String Theory Limits and Dualities

Voor Heit en Mem.

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Contents

Chapter 1

Introduction

At this moment the physics of elementary particles is well described by the Standard Model. At the smallest scales we can experimentally probe (~ 100 GeV), the Standard Model predicts the outcomes of (scattering) experiments with incredible accuracy. The Standard Model accommodates the (observed) constituents of matter, the quarks and leptons, and the vector particles responsible for the mediation of the strong and electroweak forces. The only ingredient of the Standard Model which still lacks experimental support is the Higgs boson particle, which is thought to be responsible for the breaking of the electroweak gauge symmetry and the masses of the different particles. One can therefore conclude that at this moment there is no direct experimental need to construct and investigate theoretical models that go beyond the Standard Model[1].

On the other hand there are many theoretical reasons to go beyond the Standard Model. The pillars of contemporary theoretical physics are quantum mechanics and general relativity. The Standard Model is a collection of quantum (gauge) field theories which can be considered to be a merger of quantum mechanics and *special* relativity, describing physics at small scales and relativistic energies. General relativity has proven its accuracy on large scales describing very massive objects. When we keep increasing the energy scales and at the same time keep decreasing our length scales, we expect new physics which is not described by either general relativity or the Standard Model. General relativity breaks down at short distances and in the Standard Model or in quantum field theory we should incorporate the effects of (quantum) gravitational interactions. The typical energy

[1] We have to remark here that recent experiments have most probably excluded the possibility that all neutrinos are massless, which requires a modification of the Standard Model. Still, in the context of this thesis we would like to consider this a minor modification.

scale at which this happens is called the Planck mass ($\sim 10^{19}$ GeV). This scale is way beyond the experimental energies currently accessible and one might wonder whether it will ever be possible to attain these energies in a controlled experiment.

This does not mean, however, that this problem is of purely academic interest. Immediately after the big bang ($\sim 10^{-43}$ s), from which our observable universe evolved, the energies were of the order of the Planck scale and the physics during that (short) time determined the further development of the universe. It is therefore of direct interest in cosmology to search for a theory that is able to unify or merge quantum mechanics and general relativity.

Other arguments are of a more theoretical and/or aesthetic nature. The Standard Model contains a huge amount of parameters which have to be determined by experiments, e.g. masses, coupling constants, angles and so on. One would expect (or perhaps one likes to expect) that a fundamental theory of nature will not allow too many adjustable parameters. In the best case scenario, we would like our theory to be unique. Many people therefore like to think of the standard model as an effective theory, only applicable at our current available energy scales [1]. This point of view is backed up by considering the running of the coupling constants of the different gauge theories that are part of the Standard Model. When plotting the coupling constants as a function of the energy scale, one finds that at a particular high energy scale, referred to as the Grand Unified Theory (GUT) energy scale ($\sim 10^{14}$ GeV), the three coupling constants all seem to meet in (approximately) the same point. This suggests a possible unified description at and above the GUT energy scale.

The GUT energy scale lies several orders of magnitude below the Planck scale, so this unified theory would not involve quantum gravity. However, for the three gauge theory coupling constants to meet at the same point, the theoretical concept of supersymmetry improves on the approximate result without supersymmetry. Supersymmetry is a symmetry that connects bosons and fermions: starting with a bosonic particle one can perform a supersymmetry transformation and end up with a fermionic particle. In our world supersymmetry, if it exists, must be a broken symmetry, because we have not detected any supersymmetric partners of the Standard Model particles. Global supersymmetric quantum field theories have slightly different properties than their non–supersymmetric counterparts. Most importantly, in the perturbative expansion cancellations take place between bosons and fermions, generically making supersymmetric quantum theories better behaved. Local supersymmetric theories automatically include supergravity, the supersymmetric version of general relativity. So Grand Unified Theory, supersymmetry and supergravity are intimately linked, which again suggests a possible unified description of gauge field theories and quantum gravity at the Planck scale.

From another point of view the basic fact that general relativity is a classical field theory is unsatisfactory. At scales around the Planck length, quantum gravitational effects are bound to become important and we will need a quantum theory of gravity. However, so far general relativity has resisted all standard methods of quantization and is said to be non–renormalizable (for an overview see [2]). One may think this is just a technical problem, but there are fundamental interpretational problems as well when trying to quantize general relativity because we are trying to quantize space and time. Many theoretical physicists agree that in order to deal with the problem of quantization of space and time a radically new approach is called for. This is also emphasized by the confusing properties of black holes in general relativity, which in a semiclassical approach are not black at all and emit black body Hawking radiation. Studying these objects in general relativity, it turns out that one can formulate black hole laws that are strikingly similar to the laws of thermodynamics. For example, one can assign a temperature and an entropy to a black hole. At this moment one of the key questions in theoretical physics is to try to understand what the fundamental degrees of freedom are that make up the entropy of the black hole and how these thermodynamic degrees of freedom arise from (quantum) general relativity.

At this moment, string theory is the only theoretical construction that can deal with quantum gravity[2] albeit in a perturbative, background dependent way. String theory needs supersymmetry and extra spacetime dimensions to be set up consistently (free of anomalies). In fact, there exist five different superstring theories which are all living in ten spacetime dimensions and which are distinguished by the number of supersymmetries and by the kind of strings (open and/or closed). The construction of these five different anomaly free string theories is referred to as the first string revolution (1984 – 1985). Through the method of compactification one can try to make contact with our observed universe, containing four extended spacetime dimensions. String theory not only contains (quantum) gravity, but gauge theories also appear naturally. All these ingredients, extra spacetime dimensions, supersymmetry, quantum gravity and gauge theories, which are part of any consistent string theory, make them interesting and promising candidates for a unified theory [4].

One of the biggest problems of string theory is the fact that only a perturbative, background dependent formulation exists. This makes it very hard to gain any information about non–perturbative string theory. To make contact with our ob-

[2]As Weinberg said [3]: “String theory is the only game in town”.

servable universe this is an important complication because it turns out that there exist millions of ways to compactify to four dimensions. Perturbative string theory will not predict which compactification is actually going to be preferred and it is expected that non–perturbative information is needed in order to determine the energetically favored compactification (vacuum). Another way of saying this is that string theory, as it is formulated at this moment, is incapable of dynamically generating its own vacuum.

An immediate drawback of string theory as a candidate of a unified theory is the fact that there exist five of them. Ideally, one would like a single unique structure. The developments in string theory over the last ten years seem to indicate that this is in fact true. The reason why this was not recognized before again has its roots in the fact that string theories are only defined perturbatively. Only in the last ten years was it discovered that all five string theories are related by duality transformations. The concept of duality, a general term used to describe a relation between two physical theories or mathematical structures, has become very important in string theory [5]. Through the concept of duality it has now been recognized that all five string theory formulations describe different perturbative corners of a single unique theoretical structure, which has been named M–theory[3] [6]. M–theory is supposed to live in eleven spacetime dimensions and its low energy limit is described by eleven–dimensional supergravity, which is the maximal spacetime dimension for a supergravity theory with Minkowski signature. The theoretical tools used in establishing these duality relations were supersymmetry and the use of D–brane string solitons, which enabled one to study string theory beyond the perturbative regime.

One of the main topics of research following the second superstring revolution (1994 – 1996) was to find a formulation of M–theory. Following up on work done on a regularized quantization of supermembranes, Matrix theory emerged as a possible non–perturbative candidate capturing the dynamics of discrete light–cone quantized M–theory. Although this formulation of M–theory certainly lacks general covariance and is not background independent, it was the first time a non–perturbative description of quantum gravity had been put forward. The surprising thing about Matrix theory is that it is a $0+1$–dimensional quantum mechanics model of N particles. It can be obtained by considering a low energy limit of string theory in the background of N D0–brane solitons. These non–trivial low energy limits of string theory in the background of D–brane solitons were studied further and have led to all kinds of interesting relations between gauge theories

[3] The M stands for anything you like, for example Mother, Membranes or Mystery.

and string models containing gravity. This thesis discusses certain aspects of these string theory limits and the duality relations between gauge theories and closed superstring models containing gravity. One of the main goals of this thesis is to show that *particle* limits of string theory can be setup very generally and lead to interesting connections or correspondences between quantum field theories and gravity. From that point of view string theory can be regarded as a very useful tool to learn more about quantum particle theories.

The organization of this thesis is as follows. In chapter 2 we introduce the basic ingredients of string theory. Chapter 3 discusses supersymmetry algebras and the concept of BPS states. In the same chapter we also introduce the different eleven– and ten–dimensional supergravity models, the BPS soliton solutions of these models and present the concepts of string duality and M–theory. This will provide the necessary background material to move on to chapters 4 and 5. Chapter 4 introduces Matrix theory as a candidate description of M–theory and discusses the construction of (part of) the BPS spectrum in Matrix theory. This BPS spectrum is then compared to (part of) the BPS spectrum of M–theory. In chapter 5 we discuss a general approach to consider non–trivial low energy limits of string theory in the background of p–brane soliton solutions. The resulting duality relations are discussed and some new examples are presented. Finally in the concluding chapter 6 we summarize and discus our results and try to establish a connection between Matrix theory and the dualities obtained in chapter 5.

Chapter 2

String Theory

This chapter discusses the basics of string theory. We will start with a treatment of the bosonic string and give an extended derivation of the closed string spectrum. This will enable us to discuss the open bosonic and superstring spectra rather succinctly. Bosonic string T–duality will then naturally lead to the introduction of D–branes. In the following sections we will then introduce the different consistent closed and open superstring theories and the appearance of D–branes in those theories. For a traditional introduction to string theory we refer to [7]. Another good introduction, with more emphasis on conformal field theory techniques, is given in [8]. For an introduction into the concept of D–branes we refer to the review paper [9] and for a modern introduction into string theory, including D–branes, we refer to [10]. Throughout this thesis we will use units in which $\hbar = c = 1$ and we denote the Planck length and mass in D dimensions by $l_p^{(D)}$ and $m_p^{(D)}$ respectively.

