level Feynman diagrams contributing to the scattering of an initial charged scalar and a charged fermion into its charge-conjugated final state. The unlabeled intermediate state is a neutral fermion. There are four more diagrams, obtained from these by crossing the initial and final scalar lines.

neutral scalar. The $(\frac{1}{2},0)$ fields ξ and η have opposite charges $Q = +1$ and -1 respectively, and interact with the scalar ϕ according to

$$\mathcal{L}_{\text{int}} = -\phi[\kappa\xi\eta + \kappa^*\bar{\xi}\bar{\eta}]\,, \tag{2.82}$$

where κ is a coupling parameter. Then, for the elastic scattering of a $Q = +1$ fermion and a scalar, the diagrams of Fig. 2.14 contribute at tree-level plus another four diagrams (not shown) where the initial and final state scalars are crossed. Since these diagrams match precisely those of Fig. 2.12, one obtains the same matrix element, eq. (2.80), previously obtained for the scattering of a neutral scalar and neutral two-component fermion, with the replacement of λ with κ and m_χ with the mass of the charged fermion, m.

Consider next the scattering of a charged Dirac fermion and a charged scalar, where both the scalar and fermion have the same absolute value of the charge. As usual, we denote the charged $Q = +1$ fermion by the pair of two-component fermions ξ and η and the (intermediate state) neutral two-component fermion by χ. The charged $Q = \pm 1$ scalar is represented by the scalar field ϕ and its complex conjugate, and the corresponding interaction Lagrangian takes the form:

$$\mathcal{L}_{\text{int}} = -\phi^*[\kappa_1\xi\chi + \kappa_2^*\bar{\eta}\bar{\chi}] - \phi[\kappa_2\eta\chi + \kappa_1^*\bar{\xi}\bar{\chi}]\,. \tag{2.83}$$

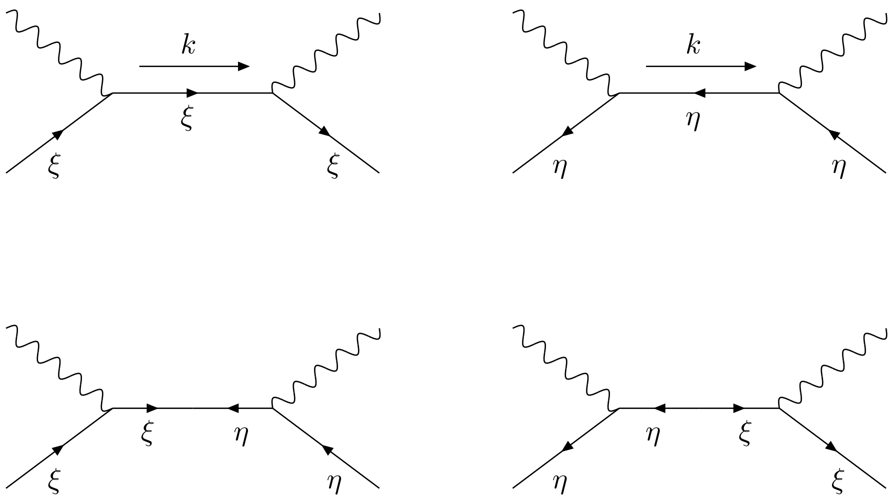


Fig. 2.16. Tree-level Feynman diagrams contributing to the elastic scattering of a neutral vector boson and a charged Dirac fermion. There are four more diagrams, obtained from these by crossing the initial and final vector lines.

One scattering process that deserves special attention is the scattering of an initial boson-fermion state into its charge-conjugated final state via the exchange of a neutral fermion. The relevant diagrams are shown in Fig. 2.15 plus the corresponding diagrams with the intial and final scalars crossed. The derivation is similar to the ones given previously, and we end up with

$$\begin{aligned} i\mathcal{M} = \frac{-i}{s-m_\chi^2}\Big\{ & \kappa_1\kappa_2^*[\bar{x}(\vec{\boldsymbol{p}}_{\boldsymbol{2}},s_2)\,\overline{\sigma}\!\cdot\! k\, x(\vec{\boldsymbol{p}}_{\boldsymbol{1}},s_1) + y(\vec{\boldsymbol{p}}_{\boldsymbol{2}},s_2)\,\sigma\!\cdot\! k\,\bar{y}(\vec{\boldsymbol{p}}_{\boldsymbol{1}},s_1)] \\ & + m\left[\kappa_1^2 y(\vec{\boldsymbol{p}}_{\boldsymbol{2}},s_2)x(\vec{\boldsymbol{p}}_{\boldsymbol{1}},s_1) + (\kappa_2^*)^2\bar{x}(\vec{\boldsymbol{p}}_{\boldsymbol{2}},s_2)\bar{y}(\vec{\boldsymbol{p}}_{\boldsymbol{1}},s_1)\right]\Big\} \\ & +(\text{crossed})\,, \end{aligned} \tag{2.84}$$

where m is the mass of the charged fermion and the four-momentum k is defined as shown in Fig. 2.15.

The scattering of a charged fermion and a neutral spin-1 vector boson can be similarly treated. For example, consider the amplitude for the elastic scattering of a charged fermion and a neutral vector boson. Using the interaction Lagrangian given in eq. (2.78), the relevant diagrams are those shown in Fig. 2.16, plus four diagrams (not shown) obtained from these by crossing the initial and final state vectors. Applying the Feynman

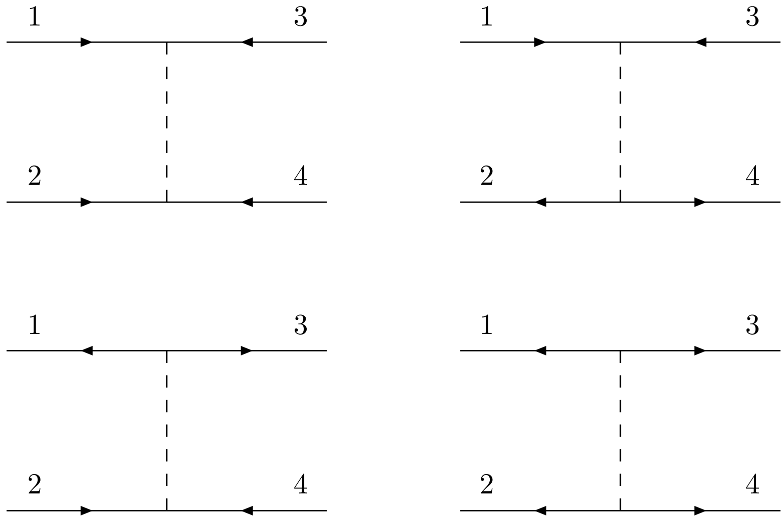


Fig. 2.17. Tree-level Feynman diagrams contributing to the elastic scattering of identical neutral Majorana fermions via scalar exchange in the t-channel. Additionally, there are four u-channel diagrams obtained from these by crossing either the initial or final fermion lines. Finally, one must also evaluate four s-channel diagrams in which the two-component fermions 1 and 2 annihilate into an intermediate scalar which subsequently decays into two-component fermions 3 and 4.

rules following from eq. (2.74) as before, one obtains

$$\begin{aligned} i\mathcal{M} = \frac{-i}{s-m^2}\Bigg\{ & G_\xi^2\, \bar{x}(\vec{\boldsymbol{p}}_{\boldsymbol{2}}, s_2)\, \overline{\sigma}\cdot\varepsilon_2^*\, \sigma\cdot p\, \overline{\sigma}\cdot\varepsilon_1\, x(\vec{\boldsymbol{p}}_{\boldsymbol{1}}, s_1) \\ & + G_\eta^2\, y(\vec{\boldsymbol{p}}_{\boldsymbol{2}}, s_2)\, \sigma\cdot\varepsilon_2^*\, \overline{\sigma}\cdot p\, \sigma\cdot\varepsilon_1 \bar{y}\,(\vec{\boldsymbol{p}}_{\boldsymbol{1}}, s_1) \\ -mG_\xi G_\eta \Big[& y(\vec{\boldsymbol{p}}_{\boldsymbol{2}}, s_2)\, \sigma\cdot\varepsilon_2^*\, \overline{\sigma}\cdot\varepsilon_1\, x(\vec{\boldsymbol{p}}_{\boldsymbol{1}}, s_1) + \bar{x}(\vec{\boldsymbol{p}}_{\boldsymbol{2}}, s_2)\, \overline{\sigma}\cdot\varepsilon_2^*\, \sigma\cdot\varepsilon_1\, \bar{y}(\vec{\boldsymbol{p}}_{\boldsymbol{1}}, s_1)\Big]\Bigg\} \\ & + (\text{crossed}) \end{aligned} \tag{2.85}$$

and the masses and the assignments of momenta and spins are as before. The computation of the amplitude for the scattering of a charged fermon and a charged vector boson is straightforward and will not be given explicitly here.

Finally, let us work out an example with four external-state fermions. Consider the case of elastic scattering of two identical Majorana fermions due to scalar exchange, governed by the interaction of eq. (2.71). The t-channel diagrams for scattering initial fermions labeled $1, 2$ into final state fermions labeled $3, 4$ are shown in Fig. 2.17. There are also four

u-channel and four s-channel annihilation diagrams (not shown). The resulting matrix element is:

$$\begin{aligned}
i\mathcal{M} = \frac{-i}{s-m_\phi^2}\Big\{&\lambda^2(x_1x_2)(y_3y_4)+(\lambda^*)^2(\bar{y}_1\bar{y}_2)(\bar{x}_3\bar{x}_4)\\
&+|\lambda|^2\left[(x_1x_2)(\bar{x}_3\bar{x}_4)+(\bar{y}_1\bar{y}_2)(y_3y_4)\right]\Big\}\\
+(-1)\frac{-i}{t-m_\phi^2}\Big\{&\lambda^2(y_3x_1)(y_4x_2)+(\lambda^*)^2(\bar{x}_3\bar{y}_1)(\bar{x}_4\bar{y}_2)\\
&+|\lambda|^2\left[(\bar{x}_3\bar{y}_1)(y_4x_2)+(y_3x_1)(\bar{x}_4\bar{y}_2)\right]\Big\}\\
+\frac{-i}{u-m_\phi^2}\Big\{&\lambda^2(y_4x_1)(y_3x_2)+(\lambda^*)^2(\bar{x}_4\bar{y}_1)(\bar{x}_3\bar{y}_2)\\
&+|\lambda|^2\left[(\bar{x}_4\bar{y}_1)(y_3x_2)+(y_4x_1)(\bar{x}_3\bar{y}_2)\right]\Big\},
\end{aligned} \tag{2.86}$$

where $x_i \equiv x(\vec{\boldsymbol{p}}_{\boldsymbol{i}}, s_i)$, $y_i \equiv y(\vec{\boldsymbol{p}}_{\boldsymbol{i}}, s_i)$, m_ϕ is the mass of the exchanged scalar, $s=(p_1+p_2)^2$, $t=(p_1-p_3)^2$ and $u=(p_1-p_4)^2$. The relative minus sign (in parentheses) between the t-channel diagram and the s and u-channel diagrams is obtained by observing that 3142 is an odd permutation and 4132 is an even permutation of 1234.[7]

2.9 Conventions for fermion and anti-fermion names and fields

Dirac fermions present a problem for labeling Feynman diagrams. In the two-component language, a Dirac fermion is described by two distinct $(\frac{1}{2},0)$ fields ξ and η. This field represents both the particle and anti-particle which are not identical. This is best illustrated by considering the electron which is a Dirac fermion. Let us denote the corresponding two-component fields by $e \equiv \xi$ and $e^c \equiv \eta$.[8] We shall also denote the one-particle electron and positron states by e^- and e^+ respectively. In a given Feynman diagram, one has the option of labelling graphs by particle names or field names; each choice has advantages and disadvantages. It is convenient to display a translation between the two labeling conventions:

Note that the two-component fields e and $\overline{e^c}$ both represent the negatively charged electron, conventionally denoted by e^-, whereas both e^c and $\overline{e}$

[7] Note that we would have obtained the same sign for the u-channel diagram had we crossed the initial state fermion lines instead of the final state fermion lines.

[8] Equivalently, if one considers the four-component Dirac field Ψ_e, then $e = P_L\Psi_e$ and $e^c = P_L\Psi_e^c$ can be thought of as two left-handed fields that are combined to form the Dirac electron field.

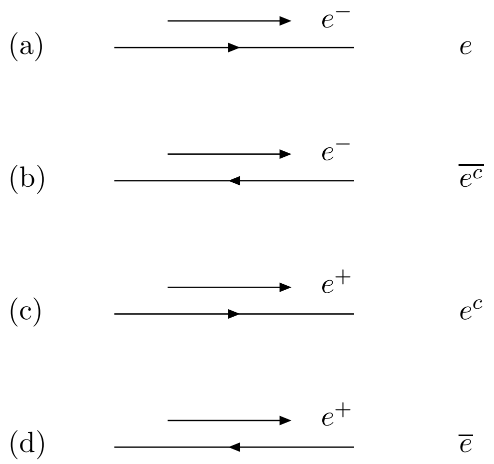


Fig. 2.18. The translation between the particle name and the two-component field name conventions for external lines in a Feynman diagram. The diagrams represent an electron or positron whose momentum points from left to right (as specified by the arrow above each line). The corresponding two-component field label is indicated to the right of each line.

represent the positively charged positron, conventionally denoted by e^+.

One also needs to establish a convention for labeling the two-component fermion fields that appear in the Feynman rules. As an example, consider the two-component Feynman rules for the QED coupling of electrons and positrons to the photon, which are exhibited in Fig. 2.19.
In general, two-component fermion lines with dotted indices always correspond to arrows going away from the vertex, and two-component fermion lines with undotted indices always correspond to arrows going toward the vertex. When employing the Feynman rules of Fig. 2.19, the choice of the two-component field label depends on the direction of momentum flow of the corresponding fermion, following the prescription of Fig. 2.18. For example, if the direction of the momentum flow in Fig. 2.19 follows the direction of the arrows of the two-component fermion fields, then one should label the two-component fermion lines with unbarred fields. That is, the Feynman rules are labeled with two-component (unbarred) fields; they can be applied to processes involving either fermions or anti-fermions (following the prescription of Fig. 2.18).

A simple example should make this clear. Consider the s-channel tree-level Feynman diagrams that contribute to Bhabha scattering ($e^-e^+ \to e^-e^+$). If we label the external fermion lines accoridng to the corresponding particle names, the result is shown in Figure 2.20. However, when using the particle name convention, one must discern

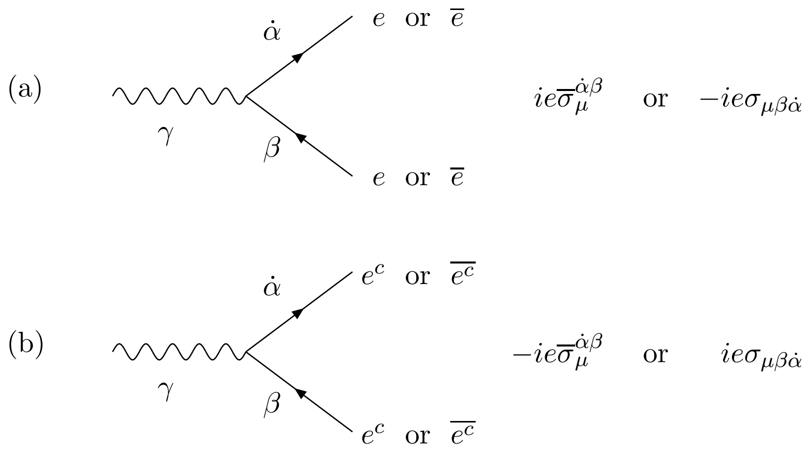


Fig. 2.19. The two-component Feynman rules for the QED vertex. The choice of the two-component field label depends on the direction of momentum flow (which is independent of the direction of the arrow on the fermion lines), following the prescription of Fig. 2.18.

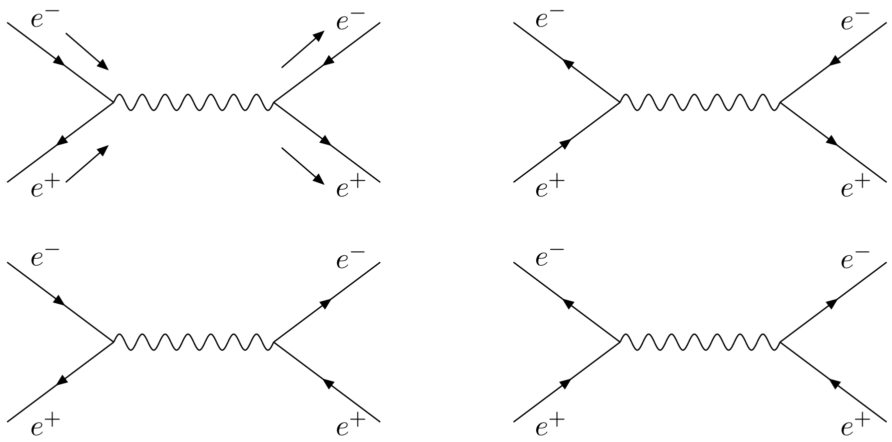


Fig. 2.20. Tree-level s-channel Feynman diagrams for $e^-e^+ \to e^-e^+$, with the external lines labeled according to the particle names. The momentum flow of the external particles is indicated by the arrows above the corresponding fermion lines in the upper left diagram.

the identity of the external two-component fermion fields by carefully observing the direction of the arrow of each fermion line. If the arrow coincides with the direction of propagation, then we identify the electron [positron] line with e [e^c]. If the arrow is opposite to the direction of

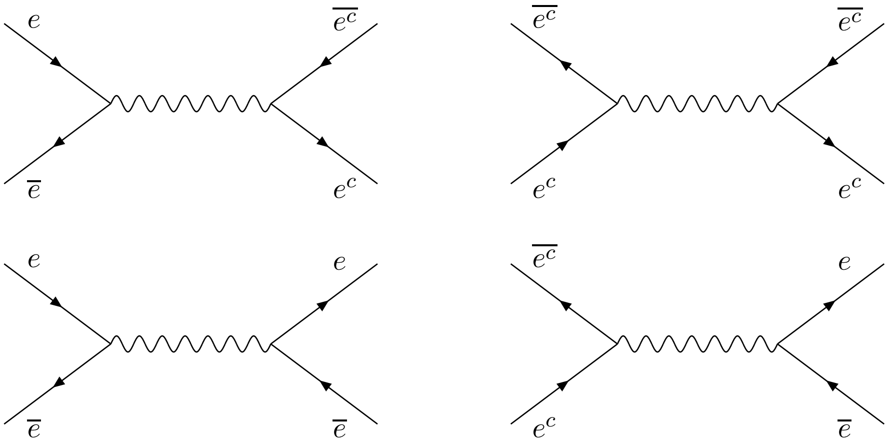


Fig. 2.21. Tree-level s-channel Feynman diagrams for $e^+e^- \to e^+e^-$, with the external lines labeled according to the two-component fermion fields. Note that all momenta flow from left to right.

propagation, then we identify the electron [positron] line with $\overline{e^c}$ [$\overline{e}$], in accord with the prescription of Fig. 2.18. Thus, in order to employ the two-component QED Feynman rules given in Fig. 2.19, we relabel the graphs of eq. (2.20) by employing the two-component fermion field labels, as shown in Fig. 2.21. One can now employ the Feynman rules of Fig. 2.19 directly to compute the invariant amplitude. Note that the choice of rule involving either the fields e and $\overline{e}$ in Fig. 2.19(a) or the fields e^c and $\overline{e^c}$ in Fig. 2.19(b) is unambiguous.

Problems

1. In eq. (2.14), we introduced the eigeinstates of the helicity operator. Derive the following explicit formula for $\zeta_\lambda(\hat{\boldsymbol{p}})$:

$$\zeta_\lambda(\hat{\boldsymbol{p}}) = \exp\left(-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2\right)\, \zeta_\lambda(\hat{\boldsymbol{z}})\,, \tag{2.87}$$

where $\hat{\boldsymbol{n}}=(-\sin\phi\,,\,\cos\phi\,,\,0\,)$.

HINT: Obtain $\zeta_\lambda(\hat{\boldsymbol{p}})$ from $\zeta_\lambda(\hat{\boldsymbol{z}})$ by employing the spin-1/2 rotation operator corresponding to a rotation from the $\hat{\boldsymbol{z}}$-direction to the direction of $\hat{\boldsymbol{p}}$ (characterized by polar angle θ and azimuthal angle ϕ).

2. Suppose that $x_\alpha(\vec{\boldsymbol{p}},\lambda)$ and $y_\alpha(\vec{\boldsymbol{p}},\lambda)$ satisfy eqs. (2.7)–(2.10) and eqs. (2.20)–(2.23).

(a) Using the identity $(p\cdot\overline{\sigma})(p\cdot\sigma) = (p\cdot\sigma)(p\cdot\overline{\sigma}) = p^2$, show that both x_α and y_α must satisfy the mass-shell condition, $p^2 = m^2$ (or equivalently, $p^0 = E_{\boldsymbol{p}}$).

(b) Show that eqs. (2.20)–(2.23) imply that

$$(s\cdot\sigma)_{\alpha\dot{\alpha}}(s\cdot\overline{\sigma})^{\dot{\alpha}\beta} = -\delta_\alpha{}^\beta\,, \qquad (s\cdot\overline{\sigma})^{\dot{\alpha}\alpha}(s\cdot\sigma)_{\alpha\dot{\beta}} = -\delta^{\dot{\alpha}}{}_{\dot{\beta}}\,. \tag{2.88}$$

and conclude that $s\cdot s = -1$.

(c) Using the identity: $p\cdot\sigma\, s\cdot\overline{\sigma} + s\cdot\sigma\, p\cdot\overline{\sigma} = p\cdot\overline{\sigma}\, s\cdot\sigma + s\cdot\overline{\sigma}\, p\cdot\sigma = 2p\cdot s$, show that both x_α and y_α must satisfy the condition $p\cdot s = 0$.

3. (a) Verify that $\sqrt{p\cdot\sigma}\sqrt{p\cdot\overline{\sigma}} = m$.

(b) The two-component helicity spinor ζ_λ is defined by eq. (2.11). Using the explicit forms for ζ_λ and $s^\mu(\lambda)$, prove that:

$$\sqrt{p\cdot\sigma}\left[s(\lambda)\cdot\overline{\sigma}\right]\sqrt{p\cdot\sigma}\;\zeta_\lambda = m\zeta_\lambda\,. \tag{2.89}$$

Show that the same result can be obtained from the first equation in eq. (2.20) by using the explicit forms for x_β and $y^{\dot{\alpha}}$.

(c) Noting that $s(\lambda)$ is proportional to λ, use the result of part (b) to prove eq. (2.29).

4. Consider a massive fermion of mass m with four momentum $(E\,;\,\vec{\boldsymbol{p}})$. Define a set of three four-vectors s^a_μ $(a = 1, 2, 3)$ such that the s^a and p/m

form an orthonormal set of four-vectors. That is,

$$p \cdot s^a = 0\,, \tag{2.90}$$
$$s^a \cdot s^b = -\delta^{ab}\,, \tag{2.91}$$
$$s^a_\mu\, s^a_\nu = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}\,, \tag{2.92}$$

where the sum over the repeated index a is implicitly assumed. A convenient choice for the s^a is

$$\begin{aligned} s^{1\mu} &= (0\,;\ \cos\theta\cos\phi,\ \cos\theta\sin\phi,\ -\sin\theta)\,,\\ s^{2\mu} &= (0\,;\ -\sin\phi,\ \cos\phi,\ 0)\,,\\ s^{3\mu} &= \left(\frac{|\vec{\boldsymbol{p}}|}{m}\,;\ \frac{E}{m}\hat{\boldsymbol{p}}\right)\,, \end{aligned} \tag{2.93}$$

in a coordinate system where $\hat{\boldsymbol{p}} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$. Note that s^3 is identical to the positive helicity spin vector; that is, $s \equiv 2\lambda s^3$ [see eq. (2.15)].

(a) Derive the following properties:

$$\epsilon^{\mu\nu\lambda\sigma} p_\mu s^1_\nu s^2_\lambda s^3_\sigma = -m\,, \tag{2.94}$$

$$s^a_\mu s^b_\nu - s^a_\nu s^b_\mu = \frac{\epsilon^{abc}\epsilon_{\mu\nu\gamma\delta}(s^c)^\gamma p^\delta}{m}\,, \tag{2.95}$$

$$[(\sigma\cdot s^a)\,(\overline{\sigma}\cdot s^b)]_\alpha{}^\beta = -\delta^{ab}\delta_\alpha{}^\beta - \frac{i\epsilon^{abc}}{m}[(\sigma\cdot p)\,(\overline{\sigma}\cdot s^c)]_\alpha{}^\beta\,, \tag{2.96}$$

$$[(\overline{\sigma}\cdot s^a)\,(\sigma\cdot s^b)]^{\dot\alpha}{}_{\dot\beta} = -\delta^{ab}\delta^{\dot\alpha}{}_{\dot\beta} + \frac{i\epsilon^{abc}}{m}[(\overline{\sigma}\cdot p)\,(\sigma\cdot s^c)]^{\dot\alpha}{}_{\dot\beta}\,. \tag{2.97}$$

(b) Prove that the helicity two-component spinors, $x(\vec{\boldsymbol{p}},\lambda)$ and $y(\vec{\boldsymbol{p}},\lambda)$ satisfy the following equations:

$$(s^a\!\cdot\overline{\sigma})^{\dot\alpha\beta}x_\beta(\vec{\boldsymbol{p}},\lambda') = \tau^a_{\lambda\lambda'}\bar{y}^{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)\,, \qquad (s\!\cdot\sigma)_{\alpha\dot\beta}\bar{y}^{\dot\beta}(\vec{\boldsymbol{p}},\lambda') = -\tau^a_{\lambda\lambda'}x_\alpha(\vec{\boldsymbol{p}},\lambda)\,,$$

$$(s\!\cdot\sigma)_{\alpha\dot\beta}\bar{x}^{\dot\beta}(\vec{\boldsymbol{p}},\lambda') = -\tau^a_{\lambda\lambda'}y_\alpha(\vec{\boldsymbol{p}},\lambda)\,, \qquad (s\!\cdot\overline{\sigma})^{\dot\alpha\beta}y_\beta(\vec{\boldsymbol{p}},\lambda') = \tau^a_{\lambda\lambda'}\bar{x}^{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)\,,$$

$$x^\alpha(\vec{\boldsymbol{p}},\lambda')(s\!\cdot\sigma)_{\alpha\dot\beta} = -\tau^a_{\lambda\lambda'}\bar{y}_{\dot\beta}(\vec{\boldsymbol{p}},\lambda)\,, \qquad \bar{y}_{\dot\alpha}(\vec{\boldsymbol{p}},\lambda')(s\!\cdot\overline{\sigma})^{\dot\alpha\beta} = \tau^a_{\lambda\lambda'}x^\beta(\vec{\boldsymbol{p}},\lambda)\,,$$

$$\bar{x}_{\dot\alpha}(\vec{\boldsymbol{p}},\lambda')(s\!\cdot\overline{\sigma})^{\dot\alpha\beta} = \tau^a_{\lambda\lambda'}y^\beta(\vec{\boldsymbol{p}},\lambda)\,, \qquad y^\alpha(\vec{\boldsymbol{p}},\lambda')(s\!\cdot\sigma)_{\alpha\dot\beta} = -\tau^a_{\lambda\lambda'}\bar{x}_{\dot\beta}(\vec{\boldsymbol{p}},\lambda)\,,$$

where the τ^a are the Pauli matrices,[9] and there is an implicit sum over the repeated label $\lambda = \pm\frac{1}{2}$. These equations generalize the results of eqs. (2.20)–(2.23).

[9] The first (second) row and column of the τ-matrices correspond to $\lambda = 1/2$ $(-1/2)$. Thus, for example, $\tau^3_{\lambda\lambda'} = 2\lambda\delta_{\lambda\lambda'}$ (no sum over λ).

(c) Using the results of part (b), derive the following generalizations of eqs. (2.31)–(2.34):

$$x_\alpha(\vec{\boldsymbol{p}},\lambda')\bar{x}_{\dot\beta}(\vec{\boldsymbol{p}},\lambda) = \tfrac{1}{2}(p_\mu\delta_{\lambda'\lambda} - m s^a_\mu \tau^a_{\lambda'\lambda})\sigma^\mu_{\alpha\dot\beta}\,, \tag{2.98}$$

$$\bar{y}^{\dot\alpha}(\vec{\boldsymbol{p}},\lambda')y^\beta(\vec{\boldsymbol{p}},\lambda) = \tfrac{1}{2}(p_\mu\delta_{\lambda'\lambda} + m s^a_\mu \tau^a_{\lambda'\lambda})\overline{\sigma}^{\mu\dot\alpha\beta}\,, \tag{2.99}$$

$$x_\alpha(\vec{\boldsymbol{p}},\lambda')y^\beta(\vec{\boldsymbol{p}},\lambda) = \tfrac{1}{2}\left(m\delta_{\lambda'\lambda}\delta_\alpha{}^\beta - [(\sigma\cdot s^a\tau^a_{\lambda'\lambda})\,(\overline{\sigma}\cdot p)]_\alpha{}^\beta\right)\,, \tag{2.100}$$

$$\bar{y}^{\dot\alpha}(\vec{\boldsymbol{p}},\lambda')\bar{x}_{\dot\beta}(\vec{\boldsymbol{p}},\lambda) = \tfrac{1}{2}\left(m\delta_{\lambda'\lambda}\delta^{\dot\alpha}{}_{\dot\beta} + [(\overline{\sigma}\cdot s^a\tau^a_{\lambda'\lambda})\,(\sigma\cdot p)]^{\dot\alpha}{}_{\dot\beta}\right)\,. \tag{2.101}$$

These equations are the two-component versions of the Bouchiat-Michel formulas. (See Chapter 3, problem 6 for the four-component spinor version of these formulas.)

(d) Take the $m \to 0$ limit and derive the relevant Bouchiat-Michel formulas for a massless helicity two-component spinor. [*HINT:* Recall that $s^3 = p/m + \mathcal{O}(m/E)$ as $m \to 0$.]

3
From Two-Component to Four-Component Spinors

To motivate the introduction of four component spinors for spin-1/2 particles, it is instructive to compare real and complex scalar fields that describe spin-0 particles. Consider a free scalar field theory with two real, mass-degenerate scalar fields $\phi_1(x)$ and $\phi_2(x)$. The corresponding Lagrangian is given by:

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi_1)(\partial^\mu\phi_1) + \tfrac{1}{2}(\partial_\mu\phi_2)(\partial^\mu\phi_2) - \tfrac{1}{2}m^2(\phi_1^2 + \phi_2^2)\,, \tag{3.1}$$

where m^2 is assumed to be positive. In this case, we can rewrite the theory in terms of a single complex scalar field:

$$\Phi = \frac{1}{\sqrt{2}}\left(\phi_1 + i\phi_2\right)\,, \qquad \Phi^* = \frac{1}{\sqrt{2}}\left(\phi_1 - i\phi_2\right)\,. \tag{3.2}$$

In terms of the complex field, eq. (3.1) takes the following form:

$$\mathcal{L} = (\partial_\mu\Phi)^*(\partial^\mu\Phi) - m^2\Phi^*\Phi\,. \tag{3.3}$$

Note that eq. (3.1) exhibits a global O(2) symmetry, $\phi_i \to C_{ij}\phi_j$. Corresponding to this symmetry is a conserved Noether current

$$J^\mu = \phi_1\partial^\mu\phi_2 - \phi_2\partial^\mu\phi_1\,, \tag{3.4}$$

with a corresponding conserved charge, $Q = \int J^0 d^3x$. The complex field $\phi(x)$ creates and annihilates particles of definite charge Q. Comparing these results to eq. (1.102) with $m_1 = m_2$ and eqs. (1.103)–(1.105), we see that in a theory of mass degenerate spin-1/2 fermions, the role of the real fields ϕ_i are taken by the χ_i. The fermion fields of definite (and opposite) charge are ξ and η. Thus, we combine ξ and $\bar{\eta}$ into one complex four-component spinor field Ψ, which plays the same role as the complex scalar field ϕ.

3.1 Four-component spinors

We now spell out in more detail the four-component Dirac fermion notation. A four component Dirac spinor field, $\Psi(x)$, is made up of two mass-degenerate two-component spinor fields, $\xi(x)$ and $\eta(x)$ as follows:

$$\Psi(x) \equiv \begin{pmatrix} \xi_\alpha(x) \\ \bar{\eta}^{\dot{\alpha}}(x) \end{pmatrix} . \tag{3.5}$$

The field equation satisfied by $\Psi(x)$ is obtained from eq. (1.106). This is the free-field Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0 , \tag{3.6}$$

where the 4×4 Dirac gamma matrices, γ^μ, can be expressed in 2×2 block form as follows:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot{\beta}} \\ \overline{\sigma}^{\mu\dot{\alpha}\beta} & 0 \end{pmatrix} . \tag{3.7}$$

Using eqs. (1.67) and (1.68), one immediately obtains the defining property of the Dirac matrices:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} . \tag{3.8}$$

There are many other representations for the Dirac matrices that satisfy eq. (3.8). The explicit form given in eq. (3.7) is called the *chiral* representation, which corresponds to a basis choice in which γ_5 is diagonal:

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\delta_\alpha{}^\beta & 0 \\ 0 & \delta^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix} . \tag{3.9}$$

It is convenient to define chiral projections operators:

$$P_L \equiv \tfrac{1}{2}(1 - \gamma_5) , \qquad P_R \equiv \tfrac{1}{2}(1 + \gamma_5) , \tag{3.10}$$

so that

$$\Psi_L(x) \equiv P_L\Psi(x) = \begin{pmatrix} \xi_\alpha(x) \\ 0 \end{pmatrix} , \quad \Psi_R(x) \equiv P_R\Psi(x) = \begin{pmatrix} 0 \\ \bar{\eta}^{\dot{\alpha}}(x) \end{pmatrix} . \tag{3.11}$$

In addition, we introduce:[1]

$$\tfrac{1}{2}\Sigma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} \sigma^{\mu\nu}{}_\alpha{}^\beta & 0 \\ 0 & \overline{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix} . \tag{3.12}$$

[1] In most textbooks, $\Sigma^{\mu\nu}$ is called $\sigma^{\mu\nu}$. Here, we use the former symbol so that there is no confusion with the two-component definition of $\sigma^{\mu\nu}$ given in eq. (1.76).

The duality conditions satisfied by $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$ [eq. (1.78)] imply that:

$$\gamma_5\Sigma^{\mu\nu} = \tfrac{1}{2}i\epsilon^{\mu\nu\rho\kappa}\Sigma_{\rho\kappa}\,. \tag{3.13}$$

Given a four-component spinor field Ψ, we introduce four related spinor fields: the Dirac adjoint field $\overline{\Psi}$, the space-reflected field Ψ^p, the time-reversed field Ψ^t and the charge conjugate field Ψ^c, which are respectively given by

$$\overline{\Psi}(x) \equiv \Psi^\dagger(x)A = \left(\eta^\alpha(x)\,,\,\bar{\xi}_{\dot{\alpha}}(x)\right)\,, \tag{3.14}$$

$$\Psi^p(x) \equiv i\gamma^0\Psi(x) = \begin{pmatrix} i\sigma^0_{\alpha\dot{\beta}}\overline{\eta}^{\dot{\beta}}(x) \\ i\overline{\sigma}^{0\dot{\alpha}\beta}\xi_\beta(x)\end{pmatrix}\,, \tag{3.15}$$

$$\Psi^t(x) \equiv -\gamma^0B^{-1}\overline{\Psi}^T(x) = -\gamma^0B^{-1}A^T\Psi^*(x) = \begin{pmatrix} \sigma^0_{\alpha\dot{\beta}}\overline{\xi}^{\dot{\beta}}(x) \\ -\overline{\sigma}^{0\dot{\alpha}\beta}\eta_\beta(x)\end{pmatrix}\,, \tag{3.16}$$

$$\Psi^c(x) \equiv C\overline{\Psi}^T(x) = CA^T\Psi^*(x) = \begin{pmatrix}\eta_\alpha(x) \\ \bar{\xi}^{\dot{\alpha}}(x)\end{pmatrix}\,. \tag{3.17}$$

That is, in the chiral representation, A, B and C are explicitly given by

$$A = \begin{pmatrix} 0 & \delta^{\dot{\alpha}}{}_{\dot{\beta}} \\ \delta_\alpha{}^\beta & 0\end{pmatrix}\,, \qquad B = \begin{pmatrix}\epsilon^{\alpha\beta} & 0 \\ 0 & -\epsilon_{\dot{\alpha}\dot{\beta}}\end{pmatrix}\,, \qquad C = \begin{pmatrix}\epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}}\end{pmatrix}\,. \tag{3.18}$$

Note the numerical equalities:[2]

$$A = \gamma^0\,, \qquad B = \gamma^1\gamma^3\,, \qquad C = i\gamma^0\gamma^2\,, \tag{3.19}$$

although these identifications do not respect the structure of the undotted and dotted indices specified in eq. (3.18) In calculations that involve translations between two-component and four-component notation, the expressions given in eq. (3.18) should be used. In calculations involving only four-component notation, there is no harm in using the numerical values for the matrices noted above.

Although the explicit forms for A, B, and C were given in the chiral representation, they can be defined independently of the gamma matrix representation. In general, these matrices must satisfy[1]

$$A\gamma^\mu A^{-1} = \gamma^{\mu\dagger}\,, \tag{3.20}$$

$$B\gamma^\mu B^{-1} = \gamma^{\mu T}\,, \tag{3.21}$$

$$C^{-1}\gamma^\mu C = -\gamma^{\mu T}\,. \tag{3.22}$$

[2] These identifications have been obtained specifically in the chiral representation of the Dirac matrices. As shown in Appendix A, eq. (3.19) also holds in the Dirac representation but not in the Majorana representation.

We now impose the following additional conditions:[3]

$$\Psi = A^{-1}\overline{\Psi}^\dagger\,, \qquad (\Psi^t)^t = -\Psi\,, \qquad (\Psi^c)^c = \Psi\,. \tag{3.23}$$

The first of these conditions together with eq. (3.14) is equivalent to the statement that $\overline{\Psi}\Psi$ is hermitian. The second condition corresponds to the result previously noted that when the time-reversal operator is applied twice to a fermion field it yields $T^2 = -1$. Finally, the third condition corresponds to the statement that the charge conjugation operator applied twice is equal to the identity operator. Using eqs. (3.20)–(3.23) and the defining property of the gamma matrices [eq. (3.8)], one can show that (independently of the gamma matrix representation) the matrices A, B and C must satisfy the following relations:

$$A^\dagger = A\,, \qquad B^T = -B\,, \qquad C^T = -C\,, \tag{3.24}$$

$$CB = -\gamma_5\,, \quad BA^{-1} = -(AB^{-1})^*\,, \quad (AC)^{-1} = (AC)^*\,. \tag{3.25}$$

We now examine the Dirac bilinear covariants, which are quantities that are quadratic in the Dirac spinor field and transform irreducibly as Lorentz tensors. These are easily constructed from the corresponding quantities that are quadratic in the two-component fermion fields. To construct a translation table between the two-component form and the four-component forms for the bilinear covariants, we first introduce two Dirac spinor fields:

$$\Psi_1(x) \equiv \begin{pmatrix} \xi_1(x) \\ \bar\eta_1(x) \end{pmatrix}\,, \qquad \Psi_2(x) \equiv \begin{pmatrix} \xi_2(x) \\ \bar\eta_2(x) \end{pmatrix}\,, \tag{3.26}$$

where spinor indices have been suppressed on the two-component fields $\xi_i(x)$ and $\bar\eta_i(x)$. The results are then exhibited in Table 3.1.

It then follows that:

$$\overline{\Psi}_1\Psi_2 = \eta_1\xi_2 + \bar\xi_1\bar\eta_2 \tag{3.27}$$

$$\overline{\Psi}_1\gamma_5\Psi_2 = -\eta_1\xi_2 + \bar\xi_1\bar\eta_2 \tag{3.28}$$

$$\overline{\Psi}_1\gamma^\mu\Psi_2 = \bar\xi_1\overline{\sigma}^\mu\xi_2 - \bar\eta_2\overline{\sigma}^\mu\eta_1 \tag{3.29}$$

$$\overline{\Psi}_1\gamma^\mu\gamma_5\Psi_2 = -\bar\xi_1\overline{\sigma}^\mu\xi_2 - \bar\eta_2\overline{\sigma}^\mu\eta_1 \tag{3.30}$$

$$\overline{\Psi}_1\Sigma^{\mu\nu}\Psi_2 = 2(\eta_1\sigma^{\mu\nu}\xi_2 + \bar\xi_1\overline{\sigma}^{\mu\nu}\bar\eta_2) \tag{3.31}$$

$$\overline{\Psi}_1\Sigma^{\mu\nu}\gamma_5\Psi_2 = -2(\eta_1\sigma^{\mu\nu}\xi_2 - \bar\xi_1\overline{\sigma}^{\mu\nu}\bar\eta_2)\,. \tag{3.32}$$

[3] The defining relations [eqs. (3.20)–(3.22)] and the conditions given in eq. (3.23) does not quite fix the values of the matrices A, B and C [see eq. (A.39)]. Nevertheless, the remaining freedom in defining these matrices has no effect on any of the results in this section.

Table 3.1. A translation table relating bilinear covariants in two-component and four-component notation.

$\overline{\Psi}_1 P_L \Psi_2 = \eta_1 \xi_2$	$\overline{\Psi}{}^c_1 P_L \Psi^c_2 = \xi_1 \eta_2$
$\overline{\Psi}_1 P_R \Psi_2 = \bar{\xi}_1 \bar{\eta}_2$	$\overline{\Psi}{}^c_1 P_R \Psi^c_2 = \bar{\eta}_1 \bar{\xi}_2$
$\overline{\Psi}{}^c_1 P_L \Psi_2 = \xi_1 \xi_2$	$\overline{\Psi}_1 P_L \Psi^c_2 = \eta_1 \eta_2$
$\overline{\Psi}_1 P_R \Psi^c_2 = \bar{\xi}_1 \bar{\xi}_2$	$\overline{\Psi}{}^c_1 P_R \Psi_2 = \bar{\eta}_1 \bar{\eta}_2$
$\overline{\Psi}_1 \gamma^\mu P_L \Psi_2 = \bar{\xi}_1 \overline{\sigma}^\mu \xi_2$	$\overline{\Psi}{}^c_1 \gamma^\mu P_L \Psi^c_2 = \bar{\eta}_1 \overline{\sigma}^\mu \eta_2$
$\overline{\Psi}{}^c_1 \gamma^\mu P_R \Psi^c_2 = \xi_1 \sigma^\mu \bar{\xi}_2$	$\overline{\Psi}_1 \gamma^\mu P_R \Psi_2 = \eta_1 \sigma^\mu \bar{\eta}_2$
$\overline{\Psi}_1 \Sigma^{\mu\nu} P_L \Psi_2 = 2\, \eta_1 \sigma^{\mu\nu} \xi_2$	$\overline{\Psi}{}^c_1 \Sigma^{\mu\nu} P_L \Psi^c_2 = 2\, \xi_1 \sigma^{\mu\nu} \eta_2$
$\overline{\Psi}_1 \Sigma^{\mu\nu} P_R \Psi_2 = 2\, \bar{\xi}_1 \overline{\sigma}^{\mu\nu} \bar{\eta}_2$	$\overline{\Psi}{}^c_1 \Sigma^{\mu\nu} P_R \Psi^c_2 = 2\, \bar{\eta}_1 \overline{\sigma}^{\mu\nu} \bar{\xi}_2$

Note that the Lorentz vector bilinear covariants are written in terms of $\overline{\sigma}^\mu$. Of course, one could also rewrite these expressions by eliminating $\overline{\sigma}^\mu$ in favor of σ^μ by using eq. (1.72). We expect a total of sixteen independent components among the bilinear covariants corresponding to the sixteen-dimensional space of 4×4 matrices. From eq. (3.13), we see that $\Sigma^{\mu\nu}$ and $\Sigma^{\mu\nu}\gamma_5$ are not independent. That is, the sixteen-dimensional space is spanned by I_4, γ_5, γ^μ, $\gamma^\mu\gamma_5$ and $\Sigma^{\mu\nu}$, which matches the counting presented at the end of section 1.4.

When $\Psi_2 = \Psi_1$, the bilinear covariants listed in eqs. (3.27)–(3.32) are either hermitian or anti-hermitian. We shall always work in a convention where $A^\dagger = A$. Then, $\bar{\Psi}\mathcal{O}\Psi$ is hermitian if

$$A\mathcal{O}A^{-1} = \mathcal{O}^\dagger \,. \tag{3.33}$$

Thus, one obtains hermitian bilinear covariants if $\mathcal{O} = I_4$, $i\gamma_5$, γ^μ, $\gamma^\mu\gamma_5$, $\Sigma^{\mu\nu}$ or $i\Sigma^{\mu\nu}\gamma_5$.

One can also describe self-conjugate fermions in four-component fermion notation. These are the Majorana fermion fields, which are the analogues of the real scalar fields. That is, for a complex scalar field, one can reduces the number of degrees of freedom by a factor of two by imposing the reality condition $\phi^* = \phi$. For a Dirac fermion Ψ, one can likewise reduce the number of degrees of freedom by a factor of two by imposing the condition $\Psi^c = \Psi$.[4] Thus, we define a

[4] Note that $\Psi^c = \Psi$ is a Lorentz covariant condition since it sets equal two quantities with the same Lorentz transformation properties [see eq. (3.44)]. The same is not true in general for the condition $\Psi^* = \Psi$, except in special representations of the gamma matrices (such as the Majorana representation).

Majorana four-component fermion by imposing the Majorana condition: $\Psi_M = \Psi_M^c = C\overline{\Psi}_M^T$. Explicitly, this condition in the chiral representation implies that Ψ_M takes the following form:

$$\Psi_M(x) \equiv \begin{pmatrix} \chi_\alpha(x) \\ \bar{\chi}^{\dot{\alpha}}(x) \end{pmatrix} . \tag{3.34}$$

The results of Table 3.1 and eqs. (3.27)–(3.32) also apply to Majorana fermions. However, due to the Majorana condition, the bilinear covariants of eqs. (3.27)–(3.32) are either hermitian or anti-hermitian even if $\Psi_1 \neq \Psi_2$. In general, $\bar{\Psi}_{M1}\mathcal{O}\Psi_{M2}$ is hermitian if

$$AC\mathcal{O}^T(AC)^{-1} = \mathcal{O}^\dagger . \tag{3.35}$$

To derive eq. (3.35), we have made use of eqs. (3.24) and (3.25), the Majorana condition and the anti-commutativity of the fermion fields. Thus, one obtains $(\bar{\Psi}_{M1}\mathcal{O}\Psi_{M2})^\dagger = \bar{\Psi}_{M1}\mathcal{O}\Psi_{M2}$ if $\mathcal{O} = I_4$, $i\gamma_5$, $i\gamma^\mu$, $\gamma^\mu\gamma_5$, $i\Sigma^{\mu\nu}$ or $\Sigma^{\mu\nu}\gamma_5$. In addition, we obtain the following relations among the Majorana bilinear covariants:

$$\overline{\Psi}_{M1}\Psi_{M2} = \overline{\Psi}_{M2}\Psi_{M1} \,, \tag{3.36}$$
$$\overline{\Psi}_{M1}\gamma_5\Psi_{M2} = \overline{\Psi}_{M2}\gamma_5\Psi_{M1} \,, \tag{3.37}$$
$$\overline{\Psi}_{M1}\gamma^\mu\Psi_{M2} = -\overline{\Psi}_{M2}\gamma^\mu\Psi_{M1} \,, \tag{3.38}$$
$$\overline{\Psi}_{M1}\gamma^\mu\gamma_5\Psi_{M2} = \overline{\Psi}_{M2}\gamma^\mu\gamma_5\Psi_{M1} \,, \tag{3.39}$$
$$\overline{\Psi}_{M1}\Sigma^{\mu\nu}\Psi_{M2} = -\overline{\Psi}_{M2}\Sigma^{\mu\nu}\Psi_{M1} \,, \tag{3.40}$$
$$\overline{\Psi}_{M1}\Sigma^{\mu\nu}\gamma_5\Psi_{M2} = -\overline{\Psi}_{M2}\Sigma^{\mu\nu}\gamma_5\Psi_{M1} \,. \tag{3.41}$$

We can combine eqs. (3.38) and (3.39) to obtain:

$$\overline{\Psi}_{M1}\gamma^\mu P_L\Psi_{M2} = -\overline{\Psi}_{M2}\gamma^\mu P_R\Psi_{M1} \,. \tag{3.42}$$

In particular, if $\Psi_{M1} = \Psi_{M2} \equiv \Psi_M$, then eqs. (3.36)–(3.41) imply that:

$$\overline{\Psi}_M\gamma^\mu\Psi_M = \overline{\Psi}_M\Sigma^{\mu\nu}\Psi_M = \overline{\Psi}_M\Sigma^{\mu\nu}\gamma_5\Psi_M = 0 \,, \tag{3.43}$$

leaving only six independent components of the non-vanishing bilinear covariants: $\overline{\Psi}_M\Psi_M$, $\overline{\Psi}_M\gamma_5\Psi_M$ and $\overline{\Psi}_M\gamma_\mu\gamma_5\Psi_M$. This matches the counting exhibited at the end of section 1.4.

The behavior of the four-component fermion fields under proper and improper Lorentz transformations is easily obtained from the corresponding results for the two-component fermion fields. Consider first the transformation of the four-component spinor field under proper orthochronous Lorentz transformations. The transformations of the two-component spinors in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations are given by eqs. (1.31) and (1.32) and eqs. (1.40) and (1.41) respectively. Using the results of eqs. (1.82)–(1.84), it follows that when $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$,

$$\Psi'(x) = S\Psi(\Lambda^{-1}x)\,, \qquad \overline{\Psi}'(x) = \overline{\Psi}(\Lambda^{-1}x)S^{-1}\,, \tag{3.44}$$

where

$$S \equiv \begin{pmatrix} M_{\alpha}{}^{\beta} & 0 \\ 0 & (M^{-1})^{\dagger\,\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix} = \exp\left(\frac{-i}{4}\theta^{\mu\nu}\Sigma_{\mu\nu}\right)\,. \tag{3.45}$$

The transformation law for $\overline{\Psi}$ is easily derived from that of Ψ by making use of the relation $A^{-1}S^{\dagger}A = S^{-1}$ [the latter is a consequence of eq. (3.20)]. It then follows immediately that $\overline{\Psi}(x)\Psi(x)$ is a Lorentz scalar. Similarly, one can prove that the other bilinear covariants transform like definite Lorentz tensors. This requires one further result:

$$S^{-1}\gamma^{\mu}S = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}\,, \tag{3.46}$$

which is the four-component version of eqs. (1.70) and (1.75). It then follows that $\overline{\Psi}\gamma^{\mu}\Psi$ transforms as a Lorentz four-vector, $\overline{\Psi}\Sigma^{\mu\nu}\Psi$ transforms as an antisymmetric second rank tensor, *etc.*

The behavior of the four-component fermion field $\Psi(x)$ under P, T and C is likewise easily obtained from the corresponding behavior of the two component fermion fields (written in the basis of the fields of definite charge). Using the results of eqs. (1.147) and (1.148) for P, eqs. (1.175) and (1.176) for T and eqs. (1.186) and (1.187) for C, it follows that:[5]

$$\mathcal{P}\Psi(x)\mathcal{P}^{-1} = \eta_P i\gamma^0\Psi(x_P) = \eta_P\Psi^p(x_P)\,, \tag{3.47}$$

$$\mathcal{T}\Psi(x)\mathcal{T}^{-1} = \eta_T(\gamma^0A^{-1})^TB\Psi(x_T) = \eta_T\Psi^{t*}(x_T)\,, \tag{3.48}$$

$$\mathcal{C}\Psi(x)\mathcal{C}^{-1} = \eta_C C\overline{\Psi}^T(x) = \eta_C CA^T\Psi^*(x) = \eta_C\Psi^c(x)\,. \tag{3.49}$$

For convenience, we also list the corresponding transformation laws for the Dirac adjoint field:

$$\mathcal{P}\overline{\Psi}(x)\mathcal{P}^{-1} = -i\eta_P^*\overline{\Psi}(x_p)\gamma^0\,, \tag{3.50}$$

$$\mathcal{T}\overline{\Psi}(x)\mathcal{T}^{-1} = \eta_T^*\overline{\Psi}(x_T)B^{-1}(A^{\dagger}\gamma^0)^T\,, \tag{3.51}$$

$$\mathcal{C}\overline{\Psi}(x)\mathcal{C}^{-1} = -\eta_C^*\Psi^T(x)C^{-1}\,. \tag{3.52}$$

Thus, *e.g.*, the hermitian bilinear covariant $\overline{\Psi}(x)\Psi(x)$ is a scalar quantity under P, T and C, respectively. These results apply to both Dirac and

[5] Once again, we remind the reader that we have deviated from the standard textbook conventions in eq. (3.47) by separating out the factor of i from the phase η_P. As a result, in our notation $\eta_P = \pm 1$ for a Majorana fermion, whereas in the standard convention $\eta_P = \pm i$ after the factor of i is absorbed into the definition of η_P.

Majorana four-component fermion fields. For the transformation of Dirac fields, the phases η_P, η_T and η_C are arbitrary. When applied to Majorana fields, the relation $\Psi_M = C\overline{\Psi}_M^T$ imposes constraints on these phases. In particular, it is straightforward to show that the transformation of $\Psi^c(x)$ under P, T and C is given by eqs. (3.47)–(3.52) with the replacement of the phases η_P, η_T and η_C by their complex conjugates. Hence, for Majorana fields, which satisfy $\Psi^c_M = \Psi_M$, it follows that the phases η_P, η_T and η_C are real (and equal to ± 1).

The transformation law under CPT is particularly simple (after some algebraic manipulation):[6]

$$\begin{aligned}
\mathcal{CPT}\,\Psi(x)(\mathcal{CPT})^{-1} &= \eta_T(\gamma^0 A^{-1})^T B\,\mathcal{CP}\Psi(x_T)(\mathcal{CP})^{-1} \\
&= i\eta_{CPT}(\gamma^0 A^{-1})^T B\gamma^0 C A^T \Psi^*(-x) \\
&= i\eta_{CPT}(ACBA^{-1})^T\Psi^*(-x)\,, \qquad (3.53)
\end{aligned}$$

where as before $\eta_{CPT} \equiv \eta_C\eta_P\eta_T$. Recalling that $CB = -\gamma_5$ and using eq. (3.20), we end up with:

$$\mathcal{CPT}\,\Psi(x)(\mathcal{CPT})^{-1} = \quad i\eta_{CPT}[\gamma_5\Psi(-x)]^*\,, \qquad (3.54)$$

$$\mathcal{CPT}\,\overline{\Psi}(x)(\mathcal{CPT})^{-1} = -i\eta^*_{CPT}[A^\dagger\gamma_5\Psi(-x)]^T\,. \qquad (3.55)$$

It is now straightforward to work out the properties of the bilinear covariants under C, P, T, CP and CPT. For example, consider the CPT-transformation properties of the hermitian bilinear covariants $\bar{\Psi}\mathcal{O}\Psi$ and $\bar{\Psi}_{M1}\mathcal{O}\Psi_{M2}$ where Ψ is a Dirac field and Ψ_{Mi} ($i = 1, 2$) are Majorana fields:

$$\mathcal{CPT}\;\bar{\Psi}\mathcal{O}\Psi\,(\mathcal{CPT})^{-1} = \bar{\Psi}\gamma_5\mathcal{O}\gamma_5\Psi\,, \qquad (3.56)$$

$$\mathcal{CPT}\;\bar{\Psi}_{M1}\mathcal{O}\Psi_{M2}\,(\mathcal{CPT})^{-1} = \bar{\Psi}_{M1}\gamma_5\mathcal{O}\gamma_5\Psi_{M2}\,, \qquad (3.57)$$

where $\mathcal{O}$ satisfies eq. (3.33) [eq. (3.35)] in the case of the Dirac [Majorana] bilinear covariants. In deriving these results, we have used the anti-unitary property of $\mathcal{CPT}$ and the anti-commutativity of the fermion fields.[7] In addition, eq. (3.57) was derived under the assumption that η_{CPT} is the same for all fermion fields [see eq. (1.200) and the associated discussion]. Indeed, eqs. (3.56) and (3.57) are consistent with the results of eqs. (1.201)–(1.203).

[6] Recall that for any c-number quantity z (in contrast to an operator acting on the Hilbert space), $\mathcal{P}z\mathcal{P}^{-1} = z$, $\mathcal{C}z\mathcal{C}^{-1} = z$ and $\mathcal{T}z\mathcal{T}^{-1} = z^*$.

[7] Strictly speaking, the bilinear covariants are always assumed to be normal-ordered expressions. As a result, it is legitimate to treat the fermion fields as anti-commuting (that is, by ignoring the delta function that appears in the equal-time anti-commutation relations).

3.2 Lagrangians for free four-component fermions

We briefly look at the Lagrangians for a free fermion field theory in four-component notation. These can be obtained from the two-component free-fermion field Lagrangians given in section 1.5 using the results of Table 3.1. For example, for a theory of a single two-component fermion, we employ the Majorana four-component fermion [eq. (3.34)] and obtain the following Lagrangian:

$$\mathcal{L} = \tfrac{1}{2} i \overline{\Psi}_M \gamma^\mu \partial_\mu \Psi_M - \tfrac{1}{2} m \overline{\Psi}_M \Psi_M \,. \tag{3.58}$$

This result was obtained from eq. (1.99) under the assumption that the mass parameter m is real.[8] The corresponding field equations [eqs. (1.100) and (1.101)] in four-component notation are the free-field Dirac equation and its conjugate:

$$(i\gamma^\mu \partial_\mu - m)\Psi_M = 0\,, \qquad \overline{\Psi}_M (i\gamma^\mu \overleftarrow{\partial_\mu} + m) = 0\,. \tag{3.59}$$

In a theory of a pair of two-component fermions of equal mass, we work in the basis of definite charged fields ξ and η. Combining these into a single Dirac field Ψ, the corresponding free-field Lagrangian [eq. (1.105)] yields the free-field Dirac Lagrangian in four-component notation:

$$\mathcal{L} = i\overline{\Psi}\gamma^\mu \partial_\mu \Psi - m \overline{\Psi}\,\Psi\,. \tag{3.60}$$

The corresponding field equations are again the free-field Dirac equation [eq. (3.6)] and its conjugate. In section 1.5, we learned that a charged Dirac fermion was equivalent to a pair of mass-degenerate Majorana fermions. This identification carries over to four-component notation as follows. Given the four-component Dirac spinor $\Psi(x)$ [eq. (3.5)], we construct a pair of four-component Majorana fermions:

$$\Psi_{M1} = \frac{1}{\sqrt{2}}\left(\Psi + \Psi^c\right)\,, \tag{3.61}$$

$$\Psi_{M2} = \frac{-i}{\sqrt{2}}\left(\Psi - \Psi^c\right)\,. \tag{3.62}$$

From the definition of the charge conjugated spinor [eq. (3.17)], it follows that $(i\Psi)^c = -i\Psi^c$. Thus we see that $\Psi^c_{Mi} = \Psi_{Mi}$ (for $i = 1, 2$) are self-conjugate four-component spinors.[9] One can rewrite the Dirac

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[8] In the case of a complex mass parameter m, the term $-\frac{1}{2}m\overline{\Psi}_M\Psi_M$ in eq. (3.59) would be replaced by $-\frac{1}{2}\left[(\mathrm{Re}\ m)\overline{\Psi}_M\,\Psi_M - i(\mathrm{Im}\ m)\overline{\Psi}_M\gamma_5\Psi_M\right]$.

[9] Moreover, if $\mathcal{C}\Psi\mathcal{C}^{-1} = \eta_C\Psi^c$, with the phase η_C chosen real ($|\eta_C| = 1$), then $\mathcal{C}\Psi_{M1}\mathcal{C}^{-1} = \eta_C\Psi_{M1}$ and $\mathcal{C}\Psi_{M2}\mathcal{C}^{-1} = -\eta_C\Psi_{M2}$ [*cf.* eq. (1.193) for the corresponding result in two-component notation].

Lagrangian [eq. (3.60)] in terms of the Majorana fields Ψ_{Mi}:

$$\mathcal{L} = \tfrac{1}{2}i(\overline{\Psi}_{M1}\gamma^\mu\partial_\mu\Psi_{M1} + \overline{\Psi}_{M2}\gamma^\mu\partial_\mu\Psi_{M2}) - \tfrac{1}{2}m(\overline{\Psi}_{M1}\,\Psi_{M1} + \overline{\Psi}_{M2}\,\Psi_{M2})\,. \tag{3.63}$$

In deriving eq. (3.63), we have used eqs. (3.36) and (3.38) and dropped a term that is a total divergence (which does not contribute to the action).

The complex four-component Dirac field, which represents a charged spin-1/2 particle, possesses eight off-shell degrees of freedom. In particular, Ψ and $\overline{\Psi}$ are initially independent degrees of freedom. After imposing the field equations [eq. (3.6)], the four complex components are no longer independent, resulting in four physical on-shell degrees of freedom. As noted above, the complex four-component Dirac field can also be employed to describe a neutral self-conjugate spin-1/2 Majorana fermion. In this case, one must impose an additional constraint: $\Psi^c(x) = \Psi(x)$. This Majorana constraint reduces the number of degrees of freedom of the Dirac fermion by a factor of two, resulting in four off-shell and two physical on-shell degrees of freedom. Of course, the counting of degrees of freedom above reproduces the results of the previous analysis based on two-component fermion fields [see section 1.5].

Models of N free four-component fermions, with non-diagonal mass matrices are easily constructed. The diagonalization procedure to identify the mass eigenstates is easily obtained by converting the two-component fermion mass diagonalization methods of Section 1.6 to four-component notation. We leave the details as an exercise for the reader.

