SUPERGRAVITY COUPLINGS: A GEOMETRIC FORMULATION

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Contents

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Abstract

This report provides a pedagogical introduction to the description of the general Poincaré supergravity/matter/Yang–Mills couplings using methods of Kähler superspace geometry. At a more advanced level this approach is generalized to include tensor field and Chern–Simons couplings in supersymmetry and supergravity, relevant in the context of weakly and strongly coupled string theories.

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1. Introduction

Since its appearance in string theory [117,118,132,43,140], in elementary particle physics [94,149] and in quantum field theory [154–156,102], supersymmetry has become a central issue in the quest for unification of the fundamental forces of Nature.

Mathematically, supersymmetry transformations fall in the category of graded Lie groups, with commuting and anticommuting parameters [12,37]. In addition to the generators of Lorentz transformations and translations in a D-dimensional space–time, the supersymmetry algebra contains one or more spinor supercharges ("simple" or "N-extended" supersymmetry). As a consequence of the particular algebraic structure, Wigner's analysis of unitary representations [161] can be generalized to the supersymmetric case [136,116,76,66], giving rise to the notion of supermultiplets which combine bosons and fermions.

Although theoretically very appealing, no explicit sign of such a Bose–Fermi symmetry has been observed experimentally. This does not prevent experimental physicists to put supersymmetric versions of the standard model [119,103] to the test [127,128]. So far they turn out to be compatible with data.

On a more fundamental level, in the context of recent developments in string/brane theory [139,141,95,129], supergravity in 11 dimensions [116,40] seems to play an important role. Such a string, or membrane theory is expected to manifest itself in a four-dimensional point particle limit as some locally supersymmetric effective theory.

The basic structure of a generic $D = 4$, $N = 1$ effective theory is provided by supergravity [50,75] coupled to various lower spin multiplets. The off-shell supergravity multiplet is usually taken to be the one with minimal auxiliary field content [147,67], the so-called *minimal supergravity multiplet*.[4]

Chiral multiplets are expected to appear in the form of some non-linear sigma model. Supersymmetry requires a Kähler structure [164]: the complex scalar fields of the chiral multiplets are coordinates of a Kähler manifold [74,9,4,8]. At the same time they may be subject to Yang–Mills gauge transformations, requiring the coupling to supersymmetric *Yang–Mills multiplets* [70,135].

The general theory, combining minimal supergravity, chiral matter and supersymmetric Yang–Mills theory has been worked out in [38–40,42]. In this construction, generalized rescalings, compatible with supersymmetry, had to be carried out to establish the canonical normalization of the Einstein term. In its final form, this theory exhibits chiral Kähler phase transformations. Alternatively, using conformal tensor calculus and particular gauge conditions [110,109], the cumbersome Weyl rescalings could be avoided.

But string/membrane theory requires more fields and more structures – *linear multiplets* [71,143] and *3-form multiplets* [82], together with Chern–Simons terms of the gauge and gravitational types should be included. They are relevant for string corrections to gauge couplings [52,5,28,112,142,47–49,27], in particular non-holomorphic gauge coupling functions, and for effective descriptions of gaugino condensation [162], as well as for a supersymmetric implementation of the consequences of the Green–Schwarz mechanism [96] in an effective theory [32,69].

[4] Other possibilities, such as the *new minimal supergravity multiplet* [3,145] and the *non-minimal supergravity multiplet* [23,144] are less popular [124–126] in this context.

It is clear that a systematic approach should be employed to cope with such complex structures. This report provides a presentation of the geometric superspace approach.

The notion of superspace is based on the concept of superfields [134,71,138]: space–time is promoted to superspace in adding anticommuting parameters and superfields are functions of space–time coordinates and the anticommuting coordinates. Supersymmetry transformations are realized as differential operations involving spinor derivatives.

Implementing the machinery of differential geometry, like differential forms, exterior derivatives, interior product, etc., on superspace gives rise to superspace geometry. In this framework supersymmetry and general coordinate transformations are described in a unified way as certain diffeomorphisms. Both the graviton and its superpartner, the gravitino, are identified in the frame differential form of superspace.

The superspace formulation of supergravity [100,157–159,101,163] and supersymmetric gauge theory [150,151] is by now standard textbook knowledge [80,153]. A characteristic feature of this formulation is that the structure group in superspace is represented by the vector and spinor representations of the Lorentz group.

This superspace geometry may be modified by adding a chiral $U(1)$ to the structure group transformations, accompanied by the corresponding gauge potential differential form. Associated with this Abelian gauge group is an unconstrained pre-potential superfield. By itself, this structure is called $U(1)$ superspace [105], it allows to obtain the known supergravity multiplets mentioned above: minimal, new minimal and non-minimal, upon applying suitable restrictions [115].

The superspace description of the supergravity–matter coupling is obtained from $U(1)$ superspace as well: in this case the chiral $U(1)$ is replaced by superfield Kähler transformations. At the same time the unconstrained pre-potential is identified with the superfield Kähler potential [21,18,98,99]. In this formulation, called *Kähler superspace* geometry, or $U_K(1)$ superspace geometry, the Kähler phase transformations are implemented ab initio at a geometric level, the Kähler weights of all the super- and component fields are given intrinsically and no rescalings are needed in the construction of the supersymmetric action. The Kähler superspace formulation is related to the Kähler–Weyl formalism [152] in a straightforward way [18].

The construction of the general supergravity/matter/Yang–Mills system using the Kähler superspace formulation is the central issue of this report.

In Section 2 we review rigid superspace geometry in some detail, including supersymmetric gauge theory. Notational details are presented in Appendix A. Section 3 contains a detailed account of the Kähler superspace construction. A collection of elements of $U(1)$ superspace can be found in Appendix B. A more general setting which includes Kähler gauged isometries is treated in Appendix C. Derivation of the superfield equations of motion is reviewed in Appendix D.

In Section 4 we define component fields, their supersymmetry transformations and construct the complete component field action. The Kähler superspace formulation is particularly convenient when the supergravity/matter/Yang–Mills system is to be extended to contain linear multiplets, Chern–Simons forms and 3-form multiplets, as explained in detail in Sections 5 and 6. Appendices E and F contain complements to these sections.

This report is not intended to provide a review of supersymmetry and its applications. It is rather focused on a quite special issue, the description of $D = 4$, $N = 1$ supergravity couplings in geometric terms, more precisely in terms of superspace geometry. We have made an effort to furnish a self-contained and exhaustive presentation of this highly technical subject.

Even when restricted to $D = 4$, $N = 1$, there are many topics we have not mentioned, among them supersymmetry breaking, quantization, anomalies and their cohomological BRS construction, conformal supergravity or gravitational Chern–Simons forms.

Similar remarks apply to the bibliography. The references cited are rather restricted to those directly related to the technical aspects of differential geometry in superspace applied to supergravity couplings. Even though we cannot claim to have a complete bibliographical list and apologize in advance for any undue omissions.

2. Rigid superspace geometry

We gather, here, some of the basic features of superspace geometry which will be useful later on. In Section 2.1 we begin with a list of the known off-shell multiplets in $D = 4$, $N = 1$ supersymmetry, recall the properties of rigid superspace endowed with constant torsion, and define supersymmetry transformations in this geometric framework. Next, supersymmetric Abelian gauge theory is reviewed in detail in Section 2.2 as an illustration of the methods of superspace geometry and also in view of its important role in the context of supergravity/matter coupling. Although very similar in structure, the non-Abelian case is presented separately in Section 2.3. In Section 2.4 we emphasize the similarity of Kähler transformations with the Abelian gauge structure, in particular the interpretation of the kinetic matter action as a composite D-term.

2.1. Prolegomena

2.1.1. $D = 4, N = 1$ *supermultiplet catalogue*

Since the supersymmetry algebra is an extension of the Poincaré algebra, Wigner's analysis [161] can be generalized to classify unitary representations [136,116,76,66] in terms of physical states. On the other hand, field theories are usually described in terms of local fields. As on-shell representations of supersymmetry combine different spins (resp. helicities), supermultiplets of local fields will contain components in different representations of the Lorentz group. A multiplet of a given helicity content can have several incarnations in terms of local fields. In the simplest case, the massless helicity $(1/2, 0)$ multiplet may be realized in three different ways, the *chiral multiplet*, sometimes also called scalar multiplet [156], the *linear multiplet* [71,143] or the *3-form multiplet* [82], which will be displayed below. At helicity $(1, 1/2)$ only one realization is known: the usual gauge multiplet [155]. The $(3/2, 1)$ multiplet has a number of avatars as well [122,123,81,87,83]. Finally, the $(2, 3/2)$ multiplet, which contains the graviton, is known in three versions: the minimal multiplet [147,67], the new minimal multiplet [3,145] and the non-minimal multiplet [23,144]. This exhausts the list of massless multiplets in $D = 4$, $N = 1$ supersymmetry in the sense of irreducible multiplets. The massive multiplet of spin content $(1, 1/2, 1/2, 0)$ which will be presented below may be understood as a combination of a gauge and a chiral multiplet. We just display the content of some of the off-shell supermultiplets that we shall use in the sequel, indicating the number of bosonic (**b**) and fermionic (**f**) degrees of freedom (the vertical bar separates auxiliary fields from physical ones).

- *The chiral/scalar multiplet*:

$$\phi \sim (A, \chi_\alpha \,|\, F) \begin{cases} A, & 2\mathbf{b}, \quad \text{complex scalar}\ , \\ \chi_\alpha, & 4\mathbf{f}, \quad \text{Weyl spinor}\ , \\ F, & 2\mathbf{b}, \quad \text{complex scalar}\ . \end{cases}$$

The conjugate multiplet $\bar{\phi} \sim (\bar{A}, \bar{\chi}^{\dot{\alpha}} \,|\, \bar{F})$, consists of the complex conjugate component fields. It has the same number of degree of freedom.

- *The generic vector multiplet*:

$$V \sim \left(C, \frac{\varphi_\alpha}{\bar{\varphi}^{\dot{\alpha}}}, H, V_m, \frac{\lambda_\alpha}{\bar{\lambda}^{\dot{\alpha}}}, D \right) \begin{cases} C, & 1\mathbf{b}, \quad \text{real scalar}\ , \\ \varphi_\alpha, \bar{\varphi}^{\dot{\alpha}}, & 4\mathbf{f}, \quad \text{Majorana spinor}\ , \\ H, & 2\mathbf{b}, \quad \text{complex scalar}\ , \\ V_m, & 4\mathbf{b}, \quad \text{real vector}\ , \\ \lambda_\alpha, \bar{\lambda}^{\dot{\alpha}}, & 4\mathbf{f}, \quad \text{Majorana spinor}\ , \\ D, & 1\mathbf{b}, \quad \text{real scalar}\ . \end{cases}$$

This vector multiplet can occur in two ways in physical models: as a massive vector field and its supersymmetric partners or as a gauge multiplet. In the massive vector case all dynamical fields have the same mass, the Majorana spinors, φ_α, $\bar{\varphi}^{\dot{\alpha}}$ and λ_α, $\bar{\lambda}^{\dot{\alpha}}$ combine into a Dirac spinor; the auxiliary sector contains one real and one complex scalar:

$$V_{\text{massive}} \sim (C, V_m, \Psi \,|\, H, D) \begin{cases} C, & 1\mathbf{b}, \quad \text{real scalar}\ , \\ \Psi, & 8\mathbf{f}, \quad \text{Dirac spinor}\ , \\ V_m, & 4\mathbf{b}, \quad \text{real vector}\ , \\ H, & 2\mathbf{b}, \quad \text{complex scalar}\ , \\ D, & 1\mathbf{b}, \quad \text{real scalar}\ . \end{cases}$$

The gauge multiplet contains less dynamical degrees of freedom due to gauge transformations which have the structure of scalar multiplets. One is left with a massless vector, a Majorana spinor (the gaugino) and an auxiliary scalar:

$$V_{\text{gauge}} \sim \left(V_m, \frac{\lambda_\alpha}{\bar{\lambda}^{\dot{\alpha}}} \,\middle|\, D \right) \begin{cases} V_m, & 3\mathbf{b}, \quad \text{gauge vector}\ , \\ \lambda_\alpha, \bar{\lambda}^{\dot{\alpha}}, & 4\mathbf{f}, \quad \text{Majorana spinor}\ , \\ D, & 1\mathbf{b}, \quad \text{real scalar}\ . \end{cases}$$

- *The 2-form (or linear) multiplet*:

$$L_{\text{linear}} \sim \left(L, \frac{\Lambda_\alpha}{\bar{\Lambda}^{\dot{\alpha}}}, b_{mn} \right) \begin{cases} L, & 1\mathbf{b}, \quad \text{real scalar}\ , \\ \Lambda_\alpha, \bar{\Lambda}^{\dot{\alpha}}, & 4\mathbf{f}, \quad \text{Majorana spinor}\ , \\ b_{mn}, & 3\mathbf{b}, \quad \text{antisym. tensor}\ . \end{cases}$$

The number of physical degrees of freedom of b_{mn} is $3 = 6 - 4 + 1$. This multiplet contains no auxiliary field.

- *The 3-form* (*or constrained chiral*) *multiplet*:

$$C_{(3)} \sim \left(Y, {\eta_\alpha \atop \bar\eta^{\dot\alpha}}, C_{lmn} \,|\, H \right) \begin{cases} Y, & 2\mathbf{b}, \quad \text{complex scalar}\,, \\ \eta_\alpha, \bar\eta^{\dot\alpha}, & 4\mathbf{f}, \quad \text{Majorana spinor}\,, \\ C_{lmn} & 1\mathbf{b}, \quad \text{antisym. tensor}\,, \\ H, & 1\mathbf{b}, \quad \text{real scalar}\,. \end{cases}$$

The number of physical degrees of freedom of C_{lmn} is $1 = 4 - 6 + 4 - 1$.

Although this section is devoted to rigid superspace, to be complete, we include here the list of multiplets appearing in supergravity:

- *The minimal multiplet* (12 + 12):

$$\left(e_m{}^a, {\psi_m{}^\alpha \atop \bar\psi_{m\dot\alpha}} \,|\, b_a, M \right) \begin{cases} e_m{}^a, & 6\mathbf{b}, \quad \text{graviton}\,, \\ \psi_m{}^\alpha, \bar\psi_{m\dot\alpha}, & 12\mathbf{f}, \quad \text{gravitino}\,, \\ b_a, & 4\mathbf{b}, \quad \text{real vector}\,, \\ M, & 2\mathbf{b}, \quad \text{complex scalar}\,. \end{cases}$$

- *The new minimal multiplet* (12 + 12):

$$\left(e_m{}^a, {\psi_m{}^\alpha \atop \bar\psi_{m\dot\alpha}} \,|\, V_m, b_{mn} \right) \begin{cases} e_m{}^a, & 6\mathbf{b}, \quad \text{graviton}\,, \\ \psi_m{}^\alpha, \bar\psi_{m\dot\alpha}, & 12\mathbf{f}, \quad \text{gravitino}\,, \\ V_m, & 3\mathbf{b}, \quad \text{gauge vector}\,, \\ b_{mn}, & 3\mathbf{b}, \quad \text{antisym. tensor}\,. \end{cases}$$

- *The non-minimal multiplet* (20 + 20):

$$\left(e_m{}^a, {\psi_m{}^\alpha \atop \bar\psi_{m\dot\alpha}} \,|\, b_a, c_a, {\chi_\alpha \atop \bar\chi^{\dot\alpha}}, {T_\alpha \atop \bar T^{\dot\alpha}}, S \right) \begin{cases} e_m{}^a, & 6\mathbf{b}, \quad \text{graviton}\,, \\ \psi_m{}^\alpha, \bar\psi_{m\dot\alpha}, & 12\mathbf{f}, \quad \text{gravitino}\,, \\ b_a, & 4\mathbf{b}, \quad \text{real vector}\,, \\ c_a, & 4\mathbf{b}, \quad \text{real vector}\,, \\ \chi_\alpha, \bar\chi^{\dot\alpha}, & 4\mathbf{f}, \quad \text{Majorana}\,, \\ T_\alpha, \bar T^{\dot\alpha} & 4\mathbf{f}, \quad \text{Majorana}\,, \\ S, & 2\mathbf{b}, \quad \text{complex scalar}\,. \end{cases}$$

In this report we will only be concerned with the minimal supergravity multiplet.

We conclude the list of known $N = 1$ supermuliplets with the (3/2, 1) multiplet [122,46,68,72]. It describes physical states of helicities 3/2 and 1, its off-shell realization contains 20 bosonic and

20 fermionic component fields.

- *The* (3/2, 1) *multiplet* (20 + 20):

$$\left(B_m \begin{matrix}\Gamma_m{}^\alpha \\ \bar\Gamma_{m\dot\alpha}\end{matrix} \middle| \begin{matrix}\rho^\alpha \\ \bar\rho^{\dot\alpha}\end{matrix}, P, J, Y_a, T_{ba}, \begin{matrix}\Sigma^\alpha \\ \bar\Sigma_{\dot\alpha}\end{matrix}\right) \begin{cases} B_m, & 3\mathbf{b}, \quad \text{gauge vector}\ , \\ \Gamma_m{}^\alpha, \bar\Gamma_{m\dot\alpha}, & 12\mathbf{f}, \quad \text{Rarita–Schwinger}\ , \\ \rho_\alpha, \bar\rho^{\dot\alpha}, & 4\mathbf{f}, \quad \text{Majorana}\ , \\ P, & 1\mathbf{b}, \quad \text{real scalar}\ , \\ J, & 2\mathbf{b}, \quad \text{complex scalar}\ , \\ Y_a, & 4\mathbf{b}, \quad \text{complex vector}\ , \\ T_{ba}, & 6\mathbf{b}, \quad \text{antisym. tensor}\ , \\ \Sigma^\alpha, \bar\Sigma_{\dot\alpha}, & 4\mathbf{f}, \quad \text{Majorana}\ . \end{cases}$$

The component field content displayed here corresponds to the de Wit–van Holten multiplet [46]. It is related to the Ogievetsky–Sokatchev multiplet [122] by a duality relation [111,84], similar to that between chiral and linear multiplet. Superspace descriptions are discussed in [87,83,84].

2.1.2. *Superfields and multiplets*

The anticommutation relation

$$\{Q_\alpha, \bar Q^{\dot\alpha}\} = 2(\sigma^a\varepsilon)_\alpha{}^{\dot\alpha} P_a\ , \tag{2.1.1}$$

which relates the generators Q_α and $\bar Q^{\dot\alpha}$ of supersymmetry transformations to translations P_a in space–time is at the heart of the supersymmetry algebra. Superspace geometry, on the other hand, is based on the notion of superfields which are functions depending on space–time coordinates x^m as well as on spinor, anticommuting variables θ^α and $\bar\theta_{\dot\alpha}$. Due to the anticommutativity, superfields are polynomials of finite degree in the spinor variables. Coefficients of the monomials in $\theta^\alpha, \bar\theta_{\dot\alpha}$ are called *component fields*.

Supersymmetry transformations of superfields are generated by the differential operators

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\bar\theta_{\dot\alpha}(\bar\sigma^m\varepsilon)^{\dot\alpha}{}_\alpha \frac{\partial}{\partial x^m}\ , \tag{2.1.2}$$

$$\bar Q^{\dot\alpha} = \frac{\partial}{\partial\bar\theta_{\dot\alpha}} - i\theta^\alpha(\sigma^m\varepsilon)_\alpha{}^{\dot\alpha} \frac{\partial}{\partial x^m} \tag{2.1.3}$$

which, of course, together with $P_a = -i\partial/\partial x^a$ satisfy (2.1.1) as well. A general superfield, however, does not necessarily provide an irreducible representation of supersymmetry.

The differential operators

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\bar\theta_{\dot\alpha}(\bar\sigma^m\varepsilon)^{\dot\alpha}{}_\alpha \frac{\partial}{\partial x^m}\ , \tag{2.1.4}$$

$$D^{\dot\alpha} = \frac{\partial}{\partial\bar\theta_{\dot\alpha}} + i\theta^\alpha(\sigma^m\varepsilon)_\alpha{}^{\dot\alpha} \frac{\partial}{\partial x^m}\ , \tag{2.1.5}$$

anticommute with the supersymmetry generators, i.e. they are covariant with respect to supersymmetry transformations and satisfy, by definition, the anticommutation relations

$$\{D_\alpha, D^{\dot\alpha}\} = 2i(\sigma^m\varepsilon)_\alpha{}^{\dot\alpha}\frac{\partial}{\partial x^m}\,, \tag{2.1.6}$$

$$\{D_\alpha, D_\beta\} = 0, \quad \{D^{\dot\alpha}, D^{\dot\beta}\} = 0\,. \tag{2.1.7}$$

These spinor covariant derivatives can be employed to define *constrained superfields* which may be used to define irreducible field representations of the supersymmetry algebra.

The most important ones are

- The chiral superfields $\phi, \bar\phi$ are complex superfields, subject to the constraints

$$D^{\dot\alpha}\phi = 0, \quad D_\alpha\bar\phi = 0\,. \tag{2.1.8}$$

  They are usually employed to describe supersymmetric matter multiplets.
- The superfields $W^\alpha, W_{\dot\alpha}$, subject to the constraints

$$D_\alpha W_{\dot\alpha} = 0, \quad D^{\dot\alpha}W^\alpha = 0\,, \tag{2.1.9}$$

$$D^\alpha W_\alpha = D_{\dot\alpha}W^{\dot\alpha} \tag{2.1.10}$$

  are related to the field strength tensor and play a key role in the description of supersymmetric gauge theories.
- The linear superfield L, subject to the linearity constraints[5]

$$D^2L = 0, \quad \bar D^2L = 0\,. \tag{2.1.11}$$

  As explained above, it describes the supermultiplet of an antisymmetric tensor or 2-form gauge potential, as such it plays a key role in describing moduli fields in superstring effective theories.
- The 3-form superfields $Y, \bar Y$, are chiral superfields ($D_{\dot\alpha}Y = 0, D_\alpha\bar Y = 0$) with a further constraint

$$D^2Y - \bar D^2\bar Y = \frac{8i}{3}\varepsilon^{klmn}\Sigma_{klmn}\,, \tag{2.1.12}$$

  with Σ_{klmn}, the field strength of the 3-form. These superfields are relevant in the context of gaugino condensation and of Chern–Simons forms couplings.

The superfields L and $W^\alpha, W_{\dot\alpha}$ are invariant under the respective gauge transformations, they can be viewed as some kind of invariant field strengths. As is well known, geometric formulations of 1-, 2- and 3-form gauge theories in superspace exist such that indeed W^α, $W_{\dot\alpha}$, L and Y, $\bar Y$ are properly identified as field strength superfields with (2.1.9)–(2.1.12) constituting the corresponding Bianchi identities.

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[5] With the usual notations $D^2 = D^\alpha D_\alpha$ and $\bar D^2 = D_{\dot\alpha}D^{\dot\alpha}$, which will be used throughout this paper.

2.1.3. Geometry and supersymmetry

In order to prepare the ground for a geometric superspace formulation of such theories one introduces a local frame for rigid superspace. It is suggestive to re-express (2.1.4)–(2.1.7) in terms of supervielbein (a generalization of Cartan's local frame) and torsion in a superspace of coordinates $z^M \sim (x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}})$, derivatives $\partial_M \sim (\partial/\partial x^m, \partial/\partial\theta^\mu, \partial/\partial\bar{\theta}_{\dot{\mu}})$ and differentials $dz^M \sim (dx^m, d\theta^\mu, d\bar{\theta}_{\dot{\mu}})$. The latter may be viewed as the tangent and cotangent frames of superspace, respectively. The supervielbein 1-form of rigid superspace is

$$E^A = dz^M E_M{}^A \,, \tag{2.1.13}$$

with

$$E_M{}^A = \begin{pmatrix} \delta_m{}^a & 0 & 0 \\ -\,\mathrm{i}(\bar{\theta}\bar{\sigma}^a\varepsilon)_\mu & \delta_\mu{}^\alpha & 0 \\ -\,\mathrm{i}(\theta\sigma^a\varepsilon)^{\dot{\mu}} & 0 & \delta^{\dot{\mu}}{}_{\dot{\alpha}} \end{pmatrix}. \tag{2.1.14}$$

The inverse vielbein $E_A{}^M$, defined by the relations

$$E_M{}^A(z)E_A{}^N(z) = \delta_M{}^N, \quad E_A{}^M(z)E_M{}^B(z) = \delta_A{}^B \,,$$

reads

$$E_A{}^M = \begin{pmatrix} \delta_a{}^m & 0 & 0 \\ \mathrm{i}(\bar{\theta}\bar{\sigma}^m\varepsilon)_\alpha & \delta_\alpha{}^\mu & 0 \\ \mathrm{i}(\theta\sigma^m\varepsilon)^{\dot{\alpha}} & 0 & \delta^{\dot{\alpha}}{}_{\dot{\mu}} \end{pmatrix}. \tag{2.1.15}$$

The torsion 2-form in rigid superspace is defined as the exterior derivative of the vielbein 1-form:

$$dE^A = T^A = \tfrac{1}{2}E^B E^C T_{CB}{}^A \,. \tag{2.1.16}$$

Now, for the differential operators $D_A = (\partial/\partial x^a, D_\alpha, D^{\dot{\alpha}})$ we have

$$D_A = E_A{}^M \partial_M \,, \tag{2.1.17}$$

$$(D_C, D_B) = -\,T_{CB}{}^A D_A \,, \tag{2.1.18}$$

with the graded commutator defined as $(D_C, D_B) = D_C D_B - (-)^{bc} D_B D_C$ with $b = 0$ for a vector and $b = 1$ for a spinor index. The fact that the same torsion coefficient appears in (2.1.18) and in (2.1.18) reflects the fact that $dd = 0$ in superspace. To be more precise consider the action of dd on some generic 0-form superfield Φ. Application of d to the expression $d\Phi = E^B D_B \Phi$, in combination with the rules of superspace exterior calculus, i.e. $dd\Phi = dE^B D_B \Phi = E^B E^C D_C D_B \Phi + (dE^A) D_A \Phi$, and the definitions introduced so far gives immediately

$$dd\Phi = \tfrac{1}{2}E^B E^C((D_C, D_B)\Phi + T_{CB}{}^A D_A \Phi) \,, \tag{2.1.19}$$

establishing the assertion. A glance at the differential algebra of the D_A's, in particular (2.1.6), shows then that the only non-vanishing torsion component is

$$T_\gamma{}^{\dot{\beta}a} = -\,2\mathrm{i}(\sigma^a\varepsilon)_\gamma{}^{\dot{\beta}} \,. \tag{2.1.20}$$

Given the relation between supersymmetry transformations and the "square root" of space–time translations (2.1.1), we would like to interpret them as diffeomorphisms in superspace. The action of diffeomorphisms on geometric objects such as vector and tensor fields or differential forms is encoded in the Lie derivative, which can be defined in terms of basic operations of a differential algebra (suitably extended to superspace), i.e. the exterior derivative, d, and the interior product, $\imath_\zeta$, such that

$$L_\zeta = \imath_\zeta d + d\imath_\zeta \ . \tag{2.1.21}$$

The interior product, for instance, of a vector field ζ with the vielbein 1-form is

$$\imath_\zeta E^A = \zeta^M E_M{}^A = \zeta^A \ . \tag{2.1.22}$$

The definition of differential forms in superspace (or superforms) and the conventions for the differential calculus are those of Wess and Bagger [153] – cf. Appendix A.1 below for a summary. Then, on superforms d acts as an antiderivation of degree $+1$, the exterior derivative of a p-form is a $(p+1)$-form. Likewise, $\imath_\zeta$ acts as an antiderivation of degree -1 so that the Lie derivative L_ζ, defined by (2.1.21), does not change the degree of differential forms. This geometric formulation will prove to be very efficient to construct more general supersymmetric or supergravity theories involving p-form fields.

For the vielbein itself, combination of (2.1.16) and (2.1.22) yields

$$L_\zeta E^A = d\zeta^A + \imath_\zeta T^A \ . \tag{2.1.23}$$

On a 0-form superfield, Φ, the Lie derivative acts according to

$$L_\zeta \Phi = \imath_\zeta d\Phi = \zeta^A D_A \Phi = \zeta^M \partial_M \Phi \ . \tag{2.1.24}$$

The Lie derivative L_ξ with respect to the particular vector field

$$\xi^M = (\mathrm{i}\theta^\alpha(\sigma^m\varepsilon)_\alpha{}^{\dot\alpha}\bar\xi_{\dot\alpha} + \mathrm{i}\bar\theta_{\dot\alpha}(\bar\sigma^m\varepsilon)^{\dot\alpha}{}_\alpha\xi^\alpha, \xi^\mu, \bar\xi_{\dot\mu}) \ , \tag{2.1.25}$$

leaves the vielbein 1-form (2.1.13), (2.1.14) invariant, i.e.

$$L_\xi E^A = 0 \ . \tag{2.1.26}$$

This is most easily seen in terms of $\xi^A = \imath_\xi E^A$, which is explicitly given as

$$\xi^A = (2\mathrm{i}(\theta\sigma^a\bar\xi) + 2\mathrm{i}(\bar\theta\bar\sigma^a\xi), \xi^\alpha, \bar\xi_{\dot\alpha}) \ . \tag{2.1.27}$$

Recall that $L_\xi E^A = d\xi^A + \imath_\xi T^A$. This shows immediately that for the spinor components the equation is satisfied, because $\xi^{\underset{\cdot}{\alpha}}$ is constant and $T^{\underset{\cdot}{\alpha}}$ vanishes. As to the vector part one keeps in mind that in $d\xi^a = E^B D_B \xi^a$ only the derivatives with respect to θ, $\bar\theta$ contribute and compare the result

$$d\xi^a = 2\mathrm{i}E^\alpha(\sigma^a\varepsilon)_\alpha{}^{\dot\alpha}\bar\xi_{\dot\alpha} + 2\mathrm{i}E_{\dot\alpha}(\bar\sigma^a\varepsilon)^{\dot\alpha}{}_\alpha\xi^\alpha$$

to the expression for the interior product acting on $T^a = 2\mathrm{i}E_{\dot\beta}E^\gamma(\sigma^a\varepsilon)_\gamma{}^{\dot\beta}$, i.e.

$$\imath_\xi T^a = 2\mathrm{i}E^\gamma(\sigma^a\varepsilon)_\alpha{}^{\dot\alpha}\bar\xi_{\dot\alpha} + 2\mathrm{i}E_{\dot\alpha}(\bar\sigma^a\varepsilon)^{\dot\alpha}{}_\alpha\xi^\alpha \ .$$

The Lie derivative of a generic superfield Φ in terms of the particular vector field ξ^A defined in (2.1.27) is given as

$$L_\xi \Phi = \xi^A D_A \Phi = (\xi^\alpha Q_\alpha + \bar{\xi}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}})\Phi , \tag{2.1.28}$$

reproducing the infinitesimal supersymmetry transformation with Q_α and $\bar{Q}^{\dot{\alpha}}$ as defined in (2.1.2) and (2.1.3):

- *Supersymmetry transformations can be identified as diffeomorphisms of parameters $\xi^\alpha, \bar{\xi}_{\dot{\alpha}}$ which leave E^A invariant.*

Combining such a supersymmetry transformation with a translation of parameter ε^a, we obtain

$$L_\varepsilon \Phi + L_\xi \Phi = \varepsilon^a \partial_a \Phi + \xi^A D_A \Phi = (\varepsilon^a + \xi^a)\partial_a \Phi + \xi^\alpha D_\alpha \Phi + \bar{\xi}_{\dot{\alpha}} D^{\dot{\alpha}} \Phi . \tag{2.1.29}$$

The transformations with the particular choice $\varepsilon^a = -\xi^a$ of a ξ dependent space–time translation, will be called *supertranslations*. They are given as

$$\delta\Phi = (\xi^\alpha D_\alpha + \bar{\xi}_{\dot{\alpha}} D^{\dot{\alpha}})\Phi . \tag{2.1.30}$$

These special transformations will be used in the formulation of supersymmetric theories (and in particular in supergravity [163]). Let us stress that for $\theta = \bar{\theta} = 0$, supersymmetry transformations and supertranslations coincide. The components of a superfield are traditionally defined as coefficients in an expansion with respect to θ and $\bar{\theta}$. In the geometric approach presented here, component fields are defined as lowest components of superfields. Higher components are obtained by successive applications of *covariant* derivatives and subsequent projection to $\theta = \bar{\theta} = 0$. Component fields defined this way are naturally related by supertranslations. The basic operational structure is the algebra of covariant derivatives.

2.2. Abelian gauge structure

2.2.1. Abelian gauge potential

In analogy to usual gauge theory, gauge potentials in supersymmetric gauge theories are defined as 1-forms in superspace

$$A = E^A A_A = E^a A_a + E^\alpha A_\alpha + E_{\dot{\alpha}} A^{\dot{\alpha}} . \tag{2.2.1}$$

The coefficients $A_a, A_\alpha, A^{\dot{\alpha}}$ are, by themselves, superfields. Since we consider here an Abelian gauge theory, A transforms under gauge transformations as

$$A \mapsto A - g^{-1} dg . \tag{2.2.2}$$

The gauge transformation parameters g are 0-form superfields and the invariant field strength is a 2-form,

$$F = dA = \tfrac{1}{2} E^A E^B F_{BA} . \tag{2.2.3}$$

Observe that, following (2.1.16) a torsion term appears in its explicit expression:

$$F_{BA} = D_B A_A - (-)^{ab} D_A A_B + T_{BA}{}^C A_C \ . \tag{2.2.4}$$

By definition, (2.2.3), the field strength satisfies the Bianchi identity

$$dF = 0 \ . \tag{2.2.5}$$

Consider next a covariant (0-form) superfield Φ of weight $w(\Phi)$ under Abelian superfield gauge transformations, i.e.

$$\Phi \overset{g}{\mapsto} g^{w(\Phi)} \Phi \ . \tag{2.2.6}$$

Its covariant (exterior) derivative,

$$\mathscr{D}\Phi = E^A \mathscr{D}_A \Phi \ , \tag{2.2.7}$$

is defined as[6]

$$\mathscr{D}\Phi = d\Phi + w(\Phi) A \Phi \ . \tag{2.2.8}$$

Covariant differentiation of (2.2.7) yields in turn ($w(\mathscr{D}\Phi) = w(\Phi)$)

$$\mathscr{D}\mathscr{D}\Phi = w(\Phi) F \Phi \ , \tag{2.2.9}$$

leading to the graded commutator

$$(\mathscr{D}_B, \mathscr{D}_A)\Phi = w(\Phi) F_{BA} \Phi - T_{BA}{}^C \mathscr{D}_C \Phi \ . \tag{2.2.10}$$

Supertranslations in superspace *and* infinitesimal superfield gauge transformations, $g \approx 1 + \alpha$, with α a real superfield, change A and Φ into $A' = A + \delta A$ and $\Phi' = \Phi + \delta\Phi$ such that

$$\delta A = \imath_\xi F - d(\alpha - \imath_\xi A) \tag{2.2.11}$$

and

$$\delta \Phi = \imath_\xi \mathscr{D}\Phi + w(\Phi)(\alpha - \imath_\xi A)\Phi \ . \tag{2.2.12}$$

The combination of a supertranslation and of a *compensating* gauge transformation of superfield parameter $\alpha = \imath_\xi A$ gives rise to remarkably simple transformation laws. This parametrization is particularly useful for the definition of component fields and their supersymmetry transformations. We shall call these special transformations: *Wess–Zumino transformations*, they are given as

$$\delta_{\mathrm{WZ}} \Phi = \imath_\xi \mathscr{D}\Phi, \quad \delta_{\mathrm{WZ}} A = \imath_\xi F \ . \tag{2.2.13}$$

Let us stress that the formalism developed here is well adapted to describe supersymmetry transformations of differential forms.

[6] If Φ_p is a p-form, we define it as $\mathscr{D}\Phi_p = d\Phi_p + (-)^p w(\Phi_p) A \Phi_p$.

So far, Φ was considered as some generic superfield. Matter fields are described in terms of chiral superfields. In the context of a gauge structure the chirality conditions are most conveniently defined in terms of covariant derivatives. A superfield ϕ is called *covariantly chiral* and a superfield $\bar{\phi}$ is called *covariantly antichiral*, if they satisfy the conditions

$$\mathscr{D}^{\dot{\alpha}}\phi = 0, \quad \mathscr{D}_\alpha \bar{\phi} = 0 . \tag{2.2.14}$$

Observe that usually they are supposed to have opposite weights $w(\bar{\phi}) = -w(\phi)$. Consistency of the covariant chirality constraints (2.2.14) with the graded commutation relations (2.2.10) implies then

$$F^{\dot{\beta}\dot{\alpha}} = 0, \quad F_{\beta\alpha} = 0 . \tag{2.2.15}$$

Moreover, due to the (constant) torsion term in (2.2.4), i.e.

$$F_\beta{}^{\dot{\alpha}} = D_\beta A^{\dot{\alpha}} + D^{\dot{\alpha}} A_\beta - 2\mathrm{i}(\sigma^a \varepsilon)_\beta{}^{\dot{\alpha}} A_a , \tag{2.2.16}$$

the condition

$$F_\beta{}^{\dot{\alpha}} = 0 \tag{2.2.17}$$

amounts to a mere covariant redefinition of the vector superfield gauge potential A_a. Given constraints (2.2.15) on $F_{\beta\alpha}$ and $F^{\dot{\beta}\dot{\alpha}}$, the properties of the remaining components $F_{\beta a}$, $F^{\dot{\beta}}{}_a$ and F_{ba} of the superfield strength F_{BA} are easily derived from the Bianchi identities (2.2.5) which read[7]

$$\oint_{(CBA)} (D_C F_{BA} + T_{CB}{}^D F_{DA}) = 0 . \tag{2.2.18}$$

It turns out that the whole geometric structure which describes supersymmetric gauge theories can be formulated only in terms of the superfields W_α and $W^{\dot{\alpha}}$ such that

$$F_{\beta a} = +\mathrm{i}\sigma_{a\beta\dot{\beta}} W^{\dot{\beta}} , \tag{2.2.19}$$

$$F^{\dot{\beta}}{}_a = -\mathrm{i}\bar{\sigma}_a^{\dot{\beta}\beta} W_\beta , \tag{2.2.20}$$

$$F_{ba} = \tfrac{1}{2}(\bar{\sigma}_{ba})^{\dot{\beta}}{}_{\dot{\alpha}} D^{\dot{\alpha}} W_{\dot{\beta}} - \tfrac{1}{2}(\sigma_{ba})_\beta{}^\alpha D_\alpha W^\beta . \tag{2.2.21}$$

Furthermore, the Bianchi identities imply restrictions (2.1.9) and (2.1.10). In this sense these equations have an interpretation as Bianchi identities, providing a condensed version of (2.2.18).

2.2.2. Solution of constraints and pre-potentials

Eq. (2.2.18) is the supersymmetric analogue of the geometric part of Maxwell's equations

$$\partial_c F_{ba} + \partial_a F_{cb} + \partial_b F_{ac} = 0 , \tag{2.2.22}$$

---

[7] $\oint_{(CBA)}$ stands for the graded cyclic permutation on the super-indices CBA, explicitly defined as $\oint_{(CBA)} CBA = CBA + (-)^{a(c+b)} ACB + (-)^{(b+a)c} BAC$.

which are solved in terms of a vector potential, A_a, such that $F_{ba} = \partial_b A_a - \partial_a A_b$. In the supersymmetric case a similar mechanism takes place, via the explicit solution of constraints (2.1.9) and (2.1.10). To be more precise these solutions can be written in terms of superfields T and U as

$$A_\alpha = -T^{-1} D_\alpha T = -D_\alpha \log T \,, \tag{2.2.23}$$

$$A^{\dot\alpha} = -U^{-1} D^{\dot\alpha} U = -D^{\dot\alpha} \log U \,. \tag{2.2.24}$$

Indeed one obtains from (2.2.19) and (2.2.20)

$$W_\alpha = +\tfrac{1}{8}\bar{D}^2 D_\alpha \log(TU^{-1}), \quad W^{\dot\alpha} = +\tfrac{1}{8} D^2 D^{\dot\alpha} \log(TU^{-1}) \,, \tag{2.2.25}$$

which is easily seen to satisfy (2.1.9) and (2.1.10). The superfields T and U are called pre-potentials; they are subject to gauge transformations which have to be consistent with the gauge transformations (2.2.2) of the potentials. However due to the special form of solutions (2.2.23) and (2.2.24), we have the freedom to make extra chiral (resp. antichiral) transformations, explicitly

$$T \mapsto \bar{\mathbf{P}} T g \,, \tag{2.2.26}$$

$$U \mapsto \mathbf{Q} U g \,. \tag{2.2.27}$$

The new superfields $\bar{\mathbf{P}}$ and $\mathbf{Q}$ parametrize so-called pre-gauge transformations which do not show up in the transformation laws of the potentials themselves due to their chirality properties

$$D_\alpha \bar{\mathbf{P}} = 0, \quad D^{\dot\alpha} \mathbf{Q} = 0 \,. \tag{2.2.28}$$

The terminology originates from the fact that, due to the covariant constraints, the gauge potentials can be expressed in terms of more fundamental *unconstrained* quantities, the pre-potentials, which in turn give rise to new gauge structures, the pre-gauge transformations.

The pre-potentials serve to mediate between quantities subject to different types of gauge (pre-gauge) transformations g ($\bar{\mathbf{P}}$ and $\mathbf{Q}$) and we can build combinations of these which are sensitive to all these transformations. For instance, the composite field $T^a U^b$ transforms under gauge and pre-gauge transformations as follows:

$$(T^a U^b) \mapsto (T^a U^b) \bar{\mathbf{P}}^a \mathbf{Q}^b g^{a+b}. \tag{2.2.29}$$

Now if we consider a generic superfield Φ of weight $w(\Phi)$ as in (2.2.6) and define

$$\Phi(a,b) = (T^a U^b)^{-w(\Phi)} \Phi \,, \tag{2.2.30}$$

this new superfield $\Phi(a,b)$ is inert under g superfield gauge transformations if $a + b = 1$, but still transforms under chiral and antichiral superfield gauge transformations $\mathbf{Q}$ and $\bar{\mathbf{P}}$ as

$$\Phi(a,b) \mapsto [g^{(a+b-1)} \bar{\mathbf{P}}^a \mathbf{Q}^b]^{-w(\Phi)} \Phi(a,b) \,. \tag{2.2.31}$$

$\Phi(a,b)$ will be said to be in the (a,b)-basis with respect to $\bar{\mathbf{P}}$ and $\mathbf{Q}$ superfield pre-gauge transformations. It is convenient to introduce the corresponding definitions for the gauge potential as well

$$\begin{aligned} A(a,b) &= A + (T^a U^b)^{-1} d(T^a U^b) \\ &= A + a\, d \log T + b\, d \log U \,. \end{aligned} \tag{2.2.32}$$

It should be clear that $F(a,b) \equiv dA(a,b) = F = dA$, in any basis and thus that the superfields $W^{\alpha}, W_{\dot\alpha}$ are basis independent. It is interesting to note that we can write

$$A_{\alpha}(a,b) = (a - \tfrac{1}{2})D_{\alpha} \log T + (b - \tfrac{1}{2})D_{\alpha} \log U - \tfrac{1}{2}D_{\alpha} \log W \ ,$$
$$A^{\dot\alpha}(a,b) = (a - \tfrac{1}{2})D^{\dot\alpha} \log T + (b - \tfrac{1}{2})D^{\dot\alpha} \log U + \tfrac{1}{2}D^{\dot\alpha} \log W \ , \tag{2.2.33}$$

where the superfield $W = (TU^{-1})$ is inert under g gauge transformations (2.2.29), basis independent and transforms as

$$W \mapsto \bar{\mathbf{P}} W \mathbf{Q}^{-1} \ . \tag{2.2.34}$$

Therefore, we can gauge away the T and U terms in the expressions for $A_{\alpha}(a,b)$ and $A^{\dot\alpha}(a,b)$, but not the W one. The covariant derivative in the (a,b)-basis is then defined as

$$\mathscr{D}\Phi(a,b) = d\Phi(a,b) + w(\Phi)A(a,b)\Phi(a,b) \tag{2.2.35}$$

and transforms in accordance with (2.2.30):

$$\mathscr{D}\Phi(a,b) = (T^a U^b)^{-w(\Phi)} \mathscr{D}\Phi \ . \tag{2.2.36}$$

Again $\mathscr{D}\Phi(a,b)$ is inert under g gauge transformations if $a + b = 1$, so hereafter we will stick to this case and omit the label b, unless specified. Observe now that

$$(a,b) = (\tfrac{1}{2},\tfrac{1}{2}) \Rightarrow A_{\alpha}(\tfrac{1}{2}) = -\tfrac{1}{2}D_{\alpha} \log W, \quad A^{\dot\alpha}(\tfrac{1}{2}) = +\tfrac{1}{2}D^{\dot\alpha} \log W \ ,$$
$$(a,b) = (1,0) \Rightarrow A_{\alpha}(1) = 0, \qquad A^{\dot\alpha}(1) = +D^{\dot\alpha} \log W \ ,$$
$$(a,b) = (0,1) \Rightarrow A_{\alpha}(0) = -D_{\alpha} \log W, \quad A^{\dot\alpha}(0) = 0 \ . \tag{2.2.37}$$

The three particular bases presented in (2.2.37) are useful in different situations. Later on, in the discussion of Kähler transformations and in the construction of supergravity/matter couplings, we shall identify spinor components of the Kähler $U(1)$ connection with spinor derivatives of the Kähler potential, namely

$$A_{\alpha} = \tfrac{1}{4}D_{\alpha}K, \quad A^{\dot\alpha} = -\tfrac{1}{4}D^{\dot\alpha}K \ . \tag{2.2.38}$$

Such an identification is easily made in the $(\tfrac{1}{2},\tfrac{1}{2})$ base, called the vector basis: setting

$$W \equiv \exp(-K/2) \ , \tag{2.2.39}$$

we obtain (2.2.38). Moreover, if we parametrize $\bar{\mathbf{P}} = \exp(-\bar{F}/2)$ and $\mathbf{Q} = \exp(F/2)$ (we take $\bar{F}$ and F since K is real) we obtain, given (2.2.34),

$$K \mapsto K + F + \bar{F} \ , \tag{2.2.40}$$

the usual form of Kähler transformations. A generic superfield Φ, in this base, transforms as

$$\Phi(\tfrac{1}{2}) \mapsto e^{-(i/2)w(\Phi)\mathrm{Im}\,F}\Phi(\tfrac{1}{2}) \ . \tag{2.2.41}$$

In addition for the connection we obtain

$$A(\tfrac{1}{2}) \mapsto A(\tfrac{1}{2}) + \frac{i}{2} d \,\mathrm{Im}\, F \,, \tag{2.2.42}$$

where the vector component is, using (2.2.17),

$$A(\tfrac{1}{2})_a = \frac{i}{16} \bar{\sigma}_a^{\dot{\alpha}\alpha} [D_\alpha, D_{\dot{\alpha}}] K \,. \tag{2.2.43}$$

In other contexts (anomalies and Chern–Simons forms study) the $(0,1)$ and $(1,0)$ bases are relevant; we name them respectively chiral and antichiral bases. Indeed, let us consider the covariant chiral superfield ϕ, with $w(\phi) = +w$, in the $(0,1)$-basis the superfield $\phi(0) = U^{-w}\phi$ transforms under **Q**-transformations only,

$$\phi(0) \mapsto \mathbf{Q}^{-w} \phi(0) \,, \tag{2.2.44}$$

whereas the gauge potential has the property $A^{\dot{\alpha}}(0) = 0$. Then, in this basis, the covariant chirality constraint for ϕ, (2.2.14), takes a very simple form for $\phi(0)$: $D^{\dot{\alpha}}\phi(0) = 0$. Analogous arguments hold for $\bar{\phi}$, with weight $w(\bar{\phi}) = -w$, in the $(0,1)$-basis, i.e. $D_\alpha \bar{\phi}(1) = 0$. So it is $\phi(0)$ and $\bar{\phi}(1)$ which are actually the "traditional" chiral superfields, our ϕ and $\bar{\phi}$ are different objects, they are *covariant* (anti)chiral superfields. We emphasize this point because to build the matter action coupled to gauge fields we shall simply use the density

$$\bar{\phi}\phi = \bar{\phi}(1) W^w \phi(0) = \bar{\phi}(1) \mathrm{e}^{2wV} \phi(0) \,, \tag{2.2.45}$$

where we have defined

$$W \equiv \mathrm{e}^{2V} \,. \tag{2.2.46}$$

We thus recover the standard formulation of the textbooks in terms of non-covariantly chiral superfields $\phi(0), \bar{\phi}(1)$, with V the usual vector superfield; this is illustrated in Section 2.2.4. The chiral and the antichiral bases are related among themselves by means of the superfield W, $\phi(0) = W^w \phi(1)$.

Similarly, $A(1)$ and $A(0)$ are related by a gauge-like transformation

$$A(0) = A(1) - W^{-1} dW \,. \tag{2.2.47}$$

Finally, the basis independent superfields W^α and $W_{\dot{\alpha}}$ are easily obtained as

$$W_\alpha = \tfrac{1}{4} \bar{D}^2 D_\alpha V, \quad W^{\dot{\alpha}} = \tfrac{1}{4} D^2 D^{\dot{\alpha}} V \,, \tag{2.2.48}$$

which is nothing but the solution to the reduced Bianchi identities (2.1.9) and (2.1.10).

2.2.3. *Components and Wess–Zumino transformations*

Component fields are systematically defined as lowest components of superfields, expansion in terms of anticommuting parameters is replaced by successive application of covariant derivatives.

In this approach the component fields of a chiral multiplet ϕ of weight w are defined as

$$\phi| = A(x), \quad \mathscr{D}_\alpha \phi| = \sqrt{2}\chi_\alpha(x), \quad \mathscr{D}^\alpha \mathscr{D}_\alpha \phi| = -4F(x) , \tag{2.2.49}$$

whereas those of the gauge supermultiplet are identified as

$$A_m| = \mathrm{i}a_m, \quad W^{\dot\beta}| = \mathrm{i}\bar\lambda^{\dot\beta}, \quad W_\beta| = -\mathrm{i}\lambda_\beta, \quad \mathscr{D}^\alpha W_\alpha| = -2\mathbf{D} . \tag{2.2.50}$$

Their Wess–Zumino transformations are obtained from (2.2.13) in identifying Φ successively with ϕ, $\mathscr{D}_\alpha\phi$ and $\mathscr{D}^\alpha\mathscr{D}_\alpha\phi$. We obtain

$$\delta_{\mathrm{WZ}} A = \sqrt{2}\xi\chi , \tag{2.2.51}$$

$$\delta_{\mathrm{WZ}} \chi_\alpha = +\mathrm{i}\sqrt{2}(\bar\xi\bar\sigma^m\varepsilon)_\alpha \mathscr{D}_m A + \sqrt{2}\xi_\alpha F , \tag{2.2.52}$$

$$\delta_{\mathrm{WZ}} F = \mathrm{i}\sqrt{2}(\bar\xi\bar\sigma^m)^\alpha \mathscr{D}_m \chi_\alpha + 2\mathrm{i}w(\bar\xi\bar\lambda)A . \tag{2.2.53}$$

The covariant derivatives arise in a very natural way due to our geometric construction; they are given as

$$\mathscr{D}_m A = (\partial_m + \mathrm{i}wa_m)A, \quad \mathscr{D}_m \chi_\alpha = (\partial_m + \mathrm{i}wa_m)\chi_\alpha , \tag{2.2.54}$$

$$\mathscr{D}_m \bar A = (\partial_m - \mathrm{i}wa_m)\bar A, \quad \mathscr{D}_m \bar\chi^{\dot\alpha} = (\partial_m - \mathrm{i}wa_m)\bar\chi^{\dot\alpha} . \tag{2.2.55}$$

As to the gauge supermultiplet, the supersymmetry transformation of the component field gauge potential A_m is obtained from the Wess–Zumino transformation of the 1-form A in (2.2.13), projected to the lowest vector component, with the result

$$\delta_{\mathrm{WZ}} a_m = \mathrm{i}(\xi\sigma_m\bar\lambda) + \mathrm{i}(\bar\xi\bar\sigma_m\lambda) . \tag{2.2.56}$$

The corresponding equations of the gaugino component fields are obtained replacing Φ with W_α and $W^{\dot\alpha}$

$$\delta_{\mathrm{WZ}} \lambda^\alpha = -(\xi\sigma^{mn})^\alpha f_{mn} + \mathrm{i}\xi^\alpha \mathbf{D} , \tag{2.2.57}$$

$$\delta_{\mathrm{WZ}} \bar\lambda_{\dot\alpha} = -(\bar\xi\bar\sigma^{mn})_{\dot\alpha} f_{mn} - \mathrm{i}\bar\xi_{\dot\alpha} \mathbf{D} , \tag{2.2.58}$$

where $f_{mn} = \partial_m a_n - \partial_n a_m = -\mathrm{i}F_{mn}|$ and we used the Abelian versions of (B.5.20) and (B.5.21). Finally, for the auxiliary component we have

$$\delta_{\mathrm{WZ}} \mathbf{D} = -\xi\sigma^m\partial_m\bar\lambda + \bar\xi\bar\sigma^m\partial_m\lambda . \tag{2.2.59}$$

Observe that these are the supersymmetry transformations which would have been obtained in the Wess–Zumino gauge of the traditional approach. This is due to the definition of Wess–Zumino transformation in terms of particular compensating gauge transformation. In this way the Wess–Zumino gauge is realized in a geometric manner.

We should like to comment briefly on the implementation of R-transformations [137], [59–61], related to a phase freedom on the superspace anticommuting coordinates, in the language employed here. As the role of $\theta, \bar{\theta}$ is now taken by the covariant spinor derivatives, we assign to the latters R-parity charges of opposite sign to those of the corresponding θ's. This way it is easy to recover the usual arguments in the discussion of properties and consequences of R-transformations in supersymmetric theories.

2.2.4. Component field actions

We have seen how component fields and their Wess–Zumino transformations are obtained from the algebra of covariant superspace derivatives and projections to lowest superfield components. This kind of mechanism is applied to the construction of supersymmetric component field actions as well.

Let us explain this with the example of the kinetic action of the chiral matter multiplet. The key idea is to consider the D-term of the gauge invariant superfield $\phi\bar{\phi}$, given as the lowest component of the superfield $D^2\bar{D}^2\bar{\phi}\phi$. To be exact, this definition differs from the earlier one by a total space–time derivative, irrelevant in the construction of invariant actions. The explicit component field action is obtained expanding the product of spinor derivatives and using the Leibniz rule. When acting on ϕ or $\bar{\phi}$ individually the ordinary covariant derivatives, D_A, transmute into gauge covariant derivatives, $\mathscr{D}_A$, giving rise to the expansion

$$D^2\bar{D}^2(\bar{\phi}\phi) = \phi\mathscr{D}^2\bar{\mathscr{D}}^2\bar{\phi} + 2(\mathscr{D}^\alpha\phi)\mathscr{D}_\alpha\bar{\mathscr{D}}^2\bar{\phi} + (\mathscr{D}^2\phi)\bar{\mathscr{D}}^2\bar{\phi} \; . \tag{2.2.60}$$

At this point the algebra of covariant derivatives intervenes. The relations

$$\mathscr{D}_\alpha\bar{\mathscr{D}}^2\bar{\phi} = -4\mathrm{i}\sigma^a_{\alpha\dot{\alpha}}\mathscr{D}_a\bar{\mathscr{D}}^{\dot{\alpha}}\bar{\phi} - 8wW_\alpha\bar{\phi} \; , \tag{2.2.61}$$

$$\mathscr{D}^2\bar{\mathscr{D}}^2\bar{\phi} = 16\mathscr{D}^a\mathscr{D}_a\bar{\phi} - 16wW_{\dot{\alpha}}\bar{\mathscr{D}}^{\dot{\alpha}}\bar{\phi} - 8w\bar{\phi}D^\alpha W_\alpha \; , \tag{2.2.62}$$

illustrate how gauge covariant derivatives will appear in the component field formalism in a completely natural way by construction. This should be contrasted with the method using explicit expansions in the anticommuting coordinates of superspace. In the approach pursued here, the component field action is simply obtained from combining (2.2.61) and (2.2.62) with (2.2.60) and projecting to lowest components, with the result

$$\begin{aligned}\frac{1}{16}D^2\bar{D}^2(\bar{\phi}\phi)| = &-\mathscr{D}^m A\mathscr{D}_m\bar{A} - \frac{\mathrm{i}}{2}(\chi\sigma^m\mathscr{D}_m\bar{\chi} + \bar{\chi}\bar{\sigma}^m\mathscr{D}_m\chi) \\ &+ F\bar{F} + w\mathbf{D}A\bar{A} + \mathrm{i}w\sqrt{2}(\bar{A}\lambda\chi - A\bar{\lambda}\bar{\chi}) \; .\end{aligned} \tag{2.2.63}$$

In this approach $D^2\bar{D}^2$ plays the role of the volume element of superspace. Again, as in the derivation of the Wess–Zumino transformations (2.2.52) and (2.2.53), the covariant space–time derivatives appear in a very natural way as a consequence of use of covariant differential calculus, without recourse to the introduction of the vector superfield V. The relation between the present formulation and the traditional one is established in Section 2.2.2.

The kinetic terms of the gauge multiplet are derived from the superfield $W^\alpha W_\alpha$ and its complex conjugate $W_{\dot{\alpha}}W^{\dot{\alpha}}$. As $W^\alpha W_\alpha$ is chiral, and $W_{\dot{\alpha}}W^{\dot{\alpha}}$ antichiral, this will be achieved by a F-term construction. The relevant superfields we have to consider are therefore $D^2(W^\alpha W_\alpha)$ and $\bar{D}^2(W_{\dot{\alpha}}W^{\dot{\alpha}})$. In the explicit evaluation we will make use of certain superfield building blocks, which are the

Abelian flat superspace versions of (B.5.20), (B.5.21) and (B.5.28), (B.5.29). Simple spinor derivatives of the gaugino superfields are given as

$$D_\beta W_\alpha = -(\sigma^{ba}\varepsilon)_{\beta\alpha} F_{ba} - \varepsilon_{\beta\alpha}\mathbf{D} , \tag{2.2.64}$$

$$\bar{D}_{\dot\beta} \bar{W}_{\dot\alpha} = -(\varepsilon\bar\sigma^{ba})_{\dot\beta\dot\alpha} F_{ba} + \varepsilon_{\dot\beta\dot\alpha}\mathbf{D} \tag{2.2.65}$$

with

$$\mathbf{D} = -\tfrac{1}{2} D^\alpha W_\alpha$$

the D-term superfield. Double spinor derivatives arising in the construction are

$$D^2 W_\alpha = 4\mathrm{i}\sigma^m_{\alpha\dot\alpha}\partial_m \bar{W}^{\dot\alpha}, \quad \bar{D}^2 \bar{W}^{\dot\alpha} = 4\mathrm{i}\bar\sigma^{m\dot\alpha\alpha}\partial_m W_\alpha . \tag{2.2.66}$$

It is then straightforward to derive

$$D^2(W^\alpha W_\alpha) = -2F^{ba}F_{ba} + 8\mathrm{i} W^\alpha \sigma^a_{\alpha\dot\alpha}\partial_a \bar{W}^{\dot\alpha} - 4\mathbf{D}^2 - \mathrm{i}\varepsilon^{dcba}F_{dc}F_{ba} , \tag{2.2.67}$$

$$\bar{D}^2(\bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha}) = -2F^{ba}F_{ba} + 8\mathrm{i} \bar{W}_{\dot\alpha}\bar\sigma^{a\dot\alpha\alpha}\partial_a W_\alpha - 4\mathbf{D}^2 + \mathrm{i}\varepsilon^{dcba}F_{dc}F_{ba} . \tag{2.2.68}$$

Projection to lowest components identifies the component field kinetic terms of the gauge multiplet in[8]

$$-\frac{1}{16}(D^2 W^\alpha W_\alpha + \bar{D}^2 \bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha})| = -\frac{1}{4} f^{mn} f_{mn} - \frac{\mathrm{i}}{2}\lambda\sigma^m\partial_m\bar\lambda - \frac{\mathrm{i}}{2}\bar\lambda\bar\sigma^m\partial_m\lambda + \frac{1}{2}\mathbf{D}^2 , \tag{2.2.69}$$

whereas the orthogonal combination yields a total space–time derivative.

So far, we have illustrated the construction of the component field Lagrangian for a chiral matter multiplet with an Abelian gauge multiplet. The discussion of the F-term construction of mass term and self-interactions of the matter multiplet, arising from the chiral superpotential and its complex conjugate will be postponed to more interesting situations.

As is clear from its supersymmetry transformation law, the component field $\mathbf{D}$ may be added to the supersymmetric action – this is the genuine Fayet–Iliopoulos D-term. In the terminology employed here, it arises from projection to the lowest component of the D-term superfield

$$\mathbf{D} = -\tfrac{1}{8} D^\alpha \bar{D}^2 D_\alpha V . \tag{2.2.70}$$

From this point of view, gauge invariance

$$V \mapsto V + \mathrm{i}(\Lambda - \bar\Lambda) \tag{2.2.71}$$

is ensured due to the fact that chiral and antichiral superfields are annihilated by the superspace volume element $D^\alpha \bar{D}^2 D_\alpha = \bar{D}_{\dot\alpha} D^2 \bar{D}^{\dot\alpha}$.

[8] The gauge coupling g may be restored explicitly: rescaling the components of the gauge multiplet by g and the pure gauge action by g^{-2}.

As we have noted above, the kinetic term of the chiral matter multiplet may be viewed as a D-term as well, identifying V with $\phi\bar{\phi}$. In this case the gauge invariance (2.2.71) indicates that the addition of holomorphic or anti-holomorphic superfield functions $F(\phi)$ or $\bar{F}(\bar{\phi})$ will not change the Lagrangian.

We have described here the simplest case of a supersymmetric gauge theory, a single chiral multiplet interacting with an Abelian gauge multiplet.

Mass terms and self-interactions of the chiral multiplet, on the other hand, would arise from a F-term construction applied to ϕ^2 and ϕ^3 and their complex conjugates, for power-counting renormalizable theories, or to holomorphic and antiholomorphic functions $W(\phi)$ and $\bar{W}(\bar{\phi})$ in more general situations. In the simplest case of a single chiral superfield with non-vanishing Abelian charge, as discussed here, this kind of superpotential terms are incompatible with gauge invariance. The construction of a non-trivial invariant superpotential requires several chiral superfields with suitably adjusted weights under gauge transformations.

For the sake of pedagogical simplicity, we will now describe the superpotential term for a single chiral superfield, restricting ourselves to the case of a self-interacting scalar multiplet in the absence of gauge couplings.

The F-term construction amounts to evaluate D^2W and project to lowest superfield components, resulting in

$$-\frac{1}{4}D^2W(\phi)| = -\frac{1}{2}\frac{\partial^2 W}{\partial A^2}(\chi\chi) + \frac{\partial W}{\partial A}F \tag{2.2.72}$$

for W and

$$-\frac{1}{4}\bar{D}^2\bar{W}(\bar{\phi})| = -\frac{1}{2}\frac{\partial^2 \bar{W}}{\partial \bar{A}^2}(\bar{\chi}\bar{\chi}) + \frac{\partial \bar{W}}{\partial \bar{A}}\bar{F}\,. \tag{2.2.73}$$

In the component field expressions, the holomorphic function W is to be considered as a function of the complex scalar A and correspondingly $\bar{W}$ as a function of $\bar{A}$. Combining the superpotential terms with the kinetic terms (2.2.63), for $w = 0$, and eliminating the auxiliary fields, $F, \bar{F}$, through their algebraic equations of motion, $F = -\partial\bar{W}/\partial\bar{A}$, we obtain the on-shell Lagrangian

$$-\partial^m A\partial_m\bar{A} - \frac{\mathrm{i}}{2}(\chi\sigma^m\partial_m\bar{\chi} + \bar{\chi}\bar{\sigma}^m\partial_m\chi) - \frac{1}{2}\frac{\partial^2 W}{\partial A^2}(\chi\chi) - \frac{1}{2}\frac{\partial^2 \bar{W}}{\partial \bar{A}^2}(\bar{\chi}\bar{\chi}) - \left|\frac{\partial W}{\partial A}\right|^2 \tag{2.2.74}$$

for a single self-interacting scalar multiplet, the last term being just the usual scalar potential contribution.

2.3. Supersymmetric Yang–Mills theories

The interplay between chiral, antichiral and real gauge transformation formulations, as encountered in the Abelian case, persists in the case of supersymmetric Yang–Mills theory. These properties are not only of academic interest, but quite useful, if not indispensable in contexts like Chern–Simons couplings or supersymmetric chiral anomalies.

The 1-form Yang–Mills gauge potential is now Lie algebra valued,

$$\mathscr{A} = E^A \mathscr{A}_A = E^A \mathscr{A}_A^{(r)} \mathbf{T}_{(r)} \ , \tag{2.3.1}$$

the generators $\mathbf{T}_{(r)}$ fulfill the commutation relations

$$[\mathbf{T}_{(r)}, \mathbf{T}_{(s)}] = \mathrm{i} c_{(r)(s)}{}^{(t)} \mathbf{T}_{(t)} \ . \tag{2.3.2}$$

Under a gauge transformation, parametrized by a matrix superfield $\boldsymbol{g}$, the gauge potential $\mathscr{A}$ transforms as

$$\mathscr{A} \mapsto \boldsymbol{g}^{-1} \mathscr{A} \boldsymbol{g} - \boldsymbol{g}^{-1} d\boldsymbol{g} \ . \tag{2.3.3}$$

Observe that this corresponds to a gauge transformation in the *real basis*, i.e. the parameters of the gauge transformations are real unconstrained superfields. The covariant field strength is defined by

$$\mathscr{F} = d\mathscr{A} + \mathscr{A}\mathscr{A} \tag{2.3.4}$$

and transforms covariantly

$$\mathscr{F} \mapsto \boldsymbol{g}^{-1} \mathscr{F} \boldsymbol{g} \ . \tag{2.3.5}$$

Its components are given by

$$\mathscr{F}_{BA} = D_B \mathscr{A}_A - (-)^{ab} D_A \mathscr{A}_B - (\mathscr{A}_B, \mathscr{A}_A) + T_{BA}{}^C \mathscr{A}_C \ , \tag{2.3.6}$$

exhibiting now, in addition to the derivative terms and the torsion term, the graded commutator $(\mathscr{A}_B, \mathscr{A}_A)$.

Due to its definition, the field strength, $\mathscr{F}$, satisfies Bianchi identities

$$\mathscr{D}\mathscr{F} = d\mathscr{F} - \mathscr{A}\mathscr{F} + \mathscr{F}\mathscr{A} = 0 \ . \tag{2.3.7}$$

Consider next generic superfields Φ and $\bar{\Phi}$ of gauge transformation

$$\bar{\Phi} \mapsto \bar{\Phi}\boldsymbol{g}, \quad \Phi \mapsto \boldsymbol{g}^{-1}\Phi \ , \tag{2.3.8}$$

so that $\bar{\Phi}\Phi$ is invariant. Covariant exterior derivatives $\mathscr{D}\Phi = E^A \mathscr{D}_A \Phi$ are defined as

$$\mathscr{D}\bar{\Phi} = d\bar{\Phi} + \bar{\Phi}\mathscr{A}, \quad \mathscr{D}\Phi = d\Phi - \mathscr{A}\Phi \ . \tag{2.3.9}$$

Double exterior covariant derivatives

$$\mathscr{D}\mathscr{D}\bar{\Phi} = +\bar{\Phi}\mathscr{F}, \quad \mathscr{D}\mathscr{D}\Phi = -\mathscr{F}\Phi$$

give rise to

$$(\mathscr{D}_B, \mathscr{D}_A)\Phi = -\mathscr{F}_{BA}\Phi - T_{BA}{}^C \mathscr{D}_C \Phi \ , \tag{2.3.10}$$

$$(\mathscr{D}_B, \mathscr{D}_A)\bar{\Phi} = +\bar{\Phi}\mathscr{F}_{BA} - T_{BA}{}^C \mathscr{D}_C \bar{\Phi} \ . \tag{2.3.11}$$

In this framework, matter fields are described by covariantly chiral superfields, i.e. we specialize the generic superfields Φ and $\bar{\Phi}$ to matter superfields ϕ and $\bar{\phi}$, which still transform under (2.3.8), but are required to be covariantly chiral and antichiral, respectively, i.e.

$$\mathscr{D}^{\dot{\alpha}}\phi = 0, \quad \mathscr{D}_{\alpha}\bar{\phi} = 0 \ . \tag{2.3.12}$$

Compatibility of these conditions with the graded commutation relations (2.3.10) and (2.3.11) above suggest to impose the constraints

$$\mathscr{F}^{\dot{\beta}\dot{\alpha}} = 0, \quad \mathscr{F}_{\beta\alpha} = 0 \ , \tag{2.3.13}$$

called *representation preserving constraints*. Furthermore, in view of the explicit expression

$$\mathscr{F}_{\beta}{}^{\dot{\alpha}} = D_{\beta}\mathscr{A}^{\dot{\alpha}} + D^{\dot{\alpha}}\mathscr{A}_{\beta} - \{\mathscr{A}_{\beta}, \mathscr{A}^{\dot{\alpha}}\} - 2\mathrm{i}(\sigma^{c}\varepsilon)_{\beta}{}^{\dot{\alpha}}\mathscr{A}_{c} \ , \tag{2.3.14}$$

the constraint

$$\mathscr{F}_{\beta}{}^{\dot{\alpha}} = 0 \tag{2.3.15}$$

just corresponds to a linear covariant redefinition of the vector component, $\mathscr{A}_a$, of the connection superfield. For this reason it is called a *conventional constraint*.

As in the Abelian case, the constraints are solved in terms of pre-potentials. The representation preserving constraints (2.3.13) suggest to express the spinor components of $\mathscr{A}$ as

$$\mathscr{A}_{\alpha} = -\mathscr{T}^{-1}D_{\alpha}\mathscr{T}, \quad \mathscr{A}^{\dot{\alpha}} = -\mathscr{U}^{-1}D^{\dot{\alpha}}\mathscr{U} \ , \tag{2.3.16}$$

in terms of pre-potential superfields $\mathscr{U}$ and $\mathscr{T}$. Their gauge transformations should be adjusted such that they reproduce those of the gauge potentials themselves, that is

$$\mathscr{T} \mapsto \bar{\mathscr{P}}\mathscr{T}\boldsymbol{g}, \quad \mathscr{U} \mapsto \mathscr{Q}\mathscr{U}\boldsymbol{g} \ . \tag{2.3.17}$$

Here, $\bar{\mathscr{P}}$ and $\mathscr{Q}$ denote the pre-gauge transformations and are, respectively, antichiral and chiral superfields.

Recall that $\mathscr{A}$ is the gauge potential in the real basis of gauge transformations; by construction, it is inert under the chiral and antichiral pre-gauge transformations. On the other hand, pre-potential-dependent redefinitions of $\mathscr{A}$, which have the form of gauge transformations,

$$\mathscr{A}(1) = \mathscr{T}\mathscr{A}\mathscr{T}^{-1} - \mathscr{T}d\mathscr{T}^{-1} \ , \tag{2.3.18}$$

$$\mathscr{A}(0) = \mathscr{U}\mathscr{A}\mathscr{U}^{-1} - \mathscr{U}d\mathscr{U}^{-1} \ , \tag{2.3.19}$$

give rise to new gauge potentials which are inert under the original $\boldsymbol{g}$ gauge transformations and transform under chiral (resp. antichiral) gauge transformations, i.e.

$$\mathscr{A}(1) \mapsto \bar{\mathscr{P}}\mathscr{A}(1)\bar{\mathscr{P}}^{-1} - \bar{\mathscr{P}}d\bar{\mathscr{P}}^{-1} \ , \tag{2.3.20}$$

$$\mathscr{A}(0) \mapsto \mathscr{Q}\mathscr{A}(0)\mathscr{Q}^{-1} - \mathscr{Q}d\mathscr{Q}^{-1} \ . \tag{2.3.21}$$

The connections $\mathscr{A}(1) = E^A \mathscr{A}_A(1)$ and $\mathscr{A}(0) = E^A \mathscr{A}_A(0)$ take a particularly simple form

$$\mathscr{A}^{\dot\alpha}(1) = -\mathscr{W} D^{\dot\alpha} \mathscr{W}^{-1}, \quad \mathscr{A}_\alpha(1) = 0, \quad \mathscr{A}_{\alpha\dot\alpha}(1) = \frac{i}{2} D_\alpha \mathscr{A}_{\dot\alpha}(1) , \tag{2.3.22}$$

$$\mathscr{A}_\alpha(0) = -\mathscr{W}^{-1} D_\alpha \mathscr{W}, \quad \mathscr{A}^{\dot\alpha}(0) = 0, \quad \mathscr{A}_{\alpha\dot\alpha}(0) = \frac{i}{2} D_{\dot\alpha} \mathscr{A}_\alpha(0) , \tag{2.3.23}$$

expressed in terms of the combination

$$\mathscr{W} = \mathscr{T} \mathscr{U}^{-1} \tag{2.3.24}$$

with gauge transformations

$$\mathscr{W} \mapsto \bar{\mathscr{P}} \mathscr{W} \mathscr{Q}^{-1} . \tag{2.3.25}$$

The corresponding change of basis on the covariant chiral and antichiral superfields ϕ and $\bar\phi$ is achieved via the redefinitions which have the form of gauge transformations as well, such that

$$\phi(1) = \mathscr{T} \phi, \quad \bar\phi(1) = \bar\phi \mathscr{T}^{-1} , \tag{2.3.26}$$

$$\phi(0) = \mathscr{U} \phi, \quad \bar\phi(0) = \bar\phi \mathscr{U}^{-1} . \tag{2.3.27}$$

In this case, we also obtain particularly simple chirality conditions for $\phi(0)$ and $\bar\phi(1)$. The invariant combination $\bar\phi\phi$ behaves under this change of bases as

$$\bar\phi\phi = \bar\phi(1) \mathscr{W} \phi(0) . \tag{2.3.28}$$

The right-hand side of this equation corresponds to the traditional formulation in terms of simply chiral (resp. antichiral) fields, explicitly

$$D^{\dot\alpha} \phi(0) = 0, \quad D_\alpha \bar\phi(1) = 0 . \tag{2.3.29}$$

The superfield $\mathscr{W}$ provides the bridge between the chiral and antichiral bases. Setting $\mathscr{W} = \exp 2V$, we recover the usual description of supersymmetric Yang–Mills theories.

As before, the components of the field strength $\mathscr{F}_{\beta a}$, $\mathscr{F}^{\dot\beta}{}_a$ and $\mathscr{F}_{ba}$ can be expressed in terms of two superfields $\mathscr{W}_\alpha$, $\mathscr{W}^{\dot\alpha}$ and their spinor derivatives, namely

$$\mathscr{F}_{\beta a} = +i\sigma_{a\beta\dot\beta} \mathscr{W}^{\dot\beta} , \tag{2.3.30}$$

$$\mathscr{F}^{\dot\beta}{}_a = -i\bar\sigma_a^{\dot\beta\beta} \mathscr{W}_\beta , \tag{2.3.31}$$

$$\mathscr{F}_{ba} = \tfrac{1}{2}(\varepsilon\sigma_{ba})^{\beta\alpha} \mathscr{D}_\alpha \mathscr{W}_\beta + \tfrac{1}{2}(\bar\sigma_{ba}\varepsilon)^{\dot\beta\dot\alpha} \mathscr{D}_{\dot\alpha} \mathscr{W}_{\dot\beta} . \tag{2.3.32}$$

The gaugino superfields $\mathscr{W}_\alpha$ and $\mathscr{W}^{\dot\alpha}$ fulfill

$$\mathscr{D}_\alpha \mathscr{W}^{\dot\alpha} = 0, \quad \mathscr{D}^{\dot\alpha} \mathscr{W}_\alpha = 0 , \tag{2.3.33}$$

$$\mathscr{D}^\alpha \mathscr{W}_\alpha = \mathscr{D}_{\dot\alpha} \mathscr{W}^{\dot\alpha} , \tag{2.3.34}$$

as a result of Bianchi identities.

The superfields $\mathscr{W}_\alpha$ and $\mathscr{W}^{\dot\alpha}$ are the building blocks of the kinetic terms for the supersymmetric Yang–Mills action. Recall the field content of the Yang–Mills gauge multiplet: it consists of the gauge potentials $\boldsymbol{a}_m(x)$, the gauginos $\lambda(x), \bar\lambda(x)$, which are Majorana spinors, and the auxiliary scalars $\mathbf{D}(x)$. All these component fields are Lie-algebra valued, they are identified in the gaugino superfields $\mathscr{W}^\alpha$ and $\mathscr{W}_{\dot\alpha}$, subject to the constraint conditions (2.3.33) and (2.3.34).

The component fields are defined as lowest components of superfields; for the gauge potential we have

$$\mathscr{A}_m| = \mathrm{i}\boldsymbol{a}_m \;, \tag{2.3.35}$$

whereas the gaugino component fields are defined as the lowest components of the gaugino superfields themselves,

$$\mathscr{W}_\alpha| = -\mathrm{i}\lambda_\alpha, \quad \mathscr{W}^{\dot\alpha}| = \mathrm{i}\bar\lambda^{\dot\alpha} \,. \tag{2.3.36}$$

The Yang–Mills field strength $\boldsymbol{f}_{mn} = \partial_m \boldsymbol{a}_n - \partial_n \boldsymbol{a}_m - \mathrm{i}[\boldsymbol{a}_m, \boldsymbol{a}_n]$ and the auxiliary field $\mathbf{D}(x)$ appear at the linear level in the superfield expansion

$$\begin{aligned} \mathscr{D}_\beta \mathscr{W}_\alpha| &= -\mathrm{i}(\sigma^{mn}\varepsilon)_{\beta\alpha}\boldsymbol{f}_{mn} - \varepsilon_{\beta\alpha}\,\mathbf{D}(x) \,, \\ \mathscr{D}_{\dot\beta} \mathscr{W}_{\dot\alpha}| &= -\mathrm{i}(\varepsilon\bar\sigma^{mn})_{\dot\beta\dot\alpha}\boldsymbol{f}_{mn} + \varepsilon_{\dot\beta\dot\alpha}\,\mathbf{D}(x) \,, \end{aligned} \tag{2.3.37}$$

this means that the auxiliary field is identified as

$$\mathscr{D}^\alpha \mathscr{W}_\alpha| = \mathscr{D}_{\dot\alpha} \mathscr{W}^{\dot\alpha}| = -2\mathbf{D}(x) \,. \tag{2.3.38}$$

The Lagrangian for pure Yang–Mills gauge theory is then given by (we often use the shorthand notation $\mathscr{W}^2 = \mathscr{W}^\alpha \mathscr{W}_\alpha$ and $\bar{\mathscr{W}}^2 = \mathscr{W}_{\dot\alpha}\mathscr{W}^{\dot\alpha}$)

$$\mathscr{L} = -\tfrac{1}{16}D^2\,\mathrm{tr}(\mathscr{W}^2) - \tfrac{1}{16}\bar D^2\,\mathrm{tr}(\bar{\mathscr{W}}^2) \,. \tag{2.3.39}$$

As in the Abelian case, the gauge invariant product $\bar\phi\phi$ provides both the kinetic terms for matter superfields and their minimal supersymmetric coupling to Yang–Mills fields.

2.4. Supersymmetry and Kähler manifolds

As explained by Zumino, supersymmetric non-linear sigma models have necessarily a Kähler structure [164]. The complex scalars of the chiral matter multiplets have an interpretation as complex coordinates of a Kähler manifold and the supersymmetric component field Lagrangian is given as[9]

$$\begin{aligned} \mathscr{L}_{\text{Kähler}} = &- g_{i\bar\jmath}\eta^{mn}\partial_m A^i \partial_n \bar A^{\bar\jmath} - \frac{\mathrm{i}}{2} g_{i\bar\jmath}(\chi^i\sigma^m \mathscr{D}_m \bar\chi^{\bar\jmath}) + \frac{\mathrm{i}}{2} g_{i\bar\jmath}(\mathscr{D}_m\chi^i\sigma^m\bar\chi^{\bar\jmath}) \\ &+ \frac{1}{4}R_{i\bar\imath j\bar\jmath}(\chi^i\chi^j)(\bar\chi^{\bar\imath}\bar\chi^{\bar\jmath}) + g_{i\bar\jmath}F^i\bar F^{\bar\jmath} \,. \end{aligned} \tag{2.4.1}$$

[9] For the sake of clarity, we consider, here, only matter multiplets without gauge couplings. Couplings to Yang–Mills theory will be constructed later on, in the context of the complete supergravity/matter/Yang–Mills system in Section 4, and gauged isometries described in Appendix C.

As a function of the scalar fields A^i and $\bar{A}^{\bar{j}}$, the Kähler metric $g_{i\bar{j}}$ derives from a Kähler potential. The covariant derivatives

$$\mathscr{D}_m \chi^i_\alpha = \partial_m \chi^i_\alpha + \Gamma^i{}_{kl} \partial_m A^k \chi^l_\alpha , \quad \mathscr{D}_m \bar{\chi}^{\bar{j}\dot{\alpha}} = \partial_m \bar{\chi}^{\bar{j}\dot{\alpha}} + \Gamma^{\bar{i}}{}_{\bar{k}\bar{l}} \partial_m A^{\bar{k}} \bar{\chi}^{\bar{l}\dot{\alpha}} , \tag{2.4.2}$$

contain the Levi–Civita symbols ($g_{i\bar{\imath},k}$ denotes the derivative of $g_{i\bar{\imath}}$ with respect to A^k)

$$\Gamma^i{}_{kl} = g^{i\bar{\imath}} g_{k\bar{\imath},l} , \quad \Gamma^{\bar{\imath}}{}_{\bar{k}\bar{l}} = g^{i\bar{\imath}} g_{i\bar{k},\bar{l}} , \tag{2.4.3}$$

whereas the quartic spinor terms exhibit the curvature tensor of the Kähler manifold,

$$R_{i\bar{\imath}j\bar{j}} = g_{i\bar{\imath},j\bar{j}} - g_{k\bar{k}} \Gamma^k{}_{ij} \Gamma^{\bar{k}}{}_{\bar{\imath}\bar{j}} . \tag{2.4.4}$$

The auxiliary fields, here, correspond to those of the diagonalized version in [164]; more details will be given below. The supersymmetry transformations of the chiral multiplet, which leave the action invariant are given as

$$\delta A^i = \sqrt{2} \xi \chi^i , \tag{2.4.5}$$

$$\delta \chi^i_\alpha = + \mathrm{i} \sqrt{2} (\bar{\xi} \bar{\sigma}^m \varepsilon)_\alpha \partial_m A^i + \sqrt{2} \xi_\alpha F^i , \tag{2.4.6}$$

$$\delta F^i = \mathrm{i} \sqrt{2} (\bar{\xi} \bar{\sigma}^m)^\alpha \mathscr{D}_m \chi^i_\alpha . \tag{2.4.7}$$

As pointed out by Zumino in the same paper, the structure of the supersymmetric non-linear sigma model is most conveniently understood in the language of superfields. As he explained, the lowest component of the superfield

$$\mathscr{L}_{\text{Kähler}} = \tfrac{1}{16} D^\alpha \bar{D}^2 D_\alpha K(\phi, \bar{\phi}) ,$$

reproduces exactly the component field Lagrangian given above. In other words, the kinetic Lagrangian may be understood as a Fayet–Iliopoulos D-term. The Kähler metric, defined as the lowest superfield component of (using the same symbols for the component and the superfield)

$$g_{k\bar{k}} = \frac{\partial^2 K}{\partial \phi^k \partial \bar{\phi}^{\bar{k}}} , \tag{2.4.8}$$

the Levi–Civita symbol and the Kähler curvature appear in the process of successive application of spinor derivatives and subsequent projection to lowest components. Chirality of the matter superfields and the fact that the differential operator $D^\alpha \bar{D}^2 D_\alpha = D_{\dot{\alpha}} D^2 D^{\dot{\alpha}}$ annihilates chiral superfields, imply invariance under the superfield Kähler transformations

$$K(\phi, \bar{\phi}) \mapsto K(\phi, \bar{\phi}) + F(\phi) + \bar{F}(\bar{\phi}) . \tag{2.4.9}$$

This shows that, in fact, the Kähler manifold is spanned by the chiral (resp. antichiral) matter superfields ϕ^i and $\bar{\phi}^{\bar{j}}$, i.e. a mapping from superspace into the Kähler manifold. Complex structure on the one hand, in Kähler geometry and chirality conditions on the other hand, in supersymmetry, give rise to intriguing analogies [98].

In the following, we will elaborate somewhat more on these geometric superspace aspects, which will be of essential importance later on in the context of supergravity/matter coupling. The properties of the pre-potential V in the $(\frac{1}{2},\frac{1}{2})$ basis of Abelian gauge theory – cf. Section 2.2.2 – suggest to interpret $K(\phi,\bar{\phi})$ as a particular, superfield dependent, pre-potential.[10]

Replacing the unconstrained pre-potential V by the Kähler potential $K(\phi,\bar{\phi})$ we define[11]

$$A_\alpha = +\tfrac{1}{4}D_\alpha K = +\tfrac{1}{4}K_k D_\alpha \phi^k \,, \tag{2.4.10}$$

$$A^{\dot{\alpha}} = -\tfrac{1}{4}D^{\dot{\alpha}} K = -\tfrac{1}{4}K_{\bar{k}} D^{\dot{\alpha}} \bar{\phi}^{\bar{k}} \,. \tag{2.4.11}$$

Here K_k (resp. $K_{\bar{k}}$) denote derivatives of the Kähler potential with respect to the superfield coordinates ϕ^k and $\bar{\phi}^{\bar{k}}$. Following the construction in Abelian gauge theory we define furthermore

$$A_{\alpha\dot{\alpha}} = \frac{\mathrm{i}}{2}(D_\alpha A_{\dot{\alpha}} + D_{\dot{\alpha}} A_\alpha) \,. \tag{2.4.12}$$

This corresponds to a conventional constraint. Substituting for A_α and $A_{\dot{\alpha}}$ yields

$$A_a = \frac{1}{4}(K_i \partial_a \phi^i - K_{\bar{j}} \partial_a \bar{\phi}^{\bar{j}}) + \frac{\mathrm{i}}{8}\bar{\sigma}_a^{\dot{\alpha}\alpha} g_{i\bar{j}} D_\alpha \phi^i D_{\dot{\alpha}} \bar{\phi}^{\bar{j}} \,. \tag{2.4.13}$$

The expressions for $A_\alpha, A^{\dot{\alpha}}$ and A_a can be subsumed compactly in superform language,

$$A = \frac{1}{4}(K_i d\phi^i - K_{\bar{j}} d\bar{\phi}^{\bar{j}}) + \frac{\mathrm{i}}{8}E^a \bar{\sigma}_a^{\dot{\alpha}\alpha} g_{i\bar{j}} D_\alpha \phi^i D_{\dot{\alpha}} \bar{\phi}^{\bar{j}} \,. \tag{2.4.14}$$

Let us note that this potential, A, transforms as it should (i.e. as a connection) under Kähler transformations,

$$A \mapsto A + \frac{\mathrm{i}}{2} d\, \mathrm{Im}\, F \,. \tag{2.4.15}$$

We can now apply the machinery of Abelian gauge structure in superspace to determine the component field action as the corresponding D-term. First, applying the exterior derivative to A gives the composite field strength 2-form

$$F = dA = \frac{1}{2} g_{i\bar{j}}\, d\phi^i d\bar{\phi}^{\bar{j}} + \frac{\mathrm{i}}{8} d(E^a \bar{\sigma}_a^{\dot{\alpha}\alpha} g_{i\bar{j}} D_\alpha \phi^i D_{\dot{\alpha}} \bar{\phi}^{\bar{j}}) \,. \tag{2.4.16}$$

[10] It should however be noted that, in distinction to Section 2.2.2, there are no phase transformations on the matter fields corresponding to Kähler transformations. In the language of Section 2.2.2, all the matter fields have weight zero. Non-trivial Kähler phase transformations will only appear later on in the coupling of matter to supergravity.

[11] Normalizations are chosen for later convenience in the supergravity/matter system.

As in the generic Abelian case, the coefficients of F are expressed in terms of a single Weyl spinor and its complex conjugate, in particular

$$F_{\beta a} = +\frac{i}{2}\sigma_{a\beta\dot{\beta}}\bar{X}^{\dot{\beta}}, \quad F^{\dot{\beta}}{}_{a} = -\frac{i}{2}\bar{\sigma}_{a}^{\dot{\beta}\beta}X_{\beta} . \tag{2.4.17}$$

On the one hand, X_α and $\bar{X}^{\dot{\alpha}}$ are given in terms of the Kähler potential

$$X_\alpha = -\tfrac{1}{8}\bar{D}^2 D_\alpha K, \quad \bar{X}^{\dot{\alpha}} = -\tfrac{1}{8}D^2 D^{\dot{\alpha}} K , \tag{2.4.18}$$

on the other hand, identifying (2.4.17) in (2.4.16), we obtain

$$X_\alpha = -\frac{i}{2}g_{i\bar{j}}\sigma^a_{\alpha\dot{\alpha}}\partial_a\phi^i D^{\dot{\alpha}}\bar{\phi}^{\bar{j}} + \frac{1}{2}g_{i\bar{j}}D_\alpha\phi^i\bar{F}^{\bar{j}} , \tag{2.4.19}$$

$$\bar{X}^{\dot{\alpha}} = -\frac{i}{2}g_{k\bar{k}}\bar{\sigma}^{a\dot{\alpha}\alpha}\partial_a\bar{\phi}^{\bar{j}}D_\alpha\phi^i + \frac{1}{2}g_{i\bar{j}}D^{\dot{\alpha}}\bar{\phi}^{\bar{j}}F^i . \tag{2.4.20}$$

Here we used the definitions

$$F^i = -\tfrac{1}{4}\mathscr{D}^\alpha D_\alpha\phi^i, \quad \bar{F}^{\bar{j}} = -\tfrac{1}{4}\mathscr{D}_{\dot{\alpha}}D^{\dot{\alpha}}\bar{\phi}^{\bar{j}} \tag{2.4.21}$$

with second covariant derivatives defined as

$$\mathscr{D}_B D_\alpha\phi^i = D_B D_\alpha\phi^i + \Gamma^i{}_{kl}D_B\phi^k D_\alpha\phi^l , \tag{2.4.22}$$

$$\mathscr{D}_B D^{\dot{\alpha}}\bar{\phi}^{\bar{j}} = D_B D^{\dot{\alpha}}\bar{\phi}^{\bar{j}} + \Gamma^{\bar{j}}{}_{\bar{k}\bar{l}}D_B\bar{\phi}^{\bar{k}}D^{\dot{\alpha}}\bar{\phi}^{\bar{l}} , \tag{2.4.23}$$

assuring covariance with respect to Kähler transformations and (ungauged) isometries of the Kähler manifold. Observe that, in terms of these definitions, the component field Lagrangian will come out to be diagonal in the auxiliary fields [164]. Due to their definition, the superfields $X_\alpha, \bar{X}^{\dot{\alpha}}$ have the following properties:

$$D^{\dot{\alpha}}X_\alpha = 0, \quad D_\alpha\bar{X}^{\dot{\alpha}} = 0, \tag{2.4.24}$$

$$D^\alpha X_\alpha = D_{\dot{\alpha}}\bar{X}^{\dot{\alpha}} . \tag{2.4.25}$$

It is then easy to obtain the superfield expression of the Kähler D-term

$$-\frac{1}{2}D^\alpha X_\alpha = -\eta^{ab}g_{i\bar{j}}\partial_b\phi^i\partial_a\bar{\phi}^{\bar{j}} - \frac{i}{4}g_{i\bar{j}}\sigma^a_{\alpha\dot{\alpha}}D^\alpha\phi^i\mathscr{D}_a D^{\dot{\alpha}}\bar{\phi}^{\bar{j}}$$
$$- \frac{i}{4}g_{i\bar{j}}\sigma^a_{\alpha\dot{\alpha}}D^{\dot{\alpha}}\bar{\phi}^{\bar{j}}\mathscr{D}_a D^\alpha\phi^i + g_{i\bar{j}}F^i\bar{F}^{\bar{j}} + \frac{1}{16}R_{i\bar{i}j\bar{j}}D^\alpha\phi^i D_\alpha\phi^j D_{\dot{\alpha}}\bar{\phi}^{\bar{i}}D^{\dot{\alpha}}\bar{\phi}^{\bar{j}} \tag{2.4.26}$$

with covariant derivatives defined above in (2.4.22), (2.4.23) and the curvature tensor given in (2.4.4). Projection of this last equation to lowest superfield components results in the component field Lagrangian (2.4.1). The construction presented here will be generalized later on and applied to the full supergravity/matter/Yang–Mills system.

3. Matter in curved superspace

The formulation of supersymmetry as a local symmetry naturally leads to supergravity, where the graviton, of helicity 2, has a fermionic partner, the gravitino, of helicity 3/2. The corresponding local fields are the vierbein $e_m{}^a(x)$ and the Rarita–Schwinger field $\psi_m{}^\alpha(x)$, $\bar{\psi}_{m\dot{\alpha}}(x)$. As mentioned in Section 2.1.1, the different $D = 4, N = 1$ supergravity multiplets (minimal, new minimal and non-minimal) all contain the graviton and the gravitino, but differ by their systems of auxiliary fields.

In the geometric formulation of supergravity, the vierbein $e_m{}^a(x)$ is generalized to the frame superfield $E_M{}^A$ in superspace, describing the graviton and the gravitino in a unified way. The three different supergravity multiplets, as well as the coupling of minimal supergravity to matter, which will be presented here, are then derived from a superspace geometry in suitably choosing the structure group and torsion constraints.

The choice of a structure group, which we take to be the product of Lorentz and chiral $U(1)$ transformations, already determines the properties of superspace geometry to a large extent.

Further specification derives from requiring appropriate covariant constraints on the torsion and curvature tensors, which, given the extension of the notion of space–time to superspace, acquire a plethora of new components. One distinguishes between *geometric* and *dynamical* constraints. Geometric constraints help to restrict the properties of superspace geometry without leading to any dynamics, i.e. to any equation of motion. Dynamical constraints may then be imposed as further restrictions which imply equations of motion.

Geometric constraints come in two categories: first, the so-called conventional constraints which are used to absorb part of the torsion in covariant redefinitions of the Lorentz and $U(1)$ connection and of the frame of superspace; second, the so-called representation preserving constraints, which arise from consistency conditions for covariant chiral superfields (essential for the description of supergravity/matter couplings) with their commutation relations.

Different supergravity multiplets (minimal, new minimal or non-minimal) are obtained from different kinds of geometric constraints.

As emphasized in the introduction, we will only consider the *minimal multiplet of supergravity*, whose superspace description is briefly recalled in Section 3.1.

We will then show in some detail how supergravity/matter/Yang–Mills couplings are obtained from a unified geometric setting by including superfield Kähler transformations in the structure group.

In Section 3.2 we show explicitly how this formulation can be obtained from the conventional one, [38–40,42], by means of field-dependent superfield rescalings. This leads in a natural way to the identification of the supergravity/matter system as a special case of $U(1)$ superspace geometry whose structure is reviewed in Section 3.3. In Section 3.4, we identify Kähler superspace as a special case of $U(1)$ superspace geometry, define supergravity transformations and present invariant actions and equations of motion at the superfield level.

3.1. Minimal supergravity

In supergravity, the dynamical degrees of freedom are the graviton and the gravitino. They are identified as the local frame of space–time or vierbein, $e_m{}^a(x)$, and the Rarita–Schwinger [133] field $\psi_m{}^\alpha(x)$, $\bar{\psi}_{m\dot{\alpha}}(x)$. The supergravity action [50,75] is then defined as a certain combination of the Einstein and Rarita–Schwinger actions, invariant under space–time-dependent supersymmetry transformations relating the graviton and the gravitino. The commutators of these transformations only close on-shell, i.e. modulo equations of motion. In minimal supergravity [147,67], a complex scalar M, $\bar{M}$ and a real vector b_a are added as auxiliary fields to avoid the appearance of the equations of motion at the geometric level and to define an off-shell theory.

The formulation of supergravity in superspace [2,157] provides a unified description of the vierbein and the Rarita–Schwinger fields. They are identified in a common geometric object, the local frame of superspace,

$$E^A = dz^M E_M{}^A(z) \; , \tag{3.1.1}$$

defined as a 1-form over superspace, with coefficient superfields $E_M{}^A(z)$, generalizing the usual frame, $e^a = dx^m e_m{}^a(x)$, which is a space–time differential form. Vierbein and Rarita–Schwinger fields are identified as lowest superfield components, such that

$$e_m{}^a(x) = E_m{}^a|, \quad \tfrac{1}{2}\psi_m{}^\alpha(x) = E_m{}^\alpha|, \quad \tfrac{1}{2}\bar{\psi}_{m\dot{\alpha}}(x) = E_{m\dot{\alpha}}| \; . \tag{3.1.2}$$

Correspondingly, as in ordinary gravity, one introduces supercoordinate transformations, thus unifying the usual general coordinate transformations and the local supersymmetry transformations as their vector and spinor parts, respectively. Local Lorentz transformations act through their vector and spinor representations on E^a and $E^\alpha, E_{\dot{\alpha}}$.

Covariant derivatives with respect to local Lorentz transformations are constructed by means of the spin connection, which is a 1-form in superspace as well,

$$\phi_B{}^A = dz^M \phi_{MB}{}^A(z) \; . \tag{3.1.3}$$

It takes values in the Lie algebra of the Lorentz group such that its spinor components are given in terms of the vector ones as

$$\phi_\beta{}^\alpha = -\tfrac{1}{2}(\sigma^{ba})_\beta{}^\alpha \phi_{ba}, \quad \phi^{\dot{\beta}}{}_{\dot{\alpha}} = -\tfrac{1}{2}(\bar{\sigma}^{ba})^{\dot{\beta}}{}_{\dot{\alpha}} \phi_{ba} \; . \tag{3.1.4}$$

These are the basic geometric objects in the superspace description of supergravity. The covariant exterior derivative of the frame in superspace,

$$T^A = dE^A + E^B \phi_B{}^A \; , \tag{3.1.5}$$

defines torsion in superspace as a 2-form

$$T^A = \tfrac{1}{2} E^B E^C T_{CB}{}^A \; . \tag{3.1.6}$$

Likewise, the covariant expression

$$R_B{}^A = d\phi_B{}^A + \phi_B{}^C \phi_C{}^A \tag{3.1.7}$$

defines the curvature 2-form in superspace

$$R_B{}^A = \tfrac{1}{2} E^C E^D R_{DCB}{}^A \ . \tag{3.1.8}$$

It is a special feature of supergravity that the curvature tensor is completely expressed in terms of the torsion and its derivatives [53]. We do not intend here to give a complete and detailed review of this geometric structure; for a detailed exposition we refer to [153].

Recall that superspace torsion is subject to covariant constraints [88,160] which imply that all the coefficients of torsion are given in terms of the covariant supergravity superfields

$$R, \quad R^\dagger, \quad G_a, \quad W_{\underset{\smile}{\gamma\beta\alpha}}, \quad W_{\underset{\smile}{\dot\gamma\dot\beta\dot\alpha}} \tag{3.1.9}$$

and their covariant derivatives. To be more explicit, the non-vanishing components of superspace torsion are

$$T_\gamma{}^{\dot\beta a} = -2\mathrm{i}(\sigma^a \varepsilon)_\gamma{}^{\dot\beta} \ , \tag{3.1.10}$$

$$T_{\gamma b \dot\alpha} = -\mathrm{i}\sigma_{b\gamma\dot\alpha} R^\dagger, \quad T^{\dot\gamma}{}_b{}^\alpha = -\mathrm{i}\bar\sigma_b^{\dot\gamma\alpha} R \ , \tag{3.1.11}$$

$$T_{\gamma b}{}^\alpha = \frac{\mathrm{i}}{2} G^c (\sigma_c \bar\sigma_b)_\gamma{}^\alpha + \frac{3\mathrm{i}}{2}\delta_\gamma{}^\alpha G_b, \quad T^{\dot\gamma}{}_{b\dot\alpha} = -\frac{\mathrm{i}}{2} G^c(\bar\sigma_c \sigma_b)^{\dot\gamma}{}_{\dot\alpha} - \frac{3\mathrm{i}}{2}\delta^{\dot\gamma}{}_{\dot\alpha} G_b \ . \tag{3.1.12}$$

As for $T_{cb}{}^\alpha$ and $T_{cb\dot\alpha}$, they will be interpreted later on as the covariant Rarita–Schwinger field strength superfields. They involve the superfields $W_{\underset{\smile}{\gamma\beta\alpha}}$ and $W_{\underset{\smile}{\dot\gamma\dot\beta\dot\alpha}}$ called Weyl tensor superfields, because they occur in the decomposition of these Rarita–Schwinger superfields in very much the same way as the usual Weyl tensor occurs in the decomposition of the covariant curvature tensor.