2.1 The bosonic string

In developing the theory of strings we will follow the same route as in a development of point particle theory. We will start writing down an action describing the motion of a classical bosonic string moving in a flat D–dimensional target spacetime and we will use mostly plus signature $\eta^{\mu\nu} = diag(-1,1,\ldots,1)$. This string will sweep out a two-dimensional worldsheet Σ in the target spacetime as opposed to the one-dimensional worldline of a particle, see Figure 2.1. Generalizing the action principle of a particle, we obtain the Nambu-Goto action of a classical string

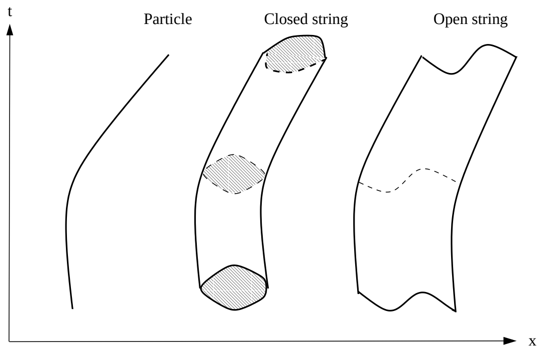

Figure 2.1: Swept out trajectories by particles, closed and open strings respectively.

which is proportional to the (relativistic) area of the worldsheet

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int_\Sigma d^2\sigma\,\sqrt{|\det\left(\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}\right)|}\,. \tag{2.1}$$

In this equation σ^a denote worldsheet coordinates ($a = 0, 1$) and $X^\mu(\sigma^a)$ denote spacetime coordinates ($\mu = 0, 1, \ldots, D-1$) which are functions of the worldsheet coordinates. Partial derivatives with respect to worldsheet coordinates are denoted by ∂_a. The minus sign in front of the action is there to make sure that kinetic energy on the worldsheet has the proper positive sign.

The scale is set by the parameter α' which has dimensions $[length]^2$ and can be interpreted as the square of the size of the string (which is denoted by l_s). Historically this parameter was called the Regge slope parameter, referring to the first introduction of string theory as a candidate for the strong interactions when α' was tuned to explain the linear relation between the squared mass and the angular momentum of observed resonances. Nowadays, as a potential candidate for a consistent description of quantum gravity α' should be of the order of the Planck length. The quantity $T = \frac{1}{2\pi\alpha'}$ is the tension of the string.

The appearance of the square–root in this action (2.1) is problematic. It makes the quantum analysis of this theory rather difficult. However, these problems can be avoided by introducing a non–dynamical auxiliary field $h^{ab}(\sigma^c)$, which should be interpreted as the metric on the worldsheet Σ. This enables us to introduce an equivalent action (at least classically) describing the embedding of a general curved surface with metric h^{ab} into D–dimensional flat spacetime

$$S_P = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\,\sqrt{|h|}\,h^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}\,. \tag{2.2}$$

In this action, usually called the Polyakov action, h is defined to be the determinant of the worldsheet metric h_{ab}. Variation of this action with respect to h^{ab} defines the two-dimensional energy momentum tensor T_{ab} and we obtain

$$T_{ab} \equiv \partial_a X^\mu \partial_b X_\mu - \tfrac{1}{2} h_{ab} h^{cd} \partial_c X^\mu \partial_d X_\mu = 0 \,. \tag{2.3}$$

Multiplication with h^{ab} shows that the trace of the energy momentum tensor $T^a{}_a$ vanishes identically as well. Taking determinants in (2.3) and taking a square–root we obtain

$$\sqrt{|\det(\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu})|} = \tfrac{1}{2}\sqrt{|h|} h^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \,, \tag{2.4}$$

relating the Nambu-Goto (2.1) and the Polyakov action (2.2). The Polyakov action only equals the Nambu-Goto action after using the (classical) equations of motion. The Polyakov action will be our starting point for analyzing the classical and, more importantly, the quantum behavior of the string.

Before we look at solutions to the string equations of motion derived from (2.2) it is worth while discussing the symmetries of the Polyakov action. First of all, by construction the Polyakov action is invariant under the following symmetry transformations.

- Target spacetime Poincaré transformations

  $$X^{\mu\prime} = \Lambda^\mu{}_\nu X^\nu + a^\mu \,, \tag{2.5}$$

  which are induced by choosing the background to be flat Minkowski spacetime. We will discuss general covariant backgrounds later.

- Worldsheet reparametrisations

  $$\sigma^{a\prime} = f^a(\sigma^b) \,, \tag{2.6}$$

  where f^a are arbitrary functions. How we choose to parametrise the string should not have any physical consequences.

Finally there is a worldsheet symmetry which depends crucially on the fact that we are dealing with one-dimensional objects.

- Weyl rescaling

  $$h^{ab\prime} = \Lambda(\sigma^c) h^{ab} \,, \tag{2.7}$$

meaning that the theory on the worldsheet is scale invariant. Only in two dimensions is the combination $\sqrt{|h|}h^{ab}$ invariant under conformal transformations (2.7). This symmetry will have important consequences for the quantum analysis of the theory.

Using the local worldsheet symmetries we can choose the arbitrary worldsheet metric h^{ab} to take on a useful form (compare with fixing a gauge in gauge theory). We can always use worldsheet reparametrisations and scale invariance to make the worldsheet metric equal to $\eta^{ab} = diag(-1,1)$ locally (we will not discuss the global aspects of such a gauge choice). So we can now write the Polyakov action as

$$S_P = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\, \eta^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}\,. \tag{2.8}$$

Notice that we have not used up all worldsheet symmetries. In particular the above action (2.8) is still invariant under special worldsheet coordinate transformations $\sigma^a = f^a(\sigma^c)$ that change the metric with a scale factor

$$\eta^{ab'} = \partial_c f^a\, \partial_d f^b\, \eta^{cd} = \Lambda(\sigma^c)\eta^{ab}\,. \tag{2.9}$$

Coordinate transformations with this property are called conformal transformations (they preserve angles) and the theory on the worldsheet is called a conformal field theory (CFT). The powerful techniques of conformal field theory in two (worldsheet) dimensions can now be used to analyze this string theory [8]. However, in this thesis we will not discuss these techniques and we refer the reader to [11] for a good review on conformal field theory and its applications.

Varying the action (2.8) with respect to X^μ and integrating by parts we obtain

$$\delta S = \frac{1}{2\pi\alpha'}\int_\Sigma d^2\sigma\, \delta X^\mu \partial^2 X_\mu - \frac{1}{2\pi\alpha'}\int_{\partial\Sigma} d\sigma\, \delta X^\mu \partial_n X_\mu\,. \tag{2.10}$$

Demanding the variation of the action to be zero, the first part of (2.10) just gives Laplace's equation for the worldsheet scalars X_μ

$$\partial^2 X_\mu = 0\,. \tag{2.11}$$

The second part of the variation in (2.10) is a boundary term where ∂_n denotes the derivative normal to the boundary. To get rid of these boundary terms we now have to make a distinction between closed and open strings.

- Closed strings can be parametrized in a spatial direction by $\sigma \in [0, 2\pi]$ and in a timelike direction by τ. They are defined by the "boundary condition"

$$X^{\mu}(\tau, 0) \equiv X^{\mu}(\tau, 2\pi) . \tag{2.12}$$

  Closed strings have no spacelike boundary and therefore the boundary term in (2.10) vanishes[1].

- Open strings do have a boundary and therefore we have to get rid of the boundary term in (2.10) by imposing boundary conditions on the open string endpoints. Introducing a spatial coordinate $\sigma \in [0, \pi]$ this can be done in two distinct ways.

  – Neumann boundary conditions

$$\partial_n X^{\mu}|_{\sigma=0,\pi} = 0 . \tag{2.13}$$

    This boundary condition is Poincaré invariant and therefore was considered to be the only physically acceptable one. It just means that there is no momentum flow off the string.

  – Dirichlet boundary conditions

$$\delta X^{\nu}|_{\sigma=0,\pi} = 0 \Rightarrow X^{\nu}|_{\sigma=0,\pi} = constant . \tag{2.14}$$

    This boundary condition was considered pathological in the past because of lack of Poincaré invariance. It made an impressive comeback the last five years when it was discovered that these open strings describe solitonic state vacua (D–branes) of closed string theories. We will discuss the arguments for that in the section on T–duality and D–branes. Physically the Dirichlet boundary condition (2.14) tells us that the endpoints of the open string are fixed on some hyperplane. If all spacetime coordinates X^{ν} are fixed, including the timelike coordinate,then the string endpoints are stuck to a spacetime event, and we speak of a D–instanton. We are free to impose Dirichlet boundary conditions on some coordinates (say n) and Neumann on others $(D-n)$ to obtain a D–brane with $p = (D-n-1)$ spatial dimensions.

[1]Remember that the variation δX^{μ} vanishes at the timelike boundaries by definition.

Of course we still have the equations of motion for the metric (2.3), which after gauge fixing become the constraints

$$T_{10} = T_{01} = \tfrac{1}{2}\dot{X}^\mu X'_\mu = 0 \quad , \quad T_{00} = T_{11} = \frac{1}{4}\left(\dot{X}^2 + X'^2\right) = 0\,, \tag{2.15}$$

which can also be written as

$$\left(\dot{X} \pm X'\right)^2 = 0\,, \tag{2.16}$$

where we left out the spacetime indices. The dot represents a partial derivative with respect to a timelike worldsheet coordinate τ and a prime represents a partial derivative with respect to the spacelike worldsheet coordinate σ normal to τ. These equations (2.15) go under the name of Virasoro constraints.