3.3 Properties of the four-component spinor wave functions

The results of Sections 2.1 and 2.3 can easily be translated into four-component fermion notation using the results of Section 3.1. For example, the free-field Lagrangian for a massive (neutral) Majorana four-component spinor field $\Psi_M(x)$ is given by eq. (3.58), and yields the field equations given by eq. (3.59). As a result, $\Psi_M(x)$ can be expanded in a Fourier series:[10]

$$\Psi_M(x) = \sum_\lambda \int \frac{d^3\boldsymbol{p}}{(2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}} \times \left[u(\vec{\boldsymbol{p}},\lambda)a(\vec{\boldsymbol{p}},\lambda)e^{-ip\cdot x} + v(\vec{\boldsymbol{p}},\lambda)a^\dagger(\vec{\boldsymbol{p}},\lambda)e^{ip\cdot x}\right], \tag{3.64}$$

[10] One can also obtain the results of this section by considering the free-field Lagrangian for a massive Dirac four-component spinor field $\Psi(x)$ [see eq. (3.60)]. By virtue of the field equations, the Fourier mode expansion of $\Psi(x)$ is given by eq. (3.64), with $a^\dagger(\vec{\boldsymbol{p}},\lambda)$ replaced by $b^\dagger(\vec{\boldsymbol{p}},\lambda)$ [*c.f.* eqs. (2.49) and (2.50)].

where as in Section 2.1 $E_{\boldsymbol{p}} \equiv (|\vec{\boldsymbol{p}}|^2 + m^2)^{1/2}$, and the creation and annihilation operators $a^\dagger$ and a satisfy anticommutation relations given by eq. (2.2). We again emphasize that the spinor wave functions u and v are *commuting* quantities.

The results of Sections 2.1 and 2.2 can then be used to relate the two-component spinor wave functions $x_\alpha(\vec{\boldsymbol{p}}, \lambda)$ and $y_\alpha(\vec{\boldsymbol{p}}, \lambda)$ to the more traditional four-component spinor wave functions $u(\vec{\boldsymbol{p}})$ and $v(\vec{\boldsymbol{p}})$. Specifically,

$$u(\vec{\boldsymbol{p}}, s) = \begin{pmatrix} x_\alpha(\vec{p}, s) \\ \bar{y}^{\dot{\alpha}}(\vec{\boldsymbol{p}}, s) \end{pmatrix}, \qquad \bar{u}(\vec{\boldsymbol{p}}, s) = \left(y^\alpha(\vec{\boldsymbol{p}}, s),\ \bar{x}_{\dot{\alpha}}(\vec{\boldsymbol{p}}, s)\right), \tag{3.65}$$

$$v(\vec{\boldsymbol{p}}, s) = \begin{pmatrix} y_\alpha(\vec{\boldsymbol{p}}, s) \\ \bar{x}^{\dot{\alpha}}(\vec{\boldsymbol{p}}, s) \end{pmatrix}, \qquad \bar{v}(\vec{\boldsymbol{p}}, s) = \left(x^\alpha(\vec{\boldsymbol{p}}, s),\ \bar{y}_{\dot{\alpha}}(\vec{\boldsymbol{p}}, s)\right), \tag{3.66}$$

where $v(\vec{\boldsymbol{p}}, s) = C\bar{u}(\vec{\boldsymbol{p}}, s)^T$ and C is the charge conjugation matrix [see eq. (3.18)]. Likewise, $u(\vec{\boldsymbol{p}}, s) = C\bar{v}(\vec{\boldsymbol{p}}, s)^T$ One can check that u and v satisfy the Dirac equations[11]

$$(\not{p} - m)\, u(\vec{\boldsymbol{p}}, s) = (\not{p} + m)\, v(\vec{\boldsymbol{p}}, s) = 0\,, \tag{3.67}$$

$$\bar{u}(\vec{\boldsymbol{p}}, s)\, (\not{p} - m) = \bar{v}(\vec{\boldsymbol{p}}, s)\, (\not{p} + m) = 0\,, \tag{3.68}$$

corresponding to eqs. (2.7)–(2.10), and

$$(\gamma_5 \not{s} - 1)\, u(\vec{\boldsymbol{p}}, s) = (\gamma_5 \not{s} - 1)\, v(\vec{\boldsymbol{p}}, s) = 0\,, \tag{3.69}$$

$$\bar{u}(\vec{\boldsymbol{p}}, s)\, (\gamma_5 \not{s} - 1) = \bar{v}(\vec{\boldsymbol{p}}, s)\, (\gamma_5 \not{s} - 1) = 0\,, \tag{3.70}$$

corresponding to eqs. (2.20)–(2.23). For massive fermions, eqs. (2.31)–(2.34) correspond to

$$u(\vec{\boldsymbol{p}}, s)\bar{u}(\vec{\boldsymbol{p}}, s) = \tfrac{1}{2}(1 + \gamma_5 \not{s})\, (\not{p} + m)\,, \tag{3.71}$$

$$v(\vec{\boldsymbol{p}}, s)\bar{v}(\vec{\boldsymbol{p}}, s) = \tfrac{1}{2}(1 + \gamma_5 \not{s})\, (\not{p} - m)\,. \tag{3.72}$$

To apply the above formulas to the massless case, recall that in the $m \to 0$ limit, $s = 2\lambda p/m + \mathcal{O}(m/E)$. Inserting this result in eqs. (3.67) and (3.69), it follows that the massless helicity spinors are eigenstates of γ_5

$$\gamma_5 u(\vec{\boldsymbol{p}}, s) = 2\lambda u(\vec{\boldsymbol{p}}, s)\,, \tag{3.73}$$

$$\gamma_5 v(\vec{\boldsymbol{p}}, s) = -2\lambda v(\vec{\boldsymbol{p}}, s)\,. \tag{3.74}$$

Applying the same limiting procedure to eqs. (3.71) and (3.72) and using the mass-shell condition ($\not{p}\not{p} = p^2 = m^2$), one obtains the helicity

[11] We use the standard Feynman slash notation: $\not{p} \equiv \gamma_\mu p^\mu$.

projection operators for a massless spin-1/2 particle

$$u(\vec{\boldsymbol{p}}, s)\bar{u}(\vec{\boldsymbol{p}}, s) = \tfrac{1}{2}(1 + 2\lambda\gamma_5)\,\not{p}\,, \tag{3.75}$$

$$v(\vec{\boldsymbol{p}}, s)\bar{v}(\vec{\boldsymbol{p}}, s) = \tfrac{1}{2}(1 - 2\lambda\gamma_5)\,\not{p}\,, \tag{3.76}$$

which correspond to eqs. (2.39)–(2.42). Finally, the spin-sum identities

$$\sum_s u(\vec{\boldsymbol{p}}, s)\bar{u}(\vec{\boldsymbol{p}}, s) = \not{p} + m\,, \tag{3.77}$$

$$\sum_s v(\vec{\boldsymbol{p}}, s)\bar{v}(\vec{\boldsymbol{p}}, s) = \not{p} - m\,, \tag{3.78}$$

$$\sum_s u(\vec{\boldsymbol{p}}, s)v^T(\vec{\boldsymbol{p}}, s) = (\not{p} + m)C^T\,, \tag{3.79}$$

$$\sum_s \bar{u}^T(\vec{\boldsymbol{p}}, s)\bar{v}(\vec{\boldsymbol{p}}, s) = C^{-1}(\not{p} - m)\,, \tag{3.80}$$

$$\sum_s \bar{v}^T(\vec{\boldsymbol{p}}, s)\bar{u}(\vec{\boldsymbol{p}}, s) = C^{-1}(\not{p} + m)\,, \tag{3.81}$$

$$\sum_s v(\vec{\boldsymbol{p}}, s)u^T(\vec{\boldsymbol{p}}, s) = (\not{p} - m)C^T\,, \tag{3.82}$$

correspond to eqs. (2.43)–(2.46).

Finally, we note the following useful properties satisfied by the four-component spinor wave functions. Starting from eqs. (3.65) and (3.66) and applying the identities eqs. (B.42)–(B.49) for commuting spinor wave functions, one easily verifies that:

$$\begin{aligned}
\bar{v}(\vec{\boldsymbol{p}}, s)\Gamma v(\vec{\boldsymbol{p}}', s') &= -\eta_\Gamma \bar{u}(\vec{\boldsymbol{p}}', s')\Gamma u(\vec{\boldsymbol{p}}, s)\,,\\
\bar{v}(\vec{\boldsymbol{p}}, s)\Gamma u(\vec{\boldsymbol{p}}', s') &= -\eta_\Gamma \bar{v}(\vec{\boldsymbol{p}}', s')\Gamma u(\vec{\boldsymbol{p}}, s)\,,\\
\bar{u}(\vec{\boldsymbol{p}}, s)\Gamma v(\vec{\boldsymbol{p}}', s') &= -\eta_\Gamma \bar{u}(\vec{\boldsymbol{p}}', s')\Gamma v(\vec{\boldsymbol{p}}, s)\,,
\end{aligned} \tag{3.83}$$

where $\eta_\Gamma = +1$ for $\Gamma = 1, \gamma_5, \gamma^\mu\gamma_5$ and $\eta_\Gamma = -1$ for $\Gamma = \gamma^\mu, \Sigma^{\mu\nu}$, $\Sigma^{\mu\nu}\gamma_5$.

3.4 Feynman rules for four-component Majorana fermions

Starting with the two-component fermion Feynman rules developed in Sections 2.4—2.7, it is strightforward to generate the corresponding set of rules for four-component fermions. The resulting rules for Dirac fermions simply reproduce the well-known Feynman rules developed in most quantum field theory textbooks. However, one also can develop Feynman rules for four-component (neutral) Majorana fermions, which are much less known. It is rather odd that these rules are not more common in the standard textbooks. After all, there are no significant complications that arise when comparing Feynman rules for charged and neutral scalars. A possible explanation is that Feynman rules for (charged) Dirac fermions were slightly simplified by making use of the fermion-number conservation that is connected to the underlying charge conservation. In this section, we shall see that a minor adjustment of the Feynamn rules for Dirac fermions will allow us to treat neutral Majorana fermions on an equal footing.

We begin by considering a set of neutral and charged fermions interactiong with a neutral scalar or vector boson. Starting from the two-component fermion Lagrangian given by eq. (2.66), we combine the χ_i fields into four-component Majorana fields Ψ_{Mi} and the oppositely charged ξ_j and η_j fields into four-component Dirac fields Ψ_j. Converting eq. (2.66) to four-component notation yields[12]

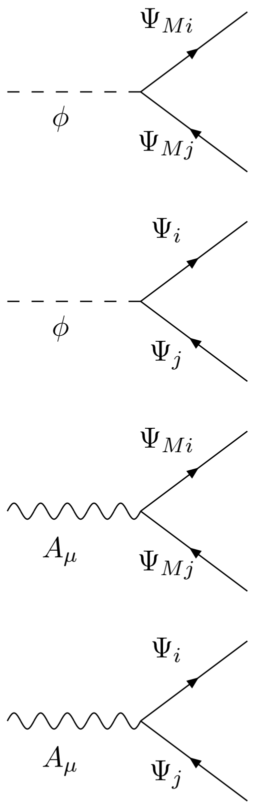
$$\begin{aligned}\mathcal{L}_{\rm int} = &-\tfrac{1}{2}(\lambda^{ij}\overline{\Psi}_{Mi}P_L\Psi_{Mj} + \lambda_{ij}\overline{\Psi}_{Mi}P_R\Psi_{Mj})\phi \\ &-(\kappa^{ji}\overline{\Psi}_i P_L\Psi_j + \kappa_{ij}\overline{\Psi}_i P_R\Psi_j)\phi \\ &-\left[(G_\xi)_i{}^j\overline{\Psi}_{Mi}\gamma^\mu P_L\Psi_{Mj} + (G_\eta)_i{}^j\overline{\Psi}_i\gamma^\mu P_L\Psi_j - (G_\chi)_j{}^i\overline{\Psi}_i\gamma^\mu P_R\Psi_j\right]A_\mu\,,\end{aligned} \tag{3.84}$$

where λ is a complex symmetric matrix, κ is an arbitrary complex matrix and G_chi, G_ξ and G_η are hermitian matrices. It is convenient to use eq. (3.42) to rewrite the term proportional to $(G_\chi)_i{}^j$ in eq. (3.84) as follows

$$(G_\chi)_i{}^j\overline{\Psi}_{Mi}\gamma^\mu P_L\Psi_{Mj} = \tfrac{1}{2}\overline{\Psi}_{Mi}\gamma^\mu\left[(G_\chi)_i{}^jP_L - (G_\chi)_j{}^iP_R\right]\Psi_{Mj}\,. \tag{3.85}$$

Using standard four-component methods, the Feynman rules for the vertices are easily obtained and displayed in Fig. 3.1. Note that the arrows on the Dirac fermion lines depict the flow of the conserved charge. A Majorana fermion is self-conjugate, so its arrow simply reflects the structure of the interaction Lagrangian; *i.e.*, $\overline{\Psi}_M$ $[\Psi_M]$ is represented by an arrow pointing out of [into] the vertex. The arrow directions

[12] The convention for flavor indices [see Section A.2] in which raising all lowered indices (or vice versa) is equivalent to complex conjugation is less useful in this context, since a four-component fermion field is made up of an unbarred two-component fermion field with a lowered flavor index and a barred two-component fermion field with a raised flavor index. Henceforth, we shall (arbitrarily) take all flavor indices attached to four-component fermion fields as lowered indices. Nevertheless, we continue to distinguish λ_{ij} and $\lambda^{ij} \equiv \lambda^*_{ij}$, *etc.* However, in this case one must expand the rule for the summation over repeated indices by performing the sums irrespective of whether an index is in the lowered or raised position.



$$-i(\lambda^{ij} P_L + \lambda_{ij} P_R)$$

$$-i(\kappa^{ji} P_L + \kappa_{ij} P_R)$$

$$-i\gamma_\mu[(G_\chi)_i{}^j P_L - (G_\chi)_j{}^i P_R]$$

$$-i\gamma_\mu[(G_\xi)_i{}^j P_L - (G_\eta)_j{}^i P_R]$$

Fig. 3.1. Feynman rules for four-component fermion interactions with neutral scalar and vector bosons.

determine the placement of the u and v spinors in an invariant amplitude (as specified at the end of this section).

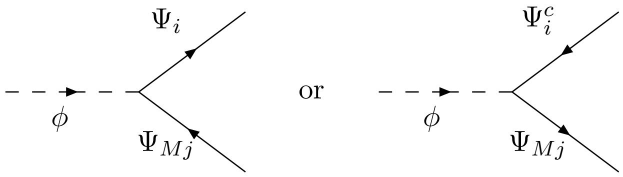
One can also treat the interaction of fermions with charged bosons. We may then rewrite the interaction Lagrangian given in eq. (2.67) in four-component notation:

$$\begin{aligned}\mathcal{L}_{\text{int}} = &-\tfrac{1}{2}\left[(\kappa_2)^{ij}\overline{\Psi}_i P_L \Psi_{Mj} + (\kappa_1)_{ij}\overline{\Psi}_i P_R \Psi_{Mj}\right]\phi \\ &- \left[(G_1)_i{}^j\overline{\Psi}_i\gamma^\mu P_L\Psi_{Mj} - (G_2)_j{}^i\overline{\Psi}_i\gamma^\mu P_R\Psi_{Mj}\right]W_\mu + \text{h.c.}\end{aligned} \tag{3.86}$$

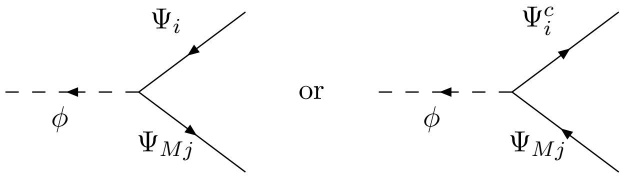
There is an equivalent form of eq. (3.86) where the interaction Lagrangian is written in terms of charge-conjugated fields. In general,[13]

$$\overline{\Psi}{}^c_i\Gamma\Psi^c_j = \overline{\Psi}_j C\Gamma^T C^{-1}\Psi_i = \eta_\Gamma\overline{\Psi}_j\Gamma\Psi_i\,, \tag{3.87}$$

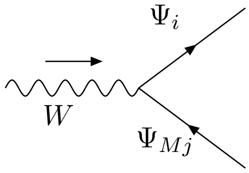
[13] In deriving eq. (3.87), we used $C^T = -C$ and noted that the fermion fields anticommute.



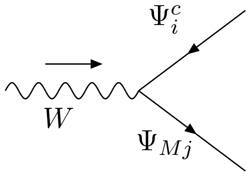
$$-i(\kappa^{2ij}P_L + \kappa_{1ij}P_R)$$



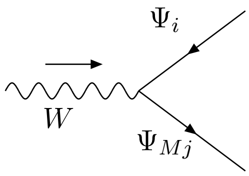
$$-i(\kappa^{1ij}P_L + \kappa_{2ij}P_R)$$



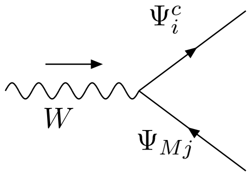
$$-i\gamma^\mu({G_{1i}}^j P_L - {G_{2j}}^i P_R)$$



$$i\gamma^\mu({G_{1i}}^j P_R - {G_{2j}}^i P_L)$$



$$-i\gamma^\mu({G_{1j}}^i P_L - {G_{2i}}^j P_R)$$



$$i\gamma^\mu({G_{1i}}^j P_R - {G_{2j}}^i P_L)$$

Fig. 3.2. Feynman rules for four-component fermion interactions with charged scalar and vector bosons. The arrows on the boson and Dirac fermion lines indicate the direction of charge flow.

where the sign $\eta_\Gamma = +1$ for $\Gamma = 1, \gamma_5, \gamma^\mu\gamma_5$ and $\eta_\Gamma = -1$ for $\Gamma = \gamma^\mu, \Sigma^{\mu\nu}$, $\Sigma^{\mu\nu}\gamma_5$. Noting that Majorana fermions are self-conjugate, the Feynman rules for the interactions of neutral and charged fermions with charged bosons can take two possible forms, as shown in Fig. 3.2. One is free to choose either a Ψ or Ψ^c line to represent a Dirac fermion at any place in a given Feynman graph. The direction of the arrow on the Ψ or Ψ^c line indicates the corresponding direction of charge flow.[14] Moreover, the structure of the interaction Lagrangian implies that the arrow directions on fermion lines flow continuously through the diagram. This requirement then determines the direction of the arrows on Majorana fermion lines.

For a given process, there may be a number of distinct choices for the arrow directions on the Majorana fermion lines, which may depend on whether one represents a given Dirac fermion by Ψ or Ψ^c. However, different choices do *not* lead to independent Feynman diagrams.[15] When computing an invariant amplitude, one first writes down the relevant Feynman diagrams with no arrows on any Majorana fermion line. The number of distinct graphs contributing to the process is then determined. Finally, one makes some choice for how to distribute the arrows on the Majorana fermion lines and how to label Dirac fermion lines (either as the field or its conjuagate) in a manner consistent with the rules of Figs. 3.1 and 3.2. The end result for the invariant amplitude (apart from an overall unobservable phase) does not depend on the choices made for the direction of the fermion arrows.

Using the above procedure, the Feynman rules for the external fermion wave functions are the same for Dirac and Majorana fermions:

- $u(\vec{\boldsymbol{p}}, s)$: incoming Ψ [or Ψ^c] with momentum $\vec{\boldsymbol{p}}$ parallel to the arrow direction,
- $\bar{u}(\vec{\boldsymbol{p}}, s)$: outgoing Ψ [or Ψ^c] with momentum $\vec{\boldsymbol{p}}$ parallel to the arrow direction,
- $v(\vec{\boldsymbol{p}}, s)$: outgoing Ψ [or Ψ^c] with momentum $\vec{\boldsymbol{p}}$ anti-parallel to the arrow direction,
- $\bar{v}(\vec{\boldsymbol{p}}, s)$: incoming Ψ [or Ψ^c] with momentum $\vec{\boldsymbol{p}}$ anti-parallel to the arrow direction.

The proof that the above rules for external wave functions apply unambiguously to Majorana fermions is straightforward. Simply insert

[14] Since the charge of Ψ^c is opposite to that of Ψ, the corresponding arrow direction of the two lines point in opposite directions.

[15] In contrast, the two-component Feynman rules developed in Sections 2.4—2.7 require that two vertices differing by the direction of the arrows on the two-component fermion lines must both be included in the calculation of the matrix element.

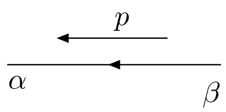
the plane wave expansion of the Majorana field [eq. (3.64)] into eq. (3.84), and evaluate matrix elements for, *e.g.*, the decay of a scalar or vector particle into a pair of Majorana fermions.

Finally, we note that either Ψ or Ψ^c can be used to represent the propagation of a (virtual) Dirac fermion. Here, there is no ambiguity in the propagator Feynman rule, since free Dirac fermion fields satisfy

$$\left\langle 0|T(\Psi_\alpha(x)\overline{\Psi}_\beta(y))|0\right\rangle = \left\langle 0|T(\Psi^c_\alpha(x)\overline{\Psi}^c_\beta(y))|0\right\rangle . \tag{3.88}$$

As a result, the Feynman rules for the propagator of a Ψ and Ψ^c line are identical.

The Feynman rule for a four-component Majorana or Dirac fermion takes the following form:



$$\frac{i(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i\epsilon}$$

Fig. 3.3. The Feynman rule for the virtual propagation of either a Majorana or Dirac fermion. The four-component spinor labels α and β are explicitly indicated.

We reiterate that the directions of the arrows that appear in both internal and external fermion lines flow continuously through any Feynman diagram, if the procedures outline above are followed. Then, one may determine the contribution to an invariant amplitude from a given Feynman diagram by traversing the diagram in a direction *opposite* to the arrow directions. By this procedure, all 4×4 spinor matrix quantities appear in their natural order, and one may drop the explicit four-component spinor labels.

3.5 Simple examples of Feynman diagrams revisited

We now reconsider the matrix elements for scalar and vector particle decays into fermion pairs and $2 \to 2$ elastic scattering of a fermion off a scalar and vector boson, respectively. We shall compute the matrix elements using the Feynman rules of fig. 3.1, and check that the results agree with the ones obtained by two-component methods of Sections 2.4—2.7.

The matrix element for the decay $\phi \to \Psi_M(\vec{\boldsymbol{p}}_1, s_1)\Psi_M(\vec{\boldsymbol{p}}_2, s_2)$ is given by

$$i\mathcal{M} = -i\bar{u}(\vec{\boldsymbol{p}}_1, s_1)(\lambda P_L + \lambda^* P_R)v(\vec{\boldsymbol{p}}_2, s_2)\,. \tag{3.89}$$

One can easily check that this result matches with eq. (2.72), which was derived using two-component techniques. Note that if one had chosen to switch the two final states (equivalent to switching the directions of the Majorana fermion arrows), then the resulting matrix element would simply exhibit an overall sign change [due to the results of eq. (3.83)].[16] Similarly, for $\phi \to \Psi_{Mi}\Psi_{Mj}$ $(i \neq j)$ or for the decay into a pair of Dirac fermions, $\phi \to \overline{\Psi}\Psi$, one again obtains the invariant matrix element given in eq. (3.89).

For the decay $A_\mu \to \Psi_M(\vec{\boldsymbol{p}}_1, s_1)\Psi_M(\vec{\boldsymbol{p}}_2, s_2)$, one obtains:

$$i\mathcal{M} = iG_\xi \bar{u}(\vec{\boldsymbol{p}}_1, s_1)\gamma^\mu\gamma_5 v(\vec{\boldsymbol{p}}_2, s_2)\varepsilon_\mu \,. \qquad (3.90)$$

One can easily check that this result matches with eq. (2.75). For the decay into non-identical Majorana fermions, $A_\mu \to \Psi_{Mi}\Psi_{Mj}$ $(i \neq j)$, we can use the Feynman rules of Fig. 3.1 to obtain:

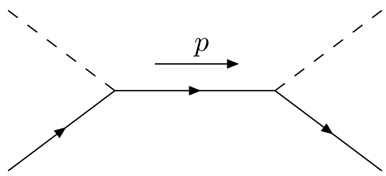
$$i\mathcal{M} = -i\bar{u}(\vec{\boldsymbol{p}}_i, s_i)\gamma^\mu \left[(G_\xi)_i{}^j P_L - (G_\xi)_j{}^i P_R\right] v(\vec{\boldsymbol{p}}_j, s_j)\varepsilon_\mu \,, \qquad (3.91)$$

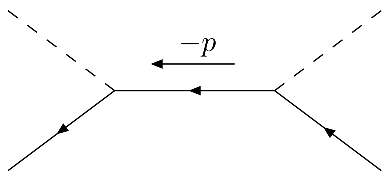
Again, we note that if one had chosen to switch the two final states (equivalent to switching the directions of the Majorana fermion arrows), then the resulting matrix element would simply exhibit an overall sign change [due to the results of eq. (3.83)]. Finally, for the decay of the vector particle into a Dirac fermion-antifermion pair, $A_\mu \to \overline{\Psi}\Psi$, the matrix element is given by:

$$i\mathcal{M} = -i\bar{u}(\vec{\boldsymbol{p}}_1, s_1)\gamma^\mu(G_\chi P_L - G_\eta P_R)v(\vec{\boldsymbol{p}}_2, s_2)\varepsilon_\mu \,, \qquad (3.92)$$

which matches the result of eq. (2.79).

Turning to the elastic scattering of a neutral Majorana fermion and a neutral scalar, we shall examine two equivalent ways for computing the amplitude. Following the rules previously stated, there are two possible choices for the direction of arrows on the Majorana fermion lines. Thus, one may evaluate either one of the following two diagrams:





plus a second diagram in each case (not shown) where the initial and final state scalars are crossed. The contribution of the first diagram above to

[16] The overall sign change is a consequence of the Fermi-Dirac statistics, and corresponds to changing which order one uses to construct the two particle final state.

the matrix element for $\phi\Psi_M \to \phi\Psi_M$ is given by:

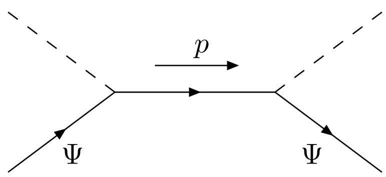
$$\frac{-i}{s-m^2}\,\bar{u}(\vec{\boldsymbol{p}}_2, s_2)(\lambda P_L + \lambda^* P_R)(\not{p} + m)(\lambda P_L + \lambda^* P_R)u(\vec{\boldsymbol{p}}_1, s_1)$$
$$= \frac{-i}{s-m^2}\,\bar{u}(\vec{\boldsymbol{p}}_2, s_2)\left[|\lambda|^2\not{p} + \left(\lambda^2 P_L + (\lambda^*)^2 P_R\right) m\right] u(\vec{\boldsymbol{p}}_1, s_1)\,, \quad (3.93)$$

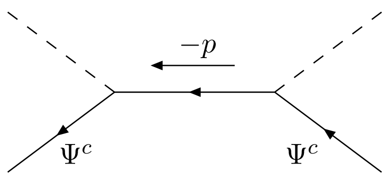
where m is the Majorana fermion mass, s is the centre-of-mass energy squared. Using eqs. (3.7) and (3.65), one recovers the results of eq. (2.80). Had we chosen to evaluate the second diagram instead, the resulting contribution to the invariant amplitude would have been given by:

$$\frac{-i}{s-m^2}\,\bar{v}(\vec{\boldsymbol{p}}_1, s_1)\left[-|\lambda|^2\not{p} + \left(\lambda^2 P_L + (\lambda^*)^2 P_R\right) m\right] v(\vec{\boldsymbol{p}}_2, s_2)\,. \quad (3.94)$$

Using eq. (3.83), one can quickly verify that the amplitude computed in eq. (3.94) is just the negative of eq. (3.93). This is expected, since the order of spinor wave functions (12) in eq. (3.94) is an odd permuation (21) of the order of spinor wave functions in eq. (3.93). As in the two-component Feynman rules, the overall sign of the amplitude is arbitrary, but the relative signs of either of the diagrams above and the corresponding crossed diagram is not ambiguous. This relative sign is given by the relative sign of the permuation of spinor wave functions appearing in the contributions to the invariant amplitude.

Next, we consider the elastic scattering of a charged fermion and a neutral scalar. Again, we examine two equivalent ways for computing the amplitude. Following the rules previously stated, there are two possible choices for the direction of arrows on the fermion lines, depending on whether we represent the fermion by Ψ or Ψ^c. Thus, we may evaluate either one of the following two diagrams:

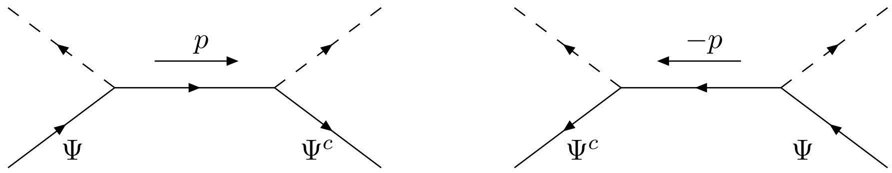




plus a second diagram in each case (not shown) where the initial and final state scalars are crossed. Evaluating the first diagram above, the matrix element for $\phi\Psi \to \phi\Psi$ is given by eq. (3.93), with λ replaced by κ. Had we chosen to evaluate the second diagram instead, the resulting amplitude would have been given by eq. (3.94), with λ replaced by κ. Thus, the discussion above in the case of neutral fermion scattering processes also applies to charged fermion scattering processes.

In processes that only involve vertices with two Dirac fields, one can always choose to avoid employing charge-conjugated Dirac fermion lines.

This may no longer hold true for processes that involve a vertex with one Dirac and one Majorana fermion. For example, consider the scattering of a (charged) Dirac fermion and a charged scalar via the exchange of a neutral Majorana fermion, in which the charge of the outgoing fermion is opposite to that of the incoming fermion. If one attempts to draw the relevant Feynman diagram with no charge-conjuagted Dirac fermion lines, one finds that there is no possible choice of arrow directions for the Majorana fermion that is consistent with the the vertex rules of Fig. 3.2. The resolution is simple: one can choose the incoming line to be Ψ and the outgoing line to be Ψ^c or vice versa. Thus, the two possible choices are given by:

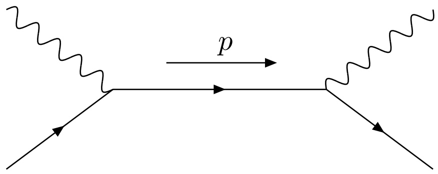


plus a second diagram in each case (not shown) in which the intial and final scalars are crossed. The contribution of the first diagram above to the matrix element for $\phi^-\Psi \to \phi^+\Psi^c$ is given by:

$$\begin{aligned} &\frac{-i}{s-m^2}\,\bar{u}(\vec{\boldsymbol{p}}_2, s_2)(\kappa_1 P_L + \kappa_2^* P_R)(\not{p} + m)(\kappa_1 P_L + \kappa_2^* P_R)u(\vec{\boldsymbol{p}}_1, s_1) \\ &= \frac{-i}{s-m^2}\,\bar{u}(\vec{\boldsymbol{p}}_2, s_2)\left[\kappa_1\kappa_2^*\not{p} + \left(\kappa_1^2 P_L + (\kappa_2^*)^2 P_R\right) m\right] u(\vec{\boldsymbol{p}}_1, s_1)\,, \qquad (3.95) \end{aligned}$$

where m is the mass of the s-channel exchanged Majorana fermion. One can check that this is equivalent to eq. (2.84) obtained via the two-component methods. Had we evaluated the second diagram above, then after using eq. (3.83), one finds that the resulting invariant amplitude is just the negative of eq. (3.95), as expected. As before, the relative sign between either diagram and its crossed version is not ambiguous.

In the case of elastic scattering of a fermion and a neutral vector boson, the two contributing diagrams are



plus a second diagram (not shown) where the initial and final state vector bosons are crossed. Consider first the scattering of a neutral Majorana fermion of mass m. Using the Feynman rules of Fig. 3.1 and noting that G_ξ is real (here, the flavor index runs over only one component), we see that the Feynman rule for the $A_\mu\overline{\Psi}_M\Psi_M$ vertex is given by $iG_\xi\gamma^\mu\gamma_5$. Hence, the corresponding matrix element is given by

$$i\mathcal{M} = \frac{-iG_\xi^2}{s-m^2}\bar{u}(\vec{\boldsymbol{p}}_2,s_2)\,\gamma\!\cdot\!\varepsilon_2^*\,(\not{p}-m)\,\gamma\!\cdot\!\varepsilon_1 u(\vec{\boldsymbol{p}}_1,s_1) + (\text{crossed}), \qquad (3.96)$$

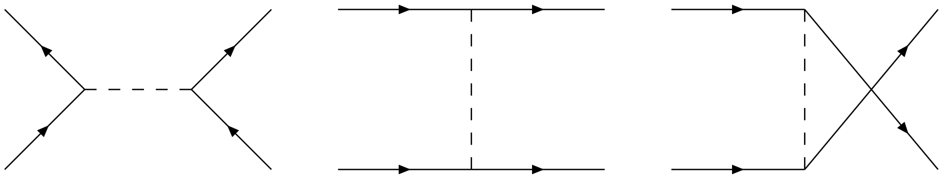
where we have used $\gamma^\nu\gamma_5(\not{p}+m)\gamma^\mu\gamma_5 = \gamma^\nu(\not{p}-m)\gamma^\mu$. Using eqs. (3.7) and (3.65), one easily recovers the results of eq. (2.81).

Next, consider the elastic scattering of a Dirac fermion and a neutral vector boson (Compton scattering). The corresponding matrix element is given by

$$\begin{aligned} i\mathcal{M} &= \frac{-i}{s-m^2}\bar{u}(\vec{\boldsymbol{p}}_2,s_2)\,\gamma\!\cdot\!\varepsilon_2^*\,(G_\chi P_L - G_\eta P_R)(\not{p}+m)\,\gamma\!\cdot\!\varepsilon_1 \\ &\qquad\qquad \times(G_\chi P_L - G_\eta P_R)u(\vec{\boldsymbol{p}}_1,s_1) + (\text{crossed}) \\ &= \frac{-i}{s-m^2}\bar{u}(\vec{\boldsymbol{p}}_2,s_2)\,\gamma\!\cdot\!\varepsilon_2^*\left[(G_\chi^2 P_L + G_\eta^2 P_R)\not{p} - G_\chi G_\eta m\right]\,\gamma\!\cdot\!\varepsilon_1 u(\vec{\boldsymbol{p}}_1,s_1) \\ &\qquad\qquad +(\text{crossed})\,. \end{aligned} \qquad (3.97)$$

One can easily check that this result coincides with that of eq. (2.85).

Finally, we examine the elastic scattering of two identical Majorana fermions via scalar exchange. The three contributing diagrams are:



and the corresponding matrix element is given by

$$\begin{aligned} i\mathcal{M} &= \frac{-i}{s-m_\phi^2}\left[\bar{v}_1(\lambda P_L + \lambda^* P_R)u_2\,\bar{u}_3(\lambda P_L + \lambda^* P_R)v_4\right] \\ &\quad + (-1)\frac{-i}{t-m_\phi^2}\left[\bar{u}_3(\lambda P_L + \lambda^* P_R)u_1\,\bar{u}_4(\lambda P_L + \lambda^* P_R)u_2\right] \\ &\quad + \frac{-i}{u-m_\phi^2}\left[\bar{u}_4(\lambda P_L + \lambda^* P_R)u_1\,\bar{u}_3(\lambda P_L + \lambda^* P_R)u_2\right]\,, \end{aligned} \qquad (3.98)$$

where $u_i \equiv u(\vec{\boldsymbol{p}}_i,s_i)$, $v_j \equiv u(\vec{\boldsymbol{p}}_j,s_j)$ and m_ϕ is the exchanged scalar mass. The relative minus sign of the t-channel graph relative to the other two is obtained by noting that 3142 [4132] is an odd [even] permutation of 1234. Using eqs. (3.7) and (3.65), one easily recovers the results of eq. (2.86).

Problems

1. (a) Translate the results of problem 3(a) of Chapter 1 to four-component notation. Show that the resulting condition for P takes the form:

$$S_P^{-1}\gamma^\mu S_P = (\Lambda_P)^\mu{}_\nu\gamma^\nu , \tag{3.99}$$

where

$$S_P \equiv \begin{pmatrix} 0 & P \\ \overline{P} & 0 \end{pmatrix} . \tag{3.100}$$

Verify eq. (3.99) directly in the four-component language by noting that $\Psi^p(x) = iS_P\Psi^(x)$, where $S_P \equiv \gamma^0$ [see eq. (3.15)].

(b) Translate the results of problem 4(a) of Chapter 1 to four-component notation. Show that the resulting condition for T takes the form:

$$S_T^{-1}\gamma^\mu S_T = -(\Lambda_T)^\mu{}_\nu\gamma^{\nu\,*} , \tag{3.101}$$

where

$$S_T \equiv \begin{pmatrix} T & 0 \\ 0 & -\overline{T} \end{pmatrix} . \tag{3.102}$$

Verify eq. (3.101) directly in the four-component language by noting that $\Psi^t(x) = S_T\Psi^*(x)$, where $S_T \equiv -\gamma^0 B^{-1}A^T$ [see eq. (3.16)].

2. (a) The Racah time-reversed spinor is defined by $\Psi'(x') = [\Psi^c(x)]^t$, where $x'^\mu = (\Lambda_T)^\mu{}_\nu x^\nu$. Without assuming the chiral representation, show that $[\Psi^c(x)]^t = S_{CT}\Psi(x)$, where $S_{CT} = \gamma^0\gamma_5$. Then, verify that:

$$S_{CT}^{-1}\gamma^\mu S_{CT} = (\Lambda_T)^\mu{}_\nu\gamma^\nu . \tag{3.103}$$

This is the direct analogue of the parity transformation given in eq. (3.99).

(b) What is the two-component version of Racah time reflection? Modify the results of problem 3(b) accordingly.

3. (a) Explain what is wrong with the following computation:

$$\begin{aligned}\mathcal{CPT}\Psi(x)(\mathcal{CPT})^{-1} &= \mathcal{CP}[\eta_T(\gamma^0A^{-1})^TB\,Psi(x_T)](\mathcal{CP})^{-1} \\ &= \mathcal{C}[i\eta_P\eta_T\gamma^0(\gamma^0A^{-1})^TB\Psi(-x)]\mathcal{C}^{-1} \\ &= \eta_C CA^T[i\eta_P\eta_T\gamma^0(\gamma^0A^{-1})^TB\Psi(-x)]^* , \\ &= -i\eta_C\eta_P^*\eta_T^*CA^T\gamma^{0*}(\gamma^0A^{-1})^\dagger B^*\Psi^*(-x) ,\end{aligned}$$

where Ψ is a four-component Dirac field. In particular, by writing out the various matrices in the chiral representation and keeping the two-component spinor labels explicit, show that the last line above does not make any sense. Compare with eq. (3.53) in the text and explain why the latter provides the correct method of computation.

(b) The field Ψ transforms under charge conjugation according to eq. (3.49). Compare the following two computations:

$$\mathcal{C}i\Psi\mathcal{C}^{-1} = (\mathcal{C}\,i\,\mathcal{C}^{-1})(\mathcal{C}\Psi\mathcal{C}^{-1}) = i\eta_C CA^T\Psi^*\,, \tag{3.104}$$

$$\mathcal{C}i\Psi\mathcal{C}^{-1} = \eta_C CA^T(i\Psi)^* = -i\eta_C CA^T\Psi^*\,. \tag{3.105}$$

Which computation is correct? Explain.

4. (a) Work out the transformation laws for $\Psi^t(x)$ and $\overline{\Psi}^t(x)$ under P, T and C, respectively.

(b) Work out the transformation laws for $\Psi(x)$ and $\overline{\Psi}(x)$ under a CP transformation.

5. In Section 1.6, a field theory of n free two-component fermion fields is constructed with an arbitrary mass matrix M^{ij}. Convert the results of this section to four-component spinor notation. Write down the most general model of n four-component free fermion fields and discuss the mass diagonalization procedure.

6. (a) Using the results of problem 4(c), prove the four-component Bouchiat-Michel formulas:

$$u(\vec{\boldsymbol{p}},\lambda')\bar{u}(\vec{\boldsymbol{p}},\lambda) = \tfrac{1}{2}\left[\delta_{\lambda\lambda'} + \gamma_5 \not{s}^a \tau^a_{\lambda\lambda'}\right](\not{p}+m)\,, \tag{3.106}$$

$$v(\vec{\boldsymbol{p}},\lambda')\bar{v}(\vec{\boldsymbol{p}},\lambda) = \tfrac{1}{2}\left[\delta_{\lambda'\lambda} + \gamma_5 \not{s}^a \tau^a_{\lambda'\lambda}\right](\not{p}-m)\,, \tag{3.107}$$

where the s^a are defined byeqs. (2.93)–(2.93) and the τ^a are the usual Pauli matrices.

(b) Take the $m \to 0$ limit in part (a) and derive the relevant Bouchiat-Michel formulas for a massless helicity four-component spinor. [*HINT:* Recall that $s^3 = p/m + \mathcal{O}(m/E)$ as $m \to 0$.]

References

[1] J.M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons*, second expanded edition (Springer-Verlag, New York, 1976), Appendix A2-2; D. Bailin, in ref. [6], chapter 2.

4
Gauge Theories and the Standard Model

In the first chapter, we focused on quantum field theories of free fermions. If we wish to construct renormalizable interacting quantum field theories, one must introduce additional fields. The requirement of renormalizability imposes two constraints. First, the terms of the interaction Lagrangian must be no higher than dimension-four. Thus, no (perturbatively) renormalizable interacting theory that consists only of spin-1/2 fields exists, since the simplest interaction term involving fermions is a dimension-six four-fermion interaction. Renormalizable interacting theories consisting of scalars, fermions and spin-one bosons can be constructed. The vector bosons must either be abelian vector fields or non-abelian gauge fields. This exhausts all possible renormalizable field theories.

The Standard Model is a spontaneously broken non-abelian gauge theory containing elementary scalars, fermions and spin-one gauge bosons. Typically, one refers to the spin-0 and spin-1/2 fields (which are either neutral or charged with respect to the underlying gauge group) as *matter fields*, whereas the spin-1 gauge bosons are called *gauge fields*. In this chapter we review the ingredients for constructing non-abelian (Yang-Mills) gauge theories and its breaking via the dynamics of a sector of self-interacting scalar fields. The Standard Model of fundamental particles and interactions is then exhibited, and some of its properties are described.

4.1 Abelian Gauge field theory

The first (and simplest) known gauge theory is quantum electrodynamics (QED). This was a very successful theory that described the interactions of electrons, positrons and photons. The Lagrangian of QED is given by:

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\overline{\xi}\,\overline{\sigma}^{\mu}D_{\mu}\xi + i\overline{\eta}\,\overline{\sigma}^{\mu}D_{\mu}\eta - m(\xi\eta + \overline{\xi}\,\overline{\eta})\,, \tag{4.1}$$

where the electromagnetic field strength tensor $F_{\mu\nu}$ is defined in terms of the gauge field A_μ as

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{4.2}$$

and the covariant derivative D_μ is defined as:

$$D_\mu\psi(x) \equiv (\partial_\mu + ieq_\psi A_\mu)\psi(x)\,, \tag{4.3}$$

where $\psi(x) = \xi(x)$ or $\eta(x)$ with $q_\xi = -1$ and $q_\eta = +1$. Note that we have written eq. (4.1) in terms of the two-component fermion fields ξ and η. The identification of these fields with the electron and positron (with corresponding electric charges in units of $e > 0$ are given by $q_\xi = -1$ and $q_\eta = +1$) has been given in Section 2.9.[1]

The QED Lagrangian consists of a sum of the kinetic energy term for the gauge (photon) field, $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and the Dirac Lagrangian with the ordinary derivative ∂_μ replaced by a covariant derivative D_μ. This Lagrangian is invariant under a local U(1) gauge transformation:

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\Lambda(x)\,, \tag{4.4}$$
$$\xi(x) \to \exp[ie\Lambda(x)]\xi(x)\,, \tag{4.5}$$
$$\eta(x) \to \exp[-ie\Lambda(x)]\eta(x)\,, \tag{4.6}$$

relecting the fact that the U(1)-charges of ξ and η fields are $-e$ and $+e$, respectively. The Feynman rule for electrons interacting with photons is obtained by taking $G_\psi = eq_\psi$ for $\psi = \xi$ and η. That is, we employ $G_\xi = -G_\eta = -e$ in the two-component Feynamn rules displayed in Fig. 2.6. One can easily check that the corresponding four-component Feynman rule of Fig. 3.1 yields the well-known rule for the $e^+e^-\gamma$ vertex of QED.

One can extend the theory above by including charged scalars among the possible matter fields. For example, a scalar field $\Phi(x)$ of definite U(1) charge q_Φ will transform under the local U(1) gauge transformation:

$$\Phi(x) \to \exp[-ieq_\Phi\Lambda(x)]\Phi(x)\,. \tag{4.7}$$

A gauge invariant Lagrangian involving the scalar fields can be obtained from the free-field scalar Lagrangian of eq. (3.3), by replacing $\partial_\mu\Phi(x)$ with

[1] It is a simple matter to rewrite eq. (4.1) in terms of the four-component spinor electron field. Simply replace $\partial_\mu\Psi(x)$ with $D_\mu\Psi(x)$ in the Dirac Lagrangian [eq. (3.60)] with $q_\Psi = -1$ and add the kinetic energy term for the gauge fields.

$D_\mu\Phi(x) \equiv (\partial_\mu + ieq_\Phi)\Phi(x)$. One may also add gauge-invariant Yukawa interactions of the form

$$\mathcal{L}_{\rm Y} = -y_{ijk}\Phi_i\psi_j\psi_k + \text{h.c.}\,, \tag{4.8}$$

where Φ_i consist of either neutral or charged scalar fields and ψ_i consist of of neutral Majorana (χ) or charged Dirac pairs (ξ and η) two-component fermion fields. The (complex) Yukawa couplings y_{ijk} vanish unless the condition $q_{\Phi_i} + q_{\psi_j} + q_{\psi_k} = 0$ is satisfied, as required by gauge invariance.

4.2 Non-abelian gauge groups and their Lie algebras

Abelian gauge field theory can be generalized by replacing the abelian U(1) gauge group of QED with a non-abelian gauge group G. We again consider possible matter fields—multiplets of scalar fields $\Phi_i(x)$ and two-component fermion fields $\psi_i(x)$ [or equivalently, four component fermions $\Psi_i(x)$] that are either neutral or charged with respect to G.

The symmetry group G can be expressed in general as a direct product of simple gauge group factors and some number of U(1) factors. The list of all possible simple Lie groups are known and consist of SU(n), SO(n), Sp(n) and five exceptional groups (G_2, F_4, E_6, E_7 and E_8). Given some matter field (either scalar or fermion), which we generically designate by $\phi_i(x)$, the gauge transformation under which the Lagrangian is invariant, is given by:

$$\phi_i(x) \to U_i{}^j(g)\phi_j(x)\,, \qquad i,j = 1,2,\ldots,d_R\,, \tag{4.9}$$

where g is an element of G (that is, g is a specific gauge transformation) and $U(g)$ is a (possibly reducible) unitary representation of G of dimension d_R. One is always free to redefine the fields via $\phi(x) \to V\phi(x)$, where V is any fixed unitary matrix (independent of the choice of g). The gauge transformation law for the redefined $\phi(x)$ now has $U(g)$ replaced by $V^{-1}U(g)V$. If $U(g)$ is a reducible representation, then it is possible to find a V such that the $U(g)$ for *all* group lements g assume a block diagonal form. Otherwise, the representation $U(g)$ is irredicible. Henceforth, we will assume that matter field representations are broken down into their irreducible pieces. Irreducible representations imply that the corresponding multiplets transform only among themselves, and thus we can focus on these pieces separately without loss of generality.

The *local* gauge transformation $U(g)$ is also a function of space-time position, x^μ. Explicitly,

$$U(g(x)) = \exp[-ig\Lambda^a(x)T^a]\,, \tag{4.10}$$

where there is an implicit sum over the repeated index $a = 1, 2, \ldots, d_G$. The T^a are a set of d_G linearly independent hermitian matrices[2] called generators of the Lie group, and the corresponding $\Lambda^a(x)$ are arbitrary x-dependent functions. The constant g (which is analogous to e of the abelian theory) is called the gauge coupling.

In the next section, we will see that the gauge fields transform under the adjoint representation of the (global) gauge group. If G is semi-simple (that is, a direct product of simple Lie groups), then the adjoint representation is irreducible. The explicit matrix elements of the adjoint representation generators are given by

$$(T^a)_{bc} = -if_{abc} \qquad a, b, c = 1, 2, \ldots, d_G\,. \tag{4.11}$$

Thus, the dimension of the adjoint representation matrices coindices with the number of generators, d_G. Consequently, we shall often refer to the indices a, b, c as adjoint indices.

Lie group theory teaches us that the number of linearly independent generators, d_G, depends only on the abstract definition of G (and not on the choice of representation). Thus, d_G is also called the dimension of the Lie group G. Moreover, the commutator of two generators is a linear combination of generators:

$$[T^a, T^b] = if^{abc}T^c\,, \tag{4.12}$$

where the f^{abc} are called the structure constants of the Lie group. In studying the stucture of gauge field theories, nearly all the information of interest can be ascertained by focusing on infinitesimal gauge transformations. In this case,

$$U(g(x)) \simeq I_{d_R} - ig\Lambda^a(x)T^a\,, \tag{4.13}$$

and the infinitesimal gauge transformation takes the form $\phi_i(x) \to \phi_i(x) + \delta\phi_i(x)$, where

$$\delta\phi_i(x) = -ig\Lambda^a(x)(T^a)_i{}^j\phi_j(x)\,. \tag{4.14}$$

From the generators T^a, one can reconstruct the group elements $U(g(x))$, so it is sufficient to focus on the infinitesimal group transformations. The group generators T^a span a real vector space, whose general element is c^aT^a, where the c^a are real numbers.[3] Given, eq. (4.12), the commutator

[2] The condition of linear independence means that $c^aT^a = 0$ (implicit sum over a) implies that $c^a = 0$ for all a.

[3] With the T^a hermitian, we require the c^a to be real in order that $U(g)$ be unitary. Then, the T^a span a *real* Lie algebra. Mathematicians consider the elements of the real Lie algebra to be ic^aT^a, with anti-hermitian generators iT^a. Note that for real Lie algebras, the representation matrices for T^a (or iT^a) may be complex or quaternionic.

of any two vectors (*e.g.*, $[c^a T^a, d^b T^b]$) is well defined. Thus, one can formally define a "vector product" of any two elements of the vector space as the commutator of the two vectors. Then, this vector space is also an algebra, called a Lie algebra. Henceforth, the Lie algebra corresponding to the Lie group G will be designated by $\mathfrak{g}$. A Lie algebra has one additional important property:

$$\big[T^a, [T^b, T^c]\big] + \big[T^b, [T^c, T^a]\big] + \big[T^c, [T^a, T^b]\big] = 0\,. \tag{4.15}$$

This is called the Jacobi identity, and it is clearly satisfied by any three elements of the Lie algebra.

The choice of basis vectors (or generators) T^a is arbitrary. Moreover, the values of the structure constants f^{abc} also depend on this choice of basis. Nevertheless, there is a canonical choice which we now adopt. The generators are chosen such that:

$$\mathrm{Tr}\;(T^a T^b) = T_R \delta^{ab}\,, \tag{4.16}$$

where T_R depends on the irreducible representation of the T^a. Having chosen this basis, there is no distinction between upper and lower adjoint indices.[4] This basis choice is also convenient since in this basis, the f^{abc} are completely antisymmetric under the interchange of a, b and c.[5] One can show that once T_R is chosen for any one non-trivial irreducible representation R, then the value of T_R for any other irreducible representation is fixed. Corresponding to each simple real (compact) Lie algebra, one can identify one particular irreducible representation, called the *defining* representation (sometimes, but less accurately, called the fundamental representation); the most useful examples are listed in Table 4.1. For the defining (or fundamental) representation (which is indicated by $R = F$), the conventional value for T_R is taken to be:

$$T_F = \tfrac{1}{2}\,. \tag{4.17}$$

As noted above, the basis choice of eq. (4.16) with the normalization convention given by eq. (4.17) fixes the value of T_R for an arbitrary irreducible representation R. The quantity T_R/T_F is called the *index* of the representation R.

For the record, we mention two other properties of Lie algebras that will be useful in this book. First, a Casimir operator is defined to be

[4] More generally, in an arbitrary basis, Tr $(T^a T^b) = T_R g^{ab}$, where g^{ab} is the Cartan-Killing form (which can be used to raise and lower adjoint indices).

[5] By definition, f^{abc} is antisymmetric under the interchange of a and b. But the complete antisymmetry under the interchange of all three indices requires eq. (4.16) to be satisfied.

Table 4.1. Simple real compact Lie algebras, $\mathfrak{g}$, of dimension d_G and rank r_G. Note that $[\frac{1}{2}n]$ indicates the greatest integer less than $\frac{1}{2}n$. The definining representation refers to arbitrary linear combinations ic^aT^a, where the c^a are real and the T^a are the generators of G.

$\mathfrak{g}$	d_G	r_G	defining represnetation
$\mathfrak{so}(n)$	$\frac{1}{2}n(n-1)$	$[\frac{1}{2}n]$	$n \times n$ real antisymmetric
$\mathfrak{su}(n)$	n^2-1	$n-1$	$n \times n$ traceless complex anti-hermtian
$\mathfrak{sp}(n)$	$n(2n+1)$	n	$n \times n$ quaternionic anti-hermitian

an operator that commutes with all the generators T^a of G. Then, there exists a quadratic Casimir operator that is defined by:

$$(T^a)_i{}^k(T^a)_k{}^j = C_R\delta_i{}^j\,. \tag{4.18}$$

Any operator that commutes with all the generators must be a multiple of the identity (by Schur's lemma). The coefficient of $\delta_i{}^j$ depends on the representation R and is denoted by C_R. The following theorem is then easily proven:

$$T_R d_G = C_R d(R)\,, \tag{4.19}$$

where $d(R)$ is the dimension of the representation R. Note that for the adjoint representation ($R = A$), $d(A) = d_G$, so that $C_A = T_A$. As an example, for SU(n), Table 4.1 yields $d_G = n^2 - 1$ and $d(F) = n$. Using eq. (4.17), one the obtains $C_F = (n^2-1)/(2n)$. From an explicit representation of the f^{abc} for SU(n), one can also derive $C_A = T_A = n$.

Second, for any semi-simple Lie algebra $\mathfrak{g}$ (that is, a direct sum of simple Lie algebras), given an irreducible representation of the hermitian generators T^a, one can always find an equivalent representation $V^{-1}T^aV$ for some unitary matrix V. There exists some choice of V (not necessarily unique) that maximizes the number of diagonal generators, $V^{-1}T^aV$. This number, called the *rank* of $\mathfrak{g}$, is independent of the choice of representation, and is a property of the abstract Lie algebra. The ranks of the classical Lie algebras are given in Table 4.1.

4.3 Non-abelian gauge field theory

In order to construct a non-abelian gauge theory, we follow the recipe introducted in the case of abelian gauge theory. Namely, we introduce a gauge field A_μ and a covariant derivative D_μ. By replacing ∂_μ with D_μ in the kinetic energy terms of the matter fields and introducing an appropriate transformation law for A_μ, the resulting matter kinetic energy terms are invariant under local gauge transformations.

As an example, consider a scalar field theory with Lagrangian

$$\mathcal{L} = (\partial_\mu \phi_i^*)(\partial^\mu \phi_i) - V(\phi, \phi^*)\,, \tag{4.20}$$

where the scalar potential V is invariant under gauge transformations, $\phi_i(x) \to U_i{}^j(g)\phi_j(x)$; that is,

$$V(U\phi, (U\phi)^*) = V(\phi, \phi^*)\,. \tag{4.21}$$

Although $\mathcal{L}$ is invariant under global (*i.e.*, x-independent) gauge transformations, the kinetic energy term is not invariant under local gauge transformations due to the presence of the derivative. In particular, under local gauge transformations $\partial_\mu\phi \to \partial_\mu(U\phi) = U\partial_\mu\phi + (\partial_\mu U)\phi$. We therefore introduce the covariant derivative acting on a matter field that transforms according to some representation R of the symmetry group G:

$$(D^\mu)_i{}^j = \delta_I{}^j\partial^\mu + igA_\mu^a(x)(T^a)_i{}^j \qquad a = 1, 2, \ldots, d_G\,, \tag{4.22}$$

where the flavor indices i, $j = 1, 2, \ldots, d(R)$. By making a suitable transformation law for $A_mu^a(x)$, one can arrange $D_\mu\phi$ to transform under a local gauge transformation as

$$D_\mu\phi \to UD_\mu\phi\,, \tag{4.23}$$

in which case,

$$\mathcal{L} = (D_\mu\phi_i)^*(D^\mu\phi_i) - V(\phi, \phi^*) \tag{4.24}$$

is invariant under local gauge transformations.

The transformation law for $A_\mu^a(x)$ is most easily expressed for the matrix-valued gauge field[6]

$$A_mu(x) \equiv A_\mu^a(x)T^a\,. \tag{4.25}$$

Under local gauge transformations, the matrix-valued gauge field transforms as

$$A_\mu \to UA_\mu U^{-1} - \frac{i}{g}U(\partial_\mu U^{-1})\,. \tag{4.26}$$

Using eq. (4.26), one quickly shows that $D_\mu\phi$ transforms as expected:

$$\begin{aligned} D_\mu\phi \equiv (\partial_\mu + igA_\mu)\phi \to & \left[\partial_\mu + igUA_\mu U^{-1} + U(\partial_\mu U^{-1})\right]U\phi \\ &= UD_\mu\phi + \left[\partial_\mu U + U(\partial_\mu U^{-1})U\right]\phi \\ &= UD_\mu\phi\,. \end{aligned} \tag{4.27}$$

[6] The matrix-valued gauge field that one employs in the covariant derivative depends on the representation of the matter fields on which the covariant derivative acts.

In the last step, we noted that

$$\partial_\mu U + U(\partial_\mu U^{-1})U = \left[(\partial_\mu U)U^{-1} + U(\partial_\mu U^{-1})\right] U \tag{4.28}$$
$$= \left[\partial_\mu (UU^{-1})\right] U = 0\,, \tag{4.29}$$

since $UU^{-1} = I$.

It is also useful to exhibit the infinitesimal version of the transformation law, by taking the ininitesimal form for U [eq. (4.13)]. This allows us to directly write down the transformation law for $A^a_\mu(x)$. Namely, $A^a_\mu(x) \to A^a_\mu(x) + \delta A^a_\mu(x)$, where

$$\delta A^a_\mu = g f^{abc} \Lambda^b A^c_\mu + \partial_\mu \Lambda^a \equiv D^{ab}_\mu \Lambda^b\,, \tag{4.30}$$

where

$$D^{ab}_\mu \equiv \delta^{ab}\partial_\mu + g f^{abc} A^c_\mu \tag{4.31}$$

is the covariant derivative operator applied to fields in the adjoint representation (*i,e,*, insert eq. (4.11) for the T^a in eq. (4.22)). In particular, under global gauge transformations, the gauge fields $A^a_\mu(x)$ transform under the adjoint representation of the gauge group.

To complete the construction of the non-abelian gauge field theory, we must introduce a gauge invaraint kinetic energy term for the gauge fields. To motivate the definition of the gauge field strength tensor, we consider $[D_\mu, D_\nu]$ acting on ϕ,

$$[D_\mu, D_\nu]\phi = [\partial_\mu + igA_\mu, \partial_\nu + igA_\nu]\phi$$
$$= ig\left\{\partial_\mu A_\nu + \partial_\nu A_\mu + ig[A_\mu, A_\nu]\right\}\phi\,. \tag{4.32}$$

Thus, we define the matrix-valued gauge field strength tensor $F_{\mu\nu} \equiv F^a_{\mu\nu}T^a$ as follows:

$$[D_\mu, D_\nu] \equiv igF_{\mu\nu}\,. \tag{4.33}$$

Using eq. (4.32) and the commutation relations of the Lie group generators, it follows that

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^\mu_b A^\nu_c\,. \tag{4.34}$$

Under gauge transformations, the transformation law for $F_{\mu\nu}$ is easily obtained. Starting with $\phi \to U\phi$, $D_\mu\phi \to UD_\mu\phi$ (the latter is the essential property of the covariant derivative), and $[D_\mu, D_\nu]\,\phi \to U[D_\mu, D_\nu]\phi$, one easily derives $F_{\mu\nu}\phi \to UF_{\mu\nu}\phi = (UF_{\mu\nu}U^{-1})U\phi$. That is,

$$F_{\mu\nu} \to UF_{\mu\nu}U^{-1}\,, \tag{4.35}$$

which is the transformation law for an adjoint field. Namely, under an infinitesimal gauge transformation,

$$\delta F^a_{\mu\nu}(x) = g f^{abc}\Lambda^b F A^c_{\mu\nu}(x)\,. \tag{4.36}$$

Note that inAbelian gauge theory, $UF_{\mu\nu}U^{-1} = UU^{-1}F_{\mu\nu} = F_{\mu\nu}$ so that $F^{\mu\nu}$ is gauge invariant (*i.e.*, neutral under the gauge group). For non-Abelian gauge group, $F_{\mu\nu}$ transforms non-trivially; *i.e.*, it carries non-trivial gauge charge.

We can now construct a gauge-invariant kinetic energy term for the gauge fields:

$$\mathcal{L}_{\text{gauge}} = \frac{-1}{4T_R}\text{Tr }(F_{\mu\nu}F^{\mu\nu}) \tag{4.37}$$

Using eq. (4.35), it is clear that $\mathcal{L}_{\text{gauge}}$ is gauge invarinat due to the invariance of the trace under cyclic permutation of the quantities inside the trace. Using Tr $(F_{\mu\nu}F^{\mu\nu}) = F^a_{\mu\nu}F^{\mu\nu b}\text{Tr }(T^aT^b) == T_R F^a_{\mu\nu}F^{\mu\nu a}$, the form for $\mathcal{L}_{\text{gauge}}$ does not depend on R and we end up with:

$$\begin{aligned}\mathcal{L}_{\text{gauge}} &= -\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}\\ &= -\tfrac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc}A^b_\mu A^c_\nu)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a} - g f^{ade}A^\mu_d A^\nu_e)\,.\end{aligned} \tag{4.38}$$

Thus, for non-abelian gauge theory (in contrast to abelian gauge theory), the gauge kinetic energy term generates three-point and four-point self-interactions among the gauge fields.

To summarize, if $\mathcal{L}_{\text{matter}}$ is invariant under a group G of global (gauge) transformations, then

$$\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{matter}}(\partial_\mu \to D_\mu) \tag{4.39}$$

is invariant under a group G of local gauge transformations. Above, $\mathcal{L}_{\text{matter}}$ contains a sum of kinetic energy terms of the various scalar and fermion matter multiplets, each of which transforms under some irreducible representation of the gauge group. In these terms, we replace the ordinary derivative with $D_\mu = \partial_\mu + igA^a_\mu T^a$ and use the matrix representation T^a appropriate for each of the matter field multiplets.

Note that there is no mass term for the gauge field, since the term:

$$\mathcal{L}_{\text{mass}} = \tfrac{1}{2}m^2 A^a_\mu A^{\mu a} \tag{4.40}$$

would violate the local gauge invariance. This is a tree-level result; in the next section we will discusee whether this result persists to all orders in perturbation theory.