The auxiliary component fields mentioned above appear as lowest components in the basic superfields R, $R^\dagger$ and G_a such that

$$M(x) = -6R|, \quad \bar M(x) = -6R^\dagger|, \quad b_a = -3G_a| \ . \tag{3.1.13}$$

Consistency of the superspace Bianchi identities with the special form of the torsion components displayed so far implies the chirality conditions

$$\mathscr{D}_\alpha R^\dagger = 0, \quad \mathscr{D}_{\dot\alpha} R = 0 \ , \tag{3.1.14}$$

$$\mathscr{D}_\alpha W_{\underset{\smile}{\dot\gamma\dot\beta\dot\alpha}} = 0, \quad \mathscr{D}_{\dot\alpha} W_{\underset{\smile}{\gamma\beta\alpha}} = 0 \ , \tag{3.1.15}$$

as well as the relations

$$\mathscr{D}_\alpha R = \mathscr{D}^{\dot\alpha} G_{\alpha\dot\alpha}, \quad \mathscr{D}^{\dot\alpha} R^\dagger = -\mathscr{D}_\alpha G^{\alpha\dot\alpha} \ . \tag{3.1.16}$$

Moreover,

$$\mathscr{D}^2 R + \bar{\mathscr{D}}^2 R^\dagger = -\tfrac{2}{3}\mathscr{R} + 4G^a G_a + 32R^\dagger R \ , \tag{3.1.17}$$

where $\mathscr{R} \equiv R_{ab}{}^{ab}$ is the curvature scalar superfield. This relation is at the heart of the construction of the supersymmetric component field action. On the other hand, the orthogonal combination

$$\mathscr{D}^2 R - \bar{\mathscr{D}}^2 R^\dagger = 4\mathrm{i}\mathscr{D}_a G^a \tag{3.1.18}$$

is a consequence of (3.1.16), it has an intriguing resemblance with the 3-form constraint in superspace – cf. (6.1.2).

The component field Lagrangian is obtained from the superspace integral [158]

$$\mathscr{L}_{\text{supergravity}} = \int E \,, \tag{3.1.19}$$

where $\int E$ stands for $\int d^2\theta d^2\bar{\theta} E$, and E denotes the superdeterminant of $E_M{}^A$. Integration over $d^2\theta d^2\bar{\theta}$ yields the usual curvature scalar term, $-\frac{1}{2}e\mathscr{R}$, together with all the other terms necessary for the supersymmetric completion, with the usual canonical normalization.

3.2. Superfield rescaling

In the conventional superfield approach [152] to the coupling of matter fields to supergravity, the superspace action for the kinetic terms is taken to be

$$\mathscr{L}_{\text{kin}} = -3\int E \mathrm{e}^{-(1/3)K(\phi,\bar{\phi})} \,. \tag{3.2.1}$$

Given (3.1.19) we may hope that, by a suitable modification of the superspace geometry, the factor $\exp(-K(\phi,\bar{\phi})/3)$ can be absorbed into E; however this will be possible only if there are symmetries which allow such a modification, so let us analyze the situation in that respect. Supersymmetry transformations as well as general coordinate transformations are encoded in the diffeomorphisms of superspace; precisely the action (3.2.1) is invariant under superdiffeomorphisms and thereby under supersymmetry and general coordinate transformations. The superspace geometry relevant to (3.2.1) is that of the so-called minimal supergravity multiplet. The structure group in superspace in this case is the Lorentz group. By construction, (3.2.1) is Lorentz invariant.

In addition to superdiffeomorphisms and Lorentz transformations, which are symmetries of the kinetic action (3.2.1), superspace geometry allows also for a generalization of dilatation transformations to the supersymmetric case, which are known as super-Weyl or Howe–Tucker transformations [106]. These are defined as transformations of the frame in superspace and of the Lorentz superfield connection which respect the torsion constraints and reduce to ordinary dilatations when supersymmetry is switched off.

As a result, for the minimal supergravity multiplet, they change the frame of superspace in such a way that

$$E_M{}^a \mapsto E_M{}^a \mathrm{e}^{\Sigma+\bar{\Sigma}} \tag{3.2.2}$$

$$E_M{}^\alpha \mapsto \mathrm{e}^{2\bar{\Sigma}-\Sigma}\left(E_M{}^\alpha + \frac{\mathrm{i}}{2} E_M{}^b (\varepsilon\sigma_b)^\alpha{}_{\dot{\alpha}} \mathscr{D}^{\dot{\alpha}} \bar{\Sigma}\right), \tag{3.2.3}$$

$$E_{M\dot{\alpha}} \mapsto \mathrm{e}^{2\Sigma-\bar{\Sigma}}\left(E_{M\dot{\alpha}} + \frac{\mathrm{i}}{2} E_M{}^b (\varepsilon\bar{\sigma}_b)_{\dot{\alpha}}{}^\alpha \mathscr{D}_\alpha \Sigma\right). \tag{3.2.4}$$

The chirality conditions

$$\mathscr{D}_\alpha \bar{\Sigma} = 0, \quad \mathscr{D}^{\dot{\alpha}} \Sigma = 0 , \tag{3.2.5}$$

of the superfield parameters $\bar{\Sigma}$ and Σ are a characteristic feature of the superspace geometry of minimal supergravity, i.e. of the torsion constraints which model it.

As a consequence of (3.2.2)–(3.2.4), the superdeterminant of the frame in superspace is subject to the following super-Weyl transformations:

$$E \mapsto E e^{2(\Sigma + \bar{\Sigma})} . \tag{3.2.6}$$

Since the Kähler potential $K(\phi, \bar{\phi})$ is inert under super-Weyl transformations, (3.2.6) indicates that the kinetic action (3.2.1) is not invariant.

However, $K(\phi, \bar{\phi})$ is subject to Kähler transformations

$$K(\phi, \bar{\phi}) \mapsto K(\phi, \bar{\phi}) + F(\phi) + \bar{F}(\bar{\phi}) , \tag{3.2.7}$$

which by themselves are not an invariance of (3.2.1) either. Then, it is easy to see that the kinetic superfield action is Kähler invariant, if together with (3.2.7), a *compensating* super-Weyl transformation [152] of parameters

$$\Sigma = \tfrac{1}{6} F(\phi), \quad \bar{\Sigma} = \tfrac{1}{6} \bar{F}(\bar{\phi}) \tag{3.2.8}$$

is performed.

In this way, a Kähler invariant action in superspace is obtained which contains the kinetic terms for supergravity and matter superfields and leads to the correct result in the flat superspace limit.

On the other hand, the component field action which derives from (3.2.1) in the conventional approach, yields the correctly normalized Einstein action only after a field-dependent rescaling of the component fields [153]. The correct Kähler transformations of the various component fields are then identified on the rescaled fields.

These complications can be avoided, however, if one starts right away from Kähler superspace as explained below. In particular, Kähler transformations are then consistently introduced at the *superfield level*. Another way to understand this is to perform the rescalings directly in terms of superfields: this will give the explicit relation between the conventional superfield approach described just above and our Kähler superspace construction.

The aim is therefore to absorb the exponential of the Kähler potential in (3.2.1) by means of a superfield rescaling of the frame in superspace. The first attempt might have been to employ a super-Weyl transformation. However, this does not work because the combination, $\Sigma + \bar{\Sigma}$, of chiral and antichiral superfield in (3.2.6) is not sufficient to absorb the more general real superfield $K(\phi, \bar{\phi})$ in (3.2.1). On the other hand, the chirality (resp. anti-chirality) conditions on Σ (resp. $\bar{\Sigma}$) are consequences of the invariance of the torsion constraints under transformations (3.2.2)–(3.2.4). If one is willing to give up this requirement, more general rescalings are possible, at the price of changing the torsion constraints and thus the superspace structure. We are therefore led to study more general transformations of the frame (and of the Lorentz connection) together with their consequences for the corresponding coefficients of the torsion 2-form. To be more precise, note that the arbitrary transformations of the vielbein $E_M{}^A$ and of the Lorentz connection $\phi_{MB}{}^A$

$$E'_M{}^A = E_M{}^{\underline{A}} X_{\underline{A}}{}^A \ , \tag{3.2.9}$$

$$\phi'_{MB}{}^A = \phi_{MB}{}^A + \chi_{MB}{}^A \ , \tag{3.2.10}$$

change the torsion coefficients as

$$\begin{aligned} T'_{CB}{}^A = (-)^{\underline{c}(b+\underline{b})} X^{-1}{}_C{}^{\underline{C}} X^{-1}{}_B{}^{\underline{B}} (T_{\underline{CB}}{}^{\underline{A}} X_{\underline{A}}{}^A + \mathscr{D}_{\underline{C}} X_{\underline{B}}{}^A - (-)^{\underline{cb}} \mathscr{D}_{\underline{B}} X_{\underline{C}}{}^A) \\ + X^{-1}{}_C{}^{\underline{C}} \chi_{\underline{C}B}{}^A - (-)^{cb} X^{-1}{}_B{}^{\underline{B}} \chi_{\underline{B}C}{}^A \ . \end{aligned} \tag{3.2.11}$$

For our present purpose it is sufficient to consider the superfield rescalings

$$X_B{}^A = \begin{pmatrix} \delta_b{}^a X\bar{X} & X_b{}^\alpha & X_{b\dot{\alpha}} \\ 0 & \delta_\beta{}^\alpha X & 0 \\ 0 & 0 & \delta^{\dot{\beta}}{}_{\dot{\alpha}} \bar{X} \end{pmatrix} . \tag{3.2.12}$$

The superfield X and its complex conjugate $\bar{X}$ are arbitrary, furthermore

$$X_b{}^\alpha = \frac{\mathrm{i}}{2} (\varepsilon\sigma_b)^\alpha{}_{\dot{\alpha}} \bar{X}^{-1} \mathscr{D}^{\dot{\alpha}} (X\bar{X}) \ , \tag{3.2.13}$$

$$X_{b\dot{\alpha}} = \frac{\mathrm{i}}{2} (\varepsilon\bar{\sigma}_b)_{\dot{\alpha}}{}^\alpha X^{-1} \mathscr{D}_\alpha (X\bar{X}) \ . \tag{3.2.14}$$

Observe that (3.2.12)–(3.2.14) differ from (3.2.2)–(3.2.4) only by the fact that X and $\bar{X}$ are, contrary to Σ and $\bar{\Sigma}$, not subjected to any restrictions.[12] What are the effects of the superfield rescalings (3.2.12)–(3.2.14) on the various torsion coefficients? First of all, note that these transformations leave the torsion constraints

$$T_{\gamma\beta}{}^a = 0, \quad T^{\dot{\gamma}\dot{\beta}a} = 0 \ , \tag{3.2.15}$$

$$T_{\gamma\beta\dot{\alpha}} = 0, \quad T^{\dot{\gamma}\dot{\beta}\alpha} = 0 \tag{3.2.16}$$

and

$$T_\gamma{}^{\dot{\beta}a} = -2\mathrm{i}(\sigma^a\varepsilon)_\gamma{}^{\dot{\beta}} \tag{3.2.17}$$

unchanged. It is well known that the torsion constraints

$$T_{\gamma b}{}^a = 0, \quad T^{\dot{\gamma}}{}_b{}^a = 0 \tag{3.2.18}$$

and

$$T_{cb}{}^a = 0 \ , \tag{3.2.19}$$

[12] Eqs. (3.2.2)–(3.2.4) can be obtained from (3.2.12)–(3.2.14) by restricting X and $\bar{X}$ to be given as $X = \exp(2\bar{\Sigma} - \Sigma)$ and $\bar{X} = \exp(2\Sigma - \bar{\Sigma})$.

allow to determine the Lorentz connection in superspace completely in terms of $E_M{}^A$. Likewise, the requirement that (3.2.18) and (3.2.19) are left invariant under (3.2.12)–(3.2.14) determines $\chi_{Cb}{}^a$ in terms of X and $\bar{X}$,

$$\chi_{\gamma ba} = 2(\sigma_{ba})_\gamma{}^\varphi (X\bar{X})^{-1} \mathscr{D}_\varphi (X\bar{X}) \,, \tag{3.2.20}$$

$$\chi^{\dot{\gamma}}{}_{ba} = 2(\bar{\sigma}_{ba})^{\dot{\gamma}}{}_{\dot{\varphi}} (X\bar{X})^{-1} \mathscr{D}^{\dot{\varphi}} (X\bar{X}) \,, \tag{3.2.21}$$

$$\begin{aligned}\chi_{cba} = {} & \eta_{ca}(X\bar{X})^{-1}\mathscr{D}_b(X\bar{X}) - \eta_{cb}(X\bar{X})^{-1}\mathscr{D}_a(X\bar{X}) \\ & + \tfrac{1}{2}\varepsilon_{dcba}(X\bar{X})^{-1}\mathscr{D}_\varphi(X\bar{X})(\varepsilon\sigma^d)^{\varphi}{}_{\dot{\varphi}}(X\bar{X})^{-1}\mathscr{D}^{\dot{\varphi}}(X\bar{X}) \,.\end{aligned} \tag{3.2.22}$$

This means that $X_B{}^A$ and $\chi_{CB}{}^A$ are now completely fixed in terms of the unconstrained superfields X and $\bar{X}$.

However, the remaining torsion constraints,

$$T_{\gamma\beta}{}^\alpha = 0, \quad T^{\dot{\gamma}}{}_\beta{}^\alpha = 0 \tag{3.2.23}$$

and

$$T_\gamma{}^{\dot{\beta}}{}_{\dot{\alpha}} = 0, \quad T^{\dot{\gamma}\dot{\beta}}{}_{\dot{\alpha}} = 0 \tag{3.2.24}$$

are no longer conserved by the superfield rescalings (3.2.12)–(3.2.14) and (3.2.20)–(3.2.22). The new torsion coefficients take the form

$$T'_{\gamma\beta}{}^\alpha = -\delta_\beta{}^\alpha A'_\gamma - \delta_\gamma{}^\alpha A'_\beta \,, \tag{3.2.25}$$

$$T'_\gamma{}^{\dot{\beta}}{}_{\dot{\alpha}} = \delta^{\dot{\beta}}{}_{\dot{\alpha}} A'_\gamma \tag{3.2.26}$$

with A'_γ defined as

$$A'_\gamma = -X^{-1}(2X^{-1}\mathscr{D}_\gamma X + \bar{X}^{-1}\mathscr{D}_\gamma \bar{X}) \,. \tag{3.2.27}$$

The complex conjugate equations are

$$T'^{\dot{\gamma}\dot{\beta}}{}_{\dot{\alpha}} = \delta^{\dot{\beta}}{}_{\dot{\alpha}} A'^{\dot{\gamma}} + \delta^{\dot{\gamma}}{}_{\dot{\alpha}} A'^{\dot{\beta}} \,, \tag{3.2.28}$$

$$T'^{\dot{\gamma}}{}_\beta{}^\alpha = -\delta_\beta{}^\alpha A'^{\dot{\gamma}} \,, \tag{3.2.29}$$

$$A'^{\dot{\gamma}} = \bar{X}^{-1}(2\bar{X}^{-1}\mathscr{D}^{\dot{\gamma}}\bar{X} + X^{-1}\mathscr{D}^{\dot{\gamma}}X) \,. \tag{3.2.30}$$

Next, we examine the consequences of the superfield rescalings for the remaining torsion coefficients by solving the Bianchi identities in the presence of the new constraints or, equivalently, by explicit calculation from (3.2.11).[13] In either case the tensor decompositions of $T'_{\gamma b\dot{\alpha}}$ and $T'^{\dot{\gamma}}{}_b{}^\alpha$ do not change, i.e.

$$T'_{\gamma b\dot{\alpha}} = -\mathrm{i}\sigma_{b\gamma\dot{\alpha}} R'^\dagger \,, \tag{3.2.31}$$

$$T'^{\dot{\gamma}}{}_b{}^\alpha = -\mathrm{i}\bar{\sigma}_b^{\dot{\gamma}\alpha} R' \,. \tag{3.2.32}$$

[13] Solutions to the Bianchi identities in terms of R, $R^\dagger$ and G_a are presented in Section 3.3 and Appendix B.

The rescaled superfields $R'^{\dagger}$ and R' are related to the old ones by

$$R'^{\dagger} = X^{-2}\{R^{\dagger} - \tfrac{1}{4}[(X\bar{X})^{-1}\mathscr{D}^{\alpha}\mathscr{D}_{\alpha}(X\bar{X}) + Y^{\alpha}Y_{\alpha}]\} , \quad (3.2.33)$$

$$R' = \bar{X}^{-2}\{R - \tfrac{1}{4}[(X\bar{X})^{-1}\mathscr{D}_{\dot{\alpha}}\mathscr{D}^{\dot{\alpha}}(X\bar{X}) + \bar{Y}_{\dot{\alpha}}\bar{Y}^{\dot{\alpha}}]\} \quad (3.2.34)$$

with the definitions

$$Y_{\alpha} = (X\bar{X})^{-1}\mathscr{D}_{\alpha}(X\bar{X}), \quad \bar{Y}^{\dot{\alpha}} = (X\bar{X})^{-1}\mathscr{D}^{\dot{\alpha}}(X\bar{X}) . \quad (3.2.35)$$

The torsion coefficients $T_{\gamma b}{}^{\alpha}$ and $T^{\dot{\gamma}}{}_{b\dot{\alpha}}$, however, pick up additional terms under the superfield rescalings,

$$T'_{\gamma b}{}^{\alpha} = \mathrm{i}(\sigma_{cb})_{\gamma}{}^{\alpha}G'^{c} + \mathrm{i}\delta_{\gamma}{}^{\alpha}G'_{b} - \frac{\mathrm{i}}{4}\delta_{\gamma}{}^{\alpha}\bar{\sigma}_{b}^{\dot{\beta}\beta}\{(X^{-1}\mathscr{D}_{\beta} + A'_{\beta})A'_{\dot{\beta}} + (\bar{X}^{-1}\mathscr{D}_{\dot{\beta}} - A'_{\dot{\beta}})A'_{\beta}\} , \quad (3.2.36)$$

$$T'^{\dot{\gamma}}{}_{b\dot{\alpha}} = \mathrm{i}(\sigma_{bc})^{\dot{\gamma}}{}_{\dot{\alpha}}G'^{c} - \mathrm{i}\delta^{\dot{\gamma}}{}_{\dot{\alpha}}G'_{b} + \frac{\mathrm{i}}{4}\delta^{\dot{\gamma}}{}_{\dot{\alpha}}\bar{\sigma}_{c}^{\dot{\beta}\beta}\{(X^{-1}\mathscr{D}_{\beta} + A'_{\beta})A'_{\dot{\beta}} + (\bar{X}^{-1}\mathscr{D}_{\dot{\beta}} - A'_{\dot{\beta}})A'_{\beta}\} . \quad (3.2.37)$$

The rescaled superfield $G'_{\beta\dot{\beta}} = \sigma^{b}_{\beta\dot{\beta}}G'_{b}$ is defined as

$$G'_{\beta\dot{\beta}} = (X\bar{X})^{-1}\{G_{\beta\dot{\beta}} - \tfrac{1}{2}[\mathscr{D}_{\beta}, \mathscr{D}_{\dot{\beta}}]\log(X\bar{X}) + Y_{\beta}\bar{Y}_{\dot{\beta}}\} . \quad (3.2.38)$$

The purpose of this detailed presentation of superfield rescalings and their consequences for the superspace torsion is twofold. First of all, in the case $A'_{\alpha} = 0$, $A'_{\dot{\alpha}} = 0$ the usual super-Weyl or Howe–Tucker transformations, which leave the torsion constraints invariant, are reproduced. Second, if X and $\bar{X}$ are kept arbitrary, the supervolume E of the moving frame in superspace changes as

$$E' = E(X\bar{X})^{2} . \quad (3.2.39)$$

This shows that for the particular field-dependent rescalings of parameters

$$X = \bar{X} = \mathrm{e}^{-(1/12)K(\phi,\bar{\phi})} , \quad (3.2.40)$$

the kinetic action (3.2.1) takes the form

$$\mathscr{L}_{\mathrm{kin}} = -3\int E' . \quad (3.2.41)$$

That is, the kinetic Lagrangian action is the integral over a new superspace defined with the supervolume E'. In addition, in this case, from (3.2.40) and (3.2.27), (3.2.30) one obtains

$$A'_{\gamma} = +\tfrac{1}{4}\mathscr{D}'_{\gamma}K(\phi,\bar{\phi}) , \quad (3.2.42)$$

$$A'^{\dot{\gamma}} = -\tfrac{1}{4}\mathscr{D}'^{\dot{\gamma}}K(\phi,\bar{\phi}) . \quad (3.2.43)$$

The primed spinor derivatives are, of course, given as

$$\mathscr{D}'_\gamma = X^{-1}\mathscr{D}_\gamma, \quad \mathscr{D}'^{\dot\gamma} = \bar{X}^{-1}\mathscr{D}^{\dot\gamma} . \tag{3.2.44}$$

At this stage it is very suggestive to interpret the additional terms in (3.2.25), (3.2.26) and (3.2.28), (3.2.29) not as unfortunate contributions to the torsion but rather as superfield gauge potentials associated to the structure group of a modified superspace geometry which realizes Kähler transformations as field-dependent chiral rotations. To see this more clearly observe that the new frame is related to the old one by

$$E'_M{}^a = \mathrm{e}^{-(1/6)K(\phi,\bar\phi)} E_M{}^a , \tag{3.2.45}$$

$$E'_M{}^\alpha = \mathrm{e}^{-(1/12)K(\phi,\bar\phi)}\left(E_M{}^\alpha - \frac{\mathrm{i}}{12} E_M{}^b (\varepsilon\sigma_b)^\alpha{}_{\dot\alpha} \mathscr{D}^{\dot\alpha} K(\phi,\bar\phi)\right) , \tag{3.2.46}$$

$$E'_{M\dot\alpha} = \mathrm{e}^{-(1/12)K(\phi,\bar\phi)}\left(E_{M\dot\alpha} - \frac{\mathrm{i}}{12} E_M{}^b (\varepsilon\bar\sigma_b)_{\dot\alpha}{}^\alpha \mathscr{D}_\alpha K(\phi,\bar\phi)\right) . \tag{3.2.47}$$

It is then easy to see that under the combination of Kähler transformations and compensating super-Weyl transformations these new variables transform homogeneously

$$E'_M{}^a \mapsto E'_M{}^a , \tag{3.2.48}$$

$$E'_M{}^\alpha \mapsto \mathrm{e}^{-(\mathrm{i}/2)\mathrm{Im}\,F} E'_M{}^\alpha , \tag{3.2.49}$$

$$E'_{M\dot\alpha} \mapsto \mathrm{e}^{+(\mathrm{i}/2)\mathrm{Im}\,F} E'_{M\dot\alpha} . \tag{3.2.50}$$

Indeed, these transformations represent chiral rotations of parameter $-\mathrm{i}/2\,\mathrm{Im}\,F$ and chiral weights $w(E_M{}^a) = 0$, $w(E_M{}^\alpha) = 1$, $w(E_{M\dot\alpha}) = -1$. Likewise, by the same mechanism, the superfields R', $R'^\dagger$ and G'_b undergo chiral rotations of weights $w(R') = 2$, $w(R'^\dagger) = -2$ and $w(G'_b) = 0$.

The corresponding gauge potential 1-form in superspace is then identified to be

$$A' = E'^a A'_a + E'^\alpha A'_\alpha + E'_{\dot\alpha} A'^{\dot\alpha} \tag{3.2.51}$$

with field strength $F' = dA'$. The spinor coefficients A'_α and $A'^{\dot\alpha}$ are given by (3.2.42) and (3.2.43) and give rise to

$$F'_{\beta\alpha} = 0, \quad F'^{\dot\beta\dot\alpha} = 0 . \tag{3.2.52}$$

The equation for the field strength $F'_\beta{}^{\dot\alpha}$ allows to determine the vector component

$$A'_{\alpha\dot\alpha} = \frac{\mathrm{i}}{2}(\mathscr{D}'_\alpha + A'_\alpha)A'_{\dot\alpha} + \frac{\mathrm{i}}{2}(\mathscr{D}'_{\dot\alpha} - A'_{\dot\alpha})A'_\alpha - \frac{\mathrm{i}}{2}F'_{\alpha\dot\alpha} . \tag{3.2.53}$$

Comparing (3.2.53) to (3.2.36), (3.2.37) and substituting appropriately yields

$$T'_{\gamma b}{}^{\alpha} = \mathrm{i}(\sigma_{cb})_{\gamma}{}^{\alpha} G'^{c} + \mathrm{i}\delta_{\gamma}{}^{\alpha}(G'_b + \tfrac{1}{2}F'_b) + \delta_{\gamma}{}^{\alpha} A'_b \ , \tag{3.2.54}$$

$$T'^{\dot\gamma}{}_{b\dot\alpha} = \mathrm{i}(\bar\sigma_{bc})^{\dot\gamma}{}_{\dot\alpha} G'^{c} - \mathrm{i}\delta^{\dot\gamma}{}_{\dot\alpha}(G'_b + \tfrac{1}{2}F'_b) - \delta^{\dot\gamma}{}_{\dot\alpha} A'_b \ . \tag{3.2.55}$$

Note that in this construction, A'_b and F'_b always appear in the combination $A'_b + \mathrm{i}/2F'_b$.

As a consequence of their definition, the coefficients of the connection 1-form A' change under transformations (3.2.7) and (3.2.8) as

$$A'_\alpha \mapsto \mathrm{e}^{+(\mathrm{i}/2)\mathrm{Im}\,F}\left(A'_\alpha + \frac{\mathrm{i}}{2}\mathscr{D}'_\alpha \mathrm{Im}\,F\right) , \tag{3.2.56}$$

$$A'^{\dot\alpha} \mapsto \mathrm{e}^{-(\mathrm{i}/2)\mathrm{Im}\,F}\left(A'^{\dot\alpha} + \frac{\mathrm{i}}{2}\mathscr{D}'^{\dot\alpha} \mathrm{Im}\,F\right) , \tag{3.2.57}$$

$$A'_a \mapsto \left(A'_a + \frac{\mathrm{i}}{2}\mathscr{D}'_a \mathrm{Im}\,F\right) . \tag{3.2.58}$$

Taking into account the properties of the rescaled frame, the transformation law for the 1-form A' in superspace becomes simply

$$A' \mapsto A' + \frac{\mathrm{i}}{2} d\,\mathrm{Im}\,F \ . \tag{3.2.59}$$

To summarize, the matter field-dependent superfield rescalings of frame and Lorentz connection, which might have appeared embarrassing in the first place, because they changed the geometric structure, actually led to a very elegant and powerful description of matter fields in the presence of supergravity. The most remarkable feature is that, in the supersymmetric case, matter and gravitation lend themselves concisely to a unified geometric description. Due to the close analogy between the Kähler potential and the pre-potential of supersymmetric gauge theory it is possible to include Kähler transformations in the structure group of superspace geometry. They are realized by chiral rotations as explained in detail above and the Kähler potential takes the place of the corresponding pre-potential. The superspace potentials can then be used to construct Kähler covariant spinor and vector derivatives, Kähler transformations are thus defined from the beginning at the full superfield level and imbedded in the geometry of superspace.

Furthermore, we have seen in (3.2.41), that the kinetic action for both supergravity and matter fields is given by minus three times the volume of superspace. Its expansion in terms of component fields gives immediately the correctly normalized kinetic terms for all the component fields without any need for rescalings or complicated integrations by parts at the component field level.

3.3. $U(1)$ *superspace geometry*

The result of the construction in the preceding section has a natural explanation in the framework of $U(1)$ superspace geometry, which will be reviewed in this section. In this approach, the conventional superspace geometry is enlarged to include a chiral $U(1)$ factor in the structure group. As a consequence, the basic superfields of the new geometry are the supervielbein $E_M{}^A(z)$ and the Lorentz gauge connection $\phi_{MB}{}^A(z)$ together with a gauge potential $A_M(z)$ for chiral $U(1)$ transformations. These superfields define coefficients of 1-forms in superspace such that

$$E^A = dz^M E_M{}^A(z) , \tag{3.3.1}$$

$$\phi_B{}^A = dz^M \phi_{MB}{}^A(z) , \tag{3.3.2}$$

$$A = dz^M A_M(z) . \tag{3.3.3}$$

Torsion and field strengths are then defined with the help of the exterior derivative d in superspace

$$T^A = dE^A + E^B \phi_B{}^A + w(E^A) E^A A , \tag{3.3.4}$$

$$R_B{}^A = d\phi_B{}^A + \phi_B{}^C \phi_C{}^A , \tag{3.3.5}$$

$$F = dA . \tag{3.3.6}$$

The chiral $U(1)$ weights $w(E^A)$ are defined as

$$w(E^a) = 0, \quad w(E^\alpha) = 1, \quad w(E_{\dot\alpha}) = -1 . \tag{3.3.7}$$

The non-vanishing parts $\phi_b{}^a$, $\phi_\beta{}^\alpha$, $\phi^\beta{}_{\dot\alpha}$ of $\phi_B{}^A$ (the Lorentz connection) are related among each other as usual,

$$\phi_\beta{}^\alpha = -\tfrac{1}{2}(\sigma^{ba})_\beta{}^\alpha \phi_{ba}, \quad \phi^\beta{}_{\dot\alpha} = -\tfrac{1}{2}(\bar\sigma^{ba})^\beta{}_{\dot\alpha} \phi_{ba} . \tag{3.3.8}$$

As is well known [53], for this choice of structure group, the Lorentz curvature and $U(1)$ field strength,

$$R_B{}^A = \tfrac{1}{2} E^C E^D R_{DCB}{}^A , \tag{3.3.9}$$

$$F = \tfrac{1}{2} E^C E^D F_{DC} \tag{3.3.10}$$

are completely defined in terms of the coefficients of the torsion 2-form,

$$T^A = \tfrac{1}{2} E^B E^C T_{CB}{}^A \tag{3.3.11}$$

and covariant derivatives thereof as a consequence of the superspace Bianchi identities,

$$\mathscr{D} T^A - E^B R_B{}^A - w(E^A) E^A F = 0 . \tag{3.3.12}$$

In the present case, covariant derivatives are understood to be covariant with respect to both Lorentz and $U(1)$ transformations. The covariant derivative of a generic superfield χ_A of chiral weight $w(\chi_A)$ is defined as

$$\mathscr{D}_B \chi_A = E_B{}^M \partial_M \chi_A - \phi_{BA}{}^C \chi_C + w(\chi_A) A_B \chi_A \tag{3.3.13}$$

with (graded) commutator

$$(\mathscr{D}_C, \mathscr{D}_B)\chi_A = -T_{CB}{}^F \mathscr{D}_F \chi_A - R_{CBA}{}^F \chi_F + w(\chi_A) F_{CB} \chi_A . \tag{3.3.14}$$

The chiral weights of the various objects are related to that of the vielbein, E^A, in a simple way, e.g.

$$\begin{aligned} w(\mathscr{D}_A) &= -w(E^A) , \\ w(T_{CB}{}^A) &= w(E^A) - w(E^B) - w(E^C) , \\ w(R_{CBA}{}^F) &= -w(E^B) - w(E^C) . \end{aligned} \tag{3.3.15}$$

Finally, the vielbein E^A, the covariant derivative $\mathscr{D}_A$ and the $U(1)$ gauge potential A_A change under chiral $U(1)$ structure group transformations g as

$$E^A \mapsto E^A g^{w(E^A)} , \tag{3.3.16}$$

$$\mathscr{D}_A \mapsto g^{-w(E^A)} \mathscr{D}_A , \tag{3.3.17}$$

$$A_A \mapsto g^{-w(E^A)} (A_A - g^{-1} E_A{}^M \partial_M g) . \tag{3.3.18}$$

As said in the introduction, the choice of structure group largely determines the $U(1)$ superspace geometry, which is further specified by appropriate covariant torsion constraints. For instance, combination of the covariant chirality conditions with the commutation relation (3.3.14) suggests

$$T_{\gamma\beta}{}^a = 0, \quad T^{\dot\gamma\dot\beta a} = 0 . \tag{3.3.19}$$

For a more complete presentation, we refer to [92], and references therein. Here, we content ourselves to sketch out the essential features of the resulting structure in superspace.

First of all, we note that all the coefficients of torsion and of Lorentz and $U(1)$ field strengths are given in terms of the covariant superfields $R, R^\dagger$ (resp. chiral and antichiral) and G_a (real) of canonical dimension 1 and of the Weyl spinor superfields $W_{\underbrace{\gamma\beta\alpha}}$ and $W_{\underbrace{\dot\gamma\dot\beta\dot\alpha}}$ of canonical dimension 3/2.

Moreover, the only non-vanishing component at dimension zero is the constant torsion already present in rigid superspace,

$$T_\gamma{}^{\dot\beta a} = -2\mathrm{i}(\sigma^a \varepsilon)_\gamma{}^{\dot\beta} . \tag{3.3.20}$$

We then proceed in the order of increasing canonical dimension. At dimension 1/2, all the torsion coefficients vanish whereas at dimension 1 the above-mentioned superfields $R, R^\dagger$ and G_a are identified as

$$T_{\gamma b \dot\alpha} = -\mathrm{i}\sigma_{b\gamma\dot\alpha} R^\dagger, \quad T_{\gamma b}{}^\alpha = \frac{\mathrm{i}}{2}(\sigma_c \bar\sigma_b)_\gamma{}^\alpha G^c , \tag{3.3.21}$$

$$T^{\dot\gamma}{}_b{}^\alpha = -\mathrm{i}\bar\sigma_b^{\dot\gamma\alpha} R, \quad T^{\dot\gamma}{}_{b\dot\alpha} = -\frac{\mathrm{i}}{2}(\bar\sigma_c \sigma_b)^{\dot\gamma}{}_{\dot\alpha} G^c . \tag{3.3.22}$$

The purely vector torsion is taken to vanish

$$T_{cb}{}^{a} = 0 \; . \tag{3.3.23}$$

At dimension 3/2, the super-covariant Rarita–Schwinger (super)field strengths $T_{cb}{}^{\alpha}$ and $T_{cb\dot{\alpha}}$ are most conveniently displayed in spinor notation

$$T_{\gamma\dot{\gamma}\,\beta\dot{\beta}}{}^{A} = \sigma^{c}_{\gamma\dot{\gamma}}\,\sigma^{b}_{\beta\dot{\beta}}\,T_{cb}{}^{A} \; . \tag{3.3.24}$$

Together with $G_{\alpha\dot{\alpha}} = \sigma^{a}_{\alpha\dot{\alpha}} G_a$ we obtain

$$T_{\gamma\dot{\gamma}\,\beta\dot{\beta}\,\alpha} = +2\varepsilon_{\dot{\gamma}\dot{\beta}} W_{\underbrace{\gamma\beta\alpha}} + \tfrac{2}{3}\varepsilon_{\dot{\gamma}\dot{\beta}}(\varepsilon_{\alpha\beta}S_{\gamma} + \varepsilon_{\alpha\gamma}S_{\beta}) - 2\varepsilon_{\gamma\beta}T_{\underbrace{\dot{\gamma}\dot{\beta}}\alpha} \, , \tag{3.3.25}$$

$$T_{\underbrace{\dot{\gamma}\dot{\beta}}\alpha} = -\tfrac{1}{4}(\mathscr{D}_{\dot{\gamma}}G_{\alpha\dot{\beta}} + \mathscr{D}_{\dot{\beta}}G_{\alpha\dot{\gamma}}) \, , \tag{3.3.26}$$

$$S_{\gamma} = -\mathscr{D}_{\gamma}R + \tfrac{1}{4}\mathscr{D}^{\dot{\gamma}}G_{\gamma\dot{\gamma}} \tag{3.3.27}$$

and

$$T_{\gamma\dot{\gamma}\,\beta\dot{\beta}\,\dot{\alpha}} = -2\varepsilon_{\gamma\beta} W_{\underbrace{\dot{\gamma}\dot{\beta}\dot{\alpha}}} - \tfrac{2}{3}\varepsilon_{\gamma\beta}(\varepsilon_{\dot{\alpha}\dot{\beta}}S_{\dot{\gamma}} + \varepsilon_{\dot{\alpha}\dot{\gamma}}S_{\dot{\beta}}) + 2\varepsilon_{\dot{\gamma}\dot{\beta}}T_{\underbrace{\gamma\beta}\dot{\alpha}} \, , \tag{3.3.28}$$

$$T_{\underbrace{\gamma\beta}\dot{\alpha}} = +\tfrac{1}{4}(\mathscr{D}_{\gamma}G_{\beta\dot{\alpha}} + \mathscr{D}_{\beta}G_{\gamma\dot{\alpha}}) \, , \tag{3.3.29}$$

$$S_{\dot{\gamma}} = +\mathscr{D}_{\dot{\gamma}}R^{\dagger} - \tfrac{1}{4}\mathscr{D}^{\gamma}G_{\gamma\dot{\gamma}} \; . \tag{3.3.30}$$

The $U(1)$ weights of the basic superfields appearing in (3.3.21), (3.3.22) and (3.3.27), (3.3.30) are

$$\begin{aligned} &w(R) = 2, \quad w(R^{\dagger}) = -2 \, , \\ &w(G_a) = 0 \, , \\ &w(W_{\underbrace{\gamma\beta\alpha}}) = 1, \quad w(W_{\underbrace{\dot{\gamma}\dot{\beta}\dot{\alpha}}}) = -1 \; . \end{aligned} \tag{3.3.31}$$

As already mentioned above, the coefficients of Lorentz curvatures and $U(1)$ field strengths are expressed in terms of these few superfields. At dimension one we obtain

$$R_{\delta\gamma\, ba} = 8(\sigma_{ba}\varepsilon)_{\delta\gamma}R^{\dagger} \, , \tag{3.3.32}$$

$$R^{\dot{\delta}\dot{\gamma}}{}_{ba} = 8(\bar{\sigma}_{ba}\varepsilon)^{\dot{\delta}\dot{\gamma}}R \, , \tag{3.3.33}$$

$$R_{\delta}{}^{\dot{\gamma}}{}_{ba} = -2\mathrm{i}G^{d}(\sigma^{c}\varepsilon)_{\delta}{}^{\dot{\gamma}}\varepsilon_{dcba} \, , \tag{3.3.34}$$

for the Lorentz curvatures whereas the chiral $U(1)$ field strengths are given by

$$F_{\beta\alpha} = 0, \quad F^{\dot{\beta}\dot{\alpha}} = 0 \, , \tag{3.3.35}$$

$$F_{\beta}{}^{\dot{\alpha}} = 3(\sigma^{a}\varepsilon)_{\beta}{}^{\dot{\alpha}}G_a \; . \tag{3.3.36}$$

At dimension 3/2, we find

$$R_{\dot\delta cba} = \mathrm{i}\sigma_{c\delta\dot\delta} T_{ba}{}^{\delta} + \mathrm{i}\sigma_{b\delta\dot\delta} T_{ca}{}^{\delta} + \mathrm{i}\sigma_{a\delta\dot\delta} T_{bc}{}^{\delta} , \tag{3.3.37}$$

$$R^{\delta}{}_{c\,ba} = \mathrm{i}\bar\sigma_c^{\dot\delta\delta} T_{ba\dot\delta} + \mathrm{i}\bar\sigma_b^{\dot\delta\delta} T_{ca\dot\delta} + \mathrm{i}\bar\sigma_a^{\dot\delta\delta} T_{bc\dot\delta} \tag{3.3.38}$$

and

$$F_{\delta c} = \frac{3\mathrm{i}}{2}\mathscr{D}_\delta G_c + \frac{\mathrm{i}}{2}\sigma_{c\delta\dot\delta}\bar X^{\dot\delta}, \quad F^{\dot\delta}{}_c = \frac{3\mathrm{i}}{2}\mathscr{D}^{\dot\delta} G_c - \frac{\mathrm{i}}{2}\bar\sigma_c^{\dot\delta\delta} X_\delta \tag{3.3.39}$$

with the definitions

$$X_\delta = \mathscr{D}_\delta R - \mathscr{D}^{\dot\delta} G_{\delta\dot\delta}, \quad \bar X^{\dot\delta} = \mathscr{D}^{\dot\delta} R^\dagger + \mathscr{D}_\delta G^{\delta\dot\delta} . \tag{3.3.40}$$

Finally, having expressed torsions, curvatures and $U(1)$ field strengths in terms of few covariant superfields, the Bianchi identities themselves are now represented by a small set of rather simple conditions, such as

$$\mathscr{D}_\alpha \bar W_{\underbrace{\dot\gamma\dot\beta\dot\alpha}} = 0, \quad \mathscr{D}_{\dot\alpha} W_{\underbrace{\gamma\beta\alpha}} = 0 \tag{3.3.41}$$

or

$$\mathscr{D}_\alpha T_{cb}{}^{\alpha} + \mathscr{D}^{\dot\alpha} T_{cb\dot\alpha} = 0 \tag{3.3.42}$$

for these superfields. A detailed account of these relations is given in Appendix B.2.

Let us stress, that the complex superfield R, subject to chirality conditions

$$\mathscr{D}_\alpha R^\dagger = 0, \quad \mathscr{D}^{\dot\alpha} R = 0 , \tag{3.3.43}$$

plays a particularly important role, it contains the curvature scalar in its superfield expansion. As in our language superfield expansions are replaced by successive applications of spinor derivatives, the relevant relation is

$$\mathscr{D}^2 R + \bar{\mathscr{D}}^2 R^\dagger = -\tfrac{2}{3} R_{ba}{}^{ba} - \tfrac{2}{3}\mathscr{D}^\alpha X_\alpha + 4 G^a G_a + 32 R R^\dagger . \tag{3.3.44}$$

Interestingly enough the curvature scalar is necessarily accompanied by the D-term superfield $\mathscr{D}^\alpha X_\alpha = -2\mathbf{D}$ of the $U(1)$ gauge sector, described in terms of the gaugino superfields X_α and $\bar X^{\dot\alpha}$ subject to the usual chirality and reality conditions

$$\mathscr{D}_\alpha \bar X^{\dot\alpha} = 0, \quad \mathscr{D}^{\dot\alpha} X_\alpha = 0 , \tag{3.3.45}$$

$$\mathscr{D}^\alpha X_\alpha - \mathscr{D}_{\dot\alpha} \bar X^{\dot\alpha} = 0 . \tag{3.3.46}$$

This shows very clearly that generic $U(1)$ superspace provides the natural framework for the description of gauged R-transformations [73,10,146,35,30]. Relation (3.3.44) shows that supersymmetric completion of the (canonically normalized) curvature scalar action induces a Fayet–Iliopoulos term for gauged R-transformations.

At this point we wish to make a digression to indicate how the superspace geometry described above can be related to that of [115] *and restricted to the superspace geometry relevant to the minimal supergravity multiplet. To this end, call* A_0 *the* $U(1)$ *gauge potential of the superspace geometry described here and* A_1 *the* $U(1)$ *gauge potential of* [115]. *The two (equivalent) descriptions are related through*

$$A_1 = A_0 - \frac{3i}{2} E^a G_a \ . \tag{3.3.47}$$

On the other hand, the superspace geometry of [153] *is recovered by*

$$A_1 = 0, \quad X_\alpha = 0, \quad \bar{X}^{\dot\alpha} = 0 \ , \tag{3.3.48}$$

giving rise (among other things) to

$$T^{o.m}{}_{\gamma b}{}^{\alpha} = +\frac{3i}{2}\delta_\gamma{}^\alpha G_b + \frac{i}{2} G^c (\sigma_c \bar\sigma_b)_\gamma{}^\alpha \ , \tag{3.3.49}$$

$$T^{o.m\dot\gamma}{}_{b\dot\alpha} = -\frac{3i}{2}\delta^{\dot\gamma}{}_{\dot\alpha} G_b - \frac{i}{2} G^c (\bar\sigma_c \bar\sigma_b)^{\dot\gamma}{}_{\dot\alpha} \tag{3.3.50}$$

and

$$\mathscr{D}_\alpha R = \mathscr{D}^{\dot\alpha} G_{\alpha\dot\alpha}, \quad \mathscr{D}^{\dot\alpha} R^\dagger = -\mathscr{D}_\alpha G^{\alpha\dot\alpha} \ . \tag{3.3.51}$$

In this sense $U(1)$ superspace is the underlying framework for both minimal supergravity and its coupling to matter. Note, en passant, that in [115] the other two supergravity multiplets, non-minimal and new minimal, have been derived from generic $U(1)$ superspace as well.

3.4. Formulation in Kähler superspace

As pointed out earlier, the description of supersymmetric non-linear sigma models [164] as well as the construction of supergravity/matter couplings [41,42,38,39,6,7,21,20] is based on an intriguing analogy between Kähler geometry and supersymmetric gauge theory, which are both defined by means of differential constraints. In Kähler geometry the fundamental 2-form of complex geometry is required to be closed whereas supersymmetric gauge theory is characterized by covariant constraints as explained in Section 2.3. The constraints imply that the Kähler metric is expressed in terms of derivatives of the Kähler potential whereas, on the other hand, the superspace gauge potential is expressed in terms of a pre-potential. Pre-potential transformations, which are chiral superfields should then be compared to Kähler transformations which are holomorphic functions of the complex coordinates.

Matter superfields, on the other hand, are given by chiral superfields. It remains to promote the complex coordinates of the Kähler manifold to chiral superfields: holomorphic functions of chiral superfields are still chiral superfields. Correspondingly, the Kähler potential becomes a function of the chiral and antichiral superfield coordinates. The geometry of the supersymmetry coupling is then obtained by replacing the gauge potential in $U(1)$ superspace by the superfield Kähler potential [21,20,98].

In Section 3.4.1 we present the basic features of this geometric structure in a self-contained manner. In Section 3.4.2 we include Yang–Mills interactions (cf. Appendix B for their formulation in $U(1)$ superspace). Gauged superfield isometries of the Kähler metric are treated in Appendix C. We also study carefully the supergravity transformations of the whole system. Finally in Section 3.4.3 invariant superfield actions and the corresponding superfield equations of motion will be discussed.

3.4.1. Definition and properties of Kähler superspace

Kähler superspace geometry is defined as $U(1)$ superspace geometry, presented in Section 3.3, with suitable identification of the $U(1)$ pre-potential and pre-gauge transformations with the Kähler potential and Kähler transformations. The relevant version of $U(1)$ superspace geometry is the one where the $U(1)$ structure group transformations are realized in terms of chiral and antichiral superfields as described in (2.2.2) for the $(\frac{1}{2},\frac{1}{2})$ basis, where most of the work has already been done. As a matter of fact, the structures developed there in the framework of rigid superspace are very easily generalized to the present case of curved $U(1)$ superspace geometry. To begin with, the solution of (3.3.35) is given as

$$A_\alpha = -T^{-1}E_\alpha{}^M \partial_M T \ , \tag{3.4.1}$$

$$A^{\dot\alpha} = -U^{-1}E^{\dot\alpha M}\partial_M U \tag{3.4.2}$$

with $E_A{}^M$ now the full (inverse) frame of $U(1)$ superspace geometry. As anticipated in Section 2.2.2 the geometric structure relevant to the superspace formulation of supergravity/matter coupling is the basis $(a,b)=(\frac{1}{2},\frac{1}{2})$. In this basis one has

$$A_\alpha(\tfrac{1}{2}) = -\tfrac{1}{2}W^{-1}E_\alpha{}^M(\tfrac{1}{2})\partial_M W \ , \tag{3.4.3}$$

$$A^{\dot\alpha}(\tfrac{1}{2}) = +\tfrac{1}{2}W^{-1}E^{\dot\alpha M}(\tfrac{1}{2})\partial_M W \ , \tag{3.4.4}$$

where $W = T\,U^{-1}$ transforms as given in (2.2.34). For the vielbein we have

$$E^A(\tfrac{1}{2}) \mapsto [\bar{\mathbf{P}}\mathbf{Q}]^{-w(A)/2} E^A(\tfrac{1}{2}) \tag{3.4.5}$$

and

$$A_\alpha(\tfrac{1}{2}) \mapsto (\bar{\mathbf{P}}\,\mathbf{Q})^{1/2}[A_\alpha(\tfrac{1}{2}) + \tfrac{1}{2}E_\alpha{}^M(\tfrac{1}{2})\partial_M \log \mathbf{Q}] \ , \tag{3.4.6}$$

$$A^{\dot\alpha}(\tfrac{1}{2}) \mapsto (\bar{\mathbf{P}}\mathbf{Q})^{-1/2}[A^{\dot\alpha}(\tfrac{1}{2}) + \tfrac{1}{2}E^{\dot\alpha M}(\tfrac{1}{2})\partial_M \log \bar{\mathbf{P}}] \ . \tag{3.4.7}$$

In order to make contact with the superspace structures obtained in Section 2.2, we relate W to the Kähler potential $K(\phi,\bar{\phi})$ and $\bar{\mathbf{P}}$ and $\mathbf{Q}$ to the Kähler transformations $F(\phi)$ and $F(\bar{\phi})$. It is very easy to convince oneself that the identifications

$$W = \exp(-K(\phi,\bar{\phi})/2)\ , \tag{3.4.8}$$

$$\bar{\mathbf{P}} = \exp(-\bar{F}(\bar{\phi})/2)\ , \tag{3.4.9}$$

$$\mathbf{Q} = \exp(+F(\phi)/2) \tag{3.4.10}$$

reproduce exactly the geometric structures obtained at the end of Section 3.2 after superfield rescalings. The primed quantities defined there are identical with the $U(1)$ superspace geometry in the $(\frac{1}{2},\frac{1}{2})$ basis after identifications (3.4.8)–(3.4.10), i.e.

$$E'^A = E^A(\tfrac{1}{2})\ , \tag{3.4.11}$$

$$A' = A(\tfrac{1}{2})\ . \tag{3.4.12}$$

In particular, from (2.2.34) we recover the Kähler transformations

$$K(\phi,\bar{\phi}) \mapsto K(\phi,\bar{\phi}) + F(\phi) + \bar{F}(\bar{\phi})\ . \tag{3.4.13}$$

Moreover, (3.4.3) and (3.4.4) reproduce (3.2.42), (3.2.43), and (3.4.6), (3.4.7) correspond exactly to (3.2.56) and (3.2.57).

We have thus constructed the superspace geometry relevant for the description of supergravity/matter couplings and at the same time established the equivalence with the more traditional formulation.

In this new kind of superspace geometry, called *Kähler superspace geometry*, or $U_K(1)$ *superspace geometry*, the complete action for the kinetic terms of both supergravity and matter fields is given by the superdeterminant of the frame in superspace. Expression of this superfield action in terms of component fields leads to the correctly normalized component field actions without any need for rescalings. Invariance under superfield Kähler transformations is achieved ab initio without any need for compensating transformations.

The local frame E^A is subject to both Lorentz and Kähler transformations in a well-defined way. Covariance of the torsion 2-form is achieved with the help of gauge potentials $\phi_B{}^A$ and A for Lorentz and Kähler transformations, respectively:

$$T^A = dE^A + E^B\phi_B{}^A + w(E^A)E^A A\ . \tag{3.4.14}$$

The complete expression is the same as in $U(1)$ superspace geometry, except that the chiral gauge potential is no longer an independent field but rather expressed in terms of the Kähler potential $K(\phi,\bar{\phi})$. Hence, this superspace torsion contains at the same time supergravity and matter fields! The Kähler transformations of A are induced from those of the Kähler potential, i.e.

$$K(\phi,\bar{\phi}) \mapsto K(\phi,\bar{\phi}) + F(\phi) + \bar{F}(\bar{\phi}) \tag{3.4.15}$$

to be

$$A \mapsto A + \frac{i}{2} d \,\mathrm{Im}\, F \,. \tag{3.4.16}$$

At the same time the frame is required to undergo the chiral rotation

$$E^A \mapsto E^A e^{-(i/2)w(E^A)\mathrm{Im}\, F} \,, \tag{3.4.17}$$

ensuring a covariant transformation law of the superspace torsion,

$$T^A \mapsto T^A e^{-(i/2)w(E^A)\mathrm{Im}\, F} \,. \tag{3.4.18}$$

Its coefficients are subject to the same constraints as those of $U(1)$ superspace and therefore the tensor decompositions as obtained from the analysis of superspace Bianchi identities remain valid. For details we refer to Appendix B.

We shall, however, present in detail the structure of the $U(1)$ gauge sector, in particular the special properties which arise from the parametrization of A in terms of the Kähler potential $K(\phi, \bar{\phi})$, namely

$$A_\alpha = \tfrac{1}{4} E_\alpha{}^M \partial_M K(\phi, \bar{\phi}), \qquad A^{\dot{\alpha}} = -\tfrac{1}{4} E^{\dot{\alpha} M} \partial_M K(\phi, \bar{\phi}) \,, \tag{3.4.19}$$

$$A_{\alpha\dot{\alpha}} - \frac{3i}{2} G_{\alpha\dot{\alpha}} = \frac{i}{2}(\mathscr{D}_\alpha A_{\dot{\alpha}} + \mathscr{D}_{\dot{\alpha}} A_\alpha) \,. \tag{3.4.20}$$

It follows that its field strength 2-form, $F = dA$, has the spinor coefficients

$$F_{\beta\alpha} = 0, \quad F^{\dot{\beta}\dot{\alpha}} = 0, \quad F_\beta{}^{\dot{\alpha}} = 3(\sigma^a \varepsilon)_\beta{}^{\dot{\alpha}} G_a \,. \tag{3.4.21}$$

Of course, this reproduces the structure of the constraints already encountered in $U(1)$ superspace which implies also

$$F_{\beta a} - \frac{3i}{2} \mathscr{D}_\beta G_a = +\frac{i}{2} \sigma_{a\beta\dot{\beta}} \bar{X}^{\dot{\beta}} \,, \tag{3.4.22}$$

$$F^{\dot{\beta}}{}_a - \frac{3i}{2} \mathscr{D}^{\dot{\beta}} G_a = -\frac{i}{2} \bar{\sigma}_a^{\dot{\beta}\beta} X_\beta \tag{3.4.23}$$

with

$$X_\alpha = \mathscr{D}_\alpha R - \mathscr{D}^{\dot{\alpha}} G_{\alpha\dot{\alpha}} \,, \tag{3.4.24}$$

$$\bar{X}^{\dot{\alpha}} = \mathscr{D}^{\dot{\alpha}} R^\dagger + \mathscr{D}_\alpha G^{\alpha\dot{\alpha}} \,. \tag{3.4.25}$$

In the absence of matter, the superfields $X_\alpha, \bar{X}^{\dot{\alpha}}$ vanish and we are left with standard superspace supergravity. In the presence of matter they are given in terms of the Kähler potential as

$$X_\alpha = -\tfrac{1}{8}(\bar{\mathscr{D}}^2 - 8R)\mathscr{D}_\alpha K(\phi, \bar{\phi}) \,, \tag{3.4.26}$$

$$\bar{X}^{\dot{\alpha}} = -\tfrac{1}{8}(\mathscr{D}^2 - 8R^\dagger)\mathscr{D}^{\dot{\alpha}} K(\phi, \bar{\phi}) \,. \tag{3.4.27}$$

These expressions are simply a consequence of the explicit definitions given so far.

In an alternative, slightly more illuminating way, we may write A as[14]

$$A = \frac{1}{4}(K_k d\phi^k - K_{\bar{k}} d\bar{\phi}^{\bar{k}}) + \frac{i}{8}E^a(12G_a + \bar{\sigma}_a^{\dot{\alpha}\alpha} g_{k\bar{k}} \mathscr{D}_\alpha \phi^k \mathscr{D}_{\dot{\alpha}} \bar{\phi}^{\bar{k}}) , \tag{3.4.28}$$

where K_k and $K_{\bar{k}}$ stand for the derivatives of the Kähler potential with respect to ϕ^k and $\bar{\phi}^{\bar{k}}$, this way of writing A is more in line with Kähler geometry. The exterior derivative of A,

$$F = dA = \frac{1}{2} g_{k\bar{k}} d\phi^k d\bar{\phi}^{\bar{k}} + \frac{i}{8} d[E^a(12G_a + \bar{\sigma}_a^{\dot{\alpha}\alpha} g_{k\bar{k}} \mathscr{D}_\alpha \phi^k \mathscr{D}_{\dot{\alpha}} \bar{\phi}^{\bar{k}})] , \tag{3.4.29}$$

yields the superspace analogue of the fundamental form in ordinary Kähler geometry, with complex coordinates replaced by chiral superfields (the additional term is not essential and could have been absorbed in a redefinition of the vector component of A).

This form of F is also very convenient to derive directly the explicit expression of X_α and of $\bar{X}^{\dot{\alpha}}$ in terms of the matter superfields, avoiding explicit evaluation of the spinor derivatives in (3.4.27) and (3.4.28). A straightforward identification in $F_{\beta a}$ (resp. $F^{\dot{\beta}}{}_a$) shows that

$$X_\alpha = -\frac{i}{2} g_{k\bar{k}} \sigma^a_{\alpha\dot{\alpha}} \mathscr{D}_a \phi^k \mathscr{D}^{\dot{\alpha}} \bar{\phi}^{\bar{k}} + \frac{1}{2} g_{k\bar{k}} \mathscr{D}_\alpha \phi^k \bar{F}^{\bar{k}} , \tag{3.4.30}$$

$$\bar{X}^{\dot{\alpha}} = -\frac{i}{2} g_{k\bar{k}} \bar{\sigma}^{a\dot{\alpha}\alpha} \mathscr{D}_a \bar{\phi}^{\bar{k}} \mathscr{D}_\alpha \phi^k + \frac{1}{2} g_{k\bar{k}} \mathscr{D}^{\dot{\alpha}} \bar{\phi}^{\bar{k}} F^k . \tag{3.4.31}$$

Here we have used the definitions

$$F^k = -\tfrac{1}{4}\mathscr{D}^2 \phi^k, \qquad \bar{F}^{\bar{k}} = -\tfrac{1}{4}\bar{\mathscr{D}}^2 \bar{\phi}^{\bar{k}} . \tag{3.4.32}$$

The covariant derivatives are defined as

$$\mathscr{D}_\alpha \phi^k = E_\alpha{}^M \partial_M \phi^k, \qquad \mathscr{D}^{\dot{\alpha}} \bar{\phi}^{\bar{k}} = E^{\dot{\alpha}M} \partial_M \bar{\phi}^{\bar{k}} , \tag{3.4.33}$$

$$\mathscr{D}_B \mathscr{D}_\alpha \phi^k = E_B{}^M \partial_M \mathscr{D}_\alpha \phi^k - \phi_{B\alpha}{}^\varphi \mathscr{D}_\varphi \phi^k - A_B \mathscr{D}_\alpha \phi^k + \Gamma^k{}_{ij} \mathscr{D}_B \phi^i \mathscr{D}_\alpha \phi^j , \tag{3.4.34}$$

$$\mathscr{D}_B \mathscr{D}^{\dot{\alpha}} \bar{\phi}^{\bar{k}} = E_B{}^M \partial_M \mathscr{D}^{\dot{\alpha}} \bar{\phi}^{\bar{k}} - \phi_B{}^{\dot{\alpha}}{}_{\dot{\varphi}} \mathscr{D}^{\dot{\varphi}} \bar{\phi}^{\bar{k}} + A_B \mathscr{D}^{\dot{\alpha}} \bar{\phi}^{\bar{k}} + \Gamma^{\bar{k}}{}_{\bar{i}\bar{j}} \mathscr{D}_B \bar{\phi}^{\bar{i}} \mathscr{D}^{\dot{\alpha}} \bar{\phi}^{\bar{j}} , \tag{3.4.35}$$

assuring covariance with respect to Lorentz and Kähler transformations and (ungauged) isometries of the Kähler metric. The Levi–Civita symbols

$$\Gamma^k{}_{ij} = g^{k\bar{l}} g_{i\bar{l},j}, \qquad \Gamma^{\bar{k}}{}_{\bar{i}\bar{j}} = g^{l\bar{k}} g_{l\bar{i},\bar{j}} \tag{3.4.36}$$

[14] Note that the term containing G_a originates from our particular choice of constraint (3.3.36), i.e. $F_\beta{}^{\dot{\alpha}} = 3(\sigma^a \varepsilon)_\beta{}^{\dot{\alpha}} G_a$.

are now, of course, functions of the matter superfields. Do not forget that, due to their geometric origin, the superfields $X_\alpha, \bar{X}^{\dot\alpha}$ have the properties

$$\mathscr{D}^{\dot\alpha} X_\alpha = 0, \quad \mathscr{D}_\alpha \bar{X}^{\dot\alpha} = 0 , \tag{3.4.37}$$

$$\mathscr{D}^\alpha X_\alpha = \mathscr{D}_{\dot\alpha} \bar{X}^{\dot\alpha} . \tag{3.4.38}$$

As we shall see later on, the lowest components of the superfields $X_\alpha, \bar{X}^{\dot\alpha}$, as well as that of $\mathscr{D}^\alpha X_\alpha$, appear in the construction of the component field action. In order to prepare the ground for this construction we display here the superfield expression of the Kähler D-term. It is

$$\begin{aligned} -\frac{1}{2}\mathscr{D}^\alpha X_\alpha = & -g_{k\bar{k}}\eta^{ab}\mathscr{D}_b\phi^k\mathscr{D}_a\bar{\phi}^{\bar{k}} - \frac{\mathrm{i}}{4}g_{k\bar{k}}\sigma^a_{\alpha\dot\alpha}\mathscr{D}^\alpha\phi^k\mathscr{D}_a\mathscr{D}^{\dot\alpha}\bar{\phi}^{\bar{k}} \\ & -\frac{\mathrm{i}}{4}g_{k\bar{k}}\sigma^a_{\alpha\dot\alpha}\mathscr{D}^{\dot\alpha}\bar{\phi}^{\bar{k}}\mathscr{D}_a\mathscr{D}^\alpha\phi^k + g_{k\bar{k}}F^k\bar{F}^{\bar{k}} + \frac{1}{16}R_{k\bar{k}j\bar{j}}\mathscr{D}^\alpha\phi^k\mathscr{D}_\alpha\phi^j\mathscr{D}_{\dot\alpha}\bar{\phi}^{\bar{k}}\mathscr{D}^{\dot\alpha}\bar{\phi}^{\bar{j}} \end{aligned} \tag{3.4.39}$$

with covariant derivatives as defined above in (3.4.34) and (3.4.35). The Riemann tensor is given as

$$R_{k\bar{k}j\bar{j}} = g_{k\bar{k},j\bar{j}} - g^{l\bar{l}}g_{k\bar{l},j}g_{l\bar{k},\bar{j}} . \tag{3.4.40}$$

The terminology employed here concerning the notion of a D-term may appear unusual but it is perfectly adapted to the construction in curved superspace, where explicit superfield expansions are replaced by successively taking covariant spinor derivatives and projecting to lowest superfield components. In this sense the lowest component of the superfield $\mathscr{D}^\alpha X_\alpha$ indeed provides the complete and invariant geometric definition of the component field D-term.

In our geometric formulation, this Kähler D-term appears very naturally in the superfield expansions of the superfields $R, R^\dagger$ of the supergravity sector. To see this in more detail, recall first of all the chirality properties,

$$\mathscr{D}_\alpha R^\dagger = 0, \quad \mathscr{D}^{\dot\alpha} R = 0 \tag{3.4.41}$$

with $R, R^\dagger$ having chiral weights $w(R) = 2$ and $w(R^\dagger) = -2$, respectively. For the spinor derivatives of the opposite chirality the Bianchi identities imply

$$\mathscr{D}_\alpha R = -\tfrac{1}{3}X_\alpha - \tfrac{2}{3}(\sigma^{cb}\varepsilon)_{\alpha\varphi}T_{cb}{}^\varphi , \tag{3.4.42}$$

$$\mathscr{D}^{\dot\alpha} R^\dagger = -\tfrac{1}{3}\bar{X}^{\dot\alpha} - \tfrac{2}{3}(\bar\sigma^{cb}\varepsilon)^{\dot\alpha\dot\varphi}T_{cb\dot\varphi} . \tag{3.4.43}$$

Applying once more suitable spinor derivatives and making use of the Bianchi identities yields

$$\mathscr{D}^2 R + \bar{\mathscr{D}}^2 R^\dagger = -\tfrac{2}{3}R_{ba}{}^{ba} - \tfrac{2}{3}\mathscr{D}^\alpha X_\alpha + 4G^a G_a + 32RR^\dagger . \tag{3.4.44}$$

This relation will turn out to be crucial for the construction of the component field action.

3.4.2. The supergravity/matter/Yang–Mills system

Having established Kähler superspace geometry as a general framework for the coupling of supergravity to matter, it is quite natural to include couplings to supersymmetric Yang–Mills theory as well. In terms of superspace the basic geometric objects for this construction are

- $E^A = dz^M E_M{}^A$, the frame of superspace,
- $\phi^k, \bar{\phi}^{\bar{k}}$, the chiral matter superfields,
- $\mathscr{A}^{(r)} = dz^M \mathscr{A}_M^{(r)}$, the Yang–Mills potential.

As we have already pointed out in Section 2.3, Yang–Mills couplings of supersymmetric matter are described in terms of covariantly chiral superfields. It remains to couple the matter/Yang–Mills system as described in Section 2.3 to supergravity, in combination with the structure of Kähler superspace. This is very easy. All we have to do is to write all the equations of Section 2.3 in the background of Kähler superspace. This will define the underlying geometric structure of the *supergravity/matter/Yang–Mills system.*[15]

As to the geometry of the supergravity/matter sector, the Kähler potential is now understood to be given in terms of covariantly chiral superfields. As a consequence, the composite $U(1)$ Kähler connection A, given before in (3.4.28), becomes now

$$A = \frac{1}{4} K_k \mathscr{D}\phi^k - \frac{1}{4} K_{\bar{k}} \mathscr{D}\bar{\phi}^{\bar{k}} + \frac{\mathrm{i}}{8} E^a (12 G_a + \bar{\sigma}_a^{\dot{\alpha}\alpha} g_{k\bar{k}} \mathscr{D}_\alpha \phi^k \mathscr{D}_{\dot{\alpha}} \bar{\phi}^{\bar{k}}) , \tag{3.4.45}$$

simply as a consequence of covariant chirality conditions, expressions (3.4.19) and (3.4.20) for the components A_A being still valid. The covariant exterior derivatives

$$\mathscr{D}\phi^k = d\phi^k - \mathscr{A}^{(r)}(\mathbf{T}_{(r)}\phi)^k, \qquad \mathscr{D}\bar{\phi}^{\bar{k}} = d\bar{\phi}^{\bar{k}} + \mathscr{A}^{(r)}(\bar{\phi}\mathbf{T}_{(r)})^{\bar{k}} , \tag{3.4.46}$$

appearing here are now defined in the background of Kähler superspace. The superfields X_α, $\bar{X}^{\dot{\alpha}}$, previously given in (3.4.30) and (3.4.31), are still identified as the field strength components $F_{\beta a}$ (resp. $F^{\dot{\beta}}{}_a$). They take now the form

$$X_\alpha = -\frac{\mathrm{i}}{2} g_{k\bar{k}} \mathscr{D}_a \phi^k \sigma^a_{\alpha\dot{\alpha}} \mathscr{D}^{\dot{\alpha}} \bar{\phi}^{\bar{k}} + \frac{1}{2} g_{k\bar{k}} \bar{F}^{\bar{k}} \mathscr{D}_\alpha \phi^k - \frac{1}{2} \mathscr{W}_\alpha^{(r)} \mathscr{K}_{(r)} , \tag{3.4.47}$$

$$\bar{X}^{\dot{\alpha}} = -\frac{\mathrm{i}}{2} g_{k\bar{k}} \mathscr{D}_a \bar{\phi}^{\bar{k}} \bar{\sigma}^{a\dot{\alpha}\alpha} \mathscr{D}_\alpha \phi^k + \frac{1}{2} g_{k\bar{k}} F^k \mathscr{D}^{\dot{\alpha}} \bar{\phi}^{\bar{k}} - \frac{1}{2} \mathscr{W}^{(r)\dot{\alpha}} \mathscr{K}_{(r)} . \tag{3.4.48}$$

The derivatives are covariant with respect to the Yang–Mills gauge structure and we have defined

$$\mathscr{K}_{(r)} = K_k (\mathbf{T}_{(r)}\phi)^k + K_{\bar{k}} (\bar{\phi}\mathbf{T}_{(r)})^{\bar{k}} . \tag{3.4.49}$$

[15] More generally, the complex manifold of chiral matter superfields, in the sense of Kähler geometry, could be endowed with gauged isometries, compatible with supersymmetry. We have deferred the description of the corresponding geometric structure in superspace to Appendix C, see also [18].

Likewise, the Kähler D-term superfield – cf. (3.4.39),

$$-\frac{1}{2}\mathscr{D}^\alpha X_\alpha = -g_{k\bar{k}}\eta^{ab}\mathscr{D}_a\phi^k\mathscr{D}_b\bar{\phi}^{\bar{k}} - \frac{i}{4}g_{k\bar{k}}\sigma^a_{\alpha\dot{\alpha}}\mathscr{D}^\alpha\phi^k\mathscr{D}_a\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{k}}$$
$$-\frac{i}{4}g_{k\bar{k}}\sigma^a_{\alpha\dot{\alpha}}\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{k}}\mathscr{D}_a\mathscr{D}^\alpha\phi^k + g_{k\bar{k}}F^k\bar{F}^{\bar{k}} + \frac{1}{16}R_{j\bar{j}k\bar{k}}\mathscr{D}^\alpha\phi^k\mathscr{D}_\alpha\phi^j\mathscr{D}_{\dot{\alpha}}\bar{\phi}^{\bar{k}}\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{j}}$$
$$-g_{k\bar{k}}(\bar{\phi}\mathbf{T}_{(r)})^{\bar{k}}\mathscr{W}^{(r)}_\alpha\mathscr{D}^\alpha\phi^k + g_{k\bar{k}}(\mathbf{T}_{(r)}\phi)^k\mathscr{W}^{(r)}_{\dot{\alpha}}\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{k}}$$
$$+\frac{1}{4}\mathscr{D}^\alpha\mathscr{W}^{(r)}_\alpha[K_k(\mathbf{T}_{(r)}\phi)^k + K_{\bar{k}}(\bar{\phi}\mathbf{T}_{(r)})^{\bar{k}}], \tag{3.4.50}$$

receives additional terms due to the Yang–Mills couplings. Observe that covariant derivatives refer to all symmetries, definitions (3.4.34) and (3.4.35) are replaced by

$$\mathscr{D}_B\mathscr{D}_\alpha\phi^k = E_B{}^M\partial_M\mathscr{D}_\alpha\phi^k - \phi_{B\alpha}{}^\varphi\mathscr{D}_\varphi\phi^k - \mathscr{A}^{(r)}_B(\mathbf{T}_{(r)}\mathscr{D}_\alpha\phi)^k - A_B\mathscr{D}_\alpha\phi^k + \Gamma^k{}_{ij}\mathscr{D}_B\phi^i\mathscr{D}_\alpha\phi^j, \tag{3.4.51}$$

$$\mathscr{D}_B\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{k}} = E_B{}^M\partial_M\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{k}} - \phi_B{}^{\dot{\alpha}}{}_{\dot{\varphi}}\mathscr{D}^{\dot{\varphi}}\bar{\phi}^{\bar{k}} + \mathscr{A}^{(r)}_B(\mathscr{D}^{\dot{\alpha}}\bar{\phi}\mathbf{T}_{(r)})^{\bar{k}} + A_B\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{k}} + \Gamma^{\bar{k}}{}_{\bar{i}\bar{j}}\mathscr{D}_B\bar{\phi}^{\bar{i}}\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{j}} \tag{3.4.52}$$

with A_B identified in (3.4.45). In terms of these covariant derivatives the superfields F^k and $\bar{F}^{\bar{k}}$ are still defined as in (3.4.32).

Based on this geometric formulation, we can now proceed to derive supersymmetry transformations in terms of superfields, as in Appendix C.3, and in component fields, as in Section 4.3. Invariant actions in superspace and superfield equations of motion are discussed below, Section 3.4.3, and in Appendix D, whereas component field actions, derived from superspace, are given in Sections 4.4 and 4.5.

3.4.3. Superfield actions and equations of motion

Invariant actions in superspace supergravity are obtained upon integrating superspace densities over the commuting and anticommuting directions of superspace. Densities, in this case, are constructed with the help of E, the superdeterminant of $E_M{}^A$. As we have already alluded to above, the supergravity action in standard superspace geometry is just the volume of superspace. In our present situation where both supergravity and matter occur together in a generalized superspace geometry, the volume element corresponding to this superspace geometry yields the complete kinetic actions for the supergravity/matter system. To be more precise, the kinetic terms for the supergravity/matter system in our geometry are obtained from

$$\mathscr{A}_{\text{supergravity}+\text{matter}} = -3\int_* E, \tag{3.4.53}$$

where the asterisk denotes integration over space–time *and* superspace. The action of the kinetic terms of the Yang–Mills multiplet, coupled to supergravity and matter, is given as

$$\mathscr{A}_{\text{Yang–Mills}} = \frac{1}{8}\int_*\frac{E}{R}f_{(r)(s)}(\phi)\mathscr{W}^{(r)\alpha}\mathscr{W}^{(s)}_\alpha + \frac{1}{8}\int_*\frac{E}{R^\dagger}\bar{f}_{(r)(s)}(\bar{\phi})\mathscr{W}^{(r)}_{\dot{\alpha}}\mathscr{W}^{(s)\dot{\alpha}}, \tag{3.4.54}$$

whereas the superpotential coupled to supergravity is obtained from

$$\mathscr{A}_{\text{superpotential}} = \frac{1}{2}\int_{*} \frac{E}{R} e^{K/2} W(\phi) + \frac{1}{2}\int_{*} \frac{E}{R^{\dagger}} e^{K/2} \bar{W}(\bar{\phi}) \,. \tag{3.4.55}$$

Clearly, these actions are invariant under superspace coordinate transformations, what about invariance under Kähler transformations?

First of all, the superfields R and $R^{\dagger}$ have chiral weights $w(R) = 2$ and $w(R^{\dagger}) = -2$, respectively, so their Kähler transformations are

$$R \mapsto R e^{-2i\,\mathrm{Im}\,F}, \quad R^{\dagger} \mapsto R^{\dagger} e^{+2i\,\mathrm{Im}\,F} \,. \tag{3.4.56}$$

The Yang–Mills action is invariant provided the symmetric functions $f_{(r)(s)}(\phi) = f_{(s)(r)}(\phi)$ and $\bar{f}_{(r)(s)}(\bar{\phi}) = \bar{f}_{(s)(r)}(\bar{\phi})$ are inert under Kähler transformations. The superpotential terms are invariant, provided the superpotential transforms as

$$W(\phi) \mapsto e^{-F} W(\phi), \quad \bar{W}(\bar{\phi}) \mapsto e^{-\bar{F}} \bar{W}(\bar{\phi}) \,. \tag{3.4.57}$$

In this case, although neither the Kähler potential nor the superpotential are tensors with respect to Kähler transformations, the combinations

$$e^{K/2} W, \quad e^{K/2} \bar{W} \tag{3.4.58}$$

have perfectly well-defined chiral weights, namely

$$w(e^{K/2} W) = 2, \quad w(e^{K/2} \bar{W}) = -2 \,. \tag{3.4.59}$$

As to Yang–Mills symmetries, the kinetic term of the supergravity/matter system is obviously invariant, so is the superpotential term, by construction. The Yang–Mills term itself is invariant provided

$$i(\mathbf{T}_{(p)}\phi)^{k} \frac{\partial}{\partial \phi^{k}} f_{(r)(s)}(\phi) = c_{(p)(r)}{}^{(t)} f_{(t)(s)}(\phi) + c_{(p)(s)}{}^{(t)} f_{(t)(r)}(\phi) \,, \tag{3.4.60}$$

$$-\, i(\bar{\phi}\mathbf{T}_{(p)})^{\bar{k}} \frac{\partial}{\partial \bar{\phi}^{\bar{k}}} \bar{f}_{(r)(s)}(\bar{\phi}) = c_{(p)(r)}{}^{(t)} \bar{f}_{(t)(s)}(\bar{\phi}) + c_{(p)(s)}{}^{(t)} \bar{f}_{(t)(r)}(\bar{\phi}) \,, \tag{3.4.61}$$

that is, provided $f_{(r)(s)}(\phi)$ and $\bar{f}_{(r)(s)}(\bar{\phi})$ transform as the symmetric product of two adjoint representations of the Yang–Mills structure group.

We still have to justify that the superfield actions presented above indeed correctly describe the dynamics of the supergravity/matter system. One way to do so is to simply work out the corresponding component field actions – this will be done in the next chapter. Another possibility is to derive the superfield equations of motion – this will be done here. To begin with, the variation of the action $\mathscr{A} = \int d^4x \mathscr{L}(x)$ for the supergravity/matter kinetic terms can be written as

$$\delta\mathscr{A}_{\text{supergravity}+\text{matter}} = -3\int_* EH_A{}^A(-)^a\ , \tag{3.4.62}$$

where we have defined

$$H_B{}^A = E_B{}^M \delta E_M{}^A\ . \tag{3.4.63}$$

This is not the end of the story, however. The vielbein variations by themselves are not suitable, because of the presence of the torsion constraints. Solving the variational equations of the torsion constraints allows to express the vielbein variations in terms of unconstrained superfields and to derive the correct superfield equations of motion [158]. In our case the matter fields must be taken into account as well. Again, their variations themselves are not good – we have to solve first the variational equations for the chirality constraints to identify the unconstrained variations. Similar remarks hold for the Yang–Mills sector. In Appendix D a detailed derivation of the equations of motion is presented; here we content ourselves to state the results:

The complete action is given as

$$\mathscr{A} = \mathscr{A}_{\text{supergravity}+\text{matter}} + \mathscr{A}_{\text{Yang–Mills}} + \mathscr{A}_{\text{superpotential}}\ . \tag{3.4.64}$$

The superfield equations of motion are then

- *Supergravity sector*:

$$R - \tfrac{1}{2}e^{K/2}W(\phi) = 0\ , \tag{3.4.65}$$

$$R^\dagger - \tfrac{1}{2}e^{K/2}\bar{W}(\bar{\phi}) = 0\ , \tag{3.4.66}$$

$$G_b + \tfrac{1}{8}\bar{\sigma}_b^{\dot{\alpha}\alpha} g_{k\bar{k}} \mathscr{D}_\alpha \phi^k \mathscr{D}_{\dot{\alpha}} \bar{\phi}^{\bar{k}} - \tfrac{1}{8}\bar{\sigma}_b^{\dot{\alpha}\alpha}(f + \bar{f})_{(r)(s)} \mathscr{W}_\alpha^{(r)} \mathscr{W}_{\dot{\alpha}}^{(s)} = 0\ . \tag{3.4.67}$$

- *Yang–Mills sector*:

$$\frac{1}{2} f_{(r)(s)}(\phi) \mathscr{D}^\alpha \mathscr{W}_\alpha^{(s)} - \frac{1}{2}\frac{\partial f_{(r)(s)}}{\partial \phi^k} \mathscr{D}_\alpha \phi^k \mathscr{W}^{(s)\alpha} + \frac{1}{2}[K_k(\mathbf{T}_{(r)}\phi)^k + K_{\bar{k}}(\bar{\phi}\mathbf{T}_{(r)})^{\bar{k}}] + \text{h.c.} = 0\ . \tag{3.4.68}$$

- *Matter sector*:

$$g_{k\bar{k}}\bar{F}^{\bar{k}} + \frac{1}{4}\frac{\partial f_{(r)(s)}}{\partial \phi^k} \mathscr{W}^{(r)\alpha} \mathscr{W}_\alpha^{(s)} + e^{K/2} W \frac{\partial}{\partial \phi^k} \log(e^K W) = 0\ , \tag{3.4.69}$$

$$g_{k\bar{k}} F^k + \frac{1}{4}\frac{\partial \bar{f}_{(r)(s)}}{\partial \bar{\phi}^{\bar{k}}} \mathscr{W}_{\dot{\alpha}}^{(r)} \mathscr{W}^{(s)\dot{\alpha}} + e^{K/2} \bar{W} \frac{\partial}{\partial \bar{\phi}^{\bar{k}}} \log(e^K \bar{W}) = 0\ . \tag{3.4.70}$$

The lowest components in the superfield expansion provide the algebraic equations for the auxiliary fields. The equations of motion of all the other component fields of the supergravity/matter system are contained at higher orders in the superfield expansion. They are most easily obtained by suitably applying spinor derivatives and projecting afterwards to lowest superfield components.

4. Component field formalism

The superspace approach presented in the previous section provides a concise and coherent framework for the component field construction of the general supergravity/matter/Yang–Mills system. Supersymmetry and Kähler transformations of the component fields derive directly from the geometric structure, the corresponding invariant component field action has a canonically normalized curvature scalar term, without any need of component field Weyl rescalings. This should be contrasted with the original component field approach [41,42,38,39], where normalization of the action and invariance under Kähler phase transformations appeared only after a Weyl rescaling of the component fields or, equivalently, a conformal gauge fixing [109,110].

Anticipating on our results, we will see that the supergravity/matter Lagrangian (3.4.53), when projected to component fields, exhibits the kinetic Lagrangian density of the matter sector as a Fayet–Iliopoulos D-term, i.e. it has the decomposition

$$\mathscr{L}_{\text{supergravity}+\text{matter}} = \mathscr{L}_{\text{supergravity}} + e\mathbf{D}_{\text{matter}} . \tag{4.1}$$

Here e denotes the usual vierbein determinant $e = \det(e_m{}^a)$ and $\mathbf{D}_{\text{matter}}$ is the D-term pertaining to the Abelian Kähler gauge structure of the previous section. More precisely, the component field D-term derived from Kähler superspace has the form

$$\mathbf{D}_{\text{matter}} = -\frac{1}{2}\mathscr{D}^\alpha X_\alpha| + \frac{\mathrm{i}}{2}\psi_m{}^\alpha \sigma^m_{\alpha\dot\alpha} \bar X^{\dot\alpha}| + \frac{\mathrm{i}}{2}\bar\psi_{m\dot\alpha}\bar\sigma^{m\dot\alpha\alpha} X_\alpha| , \tag{4.2}$$

where the vertical bars denote projections to lowest superfield components of the superfields given, respectively, in (3.4.50), (3.4.47) and (3.4.48). Recall that a D-term in global supersymmetry may be understood as the lowest component of the superfield $D^\alpha X_\alpha$ with

$$X_\alpha = -\tfrac{1}{8}\bar D^2 D_\alpha K(\phi, \bar\phi) . \tag{4.3}$$

In this sense the Kähler superspace construction is the natural generalization of Zumino's construction [164] of supersymmetric sigma models.

In Section 4.1 we identify component fields and provide a method to derive super-covariant component field strength and space–time derivatives. In Section 4.2 we discuss some more of the basic building blocks useful for the component field formulation, in particular for the geometric derivation of supersymmetry transformations of all the component fields, which are given explicitly in Section 4.3, and the component field actions, constructed in Sections 4.4 and 4.5.

4.1. Definition of component fields

As explained already in Section 2, component fields are obtained as projections to lowest components of superfields. A supermultiplet is defined through successive application of covariant spinor derivatives and subsequent projection to lowest components, as for instance for the chiral multiplet in Section 2.2.3. Defined in this manner the component fields are related in a natural way by Wess–Zumino transformations. The structure of a supersymmetric theory, in particular the construction of invariant actions, as in Section 2.2.4, is then completely determined by the algebra of covariant derivatives. This approach avoids cumbersome expansions in the anticommuting variables and provides a geometric realization of the Wess–Zumino gauge. It is of particular importance in the case of the component field formalism for supergravity, as will be pointed out here.

In a first step we are going to identify the vierbein and the Rarita–Schwinger fields. They appear as the dx^m coefficients of the differential form $E^A = dz^M E_M{}^A$. It is therefore convenient to define systematically an operation which projects at the same time on the dx^m coefficients and on lowest superfield components, called the *double-bar projection* [11]. To be more precise, we define

$$E^a\| = e^a = dx^m e_m{}^a(x) , \tag{4.1.1}$$

$$E^\alpha\| = e^\alpha = \tfrac{1}{2} dx^m \psi_m{}^\alpha(x), \qquad E_{\dot\alpha}\| = e_{\dot\alpha} = \tfrac{1}{2} dx^m \bar\psi_{m\dot\alpha}(x) . \tag{4.1.2}$$

This identifies the vierbein field $e_m{}^a(x)$ and thereby the usual metric tensor

$$g_{mn} = e_m{}^a e_n{}^b \eta_{ab} , \tag{4.1.3}$$

as well as the gravitino field $\psi_m{}^\alpha$, $\bar\psi_{m\dot\alpha}$, which is at the same time a vector and a Majorana spinor. The factors 1/2 are included for later convenience in the construction of the Rarita–Schwinger action.

The definition of component fields as lowest superfield components defines unambiguously their chiral $U_K(1)$ weights due to the geometric construction of the previous section. As a consequence, the vierbein has vanishing weight whereas the Rarita–Schwinger field is assigned chiral weights

$$w(\psi_m{}^\alpha) = +1, \qquad w(\bar\psi_{m\dot\alpha}) = -1 . \tag{4.1.4}$$

The remaining component fields are defined as

$$R| = -\tfrac{1}{6}M, \quad R^\dagger| = -\tfrac{1}{6}\bar M, \quad G_a| = -\tfrac{1}{3}b_a \tag{4.1.5}$$

with chiral $U_K(1)$ weights

$$w(M) = +2, \quad w(\bar M) = -2, \quad w(b_a) = 0 . \tag{4.1.6}$$

The vierbein and Rarita–Schwinger fields together with M, $\bar M$ and b_a are the components of the supergravity sector, M, $\bar M$ and b_a will turn out to describe non-propagating, or auxiliary fields.

Supergravity in terms of component fields is quite complex. However, when derived from superspace geometry a number of elementary building blocks arise in a natural way, allowing to gather complicated expressions involving the basic component fields and their derivatives in a compact and concise way.

As a first example we consider the spin connection. In ordinary gravity with vanishing torsion, the spin connection is given in terms of the vierbein and its derivatives. In the supergravity case it acquires additional contributions, as we explain now. To begin with, consider the torsion component $T^a = dE^a + E^b \phi_b{}^a$, which is a superspace 2-form. The component field spin connection is identified upon applying the double-bar projection to $\phi_b{}^a$,

$$\phi_b{}^a|| = \omega_b{}^a = dx^m \omega_{mb}{}^a(x) \ . \tag{4.1.7}$$

Defining

$$\phi_\beta{}^\alpha|| = \omega_\beta{}^\alpha = dx^m \omega_{m\beta}{}^\alpha(x), \qquad \phi^{\dot\beta}{}_{\dot\alpha}|| = \omega^{\dot\beta}{}_{\dot\alpha} = dx^m \omega_m{}^{\dot\beta}{}_{\dot\alpha}(x) \tag{4.1.8}$$

for the spinor components, (3.1.4) gives rise to the usual relations

$$\omega_{m\beta}{}^\alpha = -\tfrac{1}{2}(\sigma^{ba})_\beta{}^\alpha \omega_{mba}, \quad \omega_m{}^{\dot\beta}{}_{\dot\alpha} = -\tfrac{1}{2}(\bar\sigma^{ba})^{\dot\beta}{}_{\dot\alpha}\omega_{mba} \ . \tag{4.1.9}$$

Then, applying the double-bar projection to the full torsion yields

$$T^a|| = \tfrac{1}{2} dx^m dx^n T_{nm}{}^a| = de^a + e^b \omega_b{}^a = De^a \ . \tag{4.1.10}$$

In this expression the exterior derivative is purely space–time. Using moreover

$$T_{nm}{}^a| = \mathscr{D}_n e_m{}^a - \mathscr{D}_m e_n{}^a \ , \tag{4.1.11}$$

the component field covariant derivative of the vierbein is identified as

$$\mathscr{D}_n e_m{}^a = \partial_n e_m{}^a + e_m{}^b \omega_{nb}{}^a \ . \tag{4.1.12}$$

Seemingly this is the same expression as in ordinary gravity, so how does supersymmetry modify it? To this end, we note that the double-bar projection can be employed in an alternative way, in terms of the covariant component field differentials e^A defined above. Taking into account the torsion constraints, in particular $T_{cb}{}^a = 0$, this reads simply

$$T^a|| = e_\beta e^\gamma T_\gamma{}^{\dot\beta a}| \ , \tag{4.1.13}$$

where only the constant torsion coefficient $T_\gamma{}^{\dot\beta a} = -2i(\sigma^a \varepsilon)_\gamma{}^{\dot\beta}$ survives. Combining the two alternative expressions for $T^a||$ gives rise to

$$\mathscr{D}_n e_m{}^a - \mathscr{D}_m e_n{}^a = \frac{i}{2}(\psi_n \sigma^a \bar\psi_m - \psi_m \sigma^a \bar\psi_n) \ . \tag{4.1.14}$$

In view of the explicit form of the covariant derivatives, it is a matter of straightforward algebraic manipulations to arrive at ($\sigma_m = e_m{}^a \sigma_a$)

$$\begin{aligned}\omega_{mnp} = e_{pa} e_n{}^b \omega_{mb}{}^a = {} & \tfrac{1}{2}(e_m{}^a \partial_n e_{pa} - e_p{}^a \partial_m e_{na} - e_n{}^a \partial_p e_{ma}) - \tfrac{1}{2}(e_m{}^a \partial_p e_{na} - e_n{}^a \partial_m e_{pa} - e_p{}^a \partial_n e_{ma}) \\ & + \frac{i}{4}(\psi_p \sigma_m \bar\psi_n - \psi_m \sigma_n \bar\psi_p - \psi_n \sigma_p \bar\psi_m) - \frac{i}{4}(\psi_n \sigma_m \bar\psi_p - \psi_m \sigma_p \bar\psi_n - \psi_p \sigma_n \bar\psi_m) \ .\end{aligned} \tag{4.1.15}$$

This shows how $\omega_{mb}{}^a$ is expressed in terms of the vierbein, its derivatives and, in the supersymmetric case, with additional terms quadratic in the gravitino (Rarita–Schwinger) field.

The Rarita–Schwinger component field strength is given terms of the covariant derivative of the gravitino field. As a consequence of the non-vanishing chiral $U_K(1)$ weights (4.1.4), contributions from the matter sector arise due to the presence of the component

$$A|| = dx^m A_m(x) , \tag{4.1.16}$$

of the $U_K(1)$ gauge potential. In order to work out the explicit form of $A_m(x)$, the double-bar projection must be applied to the superspace 1-form

$$A = \frac{1}{4} K_k \mathscr{D}\phi^k - \frac{1}{4} K_{\bar{k}} \mathscr{D}\bar{\phi}^{\bar{k}} + \frac{\mathrm{i}}{8} E^a (12 G_a + \bar{\sigma}_a^{\dot{\alpha}\alpha} g_{k\bar{k}} \mathscr{D}_\alpha \phi^k \mathscr{D}_{\dot{\alpha}} \bar{\phi}^{\bar{k}}) \tag{4.1.17}$$

as given in (3.4.45). This in turn means that we need to define first matter and Yang–Mills component fields and their covariant derivatives. Recall that the exterior Yang–Mills covariant derivatives are defined as

$$\mathscr{D}\phi^k = d\phi^k - \mathscr{A}^{(r)}(\mathbf{T}_{(r)}\phi)^k, \qquad \mathscr{D}\bar{\phi}^{\bar{k}} = d\bar{\phi}^{\bar{k}} + \mathscr{A}^{(r)}(\bar{\phi}\mathbf{T}_{(r)})^{\bar{k}} . \tag{4.1.18}$$

This shows that, for the definition of the component field Kähler connection A_m, we need at the same time the component fields for the matter and Yang–Mills sectors. The components of chiral, resp. antichiral superfields ϕ^k (resp. $\bar{\phi}^{\bar{k}}$) are defined as

$$\phi^k| = A^k, \quad \mathscr{D}_\alpha \phi^k| = \sqrt{2}\chi_\alpha^k, \quad \mathscr{D}^\alpha \mathscr{D}_\alpha \phi^k| = -4F^k , \tag{4.1.19}$$

$$\bar{\phi}^{\bar{k}}| = \bar{A}^{\bar{k}}, \quad \mathscr{D}_{\dot{\alpha}} \bar{\phi}^{\bar{k}}| = \sqrt{2}\bar{\chi}_{\dot{\alpha}}^{\bar{k}}, \quad \mathscr{D}_{\dot{\alpha}} \mathscr{D}^{\dot{\alpha}} \bar{\phi}^{\bar{k}}| = -4\bar{F}^{\bar{k}} \tag{4.1.20}$$

with indices $k, \bar{k}$ referring to the Kähler manifold (not to be confused with space–time indices). As to the Yang–Mills potential we define

$$\mathscr{A}|| = \mathrm{i}\boldsymbol{a} = \mathrm{i}dx^m \boldsymbol{a}_m , \tag{4.1.21}$$

whereas the remaining covariant components of the Yang–Mills multiplet are defined as

$$\mathscr{W}^{\dot{\beta}}| = \mathrm{i}\bar{\lambda}^{\dot{\beta}}, \quad \mathscr{W}_\beta| = -\mathrm{i}\lambda_\beta, \quad \mathscr{D}^\alpha \mathscr{W}_\alpha| = -2\mathbf{D} . \tag{4.1.22}$$

Recall that all the components of this multiplet are Lie algebra valued, corresponding to their identification in $\mathscr{A} = \mathscr{A}^{(r)}\mathbf{T}_{(r)}$ and $\mathscr{F} = \mathscr{F}^{(r)}\mathbf{T}_{(r)}$. We can now apply the double-bar projection to A and identify $A|| = dx^m A_m(x)$, where, for reasons of notational economy, the same symbol A_m for the superfield and its lowest component, i.e. $A_m(x) = A_m|$, is used. We obtain the explicit component field form by the double-bar projection of the covariant exterior derivatives of the matter superfields, i.e.

$$D\phi^k|| = dx^m(\partial_m A^k - \mathrm{i}\boldsymbol{a}_m^{(r)}(\mathbf{T}_{(r)}A)^k), \quad D\bar{\phi}^{\bar{k}}|| = dx^m(\partial_m \bar{A}^{\bar{k}} + \mathrm{i}\boldsymbol{a}_m^{(r)}(\bar{A}\mathbf{T}_{(r)})^{\bar{k}}) ,$$

suggesting the definitions

$$\mathscr{D}_m A^k = \partial_m A^k - \mathrm{i}\boldsymbol{a}_m^{(r)}(\mathbf{T}_{(r)}A)^k, \quad \mathscr{D}_m \bar{A}^{\bar{k}} = \partial_m \bar{A}^{\bar{k}} + \mathrm{i}\boldsymbol{a}_m^{(r)}(\bar{A}\mathbf{T}_{(r)})^{\bar{k}} . \tag{4.1.23}$$

It is then straightforward to read off the explicit component field expression

$$A_m + \frac{\mathrm{i}}{2}e_m{}^a b_a = \frac{1}{4}K_k \mathscr{D}_m A^k - \frac{1}{4}K_{\bar{k}}\mathscr{D}_m A^{\bar{k}} + \frac{\mathrm{i}}{4}g_{k\bar{k}}\chi^k\sigma_m\bar{\chi}^{\bar{k}} , \tag{4.1.24}$$

this field-dependent Kähler connection will show up in any covariant derivative acting on components with non-vanishing $U_K(1)$ weights. The spinor components of the Kähler connection are field dependent as well, they are given as – cf. (3.4.20)

$$A_\alpha| = \frac{1}{2\sqrt{2}}K_k\chi_\alpha^k, \qquad A_{\dot{\alpha}}| = -\frac{1}{2\sqrt{2}}K_{\bar{k}}\bar{\chi}_{\dot{\alpha}}^{\bar{k}} . \tag{4.1.25}$$

These terms will appear explicitly in various places of component field expressions later on as well.

We can now turn to the construction of the supercovariant component field strength $T_{cb}{}^{\underline{\alpha}}|$ for the gravitino. The relevant superspace 2-forms are $T^\alpha = dE^\alpha + E^\beta\phi_\beta{}^\alpha + E^\alpha A$ and its conjugate $T_{\dot{\alpha}}$. The double-bar projection of the field strength itself is then ($\underline{\alpha} = \alpha, \dot{\alpha}$)

$$T^{\underline{\alpha}}|| = \tfrac{1}{2}dx^m dx^n T_{nm}{}^{\underline{\alpha}}| , \tag{4.1.26}$$

where

$$T_{nm}{}^{\underline{\alpha}}| = \tfrac{1}{2}(\mathscr{D}_n\psi_m{}^{\underline{\alpha}} - \mathscr{D}_m\psi_n{}^{\underline{\alpha}}) \tag{4.1.27}$$

contains the covariant derivatives

$$\mathscr{D}_n\psi_m{}^\alpha = \partial_n\psi_m{}^\alpha + \psi_m{}^\beta\omega_{n\beta}{}^\alpha + \psi_m{}^\alpha A_n , \tag{4.1.28}$$

$$\mathscr{D}_n\bar{\psi}_{m\dot{\alpha}} = \partial_n\bar{\psi}_{m\dot{\alpha}} + \bar{\psi}_{m\dot{\beta}}\omega_n{}^{\dot{\beta}}{}_{\dot{\alpha}} - \bar{\psi}_{m\dot{\alpha}}A_n . \tag{4.1.29}$$

On the other hand, we employ the double-bar projection in terms of the covariant differentials,

$$T^\alpha|| = \tfrac{1}{2}e^b e^c T_{cb}{}^\alpha| + e^b e^\gamma T_{\gamma b}{}^\alpha| + e^b e_{\dot{\gamma}} T^{\dot{\gamma}}{}_b{}^\alpha| , \tag{4.1.30}$$

and similarly for $T_{\dot{\alpha}}$. Using the explicit form of the torsion coefficients appearing here, and comparing the two alternative forms of $T^{\underline{\alpha}}||$ gives rise to the component field expressions

$$T_{cb}{}^\alpha| = \frac{1}{2}e_b{}^m e_c{}^n(\mathscr{D}_n\psi_m{}^\alpha - \mathscr{D}_m\psi_n{}^\alpha) + \frac{\mathrm{i}}{12}(e_c{}^m\psi_m\sigma_a\bar{\sigma}_b - e_b{}^m\psi_m\sigma_a\bar{\sigma}_c)^\alpha b^a$$
$$- \frac{\mathrm{i}}{12}(e_c{}^m\bar{\psi}_m\bar{\sigma}_b - e_b{}^m\bar{\psi}_m\bar{\sigma}_c)^\alpha M \tag{4.1.31}$$

and

$$T_{cb\dot{\alpha}}| = \frac{1}{2}e_b{}^m e_c{}^n(\mathscr{D}_n\bar{\psi}_{m\dot{\alpha}} - \mathscr{D}_m\bar{\psi}_{n\dot{\alpha}}) - \frac{\mathrm{i}}{12}(e_c{}^m\bar{\psi}_m\bar{\sigma}_a\sigma_b - e_b{}^m\bar{\psi}_m\bar{\sigma}_a\sigma_c)_{\dot{\alpha}}b^a$$
$$- \frac{\mathrm{i}}{12}(e_c{}^m\psi_m\sigma_b - e_b{}^m\psi_m\sigma_c)_{\dot{\alpha}}\bar{M} \tag{4.1.32}$$

for the supercovariant gravitino field strength. The contributions of the matter and Yang–Mills sector are hidden in the covariant derivatives through the definitions given above.