2.1.1 Closed string spectrum

We choose worldsheet coordinates τ and σ, where $\sigma \in [0, 2\pi]$ denotes the spatial direction of the string. The most general solution of the Laplace equation (2.11) that also satisfies the periodicity condition for the closed string (2.12) can be split into a left– (+) and right–moving (−) part

$$X^\mu(\tau,\sigma) = X^\mu_+(\tau+\sigma) + X^\mu_-(\tau-\sigma)\,, \tag{2.17}$$

where

$$\begin{aligned} X^\mu_-(\tau-\sigma) &= \tfrac{1}{2}x^\mu + \tfrac{1}{2}\sqrt{\alpha'}\,\alpha^\mu_0\,(\tau-\sigma) + \tfrac{1}{2}\sqrt{\alpha'}\,i\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in(\tau-\sigma)}\,, \\ X^\mu_+(\tau+\sigma) &= \tfrac{1}{2}x^\mu + \tfrac{1}{2}\sqrt{\alpha'}\,\tilde{\alpha}^\mu_0(\tau+\sigma) + \tfrac{1}{2}\sqrt{\alpha'}\,i\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-in(\tau+\sigma)}\,. \end{aligned} \tag{2.18}$$

The α^μ_n and $\tilde{\alpha}^\mu_n$ for $n \neq 0$ are arbitrary dimensionless Fourier modes, not to be confused with the fundamental scale parameter α', and n runs over the integers. From the periodicity condition (2.12) we conclude that

$$\tilde{\alpha}^\mu_0 - \alpha^\mu_0 = 0\,. \tag{2.19}$$

This enables us to define

$$\alpha^\mu_0 = \tilde{\alpha}^\mu_0 \equiv \sqrt{\alpha'}\,p^\mu\,, \tag{2.20}$$

for reasons that will become clear when we calculate the center of mass momentum. Because $X^\mu(\tau,\sigma)$ has to be real we conclude that x^μ and p^μ, which are constants, have to be real and find the following reality conditions for the Fourier modes

$$(\alpha_n^\mu)^* = \alpha_{-n}^\mu \quad \text{and} \quad (\tilde{\alpha}_n^\mu)^* = \tilde{\alpha}_{-n}^\mu\,, \tag{2.21}$$

where the * denotes complex conjugation. This allows us to write

$$\begin{aligned} \partial_- X_-^\mu &= \tfrac{1}{2}\sqrt{\alpha'}\sum_n \alpha_n^\mu e^{-in(\tau-\sigma)}\,, \\ \partial_+ X_+^\mu &= \tfrac{1}{2}\sqrt{\alpha'}\sum_n \tilde{\alpha}_n^\mu e^{-in(\tau+\sigma)}\,, \end{aligned} \tag{2.22}$$

where the $-$ and $+$ denote differentiation with respect to $\tau-\sigma$ and $\tau+\sigma$ respectively and $n=0$ is now included in the sum. Calculating the center of mass position of the closed string we obtain

$$x_{CM}^\mu(\tau) = \frac{1}{2\pi}\int_0^{2\pi} d\sigma\, X^\mu(\tau,\sigma) = x^\mu + \alpha'\, p^\mu\, \tau\,, \tag{2.23}$$

because all the oscillator terms integrate to zero. So we see that the constant x^μ represents the center of motion position of the string at $\tau = 0$. Calculating the center of mass momentum we find

$$\begin{aligned} p_{CM}^\mu &= T\int_0^{2\pi} d\sigma\, \dot{X}^\mu(\tau,\sigma) = T\int_0^{2\pi} d\sigma\, (\partial_- X_-^\mu + \partial_+ X_+^\mu) \\ &= T\int_0^{2\pi} d\sigma\, \tfrac{1}{2}\sqrt{\alpha'}(\alpha_0^\mu + \tilde{\alpha}_0^\mu) = p^\mu\,, \end{aligned} \tag{2.24}$$

where again all the oscillator contributions integrate to zero and we used definitions (2.20). We also used that the tension T is defined as $\frac{1}{2\pi\alpha'}$, canceling all π's and α''s in the above expression.

We conclude that the variables describing the classical motion of the string are the center of mass position x^μ and momentum p^μ plus an infinite collection of variables α_n^μ and $\tilde{\alpha}_n^\mu$. This just reflects the fact that the string can move as a whole but it can also be in an infinite number of internal vibration modes, represented by the oscillator degrees of freedom.

We are now ready to quantize the closed bosonic string. As in particle theory we can proceed in different ways. We could first solve the classical constraints (2.15) and then quantize canonically (imposing commutation relations), losing manifest Lorentz covariance. The solution of the constraints is achieved most easily in the so–called light–cone gauge, so this way of quantizing is usually referred to as light–cone quantization (LCQ). Another way would be to quantize covariantly and impose the Virasoro constraints as conditions on the states in Hilbert space, called covariant canonical quantization (CCQ). A third possibility is to use path integral quantization (PIQ) which has manifest Lorentz covariance, but works in a bigger Hilbert space which contains ghost fields. We will first proceed using CCQ, to introduce the Virasoro operators and the constraints on the physical states. A quick and physically insightful way to deduce the physical spectrum of the string will however make use of LCQ. For a discussion on path integral quantization we refer to [7, 8, 10].

As is usual in canonical quantization we will replace all fields by operators and Poisson brackets by (equal time) commutators. The Virasoro constraints (2.15) are then operator constraints which have to annihilate physical states. We impose the usual commutation relation between position $X^\mu(\tau,\sigma)$ and momentum $P^\mu(\tau,\sigma) \equiv T\dot{X}^\mu$

$$[X^\mu(\tau,\sigma), P^\nu(\tau,\sigma')] = i\delta(\sigma-\sigma')\eta^{\mu\nu}\,. \tag{2.25}$$

Using the expressions (2.17) and (2.18) this commutation relation can be translated into commutation relations involving the center of mass position and momentum and commutation relations involving the Fourier modes.

$$\begin{aligned} [x^\mu, p^\nu] &= i\eta^{\mu\nu}\,, \\ [\alpha_m^\mu, \alpha_n^\nu] &= m\,\delta_{m+n,0}\,\eta^{\mu\nu}\,, \\ [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] &= m\,\delta_{m+n,0}\,\eta^{\mu\nu}\,. \end{aligned} \tag{2.26}$$

All the other commutators vanish. Looking at these commutators we conclude that the first is just the usual one for a propagating particle, while the others are commutators familiar from the creation and annihilation operators in the harmonic oscillator for an infinite set of oscillators (except for a factor of m). We note that the reality condition (2.21) just becomes a hermiticity condition on the oscillators.

Next we have to define a Hilbert space on which these operators act. Because we just concluded that our string is nothing but an infinite set of oscillators this is not very difficult. From (2.26) we conclude that the negative frequency modes α_m^μ, $m > 0$ are lowering operators and the positive frequency modes $m < 0$ are raising operators. The ground state should now be defined as the state annihilated by all lowering operators. The complete definition of a state also involves the center of mass operators x^μ and p^μ. If we choose to diagonalise the momentum operator, every state is also characterized by momentum k^ν. The (infinitely degenerate) ground state $|0,k^\nu\rangle$ can then be defined as

$$\begin{aligned} \alpha_m^\mu |0,k^\nu\rangle &= 0 \quad \forall m > 0\,, \\ p^\mu |0,k^\nu\rangle &= k^\mu |0,k^\nu\rangle\,. \end{aligned} \tag{2.27}$$

Excited states can now be constructed by acting on the ground state with the negative frequency modes. Because of the Minkowski signature some of these states have negative (or zero) norm which could cause problems. In fact not all these states are physical because of the Virasoro constraints (2.15) and therefore we should now take a closer look at these constraints.

In worldsheet light–cone coordinates the components of the worldsheet stress energy tensor (2.3) are just

$$T_{++} = \tfrac{1}{2}\partial_+ X^\mu \partial_+ X_\mu \quad , \quad T_{--} = \tfrac{1}{2}\partial_- X^\mu \partial_- X_\mu \quad , \quad T_{+-} = T_{-+} = 0\,. \tag{2.28}$$

Defining the Virasoro operators as the Fourier modes of the worldsheet stress energy tensor we obtain

$$L_m \equiv \frac{1}{\pi\alpha'} \int_0^{2\pi} d\sigma\, T_{--} e^{im(\tau-\sigma)} \quad , \quad \tilde{L}_m \equiv \frac{1}{\pi\alpha'} \int_0^{2\pi} d\sigma\, T_{++} e^{im(\tau+\sigma)}\,, \tag{2.29}$$

and using (2.17) through (2.22) to express the Virasoro operators in terms of the oscillators we obtain

$$L_m = \tfrac{1}{2}\sum_n \alpha^\mu_{m-n}\alpha_{n\mu} \quad , \quad \tilde{L}_m = \tfrac{1}{2}\sum_n \tilde{\alpha}^\mu_{m-n}\tilde{\alpha}_{n\mu}\,. \tag{2.30}$$

In the quantum theory we should normal order this expression. This means putting positive frequency modes to the right of the negative frequency modes. As the reader can check, only L_0 and $\tilde{L}_0$ are sensitive to normal ordering and can be written as

$$L_0 = \tfrac{1}{2}{\alpha_0}^2 + \sum_{n=1}^{\infty} :\alpha^\mu_{-n}\alpha_{n\mu}: \quad , \quad \tilde{L}_0 = \tfrac{1}{2}{\tilde{\alpha}_0}^2 + \sum_{n=1}^{\infty} :\tilde{\alpha}^\mu_{-n}\tilde{\alpha}_{n\mu}:\,, \tag{2.31}$$

where the : denote normal ordering. Because the commutator of two oscillators is a constant and we do not know in advance what this constant should be, we include normal ordering constants a and $\tilde{a}$ and replace the L_0 and $\tilde{L}_0$ constraints by $(L_0 - a)$ and $(\tilde{L}_0 - \tilde{a})$. We can calculate the Virasoro algebra and find

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}\,, \tag{2.32}$$

where c is the central charge, which is equal to the target spacetime dimension d (or the number of scalars X on the worldsheet). A similar expression holds for $[\tilde{L}_m, \tilde{L}_n]$.

Using the Virasoro algebra it can be deduced that it is only consistent to impose "weakly" vanishing constraints,

$$\begin{aligned} L_m\,|phys\rangle = 0 \qquad &, \qquad \tilde{L}_m\,|phys\rangle = 0 \quad \forall\, m > 0\,, \\ (L_0 - a)\,|phys\rangle = 0 \qquad &, \qquad (\tilde{L}_0 - \tilde{a})\,|phys\rangle = 0\,. \end{aligned} \tag{2.33}$$

These constraints ensure that all expectation values of the Virasoro operators vanish

$$\langle phys'|L_n|phys\rangle = 0 \quad , \quad \langle phys'|\tilde{L}_n|phys\rangle = 0\,, \tag{2.34}$$

explaining why they are called "weakly" vanishing constraints.