4.4 Feynman rules for Gauge theories

The Feynamn rules for the self-interactions of the gauge fields and for the interactions of matter are simple to obtain. The triple and quartic gauge boson self-couplings follow from the the form of the gauge kinetic energy term [eq. (4.38)]. The interactions of the gauge bosons with matter are derived from the matter kinetic energy terms. Specifically, after replacing the ordinary derivatives with covariant derivatives, the gauge field dependence of D_μ generates cubic and quartic terms in the Lagrangian that are linear and quadratic in A_μ. For example, the Feynman rules for the interactions of gauge bosons and fermions have been given in Figs. 2.6—2.8 [see also Figs. 3.1 and 3.2].

However, an apparent problem is encountered when one tries to obtain the Feynman rule for the the gauge boson propagator. In general the rule for the tree-level propagator is obtained by inverting the operator that appears in the part of the Lagrangian that is quadratic in the fields. In the case of gauge fields, this is the kinetic energy term, which we can rewrite as follows

$$
\begin{aligned}
-\tfrac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a})& \\
= \tfrac{1}{2} A^a_\mu (g^{\mu\nu}\Box - \partial^\mu\partial^\nu) A^a_\nu \ + \ \text{total divergence}\,. &\qquad (4.41)
\end{aligned}
$$

The total divergence does not contribute to the action. But, note that $(g^{\mu\nu}\Box - \partial^\mu\partial^\nu)\partial_\nu = 0$, which implies that $g^{\mu\nu}\Box - \partial^\mu\partial^\nu$ has a zero eigenvalue and therefore is not an invertible operator.

The solution to this problem is to add the so-called gauge-fixing term and Faddeev-Popov ghost fields (adjoint fields ω^a and ω^{*a}). The justification of this procedure can be found in the standard quantum field theory textbooks (and is most easily explained using path integral techniques). Here, we take a more practical view and simply note that the following Yang-Mills (YM) Lagrangian

$$
\mathcal{L}_{\rm YM} = -\tfrac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} - \frac{1}{2\xi}(\partial_\mu A^\mu_a)^2 + \partial^\mu w^*_a D^{ab}_\mu w_b \qquad (4.42)
$$

is invariant under a BRS extended gauge symmetry, whose infinitesimal transformation laws arge given by:

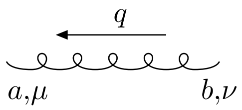
$$
\delta A^a_\mu = \epsilon D^{ab}_\mu w_b \qquad (4.43)
$$

$$
\delta w_a = \tfrac{1}{2}\epsilon g f^{abc} w_b w_c \qquad (4.44)
$$

$$
\delta w^*_a = -\frac{1}{\xi}\epsilon \partial_\mu A^\mu_a \qquad (4.45)
$$

where ϵ is an infinitesimal anticommuting parameter and w, w^* are independent (unphysical) anticommuting scalar fields. Note that the gauge transformation function $\Lambda^a(x)$ [see eq. (4.30)]has been promoted to a field, whose transformation law is given above. Eq. (4.42) generates new interaction terms involving the gauge fields and Faddeev-Popov ghosts. The Faddeev-Popov ghosts can therefore appear inside loops of Feynman diagrams. One can show that any scattering amplitudes that involve only physical particles as external states satisfy unitarity. Hence the theory based on eq. (4.42) is consistent.

In particular, due to the gauge-fixing term (the term that involves the gauge-fixing parameter ξ), one observes that the past of the Lagrangian quadratic in the gauge fields has now changed, and the propagator can now be defined. Converting to momentum space, the Feynman rule for a non-abelian gauge boson propagator is:



$$\frac{-i\delta^{ab}}{q^2+i\epsilon}\left[-g_{\mu\nu}+(1-\xi)\frac{q_\mu q_\nu}{q^2}\right]$$

The Feynman gauge ($\xi = 1$) and the Landau gauge ($\xi = 0$) are two of the more common gauge choices made in practical computations. Of course, any physical quantity must be independent of ξ. This provides a good check of Feynman diagram computations of graphs in which internal gauge bosons propagate. Note that the above considerations also apply to abelian gauge theories such as QED. In this case, one can can introduce ghost fields to exhibit the extended BRS symmetry. However, within the class of gauge fixing terms considered here, the ghost fields are non-interacting (since the photon does not carry any U(1)-charge) and hence the ghosts can be dropped. The photon propagator still takes the form above (but with the factor of δ^{ab} removed).

Finally, let us examine the question of the gauge boson mass. We know that the gauge boson is massless at tree-level. But, can one generate mass via radiative corrections? One must compute the gauge boson two-point function (which corresponds to the radiatively-corrected inverse propagator) and check to see whether the zero at $q^2 = 0$ (corresponding to a zero mass gauge boson) is shifted. Summing up the geometric series yields:

$$\mathcal{D}^{\mu\nu}(q) = D^{\mu\nu}(q) + D^{\mu\lambda}(q) i\Pi_{\lambda\rho}(q)\mathcal{D}^{\rho\nu}(q)\,, \tag{4.46}$$

where $D^{\mu\nu}(q)$ is the tree-level gauge field propagator and $i\Pi^{\mu\nu}(q)$ is the sum over all one-particle irreducible (1PI) diagrams (these are the graphs cannot be split into two separate graphs by cutting through one internal line). The Ward identity of the theory (a consequence of gauge invariance) implies that $q_\mu \Pi^{\mu\nu} = q_\nu \Pi^{\mu\nu} = 0$. It follows that one can write:

$$i\Pi^{\mu\nu}_{ab}(q) = -i(q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)\delta^{ab} \,. \tag{4.47}$$

After Multiplying on the left of eq. (4.46) by D^{-1} and on the right by $\mathcal{D}^{-1}$ (where, *e.g.*, $D^{-1}_{\mu\lambda} D^{\lambda\nu} = g^\nu_\mu$), one obtains:

$$\mathcal{D}^{-1}_{\mu\nu}(q) = D^{-1}_{\mu\nu}(q) = i\Pi_{\mu\nu}(q) \,. \tag{4.48}$$

It is convenient to decompose $D_{\mu\nu}(q)$ Decompose into transverse and longitudinal pieces:

$$D_{\mu\nu}(q) = D(q^2)\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + D^{(\ell)}(q^2)\frac{q_\mu q_\nu}{q^2} \,. \tag{4.49}$$

A similar decomposition of the inverse $D^{-1}_{\mu\nu}(q)$ and $\mathcal{D}^{-1}_{\mu\nu}(q)$ are easily obtained; for example,

$$D^{-1}_{\mu\nu}(q) = \frac{1}{D(q^2)}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{1}{D^{(\ell)}(q^2)}\frac{q_\mu q_\nu}{q^2} \,. \tag{4.50}$$

Since $\Pi_{\mu\nu}(q)$ is transverse [eq. (4.47)], one easily concludes from eq. (4.48) that

$$\mathcal{D}^{(\ell)}(q^2) = D^{(\ell)}(q^2) = \frac{-i\xi}{q^2 + i\epsilon} \tag{4.51}$$

$$\mathcal{D}^{-1}(q^2) = D^{-1}(q^2) + iq^2\Pi(q^2) \,. \tag{4.52}$$

Using $D^{-1}(q^2) = iq^2$, we conclude that[7]

$$\mathcal{D}^{\mu\nu}(q) = \frac{-i}{q^2[1 + \Pi(q^2)]}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) - \frac{i\xi q^\mu q^\nu}{q^4} \,. \tag{4.53}$$

Thus, the pole at $q^2 = 0$ is not shifted. That is, the gauge boson mass remains zero to all orders in perturbation theory.

This elegant argument has a loophole. Namely, if $\Pi(q^2)$ develops a pole at $q^2 = 0$, then the pole of $\mathcal{D}^{\mu\nu}(q)$ shifts away from zero: $\Pi(q^2) \simeq -m_v^2/q^2$ as $q^2 \to 0$ implies that $\mathcal{D}(q^2) \simeq -i/(q^2 - m_v^2)$. This requires some non-trivial dynamics to generates a massless intermediate state in $\Pi_{\mu\nu}(q)$. Such a massless state is called the Goldstone boson. The Standard Model employs the dynamics of elementary scalar fields in order to generate the Goldstone mode. We thus turn our attention to the vector boson mass generation mechanism of the Standard Model.

[7] We have dropped the explicit $+i\epsilon$ that is associated with the pole of the propagator.

4.5 Spontaneously broken gauge theories

Start with a non-abelian gauge theory with scalar (and fermion) matter, given by eq. (4.39). The corresponding scalar potential function must be gauge invariant [eq. (4.21)]. In order to identify the physical scalar degrees of freedom of this model, one must minimize the scalar potential and determine the corresponding values of the scalar fields at the potential minimum. These are the scalar vacuum expectation values. Expanding the scalar fields about their vacuum expectation values yields the tree-level scalar masses and self-couplings. However, in general the scalar fields are charged under the global symmetry group G, in which case a non-zero vacuum expectation value would be incompatible with the global symmetry.

4.5.1 Goldstone's theorem

In the absence of the gauge fields, Goldstone proved the following theorem:

Theorem: If the Lagrangian is invariant under a continuous global symmetry group G (with dimension d_G), but the vacuum state of the theory does not respect all G-transformations, then the theory exhibits spontaneous symmetry breaking. If the vacuum state respects H-transformations, then we say that the group G breaks to a subgroup H (with dimension d_H). The physical spectrum will then contain n massless scalar excitations (called Goldstone bosons) where $n \equiv d_G - d_H$.

This theorem can be proved independent of perturbation theory. However, it can be demonstrated rather easily with a tree-level computation. Let $\phi_i(x)$ be a set of N real scalar fields. The Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi_i)(\partial^\mu \phi_i) - V(\phi_i) \tag{4.54}$$

is assumed to be invariant under O(N), under which the scalar fields transform infinitesimally as

$$\delta\phi_i(x) = -ig\Lambda^a (T^a)_i^{\,j} \phi_i(x) \tag{4.55}$$

where the T^a are imaginary antisymmetric matrices. Since under the symmetry transformation, $\phi_i \to \mathcal{O}_i{}^j \phi_j$ (where $\mathcal{O}$ is orthogonal), the kinetic term is automatically invariant.

We also assume that $V(\phi)$ is invariant, which implies that

$$V(\phi + \delta\phi) \simeq V(\phi) + \frac{\partial V}{\partial \phi_i}\delta\phi_i = V(\phi)\,. \tag{4.56}$$

Step one above is simply a Taylor expansion to first order in the field variation, while step two imposes the invariance assumption. Inserting the result for $\delta\phi_i(x)$ from eq. (4.55), it follows that:

$$\frac{\partial V}{\partial \phi_i}(T^a)_i{}^j\phi_j = 0\,. \tag{4.57}$$

Suppose $V(\phi)$ has a minimum at $\langle\phi_i\rangle = v_i$, which is not invariant under the global symmetry group. That is,

$$\mathcal{O}_i{}^j v_j \simeq \left(\delta_i{}^j - ig\Lambda^a(T^a)_i{}^j\right) v_j \neq v_i\,. \tag{4.58}$$

It follows that there exists at least one a such that

$$(T^a v)_j \neq 0 \tag{4.59}$$

and we conclude that the global symmetry is spontaneously broken. Now, shift the field by its vacuum expectation value:

$$\phi_i \equiv v_i + \varphi_i \tag{4.60}$$

and express the scalar Lagrangian in terms of the φ_i

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\varphi_i)(\partial^\mu\varphi_i) - \tfrac{1}{2}M^2_{ij}\varphi_i\varphi_j + \mathcal{O}(\varphi^3)\,, \tag{4.61}$$

where

$$M^2_{ij} \equiv \left(\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\right)_{\phi_i=v_i}\,. \tag{4.62}$$

The terms cubic and higher in the φ_i do not concern us here. Since V must satisfy eq. (4.57), we can differentiate the latter with respect to ϕ_k and then set all $\phi_i = v_i$. Since $(\partial V/\partial\phi_i)_{\phi_i=v_i} = 0$ as a consequence of the minimum condition, the end result is given by

$$M^2_{ki}(T^a v)_i = 0\,. \tag{4.63}$$

Either $T^a v = 0$ or $(T^a v)_i$ is an eigenvector of M^2 with zero eigenvalue. That is, $G_a = \eta_i T^a_{ij} v_j$ is precisely the linear combination of scalar fields whose mass is zero. These are the Goldstone bosons, and there are $d_G - d_H$ independent Goldstone modes, where H is the residual symmetry group (corresponding to the maximal number of linearly independent elements of the Lie algebra that annihilate the vacuum).

4.5.2 Massive gauge bosons

If a spontaneously broken global symmetry is made local, then a remarkable mechanism called the Higgs mechanism takes place. The Goldstone bosons disappear from the spectrum, and the formerly massless gauge bosons become massive. Roughly speaking, the Goldstone bosons become the longitudinal degrees of freedom of the massive gauge bosons. This is easily demonstrated in a generalization of the tree-level analysis given above. If the O(N) scalar sector above is coupled to gauge fields, then one must replace the ordinary derivative with a covariant derivative in the scalar kinetic energy term:

$$\mathcal{L}_{\rm KE} = \tfrac{1}{2}[\partial_\mu \phi_i + igA^a_\mu (T^a)_i{}^j \phi_j][\partial^\mu \phi_i + igA^{\mu a}(T^a)_i{}^k \phi_k] \,. \tag{4.64}$$

As above, the $\phi_i(x)$ are real fields at the T^a are pure imaginary antisymmetric matrix generators corresponding to the representation of the scalar multiplet. It is convenient to define real antisymmetric matrices

$$L_a \equiv iT^a \,. \tag{4.65}$$

If we expand the ϕ_i around its vacuum expectation value [eq. (4.60)], then one immediately observes a term quadratic in the gauge fields:

$$\mathcal{L}_{\rm mass} = \tfrac{1}{2} M^2_{ab} A^a_\mu A^{\mu b} \,, \tag{4.66}$$

where

$$M^2_{ab} = g^2 (L_a v, L_b v) \tag{4.67}$$

is the gauge boson squared-mass matrix. Here, we have employed a convenient notation where:

$$(x, y) \equiv \sum_i x_i y_i \,. \tag{4.68}$$

If $L_a v \neq 0$ for at least one a, then the gauge symmetry is broken and the gauge bosons acquire mass. The gauge boson squared-mass matrix is real symmetric, so it can be diagonalized with an orthogonal similarity transformation:

$$\mathcal{O} M^2 \mathcal{O}^T = \text{diag }(0, 0, \ldots, 0, \; m_1^2, m_2^2, \ldots) \,. \tag{4.69}$$

If we define

$$\widetilde{L}_a \equiv \mathcal{O}_{ab} L_b \tag{4.70}$$

then

$$(\mathcal{O} M^2 \mathcal{O}^T)_{ab} = (\widetilde{L}_a v, \widetilde{L}_b v) = m_a^2 \delta_{ab} \tag{4.71}$$

where $m_a^2 = 0, \quad \widetilde{L}_a v = 0 \quad a = 1, \ldots, d_H$ correspond to the unbroken generators and $m_a^2 \neq 0, \quad \widetilde{L}_a v \neq 0 \quad a = d_H + 1, \ldots, d_G$ correspond to the broken generators. That is, there are $d_G - d_H$ massive gauge bosons (corresponding to the dimension of the coset space G/H). The gauge boson mass eigenstates are:

$$\widetilde{A}^a_\mu \equiv \mathcal{O}_{ab} A^b_\mu \,. \tag{4.72}$$

Note that

$$\widetilde{L}_a \widetilde{A}^a_\mu = \mathcal{O}_{ac}\mathcal{O}_{ab} L_c A^b_\mu = L_c A^c_\mu \,. \tag{4.73}$$

It is instructive to revisit the scalar Lagrangian:

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi_i + (L_a)_i{}^j A^a_\mu \phi_j)(\partial^\mu \phi_i + (L_b)_i{}^k A^b_\mu \phi_k) - V(\phi) \,. \tag{4.74}$$

Expanding the scalar fields around their vacuum expectation values, and identifying the massive gauge boson mass eigenstates,

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}(\partial_\mu \varphi_i)(\partial^\mu \varphi_i) + \tfrac{1}{2} m_a^2 \widetilde{A}^a_\mu \widetilde{A}^{\mu a} + \tfrac{1}{2}(\widetilde{L}_a \widetilde{A}^a_\mu v, \partial^\mu \varphi) + \tfrac{1}{2}(\partial_\mu \varphi, \widetilde{L}_a \widetilde{A}^a_\mu v) + \ldots \\ &= \tfrac{1}{2}(\partial_\mu \phi_i)(\partial^\mu \phi_i) + \tfrac{1}{2} m_a^2 A^a_\mu A^{\mu a} + m_a \widetilde{A}^a_\mu \partial^\mu G_a + \ldots \,,\end{aligned} \tag{4.75}$$

where

$$G_a = \frac{1}{m_a}(\widetilde{L}_a v, \eta) \,. \tag{4.76}$$

In eq. (4.75), we have not displayed terms cubic or higher in the fields. Note that the sum over the repeated index a is assumed. We recognize G_a as the Goldstone boson which appeared in the spontaneously broken global theory without gauge bosons.

The coupling $m_a A^a_\mu \partial^\mu G_a$ provides an explanation for the vector boson mass generation mechanism. Namely, this coupling yields a new interaction vertex in the theory. We can then evaluate the contribution of an intermediate Goldstone line to $i\Pi^{\mu\nu}(q)$:

$$\begin{aligned} i\Pi^{\mu\nu}(k) &= m_a^2 k^\mu (-k^\nu) \frac{i}{k^2} + \ldots \\ &= -i m_a^2 \frac{k^\mu k^\nu}{k^2} + \ldots \end{aligned} \tag{4.77}$$

This is the only source for a pole at $k^2 = 0$ at this order in perturbation theory. Gauge invariance ensures that

$$i\Pi^{\mu\nu}(k) = i(k^\mu k^\nu - k^2 g^{\mu\nu})\Pi(k^2) \tag{4.78}$$

so that

$$\Pi(k^2) \simeq \frac{-m_a^2}{k^2} \tag{4.79}$$

and

$$\mathcal{D}(k^2) = \frac{i}{k^2[1+\Pi(k^2)]} = \frac{i}{k^2 - m_a^2} \tag{4.80}$$

We say that the gauge boson "eats" or absorbs the corresponding Goldstone boson and thereby acquires mass via the Higgs mechanism.

4.5.3 The unitary and R_ξ gauges

The spontaneously broken non-abelian gauge theory Lagrangian contains the Goldstone boson field. However, as we shall now demonstrate, the Goldstone bosons are gauge artifacts that can be removed by a gauge transformation. Consider the transformation law for the shifted scalar field, $\varphi_i(x) \equiv \phi_i(x) - v_i$. Using eqs. (4.55) and (4.65) and noting that $\delta v_i = 0$,

$$\delta\varphi_i = -\Lambda(\varphi + v)\,, \tag{4.81}$$

where $\Lambda = L_a\Lambda^a = \widetilde{L}_a\widetilde{\Lambda}^a$ if we define $\widetilde{\Lambda}^a = \mathcal{O}_{ab}\Lambda^b$. Then,

$$\begin{aligned}\delta G_a &= \frac{1}{m_a}(\widetilde{L}_a v, \delta\varphi) = \frac{-1}{m_a}(\widetilde{L}_a v, \Lambda\varphi + \Lambda v) & (4.82)\\ &= \frac{1}{m_a}(\widetilde{L}_a v, \Lambda\varphi) - \frac{1}{m_a}(\widetilde{L}_a v, \widetilde{L}_b v)\widetilde{\Lambda}^b \\ &= \frac{1}{m_a}(\widetilde{L}_a v, \Lambda\varphi) - m_a\widetilde{\Lambda}_a\,, & (4.83)\end{aligned}$$

after using eq. (4.71) for the gauge boson masses. Note that the last term above, $m_a\widetilde{\Lambda}_a$ is an inhomogeneous term independent of φ. Thus, one can simply choose $\widetilde{\Lambda}_a$ such that $G_a = 0$. This is called the unitary gauge.

So far, we have not mentioned the gauge fixing term and Faddeev-Popov ghosts. In spontaneously-broken non-abelian gauge theories, the R_ξ-gauge turns out to be particularly useful.[8] The R_ξ-gauge is defined

[8] The R stands for renormalizable, and ξ is the gauge fixing parameter. In the R-gauges, the theory is manifestly renormalizable although unitarity is not manifest and must be separately proved. The exact opposite is true for the unitary gauge.

by the following gauge-fixing term:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{GF}} &= \frac{-1}{2\xi}\left(\partial^\mu \widetilde{A}^a_\mu - \xi\eta_i(\widetilde{L}_a\nu)_i\right)^2 \\
&= \frac{-1}{2\xi}(\partial^\mu \widetilde{A}^a_\mu)^2 + \tfrac{1}{2}\left(\widetilde{L}_a(\partial^\mu \widetilde{A}^a_\mu)\nu\eta\right) \\
&\qquad\qquad + \tfrac{1}{2}\left(\eta, \widetilde{L}_a(\partial^\mu \widetilde{A}^a_\mu)\nu\right) - \xi\left[\eta_i(\widetilde{L}_a\nu)_i\right]^2 \\
&= \frac{-1}{2\xi}(\partial^\mu \widetilde{A}^a_\mu)^2 + m_a G_a(\partial^\mu \widetilde{A}^a_\mu) - \frac{\xi m_a^2}{2} G_a G_a\,. \qquad (4.84)
\end{aligned}
$$

At this point, we notice that the term $m_a G_a(\partial^\mu \widetilde{A}^a_\mu)$ of eq. (4.84) combines with $m_a \widetilde{A}^a_\mu \partial^\mu G$ of eq. (4.75) to yield

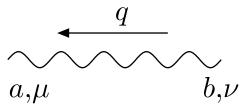
$$
m_a[G_a\partial^\mu \widetilde{A}^a_\mu + \widetilde{A}^a_\mu \partial^\mu G_a] = m_a \partial^\mu(G_a \widetilde{A}^a_\mu)\,, \qquad (4.85)
$$

which is a total divergence which can be dropped from the Lagrangian. One must also include Faddeev-Popov ghosts:

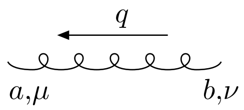
$$
\mathcal{L}_{\mathrm{FP}} = \partial^\mu w_a^* D_\mu^{ab} w_b - \xi w_a^* M_{ab}^2 w_b - \xi g^2 w_a^* w_b(\varphi, T^a T^b v)\,, \qquad (4.86)
$$

D_μ^{ab} is defined in eq. (4.31) and M_{ab}^2 is the gauge boson squared-mass matrix.

In the R_ξ gauges, the Feynman rules for the massless and massive gauge boson propagators take on the following form, respectively:



$$
\frac{i\delta^{ab}}{q^2 + i\epsilon}\left[-g_{\mu\nu} + (1-\xi_0)\frac{q_\mu q_\nu}{q^2}\right]
$$

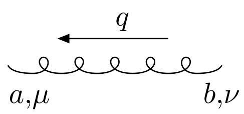


$$
\frac{-i\delta^{ab}}{q^2 - m_a^2 + i\epsilon}\left[-g_{\mu\nu} + (1-\xi)\frac{q_\mu q_\nu}{q^2 - \xi m_a^2}\right]
$$

Above, the massless gauge bosons are indicated by wavy lines, and the massive gauge bosons are indicated by curly lines. In principle $\xi_0 \neq \xi$ is possible since for $\widetilde{L}_a v = 0$ we are free to choose an arbitrary ξ-parameter. In practice, one usually sets $\xi_0 = \xi$.

In addition, we note from eq. (4.84) that the Goldstone bosons have acquired a squared-mass equal to ξm_a^2. This is an artifact of the gauge choice. Nevertheless, for a consistent computation in the R_ξ-gauge, both the Goldstone bosons and the Faddeev-Popov ghosts must be included as possible internal lines in Feynman graphs. As noted previously, any physical quantity must ultimately be independent of ξ.

As in the unbroken non-abelian gauge theory, the two most useful gauges are $\xi = 1$ (now called the 't Hooft-Feynman gauge) and $\xi = 0$ (still called the Landau gauge). The Landau gauge has an additional extra benefit that the Goldstone bosons are massless. Finally, we note that one can attempt to take the limit of $\xi \to \infty$. This corresponds to the unitary gauge, since the Goldstone boson masses become infinite and thus decouple from all Feynman graphs. Moreover, one can check that the massive gauge boson propagator reduces to



$$\frac{-i\delta^{ab}}{q^2 - m_a^2 + i\epsilon}\left[-g_{\mu\nu} + \frac{q_\mu q_\nu}{m_a^2}\right]$$

which is the expected form for a massive gauge boson propagator in a gauge-noninvariant model. The fact that the unitary gauge is a limiting case of the R_ξ-gauge played an essential role in the proof that spontaneously-broken non-abelian gauge theories are unitary and renormalizable.

4.5.4 The physical Higgs bosons

An important check of the formalism is the counting of bosonic degrees of freedom. Assume that the multiplet of scalar fields transforms according to some representation R of dimension $d(R)$ under the transformation group G (which has dimension d_G). Prior to spontaneous symmetry breaking, the models contains $d(R)$ scalar degrees of freedom and $2d_G$ vector-boson degrees of freedom. The latter consists of d_G massless gauge bosons (one for each possible value of the adjoint index a), with each massless gauge boson contributing two degrees of freedom corresponding to the two possible transverse helicities. After spontaneous symmetry breaking of G to a subgroup H (which has dimension d_H), there are $d_G - d_H$ Goldstone bosons which are unphysical (and can be removed from the spectrum by going to the unitary gauge). This leaves

$$d(R) - d_G + d_H \text{ scalar degrees of freedom} \tag{4.87}$$

which correspond to the physical Higgs bosons of the theory. We also found that there are d_H massless gauge bosons (one for each unbroken generator) and $d_G - d_H$ massive gauge bosons (one for each broken generator). But, for massive gauge bosons which possess a longitudinal helicity state, we must count three degrees of freedom. Thus, we end up with

$$3d_G - d_H \text{ vector boson degrees of freedom} \tag{4.88}$$

Adding the two yields a total of $d(R) + 2d_G$ bosonic degrees of freedom, in agreement with our previous counting.

It is instructive to check that the physical Higgs bosons cannot be removed by a gauge transformation. We divide the scalars into two classes: (i) the Goldstone bosons G_a, $a = d_H + 1, \ldots, d_G$ [see eq. (4.76)] and (ii) the scalar states orthogonal to G_a. These are the Higgs bosons:

$$\widetilde{H}_k = c_j^{(k)} \varphi_j \,, \tag{4.89}$$

where the $c_j^{(k)}$ satisfy:

$$\sum_j c_j^{(k)} (\widetilde{L}_a v)_j = 0 \,. \tag{4.90}$$

Under a gauge transformation,

$$\begin{aligned} \delta(c_j^{(k)} \eta_j) &= c_j^{(k)} \delta\varphi_j \\ &= -c_j^{(k)} (\Lambda\varphi + \Lambda v)_j \\ &= -c_j^{(k)} (\Lambda\varphi)_j \,, \end{aligned} \tag{4.91}$$

where we have noted that $c_j(\Lambda v)_j = c_j \widetilde{\Lambda}^a (\widetilde{L}_a v)_j = 0$ by orthogonality. Thus the transformation law for $\widetilde{H}_k$ is homogeneous in the scalar fields, and thus one cannot remove the Higgs boson field by a gauge transformation.

The states $\widetilde{H}_k$ are in general not mass-eigenstates. Employing the $\left\{ G_a, \widetilde{H}_k \right\}$ basis for the scalar fields, we note that the scalar boson squared-mass matrix

$$\mathcal{M}^2_{ij} = \left(\frac{\partial^2 V}{\partial \phi_i \partial \phi_j} \right)_{\phi_i = v} \tag{4.92}$$

satisfies $(M^2)_{ij} (\widetilde{L}_a v)_j = 0$. Thus, the non-derivative quadratic terms in the scalar part of the Lagrangian is

$$\mathcal{L}_{\text{scalar mass}} = -\tfrac{1}{2} (\mathcal{M}^2)_{k\ell} \widetilde{H}_k \widetilde{H}_\ell \,. \tag{4.93}$$

Diagonalizing $\mathcal{M}^2$ yields the Higgs boson eigenstates, H_k, and their respective squared-masses.

4.6 Complex representations of scalar fields

We now have nearly all the ingredients necessary to construct the Standard Model. However, in our treatment of spontaneously broken non-abelian gauge fields, we took the scalar fields to be real fields that transformed under a simple orthogonal symmetry group. The Standard Model employs complex scalars that transform under semi-simple product of unitary symmetry groups. In this section, we provide the necessary information which will allow one to treat the complex and semi-simple cases.

4.7 The Standard Model of particle physics

The Standard Model is a spontaneously broken non-abelian gauge theory based on the symmetry group SU(3)×SU(2)×U(1).

4.8 Parameter count of the Standard Model

The Standard Model Lagrangian appears to contain many parameters. However, not all these parameters can be physical. One is always free to redefine the Standard Model fields in an arbitrary manner. By suitable redefinitions, one can remove some of the apparent parameter freedom and identify the true physical independent parametric degrees of freedom.

To illustrate the procedure, let us first make a list of the parameters of the Standard Model. First, the gauge sector consists of three real gauge couplings (g_3, g_2 and g_1) and the QCD vacuum angle (θ_{QCD}). The Higgs sector consists of one Higgs squared-mass parameter and one Higgs self-coupling (m^2 and λ). Traditionally, one trades in the latter two real parameters for the vacuum expectation value ($v = 246$ GeV) and the physical Higgs mass. The fermion sector consists of three Higgs-Yukawa coupling matrices $\boldsymbol{y}_u$, $\boldsymbol{y}_d$, and $\boldsymbol{y}_e$. Initially, $\boldsymbol{y}_u$, $\boldsymbol{y}_d$, and $\boldsymbol{y}_e$ are arbitrary complex 3×3 matrices, which in total depend on 27 real and 27 imaginary degrees of freedom.

But, most of these degrees of freedom are unphysical. In particular, in the limit where $\boldsymbol{y}_u = \boldsymbol{y}_d = \boldsymbol{y}_e = 0$, the Standard Model possesses a global $\mathrm{U}(3)^5$ symmetry corresponding to three generations of the five SU(3)×SU(2)×U(1) multiplets: $(\nu_m,\ e^-_m)_L$, $(e^c_m)_L$, $(u_m, d_m)_L$, $(u^c_m)_L$, $(d^c_m)_L$, where m is the generation label. Thus, one can make global $\mathrm{U}(3)^5$ rotations on the fermion fields of the Standard Model to absorb the unphysical degrees of freedom of $\boldsymbol{y}_u$, $\boldsymbol{y}_d$, and $\boldsymbol{y}_e$. A U(3) matrix can be parameterized by three real angles and six phases, so that with the most general $\mathrm{U}(3)^5$ rotation, we can apparently remove 15 real angles and 30 phases from $\boldsymbol{y}_u$, $\boldsymbol{y}_d$, and $\boldsymbol{y}_e$. However, the $\mathrm{U}(3)^5$ rotations include four exact U(1) global symmetries of the Standard Model, namely B and the three separate lepton numbers L_e, L_μ and L_τ. Thus, one can only remove 26 phases from $\boldsymbol{y}_u$, $\boldsymbol{y}_d$, and $\boldsymbol{y}_e$. This leaves 12 real parameters (corresponding to six quark masses, three lepton masses, [9] and three CKM mixing angles) and one imaginary degree of freedom (the phase of the CKM matrix). Adding up to get the final result, one finds that the Standard Model possesses 19 independent parameters (of which 13 are associated with the flavor sector).

4.9 Grand Unification and Running Couplings

The gauge couplings of the Standard Model are functions of the energy scale.

[9] The neutrinos in the Standard Model are automatically massless and are not counted as independent degrees of freedom in the parameter count.

Part 2
Constructing Supersymmetric Theories

5
Introduction to Supersymmetry

5.1 Motivation: the Hierarchy Problem

Despite the successes of the Standard Model, it seems very likely that new physics is present at the TeV scale. The mere fact that the ratio M_{P}/M_W is so huge is already a powerful clue to the character of physics beyond the Standard Model, because of the infamous "hierarchy problem". This is not really a difficulty with the Standard Model itself, but rather a disturbing sensitivity of the Higgs potential to new physics in almost any imaginable extension of the Standard Model. The electrically neutral part of the Standard Model Higgs field is a complex scalar H with a classical potential given by

$$V = m_H^2 |H|^2 + \lambda |H|^4 \,. \tag{5.1}$$

The Standard Model requires a non-vanishing vacuum expectation value (VEV) for H at the minimum of the potential. This will occur if $m_H^2 < 0$, resulting in $\langle H \rangle = \sqrt{-m_H^2/2\lambda}$. Since we know experimentally that $\langle H \rangle =$ 174 GeV from measurements of the properties of the weak interactions, it must be that m_H^2 is very roughly of order $-(100\text{ GeV})^2$. However, m_H^2 receives enormous quantum corrections from the virtual effects of every particle that couples, directly or indirectly, to the Higgs field.

For example, in Fig. 5.1a we have a correction to m_H^2 from a loop containing a Dirac fermion f with mass m_f. If the Higgs field couples to f with a term in the Lagrangian $-\lambda_f H \overline{f} f$, then the Feynman diagram in Fig. 5.1a yields a correction

$$\Delta m_H^2 = \frac{|\lambda_f|^2}{16\pi^2} \left[-2\Lambda_{\mathrm{UV}}^2 + 6 m_f^2 \ln(\Lambda_{\mathrm{UV}}/m_f) + \ldots \right] . \tag{5.2}$$

Here Λ_{UV} is an ultraviolet momentum cutoff used to regulate the loop integral; it should be interpreted as the energy scale at which new

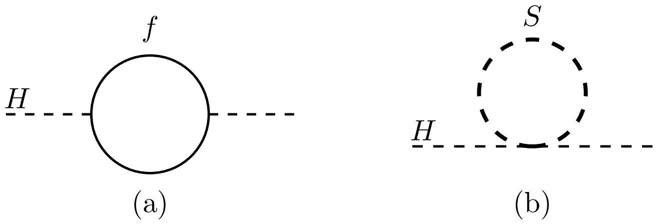


Fig. 5.1. One-loop quantum corrections to the Higgs (mass)^2 parameter m_H^2, due to (a) a virtual fermion f, and (b) a virtual scalar S.

physics enters to alter the high-energy behavior of the theory. The ellipses represent terms that depend on the precise manner in which the momentum cutoff is applied, and that approach a constant as $\Lambda_{\mathrm{UV}} \to \infty$. Each of the leptons and quarks of the Standard Model can play the role of f; for quarks, eq. (5.2) should be multiplied by 3 to account for color. The largest correction comes when f is the top quark with $\lambda_f \approx 1$. The problem is that if Λ_{UV} is of order M_{P}, say, then this quantum correction to m_H^2 is some 30 orders of magnitude larger than the aimed-for value of $m_H^2 \sim -(100\ \mathrm{GeV})^2$. This is only directly a problem for corrections to the Higgs scalar boson (mass)^2, because quantum corrections to fermion and gauge boson masses do not have the quadratic sensitivity to Λ_{UV} found in eq. (5.2). However, the quarks and leptons and the electroweak gauge bosons Z^0, $W^{\pm}$ of the Standard Model all owe their masses to $\langle H \rangle$, so that the entire mass spectrum of the Standard Model is directly or indirectly sensitive to the cutoff Λ_{UV}.

One could imagine that the solution is to simply pick an ultraviolet cutoff Λ_{UV} that is not too large. However, one still has to concoct some new physics at the scale Λ_{UV} that not only alters the propagators in the loop, but actually cuts off the loop integral. This is not easy to do in a theory whose Lagrangian does not contain more than two derivatives, and higher-derivative theories generally suffer from a failure of either unitarity or causality [2]. In string theories, this problem is ignored, and loop integrals are nevertheless cut off at high Euclidean momentum p by factors $e^{-p^2/\Lambda_{\mathrm{UV}}^2}$. However, then Λ_{UV} is a string scale that is usually[2] thought to be not very far below M_{P}. Furthermore, there is a contribution similar to eq. (5.2) from the virtual effects of any arbitrarily heavy particles that might exist.

For example, suppose there exists a heavy complex scalar particle S with mass m_S that couples to the Higgs with a Lagrangian term

[2] Some recent attacks on the hierarchy problem have assumed that the ultimate cutoff scale is actually much smaller than the apparent Planck scale.

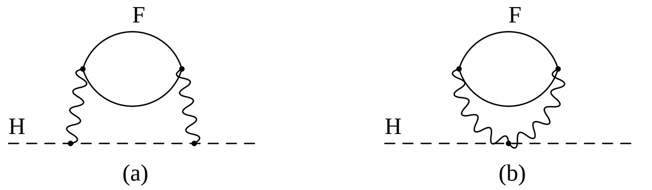


Fig. 5.2. Two-loop corrections to the Higgs (mass)^2 due to a heavy fermion.

$-\lambda_S|H|^2|S|^2$. Then the Feynman diagram in Fig. 5.1b gives a correction

$$\Delta m_H^2 = \frac{\lambda_S}{16\pi^2}\left[\Lambda_{\mathrm{UV}}^2 - 2m_S^2\ln(\Lambda_{\mathrm{UV}}/m_S) + \ldots\right]. \tag{5.3}$$

If one rejects the possibility of a physical interpretation for Λ_{UV} and uses dimensional regularization on the loop integral instead of a momentum cutoff, then there will be no Λ_{UV}^2 piece. However, even then the term proportional to m_S^2 cannot be eliminated without the physically unjustifiable tuning of a counter-term specifically for that purpose. So m_H^2 is sensitive to the masses of the *heaviest* particles that H couples to; if m_S is very large, its effects on the Standard Model do not decouple, but instead make it very difficult to understand why m_H^2 is so small.

This problem arises even if there is no direct coupling between the Standard Model Higgs boson and the unknown heavy particles. For example, suppose there exists a heavy fermion F that, unlike the quarks and leptons of the Standard Model, has vector-like quantum numbers and therefore gets a large mass m_F without coupling to the Higgs field. [In other words, an arbitrarily large mass term of the form $m_F\overline{F}F$ is not forbidden by any symmetry, including $SU(2)_L$.] In that case, no diagram like Fig. 5.1a exists for F. Nevertheless there will be a correction to m_H^2 as long as F shares some gauge interactions with the Standard Model Higgs field; these may be the familiar electroweak interactions, or some unknown gauge forces that are broken at a very high energy scale inaccessible to experiment. In either case, the two-loop Feynman diagrams in Fig. 5.2 yield a correction

$$\Delta m_H^2 = x\left(\frac{g^2}{16\pi^2}\right)^2\left[a\Lambda_{\mathrm{UV}}^2 + 48m_F^2\ln(\Lambda_{\mathrm{UV}}/m_F) + \ldots\right], \tag{5.4}$$

where g is the gauge coupling in question, and x is a group theory factor of order 1. (Specifically, x is the product of the quadratic Casimir invariant of H and the Dynkin index of F for the gauge group in question.) The coefficient a depends on the precise method of cutting off the momentum integrals. It does not arise at all if one uses dimensional regularization, but the m_F^2 contribution is always present. The numerical factor $(g^2/16\pi^2)^2$ may be quite small (of order 10^{-5} for electroweak interactions), but the important point is that these contributions to Δm_H^2 are sensitive to the largest masses and/or ultraviolet cutoff in the theory, presumably of order $M_{\rm P}$. The "natural" (mass)^2 of a fundamental Higgs scalar, including quantum corrections, therefore seems to be more like some small but appreciable fraction of $M_{\rm P}^2$ rather than the experimentally favored value! Even very indirect contributions from Feynman diagrams with three or more loops can give unacceptably large contributions to Δm_H^2. The argument above applies not just for heavy particles, but for arbitrary high-scale physical phenomena such as condensates or additional compactified space-time dimensions.

If the Higgs boson is a fundamental particle, and there really is physics far above the electroweak scale, then we have two remaining options: either we must make the rather bizarre assumption that there do not exist *any* heavy particles that couple (even indirectly or extremely weakly) to the Higgs scalar field, or else some rather striking cancellation is needed between the various contributions to Δm_H^2.

The systematic cancellation of the dangerous contributions to Δm_H^2 can only be brought about by the type of conspiracy that is better known to physicists as a symmetry. The form of eqs. (5.2) and (5.3) suggests that the new symmetry ought to relate fermions and bosons, because of the relative minus sign between fermion loop and boson loop contributions to Δm_H^2. (Note that λ_S must be positive if the scalar potential is to be bounded from below.) If each of the quarks and leptons of the Standard Model is accompanied by two complex scalars with $\lambda_S = |\lambda_f|^2$, then the $\Lambda_{\rm UV}^2$ contributions of Figs. 5.1a and 5.1b will neatly cancel. Clearly, more restrictions on the theory will be necessary to ensure that this success persists to higher orders, so that, for example, the contributions in Fig. 5.2 and eq. (5.4) from a very heavy fermion are cancelled by the two-loop effects of some very heavy bosons. Fortunately, the cancellation of all such contributions to scalar masses is not only possible, but is actually unavoidable, once we merely assume that a symmetry relating fermions and bosons, called a *supersymmetry*, exists.

5.2 Enter supersymmetry

A supersymmetry transformation turns a bosonic state into a fermionic state, and vice versa. The operator Q that generates such transformations must be an anticommuting spinor, with

$$Q|\text{Boson}\rangle = |\text{Fermion}\rangle; \qquad Q|\text{Fermion}\rangle = |\text{Boson}\rangle. \tag{5.5}$$

Spinors are intrinsically complex objects, so $\bar{Q}$ (the hermitian conjugate of Q) is also a symmetry generator. Because Q and $\bar{Q}$ are fermionic operators, they carry spin angular momentum 1/2, so it is clear that supersymmetry must be a spacetime symmetry. The possible forms for such symmetries in an interacting quantum field theory are highly restricted by the Haag-Lopuszanski-Sohnius extension of the Coleman-Mandula theorem. For realistic theories that, like the Standard Model, have chiral fermions (i.e., fermions whose left- and right-handed pieces transform differently under the gauge group) and thus the possibility of parity-violating interactions, this theorem implies that the generators Q and $\bar{Q}$ must satisfy an algebra of anticommutation and commutation relations with the schematic form

$$\{Q, \bar{Q}\} = P^\mu, \tag{5.6}$$

$$\{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0, \tag{5.7}$$

$$[P^\mu, Q] = [P^\mu, \bar{Q}] = 0, \tag{5.8}$$

where P^μ is the momentum generator of spacetime translations. Here we have ruthlessly suppressed the spinor indices on Q and $\bar{Q}$; we will, in Chapter 6, derive the precise version of eqs. (5.6)-(5.8) with indices restored. In the meantime, we simply note that the appearance of P^μ on the right-hand side of eq. (5.6) is unsurprising, since it transforms under Lorentz boosts and rotations as a spin-1 object while Q and $\bar{Q}$ on the left-hand side each transform as spin-1/2 objects.

The single-particle states of a supersymmetric theory fall into irreducible representations of the supersymmetry algebra, called *supermultiplets*. Each supermultiplet contains both fermion and boson states, which are commonly known as *superpartners* of each other. By definition, if $|\Omega\rangle$ and $|\Omega'\rangle$ are members of the same supermultiplet, then $|\Omega'\rangle$ is proportional to some combination of Q and $\bar{Q}$ operators acting on $|\Omega\rangle$, up to a spacetime translation or rotation. The $(\text{mass})^2$ operator P^2 commutes with the operators Q, $\bar{Q}$, and with all spacetime rotation and translation operators, so it follows immediately that particles that inhabit the same irreducible supermultiplet must have equal eigenvalues of P^2, and therefore equal masses.

The supersymmetry generators $Q, \bar{Q}$ also commute with the generators of gauge transformations. Therefore particles in the same supermultiplet must also be in the same representation of the gauge group, and so must have the same electric charges, weak isospin, and color degrees of freedom.

Each supermultiplet contains an equal number of fermion and boson degrees of freedom. To prove this, consider the operator $(-1)^{2s}$ where s is the spin angular momentum. By the spin-statistics theorem, this operator has eigenvalue $+1$ acting on a bosonic state and eigenvalue -1 acting on a fermionic state. Any fermionic operator will turn a bosonic state into a fermionic state and vice versa. Therefore $(-1)^{2s}$ must anticommute with every fermionic operator in the theory, and in particular with Q and $\bar{Q}$. Now consider the subspace of states $|i\rangle$ in a supermultiplet that have the same eigenvalue p^μ of the four-momentum operator P^μ. In view of eq. (5.8), any combination of Q or $\bar{Q}$ acting on $|i\rangle$ will give another state $|i'\rangle$ that has the same four-momentum eigenvalue. Therefore one has a completeness relation $\sum_i |i\rangle\langle i| = 1$ within this subspace of states. Now one can take a trace over all such states of the operator $(-1)^{2s}P^\mu$ (including each spin helicity state separately):

$$\begin{aligned}
\sum_i \langle i|(-1)^{2s}P^\mu|i\rangle &= \sum_i \langle i|(-1)^{2s}Q\bar{Q}|i\rangle + \sum_i \langle i|(-1)^{2s}\bar{Q}Q|i\rangle \\
&= \sum_i \langle i|(-1)^{2s}Q\bar{Q}|i\rangle + \sum_{i,j} \langle i|(-1)^{2s}\bar{Q}|j\rangle\langle j|Q|i\rangle \\
&= \sum_i \langle i|(-1)^{2s}Q\bar{Q}|i\rangle + \sum_j \langle j|Q(-1)^{2s}\bar{Q}|j\rangle \\
&= \sum_i \langle i|(-1)^{2s}Q\bar{Q}|i\rangle - \sum_j \langle j|(-1)^{2s}Q\bar{Q}|j\rangle \\
&= 0.
\end{aligned} \tag{5.9}$$

The first equality follows from the supersymmetry algebra relation eq. (5.6); the second and third from use of the completeness relation; and the fourth from the fact that $(-1)^{2s}$ must anticommute with Q. Now $\sum_i \langle i|(-1)^{2s}P^\mu|i\rangle = p^\mu \ \mathrm{Tr}[(-1)^{2s}]$ is just proportional to the number of bosonic degrees of freedom n_B minus the number of fermionic degrees of freedom n_F in the trace, so that

$$n_B = n_F \tag{5.10}$$

must hold for a given $p^\mu \neq 0$ in each supermultiplet.

The simplest possibility for a supermultiplet consistent with eq. (5.10) has a single Weyl fermion (with two spin helicity states, so $n_F = 2$) and two real scalars (each with $n_B = 1$). It is natural to assemble the two real scalar degrees of freedom into a complex scalar field; as we will see below this provides for convenient formulation of the supersymmetry algebra, Feynman rules, supersymmetry violating effects, etc. This combination of a two-component Weyl fermion and a complex scalar field is called a *chiral* or *matter* or *scalar* supermultiplet.

The next-simplest possibility for a supermultiplet contains a spin-1 vector boson. If the theory is to be renormalizable, this must be a gauge boson that is massless, at least before the gauge symmetry is spontaneously broken. A massless spin-1 boson has two helicity states, so the number of bosonic degrees of freedom is $n_B = 2$. Its superpartner is therefore a massless spin-1/2 Weyl fermion, again with two helicity states, so $n_F = 2$. (If one tried instead to use a massless spin-3/2 fermion, the theory would not be renormalizable.) Gauge bosons must transform as the adjoint representation of the gauge group, so their fermionic partners, called *gauginos*, must also. Since the adjoint representation of a gauge group is always its own conjugate, the gaugino fermions must have the same gauge transformation properties for left-handed and for right-handed components. Such a combination of spin-1/2 gauginos and spin-1 gauge bosons is called a *gauge* or *vector* supermultiplet.

If we include gravity, then the spin-2 graviton (with 2 helicity states, so $n_B = 2$) has a spin-3/2 superpartner called the gravitino. The gravitino would be massless if supersymmetry were unbroken, and so it has $n_F = 2$ helicity states.

There are other possible combinations of particles with spins that can satisfy eq. (5.10). However, these are always reducible to combinations of chiral and gauge supermultiplets if they have renormalizable interactions, except in certain theories with "extended" supersymmetry. Theories with extended supersymmetry have more than one distinct copy of the supersymmetry generators $Q, \bar{Q}$. Such models are mathematically amusing, but evidently do not have any phenomenological prospects. The reason is that extended supersymmetry in four-dimensional field theories cannot allow for chiral fermions or parity violation as observed in the Standard Model. So we will not discuss such possibilities further, although extended supersymmetry in higher-dimensional field theories might describe the real world if the extra dimensions are compactified, and extended supersymmetry in four dimensions provides interesting toy models. The ordinary, non-extended, phenomenologically-viable type of supersymmetric model is sometimes called $N = 1$ supersymmetry, with N referring to the number of supersymmetries (the number of distinct copies of $Q, \bar{Q}$).

In a supersymmetric extension of the Standard Model, each of the known fundamental particles is therefore in either a chiral or gauge supermultiplet, and must have a superpartner with spin differing by 1/2 unit. The first step in understanding the exciting phenomenological consequences of this prediction is to decide exactly how the known particles fit into supermultiplets, and to give them appropriate names. A crucial observation here is that only chiral supermultiplets can contain fermions whose left-handed parts transform differently under the gauge group than their right-handed parts. All of the Standard Model fermions (the known quarks and leptons) have this property, so they must be members of chiral supermultiplets.[3] The names for the spin-0 partners of the quarks and leptons are constructed by prepending an "s", which is short for scalar. So, generically they are called *squarks* and *sleptons* (short for "scalar quark" and "scalar lepton"). The left-handed and right-handed pieces of the quarks and leptons are separate two-component Weyl fermions with different gauge transformation properties in the Standard Model, so each must have its own complex scalar partner. The symbols for the squarks and sleptons are the same as for the corresponding fermion, but with a tilde ($\widetilde{\ }$) used to denote the superpartner of a Standard Model particle. For example, the superpartners of the left-handed and right-handed parts of the electron Dirac field are called left- and right-handed selectrons, and are denoted $\widetilde{e}_L$ and $\widetilde{e}_R$. It is important to keep in mind that the "handedness" here does not refer to the helicity of the selectrons (they are spin-0 particles) but to that of their superpartners. A similar nomenclature applies for smuons and staus: $\widetilde{\mu}_L$, $\widetilde{\mu}_R$, $\widetilde{\tau}_L$, $\widetilde{\tau}_R$. The Standard Model neutrinos (neglecting their very small masses) are always left-handed, so the sneutrinos are denoted generically by $\widetilde{\nu}$, with a possible subscript indicating which lepton flavor they carry: $\widetilde{\nu}_e$, $\widetilde{\nu}_\mu$, $\widetilde{\nu}_\tau$. Finally, a complete list of the squarks is $\widetilde{q}_L$, $\widetilde{q}_R$ with $q = u, d, s, c, b, t$. The gauge interactions of each of these squark and slepton fields are the same as for the corresponding Standard Model fermions; for instance, the left-handed squarks $\widetilde{u}_L$ and $\widetilde{d}_L$ couple to the W boson while $\widetilde{u}_R$ and $\widetilde{d}_R$ do not.

It seems clear that the Higgs scalar boson must reside in a chiral supermultiplet, since it has spin 0. Actually, it turns out that just one chiral supermultiplet is not enough. One way to see this is to note that if there were only one Higgs chiral supermultiplet, the electroweak gauge symmetry would suffer a gauge anomaly, and would be inconsistent as a quantum theory. This is because the conditions for cancellation of gauge anomalies include $\mathrm{Tr}[T_3^2 Y] = \mathrm{Tr}[Y^3] = 0$, where T_3 and Y are the third component of weak isospin and the weak hypercharge, respectively, in a normalization where the ordinary electric charge is $Q_{\mathrm{EM}} = T_3 + Y$. The traces run over all of the left-handed Weyl fermionic degrees of freedom in the theory. In the Standard Model, these conditions are already satisfied, somewhat miraculously, by the known quarks and leptons. Now, a fermionic partner of a Higgs chiral supermultiplet must be a weak isodoublet with weak hypercharge $Y = 1/2$ or $Y = -1/2$. In either case alone, such a fermion will make a non-zero contribution to the traces and spoil the anomaly cancellation. This can be avoided if there are two Higgs supermultiplets, one with each of $Y = \pm 1/2$, so that the total contribution

[3] In particular, one cannot attempt to make a spin-1/2 neutrino be the superpartner of the spin-1 photon; the neutrino is in a doublet, and the photon is neutral, under weak isospin.

to the anomaly traces from the two fermionic members of the Higgs chiral supermultiplets vanishes by cancellation. As we will see in section 8.1, both of these are also necessary for another completely different reason: because of the structure of supersymmetric theories, only a $Y = +1/2$ Higgs chiral supermultiplet can have the Yukawa couplings necessary to give masses to charge $+2/3$ up-type quarks (up, charm, top), and only a $Y = -1/2$ Higgs can have the Yukawa couplings necessary to give masses to charge $-1/3$ down-type quarks (down, strange, bottom) and to the charged leptons. We will call the $SU(2)_L$-doublet complex scalar fields with $Y = 1/2$ and $Y = -1/2$ by the names H_u and H_d, respectively.[4] The weak isospin components of H_u with $T_3 = (+1/2,\ -1/2)$ have electric charges 1, 0 respectively, and are denoted $(H_u^+,\ H_u^0)$. Similarly, the $SU(2)_L$-doublet complex scalar H_d has $T_3 = (+1/2,\ -1/2)$ components $(H_d^0,\ H_d^-)$. The neutral scalar that corresponds to the physical Standard Model Higgs boson is in a linear combination of H_u^0 and H_d^0; we will discuss this further in section 8.5. The generic nomenclature for a spin-1/2 superpartner is to append "-ino" to the name of the Standard Model particle, so the fermionic partners of the Higgs scalars are called higgsinos. They are denoted by $\widetilde{H}_u$, $\widetilde{H}_d$ for the $SU(2)_L$-doublet left-handed Weyl spinor fields, with weak isospin components $\widetilde{H}_u^+$, $\widetilde{H}_u^0$ and $\widetilde{H}_d^0$, $\widetilde{H}_d^-$.

We have now found all of the chiral supermultiplets of a minimal phenomenologically viable extension of the Standard Model. They are summarized in Table 1, classified according to their transformation properties under the Standard Model gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$, which combines u_L, d_L and ν, e_L degrees of freedom into $SU(2)_L$ doublets. Here we follow a standard convention that all chiral supermultiplets are defined in terms of left-handed Weyl spinors, so that the *conjugates* of the right-handed quarks and leptons (and their superpartners) appear in Table 1. This protocol for defining chiral supermultiplets turns out to be very useful for constructing supersymmetric Lagrangians, as we will see in Chapter 6. It is also useful to have a symbol for each of the chiral supermultiplets as a whole; these are indicated in the second column of Table 5.1. Thus, for example, Q stands for the $SU(2)_L$-doublet chiral supermultiplet containing $\widetilde{u}_L, u_L$ (with weak isospin component $T_3 = +1/2$), and $\widetilde{d}_L, d_L$ (with $T_3 = -1/2$), while $\overline{u}$ stands for the $SU(2)_L$-singlet supermultiplet containing $\widetilde{u}_R^*, u_R^\dagger$. There are three families for each of the quark and lepton supermultiplets; Table 5.1 lists the first-family representatives. Below, a family index $i = 1, 2, 3$ will be affixed to the chiral supermultiplet names $(Q_i,\ \overline{u}_i, \ldots)$ when needed,

[4] Other notations appearing in the literature have H_1, H_2 or $H, \overline{H}$ instead of H_u, H_d. The notation used here has the virtue of making it easy to remember which Higgs is responsible for giving masses to which type of quarks.

Table 5.1. Chiral supermultiplets in the Minimal Supersymmetric Standard Model. The spin-0 fields are complex scalars, and the spin-1/2 fields are left-handed two-component Weyl fermions.

Names		spin 0	spin 1/2	$SU(3)_C$, $SU(2)_L$, $U(1)_Y$
squarks, quarks	Q	$(\widetilde{u}_L \ \ \widetilde{d}_L)$	$(u_L \ \ d_L)$	$(\,\mathbf{3},\ \mathbf{2}\,,\ \frac{1}{6})$
($\times 3$ families)	$\overline{u}$	$\widetilde{u}^*_R$	$\bar{u}_R$	$(\,\overline{\mathbf{3}},\ \mathbf{1},\ -\frac{2}{3})$
	$\overline{d}$	$\widetilde{d}^*_R$	$\bar{d}_R$	$(\,\overline{\mathbf{3}},\ \mathbf{1},\ \frac{1}{3})$
sleptons, leptons	L	$(\widetilde{\nu}\ \widetilde{e}_L)$	$(\nu \ \ e_L)$	$(\,\mathbf{1},\ \mathbf{2}\,,\ -\frac{1}{2})$
($\times 3$ families)	$\overline{e}$	$\widetilde{e}^*_R$	$\bar{e}_R$	$(\,\mathbf{1},\ \mathbf{1},\ 1)$
Higgs, higgsinos	H_u	$(H_u^+ \ \ H_u^0)$	$(\widetilde{H}_u^+ \ \ \widetilde{H}_u^0)$	$(\,\mathbf{1},\ \mathbf{2}\,,\ +\frac{1}{2})$
	H_d	$(H_d^0 \ \ H_d^-)$	$(\widetilde{H}_d^0 \ \ \widetilde{H}_d^-)$	$(\,\mathbf{1},\ \mathbf{2}\,,\ -\frac{1}{2})$

e.g. $(\overline{e}_1, \overline{e}_2, \overline{e}_3) = (\overline{e}, \overline{\mu}, \overline{\tau})$. The bar on $\overline{u}$, $\overline{d}$, $\overline{e}$ fields is part of the name, and does not denote any kind of conjugation.

It is interesting to note that the Higgs chiral supermultiplet H_d (containing H_d^0, H_d^-, $\widetilde{H}_d^0$, $\widetilde{H}_d^-$) has exactly the same Standard Model gauge quantum numbers as the left-handed sleptons and leptons L_i, e.g. $(\widetilde{\nu}$, $\widetilde{e}_L$, ν, $e_L)$. Naively, one might therefore suppose that we could have been more economical in our assignment by taking a neutrino and a Higgs scalar to be superpartners, instead of putting them in separate supermultiplets. This would amount to the proposal that the Higgs boson and a sneutrino should be the same particle. This is a nice try that played a key role in some of the first attempts to connect supersymmetry to phenomenology, but it is now known to not work. Even ignoring the anomaly cancellation problem mentioned above, many insoluble phenomenological problems would result, including lepton number non-conservation and a mass for at least one of the neutrinos in gross violation of experimental bounds. Therefore, all of the superpartners of Standard Model particles are really new particles, and cannot be identified with some other Standard Model state.

The vector bosons of the Standard Model clearly must reside in gauge supermultiplets. Their fermionic superpartners are generically referred to as gauginos. The $SU(3)_C$ color gauge interactions of QCD are mediated by the gluon, whose spin-1/2 color-octet supersymmetric partner is the

Table 5.2. Gauge supermultiplets in the Minimal Supersymmetric Standard Model.

Names	spin 1/2	spin 1	$SU(3)_C$, $SU(2)_L$, $U(1)_Y$
gluino, gluon	$\widetilde{g}$	g	$(\,\mathbf{8},\ \mathbf{1}\ ,\ 0)$
winos, W bosons	$\widetilde{W}^{\pm}\ \ \widetilde{W}^0$	$W^{\pm}\ \ W^0$	$(\,\mathbf{1},\ \mathbf{3}\ ,\ 0)$
bino, B boson	$\widetilde{B}^0$	B^0	$(\,\mathbf{1},\ \mathbf{1}\ ,\ 0)$

gluino. As usual, a tilde is used to denote the supersymmetric partner of a Standard Model state, so the symbols for the gluon and gluino are g and $\widetilde{g}$ respectively. The electroweak gauge symmetry $SU(2)_L \times U(1)_Y$ has associated with it spin-1 gauge bosons W^+, W^0, W^- and B^0, with spin-1/2 superpartners $\widetilde{W}^+, \widetilde{W}^0, \widetilde{W}^-$ and $\widetilde{B}^0$, called *winos* and *bino*. After electroweak symmetry breaking, the W^0, B^0 gauge eigenstates mix to give mass eigenstates Z^0 and γ. The corresponding gaugino mixtures of $\widetilde{W}^0$ and $\widetilde{B}^0$ are called zino ($\widetilde{Z}^0$) and photino ($\widetilde{\gamma}$); if supersymmetry were unbroken, they would be mass eigenstates with masses m_Z and 0. Table 2 summarizes the gauge supermultiplets of a minimal supersymmetric extension of the Standard Model.

The chiral and gauge supermultiplets in Tables 1 and 2 make up the particle content of the Minimal Supersymmetric Standard Model (MSSM). The most obvious and interesting feature of this theory is that none of the superpartners of the Standard Model particles has been discovered as of this writing. If supersymmetry were unbroken, then there would have to be selectrons $\widetilde{e}_L$ and $\widetilde{e}_R$ with masses exactly equal to $m_e = 0.511...$ MeV. A similar statement applies to each of the other sleptons and squarks, and there would also have to be a massless gluino and photino. These particles would have been extraordinarily easy to detect long ago. Clearly, therefore, supersymmetry is a broken symmetry in the vacuum state chosen by Nature.