Yet another important object in the component field formulation is the supercovariant version of the curvature scalar, identified as ${R_{ab}}^{ab}|$. We use the same method as before for its evaluation; the relevant superspace quantity is the curvature 2-form

$$ {R_b}^a = d{\phi_b}^a + {\phi_b}^c{\phi_c}^a \ . \tag{4.1.33} $$

The double-bar projection yields

$$ {R_b}^a\| = \tfrac{1}{2}dx^m dx^n {R_{nmb}}^a| \ , \tag{4.1.34} $$

where ${R_{nmb}}^a|$ is given in terms of ${\omega_{mb}}^a$. Note that, in distinction to ordinary gravity, the explicit form of ${\omega_{mb}}^a$, given above in (4.1.15) contains quadratic gravitino terms, which will give rise to complicated additional contributions in ${R_{nmb}}^a|$. Fortunately enough, in the present formulation, the projection technique takes care of these complications automatically in a concise way. As to the curvature scalar, we use the notation

$$ \mathscr{R}(x) = {e_a}^n {e_b}^m {R_{nm}}^{ab}| \ . \tag{4.1.35} $$

The relation between ${R_{ab}}^{ab}|$ and $\mathscr{R}(x)$ is once more obtained after employing the double-bar projection in terms of covariant differentials, i.e.

$$ {R_b}^a\| = \tfrac{1}{2}e^c e^d {R_{dcb}}^a| + e^c e^{\underline{\delta}} {R_{\underline{\delta}cb}}^a| + \tfrac{1}{2}e^{\underline{\gamma}} e^{\underline{\delta}} {R_{\underline{\delta}\underline{\gamma}b}}^a| \ , \tag{4.1.36} $$

Although our formalism is quite compact it requires still some algebra (the values of the curvature tensor components present on the right-hand side can be found in Appendix B.3) to arrive at the result

$$ \begin{aligned} {R_{ab}}^{ab}| = {} & \mathscr{R} + 2\mathrm{i}{e_b}^m(\psi_m\sigma_a\varepsilon)^{\dot\varphi}{T^{ab}}_{\dot\varphi}| + 2\mathrm{i}{e_b}^m(\sigma_a\bar\psi_m)_\varphi T^{ab\varphi}| \\ & - \frac{1}{3}\bar M\psi_m\sigma^{mn}\psi_n - \frac{1}{3}M\bar\psi_m\bar\sigma^{mn}\bar\psi_n - \frac{\mathrm{i}}{3}\varepsilon^{klmn}b_k\psi_l\sigma_m\bar\psi_n \ . \end{aligned} \tag{4.1.37} $$

Observe that this simple looking expression hides quite a number of complicated terms, in particular Rarita–Schwinger fields up to fourth order as well as contributions from the matter and Yang–Mills sectors.

Fully covariant derivatives for the components of the chiral superfields (to make things clear we write the spin term, the $U_K(1)$ term, the Yang–Mills term and the one with Kähler Levi–Civita symbol – in this order) are defined as

$$ \mathscr{D}_m\chi^k_\alpha = \partial_m\chi^k_\alpha - {\omega_{m\alpha}}^\varphi\chi^k_\varphi - A_m\chi^k_\alpha - \mathrm{i}\boldsymbol{a}^{(r)}_m(\mathbf{T}_{(r)}\chi_\alpha)^k + \chi^i_\alpha{\Gamma^k}_{ij}\mathscr{D}_m A^j \ , \tag{4.1.38} $$

$$ \mathscr{D}_m\bar\chi^{\dot\alpha\bar k} = \partial_m\bar\chi^{\dot\alpha\bar k} - {\omega_m}^{\dot\alpha}{}_{\dot\varphi}\bar\chi^{\dot\varphi\bar k} + A_m\bar\chi^{\dot\alpha\bar k} + \mathrm{i}\boldsymbol{a}^{(r)}_m(\bar\chi^{\dot\alpha}\mathbf{T}_{(r)})^{\bar k} + \bar\chi^{\dot\alpha\bar\imath}{\Gamma^{\bar k}}_{\bar\imath\bar\jmath}\mathscr{D}_m\bar A^{\bar\jmath} \ . \tag{4.1.39} $$

In the Yang–Mills sector we apply the double-bar projection to the field strength

$$\mathscr{F} = d\mathscr{A} + \mathscr{A}\mathscr{A} = \tfrac{1}{2}E^A E^B \mathscr{F}_{BA} \ . \tag{4.1.40}$$

Taking into account coefficients

$$\mathscr{F}_{\beta a}| = -(\sigma_a \bar{\lambda})_\beta , \qquad \mathscr{F}^{\dot\beta}{}_a| = -(\bar{\sigma}_a \lambda)^{\dot\beta} , \tag{4.1.41}$$

given in terms of the gaugino field, we establish the expression

$$\begin{aligned}\mathscr{F}_{ba}| = {} & \mathrm{i} e_b{}^n e_a{}^m (\partial_n \boldsymbol{a}_m - \partial_m \boldsymbol{a}_n - \mathrm{i}[\boldsymbol{a}_n, \boldsymbol{a}_m]) + \tfrac{1}{2} e_b{}^n (\psi_n \sigma_a \bar{\lambda}) - \tfrac{1}{2} e_a{}^n (\psi_n \sigma_b \bar{\lambda}) \\ & + \tfrac{1}{2} e_b{}^n (\bar{\psi}_n \bar{\sigma}_a \lambda) - \tfrac{1}{2} e_a{}^n (\bar{\psi}_n \bar{\sigma}_b \lambda)\end{aligned} \tag{4.1.42}$$

for the supercovariant field strength. The covariant derivatives of the gaugino field read

$$\mathscr{D}_m \lambda_\alpha = \partial_m \lambda_\alpha - \omega_{m\alpha}{}^\varphi \lambda_\varphi + \mathrm{i}[\boldsymbol{a}_m, \lambda_\alpha] + A_m \lambda_\alpha \ , \tag{4.1.43}$$

$$\mathscr{D}_m \bar{\lambda}^{\dot\alpha} = \partial_m \bar{\lambda}^{\dot\alpha} - \omega_m{}^{\dot\alpha}{}_{\dot\varphi} \bar{\lambda}^{\dot\varphi} + \mathrm{i}[\boldsymbol{a}_m, \bar{\lambda}^{\dot\alpha}] - A_m \bar{\lambda}^{\dot\alpha} \ . \tag{4.1.44}$$

4.2. Some basic building blocks

We indicated above that one of the necessary tasks to obtain the Lagrangian is to derive the components of the chiral superfields $X_\alpha, \bar{X}^{\dot\alpha}$. Their superfield explicit form was already derived – cf. (3.4.47) and (3.4.48) – but for the sake of simplicity, we give them here again,

$$X_\alpha = -\frac{\mathrm{i}}{2} g_{i\bar{j}} \mathscr{D}_{\alpha\dot\varphi} \phi^i \mathscr{D}^{\dot\varphi} \bar{\phi}^{\bar{j}} + \frac{1}{2} \bar{F}^{\bar{j}} g_{i\bar{j}} \mathscr{D}_\alpha \phi^i - \frac{1}{2} \mathscr{W}^{(r)}_\alpha [K_k (\mathbf{T}_{(r)} \phi)^k + K_{\bar{k}} (\bar{\phi} \mathbf{T}_{(r)})^{\bar{k}}] \ ,$$

$$\bar{X}^{\dot\alpha} = -\frac{\mathrm{i}}{2} g_{i\bar{j}} \mathscr{D}^{\varphi\dot\alpha} \bar{\phi}^{\bar{j}} \mathscr{D}_\varphi \phi^i + \frac{1}{2} F^i g_{i\bar{j}} \mathscr{D}^{\dot\alpha} \bar{\phi}^{\bar{j}} - \frac{1}{2} \mathscr{W}^{(r)\dot\alpha} [K_k (\mathbf{T}_{(r)} \phi)^k + K_{\bar{k}} (\bar{\phi} \mathbf{T}_{(r)})^{\bar{k}}] \ .$$

One infers – cf. (3.4.50)

$$\begin{aligned}-\frac{1}{2}\mathscr{D}^\alpha X_\alpha = {} & - g_{i\bar{j}} \eta^{ab} \mathscr{D}_a \phi^i \mathscr{D}_b \bar{\phi}^{\bar{j}} - \frac{\mathrm{i}}{4} g_{i\bar{j}} \sigma^a_{\alpha\dot\alpha} \mathscr{D}^\alpha \phi^i \mathscr{D}_a \mathscr{D}^{\dot\alpha} \bar{\phi}^{\bar{j}} \\ & - \frac{\mathrm{i}}{4} g_{i\bar{j}} \sigma^a_{\alpha\dot\alpha} \mathscr{D}^{\dot\alpha} \bar{\phi}^{\bar{j}} \mathscr{D}_a \mathscr{D}^\alpha \phi^i + g_{i\bar{j}} F^i \bar{F}^{\bar{j}} \\ & + \tfrac{1}{16} R_{j\bar{j}k\bar{k}} \mathscr{D}^\alpha \phi^k \mathscr{D}_\alpha \phi^j \mathscr{D}_{\dot\alpha} \bar{\phi}^{\bar{k}} \mathscr{D}^{\dot\alpha} \bar{\phi}^{\bar{j}} \\ & - g_{i\bar{j}} (\bar{\phi} \mathbf{T}_{(r)})^{\bar{j}} \mathscr{W}^{(r)}_\alpha \mathscr{D}^\alpha \phi^i + g_{i\bar{j}} (\mathbf{T}_{(r)} \phi)^i \mathscr{W}^{(r)}_{\dot\alpha} \mathscr{D}^{\dot\alpha} \bar{\phi}^{\bar{j}} \\ & + \tfrac{1}{4} \mathscr{D}^\alpha \mathscr{W}^{(r)}_\alpha [K_k (\mathbf{T}_{(r)} \phi)^k + K_{\bar{k}} (\bar{\phi} \mathbf{T}_{(r)})^{\bar{k}}] \ ,\end{aligned} \tag{4.2.1}$$

where

$$R_{i\bar{i}j\bar{j}} = \partial_i \partial_{\bar{i}} g_{j\bar{j}} - g^{k\bar{k}} g_{i\bar{k},j} g_{k\bar{i},\bar{j}} = \partial_i \partial_{\bar{i}} g_{j\bar{j}} - \Gamma^k{}_{ij} g_{k\bar{i},\bar{j}} \ . \tag{4.2.2}$$

We see that the main effort is to obtain the component field expressions of supercovariant derivatives. Special attention should be paid to the supercovariant derivatives with respect to Lorentz indices. As an example, we detail the computation of $\mathscr{D}_a\phi^i|$. The starting point is the superspace exterior derivative $\mathscr{D}\phi^i$, whose double-bar projection reads

$$\mathscr{D}\phi^i\| = dx^m \mathscr{D}_m A^i(x) . \tag{4.2.3}$$

On the other hand, in terms of covariant differentials and due to the chirality of ϕ^i, we have

$$\mathscr{D}\phi^i\| = e^a \mathscr{D}_a \phi^i| + \sqrt{2} e^\alpha \chi_\alpha . \tag{4.2.4}$$

Combination of these two equations gives immediately

$$\mathscr{D}_a\phi^i| = e_a{}^m\left(\mathscr{D}_m A^i - \frac{1}{\sqrt{2}}\psi_m{}^\alpha \chi^i_\alpha\right) . \tag{4.2.5}$$

Similarly,

$$\mathscr{D}_a\bar{\phi}^{\bar{j}}| = e_a{}^m\left(\mathscr{D}_m \bar{A}^{\bar{j}} - \frac{1}{\sqrt{2}}\bar{\psi}_{m\dot{\alpha}} \bar{\chi}^{\bar{j}\dot{\alpha}}\right) . \tag{4.2.6}$$

The lowest components of the superfields $X_\alpha, \bar{X}^{\dot{\alpha}}$ are then obtained as

$$\begin{aligned} X_\alpha| = & -\frac{\mathrm{i}}{\sqrt{2}} g_{k\bar{k}} \sigma^m_{\alpha\dot{\alpha}} \bar{\chi}^{\bar{k}\dot{\alpha}}\left(\mathscr{D}_m A^k - \frac{1}{\sqrt{2}}\psi_m{}^\beta \chi^k_\beta\right) \\ & + \frac{1}{\sqrt{2}} g_{k\bar{k}} \chi^k_\alpha \bar{F}^{\bar{k}} + \frac{\mathrm{i}}{2}\lambda^{(r)}_\alpha [K_k(\mathbf{T}_{(r)}A)^k + K_{\bar{k}}(\bar{A}\mathbf{T}_{(r)})^{\bar{k}}] , \end{aligned} \tag{4.2.7}$$

$$\begin{aligned} \bar{X}^{\dot{\alpha}}| = & -\frac{\mathrm{i}}{\sqrt{2}} g_{k\bar{k}} \bar{\sigma}^{m\dot{\alpha}\alpha} \chi^k_\alpha\left(\mathscr{D}_m \bar{A}^{\bar{k}} - \frac{1}{\sqrt{2}}\bar{\psi}_{m\beta} \bar{\chi}^{\bar{k}\beta}\right) \\ & + \frac{1}{\sqrt{2}} g_{k\bar{k}} \bar{\chi}^{\bar{k}\dot{\alpha}} F^k - \frac{\mathrm{i}}{2}\bar{\lambda}^{(r)\dot{\alpha}} [K_k(\mathbf{T}_{(r)}A)^k + K_{\bar{k}}(\bar{A}\mathbf{T}_{(r)})^{\bar{k}}] . \end{aligned} \tag{4.2.8}$$

As to $-\frac{1}{2}\mathscr{D}^\alpha X_\alpha|$, we infer that the first term in (4.2.1) reads

$$\begin{aligned} -g_{i\bar{j}}\eta^{ab}\mathscr{D}_a\phi^i\mathscr{D}_b\bar{\phi}^{\bar{j}}| = & -g_{i\bar{j}} g^{mn}\mathscr{D}_m A^i \mathscr{D}_n \bar{A}^{\bar{j}} + \frac{1}{\sqrt{2}} g_{i\bar{j}} g^{mn}\mathscr{D}_m A^i \bar{\psi}_{n\dot{\alpha}} \bar{\chi}^{\bar{j}\dot{\alpha}} \\ & + \frac{1}{\sqrt{2}} g_{i\bar{j}} g^{mn}\mathscr{D}_m \bar{A}^{\bar{j}} \psi_n{}^\alpha \chi^i{}_\alpha - \tfrac{1}{2} g_{i\bar{j}} g^{mn} \psi_m{}^\alpha \chi^i{}_\alpha \bar{\psi}_{n\dot{\alpha}} \bar{\chi}^{\bar{j}\dot{\alpha}} . \end{aligned} \tag{4.2.9}$$

We see that this term provides the kinetic term for the scalar components of the (anti)chiral matter supermultiplets (as promised, $\mathbf{D}_{\mathrm{matter}}$ contains all the derivative interactions of such fields). Likewise,

$$\mathscr{D}_a \mathscr{D}^{\dot\alpha} \bar\phi^{\bar j}| = e_a{}^m \left[\sqrt{2} \mathscr{D}_m \bar\chi^{\dot\alpha \bar j} - \bar\psi_m{}^{\dot\alpha} \bar F^{\bar j} - \mathrm{i}(\psi_m \sigma^n \varepsilon)^{\dot\alpha} \left(\mathscr{D}_n \bar A^{\bar j} - \frac{1}{\sqrt{2}} \bar\psi_{n\dot\varphi} \bar\chi^{\dot\varphi \bar j} \right) \right], \tag{4.2.10}$$

$$\mathscr{D}_a \mathscr{D}^{\alpha} \phi^i| = e_a{}^m \left[\sqrt{2} \mathscr{D}_m \chi^{\alpha i} - \psi_m{}^{\alpha} F^i + \mathrm{i}(\bar\psi_m \bar\sigma^n)^{\alpha} (\mathscr{D}_n A^i - \frac{1}{\sqrt{2}} \psi_n{}^{\varphi} \chi^i_{\varphi}) \right]. \tag{4.2.11}$$

Hence the second term in (4.2.1) yields

$$\begin{aligned} & -\frac{\mathrm{i}}{4} g_{i\bar j}\, \sigma^a_{\alpha\dot\alpha} \mathscr{D}^{\alpha} \phi^i \mathscr{D}_a \mathscr{D}^{\dot\alpha} \bar\phi^{\bar j}| - \frac{\mathrm{i}}{4} g_{i\bar j}\, \sigma^a_{\alpha\dot\alpha} \mathscr{D}^{\dot\alpha} \bar\phi^{\bar j} \mathscr{D}_a \mathscr{D}^{\alpha} \phi^i| \\ & = -\frac{\mathrm{i}}{2} \chi^{\alpha i} g_{i\bar j}\, \sigma^m_{\alpha\dot\alpha} \mathscr{D}_m \bar\chi^{\dot\alpha \bar j} + \frac{\mathrm{i}}{2} (\mathscr{D}_m \chi^{\alpha i}) \sigma^m_{\alpha\dot\alpha} g_{i\bar j} \bar\chi^{\dot\alpha \bar j} + \frac{\mathrm{i}}{2\sqrt{2}} (\chi^i \sigma^m \bar\psi_m) g_{i\bar j} \bar F^{\bar j} - \frac{\mathrm{i}}{2\sqrt{2}} (\psi_m \sigma^m \bar\chi^{\bar j}) g_{i\bar j} F^i \\ & \quad - \frac{1}{2\sqrt{2}} (\psi_m \sigma^n \bar\sigma^m \chi^i) g_{i\bar j} \left(\mathscr{D}_n \bar A^{\bar j} - \frac{1}{\sqrt{2}} \bar\psi_{n\dot\varphi} \bar\chi^{\dot\varphi \bar j} \right) - \frac{1}{2\sqrt{2}} (\bar\psi_m \bar\sigma^n \sigma^m \bar\chi^{\bar j}) g_{i\bar j} \left(\mathscr{D}_n A^i - \frac{1}{\sqrt{2}} \psi_n{}^{\varphi} \chi^i_{\varphi} \right). \end{aligned} \tag{4.2.12}$$

We stress the presence of the kinetic term for the fermionic component of the matter supermultiplet.

Altogether we obtain from (4.2.1)

$$\begin{aligned} -\frac{1}{2} \mathscr{D}^{\alpha} X_{\alpha}| = & - g_{i\bar j} g^{mn} \mathscr{D}_m A^i \mathscr{D}_n \bar A^{\bar j} - \frac{\mathrm{i}}{2} \chi^{\alpha i} g_{i\bar j} \sigma^m_{\alpha\dot\alpha} \mathscr{D}_m \bar\chi^{\bar j \dot\alpha} \\ & + \frac{\mathrm{i}}{2} (\mathscr{D}_m \chi^{\alpha i}) \sigma^m_{\alpha\dot\alpha} g_{i\bar j} \bar\chi^{\dot\alpha \bar j} + g_{i\bar j} F^i \bar F^{\bar j} + \frac{1}{2} g_{i\bar j} g^{mn} (\psi_m \chi^i)(\bar\psi_n \bar\chi^{\bar j}) \\ & + \tfrac{1}{4} R_{i\bar i j\bar j} (\chi^i \chi^j)(\bar\chi^{\bar i} \bar\chi^{\bar j}) - \mathrm{i}\sqrt{2} (\chi^i \lambda^{(r)}) g_{i\bar j} (\bar A \mathbf{T}_{(r)})^{\bar j} + \mathrm{i}\sqrt{2} (\bar\chi^{\bar j} \bar\lambda^{(r)}) g_{i\bar j} (\mathbf{T}_{(r)} A)^i \\ & - \tfrac{1}{2} \mathbf{D}^{(r)} [K_k (\mathbf{T}_{(r)} A)^k + K_{\bar k} (\bar A \mathbf{T}_{(r)})^{\bar k}] \\ & - \frac{\mathrm{i}}{2\sqrt{2}} (\bar\psi_m \bar\sigma^m \chi^i) g_{i\bar j} \bar F^{\bar j} - \frac{\mathrm{i}}{2\sqrt{2}} (\psi_m \sigma^m \bar\chi^{\bar j}) g_{i\bar j} F^i \\ & - \frac{1}{2\sqrt{2}} (\bar\psi_m \bar\sigma^n \sigma^m \bar\chi^{\bar j} - 2 \bar\psi_m \bar\chi^{\bar j} g^{nm}) g_{i\bar j} \left(\mathscr{D}_n A^i - \frac{1}{\sqrt{2}} \psi_n \chi^i \right) \\ & - \frac{1}{2\sqrt{2}} (\psi_m \sigma^n \bar\sigma^m \chi^i - 2 \psi_m \chi^i g^{nm}) g_{i\bar j} \left(\mathscr{D}_n \bar A^{\bar j} - \frac{1}{\sqrt{2}} \bar\psi_n \bar\chi^{\bar j} \right). \end{aligned} \tag{4.2.13}$$

It is straightforward to obtain the other terms in $\mathbf{D}_{\text{matter}}$, the final result reads

$$\begin{aligned}\mathbf{D}_{\text{matter}} = &- g_{i\bar{j}} g^{mn} \mathscr{D}_m A^i \mathscr{D}_n \bar{A}^{\bar{j}} - \frac{i}{2} g_{i\bar{j}} (\chi^i \sigma^m \nabla_m \bar{\chi}^{\bar{j}}) + \frac{i}{2} g_{i\bar{j}} (\nabla_m \chi^i \sigma^m \bar{\chi}^{\bar{j}}) \\ &+ g_{i\bar{j}} F^i \bar{F}^{\bar{j}} + \tfrac{1}{4} R_{i\bar{i}j\bar{j}} (\chi^i \chi^j)(\bar{\chi}^{\bar{i}} \bar{\chi}^{\bar{j}}) - \tfrac{1}{2} g_{i\bar{j}} (\chi^i \sigma^a \bar{\chi}^{\bar{j}}) b_a \\ &- \frac{1}{\sqrt{2}} (\bar{\psi}_m \bar{\sigma}^n \sigma^m \bar{\chi}^{\bar{j}}) g_{i\bar{j}} \mathscr{D}_n A^i - \frac{1}{\sqrt{2}} (\psi_m \sigma^n \bar{\sigma}^m \chi^i) g_{i\bar{j}} \mathscr{D}_n \bar{A}^{\bar{j}} \\ &- \frac{i}{2} g_{i\bar{j}} \varepsilon^{klmn} (\chi^i \sigma_k \bar{\chi}^{\bar{j}})(\psi_l \sigma_m \bar{\psi}_n) - \frac{1}{2} g_{i\bar{j}} g^{mn} (\psi_m \chi^i)(\bar{\psi}_n \bar{\chi}^{\bar{j}}) \\ &- i\sqrt{2} (\chi^i \lambda^{(r)}) g_{i\bar{j}} (\bar{A} \mathbf{T}_{(r)})^{\bar{j}} + i\sqrt{2} (\bar{\chi}^{\bar{j}} \bar{\lambda}^{(r)}) g_{i\bar{j}} (\mathbf{T}_{(r)} A)^i \\ &- \tfrac{1}{2} [\mathbf{D}^{(r)} + \tfrac{1}{2} (\bar{\psi}_m \bar{\sigma}^m \lambda^{(r)} - \psi_m \sigma^m \bar{\lambda}^{(r)})] [K_k (\mathbf{T}_{(r)} A)^k + K_{\bar{k}} (\bar{A} \mathbf{T}_{(r)})^{\bar{k}}] \,. \end{aligned} \tag{4.2.14}$$

In this expression, the covariant derivatives $\mathscr{D}_m A^i, \mathscr{D}_m \bar{A}^{\bar{j}}$ are defined in (4.1.23). The derivatives $\nabla_m \chi^i_\alpha, \nabla_m \bar{\chi}^{\bar{j}\dot{\alpha}}$ differ from $\mathscr{D}_m \chi^i_\alpha, \mathscr{D}_m \bar{\chi}^{\bar{j}\dot{\alpha}}$ already introduced in (4.1.38) and (4.1.39) by the contribution of $(i/2) e_m{}^a b_a$ to A_m – cf. (4.1.24). This allows to keep track of the complete dependence in the auxiliary field b_a in order to solve its equation of motion later. Explicitly,

$$\begin{aligned}\nabla_m \chi^i_\alpha = &\, \partial_m \chi^i_\alpha - \omega_{m\alpha}{}^\varphi \chi^i_\varphi - i \boldsymbol{a}_m^{(r)} (\mathbf{T}_{(r)} \chi_\alpha)^i - \tfrac{1}{4} (K_j \mathscr{D}_m A^j - K_{\bar{j}} \mathscr{D}_m \bar{A}^{\bar{j}}) \chi^i_\alpha \\ &- \frac{i}{4} g_{j\bar{k}} (\chi^j \sigma_m \bar{\chi}^{\bar{k}}) \chi^i_\alpha + \chi^j_\alpha \Gamma^i{}_{jk} \mathscr{D}_m A^k \,, \end{aligned} \tag{4.2.15}$$

$$\begin{aligned}\nabla_m \bar{\chi}^{\bar{j}\dot{\alpha}} = &\, \partial_m \bar{\chi}^{\bar{j}\dot{\alpha}} - \omega_m{}^{\dot{\alpha}}{}_{\dot{\varphi}} \bar{\chi}^{\bar{j}\dot{\varphi}} + i \boldsymbol{a}_m^{(r)} (\bar{\chi}^{\dot{\alpha}} \mathbf{T}_{(r)})^{\bar{j}} + \tfrac{1}{4} (K_k \mathscr{D}_m A^k - K_{\bar{k}} \mathscr{D}_m \bar{A}^{\bar{k}}) \bar{\chi}^{\bar{j}\dot{\alpha}} \\ &+ \frac{i}{4} g_{j\bar{k}} (\chi^j \sigma_m \bar{\chi}^{\bar{k}})\, \bar{\chi}^{\bar{j}\dot{\alpha}} + \bar{\chi}^{\dot{\alpha}\bar{i}} \Gamma^{\bar{j}}{}_{\bar{i}\bar{k}} \mathscr{D}_m \bar{A}^{\bar{k}} \,. \end{aligned} \tag{4.2.16}$$

Finally, using the set of equations

$$\begin{aligned} \mathscr{D}_\beta \mathscr{W}_\alpha + \mathscr{D}_\alpha \mathscr{W}_\beta &= -2(\sigma^{ba}\varepsilon)_{\beta\alpha} \mathscr{F}_{ba} \,, \\ \mathscr{D}_\beta \mathscr{W}_\alpha - \mathscr{D}_\alpha \mathscr{W}_\beta &= +\varepsilon_{\beta\alpha} \mathscr{D}^\varphi \mathscr{W}_\varphi \,, \end{aligned} \tag{4.2.17}$$

$$\begin{aligned} \mathscr{D}_{\dot\beta} \mathscr{W}_{\dot\alpha} + \mathscr{D}_{\dot\alpha} \mathscr{W}_{\dot\beta} &= -2(\varepsilon\bar{\sigma}^{ba})_{\dot\beta\dot\alpha} \mathscr{F}_{ba} \,, \\ \mathscr{D}_{\dot\beta} \mathscr{W}_{\dot\alpha} - \mathscr{D}_{\dot\alpha} \mathscr{W}_{\dot\beta} &= -\varepsilon_{\dot\beta\dot\alpha} \mathscr{D}_{\dot\varphi} \mathscr{W}^{\dot\varphi} \,, \end{aligned} \tag{4.2.18}$$

we obtain, along the same lines as before, the lowest components of the supercovariant derivative of the Yang–Mills superfields ($\mathscr{F}_{ba}|$ has been given in (4.1.42)),

$$\mathscr{D}_a \mathscr{W}^\alpha| = e_a{}^m[-\mathrm{i}\mathscr{D}_m \lambda^\alpha - \tfrac{1}{2}(\mathrm{i}\boldsymbol{f}_{pq} + \psi_p \sigma_q \bar{\lambda} + \bar{\psi}_p \bar{\sigma}_q \lambda)(\psi_m \sigma^{pq})^\alpha - \tfrac{1}{2}\mathbf{D}\psi_m{}^\alpha] ,$$
$$\mathscr{D}_a \mathscr{W}_{\dot\alpha}| = e_a{}^m[+\mathrm{i}\mathscr{D}_m \bar{\lambda}_{\dot\alpha} + \tfrac{1}{2}(\mathrm{i}\boldsymbol{f}_{pq} + \psi_p \sigma_q \bar{\lambda} + \bar{\psi}_p \bar{\sigma}_q \lambda)(\bar{\psi}_m \bar{\sigma}^{pq})_{\dot\alpha} - \tfrac{1}{2}\mathbf{D}\bar{\psi}_{m\dot\alpha}] , \tag{4.2.19}$$

where

$$\boldsymbol{f}_{mn} = \partial_m \boldsymbol{a}_n - \partial_n \boldsymbol{a}_m - \mathrm{i}[\boldsymbol{a}_m, \boldsymbol{a}_n] \tag{4.2.20}$$

and the covariant derivatives $\mathscr{D}_m \lambda^\alpha, \mathscr{D}_m \bar{\lambda}_{\dot\alpha}$ are defined in (4.1.43) and (4.1.44).

4.3. Supersymmetry transformations

In the superspace formalism, supersymmetry transformations are identified as special cases of superspace diffeomorphisms. The general form of these diffeomorphisms is given in Appendix C.3 and we will use the results obtained there.

Before writing these transformations at the component field level, we would like to stress a point of some importance in the process of generalizing supergravity transformations to the Kähler superspace. For this we need the transformation law of the vielbein and of a generic (spinless) superfield Φ under diffeomorphisms (ξ^C), Lorentz ($\Lambda_B{}^A$) and Kähler (Λ) transformations

$$\delta E_M{}^A = \mathscr{D}_M \xi^A + E_M{}^B \xi^C T_{CB}{}^A + E_M{}^B(\Lambda_B{}^A - \xi^C \phi_{CB}{}^A) + w(E^A)(\Lambda - \xi^C A_C) E_M{}^A , \tag{4.3.1}$$

$$\delta \Phi = \xi^B \mathscr{D}_B \Phi + w(\Phi)(\Lambda - \xi^C A_C)\Phi . \tag{4.3.2}$$

Supergravity transformations are defined [153] by compensating the term $\xi^C \phi_{CB}{}^A$ with a field-dependent Lorentz transformation

$$\Lambda_B{}^A = \xi^C \phi_{CB}{}^A . \tag{4.3.3}$$

The point is that the same procedure cannot be followed for the Kähler transformation since Λ is fixed to be of the form

$$\Lambda = -\frac{F(\phi) - \bar{F}(\bar{\phi})}{4} \tag{4.3.4}$$

and generic terms proportional to the Kähler connection appear in the supergravity transformations, weighted by the Kähler weight of the field considered.

Supergravity transformations, denoted by the symbol δ_{WZ}, are discussed in detail in Appendix C. As in the remainder of this section we will be exclusively concerned with supergravity transformations, we will drop from now on the subscript in δ_{WZ}, supergravity variations will be denoted δ.

- *Supergravity sector*: The transformations of vierbein and gravitino are derived from (C.3.32), which reads

$$\delta E_M{}^A = \mathscr{D}_M \xi^A + E_M{}^B \xi^C T_{CB}{}^A - \tfrac{1}{4} w(E^A) E_M{}^A \xi^B (K_k \mathscr{D}_B \phi^k - K_{\bar{k}} \mathscr{D}_B \bar{\phi}^{\bar{k}})$$
$$- \frac{\mathrm{i}}{8} w(E^A) E_M{}^A \xi^b (12 G_b + \bar{\sigma}_b^{\dot\alpha\alpha} g_{k\bar{k}} \mathscr{D}_\alpha \phi^k \mathscr{D}_{\dot\alpha} \bar{\phi}^{\bar{k}}) . \tag{4.3.5}$$

Projecting to lowest components and using (4.1.1), (4.1.2) and (4.1.5), together with the torsions summarized in Appendix B, and

$$\xi^a| = 0, \qquad \xi^\alpha| = \xi^\alpha, \qquad \xi_{\dot\alpha}| = \xi_{\dot\alpha} \tag{4.3.6}$$

gives rise to

$$\delta e_m{}^a = \mathrm{i}\xi\sigma^a\bar\psi_m + \mathrm{i}\bar\xi\bar\sigma^a\psi_m \ , \tag{4.3.7}$$

$$\delta\psi_m{}^\alpha = 2\mathscr{D}_m\xi^\alpha - \frac{\mathrm{i}}{3}(\xi\sigma^a\bar\sigma_m)^\alpha b_a + \frac{\mathrm{i}}{3}(\bar\xi\bar\sigma_m)^\alpha M - \frac{1}{2\sqrt{2}}\psi_m{}^\alpha(K_i\xi\chi^i - K_{\bar\jmath}\bar\xi\bar\chi^{\bar\jmath}) \ , \tag{4.3.8}$$

$$\delta\bar\psi_{m\dot\alpha} = 2\mathscr{D}_m\bar\xi_{\dot\alpha} + \frac{\mathrm{i}}{3}(\bar\xi\bar\sigma^a\sigma_m)_{\dot\alpha}b_a + \frac{\mathrm{i}}{3}(\xi\sigma_m)_{\dot\alpha}\bar M + \frac{1}{2\sqrt{2}}\bar\psi_{m\dot\alpha}(K_i\xi\chi^i - K_{\bar\jmath}\bar\xi\bar\chi^{\bar\jmath}) \tag{4.3.9}$$

with

$$\mathscr{D}_m\xi^\alpha = \partial_m\xi^\alpha + \xi^\beta\omega_{m\beta}{}^\alpha + \xi^\alpha A_m \ , \tag{4.3.10}$$

$$\mathscr{D}_m\bar\xi_{\dot\alpha} = \partial_m\bar\xi_{\dot\alpha} + \bar\xi_{\dot\beta}\omega_m{}^{\dot\beta}{}_{\dot\alpha} - \bar\xi_{\dot\alpha}A_m \tag{4.3.11}$$

and A_m given in (4.1.24). For future use, note that the determinant of the vielbein transforms as

$$\delta e = e e_a{}^m\delta e_m{}^a = e(\mathrm{i}\xi\sigma^m\bar\psi_m + \mathrm{i}\bar\xi\bar\sigma^m\psi_m) \tag{4.3.12}$$

and the $\sigma_m, \bar\sigma^m$ matrices as

$$\delta\sigma_{m\alpha\dot\alpha} = \delta(e_m{}^a\sigma_{a\alpha\dot\alpha}) = +\mathrm{i}\sigma_{n\alpha\dot\alpha}(\xi\sigma^n\bar\psi_m + \bar\xi\bar\sigma^n\psi_m) \ , \tag{4.3.13}$$

$$\delta\bar\sigma^{m\dot\alpha\alpha} = \delta(\bar\sigma^{a\dot\alpha\alpha}e_a{}^m) = -\mathrm{i}\bar\sigma^{n\dot\alpha\alpha}(\xi\sigma^m\bar\psi_n + \bar\xi\bar\sigma^m\psi_n) \ . \tag{4.3.14}$$

The supersymmetry transformations of the components $M, \bar M$ and b_a are derived from the supergravity transformations (C.3.35)

$$\begin{aligned}\delta\Phi &= \xi^A\mathscr{D}_A\Phi - \tfrac{1}{4}w(\Phi)\xi^A(K_k\mathscr{D}_A\phi^k - K_{\bar k}\mathscr{D}_A\bar\phi^{\bar k})\Phi \\ &\quad - \frac{\mathrm{i}}{8}w(\Phi)\xi^b(12G_b + \bar\sigma_b^{\dot\alpha\alpha}g_{k\bar k}\mathscr{D}_\alpha\phi^k\mathscr{D}_{\dot\alpha}\bar\phi^{\bar k})\Phi\end{aligned} \tag{4.3.15}$$

of the generic superfield Φ after suitable specification. In a first step, projection to lowest components yields

$$\delta\Phi| = \xi^\alpha\mathscr{D}_\alpha\Phi| + \bar\xi_{\dot\alpha}\mathscr{D}^{\dot\alpha}\Phi| - \frac{1}{2\sqrt{2}}w(\Phi)(K_k\xi\chi^k - K_{\bar k}\bar\xi\bar\chi^{\bar k})\Phi| \ . \tag{4.3.16}$$

Substituting $R, R^\dagger$ and G_a for Φ and using the information given in Appendix B, in particular (B.4.3)–(B.4.6), it is straightforward to arrive at the transformation laws

$$\begin{aligned}\delta M = & -\mathrm{i}\sqrt{2} g_{i\bar{j}} (\xi\sigma^m\bar{\chi}^{\bar{j}})\left(\mathscr{D}_m A^i - \frac{1}{\sqrt{2}}\psi_m\chi^i\right) + \sqrt{2} g_{i\bar{j}}(\xi\chi^i)\bar{F}^{\bar{j}} \\ & + \mathrm{i}(\xi\lambda^{(r)})[K_k(\mathbf{T}_{(r)}A)^k + K_{\bar{k}}(\bar{A}\mathbf{T}_{(r)})^{\bar{k}}] - \frac{1}{\sqrt{2}} M(K_i\xi\chi^i - K_{\bar{j}}\bar{\xi}\bar{\chi}^{\bar{j}}) \\ & + 4(\xi\sigma^{nm}\mathscr{D}_n\psi_m) - \mathrm{i}(\xi\sigma^m\bar{\sigma}_a\psi_m)b^a - \mathrm{i}(\xi\sigma^m\bar{\psi}_m)M\ , \end{aligned} \tag{4.3.17}$$

$$\begin{aligned}\delta\bar{M} = & -\mathrm{i}\sqrt{2} g_{i\bar{j}} (\bar{\xi}\bar{\sigma}^m\chi^i)\left(\mathscr{D}_m \bar{A}^{\bar{j}} - \frac{1}{\sqrt{2}}\bar{\psi}_m\bar{\chi}^{\bar{j}}\right) + \sqrt{2} g_{i\bar{j}} F^i(\bar{\xi}\bar{\chi}^{\bar{j}}) \\ & - \mathrm{i}(\bar{\xi}\bar{\lambda}^{(r)})[K_k(\mathbf{T}_{(r)}A)^k + K_{\bar{k}}(\bar{A}\mathbf{T}_{(r)})^{\bar{k}}] + \frac{1}{\sqrt{2}} \bar{M}(K_i\xi\chi^i - K_{\bar{j}}\bar{\xi}\bar{\chi}^{\bar{j}}) \\ & + 4(\bar{\xi}\bar{\sigma}^{nm}\mathscr{D}_n\bar{\psi}_m) + i(\bar{\xi}\bar{\sigma}^m\sigma_a\psi_m)b^a - \mathrm{i}(\bar{\xi}\bar{\sigma}^m\psi_m)\bar{M}\ , \end{aligned} \tag{4.3.18}$$

$$\begin{aligned}\delta b_a = & \tfrac{1}{2}(\xi\sigma_a\bar{\sigma}^{nm} - 3\xi\sigma^{nm}\sigma_a)\mathscr{D}_n\bar{\psi}_m - \tfrac{1}{2}(\bar{\xi}\bar{\sigma}_a\sigma^{nm} - 3\bar{\xi}\bar{\sigma}^{nm}\bar{\sigma}_a)\mathscr{D}_n\psi_m \\ & - \frac{\mathrm{i}}{2} e_a{}^m(\xi\sigma^d\bar{\psi}_m + \bar{\xi}\bar{\sigma}^d\psi_m)b_d - \frac{\mathrm{i}}{2} e_a{}^m(\bar{\psi}_m\bar{\xi})M + \frac{\mathrm{i}}{2} e_a{}^m(\xi\psi_m)\bar{M} \\ & - \frac{\mathrm{i}}{\sqrt{2}} g_{k\bar{k}}(\xi\sigma_a\bar{\sigma}^m\chi^k)\left(\mathscr{D}_m\bar{A}^{\bar{k}} - \frac{1}{\sqrt{2}}\bar{\psi}_m\bar{\chi}^{\bar{k}}\right) + \frac{1}{\sqrt{2}} g_{k\bar{k}}(\xi\sigma_a\bar{\chi}^{\bar{k}})F^k \\ & - \frac{\mathrm{i}}{\sqrt{2}} g_{k\bar{k}}(\bar{\xi}\bar{\sigma}_a\sigma^m\bar{\chi}^{\bar{k}})\left(\mathscr{D}_m A^k - \frac{1}{\sqrt{2}}\psi_m\chi^k\right) + \frac{1}{\sqrt{2}} g_{k\bar{k}}(\bar{\xi}\bar{\sigma}_a\chi^k)\bar{F}^{\bar{k}} \\ & - \frac{\mathrm{i}}{2}(\xi\sigma_a\bar{\lambda}^{(r)} - \bar{\xi}\bar{\sigma}_a\lambda^{(r)})[K_k(\mathbf{T}_{(r)}A)^k + K_{\bar{k}}(\bar{A}\mathbf{T}_{(r)})^{\bar{k}}]\ . \end{aligned} \tag{4.3.19}$$

- *Matter sector*: Let us first discuss the chiral superfield ϕ^i. The supersymmetry transformation of the component field A^i is derived from (C.3.33)

$$\delta\phi^i = \xi^A\mathscr{D}_A\phi^i\ , \tag{4.3.20}$$

upon straightforward projection to lowest components. As to the components χ^i_α and F^i the situation is slightly more involved. They are identified in the lowest components of the superfields $\mathscr{D}_\alpha\phi^i$ and $\mathscr{D}^\alpha\mathscr{D}_\alpha\phi^i$ of respective chiral weights -1 and -2. They are particular cases of a generic superfield of the type $\mathbf{U}^i$, with some chiral weight. The relevant equations in Appendix C are (C.3.27)–(C.3.31) and (C.3.36). We have to consider a superfield $\mathbf{U}^i$

(which is actually a mixture of the superfields Φ and $\mathbf{U}^i$ of Appendix C) with supergravity transformation

$$\delta\mathbf{U}^i = \xi^A\mathscr{D}_A\mathbf{U}^i + \Gamma^i{}_{jk}\xi^A\mathscr{D}_A\phi^j\mathbf{U}^k - \tfrac{1}{4}w(\mathbf{U}^i)\mathbf{U}^i\xi^A(K_k\mathscr{D}_A\phi^k - K_{\bar{k}}\mathscr{D}_A\bar{\phi}^{\bar{k}})$$
$$- \frac{\mathrm{i}}{8}w(\mathbf{U}^i)\mathbf{U}^i\xi^b(12G_b + \bar{\sigma}_b^{\dot{\alpha}\alpha}g_{k\bar{k}}\mathscr{D}_\alpha\phi^k\mathscr{D}_{\dot{\alpha}}\bar{\phi}^{\bar{k}}) \,. \tag{4.3.21}$$

This provides the supergravity transformations for χ^i_α and F^i, once $\mathbf{U}^i$ is replaced by $\mathscr{D}_\alpha\phi^i$ and $\mathscr{D}^\alpha\mathscr{D}_\alpha\phi^i$, and the result projected to lowest components. Intermediate steps in the computation involve the covariant derivative relations

$$\mathscr{D}^{\dot{\beta}}\mathscr{D}_\alpha\phi^i = 2\mathrm{i}(\sigma^a\varepsilon)_\alpha{}^{\dot{\beta}}\mathscr{D}_a\phi^i \,, \tag{4.3.22}$$

$$\mathscr{D}_\beta\mathscr{D}^\alpha\mathscr{D}_\alpha\phi^i = \tfrac{2}{3}\{\mathscr{D}_\beta, \mathscr{D}^\alpha\}\mathscr{D}_\alpha\phi^i = 8R^\dagger\mathscr{D}_\beta\phi^i \,, \tag{4.3.23}$$

$$\mathscr{D}^{\dot{\beta}}\mathscr{D}^\alpha\mathscr{D}_\alpha\phi^i = -4\mathrm{i}(\sigma^a\varepsilon)_\alpha{}^{\dot{\beta}}\mathscr{D}_a\mathscr{D}^\alpha\phi^i + 4(\sigma^a\varepsilon)_\alpha{}^{\dot{\beta}}G_a\mathscr{D}^\alpha\phi^i$$
$$+ R^i{}_{jk\bar{k}}\mathscr{D}^{\dot{\beta}}\bar{\phi}^{\bar{k}}\mathscr{D}^\alpha\phi^k\mathscr{D}_\alpha\phi^j - 8\mathscr{W}^{(r)\dot{\beta}}(\mathbf{T}_{(r)}\phi)^i \,. \tag{4.3.24}$$

As a final result we obtain the component field transformations

$$\delta A^i = \sqrt{2}\xi\chi^i \,, \tag{4.3.25}$$

$$\delta\chi^i_\alpha = \mathrm{i}\sqrt{2}(\bar{\xi}\bar{\sigma}^m\varepsilon)_\alpha\left(\mathscr{D}_m A^i - \frac{1}{\sqrt{2}}\psi_m\chi^i\right) + \sqrt{2}\xi_\alpha F^i$$
$$+ \frac{1}{\sqrt{2}}\xi_\alpha\Gamma^i{}_{jk}(\chi^j\chi^k) + \frac{1}{2\sqrt{2}}\chi^i_\alpha(K_k\xi\chi^k - K_{\bar{k}}\bar{\xi}\bar{\chi}^{\bar{k}}) \,, \tag{4.3.26}$$

$$\delta F^i = \mathrm{i}\sqrt{2}(\bar{\xi}\bar{\sigma}^m\nabla_m\chi^i) - \mathrm{i}(\bar{\xi}\bar{\sigma}^m\psi_m)F^i + (\bar{\xi}\bar{\sigma}^m\sigma^n\bar{\psi}_m)\left(\mathscr{D}_n A^i - \frac{1}{\sqrt{2}}\psi_n\chi^i\right)$$
$$+ \frac{\sqrt{2}}{3}\bar{M}\xi\chi^i + \frac{\sqrt{2}}{3}(\bar{\xi}\bar{\sigma}^a\chi^i)b_a - 2\mathrm{i}\bar{\xi}\bar{\lambda}^{(r)}(\mathbf{T}_{(r)}A)^i$$
$$+ \sqrt{2}\Gamma^i{}_{jk}(\xi\chi^j)F^k - \frac{1}{\sqrt{2}}R^i{}_{jk\bar{k}}(\chi^j\chi^k)(\bar{\xi}\bar{\chi}^{\bar{k}}) + \frac{1}{\sqrt{2}}F^i(K_k\xi\chi^k - K_{\bar{k}}\bar{\xi}\bar{\chi}^{\bar{k}}) \,, \tag{4.3.27}$$

where the relevant covariant derivatives are given in (4.1.23) and (4.2.15). The supersymmetry transformations for a general chiral superfield of non-zero weight w will be given in the next subsection – cf. (4.4.10)–(4.4.12).

Similarly, for an antichiral superfield $\bar{\phi}^{\bar{j}}$ of supergravity transformation

$$\delta_\xi\bar{\phi}^{\bar{\imath}} = \xi^A\mathscr{D}_A\bar{\phi}^{\bar{\imath}} \,, \tag{4.3.28}$$

we use the relations

$$\mathscr{D}^{\beta}\mathscr{D}_{\alpha}\bar{\phi}^{\bar{\jmath}} = 2\mathrm{i}(\sigma^a\varepsilon)_{\alpha}{}^{\beta}\mathscr{D}_a\bar{\phi}^{\bar{\jmath}} , \tag{4.3.29}$$

$$\mathscr{D}^{\beta}\mathscr{D}_{\dot{\alpha}}\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{\jmath}} = 8R\mathscr{D}^{\beta}\bar{\phi}^{\bar{\jmath}} , \tag{4.3.30}$$

$$\begin{aligned}\mathscr{D}_{\beta}\mathscr{D}_{\dot{\alpha}}\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{\jmath}} = & -4\mathrm{i}\sigma^a_{\beta\dot{\beta}}\mathscr{D}_a\mathscr{D}^{\dot{\beta}}\bar{\phi}^{\bar{\jmath}} - 4\sigma^a_{\beta\dot{\beta}}G_a\mathscr{D}^{\dot{\beta}}\bar{\phi}^{\bar{\jmath}} \\ & + R^{\bar{\jmath}}{}_{\bar{\imath}\bar{k}k}\mathscr{D}_{\beta}\phi^k\mathscr{D}_{\dot{\alpha}}\bar{\phi}^{\bar{k}}\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{\imath}} + 8\mathscr{W}^{(r)}_{\beta}(\bar{\phi}\mathbf{T}_{(r)})^{\bar{\jmath}} ,\end{aligned} \tag{4.3.31}$$

to arrive at the component field transformations

$$\delta\bar{A}^{\bar{\jmath}} = \sqrt{2}\bar{\xi}\bar{\chi}^{\bar{\jmath}} , \tag{4.3.32}$$

$$\begin{aligned}\delta\bar{\chi}^{\dot{\alpha}\bar{\jmath}} = {} & \mathrm{i}\sqrt{2}(\xi\sigma^m\varepsilon)^{\dot{\alpha}}\left(\mathscr{D}_m\bar{A}^{\bar{\jmath}} - \frac{1}{\sqrt{2}}\bar{\psi}_n\bar{\chi}^{\bar{\jmath}}\right) + \sqrt{2}\bar{\xi}^{\dot{\alpha}}\bar{F}^{\bar{\jmath}} \\ & + \frac{1}{\sqrt{2}}\bar{\xi}^{\dot{\alpha}}\Gamma^{\bar{\imath}}{}_{\bar{\jmath}\bar{k}}(\bar{\chi}^{\bar{\jmath}}\bar{\chi}^{\bar{k}}) - \frac{1}{2\sqrt{2}}\bar{\chi}^{\dot{\alpha}\bar{\jmath}}(K_k\xi\chi^k - K_{\bar{k}}\bar{\xi}\bar{\chi}^{\bar{k}}) ,\end{aligned} \tag{4.3.33}$$

$$\begin{aligned}\delta\bar{F}^{\bar{\jmath}} = {} & \mathrm{i}\sqrt{2}(\xi\sigma^m\nabla_m\bar{\chi}^{\bar{\jmath}}) - \mathrm{i}(\xi\sigma^m\bar{\psi}_m)\bar{F}^{\bar{\jmath}} + (\xi\sigma^m\bar{\sigma}^n\psi_m)\left(\mathscr{D}_n\bar{A}^{\bar{\jmath}} - \frac{1}{\sqrt{2}}\bar{\psi}_n\bar{\chi}^{\bar{\jmath}}\right) \\ & + \frac{\sqrt{2}}{3}M\bar{\xi}\bar{\chi}^{\bar{\jmath}} + \frac{\sqrt{2}}{3}(\xi\sigma^a\bar{\chi}^{\bar{\jmath}})b_a + 2\mathrm{i}\xi\lambda^{(r)}(\bar{A}\mathbf{T}_{(r)})^{\bar{\jmath}} \\ & + \sqrt{2}\Gamma^{\bar{\jmath}}{}_{\bar{\imath}\bar{k}}(\bar{\xi}\bar{\chi}^{\bar{\imath}})\bar{F}^{\bar{k}} - \frac{1}{\sqrt{2}}R^{\bar{\jmath}}{}_{\bar{\imath}\bar{k}k}(\bar{\chi}^{\bar{\imath}}\bar{\chi}^{\bar{k}})(\xi\chi^k) - \frac{1}{\sqrt{2}}\bar{F}^{\bar{\jmath}}(K_k\xi\chi^k - K_{\bar{k}}\bar{\xi}\bar{\chi}^{\bar{k}}) ,\end{aligned} \tag{4.3.34}$$

after suitable projection to lowest components.

- *Yang–Mills sector*:[16] As to the supergravity transformation of the gauge potential $\boldsymbol{a}_m = -\mathrm{i}\mathscr{A}_m|$, we project (C.3.23)

$$\delta\mathscr{A}_M = E_M{}^B\xi^C\mathscr{F}_{CB}{}^A \tag{4.3.35}$$

to lowest components and use (4.1.42) to obtain

$$\delta\boldsymbol{a}_m = \mathrm{i}(\xi\sigma_m\bar{\lambda}) + \mathrm{i}(\bar{\xi}\bar{\sigma}_m\lambda) . \tag{4.3.36}$$

Concerning the fermionic components $\lambda^{\alpha}, \bar{\lambda}_{\dot{\alpha}}$ defined in (4.1.22), the supersymmetry transformations are obtained after identification of Φ in (4.3.15) with $\mathscr{W}^{\alpha}$ (resp. $\bar{\mathscr{W}}_{\dot{\alpha}}$) and subsequent

[16] All the fields below belong to the adjoint representation of the Yang–Mills group, $(\boldsymbol{a}_m, \lambda, \bar{\lambda}, \mathbf{D}) = (\boldsymbol{a}_m^{(r)}, \lambda^{(r)}, \bar{\lambda}^{(r)}, \mathbf{D}^{(r)}) \cdot \mathbf{T}_{(r)}$.

projection to lowest components. Using (4.2.17), (4.2.18) and the explicit form of $\mathscr{F}_{ba}|$ in (4.1.42), we obtain

$$\delta\lambda^\alpha = (\xi\sigma^{mn})^\alpha(-\boldsymbol{f}_{mn} + \mathrm{i}\psi_m\sigma_n\bar{\lambda} + \mathrm{i}\bar{\psi}_m\bar{\sigma}_n\lambda) + \mathrm{i}\xi^\alpha\mathbf{D} - \frac{1}{2\sqrt{2}}\lambda^\alpha(K_k\xi\chi^k - K_{\bar{k}}\bar{\xi}\bar{\chi}^{\bar{k}}) , \tag{4.3.37}$$

$$\delta\bar{\lambda}_{\dot{\alpha}} = (\bar{\xi}\bar{\sigma}^{mn})_{\dot{\alpha}}(-\boldsymbol{f}_{mn} + \mathrm{i}\psi_m\sigma_n\bar{\lambda} + \mathrm{i}\bar{\psi}_m\bar{\sigma}_n\lambda) - \mathrm{i}\bar{\xi}_{\dot{\alpha}}\mathbf{D} + \frac{1}{2\sqrt{2}}\bar{\lambda}_{\dot{\alpha}}(K_k\xi\chi^k - K_{\bar{k}}\bar{\xi}\bar{\chi}^{\bar{k}}) \tag{4.3.38}$$

with $\boldsymbol{f}_{mn}$ defined in (4.2.20). Finally, the transformation

$$\begin{aligned}\delta\mathbf{D} = & -\xi\sigma^m\mathscr{D}_m\bar{\lambda} + \bar{\xi}\bar{\sigma}^m\mathscr{D}_m\lambda + \frac{\mathrm{i}}{2}(\bar{\psi}_m\bar{\sigma}^m\xi + \psi_m\sigma^m\bar{\xi})\mathbf{D} \\ & + \tfrac{1}{2}(\bar{\psi}_m\bar{\sigma}^{kl}\bar{\sigma}^m\xi - \psi_m\sigma^{kl}\sigma^m\bar{\xi})(\boldsymbol{f}_{kl} - \mathrm{i}\psi_k\sigma_l\bar{\lambda} - \mathrm{i}\bar{\psi}_k\bar{\sigma}_l\lambda) ,\end{aligned} \tag{4.3.39}$$

of the auxiliary field is obtained along the same lines.

4.4. Generic component field action

Although superfield actions, as discussed in Section 3.4, are quite compact, and invariance under supersymmetry transformations is rather transparent, their component field expansions are notoriously complicated. In Section 3.4 we have seen that the chiral volume element provides the generalization of the F-term construction to the case of local supersymmetry. The superfield actions for the supergravity/matter system, the Yang–Mills kinetic terms and the superpotential in(3.4.53)–(3.4.55) are all of the generic form

$$\mathscr{A}(\mathbf{r},\bar{\mathbf{r}}) = \int_* \frac{E}{R}\mathbf{r} + \text{h.c} . \tag{4.4.1}$$

with $\mathbf{r}$ a chiral superfield of $U(1)$ weight $w(\mathbf{r}) = 2$. The various superfield actions are then obtained from identifying $\mathbf{r}$, respectively, with

$$\mathbf{r}_{\text{supergravity + matter}} = -3R , \tag{4.4.2}$$

$$\mathbf{r}_{\text{Yang–Mills}} = \tfrac{1}{8}f_{(r)(s)}(\phi)\mathscr{W}^{(r)\alpha}\mathscr{W}^{(s)}_\alpha \tag{4.4.3}$$

and

$$\mathbf{r}_{\text{superpotential}} = \mathrm{e}^{K/2}W(\phi) . \tag{4.4.4}$$

We will proceed, in a first step, with the construction of a locally supersymmetric component field action a generic chiral superfield $\mathbf{r}$, starting from the definition

$$\mathscr{A}(\mathbf{r},\bar{\mathbf{r}})| = \int_* \frac{E}{R}\mathbf{r}| + \text{h.c.} = \int d^4x\mathscr{L}(\mathbf{r},\bar{\mathbf{r}}) . \tag{4.4.5}$$

In the following, we will determine $\mathscr{L}(\mathbf{r},\bar{\mathbf{r}})$ as a suitably modified F-term for the superfield $\mathbf{r}$. Defining the components of $\mathbf{r}$ as usual,

$$\mathbf{r} = \mathbf{r}|, \quad \mathbf{s}_\alpha = \frac{1}{\sqrt{2}}\mathscr{D}_\alpha\mathbf{r}|, \quad \mathbf{f} = -\frac{1}{4}\mathscr{D}^\alpha\mathscr{D}_\alpha\mathbf{r}| \,, \tag{4.4.6}$$

it should be clear that the F-term space–time density, i.e. the component field $e\mathbf{f}$ alone is not invariant under supergravity transformations. Calling

$$l_1 = e\mathbf{f} \,, \tag{4.4.7}$$

we allow for additional terms

$$l_2 = \lambda_2^\alpha \mathbf{s}_\alpha \,, \tag{4.4.8}$$

$$l_3 = \lambda_3 \mathbf{r} \tag{4.4.9}$$

with field-dependent coefficients $\lambda_2^\alpha, \lambda_3$ of respective $U(1)$ weights $-1, -2$. The strategy is then to use the supersymmetry transformations of the gravity sector, which are already known, and those of the generic multiplet to determine l_2 and l_3, i.e. λ_2^α and λ_3, such that $l_1 + l_2 + l_3$ is invariant under supersymmetry, up to a total space–time derivative. The reader not interested in the details of the computation can go directly to (4.4.21), (4.4.22) which summarize the results.

The supersymmetry transformation laws for the components of a superfield $\mathbf{r}$ of Kähler weight $w \equiv w(\mathbf{r})$ are obtained from the general procedure exposed in Section 4.3, they read

$$\delta\mathbf{r} = \sqrt{2}\xi\mathbf{s} - \frac{w}{2\sqrt{2}}(K_k\xi\chi^k - K_{\bar{k}}\bar{\xi}\bar{\chi}^{\bar{k}})\mathbf{r} \,, \tag{4.4.10}$$

$$\delta\mathbf{s}_\alpha = \sqrt{2}\xi_\alpha\mathbf{f} + \mathrm{i}\sqrt{2}(\sigma^m\bar{\xi})_\alpha\left(\mathscr{D}_m\mathbf{r} - \frac{1}{\sqrt{2}}\psi_m\mathbf{s} + \frac{\mathrm{i}w}{2}e_m{}^a b_a\mathbf{r}\right) - \frac{w-1}{2\sqrt{2}}(K_k\xi\chi^k - K_{\bar{k}}\bar{\xi}\bar{\chi}^{\bar{k}})\mathbf{s}_\alpha \,, \tag{4.4.11}$$

$$\begin{aligned}\delta\mathbf{f} = {} & \mathrm{i}\sqrt{2}(\bar{\xi}\bar{\sigma}^m\mathscr{D}_m\mathbf{s}) - \mathrm{i}(\bar{\xi}\bar{\sigma}^m\psi_m)\mathbf{f} + (\bar{\xi}\bar{\sigma}^m\sigma^n\bar{\psi}_m)\left(\mathscr{D}_n\mathbf{r} - \frac{1}{\sqrt{2}}\psi_n\mathbf{s} + \frac{\mathrm{i}w}{2}e_n{}^a b_a\mathbf{r}\right) \\ & + \frac{\sqrt{2}}{3}\bar{M}\xi\mathbf{s} - \frac{\sqrt{2}}{6}(3w-2)(\bar{\xi}\bar{\sigma}^a\mathbf{s})b_a + w\mathbf{r}\bar{\xi}_{\dot{\alpha}}\bar{X}^{\dot{\alpha}} - \frac{w-2}{2\sqrt{2}}(K_k\xi\chi^k - K_{\bar{k}}\bar{\xi}\bar{\chi}^{\bar{k}})\mathbf{f} \,.\end{aligned} \tag{4.4.12}$$

Thus, specifying to the case $w = 2$ and using (4.3.12) and (4.4.12), gives rise to

$$\begin{aligned}\frac{1}{e}\delta l_1 = {} & \mathrm{i}(\xi\sigma^m\bar{\psi}_m)\mathbf{f} - \frac{2\sqrt{2}}{3}(\bar{\xi}\bar{\sigma}^a\mathbf{s})b_a + \mathrm{i}\sqrt{2}(\bar{\xi}\bar{\sigma}^m\mathscr{D}_m\mathbf{s}) \\ & + (\bar{\xi}\bar{\sigma}^m\sigma^n\bar{\psi}_m)\left(\mathscr{D}_n\mathbf{r} - \frac{1}{\sqrt{2}}\psi_n\mathbf{s} + \mathrm{i}e_n{}^a b_a\mathbf{r}\right) + \frac{\sqrt{2}}{3}\bar{M}\xi\mathbf{s} + 2\mathbf{r}\bar{\xi}_{\dot{\alpha}}\bar{X}^{\dot{\alpha}} \,.\end{aligned} \tag{4.4.13}$$

A glance at the transformation law (4.4.11) shows that the first term can be cancelled in choosing

$$l_2 = \frac{\mathrm{i}e}{\sqrt{2}}(\bar{\psi}_m \bar{\sigma}^m)^\alpha \mathbf{s}_\alpha \ . \tag{4.4.14}$$

In the next step we work out the supersymmetry transformation of the sum $l_1 + l_2$. Using (4.3.9) and (4.3.14) we obtain

$$\frac{1}{e}\delta(l_1 + l_2) = \sqrt{2}\xi \mathbf{s}\bar{M} + 2\mathbf{r}\bar{\xi}_{\dot{\alpha}}\bar{X}^{\dot{\alpha}} + \mathrm{i}\sqrt{2}(\bar{\xi}\bar{\sigma}^m \mathscr{D}_m \mathbf{s}) + \mathrm{i}\sqrt{2}(\mathscr{D}_m \bar{\xi}\bar{\sigma}^m \mathbf{s})$$
$$+ 4(\bar{\xi}\bar{\sigma}^{mn}\bar{\psi}_m)\left(\mathscr{D}_n \mathbf{r} - \frac{1}{\sqrt{2}}\psi_n \mathbf{s} + \mathrm{i}e_n{}^a b_a \mathbf{r}\right) + \frac{1}{\sqrt{2}}(\bar{\psi}_m \bar{\sigma}^n \mathbf{s})(\xi\sigma^m \bar{\psi}_n + \bar{\xi}\bar{\sigma}^m \psi_n)$$
$$- \frac{1}{\sqrt{2}}(\bar{\psi}_m \bar{\sigma}^m \mathbf{s})(\xi \sigma^n \bar{\psi}_n + \bar{\xi}\bar{\sigma}^n \psi_n) \ . \tag{4.4.15}$$

Again, requiring cancellation of the first term suggests to choose

$$l_3 = -e\bar{M}\mathbf{r} \ . \tag{4.4.16}$$

Taking into account the supergravity transformation law (4.3.18), we now obtain

$$\frac{1}{e}\delta(l_1 + l_2 + l_3) = 4(\mathscr{D}_n \bar{\xi}\bar{\sigma}^{nm}\bar{\psi}_m)\mathbf{r} - \mathrm{i}(\bar{\xi}\bar{\sigma}^a \sigma^m \bar{\psi}_m) b_a \mathbf{r} - \mathrm{i}(\xi\sigma^m \bar{\psi}_m)\bar{M}\mathbf{r}$$
$$+ \mathrm{i}\sqrt{2}\mathscr{D}_m(\bar{\xi}\bar{\sigma}^m \mathbf{s}) - 4\mathscr{D}_n(\bar{\xi}\bar{\sigma}^{nm}\bar{\psi}_m \mathbf{r}) + \frac{4}{\sqrt{2}}(\bar{\xi}\bar{\sigma}^{nm}\bar{\psi}_m)(\psi_n \mathbf{s})$$
$$+ \frac{1}{\sqrt{2}}(\bar{\psi}_m \bar{\sigma}^n \mathbf{s})(\xi\sigma^m \bar{\psi}_n + \bar{\xi}\bar{\sigma}^m \psi_n) - \frac{1}{\sqrt{2}}(\bar{\psi}_m \bar{\sigma}^m \mathbf{s})(\xi\sigma^n \bar{\psi}_n + \bar{\xi}\bar{\sigma}^n \psi_n) \ . \tag{4.4.17}$$

Here, the first term can be cancelled with the help of another term of the type l_3. Indeed, the transformation law (4.3.9) suggests to take

$$l_3' = -e\mathbf{r}\bar{\psi}_m \bar{\sigma}^{mn}\bar{\psi}_n \ . \tag{4.4.18}$$

Using (4.3.9) and (4.3.13), (4.3.14), we find

$$\frac{1}{e}\delta l_3' = -4(\mathscr{D}_n \bar{\xi}\bar{\sigma}^{nm}\bar{\psi}_m)\mathbf{r} + \mathrm{i}(\bar{\xi}\bar{\sigma}^a \sigma^m \bar{\psi}_m) b_a \mathbf{r} + \mathrm{i}(\xi\sigma^m \bar{\psi}_m)\bar{M}\mathbf{r}$$
$$- \sqrt{2}(\bar{\psi}_m \bar{\sigma}^{mn}\bar{\psi}_n)(\xi \mathbf{s}) + 2\mathrm{i}\mathbf{r}(\bar{\psi}_m \bar{\sigma}^{kn}\bar{\psi}_n)(\xi\sigma^m \bar{\psi}_k + \bar{\xi}\bar{\sigma}^m \psi_k)$$
$$- \mathrm{i}\mathbf{r}(\bar{\psi}_m \bar{\sigma}^{mn}\bar{\psi}_n)(\xi\sigma^k \bar{\psi}_k + \bar{\xi}\bar{\sigma}^k \psi_k) \ . \tag{4.4.19}$$

Using the relation

$$e e_a{}^m \mathscr{D}_m v^a = \partial_m(e v^a e_a{}^m) + \frac{\mathrm{i}e}{2}(\sigma^b\varepsilon)_\beta{}^\beta v^a(e_b{}^n e_a{}^m - e_b{}^m e_a{}^n)\psi_n{}^\beta\bar{\psi}_{m\dot{\beta}} \tag{4.4.20}$$

for integration by parts at the component field level and after some algebra together with (A.2.58), we finally obtain

$$\delta(l_1 + l_2 + l_3 + l_3') = \partial_m[\mathrm{i}\sqrt{2}e(\bar{\xi}\bar{\sigma}^m\mathbf{s}) - 4e(\bar{\xi}\bar{\sigma}^{mn}\bar{\psi}_m)\mathbf{r}] \ . \tag{4.4.21}$$

This shows that the Lagrangian density

$$\mathscr{L}(\mathbf{r},\bar{\mathbf{r}}) = e(\mathbf{f}+\bar{\mathbf{f}}) + \frac{\mathrm{i}e}{\sqrt{2}}(\psi_m\sigma^m\bar{\mathbf{s}} + \bar{\psi}_m\bar{\sigma}^m\mathbf{s}) - e\bar{\mathbf{r}}(M + \psi_m\sigma^{mn}\psi_n) - e\mathbf{r}(\bar{M} + \bar{\psi}_m\bar{\sigma}^{mn}\bar{\psi}_n) \ , \tag{4.4.22}$$

constructed with components (4.4.6) of a generic chiral superfield of chiral weight $w = 2$ provides a supersymmetric action.

4.5. Invariant actions

The generic construction can now be applied to derive the component field versions of the superfield actions discussed in Section 4.4.3, namely $\mathscr{A}_{\text{supergravity+matter}}$, $\mathscr{A}_{\text{superpotential}}$ and $\mathscr{A}_{\text{Yang–Mills}}$ given respectively in Eqs. (3.4.53)–(3.4.55).

4.5.1. Supergravity and matter

Identifying the generic superfield such that

$$\mathbf{r}_{\text{supergravity+matter}} = -3R \ , \tag{4.5.1}$$

determines component fields correspondingly. The lowest component is given as

$$\mathbf{r} = \frac{M}{2} \ . \tag{4.5.2}$$

As a consequence of (3.4.42) the spinor component takes the form

$$\mathbf{s}_\alpha = \frac{1}{\sqrt{2}}X_\alpha| + \sqrt{2}(\sigma^{cb}\varepsilon)_{\alpha\varphi}T_{cb}{}^\varphi| \ . \tag{4.5.3}$$

In the construction of the component field Lagrangian this appears in the combination

$$\begin{aligned}\frac{\mathrm{i}}{\sqrt{2}}(\bar{\psi}_m\bar{\sigma}^m\mathbf{s}) = {} & \frac{\mathrm{i}}{2}\bar{\psi}_m\bar{\sigma}^m X| + ie_b^m(\bar{\psi}_m\bar{\sigma}_a\varepsilon)_\varphi T^{ab\varphi}| \\ & + \frac{1}{2}\varepsilon^{mnpq}\bar{\psi}_m\bar{\sigma}_n\mathscr{D}_p\psi_q + \frac{\mathrm{i}}{6}\varepsilon^{mnpq}\bar{\psi}_m\bar{\sigma}_n\psi_q b_p \\ & + \frac{\mathrm{i}}{6}(\psi_n\sigma^m\bar{\psi}_m - \psi_m\sigma^m\bar{\psi}_n)b^n + \frac{1}{3}\bar{\psi}_m\bar{\sigma}^{mn}\bar{\psi}_n M \ ,\end{aligned} \tag{4.5.4}$$

where we have used (A.2.46) and (4.1.31) as well as other formulas given in Appendix A. Finally, from (3.4.44) and (4.1.37), we infer

$$\mathbf{f}+\bar{\mathbf{f}} = -\tfrac{1}{4}\mathscr{R} - \mathrm{i}e_b{}^m(\bar{\psi}_m\bar{\sigma}_a\varepsilon)_\varphi T^{ab\varphi}| - \tfrac{1}{4}\mathscr{D}^\alpha X_\alpha| + \tfrac{1}{6}b^a b_a + \tfrac{1}{3}M\bar{M}$$

$$+\frac{\mathrm{i}}{12}\varepsilon^{mnpq}\bar{\psi}_m\bar{\sigma}_n\psi_q b_p + \frac{1}{6}M\bar{\psi}_m\bar{\sigma}^{mn}\bar{\psi}_n + \text{h.c}\,. \tag{4.5.5}$$

with the curvature scalar $\mathscr{R}$ defined in (4.1.35).

Recapitulating, the Lagrangian (4.4.22) becomes

$$e^{-1}\mathscr{L}_{\text{supergravity}+\text{matter}} = -\frac{1}{4}\mathscr{R} + \frac{1}{2}\varepsilon^{mnpq}\bar{\psi}_m\bar{\sigma}_n\left(\mathscr{D}_p\psi_q + \frac{\mathrm{i}}{2}b_p\psi_q\right)$$

$$-\frac{1}{6}M\bar{M} + \frac{1}{6}b^a b_a - \frac{1}{4}\mathscr{D}^\alpha X_\alpha| + \frac{\mathrm{i}}{2}\bar{\psi}_m\bar{\sigma}^m X| + \text{h.c.}$$

$$= -\tfrac{1}{2}\mathscr{R} + \tfrac{1}{2}\varepsilon^{mnpq}(\bar{\psi}_m\bar{\sigma}_n\nabla_p\psi_q - \psi_m\sigma_n\nabla_p\bar{\psi}_q) - \tfrac{1}{3}M\bar{M} + \tfrac{1}{3}b^a b_a + \mathbf{D}_{\text{matter}}\,. \tag{4.5.6}$$

The cancellation of the $\psi_m b_n \bar{\psi}_p$ terms with those coming from (4.1.24) is manifest in terms of the new covariant derivatives

$$\nabla_n\psi_m{}^\alpha = \partial_n\psi_m{}^\alpha + \psi_m{}^\beta\omega_{n\beta}{}^\alpha + \tfrac{1}{4}\psi_m{}^\alpha(K_k\mathscr{D}_n A^k - K_{\bar{k}}\mathscr{D}_n A^{\bar{k}} + \mathrm{i}g_{k\bar{k}}\chi^k\sigma_n\bar{\chi}^{\bar{k}})\,, \tag{4.5.7}$$

$$\nabla_n\bar{\psi}_{m\dot{\alpha}} = \partial_n\bar{\psi}_{m\dot{\alpha}} + \bar{\psi}_{m\dot{\beta}}\omega_n{}^{\dot{\beta}}{}_{\dot{\alpha}} - \tfrac{1}{4}\bar{\psi}_{m\dot{\alpha}}(K_k\mathscr{D}_n A^k - K_{\bar{k}}\mathscr{D}_n A^{\bar{k}} + \mathrm{i}g_{k\bar{k}}\chi^k\sigma_m\bar{\chi}^{\bar{k}})\,, \tag{4.5.8}$$

which are fully Lorentz, Kähler and gauge covariant derivatives. Finally, the expression of $\mathbf{D}_{\text{matter}}$, defined in (4.2), in terms of the component fields has been given explicitly in (4.2.15).

We now see explicitly what was stressed in the introduction to this section: the explicit dependence in the matter fields appears only through the D-term induced by the Kähler structure $e\mathbf{D}_{\text{matter}}$; the rest of the Lagrangian has the form of the standard supergravity Lagrangian. It should be kept in mind, however, that all the covariant derivatives in $\mathscr{L}_{\text{supergravity}+\text{matter}}$ are now covariant also with respect to the Kähler and Yang–Mills transformations.

4.5.2. Superpotential

We now turn to the potential term in the Lagrangian and consider

$$\mathbf{r}_{\text{superpotential}} = \mathrm{e}^{K/2}W\,. \tag{4.5.9}$$

In order to identify the corresponding component fields we have to apply covariant spinor derivatives. Since neither K nor W are tensors with respect to the Kähler phase transformations we make use of $\mathscr{D}_\alpha\mathbf{r} = E_\alpha{}^M\partial_M\mathbf{r} + 2A_\alpha\mathbf{r}$, before applying the product rule. Recall that in (C.4.8), the explicit form of A_α is given as

$$A_\alpha = \tfrac{1}{4}K_k\mathscr{D}_\alpha\phi^k\,, \tag{4.5.10}$$

in terms of the usual Yang–Mills covariant derivative. Using furthermore the requirement that W as well as K are Yang–Mills invariant, we obtain

$$E_\alpha{}^M \partial_M W = W_k \mathscr{D}_\alpha \phi^k, \quad E_\alpha{}^M \partial_M K = K_k \mathscr{D}_\alpha \phi^k . \tag{4.5.11}$$

Adding these three contributions yields

$$\mathscr{D}_\alpha \mathbf{r} = \mathrm{e}^{K/2}(K_k W + W_k)\mathscr{D}_\alpha \phi^k . \tag{4.5.12}$$

Let us note that the combination $(K_k W + W_k)$ behaves as W under Kähler transformations, i.e.

$$W \mapsto \mathrm{e}^{-F} W \quad \text{then} \quad (K_k W + W_k) \mapsto \mathrm{e}^{-F}(K_k W + W_k) . \tag{4.5.13}$$

This suggests to denote

$$(K_k W + W_k) = D_k W \tag{4.5.14}$$

and we obtain

$$\mathbf{s}_\alpha = \mathrm{e}^{K/2} \chi^k_\alpha D_k W . \tag{4.5.15}$$

The evaluation of $\mathscr{D}^\alpha \mathscr{D}_\alpha \mathbf{r}_{\text{superpotential}}$ proceeds along the same lines. Taking carefully into account the Kähler structure leads to

$$\mathscr{D}^\alpha \mathscr{D}_\alpha \mathbf{r} = + \mathrm{e}^{K/2}(K_k W + W_k)\mathscr{D}^\alpha \mathscr{D}_\alpha \phi^k + \mathrm{e}^{K/2}[(K_{ij} - K_k \Gamma^k{}_{ij} + K_i K_j)W + (W_{ij} - W_k \Gamma^k{}_{ij} + W_j K_i + W_i K_j)]\mathscr{D}^\alpha \phi^i \mathscr{D}_\alpha \phi^j . \tag{4.5.16}$$

Observe that the expression inside brackets is just equal to $(\partial_i + K_i)D_j W - \Gamma^k{}_{ij} D_k W$ and transforms as W and $D_i W$ under Kähler (the presence of the Levi–Civita symbol ensures the covariance of the derivatives with respect to Kähler manifold indices). Again, this suggests the definition

$$D_i D_j W = (\partial_i + K_i)D_j W - \Gamma^k{}_{ij} D_k W , \tag{4.5.17}$$

giving rise to the compact expression

$$\mathbf{f} = \mathrm{e}^{K/2}[F^k D_k W - \tfrac{1}{2}\chi^i \chi^j D_i D_j W] \tag{4.5.18}$$

for the F-term component field. Substituting in the generic formula (4.4.22), yields the Lagrangian

$$\begin{aligned} \mathrm{e}^{-1}\mathscr{L}_{\text{superpotential}} &= \mathrm{e}^{K/2}[F^k D_k W + \bar{F}^{\bar{k}} D_{\bar{k}} \bar{W} - M\bar{W} - \bar{M} W] \\ &- \frac{\mathrm{e}^{K/2}}{2}[\chi^i \chi^j D_i D_j W + \bar{\chi}^{\bar{i}} \bar{\chi}^{\bar{j}} D_{\bar{i}} D_{\bar{j}} \bar{W}] \\ &+ \frac{\mathrm{e}^{K/2}}{\sqrt{2}}[\mathrm{i}(\bar{\psi}_m \bar{\sigma}^m \chi^k) D_k W + \mathrm{i}(\psi_m \sigma^m \bar{\chi}^{\bar{k}}) D_{\bar{k}} \bar{W} - \sqrt{2}(\bar{\psi}_m \bar{\sigma}^{mn} \bar{\psi}_n) W - \sqrt{2}(\psi_m \sigma^{mn} \psi_n) \bar{W}] . \end{aligned} \tag{4.5.19}$$

4.5.3. Yang–Mills

Finally, to obtain the Yang–Mills Lagrangian, we start from the superfield

$$\mathbf{r}_{\text{Yang–Mills}} = \tfrac{1}{4} f_{(r)(s)} \mathscr{W}^{(r)\alpha} \mathscr{W}^{(s)}_{\alpha} \tag{4.5.20}$$

with lowest component

$$\mathbf{r} = -\tfrac{1}{4} f_{(r)(s)} (\lambda^{(r)} \lambda^{(s)}) . \tag{4.5.21}$$

Applying a covariant spinor derivative to $\mathbf{r}_{\text{Yang–Mills}}$ and using the transformation properties of $f_{(r)(s)}$ and $\bar{f}_{(r)(s)}$ as given in (3.4.60) and (3.4.61), together with (4.2.17) yields

$$\mathscr{D}_\alpha \mathbf{r}_{\text{Yang–Mills}} = -\tfrac{1}{4} f_{(r)(s)} \mathscr{W}^{(r)}_{\alpha} \mathscr{D}^{\varphi} \mathscr{W}^{(s)}_{\varphi} + \tfrac{1}{2} f_{(r)(s)} (\sigma^{ba} \mathscr{W}^{(r)})_\alpha \mathscr{F}^{(s)}_{ba}$$
$$+ \frac{1}{4} \frac{\partial f_{(r)(s)}}{\partial \phi^i} \mathscr{D}_\alpha \phi^i \mathscr{W}^{(r)\varphi} \mathscr{W}^{(s)}_{\varphi} . \tag{4.5.22}$$

It remains to project to the lowest superfield components – cf. (4.1.22), (4.1.42), (4.4.6), giving rise to

$$\mathbf{s}_\alpha = \frac{-\mathrm{i}}{2\sqrt{2}} f_{(r)(s)} [\lambda^{(r)}_\alpha \mathbf{D}^{(s)} + (\sigma^{mn} \lambda^{(r)})_\alpha (\mathrm{i} \boldsymbol{f}^{(s)}_{mn} + \psi_m \sigma_n \bar{\lambda}^{(s)} + \bar{\psi}_m \bar{\sigma}_n \lambda^{(s)})]$$
$$- \frac{1}{4} \frac{\partial f_{(r)(s)}}{\partial A^i} \chi^i_\alpha (\lambda^{(r)} \lambda^{(s)}) \tag{4.5.23}$$

with $\boldsymbol{f}^{(s)}_{pq}$ defined in (4.2.20). Similarly, using (4.2.17) and (B.5.28), we obtain

$$\mathscr{D}^\alpha \mathscr{D}_\alpha \mathbf{r} = -\frac{1}{2} f_{(r)(s)} \left(\frac{1}{2} (\mathscr{D}^\alpha \mathscr{W}^{(r)}_\alpha)(\mathscr{D}^\beta \mathscr{W}^{(s)}_\beta) + \mathscr{F}^{(r)ba} \mathscr{F}^{(s)}_{ba} + \frac{\mathrm{i}}{2} \varepsilon^{dcba} \mathscr{F}^{(r)}_{dc} \mathscr{F}^{(s)}_{ba} \right)$$
$$+ \frac{1}{2} f_{(r)(s)} \mathscr{W}^{(r)\alpha} (12 R^\dagger \mathscr{W}^{(s)}_\alpha + 4\mathrm{i} \sigma^a_{\alpha\dot{\alpha}} \mathscr{D}_a \mathscr{W}^{(s)\dot{\alpha}})$$
$$- \frac{\partial f_{(r)(s)}}{\partial \phi^i} \left(\frac{1}{2} \mathscr{D}^\alpha \phi^i \mathscr{W}^{(r)}_\alpha \mathscr{D}^\beta \mathscr{W}^{(s)}_\beta - \mathscr{D}^\alpha \phi^i (\sigma^{ba})_\alpha{}^\beta \mathscr{W}^{(r)}_\beta \mathscr{F}^{(s)}_{ba} \right)$$
$$+ \frac{1}{4} \frac{\partial f_{(r)(s)}}{\partial \phi^i} \mathscr{D}^\alpha \mathscr{D}_\alpha \phi^i \mathscr{W}^{(r)\varphi} \mathscr{W}^{(s)}_\varphi$$
$$+ \frac{1}{4} \left(\frac{\partial^2 f_{(r)(s)}}{\partial \phi^i \partial \phi^j} - \Gamma^l{}_{ij} \frac{\partial f_{(r)(s)}}{\partial \phi^l} \right) \mathscr{D}^\alpha \phi^i \mathscr{D}_\alpha \phi^j \mathscr{W}^{(r)\varphi} \mathscr{W}^{(s)}_\varphi . \tag{4.5.24}$$

One recognizes in the last line the covariant derivative of $f_{(r)(s)}$ with respect to Kähler manifold indices. The corresponding component field expression is

$$
\begin{aligned}
\boldsymbol{f} = & -\frac{1}{4} f_{(r)(s)} \Big[\frac{1}{2} \boldsymbol{f}^{(r)mn} \boldsymbol{f}^{(s)}_{mn} + \frac{\mathrm{i}}{4} \varepsilon^{mnpq} \boldsymbol{f}^{(r)}_{mn} \boldsymbol{f}^{(s)}_{pq} + 2\mathrm{i} \lambda^{(r)} \sigma^m \mathscr{D}_m \bar{\lambda}^{(s)} \\
& - \mathbf{D}^{(r)} \mathbf{D}^{(s)} + \bar{M} \lambda^{(r)} \lambda^{(s)} - (\lambda^{(r)} \sigma^m \bar{\psi}_m) \mathbf{D}^{(s)} \\
& - \mathrm{i} (\psi_m \sigma^{pq} \sigma^m \bar{\lambda}^{(r)} + \bar{\psi}_m \bar{\sigma}^{pq} \bar{\sigma}^m \lambda^{(r)} - \bar{\psi}_m \bar{\sigma}^m \sigma^{pq} \lambda^{(r)}) \boldsymbol{f}^{(s)}_{pq} \Big] \\
& + \frac{1}{4} f_{(r)(s)} \left(\frac{1}{2} \psi_m \sigma^{pq} \sigma^m \bar{\lambda}^{(r)} + \bar{\psi}_m \bar{\sigma}^{pq} \bar{\sigma}^m \lambda^{(r)} - \frac{1}{2} \bar{\psi}_m \bar{\sigma}^m \sigma^{pq} \lambda^{(r)} \right) (\psi_p \sigma_q \bar{\lambda}^{(s)} + \bar{\psi}_p \bar{\sigma}_q \lambda^{(s)}) \\
& + \frac{1}{4} \frac{\partial f_{(r)(s)}}{\partial A^i} \Big[-\sqrt{2} (\chi^i \sigma^{mn} \lambda^{(r)}) \boldsymbol{f}^{(s)}_{mn} + \mathrm{i}\sqrt{2} (\chi^i \lambda^{(r)}) \mathbf{D}^{(s)} - F^i (\lambda^{(r)} \lambda^{(s)}) \\
& + \mathrm{i}\sqrt{2} (\chi^i \sigma^{mn} \lambda^{(r)}) (\psi_m \sigma_n \bar{\lambda}^{(s)} + \bar{\psi}_m \bar{\sigma}_n \lambda^{(s)}) \Big] \\
& + \frac{1}{8} \left(\frac{\partial^2 f_{(r)(s)}}{\partial A^i \partial A^j} - \Gamma^l{}_{ij} \frac{\partial f_{(r)(s)}}{\partial A^l} \right) (\chi^i \chi^j) (\lambda^{(r)} \lambda^{(s)}) \,, \qquad (4.5.25)
\end{aligned}
$$

where the covariant derivative $\mathscr{D}_m \bar{\lambda}^{(s)}$ is defined in (4.1.44). Making heavy use of relations (A.2.42)–(A.2.51), we finally obtain

$$
\begin{aligned}
\mathrm{e}^{-1} \mathscr{L}_{\text{Yang–Mills}} = & -\tfrac{1}{4} f_{(r)(s)} \\
& \times \Big[\frac{1}{2} \boldsymbol{f}^{(r)mn} \boldsymbol{f}^{(s)}_{mn} + \frac{\mathrm{i}}{4} \varepsilon^{mnpq} \boldsymbol{f}^{(r)}_{mn} \boldsymbol{f}^{(s)}_{pq} + 2\mathrm{i} \lambda^{(r)} \sigma^m \nabla_m \bar{\lambda}^{(s)} - \mathbf{D}^{(r)} \mathbf{D}^{(s)} - (\lambda^{(r)} \sigma^a \bar{\lambda}^{(s)}) b_a \\
& - \mathrm{i} \boldsymbol{f}^{(r)mn} (\psi_m \sigma_n \bar{\lambda}^{(s)} + \bar{\psi}_m \bar{\sigma}_n \lambda^{(s)}) + \tfrac{1}{2} \varepsilon^{mnpq} \boldsymbol{f}^{(r)}_{mn} (\psi_p \sigma_q \bar{\lambda}^{(s)} - \bar{\psi}_p \bar{\sigma}_q \lambda^{(s)}) \\
& + \tfrac{1}{8} (\lambda^{(r)} \lambda^{(s)}) (4 \bar{\psi}_m \bar{\psi}^m + \bar{\psi}_m \bar{\sigma}^m \sigma^n \bar{\psi}_n) + \tfrac{1}{8} (\bar{\lambda}^{(r)} \bar{\lambda}^{(s)}) (4 \psi_m \psi^m + \psi_m \sigma^m \bar{\sigma}^n \psi_n) \\
& - \tfrac{1}{2} (g^{mp} g^{nq} - g^{mq} g^{np} - i \varepsilon^{mnpq}) (\bar{\psi}_m \bar{\sigma}_n \lambda^{(r)}) (\psi_p \sigma_q \bar{\lambda}^{(s)}) \Big] \\
& - \frac{1}{4} \frac{\partial f_{(r)(s)}}{\partial A^i} \Big[\sqrt{2} (\chi^i \sigma^{mn} \lambda^{(r)}) \boldsymbol{f}^{(s)}_{mn} - \mathrm{i}\sqrt{2} (\chi^i \lambda^{(r)}) \mathbf{D}^{(s)} + F^i (\lambda^{(r)} \lambda^{(s)}) \\
& - \mathrm{i} \frac{\sqrt{2}}{4} (\lambda^{(r)} \lambda^{(s)}) (\bar{\psi}_m \bar{\sigma}^m \chi^i) - \mathrm{i}\sqrt{2} (\psi_m \sigma_n \bar{\lambda}^{(r)}) (\chi^i \sigma^{mn} \lambda^{(s)}) \Big] \\
& + \frac{1}{8} \left(\frac{\partial^2 f_{(r)(s)}}{\partial A^i \partial A^j} - \Gamma^l{}_{ij} \frac{\partial f_{(r)(s)}}{\partial A^l} \right) (\chi^i \chi^j) (\lambda^{(r)} \lambda^{(s)}) + \text{h.c}\,. \qquad (4.5.26)
\end{aligned}
$$

The Yang–Mills field strength $\boldsymbol{f}^{(r)}_{mn}$ is defined in (4.2.20). The covariant derivatives

$$\nabla_m \lambda^{(r)}_\alpha = \partial_m \lambda^{(r)}_\alpha - \omega_{m\alpha}{}^{\varphi}\lambda^{(r)}_\varphi - \boldsymbol{a}^{(t)}_m c_{(s)(t)}{}^{(r)}\lambda^{(s)}_\alpha$$

$$+ \frac{1}{4}(K_j \mathscr{D}_m A^j - K_{\bar{j}}\mathscr{D}_m \bar{A}^{\bar{j}})\lambda^{(r)}_\alpha + \frac{\mathrm{i}}{4} g_{j\bar{k}}(\chi^j \sigma_m \bar{\chi}^{\bar{k}})\lambda^{(r)}_\alpha, \tag{4.5.27}$$

$$\nabla_m \bar{\lambda}^{(r)\dot{\alpha}} = \partial_m \bar{\lambda}^{(r)\dot{\alpha}} - \omega_m{}^{\dot{\alpha}}{}_{\dot{\varphi}}\bar{\lambda}^{(r)\dot{\varphi}} - \boldsymbol{a}^{(t)}_m c_{(s)(t)}{}^{(r)}\bar{\lambda}^{(s)\dot{\alpha}}$$

$$- \frac{1}{4}(K_k \partial_m A^k - K_{\bar{k}}\partial_m \bar{A}^{\bar{k}})\bar{\lambda}^{(r)\dot{\alpha}} - \frac{\mathrm{i}}{4} g_{j\bar{k}}(\chi^j \sigma_m \bar{\chi}^{\bar{k}})\bar{\lambda}^{(r)\dot{\alpha}} \,. \tag{4.5.28}$$

differ from the covariant derivatives $\mathscr{D}_m \lambda^{(r)}_\alpha$ and $\mathscr{D}_m \bar{\lambda}^{(r)\dot{\alpha}}$ introduced in (4.1.43) and (4.1.44) by the covariant b_a dependent term appearing in the definition of A_m, in analogy with previous definitions – cf. (4.2.15), (4.2.16) and (4.5.7), (4.5.8).

4.5.4. Recapitulation

The complete Lagrangian describing the interaction of Yang–Mills and chiral supermultiplets with supergravity is given by the sum of (4.5.6), (4.5.19), and (4.5.26), with the matter D-term given in (4.2.15). In taking the sum, we diagonalize in the auxiliary field sector, with the result

$$\begin{aligned}
\mathrm{e}^{-1}\mathscr{L} \\
= & -\tfrac{1}{2}\mathscr{R} + \tfrac{1}{2}\varepsilon^{mnpq}(\bar{\psi}_m \bar{\sigma}_n \nabla_p \psi_q - \psi_m \sigma_n \nabla_p \bar{\psi}_q) \\
& - g_{i\bar{j}}\mathscr{D}^m A^i \mathscr{D}_m \bar{A}^{\bar{j}} - \frac{\mathrm{i}}{2} g_{i\bar{j}}(\chi^i \sigma^m \nabla_m \bar{\chi}^{\bar{j}} + \bar{\chi}^{\bar{j}}\bar{\sigma}^m \nabla_m \chi^i) \\
& - \tfrac{1}{4}\mathrm{Re} f_{(r)(s)} \boldsymbol{f}^{(r)mn}\boldsymbol{f}^{(s)}_{mn} + \tfrac{1}{8}\mathrm{Im} f_{(r)(s)}\, \varepsilon_{mnpq}\boldsymbol{f}^{(r)mn}\boldsymbol{f}^{(s)pq} \\
& - \frac{\mathrm{i}}{2}[f_{(r)(s)}\lambda^{(r)}\sigma^m \nabla_m \bar{\lambda}^{(s)} + \bar{f}_{(r)(s)}\bar{\lambda}^{(r)}\bar{\sigma}^m \nabla_m \lambda^{(s)}] \\
& + 3\mathrm{e}^K |W|^2 - g^{i\bar{j}}\mathrm{e}^K D_i W\, D_{\bar{j}}\bar{W} - \frac{\mathrm{e}^{K/2}}{2}[D_i D_j W\,(\chi^i \chi^j) + D_{\bar{i}} D_{\bar{j}}\bar{W}\,(\bar{\chi}^{\bar{i}}\bar{\chi}^{\bar{j}})] \\
& + \tfrac{1}{4}(R_{i\bar{j}k\bar{l}} + \tfrac{3}{2} g_{i\bar{j}} g_{k\bar{l}})(\chi^i \chi^k)(\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{l}}) - \tfrac{3}{4} g_{i\bar{j}}\,\mathrm{Re} f_{(r)(s)}\,(\chi^i \lambda^{(r)})(\bar{\chi}^{\bar{j}}\bar{\lambda}^{(s)}) \\
& - \mathrm{i}\sqrt{2}(\chi^i \lambda^{(r)}) g_{i\bar{j}}(\bar{A}\mathbf{T}_{(r)})^{\bar{j}} + \mathrm{i}\sqrt{2}(\bar{\chi}^{\bar{j}}\bar{\lambda}^{(r)}) g_{i\bar{j}}(\mathbf{T}_{(r)}A)^i \\
& - \frac{1}{2\sqrt{2}}\left[\frac{\partial f_{(r)(s)}}{\partial A^i}(\chi^i \sigma^{mn}\lambda^{(r)}) + \frac{\partial \bar{f}_{(r)(s)}}{\partial \bar{A}^{\bar{i}}}(\bar{\chi}^{\bar{i}}\bar{\sigma}^{mn}\bar{\lambda}^{(r)})\right]\boldsymbol{f}^{(s)}_{mn}
\end{aligned}$$

$$+\frac{1}{8}\left[\left(\frac{\partial^2 f_{(r)(s)}}{\partial A^i \partial A^j} - \Gamma^l{}_{ij}\frac{\partial f_{(r)(s)}}{\partial A^l}\right)(\chi^i\chi^j) + 2g^{i\bar{j}}\mathrm{e}^{K/2}\,D_{\bar{j}}\bar{W}\,\frac{\partial f_{(r)(s)}}{\partial A^i}\right](\lambda^{(r)}\lambda^{(s)})$$

$$+\frac{1}{8}\left[\left(\frac{\partial^2 \bar{f}_{(r)(s)}}{\partial \bar{A}^{\bar{\imath}} \partial \bar{A}^{\bar{j}}} - \Gamma^{\bar{l}}{}_{\bar{\imath}\bar{j}}\frac{\partial \bar{f}_{(r)(s)}}{\partial \bar{A}^{\bar{l}}}\right)(\bar{\chi}^{\bar{\imath}}\bar{\chi}^{\bar{j}}) + 2g^{i\bar{j}}\mathrm{e}^{K/2}\,D_i W\,\frac{\partial \bar{f}_{(r)(s)}}{\partial \bar{A}^{\bar{j}}}\right](\bar{\lambda}^{(r)}\bar{\lambda}^{(s)})$$

$$+\left(\frac{3}{8}\mathrm{Re}f_{(r)(t)}\,\mathrm{Re}f_{(s)(u)} - \frac{1}{16}g^{i\bar{j}}\frac{\partial f_{(r)(s)}}{\partial A^i}\frac{\partial \bar{f}_{(t)(u)}}{\partial \bar{A}^{\bar{\imath}}}\right)(\lambda^{(r)}\lambda^{(s)})(\bar{\lambda}^{(t)}\bar{\lambda}^{(u)})$$

$$+\frac{1}{2}(\mathrm{Re}f_{(r)(s)})^{-1}\left[K_i(\mathbf{T}_{(r)}A)^i - \frac{\mathrm{i}}{\sqrt{2}}\frac{\partial f_{(r)(t)}}{\partial A^i}(\chi^i\lambda^{(t)})\right]$$

$$\times\left[K_{\bar{j}}(\bar{A}\mathbf{T}_{(s)})^{\bar{j}} + \frac{\mathrm{i}}{\sqrt{2}}\frac{\partial \bar{f}_{(s)(u)}}{\partial \bar{A}^{\bar{j}}}(\bar{\chi}^{\bar{j}}\bar{\lambda}^{(u)})\right]$$

$$-\frac{1}{\sqrt{2}}(\bar{\psi}_m\bar{\sigma}^n\sigma^m\bar{\chi}^{\bar{j}})g_{i\bar{j}}\mathscr{D}_n A^i - \frac{1}{\sqrt{2}}(\psi_m\sigma^n\bar{\sigma}^m\chi^i)g_{i\bar{j}}\mathscr{D}_n\bar{A}^{\bar{j}}$$

$$-\tfrac{1}{4}(\bar{\psi}_m\bar{\sigma}^m\lambda^{(r)} - \psi_m\sigma^m\bar{\lambda}^{(r)})[K_k(\mathbf{T}_{(r)}A)^k + K_{\bar{k}}(\bar{A}\mathbf{T}_{(r)})^{\bar{k}}]$$

$$+\mathrm{i}\frac{1}{8\sqrt{2}}\left[\frac{\partial f_{(r)(s)}}{\partial A^i}(\lambda^{(r)}\lambda^{(s)})(\bar{\psi}_m\bar{\sigma}^m\chi^i) + \frac{\partial \bar{f}_{(r)(s)}}{\partial \bar{A}^{\bar{\imath}}}(\bar{\lambda}^{(r)}\bar{\lambda}^{(s)})(\psi_m\bar{\sigma}^m\bar{\chi}^{\bar{\imath}})\right]$$

$$+\mathrm{i}\frac{1}{2\sqrt{2}}\left[\frac{\partial f_{(r)(s)}}{\partial A^i}(\psi_m\sigma_n\bar{\lambda}^{(r)})(\chi^i\sigma^{mn}\lambda^{(s)}) + \frac{\partial \bar{f}_{(r)(s)}}{\partial \bar{A}^{\bar{\imath}}}(\bar{\psi}_m\bar{\sigma}_n\lambda^{(r)})(\bar{\chi}^{\bar{\imath}}\bar{\sigma}^{mn}\bar{\lambda}^{(s)})\right]$$

$$+\frac{\mathrm{i}}{2}\mathrm{Re}f_{(r)(s)}\boldsymbol{f}^{(r)mn}\left[(\psi_m\sigma_n\bar{\lambda}^{(s)} + \bar{\psi}_m\bar{\sigma}_n\lambda^{(s)}) - \frac{\mathrm{i}}{2}\varepsilon_{mnpq}(\psi^p\sigma^q\bar{\lambda}^{(s)} - \bar{\psi}^p\bar{\sigma}^q\lambda^{(s)})\right]$$

$$+\mathrm{e}^{K/2}\left[\frac{\mathrm{i}}{\sqrt{2}}(\bar{\psi}_m\bar{\sigma}^m\chi^k)\,D_k W + \frac{\mathrm{i}}{\sqrt{2}}(\psi_m\sigma^m\bar{\chi}^{\bar{k}})\,D_{\bar{k}}\bar{W} - \bar{\psi}_m\bar{\sigma}^{mn}\bar{\psi}_n W + \psi_m\sigma^{mn}\psi_n\bar{W}\right]$$

$$-\frac{\mathrm{i}}{2}g_{i\bar{j}}\,\varepsilon^{klmn}(\chi^i\sigma_k\bar{\chi}^{\bar{j}})(\psi_l\sigma_m\bar{\psi}_n) - \frac{1}{2}g_{i\bar{j}}\,g^{mn}(\psi_m\chi^i)(\bar{\psi}_n\bar{\chi}^{\bar{j}})$$

$$+\tfrac{1}{16}\mathrm{Re}f_{(r)(s)}\,[(\lambda^{(r)}\lambda^{(s)})\,\bar{\psi}_m(3g^{mn} + 2\bar{\sigma}^{mn})\bar{\psi}_n + (\bar{\lambda}^{(r)}\bar{\lambda}^{(s)})\,\psi_m(3g^{mn} + 2\sigma^{mn})\psi_n]$$

$$+\tfrac{1}{4}[\mathrm{Re}f_{(r)(s)}\,(g^{mp}g^{nq} - g^{mq}g^{np}) + \mathrm{Im}f_{(r)(s)}\,\varepsilon^{mnpq}](\bar{\psi}_m\bar{\sigma}_n\lambda^{(r)})(\psi_p\sigma_q\bar{\lambda}^{(s)})$$

$$-\tfrac{1}{3}\mathbf{M}\bar{\mathbf{M}} + \tfrac{1}{3}\mathbf{b}^a\mathbf{b}_a + g_{i\bar{j}}\mathbf{F}^i\bar{\mathbf{F}}^{\bar{j}} + \tfrac{1}{2}\mathrm{Re}f_{(r)(s)}\hat{\mathbf{D}}^{(r)}\hat{\mathbf{D}}^{(s)}. \qquad (4.5.29)$$

The diagonalized auxiliary fields, defined as

$$\mathbf{M} = M + 3\mathrm{e}^{K/2} W \ , \tag{4.5.30}$$

$$\bar{\mathbf{M}} = \bar{M} + 3\mathrm{e}^{K/2} \bar{W} \ , \tag{4.5.31}$$

$$\mathbf{b}^a = b^a - \tfrac{3}{4} g_{i\bar{j}} (\chi^i \sigma^a \bar{\chi}^{\bar{j}}) + \tfrac{3}{4} \mathrm{Re} f_{(r)(s)} (\lambda^{(r)} \sigma^a \bar{\lambda}^{(s)}) \ , \tag{4.5.32}$$

$$\mathbf{F}^i = F^i + \mathrm{e}^{K/2} g^{i\bar{k}} D_{\bar{k}} \bar{W} + \frac{1}{4} g^{i\bar{k}} \frac{\partial \bar{f}_{(r)(s)}}{\partial \bar{A}^{\bar{k}}} (\bar{\lambda}^{(r)} \bar{\lambda}^{(s)}) \ , \tag{4.5.33}$$

$$\bar{\mathbf{F}}^{\bar{j}} = \bar{F}^{\bar{j}} + \mathrm{e}^{K/2} g^{l\bar{j}} D_l W + \frac{1}{4} g^{l\bar{j}} \frac{\partial f_{(t)(u)}}{\partial A^l} (\lambda^{(t)} \lambda^{(u)}) \ , \tag{4.5.34}$$

$$\hat{\mathbf{D}}^{(r)} = \mathbf{D}^{(r)} - (\mathrm{Re} f_{(r)(t)})^{-1} \left(K_k (\mathbf{T}_{(t)} A)^k - \frac{\mathrm{i}}{\sqrt{2}} \frac{\partial f_{(t)(v)}}{\partial A^k} (\lambda^{(v)} \chi^k) \right) \tag{4.5.35}$$

have trivial equations of motion which coincide with the lowest components of those found in (3.4.65)–(3.4.70) in superfield language.