Analyzing the L_0 and $\tilde{L}_0$ condition in (2.33), using equations (2.31) and (2.20) and the fact that $M^2 = -p^2$, we can conclude $a = \tilde{a}$ by working on the ground state. For both constraints $L_0 - a$ and $\tilde{L}_0 - a$ to hold for all closed string physical states we must also conclude that we need a level matching condition saying

$$N \equiv \sum_{n=1}^{\infty} \alpha^{\mu}_{-n}\alpha_{n\mu} = \tilde{N} \equiv \sum_{n=1}^{\infty} \tilde{\alpha}^{\mu}_{-n}\tilde{\alpha}_{n\mu}\,, \tag{2.35}$$

where we have defined level number operators $N = \tilde{N}$. This enables us to write down the mass–shell condition for the closed string states

$$M^2 = \frac{4}{\alpha'}(N - a)\,. \tag{2.36}$$

So the physical states in our theory are the states created from the ground state (2.27) using creation operators which also satisfy (2.33). What we need to show is that the spectrum obtained in this way only contains positive norm states. To show this is a very non–trivial task and we will just mention the result. The famous "no ghost" theorem states that if and only if the target spacetime dimension D is 26, will the physical spectrum be free of negative norm states (and the null states decouple).

So far we have been discussing a fully Lorentz covariant method for quantizing the string. In that setup it is not very obvious what the physical states are. To get a better look at the physical spectrum it is insightful to discuss light–cone quantization. The solution to the constraints before quantizing is achieved the easiest when we break Lorentz–covariance by choosing target spacetime light–cone coordinates X^+ and X^-

$$X^+ = \frac{1}{2}\sqrt{2}(X^0 + X^1) \quad , \quad X^- = \frac{1}{2}\sqrt{2}(X^0 - X^1)\,. \tag{2.37}$$

The result of this choice will be first of all that we can use the left–over worldsheet conformal symmetries (2.9) to gauge away all oscillators in X^+. Secondly, when we solve the constraints (2.15), the oscillators in X^- can be expressed using the oscillators α_n^i and $\tilde{\alpha}_n^i$ of the transverse coordinates X^i. This means the only physical internal excitations are the transverse α_n^i and $\tilde{\alpha}_n^i$. There is a physical way of understanding this. The string is described by a 2–dimensional surface embedded in D–dimensional target spacetime. Excitations (vibrations) in the internal directions (τ,σ) can be transformed away by coordinate transformations on the worldsheet. Therefore there should only exist $D-2$ physical (oscillator) degrees of freedom on the string. To construct the physical spectrum we only use the transverse oscillators α_n^i and $\tilde{\alpha}_n^i$ with $n<0$, keeping in mind the level matching condition (2.35). After that we only have to insert the string mass shell condition (2.36) involving L_0 and $\tilde{L}_0$, all other constraints are already solved for by using only the transversal oscillators. In principle we should check Lorentz covariance after performing the light–cone quantization. We will just quote the result that only when $D=26$ we keep Lorentz covariance, which is consistent with the result mentioned in the previous paragraph on the covariant quantization approach. We will see that the investigation of the spectrum, together with enforced Lorentz covariance, also fixes the normal ordering constant to be equal to $a=1$.

The ground state for the closed string is $|0,k^\mu\rangle$ for which we have the mass–shell condition $M^2 = \frac{-4a}{\alpha'}$. The first excited level is constructed through

$$\alpha_{-1}^i \tilde{\alpha}_{-1}^j |0,k^\mu\rangle\,, \tag{2.38}$$

for which we find the mass–shell condition

$$M^2 = \frac{4}{\alpha'}(1-a)\,. \tag{2.39}$$

The state (2.38) can be decomposed into irreducible representations of the transverse rotation group $SO(24)$. These are the symmetric traceless part, the antisymmetric and the trace part of (2.38), which can be interpreted as a small excitation of the metric tensor G_{ij}, an antisymmetric tensor field B_{ij} and a scalar Φ respectively. Lorentz invariance requires physical states to be representations of the little group of the Lorentz group $SO(25,1)$. For massless states this is $SO(24)$ and for massive states this is $SO(25)$. We just saw that the first excited level states transform in representations of $SO(24)$ and therefore they should be massless. This fixes the normal ordering constant to be equal to $a = 1$ (2.39). This also means that the ground state has negative mass and is therefore tachyonic. This is not a good sign, it signals an instability of the theory. It turns out that higher level excitations can be uniquely combined into representations of $SO(25)$, which is consistent with Lorentz invariance for massive states. The higher level excitations have masses proportional to $M^2 \propto \frac{4}{\alpha'}$. Assuming $\sqrt{\alpha'} \approx l_p^{(26)}$ shows that these masses are very large and can be neglected when taking a low–energy limit.

We want to end this section by summarizing the most important conclusions from the analysis of the bosonic string. It is important to recognize that the analysis of the string becomes non–trivial because of the constraints. Enforcing target spacetime Lorentz invariance fixes the target spacetime dimension $D = 26$ and determines the normal ordering constant $a = 1$ when using the light–cone gauge. When we couple the string to more general background fields we will learn that the constraint on the spacetime dimension is directly related, before we fix any gauge, to the conformal anomaly. The first excited states are massless and describe small excitations of an antisymmetric tensor field $B^{\mu\nu}$, a scalar Φ usually called the dilaton and, very interestingly, the gravitational field $G^{\mu\nu}$. This last observation raised hopes that string theory could describe quantum gravity. However the ground state is tachyonic which usually signals an instability of the vacuum. When discussing superstrings we will see a consistent way to get rid of the tachyon.

2.1.2 Open string spectrum

In this section we will only discuss the open string with Neumann boundary conditions in all target spacetime directions. The open string spectrum with Dirichlet boundary conditions in some directions will be discussed in the section on D–branes. Most of the previous discussion also applies for the open bosonic string with Neumann boundary conditions. There is basically just one difference. Because of the Neumann boundary conditions the right– and left–moving Fourier modes α_n^μ and $\tilde{\alpha}_n^\mu$ are no longer independent, i.e. only standing wave solutions are allowed. Solving the boundary conditions (2.13) we find the classical solution

$$X^\mu(\tau,\sigma) = x^\mu + 2\alpha' p^\mu \tau + \sqrt{2\alpha'}\, i \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in\tau} \cos n\sigma \,, \tag{2.40}$$

where σ now runs from 0 to π. Just as in the closed string the constants x^μ and p^μ are constants describing the center of mass position and momentum of the open string. In principle we can now just copy the results found for the closed bosonic string. Demanding Lorentz invariance has the same consequences, $D = 26$ and $a = 1$. Going through the same steps as for the closed string we obtain the mass–shell condition for open string states

$$M^2 = \frac{1}{\alpha'}(N-1)\,. \tag{2.41}$$

Where we have defined a level number operator for the open bosonic string

$$N = \sum_{n=1}^{\infty} \alpha_{-n}^{\mu}\alpha_{n\mu}\,. \tag{2.42}$$

So again we find a tachyonic ground state. The first excited level in the light–cone gauge can now be constructed as

$$\alpha_{-1}^{i}|0,k^\mu\rangle\,, \tag{2.43}$$

for which we again find that it is massless. We also immediately conclude that this state should be an abelian foton because (2.43) is the vector representation of $SO(24)$. For higher level n excitations there always exists a state described by a symmetric $SO(25)$ massive tensor of rank n. This also implies that the maximal spin at level n can be expressed in terms of the mass

$$J^{max} = \alpha' M^2 + 1\,. \tag{2.44}$$

Let me briefly mention a method to introduce non-abelian massless foton states in the open string theory. For an oriented open string both endpoints are distinct and we could associate labels i and $\bar{j}$ to them, transforming in the fundamental (N) and anti–fundamental ($\bar{N}$) representation of $U(N)$[2]. These labels are usually called Chan–Paton factors. Because these labels are not dynamical, the only thing that changes in the spectrum is that the states are promoted to $N \times N$ matrices, transforming in the N^2 dimensional adjoint representation of $U(N)$, as is appropriate for a non-abelian foton. We will discuss this setup in somewhat more detail when we discuss D–branes.

[2]This set up is based on the original motivation for string theory as a description of the strong interactions. The open string resembles a gluon flux tube, with a quark and anti–quark at the endpoints, on which for example the color group $SU(3)$ acts.

So we conclude that for the open string we also find a tachyonic ground state. Interestingly we find a massless (abelian) foton at the first excited level. Indeed when we discuss a low–energy limit, neglecting all massive excitations, at lowest order we will just find Maxwell theory. As for the closed bosonic string this open string theory lives in 26 dimensions. In the supersymmetric extension of the open string, as in the closed string, we will be able to get rid of the unwanted tachyon state.

2.1.3 String interactions

What we have done so far is to construct the physical quantum spectrum of the closed and open free bosonic string. We have not said anything on how different string states interact with each other. In order to construct an interacting string theory we would like to construct a string field theory, or what is usually called a second quantized theory. However up to the present nobody has been able to construct a fully satisfactory string field theory. It is known how to set up a string perturbation expansion in a small string coupling constant g_s making use of a stringy generalization of Feynman diagrams[3].

To lowest order we should consider string tree diagrams. Consider a string moving through target spacetime emitting another string. Again we could now parametrise the worldsheet, find appropriate boundary conditions and quantize the theory we obtain. Because of conformal invariance however there exists a much easier method to describe these interacting strings, involving so–called vertex operators. Using conformal transformations the string tree diagram can be transformed into a worldsheet (for the closed string this will be a sphere, for the open string this is a disc) with points inserted representing the outgoing string states. At these points operators should be inserted to describe the emission of a string state, these will be the vertex operators. For closed strings the perturbative expansion and their conformally transformed surfaces with inserted vertex operators are shown in Figure 2.2. We will proceed by giving some insight in how to construct these vertex operators.

To describe the basic elements of this construction we will just transform the tree diagram worldsheet to one with one vertex operator inserted representing a string emitting another string. In quantum mechanics we would calculate a

[3]In string theory we are in a situation where, if we compare with particle theory, we would only know the bosonic free particle propagator, and not be aware of the Klein–Gordon equation and its interacting extensions. Using the free particle propagator we could still setup the Feynman expansion to introduce perturbative interactions.