A very important clue as to the nature of supersymmetry breaking can be obtained by returning to the motivation provided by the hierarchy problem. Supersymmetry forced us to introduce two complex scalar fields for each Standard Model Dirac fermion, which is just what is needed to enable a cancellation of the quadratically divergent ($\Lambda_{\rm UV}^2$) pieces of eqs. (5.2) and (5.3). This sort of cancellation also requires that the associated dimensionless couplings should be related (e.g. $\lambda_S = |\lambda_f|^2$). The necessary relationships between couplings indeed occur in unbroken supersymmetry, as we will see in Chapter 6. In fact, unbroken supersymmetry guarantees that the quadratic divergences in scalar squared masses must vanish to all orders in perturbation theory.[5] Now, if broken supersymmetry is still to provide a solution to the hierarchy problem, then the relationships between dimensionless couplings that hold in an unbroken supersymmetric theory must be maintained. Otherwise, there would be quadratically divergent radiative corrections to the Higgs scalar masses of the form

$$\Delta m_H^2 = \frac{1}{8\pi^2}(\lambda_S - |\lambda_f|^2)\Lambda_{\rm UV}^2 + \dots. \tag{5.11}$$

We are therefore led to consider "soft" supersymmetry breaking. This means that the effective Lagrangian of the MSSM can be written in the form

$$\mathcal{L} = \mathcal{L}_{\rm SUSY} + \mathcal{L}_{\rm soft}, \tag{5.12}$$

where $\mathcal{L}_{\rm SUSY}$ contains all of the gauge and Yukawa interactions and preserves supersymmetry invariance, and $\mathcal{L}_{\rm soft}$ violates supersymmetry but contains only mass terms and couplings with *positive* mass dimension. Without further justification, soft supersymmetry breaking might seem like a rather arbitrary requirement. Fortunately, we will see in section 9 that theoretical models for supersymmetry breaking do indeed yield effective Lagrangians with just such terms for $\mathcal{L}_{\rm soft}$. If the largest mass scale associated with the soft terms is denoted $m_{\rm soft}$, then the additional non-supersymmetric corrections to the Higgs scalar $({\rm mass})^2$ must vanish in the $m_{\rm soft} \to 0$ limit, so by dimensional analysis they cannot be proportional to $\Lambda_{\rm UV}^2$. More generally, these models maintain the cancellation of quadratically divergent terms in the radiative corrections of all scalar masses, to all orders in perturbation theory. The corrections also cannot go like $\Delta m_H^2 \sim m_{\rm soft}\Lambda_{\rm UV}$, because in general the loop momentum integrals always diverge either quadratically or logarithmically, not linearly, as $\Lambda_{\rm UV} \to \infty$. So they must be of the form

$$\Delta m_H^2 = m_{\rm soft}^2 \left[\frac{\lambda}{16\pi^2}\ln(\Lambda_{\rm UV}/m_{\rm soft}) + \dots\right]. \tag{5.13}$$

Here λ is schematic for various dimensionless couplings, and the ellipses stand both for terms that are independent of $\Lambda_{\rm UV}$ and for higher loop corrections (which depend on $\Lambda_{\rm UV}$ through powers of logarithms).

[5] A simple way to understand this is to note that unbroken supersymmetry requires the degeneracy of scalar and fermion masses. Radiative corrections to fermion masses are known to diverge at most logarithmically, so the same must be true for scalar masses in unbroken supersymmetry.

Because the mass splittings between the known Standard Model particles and their superpartners are just determined by the parameters $m_{\rm soft}$ appearing in $\mathcal{L}_{\rm soft}$, eq. (5.13) tells us that the superpartner masses cannot be too huge. Otherwise, we would lose our successful cure for the hierarchy problem since the $m^2_{\rm soft}$ corrections to the Higgs scalar (mass)^2 would be unnaturally large compared to the square of the electroweak breaking scale of 174 GeV. The top and bottom squarks and the winos and bino give especially large contributions to $\Delta m^2_{H_u}$ and $\Delta m^2_{H_d}$, but the gluino mass and all the other squark and slepton masses also feed in indirectly, through radiative corrections to the top and bottom squark masses. Furthermore, in most viable models of supersymmetry breaking that are not unduly contrived, the superpartner masses do not differ from each other by more than about an order of magnitude. Using $\Lambda_{\rm UV} \sim M_{\rm P}$ and $\lambda \sim 1$ in eq. (5.13), one finds that roughly speaking $m_{\rm soft}$, and therefore the masses of at least the lightest few superpartners, should be at the most about 1 TeV or so, in order for the MSSM scalar potential to provide a Higgs VEV resulting in $m_W, m_Z = 80.4,\ 91.2$ GeV without miraculous cancellations. This is the best reason for the optimism among many theorists that supersymmetry will be discovered at the Fermilab Tevatron or the CERN Large Hadron Collider, and can be studied at a future e^+e^- linear collider.

However, it is useful to keep in mind that the hierarchy problem was *not* the historical motivation for the development of supersymmetry in the early 1970's. The supersymmetry algebra and supersymmetric field theories were originally concocted independently in various disguises bearing little resemblance to the MSSM. It is quite impressive that a theory that was developed for quite different reasons, including purely aesthetic ones, can later be found to provide a solution for the hierarchy problem.

One might also wonder whether there is any good reason why all of the superpartners of the Standard Model particles should be heavy enough to have avoided discovery so far. There is. All of the particles in the MSSM that have been detected so far have something in common; they would necessarily be massless in the absence of electroweak symmetry breaking. In particular, the masses of the $W^\pm, Z^0$ bosons and all quarks and leptons are equal to dimensionless coupling constants times the Higgs VEV $\sim$ 174 GeV, while the photon and gluon are required to be massless by electromagnetic and QCD gauge invariance. Conversely, all of the undiscovered particles in the MSSM have exactly the opposite property; each of them can have a Lagrangian mass term in the absence of electroweak symmetry breaking. For the squarks, sleptons, and Higgs scalars this follows from a general property of complex scalar fields that a mass term $m^2|\phi|^2$ is always allowed by all gauge symmetries. For the higgsinos and gauginos, it follows from the fact that they are fermions in a real representation of the gauge group. So, from the point of view of the MSSM, the discovery of the top quark in 1995 marked a quite natural milestone; the already-discovered particles are precisely those that had to be light, based on the principle of electroweak gauge symmetry. There is a single exception: one neutral Higgs scalar boson should be lighter than about 150 GeV if supersymmetry is correct, for reasons to be discussed in section 8.5.

A very important feature of the MSSM is that the superpartners listed in Tables 1 and 2 are not necessarily the mass eigenstates of the theory. This is because after electroweak symmetry breaking and supersymmetry breaking effects are included, there can be mixing between the electroweak gauginos and the higgsinos, and within the various sets of squarks and sleptons and Higgs scalars that have the same electric charge. The lone exception is the gluino, which is a color octet fermion and therefore does not have the appropriate quantum numbers to mix with any other particle. The masses and mixings of the superpartners are obviously of paramount importance to experimentalists. It is perhaps slightly less obvious that these phenomenological issues are all quite directly related to one central question that is also the focus of much of the theoretical work in supersymmetry: "How is supersymmetry broken?" The reason for this is that most of what we do not already know about the MSSM has to do with $\mathcal{L}_{\rm soft}$. The structure of supersymmetric Lagrangians allows very little arbitrariness, as we will see in Chapter 6. In fact, all of the dimensionless couplings and all but one mass term in the supersymmetric part of the MSSM Lagrangian correspond directly to parameters in the ordinary Standard Model that have already been measured by experiment. For example, we will find out that the supersymmetric coupling of a gluino to a squark and a quark is determined by the QCD coupling constant α_s. In contrast, the supersymmetry-breaking part of the Lagrangian contains many unknown parameters and, apparently, a considerable amount of arbitrariness. Each of the mass splittings between Standard Model particles and their superpartners correspond to terms in the MSSM Lagrangian that are purely supersymmetry-breaking in their origin and effect. These soft supersymmetry-breaking terms can also introduce a large number of mixing angles and CP-violating phases not found in the Standard Model. Fortunately, as we will see in section 8.4, there is already strong evidence that the supersymmetry-breaking terms in the MSSM are actually not arbitrary at all. Furthermore, the additional parameters will be measured and constrained as the superpartners are detected. From a theoretical perspective, the challenge is to explain all of these parameters with a predictive model for supersymmetry breaking.

5.3 Historical analogies

In this section, we recount some entertaining examples of historical analogies to supersymmetry. Dirac predicted the positron. Gell-Mann predicted the Ω^-. However, nobody predicted the Spanish Inquisition!

6

Supersymmetric Lagrangians

In this chapter we will describe the construction of supersymmetric Lagrangians. Our aim is to arrive at a sort of recipe that will allow us to write down the allowed interactions and mass terms of a general supersymmetric theory, so that later we can apply the results to the special case of the MSSM. We will not use the superfield language, which is often more elegant and efficient for those who know it, but which might seem rather cabalistic to some readers. Our approach is therefore intended to be rather complementary to the superfield derivations given in Chapter 7. We begin by considering the simplest example of a supersymmetric theory in four dimensions; the free Wess-Zumino Model.

6.1 A free chiral supermultiplet

The minimum fermion content of any theory in four dimensions consists of a single left-handed two-component Weyl fermion ψ. Since this is an intrinsically complex object, it seems sensible to choose as its superpartner a complex scalar field ϕ. The simplest action we can write down for these fields just consists of kinetic energy terms for each:

$$S = \int d^4x \ (\mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{fermion}}) \tag{6.1}$$

$$\mathcal{L}_{\text{scalar}} = \partial^\mu \phi^* \partial_\mu \phi, \qquad \mathcal{L}_{\text{fermion}} = i\bar{\psi}\overline{\sigma}^\mu \partial_\mu \psi. \tag{6.2}$$

This is called the massless, non-interacting *Wess-Zumino model*, and it corresponds to a single chiral supermultiplet as discussed in the Introduction.

A supersymmetry transformation should turn the scalar boson field ϕ into something involving the fermion field ψ_α. The simplest possibility for the transformation of the scalar field is

$$\delta\phi = \epsilon\psi; \qquad\qquad \delta\phi^* = \bar{\epsilon}\bar{\psi}, \tag{6.3}$$

where ϵ^α is an infinitesimal, anticommuting, two-component Weyl fermion object that parameterizes the supersymmetry transformation. Until section 9.2, we will be discussing global supersymmetry, which means that ϵ^α is a constant, satisfying $\partial_\mu \epsilon^\alpha = 0$. Since ψ has dimensions of $(\text{mass})^{3/2}$ and ϕ has dimensions of (mass), it must be that ϵ has dimensions of $(\text{mass})^{-1/2}$. Using eq. (6.3), we find that the scalar part of the Lagrangian transforms as

$$\delta\mathcal{L}_{\text{scalar}} = \epsilon\partial^\mu\psi\,\partial_\mu\phi^* + \bar{\epsilon}\partial^\mu\bar{\psi}\,\partial_\mu\phi. \tag{6.4}$$

We would like for this to be cancelled by $\delta\mathcal{L}_{\text{fermion}}$, at least up to a total derivative, so that the action will be invariant under the supersymmetry transformation. Comparing eq. (6.4) with $\mathcal{L}_{\text{fermion}}$, we see that for this to have any chance of happening, $\delta\psi$ should be linear in $\bar{\epsilon}$ and in ϕ and contain one spacetime derivative. Up to a multiplicative constant, there is only one possibility to try:

$$\delta\psi_\alpha = -i(\sigma^\mu\bar{\epsilon})_\alpha\,\partial_\mu\phi; \qquad\qquad \delta\bar{\psi}_{\dot{\alpha}} = i(\epsilon\sigma^\mu)_{\dot{\alpha}}\,\partial_\mu\phi^*. \tag{6.5}$$

With this guess, one immediately obtains

$$\delta\mathcal{L}_{\text{fermion}} = -\epsilon\sigma^\mu\overline{\sigma}^\nu\partial_\nu\psi\,\partial_\mu\phi^* + \bar{\psi}\overline{\sigma}^\nu\sigma^\mu\bar{\epsilon}\,\partial_\mu\partial_\nu\phi\,. \tag{6.6}$$

This can be put in a slightly more useful form by employing the Pauli matrix identities

$$\left[\sigma^\mu\overline{\sigma}^\nu + \sigma^\nu\overline{\sigma}^\mu\right]_\alpha^{\ \beta} = 2\eta^{\mu\nu}\delta_\alpha^\beta; \qquad \left[\overline{\sigma}^\mu\sigma^\nu + \overline{\sigma}^\nu\sigma^\mu\right]^{\dot{\alpha}}_{\ \dot{\beta}} = 2\eta^{\mu\nu}\delta^{\dot{\alpha}}_{\dot{\beta}} \tag{6.7}$$

and using the fact that partial derivatives commute ($\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$). Equation (6.6) then becomes

$$\begin{aligned}\delta\mathcal{L}_{\text{fermion}} = {} & -\epsilon\partial^\mu\psi\,\partial_\mu\phi^* - \bar{\epsilon}\partial^\mu\bar{\psi}\,\partial_\mu\phi \\ & -\partial_\mu\left(\epsilon\sigma^\nu\overline{\sigma}^\mu\psi\,\partial_\nu\phi^* - \epsilon\psi\,\partial^\mu\phi^* - \bar{\epsilon}\bar{\psi}\,\partial^\mu\phi\right).\end{aligned} \tag{6.8}$$

The first two terms here just cancel against $\delta\mathcal{L}_{\text{scalar}}$, while the remaining contribution is a total derivative. So we arrive at

$$\delta S = \int d^4x\ \ (\delta\mathcal{L}_{\text{scalar}} + \delta\mathcal{L}_{\text{fermion}}) = 0, \tag{6.9}$$

justifying our guess of the numerical multiplicative factor made in eq. (6.5).

We are not quite finished in demonstrating that the theory described by eq. (6.1) is supersymmetric. We must also show that the supersymmetry algebra closes; in other words, that the commutator of two supersymmetry transformations parameterized by spinors ϵ_1 and ϵ_2 is another symmetry of the theory. Using eq. (6.5) in eq. (6.3), one finds

$$\begin{aligned}(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\phi &\equiv \delta_{\epsilon_2}(\delta_{\epsilon_1}\phi) - \delta_{\epsilon_1}(\delta_{\epsilon_2}\phi) \\ &= -i(\epsilon_1\sigma^\mu\bar{\epsilon}_2 - \epsilon_2\sigma^\mu\bar{\epsilon}_1)\,\partial_\mu\phi. \end{aligned} \tag{6.10}$$

This is a remarkable result; in words, we have found that the commutator of two supersymmetry transformations gives us back the derivative of the original field. Since ∂_μ just corresponds to the generator of spacetime translations P_μ, eq. (6.10) implies the form of the supersymmetry algebra that was foreshadowed in eq. (5.6) of the Introduction. (We will make this statement more explicit before the end of this chapter.)

All of this will be for nothing if we do not find the same result for the fermion ψ, however. Using eq. (6.3) in eq. (6.5), we find

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\psi_\alpha = -i(\sigma^\mu\bar{\epsilon}_1)_\alpha\,\epsilon_2\partial_\mu\psi + i(\sigma^\mu\bar{\epsilon}_2)_\alpha\,\epsilon_1\partial_\mu\psi. \tag{6.11}$$

We can put this into a more useful form by applying the Fierz identity

$$\chi_\alpha\,(\xi\eta) = -\xi_\alpha\,(\eta\chi) - \eta_\alpha\,(\chi\xi) \tag{6.12}$$

with $\chi = \sigma^\mu\bar{\epsilon}_1$, $\xi = \epsilon_2$, $\eta = \partial_\mu\psi$, and again with $\chi = \sigma^\mu\bar{\epsilon}_2$, $\xi = \epsilon_1$, $\eta = \partial_\mu\psi$, followed in each case by an application of the identity eq. (1.72). The result is

$$\begin{aligned}(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\psi_\alpha &= -i(\epsilon_1\sigma^\mu\bar{\epsilon}_2 - \epsilon_2\sigma^\mu\bar{\epsilon}_1)\,\partial_\mu\psi_\alpha \\ &\quad + i\epsilon_{1\alpha}\,\bar{\epsilon}_2\overline{\sigma}^\mu\partial_\mu\psi - i\epsilon_{2\alpha}\,\bar{\epsilon}_1\overline{\sigma}^\mu\partial_\mu\psi. \end{aligned} \tag{6.13}$$

The last two terms in (6.13) vanish on-shell; that is, if the equation of motion $\overline{\sigma}^\mu\partial_\mu\psi = 0$ following from the action is enforced. The remaining piece is exactly the same spacetime translation that we found for the scalar field.

The fact that the supersymmetry algebra only closes on-shell (when the classical equations of motion are satisfied) might be somewhat worrisome, since we would like the symmetry to hold even quantum mechanically. This can be fixed by a trick. We invent a new complex scalar field F, which does not have a kinetic term. Such fields are called *auxiliary*, and they are really just book-keeping devices that allow the symmetry algebra to close off-shell. The Lagrangian density for F and its complex conjugate is just

$$\mathcal{L}_{\text{auxiliary}} = F^*F\,. \tag{6.14}$$

The dimensions of F are $(\text{mass})^2$, unlike an ordinary scalar field, which has dimensions of (mass). Equation (6.14) implies the not-very-exciting equations of motion $F = F^* = 0$. However, we can use the auxiliary fields to our advantage by including them in the supersymmetry transformation rules. In view of eq. (6.13), a plausible thing to do is to make F transform into a multiple of the equation of motion for ψ:

$$\delta F = -i\epsilon\overline{\sigma}^\mu \partial_\mu \psi; \qquad \delta F^* = i\partial_\mu \bar{\psi}\overline{\sigma}^\mu \epsilon. \tag{6.15}$$

Once again we have chosen the overall factor on the right-hand sides by virtue of foresight. Now the auxiliary part of the Lagrangian density transforms as

$$\delta\mathcal{L}_{\text{auxiliary}} = -i\epsilon\overline{\sigma}^\mu \partial_\mu \psi \, F^* + i\partial_\mu \bar{\psi}\overline{\sigma}^\mu \epsilon \, F, \tag{6.16}$$

which vanishes on-shell, but not for arbitrary off-shell field configurations. It is easy to see that by adding an extra term to the transformation law for ψ and $\bar{\psi}$:

$$\delta\psi_\alpha = -i(\sigma^\mu \bar{\epsilon})_\alpha \, \partial_\mu \phi + \epsilon_\alpha F; \qquad \delta\bar{\psi}_{\dot{\alpha}} = i(\epsilon\sigma^\mu)_{\dot{\alpha}} \, \partial_\mu \phi^* + \bar{\epsilon}_{\dot{\alpha}} F^* \tag{6.17}$$

one obtains an additional contribution to $\delta\mathcal{L}_{\text{fermion}}$, which just cancels with $\delta\mathcal{L}_{\text{auxiliary}}$ up to a total derivative term. So our "modified" theory with $\mathcal{L} = \mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{fermion}} + \mathcal{L}_{\text{auxiliary}}$ is still invariant under supersymmetry transformations. Proceeding as before, one now obtains for each of the fields $X = \phi, \phi^*, \psi, \bar{\psi}, F, F^*$,

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})X = -i(\epsilon_1\sigma^\mu\bar{\epsilon}_2 - \epsilon_2\sigma^\mu\bar{\epsilon}_1) \, \partial_\mu X \tag{6.18}$$

using eqs. (6.3), (6.15), and (6.17), but without resorting to any of the equations of motion. So we have succeeded in showing that supersymmetry is a valid symmetry of the Lagrangian off-shell.

In retrospect, one can see why we needed to introduce the auxiliary field F in order to get the supersymmetry algebra to work off-shell. On-shell, the complex scalar field ϕ has two real propagating degrees of freedom, which match with the two spin polarization states of ψ. Off-shell, however, the Weyl fermion ψ is a complex two-component object, so it has four real degrees of freedom. (Going on-shell eliminates half of the propagating degrees of freedom for ψ, because the Lagrangian is linear in time derivatives, so that the canonical momenta can be reexpressed in terms of the configuration variables without time derivatives and are not independent phase space coordinates.) To make the numbers of bosonic and fermionic degrees of freedom match off-shell as well as on-shell, we had to introduce two more real scalar degrees of freedom in the complex field F, which are eliminated when one goes on-shell. This counting is

Table 6.1. Counting of real degrees of freedom in the Wess-Zumino model.

	ϕ	ψ	F
on-shell ($n_B = n_F = 2$)	2	2	0
off-shell ($n_B = n_F = 4$)	2	4	2

summarized in Table 6.1. The auxiliary field formulation is especially useful when discussing spontaneous supersymmetry breaking, as we will see in section 9.

Invariance of the action under a symmetry transformation always implies the existence of a conserved current, and supersymmetry is no exception. The *supercurrent* J^μ_α is an anticommuting four-vector, which also carries a spinor index, as befits the current associated with a symmetry with fermionic generators. By the usual Noether procedure, one finds for the supercurrent (and its hermitian conjugate) in terms of the variations of the fields $X = \phi, \phi^*, \psi, \bar{\psi}, F, F^*$:

$$\epsilon J^\mu + \bar{\epsilon}\bar{J}^\mu \equiv \sum_X \delta X \, \frac{\delta \mathcal{L}}{\delta(\partial_\mu X)} - K^\mu, \tag{6.19}$$

where K^μ is an object whose divergence is the variation of the Lagrangian density under the supersymmetry transformation, $\delta\mathcal{L} = \partial_\mu K^\mu$. Note that K^μ is not unique; one can always replace K^μ by $K^\mu + k^\mu$, where k^μ is any vector satisfying $\partial_\mu k^\mu = 0$, for example $k^\mu = \partial^\mu \partial_\nu a^\nu - \partial_\nu \partial^\nu a^\mu$. A little work reveals that, up to the ambiguity just mentioned,

$$J^\mu_\alpha = (\sigma^\nu \overline{\sigma}^\mu \psi)_\alpha \, \partial_\nu \phi^*; \qquad \bar{J}^\mu_{\dot{\alpha}} = (\bar{\psi} \overline{\sigma}^\mu \sigma^\nu)_{\dot{\alpha}} \, \partial_\nu \phi. \tag{6.20}$$

The supercurrent and its hermitian conjugate are separately conserved:

$$\partial_\mu J^\mu_\alpha = 0; \qquad \partial_\mu \bar{J}^\mu_{\dot{\alpha}} = 0, \tag{6.21}$$

as can be verified by use of the equations of motion. From these currents one constructs the conserved charges

$$Q_\alpha = \sqrt{2} \int d^3x \, J^0_\alpha; \qquad \bar{Q}_{\dot{\alpha}} = \sqrt{2} \int d^3x \, \bar{J}^0_{\dot{\alpha}}, \tag{6.22}$$

which are the generators of supersymmetry transformations. (The factor of $\sqrt{2}$ normalization is included to agree with an arbitrary historical convention.) As quantum mechanical operators, they satisfy

$$\left[\epsilon Q + \bar{\epsilon}\bar{Q}, X\right] = -i\sqrt{2}\,\delta X \tag{6.23}$$

for any field X, up to terms that vanish on-shell. This can be verified explicitly by using the canonical equal-time commutation and anticommutation relations

$$[\phi(x), \pi(y)] = [\phi^*(x), \pi^*(y)] = i\delta^{(3)}(x-y); \tag{6.24}$$
$$\{\bar{\psi}_{\dot{\alpha}}(x), \psi_\alpha(y)\} = (\sigma_0)_{\alpha\dot{\alpha}}\, \delta^{(3)}(x-y) \tag{6.25}$$

derived from the free field theory Lagrangian eq. (6.1). Here $\pi = \partial_0\phi^*$ and $\pi^* = \partial_0\phi$ are the momenta conjugate to ϕ and ϕ^* respectively. Now the content of eq. (6.18) can be expressed in terms of canonical commutators as

$$\Big[\epsilon_2 Q + \bar{\epsilon}_2\bar{Q},\, \big[\epsilon_1 Q + \bar{\epsilon}_1\bar{Q},\, X\big]\Big] - \Big[\epsilon_1 Q + \bar{\epsilon}_1\bar{Q},\, \big[\epsilon_2 Q + \bar{\epsilon}_2\bar{Q},\, X\big]\Big] = -2(\epsilon_2\sigma^\mu\bar{\epsilon}_1 - \epsilon_1\sigma^\mu\bar{\epsilon}_2)\, i\partial_\mu X \tag{6.26}$$

up to terms that vanish on-shell. The spacetime momentum operator $P^\mu = (H, \vec{P})$, where H is the Hamiltonian and $\vec{P}$ is the three-momentum operator, is given in terms of the canonical variables by

$$H = \pi^*\pi + (\vec{\nabla}\phi^*)\cdot(\vec{\nabla}\phi) + i\bar{\psi}\vec{\sigma}\cdot\vec{\nabla}\psi \tag{6.27}$$
$$\vec{P} = -\pi\vec{\nabla}\phi - \pi^*\vec{\nabla}\phi^* - i\bar{\psi}\overline{\sigma}_0\vec{\nabla}\psi. \tag{6.28}$$

It generates spacetime translations on the fields X according to

$$[P_\mu, X] = -i\partial_\mu X. \tag{6.29}$$

Rearranging the terms in eq. (6.26) using the Jacobi identity, we therefore have

$$\Big[\big[\epsilon_2 Q + \bar{\epsilon}_2\bar{Q},\, \epsilon_1 Q + \bar{\epsilon}_1\bar{Q}\big],\, X\Big] = 2(\epsilon_2\sigma_\mu\bar{\epsilon}_1 - \epsilon_1\sigma_\mu\bar{\epsilon}_2)\,[P^\mu, X], \tag{6.30}$$

for any X, so it must be that

$$\big[\epsilon_2 Q + \bar{\epsilon}_2\bar{Q},\, \epsilon_1 Q + \bar{\epsilon}_1\bar{Q}\big] = 2(\epsilon_2\sigma_\mu\bar{\epsilon}_1 - \epsilon_1\sigma_\mu\bar{\epsilon}_2)\, P^\mu, \tag{6.31}$$

up to terms that vanish on-shell. Now by expanding out eq. (6.31), one obtains the non-schematic form of the supersymmetry algebra relations

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu, \tag{6.32}$$
$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \tag{6.33}$$

as promised in the Introduction. [The commutator in eq. (6.31) turns into anticommutators in eqs. (6.32) and (6.33) in the process of extracting the anticommuting spinors ϵ_1 and ϵ_2.] The results $[Q_\alpha, P_\mu] = 0$ and $[\bar{Q}_{\dot{\alpha}}, P_\mu] = 0$ follow immediately from eq. (6.29) and the fact that the supersymmetry transformations are global (independent of position in spacetime). This demonstration of the supersymmetry algebra in terms of the canonical generators Q and $\bar{Q}$ requires the use of the Hamiltonian equations of motion, but the symmetry itself is valid off-shell at the level of the Lagrangian, as we have already shown.

6.2 Interactions of chiral supermultiplets

In a realistic theory like the MSSM, there are many chiral supermultiplets that have both gauge and non-gauge interactions. In this section, our task is to construct the most general possible theory of masses and non-gauge interactions for particles that live in chiral supermultiplets. In the MSSM these are the quarks, squarks, leptons, sleptons, Higgs scalars and higgsino fermions. We will find that the form of the non-gauge couplings, including mass terms, is highly restricted by the requirement that the action is invariant under supersymmetry transformations. (Gauge interactions will be dealt with in the following sections.)

Our starting point is the Lagrangian density for a collection of free chiral supermultiplets labeled by an index i, which runs over all gauge and flavor degrees of freedom. Since we will want to construct an interacting theory with supersymmetry closing off-shell, each supermultiplet contains a complex scalar ϕ_i and a left-handed Weyl fermion ψ_i as physical degrees of freedom, plus a complex auxiliary field F_i, which does not propagate. The results of the previous section tell us that the free part of the Lagrangian is

$$\mathcal{L}_{\text{free}} = \partial^\mu \phi^{*i} \partial_\mu \phi_i + i\bar{\psi}^i \overline{\sigma}^\mu \partial_\mu \psi_i + F^{*i} F_i \tag{6.34}$$

where we sum over repeated indices i (not to be confused with the suppressed spinor indices), with the convention that fields ϕ_i and ψ_i always carry lowered indices, while their conjugates always carry raised indices. It is invariant under the supersymmetry transformation

$$\delta\phi_i = \epsilon\psi_i, \qquad \delta\phi^{*i} = \bar{\epsilon}\bar{\psi}^i, \tag{6.35}$$
$$\delta(\psi_i)_\alpha = -i(\sigma^\mu\bar{\epsilon})_\alpha\, \partial_\mu\phi_i + \epsilon_\alpha F_i, \tag{6.36}$$
$$\delta(\bar{\psi}^i)_{\dot{\alpha}} = i(\epsilon\sigma^\mu)_{\dot{\alpha}}\, \partial_\mu\phi^{*i} + \bar{\epsilon}_{\dot{\alpha}} F^{*i}, \tag{6.37}$$
$$\delta F_i = -i\bar{\epsilon}\overline{\sigma}^\mu \partial_\mu \psi_i, \qquad \delta F^{*i} = i\partial_\mu \bar{\psi}^i \overline{\sigma}^\mu \epsilon\, . \tag{6.38}$$

We will now find the most general set of renormalizable interactions for these fields that is consistent with supersymmetry. To begin, note that in order to be renormalizable by power counting, each term must have dynamical field content with mass dimension ≤ 4. So, the only candidates are:

$$\mathcal{L}_{\text{int}} = \left(-\tfrac{1}{2} W^{ij}\psi_i\psi_j + W^i F_i + x^{ij} F_i F_j\right) + \text{c.c.} + U, \tag{6.39}$$

where W^{ij}, W^i, x^{ij}, and U are polynomials in the scalar fields ϕ_i, ϕ^{*i}, with degrees 1, 2, 0, and 4, respectively. [Terms of the form $F_i F^*_j$ can be absorbed, by a redefinition of the auxiliary fields, into the last term in equation (6.34).]

We must now require that $\mathcal{L}_{\rm int}$ is invariant under the supersymmetry transformations, since $\mathcal{L}_{\rm free}$ was already invariant by itself. This immediately requires that the candidate term $U(\phi_i, \phi^{*i})$ must vanish. If there were such a term, then under a supersymmetry transformation eq. (6.35) it would transform into another function of the scalar fields only, multiplied by $\epsilon\psi_i$ or $\bar{\epsilon}\bar{\psi}^i$, and with no spacetime derivatives or F_i, F^{*i} fields. It is easy to see from eqs. (6.35)-(6.40) that nothing of this form can possibly be cancelled by the supersymmetry transformation of any other term in the Lagrangian. Similarly, the dimensionless coupling x^{ij} must be zero, because its supersymmetry transformation likewise cannot possibly be cancelled by any other term. So, we are left with

$$\mathcal{L}_{\rm int} = \left(-\tfrac{1}{2} W^{ij}\psi_i\psi_j + W^i F_i\right) + \text{c.c.} \tag{6.40}$$

as the only possibilities. At this point, we are not assuming that W^{ij} and W^i are related to each other in any way whatsoever. However, soon we will find out that they *are* related, which is why we have chosen the same letter for them. Notice that eq. (1.54) tells us that W^{ij} is symmetric under $i \leftrightarrow j$.

It is easiest to divide the variation of $\mathcal{L}_{\rm int}$ into several parts, which must cancel separately. First, we consider the part that contains four spinors:

$$\begin{aligned}&\delta\mathcal{L}_{\rm int}|_{4-\rm spinor} = \\ &\left[-\tfrac{1}{2}\frac{\delta W^{ij}}{\delta\phi_k}(\epsilon\psi_k)(\psi_i\psi_j) - \tfrac{1}{2}\frac{\delta W^{ij}}{\delta\phi^{*k}}(\bar{\epsilon}\bar{\psi}^k)(\psi_i\psi_j)\right] + \text{c.c.}\end{aligned} \tag{6.41}$$

The term proportional to $(\epsilon\psi_k)(\psi_i\psi_j)$ cannot cancel against any other term. Fortunately, however, the Fierz identity eq. (6.12) implies

$$(\epsilon\psi_i)(\psi_j\psi_k) + (\epsilon\psi_j)(\psi_k\psi_i) + (\epsilon\psi_k)(\psi_i\psi_j) = 0, \tag{6.42}$$

so this contribution to $\delta\mathcal{L}_{\rm int}$ vanishes identically if and only if $\delta W^{ij}/\delta\phi_k$ is totally symmetric under interchange of i, j, k. There is no such identity available for the term proportional to $(\bar{\epsilon}\bar{\psi}^k)(\psi_i\psi_j)$. Since that term cannot cancel with any other, requiring it to be absent just tells us that W^{ij} cannot contain ϕ^{*k}. In other words, W^{ij} is *analytic* (or *holomorphic*) in the complex fields ϕ_k.

Combining what we have learned so far, we can write

$$W^{ij} = M^{ij} + y^{ijk}\phi_k \tag{6.43}$$

where M^{ij} is a symmetric mass matrix for the fermion fields, and y^{ijk} is a Yukawa coupling of a scalar ϕ_k and two fermions $\psi_i\psi_j$ that must be totally symmetric under interchange of i, j, k. It is convenient to write

$$W^{ij} = \frac{\delta^2}{\delta\phi_i\delta\phi_j}W \tag{6.44}$$

where we have introduced a very useful object

$$W = \frac{1}{2}M^{ij}\phi_i\phi_j + \frac{1}{6}y^{ijk}\phi_i\phi_j\phi_k, \tag{6.45}$$

called the *superpotential*. This is not a scalar potential in the ordinary sense; in fact, it is not even real. It is instead an analytic function of the scalar fields ϕ_i treated as complex variables.

Continuing on our vaunted quest, we next consider the parts of $\delta\mathcal{L}_{\text{int}}$ that contain a spacetime derivative:

$$\delta\mathcal{L}_{\text{int}}|_\partial = \left(iW^{ij}\partial_\mu\phi_j\,\psi_i\sigma^\mu\bar{\epsilon} + iW^i\,\partial_\mu\psi_i\sigma^\mu\bar{\epsilon}\right) + \text{c.c.} \tag{6.46}$$

Here we have used the identity eq. (1.72) on the second term, which came from $(\delta F_i)W^i$. Now we can use eq. (6.44) to observe that

$$W^{ij}\partial_\mu\phi_j = \partial_\mu\left(\frac{\delta W}{\delta\phi_i}\right). \tag{6.47}$$

Therefore, eq. (6.46) will be a total derivative if and only if

$$W^i = \frac{\delta W}{\delta\phi_i} = M^{ij}\phi_j + \frac{1}{2}y^{ijk}\phi_j\phi_k\,, \tag{6.48}$$

which explains why we chose its name as we did. The remaining terms in $\delta\mathcal{L}_{\text{int}}$ are all linear in F_i or F^{*i}, and it is easy to show that they cancel, given the results for W^i and W^{ij} that we have already found.

To recap, we have found that the most general non-gauge interactions for chiral supermultiplets are determined by a single analytic function of the complex scalar fields, the superpotential W. The auxiliary fields F_i and F^{*i} can be eliminated using their classical equations of motion. The part of $\mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}$ that contains the auxiliary fields is $F_iF^{*i} + W^iF_i + W_i^*F^{*i}$, leading to the equations of motion

$$F_i = -W_i^*; \qquad F^{*i} = -W^i\,. \tag{6.49}$$

Thus the auxiliary fields are expressible algebraically (without any derivatives) in terms of the scalar fields. After making the replacement eq. (6.49) in $\mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}$, we obtain the Lagrangian density

$$\mathcal{L} = \partial^\mu\phi^{*i}\partial_\mu\phi_i + i\bar{\psi}^i\bar{\sigma}^\mu\partial_\mu\psi_i - \tfrac{1}{2}\left(W^{ij}\psi_i\psi_j + W_{ij}^*\bar{\psi}^i\bar{\psi}^j\right) - W^iW_i^* \tag{6.50}$$

(Since F_i and F^{*i} appear only quadratically in the action, the result of instead doing a functional integral over them at the quantum level has precisely the same effect.) Now that the non-propagating fields F_i, F^{*i} have been eliminated, it is clear from eq. (6.50) that the scalar potential for the theory is just given in terms of the superpotential by (recall $\mathcal{L}$ contains $-V$):

$$V(\phi,\phi^*) = W^i W_i^* = F^{*i} F_i = M_{ik}^* M^{kj} \phi^{*i} \phi_j + \frac{1}{2} M^{in} y_{jkn}^* \phi_i \phi^{*j} \phi^{*k} + \frac{1}{2} M_{in}^* y^{jkn} \phi^{*i} \phi_j \phi_k + \frac{1}{4} y^{ijn} y_{kln}^* \phi_i \phi_j \phi^{*k} \phi^{*l}\,. \tag{6.51}$$

This scalar potential is automatically bounded from below; in fact, since it is a sum of squares of absolute values (of the W^i), it is always non-negative. If we substitute the general form for the superpotential eq. (6.45) into eq. (6.50), we obtain for the full Lagrangian density

$$\mathcal{L} = \partial^\mu \phi^{*i} \partial_\mu \phi_i + i \bar{\psi}^i \overline{\sigma}^\mu \partial_\mu \psi_i - \tfrac{1}{2} M^{ij} \psi_i \psi_j - \tfrac{1}{2} M_{ij}^* \bar{\psi}^i \bar{\psi}^j - V(\phi,\phi^*) - \tfrac{1}{2} y^{ijk} \phi_i \psi_j \psi_k - \tfrac{1}{2} y_{ijk}^* \phi^{*i} \bar{\psi}^j \bar{\psi}^k. \tag{6.52}$$

Now we can compare the masses of the fermions and scalars by looking at the linearized equations of motion:

$$-\partial^\mu \partial_\mu \phi_i = M_{ik}^* M^{kj} \phi_j + \ldots, \tag{6.53}$$

$$i\overline{\sigma}^\mu \partial_\mu \psi_i = M_{ij}^* \bar{\psi}^j + \ldots, \qquad i\sigma^\mu \partial_\mu \bar{\psi}^i = M^{ij} \psi_j + \ldots. \tag{6.54}$$

One can eliminate ψ in terms of $\bar{\psi}$ and vice versa in eq. (6.54), obtaining [after use of the identity eq. (6.7)]

$$-\partial^\mu \partial_\mu \psi_i = M_{ik}^* M^{kj} \psi_j + \ldots, \qquad -\partial^\mu \partial_\mu \bar{\psi}^j = \bar{\psi}^i M_{ik}^* M^{kj} + \ldots \tag{6.55}$$

Therefore, the fermions and the bosons satisfy the same wave equation with exactly the same (mass)^2 matrix with real non-negative eigenvalues, namely $(M^2)_i^{\ j} = M_{ik}^* M^{kj}$. It follows that diagonalizing this matrix gives a collection of chiral supermultiplets each of which contains a mass-degenerate complex scalar and Weyl fermion, in agreement with the general argument in the Introduction.

6.3 Supersymmetric Gauge Theories

The propagating degrees of freedom in a gauge supermultiplet are a massless gauge boson field A_μ^a and a two-component Weyl fermion gaugino λ^a. The index a here runs over the adjoint representation of the gauge group ($a = 1 \ldots 8$ for $SU(3)_C$ color gluons and gluinos; $a = 1, 2, 3$ for

Table 6.2. Counting of real degrees of freedom for each gauge supermultiplet.

	A_μ	λ	D
on-shell ($n_B = n_F = 2$)	2	2	0
off-shell ($n_B = n_F = 4$)	3	4	1

$SU(2)_L$ weak isospin; $a = 1$ for $U(1)_Y$ weak hypercharge). The gauge transformations of the vector supermultiplet fields are then

$$\delta_{\text{gauge}} A^a_\mu = \partial_\mu \Lambda^a + g f^{abc} A^b_\mu \Lambda^c, \tag{6.56}$$

$$\delta_{\text{gauge}} \lambda^a = g f^{abc} \lambda^b \Lambda^c, \tag{6.57}$$

where Λ^a is an infinitesimal gauge transformation parameter, g is the gauge coupling, and f^{abc} are the totally antisymmetric structure constants that define the gauge group. (The special case of an abelian group like $U(1)_Y$ is obtained by just setting $f^{abc} = 0$; the corresponding gaugino is a gauge singlet in that case.)

The on-shell degrees of freedom for A^a_μ and λ^a_α amount to two bosonic and two fermionic helicity states (for each a), as required by supersymmetry. However, off-shell λ^a_α consists of two complex, or four real, fermionic degrees of freedom, while A^a_μ only has three real bosonic degrees of freedom; one degree of freedom is removed by the inhomogeneous gauge transformation eq. (6.56). So, we will need one real bosonic auxiliary field, traditionally called D^a, in order for supersymmetry to be consistent off-shell. This field also transforms as an adjoint of the gauge group [i.e., like eq. (6.57) with $\lambda \to D$] and satisfies $(D^a)^* = D^a$. Like the chiral auxiliary fields F_i, the gauge auxiliary field D^a has dimensions of (mass)^2 and thus no kinetic term, so that it can be eliminated on-shell using its algebraic equation of motion. The counting of degrees of freedom is summarized in Table 6.2.

Therefore, the Lagrangian density for a gauge supermultiplet ought to be

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + i \bar{\lambda}^a \overline{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2} D^a D^a, \tag{6.58}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{6.59}$$

is the usual Yang-Mills field strength, and

$$D_\mu \lambda^a = \partial_\mu \lambda^a + g f^{abc} A^b_\mu \lambda^c \tag{6.60}$$

is the covariant derivative of the gaugino field. One can infer the appropriate form for the supersymmetry transformation of the fields, up to multiplicative constants, from the requirements that they should be linear in the infinitesimal parameters ϵ, $\bar{\epsilon}$ with dimensions of $(\text{mass})^{-1/2}$, that δA^a_μ is real, and that δD^a should be real and proportional to the field equations for the gaugino, in analogy with the role of the auxiliary field F in the chiral supermultiplet case. Thus one can guess, up to multiplicative factors,

$$\delta A^a_\mu = \frac{1}{\sqrt{2}}\left(\bar{\epsilon}\overline{\sigma}_\mu\lambda^a + \bar{\lambda}^a\overline{\sigma}_\mu\epsilon\right), \tag{6.61}$$

$$\delta\lambda^a_\alpha = \frac{i}{2\sqrt{2}}(\sigma^\mu\overline{\sigma}^\nu\epsilon)_\alpha\, F^a_{\mu\nu} + \frac{1}{\sqrt{2}}\epsilon_\alpha\, D^a, \tag{6.62}$$

$$\delta D^a = \frac{i}{\sqrt{2}}\left(\bar{\epsilon}\overline{\sigma}^\mu D_\mu\lambda^a - D_\mu\bar{\lambda}^a\overline{\sigma}^\mu\epsilon\right). \tag{6.63}$$

The factors of $\sqrt{2}$ are chosen so that the action obtained by integrating $\mathcal{L}_{\text{gauge}}$ is invariant, and the phase of λ^a is chosen for future convenience in treating the MSSM. It is now a little bit tedious, but straightforward, to check that eq. (6.18) is modified to

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})X = -i(\epsilon_1\sigma^\mu\bar{\epsilon}_2 - \epsilon_2\sigma^\mu\bar{\epsilon}_1)D_\mu X \tag{6.64}$$

for X equal to any of the gauge-covariant fields $F^a_{\mu\nu}$, λ^a, $\bar{\lambda}^a$, D^a, as well as for arbitrary covariant derivatives acting on them. This ensures that the supersymmetry algebra eqs. (6.32)-(6.33) is realized on gauge-invariant combinations of fields in gauge supermultiplets, as they were on the chiral supermultiplets.[2] These calculations require the use of identities

$$\xi\sigma^\mu\overline{\sigma}^\nu\chi = \chi\sigma^\nu\overline{\sigma}^\mu\xi = (\bar{\chi}\overline{\sigma}^\nu\sigma^\mu\bar{\xi})^* = (\bar{\xi}\overline{\sigma}^\mu\sigma^\nu\bar{\chi})^*; \tag{6.65}$$

$$\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho = \eta^{\mu\nu}\overline{\sigma}^\rho - \eta^{\mu\rho}\overline{\sigma}^\nu + \eta^{\nu\rho}\overline{\sigma}^\mu - i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_\kappa; \tag{6.66}$$

$$\sigma^\mu_{\alpha\dot{\alpha}}\overline{\sigma}^{\dot{\beta}\beta}_\mu = 2\delta^\beta_\alpha\delta^{\dot{\beta}}_{\dot{\alpha}}. \tag{6.67}$$

If we had not included the auxiliary field D^a, then the supersymmetry algebra eq. (6.64) would hold only after using the equations of motion for λ^a and $\bar{\lambda}^a$. The auxiliary fields just satisfy the equations of motion $D^a = 0$, but this is no longer true if one couples the gauge supermultiplets to chiral supermultiplets, as we now do.

[2] The supersymmetry transformations eqs. (6.61)-(6.63) are non-linear for non-abelian gauge symmetries, since there are gauge fields contained in the covariant derivatives acting on the gaugino fields and in the field strength $F^a_{\mu\nu}$. By adding even more auxiliary fields besides D^a, one can make the supersymmetry transformations linear in the fields. The version given here in which those extra auxiliary fields have been removed by gauge transformations is called "Wess-Zumino gauge".

6.4 Gauge interactions for chiral supermultiplets

Finally we are ready to consider a general Lagrangian density for a supersymmetric theory with both chiral and gauge supermultiplets. Suppose that the chiral supermultiplets transform under the gauge group in a representation with hermitian matrices $(T^a)_i{}^j$ satisfying $[T^a, T^b] = if^{abc}T^c$. [For example, if the gauge group is $SU(2)$, then $f^{abc} = \epsilon^{abc}$, and the T^a are 1/2 times the Pauli matrices for a chiral supermultiplet transforming in the fundamental representation.] Thus

$$\delta_{\text{gauge}} X_i = ig\Lambda^a (T^a X)_i \tag{6.68}$$

for $X_i = \phi_i, \psi_i, F_i$; since supersymmetry and gauge transformations commute, the scalar, fermion, and auxiliary fields must be in the same representation of the gauge group. To have a gauge-invariant Lagrangian, we need to replace the ordinary derivatives in eq. (6.34) with covariant derivatives:

$$\partial_\mu \phi_i \to D_\mu \phi_i = \partial_\mu \phi_i - igA^a_\mu (T^a \phi)_i \tag{6.69}$$

$$\partial_\mu \phi^{*i} \to D_\mu \phi^{*i} = \partial_\mu \phi^{*i} + igA^a_\mu (\phi^* T^a)^i \tag{6.70}$$

$$\partial_\mu \psi_i \to D_\mu \psi_i = \partial_\mu \psi_i - igA^a_\mu (T^a \psi)_i . \tag{6.71}$$

Naively, this simple procedure achieves the goal of coupling the vector bosons in the gauge supermultiplet to the scalars and fermions in the chiral supermultiplets. However, we also have to consider whether there are any other interactions, allowed by gauge invariance and involving the gaugino and D^a fields, that might have to be included to make a supersymmetric Lagrangian. (If A^a_μ couples to ϕ_i and ψ_i, then it should makes sense that its superpartners λ^a and D^a do as well.)

In fact, there are three such possible interaction terms that are renormalizable (of mass dimension ≤ 4), namely

$$(\phi^* T^a \psi)\lambda^a, \qquad \bar{\lambda}^a(\bar{\psi} T^a \phi), \qquad \text{and} \qquad (\phi^* T^a \phi) D^a. \tag{6.72}$$

Now one can add them, with unknown dimensionless coupling coefficients, to the Lagrangians for the chiral and gauge supermultiplets and demand that the whole mess be real and invariant under supersymmetry transformations, up to a total derivative. Not surprisingly, this is possible only if the supersymmetry transformation laws for the matter fields are modified to include gauge-covariant rather than ordinary derivatives (and to include one strategically chosen extra term in δF_i):

$$\delta \phi_i = \epsilon \psi_i \tag{6.73}$$

$$\delta \psi_{i\alpha} = -i(\sigma^\mu \bar{\epsilon})_\alpha \, D_\mu \phi_i + \epsilon_\alpha F_i \tag{6.74}$$

$$\delta F_i = -i\bar{\epsilon}\overline{\sigma}^\mu D_\mu \psi_i \; + \; \sqrt{2} g (T^a \phi)_i \, \bar{\epsilon} \bar{\lambda}^a . \tag{6.75}$$

After some algebra one can now fix the coefficients for the terms in eq. (6.72), so that the full Lagrangian density for a renormalizable supersymmetric theory is

$$\begin{aligned}\mathcal{L} = &\mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{chiral}} \\ &-\sqrt{2}g\left[(\phi^* T^a \psi)\lambda^a + \bar{\lambda}^a(\bar{\psi} T^a \phi)\right] \\ &+g(\phi^* T^a \phi)D^a.\end{aligned} \tag{6.76}$$

Here $\mathcal{L}_{\text{chiral}}$ means the chiral supermultiplet Lagrangian found in section 6.2 [e.g., eq. (6.50) or (6.52)], but with ordinary derivatives replaced everywhere by gauge-covariant derivatives, and $\mathcal{L}_{\text{gauge}}$ was given in eq. (6.58). To prove that eq. (6.76) is invariant under the supersymmetry transformations, one must use the identity

$$W^i (T^a)_i^j \phi_j = 0. \tag{6.77}$$

This is precisely the condition that must be satisfied anyway in order for the superpotential (and thus $\mathcal{L}_{\text{chiral}}$) to be gauge invariant, since the left side is proportional to $\delta_{\text{gauge}} W$.

The last two lines in eq. (6.76) are interactions whose strengths are fixed to be gauge couplings by the requirements of supersymmetry, even though they are not gauge interactions from the point of view of an ordinary field theory. The second line is a direct coupling of gauginos to matter fields, and is the "supersymmetrization" of the usual gauge boson coupling to matter fields. The last line combines with the $(1/2)D^a D^a$ term in $\mathcal{L}_{\text{gauge}}$ to provide an equation of motion

$$D^a = -g(\phi^* T^a \phi). \tag{6.78}$$

Thus, like the auxiliary fields F_i and F^{*i}, the D^a are expressible purely algebraically in terms of the scalar fields. Replacing the auxiliary fields in eq. (6.76) using eq. (6.78), one finds that the complete scalar potential is (recall $\mathcal{L}$ contains $-V$):

$$V(\phi, \phi^*) = F^{*i} F_i + \tfrac{1}{2} \sum_a D^a D^a = W_i^* W^i + \tfrac{1}{2} \sum_a g_a^2 (\phi^* T^a \phi)^2. \tag{6.79}$$

The two types of terms in this expression are called "F-term" and "D-term" contributions, respectively. In the second term in eq. (6.79), we have now written an explicit sum $\sum_a$ to cover the case that the gauge group has several distinct factors with different gauge couplings g_a. [For instance, in the MSSM the three factors $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$ have different gauge couplings g_3, g and g'.] Since $V(\phi, \phi^*)$ is a sum of squares, it is always greater than or equal to zero for every field configuration. It is a very interesting and unique feature of supersymmetric theories that the scalar potential is completely determined by the *other* interactions in the theory. The F-terms are fixed by Yukawa couplings and fermion mass terms, and the D-terms are fixed by the gauge interactions.

By using Noether's procedure [see eq. (6.19)], one finds the conserved supercurrent

$$J^\mu_\alpha = (\sigma^\nu \overline{\sigma}^\mu \psi_i)_\alpha \, D_\nu \phi^{*i} + i(\sigma^\mu \bar{\psi}^i)_\alpha \, W_i^* \\ - \frac{1}{2\sqrt{2}} (\sigma^\nu \overline{\sigma}^\rho \sigma^\mu \bar{\lambda}^a)_\alpha \, F^a_{\nu\rho} + \frac{i}{\sqrt{2}} g \phi^* T^a \phi \, (\sigma^\mu \bar{\lambda}^a)_\alpha, \tag{6.80}$$

generalizing the expression given in eq. (6.20) for the Wess-Zumino model. This expression will be useful when we discuss certain aspects of spontaneous supersymmetry breaking in section 9.2.

6.5 Summary: How to build a supersymmetric model

In a renormalizable supersymmetric field theory, the interactions and masses of all particles are determined just by their gauge transformation properties and by the superpotential W. By construction, we found that W had to be an analytic function of the complex scalar fields ϕ_i, which are always defined to transform under supersymmetry into left-handed Weyl fermions. We should mention that in an equivalent language, W is said to be a function of chiral *superfields*. A superfield is a single object that contains as components all of the bosonic, fermionic, and auxiliary fields within the corresponding supermultiplet, e.g. $\Phi_i \supset (\phi_i, \psi_i, F_i)$. (This is analogous to the way in which one often describes a weak isospin doublet or color triplet by a multicomponent field.) The gauge quantum numbers and the mass dimension of a chiral superfield are the same as that of its scalar component. In the superfield formulation, one writes instead of eq. (6.45)

$$W = \frac{1}{2} M^{ij} \Phi_i \Phi_j + \frac{1}{6} y^{ijk} \Phi_i \Phi_j \Phi_k, \tag{6.81}$$

which means exactly the same thing. While this entails no difference in practical results, the fancier version eq. (6.81) at least serves to remind us that W determines not only the scalar interactions in the theory, but the fermion masses and Yukawa couplings as well. The derivation of all of our preceding results can be obtained somewhat more elegantly using superfield methods, which have the advantage of making invariance under supersymmetry transformations manifest. We have avoided this extra layer of notation on purpose, in favor of the more pedestrian, but hopefully more familiar, component field approach. The latter is at least more appropriate for making contact with phenomenology in a universe

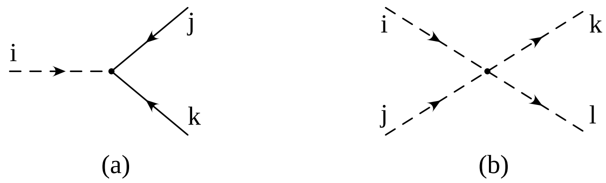


Fig. 6.1. The dimensionless non-gauge interaction vertices in a supersymmetric theory: (a) scalar-fermion-fermion Yukawa interaction y^{ijk}, (b) quartic scalar interaction $y^{ijn}y^*_{kln}$.

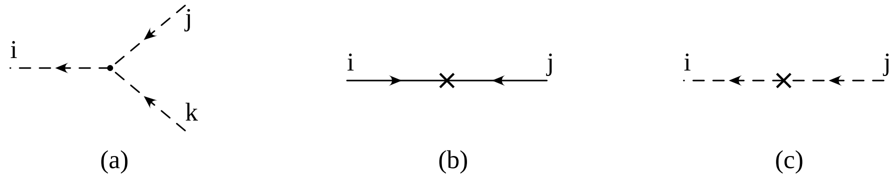


Fig. 6.2. Supersymmetric dimensionful couplings: (a) (scalar)^3 interaction vertex $M^*_{in}y^{jkn}$, (b) fermion mass term M^{ij}, (c) scalar (mass)^2 term $M^*_{ik}M^{kj}$.

with supersymmetry breaking. The only (occasional) use we will make of superfield notation is the purely cosmetic one of following the common practice of specifying superpotentials like eq. (6.81) rather than (6.45). The specification of the superpotential is really a code for the terms that it implies in the Lagrangian, so the reader may feel free to think of the superpotential either as a function $W(\phi_i)$ of the scalar fields ϕ_i or as the same function $W(\Phi_i)$ of the superfields Φ_i which contain them.

Given the supermultiplet content of the theory, the form of the superpotential is restricted by gauge invariance. In any given theory, only a subset of the couplings M^{ij} and y^{ijk} will be allowed to be non-zero. The entries of the mass matrix M^{ij} can only be non-zero for i and j such that the supermultiplets Φ_i and Φ_j transform under the gauge group in representations which are conjugates of each other. (In fact, in the MSSM there is only one such term, as we will see.) Likewise, the Yukawa couplings y^{ijk} can only be non-zero when Φ_i, Φ_j, and Φ_k transform in representations which can combine to form a singlet.

The interactions implied by the superpotential eq. (6.81) are shown [3] in Figs. 6.1 and 6.2. Those in Fig. 6.1 are all determined by the dimensionless parameters y^{ijk}. The Yukawa interaction in Fig. 6.1a corresponds to the next-to-last term in eq. (6.52). For each particular Yukawa coupling of

[3] Here, the auxiliary fields have been eliminated using their equations of motion ("integrated out") as in eq. (6.52). One can also give Feynman rules which include the auxiliary fields, although this tends to be less useful in phenomenological applications.

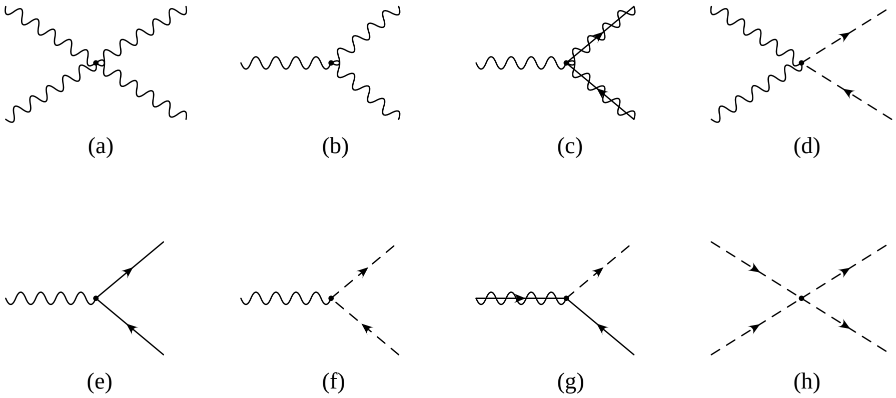


Fig. 6.3. Supersymmetric gauge interaction vertices.

$\phi_i\psi_j\psi_k$ with strength y^{ijk}, there must be equal couplings of $\phi_j\psi_i\psi_k$ and $\phi_k\psi_i\psi_j$, since y^{ijk} is completely symmetric under interchange of any two of its indices as shown in section 6.2. There is also a dimensionless coupling for $\phi_i\phi_j\phi^{*k}\phi^{*l}$, with strength $y^{ijn}y^*_{kln}$ as required by supersymmetry [see the last term in eq. (6.51)]. The arrows on both the fermion and scalar lines follow the chirality; i.e., one direction for propagation of ϕ and ψ and the other for the propagation of ϕ^* and $\bar{\psi}$. Thus there is a vertex corresponding to the one in Fig. 6.1a but with all arrows reversed, corresponding to the complex conjugate [the last term in eq. (6.52)]. The relationship between the interactions in Figs. 6.1a and 6.1b is exactly of the special type needed to cancel the quadratic divergences in quantum corrections to scalar masses, as discussed in the Introduction (compare Fig. 5.1).

In Fig. 6.2, we show the only interactions corresponding to renormalizable and supersymmetric vertices with dimensions of (mass) and (mass)^2. First, there are (scalar)^3 couplings which are entirely determined by the superpotential mass parameters M^{ij} and Yukawa couplings y^{ijk}, as indicated by the second and third terms in eq. (6.51). The propagators of the fermions and scalars in the theory are constructed in the usual way using the fermion mass M^{ij} and scalar (mass)^2 $M^*_{in}M^{nj}$. Of particular interest is the fact that the fermion mass term M^{ij} leads to a chirality-changing insertion in the fermion propagator; note the directions of the arrows in Fig. 6.2b. There is no such arrow-reversal for a scalar propagator in a theory with exact supersymmetry; as shown in Fig. 6.2c, if one treats the scalar (mass)^2 term as an insertion in the propagator, the arrow direction is preserved. Again, for each of Figures 6.2a and 6.2b there is an interaction with all arrows reversed.

In Fig. 6.3 we show in a similar manner the gauge interactions in a supersymmetric theory. Figures 6.3a,b,c occur only when the gauge group is non-abelian (e.g. for $SU(3)_C$ color and $SU(2)_L$ weak isospin in the MSSM). Figures 6.3a and 6.3b are the interactions of gauge bosons which derive from the first term in eq. (6.58). In the MSSM these are exactly the same as the well-known QCD gluon and electroweak gauge boson vertices of the Standard Model. (We do not show the interactions of ghost fields, which are necessary only for consistent loop amplitudes.) Figures 6.3c,d,e,f are just the standard interactions between gauge bosons and fermion and scalar fields which must occur in any gauge theory because of the form of the covariant derivative; they come from eqs. (6.60) and (6.69)-(6.71) inserted in the kinetic part of the Lagrangian. Figure 6.3c shows the coupling of a gaugino to a gauge boson; the gaugino line in a Feynman diagram is traditionally drawn as a solid fermion line superimposed on a gauge boson wavy line. In Fig. 6.3g we have the coupling of a gaugino to a chiral fermion and a complex scalar [the first term in the second line in eq. (6.76)]. One can think of this as the "supersymmetrization" of Figure 6.3e or 6.3f; any of these three vertices may be obtained from any other (up to a factor of $\sqrt{2}$) by replacing two of the particles by their supersymmetric partners. There is also an interaction like Fig. 6.3g but with all arrows reversed, corresponding to the complex conjugate term in the Lagrangian [the second term in the second line in eq. (6.76)]. Finally in Fig. 6.3h we have a scalar quartic interaction vertex [the last term in eq. (6.79)] which is also determined by the gauge coupling.

The results of this chapter can be used as a recipe for constructing the supersymmetric interactions for any renormalizable model. In the case of the MSSM, we already know the gauge group, particle content and the gauge transformation properties, so it only remains to decide on the superpotential. This we will do in section 8.1.

6.6 Soft supersymmetry-breaking interactions

A realistic phenomenological model must contain supersymmetry breaking. From a theoretical perspective, we expect that supersymmetry, if it exists at all, should be an exact symmetry which is spontaneously broken. In other words, the ultimate model should have a Lagrangian density which is invariant under supersymmetry, but a vacuum state which is not. In this way, supersymmetry is hidden at low energies in a manner exactly analogous to the fate of the electroweak symmetry in the ordinary Standard Model.

Many models of spontaneous symmetry breaking have indeed been proposed and we will mention the basic ideas of some of them in section 9. These always involve extending the MSSM to include new particles and interactions at very high mass scales, and there is no consensus on exactly how this should be done. However, from a practical point of view, it is extremely useful to simply parameterize our ignorance of these issues by just introducing extra terms which break supersymmetry explicitly in the effective MSSM Lagrangian. As was argued in the Introduction, the extra supersymmetry-breaking couplings should be soft (of positive mass dimension) in order to be able to naturally maintain a hierarchy between the electroweak scale and the Planck (or some other very large) mass scale. This means in particular that we should not consider any dimensionless supersymmetry-breaking couplings.

In the context of a general renormalizable theory, the possible soft supersymmetry-breaking terms in the Lagrangian are

$$\begin{aligned}\mathcal{L}_{\rm soft} = &-\tfrac{1}{2}\left(M_\lambda\,\lambda^a\lambda^a + \text{c.c.}\right) - (m^2)^i_j\phi^{j*}\phi_i \\ &-\left(\tfrac{1}{2}b^{ij}\phi_i\phi_j + \frac{1}{6}a^{ijk}\phi_i\phi_j\phi_k + \text{c.c.}\right), \end{aligned} \tag{6.82}$$

$$\mathcal{L}_{\rm maybe\ soft} = -\frac{1}{2}c_i^{jk}\phi^{*i}\phi_j\phi_k + \text{c.c.} \tag{6.83}$$

They consist of gaugino masses M_λ for each gauge group, scalar (mass)^2 terms $(m^2)^j_i$ and b^{ij}, and (scalar)^3 couplings a^{ijk} and c_i^{jk}. One might wonder why we have not included possible soft mass terms for the chiral supermultiplet fermions. The reason is that including such terms would be redundant; they can always be absorbed into a redefinition of the superpotential and the terms $(m^2)^j_i$ and c_i^{jk}. It has been shown rigorously that a softly-broken supersymmetric theory with $\mathcal{L}_{\rm soft}$ as given by eq. (6.82) is indeed free of quadratic divergences in quantum corrections to scalar masses, to all orders in perturbation theory. The situation is slightly more subtle if one tries to include the non-analytic (scalar)^3 couplings in $\mathcal{L}_{\rm maybe\ soft}$. If any of the chiral supermultiplets in the theory are completely uncharged under all gauge symmetries, then non-zero c_i^{jk} terms can lead to quadratic divergences, despite the fact that they are formally soft. Now, this constraint need *not* apply to the MSSM, which does not have any gauge-singlet chiral supermultiplets. Nevertheless, the possibility of c_i^{jk} terms is nearly always neglected. The real reason for this is that it is extremely difficult to construct any model of spontaneous supersymmetry breaking in which the c_i^{jk} are not utterly negligibly small. Equation (6.82) is therefore usually taken to be the most general soft supersymmetry-breaking Lagrangian.