We would like to end this section with one comment: it was first realized in [41,42,38,39] that the Lagrangian depends only on the combination

$$\mathscr{G} = K + \ln|W|^2 \tag{4.5.36}$$

and not independently on the Kähler potential K and the superpotential W. This can be made clear in a straightforward manner in the Kähler superspace formalism. Indeed, performing a Kähler transformation – cf. (3.2.7) – with $F = \ln W$ yields

$$\mathrm{e}^{-1} \mathscr{L}_{\text{superpotential}} = \frac{1}{2} \int \frac{E}{R} \mathrm{e}^{\mathscr{G}/2} + \text{h.c} \ . \tag{4.5.37}$$

This field-dependent redefinition, which has the form of a Kähler transformation, must of course be performed in the whole geometric structure, leading to a new superspace geometry which is completely inert under Kähler transformations. The component field expressions in this new basis, with Kähler inert components, have the same form as the previous ones, with K replaced by $\mathscr{G}$ in all the implicit dependence on the Kähler potential and W and $\bar{W}$ set to one. It was actually given in this basis in [21].

5. Linear multiplet and supergravity

The antisymmetric tensor gauge potential, $b_{mn} = -b_{nm}$, first discussed in [121], appears naturally in the context of string theory [108]. At the dynamical level it is related to a real massless scalar field through a duality transformation.

In supersymmetry, the antisymmetric tensor is part of the linear multiplet [71,143], together with a real scalar and a Majorana spinor. The duality with a massless scalar multiplet is most easily established in superfield language [111].

Postponing the discussion of the relevance of the linear multiplet and its couplings in low-energy effective superstring theory to the closing Section 7, we concentrate here on the general description of linear multiplets in superspace and couplings to the full supergravity/matter/Yang–Mills system, including Chern–Simons forms.

The basic idea of the *linear superfield formalism* is to describe a 2-form gauge potential in the background of $U_K(1)$ superspace and to promote the Kähler potential to a more general superfield function, which not only depends on the chiral matter superfields but also on linear superfields.

In order to prepare the ground, Section 5.1 provides an elementary and quite detailed introduction to the antisymmetric tensor gauge potential and to linear superfields without supergravity. Whereas the superspace geometry of the 2-form in $U_K(1)$ superspace is presented in Section 5.2, component fields are identified in Section 5.3. In Section 5.4 we explain the coupling of the linear superfield to the supergravity/matter/Yang–Mills system. Duality transformations in this general context, including Chern–Simons forms are discussed in Section 5.5, relating the *linear superfield formalism* to the *chiral superfield formalism*. In Section 5.6 we show that the linear superfield formalism provides a natural explanation of non-holomorphic gauge coupling constants. Finally, in Section 5.7 we extend our analysis to the case of several linear multiplets.

5.1. The linear multiplet in rigid superspace

5.1.1. The antisymmetric tensor gauge field

Consider first the simple case of the antisymmetric tensor gauge potential b_{mn} in four dimensions with gauge transformations parametrized by a four vector β_m such that

$$b_{mn} \mapsto b_{mn} + \partial_m \beta_n - \partial_n \beta_m \tag{5.1.1}$$

and with invariant field strength given as

$$h_{0lmn} = \partial_l b_{mn} + \partial_m b_{nl} + \partial_n b_{lm} \ . \tag{5.1.2}$$

The subscript 0 denotes here the absence of Chern–Simons forms. As a consequence of its definition the field strength satisfies the Bianchi identity

$$\varepsilon^{klmn} \partial_k h_{0lmn} = 0 \ . \tag{5.1.3}$$

The invariant kinetic action is given as

$$\mathscr{L} = \tfrac{1}{2} {}^*h_0^{m} {}^*h_{0m} \tag{5.1.4}$$

with ${}^*h_0^k = \frac{1}{3!} \varepsilon^{klmn} h_{0lmn}$ denoting the Hodge dual of the field strength tensor.

Consider next the case where a Chern–Simons term for a Yang–Mills potential $\boldsymbol{a}_m$, such as

$$Q_{lmn} = -\operatorname{tr}\left(\boldsymbol{a}_{[l}\partial_m\boldsymbol{a}_{n]} - \frac{2i}{3}\boldsymbol{a}_{[l}\boldsymbol{a}_m\boldsymbol{a}_{n]}\right) \tag{5.1.5}$$

with $[lmn] = lmn + mnl + nlm - mln - lnm - nml$, is added to the field strength,

$$h_{lmn} = h_{0\,lmn} + kQ_{lmn} . \tag{5.1.6}$$

Here k is a constant which helps keeping track of the terms induced by the inclusion of the Chern–Simons combination. The Chern–Simons term is introduced to compensate the Yang–Mills gauge transformations to the antisymmetric tensor, thus rendering the modified field strength invariant. The Bianchi identity is modified as well; it now reads

$$\varepsilon^{klmn}\partial_k h_{lmn} = -\tfrac{3}{2}k\varepsilon^{klmn}\operatorname{tr}(\boldsymbol{f}_{kl}\boldsymbol{f}_{mn}). \tag{5.1.7}$$

A dynamical theory may then be obtained from the invariant action

$$\mathscr{L} = \tfrac{1}{2}{}^*h^{m*}h_m - \tfrac{1}{4}\operatorname{tr}(\boldsymbol{f}^{mn}\boldsymbol{f}_{mn}) \tag{5.1.8}$$

with ${}^*h^k$ the dual of h_{lmn}. This action describes the dynamics of Yang–Mills potentials $\boldsymbol{a}_m(x)$ and an antisymmetric tensor gauge potential b_{mn} with effective k-dependent couplings induced through the Chern–Simons form.

This theory is dual to another one where the antisymmetric tensor is replaced by a real pseudoscalar $a(x)$ in the following sense: one starts from a first order action describing a vector $X^m(x)$, a scalar $a(x)$ and the Yang–Mills gauge potential $\boldsymbol{a}_m(x)$,

$$\mathscr{L} = (X^m - k{}^*Q^m)\partial_m a + \tfrac{1}{2}X^m X_m - \tfrac{1}{4}\operatorname{tr}(\boldsymbol{f}^{mn}\boldsymbol{f}_{mn}) , \tag{5.1.9}$$

where the gauge Chern–Simons form is included as

$${}^*Q^k = \frac{1}{3!}\varepsilon^{klmn}Q_{lmn} = -\varepsilon^{klmn}\operatorname{tr}\left(\boldsymbol{a}_l\partial_m\boldsymbol{a}_n - \frac{2i}{3}\boldsymbol{a}_l\boldsymbol{a}_m\boldsymbol{a}_n\right) . \tag{5.1.10}$$

Variation of the first-order action with respect to the field a gives rise to an equation of motion which is solved in terms of an antisymmetric tensor

$$\partial_m(X^m - k{}^*Q^m) = 0, \quad \Rightarrow X^k - k{}^*Q^k = \tfrac{1}{2}\varepsilon^{klmn}\partial_l b_{mn} . \tag{5.1.11}$$

Substituting back shows that the first term in (5.1.9) becomes a total derivative and we end up with the previous action (5.1.8) where ${}^*h^m = X^m$, describing an antisymmetric tensor gauge field coupled to a gauge Chern–Simons form.

On the other hand, varying the first order action with respect to X^m yields

$$X_m = -\partial_m a . \tag{5.1.12}$$

In this case, substitution of the equation of motion, together with the divergence equation for the Chern–Simons form, i.e.

$$\partial_k^* Q^k = -\tfrac{1}{4}\varepsilon^{klmn}\mathrm{tr}\,(\boldsymbol{f}_{kl}\boldsymbol{f}_{mn}) \tag{5.1.13}$$

gives rise to a theory describing a real scalar field with an axion coupling term

$$\mathscr{L} = -\frac{1}{2}\partial^m a(x)\partial_m a(x) - \frac{1}{4}\mathrm{tr}(\boldsymbol{f}^{mn}\boldsymbol{f}_{mn}) - \frac{k}{4}a(x)\varepsilon^{klmn}\,\mathrm{tr}(\boldsymbol{f}_{kl}\boldsymbol{f}_{mn})\ . \tag{5.1.14}$$

It is in this sense that the two actions (5.1.8) and (5.1.14) derived here from the first-order one (5.1.9) are dual to each other. They describe alternatively the dynamics of an antisymmetric tensor gauge field or of a real pseudoscalar, respectively, with special types of Yang–Mills couplings. Indeed, the pseudoscalar field is often referred to as an axion because of its couplings (5.1.14) to Yang–Mills fields (although it is not necessarily the QCD axion). Note that the kinetic term of the Yang–Mills sector is not modified in this procedure.

5.1.2. The linear superfield

As already mentioned, the linear supermultiplet consists of an antisymmetric tensor, a real scalar and a Majorana spinor. In string theories, the real scalar is the dilaton found among the massless modes of the gravity supermultiplet. As b_{mn} is the coefficient of a 2-form, we can describe its supersymmetric version by a 2-form in superspace with appropriate constrains and build the corresponding supermultiplet by solving the Bianchi identities. We shall proceed this way in Section 5.2. In superfield language it is described by a superfield L_0, subject to the constraints

$$D^2 L_0 = 0, \quad \bar{D}^2 L_0 = 0\ . \tag{5.1.15}$$

Again, the subscript 0 means that we do not include, for the moment, the coupling to Chern–Simons forms. The linear superfield L_0 contains the antisymmetric tensor only through its field strength h_{0lmn}. Indeed, the superfield L_0 is the supersymmetric analogue of h_{0lmn} (it describes the multiplet of field strengths) and the constraints (5.1.15) are the supersymmetric version of the Bianchi identities. The particular form of these constraints implies that terms quadratic in θ and $\bar{\theta}$ are not independent component fields; it is for this reason that L_0 has been called a *linear superfield* [71].

As before, component fields are identified as projections to lowest superfield components. To begin with, we identify the real scalar $L_0(x)$ of the linear multiplet as the lowest component

$$L_0| = L_0(x)\ . \tag{5.1.16}$$

The spinor derivatives of superfields are again superfields and we define the Weyl components $(\Lambda_\alpha(x), \bar{\Lambda}^{\dot{\alpha}}(x))$ of the Majorana spinor of the linear multiplet as

$$D_\alpha L_0| = \Lambda_\alpha(x), \quad D^{\dot{\alpha}} L_0| = \bar{\Lambda}^{\dot{\alpha}}(x)\ . \tag{5.1.17}$$

The antisymmetric tensor appears in L_0 via its field strength identified as

$$[D_\alpha, D_{\dot\alpha}]L_0| = -\tfrac{1}{3}\sigma_{k\alpha\dot\alpha}\varepsilon^{klmn}h_{0lmn} = -2\sigma_{k\alpha\dot\alpha}{}^*h_0^k\ , \tag{5.1.18}$$

thus completing the identification of the independent component fields contained in L_0.

The canonical supersymmetric kinetic action for the linear multiplet is then given by the square of the linear superfield integrated over superspace, i.e. in the language of projections to lowest superfield components,

$$\mathscr{L} = -\tfrac{1}{32}(D^2\bar D^2 + \bar D^2 D^2)(L_0)^2| = \frac{1}{2}{}^*h_0^m{}^*h_{0m} - \frac{1}{2}\partial^m L_0 \partial_m L_0 - \frac{\mathrm{i}}{2}\sigma^m_{\alpha\dot\alpha}(\Lambda^\alpha \partial_m \bar\Lambda^{\dot\alpha} + \bar\Lambda^{\dot\alpha}\partial_m \Lambda^\alpha)\ , \tag{5.1.19}$$

generalizing the purely bosonic action (5.1.4) given above and showing that there is no auxiliary field in the linear multiplet.

In order to construct the supersymmetric version of (5.1.8), we come now to the supersymmetric description of the corresponding Chern–Simons forms. They are described in terms of the Chern–Simons superfield Ω, which has the properties

$$\mathrm{tr}(\mathscr{W}^\alpha \mathscr{W}_\alpha) = \tfrac{1}{2}\bar D^2 \Omega, \quad \mathrm{tr}(\mathscr{W}_{\dot\alpha}\mathscr{W}^{\dot\alpha}) = \tfrac{1}{2}D^2\Omega\ . \tag{5.1.20}$$

The appearance of the differential operators D^2 and $\bar D^2$ is due to the chirality constraint (2.3.33) on the gaugino superfields $\mathscr{W}^\alpha$, $\mathscr{W}_{\dot\alpha}$, whereas the additional constraint (2.3.34) is responsible for the fact that one and the same real superfield Ω appears in both equations. The component field Chern–Simons form (5.1.5) is then identified in the lowest superfield component

$$[D_\alpha, D_{\dot\alpha}]\Omega| = -2\sigma^*_{k\alpha\dot\alpha}Q^k - 4\,\mathrm{tr}(\lambda_\alpha \bar\lambda_{\dot\alpha}) \tag{5.1.21}$$

with ${}^*Q^k$ given in (5.1.10).

Since the terms on the left-hand side in (5.1.20) are gauge invariant, it is clear that a gauge transformation adds a linear superfield to Ω. The explicit construction given in Appendix F.2, in the full supergravity context, shows that, up to a linear superfield, we may identify

$$L = L_0 + k\Omega\ , \tag{5.1.22}$$

such that L is gauge invariant. However, this superfield L satisfies now the modified linearity conditions

$$\bar D^2 L = 2k\,\mathrm{tr}(\mathscr{W}^\alpha \mathscr{W}_\alpha)\ , \tag{5.1.23}$$

$$D^2 L = 2k\,\mathrm{tr}(\mathscr{W}_{\dot\alpha}\mathscr{W}^{\dot\alpha})\ . \tag{5.1.24}$$

Again, these equations together with

$$[D_\alpha, D_{\dot\alpha}]L = -\tfrac{1}{3}\sigma_{d\alpha\dot\alpha}\varepsilon^{dcba}H_{cba} - 4k\,\mathrm{tr}(\mathscr{W}_\alpha \bar{\mathscr{W}}_{\dot\alpha}) \tag{5.1.25}$$

have an interpretation as Bianchi identities in superspace geometry. The last one shows how the usual field strength of the antisymmetric tensor together with the Chern–Simons component field appears in the superfield expansion of L,

$$[D_\alpha, D_{\dot\alpha}]L| = -\sigma_{k\alpha\dot\alpha}\varepsilon^{klmn}\left(\partial_l b_{mn} + \frac{k}{3}Q_{lmn}\right) - 4k\,\mathrm{tr}(\lambda_\alpha \bar\lambda_{\dot\alpha})\,. \tag{5.1.26}$$

The invariant action for this supersymmetric system is given as the lowest component of the superfield

$$\mathscr{L} = -\tfrac{1}{32}(D^2\bar D^2 + \bar D^2 D^2)L^2 - \tfrac{1}{16}D^2\,\mathrm{tr}\,\mathscr{W}^2 - \tfrac{1}{16}\bar D^2\,\mathrm{tr}\,\bar{\mathscr{W}}^2\,. \tag{5.1.27}$$

This action describes the supersymmetric version of the purely bosonic action (5.1.8). Its explicit component field gestalt will be displayed and commented on in a short while.

The notion of duality can be extended to supersymmetric theories as well [111]; this is most conveniently done in the language of superfields. The supersymmetric version of the first-order action (5.1.9) is given as

$$\mathscr{L} = -\tfrac{1}{32}(D^2\bar D^2 + \bar D^2 D^2)(X^2 + \sqrt{2}(X - k\Omega)(S + \bar S)) - \tfrac{1}{16}D^2\,\mathrm{tr}\,\mathscr{W}^2 - \tfrac{1}{16}\bar D^2\,\mathrm{tr}\,\bar{\mathscr{W}}^2\,. \tag{5.1.28}$$

Here, X is a real but otherwise unconstrained superfield, whereas S and $\bar S$ are chiral,

$$D_\alpha \bar S = 0, \quad \bar D^{\dot\alpha} S = 0\,. \tag{5.1.29}$$

Of course, the chiral multiplets are going to play the part of the scalar field $a(x)$ in the previous non-supersymmetric discussion.

Varying the first-order action with respect to the superfield S or, more correctly, with respect to its unconstrained pre-potential Σ, defined as $S = \bar D^2\Sigma$, the solution of the chirality constraint, shows immediately (upon integration by parts using spinor derivatives) that the superfield X must satisfy the modified linearity condition. It is therefore identified with L and we recover the action (5.1.27) above.

On the other hand, varying the first-order action (5.1.28) with respect to X yields the superfield equation of motion

$$X = -\frac{1}{\sqrt{2}}(S + \bar S)\,. \tag{5.1.30}$$

Substituting for X in (5.1.28) and observing that the terms S^2 and $\bar S^2$ yield total derivatives which are trivial upon superspace integration, we arrive at

$$\mathscr{L} = \tfrac{1}{32}(D^2\bar D^2 + \bar D^2 D^2)(\bar S S + k\sqrt{2}\Omega(S + \bar S)) - \tfrac{1}{16}D^2\,\mathrm{tr}\,\mathscr{W}^2 - \tfrac{1}{16}\bar D^2\,\mathrm{tr}\,\bar{\mathscr{W}}^2\,. \tag{5.1.31}$$

One recognizes the usual superfield kinetic term for the chiral multiplet and the Yang–Mills kinetic terms. It remains to have a closer look at the terms containing the Chern–Simons superfield. Taking into account the chirality properties for S and $\bar{S}$ and the derivative relations (5.1.20) for the Chern–Simons superfields we obtain, up to a total derivative,

$$\mathscr{L} = \tfrac{1}{32}(D^2\bar{D}^2 + \bar{D}^2 D^2)\bar{S}S - \tfrac{1}{16}D^2\,\mathrm{tr}\,\mathscr{W}^2 - \tfrac{1}{16}\bar{D}^2\,\mathrm{tr}\,\bar{\mathscr{W}}^2 + \frac{k\sqrt{2}}{8}D^2(S\,\mathrm{tr}\,\mathscr{W}^2) + \frac{k\sqrt{2}}{8}\bar{D}^2(\bar{S}\,\mathrm{tr}\,\bar{\mathscr{W}}^2)\,. \tag{5.1.32}$$

This action is the supersymmetric version of the action (5.1.14).

The component field expressions for the two dual versions (5.1.27) and (5.1.32) of the supersymmetric construction are then easily derived. In the antisymmetric tensor version, the complete invariant component field action deriving from (5.1.27) is given as

$$\begin{aligned}\mathscr{L} = {} & \frac{1}{2}{}^*h^{m*}h_m - \frac{1}{2}\partial^m L\,\partial_m L - \frac{\mathrm{i}}{2}\sigma^m_{\alpha\dot{\alpha}}(\Lambda^\alpha\partial_m\bar{\Lambda}^{\dot{\alpha}} + \bar{\Lambda}^{\dot{\alpha}}\partial_m\Lambda^\alpha) \\ & + (1+2kL)\,\mathrm{tr}\left[-\frac{1}{4}\boldsymbol{f}^{mn}\boldsymbol{f}_{mn} - \frac{\mathrm{i}}{2}\sigma^m_{\alpha\dot{\alpha}}(\lambda^\alpha\mathscr{D}_m\bar{\lambda}^{\dot{\alpha}} + \bar{\lambda}^{\dot{\alpha}}\mathscr{D}_m\lambda^\alpha) + \frac{1}{2}\hat{\mathbf{D}}\hat{\mathbf{D}}\right] \\ & - k{}^*h^m\,\mathrm{tr}(\lambda\sigma_m\bar{\lambda}) - k\Lambda\sigma^{mn}\,\mathrm{tr}(\lambda\boldsymbol{f}_{mn}) - k\bar{\Lambda}\bar{\sigma}^{mn}\,\mathrm{tr}(\bar{\lambda}\boldsymbol{f}_{mn}) \\ & - \frac{k^2}{4}(1+2kL)^{-1}[\Lambda^2\,\mathrm{tr}\,\lambda^2 + \bar{\Lambda}^2\,\mathrm{tr}\,\bar{\lambda}^2 - 2\Lambda\sigma^m\bar{\Lambda}\,\mathrm{tr}(\lambda\sigma_m\bar{\lambda})] \\ & - \frac{k^2}{2}[\mathrm{tr}\,\lambda^2\,\mathrm{tr}\,\bar{\lambda}^2 - \mathrm{tr}(\lambda\sigma^m\bar{\lambda})\mathrm{tr}(\lambda\sigma_m\bar{\lambda})]\,.\end{aligned} \tag{5.1.33}$$

This is the supersymmetric version of (5.1.8). The redefined auxiliary field

$$\hat{\mathbf{D}} = \mathbf{D} + \mathrm{i}k(1+2kL)^{-1}(\Lambda\lambda - \bar{\Lambda}\bar{\lambda}) \tag{5.1.34}$$

has trivial equation of motion.

On the other hand, in order to display the component field Lagrangian in the chiral superfield version, we recall the definition of the component field content of the chiral superfields

$$S| = S(x),\quad D_\alpha S| = \sqrt{2}\chi_\alpha(x),\quad D^2 S| = -4F(x) \tag{5.1.35}$$

and

$$\bar{S}| = \bar{S}(x),\quad D^{\dot{\alpha}}\bar{S}| = \sqrt{2}\bar{\chi}^{\dot{\alpha}}(x),\quad \bar{D}^2\bar{S}| = -4\bar{F}(x)\,. \tag{5.1.36}$$

The component field action in the dual formulation, derived from the superfield action (5.1.32) takes then the form

$$\mathscr{L} = -\partial^m \bar{S}\partial_m S - \frac{i}{2}\sigma^m_{\alpha\dot{\alpha}}(\chi^\alpha \partial_m \bar{\chi}^{\dot{\alpha}} + \bar{\chi}^{\dot{\alpha}}\partial_m \chi^\alpha) + \hat{F}\hat{\bar{F}}$$
$$+ (1 - k\sqrt{2}(S+\bar{S}))\,\mathrm{tr}\left[-\frac{1}{4}\boldsymbol{f}^{mn}\boldsymbol{f}_{mn} - \frac{i}{2}\sigma^m_{\alpha\dot{\alpha}}(\lambda^\alpha \mathscr{D}_m \bar{\lambda}^{\dot{\alpha}} + \bar{\lambda}^{\dot{\alpha}}\mathscr{D}_m \lambda^\alpha) + \frac{1}{2}\hat{\mathbf{D}}\hat{\mathbf{D}}\right]$$
$$- \frac{k}{4i\sqrt{2}}(S - \bar{S})[\varepsilon^{klmn}\,\mathrm{tr}(\boldsymbol{f}_{kl}\boldsymbol{f}_{mn}) + 4\partial_m\,\mathrm{tr}(\lambda\sigma^m\bar{\lambda})]$$
$$+ k\chi\sigma^{mn}\,\mathrm{tr}(\lambda\boldsymbol{f}_{mn}) + k\bar{\chi}\bar{\sigma}^{mn}\,\mathrm{tr}(\bar{\lambda}\boldsymbol{f}_{mn}) - \frac{k^2}{8}\,\mathrm{tr}\,\lambda^2\,\mathrm{tr}\,\bar{\lambda}^2$$
$$- \frac{k^2}{4}(1 - k\sqrt{2}(S+\bar{S}))^{-1}[\chi^2\,\mathrm{tr}\,\lambda^2 + \bar{\chi}^2\,\mathrm{tr}\,\bar{\lambda}^2 - 2(\chi\sigma^m\bar{\chi})\mathrm{tr}(\lambda\sigma_m\bar{\lambda})]\,. \tag{5.1.37}$$

This is the supersymmetric version of (5.1.14). Again, we have introduced the diagonalized combinations for the auxiliary fields

$$\hat{F} = F + \frac{k\sqrt{2}}{4}\mathrm{tr}\,\bar{\lambda}^2, \qquad \hat{\bar{F}} = \bar{F} + \frac{k\sqrt{2}}{4}\mathrm{tr}\,\lambda^2 \tag{5.1.38}$$

and

$$\hat{\mathbf{D}} = \mathbf{D} - ik[1 - k\sqrt{2}(S+\bar{S})]^{-1}(\chi\lambda - \bar{\chi}\bar{\lambda})\,. \tag{5.1.39}$$

The two supersymmetric actions (5.1.33) and (5.1.37) are dual to each other, in the precise sense of the construction performed above. In both cases the presence of the Chern–Simons form induces k-dependent effective couplings, in particular quadri-linear spinor couplings. Also, we easily recognize in the second version the axion term already encountered in the purely bosonic case discussed before.

A striking difference with the non-supersymmetric case, however, is the appearance of a k-dependent gauge coupling function, multiplying the Yang–Mills kinetic terms. This shows that supersymmetrization of (5.1.8) and (5.1.14) results not only in supplementary fermionic terms, but induces also genuinely new purely bosonic terms.

In this line of construction, one can imagine an extension of Zumino's construction of the non-linear sigma model [164,74,4], where we replace the Kähler potential $K(\phi,\bar{\phi})$ by a more general function $K(\phi,\bar{\phi},L)$ which not only depends on *complex* chiral and antichiral superfields, but also on a number of *real* linear superfields.

5.2. The geometry of the 2-form

The linear multiplet has a geometric interpretation as a 2-form gauge potential in superspace geometry. Since we wish to construct theories where the linear multiplet is coupled to the supergravity/matter system, we will formulate this 2-form geometry in the background of $U_K(1)$ superspace. The basic object is the 2-form gauge potential defined as

$$B = \tfrac{1}{2} dz^M dz^N B_{NM} \tag{5.2.1}$$

subject to gauge transformations of parameters $\beta = dz^M \beta_M$ which are themselves 1-forms in superspace

$$B \mapsto B + d\beta , \tag{5.2.2}$$

i.e.,

$$B_{NM} \mapsto B_{NM} + \partial_N \beta_M - (-)^{nm} \partial_M \beta_N . \tag{5.2.3}$$

The invariant field strength is a 3-form,

$$H = dB = \frac{1}{3!} E^A E^B E^C H_{CBA} \tag{5.2.4}$$

with E^A the frame of $U_K(1)$ superspace. As a consequence of $dd = 0$ one obtains the Bianchi identity, $dH = 0$, which fully developed reads

$$\frac{1}{4!} E^A E^B E^C E^D (4 \mathscr{D}_D H_{CBA} + 6 T_{DC}{}^F H_{FBA}) = 0 . \tag{5.2.5}$$

The linear superfield is recovered from this general structure in imposing covariant constraints on the field strength coefficients H_{CBA} such that ($\underline{\alpha} = \alpha, \dot{\alpha}$)

$$H_{\underline{\gamma}\underline{\beta}\underline{\alpha}} = 0, \quad H_{\gamma\beta a} = 0, \quad H_{\dot{\gamma}\dot{\beta}a} = 0 . \tag{5.2.6}$$

As consequences of these constraints we find (by explicitly solving them in terms of *unconstrained* pre-potentials or else working through the covariant Bianchi identities) that all the field strength components of the 2-form are expressed in terms of one real superfield. In the absence of Chern–Simons forms – cf. also Section 5.1.2, it will be denoted by L_0. It is identified in

$$H_\gamma{}^{\dot{\beta}}{}_a = -2\mathrm{i} (\sigma_a \varepsilon)_\gamma{}^{\dot{\beta}} L_0 . \tag{5.2.7}$$

Explicitly we obtain

$$H_{\gamma ba} = 2(\sigma_{ba})_\gamma{}^\varphi \mathscr{D}_\varphi L_0, \quad H^{\dot{\gamma}}{}_{ba} = 2(\bar{\sigma}_{ba})^{\dot{\gamma}}{}_{\dot{\varphi}} \mathscr{D}^{\dot{\varphi}} L_0 \tag{5.2.8}$$

and

$$-\tfrac{1}{3} \sigma_{d\alpha\dot{\alpha}} \varepsilon^{dcba} H_{cba} = ([\mathscr{D}_\alpha, \mathscr{D}_{\dot{\alpha}}] - 4\sigma^a_{\alpha\dot{\alpha}} G_a) L_0 . \tag{5.2.9}$$

This equation identifies the supercovariant field strength H_{cba} in the superfield expansion of L_0. Compatibility of the constraints imposed above with the structure of the Bianchi identities then implies the linearity conditions

$$(\bar{\mathscr{D}}^2 - 8R)L_0 = 0, \quad (\mathscr{D}^2 - 8R^\dagger)L_0 = 0 \tag{5.2.10}$$

for a linear superfield in interaction with the supergravity/matter system.

In general, when the linear multiplet is coupled to the supergravity/matter/Yang–Mills system, we will have to allow for Chern–Simons couplings as well. As gravitational Chern–Simons forms are beyond the scope of this report, we will restrict ourselves to the Yang–Mills case. Recall that the Chern–Simons 3-form of a Yang–Mills potential $\mathscr{A}$ in superspace is defined as [90]

$$\mathscr{Q}^{\mathscr{YM}} = \mathrm{tr}(\mathscr{A} d\mathscr{A} + \tfrac{2}{3}\mathscr{A}\mathscr{A}\mathscr{A}) . \tag{5.2.11}$$

Its exterior derivative yields a field strength squared term

$$d\mathscr{Q}^{\mathscr{YM}} = \mathrm{tr}(\mathscr{F}\mathscr{F}) . \tag{5.2.12}$$

The coupling to the antisymmetric tensor multiplet is obtained by incorporating this Chern–Simons form into the field strength of the 2-form gauge potential

$$H^{\mathscr{YM}} = dB + k\mathscr{Q}^{\mathscr{YM}} . \tag{5.2.13}$$

The superscript ${}^{\mathscr{YM}}$ indicates the presence of the Yang–Mills Chern–Simons form in the definition of the field strength. Note that if $\boldsymbol{a}_m$, the Yang–Mills potential and b_{mn}, the antisymmetric tensor gauge potential have the conventional dimension of a mass (the corresponding kinetic actions are then dimensionless) the constant k has dimension of an inverse mass.

Since $\mathscr{Q}^{\mathscr{YM}}$ changes under gauge transformations of the Yang–Mills connection $\mathscr{A}$ with the exterior derivative of a 2-form $\Delta(\mathscr{A}, \boldsymbol{g})$,

$$\mathscr{Q}^{\mathscr{YM}} \mapsto {}^{\boldsymbol{g}}\mathscr{Q}^{\mathscr{YM}} = \mathscr{Q}^{\mathscr{YM}} + d\Delta(\mathscr{A}, \boldsymbol{g}) , \tag{5.2.14}$$

covariance of $H^{\mathscr{YM}}$ can be achieved in assigning an inhomogeneous compensating gauge transformation

$$B \mapsto {}^{\boldsymbol{g}}B = B - k\Delta(\mathscr{A}, \boldsymbol{g}) , \tag{5.2.15}$$

to the 2-form gauge potential. Finally, the addition of the Chern–Simons forms gives rise to the modified Bianchi identities

$$dH^{\mathscr{YM}} = k\,\mathrm{tr}(\mathscr{F}\mathscr{F}) . \tag{5.2.16}$$

A question of compatibility arises when the two superspace structures are combined in the way we propose here, since the linear multiplet corresponds to a 2-form geometry with constraints on the 3-form field strength and the Yang–Mills field strength $\mathscr{F}$ is constrained as well. As it turns out [97,90], assuming the usual constraints for $\mathscr{F}$, the modified field strength $H^{\mathscr{YM}}$ may be constrained in the same way as H, without any contradiction. The most immediate way to see this is to investigate explicitly the structure of the modified Bianchi identities

$$\frac{1}{4!}E^A E^B E^C E^D (4\mathscr{D}_D H_{CBA} + 6T_{DC}{}^F H_{FBA} - 6k\,\mathrm{tr}(\mathscr{F}_{DC}\mathscr{F}_{BA})) = 0 \,. \tag{5.2.17}$$

Assuming for $H^{\mathscr{YM}}$ the same constraints as for H – cf. (5.2.6) and (5.2.7) – and replacing L_0 by $L^{\mathscr{YM}}$ on the one hand and taking into account the special properties of the $\mathscr{F}\mathscr{F}$ terms arising from the Yang–Mills constraints on the other hand, one can show that the linearity conditions (5.2.17) are replaced by the *modified linearity conditions*

$$(\mathscr{D}^2 - 8R^\dagger)L^{\mathscr{YM}} = 2k\,\mathrm{tr}\,(\mathscr{W}_{\dot\alpha}\mathscr{W}^{\dot\alpha}) \,, \tag{5.2.18}$$

$$(\bar{\mathscr{D}}^2 - 8R)L^{\mathscr{YM}} = 2k\,\mathrm{tr}(\mathscr{W}^\alpha \mathscr{W}_\alpha) \,. \tag{5.2.19}$$

Likewise, (5.2.9) acquires an additional term,

$$([\mathscr{D}_\alpha, \mathscr{D}_{\dot\alpha}] - 4\sigma^a_{\alpha\dot\alpha} G_a)L^{\mathscr{YM}} = -\tfrac{1}{3}\sigma_{d\alpha\dot\alpha}\varepsilon^{dcba} H^{\mathscr{YM}}_{cba} - 4k\,\mathrm{tr}(\mathscr{W}_\alpha \mathscr{W}_{\dot\alpha}) \,. \tag{5.2.20}$$

The special properties of $\mathscr{W}_\alpha$ allow to express the quadratic gaugino contributions in (5.2.18) and (5.2.19) in terms of a single *Chern–Simons superfield* $\Omega^{\mathscr{YM}}$,

$$\mathrm{tr}(\mathscr{W}_{\dot\alpha}\mathscr{W}^{\dot\alpha}) = \tfrac{1}{2}(\mathscr{D}^2 - 8R^\dagger)\Omega^{\mathscr{YM}}, \quad \mathrm{tr}(\mathscr{W}^\alpha \mathscr{W}_\alpha) = \tfrac{1}{2}(\bar{\mathscr{D}}^2 - 8R)\Omega^{\mathscr{YM}} \,. \tag{5.2.21}$$

The existence of $\Omega^{\mathscr{YM}}$ and its explicit construction – cf. Appendix F – rely on the similarity of Chern–Simons forms with a generic 3-form gauge potential C. The Chern–Simons form (5.2.11) plays the role of a 3-form gauge potential (5.2.14) and $\mathrm{tr}(\mathscr{F}\mathscr{F})$ corresponds to its field strength (5.2.12). Given the identification

$$\Sigma = \frac{1}{4!}E^A E^B E^C E^D \Sigma_{DCBA} = \frac{1}{4!}E^A E^B E^C E^D 6\,\mathrm{tr}(\mathscr{F}_{DC}\mathscr{F}_{BA}) \tag{5.2.22}$$

and the constraints on $\mathscr{F}$ it is immediate to deduce that indeed

$$\Sigma_{\underline{\delta}\underline{\gamma}\underline{\alpha}A} = 0 \,, \tag{5.2.23}$$

which are just the constraints of the 4-form field strength in the generic case. Anticipating part of the discussion of the next section, we observe that, as a consequence of the constraints, all the components of the generic 4-form field strength are expressible in terms of chiral superfields Y and $\bar{Y}$ ($\mathscr{D}_\alpha \bar{Y} = 0, \mathscr{D}^{\dot\alpha} Y = 0$) identified in

$$\Sigma_{\delta\gamma ba} = \tfrac{1}{2}(\sigma_{ba}\varepsilon)_{\delta\gamma}\bar{Y}, \quad \Sigma^{\dot\delta\dot\gamma}{}_{ba} = \tfrac{1}{2}(\bar\sigma_{ba}\varepsilon)^{\dot\delta\dot\gamma} Y \,. \tag{5.2.24}$$

For the remaining coefficients, i.e. $\Sigma_{\underline{\delta}cba}$ and Σ_{dcba}, respectively, we obtain then

$$\Sigma_{\delta cba} = -\tfrac{1}{16}\sigma^d_{\delta\dot\delta}\varepsilon_{dcba}\mathscr{D}^{\dot\delta}\bar{Y}, \qquad \Sigma^{\dot\delta}{}_{cba} = +\tfrac{1}{16}\bar\sigma^{d\dot\delta\delta}\varepsilon_{dcba}\mathscr{D}_\delta Y \tag{5.2.25}$$

and

$$\Sigma_{dcba} = \frac{i}{16}\varepsilon_{dcba}[(\mathscr{D}^2 - 24R^\dagger)Y - (\bar{\mathscr{D}}^2 - 24R)\bar{Y}] . \tag{5.2.26}$$

This last equation should be understood as a further constraint between the chiral superfields Y and $\bar{Y}$, thus describing the supermultiplet of a 3-form gauge potential in $U_K(1)$ superspace.

From the explicit solution of the constraints, one finds that Y and $\bar{Y}$ are given as the chiral projections of $U_K(1)$ superspace geometry acting on one and the same pre-potential Ω,

$$Y = -4(\bar{\mathscr{D}}^2 - 8R)\Omega, \quad \bar{Y} = -4(\mathscr{D}^2 - 8R^\dagger)\Omega . \tag{5.2.27}$$

Due to the same constraint structure of Σ and $\mathrm{tr}(\mathscr{F})$, this analysis applies to the case of Chern–Simons forms as well. We identify

$$Y^{\mathscr{YM}} = -8\,\mathrm{tr}(\mathscr{W}^\alpha \mathscr{W}_\alpha), \quad \bar{Y}^{\mathscr{YM}} = -8\,\mathrm{tr}(\mathscr{W}_{\dot\alpha} \mathscr{W}^{\dot\alpha}) . \tag{5.2.28}$$

Correspondingly, Ω the generic pre-potential, is identified as $\Omega^{\mathscr{YM}}$, the Chern–Simons superfield, expressed in terms of the unconstrained Yang–Mills pre-potential. A detailed account of this analysis is given in Appendix F.

It is instructive to investigate the relation between the superfields $L^{\mathscr{YM}}$ and L_0 in this context. As we have seen, L_0 and $L^{\mathscr{YM}} - k\Omega^{\mathscr{YM}}$ satisfy the same linearity conditions. As a consequence, they can be identified up to some linear superfield, i.e.

$$L^{\mathscr{YM}} = L_0 + k\beta^{\mathscr{YM}} + k\Omega^{\mathscr{YM}} . \tag{5.2.29}$$

Here $\beta^{\mathscr{YM}}$ is a pre-potential-dependent linear superfield whose explicit form, irrelevant for the present discussion, may be read off from the equations in Appendix F. Note that $\Omega^{\mathscr{YM}}$ changes under Yang–Mills gauge transformations by a linear superfield (hence (5.2.21) are unchanged), whereas the combination $\Omega^{\mathscr{YM}} + \beta^{\mathscr{YM}}$ is gauge invariant, in accordance with the gauge invariance of L_0 and $L^{\mathscr{YM}}$.

We have tried to make clear in this section that the superspace geometry of the 3-form gauge potential provides a generic framework for the discussion of Chern–Simons forms in superspace. Established in full detail for the Yang–Mills case, this property can be advantageously exploited [91] in the much more involved gravitational case, relevant in the Green–Schwarz mechanism in superstrings.

As we will consider the Yang–Mills case only, we drop the $^{\mathscr{YM}}$ superscript from now on, a superfield L being supposed to satisfy the modified linearity conditions.

5.3. Component fields

When coupled to the supergravity/matter/Yang–Mills system, the components

$$b_{mn}(x), \quad L(x), \quad \Lambda_\alpha(x), \quad \bar{\Lambda}_{\dot\alpha}(x) \tag{5.3.1}$$

of the linear multiplet are still defined as lowest superfield components, but now in the framework of $U_K(1)$ superspace geometry. For the covariant components $L(x)$, $\Lambda_\alpha(x)$, $\bar{\Lambda}_{\dot{\alpha}}(x)$, we define

$$L| = L(x), \quad \mathscr{D}_\alpha L| = \Lambda_\alpha(x), \quad \mathscr{D}^{\dot{\alpha}} L| = \bar{\Lambda}^{\dot{\alpha}}(x) \,, \tag{5.3.2}$$

whereas the antisymmetric tensor gauge field is identified as

$$B\| = b = \tfrac{1}{2} dx^m dx^n b_{nm}(x) \,. \tag{5.3.3}$$

The double-bar projection, as defined in Section 4, is particularly useful for the determination of the lowest component of H_{cba}, the supercovariant field strength of the antisymmetric tensor. Recall that the component field expression of the Chern–Simons form, in terms of $\mathscr{A}_{||} = \mathrm{i} dx^m \boldsymbol{a}_m(x)$, is given as

$$Q\| = \frac{1}{3!} dx^n dx^m dx^l Q_{lmn} = -\frac{1}{3!} dx^n dx^m dx^l \,\mathrm{tr}\left(\boldsymbol{a}_l \partial_m \boldsymbol{a}_n - \frac{2\mathrm{i}}{3} \boldsymbol{a}_l \boldsymbol{a}_m \boldsymbol{a}_n \right) . \tag{5.3.4}$$

The double-bar projection is then applied in two ways. On the one hand, we have

$$H\| = \frac{1}{3!} dx^l dx^m dx^n h_{nml} \tag{5.3.5}$$

with $h_{nml} = \partial_n b_{ml} + \partial_m b_{ln} + \partial_l b_{nm} + kQ_{nml}$. The supercovariant field strength $H_{cba|}$, on the other hand, is identified in employing the double bar projection in terms of the covariant component field differentials e^A, defined in (4.1.1), (4.1.2), and taking into account the constraints on H_{CBA}. As a result, we find

$$H\| = \frac{1}{3!} e^a e^b e^c H_{cba}| + \frac{1}{2} e^a e^b e^\gamma H_{\gamma ba}| + \frac{1}{2} e^a e^b e_{\dot{\gamma}} H^{\dot{\gamma}}_{\ ba}| + e^a e_\beta e^\gamma H_\gamma{}^\beta{}_{a|} \,. \tag{5.3.6}$$

Inserting the explicit expressions for $H_{\gamma ba}$, $H^{\dot{\gamma}}{}_{ba}$ and $H_\gamma{}^\beta{}_a$ yields then in a straightforward way

$$\frac{1}{3!} \varepsilon^{dcba} H_{cba}| = \frac{1}{3!} e_n{}^d \varepsilon^{nmlk} (h_{mlk} + 3\mathrm{i} L \psi_m \sigma_l \bar{\psi}_k) + \mathrm{i} e_n^d (\psi_m \sigma^{nm} \Lambda - \bar{\psi}_m \bar{\sigma}^{nm} \bar{\Lambda}) \,. \tag{5.3.7}$$

Note that the supercovariant field strength $H_{cba}|$, one of the basic building blocks in the construction of component field actions, exhibits terms linear and quadratic in the Rarita–Schwinger field. Details on the geometric derivation of supersymmetry transformation laws and the construction of invariant component field actions are presented in Appendix E.

5.4. Linear multiplet coupling

For the coupling of the linear multiplet to the general supergravity/matter/Yang–Mills system we may imagine to follow the same steps as before, but with the *Kähler potential* replaced by an L-dependent superfield $K(\phi, \bar{\phi}, L)$ [20,19,1], which we shall call the *kinetic potential*. Let us note that L being real, the interpretation of K as a potential of Kähler geometry is partly lost.

As we now explain, such a construction does not yield a canonically normalized Einstein term. To begin with, we note that the curvature scalar still appears in the combination

$$\mathscr{D}^2 R + \bar{\mathscr{D}}^2 R^\dagger = -\tfrac{2}{3} R_{ba}{}^{ba} - \tfrac{1}{3}(\mathscr{D}^\alpha X_\alpha + \mathscr{D}_{\dot\alpha} \bar{X}^{\dot\alpha}) + 4 G^a G_a + 32 R R^\dagger \,, \tag{5.4.1}$$

where the combination $\mathscr{D}^\alpha X_\alpha + \mathscr{D}_{\dot\alpha}\bar{X}^{\dot\alpha}$ should now be evaluated using $K = K(\phi, \bar\phi, L)$ as a starting point. This generates extra $\mathscr{D}^2 R + \bar{\mathscr{D}}^2 R^\dagger$ terms. Indeed, recall the $U_K(1)$ relations (B.4.7), (B.4.8)

$$-3\mathscr{D}_\alpha R = X_\alpha + 4S_\alpha, \quad -3\mathscr{D}^{\dot\alpha} R^\dagger = X^{\dot\alpha} - 4S^{\dot\alpha} \tag{5.4.2}$$

and the definitions

$$X_\alpha = -\tfrac{1}{8}(\bar{\mathscr{D}}^2 - 8R)\mathscr{D}_\alpha K(\phi, \bar\phi, L) \,, \tag{5.4.3}$$

$$\bar{X}^{\dot\alpha} = -\tfrac{1}{8}(\mathscr{D}^2 - 8R^\dagger)\mathscr{D}^{\dot\alpha} K(\phi, \bar\phi, L) \,. \tag{5.4.4}$$

In the L-independent case these relations serve to identify $\mathscr{D}_\alpha R$ and $\mathscr{D}^{\dot\alpha} R^\dagger$ as superfields, roughly speaking, depending through $X_\alpha, \bar{X}^{\dot\alpha}$ on the matter sector and through $S_\alpha, \bar{S}^{\dot\alpha}$ on the gravity sector. In the L-dependent case, due to the presence of R, $R^\dagger$ in the linearity conditions, successive spinor derivatives generate extra $\mathscr{D}_\alpha R$ (resp. $\mathscr{D}^{\dot\alpha} R^\dagger$) terms in the expressions of X_α (resp. $\bar{X}^{\dot\alpha}$). We can make explicit such contributions and write ($K_L \equiv \partial K/\partial L$)

$$X_\alpha = -LK_L \mathscr{D}_\alpha R + Y_\alpha, \quad \bar{X}^{\dot\alpha} = -LK_L \mathscr{D}^{\dot\alpha} R^\dagger + \bar{Y}^{\dot\alpha} \,, \tag{5.4.5}$$

where Y_α and $\bar{Y}^{\dot\alpha}$ contain all remaining contributions including those stemming from the Chern–Simons forms. Hence, in this case $\mathscr{D}_\alpha R$ and $\mathscr{D}^{\dot\alpha} R^\dagger$ are still identified as dependent superfields, but relations (5.4.2) take a modified form

$$(LK_L - 3)\mathscr{D}_\alpha R = Y_\alpha + 4S_\alpha \,, \tag{5.4.6}$$

$$(LK_L - 3)\mathscr{D}^{\dot\alpha} R^\dagger = \bar{Y}^{\dot\alpha} + 4S^{\dot\alpha} \,. \tag{5.4.7}$$

This, in turn, implies that the basic geometric relation (5.4.1) takes a modified form as well

$$\begin{aligned}(1 - \tfrac{1}{3}LK_L)(\mathscr{D}^2 R + \bar{\mathscr{D}}^2 R^\dagger) &= -\tfrac{2}{3} R_{ba}{}^{ba} + 4G^a G_a + 32 R R^\dagger \\ &\quad - \tfrac{1}{3}(\mathscr{D}^\alpha Y_\alpha + \mathscr{D}_{\dot\alpha}\bar{Y}^{\dot\alpha}) + \tfrac{1}{3}\mathscr{D}^\alpha(LK_L)\mathscr{D}_\alpha R + \tfrac{1}{3}\mathscr{D}_{\dot\alpha}(LK_L)\mathscr{D}^{\dot\alpha} R^\dagger \,. \end{aligned} \tag{5.4.8}$$

Evaluating the component field action, following the procedure of Section 4.5, we obtain an Einstein term with a field-dependent normalization $(1 - \tfrac{1}{3}LK_L)^{-1}$. In other terms, in the linear superfield formalism, a superfield action which is just the integral over the superdeterminant of the frame, leads to a non-canonical normalization of the Einstein term.

In order to have more flexibility for the normalization function we consider from now on a general superfield action

$$\mathscr{L} = -3\int E F(\phi, \bar\phi, L) \,, \tag{5.4.9}$$

where the *subsidiary function* F depends in a yet unspecified manner on the chiral and linear superfields. Observe that the kinetic potential $K(\phi,\bar{\phi},L)$ is implicit in E through the $U_K(1)$ construction. The component field version of this generalized superfield action is evaluated using the chiral superfield,

$$\mathbf{r} = -\tfrac{1}{8}(\bar{\mathscr{D}}^2 - 8R)F(\phi,\bar{\phi},L) \tag{5.4.10}$$

and its complex conjugate in the generic construction of Section 4.5. A straightforward calculation shows that in this case the Einstein term is multiplied by the *normalization function*

$$N(\phi,\bar{\phi},L) = \frac{F - LF_L}{1 - \frac{1}{3}LK_L}\ . \tag{5.4.11}$$

Requiring $N = 1$, or

$$F - LF_L = 1 - \tfrac{1}{3}LK_L\ , \tag{5.4.12}$$

ensures that we get a canonically normalized Einstein term.

Note that in the case of L-independent functions F and K, this equation implies simply $F = 1$. In the general case, the solution of (5.4.12) reads

$$F(\phi,\bar{\phi},L) = 1 + LV(\phi,\bar{\phi}) + \frac{L}{3}\int\frac{d\lambda}{\lambda}K_\lambda(\phi,\bar{\phi},\lambda)\ . \tag{5.4.13}$$

We see that the only term in $F(\phi,\bar{\phi},L)$ which is not fixed by the choice of the Kähler potential is the term $LV(\phi,\bar{\phi})$, the "integration constant" of the differential equation (5.4.12). Indeed, one can check that, in the Lagrangian (5.4.9), only a term linear in L, viz.,

$$\mathscr{L}_{\text{lin}} = -3\int ELV(\phi,\bar{\phi}) \tag{5.4.14}$$

cannot be set to 1 by a superfield rescaling since the Weyl weights of E and L sum up to zero ($\sigma(E) = -2, \sigma(L) = 2$).

As we discuss now, the real function $V(\phi,\bar{\phi})$ plays an important role in the discussion of certain anomaly cancellation mechanisms. From now on we refer to it as *linear potential*. To be more definite, consider the *effective transformation*

$$V(\phi,\bar{\phi}) \mapsto V(\phi,\bar{\phi}) + H(\phi) + \bar{H}(\bar{\phi}) \tag{5.4.15}$$

with H a chiral superfield which is a holomorphic function of the chiral matter fields. How does the Lagrangian $\mathscr{L}_{\text{lin}}$ change under such a transformation? To see this more explicitly, use integration by parts and apply the modified linearity conditions,

$$\int ELH = -\frac{1}{8}\int\frac{E}{R}H(\bar{\mathscr{D}}^2 - 8R)L = \int\frac{E}{R}H\,\mathrm{tr}(\mathscr{W}^\alpha\mathscr{W}_\alpha)\ . \tag{5.4.16}$$

Note the appearance of the chiral volume element in superspace. Therefore, (5.4.15) gives rise to the effective transformation

$$\mathscr{L}_{\text{lin}} \mapsto \mathscr{L}_{\text{lin}} + \frac{3k}{4}\int \frac{E}{R} H(\phi)\,\mathrm{tr}(\mathscr{W}^{\alpha}\mathscr{W}_{\alpha}) + \frac{3k}{4}\int \frac{E}{R^{\dagger}} \bar{H}(\bar{\phi})\,\mathrm{tr}(\mathscr{W}_{\dot{\alpha}}\mathscr{W}^{\dot{\alpha}})\ . \tag{5.4.17}$$

This shows that in the absence of Chern–Simons forms, $k = 0$, transformation (5.4.15) is a symmetry of the theory. In the presence of Chern–Simons forms it creates an Abelian anomaly term, multiplied by $H - \bar{H}$ and gives rise, at the same time, to a Yang–Mills kinetic term multiplied by $H + \bar{H}$. We will come back to this issue later on.

5.5. Duality transformations

As is well known and has been stressed in Section 5.1, the antisymmetric tensor/real scalar duality extends to the supersymmetric case, where it becomes a linear/chiral multiplet duality. This duality should now be explored for the case of a linear multiplet coupled to the general supergravity/matter/Yang–Mills system, the so-called *linear superfield formalism*, in relation to the *chiral superfield formalism*, where only chiral multiplets occur.

It is not surprising that the subsidiary function $F(\phi, \bar{\phi}, L)$, introduced in the previous subsection, be of some importance. As a matter of fact, the normalization condition (5.4.12), justified previously at the component field level will reappear in an intriguing way in the superfield duality transformation mechanism in curved superspace. Let us consider the *first-order formalism* Lagrangian

$$\mathscr{L}_{\text{FOF}} = -3\int E[F(\phi, \bar{\phi}, X) + X(S + \bar{S})]\ , \tag{5.5.1}$$

where S is a chiral superfield, $\mathscr{D}^{\dot{\alpha}} S = 0$, and X is an unconstrained superfield. The kinetic potential $K(\phi, \bar{\phi}, X)$ and the normalization function $F(\phi, \bar{\phi}, X)$ are supposed to be given in terms of this unconstrained superfield.

Variation of (5.5.1) with respect to X gives rise to

$$(S + \bar{S})(1 - \tfrac{1}{3} X K_X) = \tfrac{1}{3} F K_X - F_X\ , \tag{5.5.2}$$

where we have used

$$\delta_X E = -\tfrac{1}{3} E K_X \delta X \tag{5.5.3}$$

as derived from (D.3.3) and (D.2.89). For given F and K functions, (5.5.2) should allow to express X as a function of $\phi, \bar{\phi}$ and of $S + \bar{S}$, such that the resulting Lagrangian in the *chiral superfield formalism* is given as

$$\mathscr{L}_{\text{CSF}} = -3\int E[F(\phi, \bar{\phi}, X(\phi, \bar{\phi}, S + \bar{S})) + (S + \bar{S}) X(\phi, \bar{\phi}, S + \bar{S})]\ . \tag{5.5.4}$$

Clearly, this Lagrangian will not necessarily yield the canonical normalization of the curvature scalar term. On the other hand, we have shown in Section 3.2 that the Lagrangian, built with (anti)chiral superfields, which gives a correct Einstein term is simply

$$\mathcal{L} = -3\int E \,. \tag{5.5.5}$$

This form of (5.5.4) can be obtained in requiring

$$F(\phi, \bar{\phi}, X) + X(S + \bar{S}) = 1 \,, \tag{5.5.6}$$

where X is the solution of (5.5.2). Formally, these two equations combine into

$$F - XF_X = 1 - \tfrac{1}{3}XK_X \,. \tag{5.5.7}$$

This means that for a theory with canonical Einstein term, F and K cannot be chosen independently, they should satisfy (5.5.7), which has the same form as (5.4.12), but with L replaced by $X(\phi, \bar{\phi}, S + \bar{S})$. Likewise, $F(\phi, \bar{\phi}, X(\phi, \bar{\phi}, S + \bar{S}))$ should have the same functional dependence on X as it had before on L. These relations are of fundamental importance if we want to make meaningful comparisons between different theories (or compare, for example, the tree-level and one-loop effective actions).

Alternatively, we can vary (5.5.1) with respect to S or $\bar{S}$. Due to chirality, they can be written as

$$S = (\bar{\mathcal{D}}^2 - 8R)\Sigma, \quad \bar{S} = (\mathcal{D}^2 - 8R^\dagger)\bar{\Sigma} \,, \tag{5.5.8}$$

where Σ, $\bar{\Sigma}$ are unconstrained superfields.

Variation of (5.5.1) with respect to Σ, $\bar{\Sigma}$ yields after integration by parts:

$$(\bar{\mathcal{D}}^2 - 8R)X = 0, \quad (\mathcal{D}^2 - 8R^\dagger)X = 0 \,. \tag{5.5.9}$$

We conclude that X is a linear superfield, which we identify with L_0. An integration by parts (linear $\times$ chiral integrates to zero) then shows that (5.5.1) reproduces (5.4.9) and we are back with the *linear superfield formalism* discussed in the previous subsection.

There, however, the linear multiplet was coupled to Chern–Simons forms. How does this coupling affect the duality structure? It is clear that in the linear superfield formalism we should reproduce the modified linearity conditions. Therefore, the *first-order formalism* should include the Chern–Simons superfield Ω, such that

$$\mathcal{L}_{\mathrm{FOF}} = -3\int E[F(\phi, \bar{\phi}, X) + (X - k\Omega)(S + \bar{S})] \,. \tag{5.5.10}$$

Varying with respect to Σ, $\bar{\Sigma}$ establishes then the modified linearity conditions. On the other hand, varying[17] (5.5.10) with respect to X gives rise to the same equation (5.5.2) as before. Imposing

[17] Due to the variation law $\delta_X \Omega = \frac{1}{3}\Omega K_X \delta X$, the terms proportional to the Chern–Simons form cancel out in this equation, as expected from gauge invariance considerations.

moreover a canonical Einstein term, using (5.5.6), the Lagrangian in the *chiral superfield formalism* then reads

$$\mathscr{L}_{\mathrm{CSF}} = -3\int E[1 - k\Omega(S + \bar{S})] \, . \tag{5.5.11}$$

To put the new terms, arising from the Chern–Simons couplings, in a more familiar form, we write them as

$$\mathscr{L}_{\mathrm{CSF}} = -3\int E - \frac{3}{8}\int \frac{E}{R} S(\bar{\mathscr{D}}^2 - 8R)\Omega - \frac{3}{8}\int \frac{E}{R^\dagger}\bar{S}(\mathscr{D}^2 - 8R^\dagger)\Omega \, , \tag{5.5.12}$$

where the derivative terms vanish upon integration by parts (S and R are chiral superfields), and use (5.2.21) to obtain

$$\mathscr{L}_{\mathrm{CSF}} = -3\int E - \frac{3}{4}k\int \frac{E}{R} S\, \mathrm{tr}(\mathscr{W}^\alpha \mathscr{W}_\alpha) - \frac{3}{4}k\int \frac{E}{R^\dagger}\bar{S}\, \mathrm{tr}(\mathscr{W}_{\dot{\alpha}} \mathscr{W}^{\dot{\alpha}}) \, . \tag{5.5.13}$$

We therefore recover the standard formulation of matter coupled to supergravity with a holomorphic gauge coupling function

$$f(S) = -6kS \, . \tag{5.5.14}$$

Comparing this to (5.4.17) suggests that the effective transformations (5.4.15) should be realized in the chiral superfield formalism as field-dependent shifts of the chiral superfield S, i.e. $S \mapsto S - H(\phi)$ and $\bar{S} \mapsto \bar{S} - \bar{H}(\bar{\phi})$.

Let us stress that the duality between the linear superfield formulation and the chiral superfield formulation, discussed here for the case of one single linear superfield, extends quite obviously to the case of several linear superfields and suitable Chern–Simons couplings. We will come back to this after the next subsection.

We close this subsection on an example [21,18] which plays an important role in superstring models. We take for the Kähler potential:

$$K = K_0(\phi, \bar{\phi}) + \alpha \log L \, , \tag{5.5.15}$$

where it was already stressed that L plays the rôle of the string coupling. The corresponding solution of (5.4.9) is

$$F = 1 - \alpha/3 + LV(\phi, \bar{\phi}) \, . \tag{5.5.16}$$

The solution of (5.5.6) reads

$$\frac{\alpha}{3L} = S + \bar{S} + V(\phi, \bar{\phi}) \tag{5.5.17}$$

and

$$K(\phi, \bar{\phi}, S + \bar{S}) = K_0(\phi, \bar{\phi}) + \alpha \log \frac{\alpha}{3} - \alpha \log(S + \bar{S} + V(\phi, \bar{\phi})) \; . \tag{5.5.18}$$

It is interesting to discuss Eq. (5.5.17) in the context of the one-loop renormalization of the gauge coupling performed by Dixon et al. [52]: $S + \bar{S}$ is interpreted as the tree-level gauge coupling and $V(\phi, \bar{\phi})$ is a generic (non-holomorphic) threshold correction. We thus see that, up to a normalization factor, it is L^{-1} which must be interpreted as the renormalized gauge coupling. Thus, *the natural framework to perform the renormalization of the gauge coupling functions is the linear multiplet formulation*.

We note also that the Kähler potential in (5.5.18) is invariant under the effective transformations (5.4.15) together with $S \mapsto S - H(\phi)$ and $\bar{S} \mapsto \bar{S} - \bar{H}(\bar{\phi})$.

Adding terms of order L^n ($n \geq 2$) in (5.5.15) would include higher-order corrections, if any, but we can note here the special status played by one-loop corrections. The explicit computation of Ref. [52] indicates that, in this context, $V(\phi, \bar{\phi})$ contains a piece which is nothing else but $K_0(\phi, \bar{\phi})$. This fact has been stressed by Derendinger et al. [48] and is in agreement with the Kähler properties of $V(\phi, \bar{\phi})$ – cf. (5.4.15).

5.6. Non-holomorphic gauge couplings

In general, as explained in Section 3.4.3, supersymmetric Yang–Mills theory allows for arbitrary holomorphic gauge coupling functions in terms of the complex matter scalar fields. The corresponding invariant supergravity action (3.4.54) is given as a F-term in $U_K(1)$ superspace.

Superstring theory, in its effective low-energy limit, seems to suggest non-holomorphic gauge coupling functions [142,112] as well. From the formal point of view, such non-canonical structures arise naturally in the linear superfield formalism [19,48].

Independently of the relation to string theory, it is instructive in itself to elucidate the origin of non-holomorphic gauge couplings in the linear superfield formalism. The crucial ingredient is the coupling of Chern–Simons forms to linear multiplets, as described in Sections 5.2 and 5.4. In this context, the modified linearity conditions (5.2.18) and (5.2.19) are of utmost importance. In the following, we will point out schematically how non-holomorphic gauge couplings appear in the component field theory, starting from the geometric superspace description.

Recall that the basic object for the construction of the component field action are the chiral superfields $\mathbf{r}$ and $\bar{\mathbf{r}}$ given as

$$\mathbf{r} = -\tfrac{1}{8}(\bar{\mathscr{D}}^2 - 8R)F(\phi, \bar{\phi}, L), \quad \bar{\mathbf{r}} = -\tfrac{1}{8}(\mathscr{D}^2 - 8R^\dagger)F(\phi, \bar{\phi}, L) \; . \tag{5.6.1}$$

Working through the generic construction of Section 4.4 allows to determine unambiguously the complete component field action. As we are interested only in the gauge coupling function, it is not necessary to go through all these steps in full detail.

For the sake of a schematical discussion recall first of all that the gauge kinetic terms arise from the lowest component of the superfield

$$\mathscr{D}^2 \operatorname{tr} \mathscr{W}^2 + \bar{\mathscr{D}}^2 \operatorname{tr} \bar{\mathscr{W}}^2 \; . \tag{5.6.2}$$

On the other hand, the complete set of kinetic terms of all the component fields is identified in

$$\mathscr{D}^2\mathbf{r} + \bar{\mathscr{D}}^2\bar{\mathbf{r}} \ . \tag{5.6.3}$$

The procedure consists in evaluating the spinor derivatives in (5.6.3) and in isolating terms proportional to (5.6.2). In a first step we identify relevant terms in

$$\mathscr{D}^2\mathbf{r} + \bar{\mathscr{D}}^2\bar{\mathbf{r}} \overset{\mathrm{RT}}{=} F(\mathscr{D}^2 R + \bar{\mathscr{D}}^2 R^\dagger) - \tfrac{1}{8}(\mathscr{D}^2\bar{\mathscr{D}}^2 + \bar{\mathscr{D}}^2\mathscr{D}^2)F \ . \tag{5.6.4}$$

The symbol $\overset{\mathrm{RT}}{=}$ indicates that we only retain the terms relevant for our discussion, making the arguments more transparent. The first term on the right contains the contribution originating from the L dependence of K. Using (5.4.1), we obtain

$$\mathscr{D}^2\mathbf{r} + \bar{\mathscr{D}}^2\bar{\mathbf{r}} \overset{\mathrm{RT}}{=} -\tfrac{1}{3}F(\mathscr{D}^\alpha X_\alpha + \mathscr{D}_{\dot\alpha}\bar{X}^{\dot\alpha}) - \tfrac{1}{8}F_L(\mathscr{D}^2\bar{\mathscr{D}}^2 + \bar{\mathscr{D}}^2\mathscr{D}^2)L \ . \tag{5.6.5}$$

Next, we insert the explicit expression for X^α in terms of $K(\phi, \bar{\phi}, L)$, i.e.

$$\mathscr{D}^\alpha X_\alpha + \mathscr{D}_{\dot\alpha}\bar{X}^{\dot\alpha} \overset{\mathrm{RT}}{=} -\tfrac{1}{8}K_L(\mathscr{D}^2\bar{\mathscr{D}}^2 + \bar{\mathscr{D}}^2\mathscr{D}^2)L \ , \tag{5.6.6}$$

to arrive at the intermediate result

$$\mathscr{D}^2\mathbf{r} + \bar{\mathscr{D}}^2\bar{\mathbf{r}} \overset{\mathrm{RT}}{=} -\tfrac{1}{8}(F_L - \tfrac{1}{3}FK_L)(\mathscr{D}^2\bar{\mathscr{D}}^2 + \bar{\mathscr{D}}^2\mathscr{D}^2)L \ . \tag{5.6.7}$$

In the next step we are going to exploit the modified linearity conditions

$$\bar{\mathscr{D}}^2 L = 8RL + 2k\,\mathrm{tr}\,\mathscr{W}^2, \quad \mathscr{D}^2 L = 8R^\dagger L + 2k\,\mathrm{tr}\,\bar{\mathscr{W}}^2 \ . \tag{5.6.8}$$

As a consequence we find

$$(\mathscr{D}^2\bar{\mathscr{D}}^2 + \bar{\mathscr{D}}^2\mathscr{D}^2)L \overset{\mathrm{RT}}{=} 8L(\mathscr{D}^2 R + \bar{\mathscr{D}}^2 R^\dagger) + 2k(\mathscr{D}^2\,\mathrm{tr}\,\mathscr{W}^2 + \bar{\mathscr{D}}^2\,\mathrm{tr}\,\bar{\mathscr{W}}^2) \ . \tag{5.6.9}$$

Using once more (5.6.6), i.e.

$$\mathscr{D}^2 R + \bar{\mathscr{D}}^2 R^\dagger \overset{\mathrm{RT}}{=} \tfrac{1}{24}K_L(\mathscr{D}^2\bar{\mathscr{D}}^2 + \bar{\mathscr{D}}^2\mathscr{D}^2)L \tag{5.6.10}$$

yields

$$(\mathscr{D}^2\bar{\mathscr{D}}^2 + \bar{\mathscr{D}}^2\mathscr{D}^2)L \overset{\mathrm{RT}}{=} \frac{2k}{1 - \tfrac{1}{3}LK_L}(\mathscr{D}^2\,\mathrm{tr}\,\mathscr{W}^2 + \bar{\mathscr{D}}^2\,\mathrm{tr}\,\bar{\mathscr{W}}^2) \ . \tag{5.6.11}$$

The final result is then

$$\mathscr{D}^2\mathbf{r} + \bar{\mathscr{D}}^2\bar{\mathbf{r}} \overset{\mathrm{RT}}{=} -\frac{k}{4}\frac{F_L - \frac{1}{3}FK_L}{1-\frac{1}{3}LK_L}(\mathscr{D}^2\,\mathrm{tr}\,\mathscr{W}^2 + \bar{\mathscr{D}}^2\,\mathrm{tr}\,\bar{\mathscr{W}}^2)\,, \tag{5.6.12}$$

which allows to identify the *gauge coupling function*

$$\Gamma(\phi,\bar{\phi},L) = \frac{F_L - \frac{1}{3}FK_L}{1-\frac{1}{3}LK_L}\ . \tag{5.6.13}$$

Recall that in the standard case the gauge coupling is the sum of a holomorphic and an antiholomorphic function. In the more general formulation given here, non-holomorphic coupling functions are allowed.

At this point it is important to note that so far we did not make any reference to possible normalizations of the Einstein term, appearing in the same action. In Section 5.4 we have identified the normalization function

$$N(\phi,\bar{\phi},L) = \frac{F - LF_L}{1-\frac{1}{3}LK_L}\ . \tag{5.6.14}$$

A glance at the explicit form of Γ and N shows that they are related to F through the simple relation

$$L\Gamma + N = F\ . \tag{5.6.15}$$

Finally, the same Lagrangian contains also a kinetic term for L,

$$\frac{1}{4L}[3N_L + K_L(LN_L - N)]g^{mn}\partial_m L\partial_n L\ , \tag{5.6.16}$$

whose normalization function is expressed in terms of previously defined quantities. Note that, in view of the normalization of the curvature scalar, i.e.

$$-\frac{N}{2}\mathscr{R}\ , \tag{5.6.17}$$

it should be clear that the conformally trivial combination is obtained from the choice $N = L$; remember that L has Weyl weight $\sigma(L) = -2$.

Let us now turn to a discussion of the duality transformation in this general case, i.e. in the presence of non-trivial normalization function N, gauge coupling Γ, and subsidiary function F. The relevant first-order action is still (5.5.10). The linear superfield formalism discussed above is obtained in the usual way, varying with respect to the unconstrained pre-potentials of the chiral superfield S. The chiral superfield formalism, on the other hand, is obtained from variation with respect to X. As before, the corresponding equation of motion (5.5.2) should be understood as an expression which determines X in terms of $\phi, \bar{\phi}$ and $S + \bar{S}$. The chiral superfield formalism is then obtained from (5.5.10), but with X now a function $X(\phi, \bar{\phi}, S + \bar{S})$. As to the gauge coupling function we are back to the holomorphic case.

From what we have learned before, it should be clear that the superfields underlying the component field construction of the action are now

$$\mathbf{r} = -\frac{1}{8}(\bar{\mathscr{D}}^2 - 8R)F + \frac{k}{4}\operatorname{tr}\mathscr{W}^2, \quad \bar{\mathbf{r}} = -\frac{1}{8}(\mathscr{D}^2 - 8R^\dagger)F + \frac{k}{4}\operatorname{tr}\bar{\mathscr{W}}^2 . \tag{5.6.18}$$

It is instructive to identify the normalization function of the curvature scalar and the gauge coupling function, using a similar reasoning as before in the linear superfield formalism. Working through the successive application of spinor derivatives in $\mathscr{D}^2\mathbf{r} + \bar{\mathscr{D}}^2\bar{\mathbf{r}}$ and keeping track only of terms relevant for our purpose we find

$$\mathscr{D}^2\mathbf{r} + \bar{\mathscr{D}}^2\bar{\mathbf{r}} \overset{\mathrm{RT}}{=} -\frac{2}{3}(F + X(S + \bar{S}))R_{ba}{}^{ba} + \frac{k}{4}(S + \bar{S})(\mathscr{D}^2\operatorname{tr}\mathscr{W}^2 + \bar{\mathscr{D}}^2\operatorname{tr}\bar{\mathscr{W}}^2) . \tag{5.6.19}$$

The gauge coupling function is simply proportional to $S + \bar{S}$, in accordance with (5.5.2) and definition (5.6.13). As to the normalization function of the Einstein term we observe that, using formally (5.5.2) together with (5.6.15), means simply that

$$F + X(S + \bar{S}) = N \tag{5.6.20}$$

with the X-dependent function N written in terms of $X(\phi, \bar{\phi}, S + \bar{S})$. The determination of the normalization of the kinetic terms of S, $\bar{S}$ is left as an exercise.

5.7. Several linear multiplets

The linear superfield formalism can be easily generalized to accommodate several linear multiplets. Noting L^{I}, with $\mathrm{I} = 0, 1, \ldots, n$, the $n + 1$ copies of linear superfields we will have a set of $n + 1$ modified linearity conditions

$$(\mathscr{D}^2 - 8R^\dagger)L^{\mathrm{I}} = 2k^{\mathrm{I}}_{\mathrm{G}}\operatorname{tr}\mathscr{W}^2_{\mathrm{G}} , \tag{5.7.1}$$

$$(\bar{\mathscr{D}}^2 - 8R)L^{\mathrm{I}} = 2k^{\mathrm{I}}_{\mathrm{G}}\operatorname{tr}\bar{\mathscr{W}}^2_{\mathrm{G}} . \tag{5.7.2}$$

Here the subscript $_{\mathrm{G}}$ indicates that different linear combinations of Chern–Simons forms (Yang–Mills potentials for different gauge groups) may couple to different antisymmetric tensors.

In this general scenario the kinetic potential K and the subsidiary function F will be functions of the $n + 1$ superfields L^{I}. The superfield action

$$\mathscr{L} = -3\int E F(\phi, \bar{\phi}, L^{\mathrm{I}}) , \tag{5.7.3}$$

depends implicitly on $K(\phi, \bar{\phi}, L^{\mathrm{I}})$ through E due to the geometric construction.

The presence of several linear superfields implies that different gauge sectors may have different gauge coupling functions. The determination of the explicit form of the gauge coupling and normalization functions follows exactly the same steps as in the case of a single linear superfield, taking now the chiral superfields $\mathbf{r}$ and $\bar{\mathbf{r}}$ to be

$$\mathbf{r} = -\tfrac{1}{8}(\bar{\mathscr{D}}^2 - 8R)F(\phi, \bar{\phi}, L^{\mathrm{I}}), \quad \bar{\mathbf{r}} = -\tfrac{1}{8}(\mathscr{D}^2 - 8R^{\dagger})F(\phi, \bar{\phi}, L^{\mathrm{I}}) . \tag{5.7.4}$$

As a result, the normalization function takes the form

$$N(\phi, \bar{\phi}, L^{\mathrm{I}}) = \frac{F - L \cdot F_L}{1 - \frac{1}{3} L \cdot K_L} , \tag{5.7.5}$$

whereas the gauge coupling functions are given as

$$\Gamma_{\mathbf{G}}(\phi, \bar{\phi}, L^{\mathrm{I}}) = \left(F_{\mathrm{I}} - \frac{N}{3} K_{\mathrm{I}} \right) k_{\mathbf{G}}^{\mathrm{I}} . \tag{5.7.6}$$

We use here the notation $L \cdot F_L = L^{\mathrm{I}} F_{\mathrm{I}}$, with F_{I} denoting the derivative of F with respect to L^{I}, and the same for K. The gauge coupling and normalization functions satisfy the sum rule

$$L^{\mathrm{I}} \Gamma_{(\mathrm{I})} + N = F \tag{5.7.7}$$

with $\Gamma_{(\mathrm{I})}$ identified as $\Gamma_{\mathbf{G}} = \Gamma_{(\mathrm{I})} k_{\mathbf{G}}^{\mathrm{I}}$. The brackets indicate that the enclosed subscript does not refer to a derivative. It is also interesting to note that the kinetic term $g^{mn} \partial_m L^{\mathrm{J}} \partial_n L^{\mathrm{I}}$ is multiplied by a function

$$G_{(\mathrm{IJ})} = F_{\mathrm{IJ}} - \tfrac{1}{3}(N K_{\mathrm{IJ}} + N_{\mathrm{I}} K_{\mathrm{J}} + N_{\mathrm{J}} K_{\mathrm{I}}) . \tag{5.7.8}$$

The effective transformations in the case of a single linear multiplet generalize as well. To this end we observe first of all that a replacement

$$F(\phi, \bar{\phi}, L^{\mathrm{I}}) \mapsto F(\phi, \bar{\phi}, L^{\mathrm{I}}) + L^{\mathrm{I}} V_{(\mathrm{I})}(\phi, \bar{\phi}) \tag{5.7.9}$$

leaves the normalization function (5.7.5) as well as the sum rule (5.7.7) invariant, whereas the gauge coupling function changes as

$$\Gamma_{(\mathrm{I})}(\phi, \bar{\phi}, L^{\mathrm{I}}) \mapsto \Gamma_{(\mathrm{I})}(\phi, \bar{\phi}, L^{\mathrm{I}}) + V_{(\mathrm{I})}(\phi, \bar{\phi}) . \tag{5.7.10}$$

The counterpart of the effective action (5.4.14) in the presence of several multiplets becomes

$$\mathscr{L}_{\mathrm{lin}} = -3 \int E L^{\mathrm{I}} V_{(\mathrm{I})}(\phi, \bar{\phi}) \tag{5.7.11}$$

with effective transformations

$$V_{(\mathrm{I})}(\phi, \bar{\phi}) \mapsto V_{(\mathrm{I})}(\phi, \bar{\phi}) + H_{(\mathrm{I})}(\phi) + \bar{H}_{(\mathrm{I})}(\bar{\phi}) \tag{5.7.12}$$

giving rise to

$$\mathscr{L}_{\mathrm{lin}} \mapsto \mathscr{L}_{\mathrm{lin}} + \frac{3k}{4}\int \frac{E}{R} H_{(\mathrm{I})}(\phi) k_{\mathrm{G}}^{\mathrm{I}} \operatorname{tr} \mathscr{W}_{\mathrm{G}}^{2} + \frac{3k}{4}\int \frac{E}{R^{\dagger}} \bar{H}_{(\mathrm{I})}(\bar{\phi}) k_{\mathrm{G}}^{\mathrm{I}} \operatorname{tr} \bar{\mathscr{W}}_{\mathrm{G}}^{2} . \tag{5.7.13}$$

This shows that the case of several linear multiplets is more flexible in view of possible applications to anomaly cancellation mechanisms.

As to the duality transformations between the linear and the chiral superfield formalism we will make use of $n+1$ unconstrained real superfields X^{I} together with the real combination $S_{\mathrm{I}} + \bar{S}_{\mathrm{I}}$ of chiral superfields. The first-order action (5.5.10) generalizes then to

$$\mathscr{L}_{\mathrm{FOF}} = -3\int E[F(\phi, \bar{\phi}, X^{\mathrm{I}}) + (X^{\mathrm{I}} - k_{\mathrm{G}}^{\mathrm{I}} \Omega_{\mathrm{G}})(S_{\mathrm{I}} + \bar{S}_{\mathrm{I}})] \tag{5.7.14}$$

with Ω_{G} the Chern–Simons superfield pertaining to the gauge sector specified by the subscript $_{\mathrm{G}}$. Variation with respect to S_{I} (resp. $\bar{S}_{\mathrm{I}}$) gives back the theory in the linear superfield formalism, whereas variation with respect to X^{I} gives rise to the equation

$$(S_{\mathrm{I}} + \bar{S}_{\mathrm{I}})\left(1 - \frac{1}{3} X \cdot K_X\right) = \frac{F}{3} K_{\mathrm{I}} - F_{\mathrm{I}} . \tag{5.7.15}$$

Again, this should be understood as an equation which expresses, for given kinetic potential K and subsidiary function F, the previously unconstrained real superfields X^{I} in terms of ϕ, $\bar{\phi}$ and $S_{\mathrm{I}} + \bar{S}_{\mathrm{I}}$.

Coming back to the linear superfield formalism, we note that the particular form (5.7.5) of the normalization function N suggests to introduce projective variables for the set of linear superfields. Choosing a particular linear superfield of reference, say L^0, we define

$$L_0 = L, \quad \xi^{\mathrm{I}} = \frac{L^{\mathrm{I}}}{L^0} \tag{5.7.16}$$

with I ranging from 1 to n whenever attached to a projective variable ξ. The kinetic potential K and the subsidiary function F are now supposed to be given in terms of L and ξ^{I}. In this parametrization the normalization function N takes the form

$$N(\phi, \bar{\phi}, L, \xi^{\mathrm{I}}) = \frac{F - LF_L}{1 - \frac{1}{3} L K_L} . \tag{5.7.17}$$

Here only derivatives with respect to the particular superfield L occur. This closely resembles (5.4.11), except for the additional dependence on the projective variables ξ^{I}. Likewise, in the effective Lagrangian density one may parametrize

$$L^{\mathrm{I}} V_{(\mathrm{I})}(\phi, \bar{\phi}) = L \mathscr{V}(\phi, \bar{\phi}, \xi^{\mathrm{I}}) \tag{5.7.18}$$

with (identifying $V_{(0)} = V$)

$$\mathscr{V}(\phi, \bar{\phi}, \xi^{\mathrm{I}}) = V(\phi, \bar{\phi}) + \xi^{\mathrm{I}} V_{(\mathrm{I})}(\phi, \bar{\phi}) . \tag{5.7.19}$$

Observe that we could have chosen, instead of L^0, another superfield of reference, without changing the reasoning. Different choices are related in terms of reparametrizations in an obvious way.

As a last remark consider the linear superfield formalism for the case of a trivial coupling function $N = 1$. From the previous discussion, it should be clear that we recover the same type of differential equation (5.4.12) as in the case of a single linear multiplet

$$F - LF_L = 1 - \tfrac{1}{3}LK_L \;, \tag{5.7.20}$$

which is solved in the same way, i.e.

$$F = 1 + L\mathscr{V}(\phi, \bar{\phi}, \xi^{\mathrm{I}}) + \frac{L}{3}\int \frac{d\lambda}{\lambda} K_\lambda(\phi, \bar{\phi}, \lambda, \xi^{\mathrm{I}}) \;. \tag{5.7.21}$$

In conclusion, the linear superfield formalism for the case of several linear multiplets exhibits a quite intriguing structure which clearly should be further investigated. It would be interesting to pursue this approach in the context of duality transformations and the construction of the respective component field actions.

6. Three-form coupling to supergravity

6.1. General remarks

The 3-form supermultiplet is, besides the chiral and linear multiplet, yet another supermultiplet describing helicity $(0, 1/2)$. It consists of a three-index antisymmetric gauge potential $C_{lmn}(x)$, a complex scalar $Y(x)$, a Majorana spinor with Weyl components $\eta_\alpha(x)$, $\bar{\eta}^{\dot{\alpha}}(x)$ and a real scalar auxiliary field $H(x)$.

In superfield language [82,22] it is described by a chiral superfield

$$D^{\dot{\alpha}} Y = 0, \quad D_\alpha \bar{Y} = 0 \;, \tag{6.1.1}$$

which is subject to the additional constraint

$$D^2 \bar{Y} - \bar{D}^2 Y = \frac{8\mathrm{i}}{3} \varepsilon^{klmn} \Sigma_{klmn} \tag{6.1.2}$$

with the field strength of the 3-form gauge potential defined as

$$\Sigma_{klmn} = \partial_k C_{lmn} - \partial_l C_{mnk} + \partial_m C_{nkl} - \partial_n C_{klm} \;. \tag{6.1.3}$$

It is invariant under the transformation

$$C_{lmn} \mapsto C_{lmn} + \partial_l \Lambda_{mn} + \partial_m \Lambda_{nl} + \partial_n \Lambda_{lm} \;, \tag{6.1.4}$$

where the gauge parameters $\Lambda_{mn} = -\Lambda_{nm}$ have an interpretation as a 2-form coefficients.

The component fields of the 3-form multiplet are propagating: supersymmetry couples the rank-3 antisymmetric tensor gauge potential with *dynamical* degrees of freedom. This should be compared to the non-supersymmetric case, discussed in the context of the cosmological constant problem [104,36,58,54], where the 3-form does not imply dynamical degrees of freedom.

In Section 6.2 the superspace formulation of [82] will be adapted to the background of $U_K(1)$ superspace, providing the geometric structure underlying the coupling of the 3-form multiplet to the general supergravity/matter/Yang–Mills system (and to linear multiplets, if desired). We discuss in particular the 3-form Bianchi identities in the presence of appropriate constraints and define supergravity transformations on the superfield and component field levels.

As constraint chiral superfields, subject to the additional constraint (6.1.2), Y and $\bar{Y}$ derive from one and the same real pre-potential Ω superfield such that

$$Y = -4\bar{D}^2\Omega, \quad \bar{Y} = -4D^2\Omega . \tag{6.1.5}$$

In Appendix F we present a detailed derivation of the explicit solution of the 3-form constraints in the background of $U(1)$ superspace and identify the unconstrained pre-potential Ω in this general geometric context.

The 3-form superfields Y and $\bar{Y}$ differ from usual chiral superfields, employed for the description of matter multiplets in yet another respect: they have non-vanishing chiral weights. This property modifies considerably the possible supergravity couplings, compared to the case of vanishing chiral weights. In Section 6.3 we give a very detailed account of the couplings of the 3-form multiplet to supergravity and matter.

Although the study of the 3-form multiplet is interesting in its own right, it has an interesting application in the description of gaugino condensation. There, as a consequence of the chirality of the gaugino superfields, the composite superfields $\mathrm{tr}(\mathcal{W}^2)$ and $\mathrm{tr}(\bar{\mathcal{W}}^2)$ obey chirality conditions

$$D^{\dot{\alpha}}\,\mathrm{tr}(\mathcal{W}^2) = 0, \quad D_\alpha\,\mathrm{tr}(\bar{\mathcal{W}}^2) = 0 \tag{6.1.6}$$

as well. On the other hand, the gaugino superfields are subject to the additional constraint (6.1.2), which translates into an additional equation for the composites, corresponding to (6.1.2). At the component field level this implies the identification

$$D^2\,\mathrm{tr}(\mathcal{W}^2)| - \bar{D}^2\,\mathrm{tr}(\bar{\mathcal{W}}^2)| = \mathrm{i}\varepsilon^{klmn}\,\mathrm{tr}(\boldsymbol{f}_{kl}\boldsymbol{f}_{mn}) , \tag{6.1.7}$$

where the topological density

$$\varepsilon^{klmn}\,\mathrm{tr}(\boldsymbol{f}_{kl}\boldsymbol{f}_{mn}) = -\tfrac{2}{3}\varepsilon^{klmn}\partial_k Q_{lmn} , \tag{6.1.8}$$

plays now the role of the field-strength and the Chern–Simons form (which, under Yang–Mills transformations changes indeed by the derivative of a 2-form) the role of the 3-form gauge potential. The analogy between the Chern–Simons forms in superspace and the 3-form geometry is discussed in detail in Appendices F.2, F.3, and has already been exploited in Section 5.2.

6.2. The 3-form multiplet geometry

The superspace geometry of the 3-form multiplet has been known for some time [82]. Its coupling to the general supergravity/matter/Yang–Mills system is most conveniently described in the framework of $U_K(1)$ superspace – cf. Section 3.4. This approach is particularly useful in view of the non-trivial Kähler transformations of the 3-form superfield Y. Moreover, it provides a concise way to derive supergravity transformations of the component fields.

6.2.1. Constraints and Bianchi identities

The basic geometric object is the 3-form gauge potential

$$C = \frac{1}{3!} dz^L dz^M dz^N C_{NML} , \tag{6.2.1}$$

subject to 2-form gauge transformations of parameter $\Lambda = \frac{1}{2} dz^M dz^N \Lambda_{NM}$ such that

$$C \mapsto C + d\Lambda . \tag{6.2.2}$$

The invariant field strength

$$\Sigma = dC = \frac{1}{4!} E^A E^B E^C E^D \Sigma_{DCBA} \tag{6.2.3}$$

is a 4-form in superspace with coefficients

$$\frac{1}{4!} E^A E^B E^C E^D \Sigma_{DCBA} = \frac{1}{4!} E^A E^B E^C E^D (4\mathscr{D}_D C_{CBA} + 6 T_{DC}{}^F C_{FBA}) . \tag{6.2.4}$$

Here, the full $U_K(1)$ superspace covariant derivatives and torsions appear. Likewise, the Bianchi identity, $d\Sigma = 0$, is a 5-form with coefficients

$$\frac{1}{5!} E^A E^B E^C E^D E^E (5\mathscr{D}_E \Sigma_{DCBA} + 10 T_{ED}{}^F \Sigma_{FCBA}) = 0 . \tag{6.2.5}$$

In these formulas we have kept the covariant differentials in order to keep track of the graded tensor structure of the coefficients.

The multiplet containing the 3-form gauge potential is obtained after imposing constraints on the covariant field-strength coefficients. Following [82] we require

$$\Sigma_{\underline{\delta}\underline{\gamma}\underline{\beta}A} = 0 , \tag{6.2.6}$$

where $\underline{\alpha} \sim \alpha, \dot{\alpha}$ and $A \sim a, \alpha, \dot{\alpha}$. The consequences of these constraints can be studied by analyzing consecutively the Bianchi identities, from lower-to-higher canonical dimensions. The tensor structures of the coefficients of Σ at higher canonical dimensions are then subject to restrictions due to the constraints. In addition, covariant superfield conditions involving spinor derivatives will emerge. The constraints serve to reduce the number of independent component fields to those of the 3-form multiplet, but do not imply any dynamical equations.

As a result of this analysis (alternatively, Appendix F.1 provides the explicit solution of the constraints in terms of an unconstrained pre-potential), all the coefficients of the 4-form field strength Σ can be expressed in terms of the two superfields $\bar{Y}$ and Y, which are identified in the tensor decompositions

$$\Sigma_{\delta\gamma ba} = \tfrac{1}{2}(\sigma_{ba}\varepsilon)_{\delta\gamma}\bar{Y}, \quad \Sigma^{\dot{\delta}\dot{\gamma}}{}_{ba} = \tfrac{1}{2}(\bar{\sigma}_{ba}\varepsilon)^{\dot{\delta}\dot{\gamma}}Y \ . \tag{6.2.7}$$

As a consequence, the $U_K(1)$ weights of Y and $\bar{Y}$ are

$$w(Y) = +2, \quad w(\bar{Y}) = -2 \ . \tag{6.2.8}$$

This implies that the covariant exterior derivatives

$$\mathscr{D}Y = dY + 2AY, \quad \mathscr{D}\bar{Y} = d\bar{Y} - 2A\bar{Y} \tag{6.2.9}$$

contain $A = E^M A_M$, the $U_K(1)$ gauge potential. On the other hand, the *Weyl* weights are determined to be

$$\omega(Y) = \omega(\bar{Y}) = +3 \ . \tag{6.2.10}$$

By a special choice of conventional constraints, i.e. a covariant redefinition of C_{cba}, it is possible to impose

$$\Sigma_{\delta}{}^{\dot{\gamma}}{}_{ba} = 0 \ . \tag{6.2.11}$$

The one spinor-three vector components of Σ are given as

$$\Sigma_{\delta cba} = -\tfrac{1}{16}\sigma^{d}_{\delta\dot{\delta}}\varepsilon_{dcba}\mathscr{D}^{\dot{\delta}}\bar{Y}, \quad \Sigma^{\dot{\delta}}{}_{cba} = +\tfrac{1}{16}\bar{\sigma}^{d\dot{\delta}\delta}\varepsilon_{dcba}\mathscr{D}_{\delta}Y \ . \tag{6.2.12}$$

At the same time, the superfields $\bar{Y}$ and Y are subject to the chirality conditions

$$\mathscr{D}_{\alpha}\bar{Y} = 0, \quad \mathscr{D}^{\dot{\alpha}}Y = 0 \tag{6.2.13}$$

and are further constrained by the relation

$$\frac{8i}{3}\varepsilon^{dcba}\Sigma_{dcba} = (\mathscr{D}^2 - 24R^{\dagger})Y - (\bar{\mathscr{D}}^2 - 24R)\bar{Y} \ , \tag{6.2.14}$$

indicating that the imaginary part of the F-term of the 3-form superfield is given as the curl of the 3-form gauge potential, with a number of additional nonlinear terms due to the coupling to supergravity.

In conclusion, we have seen that all the coefficients of the superspace 4-form Σ, subject to the constraints, are given in terms of the superfields $\bar{Y}$ and Y and their spinor derivatives. It is a matter of straightforward computation to show that all the remaining Bianchi identities do not contain any new information.

6.2.2. *Component fields and supergravity transformations*

As usual, we define component fields as lowest components of superfields. First of all, the 3-form gauge potential is identified as

$$C_{klm}| = C_{klm}(x) \ . \tag{6.2.15}$$

As to the components of Y and $\bar{Y}$ we define

$$Y| = Y(x), \quad \mathscr{D}_{\alpha}Y| = \sqrt{2}\eta_{\alpha}(x) \tag{6.2.16}$$

and

$$\bar{Y}| = \bar{Y}(x), \quad \bar{\mathscr{D}}^{\dot{\alpha}}\bar{Y}| = \sqrt{2}\bar{\eta}^{\dot{\alpha}}(x) . \tag{6.2.17}$$

At the level of two covariant spinor derivatives we define the component $H(x)$ as

$$\mathscr{D}^2 Y| + \bar{\mathscr{D}}^2 \bar{Y}| = -8H(x) . \tag{6.2.18}$$

The orthogonal combination however is not an independent component field. Projection to lowest components of (6.2.14) shows that it is given as

$$\mathscr{D}^2 Y| - \bar{\mathscr{D}}^2 \bar{Y}| = -\frac{32\mathrm{i}}{3}\varepsilon^{klmn}\partial_k C_{lmn} + 2\sqrt{2}\mathrm{i}(\bar{\psi}_m\bar{\sigma}^m)^\alpha\eta_\alpha - 2\sqrt{2}\mathrm{i}(\psi_m\sigma^m)_{\dot{\alpha}}\bar{\eta}^{\dot{\alpha}}$$
$$- 4(\bar{M} + \bar{\psi}_m\bar{\sigma}^{mn}\bar{\psi}_n)Y + 4(M + \psi_m\sigma^{mn}\psi_n)\bar{Y} . \tag{6.2.19}$$

This expression provides the supercovariant component field strength of the 3-form gauge potential, displaying the modifications which arise from the coupling to supergravity: here the appearance of the Rarita–Schwinger field and the supergravity auxiliary field, in the particular combination $M\bar{Y} - \bar{M}Y$.

The component fields in the supergravity, matter and Yang–Mills sectors are defined as usual – cf. Section 6.1. Some new aspects arise in the treatment of the field-dependent $U_K(1)$ pre-potential due to the presence of the fields Y and $\bar{Y}$, carrying non-vanishing $U_K(1)$ weights. It is for this reason that we refrain from calling K a Kähler potential, we rather shall refer to the field-dependent $U_K(1)$ pre-potential as *kinetic potential*.

Before turning to the derivation of the supergravity transformations we shortly digress on the properties of the composite $U_K(1)$ connection arising from the kinetic pre-potential

$$K(\phi, Y, \bar{\phi}, \bar{Y}) ,$$

subject to Kähler transformations

$$K(\phi, Y, \bar{\phi}, \bar{Y}) \mapsto K(\phi, Y, \bar{\phi}, \bar{Y}) + F(\phi) + \bar{F}(\bar{\phi}) .$$

Requiring invariance of the kinetic potential under $U_K(1)$ transformations of the superfields Y and $\bar{Y}$, implies the relation

$$YK_Y = \bar{Y}K_{\bar{Y}} , \tag{6.2.20}$$

which we shall use systematically.[18] The composite $U_K(1)$ connection derives from the commutator term $[\mathscr{D}_\alpha, \mathscr{D}_{\dot{\alpha}}]K$, which, in the presence of the 3-form superfields is given as

$$[\mathscr{D}_\alpha, \mathscr{D}_{\dot{\alpha}}]K = 2\mathrm{i}K_k\mathscr{D}_{\alpha\dot{\alpha}}\phi^k - 2\mathrm{i}K_{\bar{k}}\mathscr{D}_{\alpha\dot{\alpha}}\bar{\phi}^{\bar{k}} + 2\mathrm{i}K_Y\mathscr{D}_{\alpha\dot{\alpha}}Y - 2\mathrm{i}K_{\bar{Y}}\mathscr{D}_{\alpha\dot{\alpha}}\bar{Y}$$
$$+ 2K_{\mathscr{A}\bar{\mathscr{A}}}\mathscr{D}_\alpha\Psi^{\mathscr{A}}\mathscr{D}_{\dot{\alpha}}\bar{\Psi}^{\bar{\mathscr{A}}} + 6(YK_Y + \bar{Y}K_{\bar{Y}})G_{\alpha\dot{\alpha}} , \tag{6.2.21}$$

[18] The special kinetic potential

$$K(\phi, \bar{\phi}, Y, \bar{Y}) = \log[X(\phi, \bar{\phi}) + Z(\phi, \bar{\phi})\bar{Y}Y] ,$$

where X and Z are functions of the matter fields, is a non-trivial example which satisfies this condition.

where we use the shorthand notation $\Psi^{\mathscr{A}} = (\phi^k, Y)$, and $\bar{\Psi}^{\bar{\mathscr{A}}} = (\bar{\phi}^{\bar{k}}, \bar{Y})$, with obvious meaning for $K_{\mathscr{A}\bar{\mathscr{A}}}$. The important point is that on the right hand the $U_K(1)$ connection, A, appears in the covariant derivatives of Y and $\bar{Y}$ due to their non-vanishing $U_K(1)$ weights. Explicitly one has

$$\mathscr{D}_{\alpha\dot{\alpha}}\bar{Y}| = \sigma^m_{\alpha\dot{\alpha}}\left(\partial_m\bar{Y} - 2A_m\bar{Y} - \frac{1}{\sqrt{2}}\bar{\psi}_{m\dot{\varphi}}\bar{\eta}^{\dot{\varphi}}\right),$$

$$\mathscr{D}_{\alpha\dot{\alpha}}Y| = \sigma^m_{\alpha\dot{\alpha}}\left(\partial_m Y + 2A_m Y - \frac{1}{\sqrt{2}}\bar{\psi}_m{}^{\varphi}\eta_{\varphi}\right).$$

Substituting in the defining equation for A_m (3.4.20) and factorizing gives then rise to

$$A_m(x) + \frac{i}{2}e_m{}^a b_a = \frac{1}{4}\frac{1}{1 - YK_Y}(K_k\mathscr{D}_m A^k - K_{\bar{k}}\mathscr{D}_m\bar{A}^{\bar{k}} + K_Y\partial_m Y - K_{\bar{Y}}\partial_m\bar{Y} + i\bar{\sigma}_m^{\dot{\alpha}\alpha}K_{\mathscr{A}\bar{\mathscr{A}}}\Psi^{\mathscr{A}}_{\alpha}\bar{\Psi}^{\bar{\mathscr{A}}}_{\dot{\alpha}}). \quad (6.2.22)$$

As above, we use the shorthand notation $\Psi^{\mathscr{A}}_{\alpha} = (\chi^k_{\alpha}, \eta_{\alpha})$ and $\bar{\Psi}^{\bar{\mathscr{A}}}_{\dot{\alpha}} = (\bar{\chi}^{\bar{k}}_{\dot{\alpha}}, \bar{\eta}_{\dot{\alpha}})$. As is easily verified by an explicit calculation, A_m defined this way transforms as it should under the $U_K(1)$ transformations given above, i.e.

$$A_m \mapsto A_m + \frac{i}{2}\partial_m \,\mathrm{Im}\, F.$$

Observe that the factor $(1 - YK_Y)^{-1}$ accounts for the non-trivial $U_K(1)$ phase transformations

$$Y \mapsto Y e^{-i\,\mathrm{Im}\,F}, \quad \bar{Y} \mapsto \bar{Y}e^{+i\,\mathrm{Im}\,F},$$

of the 3-form superfields.