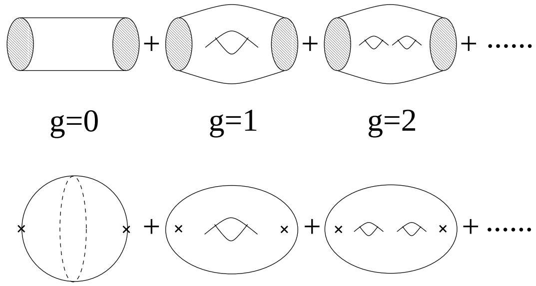

Figure 2.2: A closed string perturbative expansion before and after using conformal transformations.

probability amplitude by taking the inner product of two states as

$$A = \langle \Psi_{in} | \Psi_{out} \rangle \,. \tag{2.45}$$

We are going to do the same thing for the string theory tree diagram. Let us denote the incoming string state as $|B\rangle$ and the outgoing string state as $|B'\rangle$. To deal with the emitted state $|C\rangle$ we introduce the vertex operator $V_C(\tau,\sigma)$ which turns the string state $|B'\rangle$ after emission into the string state before emission. So then the string probability amplitude we have to calculate will be

$$A = \langle B | V_C(\tau,\sigma) | B' \rangle \,. \tag{2.46}$$

One effect of a vertex operator should always be that it decreases the momentum of the state before emission. Because

$$[p^\mu, e^{ik_\nu X^\nu(\tau,\sigma)}] = k^\mu e^{ik_\nu X^\nu(\tau,\sigma)} \,, \tag{2.47}$$

where we used (2.26), it is clear that the normal ordered operator $e^{-ik_\nu X^\nu(\tau,\sigma)}$ working on a string state will decrease the momentum by k^μ. We conclude that any vertex operator should contain such a term. The tachyon vertex operator is in fact completely determined by it. An (excited) string state is further determined by its transformation properties under the Lorentz group, and any vertex operator should contain that information. For example the foton state of an open string is described by the normal ordered vertex operator

$$V_{foton} =: \eta^\mu \dot{X}_\mu e^{-ik_\nu X^\nu} :, \tag{2.48}$$

where η^μ denotes the polarization of the foton. The conclusion will be that we can construct these vertex operators for every string state.

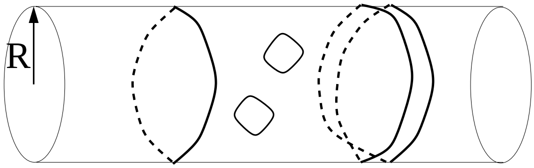

Figure 2.3: Closed strings with different winding numbers.

Inserting these vertex operators on two–dimensional Riemannian manifolds we can calculate scattering amplitudes to any order, where the order is given by the number of holes, the genus, of the two–dimensional Riemannian manifold. Note that in Figure 2.2, as opposed to perturbative field theory, there is only one diagram per given order. We also want to emphasize that the manifolds are perfectly smooth. Interactions do not involve singular points in the perturbative string expansion. In that sense string theory spreads out the interactions and this is one intuitive way of understanding why string perturbation theory is much better behaved than perturbative field theory.

Let us end this section by noting that the perturbative expansion of closed string theory is consistent by itself. Open string perturbative expansions however are not consistent on their own. This is because two open strings can join to form a closed string, and therefore open strings automatically should include closed string states when considering interactions. This will be important when we discuss T-duality and its consequences for the open string.

2.1.4 T-duality and the appearance of D–branes

So far we assumed that the target spacetime was infinitely extended flat Minkowski space $\mathbb{R}^{25,1}$. We would now like to discuss bosonic string theory in a flat background, but with one of the target spacetime coordinates periodically identified with radius R. So we assume spacetime to be the direct product $\mathbb{R}^{25} \times S^1(R)$. Let us choose X^{25} to have a period $2\pi R$. This can have two effects. First of all we know from quantum mechanics that the momentum p^{25} of the string can now only take on the values n/R with n an integer. Secondly the string can wind around the compactified dimension X^{25} an integer w number of times, as shown in Figure 2.3. This means that we have to give up condition (2.12) in the direction X^{25}. Instead we have

$$X^{25}(\tau,0) \equiv X^{25}(\tau,2\pi) + 2\pi w R\,. \tag{2.49}$$

This boundary condition tells us that in the special direction X^{25}, using the expansion (2.18), we no longer find (2.19) and (2.20) but instead

$$\begin{aligned} \tilde{\alpha}_0^{25} - \alpha_0^{25} &\equiv \frac{2}{\sqrt{\alpha'}} wR\,, \\ \alpha_0^{25} + \tilde{\alpha}_0^{25} &\equiv 2\sqrt{\alpha'}\, p^{25} = 2\sqrt{\alpha'}\,\frac{n}{R}\,. \end{aligned} \tag{2.50}$$

From these equations we deduce the following expressions for α_0^{25} and $\tilde{\alpha}_0^{25}$ separately

$$\begin{aligned} \alpha_0^{25} &= \left(\frac{n}{R} - \frac{wR}{\alpha'}\right)\sqrt{\alpha'}\,, \\ \tilde{\alpha}_0^{25} &= \left(\frac{n}{R} + \frac{wR}{\alpha'}\right)\sqrt{\alpha'}\,. \end{aligned} \tag{2.51}$$

Using the constraint Virasoro operators L_0 and $\tilde{L}_0$ and defining a 25–dimensional mass operator $M^2 = -p^\mu p_\mu$, with the index $\mu \in [0,24]$, we find the following mass–shell constraint for closed strings in this target spacetime

$$\begin{aligned} M^2 &= \frac{4}{\alpha'}\left[\frac{1}{4}(\alpha_0^{25})^2 + (N-1)\right] \\ &= \frac{4}{\alpha'}\left[\frac{1}{4}(\tilde{\alpha}_0^{25})^2 + (\tilde{N}-1)\right]. \end{aligned} \tag{2.52}$$

We note that the level number operators are defined as usual (2.35) with the spacetime indices running over the complete spacetime $[0,25]$. Instead of the level–matching condition we found for the 26–dimensional closed string (2.35) we now deduce the following condition from (2.52) and (2.51)

$$nw + (\tilde{N} - N) \equiv 0\,. \tag{2.53}$$

This reduces to the level–matching condition when $n = 0$ or $w = 0$.

Decomposing the massless states in (2.52) with $n = w = 0$ into 25–dimensional Lorentz group representations we find the same spectrum as in the 26–dimensional case except for two extra massless $U(1)_R \times U(1)_L$ foton states and a scalar, where one of the creation operators has its index in the compact direction. These are the usual Kaluza–Klein modes coming from the decomposition of the 26–dimensional metric and antisymmetric tensor. As opposed to ordinary field theory where we only have a tower of momentum modes, we find a tower of momentum modes and a tower of winding modes. In field theory we are used to the fact that when we take the limit $R \to 0$, the momentum modes become infinitely massive and therefore decouple, so the compactified dimension effectively disappears. In closed string theory however we see that in this limit all winding modes become massless, meaning the compactified dimension does not disappear but in some sense reappears.

In fact the mass spectra of the string theories at radius R and radius $\frac{\alpha'}{R}$ are identical when at the same time the winding and the Kaluza–Klein modes are interchanged $n \leftrightarrow w$. This transformation takes

$$\alpha_0^{25} \to -\alpha_0^{25} \quad , \quad \tilde{\alpha}_0^{25} \to \tilde{\alpha}_0^{25} \tag{2.54}$$

and can be generalized using (2.17) to the one–sided parity transformation working only on the right–moving $(-)$ degrees of freedom

$$X_-^{25} \to -X_-^{25} \quad , \quad X_+^{25} \to X_+^{25} \,, \tag{2.55}$$

or it can also be understood as an interchange of the worldsheet coordinates

$$\tau \leftrightarrow \sigma \,. \tag{2.56}$$

It can be proven that both closed string theories obtained in this way give us exactly the same results. This is called T–duality. The statement is that closed string theory compactified on a circle with radius R can not be distinguished from closed string theory compactified on a radius $\frac{\alpha'}{R}$. This has the important consequence that there exists a minimal length $R_{min} = \sqrt{\alpha'}$. Closed string theories with radii smaller than R_{min} can be mapped, using T–duality, to closed string theories with radii bigger than R_{min}. Exactly at this minimal length we get symmetry enhancement. Taking $|n| = |w| = 1$ we find 4 extra massless foton states because of (2.53). These states, together with the two Kaluza–Klein foton states belong to an adjoint representation of $SU(2)_L \times SU(2)_R$. This is another way of obtaining non-abelian fotons from string theory. T–duality and symmetry enhancement are considered important signs that strings see spacetime geometry very differently from the way we are used to.

Now we want to discuss open strings (with Neumann boundary conditions) and T–duality. Open strings can not wind around the compactified dimension and we only find a tower of Kaluza–Klein momentum modes. Therefore a T–duality transformation, if it exists, can not map the open string theory to the same open string theory with radius $\frac{\alpha'}{R}$. The fact that interacting open strings can only be defined by introducing closed strings as well, is convincing evidence that there should exist a T–dual open string theory. Using (2.55) or equivalently (2.56) it is in fact easy to see what happens in this case. Performing the one–sided parity transformation, i.e. $\tilde{X} = X_+ - X_-$, and introducing an extra constant c, which drops out if we sum left– and right–moving contributions, the open string expansion (2.40) in the T–dual compact coordinate becomes

$$\tilde{X}^{25}(\tau,\sigma) \equiv X_+^{25} - X_-^{25} = c + 2\alpha' p^{25}\,\sigma + \sqrt{2\alpha'}\, i \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in\tau} \sin n\sigma\,, \qquad (2.57)$$

Comparing with (2.40) we see that after the T–duality transformation the open string does not carry any center of mass momentum in the compactified direction and in the oscillator part cosines have been changed into sines. This has the important consequence that the string endpoints $\sigma = 0$ and $\sigma = \pi$ are fixed. This is in accordance with the fact that we could have also interpreted the T–duality transformation as an interchange of σ and τ (2.56) changing the Neumann boundary condition into a Dirichlet boundary condition

$$\partial_\sigma X^{25} = 0 \longrightarrow \partial_\tau \tilde{X}^{25} = 0\,. \qquad (2.58)$$

In fact all endpoints are now constrained to live on a 25–dimensional hyperplane, called a D–brane. To see this we integrate the dual coordinate over the string worldsheet coordinate σ and find

$$\tilde{X}^{25}(\pi) - \tilde{X}^{25}(0) = \int_0^\pi d\sigma\, \partial_\sigma \tilde{X}^{25} = 2\pi\alpha' p^{25} = 2\pi n \tilde{R}\,, \qquad (2.59)$$

where $\tilde{R}$ is the T–dual radius $\tilde{R} = \frac{\alpha'}{R}$. So the endpoints are fixed on the 25–dimensional D24–brane and the open strings wrap around the compact direction n times. These strings are the analogues of the closed string winding modes. In that sense T–duality again interchanges momentum and winding modes. The statement of T–duality now is that open string theory on $\mathbb{R}^{25} \times S^1(R)$ can not be distinguished from a $\mathbb{R}^{25}$ D–brane positioned on the T–dual circle $S^1(\tilde{R})$. Momentum modes of the Neumann open string map to D–brane open string winding modes, as shown in Figure 2.4.