It should be clear that $\mathcal{L}_{\rm soft}$ indeed breaks supersymmetry, since it involves only scalars and gauginos, and not their respective superpartners. In fact, the soft terms in $\mathcal{L}_{\rm soft}$ are capable of giving masses to all of the scalars and gauginos in a theory, even if the gauge bosons and fermions

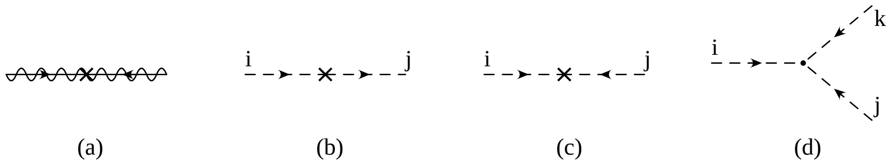


Fig. 6.4. Soft supersymmetry-breaking terms: (a) Gaugino mass insertion M_λ; (b) non-analytic scalar (mass)^2 $(m^2)^i_j$; (c) analytic scalar (mass)^2 b^{ij}; (d) (scalar)^3 coupling a^{ijk}.

in chiral supermultiplets are massless (or relatively light). The gaugino masses M_λ are always allowed by gauge symmetry. The $(m^2)^i_j$ terms are allowed for i, j such that ϕ_i, ϕ^{j*} transform in complex conjugate representations of each other under all gauge symmetries; in particular this is true of course when $i = j$, so every scalar is eligible to get a mass in this way if supersymmetry is broken. The remaining soft terms may or may not be allowed by the symmetries. In this regard it is useful to note that the b^{ij} and a^{ijk} terms have the same form as the M^{ij} and y^{ijk} terms in the superpotential [compare eq. (6.82) to eq. (6.45) or eq. (6.81)], so they will be allowed by gauge invariance if and only if a corresponding superpotential term is allowed. The Feynman diagram interactions corresponding to the allowed soft terms in eq. (6.82) are shown in Fig. 6.4. As before, for each of the interactions in Figs. 6.4a,c,d there is one with all arrows reversed, corresponding to the complex conjugate term in the Lagrangian. We will apply these general results to the specific case of the MSSM in the next chapter.

7
Superfields

Nothing here yet.

Part 3
Realistic Supersymmetric Models

8
The Minimal Supersymmetric Standard Model

In chapter 6, we have found a general recipe for constructing Lagrangians for softly broken supersymmetric theories. We are now ready to apply these general results to the MSSM. The particle content for the MSSM was described in chapter 5. In this section we will complete the model by specifying the superpotential and the soft supersymmetry-breaking terms.

8.1 The superpotential and supersymmetric interactions

The superpotential for the MSSM is given by

$$W_{\mathrm{MSSM}} = \overline{u}\mathbf{y_u}QH_u - \overline{d}\mathbf{y_d}QH_d - \overline{e}\mathbf{y_e}LH_d + \mu H_u H_d \,. \tag{8.1}$$

The objects H_u, H_d, Q, L, $\overline{u}$, $\overline{d}$, $\overline{e}$ appearing in eq. (8.1) are chiral superfields corresponding to the chiral supermultiplets in Table 1. (Alternatively, they can be just thought of as the corresponding scalar fields, as was done in section 6, but we prefer not to put the tildes on Q, L, $\overline{u}$, $\overline{d}$, $\overline{e}$ in order to reduce clutter.) The dimensionless Yukawa coupling parameters $\mathbf{y_u}, \mathbf{y_d}, \mathbf{y_e}$ are 3×3 matrices in family space. Here we have suppressed all of the gauge [$SU(3)_C$ color and $SU(2)_L$ weak isospin] and family indices. The "μ term", as it is traditionally called, can be written out as $\mu(H_u)_\alpha(H_d)_\beta\epsilon^{\alpha\beta}$, where $\epsilon^{\alpha\beta}$ is used to tie together $SU(2)_L$ weak isospin indices $\alpha, \beta = 1, 2$ in a gauge-invariant way. Likewise, the term $\overline{u}\mathbf{y_u}QH_u$ can be written out as $\overline{u}^i_a\,(\mathbf{y_u})_i{}^j\,Q^a_{j\alpha}\,(H_u)_\beta\epsilon^{\alpha\beta}$, where $i = 1, 2, 3$ is a family index, and $a = 1, 2, 3$ is a color index which is raised (lowered) in the $\mathbf{3}$ ($\overline{\mathbf{3}}$) representation of $SU(3)_C$.

The μ term in eq. (8.1) is the supersymmetric version of the Higgs boson mass in the Standard Model. It is unique, because terms $H_u^*H_u$ or $H_d^*H_d$ are forbidden in the superpotential, which must be analytic in the chiral superfields (or equivalently in the scalar fields) treated as complex variables, as shown in section 6.2. We can also see from the form of eq. (8.1) why both H_u and H_d are needed in order to give Yukawa couplings, and thus masses, to all of the quarks and leptons. Since the superpotential must be analytic, the $\overline{u}QH_u$ Yukawa terms cannot be replaced by something like $\overline{u}QH_d^*$. Similarly, the $\overline{d}QH_d$ and $\overline{e}LH_d$ terms cannot be replaced by something like $\overline{d}QH_u^*$ and $\overline{e}LH_u^*$. The analogous Yukawa couplings would be allowed in a general non-supersymmetric two Higgs doublet model, but are forbidden by the structure of supersymmetry. So we need both H_u and H_d, even without invoking the argument based on anomaly cancellation that was mentioned in section 5.2.

The Yukawa matrices determine the masses and CKM mixing angles of the ordinary quarks and leptons, after the neutral scalar components of H_u and H_d get VEVs. Since the top quark, bottom quark and tau lepton are the heaviest fermions in the Standard Model, it is often useful to make an approximation that only the $(3,3)$ family components of each of $\mathbf{y_u}$, $\mathbf{y_d}$ and $\mathbf{y_e}$ are important:

$$\mathbf{y_u} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_t \end{pmatrix}; \qquad \mathbf{y_d} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_b \end{pmatrix}; \qquad \mathbf{y_e} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_\tau \end{pmatrix}. \tag{8.2}$$

In this limit, only the third family and Higgs fields contribute to the MSSM superpotential. It is instructive to write the superpotential in terms of the separate $SU(2)_L$ weak isospin components [$Q_3 = (t\,b)$; $L_3 = (\nu_\tau\,\tau)$; $H_u = (H_u^+\,H_u^0)$; $H_d = (H_d^0\,H_d^-)$; $\overline{u}_3 = \overline{t}$; $\overline{d}_3 = \overline{b}$; $\overline{e}_3 = \overline{\tau}$], so:

$$\begin{aligned} W_{\rm MSSM} \approx\; & y_t(\overline{t}tH_u^0 - \overline{t}bH_u^+) - y_b(\overline{b}tH_d^- - \overline{b}bH_d^0) - y_\tau(\overline{\tau}\nu_\tau H_d^- - \overline{\tau}\tau H_d^0) \\ & +\mu(H_u^+H_d^- - H_u^0H_d^0). \end{aligned} \tag{8.3}$$

The minus signs inside the parentheses appear because of the antisymmetry of the $\epsilon^{\alpha\beta}$ symbol used to tie up the $SU(2)_L$ indices. The other minus signs in eq. (8.1) were chosen for convenience so that the terms $y_t\overline{t}tH_u^0$, $y_b\overline{b}bH_d^0$, and $y_\tau\overline{\tau}\tau H_d^0$, which will become the top, bottom and tau masses when H_u^0 and H_d^0 get VEVs, have positive signs in eq. (8.3).

Since the Yukawa interactions y^{ijk} in a general supersymmetric theory must be completely symmetric under interchange of i, j, k, we know that $\mathbf{y_u}$, $\mathbf{y_d}$ and $\mathbf{y_e}$ imply not only Higgs-quark-quark and Higgs-lepton-lepton couplings as in the Standard Model, but also squark-Higgsino-quark and slepton-Higgsino-lepton interactions. To illustrate this, we show in Figs. 8.1a,b,c some of the interactions which involve the top-quark Yukawa coupling y_t. Figure 8.1a is the Standard Model-like coupling of

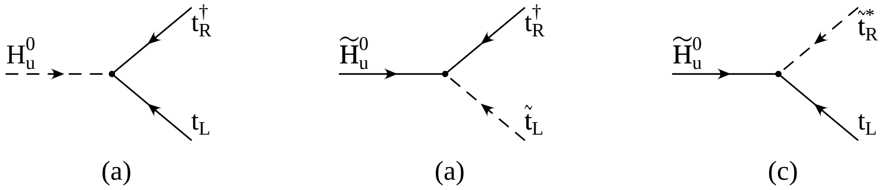


Fig. 8.1. The top-quark Yukawa coupling (a) and its supersymmetrizations (b),(c), all of strength y_t.

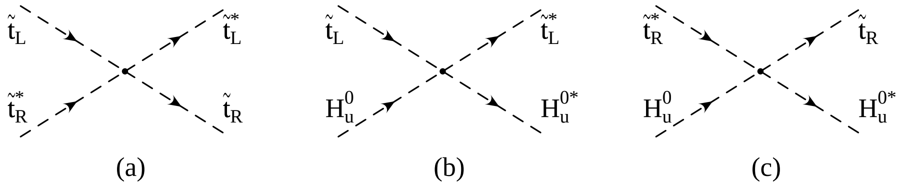


Fig. 8.2. Some of the (scalar)^4 interactions with strength proportional to y_t^2.

the top quark to the neutral complex scalar Higgs boson, which follows from the first term in eq. (8.3). For variety, here we have used t_L and $t_R^\dagger$ in place of their synonyms t and $\overline{t}$ in Fig. 8.1. In Fig. 8.1b, we have the coupling of the left-handed top squark $\widetilde{t}_L$ to the neutral higgsino field $\widetilde{H}_u^0$ and right-handed top quark, while in Fig. 8.1c the right-handed top-squark field (known either as $\widetilde{\overline{t}}$ or $\widetilde{t}_R^*$ depending on taste) couples to $\widetilde{H}_u^0$ and t_L. For each of the three interactions, there is another with $H_u^0 \to H_u^+$ and $t_L \to -b_L$, with tildes where appropriate, corresponding to the second part of the first term in eq. (8.3). All of these interactions are required by supersymmetry to have the same strength y_t. This is also an incontrovertible prediction of softly-broken supersymmetry at tree-level, since these interactions are dimensionless and can be modified by the introduction of soft supersymmetry breaking only through finite (and small) radiative corrections. A useful mnemonic is that each of Figs. 8.1a,b,c can be obtained from any of the others by changing two of the particles into their superpartners.

There are also scalar quartic interactions with strength proportional to y_t^2, as can be seen e.g. from Fig. 6.1b or the last term in eq. (6.51). Three of them are shown in Fig. 8.2. The reader is invited to check, using eq. (6.51) and eq. (8.3), that there are five more, which can be obtained by replacing $\widetilde{t}_L \to \widetilde{b}_L$ and/or $H_u^0 \to H_u^+$ in each vertex. This illustrates the remarkable economy of supersymmetry; there are many interactions determined by only a single parameter! In a similar way, the existence

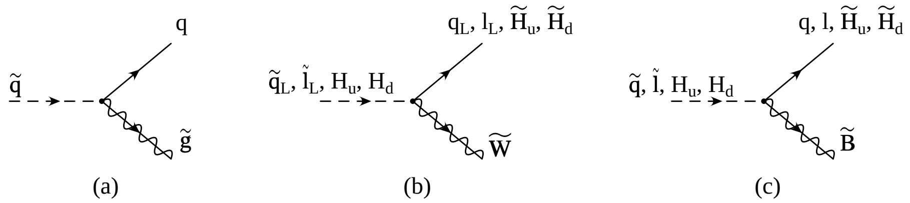


Fig. 8.3. Couplings of the gluino, wino, and bino to MSSM (scalar, fermion) pairs.

of all the other quark and lepton Yukawa couplings in the superpotential eq. (8.1) leads not only to Higgs-quark-quark and Higgs-lepton-lepton Lagrangian terms as in the ordinary Standard Model, but also to squark-higgsino-quark and slepton-higgsino-lepton terms, and scalar quartic couplings [(squark)^4, (slepton)^4, (squark)^2(slepton)^2, (squark)^2(Higgs)^2, and (slepton)^2(Higgs)^2]. If needed, these can all be obtained in terms of the Yukawa matrices $\mathbf{y_u}$, $\mathbf{y_d}$, and $\mathbf{y_e}$ as outlined above.

However, it is useful to note that the dimensionless interactions determined by the superpotential are often not the most important ones of direct interest for phenomenology. This is because the Yukawa couplings are already known to be very small, except for those of the third family (top, bottom, tau). Instead, decay and especially production processes for superpartners in the MSSM are typically dominated by the supersymmetric interactions of gauge-coupling strength, as we will explore in more detail below. The couplings of the Standard Model gauge bosons (photon, $W^{\pm}$, Z^0 and gluons) to the MSSM particles are determined completely by the gauge invariance of the kinetic terms in the Lagrangian. The gauginos also couple to (squark, quark) and (slepton, lepton) and (Higgs, higgsino) pairs as illustrated in the general case in Fig. 6.3g and the second line in eq. (6.76). For instance, each of the squark-quark-gluino couplings is given by $\sqrt{2}g_3(\widetilde{q}\,T^a q\widetilde{g} + \text{c.c.})$ where $T^a = \lambda^a/2$ $(a = 1\ldots 8)$ are the matrix generators for $SU(3)_C$, with λ^a the Gell-Mann matrices. The Feynman diagram for this interaction is shown in Fig. 8.3a. In Figs. 8.3b,c we show in a similar way the couplings of (squark, quark), (lepton, slepton) and (Higgs, higgsino) pairs to the winos and bino, with strengths proportional to the electroweak gauge couplings g and g' respectively. The winos only couple to the left-handed squarks and sleptons, and the (lepton, slepton) and (Higgs, higgsino) pairs of course do not couple to the gluino. The bino couplings for each (scalar, fermion) pair are also proportional to the weak hypercharges Y as given in Table 1. The interactions shown in Fig. 8.3 provide for decays $\widetilde{q} \to q\widetilde{g}$ and $\widetilde{q} \to \widetilde{W}q'$ and $\widetilde{q} \to \widetilde{B}q$ when the final states are kinematically allowed to be on-shell. However, a complication is that the $\widetilde{W}$ and $\widetilde{B}$ states are not mass eigenstates, because of mixing due to electroweak symmetry breaking, as we will see in section 8.6.

There are also various scalar quartic interactions in the MSSM which are uniquely determined by gauge invariance and supersymmetry, according to the last term in eq. (6.79) illustrated in Fig. 6.3h. Among them are (Higgs)^4 terms proportional to g^2 and g'^2 in the scalar potential. These are the direct generalization of the last term in the Standard Model Higgs potential, eq. (5.1), to the case of the MSSM. We will have occasion to identify them explicitly when we discuss the minimization of the MSSM Higgs potential in section 8.5.

The dimensionful terms in the supersymmetric part of the MSSM Lagrangian are all dependent on μ. Following the general result of eq. (6.52), we find that μ provides for higgsino fermion mass terms

$$-\mathcal{L}_{\text{higgsino mass}} = \mu(\widetilde{H}_u^+ \widetilde{H}_d^- - \widetilde{H}_u^0 \widetilde{H}_d^0) + \text{c.c.}, \tag{8.4}$$

as well as Higgs squared-mass terms in the scalar potential

$$-\mathcal{L}_{\text{supersymmetric Higgs mass}} = |\mu|^2 \big(|H_u^0|^2 + |H_u^+|^2 + |H_d^0|^2 + |H_d^-|^2\big). \tag{8.5}$$

Since eq. (8.5) is non-negative definite with a minimum at $H_u^0 = H_d^0 = 0$, it is clear that we cannot understand electroweak symmetry breaking without including supersymmetry-breaking (mass)^2 soft terms for the Higgs scalars, which can be negative. An explicit treatment of the Higgs scalar potential will therefore have to wait until we have introduced the soft terms for the MSSM. However, we can already see a puzzle: we expect that μ should be roughly of order 10^2 or 10^3 GeV, in order to allow a Higgs VEV of order 174 GeV without too much miraculous cancellation between $|\mu|^2$ and the negative soft (mass)^2 terms that we have not written down yet. But why should $|\mu|^2$ be so small compared to, say, $M_{\rm P}^2$, and in particular why should it be roughly of the same order as $m^2_{\rm soft}$? The scalar potential of the MSSM seems to depend on two types of dimensionful parameters which are conceptually quite distinct, namely the supersymmetry-respecting mass μ and the supersymmetry-breaking soft mass terms. Yet the observed value for the electroweak breaking scale suggests that without miraculous cancellations, both of these apparently unrelated mass scales should be within an order of magnitude or so of 100 GeV. This puzzle is called "the μ problem". Several different solutions to the μ problem have been proposed, involving extensions of the MSSM of varying intricacy. They all work in roughly the same way; the parameter μ is required or assumed to be completely absent at tree-level, and then arises from the VEV(s) of some new field(s). The latter are in turn determined by minimizing a potential which depends on soft

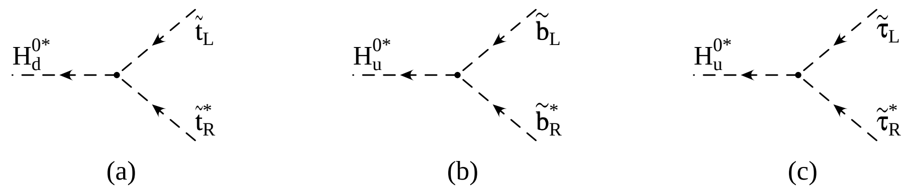


Fig. 8.4. Some of the supersymmetric $(\text{scalar})^3$ couplings proportional to $\mu^* y_t$, $\mu^* y_b$, and $\mu^* y_\tau$. When H_u^0 and H_d^0 get VEVS, these contribute to (a) $\tilde{t}_L, \tilde{t}_R$ mixing, (b) $\tilde{b}_L, \tilde{b}_R$ mixing, and (c) $\tilde{\tau}_L, \tilde{\tau}_R$ mixing.

supersymmetry-breaking terms. In this way, the value of the effective parameter μ is no longer conceptually distinct from the mechanism of supersymmetry breaking; if we can explain why $m_{\text{soft}} \ll M_{\text{P}}$, we will also be able to understand why μ is of the same order. In section 12.1 we will describe one such mechanism. Some other attractive solutions for the μ problem are proposed in Refs.[43, 44, 45] From the point of view of the MSSM, however, we can just treat μ as an independent parameter.

The μ-term and the Yukawa couplings in the superpotential eq. (8.1) combine to yield $(\text{scalar})^3$ couplings [see the second and third terms on the right-hand side of eq. (6.51)] of the form

$$\begin{aligned}\mathcal{L}_{\text{supersymmetric (scalar)}^3} = \mu^* \Big(& \tilde{\bar{u}} \mathbf{y_u} \tilde{u} H_d^{0*} + \tilde{\bar{d}} \mathbf{y_d} \tilde{d} H_u^{0*} + \tilde{\bar{e}} \mathbf{y_e} \tilde{e} H_u^{0*} \\ & + \tilde{\bar{u}} \mathbf{y_u} \tilde{d} H_d^{-*} + \tilde{\bar{d}} \mathbf{y_d} \tilde{u} H_u^{+*} + \tilde{\bar{e}} \mathbf{y_e} \tilde{\nu} H_u^{+*} \Big) + \text{c.c.} \end{aligned} \quad (8.6)$$

In Fig. 8.4 we show some of these couplings proportional to $\mu^* y_t$, $\mu^* y_b$, and $\mu^* y_\tau$ respectively. These play an important role in determining the mixing of top squarks, bottom squarks, and tau sleptons, as we will see in section 8.8.

8.2 R-parity (also known as matter parity) and its consequences

The superpotential eq. (8.1) is minimal, in the sense that it is sufficient to produce a phenomenologically viable model. However, there are other terms that one could write down that are gauge-invariant and analytic in the chiral superfields, but are not included in the MSSM because they violate either baryon number (B) or total lepton number (L). The most general gauge-invariant and renormalizable superpotential would include

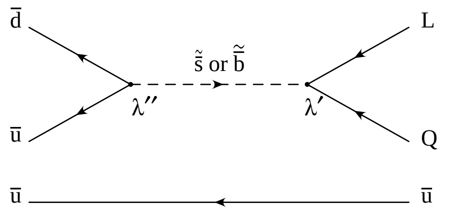


Fig. 8.5. Squarks can mediate disastrously rapid proton decay if R-parity is violated by both $\Delta\mathrm{B} = 1$ and $\Delta\mathrm{L} = 1$ interactions.

not only eq. (8.1), but also the terms

$$W_{\Delta\mathrm{L}=1} = \frac{1}{2}\lambda^{ijk} L_i L_j \overline{e}_k + \lambda'^{ijk} L_i Q_j \overline{d}_k + \mu'^{i} L_i H_u \tag{8.7}$$

$$W_{\Delta\mathrm{B}=1} = \frac{1}{2}\lambda''^{ijk} \overline{u}_i \overline{d}_j \overline{d}_k \tag{8.8}$$

where we have restored family indices $i = 1, 2, 3$. The chiral supermultiplets carry baryon number assignments $\mathrm{B} = +1/3$ for Q_i; $\mathrm{B} = -1/3$ for $\overline{u}_i, \overline{d}_i$; and $\mathrm{B} = 0$ for all others. The total lepton number assignments are $\mathrm{L} = +1$ for L_i, $\mathrm{L} = -1$ for $\overline{e}_i$, and $\mathrm{L} = 0$ for all others. Therefore, the terms in eq. (8.7) violate total lepton number by 1 unit (as well as the individual lepton flavors) and those in eq. (8.8) violate baryon number by 1 unit.

The possible existence of such terms might seem rather disturbing, since corresponding B- and L-violating processes have not been seen experimentally. The most obvious experimental constraint comes from the non-observation of proton decay, which would violate both B and L by 1 unit. If both λ' and λ'' couplings were present and unsuppressed, then the lifetime of the proton would be extremely short. For example, the Feynman diagram in Fig. 8.5 would lead to $p^+ \to e^+\pi^0$ or e^+K^0 or $\mu^+\pi^0$ or μ^+K^0 or $\nu\pi^+$ or νK^+ etc. depending on which components of λ' are largest. (The coupling λ'' must be antisymmetric in its last two flavor indices, since the color indices are contracted antisymmetrically. That is why the squark in Fig. 8.5 is $\tilde{\overline{s}}$ or $\tilde{\overline{b}}$ but not $\tilde{\overline{d}}$, for u, d quarks in the initial state.) As a rough estimate based on dimensional analysis, for example,

$$\Gamma_{p\to e^+\pi^0} \sim m^5_{\mathrm{proton}} \sum_{i=2,3} |\lambda'^{11i}\lambda''^{11i}|^2/m^4_{\tilde{d}_i}, \tag{8.9}$$

which would be a fraction of a second if the couplings were of order unity and the squarks have masses of order 1 TeV. In contrast, the decay time of the proton is measured to be in excess of 10^{33} years. Therefore, at least one of λ'^{11i} or λ''^{11i} for $i = 2, 3$ must be extremely small. Many other processes also give very strong constraints on the violation of lepton and baryon numbers; these are reviewed in these are reviewed in Chapter 11.

One could simply try to take B and L conservation as a postulate in the MSSM. However, this is clearly a step backwards from the situation in the Standard Model, where the conservation of these quantum numbers is *not* assumed, but is rather a pleasantly "accidental" consequence of the fact that there are no possible renormalizable Lagrangian terms that violate B or L. Furthermore, there is a quite general obstacle to treating B and L as fundamental symmetries of nature, since they are known to be necessarily violated by non-perturbative electroweak effects (even though those effects are calculably negligible for experiments at ordinary energies). Therefore, in the MSSM one adds a new symmetry which has the effect of eliminating the possibility of B and L violating terms in the renormalizable superpotential, while allowing the good terms in eq. (8.1). This new symmetry is called "R-parity" or equivalently "matter parity".

Matter parity is a multiplicatively conserved quantum number defined as

$$P_M = (-1)^{3(\mathrm{B}-\mathrm{L})} \tag{8.10}$$

for each particle in the theory. It is easy to check that the quark and lepton supermultiplets all have $P_M = -1$, while the Higgs supermultiplets H_u and H_d have $P_M = +1$. The gauge bosons and gauginos of course do not carry baryon number or lepton number, so they are assigned matter parity $P_M = +1$. The symmetry principle to be enforced is that a term in the Lagrangian (or in the superpotential) is allowed only if the product of P_M for all of the fields in it is $+1$. It is easy to see that each of the terms in eqs. (8.7) and (8.8) is thus forbidden, while the good and necessary terms in eq. (8.1) are allowed. This discrete symmetry commutes with supersymmetry, as all members of a given supermultiplet have the same matter parity. The advantage of matter parity is that it can in principle be an *exact* and fundamental symmetry, which B and L themselves cannot, since they are known to be violated by non-perturbative electroweak effects. So even with exact matter parity conservation in the MSSM, one expects that baryon number and total lepton number violation can occur in tiny amounts, due to nonrenormalizable terms in the Lagrangian. However, the MSSM does not have renormalizable interactions that violate B or L, with the standard assumption of matter parity conservation.

It is sometimes useful to recast matter parity in terms of R-parity, defined for each particle as

$$P_R = (-1)^{3(\mathrm{B}-\mathrm{L})+2s} \tag{8.11}$$

where s is the spin of the particle. Now, matter parity conservation and R-parity conservation are precisely equivalent, since the product of $(-1)^{2s}$ for the particles involved in any interaction vertex in a theory that conserves angular momentum is equal to $+1$. However, particles within the same supermultiplet do not have the same R-parity. In general, symmetries with the property that particles within the same multiplet have different charges are called R symmetries; they do not commute with supersymmetry. Continuous $U(1)$ R symmetries are often encountered in the model-building literature; they should not be confused with R-parity, which is a discrete Z_2 symmetry. In fact, the matter parity version of R-parity makes clear that there is really nothing intrinsically "R" about it; in other words it secretly does commute with supersymmetry, so its name is somewhat suboptimal. Nevertheless, the R-parity assignment is very useful for phenomenology because all of the Standard Model particles and the Higgs bosons have even R-parity ($P_R = +1$), while all of the squarks, sleptons, gauginos, and higgsinos have odd R-parity ($P_R = -1$).

The R-parity odd particles are known as "supersymmetric particles" or "sparticles" for short, and they are distinguished by a tilde (see Tables 1 and 2). If R-parity is exactly conserved, then there can be no mixing between the sparticles and the $P_R = +1$ particles. Furthermore, every interaction vertex in the theory contains an even number of $P_R = -1$ sparticles. This has three extremely important phenomenological consequences:

- The lightest sparticle with $P_R = -1$, called the "lightest supersymmetric particle" or LSP, must be absolutely stable. If the LSP is electrically neutral, it interacts only weakly with ordinary matter, and so can make an attractive candidate for the non-baryonic dark matter that seems to be required by cosmology.

- Each sparticle other than the LSP must eventually decay into a state that contains an odd number of LSPs (usually just one).

- In collider experiments, sparticles can only be produced in even numbers (usually two-at-a-time).

We *define* the MSSM to conserve R-parity or equivalently matter parity. While this decision seems to be well-motivated phenomenologically by proton decay constraints and the hope that the LSP will provide a good dark matter candidate, it might appear somewhat *ad hoc* from a theoretical point of view. After all, the MSSM would not suffer any internal inconsistency if we did not impose matter parity conservation. Furthermore, it is fair to ask why matter parity should be exactly conserved, given that the known discrete symmetries in the Standard

Model (ordinary parity P, charge conjugation C, time reversal T, etc.) are all known to be inexact symmetries. Fortunately, it *is* sensible to formulate matter parity as a discrete symmetry that is exactly conserved. In general, exactly conserved, or "gauged" discrete symmetries can exist provided that they satisfy certain anomaly cancellation conditions (much like continuous gauged symmetries). One particularly attractive way this could occur is if B−L is a continuous $U(1)$ gauge symmetry which is spontaneously broken at some very high energy scale. From eq. (8.10), we observe that P_M is actually a discrete subgroup of the continuous $U(1)_{\mathrm{B-L}}$ group. Therefore, if gauged $U(1)_{\mathrm{B-L}}$ is broken by scalar VEVs (or other order parameters) that carry only even integer values of 3(B−L), then P_M will automatically survive as an exactly conserved remnant. A variety of extensions of the MSSM in which exact R-parity arises in just this way have been proposed. It may also be possible to have gauged discrete symmetries which do not owe their exact conservation to an underlying continuous gauged symmetry, but rather to some other structure such as can occur in string theory. It is also possible that R-parity is broken, or is replaced by some alternative discrete symmetry. We will briefly consider these as variations on the MSSM in Chapter 11.

8.3 Soft supersymmetry breaking in the MSSM

To complete the description of the MSSM, we need to specify the soft supersymmetry breaking terms. In section 6.6, we learned how to write down the most general set of such terms in any supersymmetric theory. Applying this recipe to the MSSM, we have:

$$\begin{aligned}
-\mathcal{L}_{\text{soft}}^{\text{MSSM}} &= \tfrac{1}{2}\left(M_3\widetilde{g}\widetilde{g} + M_2\widetilde{W}\widetilde{W} + M_1\widetilde{B}\widetilde{B}\right) + \text{c.c.} \\
&\quad + \left(\widetilde{\overline{u}}\,\mathbf{a_u}\,\widetilde{Q}H_u - \widetilde{\overline{d}}\,\mathbf{a_d}\,\widetilde{Q}H_d - \widetilde{\overline{e}}\,\mathbf{a_e}\,\widetilde{L}H_d\right) + \text{c.c.} \\
&\quad + \widetilde{Q}^\dagger\,\mathbf{m_Q^2}\,\widetilde{Q} + \widetilde{L}^\dagger\,\mathbf{m_L^2}\,\widetilde{L} + \widetilde{\overline{u}}\,\mathbf{m_{\overline{u}}^2}\,\widetilde{\overline{u}}^\dagger + \widetilde{\overline{d}}\,\mathbf{m_{\overline{d}}^2}\,\widetilde{\overline{d}}^\dagger + \widetilde{\overline{e}}\,\mathbf{m_{\overline{e}}^2}\,\widetilde{\overline{e}}^\dagger \\
&\quad + m_{H_u}^2 H_u^* H_u - m_{H_d}^2 H_d^* H_d + (bH_uH_d + \text{c.c.})\,. \qquad (8.12)
\end{aligned}$$

In eq. (8.12), M_3, M_2, and M_1 are the gluino, wino, and bino mass terms. Here, and from now on, we suppress the adjoint representation gauge indices on the wino and gluino fields, and the gauge indices on all of the chiral supermultiplet fields. The second line in eq. (8.12) contains the (scalar)^3 couplings [of the type a^{ijk} in eq. (6.82)]. Each of $\mathbf{a_u}$, $\mathbf{a_d}$, $\mathbf{a_e}$ is a complex 3×3 matrix in family space, with dimensions of (mass). They are in one-to-one correspondence with the Yukawa coupling matrices in the superpotential. The third line of eq. (8.12) consists of squark and slepton mass terms of the $(m^2)_i^j$ type in eq. (6.82). Each of $\mathbf{m_Q^2}$, $\mathbf{m_{\overline{u}}^2}$, $\mathbf{m_{\overline{d}}^2}$,

$\mathbf{m^2_L}$, $\mathbf{m^2_{\overline{e}}}$ is a 3×3 matrix in family space that can have complex entries, but they must be hermitian so that the Lagrangian is real. (To avoid clutter, we do not put tildes on the $\mathbf{Q}$ in $\mathbf{m^2_Q}$, etc.) Finally, in the last line of eq. (8.12) we have supersymmetry-breaking contributions to the Higgs potential; $m^2_{H_u}$ and $m^2_{H_d}$ are (mass)^2 terms of the $(m^2)^j_i$ type, while b is the only (mass)^2 term of the type b^{ij} in eq. (6.82) that can occur in the MSSM.[2] Schematically, we can write

$$M_1,\, M_2,\, M_3,\, \mathbf{a_u},\, \mathbf{a_d},\, \mathbf{a_e} \sim m_{\rm soft}; \tag{8.13}$$

$$\mathbf{m^2_Q},\, \mathbf{m^2_L},\, \mathbf{m^2_{\overline{u}}},\, \mathbf{m^2_{\overline{d}}},\, \mathbf{m^2_{\overline{e}}},\, m^2_{H_u},\, m^2_{H_d},\, b \sim m^2_{\rm soft}, \tag{8.14}$$

with a characteristic mass scale $m_{\rm soft}$ that is not much larger than 10^3 GeV, as argued in the Introduction. The expression eq. (8.12) is the most general soft supersymmetry-breaking Lagrangian of the form eq. (6.82) that is compatible with gauge invariance and matter parity conservation.

Unlike the supersymmetry-preserving part of the Lagrangian, $\mathcal{L}^{\rm MSSM}_{\rm soft}$ introduces many new parameters that were not present in the ordinary Standard Model. A careful count reveals that there are 105 masses, phases and mixing angles in the MSSM Lagrangian that cannot be rotated away by redefining the phases and flavor basis for the quark and lepton supermultiplets, and have no counterpart in the ordinary Standard Model. Thus, in principle, supersymmetry (or more precisely, supersymmetry *breaking*) appears to introduce a tremendous arbitrariness in the Lagrangian.

8.4 Hints of an Organizing Principle

Fortunately, there is already good experimental evidence that some sort of powerful "organizing principle" must govern the soft terms. This is because most of the new parameters in eq. (8.12) involve flavor mixing or CP violation of the type that is already severely restricted by experiment.

For example, suppose that $\mathbf{m^2_{\overline{e}}}$ is not diagonal in the basis $(\widetilde{e}_R, \widetilde{\mu}_R, \widetilde{\tau}_R)$ of sleptons whose superpartners are the right-handed pieces of the Standard Model mass eigenstates e, μ, τ. In that case slepton mixing occurs, and the individual lepton numbers will not be conserved. This is true even for processes that only involve the sleptons as virtual particles. A particularly strong limit on this possibility comes from the experimental constraint on $\mu \to e\gamma$, which can occur via the one-loop diagram in Fig. 8.6(a) featuring a virtual bino and slepton. The symbol $\times$ represents an insertion of $-(\mathbf{m^2_{\overline{e}}})_{21}\widetilde{e}_R\widetilde{\mu}^*_R$ in $\mathcal{L}^{\rm MSSM}_{\rm soft}$, and the slepton-bino vertices are determined by the weak hypercharge gauge coupling [see Fig. 6.3(g) and eq. (6.76)].

[2] The parameter we call b is often seen in the literature as m^2_{12} or m^2_3 or $B\mu$.

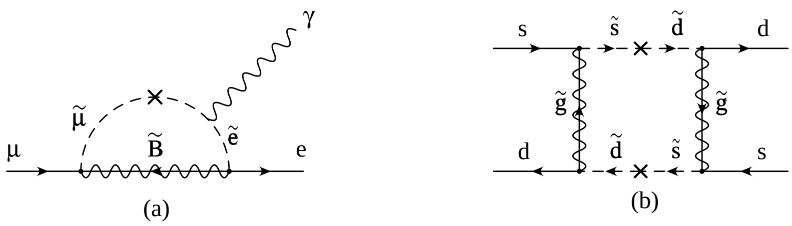


Fig. 8.6. Feynman diagrams that contribute to flavor violation in models with arbitrary soft masses: (a) $\mu \to e\gamma$ and (b) $K^0 \leftrightarrow \overline{K}^0$ mixing.

There are similar diagrams if the left-handed slepton mass matrix $\mathbf{m_L^2}$ has arbitrary off-diagonal entries. If $\mathbf{m_L^2}$ or $\mathbf{m_e^2}$ were "random", with all entries of comparable size, then the contributions to BR($\mu \to e\gamma$) would be about much larger than the present experimental upper limit of 1.2×10^{-11}, even if the sleptons are as heavy as 1 TeV. Therefore the form of the slepton mass matrices must be severely constrained to avoid large $\widetilde{e}_R, \widetilde{\mu}_R$ and $\widetilde{e}_L, \widetilde{\mu}_L$ mixings.

There are also important experimental constraints on the squark (mass)^2 matrices. The strongest of these come from the neutral kaon system. The effective hamiltonian for $K^0 \leftrightarrow \overline{K}^0$ mixing gets contributions from the diagram in Fig. 8.6b, among others, if $\mathcal{L}_{\rm soft}^{\rm MSSM}$ contains (mass)^2 terms that mix down squarks and strange squarks. The gluino-squark-quark vertices in Fig. 8.6b are all fixed by supersymmetry to be of strong interaction strength; there are similar diagrams in which the bino and winos are exchanged.[56] If the squark and gaugino masses are of order 1 TeV or less, one finds that limits on the parameters Δm_K and ϵ and ϵ'/ϵ appearing in the neutral kaon effective hamiltonian severely restrict the amount of $\widetilde{d}_L, \widetilde{s}_L$ and $\widetilde{d}_R, \widetilde{s}_R$ squark mixing, and associated CP-violating complex phases, that one can tolerate in the soft squared masses.[57] Weaker, but still interesting, constraints come from the $D^0, \overline{D}^0$ system, which limits the amounts of $\widetilde{u}_L, \widetilde{c}_L$ and $\widetilde{u}_R, \widetilde{c}_R$ mixings from the soft squark squared mass matrix, and constraints from the $B_d^0, \overline{B}_d^0$ system limit the amount of $\widetilde{d}_L, \widetilde{b}_L$ and $\widetilde{d}_R, \widetilde{b}_R$ mixing. More constraints follow from rare meson decays, notably those involving the parton-level process[58] $b \to s\gamma$. The possibility of mixings between left-handed and right-handed squarks and sleptons with the same charges is also constrained by the same processes. In the MSSM, these mixings can arise from (scalar)^3 interactions involving the $\mathbf{a_u}$, $\mathbf{a_d}$, $\mathbf{a_e}$ matrices, after the Higgs scalar obtain VEVs. For example, for down-type squarks, we have: $\widetilde{\overline{d}}\mathbf{a_d}\widetilde{Q}H_d$ + c.c. $\to$ $(\mathbf{a_d})_{12}\langle H_d^0\rangle \widetilde{d}_R^* \widetilde{s}_R$ + c.c. The experimental constraints on flavor-changing neutral currents (FCNCs) therefore imply that the corresponding off-diagonal elements of the $\mathbf{a_u}$, $\mathbf{a_d}$, $\mathbf{a_e}$ matrices cannot be too large. There are also significant constraints on CP-violating phases in the gaugino masses and (scalar)^3 soft couplings following from limits on the electric dipole moments of the neutron and electron.[59] Detailed limits can be found in the literature, but the essential lesson from experiment is that the soft supersymmetry-breaking Lagrangian cannot be arbitrary or random.

All of these potentially dangerous FCNC and CP-violating effects in the MSSM can be evaded if one assumes (or can explain!) that supersymmetry breaking should be suitably "universal". In particular, consider an idealized limit in which the squark and slepton (mass)^2 matrices are flavor-blind, each proportional to the 3×3 identity matrix in family space:

$$\begin{aligned}&\mathbf{m^2_Q} = m^2_Q \mathbf{1}; \quad \mathbf{m^2_{\overline{u}}} = m^2_{\overline{u}} \mathbf{1}; \quad \mathbf{m^2_{\overline{d}}} = m^2_{\overline{d}} \mathbf{1};\\ &\mathbf{m^2_L} = m^2_L \mathbf{1}; \quad \mathbf{m^2_{\overline{e}}} = m^2_{\overline{e}} \mathbf{1}.\end{aligned} \tag{8.15}$$

Then all squark and slepton mixing angles are rendered trivial, because squarks and sleptons with the same electroweak quantum numbers will be degenerate in mass and can be rotated into each other at will. Supersymmetric contributions to FCNC processes will therefore be very small in such an idealized limit, up to mixing induced by $\mathbf{a_u}$, $\mathbf{a_d}$, $\mathbf{a_e}$. Making the further assumption that the (scalar)^3 couplings are each proportional to the corresponding Yukawa coupling matrix:

$$\mathbf{a_u} = A_{u0}\, \mathbf{y_u}; \qquad \mathbf{a_d} = A_{d0}\, \mathbf{y_d}; \qquad \mathbf{a_e} = A_{e0}\, \mathbf{y_e} \tag{8.16}$$

will ensure that only the squarks and sleptons of the third family can have large (scalar)^3 couplings. Finally, one can avoid disastrously large CP-violating effects with the assumption that the soft parameters do not introduce new complex phases. This is automatic for $m^2_{H_u}$ and $m^2_{H_d}$, and for m^2_Q, $m^2_{\overline{u}}$ etc. if eq. (8.15) is assumed; if they were not real numbers, the Lagrangian would not be real. One can also fix μ in the superpotential and b in eq. (8.12) to be real, by an appropriate phase rotation of H_u and H_d. If one then assumes that

$$\arg(M_1),\ \arg(M_2),\ \arg(M_3),\ \arg(A_{u0}),\ \arg(A_{d0}),\ \arg(A_{e0}) = 0 \text{ or } \pi, \tag{8.17}$$

then the only CP-violating phase in the theory will be the ordinary CKM phase found in the ordinary Yukawa couplings.

Together, the conditions eqs. (8.15)-(8.17) make up a rather weak version of what is often called the assumption of *soft-breaking universality.* The MSSM with these flavor- and CP-preserving relations imposed has far fewer parameters than the most general case. There are 3 independent real gaugino masses, only 5 real squark and slepton squared mass parameters, 3 real (scalar)^3 parameters, and 4 Higgs mass parameters (one of which can be traded for the already-known electroweak breaking scale).

The soft-breaking universality relations eqs. (8.15)-(8.17) (or stronger versions of them) are presumed to be the result of some specific model for the origin of supersymmetry breaking, even though there is considerable disagreement among theorists as to what the specific model should actually be. In any case, they are indicative of an assumed underlying simplicity or symmetry of the Lagrangian at some very high energy scale Q_0, which we will call the "input scale". If we use this Lagrangian to compute masses and cross-sections and decay rates for experiments at ordinary energies near the electroweak scale, the results will involve large logarithms of order $\ln(Q_0/m_Z)$ coming from loop diagrams. As is usual in quantum field theory, the large logarithms can be conveniently resummed using renormalization group (RG) equations, by treating the couplings and masses appearing in the Lagrangian as "running" parameters. Therefore, eqs. (8.15)-(8.17) should be interpreted as boundary conditions on the running soft parameters at the RG scale Q_0 which is very far removed from direct experimental probes. We must then RG-evolve all of the soft parameters, the superpotential parameters, and the gauge couplings down to the electroweak scale or comparable scales where humans perform experiments.

At the electroweak scale, eqs. (8.15) and (8.16) will no longer hold, even if they were exactly true at the input scale Q_0. However, key flavor- and CP-conserving properties remain, to a good approximation. This is because RG corrections due to gauge interactions will respect eqs. (8.15) and (8.16), while RG corrections due to Yukawa interactions are quite small except for couplings involving the top, bottom, and tau flavors. Therefore, the (scalar)^3 couplings and scalar squared mass mixings should be quite negligible for the squarks and sleptons of the first two families. Furthermore, RG evolution does not introduce new CP-violating phases. Therefore, if universality can be arranged to hold at the input scale, supersymmetric contributions to FCNC and CP-violating observables can be acceptably small in comparison to present limits (although quite possibly measurable in future experiments).

One good reason to be optimistic that such a program can succeed is the celebrated apparent unification of gauge couplings in the MSSM. The 1-loop RG equations for the Standard Model gauge couplings g_1, g_2, g_3 are given by

$$\frac{d}{dt}g_a = \frac{1}{16\pi^2}b_a g_a^3 \qquad \Rightarrow \qquad \frac{d}{dt}\alpha_a^{-1} = -\frac{b_a}{2\pi} \qquad (a = 1, 2, 3) \quad (8.18)$$

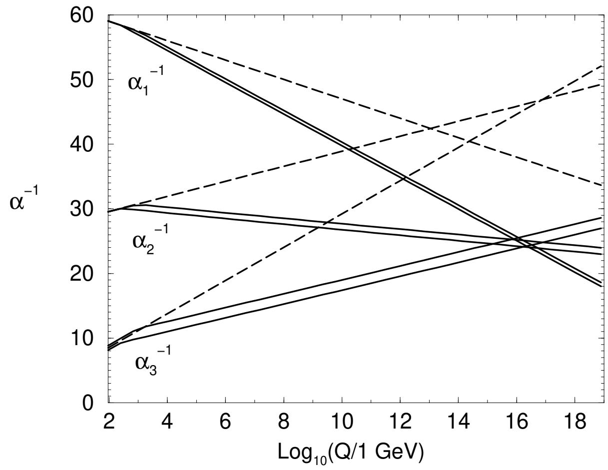


Fig. 8.7. RG evolution of the inverse gauge couplings $\alpha_a^{-1}(Q)$ in the Standard Model (dashed lines) and the MSSM (solid lines). In the MSSM case, $\alpha_3(m_Z)$ is varied between 0.113 and 0.123, and the sparticle mass thresholds between 250 GeV and 1 TeV. Two-loop effects are included.

where $t = \ln(Q/Q_0)$ with Q the RG scale. In the Standard Model, $b_a^{\rm SM} =$ (41/10, −19/6, −7), while in the MSSM one finds instead $b_a^{\rm MSSM} =$ (33/5, 1, −3). The latter set of coefficients are larger because of the virtual effects of the extra MSSM particles in loops. The normalization for g_1 here is chosen to agree with the canonical covariant derivative for grand unification of the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ into $SU(5)$ or $SO(10)$. Thus in terms of the conventional electroweak gauge couplings g and g' with $e = g\sin\theta_W = g'\cos\theta_W$, one has $g_2 = g$ and $g_1 = \sqrt{5/3}g'$. The quantities $\alpha_a = g_a^2/4\pi$ have the nice property that their reciprocals run linearly with RG scale at one-loop order. Figure 8.7 we compare the RG evolution of the α_a^{-1}, including two-loop effects, in the Standard Model (dashed lines) and the MSSM (solid lines). Unlike the Standard Model, the MSSM includes just the right particle content to ensure that the gauge couplings can unify, at a scale $M_U \sim 2 \times 10^{16}$ GeV. While the apparent unification of gauge couplings at M_U could be just an accident, it may also be taken as a strong hint in favor of a grand unified theory (GUT) or superstring models, both of which indeed predict gauge coupling unification below $M_{\rm P}$. Furthermore, if we take this hint seriously, then we can reasonably expect to be able to apply a similar RG analysis to the other MSSM couplings and soft masses as well.

It must be mentioned that there are two other possible types of explanations for the suppression of FCNCs in the MSSM, instead of the universality hypothesis of eqs. (8.15)-(8.17). One might refer to them as "irrelevancy" and "alignment" of the soft masses. The "irrelevancy" idea is that the sparticles masses are simply *extremely* heavy, so that their contributions to FCNC and CP-violating diagrams like Figs. 8.6a,b are highly suppressed. In practice, however, the degree of suppression needed typically requires $m_{\rm soft} \gg 1$ TeV for at least some of the scalar masses; this seems to go directly against the motivation for supersymmetry as a cure for the hierarchy problem as discussed in the Introduction. Nevertheless, it is possible to arrange schemes where this can work in a sensible way. The "alignment" idea is that the squark (mass)^2 matrices do not have the flavor-blindness indicated in eq. (8.15), but are arranged in flavor space to be aligned with the relevant Yukawa matrices in just such a way as to avoid large FCNC effects. The alignment models typically require rather special flavor symmetries. In any case, we will not discuss these possibilities further.

In practice, a given model for the origin of supersymmetry breaking may make predictions for the MSSM soft terms that are even stronger than eqs. (8.15)-(8.17). In Chapter 9 we will discuss the ideas that go into making such predictions and their implications for the MSSM spectrum.

8.5 Electroweak symmetry breaking and the Higgs bosons

In the MSSM, the description of electroweak symmetry breaking is slightly complicated by the fact that there are two complex Higgs doublets $H_u = (H_u^+,\ H_u^0)$ and $H_d = (H_d^0,\ H_d^-)$ rather than just one in the ordinary Standard Model. The classical scalar potential for the Higgs scalar fields in the MSSM is given by

$$\begin{aligned} V = & (|\mu|^2 + m_{H_u}^2)(|H_u^0|^2 + |H_u^+|^2) + (|\mu|^2 + m_{H_d}^2)(|H_d^0|^2 + |H_d^-|^2) \\ & + b\,(H_u^+ H_d^- - H_u^0 H_d^0) + \text{c.c.} \\ & + \frac{1}{8}(g^2 + g'^2)(|H_u^0|^2 + |H_u^+|^2 - |H_d^0|^2 - |H_d^-|^2)^2 \\ & + \tfrac{1}{2} g^2 |H_u^+ H_d^{0*} + H_u^0 H_d^{-*}|^2 . \end{aligned} \tag{8.19}$$

The terms proportional to $|\mu|^2$ come from the $|F|^2$ terms; see eq. (8.5). The terms proportional to g^2 and g'^2 are the D^2 term contributions, and may be derived from the general formula eq. (6.79) after some rearranging. Finally, the terms proportional to $m_{H_u}^2$, $m_{H_d}^2$ and b are nothing but a rewriting of the last three terms of eq. (8.12). The full scalar potential of the theory also includes many terms involving the squark and slepton fields that we can ignore here, since they do not get VEVs because they have large positive (mass)^2.

We now have to demand that the minimum of this potential should break electroweak symmetry down to electromagnetism $SU(2)_L \times U(1)_Y \to U(1)_{\rm EM}$, in accord with experiment. We can use the freedom to make gauge transformations to simplify this analysis. First, the freedom to make $SU(2)_L$ gauge transformations allows us to rotate away a possible VEV for one of the weak isospin components of one of the scalar fields; so without loss of generality we can take $H_u^+ = 0$ at the minimum of the potential. Then one finds that a minimum of the potential satisfying $\partial V/\partial H_u^+ = 0$ must also have $H_d^- = 0$. This is good, because it means that at the minimum of the potential electromagnetism is necessarily unbroken, since the charged components of the Higgs scalars cannot get VEVs. After setting $H_u^+ = H_d^- = 0$, we are left to consider the scalar potential

$$V = (|\mu|^2 + m^2_{H_u})|H_u^0|^2 + (|\mu|^2 + m^2_{H_d})|H_d^0|^2 - (b\, H_u^0 H_d^0 + {\rm c.c.}) \\ +\frac{1}{8}(g^2 + g'^2)(|H_u^0|^2 - |H_d^0|^2)^2. \tag{8.20}$$

The only term in this potential that depends on the phases of the fields is the b-term. Therefore, a redefinition of the phase of H_u or H_d can absorb any phase in b, so we can take b to be real and positive. Then it is clear that a minimum of the potential V requires that $H_u^0 H_d^0$ is also real and positive, so $\langle H_u^0 \rangle$ and $\langle H_d^0 \rangle$ must have opposite phases. We can therefore use a $U(1)_Y$ gauge transformation to make them both be real and positive without loss of generality, since H_u and H_d have opposite weak hypercharges ($\pm 1/2$). It follows that CP cannot be spontaneously broken by the Higgs scalar potential, since the VEVs and b can be simultaneously chosen real, as a convention. This means that the Higgs scalar mass eigenstates can be assigned well-defined eigenvalues of CP, at least at tree-level. (CP-violating phases in other couplings can induce loop-suppressed CP violation in the Higgs sector, but do not change the fact that b, v_u, and v_d can always be chosen real and positive.)

Note that the b-term always favors electroweak symmetry breaking. The combination of the b term and the terms $m^2_{H_u}$ and $m^2_{H_d}$ can allow for one linear combination of H_u^0 and H_d^0 to have a negative (mass)^2 near $H_u^0 = H_d^0 = 0$. This requires that

$$b^2 > (|\mu|^2 + m^2_{H_u})(|\mu|^2 + m^2_{H_d}). \tag{8.21}$$

If this inequality is not satisfied, then $H_u^0 = H_d^0 = 0$ will be a stable minimum of the potential, and electroweak symmetry breaking will not occur. A negative value for $|\mu|^2 + m^2_{H_u}$ will help eq. (8.21) to be satisfied, but it is not necessary. Furthermore, even if $m^2_{H_u} < 0$, there may be no electroweak symmetry breaking if $|\mu|$ is too large or if b is too small. Still, the large negative contributions to $m^2_{H_u}$ from the RG equation (10.18) discussed in the previous section are an important factor in ensuring that electroweak symmetry breaking can occur in models with simple boundary conditions for the soft terms.

In order for the MSSM scalar potential to be viable, it is not enough that the point $H_u^0 = H_d^0 = 0$ is destabilized by a negative (mass)^2 direction; we must also make sure that the potential is bounded from below for arbitrarily large values of the scalar fields, so that V will really have a minimum. (Recall from the discussion in sections 6.2 and 6.4 that scalar potentials in purely supersymmetric theories are automatically positive and so clearly bounded from below. But, now that we have introduced supersymmetry breaking, we must be careful.) The scalar quartic interactions in V will stabilize the potential for almost all arbitrarily large values of H_u^0 and H_d^0. However, for the special directions in field space $|H_u^0| = |H_d^0|$, the quartic contributions to V [the second line in eq. (8.20)] are identically zero. Such directions in field space are called D-flat directions, because along them the part of the scalar potential coming from D-terms vanishes. In order for the potential to be bounded from below, we need the quadratic part of the scalar potential to be positive along the D-flat directions. This requirement amounts to

$$2b < 2|\mu|^2 + m_{H_u}^2 + m_{H_d}^2. \tag{8.22}$$

Interestingly, if $m_{H_u}^2 = m_{H_d}^2$, the constraints eqs. (8.21) and (8.22) cannot both be satisfied. In models derived from the minimal supergravity or gauge-mediated boundary conditions, $m_{H_u}^2 = m_{H_d}^2$ is supposed to hold at tree level at the input scale, but the X_t contribution to the RG equation for $m_{H_u}^2$ naturally pushes it to negative or small values $m_{H_u}^2 < m_{H_d}^2$ at the electroweak scale, as we will see in section 10.1. Unless this effect is large, the parameter space in which the electroweak symmetry is broken would be quite small. So in these models electroweak symmetry breaking is actually driven purely by quantum corrections; this mechanism is therefore known as *radiative electroweak symmetry breaking*. The realization [92, 97] that this works most naturally with a large top-quark Yukawa coupling provides additional motivation for these models.

Having established the conditions necessary for H_u^0 and H_d^0 to get non-zero VEVs, we can now require that they are compatible with the observed phenomenology of electroweak symmetry breaking $SU(2)_L \times U(1)_Y \to U(1)_{\mathrm{EM}}$. Let us write

$$v_u = \langle H_u^0 \rangle, \qquad v_d = \langle H_d^0 \rangle. \tag{8.23}$$

These VEVs can be connected to the known mass of the Z^0 boson and the electroweak gauge couplings:

$$v_u^2 + v_d^2 = v^2 = 2m_Z^2/(g^2 + g'^2) \approx (174\,\mathrm{GeV})^2. \tag{8.24}$$

The ratio of the two VEVs is traditionally written as

$$\tan\beta \equiv v_u/v_d. \tag{8.25}$$

The value of $\tan\beta$ is not fixed by present experiments, but it depends on the Lagrangian parameters of the MSSM in a calculable way. Since $v_u = v\sin\beta$ and $v_d = v\cos\beta$ were taken to be real and positive, we have $0 < \beta < \pi/2$, a requirement that will be sharpened below. Now one can write down the conditions $\partial V/\partial H_u^0 = \partial V/\partial H_d^0 = 0$ under which the potential eq. (8.20) will have a minimum satisfying eqs. (8.24) and (8.25):

$$|\mu|^2 + m_{H_u}^2 = b\cot\beta + (m_Z^2/2)\cos 2\beta. \tag{8.26}$$

$$|\mu|^2 + m_{H_d}^2 = b\tan\beta - (m_Z^2/2)\cos 2\beta; \tag{8.27}$$

It is easy to check that these equations indeed satisfy the necessary conditions eqs. (8.21) and (8.22). They allow us to eliminate two of the Lagrangian parameters b and $|\mu|$ in favor of $\tan\beta$, but do not determine the phase of μ.

As an aside, we note that eqs. (8.27) and (8.26) highlight the "μ problem" already mentioned in section 8.1. Suppose we view $|\mu|^2$, b, $m_{H_u}^2$ and $m_{H_d}^2$ as input parameters, and m_Z^2 and $\tan\beta$ as output parameters obtained by solving these two equations. Then, without miraculous cancellations, we expect that all of the input parameters ought to be within an order of magnitude or two of m_Z^2. However, in the MSSM, μ is a supersymmetry-respecting parameter appearing in the superpotential, while b, $m_{H_u}^2$, $m_{H_d}^2$ are supersymmetry-breaking parameters. This has lead to a widespread belief that the MSSM must be extended at very high energies to include a mechanism that relates the effective value of μ to the supersymmetry-breaking mechanism in some way; see section 12.1 and Refs.[43, 44, 45] for examples.

The Higgs scalar fields in the MSSM consist of two complex $SU(2)_L$-doublet, or eight real, scalar degrees of freedom. When the electroweak symmetry is broken, three of them are the would-be Nambu-Goldstone bosons G^0, $G^\pm$, which become the longitudinal modes of the Z^0 and $W^\pm$ massive vector bosons. The remaining five Higgs scalar mass eigenstates consist of two CP-even neutral scalars h^0 and H^0, one CP-odd neutral scalar A^0, and a charge $+1$ scalar H^+ and its conjugate charge -1 scalar H^-. (Here we define $G^- = G^{+*}$ and $H^- = H^{+*}$.) The gauge-eigenstate fields can be expressed in terms of the mass eigenstate fields as:

$$\begin{pmatrix} H_u^0 \\ H_d^0 \end{pmatrix} = \begin{pmatrix} v_u \\ v_d \end{pmatrix} + \frac{1}{\sqrt{2}} R_\alpha \begin{pmatrix} h^0 \\ H^0 \end{pmatrix} + \frac{i}{\sqrt{2}} R_{\beta_0} \begin{pmatrix} G^0 \\ A^0 \end{pmatrix} \tag{8.28}$$

$$\begin{pmatrix} H_u^+ \\ H_d^{-*} \end{pmatrix} = R_{\beta_\pm} \begin{pmatrix} G^+ \\ H^+ \end{pmatrix} \tag{8.29}$$

where the orthogonal rotation matrices

$$R_\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \tag{8.30}$$

$$R_{\beta_0} = \begin{pmatrix} \sin\beta_0 & \cos\beta_0 \\ -\cos\beta_0 & \sin\beta_0 \end{pmatrix}, \qquad R_{\beta_\pm} = \begin{pmatrix} \sin\beta_\pm & \cos\beta_\pm \\ -\cos\beta_\pm & \sin\beta_\pm \end{pmatrix} \tag{8.31}$$

are chosen so that the quadratic part of the potential has diagonal squared-masses:

$$\begin{aligned} V = {} & \tfrac{1}{2}m_{h^0}^2(h^0)^2 + \tfrac{1}{2}m_{H^0}^2(H^0)^2 + \tfrac{1}{2}m_{G^0}^2(G^0)^2 + \tfrac{1}{2}m_{A^0}^2(A^0)^2 \\ & + m_{G^\pm}^2|G^+|^2 + m_{H^\pm}^2|H^+|^2 + \ldots . \end{aligned} \tag{8.32}$$

Then, provided that v_u, v_d minimize the tree-level potential,[3] one finds that $\beta_0 = \beta_\pm = \beta$, and $m_{G^0}^2 = m_{G^\pm}^2 = 0$, and

$$m_{A^0}^2 = 2b/\sin 2\beta \tag{8.33}$$

$$m_{h^0,H^0}^2 = \tfrac{1}{2}\Big(m_{A^0}^2 + m_Z^2 \mp \sqrt{(m_{A^0}^2 + m_Z^2)^2 - 4m_Z^2 m_{A^0}^2 \cos^2 2\beta}\Big), \tag{8.34}$$

$$m_{H^\pm}^2 = m_{A^0}^2 + m_W^2. \tag{8.35}$$

The mixing angle α is determined at tree-level by

$$\frac{\sin 2\alpha}{\sin 2\beta} = -\left(\frac{m_{H^0}^2 + m_{h^0}^2}{m_{H^0}^2 - m_{h^0}^2}\right), \qquad \frac{\tan 2\alpha}{\tan 2\beta} = \left(\frac{m_{A^0}^2 + m_Z^2}{m_{A^0}^2 - m_Z^2}\right), \tag{8.36}$$

and is traditionally chosen to be negative; it follows that $-\pi/2 < \alpha < 0$ (provided $m_{A^0} > m_Z$). The Feynman rules for couplings of the mass eigenstate Higgs scalars to the Standard Model quarks and leptons and the electroweak vector bosons, as well as to the various sparticles, have been worked out in detail in Ref.[98, 99].

The masses of A^0, H^0 and $H^\pm$ can in principle be arbitrarily large since they all grow with $b/\sin 2\beta$. In contrast, the mass of h^0 is bounded from above. It is not hard to show from eq. (8.34) that

$$m_{h^0} < |\cos 2\beta| m_Z \tag{8.37}$$

[3] It is sometimes useful to expand around VEVs v_u, v_d that do not minimize the tree-level potential, for example to minimize the loop-corrected effective potential instead. In that case, the three angles β, β_0, and $\beta_\pm$ are all slightly different, and $m_{G^0}^2$ and $m_{G^\pm}^2$ are non-zero and differ from each other by a small amount.