We turn now to the derivation of supergravity transformations. In Section 3.4.2 they were defined as combinations of superspace diffeomorphisms and field-dependent gauge transformations. In the case of the 3-form one has

$$\delta C = (\imath_{\xi}d + d\imath_{\xi})C + d\Lambda = \imath_{\xi}\Sigma + d(\Lambda + \imath_{\xi}C), \quad (6.2.23)$$

the corresponding supergravity transformation is defined as a diffeomorphism of parameter $\xi^A = \imath_{\xi}E^A$ together with a compensating infinitesimal 2-form gauge transformation of parameter $\Lambda = -\imath_{\xi}C$, giving rise to

$$\delta_{\mathrm{WZ}}C = \imath_{\xi}\Sigma = \frac{1}{3!}E^A E^B E^C \xi^D \Sigma_{DCBA}. \quad (6.2.24)$$

The supergravity transformation of the component 3-form gauge field C_{klm} is then simply obtained from the double-bar projection [11] (simultaneously to lowest superfield components and to space–time differential forms) as

$$\delta_{\mathrm{WZ}}C\| = \frac{1}{3!}dx^k dx^l dx^m \delta_{\mathrm{WZ}}C_{mlk} = \frac{1}{3!}e^A e^B e^C \xi^{\underline{\delta}}\Sigma_{\underline{\delta}CBA}|. \quad (6.2.25)$$

Taking into account the definition $e^A = E^A\|$ (4.1.1), (4.1.2) and the particular form of the coefficients of Σ we obtain

$$\delta_{\mathrm{WZ}} C_{mlk} = \frac{\sqrt{2}}{16}(\bar{\xi}\bar{\sigma}^n \eta - \xi \sigma^n \bar{\eta})\varepsilon_{nmlk} + \frac{1}{2}\oint_{mlk} [(\psi_m \sigma_{lk} \xi)\bar{Y} + (\bar{\psi}_m \bar{\sigma}_{lk} \bar{\xi})Y] . \tag{6.2.26}$$

Let us turn now to the transformations of the remaining components. To start, note that at the superfield level, one has

$$\delta_{\mathrm{WZ}} Y = \imath_\xi dY = \imath_\xi \mathscr{D} Y - 2\imath_\xi A Y , \tag{6.2.27}$$

$$\delta_{\mathrm{WZ}} \bar{Y} = \imath_\xi d\bar{Y} = \imath_\xi \mathscr{D} \bar{Y} + 2\imath_\xi A \bar{Y} . \tag{6.2.28}$$

Taking into account the explicit form of the field-dependent factor $\imath_\xi A| = \xi^{\underline{\alpha}} A_{\underline{\alpha}}|$ – compare to (4.1.25) – one finds

$$\delta_{\mathrm{WZ}} Y = \sqrt{2}\xi^\alpha \left\{\left(1 - \frac{1}{2} Y K_Y\right)\eta_\alpha - \frac{1}{2} Y K_k \chi^k_\alpha\right\} + \frac{1}{\sqrt{2}} \bar{\xi}_{\dot{\alpha}} Y \{K_{\bar{Y}} \bar{\eta}^{\dot{\alpha}} + K_{\bar{k}} \bar{\chi}^{\dot{\alpha}\bar{k}}\} ,$$

$$\delta_{\mathrm{WZ}} \bar{Y} = \sqrt{2}\bar{\xi}_{\dot{\alpha}} \left\{\left(1 - \frac{1}{2} \bar{Y} K_{\bar{Y}}\right)\bar{\eta}^{\dot{\alpha}} - \frac{1}{2} \bar{Y} K_{\bar{k}} \bar{\chi}^{\dot{\alpha}\bar{k}}\right\} + \frac{1}{\sqrt{}2} \xi^\alpha \bar{Y} \{K_Y \eta_\alpha + K_k \chi^k_\alpha\} . \tag{6.2.29}$$

It is more convenient to use a notation where one keeps the combination

$$\Xi = \xi^{\underline{\alpha}} A_{\underline{\alpha}}| = \frac{1}{2\sqrt{2}} \xi^\alpha (K_k \chi^k_\alpha + K_Y \eta_\alpha) - \frac{1}{2\sqrt{2}} \bar{\xi}_{\dot{\alpha}} (K_{\bar{k}} \bar{\chi}^{\dot{\alpha}\bar{k}} + K_{\bar{Y}} \bar{\eta}^{\dot{\alpha}}) , \tag{6.2.30}$$

giving rise to a compact form of the supersymmetry transformations

$$\delta_{\mathrm{WZ}} Y = \sqrt{2}\xi^\alpha \eta_\alpha - 2\Xi Y, \quad \delta_{\mathrm{WZ}} \bar{Y} = \sqrt{2}\bar{\xi}_{\dot{\alpha}} \bar{\eta}^{\dot{\alpha}} + 2\Xi \bar{Y} . \tag{6.2.31}$$

The transformation law for the 3-"forminos" comes out as

$$\begin{aligned}\delta_{\mathrm{WZ}} \eta_\alpha = {} & \sqrt{2}\xi_\alpha H + \frac{4\mathrm{i}\sqrt{2}}{3} \xi_\alpha \varepsilon^{klmn} \partial_k C_{lmn} + \mathrm{i}\sqrt{2}(\bar{\xi}\bar{\sigma}^m \varepsilon)_\alpha \nabla_m Y - \Xi \eta_\alpha \\ & - \frac{\mathrm{i}}{2} \xi_\alpha (\bar{\psi}_m \bar{\sigma}^m \eta - \psi_m \sigma^m \bar{\eta}) - \mathrm{i}(\bar{\xi}\bar{\sigma}^m \varepsilon)_\alpha \psi_m{}^\varphi \eta_\varphi \\ & + \frac{1}{\sqrt{2}} \xi_\alpha \{(\bar{M} + \bar{\psi}_m \bar{\sigma}^{mn} \bar{\psi}_n) Y - (M + \psi_m \sigma^{mn} \psi_n) \bar{Y}\}\end{aligned} \tag{6.2.32}$$

and

$$\delta_{\mathrm{WZ}}\bar{\eta}^{\dot{\alpha}} = \sqrt{2}\bar{\xi}^{\dot{\alpha}}H + \mathrm{i}\sqrt{2}(\xi\sigma^m\varepsilon)^{\dot{\alpha}}\nabla_m\bar{Y} - \frac{4\mathrm{i}\sqrt{2}}{3}\bar{\xi}^{\dot{\alpha}}\varepsilon^{klmn}\partial_k C_{lmn} + \Xi\bar{\eta}^{\dot{\alpha}}$$

$$+ \frac{\mathrm{i}}{2}\bar{\xi}^{\dot{\alpha}}(\bar{\psi}_m\bar{\sigma}^m\eta - \psi_m\sigma^m\bar{\eta}) - \mathrm{i}(\bar{\xi}\bar{\sigma}^m\varepsilon)_\alpha\bar{\psi}_{m\dot{\varphi}}\bar{\eta}^{\dot{\varphi}}$$

$$- \frac{1}{\sqrt{2}}\bar{\xi}^{\dot{\alpha}}\{(\bar{M} + \bar{\psi}_m\bar{\sigma}^{mn}\bar{\psi}_n)Y - (M + \psi_m\sigma^{mn}\psi_n)\bar{Y}\} . \tag{6.2.33}$$

Finally, the supergravity transformation of H is given as

$$\delta_{\mathrm{WZ}}H = \frac{1}{\sqrt{2}}(\bar{\xi}\bar{\sigma}^m)^\alpha\nabla_m\eta_\alpha + \frac{1}{2}(\bar{\xi}\bar{\sigma}^m\sigma^n\bar{\psi}_m)\left(\nabla_n Y - \frac{1}{\sqrt{2}}\psi_m{}^\varphi\eta_\varphi\right)$$

$$+ \frac{1}{\sqrt{2}}(\xi\sigma^m)_{\dot{\alpha}}\nabla_m\bar{\eta}^{\dot{\alpha}} + \frac{1}{2}(\xi\sigma^m\bar{\sigma}^n\psi_m)\left(\nabla_n\bar{Y} - \frac{1}{\sqrt{2}}\bar{\psi}_{m\dot{\varphi}}\bar{\eta}^{\dot{\varphi}}\right)$$

$$+ \frac{1}{3\sqrt{2}}\bar{M}\xi^\alpha\eta_\alpha + \frac{1}{3\sqrt{2}}M\bar{\xi}_{\dot{\alpha}}\bar{\eta}^{\dot{\alpha}} + \frac{1}{3\sqrt{2}}(\bar{\xi}\bar{\sigma}^a\eta + \xi\sigma^a\bar{\eta})b_a$$

$$+ Y\bar{\xi}_{\dot{\alpha}}\bar{X}^{\dot{\alpha}}| + \bar{Y}\xi^\alpha X_\alpha| - \frac{\mathrm{i}}{\sqrt{2}}(\bar{\xi}\bar{\sigma}^m\psi_m + \xi\sigma^m\bar{\psi}_m)H$$

$$+ \frac{2}{3}(\bar{\xi}\bar{\sigma}^p\psi_p - \xi\sigma^p\bar{\psi}_p)\varepsilon^{klmn}\partial_k C_{lmn} - \frac{1}{4\sqrt{2}}(\bar{\xi}\bar{\sigma}^n\psi_n - \xi\sigma^n\bar{\psi}_n)(\bar{\psi}_m\bar{\sigma}^m\eta - \psi_m\sigma^m\bar{\eta})$$

$$- \frac{\mathrm{i}}{4}(\bar{\xi}\bar{\sigma}^l\psi_l - \xi\sigma^l\bar{\psi}_l)\{(\bar{M} + \bar{\psi}_m\bar{\sigma}^{mn}\bar{\psi}_n)Y - (M + \psi_m\sigma^{mn}\psi_n)\bar{Y}\} . \tag{6.2.34}$$

Note that in the above equations we changed $\mathscr{D}$-derivatives into ∇-derivatives as in Section 4.2 – cf. (4.2.15), (4.2.16) – using a redefined $U_K(1)$ connection $v_m(x) = A_m(x) + (\mathrm{i}/2)e_m{}^a b_a$. This allows to keep track of the auxiliary field b_a, otherwise concealed in the numerous covariant derivatives occurring in the Lagrangian. We still have to work out the component field expressions for $X_\alpha|$ and $\bar{X}^{\dot{\alpha}}|$ from the superfields

$$X_\alpha = -\tfrac{1}{8}(\bar{\mathscr{D}}^2 - 8R)\mathscr{D}_\alpha K, \quad \bar{X}^{\dot{\alpha}} = -\tfrac{1}{8}(\mathscr{D}^2 - 8R^\dagger)\bar{\mathscr{D}}^{\dot{\alpha}}K , \tag{6.2.35}$$

given in terms of the matter and 3-form superfield-dependent kinetic potential K. This can be achieved in successively applying the spinor derivatives to K. Alternatively, one may use the expression

$$A = \frac{1}{4}K_{\mathscr{A}}\mathscr{D}\Psi^{\mathscr{A}} - \frac{1}{4}K_{\bar{\mathscr{A}}}\mathscr{D}\bar{\Psi}^{\bar{\mathscr{A}}} + \frac{i}{8}E^a\bar{\sigma}_a^{\dot{\alpha}\alpha}K_{\mathscr{A}\bar{\mathscr{A}}}\mathscr{D}_\alpha\Psi^{\mathscr{A}}\mathscr{D}_{\dot{\alpha}}\bar{\Psi}^{\bar{\mathscr{A}}}$$
$$+ \frac{3i}{2}E^aG_a\left(1 - \frac{1}{2}(YK_Y + \bar{Y}K_{\bar{Y}})\right) \tag{6.2.36}$$

for the composite $U_K(1)$ connection, take the exterior derivative $dA = F$ and identify $\bar{X}^{\dot{\alpha}}$ and X_α in the 2-form coefficients

$$F_{\beta a} = +\frac{i}{2}\sigma_{a\beta\dot{\beta}}\bar{X}^{\dot{\beta}} + \frac{3i}{2}\mathscr{D}_\beta G_a, \quad F^{\dot{\beta}}{}_a = -\frac{i}{2}\bar{\sigma}_a^{\dot{\beta}\beta}X_\beta + \frac{3i}{2}\mathscr{D}^{\dot{\beta}}G_a\ . \tag{6.2.37}$$

A straightforward calculation then yields the component field expression[19]

$$\bar{X}^{\dot{\alpha}}(1 - \bar{Y}K_{\bar{Y}})| = -\frac{i}{\sqrt{2}}K_{\mathscr{A}\bar{\mathscr{A}}}\Psi_\alpha^{\mathscr{A}}\bar{\sigma}^{m\dot{\alpha}\alpha}\left(\nabla_m\bar{\Psi}^{\bar{\mathscr{A}}} - \frac{1}{\sqrt{2}}\bar{\psi}_{m\dot{\varphi}}\bar{\Psi}^{\dot{\varphi}\bar{\mathscr{A}}}\right)$$
$$- \frac{\sqrt{2}}{8}\mathscr{D}^2\phi^k|K_{k\bar{\mathscr{A}}}\bar{\Psi}^{\dot{\alpha}\bar{\mathscr{A}}} + \frac{1}{\sqrt{2}}HK_{Y\bar{\mathscr{A}}}\bar{\Psi}^{\dot{\alpha}\bar{\mathscr{A}}} + \frac{4i}{3\sqrt{2}}\varepsilon^{klmn}\partial_kC_{lmn}\bar{\Psi}^{\dot{\alpha}\bar{\mathscr{A}}}K_{\bar{\mathscr{A}}Y}$$
$$- \frac{1}{2\sqrt{2}}K_{\bar{\mathscr{A}}\mathscr{B}\mathscr{C}}\Psi^{\alpha\mathscr{C}}\Psi_\alpha^{\mathscr{B}}\bar{\Psi}^{\dot{\alpha}\bar{\mathscr{A}}} - iK_{\bar{k}}(\bar{\lambda}^{\dot{\alpha}}\cdot\bar{A})^{\bar{k}} - \frac{i}{4}\bar{\Psi}^{\dot{\alpha}\bar{\mathscr{A}}}K_{\bar{\mathscr{A}}Y}(\bar{\psi}_m\bar{\sigma}^m\eta - \psi_m\sigma^m\bar{\eta})$$
$$+ \frac{1}{2\sqrt{2}}\bar{\Psi}^{\dot{\alpha}\bar{\mathscr{A}}}K_{\bar{\mathscr{A}}Y}\{(\bar{M} + \bar{\psi}_m\bar{\sigma}^{mn}\bar{\psi}_n)Y - (M + \psi_m\sigma^{mn}\psi_n)\bar{Y}\} \tag{6.2.38}$$

and

$$X_\alpha(1 - YK_Y)| = -\frac{i}{\sqrt{2}}K_{\mathscr{A}\bar{\mathscr{A}}}\bar{\Psi}^{\dot{\alpha}\bar{\mathscr{A}}}\bar{\sigma}^m_{\alpha\dot{\alpha}}\left(\nabla_m\Psi^{\mathscr{A}} - \frac{1}{\sqrt{2}}\psi_m{}^\varphi\Psi_\varphi^{\mathscr{A}}\right)$$
$$- \frac{\sqrt{2}}{8}\bar{\mathscr{D}}^2\bar{\phi}^{\bar{k}}|K_{\mathscr{A}\bar{k}}\Psi_\alpha^{\mathscr{A}} + \frac{1}{\sqrt{2}}HK_{\mathscr{A}\bar{Y}}\Psi_\alpha^{\mathscr{A}} - \frac{4i}{3\sqrt{2}}\varepsilon^{klmn}\partial_kC_{lmn}\Psi_\alpha^{\mathscr{A}}K_{\mathscr{A}\bar{Y}}$$
$$- \frac{1}{2\sqrt{2}}K_{\mathscr{A}\bar{\mathscr{B}}\bar{\mathscr{C}}}\bar{\Psi}_{\dot{\alpha}}^{\bar{\mathscr{C}}}\bar{\Psi}^{\dot{\alpha}\bar{\mathscr{B}}}\Psi_\alpha^{\mathscr{A}} + iK_k(\lambda_\alpha\cdot A)^k + \frac{i}{4}\Psi_\alpha^{\mathscr{A}}K_{\mathscr{A}\bar{Y}}(\bar{\psi}_m\bar{\sigma}^m\eta - \psi_m\sigma^m\bar{\eta})$$
$$- \frac{1}{2\sqrt{2}}\Psi_\alpha^{\mathscr{A}}K_{\mathscr{A}\bar{Y}}\{(\bar{M} + \bar{\psi}_m\bar{\sigma}^{mn}\bar{\psi}_n)Y - (M + \psi_m\sigma^{mn}\psi_n)\bar{Y}\}\ . \tag{6.2.39}$$

These are the component field expressions which are to be used in the transformation law of H (6.2.34). The same expressions will be needed later on in the construction of the invariant action.

[19] We make use, in the Yang–Mills sector, of the suggestive notations

$$K_{\bar{k}}(\bar{\lambda}^{\dot{\alpha}}\cdot\bar{A})^{\bar{k}} = \bar{\lambda}^{(r)\dot{\alpha}}K_{\bar{k}}(\mathbf{T}_{(r)}\bar{A})^{\bar{k}}, \quad K_k(\lambda_\alpha\cdot A)^k = \lambda_\alpha^{(r)}K_k(\mathbf{T}_{(r)}A)^k.$$

6.3. General action terms

In Section 4.5 we have explained in detail the construction of supersymmetric and $U_K(1)$ invariant component field Lagrangians starting from a generic chiral superfield $\mathbf{r}$ of $U_K(1)$ weight $w(\mathbf{r}) = +2$ and its complex conjugate $\bar{\mathbf{r}}$ of weight $w(\bar{\mathbf{r}}) = -2$. We will apply this construction to the case of 3-form superfields coupled to the supergravity/matter/Yang–Mills system. The generic Lagrangian – cf. (4.4.22) – is given as

$$\mathscr{L}(\mathbf{r},\bar{\mathbf{r}}) = \mathrm{e}(\mathbf{f}+\bar{\mathbf{f}}) + \frac{\mathrm{ie}}{\sqrt{2}}(\psi_m\sigma^m\bar{\mathbf{s}} + \bar{\psi}_m\bar{\sigma}^m\mathbf{s})$$

$$-\,\mathrm{e}\bar{\mathbf{r}}(M+\psi_m\sigma^{mn}\psi_n) - \mathrm{e}\mathbf{r}(\bar{M}+\bar{\psi}_m\bar{\sigma}^{mn}\bar{\psi}_n)\ . \tag{6.3.1}$$

Particular component field actions are then obtained by choosing $\mathbf{r}$ and $\bar{\mathbf{r}}$ appropriately. The complete action we are going to consider here will consist of three separately supersymmetric pieces,

$$\mathscr{L} = \mathscr{L}_{\text{supergravity}+\text{matter}} + \mathscr{L}_{\text{superpotential}} + \mathscr{L}_{\text{Yang–Mills}}\ . \tag{6.3.2}$$

In the following, we shall discuss one by one the three individual contributions to the total Lagrangian.

6.3.1. Supergravity and matter

The starting point is the same as in Section 4.5.1, we replace the generic superfield $\mathbf{r}$ with

$$\mathbf{r}_{\text{supergravity}+\text{matter}} = -3R\ . \tag{6.3.3}$$

The difference with Section 4.5.1 is that now the component field Lagrangian must be evaluated in the presence of the 3-form gauge field. As in Section 4.5.1 we decompose the supergravity/matter action such that

$$\mathscr{L}_{\text{supergravity}+\text{matter}} = \mathscr{L}_{\text{supergravity}} + e\mathbf{D}_{\text{matter}}\ , \tag{6.3.4}$$

where the pure supergravity part is given by the usual expression, i.e.

$$\mathscr{L}_{\text{supergravity}} = -\frac{\mathrm{e}}{2}\mathscr{R} + \frac{\mathrm{e}}{2}\varepsilon^{klmn}(\bar{\psi}_k\bar{\sigma}_l\mathscr{D}_m\psi_n - \psi_k\sigma_l\mathscr{D}_m\bar{\psi}_n) - \frac{\mathrm{e}}{3}\bar{M}M + \frac{\mathrm{e}}{3}b^a b_a,$$

except that the $U_K(1)$ covariant derivatives of the Rarita–Schwinger field contain now the new composite $U_K(1)$ connection as defined above. For the matter part, the D-term matter component field $\mathbf{D}_{\text{matter}}$ is defined in (4.0.2) in terms of the $U_K(1)$ gaugino superfield X_α. We therefore have to evaluate the superfield $\mathscr{D}^\alpha X_\alpha$ in the presence of the 3-form multiplet, i.e. apply the spinor derivative to the superfield expression

$$2\mathrm{i}X_\alpha(1-YK_Y)=K_{\mathscr{A}\bar{\mathscr{A}}}\mathscr{D}^{\dot\alpha}\bar\Psi^{\bar{\mathscr{A}}}\nabla_{\alpha\dot\alpha}\Psi^{\mathscr{A}}-\frac{\mathrm{i}}{4}K_{\mathscr{A}\bar{\mathscr{A}}}\mathscr{D}_\alpha\Psi^{\mathscr{A}}\bar{\mathscr{D}}^2\bar\Psi^{\bar{\mathscr{A}}}$$

$$-\frac{\mathrm{i}}{4}K_{\mathscr{A}\bar{\mathscr{B}}\bar{\mathscr{C}}}\mathscr{D}_{\dot\alpha}\bar\Psi^{\bar{\mathscr{C}}}\bar{\mathscr{D}}^{\dot\alpha}\bar\Psi^{\bar{\mathscr{B}}}\mathscr{D}_\alpha\Psi^{\mathscr{A}}-2\mathrm{i}K_k(\mathscr{W}_\alpha\cdot\phi)^k\ . \tag{6.3.5}$$

Remember here, that we are using the space–time covariant derivative $\nabla_{\alpha\dot\alpha}$, which by definition does not depend on the superfield $G_{\alpha\dot\alpha}$. In full detail

$$\mathscr{D}_{\alpha\dot\alpha}\bar Y=\nabla_{\alpha\dot\alpha}\bar Y-3\mathrm{i}G_{\alpha\dot\alpha}\bar Y,\quad \mathscr{D}_{\alpha\dot\alpha}Y=\nabla_{\alpha\dot\alpha}Y+3\mathrm{i}G_{\alpha\dot\alpha}Y\ , \tag{6.3.6}$$

$$\mathscr{D}_{\alpha\dot\alpha}\mathscr{D}_\beta\bar Y=\nabla_{\alpha\dot\alpha}\mathscr{D}_\beta\bar Y-\frac{3\mathrm{i}}{2}G_{\alpha\dot\alpha}\mathscr{D}_\beta\bar Y,\quad \mathscr{D}_{\alpha\dot\alpha}\mathscr{D}_\beta Y=\nabla_{\alpha\dot\alpha}\mathscr{D}_\beta Y+\frac{3\mathrm{i}}{2}G_{\alpha\dot\alpha}\mathscr{D}_\beta Y\ . \tag{6.3.7}$$

In deriving the explicit expression for $\mathscr{D}^\alpha X_\alpha$, we make systematic use of this derivative, which somewhat simplifies the calculations and is useful when passing to the component field expression later on. In applying the spinor derivative to (6.3.5) it is convenient to make use of the following relations:

$$\mathscr{D}_\alpha\mathscr{D}_{\dot\alpha}\bar Y=-2\mathrm{i}\ \nabla_{\alpha\dot\alpha}\bar Y\ , \tag{6.3.8}$$

$$\mathscr{D}_\alpha\bar{\mathscr{D}}^2\bar Y=-4\mathrm{i}\ \nabla_{\alpha\dot\alpha}\mathscr{D}^{\dot\alpha}\bar Y+2G_{\alpha\dot\alpha}\mathscr{D}^{\dot\alpha}\bar Y-8X_\alpha\bar Y\ , \tag{6.3.9}$$

$$\mathscr{D}_\alpha\bar{\mathscr{D}}^2\bar\phi^{\bar k}=-4\mathrm{i}\ \nabla_{\alpha\dot\alpha}\mathscr{D}^{\dot\alpha}\bar\phi^{\bar k}+2G_{\alpha\dot\alpha}\mathscr{D}^{\dot\alpha}\bar\phi^{\bar k}+8(\mathscr{W}_\alpha\cdot\bar\phi)^{\bar k}\ . \tag{6.3.10}$$

In order to obtain a compact form for $\mathscr{D}^\alpha X_\alpha$, we introduce $K^{\bar{\mathscr{A}}\mathscr{A}}$ as the inverse of $K_{\mathscr{A}\bar{\mathscr{A}}}$ and we define

$$-4F^{\mathscr{A}}=\mathscr{D}^2\Psi^{\mathscr{A}}+\Gamma^{\mathscr{A}}{}_{\mathscr{B}\mathscr{C}}\mathscr{D}^\alpha\Psi^{\mathscr{B}}\mathscr{D}_\alpha\Psi^{\mathscr{C}}\ , \tag{6.3.11}$$

$$-4\bar F^{\bar{\mathscr{A}}}=\bar{\mathscr{D}}^2\bar\Psi^{\bar{\mathscr{A}}}+\bar\Gamma^{\bar{\mathscr{A}}}{}_{\bar{\mathscr{B}}\bar{\mathscr{C}}}\bar{\mathscr{D}}_{\dot\alpha}\bar\Psi^{\bar{\mathscr{B}}}\bar{\mathscr{D}}^{\dot\alpha}\bar\Psi^{\bar{\mathscr{C}}} \tag{6.3.12}$$

with

$$\Gamma^{\mathscr{A}}{}_{\mathscr{B}\mathscr{C}}=K^{\bar{\mathscr{A}}\mathscr{A}}K_{\bar{\mathscr{A}}\mathscr{B}\mathscr{C}},\quad \bar\Gamma^{\bar{\mathscr{A}}}{}_{\bar{\mathscr{B}}\bar{\mathscr{C}}}=K^{\bar{\mathscr{A}}\mathscr{A}}K_{\mathscr{A}\bar{\mathscr{B}}\bar{\mathscr{C}}}\ . \tag{6.3.13}$$

Moreover, we define the new covariant derivatives

$$\hat\nabla_{\alpha\dot\alpha}\mathscr{D}^\alpha\Psi^{\mathscr{A}}=\nabla_{\alpha\dot\alpha}\mathscr{D}^\alpha\Psi^{\mathscr{A}}+\Gamma^{\mathscr{A}}{}_{\mathscr{B}\mathscr{C}}\nabla_{\alpha\dot\alpha}\Psi^{\mathscr{B}}\mathscr{D}^\alpha\Psi^{\mathscr{C}}\ , \tag{6.3.14}$$

$$\hat\nabla_{\alpha\dot\alpha}\mathscr{D}^{\dot\alpha}\bar\Psi^{\bar{\mathscr{A}}}=\nabla_{\alpha\dot\alpha}\mathscr{D}^{\dot\alpha}\bar\Psi^{\bar{\mathscr{A}}}+\bar\Gamma^{\bar{\mathscr{A}}}{}_{\bar{\mathscr{B}}\bar{\mathscr{C}}}\nabla_{\alpha\dot\alpha}\bar\Psi^{\bar{\mathscr{B}}}\mathscr{D}^{\dot\alpha}\bar\Psi^{\bar{\mathscr{C}}}\ . \tag{6.3.15}$$

Then, the superfield expression of $\mathscr{D}^\alpha X_\alpha$ becomes simply

$$2\mathrm{i}\mathscr{D}^\alpha X_\alpha(1-\bar{Y}K_{\bar{Y}}) = 4\mathrm{i}\bar{Y}K_{\mathscr{A}\bar{Y}}X^\alpha\mathscr{D}_\alpha\Psi^{\mathscr{A}} + 4\mathrm{i}YK_{Y\bar{\mathscr{A}}}\bar{X}_{\dot\alpha}\bar{\mathscr{D}}^{\dot\alpha}\bar{\Psi}^{\bar{\mathscr{A}}}$$
$$-2\mathrm{i}K_{\mathscr{A}\bar{\mathscr{A}}}\nabla^{\alpha\dot\alpha}\bar{\Psi}^{\bar{\mathscr{A}}}\nabla_{\alpha\dot\alpha}\Psi^{\mathscr{A}} - 4\mathrm{i}K_{\mathscr{A}\bar{\mathscr{A}}}F^{\mathscr{A}}\bar{F}^{\bar{\mathscr{A}}} - K_{\mathscr{A}\bar{\mathscr{A}}}\bar{\mathscr{D}}^{\dot\alpha}\bar{\Psi}^{\bar{\mathscr{A}}}\hat{\nabla}_{\alpha\dot\alpha}\mathscr{D}^\alpha\Psi^{\mathscr{A}} - K_{\mathscr{A}\bar{\mathscr{A}}}\mathscr{D}^\alpha\Psi^{\mathscr{A}}\hat{\nabla}_{\alpha\dot\alpha}\bar{\mathscr{D}}^{\dot\alpha}\bar{\Psi}^{\bar{\mathscr{A}}}$$
$$-\frac{\mathrm{i}}{4}\mathscr{R}_{\mathscr{A}\mathscr{B}\bar{\mathscr{A}}\bar{\mathscr{B}}}\mathscr{D}^\alpha\Psi^{\mathscr{A}}\mathscr{D}_\alpha\Psi^{\mathscr{B}}\bar{\mathscr{D}}_{\dot\alpha}\bar{\Psi}^{\bar{\mathscr{A}}}\bar{\mathscr{D}}^{\dot\alpha}\bar{\Psi}^{\bar{\mathscr{B}}} - 3\mathrm{i}K_{\mathscr{A}\bar{\mathscr{A}}}\mathscr{D}^\alpha\Psi^{\mathscr{A}}\bar{\mathscr{D}}^{\dot\alpha}\bar{\Psi}^{\bar{\mathscr{A}}}G_{\alpha\dot\alpha} + 2\mathrm{i}K_{\bar{k}}(\mathscr{D}^\alpha\mathscr{W}_\alpha\cdot\bar{\phi})^{\bar{k}}$$
$$-4\mathrm{i}K_{k\bar{\mathscr{A}}}\bar{\mathscr{D}}_{\dot\alpha}\bar{\Psi}^{\bar{\mathscr{A}}}(\bar{\mathscr{W}}^{\dot\alpha}\cdot\phi)^k - 4\mathrm{i}K_{\mathscr{A}\bar{k}}\mathscr{D}^\alpha\Psi^{\mathscr{A}}(\mathscr{W}_\alpha\cdot\bar{\phi})^{\bar{k}}\,. \tag{6.3.16}$$

This looks indeed very similar to the usual case (4.2.13). One of the differences however is that the F-terms and their complex conjugates for the superfields Y and $\bar{Y}$ have special forms. So we obtain for the matter part

$$(1-YK_Y)\mathbf{D}_{\mathrm{Matter}} = -\sqrt{2}X^\alpha|\Psi^{\mathscr{A}}_\alpha\bar{Y}K_{\mathscr{A}\bar{Y}} - \sqrt{2}\bar{X}_{\dot\alpha}|\bar{\Psi}^{\dot\alpha\bar{\mathscr{A}}}YK_{Y\bar{\mathscr{A}}}$$
$$-g^{mn}K_{\mathscr{A}\bar{\mathscr{A}}}\nabla_m\Psi^{\mathscr{A}}\nabla_n\Psi^{\bar{\mathscr{A}}} + K_{\mathscr{A}\bar{\mathscr{A}}}F^{\mathscr{A}}\bar{F}^{\bar{\mathscr{A}}}$$
$$-\frac{\mathrm{i}}{2}K_{\mathscr{A}\bar{\mathscr{A}}}\bar{\Psi}^{\dot\alpha\bar{\mathscr{A}}}\sigma^m_{\alpha\dot\alpha}\hat{\nabla}_m\Psi^{\alpha\mathscr{A}} - \frac{\mathrm{i}}{2}K_{\mathscr{A}\bar{\mathscr{A}}}\Psi^{\alpha\mathscr{A}}\sigma^m_{\alpha\dot\alpha}\hat{\nabla}_m\bar{\Psi}^{\dot\alpha\bar{\mathscr{A}}}$$
$$+\tfrac{1}{4}\mathscr{R}_{\mathscr{A}\mathscr{B}\bar{\mathscr{A}}\bar{\mathscr{B}}}\Psi^{\alpha\mathscr{A}}\Psi^{\mathscr{B}}_\alpha\bar{\Psi}^{\bar{\mathscr{A}}}_{\dot\alpha}\bar{\Psi}^{\dot\alpha\bar{\mathscr{B}}} - \tfrac{1}{2}K_{\mathscr{A}\bar{\mathscr{A}}}\Psi^{\alpha\mathscr{A}}\bar{\Psi}^{\dot\alpha\bar{\mathscr{A}}}b_{\alpha\dot\alpha}$$
$$-\frac{1}{\sqrt{2}}(\bar{\psi}_m\bar{\sigma}^n\sigma^m\bar{\Psi}^{\bar{\mathscr{A}}})K_{\mathscr{A}\bar{\mathscr{A}}}\nabla_n\Psi^{\mathscr{A}} - \frac{1}{\sqrt{2}}(\psi_m\sigma^n\bar{\sigma}^m\Psi^{\mathscr{A}})K_{\mathscr{A}\bar{\mathscr{A}}}\nabla_n\bar{\Psi}^{\bar{\mathscr{A}}}$$
$$-(\psi_m\sigma^{mn}\Psi^{\mathscr{A}})K_{\mathscr{A}\bar{\mathscr{A}}}(\bar{\psi}_n\bar{\Psi}^{\bar{\mathscr{A}}}) - (\bar{\psi}_m\bar{\sigma}^{mn}\bar{\Psi}^{\bar{\mathscr{A}}})K_{\mathscr{A}\bar{\mathscr{A}}}(\psi_n\Psi^{\mathscr{A}})$$
$$-\tfrac{1}{2}K_{\mathscr{A}\bar{\mathscr{A}}}g^{mn}(\psi_m\Psi^{\mathscr{A}})(\bar{\psi}_n\bar{\Psi}^{\bar{\mathscr{A}}}) - \tfrac{1}{2}\mathbf{D}^{(r)}[K_k(\mathbf{T}_{(r)}A)^k + K_{\bar{k}}(\bar{A}\mathbf{T}_{(r)})^{\bar{k}}]$$
$$+\mathrm{i}\sqrt{2}K_{k\bar{\mathscr{A}}}\bar{\Psi}^{\bar{\mathscr{A}}}_{\dot\alpha}(\bar{\lambda}^{\dot\alpha}\cdot A)^k - \mathrm{i}\sqrt{2}K_{\mathscr{A}\bar{k}}\Psi^{\alpha\mathscr{A}}(\lambda_\alpha\cdot\bar{A})^{\bar{k}}$$
$$-\tfrac{1}{2}(\bar{\psi}_m\bar{\sigma}^m)^\alpha K_k(\lambda_\alpha\cdot A)^k + \tfrac{1}{2}(\psi_m\sigma^m)_{\dot\alpha}K_{\bar{k}}(\bar{\lambda}^{\dot\alpha}\cdot\bar{A})^{\bar{k}} \tag{6.3.17}$$

with the terms in the first line given as

$$-\sqrt{2}X^\alpha|\Psi^{\mathscr{A}}_\alpha\bar{Y}K_{\mathscr{A}\bar{Y}} - \sqrt{2}\bar{X}_{\dot\alpha}|\bar{\Psi}^{\dot\alpha\bar{\mathscr{A}}}YK_{Y\bar{\mathscr{A}}}$$
$$=\frac{1}{1-YK_Y}\Bigg[+\mathrm{i}\bar{Y}K_{\mathscr{B}\bar{Y}}K_{\mathscr{A}\bar{\mathscr{A}}}\Psi^{\alpha\mathscr{B}}\bar{\Psi}^{\dot\alpha\bar{\mathscr{B}}}\sigma^m_{\alpha\dot\alpha}\left(\nabla_m\Psi^{\mathscr{A}} - \frac{1}{\sqrt{2}}\psi_m{}^\varphi\Psi^{\mathscr{A}}_\varphi\right)$$
$$+\mathrm{i}YK_{Y\bar{\mathscr{B}}}K_{\mathscr{A}\bar{\mathscr{A}}}\bar{\Psi}^{\dot\alpha\bar{\mathscr{B}}}\Psi^{\alpha\mathscr{A}}\sigma^m_{\alpha\dot\alpha}\left(\nabla_m\bar{\Psi}^{\bar{\mathscr{A}}} - \frac{1}{\sqrt{2}}\bar{\psi}_{m\dot\varphi}\bar{\Psi}^{\dot\alpha\bar{\mathscr{A}}}\right)$$
$$-YK_{Y\bar{\mathscr{B}}}K_{\mathscr{A}\bar{\mathscr{A}}}\bar{\Psi}^{\bar{\mathscr{B}}}_{\dot\alpha}\bar{\Psi}^{\dot\alpha\bar{\mathscr{A}}}F^{\mathscr{A}} - \bar{Y}K_{\mathscr{B}\bar{Y}}K_{\mathscr{A}\bar{\mathscr{A}}}\Psi^{\alpha\mathscr{B}}\Psi^{\mathscr{A}}_\alpha\bar{F}^{\bar{\mathscr{A}}}$$
$$-\mathrm{i}\sqrt{2}\bar{Y}K_{\mathscr{A}\bar{Y}}\Psi^{\alpha\mathscr{A}}K_k(\lambda_\alpha\cdot A)^k + \mathrm{i}\sqrt{2}YK_{Y\bar{\mathscr{A}}}\bar{\Psi}^{\bar{\mathscr{A}}}_{\dot\alpha}K_{\bar{k}}(\bar{\lambda}^{\dot\alpha}\cdot\bar{A})^{\bar{k}}\Bigg]\,. \tag{6.3.18}$$

6.3.2. Superpotential

In the usual case where we consider only $U_K(1)$ inert superfields like ϕ^k and $\bar{\phi}^{\bar{k}}$, the Lagrangian is obtained from identifying the generic superfield $\mathbf{r}$ with

$$\mathbf{r}_{\text{superpotential}} = e^{K/2} W \ , \tag{6.3.19}$$

as in (4.5.9) of Section 4.5.2. In the present case the superfield W is allowed to depend on the 3-form superfield as well. As we wish to maintain the transformation $W(\phi) \mapsto e^{-F} W(\phi)$ for the more general superpotential $W(\phi, Y)$, we must proceed with care due to the non-zero weight of Y. In order to distinguish this more general situation from the usual case, we use the symbol $\mathscr{P}$ for the chiral superfield of weight $w(\mathscr{P}) = 2$, defined as

$$\mathscr{P} = e^{K/2} W(\phi, Y) = \sum e^{\alpha_n K/2} W_n(\phi) Y^n \ , \tag{6.3.20}$$

where we have allowed for a parameter α_n. What happens under a Kähler transformation? Assigning a holomorphic transformation law $W_n \mapsto e^{-\beta_n F} W_n$ to the coefficient superfields, we find

$$\mathscr{P} \mapsto e^{-i \operatorname{Im} F} \mathscr{P}$$

$$e^{\alpha_n K/2} W_n(\phi) Y^n \mapsto e^{(\alpha_n \operatorname{Re} F - \beta_n F - in \operatorname{Im} F)} e^{\alpha_n K/2} W_n(\phi) Y^n \ . \tag{6.3.21}$$

Consistency with the transformations of W and Y then requires $\alpha_n = \beta_n = 1 - n$, hence

$$\mathscr{P} = e^{K/2} \sum W_n(\phi) [e^{-K/2} Y]^n \ . \tag{6.3.22}$$

This suggest to define the superfields

$$y = e^{-K/2} Y, \quad \bar{y} = e^{-K/2} \bar{Y} \tag{6.3.23}$$

as the basic variables in the construction of the superpotential term, i.e.

$$\mathscr{P} = e^{K/2} W(\phi, y), \quad \bar{\mathscr{P}} = e^{K/2} \bar{W}(\bar{\phi}, \bar{y}) \ . \tag{6.3.24}$$

Note that, by construction, y transforms as a holomorphic section. We can now proceed with the construction of $\mathscr{L}_{\text{superpotential}}$, taking $\mathscr{P}$ as starting point in the canonical procedure.

We parametrize the covariant spinor derivatives of $\mathscr{P}$ such that

$$\mathscr{D}_\alpha \mathscr{P} = \Sigma_{\mathscr{A}} \mathscr{D}_\alpha \Psi^{\mathscr{A}} \tag{6.3.25}$$

and

$$\mathscr{D}^2 \mathscr{P} = -4 \Sigma_{\mathscr{A}} F^{\mathscr{A}} + \Sigma_{\mathscr{A}\mathscr{B}} \mathscr{D}^\alpha \Psi^{\mathscr{A}} \mathscr{D}_\alpha \Psi^{\mathscr{B}} \ . \tag{6.3.26}$$

The various components of the coefficients $\Sigma_{\mathscr{A}}$ and $\Sigma_{\mathscr{A}\mathscr{B}}$ are given as

$$\Sigma_k = e^{K/2}(W_k + K_k W) - Y W_y K_k \ , \tag{6.3.27}$$

$$\Sigma_Y = e^{K/2} W K_Y + W_y (1 - Y K_Y) \tag{6.3.28}$$

and

$$\Sigma_{kl} = (e^{K/2}W - YW_y)(K_{kl} + K_k K_l) - Y(W_{ky}K_l + W_{ly}K_k) + e^{K/2}(W_{kl} + W_k K_l + W_l K_k) + e^{-K/2}Y^2 K_k K_l W_{yy} - \Sigma_{\mathscr{A}}\Gamma^{\mathscr{A}}_{kl} , \tag{6.3.29}$$

$$\Sigma_{kY} = (e^{K/2}W - YW_y)(K_{kY} + K_k K_Y) + W_{ky}(1 - YK_Y) + e^{K/2}W_k K_Y - e^{-K/2}YK_k W_{yy}(1 - YK_Y) - \Sigma_{\mathscr{A}}\Gamma^{\mathscr{A}}{}_{kY} , \tag{6.3.30}$$

$$\Sigma_{YY} = (e^{K/2}W - YW_y)(K_{YY} + K_Y K_Y) + e^{-K/2}W_{yy}(1 - YK_Y)^2 - \Sigma_{\mathscr{A}}\Gamma^{\mathscr{A}}{}_{YY} . \tag{6.3.31}$$

Complex conjugate expressions are obtained from

$$\bar{\mathscr{P}} = e^{K/2}\bar{W}(\bar{\phi}, \bar{y}) \tag{6.3.32}$$

with $\bar{y} = e^{-K/2}\bar{Y}$. Making use of the superpotential superfield and the corresponding definitions given above one derives easily the component field expression

$$\frac{1}{e}\mathscr{L}_{\text{superpotential}} = \Sigma_{\mathscr{A}}F^{\mathscr{A}} - \frac{1}{2}\Sigma_{\mathscr{A}\mathscr{B}}\Psi^{\alpha\mathscr{A}}\Psi^{\mathscr{B}}_{\alpha} + \frac{i}{\sqrt{2}}\Sigma_{\mathscr{A}}(\bar{\psi}_m\bar{\sigma}^m\Psi^{\mathscr{A}}) - e^{K/2}W(\bar{M} + \bar{\psi}_m\bar{\sigma}^{mn}\bar{\psi}_n) + \text{h.c.} \tag{6.3.33}$$

6.3.3. Yang–Mills

Finally, the Yang–Mills action is obtained in replacing the generic superfield **r** with

$$\mathbf{r}_{\text{Yang–Mills}} = \tfrac{1}{4}f_{(r)(s)}\mathscr{W}^{(r)\alpha}\mathscr{W}^{(s)}_{\alpha} \tag{6.3.34}$$

in the same way as in (4.5.20) of Section 4.5.3. Assuming the gauge coupling functions to be independent of the 3-form superfields, the resulting component field expression has the same form as in (6.2.20), which we display here in the form

$$\begin{aligned}\frac{1}{e}\mathscr{L}_{\text{Yang–Mills}} = & -\frac{1}{4}f_{(r)(s)}[\boldsymbol{f}^{(r)mn}\boldsymbol{f}^{(s)}_{mn} + 2i\lambda^{(r)}\sigma^m\nabla_m\bar{\lambda}^{(s)} + 2i\bar{\lambda}^{(s)}\bar{\sigma}^m\nabla_m\lambda^{(r)} \\ & - 2\mathbf{D}^{(r)}\mathbf{D}^{(s)} + \frac{i}{2}\varepsilon^{klmn}\boldsymbol{f}^{(r)}_{kl}\boldsymbol{f}^{(s)}_{mn} - 2(\lambda^{(r)}\sigma^a\bar{\lambda}^{(s)})b_a] \\ & -\frac{1}{4}\frac{\partial f_{(r)(s)}}{\partial A^i}[\sqrt{2}(\chi^i\sigma^{mn}\lambda^{(r)})\boldsymbol{f}^{(s)}_{mn} - \sqrt{2}(\chi^i\lambda^{(r)})\mathbf{D}^{(s)} + (\lambda^{(r)}\lambda^{(s)})F^i] \\ & -\frac{1}{4}\frac{\partial \bar{f}_{(r)(s)}}{\partial \bar{A}^{\bar{\imath}}}[\sqrt{2}(\bar{\chi}^{\bar{\imath}}\bar{\sigma}^{mn}\bar{\lambda}^{(r)})\boldsymbol{f}^{(s)}_{mn} - \sqrt{2}(\bar{\chi}^{\bar{\imath}}\bar{\lambda}^{(r)})\mathbf{D}^{(s)} + (\bar{\lambda}^{(r)}\bar{\lambda}^{(s)})\bar{F}^{\bar{\imath}}]\end{aligned}$$

$$+\frac{1}{8}\left(\frac{\partial^2 f_{(r)(s)}}{\partial A^k \partial A^l} - \frac{\partial f_{(r)(s)}}{\partial A^i} \Gamma^i{}_{kl}\right)(\chi^k \chi^l)(\lambda^{(r)} \lambda^{(s)})$$

$$+\frac{1}{8}\left(\frac{\partial^2 \bar{f}_{(r)(s)}}{\partial \bar{A}^{\bar{k}} \partial \bar{A}^{\bar{l}}} - \frac{\partial \bar{f}_{(r)(s)}}{\partial \bar{A}^{\bar{\imath}}} \bar{\Gamma}^{\bar{\imath}}{}_{\bar{k}\bar{l}}\right)(\bar{\chi}^{\bar{k}} \bar{\chi}^{\bar{l}})(\bar{\lambda}^{(r)} \bar{\lambda}^{(s)})$$

$$\text{plus } \psi_m, \bar{\psi}_m \text{ dependent terms} . \tag{6.3.35}$$

In the covariant derivatives of the gauginos

$$\nabla_m \lambda_\alpha^{(r)} = \partial_m \lambda_\alpha^{(r)} - \omega_{m\alpha}{}^{\varphi} \lambda_\varphi^{(r)} + v_m \lambda_\alpha^{(r)} - \boldsymbol{a}_m^{(t)} \lambda_\alpha^{(s)} c_{(s)(t)}{}^{(r)} , \tag{6.3.36}$$

$$\nabla_m \bar{\lambda}^{(r)\dot{\alpha}} = \partial_m \bar{\lambda}^{(r)\dot{\alpha}} - \omega_m{}^{\dot{\alpha}}{}_{\dot{\varphi}} \bar{\lambda}^{(r)\dot{\varphi}} - v_m \bar{\lambda}^{(r)\dot{\alpha}} - \boldsymbol{a}_m^{(t)} \bar{\lambda}^{(s)\dot{\alpha}} c_{(s)(t)}{}^{(r)} , \tag{6.3.37}$$

defined as in (4.2.15) and (4.2.16) the composite Kähler connection is now given in terms of (6.2.22), displaying the dependence on the 3-form multiplet. The Yang–Mills field strength tensor is given as usual

$$\boldsymbol{f}_{mn}^{(r)} = \partial_m \boldsymbol{a}_n^{(r)} - \partial_n \boldsymbol{a}_m^{(r)} + \boldsymbol{a}_m^{(s)} \boldsymbol{a}_n^{(t)} c_{(s)(t)}{}^{(r)} . \tag{6.3.38}$$

6.3.4. *Solving for the auxiliary fields*

Although this is standard stuff, we detail the calculations to make clear some subtleties related to the inclusion of the 3-form. In the different pieces of the whole Lagrangian, we isolate the contributions containing auxiliary fields and proceed sector by sector as much as possible.

Diagonalization in b_a makes use of the terms

$$\Lambda_b = \tfrac{1}{3} b^a b_a - \tfrac{1}{2} M_{\mathscr{A}\bar{\mathscr{A}}} (\Psi^{\mathscr{A}} \sigma^a \Psi^{\bar{\mathscr{A}}}) b_a + \tfrac{1}{2} f_{(r)(s)} (\lambda^{(r)} \sigma^a \bar{\lambda}^{(s)}) b_a \tag{6.3.39}$$

with

$$M_{\mathscr{A}\bar{\mathscr{A}}} = \frac{1}{1 - Y K_Y} K_{\mathscr{A}\bar{\mathscr{A}}} , \tag{6.3.40}$$

whereas the relevant terms for the Yang–Mills auxiliary sector are

$$\Lambda_{\mathbf{D}} = \frac{1}{2} f_{(r)(s)} \mathbf{D}^{(r)} \mathbf{D}^{(s)} + \frac{1}{1 - Y K_Y} \mathbf{D}^{(s)} K_{\bar{l}} (\bar{A} \mathbf{T}_{(s)})^{\bar{l}}$$

$$+ \frac{\sqrt{2}}{4} \mathbf{D}^{(s)} \left(\frac{\partial f_{(r)(s)}}{\partial A^k} (\chi^k \lambda^{(r)}) + \frac{\partial \bar{f}_{(r)(s)}}{\partial \bar{A}^{\bar{k}}} (\bar{\chi}^{\bar{k}} \bar{\lambda}^{(r)}) \right) . \tag{6.3.41}$$

The F-terms of chiral matter and the 3-form appear in the general form

$$\Lambda_{F,\bar{F}} = F^{\mathscr{A}} M_{\mathscr{A}\bar{\mathscr{A}}} \bar{F}^{\bar{\mathscr{A}}} + F^{\mathscr{A}} P_{\mathscr{A}} + \bar{P}_{\bar{\mathscr{A}}} \bar{F}^{\bar{\mathscr{A}}} \tag{6.3.42}$$

with the definitions

$$P_k = \Sigma_k - \frac{1}{4}\frac{\partial f_{(r)(s)}}{\partial A^k}(\lambda^{(r)}\lambda^{(s)}) - Y M_{Y\bar{\mathscr{B}}} M_{k\bar{\mathscr{A}}} \bar{\Psi}^{\bar{\mathscr{B}}}_{\dot{\alpha}} \bar{\Psi}^{\dot{\alpha}\bar{\mathscr{A}}} , \tag{6.3.43}$$

$$P_Y = \Sigma_Y - Y M_{Y\bar{\mathscr{B}}} M_{Y\bar{\mathscr{A}}} \bar{\Psi}^{\bar{\mathscr{B}}}_{\dot{\alpha}} \bar{\Psi}^{\dot{\alpha}\bar{\mathscr{A}}} . \tag{6.3.44}$$

We write this expression as

$$\Lambda_{F,\bar{F}} = \mathscr{F}^k M_{k\bar{k}} \bar{\mathscr{F}}^{\bar{k}} - \bar{P}_{\bar{\mathscr{A}}} M^{\bar{\mathscr{A}}\mathscr{A}} P_{\mathscr{A}} + \mathscr{F}^Y \frac{1}{M^{\bar{Y}Y}} \bar{\mathscr{F}}^{\bar{Y}} , \tag{6.3.45}$$

where $M^{\bar{\mathscr{A}}\mathscr{A}}$ is the inverse of $M_{\mathscr{A}\bar{\mathscr{A}}}$ and in particular

$$\frac{1}{M^{\bar{Y}Y}} = M_{Y\bar{Y}} - M_{Y\bar{k}} \mathfrak{M}^{\bar{k}k} M_{k\bar{Y}} \tag{6.3.46}$$

with $\mathfrak{M}^{\bar{k}k}$ the inverse of the submatrix $M_{k\bar{k}}$, related to the usual Kähler metric. Moreover,

$$\mathscr{F}^k = F^k + (\bar{P}_{\bar{k}} + F^Y M_{Y\bar{k}}) \mathfrak{M}^{\bar{k}k} , \tag{6.3.47}$$

$$\bar{\mathscr{F}}^{\bar{k}} = \bar{F}^{\bar{k}} + \mathfrak{M}^{\bar{k}k}(P_k + M_{k\bar{Y}} \bar{F}^{\bar{Y}}) \tag{6.3.48}$$

and

$$\mathscr{F}^Y = F^Y + \bar{P}_{\bar{\mathscr{A}}} M^{\bar{\mathscr{A}}Y}, \quad \bar{\mathscr{F}}^{\bar{Y}} = \bar{F}^{\bar{Y}} + M^{\bar{Y}\mathscr{A}} P_{\mathscr{A}} . \tag{6.3.49}$$

We use now the particular structure of the 3-form multiplet to further specify these F-terms. Using (6.2.18), (6.2.19), (6.3.11) and (6.3.12) we parametrize

$$\mathscr{F}^Y = H + \mathrm{i}\left(\Delta + \frac{\bar{M}Y - M\bar{Y}}{2\mathrm{i}}\right) + f^Y , \tag{6.3.50}$$

$$\bar{\mathscr{F}}^{\bar{Y}} = H - \mathrm{i}\left(\Delta + \frac{\bar{M}Y - M\bar{Y}}{2\mathrm{i}}\right) + \bar{f}^{\bar{Y}} \tag{6.3.51}$$

with

$$f^Y = -\tfrac{1}{4}\Gamma^Y{}_{\mathscr{B}\mathscr{C}}\, \mathscr{D}^\alpha \Psi^{\mathscr{B}} \mathscr{D}_\alpha \Psi^{\mathscr{C}} + \bar{P}_{\bar{\mathscr{A}}} M^{\bar{\mathscr{A}}Y} , \tag{6.3.52}$$

$$\bar{f}^{\bar{Y}} = -\tfrac{1}{4}\bar{\Gamma}^{\bar{Y}}{}_{\bar{\mathscr{B}}\bar{\mathscr{C}}}\, \mathscr{D}_{\dot{\alpha}} \bar{\Psi}^{\bar{\mathscr{B}}} \mathscr{D}^{\dot{\alpha}} \bar{\Psi}^{\bar{\mathscr{C}}} + M^{\bar{Y}\mathscr{A}} P_{\mathscr{A}} , \tag{6.3.53}$$

as well as

$$\Delta = \frac{4}{3}\varepsilon^{klmn}\partial_k C_{lmn} - \frac{1}{2\sqrt{2}}(\bar{\psi}_m \bar{\sigma}^m \eta - \psi_m \sigma^m \bar{\eta}) + \frac{1}{2\mathrm{i}}[(\bar{\psi}_m \bar{\sigma}^{mn} \bar{\psi}_n) Y - (\psi_m \sigma^{mn} \psi_n) \bar{Y}] . \tag{6.3.54}$$

In terms of these notations the last term in (6.3.45) takes then the form

$$\mathscr{F}^{Y}\frac{1}{M^{\bar{Y}Y}}\bar{\mathscr{F}}^{\bar{Y}}=\frac{1}{M^{\bar{Y}Y}}\left(H+\frac{f^{Y}+\bar{f}^{\bar{Y}}}{2}\right)^{2}+\frac{1}{M^{\bar{Y}Y}}\left(\Delta+\frac{\bar{M}Y-M\bar{Y}}{2\mathrm{i}}+\frac{f^{Y}-\bar{f}^{\bar{Y}}}{2\mathrm{i}}\right)^{2}. \tag{6.3.55}$$

In this equation the last term makes a contribution to the sector $M,\bar{M}$ and the 3-form we consider next. Except for this term, the sum of $\Lambda_b,\Lambda_{\mathbf{D}},\Lambda_{F,\bar{F}}$ will give rise to the diagonalized expression

$$\begin{aligned}\frac{1}{\mathrm{e}}\mathscr{L}(F^k,\bar{F}^{\bar{k}},b_a,\mathbf{D}^{(r)},H)&=\frac{1}{3}\hat{b}_a\hat{b}^a+\frac{1}{2}\hat{\mathbf{D}}^{(r)}f_{(r)(s)}\hat{\mathbf{D}}^{(s)}+\mathscr{F}^kM_{k\bar{k}}\bar{\mathscr{F}}^{\bar{k}}\\&+\frac{1}{M^{\bar{Y}Y}}\left(H+\frac{f^{Y}+\bar{f}^{\bar{Y}}}{2}\right)^{2}-\frac{3}{16}\mathbf{B}_a\mathbf{B}^a-\frac{1}{2}\mathbf{D}_{(r)}(f^{-1})^{(r)(s)}\mathbf{D}_{(s)}-\bar{P}_{\bar{\mathscr{A}}}M^{\bar{\mathscr{A}}\mathscr{A}}P_{\mathscr{A}}\,,\end{aligned} \tag{6.3.56}$$

where $\hat{b}_a=b_a+\mathbf{B}_a$ with

$$\mathbf{B}_a=-M_{\mathscr{A}\bar{\mathscr{A}}}(\Psi^{\mathscr{A}}\sigma_a\bar{\Psi}^{\bar{\mathscr{A}}})+f_{(r)(s)}(\lambda^{(r)}\sigma_a\bar{\lambda}^{(s)}) \tag{6.3.57}$$

and $\hat{\mathbf{D}}^{(r)}=\mathbf{D}^{(r)}+(f^{-1})^{(r)(s)}\mathbf{D}_{(s)}$ with

$$\mathbf{D}_{(r)}=-\frac{1}{1-YK_Y}K_k(\mathbf{T}_{(r)}A)^k+\frac{\sqrt{2}}{4}\left(\frac{\partial f_{(r)(s)}}{\partial A^k}(\chi^k\lambda^{(s)})+\frac{\partial\bar{f}_{(r)(s)}}{\partial\bar{A}^{\bar{k}}}(\bar{\chi}^{\bar{k}}\bar{\lambda}^{(s)})\right). \tag{6.3.58}$$

Use of the equations of motion simply sets to zero the first four terms, leaving for the Lagrangian

$$\begin{aligned}\frac{1}{\mathrm{e}}\mathscr{L}&=-\frac{3}{16}\mathbf{B}_a\mathbf{B}^a-\frac{1}{2}\mathbf{D}_{(r)}(f^{-1})^{(r)(s)}\mathbf{D}_{(s)}-\bar{P}_{\bar{Y}}\frac{1}{M_{Y\bar{Y}}}P_Y\\&-\left(\bar{P}_{\bar{k}}-\bar{P}_{\bar{Y}}\frac{M_{Y\bar{k}}}{M_{Y\bar{Y}}}\right)M^{\bar{k}k}\left(P_k-\frac{M_{k\bar{Y}}}{M_{Y\bar{Y}}}P_Y\right),\end{aligned} \tag{6.3.59}$$

where we have block diagonalized $M^{\bar{\mathscr{A}}\mathscr{A}}$.

As to the $M,\bar{M}$ dependent terms of the full action we observe that they are intricately entangled with the field strength tensor of the 3-form, a novel structure compared to the usual supergravity–matter couplings. The relevant terms for this sector are identified to be

$$\Lambda_{M,\bar{M}}=3\mathrm{e}^K|W|^2-\tfrac{1}{3}|M+3\mathrm{e}^{K/2}W|^2+\frac{1}{M^{\bar{Y}Y}}\left[\Delta-\frac{1}{2\mathrm{i}}(M\bar{Y}-\bar{M}Y)+\frac{1}{2\mathrm{i}}(f^Y-\bar{f}^{\bar{Y}})\right]^2. \tag{6.3.60}$$

One recognizes in the first two terms the usual superpotential contributions whereas the last term is new. This expression contains all the terms of the full action which depend on M, $\bar{M}$ or the 3-form C_{klm}. The question we have to answer is how far the M, $\bar{M}$ sector and the 3-form sector can be disentangled, if at all. Clearly, the dynamical consequences of this structure deserve careful investigation.

The 3-form contribution is not algebraic, so we cannot use the solution of its equation of motion in the Lagrangian [54]. One way out is to derive the equations of motion and look for an equivalent Lagrangian giving rise to the same equations of motion. Explicitly, we obtain for the 3-form

$$\partial_k \left\{ \frac{1}{M^{\bar{Y}Y}} \left[\Delta - \frac{1}{2\mathrm{i}}(M\bar{Y} - \bar{M}Y) + \frac{1}{2\mathrm{i}}(f^Y - \bar{f}^{\bar{Y}}) \right] \right\} = 0 , \tag{6.3.61}$$

solved by setting

$$\frac{1}{M^{\bar{Y}Y}} \left[\Delta - \frac{1}{2\mathrm{i}}(M\bar{Y} - \bar{M}Y) + \frac{1}{2\mathrm{i}}(f^Y - \bar{f}^{\bar{Y}}) \right] = c , \tag{6.3.62}$$

where c is a real constant. Then the e.o.m.'s for M and $\bar{M}$ read

$$M + 3\mathrm{e}^{K/2} W = -3\mathrm{i}cY, \quad \bar{M} + 3\mathrm{e}^{K/2} \bar{W} = 3\mathrm{i}c\bar{Y} . \tag{6.3.63}$$

At last, we consider the e.o.m. for e.g. $\bar{Y}$, in which we denote by $\mathscr{L}(\bar{Y})$ the many contributions of $\bar{Y}$ to the Lagrangian, except for $\Lambda_{M,\bar{M}}$,

$$\partial_m \frac{\delta \mathscr{L}(\bar{Y})}{\delta \partial_m \bar{Y}} - \frac{\delta \mathscr{L}(\bar{Y})}{\delta \bar{Y}} - \frac{\delta \Lambda_{M,\bar{M}}}{\delta \bar{Y}} = 0 . \tag{6.3.64}$$

Using (3.42) and (3.43) the last term takes the form

$$\begin{aligned} \frac{\delta \Lambda_{M,\bar{M}}}{\delta \bar{Y}} = \frac{\delta}{\delta \bar{Y}} \Big\{ & 3\mathrm{e}^K |W + \mathrm{i}cy|^2 - c^2 M^{\bar{Y}Y} - \mathrm{i}c(f^Y - \bar{f}^{\bar{Y}}) \\ & - \mathrm{i}c[(\bar{\psi}_m \bar{\sigma}^{mn} \bar{\psi}_n)Y - (\psi_m \sigma^{mn} \psi_n)\bar{Y}] - \frac{c}{\sqrt{2}}(\bar{\psi}_m \bar{\sigma}^m \eta - \psi_m \sigma^m \bar{\eta}) \Big\} . \end{aligned} \tag{6.3.65}$$

This suggests that the equations of motion can be derived from an equivalent Lagrangian obtained by dropping the 3-form contribution and shifting the superpotential W to $W + \mathrm{i}cy$. This can be seen more clearly by restricting our attention to the scalar degrees of freedom as in the next section.

6.3.5. The scalar potential

The analysis presented above allows to obtain the scalar potential of the theory as

$$\begin{aligned} V = & \left(\bar{\Sigma}_{\bar{k}} - (\bar{\Sigma}_{\bar{Y}} - \mathrm{i}c) \frac{M_{Y\bar{k}}}{M_{Y\bar{Y}}} \right) M^{\bar{k}k} \left(\Sigma_k - \frac{M_{k\bar{Y}}}{M_{Y\bar{Y}}} (\Sigma_Y + \mathrm{i}c) \right) \\ & + (\bar{\Sigma}_{\bar{Y}} - \mathrm{i}c) \frac{1}{M_{Y\bar{Y}}} (\Sigma_Y + \mathrm{i}c) - 3\mathrm{e}^K |W + \mathrm{i}cy|^2 \\ & + \frac{1}{2} \frac{1}{1 - YK_Y} K_{\bar{k}} (\mathbf{T}_{(r)} \bar{A})^{\bar{k}} (f^{-1})^{(r)(s)} \frac{1}{1 - YK_Y} K_k \, (\mathbf{T}_{(s)} A)^k . \end{aligned} \tag{6.3.66}$$

We note that the shift $W \mapsto W + \mathrm{i}cy$ induces $\Sigma_k \mapsto \Sigma_k$ and $\Sigma_Y \mapsto \Sigma_Y + \mathrm{i}c$, which are precisely the combinations which appear in (6.3.66).

In fact (6.3.66) is nothing but the scalar potential of some matter fields ϕ^k of Kähler weight 0 plus a field $Y = y\mathrm{e}^{K/2}$ of Kähler weight 2 with a superpotential $W + \mathrm{i}cy$ in the usual formulation of supergravity. In order to show this, let us consider y and $\bar{y}$ as our new field variables and define

$$K(Y, \phi, \bar{Y}, \bar{\phi}) = \mathscr{K}(y, \phi, \bar{y}, \bar{\phi}) , \tag{6.3.67}$$

Taking as an example the Kähler potential in footnote 18 with $Z = 1$, we find

$$y = Y(X + Y\bar{Y})^{-1/2}, \quad \bar{y} = \bar{Y}(X + Y\bar{Y})^{-1/2} \tag{6.3.68}$$

and therefore

$$\mathscr{K}(y, \bar{y}) = \log X(\phi, \bar{\phi}) - \log(1 - y\bar{y}) . \tag{6.3.69}$$

which is a typical Kähler potential with $SU(1,1)$ non-compact symmetry.

We can express the matrix $M_{\mathscr{A}\bar{\mathscr{A}}}$ and its inverse $M^{\bar{\mathscr{A}}\mathscr{A}}$ in terms of the derivatives of $\mathscr{K}$, namely $\mathscr{K}_{\mathscr{A}\bar{\mathscr{A}}}$ and of its inverse $\mathscr{K}^{\bar{\mathscr{A}}\mathscr{A}}$ ($\mathscr{A}$ denotes k, y as well as k, Y depending on the context). Then it appears that the expression of the scalar potential becomes very simple as we use the relevant relations. Indeed, using the following definitions

$$\hat{W} = W + \mathrm{i}cy, \quad D_{\mathscr{A}}\hat{W} = \hat{W}_{\mathscr{A}} + \mathscr{K}_{\mathscr{A}}\hat{W} , \tag{6.3.70}$$

we obtain

$$V = \mathrm{e}^{\mathscr{K}}(D_{\bar{\mathscr{A}}}\hat{W}\mathscr{K}^{\bar{\mathscr{A}}\mathscr{A}}D_{\mathscr{A}}\hat{W} - 3|\hat{W}|^2) + \tfrac{1}{2}\mathscr{K}_{\bar{k}}(\mathbf{T}_{(r)}\bar{A})^{\bar{k}}(f^{-1})^{(r)(s)}\mathscr{K}_k(\mathbf{T}_{(s)}A)^k , \tag{6.3.71}$$

which is the familiar expression of the scalar potential of the scalar fields ϕ^k and y in the standard formulation of supergravity.

7. Conclusion

Since the upsurge of supersymmetry, a number of formalisms have been developed in order to cope with the notorious complexity of this Fermi–Bose symmetry, in particular in the context of supergravity, for a sample of review articles see for instance [45,130,131,63,62,64,65,107,31,44]. Among these formalisms are tensor calculus, the superconformal compensator method and the group manifold approach. It would be an interesting undertaking to establish explicitly the relation among these different approaches and to superspace geometry, which is however certainly beyond the scope of this report.

Methods of superspace geometry are convenient in the discussion of the conceptual aspects of supersymmetric theories and useful in the derivation of component field expressions and have a wide range of applications.

In this report we have focused on the Kähler superspace approach to the construction of the general couplings of matter and Yang–Mills theory to supergravity. As a solid understanding of this subject is central for further applications and developments, we have made an effort to present the conceptual foundations and the technical ramifications in full detail. In order to demonstrate the way the geometrical formulation works, we included a detailed description of the couplings of linear and 3-form multiplets to supergravity.

There are other topics, which have been discussed in this geometric context, but which are not included in this report. Among them are the algebraic description of anomalies in supersymmetric theories [93] and the construction of the geometric BRS transformations [11].

We also refrained from a discussion of conformal supergravity and the construction of curvature-squared terms and supersymmetric topological invariants. Gravitational Chern–Simons forms, which are closely related to the 3-form geometry presented here, and their coupling to linear multiplets have a rather transparent formulation in the geometric context.

Let us also mention the systematic description of the alternative incarnations of supergravity, new minimal and non-minimal, in the framework of superspace geometry in relation with the identification of the reducible multiplet.

Finally, we have restricted ourselves to $N = 1$, $D = 4$ supersymmetry. Superspace geometry has been widely employed in the investigations of extended and higher and lower dimensional supersymmetry.

The methods discussed in this report have a potential interest for discussing effective superstring field theories and have been extensively used in this respect. We discuss in what follows some of these potential applications.

As stressed in Section 5.1, the linear multiplet plays a central role in the field theory limit of superstring theories. Its bosonic component consists of a scalar field associated with dilatation symmetry, the dilaton, and of a pseudoscalar field which has many properties in common with an axion field. Its fermionic component, sometimes called the dilatino, may be a component of the goldstino field whose presence in a supersymmetric theory is the sign of the spontaneous breakdown of supersymmetry.

The close connections of dilatation symmetry with the vanishing of the cosmological constant, of axionic couplings with the cancellation of chiral anomalies and of the goldstino with the super-Higgs mechanism certainly make the dilaton–axion–dilatino set of fields a system worthy of detailed studies. Supergravity theories provide the natural setting for such studies, given the intimate connections noted above with gravity and supersymmetry (the dilaton as a Brans–Dicke scalar, the dilatino associated with the possible breaking of local supersymmetry).

In the effective four-dimensional supergravity theory of weakly coupled 10-D string theories, the axion field does not appear as such in the spectrum. Indeed, the massless string modes include a dilaton and an antisymmetric tensor which, together with a dilatino spinor field, form a linear multiplet which plays an important role in the effective field theory. As we have seen in Section 5.1, a supersymmetric duality transformation relates this linear supermultiplet to a chiral supermultiplet [111] whose content includes the original scalar field as well as the pseudoscalar (with axion-like couplings) dual to the antisymmetric tensor.[20] However, such a transformation only establishes a relationship on shell and some relevant properties or some transparence might be lost or hidden in the chiral supermultiplet formulation. Moreover, in the context of superstring theories it appears that it is the linear multiplet, L, which plays the role of string loop expansion parameter. Therefore,

[20] Such a duality transformation may be related to a string duality in the case of some moduli fields.

stringy corrections (perturbative and non-perturbative) are naturally parametrized by L, which then allows to disentangle purely stringy effects from field theoretical ones. This is very clear in the study of gauge coupling renormalization and gaugino condensation in superstring effective theories (see below).

A classical example is the way modular invariance is realized at the quantum level in these theories. This invariance involves transformations of the moduli fields, which are described by chiral superfields in the case of a weakly coupled string theory. The corresponding invariance is realized through some Kähler transformation. The simplest example is the case of a single superfield T with Kähler potential $K(T,\bar{T}) = -3\ln(T+\bar{T})$. The modular transformation is then simply a $SL(2,Z)$ symmetry:

$$T \mapsto \frac{aT - \mathrm{i}b}{\mathrm{i}cT + d}, \quad ab - cd = 1, \quad a,b,c,d \in Z \,, \tag{7.1}$$

which amounts to the Kähler transformation

$$K \mapsto K + F + \bar{F}, \quad \text{with } F = 3\ln(\mathrm{i}cT + d) \,. \tag{7.2}$$

This invariance is violated by radiative corrections generated by quantum loops of massless particles [112,29,48]. These anomalies are cancelled by two types of counterterms. The first one is model independent and is a four-dimensional version [29,48] of the Green–Schwarz [96] anomaly cancellation mechanism. As is well known, this mechanism makes use of the presence of the antisymmetric tensor and thus, in four dimensions, it involves the linear multiplet L. The other part [52] which is model-dependent involves string threshold corrections depending on the moduli fields.

These terms play an important role when one discusses issues such as supersymmetry breaking. For example, in the classical scenario of gaugino condensation, it proves to be very useful, in order to take into account these important one-loop effects, to make a supersymmetric description of the dynamics in terms of the dilaton linear multiplet. It turns out [24,14] that, in the effective theories below the scale of condensation, a single vector superfield V incorporates the degrees of freedom of the original linear multiplet L as well as the gaugino and gauge field condensates. The one-loop terms discussed above, i.e. Green–Schwarz counterterm and moduli-dependent string threshold corrections, play an important dynamical role [15–17] in this mechanism.

As we see, one-loop terms play a crucial role in all these applications. Since supergravity is not a renormalizable theory, great care must be used in the regularization procedure. In a major effort, Gaillard and collaborators [79,113,114,77,78] have used Pauli–Villars regulators (carefully chosen not to break supersymmetry nor the symmetries of the theory) to compute the full one-loop corrections to the supergravity effective superstring theories theory in the Kähler superspace formalism.

Similar to the duality between a rank-2 antisymmetric tensor and a pseudoscalar, a rank-3 antisymmetric tensor is dual to a constant scalar field. Indeed, such a relation was considered some time ago in connection with the cosmological constant problem [104,36,58,54]. As we have seen in Section 6, the role of supersymmetry is striking when one considers the rank-3 antisymmetric tensor. Whereas in the non-supersymmetric case such a field does not correspond to any physical degree of freedom (through its equation of motion, its field strength is a constant 4-form), supersymmetry couples it with propagating fields. Indeed, the 3-form supermultiplet [82] can be described by a chiral superfield Y and an antichiral field $\bar{Y}$ subject to a further constraint (6.2.14)

$$\frac{8i}{3}\varepsilon^{dcba}\Sigma_{dcba} = (\mathscr{D}^2 - 24R^\dagger)Y - (\bar{\mathscr{D}}^2 - 24R)\bar{Y} , \tag{7.3}$$

where Σ is the gauge-invariant field strength of the rank-3 gauge potential superfield, C_{klm}, i.e. $\Sigma = dC$. Its superpartners, identified as component fields of the (anti)chiral superfield Y and $\bar{Y}$, are propagating. Supersymmetry couples the rank-3 antisymmetric tensor with *dynamical* degrees of freedom, while respecting the gauge invariance associated with the 3-form. Let us emphasize (see Appendix F) that Y is not a general chiral superfield since it must obey the constraint above (7.3), which is possible only if Y derives from a pre-potential Ω which is real:

$$\bar{Y} = -4(\mathscr{D}^2 - 8R^\dagger)\Omega, \quad Y = -4(\bar{\mathscr{D}}^2 - 8R)\Omega . \tag{7.4}$$

Rank-3 antisymmetric tensors might play an important role in several problems of interest, connected with string theories. One of them is the breaking of supersymmetry through gaugino condensation. Indeed, as we have noted above, the composite degrees of freedom are described, in the effective theory below the scale of condensation, by a vector superfield V which incorporates also the components of the fundamental linear multiplet L. The chiral superfield

$$U = -(\bar{\mathscr{D}}^2 - 8R)V , \tag{7.5}$$

has the same quantum numbers (in particular the same Kähler weight) as the superfield $W^\alpha W_\alpha$. Its scalar component, for instance, is interpreted as the gaugino condensate.

Alternatively, the vector superfield is interpreted as a "fossil" Chern–Simons field [14,13] which includes the fundamental degrees of freedom of the dilaton supermultiplet. It can be considered as a pre-potential for the chiral superfield U as in (7.4).

Another interesting appearance of the 3-form supermultiplet occurs in the context of strong-weak coupling duality. More precisely, the dual formulation of 10-D supergravity [34,85,86,120,89] appears as an effective field theory of some dual formulation of string models, such as 5-branes [55,148,56,57,26,25,51]. The Yang–Mills field strength which is a 7-form in 10 dimensions may precisely yield in 4 dimensions a 4-form field strength.

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Appendix A. Technicalities

We collect here some definitions, conventions and identities involving quantities which are frequently used in superspace calculations. We do not aim at any rigorous presentation but try to provide a compendium of formulae and relations which appear useful when performing explicit computations. We use essentially the same conventions as [153], *except* for ε_{0123} and σ^0 defined with opposite signs.

A.1. Superforms toolkit

Coordinates of curved superspace are denoted $z^M = (x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}})$ and differential elements $dz^M = (dx^m, d\theta^\mu, d\bar{\theta}_{\dot{\mu}})$, with their wedge product ($\wedge$ is understood)

$$dz^M dz^N = -(-)^{mn} dz^N dz^M \,, \tag{A.1.1}$$

m, n are the gradings of the indices M, N: 0 for the vector ones, 1 for the spinors. We define p-superforms with the following ordering convention

$$\Omega_p = \frac{1}{p!} dz^{M_1} \ldots dz^{M_p} \Omega_{M_p \ldots M_1} \,. \tag{A.1.2}$$

The coefficients $\Omega_{M_p \ldots M_1}$ are superfields and graded antisymmetric tensors in their indices, i.e.

$$\Omega_{M_1 \ldots M_i \ldots M_j \ldots M_p} = -(-)^{m_i m_j} (-)^{(m_i + m_j)(m_{j-1} + \cdots + m_i)} \Omega_{M_1 \ldots M_j \ldots M_i \ldots M_p} \,. \tag{A.1.3}$$

In agreement with (A.1.2), we define the wedge product of two (super)forms as follows:

$$\Omega_p \Omega_q = \frac{1}{p!q!} dz^{M_1} \ldots dz^{M_p} \Omega_{M_p \ldots M_1} dz^{N_1} \ldots dz^{N_q} \Omega_{N_q \ldots N_1}$$

$$= \frac{1}{p!q!} dz^{M_1} \ldots dz^{M_p} dz^{N_1} \ldots dz^{N_q} \Omega_{N_q \ldots N_1} \Omega_{M_p \ldots M_1} \,. \tag{A.1.4}$$

The exterior derivative, $d = dz^M \partial_M$ such that $d^2 = 0$, transforms a p-superform into a $(p+1)$-superform

$$d\Omega_p = \frac{1}{p!} dz^{M_1} \ldots dz^{M_p} dz^L \partial_L \Omega_{M_p \ldots M_1} \tag{A.1.5}$$

and obeys the Leibniz rule

$$d(\Omega_p \Omega_q) = \Omega_p d\Omega_q + (-)^q d\Omega_p \Omega_q \,. \tag{A.1.6}$$

The interior product, denoted $\imath_\xi$, transforms a p-superform into a $(p-1)$-superform, it depends on a vector field, e.g. ξ, with which one operates the contraction

$$\imath_\xi dz^M = \xi^M \Rightarrow \imath_\xi \Omega_p = \frac{1}{(p-1)!} dz^{M_1} \ldots dz^{M_{p-1}} \xi^{M_p} \Omega_{M_p \ldots M_1} \; . \tag{A.1.7}$$

Using the analogue of Cartan's local frame we can define quantities in the local flat tangent superspace (flat indices are traditionally $A, B, \ldots, H$; $A = a, \alpha, \dot\alpha$)

$$E^A = dz^M E_M{}^A(z), \quad dz^M = E^A E_A{}^M(z) \; . \tag{A.1.8}$$

$E_M{}^A(z)$ is called the (super)vielbein and $E_A{}^M(z)$ its inverse, they fulfill

$$E_M{}^A(z) E_A{}^N(z) = \delta_M{}^N, \quad E_A{}^M(z) E_M{}^B(z) = \delta_A{}^B \; . \tag{A.1.9}$$

The E^A's are the basis 1-forms in the tangent superspace. As we defined superforms on the dz^M basis, we can equally well define them on the E^A basis

$$\Omega_p = \frac{1}{p!} E^{A_1} \ldots E^{A_p} \Omega_{A_p \ldots A_1} \tag{A.1.10}$$

and $d = E^A D_A$. As above

$$\imath_\xi E^A = \xi^A \Rightarrow \imath_\xi \Omega_p = \frac{1}{(p-1)!} E^{A_1} \ldots E^{A_{p-1}} \xi^{A_p} \Omega_{A_p \ldots A_1} \; . \tag{A.1.11}$$

Relating the coefficients in one basis to the ones in the other implies the occurrence of many vielbeins or their inverses, e.g. for a 2-form

$$B = \tfrac{1}{2} dz^M dz^N B_{NM} = \tfrac{1}{2} E^A E_A{}^M E^B E_B{}^N B_{NM} = (-)^{b(m+a)} \tfrac{1}{2} E^A E^B E_A{}^M E_B{}^N B_{NM} \; , \tag{A.1.12}$$

so that

$$B_{BA} = (-)^{b(m+a)} E_A{}^M E_B{}^N B_{NM} \; ,$$
$$B_{NM} = (-)^{n(m+a)} E_M{}^A E_N{}^B B_{BA} \; . \tag{A.1.13}$$

A.2. Basic quantities in $SO(1,3)$ and $SL(2,C)$

In our notations, the metric tensor η_{ab} with $a, b = 0, 1, 2, 3$ is defined as

$$[\eta_{ab}] = \mathrm{diag}(-1, +1, +1, +1) \tag{A.2.1}$$

with inverse

$$\eta_{ac} \eta^{cb} = \delta_a{}^b \; . \tag{A.2.2}$$

The totally antisymmetric symbol ε_{abcd} is normalized such that

$$\varepsilon_{0123} = +1, \quad \varepsilon^{0123} = -1 . \tag{A.2.3}$$

The product of two ε-symbols is given as

$$\varepsilon^{abcd}\varepsilon_{efgh} = -\delta^{abcd}_{efgh} , \tag{A.2.4}$$

where the multi-index Kronecker delta is defined as

$$\delta^{abcd}_{efgh} \equiv \det[\delta^i_j] \tag{A.2.5}$$

with $i = a, b, c, d$ and $j = e, f, g, h$. In somewhat more explicit notation this can be written as

$$\delta^{abcd}_{efgh} = \delta^a_e \delta^{bcd}_{fgh} - \delta^a_f \delta^{bcd}_{ghe} + \delta^a_g \delta^{bcd}_{hef} - \delta^a_h \delta^{bcd}_{efg} , \tag{A.2.6}$$

$$\delta^{bcd}_{fgh} = \delta^b_f \delta^{cd}_{gh} + \delta^b_g \delta^{cd}_{hf} + \delta^b_h \delta^{cd}_{fg} , \tag{A.2.7}$$

$$\delta^{cd}_{gh} = \delta^c_g \delta^d_h - \delta^c_h \delta^d_g . \tag{A.2.8}$$

Accordingly, the respective contractions of indices yield

$$\varepsilon^{abcd}\varepsilon_{efgd} = -\delta^{abc}_{efg} , \tag{A.2.9}$$

$$\varepsilon^{abcd}\varepsilon_{efcd} = -2\delta^{ab}_{ef} , \tag{A.2.10}$$

$$\varepsilon^{abcd}\varepsilon_{ebcd} = -6\delta^a_e , \tag{A.2.11}$$

$$\varepsilon^{abcd}\varepsilon_{abcd} = -24 . \tag{A.2.12}$$

In curved space we use the totally antisymmetric tensor ε_{klmn}, defined by

$$\varepsilon_{klmn} = e_k{}^a e_l{}^b e_m{}^c e_n{}^d \varepsilon_{abcd} \tag{A.2.13}$$

with $e_k{}^a$ the moving frame. $SL(2, C)$ spinors carry undotted and dotted indices, $\alpha = 1, 2$ and $\dot{\alpha} = \dot{1}, \dot{2}$. For the case of undotted indices, the symbol $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$ is defined by

$$\varepsilon_{21} = \varepsilon^{12} = +1 . \tag{A.2.14}$$

As a consequence one has

$$\varepsilon^{\alpha\beta}\varepsilon_{\gamma\delta} = -\delta^\alpha_\gamma \delta^\beta_\delta + \delta^\alpha_\delta \delta^\beta_\gamma , \tag{A.2.15}$$

$$\varepsilon^{\alpha\beta}\varepsilon_{\beta\delta} = \delta^\alpha_\delta \tag{A.2.16}$$

together with the cyclic identity (indices β, γ, δ)

$$\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} + \varepsilon_{\alpha\delta}\varepsilon_{\beta\gamma} + \varepsilon_{\alpha\gamma}\varepsilon_{\delta\beta} = 0 . \tag{A.2.17}$$

Exactly the same definitions and identities hold if undotted indices are replaced by dotted ones, i.e. for the symbol $\varepsilon_{\dot{\alpha}\dot{\beta}} = -\varepsilon_{\dot{\beta}\dot{\alpha}}$.

The ε-symbols serve to lower and raise spinor indices. For a two-component spinor ψ_α, we define

$$\psi^\beta = \varepsilon^{\beta\alpha}\psi_\alpha, \quad \psi_\beta = \varepsilon_{\beta\alpha}\psi^\alpha . \tag{A.2.18}$$

The cyclic identity implies

$$\varepsilon_{\alpha\beta}\psi_\gamma + \varepsilon_{\gamma\alpha}\psi_\beta + \varepsilon_{\beta\gamma}\psi_\alpha = 0 . \tag{A.2.19}$$

Again, exactly the same relations hold for dotted indices. The standard convention for summation over spinor indices is

$$\psi\chi = \chi\psi = \psi^\alpha\chi_\alpha, \quad \bar\psi\bar\chi = \bar\chi\bar\psi = \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} . \tag{A.2.20}$$

The antisymmetric combination of a product of two Weyl spinors is given in terms of the ε-symbols as

$$\psi_\alpha\chi_\beta - \psi_\beta\chi_\alpha = +\varepsilon_{\alpha\beta}\psi^\varphi\chi_\varphi , \tag{A.2.21}$$

$$\bar\psi_{\dot\alpha}\bar\chi_{\dot\beta} - \bar\psi_{\dot\beta}\bar\chi_{\dot\alpha} = -\varepsilon_{\dot\alpha\dot\beta}\bar\psi_{\dot\varphi}\bar\chi^{\dot\varphi} . \tag{A.2.22}$$

Tensors $V_{\alpha\dot\alpha}$ with a pair of undotted and dotted spinor indices are equivalent to vectors V_a. The explicit relation is defined in terms of the σ-matrices, which carry the index structure $\sigma^a_{\alpha\dot\alpha}$, i.e.

$$V_{\alpha\dot\alpha} = \sigma^a_{\alpha\dot\alpha}V_a . \tag{A.2.23}$$

They are defined as

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.2.24}$$

We frequently use also the $\bar\sigma$-matrices,

$$\bar\sigma^{a\dot\alpha\alpha} = \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}\sigma^a_{\beta\dot\beta} = -(\varepsilon\sigma^a\varepsilon)^{\alpha\dot\alpha} \tag{A.2.25}$$

with numerical entries such that

$$\bar\sigma^0 = \sigma^0, \quad \bar\sigma^{1,2,3} = -\sigma^{1,2,3} . \tag{A.2.26}$$

As a consequence of (A.2.25) we have also

$$(\sigma^a\varepsilon)_\alpha{}^{\dot\alpha} = (\bar\sigma^a\varepsilon)^{\dot\alpha}{}_\alpha, \quad (\varepsilon\sigma^a)^\alpha{}_{\dot\alpha} = (\varepsilon\bar\sigma^a)_{\dot\alpha}{}^\alpha . \tag{A.2.27}$$

These matrices form a Clifford algebra, i.e.

$$(\sigma^a\bar\sigma^b + \sigma^b\bar\sigma^a)_\alpha{}^\beta = -2\eta^{ab}\delta_\alpha{}^\beta , \tag{A.2.28}$$

$$(\bar\sigma^a\sigma^b + \bar\sigma^b\sigma^a)^{\dot\alpha}{}_{\dot\beta} = -2\eta^{ab}\delta^{\dot\alpha}{}_{\dot\beta} . \tag{A.2.29}$$

The products of two σ-matrices can be written as

$$\sigma^a\bar{\sigma}^b = -\eta^{ab} + 2\sigma^{ab} , \tag{A.2.30}$$

$$\bar{\sigma}^a\sigma^b = -\eta^{ab} + 2\bar{\sigma}^{ab} . \tag{A.2.31}$$

The traceless antisymmetric combinations appearing here are defined as

$$(\sigma^{ab})_\alpha{}^\beta = \tfrac{1}{4}(\sigma^a\bar{\sigma}^b - \sigma^b\bar{\sigma}^a)_\alpha{}^\beta , \tag{A.2.32}$$

$$(\bar{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}} = \tfrac{1}{4}(\bar{\sigma}^a\sigma^b - \bar{\sigma}^b\sigma^a)^{\dot{\alpha}}{}_{\dot{\beta}} . \tag{A.2.33}$$

They are self-dual (resp. antiself-dual), i.e.

$$\varepsilon_{abcd}\sigma^{cd} = -2\mathrm{i}\sigma_{ab}, \quad \varepsilon_{abcd}\bar{\sigma}^{cd} = +2\mathrm{i}\bar{\sigma}_{ab} \tag{A.2.34}$$

and satisfy (as a consequence of vanishing trace)

$$(\varepsilon\sigma^{ab}\varepsilon)^\beta{}_\alpha = -(\sigma^{ab})_\alpha{}^\beta, \quad (\varepsilon\bar{\sigma}^{ab}\varepsilon)_{\dot{\beta}}{}^{\dot{\alpha}} = -(\bar{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}} , \tag{A.2.35}$$

$$(\varepsilon\sigma^{ab})^{\alpha\beta} = (\varepsilon\sigma^{ab})^{\beta\alpha}, \quad (\varepsilon\bar{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}} = (\varepsilon\bar{\sigma}^{ab})_{\dot{\beta}\dot{\alpha}} . \tag{A.2.36}$$

Other useful identities involving two σ-matrices are

$$\mathrm{tr}(\sigma^a\bar{\sigma}^b) = -2\eta^{ab} , \tag{A.2.37}$$

$$\sigma^a_{\alpha\dot{\alpha}}\bar{\sigma}^{\dot{\beta}\beta}_a = -2\delta^\beta_\alpha\delta^{\dot{\beta}}_{\dot{\alpha}} , \tag{A.2.38}$$

$$\sigma^a_{\alpha\dot{\alpha}}\sigma_{a\beta\dot{\beta}} = -2\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}} , \tag{A.2.39}$$

$$\bar{\sigma}^{a\dot{\alpha}\alpha}\bar{\sigma}^{\dot{\beta}\beta}_a = -2\varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}} , \tag{A.2.40}$$

which may be viewed as special cases of the "Fierz" reshuffling

$$\begin{aligned}\sigma^a_{\alpha\dot{\alpha}}\sigma^b_{\beta\dot{\beta}} = &-\tfrac{1}{2}\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\eta^{ab} + \varepsilon_{\dot{\alpha}\dot{\beta}}(\sigma^{ab}\varepsilon)_{\alpha\beta} + \varepsilon_{\alpha\beta}(\varepsilon\bar{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}} \\ &+ (\sigma^a{}_f\varepsilon)_{\alpha\beta}(\varepsilon\bar{\sigma}^{bf})_{\dot{\alpha}\dot{\beta}} + (\sigma^b{}_f\varepsilon)_{\alpha\beta}(\varepsilon\bar{\sigma}^{af})_{\dot{\alpha}\dot{\beta}} .\end{aligned} \tag{A.2.41}$$

As to the products of three σ-matrices, useful identities are

$$(\sigma^{ab}\sigma^c)_{\alpha\dot{\gamma}} = \tfrac{1}{2}(\eta^{ac}\eta^{bd} - \eta^{bc}\eta^{ad} + \mathrm{i}\varepsilon^{abcd})\sigma_{d\alpha\dot{\gamma}} , \tag{A.2.42}$$

$$(\sigma^a\bar{\sigma}^{bc})_{\alpha\dot{\gamma}} = \tfrac{1}{2}(\eta^{ac}\eta^{bd} - \eta^{ab}\eta^{cd} + \mathrm{i}\varepsilon^{abcd})\sigma_{d\alpha\dot{\gamma}} , \tag{A.2.43}$$

$$(\bar{\sigma}^{ab}\bar{\sigma}^c)^{\dot{\alpha}\gamma} = \tfrac{1}{2}(\eta^{ac}\eta^{bd} - \eta^{bc}\eta^{ad} - \mathrm{i}\varepsilon^{abcd})\bar{\sigma}^{\dot{\alpha}\gamma}_d , \tag{A.2.44}$$

$$(\bar{\sigma}^a\sigma^{bc})^{\dot{\alpha}\gamma} = \tfrac{1}{2}(\eta^{ac}\eta^{bd} - \eta^{ab}\eta^{cd} - \mathrm{i}\varepsilon^{abcd})\bar{\sigma}^{\dot{\alpha}\gamma}_d \tag{A.2.45}$$

and

$$(\sigma^a\bar{\sigma}^b\sigma^c)_{\alpha\dot{\gamma}} = (-\eta^{ab}\eta^{cd} + \eta^{ca}\eta^{bd} - \eta^{bc}\eta^{ad} + \mathrm{i}\varepsilon^{abcd})\sigma_{d\alpha\dot{\gamma}} \;, \tag{A.2.46}$$

$$(\bar{\sigma}^a\sigma^b\bar{\sigma}^c)^{\dot{\alpha}\gamma} = (-\eta^{ab}\eta^{cd} + \eta^{ca}\eta^{bd} - \eta^{bc}\eta^{ad} - \mathrm{i}\varepsilon^{abcd})\bar{\sigma}_d^{\dot{\alpha}\gamma} \;. \tag{A.2.47}$$

In explicit computations we also made repeated use of the relations

$$\sigma_{b\beta\dot{\beta}}(\sigma^{ab})_\alpha{}^\varphi = -\delta_\beta^\varphi\sigma^a_{\alpha\dot{\beta}} + \tfrac{1}{2}\delta_\alpha^\varphi\sigma^a_{\beta\dot{\beta}} \;, \tag{A.2.48}$$

$$\sigma_{b\beta\dot{\beta}}(\bar{\sigma}^{ab})^{\dot{\varphi}}{}_{\dot{\alpha}} = +\delta_{\dot{\beta}}^{\dot{\varphi}}\sigma^a_{\beta\dot{\alpha}} - \tfrac{1}{2}\delta_{\dot{\alpha}}^{\dot{\varphi}}\sigma^a_{\beta\dot{\beta}} \;, \tag{A.2.49}$$

$$\bar{\sigma}_b^{\dot{\beta}\beta}(\bar{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\varphi}} = -\delta_{\dot{\varphi}}^{\dot{\beta}}\bar{\sigma}^{a\dot{\alpha}\beta} + \tfrac{1}{2}\delta_{\dot{\varphi}}^{\dot{\alpha}}\bar{\sigma}^{a\dot{\beta}\beta} \;, \tag{A.2.50}$$

$$\bar{\sigma}_b^{\dot{\beta}\beta}(\sigma^{ab})_\varphi{}^\alpha = +\delta_\varphi^\beta\bar{\sigma}^{a\dot{\beta}\alpha} - \tfrac{1}{2}\delta_\varphi^\alpha\bar{\sigma}^{a\dot{\beta}\beta} \;, \tag{A.2.51}$$

$$\mathrm{tr}(\sigma^{ab}\sigma^{cd}) = -\tfrac{1}{2}(\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc} + \mathrm{i}\varepsilon^{abcd}) \;, \tag{A.2.52}$$

$$\mathrm{tr}(\bar{\sigma}^{ab}\bar{\sigma}^{cd}) = -\tfrac{1}{2}(\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc} - \mathrm{i}\varepsilon^{abcd}) \;, \tag{A.2.53}$$

$$[\sigma^{ab}, \sigma^{cd}] = \eta^{ac}\sigma^{bd} - \eta^{ad}\sigma^{bc} - \eta^{bc}\sigma^{ad} + \eta^{bd}\sigma^{ac} \;, \tag{A.2.54}$$

$$\{\sigma^{ab}, \sigma^{cd}\}_\alpha{}^\beta = \mathrm{tr}(\sigma^{ab}\sigma^{cd})\delta_\alpha^\beta \;, \tag{A.2.55}$$

$$(\varepsilon\sigma^{ab})^{\alpha\beta}(\sigma_{ab}\varepsilon)_{\gamma\delta} = -\delta_\gamma^\alpha\delta_\delta^\beta - \delta_\delta^\alpha\delta_\gamma^\beta \;, \tag{A.2.56}$$

$$-\frac{\mathrm{i}}{4!}\varepsilon_{abcd}(\sigma^a\bar{\sigma}^b\sigma^c\bar{\sigma}^d)_\alpha{}^\beta = \delta_\alpha^\beta, \quad \frac{\mathrm{i}}{4!}\varepsilon_{abcd}(\bar{\sigma}^a\sigma^b\bar{\sigma}^c\sigma^d)^{\dot{\alpha}}{}_{\dot{\beta}} = \delta_{\dot{\beta}}^{\dot{\alpha}} \;. \tag{A.2.57}$$

Finally, let us note that

$$\oint_{lmn}\oint_{\dot{\alpha}\dot{\beta}\dot{\gamma}}\sigma^l_{\alpha\dot{\alpha}}(\varepsilon\bar{\sigma}^{mn})_{\dot{\beta}\dot{\gamma}} = 0 \tag{A.2.58}$$

with cyclic permutations on vector and spinor indices.
In the Weyl basis the Dirac matrices are given by

$$\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \bar{\sigma}^a & 0 \end{pmatrix}. \tag{A.2.59}$$

A Majorana spinor Ψ is made of a Weyl spinor χ_α with two components, $\alpha = 1, 2$ and of its complex conjugate $\bar{\chi}^{\dot{\alpha}}, \dot{\alpha} = \dot{1}, \dot{2}$:

$$\Psi_M = \begin{pmatrix} \chi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}, \tag{A.2.60}$$

$$\bar{\Psi}_M = (\chi^\alpha, \bar{\chi}_{\dot{\alpha}}) \;. \tag{A.2.61}$$

A Dirac spinor is made of two different Weyl spinors, $\chi_\alpha, \bar{\varphi}^{\dot{\alpha}}$,

$$\Psi_D = \begin{pmatrix} \chi_\alpha \\ \bar{\varphi}^{\dot{\alpha}} \end{pmatrix}, \quad \bar{\Psi}_D = (\varphi^\alpha, \bar{\chi}_{\dot{\alpha}}) . \tag{A.2.62}$$

In the Lagrangian calculations we need to know conjugation rules

$$\begin{aligned} (\psi_1 \sigma^m \bar{\psi}_2)^\dagger &= -(\bar{\psi}_1 \bar{\sigma}^m \psi_2) = +(\psi_2 \sigma^m \bar{\psi}_1) , \\ (\psi_1 \sigma^{mn} \psi_2)^\dagger &= +(\bar{\psi}_1 \bar{\sigma}^{mn} \bar{\psi}_2) = -(\bar{\psi}_2 \bar{\sigma}^{mn} \bar{\psi}_1) \end{aligned} \tag{A.2.63}$$

and some Fierz relations

$$\begin{aligned} (\psi_1 \psi_2)(\chi_1 \chi_2) &= -\tfrac{1}{2}(\psi_1 \chi_1)(\psi_2 \chi_2) , \\ (\psi_1 \psi_2)(\bar{\chi}_1 \bar{\chi}_2) &= -\tfrac{1}{2}(\psi_1 \sigma^m \bar{\chi}_1)(\psi_2 \sigma_m \bar{\chi}_2) , \\ \psi_\alpha \bar{\chi}_{\dot{\beta}} &= -\tfrac{1}{2}\sigma^m_{\alpha\dot{\beta}}(\psi \sigma_m \bar{\chi}) . \end{aligned} \tag{A.2.64}$$

A.3. Spinor notations for tensors

We can convert vector indices into spinor indices and vice versa using σ and $\bar{\sigma}$ matrices:

$$V_{\alpha\dot{\alpha}} = \sigma^a_{\alpha\dot{\alpha}} V_a , \tag{A.3.1}$$

$$V_a = -\tfrac{1}{2}\bar{\sigma}_a^{\dot{\alpha}\alpha} V_{\alpha\dot{\alpha}} . \tag{A.3.2}$$

So the scalar product of two vectors writes

$$T_a V^a = -\tfrac{1}{2} T_{\alpha\dot{\alpha}} V^{\alpha\dot{\alpha}} . \tag{A.3.3}$$

Tensors $T_{\alpha\beta}$, $\bar{T}_{\dot{\alpha}\dot{\beta}}$ with two spinor indices have the standard decompositions

$$T_{\alpha\beta} = +\varepsilon_{\alpha\beta} T + T_{\underset{\smile}{\alpha\beta}} , \tag{A.3.4}$$

$$\bar{T}_{\dot{\alpha}\dot{\beta}} = -\varepsilon_{\dot{\alpha}\dot{\beta}} \bar{T} + \bar{T}_{\underset{\smile}{\dot{\alpha}\dot{\beta}}} \tag{A.3.5}$$

with

$$T = \tfrac{1}{2} T^\alpha{}_\alpha, \quad \bar{T} = \tfrac{1}{2} \bar{T}_{\dot{\alpha}}{}^{\dot{\alpha}} \tag{A.3.6}$$

and

$$T_{\underset{\smile}{\alpha\beta}} = \tfrac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha}) , \tag{A.3.7}$$

$$\bar{T}_{\underset{\smile}{\dot{\alpha}\dot{\beta}}} = \tfrac{1}{2}(\bar{T}_{\dot{\alpha}\dot{\beta}} + \bar{T}_{\dot{\beta}\dot{\alpha}}) . \tag{A.3.8}$$

For an antisymmetric tensor with two indices, like $F_{ba} = -F_{ab}$, in spinor notations we have

$$F_{\beta\dot\beta\alpha\dot\alpha} = \sigma^a_{\alpha\dot\alpha}\sigma^b_{\beta\dot\beta}F_{ba} = [(\sigma^{ba}\varepsilon)_{\beta\alpha}\varepsilon_{\dot\beta\dot\alpha} + (\varepsilon\bar\sigma^{ba})_{\dot\beta\dot\alpha}\varepsilon_{\beta\alpha}]F_{ba} . \tag{A.3.9}$$

Using the standard decomposition

$$F_{\beta\dot\beta\alpha\dot\alpha} \equiv -2\varepsilon_{\beta\alpha}F_{\underset{\smile}{\dot\beta\dot\alpha}} + 2\varepsilon_{\dot\beta\dot\alpha}F_{\underset{\smile}{\beta\alpha}} , \tag{A.3.10}$$

we obtain

$$F_{\underset{\smile}{\beta\alpha}} = +\tfrac{1}{2}(\sigma^{ba}\varepsilon)_{\beta\alpha}F_{ba} , \tag{A.3.11}$$

$$F_{\underset{\smile}{\dot\beta\dot\alpha}} = -\tfrac{1}{2}(\varepsilon\bar\sigma^{ba})_{\dot\beta\dot\alpha}F_{ba} \tag{A.3.12}$$

and vice versa

$$F_{ba} = (\bar\sigma_{ba}\varepsilon)^{\dot\beta\dot\alpha}F_{\underset{\smile}{\dot\alpha\dot\beta}} - (\varepsilon\sigma_{ba})^{\beta\alpha}F_{\underset{\smile}{\alpha\beta}} . \tag{A.3.13}$$

As a consequence, the kinetic term reads

$$F^{ba}F_{ba} = 2F_{\underset{\smile}{\beta\alpha}}F^{\underset{\smile}{\beta\alpha}} + 2F_{\underset{\smile}{\dot\beta\dot\alpha}}F^{\underset{\smile}{\dot\beta\dot\alpha}} . \tag{A.3.14}$$

One often uses the dual tensor defined as

$$*F^{dc} = \tfrac{1}{2}\varepsilon^{dcba}F_{ba} , \tag{A.3.15}$$

whose spinor components are

$$*F^{\delta\dot\delta\gamma\dot\gamma} = 2\mathrm{i}\varepsilon^{\dot\delta\dot\gamma}F^{\underset{\smile}{\delta\gamma}} + 2\mathrm{i}\varepsilon^{\delta\gamma}F^{\underset{\smile}{\dot\delta\dot\gamma}} . \tag{A.3.16}$$

The topological combination $*F^{ba}F_{ba}$ takes the form

$$*F^{ba}F_{ba} = 2\mathrm{i}F_{\underset{\smile}{\dot\beta\dot\alpha}}F^{\underset{\smile}{\dot\beta\dot\alpha}} - 2\mathrm{i}F^{\underset{\smile}{\beta\alpha}}F_{\underset{\smile}{\beta\alpha}} . \tag{A.3.17}$$

Along the same lines, for a symmetric tensor with two indices, $S_{ba} = S_{ab}$, one has the decomposition

$$S_{\beta\dot\beta\alpha\dot\alpha} \equiv \varepsilon_{\beta\alpha}\varepsilon_{\dot\beta\dot\alpha}S + S_{\underset{\smile}{\beta\alpha}\,\underset{\smile}{\dot\beta\dot\alpha}} = -\tfrac{1}{2}\varepsilon_{\beta\alpha}\varepsilon_{\dot\beta\dot\alpha}S^b_b + 2(\sigma^b{}_f\varepsilon)_{\beta\alpha}(\varepsilon\bar\sigma^{af})_{\dot\beta\dot\alpha}S_{ba} . \tag{A.3.18}$$

Finally, for a three-index, antisymmetric tensor, say H_{cba}, the spinor structure is most easily analyzed using its dual tensor, $*H^d$, defined as

$$*H^d = \frac{1}{3!}\varepsilon^{dcba}H_{cba}, \quad H_{abc} = \varepsilon_{abcd}*H^d . \tag{A.3.19}$$

Then owing to the spinor expression for the ε-symbol

$$\varepsilon_{\delta\dot{\delta}\ \gamma\dot{\gamma}\ \beta\dot{\beta}\ \alpha\dot{\alpha}} = 4\mathrm{i}(\varepsilon_{\delta\gamma}\varepsilon_{\beta\alpha}\varepsilon_{\dot{\delta}\dot{\beta}}\varepsilon_{\dot{\gamma}\dot{\alpha}} - \varepsilon_{\dot{\delta}\dot{\gamma}}\varepsilon_{\dot{\beta}\dot{\alpha}}\varepsilon_{\delta\beta}\varepsilon_{\gamma\alpha}) , \tag{A.3.20}$$

one obtains

$$H_{\gamma\dot{\gamma}\ \beta\dot{\beta}\ \alpha\dot{\alpha}} = 2\mathrm{i}(\varepsilon_{\dot{\gamma}\dot{\beta}}\varepsilon_{\gamma\alpha}{}^*H_{\beta\dot{\alpha}} - \varepsilon_{\gamma\beta}\varepsilon_{\dot{\gamma}\dot{\alpha}}{}^*H_{\alpha\dot{\beta}}) . \tag{A.3.21}$$

Appendix B. Elements of $U(1)$ superspace

As we have seen in the main text, $U(1)$ superspace provides the underlying structure for the geometric description of the supergravity/matterYang–Mills system. Matter fields are incorporated through well-defined specifications in the $U(1)$ gauge sector, leading to Kähler superspace geometry. Very often, however, explicit calculations are done to a large extend without taking into account the special features of Kähler superspace. For this reason we found it useful to provide a compact account of the properties of $U(1)$ superspace.