The arguments in this section can be generalized to include more compact directions. In the case of open strings this just changes the dimensionality of the T–dual D–brane. If $X^c \in [X^{25}, X^{24}, \ldots, X^{p+1}]$ are all compact coordinates then the

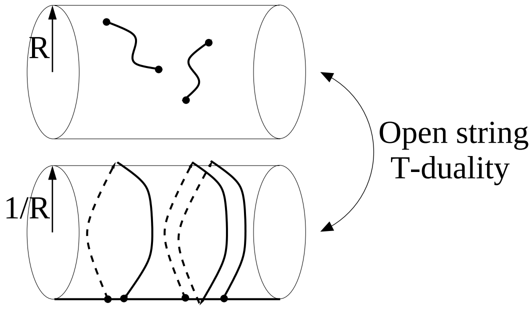

Figure 2.4: The open string T–duality transformation mapping open string momentum modes to D–brane winding modes.

T–dual theory will be described by a $(p+1)$–dimensional Dp–brane theory. In this light the conventional open string theory we started with can be considered as living on a spacetime filling D–brane (a D25-brane). An important generalization involves adding $U(N)$ Chan–Paton factors to the open string theory. Along the compact direction we could introduce vacuum expectation values for the gauge fields (usually called Wilson lines) that will break the gauge group to $U(1)^N$. We will just mention the result that the T–dual theory will involve N Dp–branes positioned in the $D-p-1$ compact direction(s). These positions (the constant c in (2.57)) can be varied and whenever any D–branes coincide we expect symmetry enhancement, e.g. $U(1)\times U(1)\to U(1)\times SU(2)$. We finally would like to point out that we can define D–branes in a decompactified spacetime by taking the limit $R\to 0$ which takes $\tilde{R}\to\infty$, formally decompactifying the T–dual target spacetime.

2.1.5 D–branes

D–branes should be understood as dynamical solitonic string objects which we can describe by open strings ending on them. They are dynamical because two D–branes can interact by exchanging closed strings, which are not confined to the D–brane worldvolume. That also means the D–brane has mass (and charge in superstring theory). From that point of view we can see the D–brane as a symmetry breaking vacuum of closed string theory[4]. In the sections on the superstring and in the next chapter we will see explicit examples of that. On the other hand we saw that D–branes appear naturally when considering T–duality for open strings.

[4]Similar to a a magnetic monopole vacuum in gauge theories.

Consider a single Dp–brane positioned in a 26–dimensional spacetime (from the T–dual perspective we consider the limit $R \to 0$ or $\tilde{R} \to \infty$). This will break the Lorentz covariance of the theory into $SO(1,p) \times SO(26-p-1)$ and the string states should fit in representations of this broken Lorentz group. The open strings describing the Dp–brane have Neumann boundary conditions in the $p+1$ Dp–brane worldvolume directions X^μ and Dirichlet boundary conditions in the $26-p-1$ transversal directions X^i. Looking at (2.59) and noting that we describe a limit where $\tilde{R} \to \infty$ we should conclude that the transversal p^i can only be zero ($n = 0$) when considering just one Dp–brane. Therefore the Dirichlet contribution to the mass–shell condition will vanish and we just find

$$M^2 = (p^i)^2 + \frac{1}{\alpha'}(N-1) = \frac{1}{\alpha'}(N-1)\,. \tag{2.60}$$

The massless states are thus defined by $\alpha^M_{-1}\,|0,k^\nu\rangle$ where $M \in [0,25]$. Decomposing these massless states into representations of the Dp–brane worldvolume Lorentz group $SO(1,p)$ we find one massless foton state

$$\alpha^\mu_{-1}\,|0,k^\nu\rangle \quad , \quad \mu \in [0,p] \tag{2.61}$$

and $26-p-1$ scalars from the Dp-brane worldvolume perspective

$$\alpha^i_{-1}\,|0,k^\nu\rangle \quad , \quad i \in [p+1,25]\,, \tag{2.62}$$

forming a representation of the transversal $SO(26-p-1)$ rotation group and therefore giving the position of the Dp–brane in target spacetime.

Now consider N Dp–branes. Open strings can stretch between the different Dp–branes. The $N = 2$ case is shown in Figure 2.5. This has the effect that p^i no longer has to be equal to zero. Denoting the distance between the Dp–branes by $\Delta^i_{ab} \equiv X^i_a(0) - X^i_b(\pi)$, where we introduced the indices a and b to label the N different Dp-branes ($\Delta_{aa} = 0$), we find for the mass shell condition (2.60)

$$M^2 = \left(\frac{\Delta_{ab}}{2\pi\alpha'}\right)^2 + \frac{1}{\alpha'}(N-1)\,. \tag{2.63}$$

For generic values of Δ^2_{ab} this will just give N copies of the single Dp–brane spectrum plus extra massive scalar and vector states coming from the open strings stretching between the Dp–branes. The masses of these states are proportional to the distance between the branes, as should be expected. Bringing Dp–branes

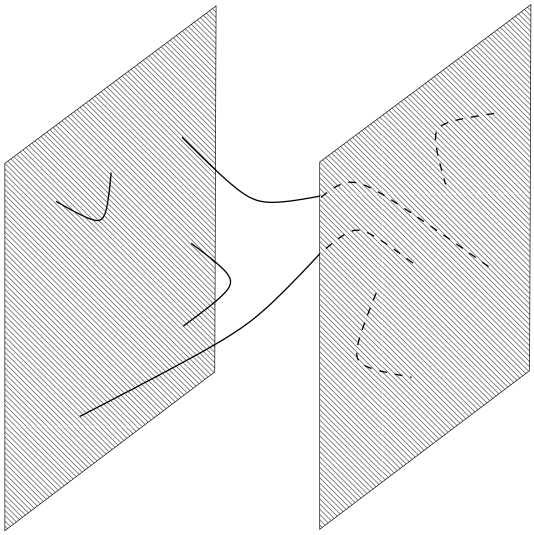

Figure 2.5: Two D–branes and some open strings ending on them.

together we find symmetry enhancement because of the extra massless states appearing in the theory. Giving every (massless) string state indices a and b labelling on which Dp–brane the two different endpoints of the open string end $|0, k^{\mu}\rangle_{ab}$, it must be clear that we find m^2 massless fotons and scalars when m Dp–branes coincide. The massless fotons and scalars form an adjoint representation of $U(m)$. That the scalars form a matrix seems very weird in the light of the interpretation as giving the position of the different Dp–branes. Because in general matrices do not commute this is an example of noncommutative geometry in string theory and again signals that strings see spacetime very differently from the classical way we are used to.

Let us next consider the interaction of D–branes with the closed strings. It is possible to calculate the scattering amplitude for two open strings producing a closed string or the amplitude for the exchange of closed strings between two D–branes, which is shown in Figure 2.6. The latter diagram is at the heart of many duality relations between closed and open strings because it can either be viewed as a closed string being exchanged between two D–branes or as an open string vacuum amplitude. Both points of view describe the same physics. From these amplitudes we can deduce the mass of the D–brane. We will just give the

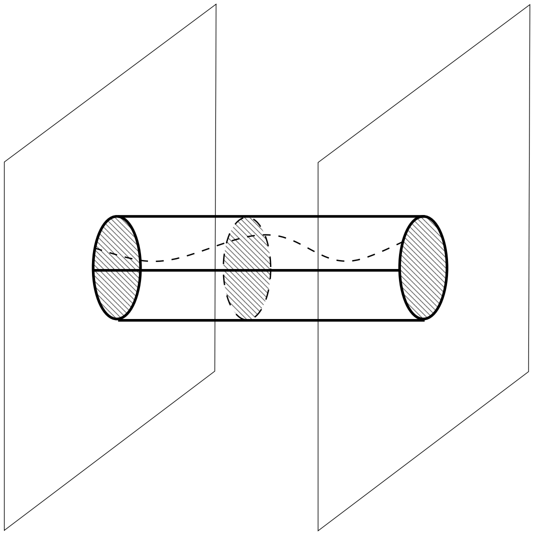

Figure 2.6: Two D–branes exchanging a closed string or equivalently a stretched open string vacuum amplitude.

important result that the mass of a D–brane is

$$M_D \propto \frac{1}{g_s} \tag{2.64}$$

This is a very important property (it will also hold for D–branes in superstring theories), as we will see later. An ordinary soliton in field theory would have a mass proportional to $1/g_s^2$, so a D–brane is somewhere in between a soliton and a perturbative state.