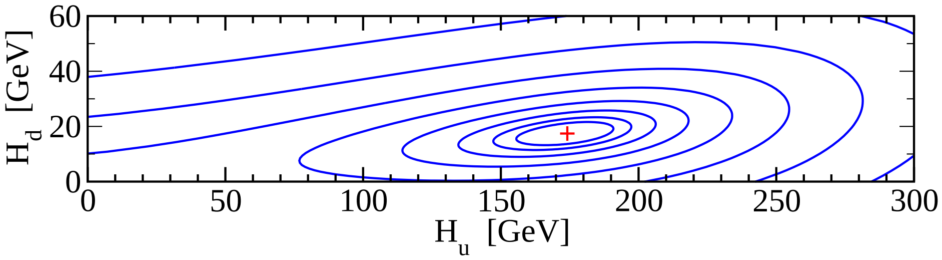


Fig. 8.8. A contour map of the Higgs potential, for a typical case with $\tan\beta \approx -\cot\alpha \approx 10$. The symbol + marks the minimum of the potential, and each curve outward represents an equipotential with $V - V_0$ doubled. Oscillations of the neutral Higgs fields along the shallow direction with $H_u^0/H_d^0 \approx 10$ correspond to the mass eigenstate h^0, and the orthogonal steeper direction to H^0.

at tree-level.[100] This corresponds to the existence of a shallow direction in the scalar potential, along the direction $(H_u^0 - v_u, H_d^0 - v_d) \propto (\cos\alpha, -\sin\alpha)$. A contour map of the potential, for a typical case with $\tan\beta \approx -\cot\alpha \approx 10$, is shown in Figure 8.8. If the tree-level inequality (8.37) were robust, the lightest Higgs boson of the MSSM would have been discovered at LEP2. However, the tree-level mass formulas given above for the Higgs mass eigenstates are subject to significant quantum corrections, which are especially important for h^0. The largest such contributions typically come from top and stop loops. In the limit of stop squark masses $m_{\tilde{t}} = m_{\tilde{t}_1} \approx m_{\tilde{t}_2}$ much greater than the top quark mass m_t, one finds a large positive one-loop radiative correction to eq. (8.34):

$$\Delta(m_{h^0}^2) = \frac{3}{2\pi^2} \cos^2\alpha \; y_t^2 m_t^2 \ln(m_{\tilde{t}}/m_t). \tag{8.38}$$

Including this and other corrections [101, 102], one can obtain only a considerably weaker, but still very interesting, bound

$$m_{h^0} \lesssim 130 \text{ GeV} \tag{8.39}$$

in the MSSM. This assumes that all of the sparticles that can contribute to $\Delta(m_{h^0}^2)$ in loops have masses that do not exceed 1 TeV. By adding extra supermultiplets to the MSSM, this bound can be made even weaker. However, assuming that none of the MSSM sparticles have masses exceeding 1 TeV and that all of the couplings in the theory remain perturbative up to the unification scale, one still finds [103]

$$m_{h^0} \lesssim 150 \text{ GeV}. \tag{8.40}$$

This bound is also weakened if, for example, the top squarks are heavier than 1 TeV, but the upper bound rises only logarithmically with the soft masses, as can be seen from eq. (8.38). Thus it is a fairly robust prediction of supersymmetry at the electroweak scale that at least one of the Higgs scalar bosons must be light. (However, if one is willing to consider arbitrary extensions of the MSSM involving non-perturbative physics near the TeV scale, none of these bounds apply.)

An interesting limit occurs when $m_{A^0} \gg m_Z$. In that case, m_{h^0} can saturate the upper bounds just mentioned with $m_{h^0} \approx m_Z|\cos 2\beta|+$ loop corrections. The particles A^0, H^0, and $H^\pm$ are much heavier and nearly degenerate, forming an isospin doublet which decouples from sufficiently low-energy experiments. The angle α is very nearly $\beta - \pi/2$. In this limit, h^0 has the same couplings to quarks and leptons and electroweak gauge bosons as would the physical Higgs boson of the ordinary Standard Model without supersymmetry. Indeed, model-building experiences have shown that it is not uncommon for h^0 to behave in a way nearly indistinguishable from a Standard Model-like Higgs boson, even if m_{A^0} is not too huge. However, it should be kept in mind that the couplings of h^0 might turn out to deviate in important ways from those of a Standard Model Higgs boson. For a given set of model parameters, it is always important to take into account the complete set of one-loop corrections and even the dominant two-loop effects in order to get reasonably accurate predictions for the Higgs masses and mixing angles.[101, 102]

In the MSSM, the masses and CKM mixing angles of the quarks and leptons are determined by the Yukawa couplings of the superpotential and the parameter $\tan\beta$. This is because the top, charm and up quarks get masses proportional to $v_u = v\sin\beta$ and the bottom, strange, and down quarks and the charge leptons get masses proportional to $v_d = v\cos\beta$. Therefore one finds at tree-level

$$m_t = y_t v_u = y_t v \sin\beta, \tag{8.41}$$
$$m_b = y_b v_d = y_b v \cos\beta, \tag{8.42}$$
$$m_\tau = y_\tau v_d = y_\tau v \cos\beta, \tag{8.43}$$

with $v \approx 175$ GeV. These relations hold for the running masses of t, b, τ rather than the physical pole masses, which are significantly larger. Including those corrections, one can relate the Yukawa couplings to $\tan\beta$ and the known fermion masses and CKM mixing angles. It is now clear why we have not neglected y_b and y_τ, even though $m_b, m_\tau \ll m_t$. To a first approximation,

$$y_b/y_t = (m_b/m_t)\tan\beta \tag{8.44}$$
$$y_\tau/y_t = (m_\tau/m_t)\tan\beta, \tag{8.45}$$

so that y_b and y_τ cannot be neglected if $\tan\beta$ is much larger than 1. In fact, there are good theoretical motivations for considering models with large $\tan\beta$. For example, models based on the GUT gauge group $SO(10)$ (or certain of its subgroups) can unify the running top, bottom and tau Yukawa couplings at the unification scale; this requires $\tan\beta$ to be very roughly of order m_t/m_b.

Note that if one tries to make $\sin\beta$ too small, then y_t will become nonperturbatively large. Requiring that y_t does not blow up above the electroweak scale, one finds that $\tan\beta \gtrsim 1.2$ or so, depending on the mass of the top quark, the QCD coupling, and other fine details. In principle, one can also determine a lower bound on $\cos\beta$ by requiring that y_b and y_τ are not nonperturbatively large. This gives a rough upper bound of $\tan\beta \lesssim 65$. However, this is complicated slightly by the fact that the bottom quark mass gets significant one-loop corrections in the large $\tan\beta$ limit [106]. One can obtain a slightly stronger upper bound on $\tan\beta$ in models where $m^2_{H_u} = m^2_{H_d}$ at the input scale, by requiring that y_b does not significantly exceed y_t. In the following, we will see that the parameter $\tan\beta$ has an important effect on the masses and mixings of the MSSM sparticles.

8.6 Neutralinos and charginos

The higgsinos and electroweak gauginos mix with each other because of the effects of electroweak symmetry breaking. The neutral higgsinos ($\widetilde{H}^0_u$ and $\widetilde{H}^0_d$) and the neutral gauginos ($\widetilde{B}$, $\widetilde{W}^0$) combine to form four neutral mass eigenstates called *neutralinos*. The charged higgsinos ($\widetilde{H}^+_u$ and $\widetilde{H}^-_d$) and winos ($\widetilde{W}^+$ and $\widetilde{W}^-$) mix to form two mass eigenstates with charge ± 1 called *charginos*. We will denote [4] the neutralino and chargino mass eigenstates by $\widetilde{N}_i$ $(i = 1, 2, 3, 4)$ and $\widetilde{C}^\pm_i$ $(i = 1, 2)$. By convention, these are labelled in ascending order, so that $m_{\widetilde{N}_1} < m_{\widetilde{N}_2} < m_{\widetilde{N}_3} < m_{\widetilde{N}_4}$ and $m_{\widetilde{C}_1} < m_{\widetilde{C}_2}$. The lightest neutralino, $\widetilde{N}_1$, is usually assumed to be the LSP, unless there is a lighter gravitino or unless R-parity is not conserved, because it is the only MSSM particle that can make a good cold dark matter candidate. In this section, we will describe the mass spectrum and mixing of the neutralinos and charginos in the MSSM.

In the gauge-eigenstate basis $\psi^0 = (\widetilde{B}, \widetilde{W}^0, \widetilde{H}^0_d, \widetilde{H}^0_u)$, the neutralino mass terms in the Lagrangian are

$$\mathcal{L}_{\text{neutralino mass}} = -\tfrac{1}{2}(\psi^0)^T \mathbf{M}_{\widetilde{N}} \psi^0 + \text{c.c.} \tag{8.46}$$

[4] Other common notations use $\widetilde{\chi}^0_i$ or $\widetilde{Z}_i$ for neutralinos, and $\widetilde{\chi}^\pm_i$ or $\widetilde{W}^\pm_i$ for charginos.

where

$$\mathbf{M}_{\widetilde{N}} = \begin{pmatrix} M_1 & 0 & -g' v_d/\sqrt{2} & g' v_u/\sqrt{2} \\ 0 & M_2 & g v_d/\sqrt{2} & -g v_u/\sqrt{2} \\ -g' v_d/\sqrt{2} & g v_d/\sqrt{2} & 0 & -\mu \\ g' v_u/\sqrt{2} & -g v_u/\sqrt{2} & -\mu & 0 \end{pmatrix}. \tag{8.47}$$

The entries M_1 and M_2 in this matrix come directly from the MSSM soft Lagrangian [see eq. (8.12)] while the entries $-\mu$ are the supersymmetric higgsino mass terms [see eq. (8.4)]. The terms proportional to g, g' are the result of Higgs-higgsino-gaugino couplings [see eq. (6.76) and Fig. 6.3g], with the Higgs scalars getting their VEVs [eqs. (8.24), (8.25)]. One can also write this as

$$\mathbf{M}_{\widetilde{N}} = \begin{pmatrix} M_1 & 0 & -c_\beta\, s_W\, m_Z & s_\beta\, s_W\, m_Z \\ 0 & M_2 & c_\beta\, c_W\, m_Z & -s_\beta\, c_W\, m_Z \\ -c_\beta\, s_W\, m_Z & c_\beta\, c_W\, m_Z & 0 & -\mu \\ s_\beta\, s_W\, m_Z & -s_\beta\, c_W\, m_Z & -\mu & 0 \end{pmatrix}. \tag{8.48}$$

Here we have introduced abbreviations $s_\beta = \sin\beta$, $c_\beta = \cos\beta$, $s_W = \sin\theta_W$, and $c_W = \cos\theta_W$. The mass matrix $\mathbf{M}_{\widetilde{N}}$ can be diagonalized by a unitary matrix $\mathbf{N}$ with

$$\widetilde{N}_i = \mathbf{N}_{ij}\psi^0_j, \tag{8.49}$$

so that

$$\mathbf{M}^{\rm diag}_{\widetilde{N}} = \mathbf{N}^*\mathbf{M}_{\widetilde{N}}\mathbf{N}^{-1} \tag{8.50}$$

has positive real entries $m_{\widetilde{N}_1}$, $m_{\widetilde{N}_2}$, $m_{\widetilde{N}_3}$, $m_{\widetilde{N}_4}$ on the diagonal. These are the absolute values of the eigenvalues of $\mathbf{M}_{\widetilde{N}}$, or equivalently the square roots of the eigenvalues of $\mathbf{M}^\dagger_{\widetilde{N}}\mathbf{M}_{\widetilde{N}}$. The indices (i, j) on $\mathbf{N}_{ij}$ are (mass, gauge) eigenstate labels. The mass eigenvalues and the mixing matrix $\mathbf{N}_{ij}$ can be given in closed form in terms of the parameters M_1, M_2, μ and $\tan\beta$, but the results are very complicated and not very illuminating.

In general, the parameters M_1, M_2, and μ can have arbitrary complex phases. In the broad class of minimal supergravity or gauge-mediated models satisfying the gaugino unification conditions eq. (9.27) or (9.40), M_2 and M_1 will have the same complex phase, which is preserved by RG evolution eq. (10.5). In that case, a redefinition of the phases of $\widetilde{B}$ and $\widetilde{W}$ allows us to make M_1 and M_2 both real and positive. The phase of μ is then really a physical parameter and cannot be rotated away. [We have already used up the freedom to redefine the phases of the Higgs fields, since we have picked b and $\langle H_u^0\rangle$ and $\langle H_d^0\rangle$ to be real and positive, to guarantee that the off-diagonal entries in eq. (8.48) proportional to m_Z are real.] However, if μ is not real, then there can be potentially disastrous CP-violating effects in low-energy physics, including electric dipole moments for both the electron and the neutron. Therefore, it is usual (although not mandatory because of the possibility of nontrivial cancellations) to assume that μ is real in the same set of phase conventions that make M_1, M_2, b, $\langle H_u^0\rangle$ and $\langle H_d^0\rangle$ real and positive. The sign of μ is still undetermined by this constraint.

In models which satisfy eq. (10.7), one has the nice prediction

$$M_1 \approx \frac{5}{3}\tan^2\theta_W\, M_2 \approx 0.5 M_2 \tag{8.51}$$

at the electroweak scale. If so, then the neutralino masses and mixing angles depend on only three unknown parameters. This assumption is sufficiently theoretically compelling that it has been made in almost all phenomenological studies; nevertheless it should be recognized as an assumption, to be tested someday by experiment.

Specializing further, there is a not-unlikely limit in which electroweak symmetry breaking effects can be viewed as a small perturbation on the neutralino mass matrix. If

$$m_Z \ll |\mu \pm M_1|, |\mu \pm M_2| \tag{8.52}$$

then the neutralino mass eigenstates are very nearly a "bino-like" $\widetilde{N}_1 \approx \widetilde{B}$; a "wino-like" $\widetilde{N}_2 \approx \widetilde{W}^0$; and "higgsino-like" $\widetilde{N}_3, \widetilde{N}_4 \approx (\widetilde{H}_u^0 \pm \widetilde{H}_d^0)/\sqrt{2}$, with mass eigenvalues:

$$m_{\widetilde{N}_1} = M_1 - \frac{m_Z^2 s_W^2 (M_1 + \mu\sin 2\beta)}{\mu^2 - M_1^2} + \ldots \tag{8.53}$$

$$m_{\widetilde{N}_2} = M_2 - \frac{m_W^2 (M_2 + \mu\sin 2\beta)}{\mu^2 - M_2^2} + \ldots \tag{8.54}$$

$$m_{\widetilde{N}_3}, m_{\widetilde{N}_4} = |\mu| + \frac{m_Z^2(1-\epsilon\sin 2\beta)(|\mu| + M_1 c_W^2 + M_2 s_W^2)}{2(|\mu|+M_1)(|\mu|+M_2)} + \ldots, \tag{8.55}$$

$$|\mu| + \frac{m_Z^2(1+\epsilon\sin 2\beta)(|\mu| - M_1 c_W^2 - M_2 s_W^2)}{2(|\mu|-M_1)(|\mu|-M_2)} + \ldots \tag{8.56}$$

where we have assumed μ is real with sign $\epsilon = \pm 1$. The labeling of the mass eigenstates $\widetilde{N}_1$ and $\widetilde{N}_2$ assumes $M_1 < M_2 < |\mu|$; otherwise the subscripts may need to be rearranged. It turns out that a "bino-like" LSP $\widetilde{N}_1$ can very easily have the right cosmological abundance to make a good dark matter candidate, so the large $|\mu|$ limit may be preferred from that point of view. In addition, this limit tends to emerge from minimal supergravity boundary conditions on the soft parameters, which often require $|\mu|$ to be larger than M_1 and M_2 in order to get correct electroweak symmetry breaking.

The chargino spectrum can be analyzed in a similar way. In the gauge-eigenstate basis $\psi^{\pm} = (\widetilde{W}^+, \widetilde{H}_u^+, \widetilde{W}^-, \widetilde{H}_d^-)$, the chargino mass terms in the Lagrangian are

$$\mathcal{L}_{\text{chargino mass}} = -\tfrac{1}{2}(\psi^{\pm})^T \mathbf{M}_{\widetilde{C}} \psi^{\pm} + \text{c.c.} \tag{8.57}$$

where, in 2×2 block form,

$$\mathbf{M}_{\widetilde{C}} = \begin{pmatrix} \mathbf{0} & \mathbf{X}^T \\ \mathbf{X} & \mathbf{0} \end{pmatrix} \tag{8.58}$$

with

$$\mathbf{X} = \begin{pmatrix} M_2 & gv_u \\ gv_d & \mu \end{pmatrix} = \begin{pmatrix} M_2 & \sqrt{2}s_\beta\, m_W \\ \sqrt{2}c_\beta\, m_W & \mu \end{pmatrix}. \tag{8.59}$$

The mass eigenstates are related to the gauge eigenstates by two unitary $2{\times}2$ matrices $\mathbf{U}$ and $\mathbf{V}$ according to

$$\begin{pmatrix} \widetilde{C}_1^+ \\ \widetilde{C}_2^+ \end{pmatrix} = \mathbf{V} \begin{pmatrix} \widetilde{W}^+ \\ \widetilde{H}_u^+ \end{pmatrix}; \qquad \begin{pmatrix} \widetilde{C}_1^- \\ \widetilde{C}_2^- \end{pmatrix} = \mathbf{U} \begin{pmatrix} \widetilde{W}^- \\ \widetilde{H}_d^- \end{pmatrix}. \tag{8.60}$$

Note that the mixing matrix for the positively charged left-handed fermions is different from that for the negatively charged left-handed fermions. They are chosen so that

$$\mathbf{U}^* \mathbf{X} \mathbf{V}^{-1} = \begin{pmatrix} m_{\widetilde{C}_1} & 0 \\ 0 & m_{\widetilde{C}_2} \end{pmatrix}, \tag{8.61}$$

with positive real entries $m_{\widetilde{C}_i}$. Because these are only $2{\times}2$ matrices, it is not hard to solve for the masses explicitly:

$$\begin{aligned} m^2_{\widetilde{C}_1}, m^2_{\widetilde{C}_2} = \frac{1}{2}\Big[&|M_2|^2 + |\mu|^2 + 2m_W^2 \\ &\mp \sqrt{(|M_2|^2 + |\mu|^2 + 2m_W^2)^2 - 4|\mu M_2 - m_W^2 \sin 2\beta|^2}\Big]. \end{aligned} \tag{8.62}$$

These are the (doubly degenerate) eigenvalues of the 4×4 matrix $\mathbf{M}^\dagger_{\widetilde{C}} \mathbf{M}_{\widetilde{C}}$, or equivalently the eigenvalues of $\mathbf{X}^\dagger \mathbf{X}$, since

$$\mathbf{V} \mathbf{X}^\dagger \mathbf{X} \mathbf{V}^{-1} = \mathbf{U}^* \mathbf{X} \mathbf{X}^\dagger \mathbf{U}^T = \begin{pmatrix} m^2_{\widetilde{C}_1} & 0 \\ 0 & m^2_{\widetilde{C}_2} \end{pmatrix}. \tag{8.63}$$

(But, they are *not* the squares of the eigenvalues of $\mathbf{X}$.) In the limit of eq. (8.52) with real M_2 and μ, one finds that the charginos mass eigenstates consist of a wino-like $\widetilde{C}_1^{\pm}$ and and a higgsino-like $\widetilde{C}_2^{\pm}$, with masses

$$m_{\widetilde{C}_1} = M_2 - \frac{m_W^2(M_2 + \mu \sin 2\beta)}{\mu^2 - M_2^2} + \ldots \tag{8.64}$$

$$m_{\widetilde{C}_2} = |\mu| + \frac{m_W^2(|\mu| + \epsilon M_2 \sin 2\beta)}{\mu^2 - M_2^2} + \ldots . \tag{8.65}$$

Here again the labeling assumes $M_2 < |\mu|$, and ϵ is the sign of μ. Amusingly, the lighter chargino $\widetilde{C}_1$ is nearly degenerate with the second lightest neutralino $\widetilde{N}_2$ in this limit, but this is not an exact result. Their higgsino-like colleagues $\widetilde{N}_3$, $\widetilde{N}_4$ and $\widetilde{C}_2$ have masses of order $|\mu|$. The case of $M_1 \approx 0.5 M_2 \ll |\mu|$ is not uncommonly found in viable models following from the boundary conditions in section 9, and it has been elevated to the status of a benchmark scenario in many phenomenological studies. However it cannot be overemphasized that such expectations are not mandatory.

In practice, the masses and mixing angles for the neutralinos and charginos are best computed numerically. The corresponding Feynman rules may be inferred in terms of $\mathbf{N}$, $\mathbf{U}$ and $\mathbf{V}$ from the MSSM Lagrangian as discussed above; they are collected in Refs.[21, 98]

8.7 The gluino

The gluino is a color octet fermion, so it cannot mix with any other particle in the MSSM, even if R-parity is violated. In this regard, it is unique among all of the MSSM sparticles. In the models following from minimal supergravity or gauge-mediated boundary conditions, the gluino mass parameter M_3 is related to the bino and wino mass parameters M_1 and M_2 by eq. (10.7)

$$M_3 = \frac{\alpha_s}{\alpha} \sin^2\theta_W \, M_2 = \frac{3}{5}\frac{\alpha_s}{\alpha} \cos^2\theta_W \, M_1 \tag{8.66}$$

at any RG scale, up to small two-loop corrections. If we use values $\alpha_s = 0.118$, $\alpha = 1/128$, $\sin^2\theta_W = 0.23$, then one finds the rough prediction

$$M_3 : M_2 : M_1 \approx 7 : 2 : 1 \tag{8.67}$$

at the electroweak scale. In particular, we suspect that the gluino should be much heavier than the lighter neutralinos and charginos.

For more precise estimates, one must take into account the fact that the parameter M_3 is really a running mass which has an implicit dependence on the RG scale Q. Because the gluino is a strongly interacting particle, M_3 runs rather quickly with Q [see eq. (10.5)]. A more useful quantity physically is the RG scale-independent mass $m_{\widetilde{g}}$ at which the renormalized gluino propagator has a pole. Including one-loop corrections to the gluino propagator due to gluon exchange and quark-squark loops, one finds that the pole mass is given in terms of the running mass in the $\overline{\mathrm{DR}}$ scheme by

$$m_{\widetilde{g}} = M_3(Q)\Big(1 + \frac{\alpha_s}{4\pi}\big[15 + 6\ln(Q/M_3) + \sum A_{\widetilde{q}}\big]\Big) \tag{8.68}$$

where

$$A_{\widetilde{q}} = \int_0^1 dx\, x \ln\big[x m_{\widetilde{q}}^2/M_3^2 + (1-x)m_q^2/M_3^2 - x(1-x)\big]. \tag{8.69}$$

The sum in eq. (8.68) is over all 12 squark-quark supermultiplets, and we have neglected small effects due to squark mixing. It is easy to check that requiring $m_{\widetilde{g}}$ to be independent of Q in eq. (8.68) reproduces the one-loop RG equation for $M_3(Q)$ in eq. (10.5). The correction terms proportional to α_s in eq. (8.68) can be quite significant, so that $m_{\widetilde{g}}/M_3(M_3)$ can exceed unity by 25% or more. The reasons for this are that the gluino is strongly interacting, with a large group theory factor [the 15 in eq. (8.68)] due to its color octet nature, and that it couples to all the squark-quark pairs. Of course, there are similar corrections which relate the running masses of all the other MSSM particles to their physical masses. These have been systematically evaluated at one-loop order in Ref.[107] They are more complicated in form and usually numerically smaller than for the gluino, but in some cases they could be quite important in future efforts to connect a given candidate model for the soft terms to experimentally measured masses and mixing angles of the MSSM particles.

8.8 Squarks and sleptons

In principle, any scalars with the same electric charge, R-parity, and color quantum numbers can mix with each other. This means that with completely arbitrary soft terms, the mass eigenstates of the squarks and sleptons of the MSSM should be obtained by diagonalizing three 6×6 (mass)^2 matrices for up-type squarks ($\widetilde{u}_L$, $\widetilde{c}_L$, $\widetilde{t}_L$, $\widetilde{u}_R$, $\widetilde{c}_R$, $\widetilde{t}_R$), down-type squarks ($\widetilde{d}_L$, $\widetilde{s}_L$, $\widetilde{b}_L$, $\widetilde{d}_R$, $\widetilde{s}_R$, $\widetilde{b}_R$), and charged sleptons ($\widetilde{e}_L$, $\widetilde{\mu}_L$, $\widetilde{\tau}_L$, $\widetilde{e}_R$, $\widetilde{\mu}_R$, $\widetilde{\tau}_R$), and one 3×3 matrix for sneutrinos ($\widetilde{\nu}_e$, $\widetilde{\nu}_\mu$, $\widetilde{\nu}_\tau$). Fortunately, the general hypothesis of flavor-blind soft parameters eqs. (8.15) and (8.16) predicts that most of these mixing angles are very small. The third-family squarks and sleptons can have very different masses compared to their first- and second-family counterparts, because of the effects of large Yukawa (y_t, y_b, y_τ) and soft (a_t, a_b, a_τ) couplings in the RG equations

(10.20)-(10.24). Furthermore, they can have substantial mixing in pairs $(\widetilde{t}_L, \widetilde{t}_R)$, $(\widetilde{b}_L, \widetilde{b}_R)$ and $(\widetilde{\tau}_L, \widetilde{\tau}_R)$. In contrast, the first- and second-family squarks and sleptons have negligible Yukawa couplings, so they end up in 7 very nearly degenerate, unmixed pairs $(\widetilde{e}_R, \widetilde{\mu}_R)$, $(\widetilde{\nu}_e, \widetilde{\nu}_\mu)$, $(\widetilde{e}_L, \widetilde{\mu}_L)$, $(\widetilde{u}_R, \widetilde{c}_R)$, $(\widetilde{d}_R, \widetilde{s}_R)$, $(\widetilde{u}_L, \widetilde{c}_L)$, $(\widetilde{d}_L, \widetilde{s}_L)$. As we have already discussed in section 8.4, this avoids the problem of disastrously large virtual sparticle contributions to FCNC processes.

Let us first consider the spectrum of first- and second-family squarks and sleptons. In models fitting into both of the broad categories of minimal supergravity [eq. (9.28)] or gauge-mediated [eq. (9.42)] boundary conditions, their running masses can be conveniently parameterized in the following way:

$$m^2_{Q_1} = m^2_{Q_2} = m_0^2 + K_3 + K_2 + \frac{1}{36}K_1, \tag{8.70}$$

$$m^2_{\overline{u}_1} = m^2_{\overline{u}_2} = m_0^2 + K_3 \qquad + \frac{4}{9}K_1, \tag{8.71}$$

$$m^2_{\overline{d}_1} = m^2_{\overline{d}_2} = m_0^2 + K_3 \qquad + \frac{1}{9}K_1, \tag{8.72}$$

$$m^2_{L_1} = m^2_{L_2} = m_0^2 \qquad + K_2 + \frac{1}{4}K_1, \tag{8.73}$$

$$m^2_{\overline{e}_1} = m^2_{\overline{e}_2} = m_0^2 \qquad\qquad + K_1. \tag{8.74}$$

In minimal supergravity models, m_0^2 is the common scalar (mass)^2 which appears in eq. (9.28). It can be 0 in the "no-scale" limit, but it could also be the dominant source of the scalar masses. The contributions K_3, K_2 and K_1 are due to the RG running proportional to the gaugino masses; see eq. (10.14). They are strictly positive. A key point is that the same K_3, K_2 and K_1 appear everywhere in eqs. (8.70)-(8.74), since all of the chiral supermultiplets couple to the same gauginos with the same gauge couplings. The different coefficients in front of K_1 just correspond to the various values of weak hypercharge squared for each scalar. The quantities K_1, K_2, K_3 depend on the RG scale Q at which they are evaluated. Explicitly, they are found by solving eq. (10.14):

$$K_a(Q) = \begin{Bmatrix} 3/5 \\ 3/4 \\ 4/3 \end{Bmatrix} \times \frac{1}{2\pi^2}\int_{\ln Q}^{\ln Q_0} dt\; g_a^2(t)\,|M_a(t)|^2 \qquad (a = 1,2,3) \tag{8.75}$$

Here Q_0 is the input RG scale at which the boundary condition eq. (9.28) is applied, and Q should be taken to be evaluated near the squark and slepton mass under consideration, presumably less than about 1 TeV or so. The values of the running parameters $g_a(Q)$ and $M_a(Q)$ can be found using eqs. (8.18) and (10.7). If the input scale is approximated by the apparent scale of gauge coupling unification $Q_0 = M_U \approx 2 \times 10^{16}$ GeV, one finds that numerically

$$K_1 \approx 0.15 m_{1/2}^2; \qquad K_2 \approx 0.5 m_{1/2}^2; \qquad K_3 \approx (4.5 \text{ to } 6.5) m_{1/2}^2. \quad (8.76)$$

for Q near 1 TeV. Here $m_{1/2}$ is the common gaugino mass parameter at the unification scale. Note that $K_3 \gg K_2 \gg K_1$; this is a direct consequence of the relative sizes of the gauge couplings g_3, g_2, and g_1. The large uncertainty in K_3 is due in part to the experimental uncertainty in the QCD coupling constant, and in part to the uncertainty in where to choose Q, since K_3 runs rather quickly below 1 TeV. If the gauge couplings and gaugino masses are unified between M_U and M_P, as would occur in a GUT model, then the effect of RG running for $M_U < Q < M_P$ can be absorbed into a redefinition of m_0^2. Otherwise, it adds a further uncertainty which is roughly proportional to $\ln(M_P/M_U)$, compared to the larger contributions in eq. (8.75) which go roughly like $\ln(M_U/1 \text{ TeV})$.

In gauge-mediated models, the same parameterization eqs. (8.70)-(8.74) holds, but m_0^2 is always 0. At the input scale Q_0, each MSSM scalar gets contributions to its (mass)^2 which depend only on its gauge interactions, as in eq. (9.42). It is not hard to see that in general these contribute in exactly the same pattern as K_1, K_2, and K_3 in eq. (8.70)-(8.74). The subsequent evolution of the scalar squared masses down to the electroweak scale again just yields more contributions to the K_1, K_2, and K_3 parameters. It is somewhat more difficult to give meaningful numerical estimates for these parameters in gauge-mediated models than in the minimal supergravity models, because of uncertainties in the messenger mass scale(s) and in the multiplicities of the messenger fields. However, in the gauge-mediated case one quite generally expects that the numerical values of the ratios K_3/K_2, K_3/K_1 and K_2/K_1 should be even larger than in eq. (8.76). There are two reasons for this. First, the running squark squared masses start off larger than slepton squared masses already at the input scale in gauge-mediated models, rather than having a common value m_0^2. Furthermore, in the gauge-mediated case, the input scale Q_0 is typically much lower than M_P or M_U, so that the RG evolution gives relatively more weight to smaller RG scales where the hierarchies $g_3 > g_2 > g_1$ and $M_3 > M_2 > M_1$ are already in effect.

In general, one therefore expects that the squarks should be considerably heavier than the sleptons, with the effect being more pronounced in gauge-mediated supersymmetry breaking models than in minimal supergravity models. For any specific choice of model, this effect can be easily quantified with an RG analysis. The hierarchy $m_{\text{squark}} > m_{\text{slepton}}$ tends to hold even in models which do not really fit into any of the categories outlined in section 9, because the RG contributions to squark masses from the gluino are always present and usually quite large, since QCD has a larger gauge coupling than the electroweak interactions.

There is also a "hyperfine" splitting in the squark and slepton mass spectrum produced by electroweak symmetry breaking. Each squark and slepton ϕ will get a contribution Δ_ϕ to its (mass)^2, coming from the $SU(2)_L$ and $U(1)_Y$ D-term quartic interactions [see the last term in eq. (6.79)] of the form (squark)^2(Higgs)^2 and (slepton)^2(Higgs)^2, when the neutral Higgs scalars H_u^0 and H_d^0 get VEVs. They are model-independent for a given value of $\tan\beta$, and are given by

$$\Delta_\phi = (T_3^\phi - Q_{\rm EM}^\phi \sin^2\theta_W)\cos 2\beta\, m_Z^2, \tag{8.77}$$

where T_3^ϕ and $Q_{\rm EM}^\phi$ are the third component of weak isospin and the electric charge of the chiral supermultiplet to which ϕ belongs. [For example, $\Delta_u = (\frac{1}{2} - \frac{2}{3}\sin^2\theta_W)\cos 2\beta\, m_Z^2$ and $\Delta_{\overline{u}} = (\frac{2}{3}\sin^2\theta_W)\cos 2\beta\, m_Z^2$]. These D-term contributions are typically smaller than the m_0^2 and K_1, K_2, K_3 contributions, but should not be neglected. They split apart the components of the $SU(2)_L$-doublet sleptons and squarks $L_1 = (\widetilde{\nu}_e, \widetilde{e}_L)$, etc. Including them, the first-family squark and slepton masses are now given by:

$$m^2_{\widetilde{d}_L} = m_0^2 + K_3 + K_2 + \frac{1}{36}K_1 + \Delta_d, \tag{8.78}$$

$$m^2_{\widetilde{u}_L} = m_0^2 + K_3 + K_2 + \frac{1}{36}K_1 + \Delta_u, \tag{8.79}$$

$$m^2_{\widetilde{u}_R} = m_0^2 + K_3 \qquad + \frac{4}{9}K_1 + \Delta_{\overline{u}}, \tag{8.80}$$

$$m^2_{\widetilde{d}_R} = m_0^2 + K_3 \qquad + \frac{1}{9}K_1 + \Delta_{\overline{d}}, \tag{8.81}$$

$$m^2_{\widetilde{e}_L} = m_0^2 \qquad + K_2 + \frac{1}{4}K_1 + \Delta_e, \tag{8.82}$$

$$m^2_{\widetilde{\nu}} = m_0^2 \qquad + K_2 + \frac{1}{4}K_1 + \Delta_\nu, \tag{8.83}$$

$$m^2_{\widetilde{e}_R} = m_0^2 \qquad\qquad + K_1 + \Delta_{\overline{e}}, \tag{8.84}$$

with identical formulas for the second-family squarks and sleptons. The mass splittings for the left-handed squarks and sleptons are governed by model-independent sum rules

$$m^2_{\widetilde{e}_L} - m^2_{\widetilde{\nu}_e} = m^2_{\widetilde{d}_L} - m^2_{\widetilde{u}_L} = -\cos 2\beta\, m_W^2. \tag{8.85}$$

Since $\cos 2\beta < 0$ in the allowed range $\tan\beta > 1$, it follows that $m_{\widetilde{e}_L} > m_{\widetilde{\nu}_e}$ and $m_{\widetilde{d}_L} > m_{\widetilde{u}_L}$, with the magnitude of the splittings constrained by electroweak symmetry breaking.

Let us next consider the masses of the top squarks, for which there are several non-negligible contributions. First, there are (mass)^2 terms for $\widetilde{t}_L^*\widetilde{t}_L$ and $\widetilde{t}_R^*\widetilde{t}_R$ which are just equal to $m^2_{Q_3} + \Delta_u$ and $m^2_{\overline{u}_3} + \Delta_{\overline{u}}$, respectively, just as for the first- and second-family squarks. Second, there are contributions equal to m_t^2 for each of $\widetilde{t}_L^*\widetilde{t}_L$ and $\widetilde{t}_R^*\widetilde{t}_R$. These come from F-terms in the scalar potential of the form $y_t^2 H_u^{0*}H_u^0\widetilde{t}_L^*\widetilde{t}_L$ and $y_t^2 H_u^{0*}H_u^0\widetilde{t}_R^*\widetilde{t}_R$ (see Figs. 8.2b and 8.2c), with the Higgs fields replaced by their VEVs. These contributions are of course present for all of the squarks and sleptons, but they are much too small to worry about except in the case of the top squarks. Third, there are contributions to the scalar potential from F-terms of the form $-\mu y_t \widetilde{\bar{t}}\widetilde{t}H_d^{0*} + \text{c.c.}$; see eqs. (8.6) and Fig. 8.4a. These become $-\mu v y_t \cos\beta\, \widetilde{t}_R^*\widetilde{t}_L + \text{c.c.}$ when H_d^0 is replaced by its VEV. Finally, there are contributions to the scalar potential from the soft (scalar)^3 couplings $a_t\widetilde{\bar{t}}\widetilde{Q}_3 H_u^0 + \text{c.c.}$ [see the first term of the second line of eq. (8.12) and eq. (10.8)], which become $a_t v\sin\beta\,\widetilde{t}_L\widetilde{t}_R^* + \text{c.c.}$ when H_u^0 is replaced by its VEV. Putting these all together, we have a (mass)^2 matrix for the top squarks, which in the gauge-eigenstate basis $(\widetilde{t}_L,\, \widetilde{t}_R)$ is given by

$$-\mathcal{L} \supset \begin{pmatrix} \widetilde{t}_L^* & \widetilde{t}_R^* \end{pmatrix} \mathbf{m^2_{\widetilde{t}}} \begin{pmatrix} \widetilde{t}_L \\ \widetilde{t}_R \end{pmatrix} \tag{8.86}$$

where

$$\mathbf{m^2_{\widetilde{t}}} = \begin{pmatrix} m^2_{Q_3} + m_t^2 + \Delta_u & v(a_t\sin\beta - \mu y_t\cos\beta) \\ v(a_t\sin\beta - \mu y_t\cos\beta) & m^2_{\overline{u}_3} + m_t^2 + \Delta_{\overline{u}} \end{pmatrix}. \tag{8.87}$$

This matrix can be diagonalized to give mass eigenstates

$$\begin{pmatrix} \widetilde{t}_1 \\ \widetilde{t}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_{\widetilde{t}} & \sin\theta_{\widetilde{t}} \\ -\sin\theta_{\widetilde{t}} & \cos\theta_{\widetilde{t}} \end{pmatrix} \begin{pmatrix} \widetilde{t}_L \\ \widetilde{t}_R \end{pmatrix} \tag{8.88}$$

with $m^2_{\widetilde{t}_1} < m^2_{\widetilde{t}_2}$ being the eigenvalues of eq. (8.87) and $0 \le \theta_{\widetilde{t}} \le \pi$. Because of the large RG effects proportional to X_t in eq. (10.20) and eq. (10.21), at the electroweak scale one finds that $m^2_{\overline{u}_3} < m^2_{Q_3}$, and both of these quantities are usually significantly smaller than the squark squared masses for the first two families. The diagonal terms m_t^2 in eq. (8.87) tend to mitigate this effect somewhat, but the off-diagonal entries will typically induce a significant mixing which always reduces the lighter top-squark (mass)^2 eigenvalue. For this reason, it is often found in models that $\widetilde{t}_1$ is the lightest squark of all.

A very similar analysis can be performed for the bottom squarks and charged tau sleptons, which in their respective gauge-eigenstate bases ($\widetilde{b}_L$, $\widetilde{b}_R$) and ($\widetilde{\tau}_L$, $\widetilde{\tau}_R$) have (mass)^2 matrices:

$$\mathbf{m}^{\mathbf{2}}_{\widetilde{\mathbf{b}}} = \begin{pmatrix} m^2_{Q_3} + \Delta_d & v(a_b\cos\beta - \mu y_b \sin\beta) \\ v(a_b\cos\beta - \mu y_b \sin\beta) & m^2_{\overline{d}_3} + \Delta_{\overline{d}} \end{pmatrix}; \qquad (8.89)$$

$$\mathbf{m}^{\mathbf{2}}_{\widetilde{\tau}} = \begin{pmatrix} m^2_{L_3} + \Delta_e & v(a_\tau\cos\beta - \mu y_\tau \sin\beta) \\ v(a_\tau\cos\beta - \mu y_\tau \sin\beta) & m^2_{\overline{e}_3} + \Delta_{\overline{e}} \end{pmatrix}. \qquad (8.90)$$

These can be diagonalized to give mass eigenstates $\widetilde{b}_1, \widetilde{b}_2$ and $\widetilde{\tau}_1, \widetilde{\tau}_2$ in exact analogy with eq. (8.88).

The magnitude and importance of mixing in the sbottom and stau sectors depends on how large $\tan\beta$ is. If $\tan\beta$ is not too large (in practice, this usually means less than about 10 or so, depending on the situation under study), the sbottoms and staus do not get a very large effect from the mixing terms and the RG effects due to X_b and X_τ, because $y_b, y_\tau \ll y_t$ from eqs. (8.44,8.45). In that case the mass eigenstates are very nearly the same as the gauge eigenstates $\widetilde{b}_L$, $\widetilde{b}_R$, $\widetilde{\tau}_L$ and $\widetilde{\tau}_R$. The latter three, and $\tilde{\nu}_\tau$, will be nearly degenerate with their first- and second-family counterparts with the same $SU(3)_C \times SU(2)_L \times U(1)_Y$ quantum numbers. However, even in the case of small $\tan\beta$, $\widetilde{b}_L$ will feel the effects of the large top Yukawa coupling because it is part of the doublet $\widetilde{Q}_3$ which contains $\widetilde{t}_L$. In particular, from eq. (10.20) we see that X_t acts to decrease $m^2_{\widetilde{Q}_3}$ as it is RG-evolved down from the input scale to the electroweak scale. Therefore the mass of $\widetilde{b}_L$ can be significantly less than the masses of $\widetilde{d}_L$ and $\widetilde{s}_L$.

For larger values of $\tan\beta$, the mixing in eqs. (8.89) and (8.90) can be quite significant, because y_b, y_τ and a_b, a_τ are non-negligible. Just as in the case of the top squarks, the lighter sbottom and stau mass eigenstates (denoted $\widetilde{b}_1$ and $\widetilde{\tau}_1$) can be significantly lighter than their first- and second-family counterparts. Furthermore, $\widetilde{\nu}_\tau$ can be significantly lighter than the nearly degenerate $\widetilde{\nu}_e$, $\widetilde{\nu}_\mu$.

The requirement that the third-family squarks and sleptons should all have positive (mass)^2 implies limits on the sizes of $a_t\sin\beta - \mu y_t \cos\beta$, $a_b\cos\beta - \mu y_b\sin\beta$, and $a_\tau\cos\beta - \mu y_\tau\sin\beta$. If they are too large, the smaller eigenvalue of eq. (8.87), (8.89) or (8.90) will be driven negative, implying that a squark or charged slepton gets a VEV, breaking $SU(3)_C$ or electromagnetism. Since this is clearly unacceptable, one can put bounds on the (scalar)^3 couplings, or equivalently on the parameter A_0 in minimal supergravity models. Even if all of the (mass)^2 eigenvalues are

Table 8.1. Undiscovered particles in the Minimal Supersymmetric Standard Model

Names	Spin	P_R	Mass Eigenstates	Gauge Eigenstates
Higgs bosons	0	+1	$h^0\ H^0\ A^0\ H^\pm$	$H_u^0\ H_d^0\ H_u^+\ H_d^-$
squarks	0	−1	$\widetilde{u}_L\ \widetilde{u}_R\ \widetilde{d}_L\ \widetilde{d}_R$	" "
			$\widetilde{s}_L\ \widetilde{s}_R\ \widetilde{c}_L\ \widetilde{c}_R$	" "
			$\widetilde{t}_1\ \widetilde{t}_2\ \widetilde{b}_1\ \widetilde{b}_2$	$\widetilde{t}_L\ \widetilde{t}_R\ \widetilde{b}_L\ \widetilde{b}_R$
sleptons	0	−1	$\widetilde{e}_L\ \widetilde{e}_R\ \widetilde{\nu}_e$	" "
			$\widetilde{\mu}_L\ \widetilde{\mu}_R\ \widetilde{\nu}_\mu$	" "
			$\widetilde{\tau}_1\ \widetilde{\tau}_2\ \widetilde{\nu}_\tau$	$\widetilde{\tau}_L\ \widetilde{\tau}_R\ \widetilde{\nu}_\tau$
neutralinos	1/2	−1	$\widetilde{N}_1\ \widetilde{N}_2\ \widetilde{N}_3\ \widetilde{N}_4$	$\widetilde{B}^0\ \widetilde{W}^0\ \widetilde{H}_u^0\ \widetilde{H}_d^0$
charginos	1/2	−1	$\widetilde{C}_1^\pm\ \widetilde{C}_2^\pm$	$\widetilde{W}^\pm\ \widetilde{H}_u^+\ \widetilde{H}_d^-$
gluino	1/2	−1	$\widetilde{g}$	" "
gravitino/ goldstino	3/2	−1	$\widetilde{G}$	" "

positive, the presence of large (scalar)^3 couplings can yield global minima of the scalar potential with non-zero squark and/or charged slepton VEVs which are disconnected from the vacuum which conserves $SU(3)_C$ and electromagnetism. However, it is not always clear whether the non-existence of such disconnected global minima should really be taken as a constraint, because the tunneling rate from our "good" vacuum to the "bad" vacua can easily be much longer than the age of the universe.

8.9 Summary: the MSSM sparticle spectrum

In the MSSM there are 32 distinct masses corresponding to undiscovered particles, not including the gravitino. In this section we have explained how the masses and mixing angles for these particles can be computed, given an underlying model for the soft terms at some input scale. Assuming only that the mixing of first- and second-family squarks and sleptons is negligible, the mass eigenstates of the MSSM are listed in Table 8.1. A complete set of Feynman rules for the interactions of these particles with each other and with the Standard Model quarks, leptons, and gauge bosons can be found in Refs.[21, 98] Specific models for the soft terms typically predict the masses and the mixing angles angles for the MSSM in terms of far fewer parameters. For example, in the minimal supergravity models, one has only the parameters m_0^2, $m_{1/2}$, A_0, μ, and b which are not already measured by experiment. On the other hand, in gauge-mediated supersymmetry breaking models, the free parameters include at least the scale Λ, the typical messenger mass scale $M_{\rm mess}$, the integer number N_5 of copies of the minimal messengers, the goldstino decay constant $\langle F \rangle$, and the Higgs mass parameters μ and b. After RG evolving the soft terms down to the electroweak scale, one can impose that the scalar potential gives correct electroweak symmetry breaking. This allows us to trade $|\mu|$ and b (or B_0) for one parameter $\tan\beta$, as in eqs. (8.27)-(8.26). So, to a reasonable approximation, the entire mass spectrum in minimal supergravity models is determined by only five unknown parameters: m_0^2, $m_{1/2}$, A_0, $\tan\beta$, and Arg(μ), while in the simplest gauge-mediated supersymmetry breaking models one can pick parameters Λ, $M_{\rm mess}$, N_5, $\langle F \rangle$, $\tan\beta$, and Arg(μ). Both frameworks are highly predictive. Of course, it is easy to imagine that the essential physics of supersymmetry breaking is not captured by either of these two scenarios in their minimal forms.

While it would be a mistake to underestimate the uncertainties in the MSSM mass and mixing spectrum, it is also useful to keep in mind some general lessons that recur in various different scenarios. Indeed, there has emerged a sort of folklore concerning likely features of the MSSM spectrum, which is partly based on theoretical bias and partly on the constraints inherent in any supersymmetric theory. We remark on these features mainly because they represent the prevailing prejudice among supersymmetry theorists, which is certainly a useful thing for the reader to know even if he or she wisely decides to remain skeptical. For example, it is perhaps not unlikely that:

- The LSP is the lightest neutralino $\widetilde{N}_1$, unless the gravitino is lighter or R-parity is not conserved. If $\mu > M_1, M_2$, then $\widetilde{N}_1$ is likely to be bino-like, with a mass roughly 0.5 times the masses of $\widetilde{N}_2$ and $\widetilde{C}_1$. In the opposite case $\mu < M_1, M_2$, then $\widetilde{N}_1$ has a large higgsino content and $\widetilde{N}_2$ and $\widetilde{C}_1$ are not much heavier.

- The gluino will be much heavier than the lighter neutralinos and charginos. This is certainly true in the case of the "standard" gaugino mass relation eq. (10.7); more generally, the running gluino mass parameter grows relatively quickly as it is RG-evolved into the infrared because the QCD coupling is larger than the electroweak gauge couplings. So even if there are big corrections to the gaugino mass boundary conditions eqs. (9.27) or (9.40), the gluino mass parameter M_3 is likely to come out larger than M_1 and M_2.

- The squarks of the first and second families are nearly degenerate and much heavier than the sleptons. This is because each squark mass gets the same large positive-definite radiative corrections from loops involving the gluino. The left-handed squarks $\widetilde{u}_L$, $\widetilde{d}_L$, $\widetilde{s}_L$ and $\widetilde{c}_L$ are likely to be heavier than their right-handed counterparts $\widetilde{u}_R$, $\widetilde{d}_R$, $\widetilde{s}_R$ and $\widetilde{c}_R$, because of the effect of K_2 in eqs. (8.78)-(8.84).

- The squarks of the first two families cannot be lighter than about 0.8 times the mass of the gluino in minimal supergravity models, and about 0.6 times the mass of the gluino in the simplest gauge-mediated models as discussed in section 9.4 if the number of messenger squark pairs is $N_5 \leq 4$. In the minimal supergravity case this is because the gluino mass feeds into the squark masses through RG evolution; in the gauge-mediated case it is because the gluino and squark masses are tied together by eqs. (9.40) and (9.42) [multiplied by N_5, as explained at the end of section 9.4].

- The lighter stop $\widetilde{t}_1$ and the lighter sbottom $\widetilde{b}_1$ are probably the lightest squarks. This is because stop and sbottom mixing effects and the effects of X_t and X_b in eqs. (10.20)-(10.22) both tend to decrease the lighter stop and sbottom masses.

- The lightest charged slepton is probably a stau $\widetilde{\tau}_1$. The mass difference $m_{\widetilde{e}_R} - m_{\widetilde{\tau}_1}$ is likely to be significant if $\tan\beta$ is large, because of the effects of a large tau Yukawa coupling. For smaller $\tan\beta$, $\widetilde{\tau}_1$ is predominantly $\widetilde{\tau}_R$ and it is not so much lighter than $\widetilde{e}_R$, $\widetilde{\mu}_R$.

- The left-handed charged sleptons $\widetilde{e}_L$ and $\widetilde{\mu}_L$ are likely to be heavier than their right-handed counterparts $\widetilde{e}_R$ and $\widetilde{\mu}_R$. This is because of the effect of K_2 in eq. (8.82). (Note also that $\Delta_e - \Delta_{\overline{e}}$ is positive but very small because of the numerical accident $\sin^2\theta_W \approx 1/4$.)

- The lightest neutral Higgs boson h^0 should be lighter than about 150 GeV, and may be much lighter than the other Higgs scalar mass eigenstates A^0, $H^\pm$, H^0.

In Figure 8.9 we show a qualitative sketch of a sample MSSM mass spectrum which illustrates these features. Variations in the model parameters can have important and predictable effects. For example, taking larger (smaller) m_0^2 in minimal supergravity models will tend to move the entire spectrum of squarks, sleptons and the Higgs scalars A^0, $H^\pm$, H^0 higher (lower) compared to the neutralinos, charginos and gluino; taking larger values of $\tan\beta$ with other model parameters held fixed will usually tend to lower $\widetilde{b}_1$ and $\widetilde{\tau}_1$ masses compared to those of the other

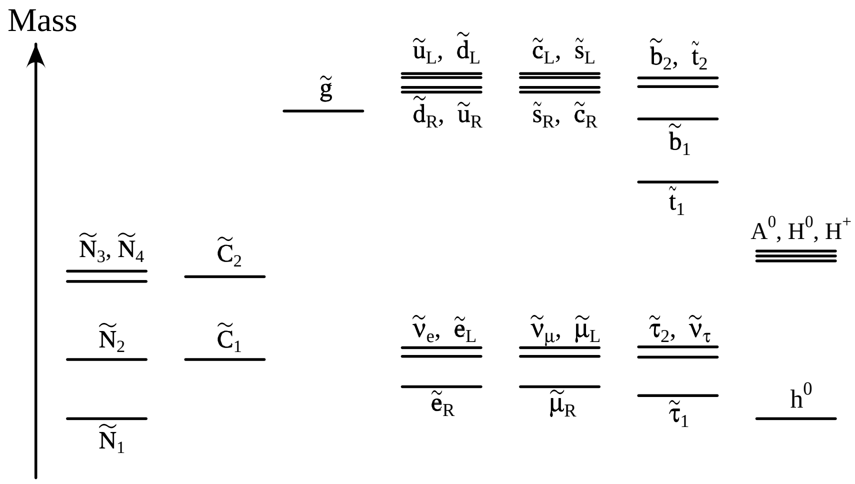


Fig. 8.9. A schematic sample spectrum for the undiscovered particles in the MSSM. This spectrum is presented for entertainment purposes only. No warranty, expressed or implied, guarantees that this spectrum looks anything like the real world.

sparticles, etc. The important point is that by measuring the masses and mixing angles of the MSSM particles we will be able to gain a great deal of information which can rule out or bolster evidence for competing proposals for the origin of supersymmetry breaking. Testing the various possible organizing principles will provide the high-energy physicists of the next millennium with an exciting challenge.

9
Origins of supersymmetry breaking

9.1 General considerations for supersymmetry breaking

In the MSSM, supersymmetry breaking is simply introduced explicitly. However, we have seen that the soft parameters cannot be arbitrary. In order to understand how patterns like eqs. (8.15), (8.16) and (8.17) can emerge, it is necessary to consider models in which supersymmetry is spontaneously broken. By definition, this means that the vacuum state $|0\rangle$ is not invariant under supersymmetry transformations, so $Q_\alpha|0\rangle \neq 0$ and $Q^\dagger_{\dot\alpha}|0\rangle \neq 0$. Now, in global supersymmetry, the Hamiltonian operator H can be related to the supersymmetry generators through the algebra eq. (6.32):

$$H = P^0 = \frac{1}{4}(Q_1 Q_1^\dagger + Q_1^\dagger Q_1 + Q_2 Q_2^\dagger + Q_2^\dagger Q_2). \tag{9.1}$$

If supersymmetry is unbroken in the vacuum state, it follows that $H|0\rangle = 0$ and the vacuum has zero energy. Conversely, if supersymmetry is spontaneously broken in the vacuum state, then the vacuum must have positive energy, since

$$\langle 0|H|0\rangle = \frac{1}{4}\Big(\|Q_1|0\rangle\|^2 + \|Q_1^\dagger|0\rangle\|^2 + \|Q_2|0\rangle\|^2 + \|Q_2^\dagger|0\rangle\|^2\Big) > 0 \tag{9.2}$$

if the Hilbert space is to have positive norm. If spacetime-dependent effects and fermion condensates can be neglected, then $\langle 0|H|0\rangle = \langle 0|V|0\rangle$, where V is the scalar potential in eq. (6.79). Therefore supersymmetry will be spontaneously broken if F_i and/or D^a does not vanish in the ground state. Note that if any state exists in which all F_i and D^a vanish, then it will have zero energy, implying that supersymmetry cannot be spontaneously broken in the true ground state. Therefore the way to achieve spontaneous supersymmetry breaking is to look for models in which the equations $F_i = 0$ and $D^a = 0$ cannot be simultaneously satisfied for *any* values of the fields.

Supersymmetry breaking with non-zero D-terms can be achieved through the Fayet-Iliopoulos mechanism. If the gauge symmetry includes a $U(1)$ factor, then one can introduce a term linear in the corresponding auxiliary field of the gauge supermultiplet:

$$\mathcal{L}_{\text{Fayet}-\text{Iliopoulos}} = \kappa D \tag{9.3}$$

where κ is a constant parameter with dimensions of (mass)^2. This term is gauge-invariant and supersymmetric by itself. [Note that the supersymmetry transformation δD in eq. (6.63) is a total derivative for a $U(1)$ gauge symmetry.] If we include it in the Lagrangian, then D may get a non-zero VEV, depending on the other interactions of the scalar fields that are charged under the $U(1)$. To see this, we can write the relevant part of the scalar potential using eqs. (6.58) and (6.76) as

$$V = \frac{1}{2}D^2 - \kappa D + gD\sum_i q_i \phi^{*i}\phi_i \tag{9.4}$$

where the q_i are the charges of the scalar fields ϕ_i under the $U(1)$ gauge group in question. The presence of the Fayet-Iliopoulos term modifies the equation of motion eq. (6.78) to

$$D = \kappa - g\sum_i q_i \phi^{*i}\phi_i. \tag{9.5}$$

Now suppose that the scalar fields ϕ_i have other interactions (such as large superpotential mass terms) which prevent them from getting VEVs. Then the auxiliary field D will be forced to get a VEV equal to κ, and supersymmetry will be broken. This mechanism cannot work for non-abelian gauge groups, however, since the analog of eq. (9.3) would not be gauge-invariant.

In the MSSM, one can imagine that the D term for $U(1)_Y$ has a Fayet-Iliopoulos term which is the principal source of supersymmetry breaking. Unfortunately, this would be an immediate disaster, because at least some of the squarks and sleptons would just get non-zero VEVs (breaking color, electromagnetism, and/or lepton number, but not supersymmetry) in order to satisfy eq. (9.5), because they do not have superpotential mass terms. This means that a Fayet-Iliopoulos term for $U(1)_Y$ must be subdominant compared to other sources of supersymmetry breaking in the MSSM, if not absent altogether. One could also attempt to trigger supersymmetry breaking with a Fayet-Iliopoulos term for some other $U(1)$ gauge symmetry which is as yet unknown because it is spontaneously broken at a very high mass scale or because it does not couple to the Standard Model particles. However, if this is the ultimate source for supersymmetry breaking, it proves difficult to give appropriate masses to all of the MSSM particles, especially the gauginos. In any case, we will not discuss D-term breaking as the ultimate origin of supersymmetry violation any further, although it may not be ruled out.

Models where supersymmetry breaking is due to non-zero F-terms, called O'Raifeartaigh models, may have brighter phenomenological prospects. The idea is to pick a set of chiral supermultiplets $\Phi_i \supset (\phi_i, \psi_i, F_i)$ and a superpotential W in such a way that the equations $F_i = -\delta W^*/\delta\phi^{*i} = 0$ have no simultaneous solution. Then $V = \sum_i |F_i|^2$ will have to be positive at its minimum, ensuring that supersymmetry is broken. The simplest example which does this has three chiral supermultiplets with

$$W = -k\Phi_1 + m\Phi_2\Phi_3 + \frac{y}{2}\Phi_1\Phi_3^2. \tag{9.6}$$

Note that W contains a linear term, with k having dimensions of $(\text{mass})^2$. This is only possible if Φ_1 is a gauge singlet. In section 6 we cheated and did not mention such a term, because we knew that the MSSM contains no such singlet chiral supermultiplet. Nevertheless, it should be clear from retracing the derivation in section 6.2 that such a term is allowed if a gauge-singlet chiral supermultiplet is added to the theory. In fact, a linear term is absolutely necessary to achieve F-term breaking, since otherwise setting all $\phi_i = 0$ will always give a supersymmetric global minimum with all $F_i = 0$. Without loss of generality, we can choose k, m, and y to be real and positive (by a phase rotation of the fields). The scalar potential following from eq. (9.6) is

$$V = |F_1|^2 + |F_2|^2 + |F_3|^2; \tag{9.7}$$

$$F_1 = k - \frac{y}{2}\phi_3^{*2}; \qquad F_2 = -m\phi_3^*; \qquad F_3 = -m\phi_2^* - y\phi_1^*\phi_3^*. \tag{9.8}$$

Clearly, $F_1 = 0$ and $F_2 = 0$ are not compatible, so supersymmetry must indeed be broken. If $m^2 > yk$ (which we assume from now on), then it is easy to show that the absolute minimum of the potential is at $\phi_2 = \phi_3 = 0$ with ϕ_1 undetermined, so $F_1 = k$ and $V = k^2$ at the minimum of the potential. The fact that ϕ_1 is undetermined is an example of a "flat direction" in the scalar potential; this is a common feature of supersymmetric models.[2]

[2] More generally, "flat directions" are non-compact lines and surfaces in the space of scalar fields along which the scalar potential vanishes. The classical scalar potential of the MSSM would have many flat directions if supersymmetry were not broken.

If we presciently choose to expand V around $\phi_1 = 0$, the mass spectrum of the theory consists of 6 real scalars with tree-level squared masses

$$0,\ 0,\ m^2,\ m^2,\ m^2 - yk,\ m^2 + yk. \tag{9.9}$$

Meanwhile, there are 3 Weyl fermions with masses

$$0,\ m,\ m. \tag{9.10}$$

The non-degeneracy of scalars and fermions is a clear sign that supersymmetry has been spontaneously broken. The 0 eigenvalues in eqs. (9.9) and (9.10) correspond to the complex scalar ϕ_1 and its fermionic partner ψ_1. However, ϕ_1 and ψ_1 have different reasons for being massless. The masslessness of ϕ_1 corresponds to the existence of the flat direction, since any value of ϕ_1 gives the same energy at tree-level. This flat direction is an accidental feature of the classical scalar potential, and in this case it is removed ("lifted") by quantum corrections. This can be seen by computing the Coleman-Weinberg one-loop effective potential. After some calculation, one finds the result that the global minimum is indeed fixed at $\phi_1 = \phi_2 = \phi_3 = 0$, with the complex scalar ϕ_1 receiving a small positive-definite (mass)^2 equal to

$$\begin{aligned} m^2_{\phi_1} = \frac{1}{32\pi^2}\Big[&\Big(\frac{ym^4}{k} + y^3 k\Big)\ln\Big(\frac{m^2+yk}{m^2-yk}\Big) \\ &+2y^2m^2\Big(\ln[1 - \frac{y^2k^2}{m^4}] - 1\Big)\Big]. \end{aligned} \tag{9.11}$$

[In the limit $yk \ll m^2$, this reduces to $m^2_{\phi_1} = y^4k^2/(48\pi^2 m^2)$.] In contrast, the Weyl fermion ψ_1 remains exactly massless because of a general feature of all models with spontaneously broken supersymmetry. To understand this, recall that the spontaneous breaking of any global symmetry always gives rise to a massless Nambu-Goldstone mode with the same quantum numbers as the broken symmetry generator. In the case of supersymmetry, the broken generator is the fermionic charge Q_α, so the Nambu-Goldstone particle must be a massless neutral Weyl fermion called the *goldstino*. In the O'Raifeartaigh model example, ψ_1 is the goldstino because it is the fermionic partner of the auxiliary field F_1 which got a VEV. (We will prove these statements in a more general context in section 9.2.)