B.1. General definitions

The basic superfields are the supervielbein $E_M{}^A(z)$, the Lorentz connection $\phi_{MB}{}^A(z)$ and the gauge potential $A_M(z)$ for chiral $U(1)$ transformations. These superfields are coefficients of 1-forms in superspace,

$$E^A = dz^M E_M{}^A(z) , \tag{B.1.1}$$

$$\phi_B^A = dz^M \phi_{MB}{}^A(z) , \tag{B.1.2}$$

$$A = dz^M A_M(z) . \tag{B.1.3}$$

Torsion curvatures and $U(1)$ field strengths are then defined as

$$T^A = dE^A + E^B\phi_B{}^A + w(E^A)E^A A , \tag{B.1.4}$$

$$R_B{}^A = d\phi_B{}^A + \phi_B{}^C\phi_C{}^A , \tag{B.1.5}$$

$$F = dA . \tag{B.1.6}$$

The chiral $U(1)$ weights $w(E^A)$ are given as

$$w(E^a) = 0, \quad w(E^\alpha) = 1, \quad w(E_{\dot{\alpha}}) = -1 . \tag{B.1.7}$$

Torsion, Lorentz curvature and $U(1)$ field strength are 2-forms in superspace,

$$T^A = \tfrac{1}{2}E^B E^C T_{CB}{}^A , \tag{B.1.8}$$

$$R_B^A = \tfrac{1}{2}E^C E^D R_{DCB}{}^A , \tag{B.1.9}$$

$$F = \tfrac{1}{2}E^C E^D F_{DC} . \tag{B.1.10}$$

They satisfy Bianchi identities

$$\mathscr{D}T^A - E^B R_B{}^A - w(E^A)E^A F = 0 \ . \tag{B.1.11}$$

A more explicit form of the Bianchi identities is

$$\oint_{(DCB)} (\mathscr{D}_D T_{CB}{}^A + T_{DC}{}^F T_{FB}{}^A - R_{DCB}{}^A - w(E^A) F_{DC} \delta_B^A) = 0 \tag{B.1.12}$$

with the graded cyclic combination of superindices D, C, B defined as

$$\oint_{(DCB)} DCB = DCB + (-)^{b(d+c)} BDC + (-)^{d(b+c)} CBD \ . \tag{B.1.13}$$

Covariant derivatives are always understood to be maximally covariant, unless explicitly otherwise stated. In our present case this means covariance with respect to both, Lorentz and $U(1)$ transformations. As an example, take the generic 0-form superfield χ_A of chiral weight $w(\chi_A)$. Its covariant derivative is defined as

$$\mathscr{D}_B \chi_A = E_B{}^M \partial_M \chi_A - \phi_{BA}{}^C \chi_C + w(\chi_A) A_B \chi_A \tag{B.1.14}$$

with graded commutator

$$(\mathscr{D}_C, \mathscr{D}_B)\chi_A = -T_{CB}{}^F \mathscr{D}_F \chi_A - R_{CBA}{}^F \chi_F + w(\chi_A) F_{CB} \chi_A \ . \tag{B.1.15}$$

The chiral weights of the various quantities are given as

$$w(\mathscr{D}_A) = -w(E^A), \quad w(T_{CB}{}^A) = w(E^A) - w(E^B) - w(E^C) \ , \tag{B.1.16}$$

$$w(R_{CBA}{}^F) = -w(E^B) - w(E^C) \ . \tag{B.1.17}$$

B.2. Torsion tensor components

For a discussion of the $U(1)$ superspace torsion constraints we refer to the main text and to the original literature. Here we content ourselves to note that all the coefficients of torsion, curvature and $U(1)$ field strength are given in terms of the few superfields

$$R,\ R^\dagger,\ G_a,\ W_{\underset{\smile}{\gamma\beta\alpha}},\ W_{\underset{\smile}{\dot\gamma\dot\beta\dot\alpha}} \tag{B.2.1}$$

and their superspace derivatives. The chiral weights of these superfields are determined according to their appearance in the torsion coefficients (see below), i.e.

$$w(R) = +2, \quad w(R^\dagger) = -2, \quad w(G_a) = 0 \ , \tag{B.2.2}$$

$$w(W_{\underset{\smile}{\gamma\beta\alpha}}) = +1, \quad w(W_{\underset{\smile}{\dot\gamma\dot\beta\dot\alpha}}) = -1 \ . \tag{B.2.3}$$

We present the torsion tensor components in order of increasing canonical dimension (remember that $[x] = -1$ and $[\theta] = -\frac{1}{2}$). We try to be as exhaustive as possible. In particular, in many places we give the results in vector as well as in spinor notation, with $\underset{\sim}{\alpha} \sim (\alpha, \dot{\alpha})$ defined as usual.

- *Dimension* 0:

$$T_{\gamma\beta}{}^{a} = 0, \quad T^{\dot{\gamma}\dot{\beta}a} = 0\ , \tag{B.2.4}$$

$$T_{\gamma}{}^{\dot{\beta}a} = -2\mathrm{i}(\sigma^{a}\varepsilon)_{\gamma}{}^{\dot{\beta}}\ . \tag{B.2.5}$$

- *Dimension* $\frac{1}{2}$:

$$T_{\underline{\gamma\beta}}{}^{\underset{\sim}{\alpha}}, = 0, \quad T_{\underset{\sim}{\gamma}b}{}^{a} = 0\ . \tag{B.2.6}$$

- *Dimension* 1: At this level appear the superfields $R, R^{\dagger}$ and G_a, i.e.

$$T_{\gamma b}{}^{\alpha} = \tfrac{\mathrm{i}}{2}(\bar{\sigma}_c\sigma_b)_{\gamma}{}^{\alpha}G^c \rightsquigarrow T_{\gamma\ \beta\dot{\beta}\ \alpha} = \mathrm{i}\varepsilon_{\beta\alpha}G_{\gamma\dot{\beta}}\ , \tag{B.2.7}$$

$$T^{\dot{\gamma}}{}_{b\dot{\alpha}} = -\tfrac{\mathrm{i}}{2}(\bar{\sigma}_c\sigma_b)^{\dot{\gamma}}{}_{\dot{\alpha}}G^c \rightsquigarrow T_{\dot{\gamma}\ \beta\dot{\beta}\ \dot{\alpha}} = \mathrm{i}\varepsilon_{\dot{\beta}\dot{\alpha}}G_{\beta\dot{\gamma}}\ , \tag{B.2.8}$$

$$T_{\gamma b\dot{\alpha}} = -\mathrm{i}\sigma_{b\gamma\dot{\alpha}}R^{\dagger} \rightsquigarrow T_{\gamma\ \beta\dot{\beta}\ \dot{\alpha}} = -2\mathrm{i}\varepsilon_{\gamma\beta}\varepsilon_{\dot{\beta}\dot{\alpha}}R^{\dagger}\ , \tag{B.2.9}$$

$$T^{\dot{\gamma}}{}_{b}{}^{\alpha} = -\mathrm{i}\bar{\sigma}_{b}^{\dot{\gamma}\alpha}R \rightsquigarrow T_{\dot{\gamma}\ \beta\dot{\beta}\ \alpha} = -2\mathrm{i}\varepsilon_{\dot{\gamma}\dot{\beta}}\varepsilon_{\beta\alpha}R\ , \tag{B.2.10}$$

$$T_{cb}{}^{a} = 0 \rightsquigarrow T_{\gamma\dot{\gamma}\ \beta\dot{\beta}\ \alpha\dot{\alpha}} = 0\ . \tag{B.2.11}$$

- *Dimension* $\frac{3}{2}$: Here, the basic objects are $T_{cb}{}^{\underset{\sim}{\alpha}}$, expressed in terms of the Weyl spinor superfields $W_{\underset{\smile}{\gamma\beta\alpha}}$, $W_{\underset{\smile}{\dot{\gamma}\dot{\beta}\dot{\alpha}}}$ and of spinor derivatives of the superfields R, $R^{\dagger}$ and G_a. These properties are most clearly exhibited using spinor notation, i.e.

$$T_{cb}{}^{\underset{\sim}{\alpha}} \rightsquigarrow T_{\gamma\dot{\gamma}\beta\dot{\beta}}{}^{\underset{\sim}{\alpha}} = 2\varepsilon_{\dot{\gamma}\dot{\beta}}T_{\underset{\smile}{\gamma\beta}}{}^{\underset{\sim}{\alpha}} - 2\varepsilon_{\gamma\beta}T_{\underset{\smile}{\dot{\gamma}\dot{\beta}}}{}^{\underset{\sim}{\alpha}} \tag{B.2.12}$$

with further tensor decompositions

$$T_{\underset{\smile}{\gamma\beta}\alpha} = W_{\underset{\smile}{\gamma\beta\alpha}} + \tfrac{1}{3}(\varepsilon_{\alpha\gamma}S_{\beta} + \varepsilon_{\alpha\beta}S_{\gamma})\ , \tag{B.2.13}$$

$$T_{\underset{\smile}{\dot{\gamma}\dot{\beta}}\dot{\alpha}} = W_{\underset{\smile}{\dot{\gamma}\dot{\beta}\dot{\alpha}}} + \tfrac{1}{3}(\varepsilon_{\dot{\alpha}\dot{\gamma}}S_{\dot{\beta}} + \varepsilon_{\dot{\alpha}\dot{\beta}}S_{\dot{\gamma}})\ . \tag{B.2.14}$$

The various tensors appearing here are defined as

$$S_{\beta} = T_{\underset{\smile}{\gamma\beta}}{}^{\gamma} = +\tfrac{1}{4}\mathscr{D}^{\dot{\beta}}G_{\beta\dot{\beta}} - \mathscr{D}_{\beta}R = \tfrac{1}{2}(T_{cb}\sigma^{cb}\varepsilon)_{\beta}\ , \tag{B.2.15}$$

$$S_{\dot{\beta}} = T_{\underset{\smile}{\dot{\gamma}\dot{\beta}}}{}^{\dot{\gamma}} = -\tfrac{1}{4}\mathscr{D}^{\beta}G_{\beta\dot{\beta}} + \mathscr{D}_{\dot{\beta}}R^{\dagger} = \tfrac{1}{2}(T_{cb}\bar{\sigma}^{cb})_{\dot{\beta}} \tag{B.2.16}$$

and

$$T_{\underset{\smile}{\dot{\gamma}\dot{\beta}}\alpha} = -\tfrac{1}{4}(\mathscr{D}_{\dot{\gamma}}G_{\alpha\dot{\beta}} + \mathscr{D}_{\dot{\beta}}G_{\alpha\dot{\gamma}})\ , \tag{B.2.17}$$

$$T_{\underset{\smile}{\gamma\beta}\dot{\alpha}} = +\tfrac{1}{4}(\mathscr{D}_{\gamma}G_{\beta\dot{\alpha}} + \mathscr{D}_{\beta}G_{\gamma\dot{\alpha}})\ . \tag{B.2.18}$$

B.3. Curvature and $U(1)$ field strength components

The curvature 2-form takes its values in the Lie algebra of the Lorentz group. Vector and spinor components are therefore related by means of the canonical decomposition

$$R_{DC\,b}{}^{a} \rightsquigarrow R_{DC\beta\dot{\beta}\alpha\dot{\alpha}} = 2\varepsilon_{\dot{\beta}\dot{\alpha}} R_{DC\underset{\smile}{\beta\alpha}} - 2\varepsilon_{\beta\alpha} R_{DC\underset{\smile}{\dot{\beta}\dot{\alpha}}} , \tag{B.3.1}$$

as defined in Appendix A. Indices D and C are superspace 2-form indices. As a general feature of superspace geometry, the components of curvature and $U(1)$ field strengths are completely determined from the torsion components and their covariant derivatives. We proceed again in order of increasing canonical dimension.

- *Dimension* 1: Here, the 2-form indices D and C are spinor indices:

$$R_{\delta\gamma\,ba} = 8(\sigma_{ba}\varepsilon)_{\delta\gamma} R^{\dagger} , \tag{B.3.2}$$

$$R^{\dot{\delta}\dot{\gamma}}{}_{ba} = 8(\bar{\sigma}_{ba}\varepsilon)^{\dot{\delta}\dot{\gamma}} R . \tag{B.3.3}$$

$$R_{\delta}{}^{\dot{\gamma}}{}_{ba} = 2\mathrm{i} G^{c}(\sigma^{d}\varepsilon)_{\delta}{}^{\dot{\gamma}} \varepsilon_{dcba} . \tag{B.3.4}$$

In spinor notation these components become, respectively,

$$R_{\delta\gamma\underset{\smile}{\beta\alpha}} = 4(\varepsilon_{\delta\beta}\varepsilon_{\gamma\alpha} + \varepsilon_{\delta\alpha}\varepsilon_{\gamma\beta}) R^{\dagger} , \tag{B.3.5}$$

$$R_{\delta\gamma\underset{\smile}{\dot{\beta}\dot{\alpha}}} = 0 , \tag{B.3.6}$$

$$R_{\dot{\delta}\dot{\gamma}\underset{\smile}{\beta\alpha}} = 0 , \tag{B.3.7}$$

$$R_{\dot{\delta}\dot{\gamma}\underset{\smile}{\dot{\beta}\dot{\alpha}}} = 4(\varepsilon_{\dot{\delta}\dot{\beta}}\varepsilon_{\dot{\gamma}\dot{\alpha}} + \varepsilon_{\dot{\delta}\dot{\alpha}}\varepsilon_{\dot{\gamma}\dot{\beta}}) R , \tag{B.3.8}$$

$$R_{\delta\dot{\gamma}\underset{\smile}{\beta\alpha}} = -\varepsilon_{\delta\beta} G_{\alpha\dot{\gamma}} - \varepsilon_{\delta\alpha} G_{\beta\dot{\gamma}} , \tag{B.3.9}$$

$$R_{\delta\dot{\gamma}\underset{\smile}{\dot{\beta}\dot{\alpha}}} = -\varepsilon_{\dot{\gamma}\dot{\beta}} G_{\delta\dot{\alpha}} - \varepsilon_{\dot{\gamma}\dot{\alpha}} G_{\delta\dot{\beta}} , \tag{B.3.10}$$

The $U(1)$ field strengths are

$$F_{\delta\gamma} = 0, \quad F^{\dot{\delta}\dot{\gamma}} = 0, \quad F_{\delta}{}^{\dot{\gamma}} = 3(\sigma^{a}\varepsilon)_{\delta}{}^{\dot{\gamma}} G_{a} . \tag{B.3.11}$$

- *Dimension* $\frac{3}{2}$: Bianchi identities tell us directly that the relevant curvatures are given in terms of torsion as

$$R_{\delta cba} = \mathrm{i}\sigma_{c\delta\dot{\delta}} T_{ba}{}^{\dot{\delta}} - \mathrm{i}\sigma_{b\delta\dot{\delta}} T_{ac}{}^{\dot{\delta}} - \mathrm{i}\sigma_{a\delta\dot{\delta}} T_{cb}{}^{\dot{\delta}} , \tag{B.3.12}$$

$$R^{\dot{\delta}}{}_{c\,ba} = \mathrm{i}\bar{\sigma}_{c}^{\dot{\delta}\delta} T_{ba\delta} - \mathrm{i}\bar{\sigma}_{b}^{\dot{\delta}\delta} T_{ac\delta} - \mathrm{i}\bar{\sigma}_{a}^{\dot{\delta}\delta} T_{cb\delta} . \tag{B.3.13}$$

In spinor notation one obtains, respectively,

$$R_{\delta\,\gamma\dot{\gamma}\,\underset{\smile}{\beta\alpha}} = +\mathrm{i}\sum_{\beta\alpha}(\varepsilon_{\delta\alpha}T_{\underset{\smile}{\gamma\beta}\dot{\gamma}} + \varepsilon_{\delta\gamma}T_{\underset{\smile}{\beta\alpha}\dot{\gamma}} - \varepsilon_{\delta\beta}\varepsilon_{\gamma\alpha}S_{\dot{\gamma}})\ , \tag{B.3.14}$$

$$R_{\delta\,\gamma\dot{\gamma}\,\underset{\smile}{\dot{\beta}\dot{\alpha}}} = +4\mathrm{i}\varepsilon_{\delta\gamma}W_{\underset{\smile}{\dot{\gamma}\dot{\beta}\dot{\alpha}}} + \mathrm{i}\sum_{\dot{\beta}\dot{\alpha}}\varepsilon_{\dot{\gamma}\dot{\beta}}\left(T_{\underset{\smile}{\delta\gamma}\dot{\alpha}} + \frac{1}{3}\varepsilon_{\delta\gamma}S_{\dot{\alpha}}\right)\ , \tag{B.3.15}$$

$$R_{\dot{\delta}\,\gamma\dot{\gamma}\,\underset{\smile}{\beta\alpha}} = -4\mathrm{i}\varepsilon_{\dot{\delta}\dot{\gamma}}W_{\underset{\smile}{\gamma\beta\alpha}} - \mathrm{i}\sum_{\beta\alpha}\varepsilon_{\gamma\beta}\left(T_{\underset{\smile}{\dot{\delta}\dot{\gamma}}\alpha} + \frac{1}{3}\varepsilon_{\dot{\delta}\dot{\gamma}}S_{\alpha}\right)\ , \tag{B.3.16}$$

$$R_{\dot{\delta}\,\gamma\dot{\gamma}\,\underset{\smile}{\dot{\beta}\dot{\alpha}}} = -\mathrm{i}\sum_{\dot{\beta}\dot{\alpha}}(\varepsilon_{\dot{\delta}\dot{\alpha}}T_{\underset{\smile}{\dot{\gamma}\dot{\beta}}\gamma} + \varepsilon_{\dot{\delta}\dot{\gamma}}T_{\underset{\smile}{\dot{\beta}\dot{\alpha}}\gamma} - \varepsilon_{\dot{\delta}\dot{\beta}}\varepsilon_{\dot{\gamma}\dot{\alpha}}S_{\gamma})\ . \tag{B.3.17}$$

Using the explicit form of the torsion coefficients as defined in the previous subsection, these curvatures may also be written as

$$R_{\delta\,\gamma\dot{\gamma}\,\underset{\smile}{\beta\alpha}} = \mathrm{i}\sum_{\beta\alpha}\left(\frac{1}{2}\varepsilon_{\delta\gamma}\mathscr{D}_{\beta}G_{\alpha\dot{\gamma}} + \frac{1}{2}\varepsilon_{\delta\beta}\mathscr{D}_{\gamma}G_{\alpha\dot{\gamma}} - \varepsilon_{\delta\beta}\varepsilon_{\gamma\alpha}\mathscr{D}_{\dot{\gamma}}R^{\dagger}\right)\ , \tag{B.3.18}$$

$$R_{\delta\,\gamma\dot{\gamma}\,\underset{\smile}{\dot{\beta}\dot{\alpha}}} = 4\mathrm{i}\varepsilon_{\delta\gamma}W_{\underset{\smile}{\dot{\gamma}\dot{\beta}\dot{\alpha}}} + \mathrm{i}\sum_{\dot{\beta}\dot{\alpha}}\varepsilon_{\dot{\gamma}\dot{\alpha}}\left(\frac{1}{3}\varepsilon_{\delta\gamma}\bar{X}_{\dot{\beta}} + \frac{1}{2}\mathscr{D}_{\delta}G_{\gamma\dot{\beta}}\right)\ , \tag{B.3.19}$$

$$R_{\dot{\delta}\,\gamma\dot{\gamma}\,\underset{\smile}{\beta\alpha}} = -4\mathrm{i}\varepsilon_{\dot{\delta}\dot{\gamma}}W_{\underset{\smile}{\gamma\beta\alpha}} + \mathrm{i}\sum_{\beta\alpha}\varepsilon_{\gamma\alpha}\left(\frac{1}{3}\varepsilon_{\dot{\delta}\dot{\gamma}}X_{\beta} + \frac{1}{2}\mathscr{D}_{\dot{\delta}}G_{\beta\dot{\gamma}}\right)\ , \tag{B.3.20}$$

$$R_{\dot{\delta}\,\gamma\dot{\gamma}\,\underset{\smile}{\dot{\beta}\dot{\alpha}}} = \mathrm{i}\sum_{\dot{\beta}\dot{\alpha}}\left(\frac{1}{2}\varepsilon_{\dot{\delta}\dot{\gamma}}\mathscr{D}_{\dot{\beta}}G_{\gamma\dot{\alpha}} + \frac{1}{2}\varepsilon_{\dot{\delta}\dot{\beta}}\mathscr{D}_{\dot{\gamma}}G_{\gamma\dot{\alpha}} - \varepsilon_{\dot{\delta}\dot{\beta}}\varepsilon_{\dot{\gamma}\dot{\alpha}}\mathscr{D}_{\gamma}R\right)\ . \tag{B.3.21}$$

Here symmetric sums over indices α, β (resp. $\dot{\alpha}, \dot{\beta}$) are understood in an obvious way and we have used the definitions

$$X_{\beta} = \mathscr{D}_{\beta}R - \mathscr{D}^{\dot{\beta}}G_{\beta\dot{\beta}}\ , \tag{B.3.22}$$

$$\bar{X}_{\dot{\beta}} = \mathscr{D}_{\dot{\beta}}R^{\dagger} - \mathscr{D}^{\beta}G_{\beta\dot{\beta}}\ . \tag{B.3.23}$$

These superfields are naturally identified in the $U(1)$ field strengths

$$F_{\delta c} = \frac{3\mathrm{i}}{2}\mathscr{D}_{\delta}G_{c} + \frac{\mathrm{i}}{2}\sigma_{c\delta\dot{\delta}}\bar{X}^{\dot{\delta}}\ , \tag{B.3.24}$$

$$F^{\dot{\delta}}{}_{c} = \frac{3\mathrm{i}}{2}\mathscr{D}^{\dot{\delta}}G_{c} - \frac{\mathrm{i}}{2}\bar{\sigma}_{c}^{\dot{\delta}\delta}X_{\delta}\ , \tag{B.3.25}$$

which, in spinor notation, read

$$F_{\delta\,\gamma\dot\gamma} = \frac{3i}{2}\mathcal{D}_\delta G_{\gamma\dot\gamma} + i\varepsilon_{\delta\gamma}\bar{X}_{\dot\gamma}\ , \tag{B.3.26}$$

$$F_{\dot\delta\,\gamma\dot\gamma} = \frac{3i}{2}\mathcal{D}_{\dot\delta} G_{\gamma\dot\gamma} + i\varepsilon_{\dot\delta\dot\gamma}X_{\gamma}\ . \tag{B.3.27}$$

- *Dimension* 2: The curvature tensor $R_{dcb}{}^{a}$ has the property

$$R_{dc\,ba} = R_{ba\,dc}\ . \tag{B.3.28}$$

Its decomposition in spinor notations is given as

$$R_{\delta\dot\delta\gamma\dot\gamma\beta\dot\beta\alpha\dot\alpha} = +4\varepsilon_{\dot\delta\dot\gamma}(\varepsilon_{\dot\beta\dot\alpha}\chi_{\underbrace{\delta\gamma}\,\underbrace{\beta\alpha}} - \varepsilon_{\beta\alpha}\psi_{\underbrace{\delta\gamma}\,\underbrace{\dot\beta\dot\alpha}}) + 4\varepsilon_{\delta\gamma}(\varepsilon_{\beta\alpha}\chi_{\underbrace{\dot\delta\dot\gamma}\,\underbrace{\dot\beta\dot\alpha}} - \varepsilon_{\dot\beta\dot\alpha}\psi_{\underbrace{\dot\delta\dot\gamma}\,\underbrace{\beta\alpha}})\ , \tag{B.3.29}$$

where

$$\chi_{\underbrace{\delta\gamma}\,\underbrace{\beta\alpha}} = \chi_{\underbrace{\delta\gamma\beta\alpha}} = +(\varepsilon_{\delta\beta}\varepsilon_{\gamma\alpha} + \varepsilon_{\delta\alpha}\varepsilon_{\gamma\beta})\chi\ , \tag{B.3.30}$$

$$\chi_{\underbrace{\dot\delta\dot\gamma}\,\underbrace{\dot\beta\dot\alpha}} = \chi_{\underbrace{\dot\delta\dot\gamma\dot\beta\dot\alpha}} + (\varepsilon_{\dot\delta\dot\beta}\varepsilon_{\dot\gamma\dot\alpha} + \varepsilon_{\dot\delta\dot\alpha}\varepsilon_{\dot\gamma\dot\beta})\chi \tag{B.3.31}$$

and

$$\chi = \tfrac{1}{24}R_{ba}{}^{ba}\ . \tag{B.3.32}$$

The tensors appearing in the spinor decomposition of the curvature are, respectively,

$T_{cb}{}^{\alpha}$, $T_{cb\dot\alpha}$	the Rarita–Schwinger field strength ,
$R_{dcb}{}^{a}$	the Lorentz curvature ,
X_α, $\bar{X}^{\dot\alpha}$	the $U(1)$ superfield .

Here $\chi_{\underbrace{\delta\gamma\beta\gamma}}$ and $\chi_{\underbrace{\dot\delta\dot\gamma\dot\beta\dot\alpha}}$ describe the Weyl tensor in spinor notation, whereas $\psi_{\underbrace{\delta\dot\gamma}\,\underbrace{\beta\alpha}}$ and χ correspond, respectively, to the Ricci $\mathcal{R}_{dc} = R_{dab}{}^{a}$ tensor and to the curvature scalar $\mathcal{R} = R_{ba}{}^{ba}$. These superfields are related to the basic superfields obtained in the preceding section in the following way:

$$\chi_{\underbrace{\delta\gamma\beta\alpha}} = \tfrac{1}{4}(\mathcal{D}_\delta W_{\underbrace{\gamma\beta\alpha}} + \mathcal{D}_\gamma W_{\underbrace{\beta\alpha\delta}} + \mathcal{D}_\beta W_{\underbrace{\alpha\delta\gamma}} + \mathcal{D}_\alpha W_{\underbrace{\delta\gamma\beta}})\ , \tag{B.3.33}$$

$$\chi_{\underbrace{\dot\delta\dot\gamma\dot\beta\dot\alpha}} = \tfrac{1}{4}(\mathcal{D}_{\dot\delta} W_{\underbrace{\dot\gamma\dot\beta\dot\alpha}} + \mathcal{D}_{\dot\gamma} W_{\underbrace{\dot\beta\dot\alpha\dot\delta}} + \mathcal{D}_{\dot\beta} W_{\underbrace{\dot\alpha\dot\delta\dot\gamma}} + \mathcal{D}_{\dot\alpha} W_{\underbrace{\dot\delta\dot\gamma\dot\beta}})\ , \tag{B.3.34}$$

$$\psi_{\underbrace{\delta\gamma}\,\underbrace{\dot\beta\dot\alpha}} = \frac{1}{8}\sum_{\delta\gamma}\sum_{\dot\beta\dot\alpha}(G_{\delta\dot\beta}G_{\gamma\dot\alpha} - \tfrac{1}{2}[\mathcal{D}_\delta, \mathcal{D}_{\dot\beta}]G_{\gamma\dot\alpha}) \tag{B.3.35}$$

and

$$\chi = -\tfrac{1}{12}(\mathcal{D}^\alpha \mathcal{D}_\alpha R + \mathcal{D}_{\dot\alpha}\mathcal{D}^{\dot\alpha} R^\dagger) + \tfrac{1}{48}[\mathcal{D}^\alpha, \mathcal{D}^{\dot\alpha}] G_{\alpha\dot\alpha} - \tfrac{1}{8} G^{\alpha\dot\alpha} G_{\alpha\dot\alpha} + 2RR^\dagger . \tag{B.3.36}$$

The $U(1)$ field strength F_{dc} with canonical spinor decomposition

$$F_{\delta\dot\delta\gamma\dot\gamma} = 2\varepsilon_{\dot\delta\dot\gamma} F_{\underset{\smile}{\delta\gamma}} - 2\varepsilon_{\delta\gamma} F_{\underset{\smile}{\dot\delta\dot\gamma}} \tag{B.3.37}$$

can be expressed as

$$F_{\underset{\smile}{\delta\gamma}} = +\frac{1}{8}\sum_{\delta\gamma} (\mathcal{D}_\delta \mathcal{D}^\delta G_{\gamma\dot\delta} + 3\mathrm{i}\mathcal{D}_\delta{}^{\dot\delta} G_{\gamma\dot\delta}) , \tag{B.3.38}$$

$$F_{\underset{\smile}{\dot\delta\dot\gamma}} = -\frac{1}{8}\sum_{\dot\delta\dot\gamma} (\mathcal{D}_{\dot\delta} \mathcal{D}^{\dot\delta} G_{\delta\dot\gamma} + 3\mathrm{i}\mathcal{D}^{\delta}{}_{\dot\delta} G_{\delta\dot\gamma}) . \tag{B.3.39}$$

B.4. Derivative relations

Superspace constraints, via the Bianchi identities, imply covariant restrictions on the basic superfields encountered in the previous subsections. Most important are the chirality conditions

$$\mathcal{D}_\alpha R^\dagger = 0, \qquad \mathcal{D}^{\dot\alpha} R = 0 \tag{B.4.1}$$

and

$$\mathcal{D}_\alpha W_{\underset{\smile}{\dot\gamma\dot\beta\dot\alpha}} = 0 , \qquad \mathcal{D}^{\dot\alpha} W_{\underset{\smile}{\gamma\beta\alpha}} = 0 . \tag{B.4.2}$$

Superfield expansions are defined in terms of covariant derivatives. We have seen that the geometry of $U(1)$ superspace can be expressed in terms of some basic superfields and their covariant derivatives. Conversely, this means that tensors like $T_{cb}{}^\alpha$, $T_{cb\dot\alpha}$, $R_{dcb}{}^a$, X_α, $\bar X^{\dot\alpha}$ are located in the superfield expansions of these basic superfields. At dimension $\frac{3}{2}$ the relevant equations are

$$\mathcal{D}_\beta R = -\tfrac{1}{3} X_\beta - \tfrac{2}{3}(T_{cb}\sigma^{cb}\varepsilon)_\beta , \tag{B.4.3}$$

$$\mathcal{D}^{\dot\beta} R^\dagger = -\tfrac{1}{3}\bar X^{\dot\beta} - \tfrac{2}{3}(T_{cb}\bar\sigma^{cb}\varepsilon)^{\dot\beta} \tag{B.4.4}$$

and

$$\mathcal{D}_\beta G_a = -\tfrac{1}{2}(T_{cb}\bar\sigma_a\sigma^{cb}\varepsilon)_\beta + \tfrac{1}{6}(T_{cb}\bar\sigma^{cb}\bar\sigma_a\varepsilon)_\beta - \tfrac{1}{3}(\bar X\bar\sigma_a\varepsilon)_\beta , \tag{B.4.5}$$

$$\mathcal{D}^{\dot\beta} G_a = +\tfrac{1}{2}(T_{cb}\sigma_a\bar\sigma^{cb}\varepsilon)^{\dot\beta} - \tfrac{1}{6}(T_{cb}\sigma^{cb}\sigma_a\varepsilon)^{\dot\beta} + \tfrac{1}{3}(X\sigma_a\varepsilon)^{\dot\beta} . \tag{B.4.6}$$

Note that, in order to compactify the notation, we have suppressed a number of spinor indices. They are easily (and without ambiguity) restored with reference to the index structures of σ-matrices explicitly defined in Appendix A. In spinor notation, these relations may equivalently be written as

$$\mathscr{D}_\beta R = -\tfrac{1}{3}X_\beta - \tfrac{4}{3}S_\beta \ , \tag{B.4.7}$$

$$\mathscr{D}^{\dot\beta} R^\dagger = -\tfrac{1}{3}X^{\dot\beta} + \tfrac{4}{3}S^{\dot\beta} \tag{B.4.8}$$

and

$$\mathscr{D}_\beta G_{\alpha\dot\alpha} = +2T_{\underset{\smile}{\beta\alpha}\dot\alpha} + \tfrac{2}{3}\varepsilon_{\beta\alpha}S_{\dot\alpha} - \tfrac{2}{3}\varepsilon_{\beta\alpha}\bar X_{\dot\alpha} \ , \tag{B.4.9}$$

$$\mathscr{D}_{\dot\beta} G_{\alpha\dot\alpha} = -2T_{\underset{\smile}{\dot\beta\dot\alpha}\alpha} - \tfrac{2}{3}\varepsilon_{\dot\beta\dot\alpha}S_\alpha - \tfrac{2}{3}\varepsilon_{\dot\beta\dot\alpha}X_\alpha \ . \tag{B.4.10}$$

In the $U(1)$ gauge sector, at dimension 2, one has

$$\mathscr{D}_\alpha \bar X_{\dot\alpha} = 0, \quad \mathscr{D}_{\dot\alpha} X_\alpha = 0 \tag{B.4.11}$$

and

$$\mathscr{D}^\alpha X_\alpha = \mathscr{D}_{\dot\alpha}\bar X^{\dot\alpha} \ . \tag{B.4.12}$$

Substituting for X_α, $\bar X^{\dot\alpha}$ yields the equivalent equations

$$\mathscr{D}^\varphi \mathscr{D}_\varphi G_a = 4\mathrm{i}\mathscr{D}_a R^\dagger, \quad \mathscr{D}_{\dot\varphi}\mathscr{D}^{\dot\varphi} G_a = -4\mathrm{i}\mathscr{D}_a R \tag{B.4.13}$$

and

$$\mathscr{D}^\alpha \mathscr{D}_\alpha R - \mathscr{D}_{\dot\alpha}\mathscr{D}^{\dot\alpha} R^\dagger = 4\mathrm{i}\mathscr{D}_a G^a \ . \tag{B.4.14}$$

The orthogonal combination is given as

$$\mathscr{D}^2 R + \bar{\mathscr{D}}^2 R^\dagger = -\tfrac{2}{3}R_{ba}{}^{ba} - \tfrac{2}{3}\mathscr{D}^\alpha X_\alpha + 4G^a G_a + 32RR^\dagger \ . \tag{B.4.15}$$

As a consequence of the chirality conditions, the mixed second spinor derivatives on R, $R^\dagger$ are

$$\mathscr{D}_{\dot\alpha}\mathscr{D}_\alpha R = -2\mathrm{i}\mathscr{D}_{\alpha\dot\alpha} R - 6G_{\alpha\dot\alpha} R \ , \tag{B.4.16}$$

$$\mathscr{D}_\alpha \mathscr{D}_{\dot\alpha} R^\dagger = -2\mathrm{i}\mathscr{D}_{\alpha\dot\alpha} R^\dagger + 6G_{\alpha\dot\alpha} R^\dagger \ . \tag{B.4.17}$$

The relation

$$\begin{aligned}[\mathscr{D}_\beta, \mathscr{D}_{\dot\beta}] G_{\alpha\dot\alpha} = &-4\psi_{\underset{\smile}{\beta\alpha}\,\underset{\smile}{\dot\beta\dot\alpha}} + 2G_{\beta\dot\beta}G_{\alpha\dot\alpha} + 4(\varepsilon_{\beta\alpha}F_{\underset{\smile}{\dot\beta\dot\alpha}} + \varepsilon_{\dot\beta\dot\alpha}F_{\underset{\smile}{\beta\alpha}}) \\ &+ 2\mathrm{i}\varepsilon_{\beta\alpha}\mathscr{D}^\varphi{}_{\dot\beta} G_{\varphi\dot\alpha} - 2\mathrm{i}\varepsilon_{\dot\beta\dot\alpha}\mathscr{D}_\beta{}^{\dot\varphi} G_{\alpha\dot\varphi} + \varepsilon_{\beta\alpha}\varepsilon_{\dot\beta\dot\alpha}(8RR^\dagger + 2G^c G_c - \tfrac{2}{3}\mathscr{D}^\varphi X_\varphi - 4\chi) \ ,\end{aligned} \tag{B.4.18}$$

may be equivalently written as

$$[\mathscr{D}_\alpha, \mathscr{D}_{\dot\alpha}] G_a = -(\sigma_a)_{\alpha\dot\alpha}(4RR^\dagger + G^b G_b + \tfrac{1}{6}\mathscr{R}) + (\sigma^b)_{\alpha\dot\alpha}(\mathscr{R}_{ba} + 2G_a G_b + \varepsilon_{abcd}\mathscr{D}^c G^d) \ . \tag{B.4.19}$$

As to the Weyl spinor superfields, their non-trivial spinor derivatives are determined to be

$$\mathscr{D}_\delta W_{\underbrace{\gamma\beta\alpha}} = \chi_{\underbrace{\delta\gamma\beta\alpha}} + \tfrac{1}{4}\varepsilon_{\delta\gamma}\mathscr{D}^\varphi W_{\underbrace{\varphi\beta\alpha}} + \tfrac{1}{4}\varepsilon_{\delta\beta}\mathscr{D}^\varphi W_{\underbrace{\varphi\alpha\gamma}} + \tfrac{1}{4}\varepsilon_{\delta\alpha}\mathscr{D}^\varphi W_{\underbrace{\varphi\gamma\beta}} , \tag{B.4.20}$$

$$\mathscr{D}_\delta W_{\underbrace{\dot\gamma\dot\beta\dot\alpha}} = \chi_{\underbrace{\delta\dot\gamma\dot\beta\dot\alpha}} + \tfrac{1}{4}\varepsilon_{\delta\dot\gamma}\mathscr{D}^{\dot\varphi} W_{\underbrace{\dot\varphi\dot\beta\dot\alpha}} + \tfrac{1}{4}\varepsilon_{\delta\dot\beta}\mathscr{D}^{\dot\varphi} W_{\underbrace{\dot\varphi\dot\alpha\dot\gamma}} + \tfrac{1}{4}\varepsilon_{\delta\dot\alpha}\mathscr{D}^{\dot\varphi} W_{\underbrace{\dot\varphi\dot\gamma\dot\beta}} \tag{B.4.21}$$

with

$$\mathscr{D}^\varphi W_{\underbrace{\varphi\beta\alpha}} = -\frac{1}{6}\sum_{\beta\alpha} (\mathscr{D}_\beta \mathscr{D}^{\dot\varphi} G_{\alpha\dot\varphi} + 3\mathrm{i}\mathscr{D}_\beta{}^{\dot\varphi} G_{\alpha\dot\varphi}) = -\frac{4}{3} F_{\underbrace{\beta\alpha}} , \tag{B.4.22}$$

$$\mathscr{D}^{\dot\varphi} W_{\underbrace{\dot\varphi\dot\beta\dot\alpha}} = +\frac{1}{6}\sum_{\dot\beta\dot\alpha} (\mathscr{D}_{\dot\beta} \mathscr{D}^{\varphi} G_{\varphi\dot\alpha} + 3\mathrm{i}\mathscr{D}^\varphi{}_{\dot\beta} G_{\varphi\dot\alpha}) = -\frac{4}{3} F_{\underbrace{\dot\beta\dot\alpha}} . \tag{B.4.23}$$

Observe that these relations may also be identified in the more compact identity

$$\mathscr{D}^\alpha T_{cb\,\alpha} + \mathscr{D}_{\dot\alpha} T_{cb}{}^{\dot\alpha} = 0 . \tag{B.4.24}$$

B.5. Yang–Mills in $U(1)$ superspace

As in Section 2.3, the Yang–Mills connection and its curvature are Lie algebra valued forms in $U(1)$ superspace,

$$\mathscr{A} = E^A \mathscr{A}_A^{(r)} \mathbf{T}_{(r)} = \mathscr{A}^{(r)} \mathbf{T}_{(r)} , \tag{B.5.1}$$

$$\mathscr{F} = \tfrac{1}{2} E^A E^B \mathscr{F}_{BA}^{(r)} \mathbf{T}_{(r)} = \mathscr{F}^{(r)} \mathbf{T}_{(r)} \tag{B.5.2}$$

with $\mathscr{F} = d\mathscr{A} + \mathscr{A}\mathscr{A}$, or

$$\mathscr{F}^{(r)} = d\mathscr{A}^{(r)} + \frac{\mathrm{i}}{2}\mathscr{A}^{(p)}\mathscr{A}^{(q)} c_{(p)(q)}{}^{(r)} . \tag{B.5.3}$$

The Bianchi identities are

$$\mathscr{D}\mathscr{F} = d\mathscr{F} - \mathscr{A}\mathscr{F} + \mathscr{F}\mathscr{A} = 0 , \tag{B.5.4}$$

i.e.

$$\mathscr{D}\mathscr{F}^{(r)} = d\mathscr{F}^{(r)} - \mathrm{i}\mathscr{A}^{(p)}\mathscr{F}^{(q)} c_{(p)(q)}{}^{(r)} = 0 . \tag{B.5.5}$$

More explicitly, decomposing on the covariant superspace basis this 3-form, we obtain

$$\oint_{(CBA)} (\mathscr{D}_C \mathscr{F}_{BA} + T_{CB}{}^F \mathscr{F}_{FA}) = 0 . \tag{B.5.6}$$

In the discussion of the Yang–Mills Bianchi identities the complete structure of $U(1)$ superspace as presented in this appendix must be taken into account, derivatives are now covariant with respect Lorentz, chiral $U(1)$ and Yang–Mills gauge transformations. The covariant constraints[21]

$$\mathcal{F}^{\dot{\alpha}\dot{\beta}} = 0, \quad \mathcal{F}_{\beta\alpha} = 0, \quad \mathcal{F}_{\beta}{}^{\dot{\alpha}} = 0\ , \tag{B.5.7}$$

together with the Bianchi identities restrict the form of the remaining components of the Yang–Mills field strength such that

$$\mathcal{F}_{\beta a} = +\,\mathrm{i}(\sigma_a)_{\beta\dot{\beta}} \mathcal{W}^{\dot{\beta}}\ , \tag{B.5.8}$$

$$\mathcal{F}^{\dot{\beta}}{}_a = -\,\mathrm{i}(\bar{\sigma}_a)^{\dot{\beta}\beta} \mathcal{W}_\beta\ , \tag{B.5.9}$$

$$\mathcal{F}_{ba} = \tfrac{1}{2}(\varepsilon\sigma_{ba})^{\beta\alpha}\mathcal{D}_\alpha \mathcal{W}_\beta + \tfrac{1}{2}(\bar{\sigma}_{ba}\varepsilon)^{\dot{\beta}\dot{\alpha}}\mathcal{D}_{\dot{\alpha}}\mathcal{W}_{\dot{\beta}}\ . \tag{B.5.10}$$

The Yang–Mills superfields

$$\mathcal{W}_\alpha = \mathcal{W}_\alpha^{(r)}\mathbf{T}_{(r)}, \quad \mathcal{W}^{\dot{\alpha}} = \mathcal{W}^{(r)\dot{\alpha}}\mathbf{T}_{(r)} \tag{B.5.11}$$

with respective chiral weights, $+1$ and -1, are subject to the reduced set of Bianchi identities

$$\mathcal{D}_\alpha \mathcal{W}^{\dot{\alpha}} = 0, \quad \mathcal{D}^{\dot{\alpha}}\mathcal{W}_\alpha = 0\ , \tag{B.5.12}$$

$$\mathcal{D}^\alpha \mathcal{W}_\alpha = \mathcal{D}_{\dot{\alpha}}\mathcal{W}^{\dot{\alpha}}\ . \tag{B.5.13}$$

We also define the D-term superfield $\mathbf{D}^{(r)}$ as

$$\mathbf{D}^{(r)} = -\,\tfrac{1}{2}\mathcal{D}^\alpha \mathcal{W}_\alpha^{(r)} \tag{B.5.14}$$

with vanishing chiral weight, $w(\mathbf{D}^{(r)}) = 0$. In spinor notation the components of the field strength are given as

$$\mathcal{F}_{\dot{\beta}\ \alpha\dot{\alpha}} = 2\mathrm{i}\varepsilon_{\dot{\beta}\dot{\alpha}}\mathcal{W}_\alpha\ , \tag{B.5.15}$$

$$\mathcal{F}_{\beta\ \alpha\dot{\alpha}} = 2\mathrm{i}\varepsilon_{\beta\alpha}\mathcal{W}_{\dot{\alpha}} \tag{B.5.16}$$

and

$$\mathcal{F}_{\beta\dot{\beta}\ \alpha\dot{\alpha}} = 2\varepsilon_{\dot{\beta}\dot{\alpha}}\mathcal{F}_{\underset{\smile}{\beta\alpha}} - 2\varepsilon_{\beta\alpha}\mathcal{F}_{\underset{\smile}{\dot{\beta}\dot{\alpha}}} \tag{B.5.17}$$

with

$$\mathcal{F}_{\underset{\smile}{\beta\alpha}} = -\,\tfrac{1}{4}(\mathcal{D}_\beta \mathcal{W}_\alpha + \mathcal{D}_\alpha \mathcal{W}_\beta)\ , \tag{B.5.18}$$

$$\mathcal{F}_{\underset{\smile}{\dot{\beta}\dot{\alpha}}} = +\,\tfrac{1}{4}(\mathcal{D}_{\dot{\beta}}\mathcal{W}_{\dot{\alpha}} + \mathcal{D}_{\dot{\alpha}}\mathcal{W}_{\dot{\beta}})\ . \tag{B.5.19}$$

[21] The explicit solution of the constraints, as explained in Section 2.3, in particular the construction of the chiral and antichiral basis, carries straightforwardly over to $U(1)$ superspace.

Conversely, the non-trivial spinor derivatives of the Yang–Mills superfields are given as

$$\mathscr{D}_\beta \mathscr{W}^{(r)}_\alpha = -(\sigma^{ba}\varepsilon)_{\beta\alpha}\mathscr{F}^{(r)}_{ba} - \varepsilon_{\beta\alpha}\mathbf{D}^{(r)} , \tag{B.5.20}$$

$$\mathscr{D}^{\dot\beta} \mathscr{W}^{(r)\dot\alpha} = -(\varepsilon\bar\sigma^{ba})^{\dot\beta\dot\alpha}\mathscr{F}^{(r)}_{ba} + \varepsilon^{\dot\beta\dot\alpha}\mathbf{D}^{(r)} \tag{B.5.21}$$

and those of the D-term superfield are

$$\mathscr{D}_\alpha \mathbf{D}^{(r)} = \mathrm{i}\sigma^a_{\alpha\dot\alpha}\mathscr{D}_a\mathscr{W}^{(r)\dot\alpha} , \tag{B.5.22}$$

$$\mathscr{D}^{\dot\alpha} \mathbf{D}^{(r)} = \mathrm{i}\bar\sigma^{a\dot\alpha\alpha}\mathscr{D}_a\mathscr{W}^{(r)}_\alpha . \tag{B.5.23}$$

The covariant derivative appearing here is defined as

$$\mathscr{D}_A \mathbf{D}^{(r)} = E_A{}^M\partial_M \mathbf{D}^{(r)} - \mathrm{i}\mathscr{A}^{(p)}_A\mathbf{D}^{(q)}c_{(p)(q)}{}^{(r)} . \tag{B.5.24}$$

Recall that the graded commutator of two covariant derivatives is

$$(\mathscr{D}_B, \mathscr{D}_A)\mathbf{D}^{(r)} = -T_{BA}{}^F\mathscr{D}_F\mathbf{D}^{(r)} - \mathrm{i}\mathscr{F}^{(p)}_{BA}\mathbf{D}^{(q)}c_{(p)(q)}{}^{(r)} . \tag{B.5.25}$$

In the case of the Yang–Mills superfields additional terms appear due to their non-trivial Lorentz and $U(1)$ structures:

$$(\mathscr{D}_C, \mathscr{D}_B)\mathscr{W}^{(r)}_\alpha = -T_{CB}{}^F\mathscr{D}_F\mathscr{W}^{(r)}_\alpha - \mathrm{i}\mathscr{F}^{(p)}_{CB}\mathscr{W}^{(q)}_\alpha c_{(p)(q)}{}^{(r)} - R_{CB\alpha}{}^\varphi\mathscr{W}^{(r)}_\varphi + F_{CB}\mathscr{W}^{(r)}_\alpha , \tag{B.5.26}$$

$$(\mathscr{D}_C, \mathscr{D}_B)\mathscr{W}^{(r)\dot\alpha} = -T_{CB}{}^F\mathscr{D}_F\mathscr{W}^{(r)\dot\alpha} - \mathrm{i}\mathscr{F}^{(p)}_{CB}\mathscr{W}^{(q)\dot\alpha}c_{(p)(q)}{}^{(r)} - R_{CB}{}^{\dot\alpha}{}_{\dot\varphi}\mathscr{W}^{(r)\dot\varphi} - F_{CB}\mathscr{W}^{(r)\dot\alpha} . \tag{B.5.27}$$

In the evaluation of (B.5.22) and (B.5.23) these relations are used in combination with (B.5.12) and (B.5.13). Further useful relations are

$$\mathscr{D}^2\mathscr{W}^{(r)}_\alpha = 4\mathrm{i}\sigma^a_{\alpha\dot\alpha}\mathscr{D}_a\mathscr{W}^{(r)\dot\alpha} + 12R^\dagger\mathscr{W}^{(r)}_\alpha , \tag{B.5.28}$$

$$\bar{\mathscr{D}}^2\mathscr{W}^{(r)\dot\alpha} = 4\mathrm{i}\bar\sigma^{a\dot\alpha\alpha}\mathscr{D}_a\mathscr{W}^{(r)}_\alpha + 12R\mathscr{W}^{(r)\dot\alpha} . \tag{B.5.29}$$

Appendix C. Gauged isometries

In the general supergravity/matter/Yang–Mills system the chiral matter superfields parametrize a Kähler manifold. These structures are quite well understood in the geometric framework of Kähler superspace. In general, from the point of view of differential geometry, Kähler manifolds admit non-linear isometry transformations, which can be gauged using suitable Yang–Mills potentials.

This appendix provides a description of gauged isometries compatible with superspace. Of course, the relevant language makes use of superfields. In a first subsection we develop the general formalism on a manifold parametrized by complex superfields, not yet necessarily subject to chirality conditions. The second subsection shows how Kähler superspace can be modified to take care of gauged isometries. The resulting geometric structure is called *isometric Kähler superspace*. In the third subsection we derive the supergravity transformations in this context and in the fourth and last subsection we establish the relation with Yang–Mills transformations of the matter superfields, which correspond to linear isometry transformations.

C.1. Isometries and superfields

As a starting point we consider a complex manifold spanned by complex superfields ϕ^k and their complex conjugates $\bar{\phi}^{\bar{k}}$. Following [6,7] we define infinitesimal variations

$$\delta\phi^k = -\alpha^{(r)} V_{(r)}\phi^k, \qquad \delta\bar{\phi}^{\bar{k}} = -\alpha^{(r)} \bar{V}_{(r)}\bar{\phi}^{\bar{k}}, \tag{C.1.1}$$

of generators $V_{(r)}$ and $\bar{V}_{(r)}$ which depend holomorphically (resp. anti-holomorphically) on the superfield coordinates

$$V_{(r)} = {V_{(r)}}^k(\phi)\frac{\partial}{\partial\phi^k}, \qquad \bar{V}_{(r)} = {\bar{V}_{(r)}}^{\bar{k}}(\bar{\phi})\frac{\partial}{\partial\bar{\phi}^{\bar{k}}} \tag{C.1.2}$$

and which satisfy commutation relations

$$[V_{(r)}, V_{(s)}] = {c_{(r)(s)}}^{(t)} V_{(t)}, \tag{C.1.3}$$

$$[\bar{V}_{(r)}, \bar{V}_{(s)}] = {c_{(r)(s)}}^{(t)} \bar{V}_{(t)}, \tag{C.1.4}$$

$$[V_{(r)}, \bar{V}_{(s)}] = 0. \tag{C.1.5}$$

In addition to holomorphy properties, solution of the Killing equations of the hermitean metric implies the appearance of Killing potentials, $G_{(r)}(\phi, \bar{\phi})$, such that

$$g_{k\bar{k}} {\bar{V}_{(r)}}^{\bar{k}} = +\mathrm{i}\frac{\partial G_{(r)}}{\partial\phi^k}, \qquad g_{k\bar{k}} {V_{(r)}}^k = -\mathrm{i}\frac{\partial G_{(r)}}{\partial\bar{\phi}^{\bar{k}}}. \tag{C.1.6}$$

In the case of Kähler geometry, i.e.

$$g_{k\bar{k}} = \frac{\partial^2 K(\phi, \bar{\phi})}{\partial\phi^k \partial\bar{\phi}^{\bar{k}}}, \tag{C.1.7}$$

these equations in turn are solved in terms of holomorphic (resp. anti-holomorphic) functions $F_{(r)}(\phi)$ (resp. $\bar{F}_{(r)}(\bar{\phi})$) – which one might call Killing pre-potentials – such that

$$G_{(r)} = \frac{\mathrm{i}}{2}(V_{(r)} - \bar{V}_{(r)})K - \frac{\mathrm{i}}{2}(F_{(r)} - \bar{F}_{(r)}) \tag{C.1.8}$$

and

$$(V_{(r)} + \bar{V}_{(r)})K = F_{(r)} + \bar{F}_{(r)}. \tag{C.1.9}$$

As a consequence of the commutation relations for $V_{(r)}, \bar{V}_{(r)}$, the pre-potentials $F_{(r)}$ and $\bar{F}_{(r)}$ satisfy consistency conditions

$$V_{(r)}F_{(s)} - V_{(s)}F_{(r)} = c_{(r)(s)}{}^{(t)}F_{(t)} + \mathrm{i}C_{(r)(s)} , \tag{C.1.10}$$

$$\bar{V}_{(r)}\bar{F}_{(s)} - \bar{V}_{(s)}\bar{F}_{(r)} = c_{(r)(s)}{}^{(t)}\bar{F}_{(t)} - \mathrm{i}C_{(r)(s)} \tag{C.1.11}$$

with antisymmetric separation constants

$$C_{(r)(s)} = -C_{(s)(r)} . \tag{C.1.12}$$

Moreover, multiplying Eqs. (C.1.6), which define the Killing potential $G_{(r)}$, appropriately with $V_{(r)}{}^k$ (resp. $\bar{V}_{(r)}{}^{\bar{k}}$) one obtains

$$V_{(r)}G_{(s)} + \bar{V}_{(s)}G_{(r)} = 0 . \tag{C.1.13}$$

Other useful relations in this context are

$$V_{(r)}G_{(s)} - V_{(s)}G_{(r)} = c_{(r)(s)}{}^{(t)}G_{(t)} + C_{(r)(s)} , \tag{C.1.14}$$

$$\bar{V}_{(r)}G_{(s)} - \bar{V}_{(s)}G_{(r)} = c_{(r)(s)}{}^{(t)}G_{(t)} + C_{(r)(s)} , \tag{C.1.15}$$

$$(V_{(r)} + \bar{V}_{(r)})G_{(s)} = c_{(r)(s)}{}^{(t)}G_{(t)} + C_{(r)(s)} . \tag{C.1.16}$$

In the following, we shall restrict ourselves to cases where it is possible to take

$$C_{(r)(s)} = 0 \tag{C.1.17}$$

and discuss gauged isometries, i.e. variations of ϕ^k and $\bar{\phi}^{\bar{k}}$ where the parameters $\alpha^{(r)}$ are unconstrained real superfields. Covariant derivatives are then constructed with the help of superfield gauge potentials which are 1-forms in superspace

$$\mathscr{A}^{(r)} = E^A \mathscr{A}_A^{(r)} \tag{C.1.18}$$

subject to gauge variations

$$\delta\mathscr{A}^{(r)} = \alpha^{(p)}\mathscr{A}^{(q)}c_{(p)(q)}{}^{(r)} - \mathrm{i}d\alpha^{(r)} . \tag{C.1.19}$$

The covariant exterior derivatives of the matter superfields are defined as

$$\mathscr{D}\phi^k = (d + \mathrm{i}\mathscr{A}^{(r)}V_{(r)})\phi^k , \tag{C.1.20}$$

$$\mathscr{D}\bar{\phi}^{\bar{k}} = (d + \mathrm{i}\mathscr{A}^{(r)}\bar{V}_{(r)})\bar{\phi}^{\bar{k}} . \tag{C.1.21}$$

By construction, they change covariantly under gauged isometries, i.e.

$$\delta\mathscr{D}\phi^k = -\alpha^{(r)}\frac{\partial V_{(r)}{}^k}{\partial\phi^l}\mathscr{D}\phi^l , \tag{C.1.22}$$

$$\delta\mathscr{D}\bar{\phi}^{\bar{k}} = -\alpha^{(r)}\frac{\partial \bar{V}_{(r)}{}^{\bar{k}}}{\partial\bar{\phi}^{\bar{l}}}\mathscr{D}\bar{\phi}^{\bar{l}} . \tag{C.1.23}$$

Of course, the covariant exterior derivative is no longer nilpotent, its square being related to the field strength

$$\mathscr{F}^{(r)} = d\mathscr{A}^{(r)} + \frac{i}{2}\mathscr{A}^{(p)}\mathscr{A}^{(q)}c_{(p)(q)}{}^{(r)} , \tag{C.1.24}$$

such that

$$\mathscr{D}\mathscr{D}\phi^k = i\mathscr{F}^{(r)}V_{(r)}\phi^k , \tag{C.1.25}$$

$$\mathscr{D}\mathscr{D}\bar{\phi}^{\bar{k}} = i\mathscr{F}^{(r)}\bar{V}_{(r)}\bar{\phi}^{\bar{k}} . \tag{C.1.26}$$

In a somewhat more explicit notation, i.e.

$$\mathscr{D}\phi^k = E^A\mathscr{D}_A\phi^k, \quad \mathscr{D}\bar{\phi}^{\bar{k}} = E^A\mathscr{D}_A\bar{\phi}^{\bar{k}} \tag{C.1.27}$$

and

$$\mathscr{F}^{(r)} = \tfrac{1}{2}E^AE^B\mathscr{F}^{(r)}_{BA} , \tag{C.1.28}$$

this yields the graded commutation relations

$$(\mathscr{D}_B, \mathscr{D}_A)\phi^k = -T_{BA}{}^C\mathscr{D}_C\phi^k + i\mathscr{F}^{(r)}_{BA}V_{(r)}\phi^k , \tag{C.1.29}$$

$$(\mathscr{D}_B, \mathscr{D}_A)\bar{\phi}^{\bar{k}} = -T_{BA}{}^C\mathscr{D}_C\bar{\phi}^{\bar{k}} + i\mathscr{F}^{(r)}_{BA}\bar{V}_{(r)}\bar{\phi}^{\bar{k}} . \tag{C.1.30}$$

C.2. Isometric Kähler superspace

The composite Kähler gauge potential was defined in terms of chiral matter superfields as a 1-form in superspace such that

$$A = \frac{1}{4}(K_k d\phi^k - K_{\bar{k}}d\bar{\phi}^{\bar{k}}) + \frac{i}{8}E^a(12G_a + \bar{\sigma}_a^{\dot{\alpha}\alpha}g_{k\bar{k}}\mathscr{D}_\alpha\phi^k\mathscr{D}_{\dot{\alpha}}\bar{\phi}^{\bar{k}}) . \tag{C.2.1}$$

Consider now the spinor derivatives to be covariant with respect to gauged isometries, as defined above, rendering the last term invariant. However, the term

$$\Delta = K_k d\phi^k - K_{\bar{k}}d\bar{\phi}^{\bar{k}} \tag{C.2.2}$$

changes under gauged isometry transformations as

$$\delta\Delta = -2i d\,\mathrm{Im}(\alpha^{(r)}F_{(r)}) + 2i(d\alpha^{(r)})G_{(r)} . \tag{C.2.3}$$

This can be verified using the relations presented so far. Interestingly enough, the first term has the form of a gauge transformation, it closely resembles a Kähler transformation. As to the second term, it is easy to see that it corresponds to

$$\delta(\mathscr{A}^{(r)}G_{(r)}) = -\mathrm{i}(d\alpha^{(r)})G_{(r)}\ . \tag{C.2.4}$$

Therefore, the combination

$$\tilde{\Delta} = \Delta + 2\mathscr{A}^{(r)}G_{(r)} \tag{C.2.5}$$

transforms as a gauge field, both under gauged isometries and under Kähler transformations, i.e.

$$\delta\tilde{\Delta} = 2\mathrm{i}\, d[\mathrm{Im}(F - \alpha^{(r)}F_{(r)})]\ . \tag{C.2.6}$$

This is completely in line with our understanding of supergravity/matter couplings, i.e. gauged isometries can be reconciled with the structure of Kähler superspace provided we replace Δ by $\tilde{\Delta}$ and require that the frame of superspace changes under a gauged isometry as well such that

$$\delta E^A = -\frac{\mathrm{i}}{2}w(E^A)E^A\,\mathrm{Im}(-\alpha^{(r)}F_{(r)})\ . \tag{C.2.7}$$

This leads us to the definition of *isometric Kähler superspace*, with a modified composite gauge potential

$$\mathfrak{A} = \tfrac{1}{4}K_k d\phi^k - \tfrac{1}{4}K_{\bar{k}}d\bar{\phi}^{\bar{k}} + \tfrac{1}{2}\mathscr{A}^{(r)}G_{(r)} + \frac{\mathrm{i}}{8}E^a(12G_a + \bar{\sigma}_a^{\dot{\alpha}\alpha}g_{k\bar{k}}\mathscr{D}_\alpha\phi^k\mathscr{D}_{\dot{\alpha}}\bar{\phi}^{\bar{k}}) \tag{C.2.8}$$

in the $U(1)$ sector, giving rise to the torsion 2-form

$$\mathfrak{T}^A = dE^A + E^B\phi_B{}^A + w(E^A)E^A\mathfrak{A}\ , \tag{C.2.9}$$

invariant under Kähler transformations and gauged isometries. Gauged isometries appear in the structure group of superspace via (C.2.7) in close analogy with Kähler transformations. Covariance with respect to these transformations is obtained with the help of the modified gauge potential defined in (C.2.8) and the usual rules of Kähler superspace. Furthermore, following definitions (C.1.20) and (C.1.21), the matter superfields are defined to be *covariantly chiral*, i.e.

$$\mathscr{D}_\alpha\bar{\phi}^{\bar{k}} = (E_\alpha{}^M\partial_M + \mathrm{i}\mathscr{A}_\alpha^{(r)}\bar{V}_{(r)})\bar{\phi}^{\bar{k}} = 0\ , \tag{C.2.10}$$

$$\mathscr{D}^{\dot{\alpha}}\phi^k = (E^{\dot{\alpha}M}\partial_M + \mathrm{i}\mathscr{A}^{(r)\dot{\alpha}}V_{(r)})\phi^k = 0\ . \tag{C.2.11}$$

Likewise, in the definition of $\mathfrak{A}$ – cf. (C.2.8), one has

$$\mathscr{D}_\alpha\phi^k = (E_\alpha{}^M\partial_M + \mathrm{i}\mathscr{A}_\alpha^{(r)}V_{(r)})\phi^k\ , \tag{C.2.12}$$

$$\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{k}} = (E^{\dot{\alpha}M}\partial_M + \mathrm{i}\mathscr{A}^{(r)\dot{\alpha}}\bar{V}_{(r)})\bar{\phi}^{\bar{k}}\ . \tag{C.2.13}$$

The superspace geometry we have established here describes supergravity and matter and accounts consistently for Kähler transformations and for gauged isometries of the Kähler metric (of which Yang–Mills symmetries are a particular case).

The field strength superfields $X_\alpha, \bar{X}^{\dot{\alpha}}$, already discussed in the ungauged case, receive now additional contributions (hereafter we shall denote them $\mathfrak{X}_\alpha$ and $\bar{\mathfrak{X}}^{\dot{\alpha}}$), involving the Yang–Mills field strength, $\mathscr{F}^{(r)}$, and the Killing potential, $G_{(r)}$. To see this, apply the exterior derivative to $\tilde{\Delta}$ to obtain

$$d\tilde{\Delta} = 2g_{k\bar{k}}\mathscr{D}\phi^k\mathscr{D}\bar{\phi}^{\bar{k}} + 2\mathscr{F}^{(r)}G_{(r)} \ . \tag{C.2.14}$$

Due to

$$\mathfrak{A} = \frac{1}{4}\tilde{\Delta} + \frac{i}{8}E^a(12G_a + \bar{\sigma}_a^{\dot{\alpha}\alpha}g_{k\bar{k}}\mathscr{D}_\alpha\phi^k\mathscr{D}_{\dot{\alpha}}\bar{\phi}^{\bar{k}}) \ , \tag{C.2.15}$$

the relation between $d\tilde{\Delta}$ and $\mathfrak{F} = d\mathfrak{A}$ is obvious. As before, the superfields $\mathfrak{X}_\alpha$ and $\bar{\mathfrak{X}}^{\dot{\alpha}}$ are identified in the field strengths $\mathfrak{F}^{\beta}{}_a$ and $\mathfrak{F}_{\beta a}$ as

$$\mathfrak{X}_\alpha = -\frac{i}{2}g_{k\bar{k}}\sigma^a_{\alpha\dot{\alpha}}\mathscr{D}_a\phi^k\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{k}} + \frac{1}{2}g_{k\bar{k}}\mathscr{D}_\alpha\phi^k\bar{F}^{\bar{k}} + \mathscr{W}^{(r)}_\alpha G_{(r)} \ , \tag{C.2.16}$$

$$\bar{\mathfrak{X}}^{\dot{\alpha}} = -\frac{i}{2}g_{k\bar{k}}\bar{\sigma}^{a\dot{\alpha}\alpha}\mathscr{D}_a\bar{\phi}^{\bar{k}}\mathscr{D}_\alpha\phi^k + \frac{1}{2}g_{k\bar{k}}\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{k}}F^k + \mathscr{W}^{(r)\dot{\alpha}}G_{(r)} \ . \tag{C.2.17}$$

In distinction to the ungauged case all derivatives are now fully covariant with respect to gauged isometries. F^k and $\bar{F}^{\bar{k}}$ are still defined as

$$F^k = -\tfrac{1}{4}\mathscr{D}^\alpha\mathscr{D}_\alpha\phi^k, \quad \bar{F}^{\bar{k}} = -\tfrac{1}{4}\mathscr{D}_{\dot{\alpha}}\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{k}} \ , \tag{C.2.18}$$

but the covariant derivatives of $\mathscr{D}_\alpha\phi^k$ and $\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{k}}$ appearing in this definition contain now new terms which take into account the gauged isometries, explicitly

$$\mathscr{D}_B\mathscr{D}_\alpha\phi^k = E_B{}^M\partial_M\mathscr{D}_\alpha\phi^k - \phi_{B\alpha}{}^\varphi\mathscr{D}_\varphi\phi^k + i\mathscr{A}_B^{(r)}\frac{\partial V_{(r)}{}^k}{\partial\phi^l}\mathscr{D}_\alpha\phi^l - \mathfrak{A}_B\mathscr{D}_\alpha\phi^k + \Gamma^k{}_{ij}\mathscr{D}_B\phi^i\mathscr{D}_\alpha\phi^j \ , \tag{C.2.19}$$

$$\mathscr{D}_B\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{k}} = E_B{}^M\partial_M\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{k}} - \phi_B{}^{\dot{\alpha}}{}_{\dot{\varphi}}\mathscr{D}^{\dot{\varphi}}\bar{\phi}^{\bar{k}} + i\mathscr{A}_B^{(r)}\frac{\partial\bar{V}_{(r)}{}^{\bar{k}}}{\partial\bar{\phi}^{\bar{l}}}\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{l}} + \mathfrak{A}_B\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{k}} + \Gamma^{\bar{k}}{}_{\bar{i}\bar{j}}\mathscr{D}_B\bar{\phi}^{\bar{i}}\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{j}} \ . \tag{C.2.20}$$

The Yang–Mills superfields appearing in (C.2.16) and (C.2.17) are identified in the field strength $\mathscr{F}^{(r)}$, i.e.

$$\mathscr{F}^{(r)}_{\beta a} = i\sigma_{a\beta\dot{\beta}}\mathscr{W}^{(r)\dot{\beta}}, \qquad \mathscr{F}^{(r)\dot{\beta}}_{a} = i\bar{\sigma}_a^{\dot{\beta}\beta}\mathscr{W}^{(r)}_\beta \tag{C.2.21}$$

and satisfy relations (B.5.12) and (B.5.13). Since the Yang–Mills gauge potentials are now defined in the framework of Kähler superspace geometry, all the chiral weights and therefore the transformation laws under Kähler transformations and gauged isometries are determined and should be taken into account in the definition of covariant derivatives.

The relevant quantity in the construction of the component field action is the Kähler D-term, defined as the lowest component of the superfield $\mathscr{D}^\alpha\mathfrak{X}_\alpha$. The geometric construction presented so far has the great advantage that full invariance is automatically ensured. The explicit form of the D-term superfield is

$$-\tfrac{1}{2}\mathscr{D}^\alpha \mathfrak{X}_\alpha = -g_{k\bar{k}}\eta^{ab}\mathscr{D}_b\phi^k\mathscr{D}_a\bar{\phi}^{\bar{k}} - \frac{i}{4}g_{k\bar{k}}\sigma^a_{\alpha\dot{\alpha}}(\mathscr{D}^\alpha\phi^k\mathscr{D}_a\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{k}} + \mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{k}}\mathscr{D}_a\mathscr{D}^\alpha\phi^k)$$
$$+ g_{k\bar{k}}F^k\bar{F}^{\bar{k}} + \tfrac{1}{16}R_{j\bar{j}k\bar{k}}\mathscr{D}^\alpha\phi^k\mathscr{D}_\alpha\phi^j\mathscr{D}_{\dot{\alpha}}\bar{\phi}^{\bar{k}}\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{j}} - i g_{k\bar{k}}\bar{V}_{(r)}{}^{\bar{k}}\mathscr{W}^{(r)}_\alpha\mathscr{D}^\alpha\phi^k$$
$$- i g_{k\bar{k}}V_{(r)}{}^k\mathscr{W}^{(r)}_{\dot{\alpha}}\mathscr{D}^{\dot{\alpha}}\bar{\phi}^{\bar{k}} - \tfrac{1}{2}(\mathscr{D}^\alpha\mathscr{W}^{(r)}_\alpha)G_{(r)}\ . \tag{C.2.22}$$

The discussion of this section shows that gauged isometries allow for a very suggestive description in the framework of Kähler superspace geometry. The results presented here in superfield form are particularly useful to extract component field expressions in a constructive and concise way as illustrated in Section 4, where we fully develop Lagrangians in component fields.

So far we have mainly dealt with matter superfields, which play the role of coordinates of the Kähler manifold, and with their covariant differentials. It will be useful to consider the more general case of a generic superfield, $\mathbf{U}^k$, of transformation law

$$\delta\mathbf{U}^k = -\alpha^{(r)}\frac{\delta V_{(r)}{}^k}{\delta\phi^l}\mathbf{U}^l\ . \tag{C.2.23}$$

For simplicity, we assume $\mathbf{U}^k$ to be a superfield (0-form) of vanishing chiral weight and scalar with respect to Lorentz transformations. The exterior covariant derivative is then defined as

$$\mathscr{D}\mathbf{U}^k = d\mathbf{U}^k + i\mathscr{A}^{(r)}\frac{\partial V_{(r)}{}^k}{\partial\phi^l}\mathbf{U}^l + \Gamma^k{}_{lm}\mathscr{D}\phi^m\mathbf{U}^l \tag{C.2.24}$$

with

$$\mathscr{D}\mathbf{U}^k = E^A\mathscr{D}_A\mathbf{U}^k\ . \tag{C.2.25}$$

Note that, as a consequence of the chirality of the matter superfields, the Levi–Civita term is absent in $\mathscr{D}^{\dot{\alpha}}\mathbf{U}^k$.

The graded commutator of two such covariant derivatives is obtained in taking the covariant exterior derivative of the 1-form $\mathscr{D}\mathbf{U}^k$, i.e.

$$\mathscr{D}\mathscr{D}\mathbf{U}^k = i\mathscr{F}^{(r)}\left(\frac{\partial V_{(r)}{}^k}{\partial\phi^l}\mathbf{U}^l + V_{(r)}{}^m\Gamma_{lm}{}^k\mathbf{U}^l\right) - g^{k\bar{k}}R_{m\bar{k}l\bar{j}}\mathscr{D}\bar{\phi}^{\bar{j}}\mathscr{D}\phi^m\mathbf{U}^l\ . \tag{C.2.26}$$

Decomposing the left-hand term according to

$$\mathscr{D}\mathscr{D}\mathbf{U}^k = E^AE^B(\mathscr{D}_B\mathscr{D}_A\mathbf{U}^k + \tfrac{1}{2}T_{BA}{}^C\mathscr{D}_C\mathbf{U}^k)\ , \tag{C.2.27}$$

allows to read off the graded commutator of two covariant derivatives of $\mathbf{U}^k$ to be

$$(\mathscr{D}_B, \mathscr{D}_A)\mathbf{U}^k = -T_{BA}{}^C \mathscr{D}_C \mathbf{U}^k + \mathrm{i}\mathscr{F}_{BA}{}^{(r)} \left(\frac{\partial V_{(r)}{}^k}{\partial \phi^l} \mathbf{U}^l + V_{(r)}{}^m \Gamma^k{}_{lm} \mathbf{U}^l \right) + g^{k\bar{k}} R_{m\bar{k}l\bar{j}} \mathbf{U}^l (\mathscr{D}_B \bar{\phi}^{\bar{j}} \mathscr{D}_A \phi^m - (-)^{ab} \mathscr{D}_A \bar{\phi}^{\bar{j}} \mathscr{D}_B \phi^m) \, . \tag{C.2.28}$$

We have considered $\mathbf{U}^k$ as a superfield inert under Lorentz and Kähler transformations. The spinor derivative $\mathscr{D}_\alpha \phi^k$ of a chiral superfield ϕ^k will transform in the same manner as $\mathbf{U}^k$ under gauged isometries but will pick up additional contributions from Lorentz and Kähler transformations.

C.3. Supergravity transformations

We have constructed a superspace geometry in terms of the basic geometric objects

- $E^A = dz^M E_M{}^A$ frame of superspace,
- $\phi^k, \bar{\phi}^{\bar{k}}$ chiral matter superfields,
- $\mathscr{A}^{(r)} = dz^M \mathscr{A}_M^{(r)}$ Yang–Mills potential.

The chiral matter superfields take their values in a Kähler manifold and we have seen that superspace geometry and Kähler geometry are intimately related. In order to describe gauged isometries of the superfield Kähler metric we have introduced the corresponding Yang–Mills potential. Infinitesimal variations of parameters

- ξ_M superspace diffeomorphisms,
- $\Lambda_B{}^A$ Lorentz transformations,
- $\alpha^{(r)}$ Yang–Mills transformations,

change the basic geometric objects such that

$$E^A \mapsto E^A + \delta E^A \, , \tag{C.3.1}$$

$$\phi^k \mapsto \phi^k + \delta\phi^k \, , \tag{C.3.2}$$

$$\bar{\phi}^{\bar{k}} \mapsto \bar{\phi}^{\bar{k}} + \delta\bar{\phi}^{\bar{k}} \, , \tag{C.3.3}$$

$$\mathscr{A}^{(r)} \mapsto \mathscr{A}^{(r)} + \delta\mathscr{A}^{(r)} \tag{C.3.4}$$

with

$$\delta E^A = L_\xi E^A + E^B \Lambda_B{}^A - \frac{\mathrm{i}}{2} w(E^A) E^A \, \mathrm{Im}(F(\phi) - \alpha^{(r)} F_{(r)}(\phi)) \, , \tag{C.3.5}$$

$$\delta\phi^k = L_\xi \phi^k - \alpha^{(r)} V_{(r)}{}^k(\phi) \, , \tag{C.3.6}$$

$$\delta\bar{\phi}^{\bar{k}} = L_\xi \bar{\phi}^{\bar{k}} - \alpha^{(r)} \bar{V}_{(r)}{}^{\bar{k}}(\bar{\phi}) \, , \tag{C.3.7}$$

$$\delta\mathscr{A}^{(r)} = L_\xi \mathscr{A}^{(r)} - \mathrm{i}d\alpha^{(r)} + \alpha^{(p)} \mathscr{A}^{(q)} c_{(p)(q)}{}^{(r)} \, . \tag{C.3.8}$$

Here, the Lie derivative in superspace is defined as

$$L_\xi = \imath_\xi d + d\imath_\xi \ . \tag{C.3.9}$$

Remarkably enough, Kähler transformations and gauged isometries appear in a well-defined way in the structure group of superspace. In the next step we wish to express these transformation laws as much as possible in terms of covariant objects – torsion, field strength and covariant derivatives – which were defined earlier as

$$\mathfrak{T}^A = \mathscr{D}E^A = dE^A + E^B\phi_B{}^A + w(E^A)E^A\mathfrak{A} \ , \tag{C.3.10}$$

$$\mathscr{D}\phi^k = d\phi^k + \mathrm{i}\mathscr{A}^{(r)}V_{(r)}{}^k(\phi) \ , \tag{C.3.11}$$

$$\mathscr{D}\bar{\phi}^{\bar{k}} = d\bar{\phi}^{\bar{k}} + \mathrm{i}\mathscr{A}^{(r)}\bar{V}_{(r)}{}^{\bar{k}}(\bar{\phi}) \ , \tag{C.3.12}$$

$$\mathscr{F}^{(r)} = d\mathscr{A}^{(r)} + \frac{\mathrm{i}}{2}\mathscr{A}^{(p)}\mathscr{A}^{(q)}c_{(p)(q)}{}^{(r)} \ . \tag{C.3.13}$$

Straightforward substitution yields

$$\delta E^A = \mathscr{D}\xi^A + \imath_\xi\mathfrak{T}^A + E^B(\Lambda_B{}^A - \imath_\xi\phi_B{}^A) - w(E^A)E^A\left[\imath_\xi\mathfrak{A} + \frac{\mathrm{i}}{2}\,\mathrm{Im}(F - \alpha^{(r)}F_{(r)})\right] , \tag{C.3.14}$$

$$\delta\phi^k = \imath_\xi\mathscr{D}\phi^k - (\alpha^{(r)} + \mathrm{i}\imath_\xi\mathscr{A}^{(r)})V_{(r)}{}^k(\phi) \ , \tag{C.3.15}$$

$$\delta\bar{\phi}^{\bar{k}} = \imath_\xi\mathscr{D}\bar{\phi}^{\bar{k}} - (\alpha^{(r)} + \mathrm{i}\imath_\xi\mathscr{A}^{(r)})\bar{V}_{(r)}{}^{\bar{k}}(\bar{\phi}) \ , \tag{C.3.16}$$

$$\delta\mathscr{A}^{(r)} = \imath_\xi\mathscr{F}^{(r)} + (\alpha^{(p)} + \mathrm{i}\imath_\xi\mathscr{A}^{(p)})\mathscr{A}^{(q)}c_{(p)(q)}{}^{(r)} - \mathrm{i}d(\alpha^{(r)} + \mathrm{i}\imath_\xi\mathscr{A}^{(r)}) \ . \tag{C.3.17}$$

Supergravity transformations δ_{WZ} are then defined as certain combinations of superspace diffeomorphisms and field-dependent compensating Lorentz and gauged isometry transformations, namely

$$\Lambda_B{}^A = \imath_\xi\phi_B{}^A \ , \tag{C.3.18}$$

$$\alpha^{(r)} = -\mathrm{i}\imath_\xi\mathscr{A}^{(r)} \ . \tag{C.3.19}$$

Taking into account the explicit form of $\mathfrak{A}$, – cf. (C.2.8), we obtain

$$\begin{aligned}\delta_{\mathrm{WZ}}\,E^A = {} & \mathscr{D}\xi^A + \imath_\xi\mathfrak{T}^A - \frac{1}{4}w(E^A)E^A(K_k\imath_\xi\mathscr{D}\phi^k - K_{\bar{k}}\imath_\xi\mathscr{D}\bar{\phi}^{\bar{k}}) \\ & - \frac{\mathrm{i}}{8}w(E^A)E^A\xi^b(12G_b + \bar{\sigma}_b^{\dot{\alpha}\alpha}g_{k\bar{k}}\mathscr{D}_\alpha\phi^k\mathscr{D}_{\dot{\alpha}}\bar{\phi}^{\bar{k}}) \ ,\end{aligned} \tag{C.3.20}$$

$$\delta_{\mathrm{WZ}}\,\phi^k = \imath_\xi\mathscr{D}\phi^k \ , \tag{C.3.21}$$

$$\delta_{\mathrm{WZ}}\,\bar{\phi}^{\bar{k}} = \imath_\xi\mathscr{D}\bar{\phi}^{\bar{k}} \ , \tag{C.3.22}$$

$$\delta_{\mathrm{WZ}}\,\mathscr{A}^{(r)} = \imath_\xi\mathscr{F}^{(r)}. \tag{C.3.23}$$

Recall that the last term in the transformation law of E^A is spurious in that it could be absorbed in covariant redefinitions of the first two terms. The interior product of ξ^M with torsion and Yang–Mills field strength is defined as

$$\imath_\xi \mathfrak{T}^A = E^B \xi^C \mathfrak{T}_{CB}{}^A \,, \tag{C.3.24}$$

$$\imath_\xi \mathscr{F}^{(r)} = E^A \xi^B \mathscr{F}^{(r)}_{BA} \,. \tag{C.3.25}$$

For later convenience we consider also generic superfields Φ and $\mathbf{U}^k$ of transformation laws

$$\delta\Phi = L_\xi d\Phi - \frac{i}{2} w(\Phi)\Phi \operatorname{Im}(F(\phi) - \alpha^{(r)} F_{(r)}(\phi)) \,, \tag{C.3.26}$$

$$\delta \mathbf{U}^k = L_\xi \mathbf{U}^k - \alpha^{(r)} \frac{\partial V_{(r)}{}^k}{\partial \phi^l} \mathbf{U}^l \tag{C.3.27}$$

and covariant derivatives

$$\mathscr{D}\Phi = d\Phi + w(\Phi)\Phi\mathfrak{A} \,, \tag{C.3.28}$$

$$\mathscr{D}\mathbf{U}^k = d\mathbf{U}^k + i\mathscr{A}^{(r)} \frac{\partial V_{(r)}{}^k}{\partial \phi^l} \mathbf{U}^l + \Gamma^k{}_{lm} \mathscr{D}\phi^l \mathbf{U}^m \,. \tag{C.3.29}$$

Straightforward substitution allows to derive the supergravity transformations

$$\begin{aligned} \delta_{\mathrm{WZ}}\, \Phi = {} & \imath_\xi \mathscr{D}\Phi - \tfrac{1}{4} w(\Phi)\Phi(K_k \imath_\xi \mathscr{D}\phi^k - K_{\bar{k}} \imath_\xi \mathscr{D}\bar{\phi}^{\bar{k}}) \\ & - \frac{i}{8} w(\Phi)\Phi \xi^a (12 G_a + \bar{\sigma}_a^{\dot{\alpha}\alpha} g_{k\bar{k}} \mathscr{D}_\alpha \phi^k \mathscr{D}_{\dot{\alpha}} \bar{\phi}^{\bar{k}}) \,, \end{aligned} \tag{C.3.30}$$

$$\delta_{\mathrm{WZ}}\, \mathbf{U}^k = \imath_\xi \mathscr{D}\mathbf{U}^k + \Gamma^k{}_{lm} \imath_\xi \mathscr{D}\phi^l \mathbf{U}^m \,. \tag{C.3.31}$$

The supergravity transformations presented so far at the full superfield level will provide the basic building blocks for the derivation of supersymmetry transformations of the component fields. We will also use these supergravity transformations in the more explicit form

$$\begin{aligned} \delta_{\mathrm{WZ}}\, E_M{}^A = {} & \mathscr{D}_M \xi^A + E_M{}^B \xi^C \mathfrak{T}_{CB}{}^A - \tfrac{1}{4} w(E^A) E_M{}^A \xi^B (K_k \mathscr{D}_B \phi^k - K_{\bar{k}} \mathscr{D}_B \bar{\phi}^{\bar{k}}) \\ & - \frac{i}{8} w(E^A) E_M{}^A \xi^b (12 G_b + \bar{\sigma}_b^{\dot{\alpha}\alpha} g_{k\bar{k}} \mathscr{D}_\alpha \phi^k \mathscr{D}_{\dot{\alpha}} \bar{\phi}^{\bar{k}}) \,, \end{aligned} \tag{C.3.32}$$

$$\delta_{\mathrm{WZ}}\, \phi^k = \xi^A \mathscr{D}_A \phi^k \,, \tag{C.3.33}$$

$$\delta_{\mathrm{WZ}}\, \bar{\phi}^{\bar{k}} = \xi^A \mathscr{D}_A \bar{\phi}^{\bar{k}} \,, \tag{C.3.34}$$

$$\begin{aligned} \delta_{\mathrm{WZ}}\, \Phi = {} & \xi^A \mathscr{D}_A \Phi - \tfrac{1}{4} w(\Phi)\Phi \xi^A (K_k \mathscr{D}_A \phi^k - K_{\bar{k}} \mathscr{D}_A \bar{\phi}^{\bar{k}}) \\ & - \frac{i}{8} w(\Phi)\Phi \xi^b (12 G_b + \bar{\sigma}_b^{\dot{\alpha}\alpha} g_{k\bar{k}} \mathscr{D}_\alpha \phi^k \mathscr{D}_{\dot{\alpha}} \bar{\phi}^{\bar{k}}) \,, \end{aligned} \tag{C.3.35}$$

$$\delta_{\mathrm{WZ}}\, \mathbf{U}^k = \xi^A \mathscr{D}_A \mathbf{U}^k + \Gamma^k{}_{lm} \xi^A \mathscr{D}_A \phi^l \mathbf{U}^m \,. \tag{C.3.36}$$

Observe the presence of the terms

$$K_k \imath_\xi \mathscr{D}\phi^k - K_{\bar{k}} \imath_\xi \mathscr{D}\bar{\phi}^{\bar{k}} = \xi^A (K_k \mathscr{D}_A \phi^k - K_{\bar{k}} \mathscr{D}_A \bar{\phi}^{\bar{k}}) \ . \tag{C.3.37}$$

The corresponding gauge transformations are field-dependent Kähler transformations and isometries, there is no free parameter which could compensate these terms unlike the case of Lorentz and Yang–Mills transformations.

C.4. The Yang–Mills case

Let us consider the situation where the gauged isometries reduce to the standard Yang–Mills transformations. This corresponds to the case where the isometries act linearly on the fields such that

$$V_{(r)}{}^k = V_{(r)} \phi^k = \ + \mathrm{i}(\mathbf{T}_{(r)} \phi)^k \ , \tag{C.4.1}$$

$$\bar{V}_{(r)}{}^{\bar{k}} = \bar{V}_{(r)} \bar{\phi}^{\bar{k}} = \ - \mathrm{i}(\bar{\phi} \mathbf{T}_{(r)})^{\bar{k}} \ , \tag{C.4.2}$$

where the $\mathbf{T}_{(r)}$, are in a suitable matrix representation of the generators of the gauge group considered, with commutation relations

$$[\mathbf{T}_{(r)}, \mathbf{T}_{(s)}] = \mathrm{i} c_{(r)(s)}{}^{(t)} \mathbf{T}_{(t)} \ , \tag{C.4.3}$$

implied by those of the $V_{(r)}$'s. Using the notation $\mathscr{A} = \mathscr{A}^{(r)} \mathbf{T}_{(r)}$, the covariant derivatives of the matter superfields take the form

$$\mathscr{D}\phi^k = (d\phi - \mathscr{A}\phi)^k, \quad \mathscr{D}\bar{\phi}^{\bar{k}} = (d\bar{\phi} + \bar{\phi}\mathscr{A})^{\bar{k}} \ . \tag{C.4.4}$$

Next, we can determine the Killing potential using (C.1.8) and (C.1.9). Since the Kähler potential is invariant under gauge transformations, (C.1.9) tells us

$$K_k (\mathbf{T}_{(r)} \phi)^k - K_{\bar{k}} (\bar{\phi} \mathbf{T}_{(r)})^{\bar{k}} = 0 = F_{(r)}(\phi) + \bar{F}_{(r)}(\bar{\phi}) \ , \tag{C.4.5}$$

implying that $F_{(r)}(\phi)$ and $\bar{F}_{(r)}(\bar{\phi})$ are just constants, which can safely be set to zero. The real Killing potential $G_{(r)}$ then becomes

$$G_{(r)} = \ + \frac{\mathrm{i}}{2} (K_k V_{(r)}{}^k - K_{\bar{k}} \bar{V}_{(r)}{}^{\bar{k}}) = \ - \frac{1}{2} [K_k (\mathbf{T}_{(r)} \phi)^k + K_{\bar{k}} (\bar{\phi} \mathbf{T}_{(r)})^{\bar{k}}] \ . \tag{C.4.6}$$

Using this information, together with the vanishing of the Killing pre-potentials, in the combination $\tilde{\Delta} = \Delta + 2\mathscr{A}^{(r)} G_{(r)}$, we obtain

$$\begin{aligned} \Delta + 2\mathscr{A}^{(r)} G_{(r)} &= K_k (d\phi^k + \mathrm{i}\mathscr{A}^{(r)} V_{(r)}{}^k) - K_{\bar{k}} (d\bar{\phi}^{\bar{k}} + \mathrm{i}\mathscr{A}^{(r)} \bar{V}_{(r)}{}^{\bar{k}}) \\ &= K_k \mathscr{D}\phi^k - K_{\bar{k}} \mathscr{D}\bar{\phi}^{\bar{k}} \ . \end{aligned} \tag{C.4.7}$$

As a consequence, we recover the Kähler connection $A = \mathfrak{A}$ of Section 3.4.2, given as

$$A = \frac{1}{4} K_k \mathscr{D}\phi^k - \frac{1}{4} K_{\bar{k}} \mathscr{D}\bar{\phi}^{\bar{k}} + \frac{i}{8} E^a (12 G_a + \bar{\sigma}_a^{\dot{\alpha}\alpha} g_{k\bar{k}} \mathscr{D}_\alpha \phi^k \mathscr{D}_{\dot{\alpha}} \bar{\phi}^{\bar{k}}) \tag{C.4.8}$$

with Yang–Mills covariant derivatives everywhere.

Finally, the supergravity transformations are directly read off from the previous discussions, Eqs. (C.3.14)–(C.3.17) and (C.3.32)–(C.3.36).

Appendix D. Superfield equations of motion

Given the geometric formulation of supersymmetric theories it is desirable to have a superfield action principle, in the sense that the variation of suitable superspace densities gives rise to superfield equations of motion.

On the other hand, the geometric descriptions of supersymmetric theories are characterized by covariant constraints (torsion constraints for supergravity, field strength constraints for Yang–Mills, 2- and 3-form gauge theories and chirality constraints for matter superfields). As a consequence, the basic building blocks initially used in the geometric construction (frame of superspace, Lorentz, Yang–Mills, 2- and 3-form gauge potentials, and chiral superfields) are no longer the fundamental objects – they are given in terms of unconstrained pre-potentials which arise from the explicit solution of the superspace constraint equations.

A possible way to formulate a superfield action principle is therefore to write superfield densities in terms of the unconstrained pre-potentials and to vary them accordingly [80]. This approach is particularly useful in the context of supergraph perturbation theory.

Another possibility [158], more closely related to superspace geometry, and which will be pursued here, is to solve directly the variational version of the constraint equations. In this way, one determines directly the variations of the basic geometric objects in terms of unconstrained entities. In this (equivalent) formulation, superspace densities are written in the usual way and the relation to component field formalism is quite transparent.