2.1.6 Strings in general background fields

So far we have been discussing bosonic strings moving in a flat Minkowski target spacetime. Analyzing the spectrum of closed strings we found three massless states describing small excitations of the (more general) metric tensor $G^{\mu\nu}$, antisymmetric tensor $B^{\mu\nu}$ and a scalar Φ and we could construct vertex operators for those states. A more general background should be interpreted as a coherent background of these massless string states[5]. Taking the deviation from a flat back-

[5]Another way of saying this would be that the massless fields have non–trivial vacuum expectation values.

ground to be small, consistency tells us that we should find the appropriate vertex operator we obtained from analyzing strings in a flat background. The generalization of (2.2) for closed strings moving in a background determined by any of the massless fields, respecting the consistency conditions from the vertex operators, is then found to be

$$S_\sigma = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\,\sqrt{g}\quad\{\quad [g^{ab}G_{\mu\nu}(X)+\varepsilon^{ab}B_{\mu\nu}(X)]\,\partial_a X^\mu \partial_b X^\nu \\ \qquad + \quad \alpha' R^{(2)}\,\Phi(X)\ \}\,. \tag{2.65}$$

In this expression ε^{ab} is the antisymmetric unit tensor, or Levi–Cevita tensor, and $R^{(2)}$ the Ricci curvature scalar of the worldsheet. It must be clear that this is no longer a free worldsheet theory because of the dependence of the (spacetime) fields on X. We now have to deal with an interacting two–dimensional quantum field theory, which is usually called a non–linear σ–model. Let us first derive an important consequence of the specific way the dilaton scalar appears in the action (2.65).

In two dimensions the Ricci scalar only depends on the topology of the worldsheet and does not influence local worldsheet dynamics. The topological Euler number (for closed strings) equals

$$\chi \equiv \frac{1}{4\pi}\int_\Sigma d^2\sigma\,\sqrt{g}R^{(2)} = 2-2h\,, \tag{2.66}$$

where h is the number of handles on the worldsheet. Suppose Φ is a constant then, in a path integral approach to calculating string scattering amplitudes, different contributions in a loop expansion are weighted by $e^{-\Phi\chi}$. Adding a handle is the same as emitting and reabsorbing a string and will lower the Euler number (2.66) by 2 and therefore add a factor of $e^{2\Phi}$ to the original amplitude. We therefore conclude that emitting a closed string state is weighted by e^{Φ} which can then be interpreted as the closed string coupling constant $g_s = e^{\Phi}$. Because one of the closed string states describes fluctuations of the dilaton scalar Φ, we arrive at the important conclusion that (closed) string theory dynamically generates its own coupling constant. For open strings we are basically led to the same conclusion that $g_s = e^{\Phi}$. However in the open string the topological Euler number is receives an extra contribution because the two–dimensional open string worldsheet has at least one boundary. In that (topologically more general) case $\chi = 2-2h-b$, where b denotes the number of boundaries. Because the first order non–trivial interacting diagram (also called tree level) for the closed string has $h=0$ and

$b = 0$ and for the open string has $h = 0$ and $b = 1$, this means that at tree level the open string diagrams are weighted with $e^{-\Phi}$, while the closed string tree level diagrams are weighted with $e^{-2\Phi}$.

From the way the antisymmetric tensor $B_{\mu\nu}$ is introduced we can easily see that the action (2.65) is invariant under

$$B_{\mu\nu} \to B_{\mu\nu}{}' = B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \,, \tag{2.67}$$

which is a (generalized) gauge transformation for a two–form potential and enables us to define a three–form field strength

$$H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]} \,. \tag{2.68}$$

This describes an obvious generalization of Maxwell theory. A one–form gauge field couples naturally to a particle, a two–form gauge field couples naturally to a string[6].

From the discussion of the bosonic string so far, it is clear that conformal invariance plays a very important role. We would like to keep quantum conformal invariance when considering strings moving in a general background. On the other hand we would also like to see that strings can not move in completely arbitrary backgrounds. Because one of the string states is a graviton we expect that in an expansion in the dimensionful parameter α', to first order the metric background should satisfy Einstein's equations.

Conformal invariance implies that the trace of the 2–dimensional worldsheet energy momentum tensor vanishes, $T^a{}_a = 0$. Calculating the trace of T^{ab} for the non–linear σ–model (2.65) we find a conformal anomaly

$$\frac{T^a_a}{\sqrt{g}} = \frac{\beta^\Phi}{48\pi^3} R^{(2)} + \frac{1}{2\pi} \left(\beta^G_{\mu\nu} g^{ab} + \beta^B_{\mu\nu} \varepsilon^{ab} \right) \partial_a X^\mu \partial_b X^\nu \,, \tag{2.69}$$

where the β–functions can be obtained perturbatively in an expansion in α' on the worldsheet. To leading non–trivial order this gives

$$\begin{aligned} \frac{\beta^G_{\mu\nu}}{\alpha'} &= R_{\mu\nu} - \frac{1}{4} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} + 2\nabla_\mu \nabla_\nu \Phi + \mathscr{O}(\alpha') \,, \\ \frac{\beta^B_{\mu\nu}}{\alpha'} &= \nabla^\rho \left(e^{-2\Phi} H_{\mu\nu\rho} \right) + \mathscr{O}(\alpha') \,, \\ \beta^\Phi &= D - 26 + 3\alpha' \left[4(\nabla\Phi)^2 - 4\Box\Phi - R + \frac{1}{12} H^2 \right] + \mathscr{O}(\alpha') \,. \end{aligned} \tag{2.70}$$

[6] To construct a non–abelian generalization of this two–form gauge field theory is still an open problem in string theory.

In these expressions $R_{\mu\nu}$ and R are the D–dimensional target spacetime Riemann tensor and Ricci curvature respectively and we also introduced the covariant derivative ∇^{μ}. We also used the curvature of the two–form gauge field, the totally antisymmetric three–form H as introduced in (2.68). These β–functions should vanish order by order if we want to cancel the anomaly and have quantum conformal invariance on the worldsheet. Looking at β^{Φ} this means $D = 26$, as we also concluded from analyzing the spectrum of the bosonic string, but in the light–cone gauge we viewed this as a consequence of demanding Lorentz–invariance. All other equations are now first order in α' and look like Einstein's field equations. In fact the first order equations can be obtained from varying the 26–dimensional target spacetime action

$$S = \frac{1}{\alpha'^{24}} \int d^{26}x \sqrt{|G|} e^{-2\Phi} \left[R + 4(\nabla\Phi)^2 - \frac{1}{12} H^2 \right] , \tag{2.71}$$

which just describes Einstein's General Relativity coupled to a massless scalar and a three–form field strength. Therefore we find, as promised, that to first order in α' quantum conformal invariance constrains the background to be a solution of the Einstein equations. We could also have concluded this from an analysis of the string tree amplitudes of the massless string states, expand these amplitudes in α' and deduce the (classical) equations of motion reproducing these quantum amplitudes, but that method is indirect and less transparent.

The same analysis can also be done for the open string and basically leads to the same conclusions. As soon as we couple a string to a more general background we obtain two perturbative expansions. First of all, the background fields are expanded in the two–dimensional worldsheet field theory coupling constant α', which is basically an expansion in the string length influencing the background fields. The limit $\alpha' \to 0$ is a particle limit and higher derivative corrections to the background fields appear because we want to account for the extendedness of the string. Besides that we have a perturbative expansion of string spacetime interactions involving the string coupling constant $g_s = e^{\Phi}$ and higher genus worldsheets.

2.2 The superstring

After this introduction of the bosonic string it is now time to get rid of the problems we encountered in the bosonic string theory. The bosonic string has two features which exclude it from being a serious candidate for a theory of quantum gravity (and all other forces). First of all the appearance of a tachyon in the spectrum is a signal of an unstable Minkowski vacuum and we would like to get rid of such a state. Secondly the bosonic string theory is not able to describe fermionic states, which is clearly a drawback for a theory that we would like to describe the real world where fermions are the basic constituents of matter.

The superstring in fact solves both of these problems. We will only briefly discuss the open and closed superstring spectrum and mainly focus our attention on how the superstring solves the bosonic string problems. Introducing spacetime interactions is done in exactly the same way as for the bosonic string. We will not discuss the non–linear σ–model extension of the superstring, because this does not essentially differ from the bosonic string. Also we will discuss more general background fields extensively in the next chapters. In general we refer to [7, 8, 10] for more details on the superstring. We will end this superstring section with a summary of the five consistent, tachyon free and target spacetime supersymmetric, superstring theories and with a section on T-duality and (supersymmetric) D–branes.

The supersymmetric generalization of the bosonic Polyakov action (2.2) has, next to bosonic symmetries, local supersymmetries and super–Weyl invariance. These symmetries can be used to gauge away the metric g^{ab} and a Rarita-Schwinger field χ_a (which is a kind of fermionic generalization of the worldsheet metric) and impose constraints on the physical states. This gauge is called the superconformal gauge and, as in the bosonic case, there could be potential quantum anomalies. In the bosonic case the absence of these anomalies restricted the dimension of spacetime to be equal to 26. The same analysis for the supersymmetric string gives $D = 10$. From now on we will assume the target spacetime to be 10–dimensional and we will be analyzing the superstring in the superconformal gauge. The superstring action then is

$$S = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\,\left[\partial_a X^\mu \partial^a X_\mu - i\,\bar{\psi}^\mu \lambda^a \partial_a \psi_\mu\right]. \tag{2.72}$$

In this action the λ^a are worldsheet Dirac matrices satisfying the Clifford algebra

$$\left\{\lambda^a, \lambda^b\right\} = -2\eta^{ab}. \tag{2.73}$$

The ψ's are 2–dimensional worldsheet Majorana spinors with the bar operation defined as usual

$$\bar{\psi} \equiv \psi^\dagger \lambda^0. \tag{2.74}$$

Notice that from the target spacetime point of view ψ^μ is a D–dimensional spacetime vector[7]. The superconformal gauge is preserved by residual gauge symmetries, the superconformal transformations and we refer to [11] to learn more about the powerful techniques of superconformal field theory. This action (2.72) is supersymmetric on the worldsheet. This means that it is invariant under the following infinitesimal supersymmetry transformations

$$\delta X^\mu = \bar{\varepsilon}\psi^\mu \quad , \quad \delta\psi^\mu = -i\lambda^a\partial_a X^\mu\varepsilon \,, \tag{2.75}$$

where the spinor ε is the infinitesimal supersymmetry transformation parameter.