The O'Raifeartaigh superpotential determines the mass scale of supersymmetry breaking $\sqrt{F_1}$ in terms of a dimensionful parameter k which is put in by hand. This is somewhat *ad hoc*, since $\sqrt{k}$ will have to be much less than $M_{\rm P}$ in order to give the right order of magnitude for the MSSM soft terms. We would like to have a mechanism which can instead generate such scales naturally. This can be done in models of dynamical supersymmetry breaking. In such theories, the small (compared to $M_{\rm P}$) mass scales associated with supersymmetry breaking arise by dimensional transmutation. In other words, they generally feature a new asymptotically-free non-Abelian gauge symmetry with a gauge coupling g which is perturbative at $M_{\rm P}$ and which gets strong in the infrared at some smaller scale $\Lambda \sim e^{-8\pi^2/|b|g_0^2} M_{\rm P}$, where g_0 is the running gauge coupling at $M_{\rm P}$ with beta function $-|b|g^3/16\pi^2$. Just as in QCD, it is perfectly natural for Λ to be many orders of magnitude below the Planck scale. Supersymmetry breaking may then be best described in terms of the effective dynamics of the strongly coupled theory. One possibility is that the auxiliary F field for a composite chiral supermultiplet (built out of the fundamental fields which transform under the new strongly-coupled gauge group) obtains a VEV. Constructing models which actually break supersymmetry in an acceptable way is a highly non-trivial business; for more information we refer the reader to Ref.[67]

The one thing that is now clear about spontaneous supersymmetry breaking (dynamical or not) is that it requires us to extend the MSSM. The ultimate supersymmetry-breaking order parameter cannot belong to any of the supermultiplets of the MSSM; a D-term VEV for $U(1)_Y$ does not lead to an acceptable spectrum, and there is no candidate gauge-singlet whose F-term could develop a VEV. Therefore one must ask what effects *are* responsible for spontaneous supersymmetry breaking, and how supersymmetry breakdown is "communicated" to the MSSM particles. It is very difficult to achieve the latter in a phenomenologically viable way working only with renormalizable interactions at tree-level. First, it is problematic to give masses to the MSSM gauginos, because supersymmetry does not allow (scalar)-(gaugino)-(gaugino) couplings which could turn into gaugino mass terms when the scalar gets a VEV. Second, at least some of the MSSM squarks and sleptons would have to be unacceptably light, and should have been discovered already. This can be understood in a general way from the existence of a sum rule which governs the tree-level squared masses of scalars and chiral fermions in theories with spontaneous supersymmetry breaking:

$$\mathrm{Tr}[M^2_{\text{real scalars}}] = 2\mathrm{Tr}[M^2_{\text{chiral fermions}}]. \tag{9.12}$$

If supersymmetry were not broken, then eq. (9.12) would follow immediately from the degeneracy of complex scalars [with two real scalar components, hence the factor of 2] and their Weyl fermion superpartners. However, eq. (9.12) still holds at tree-level when supersymmetry is broken spontaneously by F-terms and D-terms, as one can verify in general by explicitly computing the (mass)^2 matrices for arbitrary values of the

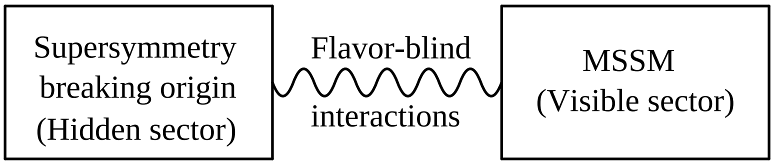


Fig. 9.1. The presumed schematic structure for supersymmetry breaking.

fields.[3] One can easily see, for example, that with the O'Raifeartaigh spectrum of eqs. (9.9) and (9.10), the sum rule eq. (9.12) is indeed satisfied. This sum rule seems to be bad news for a phenomenologically viable model, because the masses of all of the MSSM chiral fermions are already known to be small (except for the top quark and the higgsinos). Even if we could succeed in evading this, there is no reason why the resulting MSSM soft terms in this type of model should satisfy conditions like eqs. (8.15) or (8.16).

For these reasons, we expect that the MSSM soft terms arise indirectly or radiatively, rather than from tree-level renormalizable couplings to the supersymmetry-breaking order parameters. Supersymmetry breaking evidently occurs in a "hidden sector" of particles which have no (or only very small) direct couplings to the "visible sector" chiral supermultiplets of the MSSM. However, the two sectors do share some interactions which are responsible for mediating supersymmetry breaking from the hidden sector to the visible sector, where they appear as calculable soft terms. (See Fig. 9.1.) In this scenario, the tree-level sum rule eq. (9.12) need not hold for the visible sector fields, so that a phenomenologically viable superpartner mass spectrum is in principle achievable. As a bonus, if the mediating interactions are flavor-blind, then the soft terms appearing in the MSSM may automatically obey conditions like eqs. (8.15), (8.16) and (8.17).

There are two main competing proposals for what the mediating interactions might be. The first (and historically the more popular) is that they are gravitational. More precisely, they are associated with the new physics, including gravity, which enters at the Planck scale. In this *gravity-mediated supersymmetry breaking* scenario, if supersymmetry is broken in the hidden sector by a VEV $\langle F \rangle$, then the soft terms in the visible sector should be roughly of order

$$m_{\text{soft}} \sim \frac{\langle F \rangle}{M_{\text{P}}}, \tag{9.13}$$

[3] This assumes only that the trace of the $U(1)$ charges over all chiral supermultiplets in the theory vanishes ($\text{Tr}[T^a] = 0$). This holds for $U(1)_Y$ in the MSSM and more generally for any non-anomalous gauge symmetry.

by dimensional analysis. This is because we know that m_{soft} must vanish in the limit $\langle F\rangle \to 0$ where supersymmetry is unbroken, and also in the limit $M_{\text{P}} \to \infty$ (corresponding to $G_{\text{Newton}} \to 0$) in which gravity becomes irrelevant. For m_{soft} of order a few hundred GeV, one would therefore expect that the scale associated with the origin of supersymmetry breaking in the hidden sector should be roughly $\sqrt{\langle F\rangle} \sim 10^{10}$ or 10^{11} GeV. Another possibility is that the supersymmetry breaking order parameter is a gaugino condensate $\langle 0|\lambda^a\lambda^b|0\rangle = \delta^{ab}\Lambda^3 \neq 0$. If the composite field $\lambda^a\lambda^b$ is part of an auxiliary field F for some (perhaps composite) chiral superfield, then by dimensional analysis we expect supersymmetry breaking soft terms of order

$$m_{\text{soft}} \sim \frac{\Lambda^3}{M_{\text{P}}^2}, \tag{9.14}$$

with, effectively, $\langle F\rangle \sim \Lambda^3/M_{\text{P}}$. In that case, the scale associated with dynamical supersymmetry breaking should be more like $\Lambda \sim 10^{13}$ GeV.

The second main possibility is that the flavor-blind mediating interactions for supersymmetry breaking are the ordinary electroweak and QCD gauge interactions. In this *gauge-mediated supersymmetry breaking* scenario, the MSSM soft terms arise from loop diagrams involving some *messenger* particles. The messengers couple to a supersymmetry-breaking VEV $\langle F\rangle$, and also have $SU(3)_C \times SU(2)_L \times U(1)_Y$ interactions which provide a link to the MSSM. Then, using dimensional analysis, one estimates for the MSSM soft terms

$$m_{\text{soft}} \sim \frac{\alpha_a}{4\pi}\frac{\langle F\rangle}{M_{\text{mess}}} \tag{9.15}$$

where the $\alpha_a/4\pi$ is a loop factor for Feynman diagrams involving gauge interactions, and M_{mess} is a characteristic scale of the masses of the messenger fields. So if M_{mess} and $\sqrt{\langle F\rangle}$ are roughly comparable, then the scale of supersymmetry breaking can be as low as about $\sqrt{\langle F\rangle} \sim 10^4$ or 10^5 GeV (much lower than in the gravity-mediated case!) to give m_{soft} of the right order of magnitude.

9.2 The goldstino and the gravitino

As explained in the previous section, the spontaneous breaking of global supersymmetry implies the existence of a massless Weyl fermion, the goldstino. In the particular case of the O'Raifeartaigh model, the goldstino was identified to be ψ_1. More generally, we might expect that in the case of F-term or D-term breaking, the goldstino is the fermionic component of the supermultiplet whose auxiliary field obtains a VEV.

Let us make this more precise by actually proving that the goldstino exists and, in the process, identifying it. This is actually rather easy. Consider a general supersymmetric model with both gauge and chiral supermultiplets as in section 6. The fermionic degrees of freedom consist of gauginos (λ^a) and chiral fermions (ψ_i). After some of the scalar fields in the theory obtain VEVs, the fermion mass matrix will have the form:

$$\mathbf{M}_{\text{fermion}} = \begin{pmatrix} 0 & \sqrt{2}g_a(\langle\phi^*\rangle T^a)^i \\ \sqrt{2}g_a(\langle\phi^*\rangle T^a)^j & \langle W^{ij}\rangle \end{pmatrix} \tag{9.16}$$

in the (λ^a, ψ_i) basis. [The off-diagonal entries in this matrix come from the second line in eq. (6.76), and the lower right entry can be seen in eq. (6.50).] Now we simply note that $\mathbf{M}_{\text{fermion}}$ annihilates the vector

$$\widetilde{G} = \begin{pmatrix} \langle D^a\rangle/\sqrt{2} \\ \langle F_i\rangle \end{pmatrix}. \tag{9.17}$$

The first row of $\mathbf{M}_{\text{fermion}}$ annihilates $\widetilde{G}$ by virtue of the requirement eq. (6.77) that the superpotential is gauge invariant, and the second row annihilates $\widetilde{G}$ because of the condition $\langle \partial V/\partial\phi_i\rangle = 0$ which must be satisfied at the minimum of the scalar potential. Eq. (9.17) is proportional to the goldstino wavefunction; it is non-trivial if and only if at least one of the auxiliary fields has a VEV, breaking supersymmetry. So we have proven that if global supersymmetry is spontaneously broken, then the goldstino exists and has zero mass, and that its components among the various fermions in the theory are just proportional to the corresponding auxiliary field VEVs.

We can derive another very important property of the goldstino by considering the form of the conserved supercurrent eq. (6.80). Suppose for simplicity[4] that the non-vanishing auxiliary field VEV is $\langle F\rangle$ and that its goldstino superpartner is $\widetilde{G}$. Then the supercurrent conservation equation tells us that

$$0 = \partial_\mu J^\mu_\alpha = i\langle F\rangle(\sigma^\mu\partial_\mu\widetilde{G}^\dagger)_\alpha + \partial_\mu j^\mu_\alpha + \ldots \tag{9.18}$$

where j^μ_α is the part of the supercurrent which involves all of the other supermultiplets, and the ellipses represent other contributions of the goldstino supermultiplet to $\partial_\mu J^\mu_\alpha$ which we can ignore. [The first term in eq. (9.18) comes from the second term in eq. (6.80), using the equation

[4] More generally, if supersymmetry is spontaneously broken by VEVs for several auxiliary fields F_i and D^a, then one should make the replacement $\langle F\rangle \to (\sum_i |\langle F_i\rangle|^2 + \frac{1}{2}\sum_a \langle D^a\rangle^2)^{1/2}$ everywhere in the following.

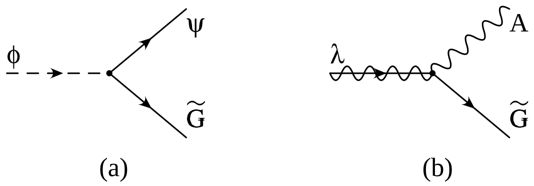


Fig. 9.2. Goldstino/gravitino interactions with superpartner pairs (ϕ, ψ) and (λ^a, A^a).

of motion $F_i = -W_i^*$ for the goldstino's auxiliary field.] This equation of motion for the goldstino field allows us to write an effective Lagrangian

$$\mathcal{L}_{\text{goldstino}} = -i\widetilde{G}^\dagger \overline{\sigma}^\mu \partial_\mu \widetilde{G} - \frac{1}{\langle F \rangle}(\widetilde{G}\partial_\mu j^\mu + \text{c.c.}) \tag{9.19}$$

which describes the interactions of the goldstino with all of the other fermion-boson pairs. In particular, since $j^\mu_\alpha = (\sigma^\nu \overline{\sigma}^\mu \psi_i)_\alpha \partial_\nu \phi^{*i} - (1/2\sqrt{2})\sigma^\nu \overline{\sigma}^\rho \sigma^\mu \lambda^{\dagger a} F^a_{\nu\rho} + \ldots$, there are goldstino-scalar-chiral fermion and goldstino-gaugino-gauge boson vertices as shown in Fig. 9.2. Since this derivation depends only on supercurrent conservation, eq. (9.19) holds independently of the details of how supersymmetry breaking is communicated from $\langle F \rangle$ to the MSSM sector fields (ϕ_i, ψ_i) and (λ^a, A^a). It may appear strange at first that the interaction terms in eq. (9.19) get larger as $\langle F \rangle$ goes to zero. However, the interaction term $\widetilde{G}\partial_\mu j^\mu$ contains two derivatives which turn out to always give a kinematic factor proportional to the (mass)^2 difference of the superpartners when they are on-shell, i.e. $m^2_{\phi_i} - m^2_{\psi_i}$ and $m^2_\lambda - m^2_A$ for Figs. 9.2a and 9.2b respectively. These can be non-zero only by virtue of supersymmetry breaking, so they must also vanish as $\langle F \rangle \to 0$, and the interaction is well-defined in that limit. Nevertheless, for fixed values of $m^2_{\phi_i} - m^2_{\psi_i}$ and $m^2_\lambda - m^2_A$, the interaction term in eq. (9.19) can be phenomenologically important if $\langle F \rangle$ is not too large.

The above remarks apply to the breaking of global supersymmetry. However, when one takes into account gravity, supersymmetry must be a local symmetry. This means that the spinor parameter ϵ^α which first appeared in section 6.1 is no longer a constant, but can vary from point to point in spacetime. The resulting locally supersymmetric theory is called *supergravity*. It necessarily unifies the spacetime symmetries of ordinary general relativity with local supersymmetry transformations. In supergravity, the spin-2 graviton has a spin-3/2 fermion superpartner called the gravitino, which we will denote $\widetilde{\Psi}^\alpha_\mu$. The gravitino has odd R-parity ($P_R = -1$), as can be seen from the definition eq. (8.11). It carries both a vector index (μ) and a spinor index (α), and transforms inhomogeneously under local supersymmetry transformations:

$$\delta\widetilde{\Psi}^{\alpha}_{\mu} = -\partial_\mu \epsilon^\alpha + \ldots \tag{9.20}$$

Thus the gravitino should be thought of as the "gauge" particle of local supersymmetry transformations [compare eq. (6.56)]. As long as supersymmetry is unbroken, the graviton and the gravitino are both massless, each with two spin helicity states. Once supersymmetry is spontaneously broken, the gravitino acquires a mass by absorbing ("eating") the goldstino, which becomes its longitudinal (helicity $\pm 1/2$) components. This is called the *super-Higgs* mechanism. It is entirely analogous to the ordinary Higgs mechanism for gauge theories, by which the $W^\pm$ and Z^0 gauge bosons in the Standard Model gain mass by absorbing the Nambu-Goldstone bosons associated with the spontaneously broken electroweak gauge invariance. The counting works, because the massive spin-3/2 gravitino now has four helicity states, of which two were originally assigned to the would-be goldstino. The gravitino mass is traditionally called $m_{3/2}$, and in the case of F-term breaking can be estimated as

$$m_{3/2} \sim \frac{\langle F \rangle}{M_{\rm P}}, \tag{9.21}$$

This follows simply from dimensional analysis, since $m_{3/2}$ must vanish in the limits that supersymmetry is restored ($\langle F \rangle \to 0$) and that gravity is turned off ($M_P \to \infty$). Equation (9.21) means that one has very different expectations for the mass of the gravitino in gravity-mediated and in gauge-mediated models, because they usually make very different predictions for $\langle F \rangle$.

In the gravity-mediated supersymmetry breaking case, the gravitino mass is comparable to the masses of the MSSM sparticles [compare eqs. (9.13) and (9.21)]. Therefore $m_{3/2}$ is expected to be at least 100 GeV or so. Its interactions will be of gravitational strength, so the gravitino will not play any role in collider physics, but it can be a very important consideration in cosmology. If it is the LSP, then it is stable and its primordial density could easily exceed the critical density, causing the universe to become matter-dominated too early. Even if it is not the LSP, the gravitino can cause problems unless its density is diluted by inflation at late times, or it decays sufficiently rapidly.

In contrast, gauge-mediated supersymmetry breaking models predict that the gravitino is much lighter than the MSSM sparticles as long as $M_{\rm mess} \ll M_{\rm P}$. This can be seen by comparing eqs. (9.15) and (9.21). The gravitino is almost certainly the LSP in this case, and all of the MSSM sparticles will eventually decay into final states that include it. Naively, one might expect that these decays are extremely slow. However, this is not necessarily true, because the gravitino inherits the *non*-gravitational interactions of the goldstino it has absorbed. This means that the gravitino, or more precisely its longitudinal (goldstino) components, can play an important role in collider physics experiments. The mass of the gravitino can generally be ignored for kinematic purposes, as can its transverse (helicity $\pm 3/2$) components which really do have only gravitational interactions. Therefore in collider phenomenology discussions one may interchangeably use the same symbol $\widetilde{G}$ for the goldstino and for the gravitino of which it is the longitudinal (helicity $\pm 1/2$) part. By using the effective Lagrangian eq. (9.19), one can compute that the decay rate of any sparticle $\widetilde{X}$ into its Standard Model partner X plus a gravitino/goldstino $\widetilde{G}$ is given by

$$\Gamma(\widetilde{X} \to X\widetilde{G}) = \frac{m_{\widetilde{X}}^5}{16\pi \langle F \rangle^2} \left(1 - \frac{m_X^2}{m_{\widetilde{X}}^2}\right)^4 . \tag{9.22}$$

This corresponds to either Fig. 9.2a or 9.2b, with $(\widetilde{X}, X) = (\phi, \psi)$ or (λ, A) respectively. One factor $(1 - m_X^2/m_{\widetilde{X}}^2)^2$ came from the derivatives in the interaction term in eq. (9.19) evaluated for on-shell final states, and another such factor comes from the kinematic phase space integral with $m_{3/2} \ll m_{\widetilde{X}}, m_X$.

If the supermultiplet containing the goldstino and $\langle F \rangle$ has canonically-normalized kinetic terms, and one requires the tree-level vacuum energy to vanish, then the estimate eq. (9.21) may be sharpened to

$$m_{3/2} = \frac{\langle F \rangle}{\sqrt{3} M_{\rm P}}. \tag{9.23}$$

In that case, one can rewrite eq. (9.22) as

$$\Gamma(\widetilde{X} \to X\widetilde{G}) = \frac{m_{\widetilde{X}}^5}{48\pi M_{\rm P}^2 m_{3/2}^2} \left(1 - \frac{m_X^2}{m_{\widetilde{X}}^2}\right)^4 , \tag{9.24}$$

and this is how the formula is sometimes presented by those who prefer to take eq. (9.23) seriously. Note that the decay width is larger for smaller $\langle F \rangle$, or equivalently for smaller $m_{3/2}$, if the other masses are fixed. If $\widetilde{X}$ is a mixture of superpartners of different Standard Model particles X, then eq. (9.22) should be multiplied by a suppression factor equal to the square of the cosine of the appropriate mixing angle. If $m_{\widetilde{X}}$ is of order 100 GeV or more, and $\sqrt{\langle F \rangle} \lesssim$ few$\times 10^6$ GeV [corresponding to $m_{3/2}$ less than roughly 1 keV according to eq. (9.23)], then the decay $\widetilde{X} \to X\widetilde{G}$ can occur quickly enough to be observed in a modern collider detector. This gives rise to some very interesting phenomenological signatures, which we will discuss further in a later Chapter.

We now turn to a slightly more systematic analysis of the way in which the MSSM soft terms arise, considering in turn the gravity-mediated and gauge-mediated scenarios.

9.3 Gravity-mediated supersymmetry breaking models

The defining feature of these models is that the hidden sector of the theory communicates with our MSSM only (or dominantly) through gravitational-strength interactions. In an effective field theory format, this means that the supergravity Lagrangian contains nonrenormalizable terms which communicate between the two sectors and which are suppressed by powers of the Planck mass, since the gravitational coupling is proportional to $1/M_{\rm P}$. These will include

$$\begin{aligned}\mathcal{L}_{\rm NR} = &-\frac{1}{M_{\rm P}} F_X \sum_a \frac{1}{2} f_a \lambda^a \lambda^a + \text{c.c.}\\ &-\frac{1}{M_{\rm P}^2} F_X F_X^* \, k^i_j \phi_i \phi^{*j}\\ &-\frac{1}{M_{\rm P}} F_X \big(\frac{1}{6} y'^{ijk} \phi_i \phi_j \phi_k + \frac{1}{2} \mu'^{ij} \phi_i \phi_j\big) + \text{c.c.}\end{aligned} \qquad (9.25)$$

where F_X is the auxiliary field for a chiral supermultiplet X in the hidden sector, and ϕ_i and λ^a are the scalar and gaugino fields in the MSSM. By themselves, the terms in eq. (9.25) are not supersymmetric, but it is possible to show that they are part of a nonrenormalizable supersymmetric Lagrangian (see Appendix) which contains other terms that we may ignore. Now if one assumes that $\langle F_X \rangle \sim 10^{10}$ or 10^{11} GeV, then $\mathcal{L}_{\rm NR}$ will give us nothing other than a Lagrangian of the form $\mathcal{L}_{\rm soft}$ in eq. (6.82), with MSSM soft terms of order a few hundred GeV. [Note that terms of the form $\mathcal{L}_{\text{maybe soft}}$ in eq. (6.83) do not arise.]

The dimensionless parameters f_a, k^i_j, y'^{ijk} and μ'^{ij} in $\mathcal{L}_{\rm NR}$ are to be determined by the underlying theory. This is a difficult enterprise in general, but a dramatic simplification occurs if one assumes a "minimal" form for the normalization of kinetic terms and gauge interactions in the full, nonrenormalizable supergravity Lagrangian (see Appendix). In that case, one finds that there is a common $f_a = f$ for the three gauginos; $k^i_j = k\delta^i_j$ is the same for all scalars; and the other couplings are proportional to the corresponding superpotential parameters, so that $y'^{ijk} = \alpha y^{ijk}$ and $\mu'^{ij} = \beta \mu^{ij}$ with universal dimensionless constants α and β. Then one finds that the soft terms in $\mathcal{L}_{\rm soft}^{\rm MSSM}$ can all be written in terms of just four parameters:

$$m_{1/2} = f\frac{\langle F_X\rangle}{M_{\rm P}}; \qquad m_0^2 = k\frac{|\langle F_X\rangle|^2}{M_{\rm P}^2}; \qquad A_0 = \alpha\frac{\langle F_X\rangle}{M_{\rm P}}; \qquad B_0 = \beta\frac{\langle F_X\rangle}{M_{\rm P}}. \tag{9.26}$$

In terms of these, one can write for the parameters appearing in eq. (8.12):

$$M_3 = M_2 = M_1 = m_{1/2}; \tag{9.27}$$

$$\mathbf{m^2_Q} = \mathbf{m^2_{\overline{u}}} = \mathbf{m^2_{\overline{d}}} = \mathbf{m^2_L} = \mathbf{m^2_{\overline{e}}} = m_0^2\,\mathbf{1}; \quad m^2_{H_u} = m^2_{H_d} = m_0^2; \tag{9.28}$$

$$\mathbf{a_u} = A_0\mathbf{y_u}; \qquad \mathbf{a_d} = A_0\mathbf{y_d}; \qquad \mathbf{a_e} = A_0\mathbf{y_e}; \tag{9.29}$$

$$b = B_0\mu. \tag{9.30}$$

It is a matter of some controversy whether the assumptions going into this parameterization are completely well-motivated on purely theoretical grounds,[5] but from a phenomenological perspective they are clearly very nice. This framework successfully evades the most dangerous types of FCNC and CP-violation as discussed in section 8.4. In particular, eqs. (9.28) and (9.29) are just stronger versions of eqs. (8.15) and (8.16), respectively. If $m_{1/2}$, A_0 and B_0 all have the same complex phase, then eq. (8.17) will also be satisfied.

Equations (9.27)-(9.30) also have the virtue of being highly predictive. [Of course, eq. (9.30) is content-free unless one can relate B_0 to the other parameters in some non-trivial way.] As discussed in section 5.4, they should be applied as RG boundary conditions at the scale $M_{\rm P}$. The RG evolution of the soft parameters down to the electroweak scale will then allow us to predict the entire MSSM spectrum in terms of just five parameters $m_{1/2}$, m_0^2, A_0, B_0, and μ (plus the already-measured gauge and Yukawa couplings of the MSSM). In practice, the approximation is usually made of starting this RG running from the unification scale $M_U \approx 2\times 10^{16}$ GeV instead of $M_{\rm P}$. The reason for this is that the apparent unification of gauge couplings gives us a strong hint that we know something about how the RG equations are behaving up to M_U, but gives us little guidance about what to expect at scales between M_U and $M_{\rm P}$. The error made in neglecting these effects is proportional to a loop suppression factor times $\ln(M_{\rm P}/M_U)$ and can be partially absorbed into a redefinition of m_0^2, $m_{1/2}$, A_0 and B_0, but in some cases can lead to important effects. The framework described in the above few paragraphs has been the subject of the bulk of phenomenological studies of supersymmetry. It is sometimes referred to as the *minimal supergravity* or *supergravity-inspired* scenario for the soft terms.

[5] The familiar flavor-blindness of gravitational interactions expressed in Einstein's equivalence principle does not, by itself, tell us anything about the form of eq. (9.25).

Particular models of gravity-mediated supersymmetry breaking can be even more predictive, relating some of the parameters $m_{1/2}$, m_0^2, A_0 and B_0 to each other and to the mass of the gravitino $m_{3/2}$. For example, three popular kinds of models for the soft terms are:

- Dilaton-dominated: $m_0^2 = m_{3/2}^2$; $m_{1/2} = -A_0 = \sqrt{3}m_{3/2}$.
- Polonyi: $m_0^2 = m_{3/2}^2$; $A_0 = (3-\sqrt{3})m_{3/2}$; $m_{1/2} = \mathcal{O}(m_{3/2})$.
- "No-scale": $m_{1/2} \gg m_0, A_0, m_{3/2}$.

The dilaton-dominated scenario arises in a particular limit of superstring theory. While it appears to be highly predictive, it can easily be generalized in other limits. The Polonyi model has the advantage of being the simplest possible model for supersymmetry breaking in the hidden sector, but it is rather *ad hoc* and does not seem to have a special place in grander schemes like superstrings. The "no-scale" limit may arise in a low-energy limit of superstrings in which the gravitino mass scale is undetermined at tree-level (hence the name). It implies that only the gaugino masses are appreciable at M_{P}. As we will see in section 10.1, RG evolution feeds $m_{1/2}$ into the squark, slepton and Higgs (mass)^2 parameters with sufficient magnitude to give acceptable phenomenology at the electroweak scale. More recent versions of the no-scale scenario, however, also can give significant A_0 and m_0^2 at M_{P}. In many cases B_0 can also be predicted in terms of the other parameters, but this is quite sensitive to model assumptions. For phenomenological studies, $m_{1/2}$, m_0^2, A_0 and B_0 are usually just taken to be convenient independent parameters of our ignorance of the supersymmetry breaking mechanism.

9.4 Gauge-mediated supersymmetry breaking models

A strong alternative to the scenario described in the previous section is provided by the gauge-mediated supersymmetry breaking proposal. The basic idea is to introduce some new chiral supermultiplets, called messengers, which couple to the ultimate source of supersymmetry breaking, and which also couple indirectly to the (s)quarks and (s)leptons and Higgs(inos) of the MSSM through the ordinary $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge boson and gaugino interactions. In this way, the ordinary gauge interactions, rather than gravity, are responsible for the appearance of soft terms in the MSSM. There is still gravitational communication between the MSSM and the source of supersymmetry breaking, of course, but that effect is now relatively unimportant compared to the gauge interaction effects.

In the simplest such model, the messenger fields are a set of chiral supermultiplets q, $\overline{q}$, ℓ, $\overline{\ell}$ which transform under $SU(3)_C \times SU(2)_L \times U(1)_Y$

as

$$q \sim (\mathbf{3}, \mathbf{1}, -\frac{1}{3}); \quad \overline{q} \sim (\overline{\mathbf{3}}, \mathbf{1}, \frac{1}{3}); \quad \ell \sim (\mathbf{1}, \mathbf{2}, \frac{1}{2}); \quad \overline{\ell} \sim (\mathbf{1}, \mathbf{2}, -\frac{1}{2}). \tag{9.31}$$

These supermultiplets contain messenger quarks $\psi_q, \psi_{\overline{q}}$ and scalar quarks $q, \overline{q}$ and messenger leptons $\psi_\ell, \psi_{\overline{\ell}}$ and scalar leptons $\ell, \overline{\ell}$. All of these particles must get very large masses so as not to have been discovered already. They manage to do so by coupling to a gauge-singlet chiral supermultiplet S through a superpotential:

$$W_{\text{mess}} = y_2 S \ell \overline{\ell} + y_3 S q \overline{q}. \tag{9.32}$$

The scalar component of S and its auxiliary (F-term) component are each supposed to acquire VEVs, denoted $\langle S \rangle$ and $\langle F_S \rangle$ respectively. This can be accomplished either by putting S into an O'Raifeartaigh-type model, or by a dynamical mechanism. Exactly how this happens is a very interesting and important question. Here, we will simply parameterize our ignorance of the precise mechanism of supersymmetry breaking by asserting that S participates in another part of the superpotential, call it W_{breaking}, which provides for supersymmetry breakdown.

Let us now consider the mass spectrum of the messenger fermions and bosons. The messenger part of the superpotential now effectively becomes $W_{\text{mess}} = y_2 \langle S \rangle \ell \overline{\ell} + y_3 \langle S \rangle q \overline{q}$. So, the fermionic messenger fields pair up to get mass terms:

$$\mathcal{L} = -(y_2 \langle S \rangle \psi_\ell \psi_{\overline{\ell}} + y_3 \langle S \rangle \psi_q \psi_{\overline{q}} + \text{c.c.}) \tag{9.33}$$

as in eq. (6.52). Meanwhile, their scalar messenger partners $\ell, \overline{\ell}$ and $q\overline{q}$ have a scalar potential given by (neglecting D-term contributions, which do not affect the following discussion):

$$\begin{aligned} V = & \left| \frac{\delta W_{\text{mess}}}{\delta \ell} \right|^2 + \left| \frac{\delta W_{\text{mess}}}{\delta \overline{\ell}} \right|^2 + \left| \frac{\delta W_{\text{mess}}}{\delta q} \right|^2 + \left| \frac{\delta W_{\text{mess}}}{\delta \overline{q}} \right|^2 \\ & + \left| \frac{\delta W_{\text{mess}}}{\delta S} + \frac{\delta W_{\text{breaking}}}{\delta S} \right|^2 \end{aligned} \tag{9.34}$$

as in eq. (6.51). Now, using the supposition that

$$\langle \delta W_{\text{breaking}} / \delta S \rangle = -\langle F_S^* \rangle \tag{9.35}$$

(with $\langle \delta W_{\text{mess}} / \delta S \rangle = 0$), and replacing S and F_S by their VEVs, one finds quadratic mass terms in the potential for the messenger scalar leptons:

$$\begin{aligned} V = & |y_2 \langle S \rangle|^2 \left(|\ell|^2 + |\overline{\ell}|^2 \right) + |y_3 \langle S \rangle|^2 \left(|q|^2 + |\overline{q}|^2 \right) \\ & - \left(y_2 \langle F_S \rangle \ell \overline{\ell} + y_3 \langle F_S \rangle q \overline{q} + \text{c.c.} \right) \\ & + \text{quartic terms.} \end{aligned} \tag{9.36}$$

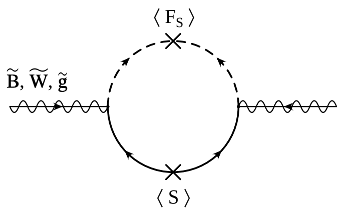


Fig. 9.3. Contributions to the MSSM gaugino masses in gauge-mediated supersymmetry breaking models arise from one-loop graphs involving virtual messenger particles.

The first line in eq. (9.36) represents supersymmetric mass terms that go along with eq. (9.33), while the second line consists of soft supersymmetry-breaking masses. The complex scalar messengers $\ell, \overline{\ell}$ thus obtain a $(\text{mass})^2$ matrix equal to:

$$\begin{pmatrix} |y_2\langle S\rangle|^2 & -y_2^*\langle F_S^*\rangle \\ -y_2\langle F_S\rangle & |y_2\langle S\rangle|^2 \end{pmatrix} \tag{9.37}$$

with squared mass eigenvalues $|y_2\langle S\rangle|^2 \pm |y_2\langle F_S\rangle|$. In just the same way, the scalars $q, \overline{q}$ get squared masses $|y_3\langle S\rangle|^2 \pm |y_3\langle F_S\rangle|$.

So far, we have found that the effect of supersymmetry breaking is to split each messenger supermultiplet pair apart:

$$\ell, \overline{\ell}: \qquad m^2_{\text{fermions}} = |y_2\langle S\rangle|^2, \qquad m^2_{\text{scalars}} = |y_2\langle S\rangle|^2 \pm |y_2\langle F_S\rangle|, \tag{9.38}$$

$$q, \overline{q}: \qquad m^2_{\text{fermions}} = |y_3\langle S\rangle|^2, \qquad m^2_{\text{scalars}} = |y_3\langle S\rangle|^2 \pm |y_3\langle F_S\rangle|. \tag{9.39}$$

The supersymmetry violation apparent in this messenger spectrum for $\langle F_S\rangle \neq 0$ is communicated to the MSSM sparticles through radiative quantum corrections. The MSSM gauginos obtain masses from the 1-loop graph shown in Fig. 9.3. The scalar and fermion lines in the loop are messenger fields. Recall that the interaction vertices in Fig. 9.3 are of gauge coupling strength even though they do not involve gauge bosons; compare Fig. 6.3g. In this way, gauge-mediation provides that $q, \overline{q}$ messenger loops give masses to the gluino and the bino, and $\ell, \overline{\ell}$ messenger loops give masses to the wino and bino fields. By computing the 1-loop diagrams one finds that the resulting MSSM gaugino masses are given by

$$M_a = \frac{\alpha_a}{4\pi}\Lambda, \qquad (a = 1, 2, 3), \tag{9.40}$$

(in the normalization discussed in section 8.4) where we have introduced a mass parameter

$$\Lambda \equiv \langle F_S\rangle / \langle S\rangle\ . \tag{9.41}$$

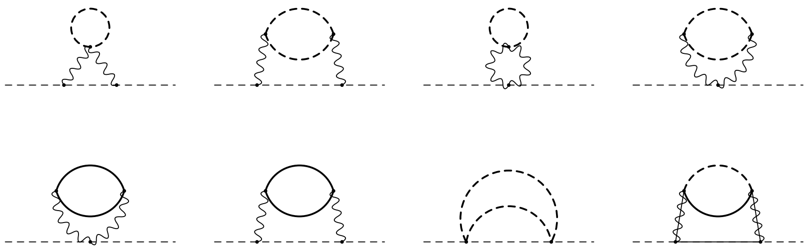


Fig. 9.4. Contributions to MSSM scalar squared masses in gauge-mediated supersymmetry breaking models arise in leading order from these two-loop Feynman graphs.

(Note that if $\langle F_S \rangle$ were 0, then $\Lambda = 0$ and the messenger scalars would be degenerate with their fermionic superpartners and there would be no contribution to the MSSM gaugino masses.) In contrast, the corresponding MSSM gauge bosons cannot get a corresponding mass shift, since they are protected by gauge invariance. So supersymmetry breaking has been successfully communicated to the MSSM ("visible sector"). To a good approximation, eq. (9.40) holds for the running gaugino masses at an RG scale Q_0 corresponding to the average characteristic mass of the heavy messenger particles, roughly of order $M_{\mathrm{mess}} \sim y_i \langle S \rangle$. The running mass parameters can then be RG-evolved down to the electroweak scale to predict the physical masses to be measured by future experiments.

The scalars of the MSSM do not get any radiative corrections to their masses at one-loop order. The leading contribution to their masses comes from the two-loop graphs shown in Fig. 9.4, with the messenger fermions (heavy solid lines) and messenger scalars (heavy dashed lines) and ordinary gauge bosons and gauginos running around the loops. By computing these graphs, one finds that each MSSM scalar ϕ gets a (mass)^2 given by:

$$m_\phi^2 = 2\Lambda^2 \left[\left(\frac{\alpha_3}{4\pi}\right)^2 C_3^\phi + \left(\frac{\alpha_2}{4\pi}\right)^2 C_2^\phi + \left(\frac{\alpha_1}{4\pi}\right)^2 C_1^\phi \right]. \qquad (9.42)$$

Here C_a^ϕ are the quadratic Casimir group theory invariants for the scalar ϕ for each gauge group. They are defined by $C_a^\phi \delta_i^j = (T^a T^a)_i^j$ where the T^a are the group generators which act on the scalar ϕ. Explicitly, they are:

$$C_3^\phi = \begin{cases} 4/3 \text{ for } \phi = \widetilde{Q}_i, \widetilde{\overline{u}}_i, \widetilde{\overline{d}}_i; \\ 0 \quad \text{ for } \phi = \widetilde{L}_i, \widetilde{\overline{e}}_i, H_u, H_d \end{cases} \tag{9.43}$$

$$C_2^\phi = \begin{cases} 3/4 \text{ for } \phi = \widetilde{Q}_i, \widetilde{L}_i, H_u, H_d; \\ 0 \quad \text{ for } \phi = \widetilde{\overline{u}}_i, \widetilde{\overline{d}}_i, \widetilde{\overline{e}}_i \end{cases} \tag{9.44}$$

$$C_1^\phi = 3Y_\phi^2/5 \text{ for each } \phi \text{ with weak hypercharge } Y_\phi. \tag{9.45}$$

The squared masses in eq. (9.42) are positive (fortunately!).

The terms $\mathbf{a_u}$, $\mathbf{a_d}$, $\mathbf{a_e}$ arise first at two-loop order, and are suppressed by an extra factor of $\alpha_a/(4\pi)$ compared to the gaugino masses. So, to a very good approximation one has, at the messenger scale,

$$\mathbf{a_u} = \mathbf{a_d} = \mathbf{a_e} = 0, \tag{9.46}$$

a significantly stronger condition than eq. (8.16). Again, eqs. (9.42) and (9.46) should be applied at an RG scale equal to the average mass of the messenger fields running in the loops. However, after evolving the RG equations down to the electroweak scale, non-zero $\mathbf{a_u}$, $\mathbf{a_d}$ and $\mathbf{a_e}$ are generated proportional to the corresponding Yukawa matrices and the non-zero gaugino masses, as we will see in section 10.1. These will only be large for the third family squarks and sleptons, in the approximation of eq. (8.2). The parameter b may also be taken to vanish near the messenger scale, but this is quite model-dependent, and in any case b will be non-zero when it is RG-evolved to the electroweak scale. In practice, b is determined by the requirement of correct electroweak symmetry breaking, as discussed below in section 8.5.

Because the gaugino masses arise at *one*-loop order and the scalar (mass)^2 contributions appear at *two*-loop order, both eq. (9.40) and (9.42) correspond to the estimate eq. (9.15) for m_{soft}, with $M_{\text{mess}} \sim y_i\langle S\rangle$. Equations (9.40) and (9.42) hold in the limit of small $\langle F_S\rangle/y_i\langle S\rangle^2$, corresponding to mass splittings within each messenger supermultiplet that are small compared to the overall messenger mass scale. The subleading corrections in an expansion in $\langle F_S\rangle/y_i\langle S\rangle^2$ turn out to be quite small unless there are very large hierarchies in the messenger sector.

The model we have described so far is often called the minimal model of gauge-mediated supersymmetry breaking. Let us now generalize it to a more complicated messenger sector. Suppose that $q, \overline{q}$ and $\ell, \overline{\ell}$ are replaced by a collection of messengers $\Phi_i, \overline{\Phi}_i$ with a superpotential

$$W_{\text{mess}} = \sum_i y_i S \Phi_i \overline{\Phi}_i. \tag{9.47}$$

The bar means that the chiral superfields $\overline{\Phi}_i$ transform as the complex conjugate representations of the Φ_i chiral superfields. Together they are said to form a "vector-like" (real) representation of the Standard Model gauge group. As before, the fermionic components of each pair Φ_i and $\overline{\Phi}_i$ pair up to get squared masses $y_i\langle S\rangle$ and their scalar partners mix to get squared masses $|y_i\langle S\rangle|^2 \pm |y_i\langle F_S\rangle|$. The MSSM gaugino mass parameters induced are now

$$M_a = \frac{\alpha_a}{4\pi}\Lambda \sum_i n_a(i) \qquad (a = 1, 2, 3) \tag{9.48}$$

where $n_a(i)$ is the Dynkin index for each $\Phi_i + \overline{\Phi}_i$, in a normalization where $n_3 = 1$ for a $\mathbf{3} + \overline{\mathbf{3}}$ of $SU(3)_C$ and $n_2 = 1$ for a pair of doublets of $SU(2)_L$. For $U(1)_Y$, one has $n_1 = 6Y^2/5$ for each messenger pair with weak hypercharges $\pm Y$. In computing n_1 one must remember to add up the contributions for each component of an $SU(3)_C$ or $SU(2)_L$ multiplet. So, for example, $(n_1, n_2, n_3) = (2/5, 0, 1)$ for $q + \overline{q}$ and $(n_1, n_2, n_3) = (3/5, 1, 0)$ for $\ell + \overline{\ell}$. Thus the total is $\sum_i(n_1, n_2, n_3) = (1, 1, 1)$ for the minimal model, so that eq. (9.48) is in agreement with eq. (9.40). On general group-theoretic grounds, n_2 and n_3 must be integers, and n_1 is always an integer multiple of $1/5$ if fractional electric charges are confined.

The MSSM scalar masses in this generalized gauge-mediation framework are now:

$$\begin{aligned} m_\phi^2 = 2\Lambda^2 \Big[&\left(\frac{\alpha_3}{4\pi}\right)^2 C_3^\phi \sum_i n_3(i) + \left(\frac{\alpha_2}{4\pi}\right)^2 C_2^\phi \sum_i n_2(i) \\ &+ \left(\frac{\alpha_1}{4\pi}\right)^2 C_1^\phi \sum_i n_1(i) \Big]. \end{aligned} \tag{9.49}$$

In writing eqs. (9.48) and (9.49) as simple sums, we have implicitly assumed that the messengers are all approximately equal in mass, with

$$M_{\text{mess}} \approx y_i\langle S\rangle. \tag{9.50}$$

This is a good approximation if the y_i are not too different from each other, because the dependence of the MSSM mass spectrum on the y_i is only logarithmic (due to RG running) for fixed Λ. However, if large hierarchies in the messenger masses are present, then the additive contributions to the gaugino and scalar masses from each individual messenger multiplet i should really instead be incorporated at the mass scale of that messenger multiplet. Then RG evolution is used to run these various contributions down to the electroweak or TeV scale; the individual messenger contributions to scalar and gaugino masses as indicated above can be thought of as threshold corrections to this RG running.

Messengers with masses far below the GUT scale will affect the running of gauge couplings and might therefore be expected to ruin the apparent unification shown in Fig. 8.7. However, if the messengers come in complete multiplets of the $SU(5)$ global symmetry[6] that contains the Standard Model gauge group and are not very different in mass, then approximate unification of gauge couplings will still occur when they are extrapolated up to the same scale M_U (but with a larger unified value for the gauge couplings at that scale). For this reason, a popular class of models is obtained by taking the messengers to consist of N_5 copies of the $\mathbf{5} + \overline{\mathbf{5}}$ of $SU(5)$, resulting in

$$N_5 = \sum_i n_1(i) = \sum_i n_2(i) = \sum_i n_3(i) \, . \tag{9.51}$$

In terms of this integer parameter N_5, eqs. (9.48) and (9.49) reduce to

$$M_a = \frac{\alpha_a}{4\pi} \Lambda N_5 \tag{9.52}$$

$$m_\phi^2 = 2\Lambda^2 N_5 \sum_{a=1}^{3} C_a^\phi \left(\frac{\alpha_a}{4\pi}\right)^2 , \tag{9.53}$$

since now there are N_5 copies of the minimal messenger sector particles running around the loops. For example, the minimal model in eq. (9.31) corresponds to $N_5 = 1$. A single copy of $\mathbf{10} + \overline{\mathbf{10}}$ of $SU(5)$ has Dynkin indices $\sum_i n_a(i) = 3$, and so can be substituted for 3 copies of $\mathbf{5} + \overline{\mathbf{5}}$. (Other combinations of messenger multiplets can also preserve the apparent unification of gauge couplings.) Note that the gaugino masses scale like N_5, while the scalar masses scale like $\sqrt{N_5}$. This means that sleptons and squarks will tend to be relatively lighter for larger values of N_5 in non-minimal models. However, if N_5 is too large, then the running gauge couplings will diverge before they can unify at M_U. For messenger masses of order 10^6 GeV or less, for example, one needs $N_5 \leq 4$.

There are many other possible generalizations of the basic gauge-mediation scenario as described above. An important general expectation in these models is that the strongly-interacting sparticles (squarks, gluino) should be heavier than weakly-interacting sparticles (sleptons, bino, winos, higgsinos) simply because of the hierarchy of gauge couplings $\alpha_3 > \alpha_2 > \alpha_1$. The common feature which makes all of these models very attractive is that the masses of the squarks and sleptons depend only on their gauge quantum numbers, leading automatically to the degeneracy

[6] This $SU(5)$ symmetry may or may not be promoted to a local gauge symmetry at the GUT scale. For our present purposes, it is used simply as a classification scheme, since the global $SU(5)$ symmetry is only approximate below the GUT scale at the messenger mass scale where gauge mediation takes place.

of squark and slepton masses needed for suppression of FCNC effects. But the most distinctive phenomenological prediction of gauge-mediated models may be the fact that the gravitino is the LSP. This can have crucial consequences for both cosmology and collider physics, as we will discuss further in a later Chapter.

10

From model assumptions to the low-energy MSSM

10.1 Renormalization Group Equations

In order to translate a set of predictions at the input scale into physically meaningful quantities which describe physics at the electroweak scale, it is necessary to evolve the gauge couplings, superpotential parameters, and soft terms using the RG equations. As a technical aside, we note that when computing RG effects and other radiative corrections in supersymmetry, it is important to choose regularization and renormalization schemes that do not violate supersymmetry. The most popular regularization method for discussing radiative corrections within the Standard Model is dimensional regularization (DREG), in which the number of spacetime dimensions is continued to $d = 4 - 2\epsilon$. Unfortunately, DREG violates supersymmetry explicitly because it introduces a mismatch between the numbers of gauge boson degrees of freedom and the gaugino degrees of freedom off-shell. This mismatch is only 2ϵ, but can be multiplied by factors up to $1/\epsilon^n$ in an n-loop calculation. In DREG, supersymmetric relations between dimensionless coupling constants ("supersymmetric Ward identities") are therefore disrespected by radiative corrections involving the finite parts of one-loop graphs and by the divergent parts of two-loop graphs. Instead, one may use the slightly different scheme known as regularization by dimensional reduction, or DRED, which does respect supersymmetry.[85] In the DRED method, all momentum integrals are still performed in $d = 4-2\epsilon$ dimensions, but the vector index μ on the gauge boson fields A^a_μ now runs over all 4 dimensions. Running couplings are then renormalized using DRED with modified minimal subtraction ($\overline{\mathrm{DR}}$) rather than the usual DREG with modified minimal subtraction ($\overline{\mathrm{MS}}$). In particular, the boundary conditions at the input scale should be applied in the supersymmetry-preserving $\overline{\mathrm{DR}}$ scheme. (See Ref.[86] for an alternative supersymmetric scheme.) One loop β-functions are always the same in the two schemes, but it is important to realize that the $\overline{\rm MS}$ scheme does violate supersymmetry, so that $\overline{\rm DR}$ is preferred [2] from that point of view. (It is also possible to work consistently within the $\overline{\rm MS}$ scheme, as long as one is careful to correctly translate all $\overline{\rm DR}$ couplings and masses into their $\overline{\rm MS}$ counterparts.[90, 91])

The MSSM RG equations in the $\overline{\rm DR}$ scheme are given in Refs.[92]– [95]; they are now known for the gauge couplings and superpotential parameters up to 3-loop order, and for the soft parameters at 2-loop order. However, for many purposes including pedagogical ones it suffices to work in the 1-loop approximation. Here, we will also use the approximation that only the third family Yukawa couplings are significant; see eq. (8.2). Then the superpotential parameters run with scale according to:

$$\frac{d}{dt}y_t = \frac{y_t}{16\pi^2}\Big[6|y_t|^2 + |y_b|^2 - \frac{16}{3}g_3^2 - 3g_2^2 - \frac{13}{15}g_1^2\Big]; \tag{10.1}$$

$$\frac{d}{dt}y_b = \frac{y_b}{16\pi^2}\Big[6|y_b|^2 + |y_t|^2 + |y_\tau|^2 - \frac{16}{3}g_3^2 - 3g_2^2 - \frac{7}{15}g_1^2\Big]; \tag{10.2}$$

$$\frac{d}{dt}y_\tau = \frac{y_\tau}{16\pi^2}\Big[4|y_\tau|^2 + 3|y_b|^2 - 3g_2^2 - \frac{9}{5}g_1^2\Big]; \tag{10.3}$$

$$\frac{d}{dt}\mu = \frac{\mu}{16\pi^2}\Big[3|y_t|^2 + 3|y_b|^2 + |y_\tau|^2 - 3g_2^2 - \frac{3}{5}g_1^2\Big]. \tag{10.4}$$

The one-loop RG equations for the gauge couplings g_1, g_2, g_3 have already been listed in eq. (8.18). Note that the β-functions (the quantities on the right side of each equation) for each supersymmetric parameter are proportional to the parameter itself. This is actually a consequence of a general and powerful result known as the *supersymmetric nonrenormalization theorem.*[96] This theorem implies that the logarithmically divergent contributions to a given process can always be written in the form of a wave-function renormalization, without any vertex renormalization.[3] It is true for any supersymmetric theory, not just the MSSM, and holds to all orders in perturbation theory. It can be proved most easily using superfield techniques. In particular, it means that once we have a theory which can explain why μ is of order 10^2 or 10^3 GeV at tree-level, we do not have to worry about μ being infected (made very large) by radiative

[2] Even the DRED scheme may not provide a supersymmetric regulator, because of ambiguities which appear at five-loop order at the latest.[87] Fortunately, this does not seem to cause any practical difficulties.[88] See also Ref.[89] for a promising proposal which avoids doing violence to the number of spacetime dimensions.

[3] Actually, there *is* vertex renormalization in the field theory in which auxiliary fields have been integrated out, but the sum of divergent contributions for a given process always has the form of wave-function renormalization. See Ref.[25] for a discussion of this point.

corrections involving the masses of some very heavy unknown particles; all such RG corrections to μ will be directly proportional to μ itself.

The one-loop RG equations for the three gaugino mass parameters in the MSSM are determined by the same quantities $b_a^{\rm MSSM}$ which appear in the gauge coupling RG eqs. (8.18):

$$\frac{d}{dt}M_a = \frac{1}{8\pi^2}b_a g_a^2 M_a \qquad (b_a = 33/5, 1, -3) \tag{10.5}$$

for $a = 1, 2, 3$. It is therefore easy to show that the three ratios M_a/g_a^2 are each constant (RG-scale independent) up to small two-loop corrections. In minimal supergravity models, we can therefore write

$$M_a(Q) = \frac{g_a^2(Q)}{g_a^2(Q_0)}m_{1/2} \qquad (a = 1, 2, 3) \tag{10.6}$$

at any RG scale $Q < Q_0$, where Q_0 is the input scale which is presumably nearly equal to M_P. Since the gauge couplings are observed to unify at $M_U \sim 0.01 M_P$, one expects [4] that $g_1^2(Q_0) \approx g_2^2(Q_0) \approx g_3^2(Q_0)$. Therefore, one finds that

$$\frac{M_1}{g_1^2} = \frac{M_2}{g_2^2} = \frac{M_3}{g_3^2} \tag{10.7}$$

at any RG scale, up to small two-loop effects and possibly larger threshold effects near M_U and M_P. The common value in eq. (10.7) is also equal to $m_{1/2}/g_U^2$ in minimal supergravity models, where g_U is the unified gauge coupling at the input scale where $m_{1/2}$ is the common gaugino mass. Interestingly, eq. (10.7) is *also* the solution to the one-loop RG equations in the case of the gauge-mediated boundary conditions eq. (9.40) applied at the messenger mass scale. This is true even though there is no such thing as a unified gaugino mass $m_{1/2}$ in the gauge-mediated case, because of the fact that the gaugino masses are proportional to the g_a^2 times a constant. So eq. (10.7) is theoretically well-motivated (but certainly not inevitable) in both frameworks. The prediction eq. (10.7) is particularly useful since the gauge couplings g_1^2, g_2^2, and g_3^2 are already quite well known at the electroweak scale from experiment. Therefore they can be extrapolated up to at least M_U, assuming that the apparent unification of gauge couplings is not a fake. The gaugino mass parameters feed into the RG equations for all of the other soft terms, as we will see.

Next we consider the 1-loop RG equations for the analytic soft parameters $\mathbf{a_u}$, $\mathbf{a_d}$, $\mathbf{a_e}$. In models obeying eq. (8.16), these matrices start

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[4] In a GUT model, it is automatic that the gauge couplings and gaugino masses are unified at all scales $Q > M_U$ and in particular at $Q \approx M_P$, because in the unified theory the gauginos all live in the same representation of the unified gauge group. In many superstring models, this is also known to be a good approximation.

off proportional to the corresponding Yukawa couplings at the input scale, and the RG evolution respects this property. With the approximation of eq. (8.2), one can therefore also write, at any RG scale,

$$\mathbf{a_u} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_t \end{pmatrix}; \qquad \mathbf{a_d} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_b \end{pmatrix}; \qquad \mathbf{a_e} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_\tau \end{pmatrix}, \tag{10.8}$$

which defines [5] running parameters a_t, a_b, and a_τ. The RG equations for these parameters and b are given by

$$\begin{aligned} 16\pi^2 \frac{d}{dt} a_t = {} & a_t \Big[18|y_t|^2 + |y_b|^2 - \frac{16}{3} g_3^2 - 3g_2^2 - \frac{13}{15} g_1^2 \Big] + 2a_b y_b^* y_t \\ & + y_t \Big[\frac{32}{3} g_3^2 M_3 + 6g_2^2 M_2 + \frac{26}{15} g_1^2 M_1 \Big]; \end{aligned} \tag{10.9}$$

$$\begin{aligned} 16\pi^2 \frac{d}{dt} a_b = {} & a_b \Big[18|y_b|^2 + |y_t|^2 + |y_\tau|^2 - \frac{16}{3} g_3^2 - 3g_2^2 - \frac{7}{15} g_1^2 \Big] + 2a_t y_t^* y_b \\ & + 2a_\tau y_\tau^* y_b + y_b \Big[\frac{32}{3} g_3^2 M_3 + 6g_2^2 M_2 + \frac{14}{15} g_1^2 M_1 \Big]; \end{aligned} \tag{10.10}$$

$$\begin{aligned} 16\pi^2 \frac{d}{dt} a_\tau = {} & a_\tau \Big[12|y_\tau|^2 + 3|y_b|^2 - 3g_2^2 - \frac{9}{5} g_1^2 \Big] + 6a_b y_b^* y_\tau \\ & + y_\tau \Big[6g_2^2 M_2 + \frac{18}{5} g_1^2 M_1 \Big]; \end{aligned} \tag{10.11}$$

$$\begin{aligned} 16\pi^2 \frac{d}{dt} b = {} & b \Big[3|y_t|^2 + 3|y_b|^2 + |y_\tau|^2 - 3g_2^2 - \frac{3}{5} g_1^2 \Big] \\ & + \mu \Big[6a_t y_t^* + 6a_b y_b^* + 2a_\tau y_\tau^* + 6g_2^2 M_2 + \frac{6}{5} g_1^2 M_1 \Big] \end{aligned} \tag{10.12}$$

in this approximation. The β-function for each of these soft parameters is *not* proportional to the parameter itself; this makes sense because couplings which violate supersymmetry are not protected by the supersymmetric nonrenormalization theorem. In particular, even if A_0 and B_0 appearing in eqs. (9.29) and (9.30) vanish at the input scale, the RG corrections proportional to gaugino masses appearing in eqs. (10.9)-(10.12) ensure that a_t, a_b, a_τ and b will still be non-zero at the electroweak scale.

Next let us consider the RG equations for the scalar masses in the MSSM. In the approximation of eqs. (8.2) and (10.8), the squarks and

[5] We must warn the reader that rescaled soft parameters $A_t = a_t/y_t$, $A_b = a_b/y_b$, and $A_\tau = a_\tau/y_\tau$ are commonly used in the literature. We do not follow this notation, because it cannot be generalized beyond the approximation of eqs. (8.2), (10.8) without introducing horrible complications such as non-polynomial RG equations, and because a_t, a_b and a_τ are the couplings that actually appear in the lagrangian anyway.

sleptons of the first two families have only gauge interactions. This means that if the scalar masses satisfy a boundary condition like eq. (8.15) at an input RG scale, then when renormalized to any other RG scale, they will still be almost diagonal, with the approximate form

$$\mathbf{m_Q^2} \approx \begin{pmatrix} m_{Q_1}^2 & 0 & 0 \\ 0 & m_{Q_1}^2 & 0 \\ 0 & 0 & m_{Q_3}^2 \end{pmatrix}; \qquad \mathbf{m_{\overline{u}}^2} \approx \begin{pmatrix} m_{\overline{u}_1}^2 & 0 & 0 \\ 0 & m_{\overline{u}_1}^2 & 0 \\ 0 & 0 & m_{\overline{u}_3}^2 \end{pmatrix}; \tag{10.13}$$

etc. The first and second family squarks and sleptons with given gauge quantum numbers remain very nearly degenerate, but the third family squarks and sleptons feel the effects of the larger Yukawa couplings and so get renormalized differently. The one-loop RG equations for the first and second family squark and slepton squared masses can be written as [6]

$$16\pi^2 \frac{d}{dt} m_\phi^2 = -\sum_{a=1,2,3} 8 g_a^2 C_a^\phi |M_a|^2 \tag{10.14}$$

for each scalar ϕ, where the $\sum_a$ is over the three gauge groups $U(1)_Y$, $SU(2)_L$ and $SU(3)_C$; M_a are the corresponding running gaugino mass parameters which are known from eq. (10.7); and the constants C_a^ϕ are the same quadratic Casimir invariants which appeared in eqs. (9.43)-(9.45). An important feature of eq. (10.14) is that the right-hand sides are strictly negative, so that the scalar (mass)^2 parameters *grow* as they are RG-evolved from the input scale down to the electroweak scale. Even if the scalars have zero or very small masses at the input scale, as in the "no-scale" boundary condition limit $m_0^2 = 0$, they will obtain large positive squared masses at the electroweak scale, thanks to the effects of the gaugino masses.

The RG equations for the (mass)^2 parameters of the Higgs scalars and third family squarks and sleptons get the same gauge contributions as in eq. (10.14), but they also have contributions due to the large Yukawa ($y_{t,b,\tau}$) and soft ($a_{t,b,\tau}$) couplings. At one-loop order, these only appear in

[6] There are also terms in the scalar (mass)^2 RG equations which are proportional to $\mathrm{Tr}[Ym^2]$ (the sum of the weak hypercharge times the soft (mass)^2 for all scalars in the theory). However, these contributions vanish in both the cases of minimal supergravity and gauge-mediated boundary conditions for the soft terms, as one can see by explicitly calculating $\mathrm{Tr}[Ym^2]$ in each case. If $\mathrm{Tr}[Ym^2]$ is zero at the input scale, then it will remain zero under RG evolution. Therefore we neglect such terms in our discussion, although they can have an important effect in more general situations.

three combinations:

$$X_t = 2|y_t|^2(m_{H_u}^2 + m_{Q_3}^2 + m_{\overline{u}_3}^2) + 2|a_t|^2, \tag{10.15}$$

$$X_b = 2|y_b|^2(m_{H_d}^2 + m_{Q_3}^2 + m_{\overline{d}_3}^2) + 2|a_b|^2, \tag{10.16}$$

$$X_\tau = 2|y_\tau|^2(m_{H_d}^2 + m_{L_3}^2 + m_{\overline{e}_3}^2) + 2|a_\tau|^2. \tag{10.17}$$

In terms of these quantities, the RG equations for the soft Higgs $(\text{mass})^2$ parameters $m_{H_u}^2$ and $m_{H_d}^2$ are

$$16\pi^2 \frac{d}{dt} m_{H_u}^2 = 3X_t - 6g_2^2|M_2|^2 - \frac{6}{5}g_1^2|M_1|^2, \tag{10.18}$$

$$16\pi^2 \frac{d}{dt} m_{H_d}^2 = 3X_b + X_\tau - 6g_2^2|M_2|^2 - \frac{6}{5}g_1^2|M_1|^2. \tag{10.19}$$

Note that X_t, X_b, and X_τ are positive, so their effect is always to *decrease* the Higgs masses as one evolves the RG equations downward from the input scale to the electroweak scale. Since y_t is the largest of the Yukawa couplings because of the experimental fact that the top quark is heavy, X_t is typically expected to be larger than X_b and X_τ. This can cause the RG-evolved $m_{H_u}^2$ to run negative near the electroweak scale, helping to destabilize the point $H_u = 0$ and so provoking a Higgs VEV which is just what we want.[7] Thus a large top Yukawa coupling favors the breakdown of the electroweak symmetry breaking because it induces negative radiative corrections to the Higgs $(\text{mass})^2$.

The third family squark and slepton $(\text{mass})^2$ parameters also get contributions which depend on X_t, X_b and X_τ. Their RG equations are given by

$$16\pi^2 \frac{d}{dt} m_{Q_3}^2 = X_t + X_b - \frac{32}{3}g_3^2|M_3|^2 - 6g_2^2|M_2|^2 - \frac{2}{15}g_1^2|M_1|^2 \tag{10.20}$$

$$16\pi^2 \frac{d}{dt} m_{\overline{u}_3}^2 = 2X_t - \frac{32}{3}g_3^2|M_3|^2 - \frac{32}{15}g_1^2|M_1|^2 \tag{10.21}$$

$$16\pi^2 \frac{d}{dt} m_{\overline{d}_3}^2 = 2X_b - \frac{32}{3}g_3^2|M_3|^2 - \frac{8}{15}g_1^2|M_1|^2 \tag{10.22}$$

$$16\pi^2 \frac{d}{dt} m_{L_3}^2 = X_\tau - 6g_2^2|M_2|^2 - \frac{3}{5}g_1^2|M_1|^2 \tag{10.23}$$

$$16\pi^2 \frac{d}{dt} m_{\overline{e}_3}^2 = 2X_\tau - \frac{24}{5}g_1^2|M_1|^2. \tag{10.24}$$

In eqs. (10.18)-(10.24), the terms proportional to $|M_3|^2$, $|M_2|^2$ and $|M_1|^2$ are just the same ones as in eq. (10.14). Note that the terms proportional

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[7] One should think of "$m_{H_u}^2$" as a parameter unto itself, and not as the square of some mythical real number m_{H_u}. Thus there is nothing strange about having $m_{H_u}^2 < 0$. However, strictly speaking $m_{H_u}^2 < 0$ is neither necessary nor sufficient for electroweak symmetry breaking; see section 8.5.

to X_t appear with smaller numerical coefficients in the $m^2_{Q_3}$ and $m^2_{\overline{u}_3}$ RG equations than they did for the Higgs scalars, and they do not appear at all in the $m^2_{\overline{d}_3}$, $m^2_{L_3}$ and $m^2_{\overline{e}_3}$ RG equations. Furthermore, the third-family squark (mass)^2 get a large positive contribution proportional to $|M_3|^2$ from the RG evolution, which the Higgs scalars do not get. These facts make it easy to understand why the Higgs scalars in the MSSM can get VEVs, but the squarks and sleptons, having large positive (mass)^2, do not. An examination of the RG equations (10.9)-(10.12), (10.14), and (10.18)-(10.24) reveals that if the gaugino mass parameters M_1, M_2, and M_3 are non-zero at the input scale, then all of the other soft terms will be generated. This is why the "no-scale" limit with $m_{1/2} \gg m_0, A_0, B_0$ can be phenomenologically viable even though the squarks and sleptons are massless at tree-level. On the other hand, if the gaugino masses were to vanish at tree-level, then they would not get any contributions to their masses at one-loop order; in that case M_1, M_2, and M_3 would be extremely small.