In this appendix we derive, as an example, the superfield equations of motion for the complete supergravity/matter/Yang–Mills system in the presence of gauged isometries. In the first two subsections, we work in generic $U(1)$ superspace, defining and solving the variational constraint equations in the first subsection and discussing superspace densities and integration by parts in the second one. The variational equations pertaining to isometric superspace are treated in Section D.4. In Section D.5 we derive the superfield equations of motion for the complete supergravity/matter/Yang–Mills system.

D.1. Integration by parts in $U(1)$ superspace

The superfield action principle for supergravity proposed by Wess and Zumino [158,163] is a generalization of usual gravity. In general relativity, especially when coupled to spinor fields, densities are constructed by means of the determinant of the vierbein, or frame. The corresponding basic superspace object is E, the superdeterminant of the frame $E_M{}^A$ in superspace. In general, a supersymmetric action will be given as the product of E with some suitable covariant superfield, integrated over superspace, i.e. over space–time and the anticommuting spinor coordinates. In the derivation of superfield equations of motion, integration by parts in superspace will be used systematically. This means that expressions like

$$\int_* E\,\mathscr{D}_\alpha v^\alpha, \quad \int_* E\,\mathscr{D}^{\dot\alpha} v_{\dot\alpha}, \quad \int_* E\,\mathscr{D}_a v^a \tag{D.1.1}$$

with v^A some generic, covariant superfield of chiral weight $w(v^A)$, should be related to pure superspace surface terms. The asterisk indicates that integration is understood over full superspace, i.e. anticommuting coordinates *and* space–time.

In order to explain the mechanism of integration by parts in some more detail let us recall first some definitions. The exterior covariant derivative $\mathscr{D}v^A = dz^M \mathscr{D}_M v^A$ being given as

$$\mathscr{D}v^A = dv^A + v^B \phi_B{}^A + w(v^A) v^A A \,, \tag{D.1.2}$$

we identify the 1-form coefficients

$$\mathscr{D}_M v^A = \partial_M v^A + (-)^{mb} v^B \phi_{MB}{}^A + w(v^A) A_M v^A \,. \tag{D.1.3}$$

Another crucial ingredient is the torsion 2-form $T^A = \frac{1}{2} dz^M dz^N T_{NM}{}^A$ defined as

$$T^A = \mathscr{D}E^A = dE^A + E^B \phi_B{}^A + w(E^A) E^A A \,. \tag{D.1.4}$$

Its components

$$T_{NM}{}^A = \mathscr{D}_N E_M{}^A - (-)^{mn} \mathscr{D}_M E_N{}^A \tag{D.1.5}$$

are given in terms of the covariant derivatives

$$\mathscr{D}_N E_M{}^A = \partial_N E_M{}^A + (-)^{(m+b)n} E_M{}^B \phi_{NB}{}^A + w(E^A) A_N E_M{}^A \,. \tag{D.1.6}$$

It is a matter of straightforward calculation to establish the superspace identity [163]

$$\partial_M (E v^A E_A{}^M)(-)^m = E[\partial_M v^A + v^N(\partial_N E_M{}^A - (-)^{mn} \partial_M E_N{}^A)] E_A{}^M (-)^m$$

Covariantizing the derivatives, this identity takes the form

$$\partial_M (E v^A E_A{}^M)(-)^m = E\mathscr{D}_A v^A (-)^a + E v^B T_{BA}{}^A (-)^a + E(w(E^A) - w(v^A)) v^A A_A \,. \tag{D.1.7}$$

This is the central point in the discussion of integration by parts in superspace. Observe that so far we did not make any use of torsion constraints. Taking into account the explicit form of the torsion coefficients in $U(1)$ superspace, one shows that the only non-vanishing contributions to the torsion term are

$$T_{b\alpha}{}^\alpha = +\mathrm{i}\, G_b, \quad T_b{}^{\dot\alpha}{}_{\dot\alpha} = -\mathrm{i}\, G_b \,, \tag{D.1.8}$$

which add to zero in the supertrace. The torsion term is therefore absent. If, in addition, we require

$$w(v^A) = w(E^A) , \tag{D.1.9}$$

we obtain

$$\partial_M(Ev^A E_A{}^M)(-)^m = E\mathscr{D}_A v^A(-)^a . \tag{D.1.10}$$

This establishes the relation alluded to above, identifying expressions like (D.1.1) as pure superspace surface terms. This relation will be frequently used in the derivation of superfield equations of motion.

D.2. Variational equations in $U(1)$ superspace

We first introduce as basic variables the variations of the vielbein, Lorentz and $U(1)$ connections modulo the effects of superspace diffeomorphisms and structure group transformations. Subsequently, we present a concise and systematic analysis of the consequences of the constraints of $U(1)$ superspace for these variables.

- *Basic definitions*: Consider the infinitesimal variations

$$\delta E^A = H^A , \tag{D.2.1}$$

$$\delta\phi_B{}^A = \Omega_B{}^A , \tag{D.2.2}$$

$$\delta A = \omega \tag{D.2.3}$$

of the frame, Lorentz and $U(1)$ gauge potential. These superspace 1-forms are parametrized in such a way that

$$H^A = E^B H_B{}^A, \qquad H_B{}^A = E_B{}^M \delta E_M{}^A , \tag{D.2.4}$$

$$\Omega_B{}^A = E^C \Omega_{CB}{}^A, \qquad \Omega_{CB}{}^A = E_C{}^M \delta\phi_{MB}{}^A , \tag{D.2.5}$$

$$\omega = E^A \omega_A, \qquad \omega_A = E_A{}^M \delta A_M . \tag{D.2.6}$$

As a consequence of these definitions the variations of torsion, curvature and $U(1)$ field strength become

$$\delta T^A = \mathscr{D}H^A + E^B\Omega_B{}^A + w(E^A)E^A\omega , \tag{D.2.7}$$

$$\delta R_B{}^A = \mathscr{D}\Omega_B{}^A , \tag{D.2.8}$$

$$\delta F = d\omega . \tag{D.2.9}$$

Here, $\mathscr{D}$ denotes the covariant exterior derivative in $U(1)$ superspace. It is straightforward to work out the explicit expressions for the coefficients of these 2-forms in superspace. The torsion variational equations

$$\delta T_{CB}{}^{A} = \mathscr{D}_C H_B{}^A - (-)^{cb}\mathscr{D}_B H_C{}^A + T_{CB}{}^F H_F{}^A - H_C{}^F T_{FB}{}^A + (-)^{cb} H_B{}^F T_{FC}{}^A + \Omega_{CB}{}^A - (-)^{cb}\Omega_{BC}{}^A + w(E^A)(\delta_B^A \omega_C - (-)^{cb}\delta_C^A \omega_B) \tag{D.2.10}$$

are of particular importance. The vielbein and gauge potential variations must leave the torsion constraints invariant. This determines the unconstrained variational superfields. The corresponding variations of curvature and $U(1)$ field strength are

$$\delta R_{DCB}{}^A = \mathscr{D}_D \Omega_{CB}{}^A - (-)^{dc}\mathscr{D}_C \Omega_{DB}{}^A + T_{DC}{}^F \Omega_{FB}{}^A - H_D{}^F R_{FCB}{}^A + (-)^{dc} H_C{}^F R_{FDB}{}^A , \tag{D.2.11}$$

$$\delta F_{DC} = \mathscr{D}_D \omega_C - (-)^{dc}\mathscr{D}_C \omega_D + T_{DC}{}^F \omega_F - H_D{}^F F_{FC} + (-)^{dc} H_C{}^F F_{FD} . \tag{D.2.12}$$

Observe that the variational superfields are determined modulo diffeomorphisms and structure group transformations, i.e. upto redefinitions of the form

$$\underline{\delta} H^A = \mathscr{L}_\xi E^A + E^B \chi_B{}^A + w(E^A) E^A \rho , \tag{D.2.13}$$

$$\underline{\delta} \Omega_B{}^A = -\mathscr{D}\chi_B{}^A + \imath_\xi R_B{}^A , \tag{D.2.14}$$

$$\underline{\delta}\omega = -d\rho + \imath_\xi F . \tag{D.2.15}$$

As a consequence, the variational equations change as

$$\underline{\delta}\delta T^A = \mathscr{L}_\xi T^A + T^B \chi_B{}^A + w(T^A) T^A \rho , \tag{D.2.16}$$

$$\underline{\delta}\delta R_B{}^A = \mathscr{L}_\xi R_B{}^A + R_B{}^C \chi_C{}^A - \chi_B{}^C R_C{}^A , \tag{D.2.17}$$

$$\underline{\delta}\delta F = \mathscr{L}_\xi F . \tag{D.2.18}$$

The covariant Lie derivative appearing here is given as

$$\mathscr{L}_\xi = \imath_\xi \mathscr{D} + \mathscr{D}\imath_\xi . \tag{D.2.19}$$

Using $\imath_\xi E^A = \xi^A$, the variation of $H_B{}^A$ reads

$$\underline{\delta} H_B{}^A = \xi^C T_{CB}{}^A + \mathscr{D}_B \xi^A + \chi_B{}^A + w(E^A)\delta_B^A \rho . \tag{D.2.20}$$

Similarly,

$$\underline{\delta}\Omega_{CB}{}^A = -\mathscr{D}_C \chi_B{}^A + \xi^D R_{DCB}{}^A , \tag{D.2.21}$$

$$\underline{\delta}\omega_A = -\mathscr{D}_A \rho + \xi^B F_{BA} . \tag{D.2.22}$$

Clearly, the variational equations of the torsion constraints are invariant under these redefinitions.

- *Torsion constraints* I: In a first step we consider the variational equations of the torsions

$$T_{\gamma\beta}{}^a = 0, \qquad T^{\dot\gamma\dot\beta a} = 0 , \tag{D.2.23}$$

$$T_\gamma{}^{\dot\beta a} = -2i(\sigma^a \varepsilon)_\gamma{}^{\dot\beta} , \tag{D.2.24}$$

$$T_{\gamma\beta\dot\alpha} = 0, \qquad T^{\dot\gamma\dot\beta\alpha} = 0 . \tag{D.2.25}$$

From (D.2.10) we read off the explicit equations

$$\delta T_{\gamma\beta}{}^{a} = \sum_{\gamma\beta} (\mathscr{D}_{\gamma} H_{\beta}{}^{a} - H_{\gamma\dot{\varphi}} T_{\beta}{}^{\dot{\varphi}a}) , \tag{D.2.26}$$

$$\delta T_{\gamma\beta\dot{\alpha}} = \sum_{\gamma\beta} (\mathscr{D}_{\gamma} H_{\beta\dot{\alpha}} - H_{\gamma}{}^{f} T_{f\beta\dot{\alpha}}) . \tag{D.2.27}$$

The pure gauge solution of (D.2.26) and (D.2.27) is

$$H_{\beta}{}^{a} = \mathscr{D}_{\beta}\bar{\Xi}^{a} + \bar{\Xi}_{\dot{\varphi}} T_{\beta}{}^{\dot{\varphi}a} , \tag{D.2.28}$$

$$H_{\beta\dot{\alpha}} = \mathscr{D}_{\beta}\bar{\Xi}_{\dot{\alpha}} + \bar{\Xi}^{f} T_{f\beta\dot{\alpha}} . \tag{D.2.29}$$

Likewise, the complex conjugate equations are solved by

$$H^{\dot{\beta}a} = \mathscr{D}^{\dot{\beta}}\Xi^{a} + \Xi^{\varphi} T_{\varphi}{}^{\dot{\beta}a} , \tag{D.2.30}$$

$$H^{\dot{\beta}\alpha} = \mathscr{D}^{\dot{\beta}}\Xi^{\alpha} + \Xi^{f} T_{f}{}^{\dot{\beta}\alpha} . \tag{D.2.31}$$

Finally, making use of the invariance of the variational equations under redefinitions of the form (D.2.20) we arrive at

$$H_{\beta}{}^{a} = \mathscr{D}_{\beta}\mathscr{V}^{a}, \qquad H^{\dot{\beta}a} = -\mathscr{D}^{\dot{\beta}}\mathscr{V}^{a} , \tag{D.2.32}$$

$$H_{\beta\dot{\alpha}} = -\mathscr{V}^{c} T_{\beta c\dot{\alpha}} = \mathrm{i}R^{\dagger}\mathscr{V}_{\beta\dot{\alpha}}, \qquad H^{\dot{\beta}\alpha} = \mathscr{V}^{c} T^{\dot{\beta}}{}_{c}{}^{\alpha} = -\mathrm{i}R\mathscr{V}^{\alpha\dot{\beta}} . \tag{D.2.33}$$

It remains to discuss the variation of (D.2.24),

$$\delta T_{\gamma}{}^{\dot{\beta}a} = \mathscr{D}_{\gamma} H^{\dot{\beta}a} + \mathscr{D}^{\dot{\beta}} H_{\gamma}{}^{a} + T_{\gamma}{}^{\dot{\beta}f} H_{f}{}^{a} - H_{\gamma}{}^{\varphi} T_{\varphi}{}^{\dot{\beta}a} - H^{\dot{\beta}}{}_{\dot{\varphi}} T_{\gamma}{}^{\dot{\varphi}a} . \tag{D.2.34}$$

We eliminate the traceless parts of $H_{\beta}{}^{\alpha}$, $H^{\dot{\beta}}{}_{\dot{\alpha}}$ by suitably choosing $\chi_{\beta}{}^{\alpha}$, $\chi^{\dot{\beta}}{}_{\dot{\alpha}}$ in (D.2.20) to arrive at

$$H_{\beta}{}^{\alpha} = \tfrac{1}{2}\delta_{\beta}{}^{\alpha} H , \tag{D.2.35}$$

$$H^{\dot{\beta}}{}_{\dot{\alpha}} = \tfrac{1}{2}\delta^{\dot{\beta}}{}_{\dot{\alpha}} \bar{H} . \tag{D.2.36}$$

As a consequence, (D.2.34) becomes

$$T_{\gamma}{}^{\dot{\beta}f} H_{f}{}^{a} - \tfrac{1}{2}(H + \bar{H}) T_{\gamma}{}^{\dot{\beta}a} - [\mathscr{D}_{\gamma}, \mathscr{D}^{\dot{\beta}}]\mathscr{V}^{a} = 0 , \tag{D.2.37}$$

showing that $H_{b}{}^{a}$ is completely determined as a function of the unconstrained superfields $H + \bar{H}$ and $\mathscr{V}^{a}$. In spinor notation this equation reads

$$H_{\beta\dot{\beta}\alpha\dot{\alpha}} = -\varepsilon_{\beta\alpha}\varepsilon_{\dot{\beta}\dot{\alpha}}(H + \bar{H}) - \frac{\mathrm{i}}{2}[\mathscr{D}_{\beta}, \mathscr{D}_{\dot{\beta}}]\mathscr{V}_{\alpha\dot{\alpha}} . \tag{D.2.38}$$

The supertrace of $H_B{}^A$ is now given as

$$H_A{}^A(-)^a = H + \bar{H} + \frac{\mathrm{i}}{4}[\mathscr{D}^\alpha, \mathscr{D}^{\dot\alpha}]\mathscr{V}_{\alpha\dot\alpha} \ . \tag{D.2.39}$$

Observe that we did not make use of the redefinitions which correspond to the chiral $U(1)$ in (D.2.20).

- *Torsion constraints* II: The variations of the torsions

$$T_{\gamma b}{}^a = 0, \qquad T_\gamma{}^{\dot\beta}{}_{\dot\alpha} = 0, \qquad T_{\gamma\beta}{}^\alpha = 0 \tag{D.2.40}$$

give rise to the equations

$$\mathscr{D}_\gamma H_b{}^a - \mathscr{D}_b H_\gamma{}^a + \Omega_{\gamma b}{}^a + T_{\gamma b}{}^F H_F{}^a - H_\gamma{}^{\underline{\varphi}} T_{\underline{\varphi} b}{}^a + H_{b\dot\varphi} T^{\dot\varphi}{}_\gamma{}^a = 0, \tag{D.2.41}$$

$$\mathscr{D}_\gamma H^{\dot\beta}{}_{\dot\alpha} + \mathscr{D}^{\dot\beta} H_{\gamma\dot\alpha} + T_\gamma{}^{\dot\beta f} H_{f\dot\alpha} - H_\gamma{}^f T_f{}^{\dot\beta}{}_{\dot\alpha} - H^{\dot\beta f} T_{f\gamma\dot\alpha} + \Omega_\gamma{}^{\dot\beta}{}_{\dot\alpha} - \delta^{\dot\beta}{}_{\dot\alpha}\omega_\gamma = 0 \ , \tag{D.2.42}$$

$$\sum_{\gamma\beta} (\mathscr{D}_\gamma H_\beta{}^\alpha - H_\gamma{}^f T_{f\beta}{}^\alpha + \Omega_{\gamma\beta}{}^\alpha + \delta^\alpha_\beta \omega_\gamma) = 0 \ . \tag{D.2.43}$$

These relations serve to express the variations $\Omega_{\gamma b}{}^a$, $H_{b\dot\alpha}$ and ω_α in terms of the so far unconstrained superfields $H, \bar{H}$ and $\mathscr{V}^a$. In this context it is convenient to define

$$\chi_b{}^a = H_b{}^a - \mathscr{D}_b \mathscr{V}^a, \tag{D.2.44}$$

$$\chi_{b\dot\alpha} = H_{b\dot\alpha} - \mathscr{V}^c T_{cb\dot\alpha} \tag{D.2.45}$$

$$\chi_\beta{}^\alpha = H_\beta{}^\alpha + \mathscr{V}^c T_{\beta c}{}^\alpha \ , \tag{D.2.46}$$

$$\chi^{\dot\beta}{}_{\dot\alpha} = H^{\dot\beta}{}_{\dot\alpha} + \mathscr{V}^c T^{\dot\beta}{}_{c\dot\alpha} \ , \tag{D.2.47}$$

$$\Pi_{\gamma b}{}^a = \Omega_{\gamma b}{}^a + \mathscr{V}^d R_{\gamma d b}{}^a \ , \tag{D.2.48}$$

$$\Sigma_\gamma = \omega_\gamma + \mathscr{V}^d F_{\gamma d} \tag{D.2.49}$$

and to write (D.2.41)–(D.2.43) in the form

$$\Pi_{\gamma b}{}^a + \mathscr{D}_\gamma \chi_b{}^a + \chi_{b\dot\varphi} T_\gamma{}^{\dot\varphi a} - 2T_{\gamma b\dot\varphi}\mathscr{D}^{\dot\varphi}\mathscr{V}^a = 0 \ , \tag{D.2.50}$$

$$\Pi_\gamma{}^{\dot\beta}{}_{\dot\alpha} - \delta^{\dot\beta}{}_{\dot\alpha}\Sigma_\gamma + T_\gamma{}^{\dot\beta d}\chi_{d\dot\alpha} + \mathscr{D}_\gamma \chi^{\dot\beta}{}_{\dot\alpha} - 2(\mathscr{D}^{\dot\beta}\mathscr{V}^d) T_{\gamma d\dot\alpha} = 0 \ , \tag{D.2.51}$$

$$\Pi_{\gamma\beta}{}^\alpha + \Pi_{\beta\gamma}{}^\alpha + \delta_\beta{}^\alpha \Sigma_\gamma + \delta_\gamma{}^\alpha \Sigma_\beta + \mathscr{D}_\gamma \chi_\beta{}^\alpha + \mathscr{D}_\beta \chi_\gamma{}^\alpha = 0 \tag{D.2.52}$$

The first of these equations allows to determine both $\Pi_{\gamma b}{}^a$ and $\chi_{b\dot{\alpha}}$. This is most easily seen in spinor notation, where (D.2.50) takes the form

$$2\varepsilon_{\dot\beta\dot\alpha}\Pi_{\underset{\smile}{\gamma\beta\alpha}} - 2\varepsilon_{\beta\alpha}\Pi_{\underset{\smile}{\gamma\dot\beta\dot\alpha}} - 4\mathrm{i}\varepsilon_{\gamma\alpha}\chi_{\beta\dot\beta\dot\alpha} + \mathscr{D}_\gamma\chi_{\beta\dot\beta\alpha\dot\alpha} + 4\mathrm{i}\varepsilon_{\gamma\beta}R^\dagger\mathscr{D}_{\dot\beta}\mathscr{V}_{\alpha\dot\alpha} = 0 \; . \tag{D.2.53}$$

Taking into account

$$\chi_{\beta\dot\beta\alpha\dot\alpha} = -\varepsilon_{\beta\alpha}\varepsilon_{\dot\beta\dot\alpha}(H+\bar{H}) - \mathrm{i}\mathscr{D}_\beta\mathscr{D}_{\dot\beta}\mathscr{V}_{\alpha\dot\alpha} - \mathrm{i}G_{\alpha\dot\beta}\mathscr{V}_{\beta\dot\alpha} + \mathrm{i}G_{\beta\dot\alpha}\mathscr{V}_{\alpha\dot\beta} \; , \tag{D.2.54}$$

we obtain

$$\Pi_{\underset{\smile}{\gamma\dot\beta\dot\alpha}} = -\frac{\mathrm{i}}{4}\mathscr{D}_\gamma\sum_{\dot\beta\dot\alpha}(\mathscr{D}^\varphi\mathscr{D}_{\dot\beta}\mathscr{V}_{\varphi\dot\alpha} - G^\varphi{}_{\dot\beta}\mathscr{V}_{\varphi\dot\alpha}) \; , \tag{D.2.55}$$

$$\Pi_{\underset{\smile}{\gamma\beta\alpha}} = \frac{\mathrm{i}}{4}\mathscr{D}_\gamma\sum_{\beta\alpha}G_\beta{}^{\dot\varphi}\mathscr{V}_{\alpha\dot\varphi} - \sum_{\beta\alpha}\varepsilon_{\gamma\beta}\mathscr{D}_\alpha\left(\frac{1}{2}(H+\bar{H}) + \frac{\mathrm{i}}{4}\mathscr{D}^\varphi\mathscr{D}^{\dot\varphi}\mathscr{V}_{\varphi\dot\varphi}\right) , \tag{D.2.56}$$

as well as

$$8\mathrm{i}\chi_{\beta\dot\beta\dot\alpha} = -4\varepsilon_{\dot\beta\dot\alpha}\mathscr{D}_\beta(H+\bar{H}) + 2\mathrm{i}\mathscr{D}_\beta\mathscr{D}^\varphi\mathscr{D}_{\dot\alpha}\mathscr{V}_{\varphi\dot\beta} - 8\mathrm{i}R^\dagger\mathscr{D}_\beta\mathscr{V}_{\dot\beta\dot\alpha} \; . \tag{D.2.57}$$

This exhausts the information contained in (D.2.50). Substituting these results reduces (D.2.51) simply to

$$\Sigma_\gamma = -\mathscr{D}_\gamma\left(H + \frac{1}{2}\bar{H} + \frac{\mathrm{i}}{4}\mathscr{D}^\varphi\mathscr{D}^{\dot\varphi}\mathscr{V}_{\varphi\dot\varphi} - \frac{\mathrm{i}}{2}\mathscr{V}^aG_a\right) \tag{D.2.58}$$

and (D.2.52) is then identically satisfied.

- *Torsion constraints* III: As to the complex conjugate torsions,

$$T^{\dot\gamma}{}_b{}^a = 0, \qquad T^{\dot\gamma}{}_\beta{}^\alpha = 0, \qquad T^{\dot\gamma\dot\beta}{}_{\dot\alpha} = 0 \; , \tag{D.2.59}$$

the variational equations read

$$\mathscr{D}^{\dot\gamma}H_b{}^a - \mathscr{D}_bH^{\dot\gamma a} + \Omega^{\dot\gamma}{}_b{}^a + T^{\dot\gamma}{}_b{}^{\varphi}H_{\underset{-}{\varphi}}{}^a + H_b{}^\varphi T_\varphi{}^{\dot\gamma a} = 0 \; , \tag{D.2.60}$$

$$\mathscr{D}^{\dot\gamma}H_\beta{}^\alpha + \mathscr{D}_\beta H^{\dot\gamma\alpha} + T_\beta{}^{\dot\gamma f}H_f{}^\alpha - H^{\dot\gamma f}T_{f\beta}{}^\alpha - H_\beta{}^fT_f{}^{\dot\gamma\alpha} + \Omega^{\dot\gamma}{}_\beta{}^\alpha + \delta^\alpha_\beta\omega^{\dot\gamma} = 0 \; , \tag{D.2.61}$$

$$\sum_{\dot\gamma\dot\beta}(\mathscr{D}^{\dot\gamma}H^{\dot\beta}{}_{\dot\alpha} - H^{\dot\gamma f}T_f{}^{\dot\beta}{}_{\dot\alpha} + \Omega^{\dot\gamma\dot\beta}{}_{\dot\alpha} - \delta^{\dot\beta}{}_{\dot\alpha}\omega^{\dot\gamma}) = 0 \; . \tag{D.2.62}$$

In this sector it is convenient to define

$$\bar{\mathscr{H}}_b{}^a = H_b{}^a + \mathscr{D}_b\mathscr{V}^a \; , \tag{D.2.63}$$

$$\bar{\mathscr{H}}_b{}^\alpha = H_b{}^\alpha + \mathscr{V}^cT_{cb}{}^\alpha \; , \tag{D.2.64}$$

$$\bar{\mathscr{H}}_\beta{}^\alpha = H_\beta{}^\alpha - \mathscr{V}^cT_{\beta c}{}^\alpha \; , \tag{D.2.65}$$

$$\bar{\mathscr{H}}^{\dot\beta}{}_{\dot\alpha} = H^{\dot\beta}{}_{\dot\alpha} - \mathscr{V}^c T_{c\dot\alpha}{}^{\dot\beta}\ , \tag{D.2.66}$$

$$\bar{\Pi}^{\dot\gamma}{}_b{}^a = \Omega^{\dot\gamma}{}_b{}^a - \mathscr{V}^d R^{\dot\gamma}{}_{db}{}^a\ , \tag{D.2.67}$$

$$\bar{\Sigma}^{\dot\gamma} = \omega^{\dot\gamma} - \mathscr{V}^d F^{\dot\gamma}{}_d\ . \tag{D.2.68}$$

With these notations and after some manipulations involving superspace Bianchi identities, (D.2.60)–(D.2.62) can be written as

$$\bar{\Pi}^{\dot\gamma}{}_b{}^a + \mathscr{D}^{\dot\gamma}\bar{\mathscr{H}}_b{}^a + \mathscr{H}_b{}^\varphi T_\varphi{}^{\dot\gamma a} + 2T^{\dot\gamma}{}_b{}^\varphi \mathscr{D}_\varphi \mathscr{V}^a = 0\ , \tag{D.2.69}$$

$$\bar{\Pi}^{\dot\gamma}{}_\beta{}^\alpha + \delta^\alpha_\beta \bar{\Sigma}^{\dot\gamma} + T_\beta{}^{\dot\gamma d}\mathscr{H}_d{}^\alpha + \mathscr{D}^{\dot\gamma}\bar{\mathscr{H}}_\beta{}^\alpha + 2(\mathscr{D}_\beta\mathscr{V}^d)T^{\dot\gamma}{}_d{}^\alpha = 0\ , \tag{D.2.70}$$

$$\bar{\Pi}^{\dot\gamma\dot\beta}{}_{\dot\alpha} + \bar{\Pi}^{\dot\beta\dot\gamma}{}_{\dot\alpha} - \delta^{\dot\beta}{}_{\dot\alpha}\bar{\Sigma}^{\dot\gamma} - \delta^{\dot\gamma}{}_{\dot\alpha}\bar{\Sigma}^{\dot\beta} + \mathscr{D}^{\dot\gamma}\bar{\mathscr{H}}^{\dot\beta}{}_{\dot\alpha} + \mathscr{D}^{\dot\beta}\bar{\mathscr{H}}^{\dot\gamma}{}_{\dot\alpha} = 0\ . \tag{D.2.71}$$

As before we employ spinor notation. Eq. (D.2.69) becomes

$$2\varepsilon_{\dot\beta\dot\alpha}\bar{\Pi}_{\dot\gamma\,\underset{\smile}{\beta\alpha}} - 2\varepsilon_{\beta\alpha}\bar{\Pi}_{\dot\gamma\,\underset{\smile}{\dot\beta\dot\alpha}} + 4\mathrm{i}\varepsilon_{\dot\gamma\dot\beta}\mathscr{H}_{\beta\beta\alpha} + \mathscr{D}_{\dot\gamma}\bar{\mathscr{H}}_{\beta\dot\beta\alpha\dot\alpha} + 4\mathrm{i}\varepsilon_{\dot\gamma\dot\beta}R\mathscr{D}_\beta\mathscr{V}_{\alpha\dot\alpha} = 0 \tag{D.2.72}$$

with

$$\bar{\mathscr{H}}_{\beta\dot\beta\alpha\dot\alpha} = -\varepsilon_{\beta\alpha}\varepsilon_{\dot\beta\dot\alpha}(H+\bar{H}) + \mathrm{i}\mathscr{D}_{\dot\beta}\mathscr{D}_\beta\mathscr{V}_{\alpha\dot\alpha} + \mathrm{i}G_{\alpha\dot\beta}\mathscr{V}_{\beta\dot\alpha} - \mathrm{i}G_{\beta\dot\alpha}\mathscr{V}_{\alpha\dot\beta}\ . \tag{D.2.73}$$

From (D.2.72) we obtain

$$\bar{\Pi}_{\dot\gamma\,\underset{\smile}{\beta\alpha}} = -\frac{\mathrm{i}}{4}\mathscr{D}_{\dot\gamma}\sum_{\beta\alpha}(\mathscr{D}^{\dot\varphi}\mathscr{D}_\beta\mathscr{V}_{\alpha\dot\varphi} + G_\beta{}^{\dot\varphi}\mathscr{V}_{\alpha\dot\varphi})\ , \tag{D.2.74}$$

$$\bar{\Pi}_{\dot\gamma\,\underset{\smile}{\dot\beta\dot\alpha}} = -\frac{\mathrm{i}}{4}\mathscr{D}_{\dot\gamma}\sum_{\dot\beta\dot\alpha} G^\varphi{}_{\dot\beta}\mathscr{V}_{\varphi\dot\alpha} + \sum_{\dot\beta\dot\alpha}\varepsilon_{\dot\gamma\dot\beta}\mathscr{D}_{\dot\alpha}\left(\frac{1}{2}(H+\bar{H}) - \frac{\mathrm{i}}{4}\mathscr{D}^{\dot\varphi}\mathscr{D}^\varphi\mathscr{V}_{\varphi\dot\varphi}\right), \tag{D.2.75}$$

as well as

$$8\mathrm{i}\bar{\mathscr{H}}_{\beta\beta\alpha} = 4\varepsilon_{\beta\alpha}\mathscr{D}_\beta(H+\bar{H}) + 2\mathrm{i}\mathscr{D}_\beta\mathscr{D}^{\dot\varphi}\mathscr{D}_\alpha\mathscr{V}_{\beta\dot\varphi} + 8\mathrm{i}R\mathscr{D}_\beta\mathscr{V}_{\alpha\beta}\ . \tag{D.2.76}$$

Eq. (D.2.70) then yields

$$\bar{\Sigma}^{\dot\gamma} = \mathscr{D}^{\dot\gamma}\left(\bar{H} + \frac{1}{2}H - \frac{\mathrm{i}}{4}\mathscr{D}^{\dot\varphi}\mathscr{D}^\varphi\mathscr{V}_{\varphi\dot\varphi} - \frac{\mathrm{i}}{2}\mathscr{V}^a G_a\right) \tag{D.2.77}$$

and (D.2.71) is identically satisfied.

This concludes our discussion of torsion constraints at dimension $= 0$ and $\frac{1}{2}$ in $U(1)$ superspace. We have found that the vielbein and connection variations are described in terms of the independent unconstrained superfields $H, \bar{H}$ and $\mathscr{V}^a$. The torsion coefficients at dimension = 1 can then be used to determine the variations of the covariant superfields $R, R^\dagger$ and G_a. For our present purpose it is sufficient to work out δR and $\delta R^\dagger$ (which are most conveniently obtained in using the corresponding curvature equations)

$$\delta R = -(\mathscr{V}^a \mathscr{D}_a + \bar{H} - \mathrm{i}\mathscr{V}^a G_a)R - \tfrac{1}{8}\mathscr{D}_{\dot{\alpha}}\mathscr{D}^{\dot{\alpha}}\left(H + \bar{H} - \frac{\mathrm{i}}{2}\mathscr{D}^{\dot{\varphi}}\mathscr{D}^{\varphi}\mathscr{V}_{\varphi\dot{\varphi}}\right), \tag{D.2.78}$$

$$\delta R^\dagger = +(\mathscr{V}^a \mathscr{D}_a - H + \mathrm{i}\mathscr{V}^a G_a)R^\dagger - \tfrac{1}{8}\mathscr{D}^{\alpha}\mathscr{D}_{\alpha}\left(H + \bar{H} + \frac{\mathrm{i}}{2}\mathscr{D}^{\varphi}\mathscr{D}^{\dot{\varphi}}\mathscr{V}_{\varphi\dot{\varphi}}\right). \tag{D.2.79}$$

- *Chiral $U(1)$ gauge sector*: The solutions of the constraints

$$F_{\beta\alpha} = 0, \qquad F^{\dot{\beta}\dot{\alpha}} = 0\ , \tag{D.2.80}$$

in the $(\frac{1}{2}, \frac{1}{2})$-basis are parametrized in terms of a pre-potential K (which, later on will be specialized to the Kähler potential) such that

$$A_\alpha = +\tfrac{1}{4}E_\alpha{}^M \partial_M K\ , \tag{D.2.81}$$

$$A^{\dot{\alpha}} = -\tfrac{1}{4}E^{\dot{\alpha}M}\partial_M K\ . \tag{D.2.82}$$

Using $\delta A = \omega$, the variation of these equations gives

$$\omega_\alpha - H_\alpha{}^B(A_B - \tfrac{1}{4}\mathscr{D}_B K) - \tfrac{1}{4}\mathscr{D}_\alpha \delta K = 0\ , \tag{D.2.83}$$

$$\omega^{\dot{\alpha}} - H^{\dot{\alpha}B}(A_B + \tfrac{1}{4}\mathscr{D}_B K) - \tfrac{1}{4}\mathscr{D}^{\dot{\alpha}}\delta K = 0\ . \tag{D.2.84}$$

Taking into account our solution for $H_A{}^B$ leads to

$$\Sigma_\alpha = +\mathscr{D}_\alpha(\tfrac{1}{4}\delta K + \mathscr{V}^b A_b - \tfrac{1}{4}\mathscr{V}^b \mathscr{D}_b K)\ , \tag{D.2.85}$$

$$\bar{\Sigma}^{\dot{\alpha}} = -\mathscr{D}^{\dot{\alpha}}(\tfrac{1}{4}\delta K + \mathscr{V}^b A_b + \tfrac{1}{4}\mathscr{V}^b \mathscr{D}_b K)\ . \tag{D.2.86}$$

Finally, comparing with (D.2.58) and (D.2.77), we arrive at the chirality conditions

$$\mathscr{D}_\alpha\left(H + \frac{1}{2}\bar{H} + \frac{1}{4}\delta K + \frac{\mathrm{i}}{4}\mathscr{D}^{\varphi}\mathscr{D}^{\dot{\varphi}}\mathscr{V}_{\varphi\dot{\varphi}} + \mathscr{V}^a\left(A_a - \frac{\mathrm{i}}{2}G_a\right) - \frac{1}{4}\mathscr{V}^a\mathscr{D}_a K\right) = 0\ , \tag{D.2.87}$$

$$\mathscr{D}^{\dot{\alpha}}\left(\bar{H} + \frac{1}{2}H + \frac{1}{4}\delta K - \frac{\mathrm{i}}{4}\mathscr{D}^{\dot{\varphi}}\mathscr{D}^{\varphi}\mathscr{V}_{\varphi\dot{\varphi}} + \mathscr{V}^a\left(A_a - \frac{\mathrm{i}}{2}G_a\right) + \frac{1}{4}\mathscr{V}^a\mathscr{D}_a K\right) = 0\ . \tag{D.2.88}$$

These chirality constraints in turn are solved with the help of chiral projection operators acting on unconstrained superfields $U, \bar{U}$ and we obtain

$$H + \bar{H} = -\frac{1}{3}\delta K - \frac{i}{6}[\mathscr{D}^\alpha, \mathscr{D}^{\dot\alpha}]\mathscr{V}_{\alpha\dot\alpha} - \frac{4}{3}\mathscr{V}^a\left(A_a - \frac{i}{2}G_a\right)$$
$$- \tfrac{2}{3}(\mathscr{D}^\alpha\mathscr{D}_\alpha - 8R^\dagger)\bar{U} - \tfrac{2}{3}(\mathscr{D}_{\dot\alpha}\mathscr{D}^{\dot\alpha} - 8R)U, \tag{D.2.89}$$

$$H - \bar{H} = 2\mathscr{D}^a\mathscr{V}_a + \mathscr{V}^a\mathscr{D}_a K - 2(\mathscr{D}^\alpha\mathscr{D}_\alpha - 8R^\dagger)\bar{U} + 2(\mathscr{D}_{\dot\alpha}\mathscr{D}^{\dot\alpha} - 8R)U\ . \tag{D.2.90}$$

In conclusion, the combinations $H - \bar{H}$ and $H + \bar{H} + \frac{1}{3}\delta K$ of variational superfields are given in terms of unconstrained superfields $U, \bar{U}$ and $\mathscr{V}_a$.

- *Yang–Mills sector*: We parametrize the variation of the Yang–Mills gauge potential in $U(1)$ superspace such that

$$\delta A^{(r)} = \Gamma^{(r)} = E^A \Gamma_A{}^{(r)}\ . \tag{D.2.91}$$

The Yang–Mills field strength, $\mathscr{F}^{(r)} = \frac{1}{2}E^A E^B \mathscr{F}_{BA}{}^{(r)}$, defined as

$$\mathscr{F}^{(r)} = dA^{(r)} + \frac{i}{2}A^{(p)}A^{(q)}f_{(p)(q)}{}^{(r)}\ , \tag{D.2.92}$$

changes under these variations as

$$\delta\mathscr{F}^{(r)} = d\Gamma^{(r)} + i\Gamma^{(p)}A^{(q)}f_{(p)(q)}{}^{(r)} = \mathscr{D}\Gamma^{(r)}\ . \tag{D.2.93}$$

The variational equations of its coefficients are

$$\delta\mathscr{F}_{BA}{}^{(r)} = \mathscr{D}_B\Gamma_A{}^{(r)} - (-)^{ab}\mathscr{D}_A\Gamma_B{}^{(r)} + T_{BA}{}^F\Gamma_F{}^{(r)} - H_B{}^F\mathscr{F}_{FA}{}^{(r)} + (-)^{ab}H_A{}^F\mathscr{F}_{FB}{}^{(r)}\ . \tag{D.2.94}$$

As in the gravitational case, we are only interested in infinitesimal variations modulo ordinary gauge variations $\varepsilon^{(r)}$, given as

$$\underline{\delta}\Gamma^{(r)} = d\varepsilon^{(r)} + i\varepsilon^{(p)}A^{(q)}f_{(p)(q)}{}^{(r)} = \mathscr{D}\varepsilon^{(r)}\ , \tag{D.2.95}$$

$$\underline{\delta}\delta\mathscr{F}^{(r)} = i\varepsilon^{(p)}\mathscr{F}^{(q)}f_{(p)(q)}{}^{(r)}\ . \tag{D.2.96}$$

The solution of the variational equations of the constraints

$$\delta\mathscr{F}_{\beta\alpha}{}^{(r)} = 0\ ,\quad \delta\mathscr{F}^{\dot\beta\dot\alpha(r)} = 0 \tag{D.2.97}$$

is expressed in terms of an unconstrained superfield $\Sigma^{(r)}$ such that

$$\Gamma_\alpha{}^{(r)} = +\mathscr{D}_\alpha\Sigma^{(r)} + \mathscr{V}^f\mathscr{F}_{f\alpha}{}^{(r)}\ , \tag{D.2.98}$$

$$\Gamma^{\dot\alpha(r)} = -\mathscr{D}^{\dot\alpha}\Sigma^{(r)} - \mathscr{V}^f\mathscr{F}_f{}^{\dot\alpha(r)}\ . \tag{D.2.99}$$

The constraint

$$\delta \mathscr{F}_{\beta}{}^{\dot\alpha(r)} = 0 \tag{D.2.100}$$

serves to express the vector component $\Gamma_a{}^{(r)}$ in terms of $\Sigma^{(r)}$ as well. It is convenient to define

$$\Lambda_a{}^{(r)} = \Gamma_a{}^{(r)} - \mathscr{V}^b \mathscr{F}_{ba}{}^{(r)} - \mathscr{D}_a \Sigma^{(r)} , \tag{D.2.101}$$

$$\bar\Lambda_a{}^{(r)} = \Gamma_a{}^{(r)} + \mathscr{V}^b \mathscr{F}_{ba}{}^{(r)} + \mathscr{D}_a \Sigma^{(r)} . \tag{D.2.102}$$

Accordingly, the solution of (D.2.100) can be written in two ways:

$$\Lambda_{\alpha\dot\alpha}{}^{(r)} = \mathrm{i}\mathscr{D}_\alpha \Gamma_{\dot\alpha}{}^{(r)} + \mathrm{i}(\mathscr{D}_{\dot\alpha}\mathscr{V}^b)\mathscr{F}_{b\alpha}{}^{(r)} , \tag{D.2.103}$$

$$\bar\Lambda_{\alpha\dot\alpha}{}^{(r)} = \mathrm{i}\mathscr{D}_{\dot\alpha} \Gamma_{\alpha}{}^{(r)} - \mathrm{i}(\mathscr{D}_{\alpha}\mathscr{V}^b)\mathscr{F}_{b\dot\alpha}{}^{(r)} . \tag{D.2.104}$$

The variations of the covariant Yang–Mills superfields $\mathscr{W}_\alpha{}^{(r)}$, $\mathscr{W}^{\dot\alpha(r)}$ are obtained from $\delta\mathscr{F}^{\beta}{}_a{}^{(r)}$, $\delta\mathscr{F}_{\beta a}{}^{(r)}$ to be

$$\begin{aligned}\delta \mathscr{W}_\alpha{}^{(r)} = & -\mathscr{V}^b \mathscr{D}_b \mathscr{W}_\alpha{}^{(r)} - (\bar H + \tfrac{1}{2}H)\mathscr{W}_\alpha{}^{(r)} + \mathrm{i}\Sigma^{(s)}\mathscr{W}_\alpha{}^{(t)} f_{(t)(s)}{}^{(r)} \\ & + \frac{\mathrm{i}}{2}(\mathscr{D}^{\dot\varphi}\mathscr{D}_\alpha \mathscr{V}_{\varphi\dot\varphi})\mathscr{W}^{\varphi(r)} - \frac{\mathrm{i}}{2}\mathscr{V}_{\alpha\dot\varphi}G^{\varphi\dot\varphi}\mathscr{W}_\varphi{}^{(r)} - \tfrac{1}{4}(\mathscr{D}_{\dot\varphi}\mathscr{D}^{\dot\varphi} - 8R)\Gamma_\alpha{}^{(r)} ,\end{aligned} \tag{D.2.105}$$

$$\begin{aligned}\delta \mathscr{W}_{\dot\alpha}{}^{(r)} = & +\mathscr{V}^b \mathscr{D}_b \mathscr{W}_{\dot\alpha}{}^{(r)} - (H + \tfrac{1}{2}\bar H)\mathscr{W}_{\dot\alpha}{}^{(r)} - \mathrm{i}\Sigma^{(s)}\mathscr{W}_{\dot\alpha}{}^{(t)} f_{(t)(s)}{}^{(r)} \\ & - \frac{\mathrm{i}}{2}(\mathscr{D}^{\varphi}\mathscr{D}_{\dot\alpha} \mathscr{V}_{\varphi\dot\varphi})\mathscr{W}^{\dot\varphi(r)} - \frac{\mathrm{i}}{2}\mathscr{V}_{\varphi\dot\alpha}G^{\varphi\dot\varphi}\mathscr{W}_{\dot\varphi}{}^{(r)} + \tfrac{1}{4}(\mathscr{D}^{\varphi}\mathscr{D}_{\varphi} - 8R^\dagger)\Gamma_{\dot\alpha}{}^{(r)} .\end{aligned} \tag{D.2.106}$$

D.3. Superspace densities

As a first application of the previous discussion, we consider the superfield action

$$\int_* E . \tag{D.3.1}$$

Recalling that the asterisk denotes integration over space–time and anticommuting coordinates, this superspace integral might be called the volume of superspace. It serves to generalize the *D*-term construction of invariant actions to local supersymmetry. Taking into account (D.2.39), the variation of the superdeterminant

$$\delta E = E H_A{}^A(-)^a \tag{D.3.2}$$

gives rise to

$$\delta \int_* E = \int_* E(H + \bar H) \tag{D.3.3}$$

with superspace surface terms neglected after integration by parts. Observe that in generic $U(1)$ superspace, the superfield $H + \bar{H}$, as given in (D.2.89), contains δK, the variation of the $U(1)$ pre-potential as an independent unconstrained variable. As a consequence, the superfield equations of motion would imply the volume of superspace to vanish. Therefore, the action (D.3.1) is not very useful in $U(1)$ superspace. However, when specified to pure Wess–Zumino superspace (resp. Kähler superspace), δK will be subject to constraints and the same action will provide the pure supergravity (resp. supergravity/matter) action.

Another useful concept in constructing superfield actions is the chiral density. It serves to generalize the F-term construction of invariant actions to the case of local supersymmetry. As a starting point consider the superspace action

$$\int_* \frac{E}{R}\,\mathscr{S} \tag{D.3.4}$$

with $\mathscr{S}$ some generic chiral superfield of weight $w(\mathscr{S}) = 2$ to ensure invariance under $U(1)$ transformations. Using the relation

$$\mathscr{S} = (\mathscr{D}_{\dot{\alpha}}\mathscr{D}^{\dot{\alpha}} - 8R)\Sigma(\mathscr{S})\,, \tag{D.3.5}$$

expressing the chiral superfield in terms of the unconstrained superfield $\Sigma(\mathscr{S})$, together with integration by parts yields

$$\int_* \frac{E}{R}\mathscr{S} = -8\int_* E\Sigma(\mathscr{S})\,. \tag{D.3.6}$$

This shows that integrating the chiral superfield $\mathscr{S}$ using the chiral density is the same as integrating its pre-potential $\Sigma(\mathscr{S})$ using the complete volume density. Note that adding a linear superfield to $\Sigma(\mathscr{S})$ does not change $\mathscr{S}$. This is coherent with relation (D.3.6), because the superspace integral of a linear superfield vanishes (this, in turn, is due to the fact that a linear superfield can be expressed in terms of spinor derivatives of unconstrained pre-potentials).

In spite of the equivalence established in (D.3.6), it is very often quite useful to work with the chiral density expression (D.3.4), and its complex conjugate

$$\int_* \frac{E}{R^\dagger}\,\bar{\mathscr{S}} \tag{D.3.7}$$

with chiral weight $w(\bar{\mathscr{S}}) = -2$ assigned to $\bar{\mathscr{S}}$. Taking into account (D.2.39), as well as (D.2.78) and (D.2.79) we find

$$\delta\int_* \frac{E}{R}\mathscr{S} = \int_* \frac{E}{R}((\delta\mathscr{S} + \mathscr{V}^a\mathscr{D}_a\mathscr{S}) + (H + 2\bar{H} - \mathrm{i}\mathscr{V}^a G_a)\mathscr{S})\,, \tag{D.3.8}$$

$$\delta\int_* \frac{E}{R^\dagger}\bar{\mathscr{S}} = \int_* \frac{E}{R^\dagger}((\bar{\delta}\mathscr{S} - \mathscr{V}^a\mathscr{D}_a\bar{\mathscr{S}}) + (\bar{H} + 2H - \mathrm{i}\mathscr{V}^a G_a)\bar{\mathscr{S}}) \tag{D.3.9}$$

with H and $\bar{H}$ determined in (D.2.89) and (D.2.90).

D.4. Variational equations in Kähler superspace

So far, in this appendix, we worked in the framework of $U(1)$ superspace. Supergravity/matter coupling is obtained in suitably specializing the $U(1)$ sector. We will present here the general case, where chiral superfields parametrize a Kähler manifold with gauged isometries. The relevant geometric framework is isometric Kähler superspace as defined in Appendix C.2.

After a discussion of the variational equations for chiral superfields and a summary of the properties of covariant isometric superspace derivatives, we will solve the variational equations pertaining to isometric superspace, thus identifying the fundamental variables relevant for the derivation of superfield equations of motions for the complete supergravity/matter/Yang–Mills system.

- *Chirality conditions*: The variational equations corresponding to the chirality conditions can be treated along the same lines as the constraint equations discussed earlier. We will first describe in some detail the procedure for the superfield ϕ^k and give the results for $\bar{\phi}^{\bar{k}}$ afterwards.
  In (C.1.20), the covariant derivative $\mathscr{D}\phi^k = E^A \mathscr{D}_A \phi^k$ has been defined as

$$\mathscr{D}\phi^k = (d + \mathrm{i}\mathscr{A}^{(r)} V_{(r)})\phi^k \,. \tag{D.4.1}$$

Its variation in terms of $\delta\phi^k$ and $\delta\mathscr{A}^{(r)} = \Gamma^{(r)}$ is given as

$$\delta\mathscr{D}\phi^k = \mathscr{D}\delta\phi^k + \mathrm{i}\Gamma^{(r)} V_{(r)}{}^k(\phi) \tag{D.4.2}$$

with the definition

$$\mathscr{D}\delta\phi^k = d\delta\phi^k + \mathrm{i}\mathscr{A}^{(r)} \frac{\partial V_{(r)}{}^k}{\partial \phi^l} \delta\phi^l \,. \tag{D.4.3}$$

Using

$$\delta\mathscr{D}\phi^k = E^A \delta\mathscr{D}_A \phi^k + E^A H_A{}^B \mathscr{D}_B \phi^k \,, \tag{D.4.4}$$

the variational equation for $\mathscr{D}_A\phi^k$ becomes

$$\delta\mathscr{D}_A\phi^k = \mathscr{D}_A\delta\phi^k + \mathrm{i}\Gamma_A^{(r)} V_{(r)}{}^k(\phi) - H_A{}^B \mathscr{D}_B \phi^k \,. \tag{D.4.5}$$

We are now in a position to study the consequences of the chirality condition $\mathscr{D}^{\dot{\alpha}}\phi^k = 0$, i.e. to determine the variations $\delta\phi^k$ of chirally constrained matter superfields in terms of unconstrained variational superfields. This is achieved in taking the $\dot{\alpha}$ component of the previous equation

$$\delta\mathscr{D}^{\dot{\alpha}}\phi^k = 0 = \mathscr{D}^{\dot{\alpha}}\delta\phi^k + \mathrm{i}\Gamma^{\dot{\alpha}(r)} V_{(r)}{}^k(\phi) - H^{\dot{\alpha}B} \mathscr{D}_B \phi^k \tag{D.4.6}$$

and making use of (D.2.99), i.e.

$$\Gamma^{\dot{\alpha}(r)} = -\mathscr{D}^{\dot{\alpha}}\Sigma^{(r)} - \mathscr{V}^b \mathscr{F}_b{}^{\dot{\alpha}(r)} \,, \tag{D.4.7}$$

in the second term. Taking into account (D.2.32) and (D.2.33) allows to write the third term in the form

$$-H^{\dot{\alpha}B}\mathscr{D}_B\phi^k = -\mathscr{D}^{\dot{\alpha}}(\mathscr{V}^b\mathscr{D}_b\phi^k) + \mathscr{V}^b[\mathscr{D}^{\dot{\alpha}},\mathscr{D}_b]\phi^k + \mathscr{V}^b T^{\dot{\alpha}}{}_b{}^{\varphi}\mathscr{D}_{\varphi}\phi^k\,. \tag{D.4.8}$$

Finally, substituting (C.1.29) for the commutator, gives rise to the chirality condition

$$\mathscr{D}^{\dot{\alpha}}\eta^k = 0 \tag{D.4.9}$$

with

$$\eta^k = \delta\phi^k + \mathscr{V}^b\mathscr{D}_b\phi^k - \mathrm{i}\Sigma^{(r)}V_{(r)}{}^k\,. \tag{D.4.10}$$

The corresponding expressions for $\delta\bar{\phi}^{\bar{k}}$ are obtained in complete analogy. There, the chirality condition

$$\mathscr{D}_{\alpha}\bar{\eta}^{\bar{k}} = 0 \tag{D.4.11}$$

is obtained for the combination

$$\bar{\eta}^{\bar{k}} = \delta\bar{\phi}^{\bar{k}} - \mathscr{V}^b\mathscr{D}_b\bar{\phi}^{\bar{k}} + \mathrm{i}\Sigma^{(r)}\bar{V}_{(r)}{}^{\bar{k}}\,. \tag{D.4.12}$$

The chirality conditions are solved in terms of unconstrained superfields $\bar{\varphi}^{\bar{k}}$ and φ^k, i.e.

$$\bar{\eta}^{\bar{k}} = (\mathscr{D}^{\alpha}\mathscr{D}_{\alpha} - 8R^{\dagger})\bar{\varphi}^{\bar{k}}\,, \tag{D.4.13}$$

$$\eta^k = (\mathscr{D}_{\dot{\alpha}}\mathscr{D}^{\dot{\alpha}} - 8R)\varphi^k\,. \tag{D.4.14}$$

- *Covariant superspace derivatives and gauged isometries*: Let $\mathbf{U}^k$ be some generic p-form in superspace, undergoing non-linear transformations

$$\delta\mathbf{U}^k = -\alpha^{(r)}\frac{\partial V_{(r)}{}^k}{\partial\phi^l}\mathbf{U}^l\,. \tag{D.4.15}$$

For simplicity, we suppose that $\mathbf{U}^k$ is inert under Lorentz and Kähler transformations. The exterior covariant derivative of this p-form is

$$\mathscr{D}\mathbf{U}^k = d\mathbf{U}^k + (-)^p\mathrm{i}\mathscr{A}^{(r)}\frac{\partial V_{(r)}{}^k}{\partial\phi^l}\mathbf{U}^l + (-)^p\Gamma^k{}_{lm}\mathscr{D}\phi^m\mathbf{U}^l \tag{D.4.16}$$

with $\Gamma^k{}_{lm}$ defined as in (2.4.3). In verifying the covariant transformation law of (D.4.16) it is convenient to use identities such as

$$(V_{(r)} + \bar{V}_{(r)})g_{k\bar{k}} + \frac{\partial V_{(r)}{}^m}{\partial\phi^k}g_{m\bar{k}} + \frac{\partial\bar{V}_{(r)}{}^{\bar{l}}}{\partial\bar{\phi}^{\bar{k}}}g_{k\bar{l}} = 0\,, \tag{D.4.17}$$

$$(V_{(r)} + \bar{V}_{(r)})g^{l\bar{l}} + \frac{\partial V_{(r)}{}^l}{\partial\phi^k}g^{k\bar{l}} - \frac{\partial\bar{V}_{(r)}{}^{\bar{l}}}{\partial\bar{\phi}^{\bar{k}}}g^{l\bar{k}} = 0 \tag{D.4.18}$$

and

$$(V_{(r)} + \bar{V}_{(r)})\Gamma^{l}{}_{mn} = \frac{\partial V_{(r)}{}^{l}}{\partial\phi^{k}}\Gamma^{k}{}_{mn} - \frac{\partial V_{(r)}{}^{k}}{\partial\phi^{m}}\Gamma^{l}{}_{kn} - \frac{\partial V_{(r)}{}^{k}}{\partial\phi^{n}}\Gamma^{l}{}_{mk} - \frac{\partial^{2} V_{(r)}{}^{l}}{\partial\phi^{m}\partial\phi^{n}} . \tag{D.4.19}$$

In the case $p = 0$, $\mathbf{U}^k$ is a superfield and its covariant derivative is given as

$$\mathscr{D}\mathbf{U}^k = E^A \mathscr{D}_A \mathbf{U}^k . \tag{D.4.20}$$

The graded commutator of two such covariant derivatives is obtained by taking the covariant exterior derivative of (D.4.20), using (D.4.16) for $p = 1$. The result is

$$\mathscr{D}\mathscr{D}\mathbf{U}^k = \mathrm{i}\mathscr{F}^{(r)}\left(\frac{\partial V_{(r)}{}^{k}}{\partial\phi^{l}}\mathbf{U}^l + V_{(r)}{}^{m}\Gamma^{k}{}_{lm}\mathbf{U}^l\right) - g^{k\bar{l}}R_{m\bar{l}l\bar{k}}\mathscr{D}\bar{\phi}^{\bar{k}}\mathscr{D}\phi^m\mathbf{U}^l . \tag{D.4.21}$$

Decomposing

$$\mathscr{D}\mathscr{D}\mathbf{U}^k = E^A E^B(\mathscr{D}_B\mathscr{D}_A\mathbf{U}^k + \tfrac{1}{2}T_{BA}{}^{C}\mathscr{D}_C\mathbf{U}^k) , \tag{D.4.22}$$

we find

$$(\mathscr{D}_B, \mathscr{D}_A)\mathbf{U}^k = -T_{BA}{}^{C}\,\mathscr{D}_C\mathbf{U}^k + \mathrm{i}\mathscr{F}^{(r)}_{BA}\left(\frac{\partial V_{(r)}{}^{k}}{\partial\phi^{l}}\mathbf{U}^l + V_{(r)}{}^{m}\Gamma^{k}{}_{lm}\,\mathbf{U}^l\right)$$
$$+ g^{k\bar{l}}R_{m\bar{l}l\bar{k}}\,\mathbf{U}^l(\mathscr{D}_B\bar{\phi}^{\bar{k}}\mathscr{D}_A\phi^m - (-)^{ab}\mathscr{D}_A\bar{\phi}^{\bar{k}}\mathscr{D}_B\phi^m) . \tag{D.4.23}$$

The spinor derivative $\mathscr{D}_\alpha\phi^k$ of a chiral superfield ϕ^k transforms in the same manner as $\mathbf{U}^k$ under gauged isometries but picks up additional contributions from Lorentz and Kähler transformations. Taking into account these modifications, we have

$$F^k = -\tfrac{1}{4}\mathscr{D}^\alpha\mathscr{D}_\alpha\phi^k \tag{D.4.24}$$

and

$$\mathscr{D}_\alpha F^k = -2R^\dagger\mathscr{D}_\alpha\phi^k . \tag{D.4.25}$$

- *Variations in isometric Kähler superspace*: As we have shown in Appendix C.2, gauged isometries can be included in the geometric description in replacing the generic $U(1)$ connection by the composite connection

$$\mathfrak{A} = \tfrac{1}{4}\tilde{\Delta} + \frac{\mathrm{i}}{8}E^a(12\,G_a + \bar{\sigma}_a^{\dot{\alpha}\alpha}g_{k\bar{k}}\mathscr{D}_\alpha\phi^k\mathscr{D}_{\dot{\alpha}}\bar{\phi}^{\bar{k}}) \tag{D.4.26}$$

with

$$\tilde{\Delta} = K_k d\phi^k - K_{\bar{k}}d\bar{\phi}^{\bar{k}} + 2\mathscr{A}^{(r)}G_{(r)} . \tag{D.4.27}$$

The resulting geometric structure in superspace is called isometric Kähler superspace. As a consequence of the particular form of the composite connection, the variational equations in the $U(1)$ sector will furnish additional information.

Recall that the field strength $\mathfrak{F} = d\mathfrak{A}$ satisfies the same constraints as that of the generic $U(1)$ connection. For this reason the generic $U(1)$ pre-potential K will be replaced by a field-dependent quantity. In standard Kähler superspace, this is just the superfield Kähler potential. In the presence of gauged isometries, the dependence on the matter sector and the Yang–Mills sector involved in the gauging of isometries will be quite intricate.

Fortunately enough, in the investigation of the variational equations, the knowledge of the explicit form of the composite pre-potential can be circumvented in considering directly the variations in terms of $\mathfrak{A}$.

The relevant object in this analysis is the variation of $\tilde{\Delta}$, which may be written as

$$\delta\tilde{\Delta} = d(K_k\delta\phi^k - K_{\bar{k}}\delta\bar{\phi}^{\bar{k}}) + 2\,g_{k\bar{k}}\mathscr{D}\phi^k\delta\bar{\phi}^{\bar{k}} - 2\,g_{k\bar{k}}\mathscr{D}\bar{\phi}^{\bar{k}}\delta\phi^k + 2\,\Gamma^{(r)}G_{(r)}\ . \tag{D.4.28}$$

We parametrize

$$\delta\tilde{\Delta} = E^A B_A \tag{D.4.29}$$

and consider the spinor coefficient

$$B_\alpha = E_\alpha{}^M\partial_M(K_k\delta\phi^k - K_{\bar{k}}\delta\bar{\phi}^{\bar{k}}) + 2\,g_{k\bar{k}}\mathscr{D}_\alpha\phi^k\delta\bar{\phi}^{\bar{k}} + 2\,\Gamma^{(r)}_\alpha G_{(r)}\ . \tag{D.4.30}$$

Taking into account the explicit expression for $\Gamma^{(r)}_\alpha$ – cf. (D.2.98), we obtain

$$\begin{aligned} B_\alpha = {}& E_\alpha{}^M\partial_M(K_k\delta\phi^k - K_{\bar{k}}\delta\bar{\phi}^{\bar{k}} + 2\Sigma^{(r)}G_{(r)}) \\ & + 2g_{k\bar{k}}\mathscr{D}_\alpha\phi^k\bar{\eta}^{\bar{k}} + 2\mathscr{V}^b(\mathscr{F}^{(r)}_{b\alpha}G_{(r)} + g_{k\bar{k}}\mathscr{D}_b\bar{\phi}^{\bar{k}}\mathscr{D}_\alpha\phi^k)\ . \end{aligned} \tag{D.4.31}$$

Remember that our aim is to determine $\delta\mathfrak{A} = \omega$, cf. (D.2.3), with the definition $\Sigma_\alpha = \omega_\alpha - \mathscr{V}^b\mathfrak{F}_{b\alpha}$, cf. (D.2.49). To this end we have to add the variation of the second term in (D.4.26) to arrive at

$$\begin{aligned} \Sigma_\alpha = {}& \tfrac{1}{4}E_\alpha{}^M\partial_M\left(K_k\delta\phi^k - K_{\bar{k}}\delta\bar{\phi}^{\bar{k}} + 2\Sigma^{(r)}G_{(r)} + 6\mathrm{i}\mathscr{V}^bG_b + \frac{\mathrm{i}}{2}\mathscr{V}^b\bar{\sigma}_b^{\dot{\alpha}\alpha}g_{k\bar{k}}\mathscr{D}_\alpha\phi^k\mathscr{D}_{\dot{\alpha}}\bar{\phi}^{\bar{k}}\right) \\ & + \tfrac{1}{2}g_{k\bar{k}}\mathscr{D}_\alpha\phi^k\bar{\eta}^{\bar{k}}\ . \end{aligned} \tag{D.4.32}$$

An explicit calculation shows that the last term in this equation can be written as a total spinor derivative as well, namely

$$\tfrac{1}{2}g_{k\bar{k}}\,\mathscr{D}_\alpha\phi^k\bar{\eta}^{\bar{k}} = E_\alpha{}^M\partial_M(2\bar{\varphi}^{\bar{k}}g_{k\bar{k}}F^k - g_{k\bar{k}}\mathscr{D}^\varphi\phi^k\mathscr{D}_\varphi\bar{\varphi}^{\bar{k}})\ . \tag{D.4.33}$$

This leads then to

$$\begin{aligned} \Sigma_\alpha = {}& \tfrac{1}{4}E_\alpha{}^M\partial_M\Bigg(K_k\delta\phi^k - K_{\bar{k}}\delta\bar{\phi}^{\bar{k}} + 2\Sigma^{(r)}G_{(r)} + 6\mathrm{i}\mathscr{V}^bG_b + \frac{\mathrm{i}}{2}\mathscr{V}^b\bar{\sigma}_b^{\dot{\alpha}\alpha}g_{k\bar{k}}\mathscr{D}_\alpha\phi^k\mathscr{D}_{\dot{\alpha}}\bar{\phi}^{\bar{k}} \\ & + 8\bar{\varphi}^{\bar{k}}g_{k\bar{k}}F^k - g_{k\bar{k}}\mathscr{D}^\varphi\phi^k\mathscr{D}_\varphi\bar{\varphi}^{\bar{k}}\Bigg)\ . \end{aligned} \tag{D.4.34}$$

This relation summarizes the consequences of the variational equations in the $U(1)$ sector which arise from the fact that $\mathfrak{A}$ is a composite connection, dependent on the Kähler and Yang–Mills sector. On the other hand, in the analysis of the consequences of the torsion constraints, – cf. (D.2.58), the superfield Σ_α had been given in terms of the, up to this point, unconstrained superfields H and $\bar{H}$, i.e.

$$\Sigma_\alpha = -E_\alpha{}^M \partial_M \left(H + \frac{1}{2}\bar{H} + \frac{\mathrm{i}}{4}\mathscr{D}^\varphi \mathscr{D}^{\dot\varphi} \mathscr{V}_{\varphi\dot\varphi} - \frac{\mathrm{i}}{2}\mathscr{V}^a G_a \right) . \tag{D.4.35}$$

Comparing the expressions in (D.4.35) and (D.4.34) leads to a chirality condition which is solved in terms of an unconstrained variational superfield $\mathscr{X}$ such that

$$\begin{aligned} H + \tfrac{1}{2}\bar{H} = & -\frac{\mathrm{i}}{4}\mathscr{D}^\varphi \mathscr{D}^{\dot\varphi}\mathscr{V}_{\varphi\dot\varphi} - \frac{1}{4}(K_k \delta\phi^k - K_{\bar k}\delta\bar\phi^{\bar k} + 2\Sigma^{(r)}G_{(r)}) \\ & - \mathrm{i}\mathscr{V}^b G_b - \frac{\mathrm{i}}{8}\mathscr{V}^b \bar\sigma_b^{\dot\alpha\alpha} g_{k\bar k}\mathscr{D}_\alpha \phi^k \mathscr{D}_{\dot\alpha}\bar\phi^{\bar k} \\ & - 2\,\bar\varphi^{\bar k} g_{k\bar k} F^k + g_{k\bar k}\mathscr{D}^\varphi \phi^k \mathscr{D}_\varphi \bar\varphi^{\bar k} + (\mathscr{D}^\varphi \mathscr{D}_\varphi - 8R^\dagger)\mathscr{X} \; . \end{aligned} \tag{D.4.36}$$

Performing the corresponding analysis for the complex conjugate sector leads to

$$\begin{aligned} \bar{H} + \tfrac{1}{2}H = & +\frac{\mathrm{i}}{4}\mathscr{D}^{\dot\varphi} \mathscr{D}^{\varphi}\mathscr{V}_{\varphi\dot\varphi} + \frac{1}{4}(K_k \delta\phi^k - K_{\bar k}\delta\bar\phi^{\bar k} - 2\Sigma^{(r)}G_{(r)}) \\ & - \mathrm{i}\mathscr{V}^b G_b - \frac{\mathrm{i}}{8}\mathscr{V}^b\, \bar\sigma_b^{\dot\alpha\alpha} g_{k\bar k}\mathscr{D}_\alpha \phi^k \mathscr{D}_{\dot\alpha}\bar\phi^{\bar k} \\ & + 2\varphi^k g_{k\bar k} \bar{F}^{\bar k} - g_{k\bar k}\mathscr{D}_{\dot\varphi} \bar\phi^{\bar k} \mathscr{D}^{\dot\varphi} \varphi^k + (\mathscr{D}_{\dot\varphi} \mathscr{D}^{\dot\varphi} - 8R)\mathscr{X}^\dagger \; . \end{aligned} \tag{D.4.37}$$

This completes our discussion of the variational equations of superspace constraints. The basic variational superfields are $\mathscr{V}_a$ and $\mathscr{X}$, $\mathscr{X}^\dagger$ for supergravity, φ^k and $\bar\varphi^{\bar k}$ for chiral matter superfields and $\Sigma^{(r)}$ for the Yang–Mills sector. Recall that the variations $\delta\phi^k$, $\delta\bar\phi^{\bar k}$ are expressed in terms of $\mathscr{V}_a$, φ^k and $\bar\varphi^{\bar k}$ according to (D.4.10) and (D.4.12)–(D.4.14). Observe that in the standard Yang–Mills case, i.e. no gauged isometries, the results (D.4.36) and (D.4.37) should reproduce those derived from (D.2.89) and (D.2.90) with δK evaluated directly as a function of chiral superfields.

D.5. Variation of the action functionals

We are now in a position to derive the superspace equations of motion for the complete supergravity/matter/Yang–Mills system. The full action

$$\mathscr{A} = \mathscr{A}_{\text{supergravity}+\text{matter}} + \mathscr{A}_{\text{Yang–Mills}} + \mathscr{A}_{\text{superpotential}} \tag{D.5.1}$$

consists of three separately supersymmetric and Kähler invariant pieces. It remains to perform the superfield variations and write down the equations of motion.

- *Variation of* $\mathscr{A}_{\text{supergravity+matter}}$: The kinetic action for the supergravity + matter system is given as

$$\mathscr{A}_{\text{supergravity+matter}} = -3\int_* E . \tag{D.5.2}$$

This is the form of the prototype action (D.3.1) discussed earlier. In its variation, cf. (D.3.3),

$$\delta\mathscr{A}_{\text{supergravity+matter}} = -3\int_* E(H+\bar{H}) , \tag{D.5.3}$$

$H+\bar{H}$ is given as the sum of (D.4.36) and (D.4.37), i.e.

$$\begin{aligned}\tfrac{3}{2}(H+\bar{H}) = {} & \frac{\mathrm{i}}{4}[\mathscr{D}^{\dot\varphi},\mathscr{D}^{\varphi}]\mathscr{V}_{\varphi\dot\varphi} - \Sigma^{(r)}G_{(r)} - 2i\mathscr{V}^{b}G_{b} - \frac{\mathrm{i}}{4}\bar{\sigma}_{b}^{\dot\alpha\alpha}g_{k\bar{k}}\mathscr{D}_{\alpha}\phi^{k}\mathscr{D}_{\dot\alpha}\bar{\phi}^{\bar{k}} \\ & - 2\bar{\varphi}^{\bar{k}}g_{k\bar{k}}F^{k} + g_{k\bar{k}}\mathscr{D}^{\varphi}\phi^{k}\mathscr{D}_{\varphi}\bar{\varphi}^{\bar{k}} + 2\varphi^{k}g_{k\bar{k}}\bar{F}^{\bar{k}} - g_{k\bar{k}}\mathscr{D}_{\dot\varphi}\bar{\phi}^{\bar{k}}\mathscr{D}^{\dot\varphi}\varphi^{k} \\ & + (\mathscr{D}^{\varphi}\mathscr{D}_{\varphi} - 8R^{\dagger})\mathscr{Z} + (\mathscr{D}_{\dot\varphi}\mathscr{D}^{\dot\varphi} - 8R)\mathscr{Z}^{\dagger} .\end{aligned} \tag{D.5.4}$$

Substituting, integrating by parts and neglecting superspace surface terms gives rise to

$$\begin{aligned}\delta\mathscr{A}_{\text{supergravity+matter}} = {} & 4\mathrm{i}\int_* E\mathscr{V}^{b}\left(G_{b} + \frac{1}{8}\bar{\sigma}_{b}^{\dot\alpha\alpha}g_{k\bar{k}}\mathscr{D}_{\alpha}\phi^{k}\mathscr{D}_{\dot\alpha}\bar{\phi}^{\bar{k}}\right) + 16\int_* E\mathscr{Z}R^{\dagger} \\ & + 16\int_* E\mathscr{Z}^{\dagger}R + 4\int_* EF^{k}g_{k\bar{k}}\bar{\varphi}^{\bar{k}} - 4\int_* E\varphi^{k}g_{k\bar{k}}\bar{F}^{\bar{k}} + 2\int_* E\Sigma^{(r)}G_{(r)} .\end{aligned} \tag{D.5.5}$$

- *Variation of* $\mathscr{A}_{\text{Yang–Mills}}$: The Yang–Mills action of (3.4.54) is obtained from the prototype action (D.3.4) in identifying $\mathscr{S}$ with

$$\mathscr{S}_{\text{Yang–Mills}} = \tfrac{1}{8}f_{(r)(s)}(\phi)\mathscr{W}^{(r)\alpha}\mathscr{W}_{\alpha}^{(s)} \tag{D.5.6}$$

and accordingly for $\bar{\mathscr{S}}$. The function $f_{(r)(s)}(\phi)$ of the chiral matter superfields is required to satisfy

$$V_{(p)}f_{(r)(s)}(\phi) = f_{(p)(r)}{}^{(q)}f_{(q)(s)}(\phi) + f_{(p)(s)}{}^{(q)}f_{(r)(q)}(\phi) , \tag{D.5.7}$$

assuring that $\mathscr{S}_{\text{Yang–Mills}}$ is indeed a chiral superfield of weight $w(\mathscr{S}_{\text{Yang–Mills}}) = 2$. Then, taking into account the variations of $\mathscr{W}_{\alpha}^{(r)}$ and ϕ^{k} as determined in this appendix, working out $\delta\mathscr{S}_{\text{Yang–Mills}}$, substituting in the general variation given in (D.3.8) and neglecting superspace surface terms yields, as an intermediate result

$$\delta\int_* \frac{E}{R}\mathscr{S}_{\text{Yang–Mills}} = \frac{1}{8}\int_* \frac{E}{R}\eta^k \frac{\partial f_{(r)(s)}}{\partial\phi^k}\mathscr{W}^{(r)\alpha}\mathscr{W}^{(s)}_\alpha + \frac{1}{2}\int_* E f_{(r)(s)}\mathscr{W}^{(r)\alpha}\Gamma^{(s)}_\alpha\,. \tag{D.5.8}$$

Using, furthermore, the explicit form of η^k and $\Gamma^{(r)}_\alpha$ gives rise to

$$\begin{aligned}\delta\int_* \frac{E}{R}\mathscr{S}_{\text{Yang–Mills}} = & -\frac{1}{2}\int_* E\,\Sigma^{(r)}\left(f_{(r)(s)}\mathscr{D}^\alpha\mathscr{W}^{(s)}_\alpha + \frac{\partial f_{(r)(s)}}{\partial\phi^k}\mathscr{D}_\alpha\phi^k\mathscr{W}^{(s)\alpha}\right)\\ & -\frac{\mathrm{i}}{2}\int_* E\,\mathscr{V}_b\,\mathscr{W}^{(r)\alpha}\sigma^b_{\alpha\dot\alpha}\mathscr{W}^{(s)\dot\alpha}f_{(r)(s)}(\phi) - \int_* E\,\varphi^k\frac{\partial f_{(r)(s)}}{\partial\phi^k}\mathscr{W}^{(r)\alpha}\mathscr{W}^{(s)}_\alpha\,. \end{aligned}\tag{D.5.9}$$

Observe that in the variation of the full Yang–Mills action

$$\mathscr{A}_{\text{Yang–Mills}} = \mathrm{Re}\int_* \frac{E}{R}\mathscr{S}_{\text{Yang–Mills}}\,, \tag{D.5.10}$$

we have to take into account the complex conjugate term as well.

- *Variation of* $\mathscr{A}_{\text{superpotential}}$: The action for the superpotential – cf. (3.4.55), is a special case of prototype action as well, in this case we identify $\mathscr{S}$ with

$$\mathscr{S}_{\text{superpotential}} = \tfrac{1}{2}\mathrm{e}^{K(\phi,\bar\phi)/2}W(\phi)\,. \tag{D.5.11}$$

In the presence of gauged isometries the condition

$$V_{(r)}W + F_{(r)}W = 0 \tag{D.5.12}$$

ensures that $\mathscr{S}_{\text{superpotential}}$ is indeed a chiral superfield of weight $w(\mathscr{S}_{\text{Superpotential}}) = 2$. An explicit calculation shows that the variation of the superpotential term is given as

$$\delta\left(\frac{1}{2}\int_* \frac{E}{R}\mathrm{e}^{K/2}W\right) = -8\int_* E\mathscr{X}^\dagger\mathrm{e}^{K/2}W - 4\int_* E\,\varphi^k\mathrm{e}^{K/2}(W_k + K_k W)\,. \tag{D.5.13}$$

For the complete superpotential action

$$\mathscr{A}_{\text{superpotential}} = \mathrm{Re}\int_* \frac{E}{R}\mathscr{S}_{\text{superpotential}}\,, \tag{D.5.14}$$

we have to take into account the complex conjugate term as well.

- *The superfield equations of motion*: In order to find the superfield equations of motion of the complete action

$$\mathscr{A} = \mathscr{A}_{\text{supergravity+matter}} + \mathscr{A}_{\text{Yang–Mills}} + \mathscr{A}_{\text{superpotential}}\,, \tag{D.5.15}$$

we simply identify the factors of the various variational superfields. From the coefficient of $\mathscr{X}^\dagger$ we obtain

$$R - \tfrac{1}{2}\mathrm{e}^{K/2}W = 0 \,. \tag{D.5.16}$$

The superfield equation corresponding to $\mathscr{V}^b$ reads

$$G_b + \tfrac{1}{8}\bar{\sigma}_b^{\dot{\alpha}\alpha} g_{k\bar{k}} \mathscr{D}_\alpha \phi^k \mathscr{D}_{\dot{\alpha}} \bar{\phi}^{\bar{k}} - \tfrac{1}{8}\bar{\sigma}_b^{\dot{\alpha}\alpha}(f + \bar{f})_{(r)(s)} \mathscr{W}_\alpha^{(r)} \bar{\mathscr{W}}_{\dot{\alpha}}^{(s)} = 0 \,. \tag{D.5.17}$$

Matter and Yang–Mills variations, respectively, give rise to the equations of motion

$$4g_{k\bar{k}}\bar{F}^{\bar{k}} + \frac{\partial f_{(r)(s)}}{\partial \phi^k} \mathscr{W}^{(r)\alpha} \mathscr{W}_\alpha^{(s)} + 4\,\mathrm{e}^{K/2}(W_k + K_k W) = 0 \tag{D.5.18}$$

and

$$\frac{1}{2} f_{(r)(s)} \mathscr{D}^\alpha \mathscr{W}_\alpha^{(s)} - \frac{1}{2}\frac{\partial f_{(r)(s)}}{\partial \phi^k} \mathscr{D}_\alpha \phi^k \mathscr{W}^{(s)\alpha} - G_{(r)} + \text{h.c.} = 0 \,. \tag{D.5.19}$$

Appendix E. Linear multiplet component field formalism

The discussion of the linear superfield formalism in Section 5 was mainly in terms of superfields. As component field expressions are notoriously heavy in notations and size we have deferred their presentation to the present appendix. We display here the complete component field action for the particular kinetic potential $K = K_0(\phi, \bar{\phi}) + \alpha \log L$ of (5.5.15) and discuss shortly the effective anomaly cancellation mechanism in terms of component fields. This appendix is designed as a complement to Section 5.

E.1. List of component fields

Component fields have been defined in various places in the main text. For the sake of clarity we give here a complete list of the component fields which will appear in the Lagrangian below:

- In the *supergravity sector* we have

  $$e_m{}^a, \quad \psi_m{}^\alpha, \quad \bar{\psi}_{m\dot{\alpha}}, \quad M, \quad \bar{M}, \quad b_a \,,$$

  the vierbein and the Rarita–Schwinger fields as dynamical variables and a complex scalar and a real vector as auxiliary fields.
- The *matter sector* is described in terms of

  $$A^k, \quad \bar{A}^{\bar{k}}, \quad \chi_\alpha^k, \quad \bar{\chi}^{\bar{k}\dot{\alpha}}, \quad F^k, \quad \bar{F}^{\bar{k}} \,,$$

  a set of complex scalars and of Majorana spinors as physical fields, together with another set of complex scalars as auxiliary fields, indices k and $\bar{k}$ referring to the Kähler variety.

- The *Yang–Mills sector* contains

$$\boldsymbol{a}_m, \quad \lambda^\alpha, \quad \bar{\lambda}_{\dot\alpha}, \quad \mathbf{D}\ ,$$

the gauge potential, the gaugino Majorana spinor and a real scalar auxiliary field, all Lie algebra valued with matricial generators $\mathbf{T}_{(r)}$ in a suitable representation.
- The *linear multiplet* consists of

$$b_{mn}, \quad L, \quad \Lambda_\alpha, \quad \bar{\Lambda}^{\dot\alpha}\ ,$$

an antisymmetric tensor gauge field, a real scalar and a Majorana spinor; it does not contain auxiliary fields. We should stress that in the actual component field Lagrangian given below the Majorana spinor always appears in the combination $\varphi_\alpha = L^{-1}\Lambda_\alpha$ and $\bar{\varphi}^{\dot\alpha} = L^{-1}\bar{\Lambda}^{\dot\alpha}$.

When derived from superspace, the component field Lagrangian contains a number of compact building blocks, which arise in a natural manner and gather complicated component field expressions in a concise way. The same structures appear in the derivation of supergravity transformations. Examples of this mechanism are the spin connection, as defined in (4.1.15) and (4.1.9), supercovariant field strength or curvature tensors like the curvature scalar in (4.1.35), the projection $R_{ab}{}^{ab}|$ in (4.1.37), or the field strength $T_{cb}{}^{\alpha}$, $T_{cb\dot\alpha}$ in (4.1.31) and (4.1.32). Other important building blocks which arise naturally are the supercovariant component field derivatives and the composite Kähler connection. This has already been described in Section 4, for the general supergravity/matter/Yang–Mills system, but is even more dramatic in the presence of linear multiplets. For the sake of illustration we will discuss two examples of supercovariant component field derivatives and the construction of the explicit form of the composite of the Kähler connection in the presence of a linear multiplet (coupled to Chern–Simons forms).

E.2. Construction of supercovariant derivatives

It might be instructive and useful to review shortly how the supercovariant component field derivatives are derived from superspace. To be definite we shall discuss here as representative examples the supercovariant derivatives of A^k and χ^k_α.

Let us begin with A^k. The starting point is the superspace covariant exterior derivative

$$D\phi^k = d\phi^k - \mathscr{A}^{(r)}(\mathbf{T}^{(r)}\phi)^k\ . \tag{E.2.1}$$

Using the double-bar projection as introduced in Section 4 one finds

$$D\phi^k|| = dx^m(\partial_m A^k - \mathrm{i}\boldsymbol{a}_m^{(r)}(\mathbf{T}_{(r)}\phi)^k)\ , \tag{E.2.2}$$

suggesting the definition

$$\mathscr{D}_m A^k = \partial_m A^k - \mathrm{i}\boldsymbol{a}_m^{(r)}(\mathbf{T}_{(r)}\phi)^k \tag{E.2.3}$$

for the component field covariant space–time derivative. On the other hand, double-bar projection in terms of covariant differentials gives

$$D\phi^k|| = dx^m\left(e_m{}^a\mathscr{D}_a\phi^k| + \frac{1}{\sqrt{2}}\psi_m{}^\alpha\chi_\alpha^k + \frac{1}{\sqrt{2}}\bar{\psi}_{m\dot{\alpha}}\bar{\chi}^{k\dot{\alpha}}\right). \tag{E.2.4}$$

The object $\mathscr{D}_a\phi^k|$ is called the supercovariant space–time derivative of A^k, explicitly given as

$$e_m{}^a\mathscr{D}_a\phi^k| = \mathscr{D}_m A^k - \frac{1}{\sqrt{2}}\psi_m{}^\alpha\chi_\alpha^k - \frac{1}{\sqrt{2}}\bar{\psi}_{m\dot{\alpha}}\bar{\chi}^{k\dot{\alpha}}. \tag{E.2.5}$$

The analogous construction for χ_α^k is slightly more involved. Here the starting point is the exterior covariant derivative

$$D\mathscr{D}_\alpha\phi^k = d\mathscr{D}_\alpha\phi^k - \phi_\alpha{}^\beta\mathscr{D}_\beta\phi^k - A\mathscr{D}_\alpha\phi^k - \mathscr{A}^{(r)}(\mathbf{T}_{(r)}\mathscr{D}_\alpha\phi)^k + \Gamma^k{}_{lj}D\phi^j\mathscr{D}_\alpha\phi^l, \tag{E.2.6}$$

which upon double-bar projection gives rise to

$$D\mathscr{D}_\alpha\phi^k|| = \sqrt{2}dx^m(\partial_m\chi_\alpha^k - \omega_{m\alpha}{}^\beta\chi_\beta^k - A_m\chi_\alpha^k - \mathrm{i}\boldsymbol{a}_m^{(r)}(\mathbf{T}_{(r)}\chi_\alpha)^k + \Gamma^k{}_{lj}\mathscr{D}_m A^j\chi_\alpha^l)$$

with $\mathscr{D}_m A^j$ defined above. This suggests to define

$$\mathscr{D}_m\chi_\alpha^k = \partial_m\chi_\alpha^k - \omega_{m\alpha}{}^\beta\chi_\beta^k - A_m\chi_\alpha^k - \mathrm{i}\boldsymbol{a}_m^{(r)}(\mathbf{T}_{(r)}\chi_\alpha)^k + \Gamma^k{}_{lj}\mathscr{D}_m A^j\chi_\alpha^l. \tag{E.2.7}$$

The double-bar projection on covariant differentials yields now

$$D\mathscr{D}_\alpha\phi^k|| = dx^m(e_m{}^a\mathscr{D}_a\mathscr{D}_\alpha\phi^k| + \tfrac{1}{2}\psi_m{}^\beta\mathscr{D}_\beta\mathscr{D}_\alpha\phi^k| + \tfrac{1}{2}\bar{\psi}_{m\dot{\beta}}\bar{\mathscr{D}}^{\dot{\beta}}\mathscr{D}_\alpha\phi^k|). \tag{E.2.8}$$

Here, the quantity $\mathscr{D}_a\mathscr{D}_\alpha\phi^k|$ is called the supercovariant component field derivative of χ_α^k. However, the two remaining terms still need some workout. Whereas the second term involves the auxiliary field F^k, the third term gives rise to the supercovariant component field derivative $\mathscr{D}_a\phi^k|$, just derived above. As a result one recovers the same form as in (4.3.11), i.e.

$$\mathscr{D}_a\mathscr{D}^\alpha\phi^i| = e_a{}^m\left(\sqrt{2}\mathscr{D}_m\chi^{\alpha i} - \psi_m{}^\alpha F^i + \mathrm{i}(\bar{\psi}_m\bar{\sigma}^n)^\alpha\left(\mathscr{D}_n A^i - \frac{1}{\sqrt{2}}\psi_n{}^\varphi\chi_\varphi^i\right)\right). \tag{E.2.9}$$

Observe, however, that this expression is different from (4.2.10), because now the composite Kähler connection A_m contains additional terms due to the linear superfield dependence of the kinetic potential.

E.3. The composite $U_K(1)$ connection

Let us first recall the identification of the spinor and vector components of the $U_K(1)$ gauge potential in terms of the kinetic potential K, adapted to the present situation, where K depends on a linear superfield as well. The relevant equations are generalizations of (3.4.20), which read now

$$A_\alpha = \tfrac{1}{4}E_\alpha{}^M \partial_M K(\phi, \bar{\phi}, L), \quad A^{\dot\alpha} = -\tfrac{1}{4}E^{\dot\alpha M}\partial_M K(\phi, \bar{\phi}, L) , \tag{E.3.1}$$

$$A_{\alpha\dot\alpha} - \frac{3\mathrm{i}}{2}G_{\alpha\dot\alpha} = \frac{\mathrm{i}}{2}(\mathscr{D}_\alpha A_{\dot\alpha} + \mathscr{D}_{\dot\alpha} A_\alpha) . \tag{E.3.2}$$

The important point to notice here is that the entities which are known a priori are the covariant components A_α, $A^{\dot\alpha}$ and A_a. As a consequence, the space–time component A_m identified in (4.1.16), i.e. $A\| = dx^m A_m(x)$, must be evaluated from the expression

$$A_m(x) = e_m{}^a A_a| + \tfrac{1}{2}\psi_m{}^\alpha A_\alpha| + \tfrac{1}{2}\bar{\psi}_{m\dot\alpha} A^{\dot\alpha}| . \tag{E.3.3}$$

Taking into account the linear multiplet couplings, Section 5, we obtain

$$\begin{aligned} A_m| + \frac{\mathrm{i}}{2} e_m^a b_a = {} & \frac{1}{4}K_k \mathscr{D}_m A^k - \frac{1}{4}K_{\bar{k}} \mathscr{D}_m \bar{A}^{\bar{k}} + \frac{\mathrm{i}}{4} g_{k\bar{k}} \chi^k \sigma_m \bar{\chi}^{\bar{k}} \\ & + \frac{\mathrm{i}\alpha}{6} e_m{}^a b_a + \frac{\mathrm{i}\alpha}{4}\frac{k}{L} {*h_m} - \frac{\mathrm{i}\alpha}{4}\frac{k}{L}\mathrm{tr}(\lambda\sigma_m\bar{\lambda}) - \frac{\mathrm{i}\alpha}{8}\varphi\sigma_m\bar{\varphi} \\ & - \frac{\alpha}{8}(\psi_n\sigma_m\bar{\sigma}^n\varphi - \bar{\psi}_n\bar{\sigma}_m\sigma^n\bar{\varphi}) - \frac{\alpha}{8}\varepsilon_{mnpq}\psi^n\sigma^p\bar{\psi}^q . \end{aligned} \tag{E.3.4}$$

Compared to the pure Kähler superspace construction, (4.1.24), i.e. without linear multiplets, a number of new terms appear. In particular, the dual field strength of the antisymmetric tensor gauge field,

$$*h^k = \frac{1}{3!}\varepsilon^{klmn} h_{lmn} \tag{E.3.5}$$

with h_{lmn} identified in (5.3.5), is given as

$$*h^k = \frac{1}{3!}\varepsilon^{klmn}\left(3\partial_n b_{ml} + k\left(\boldsymbol{a}_l \partial_m \boldsymbol{a}_n - \frac{2\mathrm{i}}{3}\boldsymbol{a}_l \boldsymbol{a}_m \boldsymbol{a}_n\right)\right) . \tag{E.3.6}$$

Instead of keeping all these terms encoded in the component field definitions of the covariant derivatives, we only retain the combination

$$v_m = \frac{1}{4}K_k \mathscr{D}_m A^k - \frac{1}{4}K_{\bar{k}} \mathscr{D}_m \bar{A}^{\bar{k}} + \frac{\mathrm{i}}{4}\, g_{k\bar{k}} \chi^k \sigma_m \bar{\chi}^{\bar{k}} \tag{E.3.7}$$

in these definitions. This renders the component field action more complicated, but shows explicitly the various couplings related to the linear multiplet. The corresponding covariant derivatives will be denoted ∇_m, they coincide with those defined in Section 4.

E.4. Genesis of the factor $LK_L - 3$

The chiral supergravity superfield R and its spinor derivatives are essential building blocks in the construction of supersymmetric actions and the derivation of supersymmetry transformations. A detailed knowledge of $\mathcal{D}_\alpha R$ and $\mathcal{D}^\alpha \mathcal{D}_\alpha R$ is crucial for the construction of supersymmetric component field actions. In Section 5.4 we have pointed out modifications to the normalization of the Einstein term in the linear superfield formalism.

We will explain here in some detail the superspace mechanism which underlies these modifications. To be definite we shall consider the superfield R. Its spinor derivative is given as

$$-3\mathcal{D}_\alpha R = X_\alpha + 4S_\alpha \, , \tag{E.4.8}$$

as a consequence of the Bianchi identities, see (B.4.7). The superfield S_α, as defined in (B.2.16), is related to the torsion $T_{cb}{}^\alpha$, while X_α is given in (5.4.3),

$$X_\alpha = -\tfrac{1}{8}(\bar{\mathcal{D}}^2 - 8R)\mathcal{D}_\alpha K(\phi, \bar{\phi}, L) \, . \tag{E.4.9}$$

Although straightforward, it will be instructive to illustrate in detail the appearance of the term $LK_L \mathcal{D}_\alpha R$ in X_α, in successively applying the spinor derivatives. In a first step, we write

$$-8X_\alpha = \bar{\mathcal{D}}^2(K_k \mathcal{D}_\alpha \phi^k) + \bar{\mathcal{D}}^2(K_L \mathcal{D}_\alpha L) \, .$$

It is clear that the linearity condition will arise from the second term, evaluation of the spinor derivatives yields

$$\bar{\mathcal{D}}^2(K_L \mathcal{D}_\alpha L) = \mathcal{D}_{\dot\alpha}(\mathcal{D}^{\dot\alpha} K_L \mathcal{D}_\alpha L) + (\mathcal{D}_{\dot\alpha} K_L)\mathcal{D}^{\dot\alpha}\mathcal{D}_\alpha L + K_L[\bar{\mathcal{D}}^2, \mathcal{D}_\alpha]L + K_L \mathcal{D}_\alpha \bar{\mathcal{D}}^2 L \, .$$

At this point the modified linearity condition (5.2.18)

$$(\bar{\mathcal{D}}^2 - 8R)L = 2k \, \mathrm{tr}(\mathcal{W}^\varphi \mathcal{W}_\varphi) \, ,$$

must be used to arrive at

$$K_L \mathcal{D}_\alpha \bar{\mathcal{D}}^2 L = 8LK_L \mathcal{D}_\alpha R + 8RK_L \mathcal{D}_\alpha L + 2k\mathcal{D}_\alpha \, \mathrm{tr}(\mathcal{W}^\varphi \mathcal{W}_\varphi) \, .$$

In this way, we recover (5.4.5) in the form

$$X_\alpha = -LK_L \mathcal{D}_\alpha R + Y_\alpha \tag{E.4.10}$$

with Y_α determined from the string of equations above. Combining this with (E.4.8) gives rise to

$$(LK_L - 3)\mathcal{D}_\alpha R = Y_\alpha + 4S_\alpha \, , \tag{E.4.11}$$

identifying $\mathcal{D}_\alpha R$ in terms of other, already known, superfields. When projected to lowest superfield components, $S_\alpha|$ will contain the supercovariant field strength of the gravitino. As to $Y_\alpha|$, one has to go through the various terms and identify properly the component field expressions. This is straightforward, but rather lengthy, and will not be done here.

E.5. Supersymmetry transformations

One of the advantages of superspace geometry is that supersymmetry transformations are defined geometrically. We have outlined in detail how this mechanism works in the case of supergravity/matter coupled to Yang–Mills in Section 4.3, based on the general formalism developed in Appendix C.3. Deriving supersymmetry transformations for component fields amounts to a bookkeeping activity in the sense that one has to apply a set of well-defined rules to extract component field properties from superspace.

The emphasis will be rather on the method of derivation of the component field transformations than their explicit gestalt (which is often quite lengthy and not very illuminating).

Here we will discuss supersymmetry transformations for component fields in the linear superfield formalism, based on the general notion of supergravity transformations extended to 2-form geometry. This will allow to derive the component field transformations for the linear multiplet, coupled to the supergravity/matter/Yang–Mills system.

At the same time, the presence of the linear superfield L in the kinetic potential $K(\phi, \bar{\phi}, L)$, which replaces the Kähler potential, will modify the supersymmetry transformations in the supergravity, matter and Yang–Mills sectors.

We will discuss here, sector by sector, how these modifications are induced from superspace geometry, before turning to the derivation of the supersymmetry transformations of the linear multiplet component fields.

- *Matter and Yang–Mills multiplets*: The supersymmetry transformations of component fields in the case of the general supergravity/matter/Yang–Mills system have been derived in Section 4. The transformations of A^k, χ^k_α, F^k are given in (4.3.25)–(4.3.27), those of $\bar{A}^k$, $\bar{\chi}^{k\dot{\alpha}}$, $\bar{F}^k$ in (4.3.32)–(4.3.34), whereas those of the Yang–Mills multiplet $\boldsymbol{a}_m$, λ^α, $\bar{\lambda}_{\dot{\alpha}}$, $\mathbf{D}$ are given in (4.3.36)–(4.3.39).

  In the linear superfield formalism, the general structure of these transformation laws remains unchanged. The modifications caused by the linear field dependence of the kinetic potential $K(\phi, \bar{\phi}, L)$ occur in two ways. First of all, whenever a covariant space–time derivative acts on a component of non-vanishing chiral weight, it should be written in terms of the new composite $U(1)$ connection (E.3.4) instead of (4.1.24).

  The second source of modifications is the term $\imath_\xi A = \xi^A A_A$, see (4.3.2), in the generic case of a component with non-vanishing chiral weight. As A_α and $A^{\dot{\alpha}}$ are now given in terms of the kinetic potential rather than the Kähler potential, new terms appear. This amounts in replacing everywhere the combination $K_k \xi \chi^k - K_{\bar{k}} \bar{\xi} \bar{\chi}^{\bar{k}}$ by

$$K_k \xi \chi^k - K_{\bar{k}} \bar{\xi} \bar{\chi}^{\bar{k}} + \frac{\alpha}{\sqrt{2}} K_L (\xi \varphi - \bar{\xi} \bar{\varphi}) \, . \tag{E.5.1}$$

  In this way, the supergravity transformations of matter and Yang–Mills fields are adapted to the linear superfield formalism.

- *Supergravity multiplet*: The mechanism just pointed out will occur for the gravitino supergravity transformations and the scalar auxiliary fields as well. Geometrically, the starting point for deriving supersymmetry transformations of the vierbein $e_m{}^a$ and the gravitino $\psi_m{}^\alpha$, $\bar{\psi}_{m\dot{\alpha}}$ is the general superspace equation (4.3.1)

$$\delta E_M{}^A = \mathscr{D}_M \xi^A + E_M{}^B \xi^C T_{CB}{}^A - w(E^A) E_M{}^A \xi^C A_C \; , \tag{E.5.2}$$

derived in Section 4.3. This relation is still valid in the linear superfield formalism. What kind of modifications arise for the component fields? Consider first the case of the vierbein $e_m{}^a$. Choosing $M = m$ and $A = a$ in (E.5.2) and projecting to lowest components reproduces the supersymmetry transformation (4.3.7). No dependence on the linear multiplet appears, the supersymmetry transformation for $e_m{}^a$ remains unchanged.

What happens in the case of the gravitino? Taking $M = m$ and $A = \alpha$ gives rise to

$$\tfrac{1}{2}\delta\psi_m{}^\alpha = \mathscr{D}_m \xi^\alpha + e_m{}^b \xi^\gamma T_{\gamma b}{}^\alpha| + e_m{}^b \bar{\xi}_{\dot{\gamma}} T^{\dot{\gamma}}{}_b{}^\alpha| - \psi_m{}^\alpha(\xi^\gamma A_\gamma| + \bar{\xi}_{\dot{\gamma}} A^{\dot{\gamma}}|) \; . \tag{E.5.3}$$

Clearly, the torsion terms are expressed in terms of the supergravity auxiliary fields as before, no modification. However, in the covariant derivative of ξ^α – cf. (4.3.10) – the composite Kähler connection $A_m|$ is now given by (E.3.4) instead of (4.1.24). Moreover, in the last term, the linear superfield dependence must be taken into account, giving rise to the second type of modification pointed out before. It is then an easy exercise to write down explicitly all the terms in the supersymmetry transformation of the gravitino in the linear superfield formalism, the result should be compared to (4.3.8) and (4.3.9).