Now that we have reviewed the effects of RG evolution from the input scale down to the electroweak or TeV scale, we are ready to work out the expected features of the MSSM spectrum in some detail. We will begin with the Higgs sector in the next section.

10.2 The Effective Potential

10.3 Radiative Corrections to Particle Masses

11
R-parity violation

Here is where we will discuss the phenomenology of R-parity violation, and discrete symmetries that could replace R parity.

Part 4
Advanced Topics

12
Origin of the μ term

12.1 The next-to-minimal supersymmetric standard model

The simplest possible extension of the particle content of the MSSM is to add a new gauge-singlet chiral supermultiplet. The resulting model is often called the next-to-minimal supersymmetric standard model (NMSSM).[130] The most general possible superpotential for this model is given by

$$W_{\rm NMSSM} = \frac{1}{6}kS^3 + \frac{1}{2}\mu_S S^2 + \lambda S H_u H_d + W_{\rm MSSM}, \tag{12.1}$$

where S stands for both the new chiral supermultiplet and its scalar component. (There could also be a term linear in S in $W_{\rm NMSSM}$, but this can always be removed by redefining S by a constant shift.)

One of the virtues of the NMSSM is that it can provide a solution to the μ problem mentioned in sections 8.1 and 8.5. To understand this, suppose we set $\mu_S = \mu = 0$ so that there are no mass terms or dimensionful parameters in the superpotential at all. Then an effective μ-term for $H_u H_d$ will still arise from the third term in eq. (12.1) if S gets a VEV, with $\mu = \lambda\langle S\rangle$. The absence of dimensionful terms in $W_{\rm NMSSM}$ can be enforced by introducing a new symmetry (in various different ways). The soft terms in the lagrangian give a contribution to the scalar potential which can be written as

$$V_{\rm soft}^{\rm NMSSM} = (\frac{1}{6}a_k S^3 + a_\lambda S H_u H_d + {\rm c.c.}) + m_S^2|S|^2 + V_{\rm soft}^{\rm MSSM}, \tag{12.2}$$

where a_k and a_λ have dimensions of mass. One may now set $b = 0$ in $V_{\rm soft}^{\rm MSSM}$, because an effective value for b will be generated, equal to $a_\lambda\langle S\rangle$. If the new parameters k, λ, a_k and a_λ are chosen correctly, then phenomenologically acceptable VEVs will be induced for S, H_u^0, and H_d^0. A correct treatment of this requires the inclusion of one-loop radiative corrections. But the important point is that the scale of the VEV $\langle S \rangle$, and therefore the effective value of μ, is then determined by the soft terms of order $m_{\rm soft}$, instead of being a free parameter which is conceptually independent of supersymmetry breaking.

The NMSSM contains, besides the particles of the MSSM, a real $P_R = +1$ scalar, a real $P_R = +1$ pseudoscalar, and a $P_R = -1$ Weyl fermion "singlino". These fields have no gauge couplings of their own, so they can only interact with Standard Model particles by mixing with the neutral MSSM fields with the same spin and charge. The real scalar mixes with the MSSM particles h^0 and H^0, and the pseudo-scalar mixes with A^0. One of the effects of replacing the μ term by the dynamical field S is to raise the upper bound on the lightest Higgs mass, for a given set of the other parameters in the theory. However, the bound in eq. (8.40) is still respected in the NMSSM (and any other perturbative extension of the MSSM), provided only that the sparticles that contribute in loops to the Higgs mass are lighter than 1 TeV or so. The odd R-parity singlino mixes with the four MSSM neutralinos, so there are really five neutralinos now. In many regions of parameter space, mixing effects involving the singlet fields are small, and they essentially just decouple. In that case, the phenomenology of the NMSSM is nearly indistinguishable from that of the MSSM. However, if any of the five NMSSM neutralinos (and especially the LSP) has a large mixing between the singlino and the usual gauginos and higgsinos, then the signatures for sparticles can be altered in important ways.[131]

12.2 Non-renormalizable operators and the μ term

Here we put discussion of the Giudice-Masiero mechanism, and models in which μ arises from non-renormalizable terms in the superpotential.

Part 5
The Appendices

Appendix A
Notation and Conventions

In this appendix, we collect our conventions for spacetime and spinor notations.

A.1 Matrix notation and the summation convention

If A is a matrix, then

$$\begin{aligned}
&A^T \text{ is the transpose of } A\,,\\
&A^* \text{ is the complex conjugate of } A\,,\\
&A^\dagger \text{ is the hermitian conjugate of } A\,,\\
&I_n \text{ is the } n\times n \text{ identity matrix}\,.
\end{aligned}$$

The summation convention can be illustrated by the rule for matrix multiplication: $(AB)_{ij} = A_{ik}B_{kj}$, where the sum over k is implicit. That is, the summation convention states that all repeated *dummy* indices are summed over. Note that a diagonal matrix can be written as

$$A_{ij} = a_i\delta_{ij} = \mathrm{diag}(\mathrm{a}_1, \mathrm{a}_2, \ldots)\,, \tag{A.1}$$

where the Kronecker delta $\delta_{ij} = 1$ if $i = j$, and otherwise is equal to 0. Since i and j are held fixed and are therefore not dummy indices, no sum over the repeated index i is implied.

A.2 Complex conjugation and the flavor index

In theories with a collection of complex fields, $\zeta_i(x)$, the flavor index i labels the individual field components. It is convenient to designate the complex conjugated fields $[\zeta_i(x)]^*$ by raising the flavor index i and omitting the complex conjugation symbol (the superscript asterisk):

$$\zeta^i(x) \equiv [\zeta_i(x)]^*\,. \tag{A.2}$$

When applied to two-component fermion fields, the $(\frac{1}{2},0)$ fermion fields, $\xi_{\alpha i}$, always have lowered flavor indices and the $(0,\frac{1}{2})$ fermion fields, $\bar{\eta}^{\dot{\alpha} i}$, always have raised indices.

This convention can be generalized to objects that possess more than one flavor index. If M^{ij} are the matrix elements of a matrix M, then we can treat the multiplet ζ_i as a vector $\boldsymbol{\zeta}$ and consider the following quantity

$$\boldsymbol{\zeta}^T M \boldsymbol{\zeta} \equiv M^{ij}\zeta_i\zeta_j \,, \tag{A.3}$$

where the summation convention has been employed such that two repeated flavor indices are summed over by contracting raised indices with lowered indices (or vice versa). In this case,[1]

$$[\boldsymbol{\zeta}^T M \boldsymbol{\zeta}]^* \equiv M_{ij}\zeta^i\zeta^j \,, \tag{A.4}$$

where we have used eq. (A.2) and have introduced:

$$M_{ij} \equiv (M^{ij})^* \,. \tag{A.5}$$

By the same convention, given a matrix of the form $G_i{}^j$, its complex conjugate would be defined by:

$$G^i{}_j \equiv (G_i{}^j)^* \,. \tag{A.6}$$

With this notation, an hermitian matrix $G_i{}^j$ satisfies the condition $G_i{}^j = G^j{}_i$.

Quantities with more than two indices are similarly treated. For example, the Yukawa couplings arise in the interactions of a scalar boson ϕ_I with pairs of two-component fermions $\psi_j\psi_k$. The complex conjugate of the Yukawa couplings y^{Ijk} are then given by $y_{Ijk} \equiv (y^{Ijk})^*$.

A.3 Spacetime notation

Contravariant four-vectors are denoted by: $a^\mu \equiv (a^0\,;\,\vec{\boldsymbol{a}}) \equiv (a^0\,;\,a^i)$. Here, we use Greek indices such as μ, ν, ρ and τ to be spacetime indices that run over $0,1,2,3$, while Latin indices such as i, j, k, and ℓ are space indices that run over $1,2,3$.

The spacetime metric is taken to be

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{A.7}$$

[1] Note that if the fields $\zeta_i(x)$ commute, then the matrix M in eq. (A.4) must be a complex symmetric matrix. Indeed, this is the case for anti-commuting fermions, by virtue of eq. (B.42).

The metric can be used to lower indices and convert contravariant four-vectors into covariant four vectors: $a_\mu \equiv g_{\mu\nu} a^\mu = (a_0\,;\, a_i) = (a^0\,;\, -\vec{\boldsymbol{a}})$. Thus, $a_0 = a^0$ and $a_i = -a^i$. To convert covariant indices into contravariant indices we use the inverse metric $g^{\mu\nu} \equiv (g^{-1})_{\mu\nu}$ which of course is numerically equal to $g_{\mu\nu}$ in flat spacetime. Equivalently:

$$g^{\mu\rho} g_{\rho\nu} = g^\mu_\nu = \delta^\mu_\nu = \begin{cases} 1\,, & \mu = \nu\,, \\ 0\,, & \mu \neq \nu\,, \end{cases} \tag{A.8}$$

where δ^μ_ν is the Kronecker delta-function.

The totally anti-commuting four-dimensional Levi-Cevita tensor $\epsilon^{\mu\nu\rho\tau}$ is equal to:

$$\epsilon^{\mu\nu\rho\tau} = \begin{cases} +1\,, & \mu\nu\rho\tau \text{ an even permutation of } 0123\,, \\ -1\,, & \mu\nu\rho\tau \text{ an odd permutation of } 0123\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{A.9}$$

That is $\epsilon^{0123} = +1$. Lowering the four indices to get $\epsilon_{\mu\nu\rho\tau}$, we see that $\epsilon_{0123} = -1$ One can also define the three-dimensional Levi-Cevita tensor

$$\epsilon^{ijk} \equiv \epsilon^{0ijk}\,. \tag{A.10}$$

Thus, $\epsilon^{123} = +1$. When the corresponding indices are lowered, one finds $\epsilon_{123} = -1$.

A.4 Two-component spinor notation

As discussed in Chapter 1, two-component spinors come in two varieties: undotted and dotted. The undotted spinor is denoted by χ_α and the dotted spinor is denoted by $\overline{\chi}^{\dot\alpha}$. We use Greek letters such as α, β and γ to denote undotted spinor indices and $\dot\alpha$, $\dot\beta$ and $\dot\gamma$ to denote dotted spinor indices. Both types of indices take on one of two values: 1 or 2. Undotted [dotted] spinor indices can be raised and lowered with a two dimensional undotted [dotted] ϵ-tensors:

$$\chi_\alpha = \epsilon_{\alpha\beta}\chi^\beta\,, \qquad \chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta\,, \qquad \bar\chi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\chi^{\dot\beta}\,, \qquad \bar\chi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\chi_{\dot\beta}\,. \tag{A.11}$$

Explicitly, the undotted ϵ-tensor is given by

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \qquad \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,, \tag{A.12}$$

so that $\epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12} = 1$, and

$$\epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \qquad \epsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,, \tag{A.13}$$

so that $\epsilon^{1\dot{2}} = -\epsilon^{\dot{2}1} = \epsilon_{\dot{2}1} = -\epsilon_{1\dot{2}} = 1$. That is, $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$ are numerically equivalent.

The ϵ-tensors satisfy:

$$\epsilon_{\alpha\beta}\epsilon^{\rho\tau} = -\delta_\alpha{}^\rho\delta_\beta{}^\tau + \delta_\alpha{}^\tau\delta_\beta{}^\rho\,, \qquad \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\rho}\dot{\tau}} = -\delta_{\dot{\alpha}}{}^{\dot{\rho}}\delta_{\dot{\beta}}{}^{\dot{\tau}} + \delta_{\dot{\alpha}}{}^{\dot{\tau}}\delta_{\dot{\beta}}{}^{\dot{\rho}}\,, \tag{A.14}$$

from which it follows that:

$$\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha{}^\gamma\,, \quad \epsilon^{\gamma\beta}\epsilon_{\beta\alpha} = \delta^\gamma{}_\alpha\,, \quad \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}} = \delta_{\dot{\alpha}}{}^{\dot{\gamma}}\,, \quad \epsilon^{\dot{\gamma}\dot{\beta}}\epsilon_{\dot{\beta}\dot{\alpha}} = \delta^{\dot{\gamma}}{}_{\dot{\alpha}} \tag{A.15}$$

A compendium of useful relations and identities in two-component notation can be found in Appendix B.

We next introduce the sigma-matrices $\sigma^\mu_{\alpha\dot{\beta}}$ and $\overline{\sigma}^{\dot{\alpha}\beta}_\mu$, which are defined by

$$\sigma^0 = \overline{\sigma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^1 = -\overline{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma^2 = -\overline{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = -\overline{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.16}$$

The relations between σ^μ and $\overline{\sigma}^\mu$ are

$$\sigma^\mu_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\overline{\sigma}^{\mu\,\dot{\beta}\beta}\,, \qquad \overline{\sigma}^{\mu\,\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\beta\dot{\beta}}\,, \tag{A.17}$$

$$\epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\overline{\sigma}^{\mu\dot{\beta}\alpha}\,, \qquad \epsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\alpha\dot{\beta}} = \epsilon_{\alpha\beta}\overline{\sigma}^{\mu\dot{\alpha}\beta}\,. \tag{A.18}$$

From the sigma-matrices, one can construct:[2]

$$\sigma^{\mu\nu}{}_\alpha{}^\beta = \frac{i}{4}(\sigma^\mu_{\alpha\dot{\rho}}\overline{\sigma}^{\nu\dot{\rho}\beta} - \sigma^\nu_{\alpha\dot{\rho}}\overline{\sigma}^{\mu\dot{\rho}\beta})\,, \tag{A.19}$$

$$\overline{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\beta}} = \frac{i}{4}(\overline{\sigma}^{\mu\dot{\alpha}\rho}\sigma^\nu_{\rho\dot{\beta}} - \overline{\sigma}^{\nu\dot{\alpha}\rho}\sigma^\mu_{\rho\dot{\beta}})\,. \tag{A.20}$$

These matrices satisfy self-duality relations:

$$\sigma^{\mu\nu} = -\tfrac{1}{2}i\epsilon^{\mu\nu\rho\kappa}\sigma_{\rho\kappa}\,, \qquad \overline{\sigma}^{\mu\nu} = \tfrac{1}{2}i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_{\rho\kappa}\,. \tag{A.21}$$

In addition, eq. (A.14) implies that

$$\epsilon_{\beta\rho}\epsilon^{\alpha\tau}\sigma^{\mu\nu}{}_\alpha{}^\beta = \sigma^{\mu\nu}{}_\rho{}^\tau\,, \qquad \epsilon^{\dot{\beta}\dot{\rho}}\epsilon_{\dot{\alpha}\dot{\tau}}\overline{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\beta}} = \overline{\sigma}^{\mu\nu\dot{\rho}}{}_{\dot{\tau}}\,, \tag{A.22}$$

$$\epsilon^{\alpha\tau}\sigma^{\mu\nu}{}_\alpha{}^\beta = \epsilon^{\rho\beta}\sigma^{\mu\nu}{}_\rho{}^\tau\,, \qquad \epsilon_{\dot{\alpha}\dot{\tau}}\overline{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\beta}} = \epsilon_{\dot{\rho}\dot{\beta}}\overline{\sigma}^{\mu\nu\dot{\rho}}{}_{\dot{\tau}}\,, \tag{A.23}$$

where we have used $\mathrm{Tr}(\sigma^{\mu\nu}) = \mathrm{Tr}(\overline{\sigma}^{\mu\nu}) = 0$.

[2] Some authors define $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$ without the factor of i.

A.5 Four-component spinors and the Dirac matrices

We begin by introducing the Dirac gamma matrices which are 4×4 matrices that satisfy:

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \,. \tag{A.24}$$

We also define the matrices:

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \,, \qquad \Sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu] \equiv \frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) \,, \tag{A.25}$$

and note the following relation:

$$\gamma_5\Sigma^{\mu\nu} = \tfrac{1}{2}i\epsilon^{\mu\nu\rho\kappa}\Sigma_{\rho\kappa} \,. \tag{A.26}$$

Finally, we introduce the chiral projection operators:

$$P_L \equiv \tfrac{1}{2}(1 - \gamma_5) \,, \qquad P_R \equiv \tfrac{1}{2}(1 + \gamma_5) \,. \tag{A.27}$$

Any 4×4 matrix can be expressed as a complex linear combination of sixteen linearly independent 4×4 matrices. A convenient basis to use consists of the following sixteen matrices: I_4, γ_5, γ^μ, $\gamma^\mu\gamma_5$ and $\Sigma^{\mu\nu}$.

Dirac matrices act on a four-component Dirac spinor field, $\Psi(x)$. Given a four-component spinor $\Psi(x)$, we define the Dirac adjoint field $\overline{\Psi}$, the space-reflected field Ψ^p, the time-reversed field Ψ^t and the charge conjugate field Ψ^c as follows:

$$\overline{\Psi}(x) \equiv \psi^\dagger A \,, \tag{A.28}$$

$$\Psi^p(x) \equiv i\gamma^0\Psi(x) \,, \tag{A.29}$$

$$\Psi^t(x) \equiv -\gamma^0 B^{-1}\overline{\Psi}^T(x) = -\gamma^0 B^{-1} A^T \Psi^*(x) \,, \tag{A.30}$$

$$\Psi^c(x) \equiv C\overline{\Psi}^T(x) = CA^T\Psi^*(x) \,, \tag{A.31}$$

where the Dirac conjugation matrix A, the time-reversal matrix B and the charge conjugation matrix C satisfy [1]:

$$A\gamma^\mu A^{-1} = \gamma^{\mu\dagger} \,, \tag{A.32}$$

$$B\gamma^\mu B^{-1} = \gamma^{\mu T} \,, \tag{A.33}$$

$$C^{-1}\gamma^\mu C = -\gamma^{\mu T} \,. \tag{A.34}$$

From these definitions and the defining properties of the gamma matrices [eq. (A.24)], one can prove that B and C are antisymmetric matrices in all representations of the gamma matrices. However, eqs. (A.32)–(A.34) do not fix the overall scale of the matrices A, B and C. Thus, there is some freedom in the definition of these matrices (independent of the chosen gamma matrix representation). We shall remove some of this freedom by imposing the following additional relations:[3]

$$(\Psi^c)^c = \Psi\,, \qquad (\Psi^t)^t = -\Psi\,. \tag{A.35}$$

After imposing eq. (A.35), one can prove that the following results are satisfied in all gamma matrix representations:

$$BA^{-1} = -(AB^{-1})^*\,, \qquad (AC)^{-1} = (AC)^*\,, \tag{A.36}$$

$$CB = -\gamma_5\,, \qquad A^\dagger A^{-1} = e^{i\theta_A} I_4\,, \tag{A.37}$$

where the value of θ_A is conventional. The standard choice, $\theta_A = 0$, corresponds to the requirement that $\overline{\Psi}(x)\Psi(x)$ is an hermitian quantity in all gamma matrix representations. In this convention, $A^\dagger = A$.

In Chapter 1, section 3.1, we introduced the chiral (or high-energy) representation for the gamma matrices, which is defined by:

$$\gamma_C^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}\,, \qquad \gamma_C^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\,. \tag{A.38}$$

This representation follows naturally from the two-component form of the Dirac Lagrangian. But, given any representation of the gamma matrices, γ^μ [and corresponding four-component Dirac spinor $\Psi(x)$], a new representation $\widetilde{\gamma}^\mu \equiv \mathcal{S}\gamma^\mu\mathcal{S}^{-1}$ also satisfies eq. (A.24), where $\mathcal{S}$ is any non-singular matrix. The Dirac spinor with respect to the new representation is given by $\widetilde{\Psi}(x) = \mathcal{S}\Psi(x)$. The matrices A, B and C are defined according to eqs. (A.32)–(A.34) with respect to the gamma matrices appropriate to the given representation. In addition, we impose eq. (A.35) and $\widetilde{A}^\dagger = \widetilde{A}$. This is still not quite sufficient to fix $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ uniquely, and we find

$$\widetilde{A} = \pm|z|(\mathcal{S}^{-1})^\dagger A\mathcal{S}^{-1}\,, \quad \widetilde{B} = z(\mathcal{S}^{-1})^T B\mathcal{S}^{-1}\,, \quad \widetilde{C} = z^{-1}\mathcal{S}C\mathcal{S}^T\,, \tag{A.39}$$

where z is an arbitrary complex number.[4] Note that there is only a phase ambiguity in the definition of $\widetilde{C}\widetilde{A}^T$. In particular,

$$\widetilde{C}\widetilde{A}^T = e^{i\zeta}\mathcal{S}CA^T\mathcal{S}^{*-1}\,. \tag{A.40}$$

[3] The condition $(\Psi^p)^p = -\Psi$ is automatically satisfied and provides no further constraint.

[4] Most modern textbooks impose one additional constraint by restricting to a class of γ-matrices for which $\gamma^{0\dagger} = \gamma^0$ and $\gamma^{i\dagger} = -\gamma^i$. All three representations discussed in this section satisfy this requirement. Representations in this class are related by a unitary transformation, $\widetilde{\gamma}^\mu = \mathcal{S}\gamma^\mu\mathcal{S}^{-1}$, with $\mathcal{S}^\dagger\mathcal{S} = I_4$. One can then show that it is possible to choose A, B and C to be unitary matrices in all the gamma matrix representations of this class. Moreover, in this convention, $|z| = 1$, so that in any gamma matrix representation of this class one is left with a sign ambiguity in the definition of A and a phase ambiguity in the definitions of B and C.

Another common representation for the gamma matrices is the Dirac (or low-energy) representation:

$$\gamma_D^\mu = \mathcal{S}\gamma_C^\mu \mathcal{S}^{-1}\,, \qquad \mathcal{S} = \frac{1}{\sqrt{2}}\begin{pmatrix} I_2 & I_2 \\ -I_2 & I_2 \end{pmatrix}\,. \tag{A.41}$$

Note that in this case, $\mathcal{S}^{-1} = \mathcal{S}^T = \mathcal{S}^\dagger$. As a result, the representations of A, B and C in terms of gamma matrices is the same for both the chiral and the Dirac representations. A convenient choice for A, B and C in terms of the gamma matrices is given by:

$$A = \gamma^0\,, \qquad B = \gamma^1\gamma^3\,, \qquad C = i\gamma^0\gamma^2\,. \tag{A.42}$$

Finally, we mention one other representation for the Dirac matrices in which all four gamma matrices are purely imaginary, *i.e.*, $\gamma^{\mu *} = -\gamma^\mu$. This is the Majorana representation:

$$\gamma_M^\mu = \mathcal{S}\gamma_D^\mu \mathcal{S}^{-1}\,, \qquad \mathcal{S} = \frac{1}{\sqrt{2}}\begin{pmatrix} I_2 & \sigma^2 \\ \sigma^2 & -I_2 \end{pmatrix}\,. \tag{A.43}$$

Here, one finds that $\mathcal{S} = \mathcal{S}^{-1} = \mathcal{S}^\dagger$. A convenient choice for A, B and C in terms of the gamma matrices is given by:

$$A = \gamma^0\,, \qquad B = \gamma^0\gamma_5\,, \qquad C = -\gamma^0\,. \tag{A.44}$$

In particular, in this convention $CA^T = I_4$, corresponding to the choice of $e^{i\zeta} = -i$ in eq. (A.40). That is, in the Majorana representation with the conventions above,

$$\Psi^c = \Psi^*\,. \tag{A.45}$$

Hence, the four-component self-conjugate Majorana fermion (which satisfies $\Psi^c = \Psi = \Psi^*$) is a real field in the Majorana representation.

Problems

1. (a) Show that if Q is a 4×4 matrix that commutes with all the γ^μ ($\mu = 0,\, 1,\, 2,\, 3$) and R is a 4×4 matrix that anticommutes with all the γ^μ, then $Q = c_1 I_4$ and $R = c_2\gamma_5$, for some complex numbers c_1 and c_2.

(b) From the defining relations for the matrices A, B and C [eqs. (A.32)–(A.34)], show that the matrices $A^{-1}A^\dagger$, $B^{-1}B^T$, $C^{-1}C^T$, $(BA^{-1})(BA^{-1})^*$ and $AC(AC)^*$ all commute with the γ^μ and the matrix CB anticommutes with the γ^μ.

(c) Using parts (a) and (b), show that $B^T = -B$ and $C^T = -C$ in all gamma matrix representations.

(d) By imposing $(\Psi^c)^c = \Psi$ and $(\Psi^t)^t = -\Psi$, derive eq. (A.36). Finally, impose the condition $\Psi = A^{-1}\overline{\Psi}^\dagger$ and conclude that $A^\dagger = A$. Verify that the choice of A, B and C is still not unique, as specified in eq. (A.39).

2. Starting with the chiral representation of the gamma matrices, transform to the Dirac representation while keeping track of the two-component spinor indices. Show that in the Dirac representation, the four-component spinor is given by

$$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix} \xi_\alpha + \sigma^0_{\alpha\dot\beta}\bar\eta^{\dot\beta} \\ \bar\eta^{\dot\alpha} - \overline{\sigma}^{0\dot\alpha\beta}\xi_\beta \end{pmatrix}, \tag{A.46}$$

and the gamma matrices are given by:

$$\gamma^0 = \begin{pmatrix} \delta_\alpha{}^\beta & 0 \\ 0 & -\delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i_{\alpha\dot\beta} \\ \overline{\sigma}^{i\dot\alpha\beta} & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & \sigma^0_{\alpha\dot\beta} \\ \overline{\sigma}^{0\dot\alpha\beta} & 0 \end{pmatrix} \tag{A.47}$$

Evaluate the matrices A, B and C in the Dirac representation and show that they are given by:

$$A = \begin{pmatrix} \overline{\sigma}^{0\dot\alpha\beta} & 0 \\ 0 & -\sigma^0_{\alpha\dot\beta} \end{pmatrix}, \quad B = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & -\epsilon_{\dot\alpha\dot\beta} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -\epsilon_{\alpha\gamma}\overline{\sigma}^{0\dot\beta\gamma} \\ \epsilon^{\dot\alpha\dot\gamma}\sigma^0_{\beta\dot\gamma} & 0 \end{pmatrix} \tag{A.48}$$

3. Starting with the Dirac representation of the gamma matrices (see problem 2), transform to the Majorana representation while keeping track of the two-component spinor indices. Show that in the Majorana representation, the four component spinor is given by

$$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix} \xi_\alpha + \sigma^2_{\alpha\dot\beta}\bar\eta^{\dot\beta} \\ -\bar\eta^{\dot\alpha} - \overline{\sigma}^{2\dot\alpha\beta}\xi_\beta \end{pmatrix}, \tag{A.49}$$

and the gamma matrices are given by:

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ -\overline{\sigma}^2 & 0 \end{pmatrix}, \quad \gamma^1 = 2i \begin{pmatrix} \sigma^{12} & 0 \\ 0 & \overline{\sigma}^{12} \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ -\overline{\sigma}^2 & 0 \end{pmatrix},$$

$$\gamma^3 = -2i \begin{pmatrix} \sigma^{23} & 0 \\ 0 & \overline{\sigma}^{23} \end{pmatrix}, \quad \gamma_5 = 2i \begin{pmatrix} \sigma^{02} & 0 \\ 0 & \overline{\sigma}^{02} \end{pmatrix}, \tag{A.50}$$

where the spinor labels in eq. (A.50) have been suppressed.

Evaluate the matrices A, B and C in the Majorana representation and show that they are given by:

$$A = -2i \begin{pmatrix} 0 & \overline{\sigma}^{02\dot{\alpha}}{}_{\dot{\beta}} \\ -\sigma^{02}{}_{\alpha}{}^{\beta} & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \epsilon^{\alpha\gamma}\sigma^2_{\gamma\dot{\beta}} \\ -\epsilon_{\dot{\alpha}\dot{\gamma}}\overline{\sigma}^{0\dot{\gamma}\beta} & 0 \end{pmatrix},$$

$$C = -i \begin{pmatrix} 0 & \epsilon_{\alpha\gamma}\overline{\sigma}^{0\dot{\beta}\gamma} \\ -\epsilon^{\dot{\alpha}\dot{\gamma}}\sigma^0_{\beta\dot{\gamma}} & 0 \end{pmatrix}. \tag{A.51}$$

The factor of $-i$ in C is conventional and has been chosen so that the numerical value of CA^T is equal to the identity matrix. Verify this last result (and display the correct spinor label structure).

Appendix B

Compendium of Useful Relations for Two-Component Notation

B.1 Sigma matrices and associated identities

Various sigma matrices—σ^μ, $\overline{\sigma}^\mu$, $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$— were defined in eqs. (A.16), (A.19) and (A.20). The following identities involving the sigma-matrices are useful.

$$\sigma^\mu_{\alpha\dot{\alpha}}\overline{\sigma}^{\dot{\beta}\beta}_\mu = 2\delta_\alpha{}^\beta\delta^{\dot{\beta}}{}_{\dot{\alpha}} \tag{B.1}$$
$$\sigma^\mu_{\alpha\dot{\alpha}}\sigma_{\mu\beta\dot{\beta}} = 2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} \tag{B.2}$$
$$\overline{\sigma}^{\mu\dot{\alpha}\alpha}\overline{\sigma}^{\dot{\beta}\beta}_\mu = 2\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} \tag{B.3}$$
$$[\sigma^\mu\overline{\sigma}_\mu]_\alpha{}^\beta = 4\delta_\alpha{}^\beta \tag{B.4}$$
$$[\overline{\sigma}^\mu\sigma_\mu]^{\dot{\alpha}}{}_{\dot{\beta}} = 4\delta^{\dot{\alpha}}{}_{\dot{\beta}} \tag{B.5}$$
$$[\sigma^\mu\overline{\sigma}^\nu + \sigma^\nu\overline{\sigma}^\mu]_\alpha{}^\beta = 2g^{\mu\nu}\delta_\alpha{}^\beta \tag{B.6}$$
$$[\overline{\sigma}^\mu\sigma^\nu + \overline{\sigma}^\nu\sigma^\mu]^{\dot{\alpha}}{}_{\dot{\beta}} = 2g^{\mu\nu}\delta^{\dot{\alpha}}{}_{\dot{\beta}} \tag{B.7}$$
$$[\sigma^\mu\overline{\sigma}^\nu\sigma^\rho\overline{\sigma}_\mu]_\alpha{}^\beta = 4g^{\nu\rho}\delta_\alpha{}^\beta \tag{B.8}$$
$$[\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho\sigma_\mu]^{\dot{\alpha}}{}_{\dot{\beta}} = 4g^{\nu\rho}\delta^{\dot{\alpha}}{}_{\dot{\beta}} \tag{B.9}$$
$$(\sigma^\mu\overline{\sigma}^\nu)_\alpha{}^\beta = -2i(\sigma^{\mu\nu})_\alpha{}^\beta + g^{\mu\nu}\delta_\alpha{}^\beta \tag{B.10}$$
$$(\overline{\sigma}^\mu\sigma^\nu)^{\dot{\alpha}}{}_{\dot{\beta}} = -2i(\overline{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} + g^{\mu\nu}\delta^{\dot{\alpha}}{}_{\dot{\beta}} \tag{B.11}$$
$$(\sigma^{\mu\nu})_\alpha{}^\beta(\sigma_{\mu\nu})_\rho{}^\kappa = 2\delta_\alpha{}^\kappa\delta_\rho{}^\beta - \delta_\alpha{}^\beta\delta_\rho{}^\kappa \tag{B.12}$$
$$(\overline{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}(\overline{\sigma}_{\mu\nu})^{\dot{\rho}}{}_{\dot{\kappa}} = 2\delta^{\dot{\alpha}}{}_{\dot{\kappa}}\delta^{\dot{\rho}}{}_{\dot{\beta}} - \delta^{\dot{\alpha}}{}_{\dot{\beta}}\delta^{\dot{\rho}}{}_{\dot{\kappa}} \tag{B.13}$$
$$(\sigma^{\mu\nu})_\alpha{}^\beta(\overline{\sigma}_{\mu\nu})^{\dot{\rho}}{}_{\dot{\kappa}} = 0 \tag{B.14}$$
$$[\sigma^{\mu\nu}\sigma_{\mu\nu}]_\alpha{}^\beta = 3\delta_\alpha{}^\beta \tag{B.15}$$
$$[\overline{\sigma}^{\mu\nu}\overline{\sigma}_{\mu\nu}]^{\dot{\alpha}}{}_{\dot{\beta}} = 3\delta^{\dot{\alpha}}{}_{\dot{\beta}}\,. \tag{B.16}$$

Eqs. (B.1), (B.12) and (B.13) are the basis for the Fierz identities, which are given in detail in Appendix A of ref. [1]. Three other useful identities of this type are:

$$\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}} = \tfrac{1}{2}\left[\sigma^\mu_{\alpha\dot{\beta}}\sigma^\nu_{\beta\dot{\alpha}} + \sigma^\mu_{\beta\dot{\alpha}}\sigma^\nu_{\alpha\dot{\beta}} - g^{\mu\nu}\sigma^\lambda_{\alpha\dot{\beta}}\sigma_{\lambda\beta\dot{\alpha}} + i\epsilon^{\mu\nu\kappa\lambda}\sigma_{\kappa\alpha\dot{\beta}}\sigma_{\lambda\beta\dot{\alpha}}\right] \tag{B.17}$$

$$\overline{\sigma}^{\mu\dot{\alpha}\alpha}\overline{\sigma}^{\nu\dot{\beta}\beta} = \tfrac{1}{2}\left[\overline{\sigma}^{\mu\dot{\alpha}\beta}\overline{\sigma}^{\nu\dot{\beta}\alpha} + \overline{\sigma}^{\mu\dot{\beta}\alpha}\overline{\sigma}^{\nu\dot{\alpha}\beta} - g^{\mu\nu}\overline{\sigma}^{\lambda\dot{\alpha}\beta}\overline{\sigma}^{\dot{\beta}\alpha}_\lambda - i\epsilon^{\mu\nu\kappa\lambda}\sigma^{\dot{\alpha}\beta}_\kappa\sigma^{\dot{\beta}\alpha}_\lambda\right] \tag{B.18}$$

$$\sigma^\mu_{\alpha\dot{\alpha}}\overline{\sigma}^{\nu\dot{\beta}\beta} = \tfrac{1}{2}g^{\mu\nu}\delta_\alpha{}^\beta\delta^{\dot{\beta}}{}_{\dot{\alpha}} - i\sigma^{\mu\nu}{}_\alpha{}^\beta\delta^{\dot{\beta}}{}_{\dot{\alpha}} + i\delta_\alpha{}^\beta\overline{\sigma}^{\mu\nu\dot{\beta}}{}_{\dot{\alpha}} + 2g_{\kappa\lambda}\sigma^{\nu\kappa}{}_\alpha{}^\beta\overline{\sigma}^{\lambda\mu\dot{\beta}}{}_{\dot{\alpha}}\,. \tag{B.19}$$

There are other identities that are quite useful, which we display here for completeness. Henceforth, we suppress the spinor indices.

$$\mathrm{Tr}[\sigma^\mu\overline{\sigma}^\nu] = \mathrm{Tr}[\overline{\sigma}^\mu\sigma^\nu] = 2g^{\mu\nu} \tag{B.20}$$

$$\mathrm{Tr}[\sigma^\mu\overline{\sigma}^\nu\sigma^\rho\overline{\sigma}^\kappa] = 2\left(g^{\mu\nu}g^{\rho\kappa} - g^{\mu\rho}g^{\nu\kappa} + g^{\mu\kappa}g^{\nu\rho} + i\epsilon^{\mu\nu\rho\kappa}\right) \tag{B.21}$$

$$\mathrm{Tr}[\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho\sigma^\kappa] = 2\left(g^{\mu\nu}g^{\rho\kappa} - g^{\mu\rho}g^{\nu\kappa} + g^{\mu\kappa}g^{\nu\rho} - i\epsilon^{\mu\nu\rho\kappa}\right) \tag{B.22}$$

$$\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}_\mu = -2\overline{\sigma}^\nu \tag{B.23}$$

$$\sigma^\mu\overline{\sigma}^\nu\sigma_\mu = -2\sigma^\nu \tag{B.24}$$

$$\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho\sigma^\kappa\overline{\sigma}_\mu = -2\overline{\sigma}^\kappa\sigma^\rho\overline{\sigma}^\nu \tag{B.25}$$

$$\sigma^\mu\overline{\sigma}^\nu\sigma^\rho\overline{\sigma}^\kappa\sigma_\mu = -2\sigma^\kappa\overline{\sigma}^\rho\sigma^\nu \tag{B.26}$$

$$\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho = g^{\mu\nu}\overline{\sigma}^\rho - g^{\mu\rho}\overline{\sigma}^\nu + g^{\nu\rho}\overline{\sigma}^\mu - i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_\kappa \tag{B.27}$$

$$\sigma^\mu\overline{\sigma}^\nu\sigma^\rho = g^{\mu\nu}\sigma^\rho - g^{\mu\rho}\sigma^\nu + g^{\nu\rho}\sigma^\mu + i\epsilon^{\mu\nu\rho\kappa}\sigma_\kappa \tag{B.28}$$

$$\sigma^{\mu\nu}\sigma^\rho = \tfrac{i}{2}(g^{\nu\rho}\sigma^\mu - g^{\mu\rho}\sigma^\nu + i\epsilon^{\mu\nu\rho\kappa}\sigma_\kappa) \tag{B.29}$$

$$\overline{\sigma}^{\mu\nu}\overline{\sigma}^\rho = \tfrac{i}{2}(g^{\nu\rho}\overline{\sigma}^\mu - g^{\mu\rho}\overline{\sigma}^\nu - i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_\kappa) \tag{B.30}$$

$$\overline{\sigma}^\mu\sigma^{\nu\rho} = \tfrac{i}{2}(g^{\mu\nu}\overline{\sigma}^\rho - g^{\mu\rho}\overline{\sigma}^\nu - i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_\kappa) \tag{B.31}$$

$$\sigma^\mu\overline{\sigma}^{\nu\rho} = \tfrac{i}{2}(g^{\mu\nu}\sigma^\rho - g^{\mu\rho}\sigma^\nu + i\epsilon^{\mu\nu\rho\kappa}\sigma_\kappa) \tag{B.32}$$

$$\begin{aligned}\sigma^{\mu\nu}\sigma^{\rho\kappa} = &-\tfrac{1}{4}(g^{\nu\rho}g^{\mu\kappa} - g^{\mu\rho}g^{\nu\kappa} + i\epsilon^{\mu\nu\rho\kappa}) \\ &+\tfrac{i}{2}(g^{\nu\rho}\sigma^{\mu\kappa} + g^{\mu\kappa}\sigma^{\nu\rho} - g^{\mu\rho}\sigma^{\nu\kappa} - g^{\nu\kappa}\sigma^{\mu\rho})\end{aligned} \tag{B.33}$$

$$\begin{aligned}\overline{\sigma}^{\mu\nu}\overline{\sigma}^{\rho\kappa} = &-\tfrac{1}{4}(g^{\nu\rho}g^{\mu\kappa} - g^{\mu\rho}g^{\nu\kappa} - i\epsilon^{\mu\nu\rho\kappa}) \\ &+\tfrac{i}{2}(g^{\nu\rho}\overline{\sigma}^{\mu\kappa} + g^{\mu\kappa}\overline{\sigma}^{\nu\rho} - g^{\mu\rho}\overline{\sigma}^{\nu\kappa} - g^{\nu\kappa}\overline{\sigma}^{\mu\rho})\,.\end{aligned} \tag{B.34}$$

Next, we consider two-component spinor identities that arise when manipulating products of two or four spinors. The heights of indices must be consistent in the sense that lowered indices must always be contracted with raised indices. Indices contracted like

$${}^\alpha{}_\alpha \qquad\qquad \text{and} \qquad\qquad {}_{\dot{\alpha}}{}^{\dot{\alpha}}\,, \tag{B.35}$$

can be suppressed. In all spinor products given in this paper, contracted indices are always to have heights that conform to eq. (1.52). For example, $\xi\eta \equiv \xi^\alpha\eta_\alpha$, $\bar{\xi}\overline{\sigma}^\mu\eta \equiv \bar{\xi}_{\dot{\alpha}}\overline{\sigma}^{\mu\dot{\alpha}\alpha}\eta_\alpha$, *etc.* The behavior of the spinor products under hermitian conjugation is as follows:

$$(\xi\eta)^\dagger = \bar{\eta}\bar{\xi} \tag{B.36}$$
$$(\xi\sigma^\mu\bar{\eta})^\dagger = \eta\sigma^\mu\bar{\xi} \tag{B.37}$$
$$(\bar{\xi}\overline{\sigma}^\mu\eta)^\dagger = \bar{\eta}\overline{\sigma}^\mu\xi \tag{B.38}$$
$$(\xi\sigma^\mu\overline{\sigma}^\nu\eta)^\dagger = \bar{\eta}\overline{\sigma}^\nu\sigma^\mu\bar{\xi} \tag{B.39}$$
$$(\xi\sigma^{\mu\nu}\eta)^\dagger = \bar{\eta}\overline{\sigma}^{\mu\nu}\bar{\xi} \tag{B.40}$$

Note that these relations are applicable both to anti-commuting and to commuting spinors.

In addition to manipulating expressions containing anti-commuting fermion fields, we often must deal with products of *commuting* spinor wave functions that arise when evaluating the Feynman rules. In the following expressions we denote the generic spinor by z_i. In the various identities listed below, an extra minus sign arises when manipulating a product of anti-commuting fermion fields. Thus, we employ the notation:

$$(-1)^A \equiv \begin{cases} +1\,, & \text{commuting spinors,} \\ -1\,, & \text{anti-commuting spinors.} \end{cases} \tag{B.41}$$

The following identities hold for the z_i:

$$z_1 z_2 = -(-1)^A z_2 z_1 \tag{B.42}$$
$$\bar{z}_1\bar{z}_2 = -(-1)^A \bar{z}_2\bar{z}_1 \tag{B.43}$$
$$z_1\sigma^\mu\bar{z}_2 = (-1)^A \bar{z}_2\overline{\sigma}^\mu z_1 \tag{B.44}$$
$$z_1\sigma^\mu\overline{\sigma}^\nu z_2 = -(-1)^A z_2\sigma^\nu\overline{\sigma}^\mu z_1 \tag{B.45}$$
$$\bar{z}_1\overline{\sigma}^\mu\sigma^\nu\bar{z}_2 = -(-1)^A \bar{z}_2\overline{\sigma}^\nu\sigma^\mu\bar{z}_1 \tag{B.46}$$
$$\bar{z}_1\overline{\sigma}^\mu\sigma^\rho\overline{\sigma}^\nu z_2 = (-1)^A z_2\sigma^\nu\overline{\sigma}^\rho\sigma^\mu\bar{z}_1 \tag{B.47}$$
$$z_1\sigma^{\mu\nu}z_2 = (-1)^A z_2\sigma^{\mu\nu}z_1 \tag{B.48}$$
$$\bar{z}_1\overline{\sigma}^{\mu\nu}\bar{z}_2 = (-1)^A \bar{z}_2\sigma^{\mu\nu}\bar{z}_1 \tag{B.49}$$

As previously noted, eqs. (B.1), (B.12) and (B.13) can be used to derive a series of Fierz identities for two-component spinor products. For example,

$$\tfrac{1}{2}(z_1\sigma^\mu\bar{z}_2)(\bar{z}_3\overline{\sigma}_\mu z_4) = (-1)^A(z_1z_4)(\bar{z}_3\bar{z}_2) \tag{B.50}$$
$$\tfrac{1}{2}(\bar{z}_1\overline{\sigma}^\mu z_2)(\bar{z}_3\overline{\sigma}_\mu z_4) = (-1)^A(\bar{z}_1\bar{z}_3)(z_4z_2) \tag{B.51}$$
$$\tfrac{1}{2}(z_1\sigma^\mu\bar{z}_2)(z_3\sigma_\mu\bar{z}_4) = (-1)^A(z_1z_3)(\bar{z}_4\bar{z}_2) \tag{B.52}$$

Many more Fierz identities can be found in Appendix A of ref. [1].

B.2 Behavior of 2-component fermion bilinears under C, P, T

In Chapter 1, we examined the behavior of two-component fermion fields under P, C and T. With these results, one can easily compute the behavior of fermion bilinears under the discrete symmetries. First we examine fermion bilinears constructed out of two component neutral fermion fields. We summarize the fermion field transformation laws:

$$\mathcal{P}\chi_\alpha(x)\mathcal{P}^{-1} = \eta_P i\sigma^0_{\alpha\dot\beta}\overline{\chi}^{\dot\beta}(x_P)\,, \tag{B.53}$$

$$\mathcal{T}\chi_\alpha(x)\mathcal{T}^{-1} = \eta_T\sigma^0_{\beta\dot\alpha}\chi^\beta(x_T)\,, \tag{B.54}$$

$$\mathcal{C}\chi_\alpha(x)\mathcal{C}^{-1} = \eta_C\chi_\alpha(x)\,, \tag{B.55}$$

where $x_P \equiv (t\,;\,-\vec{\boldsymbol{x}})$ and $x_T \equiv (-t\,;\,\vec{\boldsymbol{x}})$.

As shown in chapter 1, the phases η_P, η_T and η_C are restricted to be real (either ± 1). From these results, one easily derives:

$$\mathcal{CP}\chi_\alpha(x)(\mathcal{CP})^{-1} = \eta_{CP} i\sigma^0_{\alpha\dot\beta}\overline{\chi}^{\dot\beta}(x_P)\,, \tag{B.56}$$

$$\mathcal{CPT}\chi_\alpha(x)(\mathcal{CPT})^{-1} = -i\eta_{CPT}\overline{\chi}_{\dot\alpha}(-x)\,, \tag{B.57}$$

where $\eta_{CP} \equiv \eta_C\eta_P$ and $\eta_{CPT} \equiv \eta_C\eta_P\eta_T$. To be consistent with the CPT theorem, we demand that η_{CPT} is the same for all the fermion species. Without loss of generality, we may take:

$$\eta_{CPT} \equiv \eta_C\eta_P\eta_T = +1\,. \tag{B.58}$$

Using the above results, the behavior of the fermion bilinears under the discrete symmetries is given in Table B.1. From the results of Table B.1, one can immediately determine the transformation properties of the fermion bilinears under CPT. These are given in Table B.2.

Next, we examine the case of a charged fermion field, which is specified by a pair of two-component fermion fields ξ and η of opposite charge. We summarize the transformation laws for ξ:

$$\mathcal{P}\xi_\alpha(x)\mathcal{P}^{-1} = \eta_P i\sigma^0_{\alpha\dot\beta}\bar\eta^{\dot\beta}(x_P)\,, \tag{B.59}$$

$$\mathcal{T}\xi_\alpha(x)\mathcal{T}^{-1} = \eta_T\sigma^0_{\beta\dot\alpha}\xi^\beta(x_T)\,, \tag{B.60}$$

$$\mathcal{C}\xi_\alpha(x)\mathcal{C}^{-1} = \eta_C\eta_\alpha(x)\,, \tag{B.61}$$

with no restriction initially on the phases η_P, η_T and η_C. The corresponding transformation laws for η_α are obtained from eqs. (B.59)–(B.61) by interchanging $\xi \longleftrightarrow \eta$ and complex conjugating the phases η_P, η_T and η_C. However, to be consistent with the CPT theorem, we must again demand that η_{CPT} is independent of the fermion species.

Table B.1. Transformation properties of 2-component fermion bilinears under the discrete symmetries. The phases η_P, η_C and η_C are either $+1$ or -1. The following notation is employed: $\Lambda_P \equiv \mathrm{diag}(1,-1,-1,-1)$, $\Lambda_T \equiv \mathrm{diag}(-1,1,1,1)$, $x=(t\,;\,\boldsymbol{\vec{x}})$, $x_P=(t\,;\,-\boldsymbol{\vec{x}})$ and $x_T=(-t\,;\,\boldsymbol{\vec{x}})$.

	P	**T**
$\chi_1\chi_2(x)$	$\eta_{P1}\eta_{P2}\bar\chi_1\bar\chi_2(x_P)$	$\eta_{T1}\eta_{T2}\chi_1\chi_2(x_T)$
$\bar\chi_1\bar\chi_2(x)$	$\eta_{P1}\eta_{P2}\chi_1\chi_2(x_P)$	$\eta_{T1}\eta_{T2}\bar\chi_1\bar\chi_2(x_T)$
$\bar\chi_1\overline{\sigma}^\mu\chi_2(x)$	$\eta_{P1}\eta_{P2}(\Lambda_P)^\mu{}_\nu\chi_1\sigma^\nu\bar\chi_2(x_P)$	$-\eta_{T1}\eta_{T2}(\Lambda_T)^\mu{}_\nu\bar\chi_1\overline{\sigma}^\nu\chi_2(x_T)$
$\chi_1\sigma^\mu\bar\chi_2(x)$	$\eta_{P1}\eta_{P2}(\Lambda_P)^\mu{}_\nu\bar\chi_1\overline{\sigma}^\nu\chi_2(x_P)$	$-\eta_{T1}\eta_{T2}(\Lambda_T)^\mu{}_\nu\chi_1\sigma^\nu\bar\chi_2(x_T)$
$\chi_1\sigma^{\mu\nu}\chi_2(x)$	$\eta_{P1}\eta_{P2}(\Lambda_P)^\mu{}_\rho(\Lambda_P)^\nu{}_\tau\bar\chi_1\overline{\sigma}^{\rho\tau}\bar\chi_2(x_P)$	$-\eta_{T1}\eta_{T2}(\Lambda_T)^\mu{}_\rho(\Lambda_T)^\nu{}_\tau\chi_1\sigma^{\rho\tau}\chi_2(x_T)$
$\bar\chi_1\overline{\sigma}^{\mu\nu}\bar\chi_2(x)$	$\eta_{P1}\eta_{P2}(\Lambda_P)^\mu{}_\rho(\Lambda_P)^\nu{}_\tau\chi_1\sigma^{\rho\tau}\chi_2(x_P)$	$-\eta_{T1}\eta_{T2}(\Lambda_T)^\mu{}_\rho(\Lambda_T)^\nu{}_\tau\bar\chi_1\overline{\sigma}^{\rho\tau}\bar\chi_2(x_T)$

	C	**CP**
$\chi_1\chi_2(x)$	$\eta_{C1}\eta_{C2}\chi_1\chi_2(x)$	$\eta_{CP1}\eta_{CP2}\bar\chi_1\bar\chi_2(x_P)$
$\bar\chi_1\bar\chi_2(x)$	$\eta_{C1}\eta_{C2}\bar\chi_1\bar\chi_2(x)$	$\eta_{CP1}\eta_{CP2}\chi_1\chi_2(x_P)$
$\bar\chi_1\overline{\sigma}^\mu\chi_2(x)$	$\eta_{C1}\eta_{C2}\bar\chi_1\overline{\sigma}^\mu\chi_2(x)$	$\eta_{CP1}\eta_{CP2}(\Lambda_P)^\mu{}_\nu\chi_1\sigma^\nu\bar\chi_2(x_P)$
$\chi_1\sigma^\mu\bar\chi_2(x)$	$\eta_{C1}\eta_{C2}\chi_1\sigma^\mu\bar\chi_2(x)$	$\eta_{CP1}\eta_{CP2}(\Lambda_P)^\mu{}_\nu\bar\chi_1\overline{\sigma}^\nu\chi_2(x_P)$
$\chi_1\sigma^{\mu\nu}\chi_2(x)$	$\eta_{C1}\eta_{C2}\chi_1\sigma^{\mu\nu}\chi_2(x)$	$\eta_{CP1}\eta_{CP2}(\Lambda_P)^\mu{}_\rho(\Lambda_P)^\nu{}_\tau\bar\chi_1\overline{\sigma}^{\rho\tau}\bar\chi_2(x_P)$
$\bar\chi_1\overline{\sigma}^{\mu\nu}\bar\chi_2(x)$	$\eta_{C1}\eta_{C2}\bar\chi_1\overline{\sigma}^{\mu\nu}\bar\chi_2(x)$	$\eta_{CP1}\eta_{CP2}(\Lambda_P)^\mu{}_\rho(\Lambda_P)^\nu{}_\tau\chi_1\sigma^{\rho\tau}\chi_2(x_P)$

Table B.2. Transformation properties of 2-component fermion bilinears under CPT. The notation used below is given in the caption to Table B.1. The phase η_{CPT} is taken to be independent of the particle species as required by the CPT theorem.

	CPT
$\chi_1\chi_2(x)$	$\bar{\chi}_1\bar{\chi}_2(-x)$
$\bar{\chi}_1\bar{\chi}_2(x)$	$\chi_1\chi_2(-x)$
$\bar{\chi}_1\overline{\sigma}^\mu\chi_2(x)$	$\chi_1\sigma^\mu\bar{\chi}_2(-x)$
$\chi_1\sigma^\mu\bar{\chi}_2(x)$	$\overline{\chi}_1\overline{\sigma}^\mu\chi_2(-x)$
$\chi_1\sigma^{\mu\nu}\chi_2(x)$	$-\bar{\chi}_1\overline{\sigma}^{\mu\nu}\overline{\chi}_2(-x)$
$\bar{\chi}_1\overline{\sigma}^{\mu\nu}\overline{\chi}_2(x)$	$-\chi_1\sigma^{\mu\nu}\chi_2(-x)$

Without loss of generality, one can impose the result given in eq. (B.58). Although no further restrictions on the phases is required, it is always possible to rotate η_P and η_C by an arbitrary phase angle by redefining the parity and charge conjugation operators (by multiplying by an appropriate gauge transformation). Thus, one is free to establish a convention in which all phases are real (either ± 1), as in the case of the neutral self-conjugate fermion. In this convention, $CP = PC$, $TC = CT$ and $TP = -PT$.

Using the above results, the behavior of the charged fermion bilinears are easily determined. Here, we find it useful to list the behavior of certain linear combinations of fermion bilinears, denoted generically by $B_{12}(x)$, under P, C and T transformations. The indices 1 and 2 refer to two different charged fermions. In order to be general, we have not imposed any reality conditions on the phases η_P, η_T and η_C. The results are given in Table B.3. From the results of Table B.3, one can immediately determine the transformation properties of the corresponding fermion bilinears under CPT. These are given in Table B.4. Note that by using the translation from two-component to four-component notation given in eqs. (3.27) and (3.32), the results of Tables B.3 and B.4 reproduce the well known behavior of the Dirac bilinear covariants under P, T and C transformations.[1]

[1] To verify this assertion, recall that T and CPT are anti-unitary operators. Thus, any factor of i that appears in the bilinear covariant will change sign under these anti-unitary transformations.

Table B.3. Transformation properties of various 2-component charged fermion bilinears, $B_{12}(x)$ under the discrete symmetries. The phases η_P, η_C and η_C are arbitrary. The following notation is employed: $\Lambda_P \equiv \mathrm{diag}(1,-1,-1,-1)$, $\Lambda_T \equiv \mathrm{diag}(-1,1,1,1)$, $x=(t\,;\,\vec{\boldsymbol{x}})$, $x_P=(t\,;\,-\vec{\boldsymbol{x}})$ and $x_T=(-t\,;\,\vec{\boldsymbol{x}})$.

$B_{12}(x)$	**P**	**T**
$(\eta_1\xi_2+\overline{\xi}_1\bar{\eta}_2)(x)$	$\eta^*_{P1}\eta_{P2}B_{12}(x_P)$	$\eta^*_{T1}\eta_{T2}B_{12}(x_T)$
$(\overline{\xi}_1\bar{\eta}_2-\eta_1\xi_2)(x)$	$-\eta^*_{P1}\eta_{P2}B_{12}(x_P)$	$\eta^*_{T1}\eta_{T2}B_{12}(x_T)$
$(\bar{\xi}_1\overline{\sigma}^\mu\xi_2+\eta_1\sigma^\mu\bar{\eta}_2)(x)$	$\eta^*_{P1}\eta_{P2}(\Lambda_P)^\mu{}_\nu B^\nu_{12}(x_P)$	$-\eta^*_{T1}\eta_{T2}(\Lambda_T)^\mu{}_\nu B^\nu_{12}(x_T)$
$(\eta_1\sigma^\mu\bar{\eta}_2-\overline{\xi}_1\overline{\sigma}^\mu\xi_2)(x)$	$-\eta^*_{P1}\eta_{P2}(\Lambda_P)^\mu{}_\nu B^\nu_{12}(x_P)$	$-\eta^*_{T1}\eta_{T2}(\Lambda_T)^\mu{}_\nu B^\nu_{12}(x_T)$
$(\eta_1\sigma^{\mu\nu}\xi_2+\overline{\xi}_1\overline{\sigma}^{\mu\nu}\bar{\eta}_2)(x)$	$\eta^*_{P1}\eta_{P2}(\Lambda_P)^\mu{}_\rho(\Lambda_P)^\nu{}_\tau B^{\rho\tau}_{12}(x_P)$	$-\eta^*_{T1}\eta_{T2}(\Lambda_T)^\mu{}_\rho(\Lambda_T)^\nu{}_\tau B^{\rho\tau}_{12}(x_T)$
$(\overline{\xi}_1\overline{\sigma}^{\mu\nu}\bar{\eta}_2-\eta_1\sigma^{\mu\nu}\xi_2)(x)$	$-\eta^*_{P1}\eta_{P2}(\Lambda_P)^\mu{}_\rho(\Lambda_P)^\nu{}_\tau B^{\rho\tau}_{12}(x_P)$	$-\eta^*_{T1}\eta_{T2}(\Lambda_T)^\mu{}_\rho(\Lambda_T)^\nu{}_\tau B^{\rho\tau}_{12}(x_T)$

$B_{12}(x)$	**C**	**CP**
$(\eta_1\xi_2(x)+\overline{\xi}_1\bar{\eta}_2)(x)$	$\eta^*_{C1}\eta_{C2}B_{21}(x)$	$\eta^*_{CP1}\eta_{CP2}B_{21}(x_P)$
$(\overline{\xi}_1\bar{\eta}_2-\eta_1\xi_2)(x)$	$\eta^*_{C1}\eta_{C2}B_{21}(x)$	$-\eta^*_{CP1}\eta_{CP2}B_{21}(x_P)$
$(\bar{\xi}_1\overline{\sigma}^\mu\xi_2+\eta_1\sigma^\mu\bar{\eta}_2)(x)$	$-\eta^*_{C1}\eta_{C2}B^\mu_{21}(x)$	$-\eta^*_{CP1}\eta_{CP2}(\Lambda_P)^\mu{}_\nu B^\nu_{21}(x_P)$
$(\eta_1\sigma^\mu\bar{\eta}_2-\overline{\xi}_1\overline{\sigma}^\mu\xi_2)(x)$	$\eta^*_{C1}\eta_{C2}B^\mu_{21}(x)$	$-\eta^*_{CP1}\eta^*_{CP2}(\Lambda_P)^\mu{}_\nu B^\nu_{21}(x_P)$
$(\eta_1\sigma^{\mu\nu}\xi_2+\overline{\xi}_1\overline{\sigma}^{\mu\nu}\bar{\eta}_2)(x)$	$-\eta^*_{C1}\eta_{C2}B^{\mu\nu}_{21}(x)$	$-\eta^*_{CP1}\eta_{CP2}(\Lambda_P)^\mu{}_\rho(\Lambda_P)^\nu{}_\tau B^{\rho\tau}_{21}(x_P)$
$(\overline{\xi}_1\overline{\sigma}^{\mu\nu}\bar{\eta}_2-\eta_1\sigma^{\mu\nu}\xi_2)(x)$	$-\eta^*_{C1}\eta_{C2}B^{\mu\nu}_{21}(x)$	$\eta^*_{CP1}\eta_{CP2}(\Lambda_P)^\mu{}_\rho(\Lambda_P)^\nu{}_\tau B^{\rho\tau}_{21}(x_P)$

Table B.4. Transformation properties of various 2-component charged fermion bilinears under a CPT transformation. The notation used below is given in the caption to Table B.3. The phase η_{CPT} is taken to be independent of the particle species as required by the CPT theorem.

	CPT
$(\eta_1\xi_2(x) + \overline{\xi}_1\bar{\eta}_2)(x)$	$(\eta_2\xi_1(x) + \overline{\xi}_2\bar{\eta}_1)(-x)$
$(\overline{\xi}_1\bar{\eta}_2 - \eta_1\xi_2)(x)$	$-(\overline{\xi}_2\bar{\eta}_1 - \eta_2\xi_1)(-x)$
$(\bar{\xi}_1\overline{\sigma}^\mu\xi_2 + \eta_1\sigma^\mu\bar{\eta}_2)(x)$	$-(\bar{\xi}_2\overline{\sigma}^\mu\xi_1 + \eta_2\sigma^\mu\bar{\eta}_1)(-x)$
$(\eta_1\sigma^\mu\bar{\eta}_2 - \overline{\xi}_1\overline{\sigma}^\mu\xi_2)(x)$	$-(\eta_2\sigma^\mu\bar{\eta}_1 - \overline{\xi}_2\overline{\sigma}^\mu\xi_1)(-x)$
$(\eta_1\sigma^{\mu\nu}\xi_2 + \overline{\xi}_1\overline{\sigma}^{\mu\nu}\bar{\eta}_2)(x)$	$(\eta_2\sigma^{\mu\nu}\xi_1 + \overline{\xi}_2\overline{\sigma}^{\mu\nu}\bar{\eta}_1)(-x)$
$(\overline{\xi}_1\overline{\sigma}^{\mu\nu}\bar{\eta}_2 - \eta_1\sigma^{\mu\nu}\xi_2)(x)$	$-(\overline{\xi}_2\overline{\sigma}^{\mu\nu}\bar{\eta}_1 - \eta_2\sigma^{\mu\nu}\xi_1)(-x)$

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