Let us next turn to the auxiliary fields M, $\bar{M}$ and b_a. As we point out now, the situation is more intricate in this case. To be definite we concentrate on $M = -6R|$. Its generic supersymmetry transformation – cf. (4.3.16) – reads

$$\delta M = -6\xi^\alpha \mathscr{D}_\alpha R| - \frac{1}{\sqrt{2}} M\left(K_k \xi \chi^k - K_{\bar{k}} \bar{\xi} \bar{\chi}^{\bar{k}} + \frac{\alpha}{\sqrt{2}} K_L (\xi\varphi - \bar{\xi}\bar{\varphi}) \right) . \tag{E.5.4}$$

As to the lowest component of $\mathscr{D}_\alpha R$ we should take into account the discussion in the previous subsection, in particular (E.4.11). As a result, we find

$$\begin{aligned} \delta M = {} & \frac{1}{(\alpha - 3)} (2\xi^\alpha (\sigma^{cb}\varepsilon)_{\alpha\varphi} T_{cb}{}^\varphi| + \xi^\alpha Y_\alpha|) \\ & - \frac{1}{\sqrt{2}} M\left(K_k \xi \chi^k - K_{\bar{k}} \bar{\xi} \bar{\chi}^{\bar{k}} + \frac{\alpha}{\sqrt{2}} K_L (\xi\varphi - \bar{\xi}\varphi) \right) . \end{aligned} \tag{E.5.5}$$

This is a very compact form of a quite complicated expression. First of all the supercovariant field strength $T_{cb}{}^\varphi|$ of the gravitino is given in (4.1.31). Here, the covariant derivative (4.1.28) must now be written in terms of the composite Kähler connection constructed in (E.3.4). As to $Y_\alpha|$, its superfield form is to be determined from the string of equations of the preceding subsection and then projected to lowest components with carefully paying attention to $U(1)$ covariant space–time derivatives. The procedure is straightforward, but a bit lengthy and so is the result, which will not be presented here. Note, however, that the same quantity $Y_\alpha|$ appears in the variation of b_a as well.

- *Linear multiplet*: The linear multiplet and its couplings to the supergravity/matter/Yang–Mills system, including Chern–Simons forms, is described in the framework of 2-form geometry in superspace. In order to extract the supergravity transformations of the antisymmetric tensor we have to extend the notion of supergravity transformations to this geometric structure as well.

  Recall that invariance of the 3-form field strength $H = dB + kQ$ under Yang–Mills gauge transformations of the Chern–Simons form $Q = \mathrm{tr}(\mathscr{A}\mathscr{F} - 1/3\mathscr{A}\mathscr{A}\mathscr{A})$ is achieved in assigning a compensating Yang–Mills transformation to the 2-form gauge potential, in addition to superspace diffeomorphisms and 1-form gauge transformations $\beta = dz^M \beta_M$, such that

$$\delta B = L_\xi B + d\beta + \mathrm{i}k\,\mathrm{tr}(\boldsymbol{\alpha} d\mathscr{A}) \tag{E.5.6}$$

with $\boldsymbol{\alpha} = \alpha^{(r)}\mathbf{T}_{(r)}$. In the first term, we explicit the Lie-derivative, and use $\imath_\xi dB = \imath_\xi H - \imath_\xi Q$ with

$$\imath_\xi Q = \mathrm{tr}(\mathscr{A}\imath_\xi\mathscr{F}) + \mathrm{tr}((\imath_\xi\mathscr{A})d\mathscr{A})\,, \tag{E.5.7}$$

to arrive at

$$\delta B = \imath_\xi H - k\,\mathrm{tr}\,(\mathscr{A}\imath_\xi\mathscr{F}) + d(\beta + \imath_\xi B) + \mathrm{i}k\,\mathrm{tr}((\boldsymbol{\alpha} + \imath_\xi\mathscr{A})d\mathscr{A})\,. \tag{E.5.8}$$

Supergravity transformations, along the same lines of reasoning as in Appendix C.3 are then defined as

$$\delta_{WZ} B = \imath_\xi H - k\,\mathrm{tr}(\mathscr{A}\,\imath_\xi\mathscr{F})\,, \tag{E.5.9}$$

i.e. a combination of superspace diffeomorphisms and field-dependent compensating Yang–Mills and 1-form gauge transformations of parameters

$$\boldsymbol{\alpha} = -\imath_\xi\mathscr{A}, \quad \beta = -\imath_\xi B\,. \tag{E.5.10}$$

The supergravity transformation of the antisymmetric tensor gauge field $b_{mn}(x)$ is then obtained from (E.5.9) in applying systematically the double-bar projection, which yields

$$\begin{aligned} dx^m dx^n \tfrac{1}{2}\delta_{\mathrm{WZ}} b_{nm} = dx^m dx^n[&\xi\sigma_{nm}\Lambda + \bar\xi\bar\sigma_{nm}\bar\Lambda - \mathrm{i}L\psi_n\sigma_m\bar\xi - \mathrm{i}L\bar\psi_n\bar\sigma_m\xi \\ &+ \mathrm{i}k\,\mathrm{tr}(\boldsymbol{a}_m(\xi\sigma_n\bar\lambda + \bar\xi\bar\sigma_n\lambda))]. \end{aligned} \tag{E.5.11}$$

Supergravity transformations of $L(x)$ and Λ_α, $\bar\Lambda^{\dot\alpha}$ are obtained in the usual way, applying spinor derivatives to the superfields L and $\mathscr{D}_\alpha L$, $\mathscr{D}^{\dot\alpha}L$. As to $L(x)$ it is immediate to find $\delta L = \xi\Lambda + \bar\xi\bar\Lambda$. The case of Λ_α is slightly more interesting, let us outline the general procedure to obtain its supergravity transformation. The starting point is the superfield equation, $\delta\mathscr{D}_\alpha L = \xi^\beta\mathscr{D}_\beta\mathscr{D}_\alpha L + \xi_{\dot\beta}\mathscr{D}^{\dot\beta}\mathscr{D}_\alpha L$ written in the form

$$\delta\mathscr{D}_\alpha L = -\tfrac{1}{2}\xi_\alpha\mathscr{D}^2 L + \tfrac{1}{2}\xi_{\dot\beta}\{\mathscr{D}_\alpha, \mathscr{D}^{\dot\beta}\}L - \tfrac{1}{2}\xi_{\dot\beta}[\mathscr{D}_\alpha, \mathscr{D}^{\dot\beta}]L\,. \tag{E.5.12}$$

Using the modified linearity condition (5.2.18) and substituting for the commutator (5.2.20) gives rise to

$$\delta\mathscr{D}_\alpha L = \mathrm{i}\bar{\xi}_{\dot\alpha}(\bar\sigma^a\varepsilon)_\alpha{}^{\dot\alpha}\mathscr{D}_a L - \tfrac{1}{6}\bar\xi_{\dot\alpha}(\bar\sigma_d\varepsilon)_\alpha{}^{\dot\alpha}\varepsilon^{dcba}H_{cba} - 4\xi_\alpha R^\dagger L + 2\bar\xi_{\dot\alpha}(\bar\sigma^a\varepsilon)_\alpha{}^{\dot\alpha}G_a L$$
$$- k\xi_\alpha\,\mathrm{tr}(\mathscr{W}_{\dot\alpha}\mathscr{W}^{\dot\alpha}) + 2k\bar\xi_{\dot\alpha}\,\mathrm{tr}(\mathscr{W}_\alpha\mathscr{W}^{\dot\alpha})\ . \tag{E.5.13}$$

The supergravity transformation of Λ_α is then obtained after projecting to lowest superfield components with special care to the supercovariant component derivative $\mathscr{D}_a L|$ and field strength $H_{cba}|$.

E.6. Component field Lagrangian – I

We display here the complete component field Lagrangian for the example of Section 5, i.e. a special kinetic function of the form

$$K(\phi,\bar\phi,L) = K_0(\phi,\bar\phi) + \alpha\log L\ . \tag{E.6.1}$$

Requiring a canonical normalization function $N=1$ gives rise to a subsidiary function

$$F(\phi,\bar\phi,L) = 1 - \frac{\alpha}{3} + LV(\phi,\bar\phi) \tag{E.6.2}$$

with arbitrary linear potential $V(\phi,\bar\phi)$. The component field action is then derived from the generic procedure of Section 4.4, for the chiral superfield (5.6.1) in Section 5.6, i.e.

$$\mathbf{r} = -\tfrac{1}{8}(\bar{\mathscr{D}}^2 - 8R)F(\phi,\bar\phi,L),\quad \bar{\mathbf{r}} = -\tfrac{1}{8}(\bar{\mathscr{D}}^2 - 8R^\dagger)F(\phi,\bar\phi,L) \tag{E.6.3}$$

with F given by (E.6.2). Working through all the necessary steps leads then to the Lagrangian

$$\frac{1}{\mathrm{e}}\mathscr{L} = -\tfrac{1}{2}\mathscr{R} + \tfrac{1}{2}\varepsilon^{mnpq}(\bar\psi_m\bar\sigma_n\nabla_p\psi_q - \psi_m\sigma_n\nabla_p\bar\psi_q)$$
$$- (K_{k\bar k} - 3LV_{k\bar k})\nabla_m A^k\nabla^m\bar A^{\bar k} - \frac{\mathrm{i}}{2}(K_{k\bar k} - 3LV_{k\bar k})(\chi^k\sigma^m\nabla_m\bar\chi^{\bar k} + \bar\chi^{\bar k}\bar\sigma^m\nabla_m\chi^k)$$
$$+ \frac{\alpha}{4L^2}{*h^m}{*h_m} - \frac{\alpha}{4L^2}\partial^m L\partial_m L - \frac{\mathrm{i}\alpha}{4}(\varphi\sigma^m\nabla_m\bar\varphi + \bar\varphi\bar\sigma^m\nabla_m\varphi)$$
$$+ \frac{k}{4}\left(\frac{\alpha}{L} - 3V\right)\boldsymbol{f}^{(r)}_{mn}\boldsymbol{f}^{mn}_{(r)} + \frac{\mathrm{i}k}{2}\left(\frac{\alpha}{L} - 3V\right)(\lambda^{(r)}\sigma^m\nabla_m\bar\lambda_{(r)} + \bar\lambda^{(r)}\bar\sigma^m\nabla_m\lambda_{(r)})$$
$$- \frac{3\mathrm{i}}{2}(V_k\nabla_m A^k - V_{\bar k}\nabla_m\bar A^{\bar k}){*h^m} + \frac{\alpha(\alpha-4)}{8L}{*h^m}(\varphi\sigma_m\bar\varphi)$$

$$+\frac{1}{9}(\alpha-3)M\bar{M}-\frac{1}{9}(\alpha-3)b_a b^a+(K_{k\bar{k}}-3LV_{k\bar{k}})F^k\bar{F}^{\bar{k}}-\frac{k}{2}\left(\frac{\alpha}{L}-3V\right)\mathbf{D}^{(r)}\mathbf{D}_{(r)}$$

$$+\frac{1}{6}(\alpha-3)\left[(K_{k\bar{k}}-3LV_{k\bar{k}})(\chi^k\sigma^a\bar{\chi}^{\bar{k}})+\frac{\alpha}{2}(\varphi\sigma^a\bar{\varphi})+k\left(\frac{\alpha}{L}-3V\right)(\lambda^{(r)}\sigma^a\bar{\lambda}_{(r)})\right]b_a$$

$$-\frac{1}{2}[(K_k-3LV_k)(T_{(r)}A)^k+(\bar{A}T_{(r)})^{\bar{k}}(K_{\bar{k}}-3LV_{\bar{k}})$$

$$-3\mathrm{i}\sqrt{2}k(V_k\chi^k\lambda_{(r)}-V_{\bar{k}}\bar{\chi}^{\bar{k}}\bar{\lambda}_{(r)})+\mathrm{i}\alpha\frac{k}{L}(\bar{\lambda}_{(r)}\bar{\varphi}-\lambda_{(r)}\varphi)\Big]\mathbf{D}^{(r)}$$

$$+\frac{3L}{2}\left[\sqrt{2}V_{k\bar{k}}\bar{\varphi}\bar{\chi}^{\bar{k}}+\mathscr{V}_{k\bar{l}\bar{k}}\bar{\chi}^{\bar{l}}\bar{\chi}^{\bar{k}}-\frac{k}{L}V_k\lambda^{(r)}\lambda_{(r)}\right]F^k$$

$$+\frac{3L}{2}\left[\sqrt{2}V_{k\bar{k}}\varphi\chi^k+\mathscr{V}_{\bar{k}lk}\chi^l\chi^k-\frac{k}{L}V_{\bar{k}}\bar{\lambda}^{(r)}\bar{\lambda}_{(r)}\right]\bar{F}^{\bar{k}}$$

$$+\left[\frac{\alpha}{4L}(K_{k\bar{k}}-3LV_{k\bar{k}})+\frac{3}{2}V_{k\bar{k}}\right]{}^*h^m(\chi^k\sigma_m\bar{\chi}^{\bar{k}})+\frac{\alpha k}{4L}\left[\frac{\alpha-2}{L}-3V\right]{}^*h^m(\lambda^{(r)}\sigma_m\bar{\lambda}_{(r)})$$

$$-\frac{\mathrm{i}3L}{2}(\sqrt{2}V_{k\bar{k}}\varphi\sigma^m\bar{\chi}^{\bar{k}}+\mathscr{V}_{k\bar{k}l}\chi^l\sigma^m\bar{\chi}^{\bar{k}})\nabla_m A^k$$

$$-\frac{\mathrm{i}3L}{2}(\sqrt{2}V_{k\bar{k}}\bar{\varphi}\bar{\sigma}^m\chi^k+\mathscr{V}_{\bar{k}k\bar{l}}\bar{\chi}^{\bar{l}}\bar{\sigma}^m\chi^k)\nabla_m\bar{A}^{\bar{k}}$$

$$+[\mathrm{i}\sqrt{2}(K_{k\bar{k}}-3LV_{k\bar{k}})(\bar{\chi}^{\bar{k}}\bar{\lambda}^{(r)})-3\mathrm{i}LV_k(\bar{\varphi}\bar{\lambda}^{(r)})](T_{(r)}A)^k$$

$$-[\mathrm{i}\sqrt{2}(K_{k\bar{k}}-3LV_{k\bar{k}})(\chi^k\lambda^{(r)})-3\mathrm{i}LV_{\bar{k}}(\varphi\lambda^{(r)})](\bar{A}T_{(r)})^{\bar{k}}$$

$$+\frac{1}{4}(R_{k\bar{k}l\bar{l}}-3L\mathscr{V}_{k\bar{k}l\bar{l}})(\chi^k\chi^l)(\bar{\chi}^{\bar{k}}\bar{\chi}^{\bar{l}})+\frac{\alpha}{8}(\alpha-3)(\varphi\,\varphi)(\bar{\varphi}\bar{\varphi})$$

$$-\frac{\alpha}{8}(K_{k\bar{k}}-3LV_{k\bar{k}})(\chi^k\sigma^m\bar{\chi}^{\bar{k}})(\varphi\sigma_m\bar{\varphi})+\frac{\alpha k}{8}\left[3V-\frac{2}{L}(\alpha-2)\right](\lambda^{(r)}\sigma^m\bar{\lambda}_{(r)})(\varphi\sigma_m\bar{\varphi})$$

$$-\frac{\alpha k^2}{4L}\left(\frac{\alpha}{L}-3V-\frac{1}{L}\right)(\lambda^{(r)}\sigma^m\bar{\lambda}_{(r)})(\lambda^{(s)}\sigma_m\bar{\lambda}_{(s)})$$

$$-\frac{k}{4}\left[\frac{\alpha}{L}(K_{k\bar{k}}-3LV_{k\bar{k}})+6V_{k\bar{k}}\right](\lambda^{(r)}\sigma^m\bar{\lambda}_{(r)})(\chi^k\sigma_m\bar{\chi}^{\bar{k}})$$

$$+\frac{i3k}{2}(V_k\nabla_m A^k - V_{\bar{k}}\nabla_m \bar{A}^{\bar{k}})(\lambda^{(r)}\sigma^m\bar{\lambda}_{(r)}) - \frac{3L}{2\sqrt{2}}[\mathscr{V}_{k\bar{l}\bar{k}}(\bar{\chi}^{\bar{l}}\bar{\chi}^{\bar{k}})(\chi^k\varphi) + \mathscr{V}_{\bar{k}lk}(\chi^l\chi^k)(\bar{\chi}^{\bar{k}}\bar{\varphi})]$$

$$+\frac{3k}{4}\mathscr{V}_{kl}(\chi^k\chi^l)(\lambda^{(r)}\lambda_{(r)}) + \frac{3}{4}k\mathscr{V}_{\bar{k}\bar{l}}(\bar{\chi}^{\bar{k}}\bar{\chi}^{\bar{l}})(\bar{\lambda}^{(r)}\bar{\lambda}_{(r)})$$

$$-\frac{\alpha k}{4L}[(\bar{\lambda}^{(r)}\bar{\lambda}_{(r)})(\bar{\varphi}\bar{\varphi}) + (\lambda^{(r)}\lambda_{(r)})(\varphi\,\varphi)] - \frac{\alpha k^2}{4L^2}(\lambda^{(r)}\lambda_{(r)})(\bar{\lambda}^{(s)}\bar{\lambda}_{(s)})$$

$$-\left[\frac{\alpha k}{2L}(\varphi\sigma^{mn}\lambda_{(r)} + \bar{\varphi}\bar{\sigma}^{mn}\bar{\lambda}_{(r)}) + \frac{3k}{\sqrt{2}}(V_k\chi^k\sigma^{mn}\lambda_{(r)} + V_{\bar{k}}\bar{\chi}^{\bar{k}}\bar{\sigma}^{mn}\bar{\lambda}_{(r)})\right]\boldsymbol{f}^{(r)}_{mn}$$

$$-\frac{ik}{2}\left(\frac{\alpha}{L} - 3V\right)(\boldsymbol{f}^{(r)mn} + i{}^*\boldsymbol{f}^{(r)mn})(\psi_m\sigma_n\bar{\lambda}_{(r)}) - \frac{ik}{2}\left(\frac{\alpha}{L} - 3V\right)(\boldsymbol{f}^{(r)mn} - i{}^*\boldsymbol{f}^{(r)mn})(\bar{\psi}_m\bar{\sigma}_n\lambda_{(r)})$$

$$-\frac{1}{2}(K_k - 3LV_k)(T_{(r)}A)^k(\bar{\psi}_m\bar{\sigma}^m\lambda^{(r)}) + \frac{1}{2}(K_{\bar{k}} - 3LV_{\bar{k}})(\bar{A}T_{(r)})^{\bar{k}}(\psi_m\sigma^m\bar{\lambda}^{(r)})$$

$$-\frac{1}{\sqrt{2}}(K_{k\bar{k}} - 3LV_{k\bar{k}})[(\psi_n\sigma^m\bar{\sigma}^n\chi^k)\nabla_m\bar{A}^{\bar{k}} + (\bar{\psi}_n\bar{\sigma}^m\sigma^n\bar{\chi}^{\bar{k}})\nabla_m A^k]$$

$$-\frac{\alpha}{4L}(\psi_n\sigma^m\bar{\sigma}^n\varphi + \bar{\psi}_n\bar{\sigma}^m\sigma^n\bar{\varphi})\partial_m L + \frac{i\alpha}{4L}(\psi_n\sigma^m\bar{\sigma}^n\varphi - \bar{\psi}_n\bar{\sigma}^m\sigma^n\bar{\varphi}){}^*h_m$$

$$+\frac{i\alpha k}{8L}[(\bar{\psi}_m\bar{\sigma}^m\varphi)(\lambda^{(r)}\lambda_{(r)}) + (\psi_m\sigma^m\bar{\varphi})(\bar{\lambda}^{(r)}\bar{\lambda}_{(r)})]$$

$$-\frac{i\alpha}{16}(\alpha - 4)[(\psi_m\sigma^m\bar{\varphi})(\varphi\,\varphi) + (\bar{\psi}_m\bar{\sigma}^m\varphi)(\bar{\varphi}\bar{\varphi})]$$

$$-\frac{i3k}{4\sqrt{2}}V_k(3\psi^m\chi^k + 2\psi_n\sigma^{nm}\chi^k)(\lambda^{(r)}\sigma_m\bar{\lambda}_{(r)}) + \frac{i3k}{4\sqrt{2}}V_{\bar{k}}(3\bar{\psi}^m\bar{\chi}^{\bar{k}} + 2\bar{\psi}_n\bar{\sigma}^{nm}\bar{\chi}^{\bar{k}})(\lambda^{(r)}\sigma_m\bar{\lambda}_{(r)})$$

$$+\frac{i3k}{4\sqrt{2}}[V_k(\bar{\psi}_m\bar{\sigma}^m\chi^k)(\lambda^{(r)}\lambda_{(r)}) + V_{\bar{k}}(\psi_m\sigma^m\bar{\chi}^{\bar{k}})(\bar{\lambda}^{(r)}\bar{\lambda}_{(r)})] - \frac{i\alpha k}{4L}(\psi^m\varphi - \bar{\psi}^m\bar{\varphi})(\lambda^{(r)}\sigma_m\bar{\lambda}_{(r)})$$

$$+\frac{i\alpha k}{8}\left(\frac{\alpha}{L} - 3V - \frac{1}{L}\right)(\psi_n\sigma^m\bar{\sigma}^n\varphi - \bar{\psi}_n\bar{\sigma}^m\sigma^n\bar{\varphi})(\lambda^{(r)}\sigma_m\bar{\lambda}_{(r)})$$

$$+\frac{1}{4}(K_{k\bar{k}} - 3LV_{k\bar{k}})\left(\psi_n\sigma^m\bar{\psi}^n - \frac{i}{2}(\alpha - 4)\varepsilon^{mnpq}\psi_m\sigma_n\bar{\psi}_p\right)(\chi^k\sigma_q\bar{\chi}^{\bar{k}})$$

$$+\frac{1}{16}\left(\frac{\alpha}{L}-3V\right)[(3\psi_n\psi^n+2\psi_n\sigma^{nm}\psi_m)(\lambda^{(r)}\lambda_{(r)})+(3\bar{\psi}_n\bar{\psi}^n+2\bar{\psi}_n\bar{\sigma}^{nm}\bar{\psi}_m)(\bar{\lambda}^{(r)}\bar{\lambda}_{(r)})]$$

$$+\frac{k}{8}\left(\frac{\alpha}{L}-3V\right)\left(\oint_{npq}g^{mn}g^{pq}-\mathrm{i}(\alpha-1)\varepsilon^{mnpq}\right)(\psi_m\sigma_n\bar{\psi}_p)(\lambda^{(r)}\sigma_q\bar{\lambda}_{(r)})$$

$$+\frac{3\mathrm{i}kV}{4}\varepsilon^{mnpq}(\psi_m\sigma_n\bar{\psi}_p)(\lambda^{(r)}\sigma_q\bar{\lambda}_{(r)})-\frac{\mathrm{i}\alpha}{4L}\varepsilon^{mnpq}(\psi_m\sigma_n\bar{\psi}_p)^*h_q$$

$$+\frac{\alpha}{8}\left(g^{mn}g^{pq}+g^{mq}g^{np}-\frac{\mathrm{i}}{2}(\alpha-4)\varepsilon^{mnpq}\right)(\psi_m\sigma_n\bar{\psi}_p)(\varphi\sigma_q\bar{\varphi})$$

$$+\left[\frac{\mathrm{i}\alpha}{8}(K_{k\bar{k}}-3LV_{k\bar{k}})+\frac{3\mathrm{i}}{4}LV_{k\bar{k}}\right](\psi_n\sigma^m\bar{\sigma}^n\varphi-\bar{\psi}_n\bar{\sigma}^m\sigma^n\bar{\varphi})(\chi^k\sigma_m\bar{\chi}^{\bar{k}})$$

$$+\frac{\alpha}{8}[(\psi_m\psi^m)(\varphi\varphi)+(\bar{\psi}_m\bar{\psi}^m)(\bar{\varphi}\bar{\varphi})]+\frac{\alpha}{8}\varepsilon^{mnpq}(\psi_m\sigma_n\bar{\psi}_p)(\psi_s\sigma_q\bar{\sigma}^s\varphi-\bar{\psi}_s\bar{\sigma}_q\sigma^s\bar{\varphi})$$

$$-\frac{\alpha}{16}\varepsilon^{mnpq}\varepsilon_{mrst}(\psi_n\sigma_p\bar{\psi}_q)(\psi^r\sigma^s\bar{\psi}^t)\ . \tag{E.6.4}$$

Recall that the first term in this expression, the curvature scalar $\mathscr{R}$, is defined in (4.1.35). As mentioned above, the covariant derivatives ∇_m coincide with those defined in Section 4. For the sake of completeness, we recall here the explicit expressions. The nabla derivatives of the Rarita–Schwinger field are given in (4.1.28) and (4.1.29),

$$\nabla_n\psi_m{}^\alpha=\partial_n\psi_m{}^\alpha+\psi_m{}^\beta\omega_{n\beta}{}^\alpha+\psi_m{}^\alpha v_n\ , \tag{E.6.5}$$

$$\nabla_n\bar{\psi}_{m\dot{\alpha}}=\partial_n\bar{\psi}_{m\dot{\alpha}}+\bar{\psi}_{m\dot{\beta}}\omega_n{}^{\dot{\beta}}{}_{\dot{\alpha}}-\bar{\psi}_{m\dot{\alpha}}v_n\ , \tag{E.6.6}$$

whereas (4.1.23) and (4.1.23) define those of the matter complex scalars:

$$\nabla_m A^i=\partial_m A^i-\mathrm{i}\boldsymbol{a}_m^{(r)}(\mathbf{T}_{(r)}A)^i,\quad \nabla_m\bar{A}^{\bar{j}}=\partial_m\bar{A}^{\bar{j}}+\mathrm{i}\boldsymbol{a}_m^{(r)}(\bar{A}\mathbf{T}_{(r)})^{\bar{j}}\ . \tag{E.6.7}$$

The derivatives for the spinors in the matter sector are, (4.2.15) and (4.2.16),

$$\nabla_m\chi^i_\alpha=\partial_m\chi^i_\alpha-\omega_{m\alpha}{}^\varphi\chi^i_\varphi-\mathrm{i}\boldsymbol{a}_m^{(r)}(\mathbf{T}_{(r)}\chi_\alpha)^i-v_m\chi^i_\alpha+\chi^j_\alpha\Gamma^i{}_{jk}\nabla_m A^k\ , \tag{E.6.8}$$

$$\nabla_m\bar{\chi}^{\dot{\alpha}\bar{j}}=\partial_m\bar{\chi}^{\dot{\alpha}\bar{j}}-\omega_m{}^{\dot{\alpha}}{}_{\dot{\varphi}}\bar{\chi}^{\dot{\varphi}\bar{j}}+\mathrm{i}\boldsymbol{a}_m^{(r)}(\bar{\chi}^{\dot{\alpha}}\mathbf{T}_{(r)})^{\bar{j}}+v_m\bar{\chi}^{\dot{\alpha}\bar{j}}+\bar{\chi}^{\dot{\alpha}\bar{\imath}}\Gamma^{\bar{j}}{}_{\bar{\imath}\bar{k}}\nabla_m\bar{A}^{\bar{k}}\ , \tag{E.6.9}$$

whereas those of the Majorana spinor of the linear multiplet are given as

$$\nabla_m\varphi^i_\alpha=\partial_m\varphi^i_\alpha-\omega_{m\alpha}{}^\varphi\varphi^i_\varphi-v_m\varphi^i_\alpha,\quad \nabla_m\bar{\varphi}^{\dot{\alpha}\bar{j}}=\partial_m\bar{\varphi}^{\dot{\alpha}\bar{j}}-\omega_m{}^{\dot{\alpha}}{}_{\dot{\varphi}}\bar{\varphi}^{\dot{\varphi}\bar{j}}+v_m\bar{\varphi}^{\dot{\alpha}\bar{j}}\ . \tag{E.6.10}$$

Finally, the gaugino covariant derivatives, (4.5.27) and (4.5.28), are

$$\nabla_m \lambda_\alpha^{(r)} = \partial_m \lambda_\alpha^{(r)} - \omega_{m\alpha}{}^{\varphi} \lambda_\varphi^{(r)} - \boldsymbol{a}_m^{(t)} \lambda_\alpha^{(s)} c_{(s)(t)}{}^{(r)} + v_m \lambda_\alpha^{(r)} , \tag{E.6.11}$$

$$\nabla_m \bar{\lambda}^{(r)\dot{\alpha}} = \partial_m \bar{\lambda}^{(r)\dot{\alpha}} - \omega_m{}^{\dot{\alpha}}{}_{\dot{\varphi}} \bar{\lambda}^{(r)\dot{\varphi}} - \boldsymbol{a}_m^{(t)} \bar{\lambda}^{(s)\dot{\alpha}} c_{(s)(t)}{}^{(r)} - v_m \bar{\lambda}^{(r)\dot{\alpha}} . \tag{E.6.12}$$

As to the field strength tensors, $*h^k = \frac{1}{3!}\varepsilon^{klmn} h_{lmn}$ is given above in (E.3.6). The Yang–Mills field strength, defined in (4.2.20), reads

$$\boldsymbol{f}_{mn}^{(r)} = \partial_m \boldsymbol{a}_n^{(r)} - \partial_n \boldsymbol{a}_m^{(r)} + \boldsymbol{a}_m^{(s)} \boldsymbol{a}_n^{(t)} c_{(s)(t)}{}^{(r)} \tag{E.6.13}$$

with dual $2*\boldsymbol{f}^{(r)kl} = \varepsilon^{klmn} \boldsymbol{f}_{mn}^{(r)}$.

As to the manifold of the matter scalar fields, the basic objects are the kinetic potential K and the linear potential V. Subscripts attached to these objects denote derivatives with respect to the complex scalars. In particular, the Kähler metric $g_{k\bar{k}} = K_{k\bar{k}}$ is defined in (2.4.8), and its inverse shows up in the Levi–Civita symbols

$$\Gamma^k{}_{ij} = g^{k\bar{l}} g_{i\bar{l},j}, \qquad \Gamma^{\bar{k}}{}_{\bar{\imath}\bar{\jmath}} = g^{l\bar{k}} g_{l\bar{\imath},\bar{\jmath}} . \tag{E.6.14}$$

The curvature tensor is given as (2.4.4)

$$R_{k\bar{k}j\bar{\jmath}} = g_{k\bar{k},j\bar{\jmath}} - g^{l\bar{l}} g_{k\bar{l},j} g_{l\bar{k},\bar{\jmath}} . \tag{E.6.15}$$

As to the derivatives of the linear potential we have introduced the covariant objects

$$\mathscr{V}_{ij} = V_{ij} - \Gamma^k{}_{ij} V_k, \qquad \mathscr{V}_{\bar{\imath}\bar{\jmath}} = V_{\bar{\imath}\bar{\jmath}} - \Gamma^{\bar{k}}{}_{\bar{\imath}\bar{\jmath}} V_{\bar{k}} , \tag{E.6.16}$$

$$\mathscr{V}_{i\bar{k}j} = V_{i\bar{k}j} - \Gamma^k{}_{ij} V_{k\bar{k}}, \qquad \mathscr{V}_{\bar{\imath}k\bar{\jmath}} = V_{\bar{\imath}k\bar{\jmath}} - \Gamma^{\bar{k}}{}_{\bar{\imath}\bar{\jmath}} V_{k\bar{k}} \tag{E.6.17}$$

as well as

$$\mathscr{V}_{i\bar{\imath}j\bar{\jmath}} = V_{i\bar{\imath}j\bar{\jmath}} + \Gamma^k{}_{ij} \Gamma^{\bar{k}}{}_{\bar{\imath}\bar{\jmath}} V_{k\bar{k}} - \Gamma_i{}^k{}_j V_{\bar{\imath}k\bar{\jmath}} - \Gamma^{\bar{k}}{}_{\bar{\imath}\bar{\jmath}} V_{i\bar{k}j} . \tag{E.6.18}$$

Before turning to a discussion of the auxiliary field sector we shortly discuss the effective transformations $V \mapsto V + H + \bar{H}$. Observe that any term containing either V itself or derivatives V_k, $V_{\bar{k}}$ or V_{kl}, $V_{\bar{k}\bar{l}}$ changes under such transformations. Of particular interest is the term

$$(V_k \nabla_m A^k - V_{\bar{k}} \nabla_m \bar{A}^{\bar{k}}) * h^m ,$$

which transforms into

$$(H - \bar{H}) \boldsymbol{f}^{(r)*}_{mn} \boldsymbol{f}^{mn}_{(r)} ,$$

after integration by parts. On the other hand, the Yang–Mills kinetic term gives rise to

$$(H + \bar{H}) \boldsymbol{f}_{mn}^{(r)} \boldsymbol{f}^{mn}_{(r)} .$$

Finally, we have to comment on the structure of the auxiliary field sector. Collecting in $\mathscr{L}_{\text{aux}}$ all the terms containing auxiliary fields, that is components M, $\bar{M}$, b_a, F^k, $\bar{F}^{\bar{k}}$ and $\mathbf{D}_{(r)}$, we diagonalize in terms of new, hatted auxiliary fields which have trivial equations of motion. As a result, the auxiliary sector of the Lagrangian takes the form

$$
\begin{aligned}
e^{-1}\mathscr{L}_{\text{aux}} = & +\tfrac{1}{9}(\alpha-3)M\bar{M} - \tfrac{1}{9}(\alpha-3)\hat{b}_a\hat{b}^a \\
& + (K_{k\bar{k}} - 3LV_{k\bar{k}})\hat{F}^k\hat{\bar{F}}^{\bar{k}} - \frac{k}{2}\left(\frac{\alpha}{L} - 3V\right)\hat{\mathbf{D}}^{(r)}\hat{\mathbf{D}}_{(r)} \\
& - \left[\frac{9L^2}{4}\mathscr{V}_{\bar{j}k\bar{l}}(K_{k\bar{k}} - 3LV_{k\bar{k}})^{-1}\mathscr{V}_{j\bar{k}l} + \frac{\alpha-3}{8}(K_{j\bar{j}} - 3LV_{j\bar{j}})(K_{l\bar{l}} - 3LV_{l\bar{l}})\right](\chi^j\chi^l)(\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{l}}) \\
& - \frac{9k^2}{4}V_k(K_{k\bar{k}} - 3LV_{k\bar{k}})^{-1}V_{\bar{k}}(\lambda^{(r)}\lambda_{(r)})(\bar{\lambda}^{(s)}\bar{\lambda}_{(s)}) - \frac{9L^2}{2}V_{k\bar{j}}(K_{k\bar{k}} - 3LV_{k\bar{k}})^{-1}V_{j\bar{k}}(\varphi\chi^j)(\bar{\varphi}\bar{\chi}^{\bar{j}}) \\
& + \frac{9kL}{4}V_{\bar{k}}(K_{k\bar{k}} - 3LV_{k\bar{k}})^{-1}[\mathscr{V}_{\bar{j}k\bar{l}}(\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{l}}) + \sqrt{2}V_{k\bar{j}}(\bar{\varphi}\bar{\chi}^{\bar{j}})](\bar{\lambda}^{(s)}\bar{\lambda}_{(s)}) \\
& + \frac{9kL}{4}V_k(K_{k\bar{k}} - 3LV_{k\bar{k}})^{-1}[\mathscr{V}_{j\bar{k}l}(\chi^j\chi^l) + \sqrt{2}V_{\bar{k}j}(\varphi\chi^j)](\lambda^{(s)}\lambda_{(s)}) \\
& - \frac{9\sqrt{2}L^2}{4}\mathscr{V}_{\bar{j}k\bar{l}}(K_{k\bar{k}} - 3LV_{k\bar{k}})^{-1}V_{j\bar{k}}(\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{l}})(\chi^j\varphi) \\
& - \frac{9\sqrt{2}L^2}{4}\mathscr{V}_{j\bar{k}l}(K_{k\bar{k}} - 3LV_{k\bar{k}})^{-1}V_{k\bar{j}}(\chi^j\chi^l)(\bar{\chi}^{\bar{j}}\bar{\varphi}) \\
& + \frac{(\alpha-3)k^2}{16}\left(\frac{\alpha}{L} - 3V\right)^2(\lambda^{(r)}\sigma^m\bar{\lambda}_{(r)})(\lambda^{(s)}\sigma_m\bar{\lambda}_{(s)}) - \frac{\alpha^2(\alpha-3)}{32}(\varphi\varphi)(\bar{\varphi}\bar{\varphi}) \\
& + \frac{(\alpha-3)}{16}(K_{k\bar{k}} - 3LV_{k\bar{k}})(\chi^k\sigma_m\bar{\chi}^{\bar{k}})\left[2k\left(\frac{\alpha}{L} - 3V\right)(\lambda^{(r)}\sigma^m\bar{\lambda}_{(r)}) + \alpha(\varphi\sigma^m\bar{\varphi})\right] \\
& + \frac{\alpha(\alpha-3)k}{16}\left(\frac{\alpha}{L} - 3V\right)(\varphi\sigma^m\bar{\varphi})(\lambda^{(r)}\sigma_m\bar{\lambda}_{(r)}) \\
& + \frac{1}{8k}\left(\frac{\alpha}{L} - 3V\right)^{-1}[(K_k - 3LV_k)(T_{(r)}A)^k + (\bar{A}T_{(r)})^{\bar{k}}(K_{\bar{k}} - 3LV_{\bar{k}})]^2
\end{aligned}
$$

$$+\frac{9k}{8}\left(\frac{\alpha}{L}-3V\right)^{-1}[V_k V_j(\chi^k\chi^j)(\lambda^{(r)}\lambda_{(r)})+V_{\bar{k}}V_{\bar{j}}(\bar{\chi}^{\bar{k}}\bar{\chi}^{\bar{j}})(\bar{\lambda}^{(r)}\bar{\lambda}_{(r)})$$

$$-2V_k V_{\bar{k}}(\chi^k\sigma^m\bar{\chi}^{\bar{k}})(\lambda^{(r)}\sigma_m\bar{\lambda}_{(r)})]$$

$$+\frac{\alpha^2 k}{16L^2}\left(\frac{\alpha}{L}-3V\right)^{-1}[(\bar{\lambda}^{(r)}\bar{\lambda}_{(r)})(\bar{\varphi}\bar{\varphi})+(\lambda^{(r)}\lambda_{(r)})(\varphi\varphi)-2(\lambda^{(r)}\sigma^m\bar{\lambda}_{(r)})(\varphi\sigma_m\bar{\varphi})]$$

$$-\frac{3i}{2\sqrt{2}}\left(\frac{\alpha}{L}-3V\right)^{-1}[(K_k-3LV_k)(T_{(r)}A)^k+(\bar{A}T_{(r)})^{\bar{k}}(K_{\bar{k}}-3LV_{\bar{k}})]$$

$$\times[V_j\,\chi^j\lambda^{(r)}-V_{\bar{j}}\bar{\chi}^{\bar{j}}\bar{\lambda}^{(r)}]+\frac{i\alpha}{4L}\left(\frac{\alpha}{L}-3V\right)^{-1}$$

$$\times[(K_k-3LV_k)(T_{(r)}A)^k+(\bar{A}T_{(r)})^{\bar{k}}(K_{\bar{k}}-3LV_{\bar{k}})]$$

$$\times[\bar{\lambda}^{(r)}\bar{\varphi}-\lambda^{(r)}\varphi]+\frac{3k\alpha}{4\sqrt{2}L}\left(\frac{\alpha}{L}-3V\right)^{-1}[V_k\,(\chi^k\varphi)(\lambda^{(r)}\lambda_{(r)})+V_{\bar{k}}(\bar{\chi}^{\bar{k}}\bar{\varphi})(\bar{\lambda}^{(r)}\bar{\lambda}_{(r)})]$$

$$-\frac{3k\alpha}{4\sqrt{2}L}\left(\frac{\alpha}{L}-3V\right)^{-1}(V_k\chi^k\sigma^m\bar{\varphi}-V_{\bar{k}}\bar{\chi}^{\bar{k}}\bar{\sigma}^m\varphi)(\lambda^{(r)}\sigma_m\bar{\lambda}_{(r)})\,. \tag{E.6.19}$$

Clearly, the role of effective transformations after elimination of the auxiliary fields deserves further study.

E.7. Component field Lagrangian – II

We can merge these new contributions into the Lagrangian and eliminate trivially the auxiliary fields; this will yield a huge expression which we simplify somehow by making the following changes:

- Change the Kähler metric in the Lagrangian. Consider the Kähler potential

$$\hat{K}(\phi,\bar{\phi},L)=K(\phi,\bar{\phi},L)-3LV(\phi,\bar{\phi})=K_0(\phi,\bar{\phi})+\alpha\log L-3LV(\phi,\bar{\phi})\,, \tag{E.7.1}$$

we promote $\hat{K}_{k\bar{k}}=K_{k\bar{k}}-3LV_{k\bar{k}}$ to a metric denoted $G_{k\bar{k}}$ and define symbols and tensors in this new scheme. For instance,

$$\hat{K}_{j\bar{j}k}=g_{l\bar{j}}\Gamma^l_{jk}-3L\mathscr{V}_{j\bar{j}k}\,, \tag{E.7.2}$$

so that we can define new Christoffel symbols

$$\hat{\Gamma}^l{}_{jk} \equiv G^{l\bar{j}}\hat{K}_{j\bar{j}k} = \Gamma^l{}_{jk} - 3LG^{l\bar{j}}\mathscr{V}_{j\bar{j}k} \ , \tag{E.7.3}$$

$$\hat{\Gamma}^{\bar{l}}{}_{\bar{j}\bar{k}} \equiv G^{j\bar{l}}\hat{K}_{\bar{j}j\bar{k}} = \Gamma^{\bar{l}}{}_{\bar{j}\bar{k}} - 3LG^{j\bar{l}}\mathscr{V}_{\bar{j}\bar{k}} \tag{E.7.4}$$

and a curvature tensor

$$\hat{R}_{j\bar{j}k\bar{k}} \equiv \hat{K}_{j\bar{j}k\bar{k}} - G_{l\bar{l}}\hat{\Gamma}^l{}_{jk}\hat{\Gamma}^{\bar{l}}{}_{\bar{j}\bar{k}} = R_{j\bar{j}k\bar{k}} - 3L\mathscr{V}_{j\bar{j}k\bar{k}} - 9L^2\mathscr{V}_{j\bar{l}k}G^{l\bar{l}}\mathscr{V}_{\bar{j}l\bar{k}} \ . \tag{E.7.5}$$

We can then define the corresponding "hat" covariant derivatives like $\hat{\mathscr{V}}_{ij}, \hat{\mathscr{V}}_{i\bar{k}j}$, etc.

Finally, let us note that

$$\frac{\alpha}{L} - 3V \equiv \hat{K}_L \tag{E.7.6}$$

and that Yang–Mills invariance of $\hat{K}$ tells us

$$\hat{K}_k(T_{(r)}A)^k = (\bar{A}T_{(r)})^{\bar{k}}\hat{K}_{\bar{k}} \ , \tag{E.7.7}$$

which again simplifies the expression of the Lagrangian. With the new metric in the Kähler connection we define new covariant derivatives

$$\hat{\nabla}_m A^i \equiv \mathscr{D}_m A^i = \partial_m A^i - \mathrm{i}\boldsymbol{a}_m^{(r)}(\mathbf{T}_{(r)}A)^i,$$

$$\hat{\nabla}_m \bar{A}^{\bar{j}} \equiv \mathscr{D}_m \bar{A}^{\bar{j}} = \partial_m \bar{A}^{\bar{j}} + \mathrm{i}\boldsymbol{a}_m^{(r)}(\bar{A}\mathbf{T}_{(r)})^{\bar{j}},$$

$$\hat{\nabla}_m \chi^i_\alpha = \partial_m \chi^i_\alpha - \omega_{m\alpha}{}^{\varphi}\chi^i_\varphi - \mathrm{i}\boldsymbol{a}_m^{(r)}(\mathbf{T}_{(r)}\chi_\alpha)^i + \chi^j_\alpha\hat{\Gamma}^i_{jk}\mathscr{D}_m A^k$$

$$- \frac{1}{4}(\hat{K}_j\mathscr{D}_m A^j - \hat{K}_{\bar{j}}\mathscr{D}_m\bar{A}^{\bar{j}})\chi^i_\alpha - \frac{\mathrm{i}}{4}G_{j\bar{k}}(\chi^j\sigma_m\bar{\chi}^{\bar{k}})\chi^i_\alpha \ , \tag{E.7.8}$$

$$\hat{\nabla}_m \bar{\chi}^{\bar{j}\dot{\alpha}} = \partial_m \bar{\chi}^{\bar{j}\dot{\alpha}} - \omega_m{}^{\dot{\alpha}}{}_{\dot{\varphi}}\bar{\chi}^{\bar{j}\dot{\varphi}} + \mathrm{i}\boldsymbol{a}_m^{(r)}(\bar{\chi}^{\dot{\alpha}}\mathbf{T}_{(r)})^{\bar{j}} + \bar{\chi}^{\dot{\alpha}\bar{\imath}}\hat{\Gamma}^{\bar{j}}_{\bar{\imath}\bar{k}}\mathscr{D}_m\bar{A}^{\bar{k}}$$

$$+ \frac{1}{4}(\hat{K}_k\mathscr{D}_m A^k - \hat{K}_{\bar{k}}\mathscr{D}_m\bar{A}^{\bar{k}})\bar{\chi}^{\bar{j}\dot{\alpha}} + \frac{\mathrm{i}}{4}G_{j\bar{k}}(\chi^j\sigma_m\bar{\chi}^{\bar{k}})\bar{\chi}^{\bar{j}\dot{\alpha}} \ , \tag{E.7.9}$$

$$\hat{\nabla}_n\psi_m{}^\alpha = \partial_n\psi_m{}^\alpha + \psi_m{}^\beta\omega_{n\beta}{}^\alpha + \psi_m{}^\alpha\left(\frac{1}{4}\hat{K}_i\mathscr{D}_n A^i - \frac{1}{4}\hat{K}_{\bar{j}}\mathscr{D}_n\bar{A}^{\bar{j}} + \frac{\mathrm{i}}{4}G_{i\bar{j}}\chi^i\sigma_n\bar{\chi}^{\bar{j}}\right), \tag{E.7.10}$$

$$\hat{\nabla}_n\bar{\psi}_{m\dot{\alpha}} = \partial_n\bar{\psi}_{m\dot{\alpha}} + \bar{\psi}_{m\dot{\beta}}\omega_n{}^{\dot{\beta}}{}_{\dot{\alpha}} - \bar{\psi}_{m\dot{\alpha}}\left(\frac{1}{4}\hat{K}_i\mathscr{D}_n A^i - \frac{1}{4}\hat{K}_{\bar{j}}\mathscr{D}_n A^{\bar{j}} + \frac{\mathrm{i}}{4}G_{i\bar{j}}\chi^i\sigma_m\bar{\chi}^{\bar{j}}\right), \tag{E.7.11}$$

$$\hat{\nabla}_m\lambda^{(r)}_\alpha = \partial_m\lambda^{(r)}_\alpha - \omega_{m\alpha}{}^\varphi\lambda^{(r)}_\varphi - \boldsymbol{a}_m^{(t)}c_{(s)(t)}{}^{(r)}\lambda^{(s)}_\alpha + \frac{1}{4}(\hat{K}_j\mathscr{D}_m A^j - \hat{K}_{\bar{j}}\mathscr{D}_m\bar{A}^{\bar{j}})\lambda^{(r)}_\alpha + \frac{\mathrm{i}}{4}G_{j\bar{k}}(\chi^j\sigma_m\bar{\chi}^{\bar{k}})\lambda^{(r)}_\alpha,$$

$$\hat{\nabla}_m\bar{\lambda}^{(r)\dot{\alpha}} = \partial_m\bar{\lambda}^{(r)\dot{\alpha}} - \omega_m{}^{\dot{\alpha}}{}_{\dot{\varphi}}\bar{\lambda}^{(r)\dot{\varphi}} - \boldsymbol{a}_m^{(t)}c_{(s)(t)}{}^{(r)}\bar{\lambda}^{(s)\dot{\alpha}} - \frac{1}{4}(\hat{K}_k\partial_m A^k - \hat{K}_{\bar{k}}\partial_m\bar{A}^{\bar{k}})\bar{\lambda}^{(r)\dot{\alpha}} - \frac{\mathrm{i}}{4}G_{j\bar{k}}(\chi^j\sigma_m\bar{\chi}^{\bar{k}})\bar{\lambda}^{(r)\dot{\alpha}} \ . \tag{E.7.12}$$

- Make a shift on $^*h_m$

$$
{}^*h_m \mapsto {}^*h_m + \frac{\mathrm{i}L}{2}\Big[\varepsilon^{mnpq}(\psi_n\sigma_p\bar{\psi}_q) - \mathrm{i}G_{k\bar{k}}(\chi^k\sigma_m\bar{\chi}^{\bar{k}})
- \frac{\mathrm{i}\alpha}{2}(\varphi\sigma_m\bar{\varphi}) + \mathrm{i}k\left(\frac{2}{L} - \hat{K}_L\right)(\lambda^{(r)}\sigma_m\bar{\lambda}_{(r)})\Big]. \tag{E.7.13}
$$

Putting everything together this gives rise to the new Lagrangian

$$
\begin{aligned}
\frac{1}{\mathrm{e}}\mathscr{L} = & -\frac{1}{2}\mathscr{R} + \frac{1}{2}\varepsilon^{mnpq}(\bar{\psi}_m\bar{\sigma}_n\hat{\nabla}_p\psi_q - \psi_m\sigma_n\hat{\nabla}_p\bar{\psi}_q) \\
& - G_{k\bar{k}}\hat{\nabla}_m A^k\hat{\nabla}^m\bar{A}^{\bar{k}} - \frac{\mathrm{i}}{2}G_{k\bar{k}}(\chi^k\sigma^m\hat{\nabla}_m\bar{\chi}^{\bar{k}} + \bar{\chi}^{\bar{k}}\bar{\sigma}^m\hat{\nabla}_m\chi^k) \\
& + \frac{\alpha}{4L^2}{}^*h^{m*}h_m - \frac{\alpha}{4L^2}\partial^m L\partial_m L - \frac{\mathrm{i}\alpha}{4}(\varphi\sigma^m\hat{\nabla}_m\bar{\varphi} + \bar{\varphi}\bar{\sigma}^m\hat{\nabla}_m\varphi) \\
& + \frac{k}{4}\hat{K}_L \boldsymbol{f}_{mn}^{(r)}\boldsymbol{f}_{(r)}^{mn} + \frac{\mathrm{i}k}{2}\hat{K}_L(\lambda^{(r)}\sigma^m\hat{\nabla}_m\bar{\lambda}_{(r)} + \bar{\lambda}^{(r)}\bar{\sigma}^m\hat{\nabla}_m\lambda_{(r)}) \\
& + \frac{1}{8k}(\hat{K}_L)^{-1}[\hat{K}_k(T_{(r)}A)^k + (\bar{A}T_{(r)})^{\bar{k}}\hat{K}_{\bar{k}}]^2 \\
& - \frac{3}{2}\mathrm{i}{}^*h^m(V_k\hat{\nabla}_m A^k - V_{\bar{k}}\hat{\nabla}_m\bar{A}^{\bar{k}}) - \frac{\alpha}{2L}{}^*h^m(\varphi\sigma_m\bar{\varphi}) + \frac{3}{2}V_{k\bar{k}}{}^*h^m(\chi^k\sigma_m\bar{\chi}^{\bar{k}}) \\
& - \left[\frac{3k}{\sqrt{2}}(V_k\chi^k\sigma^{mn}\lambda_{(r)} + V_{\bar{k}}\bar{\chi}^{\bar{k}}\bar{\sigma}^{mn}\bar{\lambda}_{(r)}) + \frac{\alpha k}{2L}(\varphi\sigma^{mn}\lambda_{(r)} + \bar{\varphi}\bar{\sigma}^{mn}\bar{\lambda}_{(r)})\right]\boldsymbol{f}_{mn}^{(r)} \\
& + \mathrm{i}(T_{(r)}A)^k\left[\left(\sqrt{2}G_{k\bar{k}} + \frac{3}{\sqrt{2}}\frac{\hat{K}_k}{\hat{K}_L}V_{\bar{k}}\right)(\bar{\chi}^{\bar{k}}\bar{\lambda}^{(r)}) - \left(3LV_k - \frac{\alpha}{2L}\frac{\hat{K}_k}{\hat{K}_L}\right)(\bar{\varphi}\bar{\lambda}^{(r)})\right] \\
& - \mathrm{i}(\bar{A}T_{(r)})^{\bar{k}}\left[\left(\sqrt{2}G_{k\bar{k}} + \frac{3}{\sqrt{2}}\frac{\hat{K}_{\bar{k}}}{\hat{K}_L}V_k\right)(\chi^k\lambda^{(r)}) - \left(3LV_{\bar{k}} - \frac{\alpha}{2L}\frac{\hat{K}_{\bar{k}}}{\hat{K}_L}\right)(\varphi\lambda^{(r)})\right] \\
& + \frac{\mathrm{i}3L}{\sqrt{2}}[V_{k\bar{k}}(\bar{\chi}^{\bar{k}}\bar{\sigma}^m\varphi)\hat{\nabla}_m A^k + V_{k\bar{k}}(\chi^k\sigma^m\bar{\varphi})\hat{\nabla}_m\bar{A}^{\bar{k}}]
\end{aligned}
$$

$$+\frac{1}{4}\left(\hat{R}_{k\bar{k}l\bar{l}}+\frac{3}{2}G_{k\bar{k}}G_{l\bar{l}}\right)(\chi^k\chi^l)(\bar{\chi}^{\bar{k}}\bar{\chi}^{\bar{l}})-\frac{\alpha(\alpha+12)}{32}(\varphi\varphi)(\bar{\varphi}\bar{\varphi})$$

$$-\frac{k^2}{4}\left(\frac{\alpha}{L^2}+9V_kG^{k\bar{k}}V_{\bar{k}}\right)(\lambda^{(r)}\lambda_{(r)})(\bar{\lambda}^{(s)}\bar{\lambda}_{(s)})-\frac{3k^2}{16}\hat{K}_L^2(\lambda^{(r)}\sigma^m\bar{\lambda}_{(r)})(\lambda^{(s)}\sigma_m\bar{\lambda}_{(s)})$$

$$+\frac{3k}{4}\left(\hat{\mathscr{V}}_{kl}+\frac{3}{2}\hat{K}_L^{-1}V_kV_l\right)(\chi^k\chi^l)(\lambda^{(r)}\lambda_{(r)})+\frac{3k}{4}\left(\hat{\mathscr{V}}_{\bar{k}\bar{l}}+\frac{3}{2}\hat{K}_L^{-1}V_{\bar{k}}V_{\bar{l}}\right)(\bar{\chi}^{\bar{k}}\bar{\chi}^{\bar{l}})(\bar{\lambda}^{(r)}\bar{\lambda}_{(r)})$$

$$-\frac{\alpha k}{4L}\left(1-\frac{\alpha}{4L\hat{K}_L}\right)[(\bar{\lambda}^{(r)}\bar{\lambda}_{(r)})(\bar{\varphi}\bar{\varphi})+(\lambda^{(r)}\lambda_{(r)})(\varphi\varphi)]$$

$$-\frac{\alpha k}{16L}\hat{K}_L\left(1+\frac{2\alpha}{\hat{K}_L^2L^2}\right)(\lambda^{(r)}\sigma^m\bar{\lambda}_{(r)})(\varphi\sigma_m\bar{\varphi})$$

$$+\left[-\frac{\alpha}{16}G_{k\bar{k}}+\frac{9L^2}{4}V_{j\bar{k}}G^{j\bar{j}}V_{k\bar{j}}\right](\chi^k\sigma^m\bar{\chi}^{\bar{k}})(\varphi\sigma_m\bar{\varphi})$$

$$-\frac{3k}{8}(\hat{K}_LG_{k\bar{k}}+6\hat{K}_L^{-1}V_kV_{\bar{k}})(\chi^k\sigma^m\bar{\chi}^{\bar{k}})(\lambda^{(r)}\sigma_m\bar{\lambda}_{(r)})$$

$$+\frac{3k}{4\sqrt{2}L}(6L^2V_{\bar{k}}G^{k\bar{k}}V_{k\bar{j}}+\alpha\hat{K}_L^{-1}V_{\bar{j}})(\bar{\chi}^{\bar{j}}\bar{\varphi})(\bar{\lambda}^{(s)}\bar{\lambda}_{(s)})$$

$$+\frac{3k}{4\sqrt{2}L}(6L^2V_kG^{k\bar{k}}V_{\bar{k}j}+\alpha\hat{K}_L^{-1}V_j)(\chi^j\varphi)(\lambda^{(s)}\lambda_{(s)})$$

$$-\frac{3k\alpha}{4\sqrt{2}L}(\hat{K}_L)^{-1}[V_k(\chi^k\sigma_m\bar{\varphi})-V_{\bar{k}}(\bar{\chi}^{\bar{k}}\bar{\sigma}_m\varphi)](\lambda^{(r)}\sigma^m\bar{\lambda}_{(r)})$$

$$-\frac{3L}{2\sqrt{2}}[\hat{\mathscr{V}}_{\bar{k}j\bar{l}}(\bar{\chi}^{\bar{l}}\bar{\chi}^{\bar{k}})(\chi^j\varphi)+\hat{\mathscr{V}}_{k\bar{j}l}(\chi^l\chi^k)(\bar{\chi}^{\bar{j}}\bar{\varphi})]$$

$$-\tfrac{1}{2}\hat{K}_k(T_{(r)}A)^k(\bar{\psi}_m\bar{\sigma}^m\lambda^{(r)})+\tfrac{1}{2}\hat{K}_{\bar{k}}(\bar{A}T_{(r)})^{\bar{k}}(\psi_m\sigma^m\bar{\lambda}^{(r)})$$

$$-\frac{1}{\sqrt{2}}G_{k\bar{k}}[(\psi_n\sigma^m\bar{\sigma}^n\chi^k)\hat{\nabla}_m\bar{A}^{\bar{k}}+(\bar{\psi}_n\bar{\sigma}^m\sigma^n\bar{\chi}^{\bar{k}})\hat{\nabla}_mA^k]$$

$$-\frac{\alpha}{4L}(\psi_n\sigma^m\bar{\sigma}^n\varphi+\bar{\psi}_n\bar{\sigma}^m\sigma^n\bar{\varphi})\partial_mL+\frac{i\alpha}{4L}(\psi_n\sigma^m\bar{\sigma}^n\varphi-\bar{\psi}_n\bar{\sigma}^m\sigma^n\bar{\varphi})^*h_m$$

$$+\frac{\mathrm{i}\alpha k}{8L}[(\bar{\psi}_m\bar{\sigma}^m\varphi)(\lambda^{(r)}\lambda_{(r)})+(\psi_m\sigma^m\bar{\varphi})(\bar{\lambda}^{(r)}\bar{\lambda}_{(r)})]$$

$$-\frac{\mathrm{i}\alpha}{16}(\alpha-4)(\varphi\sigma^m\bar{\varphi})(\psi_m\varphi+\bar{\psi}_m\bar{\varphi})-\frac{\mathrm{i}3k}{4\sqrt{2}}V_k(3\chi^k\psi^m+2\chi^k\sigma^{mn}\psi_n)(\lambda^{(r)}\sigma_m\bar{\lambda}_{(r)})$$

$$+\frac{\mathrm{i}3k}{4\sqrt{2}}V_{\bar{k}}(3\bar{\chi}^{\bar{k}}\bar{\psi}^m+2\bar{\chi}^{\bar{k}}\bar{\sigma}^{mn}\bar{\psi}_n)(\lambda^{(r)}\sigma_m\bar{\lambda}_{(r)})$$

$$-\frac{\mathrm{i}3k}{4\sqrt{2}}[V_k(\chi^k\sigma^m\bar{\psi}_m)(\lambda^{(r)}\lambda_{(r)})+V_{\bar{k}}(\bar{\chi}^{\bar{k}}\bar{\sigma}^m\psi_m)(\bar{\lambda}^{(r)}\bar{\lambda}_{(r)})]$$

$$-\frac{\mathrm{i}\alpha k}{4L}(\psi^m\varphi-\bar{\psi}^m\bar{\varphi})(\lambda^{(r)}\sigma_m\bar{\lambda}_{(r)})+\frac{\mathrm{i}\alpha k}{8L}(\psi_n\sigma^m\bar{\sigma}^n\varphi-\bar{\psi}_n\bar{\sigma}^m\sigma^n\bar{\varphi})(\lambda^{(r)}\sigma_m\bar{\lambda}_{(r)})$$

$$+\frac{3\mathrm{i}}{4}LV_{k\bar{k}}(\chi^k\sigma_m\bar{\chi}^{\bar{k}})(\psi_n\sigma^m\bar{\sigma}^n\varphi-\bar{\psi}_n\bar{\sigma}^m\sigma^n\bar{\varphi})-\frac{\mathrm{i}\alpha(\alpha+4)}{32}(\psi_n\sigma^m\bar{\sigma}^n\varphi-\bar{\psi}_n\bar{\sigma}^m\sigma^n\bar{\varphi})(\varphi\sigma_m\bar{\varphi})$$

$$+\frac{1}{4}G_{k\bar{k}}(\chi^k\sigma_m\bar{\chi}^{\bar{k}})(\psi_n\sigma^m\bar{\psi}^n-\frac{\mathrm{i}}{2}\varepsilon^{mnpq}\psi_n\sigma_p\bar{\psi}_q)$$

$$-\frac{\mathrm{i}k}{2}\hat{K}_L[(\boldsymbol{f}^{(r)mn}+\mathrm{i}{}^*\boldsymbol{f}^{(r)mn})(\psi_m\sigma_n\bar{\lambda}_{(r)})+(\boldsymbol{f}^{(r)mn}-\mathrm{i}{}^*\boldsymbol{f}^{(r)mn})(\bar{\psi}_m\bar{\sigma}_n\lambda_{(r)})]$$

$$+\frac{1}{16L}\hat{K}_L[(3\psi_n\psi^n+2\psi_n\sigma^{nm}\psi_m)(\lambda^{(r)}\lambda_{(r)})+(3\bar{\psi}_n\bar{\psi}^n+2\bar{\psi}_n\bar{\sigma}^{nm}\bar{\psi}_m)(\bar{\lambda}^{(r)}\bar{\lambda}_{(r)})]$$

$$+\frac{k}{8}\hat{K}_L\left(\oint_{npq}g^{mn}g^{pq}-\mathrm{i}\varepsilon^{mnpq}\right)(\psi_m\sigma_n\bar{\psi}_p)(\lambda^{(r)}\sigma_q\bar{\lambda}_{(r)})$$

$$+\frac{\alpha}{8}[(\psi_m\psi^m)(\varphi\varphi)+(\bar{\psi}_m\bar{\psi}^m)(\bar{\varphi}\bar{\varphi})]+\frac{\alpha}{8}(g^{mn}g^{pq}+g^{mq}g^{np})(\psi_m\sigma_n\bar{\psi}_p)(\varphi\sigma_q\bar{\varphi})\,. \qquad \text{(E.7.14)}$$

Appendix F. Three-form gauge potential and Chern–Simons forms

The analogy between Chern–Simons forms and 3-form gauge potentials will be employed to determine the Chern–Simons superfield (5.2.21). To this end, we present first the explicit solution of the 4-form constraints in terms of an unconstrained superfield. Already important by itself, in the description of constrained chiral multiplets – cf. Section 6 – this analysis underlies the explicit construction of the Chern–Simons superfield. After some general remarks and definitions concerning Chern–Simons forms in superspace, the Chern–Simons superfield is determined as the counterpart of the pre-potential of the 3-form.

F.1. Explicit solution of the constraints

As shown in the main text, the constraints

$$\Sigma_{\underline{\delta}\underline{\gamma}\underline{\beta}A} = 0\ , \tag{F.1.1}$$

allow to express all the coefficients of the 4-form field strength in terms of the constrained chiral fields $Y, \bar{Y}$. The Bianchi identities in the presence of the constraints are summarized in the chirality conditions together with the additional constraint (6.1.2). Alternatively, as we will explain now, the explicit solution of the superspace constraints allows us to determine the unconstrained pre-potential of the constrained superfield. An important ingredient in this procedure will be the use of the gauge freedom of the 3-form potential, C, parametrized by a 2-form Λ,

$${}^{\Lambda}C_{CBA} = C_{CBA} + \oint_{CBA} (\mathscr{D}_C \Lambda_{BA} + T_{CB}{}^F \Lambda_{FA})\ . \tag{F.1.2}$$

As usual $\oint_{CBA}$ denotes the graded sum $CBA + (-)^{c(b+a)}BAC + (-)^{a(b+c)}ACB$. In a first step consider

$$\Sigma_{\delta\gamma\beta A} = 0\ , \tag{F.1.3}$$

which we satisfy with

$$C_{\gamma\beta A} = \mathscr{D}_A U_{\gamma\beta} + \oint_{\gamma\beta} (\mathscr{D}_\gamma U_{\beta A} + T_{A\gamma}{}^F U_{F\beta}) \tag{F.1.4}$$

and the complex conjugate

$$\Sigma^{\dot{\delta}\dot{\gamma}\dot{\beta}}{}_A = 0\ , \tag{F.1.5}$$

by

$$C^{\dot{\gamma}\dot{\beta}}{}_A = \mathscr{D}_A V^{\dot{\gamma}\dot{\beta}} + \oint^{\dot{\gamma}\dot{\beta}} (\mathscr{D}^{\dot{\gamma}} V_A{}^{\dot{\beta}} + T_A{}^{\dot{\gamma}F} V_F{}^{\dot{\beta}})\ . \tag{F.1.6}$$

Since the pre-potentials $U_{\beta A}$ and $V_A{}^{\dot{\beta}}$ should reproduce the gauge transformations of the gauge potentials $C_{\gamma\beta A}$ and $C^{\dot{\gamma}\dot{\beta}}{}_A$ we assign

$$U_{\beta A} \mapsto {}^{\Lambda}U_{\beta A} = U_{\beta A} + \Lambda_{\beta A} \tag{F.1.7}$$

and

$$V^{\dot{\beta}}{}_A \mapsto {}^{\Lambda}V^{\dot{\beta}}{}_A = V^{\dot{\beta}}{}_A + \Lambda^{\dot{\beta}}{}_A\ , \tag{F.1.8}$$

as gauge transformation laws for the pre-potentials. On the other hand, the so-called *pre-gauge transformations* are defined as the zero modes of the gauge potentials themselves, that is transformations which leave $C_{\gamma\beta A}$ and $C^{\dot{\gamma}\dot{\beta}}{}_A$ invariant. They are given as

$$U_{\beta A} \mapsto U_{\beta A} + \mathscr{D}_\beta \chi_A - (-)^a \mathscr{D}_A \chi_\beta + T_{\beta A}{}^F \chi_F \tag{F.1.9}$$

and

$$V^{\dot{\beta}}{}_A \mapsto V^{\dot{\beta}}{}_A + \mathscr{D}^{\dot{\beta}} \psi_A - (-)^a \mathscr{D}_A \psi^{\dot{\beta}} + T^{\dot{\beta}}{}_A{}^F \psi_F \ . \tag{F.1.10}$$

We parametrize the pre-potentials now as follows:

$$U_\beta{}^{\dot{\alpha}} = W_\beta{}^{\dot{\alpha}} + T_\beta{}^{\dot{\alpha} f} K_f \ , \tag{F.1.11}$$

$$V^{\dot{\beta}}{}_\alpha = W_\alpha{}^{\dot{\beta}} - T_\alpha{}^{\dot{\beta} f} K_f \tag{F.1.12}$$

and

$$U_{\beta a} = W_{\beta a} - \mathscr{D}_\beta K_a \ , \tag{F.1.13}$$

$$V^{\dot{\beta}}{}_a = W^{\dot{\beta}}{}_a + \mathscr{D}^{\dot{\beta}} K_a \ . \tag{F.1.14}$$

Explicit substitution shows that the K_a terms drop out in $C_{\gamma\beta A}$ and $C^{\dot{\gamma}\dot{\beta}}{}_A$. Denoting furthermore

$$U_{\beta\alpha} = W_{\beta\alpha} \quad \text{and} \quad V^{\dot{\beta}\dot{\alpha}} = W^{\dot{\beta}\dot{\alpha}} \ , \tag{F.1.15}$$

we arrive at

$$C_{\gamma\beta A} = \mathscr{D}_A W_{\gamma\beta} + \oint_{\gamma\beta} (\mathscr{D}_\gamma W_{\beta A} + T_{A\gamma}{}^F W_{F\beta}) \ , \tag{F.1.16}$$

$$C^{\dot{\gamma}\dot{\beta}}{}_A = \mathscr{D}_A W^{\dot{\gamma}\dot{\beta}} + \oint^{\dot{\gamma}\dot{\beta}} (\mathscr{D}^{\dot{\gamma}} W^{\dot{\beta}}{}_A + T_A{}^{\dot{\gamma} F} W_F{}^{\dot{\beta}}) \ , \tag{F.1.17}$$

i.e. a pure gauge form for the coefficients $C_{\gamma\beta A}$ and $C^{\dot{\gamma}\dot{\beta}}_A$ with the 2-form gauge parameter Λ replaced by the pre-potential 2-form

$$W = \tfrac{1}{2} E^A E^B W_{BA} \quad \text{with} \quad W_{ba} = 0 \ . \tag{F.1.18}$$

We take advantage of this fact to perform a redefinition of the 3-form gauge potentials, which has the form of a gauge transformation,

$$\hat{C} := {}^{-W}C = C - dW \ . \tag{F.1.19}$$

This leaves the field strength invariant and leads in particular to

$$\hat{C}_{\gamma\beta A} = 0 \quad \text{and} \quad \hat{C}^{\dot{\gamma}\dot{\beta}}{}_A = 0\,, \tag{F.1.20}$$

whereas the coefficient $C_\gamma{}^{\dot\beta}{}_a$ is replaced by

$$\hat{C}_\gamma{}^{\dot\beta}{}_a = C_\gamma{}^{\dot\beta}{}_a - \mathscr{D}_\gamma W^{\dot\beta}{}_a - \mathscr{D}^{\dot\beta} W_{\gamma a} - \mathscr{D}_a W_\gamma{}^{\dot\beta}\,. \tag{F.1.21}$$

We define the tensor decomposition

$$\hat{C}_\gamma{}^{\dot\beta}{}_a = T_\gamma{}^{\dot\beta f}(\eta_{fa}\Omega + \hat{W}_{[fa]} + \tilde{\Omega}_{(fa)})\,, \tag{F.1.22}$$

where $\hat{W}_{[fa]}$ is antisymmetric and $\tilde{\Omega}_{(fa)}$ symmetric and traceless, and perform another redefinition which has again the form of a gauge transformation, this time of parameter

$$\hat{W} = \tfrac{1}{2} E^a E^b \hat{W}_{[ba]}\,, \tag{F.1.23}$$

such that

$$\Omega = {}^{-\hat{W}}\hat{C} = \hat{C} - d\hat{W}\,. \tag{F.1.24}$$

Note that this reparametrization leaves $\hat{C}_{\gamma\beta A}$ and $\hat{C}^{\dot\gamma\dot\beta}{}_A$ untouched, they remain zero.

Let us summarize the preceding discussion: we started out with the 3-form gauge potential C. The constraints on its field strength led us to introduce pre-potentials. By means of pre-potential-dependent redefinitions of C, which have the form of gauge transformations (and which, therefore, leave the field strength invariant), we arrived at the representation of the 3-form gauge potential in terms of Ω, with the particularly nice properties

$$\Omega_{\gamma\beta A} = 0, \quad \Omega^{\dot\gamma\dot\beta}{}_A = 0 \tag{F.1.25}$$

and

$$\Omega_\gamma{}^{\dot\beta}{}_a = T_\gamma{}^{\dot\beta f}(\eta_{fa}\Omega + \tilde{\Omega}_{(fa)})\,. \tag{F.1.26}$$

Clearly, in this representation, calculations simplify considerably. We shall therefore, from now on, pursue the solution of the constraints in terms of Ω and turn to the equation

$$\Sigma_{\delta\gamma}{}^{\dot\beta\dot\alpha} = 0 = \oint_{\delta\gamma}^{\dot\beta\dot\alpha} T_\delta{}^{\dot\gamma f}\Omega_f{}^{\dot\beta}{}_{\dot\alpha}\,, \tag{F.1.27}$$

which tells us simply that $\tilde{\Omega}_{(ba)}$ is zero. Hence,

$$\Omega_\gamma{}^{\dot\beta}{}_a = T_\gamma{}^{\dot\beta}{}_a\Omega\,. \tag{F.1.28}$$

We turn next to the constraints

$$\Sigma_{\delta a}{}^{\dot\gamma\dot\beta} = 0 = \oint^{\dot\gamma\dot\beta} (\mathscr{D}^{\dot\gamma}\Omega_\delta{}^{\dot\beta}{}_a + T_\delta{}^{\dot\gamma f}\Omega_f{}^{\dot\beta}{}_a) \tag{F.1.29}$$

and

$$\Sigma^{\delta}{}_{\gamma\beta a} = 0 = \oint_{\gamma\beta} (\mathscr{D}_{\gamma}\Omega^{\delta}{}_{\beta a} + T_{\gamma}{}^{\delta f}\Omega_{f\beta a}) , \tag{F.1.30}$$

which, after some straightforward spinor index gymnastics give rise to

$$\Omega_{\gamma ba} = 2(\sigma_{ba})_{\gamma}{}^{\varphi}\mathscr{D}_{\varphi}\Omega , \tag{F.1.31}$$

$$\Omega^{\dot\gamma}{}_{ba} = 2(\bar\sigma_{ba})^{\dot\gamma}{}_{\dot\varphi}\mathscr{D}^{\dot\varphi}\Omega . \tag{F.1.32}$$

This completes the discussion of the solution of the constraints, we discuss next the consequences of this solution for the remaining components in Σ, i.e. $\Sigma_{\underline{\delta}\underline{\gamma}ba}$, $\Sigma_{\underline{\delta}cba}$ and Σ_{dcba}. As a first step we consider

$$\Sigma_{\delta\gamma ba} = \oint_{\delta\gamma} (\mathscr{D}_{\delta}\Omega_{\gamma ba} - T_{\delta b\dot\varphi}\Omega_{\gamma}{}^{\dot\varphi}{}_{a} + T_{\delta a\dot\varphi}\Omega_{\gamma}{}^{\dot\varphi}{}_{b}) , \tag{F.1.33}$$

and

$$\Sigma^{\dot\delta\dot\gamma}{}_{ba} = \oint^{\dot\delta\dot\gamma} (\mathscr{D}^{\dot\delta}\Omega^{\dot\gamma}{}_{ba} - T^{\dot\delta}{}_{b}{}^{\varphi}\Omega_{\varphi}{}^{\dot\gamma}{}_{a} + T^{\dot\delta}{}_{a}{}^{\varphi}\Omega_{\varphi}{}^{\dot\gamma}{}_{b}) . \tag{F.1.34}$$

Substituting for the 3-form gauge potentials as determined so far, and making appropriate use of the supergravity Bianchi identities yields

$$\Sigma_{\delta\gamma ba} = -2(\sigma_{ba}\varepsilon)_{\delta\gamma}(\mathscr{D}^2 - 8R^{\dagger})\Omega \tag{F.1.35}$$

and

$$\Sigma^{\dot\delta\dot\gamma}{}_{ba} = -2(\bar\sigma_{ba}\varepsilon)^{\dot\delta\dot\gamma}(\bar{\mathscr{D}}^2 - 8R)\Omega . \tag{F.1.36}$$

The appearance of the chiral projection operators suggests to define

$$\bar Y = -4(\mathscr{D}^2 - 8R^{\dagger})\Omega , \tag{F.1.37}$$

$$Y = -4(\bar{\mathscr{D}}^2 - 8R)\Omega . \tag{F.1.38}$$

The gauge invariant superfields Y and $\bar Y$ have chirality properties

$$\mathscr{D}_{\alpha}\bar Y = 0, \quad \mathscr{D}^{\dot\alpha}Y = 0 \tag{F.1.39}$$

and we obtain

$$\Sigma_{\delta\gamma ba} = \tfrac{1}{2}(\sigma_{ba}\varepsilon)_{\delta\gamma}\bar Y , \tag{F.1.40}$$

$$\Sigma^{\dot\delta\dot\gamma}{}_{ba} = \tfrac{1}{2}(\bar\sigma_{ba}\varepsilon)^{\dot\delta\dot\gamma} Y . \tag{F.1.41}$$

In the next step we observe that, due to the information extracted so far from the solution of the constraints, the field strength

$$\Sigma_{\delta}{}^{\dot{\gamma}}{}_{ba} = T_{\delta}{}^{\dot{\gamma}c}\Sigma_{cba} \tag{F.1.42}$$

is determined such that Σ_{cba} is totally antisymmetric in its three vector indices. As, in its explicit definition a linear term appears (due to the constant torsion term), i.e.

$$\Sigma_{\delta}{}^{\dot{\gamma}}{}_{ba} = T_{\delta}{}^{\dot{\gamma}c}\Omega_{cba} + \text{derivative and other torsion terms} , \tag{F.1.43}$$

we can absorb Σ_{cba} in a modified 3-form gauge potential

$$\underline{\Omega}_{cba} = \Omega_{cba} - \Sigma_{cba} , \tag{F.1.44}$$

such that the corresponding modified field strength vanishes, i.e.

$$\underline{\Sigma}_{\delta}{}^{\dot{\gamma}}{}_{ba} = 0 . \tag{F.1.45}$$

The outcome of this discussion is then the relation

$$([\mathscr{D}_\alpha, \mathscr{D}_{\dot{\alpha}}] - 4G_{\alpha\dot{\alpha}})\Omega = -\tfrac{1}{3}\sigma_{d\alpha\dot{\alpha}}\varepsilon^{dcba}\underline{\Omega}_{cba} , \tag{F.1.46}$$

which identifies $\underline{\Omega}_{cba}$ in the superfield expansion of the unconstrained pre-potential Ω.

Working, from now on, in terms of the modified quantities, the remaining coefficients, at canonical dimensions 3/2 and 2, i.e. $\underline{\Sigma}_{\underline{\delta}cba}$ and $\underline{\Sigma}_{dcba}$, respectively, are quite straightforwardly obtained in terms of spinor derivatives of the basic gauge invariant superfields Y and $\bar{Y}$. To be more precise, at dimension 3/2 one obtains

$$\underline{\Sigma}_{\delta\, cba} = -\tfrac{1}{16}\sigma^{d}_{\delta\dot{\delta}}\varepsilon_{dcba}\bar{\mathscr{D}}^{\dot{\delta}}\bar{Y} , \tag{F.1.47}$$

$$\underline{\Sigma}^{\dot{\delta}}{}_{cba} = +\tfrac{1}{16}\bar{\sigma}^{d\dot{\delta}\delta}\varepsilon_{dcba}\mathscr{D}_{\delta}Y \tag{F.1.48}$$

and the Bianchi identity at dimension 2 takes the simple form

$$(\mathscr{D}^2 - 24R^\dagger)Y - (\bar{\mathscr{D}}^2 - 24R)\bar{Y} = \frac{8\mathrm{i}}{3}\varepsilon^{dcba}\underline{\Sigma}_{dcba} . \tag{F.1.49}$$

As to the gauge structure of the 3-form gauge potential we note that in the transition from C to Ω, the original 2-form gauge transformations have disappeared, Ω is invariant under those. In exchange, however, as already mentioned earlier, Ω transforms under so-called pre-gauge transformations which, in turn, leave C unchanged. As a result, the residual pre-gauge transformations of the unconstrained pre-potential superfield,

$$\Omega \mapsto \Omega' = \Omega + \lambda \tag{F.1.50}$$

are parametrized in terms of a linear superfield λ which satisfies

$$(\mathscr{D}^2 - 8R^\dagger)\lambda = 0, \quad (\bar{\mathscr{D}}^2 - 8R)\lambda = 0 . \tag{F.1.51}$$

In turn, λ can be expressed in terms of an unconstrained superfield, as we know from the explicit solution of the superspace constraints of the 2-form gauge potential, actually defining the linear superfield geometrically. In other words, the pre-gauge transformations should respect the particular form of the coefficients of the 3-form Ω.

F.2. Chern–Simons forms in superspace

Under gauge transformations the Chern–Simons 3-forms change by the exterior derivative of a 2-form, which depends on the gauge parameter and the gauge potential. Due to this property one may understand the Chern–Simons form as a special case of a generic 3-form gauge potential – cf. the preceding subsection. This point of view is particularly useful for the supersymmetric case. To be as clear as possible we first recall some general properties of Chern–Simons forms in superspace.

To begin with we consider two gauge potentials $\mathscr{A}_0$ and $\mathscr{A}_1$ in superspace. Their field strength squared invariants are related through

$$\mathrm{tr}(\mathscr{F}_0\mathscr{F}_0) - \mathrm{tr}(\mathscr{F}_1\mathscr{F}_1) = d\mathscr{Q}(\mathscr{A}_0, \mathscr{A}_1) . \tag{F.2.1}$$

This is the superspace version of the Chern–Simons formula, where

$$\mathscr{F}_0 = d\mathscr{A}_0 + \mathscr{A}_0\mathscr{A}_0, \quad \mathscr{F}_1 = d\mathscr{A}_1 + \mathscr{A}_1\mathscr{A}_1 . \tag{F.2.2}$$

On the right appears the superspace Chern–Simons form

$$\mathscr{Q}(\mathscr{A}_0, \mathscr{A}_1) = 2\int_0^1 dt\, \mathrm{tr}\{(\mathscr{A}_0 - \mathscr{A}_1)\mathscr{F}_t\} , \tag{F.2.3}$$

where

$$\mathscr{F}_t = d\mathscr{A}_t + \mathscr{A}_t\mathscr{A}_t \tag{F.2.4}$$

is the field strength for the interpolating gauge potential

$$\mathscr{A}_t = (1-t)\mathscr{A}_0 + t\mathscr{A}_1 . \tag{F.2.5}$$

The Chern–Simons form is antisymmetric in its arguments, i.e.

$$\mathscr{Q}(\mathscr{A}_0, \mathscr{A}_1) = -\mathscr{Q}(\mathscr{A}_1, \mathscr{A}_0) . \tag{F.2.6}$$

In the particular case $\mathscr{A}_0 = \mathscr{A}$, $\mathscr{A}_1 = 0$, one obtains

$$\mathscr{Q}(\mathscr{A}) = \mathscr{Q}(\mathscr{A}, 0) = \mathrm{tr}(\mathscr{A}\mathscr{F} - \tfrac{1}{3}\mathscr{A}\mathscr{A}\mathscr{A}) . \tag{F.2.7}$$

We shall also make use of the identity

$$\mathscr{Q}(\mathscr{A}_0, \mathscr{A}_1) + \mathscr{Q}(\mathscr{A}_1, \mathscr{A}_2) + \mathscr{Q}(\mathscr{A}_2, \mathscr{A}_0) = d\chi(\mathscr{A}_0, \mathscr{A}_1, \mathscr{A}_2) \tag{F.2.8}$$

with

$$\chi(\mathscr{A}_0,\mathscr{A}_1,\mathscr{A}_2)=\mathrm{tr}(\mathscr{A}_0\mathscr{A}_1+\mathscr{A}_1\mathscr{A}_2+\mathscr{A}_2\mathscr{A}_0)\ . \tag{F.2.9}$$

This last relation (the so-called *triangular equation*) is particularly useful for the determination of the gauge transformation of the Chern–Simons form. The argument goes as follows: first of all, using the definition given above, one observes that

$$\mathscr{Q}({}^{g}\!\mathscr{A},0)=\mathscr{Q}(\mathscr{A},d\boldsymbol{g}\boldsymbol{g}^{-1})\ . \tag{F.2.10}$$

Combining this with the triangular equation for the special choices

$$\mathscr{A}_0=0,\quad \mathscr{A}_1=\mathscr{A},\quad \mathscr{A}_2=d\boldsymbol{g}\boldsymbol{g}^{-1}\ , \tag{F.2.11}$$

one obtains

$$\mathscr{Q}(0,\mathscr{A})+\mathscr{Q}({}^{g}\!\mathscr{A},0)+\mathscr{Q}(d\boldsymbol{g}\boldsymbol{g}^{-1},0)=d\,\mathrm{tr}(\mathscr{A}d\boldsymbol{g}\boldsymbol{g}^{-1})\ , \tag{F.2.12}$$

or, using the antisymmetry property

$$\mathscr{Q}({}^{g}\!\mathscr{A})-\mathscr{Q}(\mathscr{A})=d\,\mathrm{tr}(\mathscr{A}d\boldsymbol{g}\boldsymbol{g}^{-1})-\mathscr{Q}(d\boldsymbol{g}\boldsymbol{g}^{-1})\ . \tag{F.2.13}$$

The last term in this equation is an exact differential form in superspace as well, it can be written as

$$\mathscr{Q}(d\boldsymbol{g}\boldsymbol{g}^{-1})=d\sigma\ , \tag{F.2.14}$$

where the 2-form σ is defined as

$$\sigma=\int_0^1 dt\,\mathrm{tr}(\partial_t\boldsymbol{g}_t\boldsymbol{g}_t^{-1}d\boldsymbol{g}_t\boldsymbol{g}_t^{-1}d\boldsymbol{g}_t\boldsymbol{g}_t^{-1}) \tag{F.2.15}$$

with the interpolating group element $\boldsymbol{g}_t$ parametrized such that for $t\in[0,1]$

$$\boldsymbol{g}_0=1,\quad \boldsymbol{g}_1=\boldsymbol{g}\ . \tag{F.2.16}$$

This shows that the gauge transformation of the Chern–Simons form, which is a 3-form in superspace, is given as the exterior derivative of a 2-form

$$\mathscr{Q}({}^{g}\!\mathscr{A})-\mathscr{Q}(\mathscr{A})=d\Delta(\boldsymbol{g},\mathscr{A}) \tag{F.2.17}$$

with $\Delta=\chi-\sigma$.

The discussion so far was quite general and valid for some generic gauge potential. It does not only apply to the Yang–Mills case but to gravitational Chern–Simons forms as well.

F.3. The Chern–Simons superfield

We specialize here to the Yang–Mills case, i.e. we shall now take into account the covariant constraints on the field strength, which define supersymmetric Yang–Mills theory. It is the purpose of the present subsection to elucidate the relation between the unconstrained pre-potential, which arises in the constrained 3-form geometry, and the Chern–Simons superfield. Moreover, based on this observation and on the preceding subsections we present a geometric construction of the explicit form of the Yang–Mills Chern–Simons superfield in terms of the unconstrained pre-potential of supersymmetric Yang–Mills theory.

In this construction of the Chern–Simons superfield we will combine the knowledge acquired in the discussion of the 3-form gauge potential with the special features of Yang–Mills theory in superspace. Recall that the Chern–Simons superfield $\Omega^{\mathscr{YM}}$ is identified in the relations

$$\mathrm{tr}(\mathscr{W}_{\dot\alpha}\mathscr{W}^{\dot\alpha}) = \tfrac{1}{2}(\mathscr{D}^2 - 8R^\dagger)\Omega^{\mathscr{YM}} , \tag{F.3.1}$$

$$\mathrm{tr}(\mathscr{W}^{\alpha}\mathscr{W}_{\alpha}) = \tfrac{1}{2}(\bar{\mathscr{D}}^2 - 8R)\Omega^{\mathscr{YM}} . \tag{F.3.2}$$

The appearance of one and the same superfield under the projectors reflects the fact that the gaugino superfields $\mathscr{W}_{\underset{\sim}{\alpha}}$ are not only subject to the chirality constraints (2.3.33) but satisfy the additional condition (2.3.34). It is for this reason that the Chern–Simons form can be so neatly embedded in the geometry of the 3-form. As explained in Section 5.2 the terms on the left-hand side are located in the superspace 4-form

$$\Sigma^{\mathscr{YM}} = \mathrm{tr}(\mathscr{F}\mathscr{F}) . \tag{F.3.3}$$

Of course, the constraints on the Yang–Mills field strength induce special properties on the 4-form coefficients, in particular

$$\Sigma^{\mathscr{YM}}{}_{\underset{\sim}{\delta}\,\underset{\sim}{\gamma}\,\underset{\sim}{\beta}\,A} = 0 , \tag{F.3.4}$$

which is just the same tensor structure as the constraints on the field strength of the 3-form gauge potential. Therefore *the Chern–Simons geometry can be regarded as a special case of that of the 3-form gauge potential*. Keeping in mind this fact we obtain

$$\Sigma^{\mathscr{YM}}{}_{\delta\gamma ba} = \tfrac{1}{2}(\sigma_{ba}\varepsilon)_{\delta\gamma}\,\bar{Y}^{\mathscr{YM}} , \tag{F.3.5}$$

$$\Sigma^{\mathscr{YM}\dot\delta\dot\gamma}{}_{ba} = \tfrac{1}{2}(\bar\sigma_{ba}\varepsilon)^{\dot\delta\dot\gamma}\,Y^{\mathscr{YM}} \tag{F.3.6}$$

with

$$Y^{\mathscr{YM}} = -8\,\mathrm{tr}(\mathscr{W}^{\alpha}\mathscr{W}_{\alpha}) , \tag{F.3.7}$$

$$\bar{Y}^{\mathscr{YM}} = -8\,\mathrm{tr}(\mathscr{W}_{\dot\alpha}\mathscr{W}^{\dot\alpha}) . \tag{F.3.8}$$

These facts imply the existence and provide a method for the explicit construction of the Chern–Simons superfield: comparison of these equations with those obtained earlier in the 3-form geometry clearly suggests that the Chern–Simons superfield $\Omega^{\mathscr{YM}}$ will be the analogue of the unconstrained pre-potential superfield Ω of the 3-form. In order to establish this correspondence in full detail we translate the procedure developed in the case of the 3-form geometry to the Chern–Simons form (in the following we shall omit the $\mathscr{YM}$ superscript). The starting point for the explicit construction of the Chern–Simons superfield is the relation

$$\mathrm{tr}(\mathscr{F}\mathscr{F}) = d\mathscr{Q}(\mathscr{A}) . \tag{F.3.9}$$

In the 3-form geometry we know unambiguously the exact location of the pre-potential in superspace geometry. Since we have identified Chern–Simons as a special case of the 3-form, it is now rather straightforward to identify the Chern–Simons superfield following the same strategy. To this end we recall that the pre-potential was identified after certain field-dependent redefinitions which had the form of a gauge transformation, simplifying considerably the form of the potentials. For instance, the new potentials had the property

$$\Omega_{\gamma\beta A} = 0, \quad \Omega^{\dot{\gamma}\dot{\beta}}{}_{A} = 0 . \tag{F.3.10}$$

Note, en passant, that these redefinitions are not compulsory for the identification of the unconstrained pre-potential. They make, however, the derivation a good deal more transparent. Can these features be reproduced in the Chern–Simons framework? To answer this question we exploit a particularity of Yang–Mills in superspace, namely the existence of different types of gauge potentials corresponding to the different possible types of gauge transformations as described in Section 2.2.2. These gauge potentials are superspace 1-forms denoted by $\mathscr{A}$, $\mathscr{A}(0) = \mathfrak{a}$ and $\mathscr{A}(1) = \bar{\mathfrak{a}}$, with gauge transformations parametrized in terms of real, chiral and antichiral superfields, respectively. Moreover, the chiral and antichiral bases are related by a redefinition which has the form of a gauge transformation involving the pre-potential superfield $\mathscr{W}$

$$\mathfrak{a} = \mathscr{W}^{-1}\bar{\mathfrak{a}}\mathscr{W} - \mathscr{W}^{-1}d\mathscr{W} = {}^{\mathscr{W}}\bar{\mathfrak{a}} . \tag{F.3.11}$$

Writing the superspace Chern–Simons form in terms of $\mathfrak{a}$ shows immediately that

$$\mathscr{Q}^{\dot{\gamma}\dot{\beta}}{}_{A}(\mathfrak{a}) = 0 , \tag{F.3.12}$$

due to $\mathfrak{a}^{\dot{\alpha}} = 0$, but

$$\mathscr{Q}_{\gamma\beta A}(\mathfrak{a}) \neq 0 . \tag{F.3.13}$$

Of course, in the antichiral basis, things are just the other way round, there we have

$$\mathscr{Q}_{\gamma\beta A}(\bar{\mathfrak{a}}) = 0 . \tag{F.3.14}$$

On the other hand, due to the relation between $\mathfrak{a}$ and $\bar{\mathfrak{a}}$ and the transformation law of the Chern–Simons form (F.2.17) we have

$$\mathscr{Q}(\mathfrak{a}) - \mathscr{Q}(\bar{\mathfrak{a}}) = d\Delta(\mathscr{W}, \bar{\mathfrak{a}}) , \tag{F.3.15}$$

where now the group element, $\boldsymbol{g}$, is replaced by the pre-potential superfield $\mathscr{W}$. In some more detail, in $\Delta = \chi - \sigma$, we have

$$\chi = \chi(0, \bar{\mathfrak{a}}, \Upsilon) = \mathrm{tr}(\bar{\mathfrak{a}}\Upsilon) , \tag{F.3.16}$$

where

$$\Upsilon = d\mathscr{W}\,\mathscr{W}^{-1} = E^A \Upsilon_A \tag{F.3.17}$$

has zero-field strength

$$d\Upsilon + \Upsilon\Upsilon = 0\ . \tag{F.3.18}$$

The coefficients of the 2-form, χ, are given as

$$\chi_{BA} = \mathrm{tr}(\Upsilon_B\,\bar{\mathfrak{a}}_A - (-)^{ab}\Upsilon_A\,\bar{\mathfrak{a}}_B)\ . \tag{F.3.19}$$

For σ, we define the interpolating pre-potential $\mathscr{W}_t$

$$\Upsilon_t = d\mathscr{W}_t\,\mathscr{W}_t^{-1}\ , \tag{F.3.20}$$

such that

$$\sigma_{BA} = \int_0^1 dt\,\mathrm{tr}(\partial_t \mathscr{W}_t\,\mathscr{W}_t^{-1}(\Upsilon_{t\,B},\Upsilon_{t\,A}))\ . \tag{F.3.21}$$

Consider now

$$\mathscr{Q}_{\gamma\beta A}(\mathfrak{a}) = \mathscr{D}_A \Delta_{\gamma\beta} + \oint_{\gamma\beta} (\mathscr{D}_\gamma \Delta_{\beta A} - (-)^a T_{\gamma A}{}^F \Delta_{F\beta})\ , \tag{F.3.22}$$

following from (F.3.15), and (F.3.14) and perform a redefinition

$$\hat{\mathscr{Q}} = \mathscr{Q}(\mathfrak{a}) - d\Lambda\ , \tag{F.3.23}$$

which leaves $\mathrm{tr}(\mathscr{F}\mathscr{F})$ invariant. We then determine the 2-form Λ in terms of the coefficients of the 2-form Δ such that

$$\hat{\mathscr{Q}}_{\gamma\beta A} = 0 \tag{F.3.24}$$

and maintain, at the same time,

$$\hat{\mathscr{Q}}^{\dot\gamma\dot\beta}{}_A = 0\ . \tag{F.3.25}$$

This is achieved with the identification

$$\Lambda_{\beta A} = \Delta_{\beta A},\quad \Lambda^{\dot\beta}{}_a = -\frac{\mathrm{i}}{2}\mathscr{D}^{\dot\beta}\Delta_a,\quad \Lambda_{\beta\dot\alpha} = 0\ . \tag{F.3.26}$$

For later convenience, we put also

$$\Lambda_{ba} = \frac{\mathrm{i}}{2}(\mathscr{D}_b \Delta_a - \mathscr{D}_a \Delta_b)\ . \tag{F.3.27}$$

Here Δ_a is identified using spinor notation such that

$$\Delta_\gamma{}^{\beta} = -\frac{i}{2} T_\gamma{}^{\beta a} \Delta_a \,. \tag{F.3.28}$$

We have, of course, to perform this redefinition on all the other coefficients, in particular

$$\hat{\mathscr{Q}}_\gamma{}^{\beta}{}_a = \mathscr{Q}_\gamma{}^{\beta}{}_a(\mathfrak{a}) - \mathscr{D}^{\beta} \Xi_{\gamma a} \,. \tag{F.3.29}$$

In the derivation of this equation one uses the anticommutation relation of spinor derivatives and suitable supergravity Bianchi identities together with the definition

$$\Xi_{\gamma a} = \Delta_{\gamma a} + \frac{i}{2} \mathscr{D}_\gamma \Delta_a \,. \tag{F.3.30}$$

We parametrize

$$\hat{\mathscr{Q}}_\gamma{}^{\beta}{}_a = T_\gamma{}^{\beta}{}_a \Omega^{\mathscr{YM}} + T_\gamma{}^{\beta\, b} \hat{\mathscr{Q}}^{\mathscr{YM}}_{[ba]} \,, \tag{F.3.31}$$

where we can now identify the explicit form of the Chern–Simons superfield

$$\Omega^{\mathscr{YM}} = \mathscr{Q}(\mathfrak{a}) - \frac{i}{16} \mathscr{D}^{\dot\alpha} \Xi^{\alpha}{}_{\alpha\dot\alpha} \,. \tag{F.3.32}$$

The first term is obtained from the spinor contraction of

$$\mathscr{Q}_\gamma{}^{\beta}{}_a(\mathfrak{a}) = \mathrm{tr}(\mathfrak{a}_\gamma \mathscr{F}^{\beta}{}_a(\mathfrak{a})) = -i(\bar\sigma_a \varepsilon)^{\dot\beta}{}_{\beta}\, \mathrm{tr}(\mathfrak{a}_\gamma \mathscr{W}^{\beta}(\mathfrak{a})) \,, \tag{F.3.33}$$

i.e.

$$\mathscr{Q}(\mathfrak{a}) = \frac{i}{16} \mathscr{Q}^{\alpha\dot\alpha}{}_{\alpha\dot\alpha}(\mathfrak{a}) = -\frac{1}{4} \mathrm{tr}(\mathfrak{a}^{\alpha} \mathscr{W}_\alpha(\mathfrak{a})) \,. \tag{F.3.34}$$

It remains to read off the explicit form of the second term from the definitions above.

In closing we note that a more symmetrical form of the Chern–Simons superfield may be obtained in exploiting the relation

$$\mathscr{Q}_\gamma{}^{\beta}{}_a(\mathfrak{a}) - \mathscr{Q}_\gamma{}^{\beta}{}_a(\bar{\mathfrak{a}}) = \mathscr{D}_\gamma \Xi^{\beta}{}_a + \mathscr{D}^{\beta} \Xi_{\gamma a} + T_\gamma{}^{\beta b} \left(\Delta_{ba} + \frac{i}{2} (\mathscr{D}_b \Delta_a - \mathscr{D}_a \Delta_b) \right) \tag{F.3.35}$$

with

$$\Xi^{\beta}{}_a = \Delta^{\beta}{}_a + \frac{i}{2} \mathscr{D}^{\beta} \Delta_a \,. \tag{F.3.36}$$

Observe that different appearances of the Chern–Simons superfields should be equivalent modulo linear superfields. To establish the explicit relation of the Chern–Simons superfield presented here and that given in [33] is left as an exercise. So far, we have dealt with the superspace Chern–Simons form alone; when coupled to the linear multiplet the modified field strength is

$$H^{\mathscr{Y}\mathscr{M}} = H + k\,\mathscr{Q}^{\mathscr{Y}\mathscr{M}} \tag{F.3.37}$$

with $H = dB$. In the preceding discussion we have split $\mathscr{Q}^{\mathscr{Y}\mathscr{M}}$

$$\mathscr{Q}^{\mathscr{Y}\mathscr{M}} = \hat{\mathscr{Q}}^{\mathscr{Y}\mathscr{M}} + d\Lambda^{\mathscr{Y}\mathscr{M}}\,, \tag{F.3.38}$$

such that $\hat{\mathscr{Q}}^{\mathscr{Y}\mathscr{M}}$ has the same vanishing components as H. Defining $\mathscr{H}^{\mathscr{Y}\mathscr{M}} = H^{\mathscr{Y}\mathscr{M}} - \hat{\mathscr{Q}}^{\mathscr{Y}\mathscr{M}}$ and $\mathscr{B}^{\mathscr{Y}\mathscr{M}} = B + \Lambda^{\mathscr{Y}\mathscr{M}}$ leads to

$$\mathscr{H}^{\mathscr{Y}\mathscr{M}} = d\mathscr{B}^{\mathscr{Y}\mathscr{M}}\,. \tag{F.3.39}$$

Although $\mathscr{H}^{\mathscr{Y}\mathscr{M}}$ is no longer invariant under Yang–Mills gauge transformations, it has the same constraints as H. Therefore, the solution of the modified linearity conditions can be obtained by the same procedure as employed in the case without Chern–Simons forms.

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