Let us forget about supersymmetry for a moment and just discuss the worldsheet Majorana fermions. We choose the following representation of the Dirac matrices

$$\lambda^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \lambda^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} , \tag{2.76}$$

which satisfy the Clifford algebra (2.73). Using this representation it is easy to see that, as in the discussion of the worldsheet scalars, the fermions decompose into left– (+) and right–moving (–) Majorana–Weyl components[8]. Because we have fermions on the worldsheet we now have more possibilities for our boundary conditions. Variation of the fermions in (2.72) gives the two–dimensional Dirac equation plus a boundary term that can be written as

$$\delta S_b = \int_\Sigma d\tau \left[\psi_+^\mu\,\delta\psi_{+\mu} - \psi_-^\mu\,\delta\psi_{-\mu}\right]_{\sigma=0}^{\sigma=l} . \tag{2.77}$$

At first sight there are two different ways to solve these boundary conditions. Either we let the expression (2.77) vanish independently at the two sides $\sigma = 0$ and $\sigma = l$, or we let the expressions at both sides cancel each other. The first choice has the physical interpretation of describing the open string, because in that case both endpoints should be independent. This choice will relate ψ_+ to ψ_- at the endpoints. The other choice is the one we need to describe the closed string, where the two endpoints should be identified. Contributions from ψ_+ and ψ_- will vanish independently from each other.

[7]The introduction of worldsheet spinors in the superstring is called the Neveu–Schwarz–Ramond approach. The Green–Schwarz approach to the superstring introduces target spacetime spinors on the worldsheet. After GSO–projection, to be discussed in the next section, both approaches can be shown to be equivalent.

[8]This is why technically speaking we have $(1,1)$ supersymmetry on the worldsheet, one left–moving and one right–moving supersymmetry.

First consider the open superstring and use our convention to let σ run from 0 to π. Looking at (2.77) we conclude that we have to impose $\psi_{+}^{\mu}\,\delta\psi_{+\mu} = \psi_{-}^{\mu}\,\delta\psi_{-\mu}$ which can be rewritten as

$$\delta[(\psi_{+\mu})^2] = \delta[(\psi_{-\mu})^2]\,. \tag{2.78}$$

This means that at both ends we have $\psi_{+}^{\mu} = \pm\psi_{-}^{\mu}$. The overall sign of the components can be chosen at will, so we can define to use the $+$ sign at the endpoint $\sigma = 0$ and we are left with two options.

- Ramond boundary conditions (R)

$$\psi_{+}^{\mu}(\tau,0) = \psi_{-}^{\mu}(\tau,0) \quad , \quad \psi_{+}^{\mu}(\tau,\pi) = \psi_{-}^{\mu}(\tau,\pi)\,. \tag{2.79}$$

  Using these boundary conditions the open string states are said to be in the Ramond sector of the theory. The oscillator expansion consistent with the Ramond boundary conditions is

$$\psi_{+}^{\mu} = \frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty} d_n^{\mu} e^{-in(\tau+\sigma)}\,, \tag{2.80}$$

$$\psi_{-}^{\mu} = \frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty} d_n^{\mu} e^{-in(\tau-\sigma)}\,. \tag{2.81}$$

  Remember that the d_n are now anticommuting Fourier modes.

- Neveu–Schwarz boundary conditions (NS)

$$\psi_{+}^{\mu}(\tau,0) = \psi_{-}^{\mu}(\tau,0) \quad , \quad \psi_{+}^{\mu}(\tau,\pi) = -\,\psi_{-}^{\mu}(\tau,\pi)\,. \tag{2.82}$$

  Using these boundary conditions the open string states are said to be in the Neveu–Schwarz sector of the theory. The oscillator expansion consistent with the Neveu-Schwarz boundary conditions needs half–integer moded oscillators only giving the expansion

$$\psi_{+}^{\mu} = \frac{1}{\sqrt{2}}\sum_{r=-\infty}^{\infty} b_r^{\mu} e^{-ir(\tau+\sigma)}\,, \tag{2.83}$$

$$\psi_{-}^{\mu} = \frac{1}{\sqrt{2}}\sum_{r=-\infty}^{\infty} b_r^{\mu} e^{-ir(\tau-\sigma)}\,. \tag{2.84}$$

  So r in this expansion only runs over the half–integers $n+\frac{1}{2}$ (with n an integer).

Now let us move on to the closed string fermion boundary conditions. Left– and right–moving fermions should be independent in this case. Basically we obtain the same possibilities as for the open string, but now for both fermions independently

$$\begin{aligned} \delta[(\psi_{+\mu})^2](\tau,0) &= \delta[(\psi_{+\mu})^2](\tau,2\pi)\,, \\ \delta[(\psi_{-\mu})^2](\tau,0) &= \delta[(\psi_{-\mu})^2](\tau,2\pi)\,. \end{aligned} \tag{2.85}$$

Because we can now choose Ramond or Neveu–Schwarz boundary conditions for the left– and right–moving modes independently, we obtain four different possibilities.

- Ramond − Ramond boundary conditions (R−R).
  The oscillator expansion is exactly as in (2.80), but of course the left– and right–moving Fourier modes are now independent. The expansion is thus characterized by $\{d_m^\mu, \tilde{d}_n^\nu\}$ with $m,n \in \mathbb{Z}$.

- Ramond − Neveu–Schwarz boundary conditions (R−NS).
  Ramond boundary conditions on the right–moving fermions and Neveu–Schwarz boundary conditions on the left–moving fermions. The left–moving fermions can be expanded as in (2.83) and the right–moving as in (2.80). The expansion can be characterized by $\{d_m^\mu, \tilde{b}_s^\nu\}$ with $m,(s+\frac{1}{2}) \in \mathbb{Z}$. So the left–moving fermion is described by a half–integer moded expansion.

- Neveu–Schwarz − Ramond boundary conditions (NS−R).
  We just switched left– and right–moving fermions. The left–moving fermions can be expanded as in (2.80) and the right–moving as in (2.83). The expansion can be characterized by $\{b_r^\mu, \tilde{d}_n^\nu\}$ with $(r+\frac{1}{2}),n \in \mathbb{Z}$.

- Neveu–Schwarz − Neveu–Schwarz boundary conditions (NS−NS).
  The oscillator expansion is exactly as in (2.83), but of course the left– and right–moving Fourier modes are now independent. Both expansions are thus characterized by half–integer modes labelled by $\{b_r^\mu, \tilde{b}_s^\nu\}$ with $(r+\frac{1}{2}),(n+\frac{1}{2}) \in \mathbb{Z}$.

We are now ready to quantize the superstring. Besides the bosonic creation and annihilation operators we now have fermionic creation and annihilation operators which can be in different sectors determined by the fermionic boundary conditions. We will not explicitly carry out the quantization procedure again and just quote the results. Let us first sketch what happens in the open superstring case (the closed string can be considered a double copy of the open superstring). We find different mass–shell constraints for the two different sectors (R and NS).

$$\begin{aligned} \alpha' M_{NS}^2 &= \sum_{m=1}^{\infty} \alpha_{-m}^{\mu} \alpha_{m\mu} + \sum_{r=\frac{1}{2}}^{\infty} r\, b_{-r}^{\mu} b_{r\mu} - \tfrac{1}{2}\,, \\ \alpha' M_R^2 &= \sum_{m=1}^{\infty} \alpha_{-m}^{\mu} \alpha_{m\mu} + \sum_{n=0}^{\infty} n\, d_{-n}^{\mu} b_{n\mu}\,. \end{aligned} \tag{2.86}$$

Rather disappointingly we see that in the Neveu–Schwarz sector we still have a tachyonic state (although the mass increased by $\frac{1}{2}$). The first excited massless state can be created with $b^{\mu}_{-1/2}$ and describes a massless vector A^{μ}. The other mass levels are evenly spread with mass difference $\frac{1}{2}$.

In the Ramond sector we see that the normal ordering constant has disappeared. Therefore our Ramond–sector groundstate is already the massless state. We also note that d_0^{μ} is included in the level number operator and this operator can work on any state in the Ramond sector without changing its mass. Calculating the anticommutator of two d_0 operators (from imposing canonical anticommutators on ψ^{μ}) we find

$$\{d_0^{\mu}, d_0^{\nu}\} = \eta^{\mu\nu}\,. \tag{2.87}$$

Up to a factor of two these operators satisfy the anticommutation relations of ten–dimensional Γ–matrices. In ten dimensions these Γ–matrices are of size 32×32 and we can use them as a representation for our operator d_0. Using this representation, states in the Ramond sector have to be 32–component objects, cq. be spacetime fermions. In $D = 10$ we can take Majorana spinors. Because the massless Dirac equation relates half of the Majorana spinor components to the other half, we conclude that the massless groundstate in the Ramond sector has 16 real physical degrees of freedom.

Clearly we did not solve all the problems we had in the bosonic string. Although we did find fermions in the open superstring, we still have a tachyon in our theory. Less urgent at this point is the absence of target spacetime supersymmetry. This would mean that at every mass level the number of bosonic physical degrees of freedom matches the number of fermionic physical degrees of freedom. This clearly does not work out because the NS sector has mass levels which do not even occur in the R sector. Also at the massless level the number of degrees of freedom do not match, the bosonic vector in the NS sector has 8 physical degrees of freedom and the fermion in the R–sector has 16 physical degrees of freedom. We will next discuss the Gliozzi–Scherk–Olive (GSO) projection [12] on the spectrum which in fact projects out the tachyon and enforces target spacetime supersymmetry.

2.2.1 The GSO–projection

The GSO–projection was introduced as a projection on the spectrum which only leaves spacetime supersymmetric string states [12]. These projected string states setup an internally consistent (spacetime supersymmetric) interacting superstring theory free of tachyonic instabilities. We will discuss the basic idea of the GSO–projection for the open superstring. The generalization to closed superstrings follows immediately if we consider the closed superstring as a double copy of the open superstring.

To enforce target spacetime supersymmetry we at least have to get rid of the tachyon and the other half integer moded massive states in the NS sector. In the R sector we basically would like to project out half of the massless fermionic physical degrees of freedom to match with the massless bosonic degrees of freedom. For higher mass levels we then also need a reduction of fermionic degrees of freedom. Let us introduce a fermion number operator $(-1)^F$, where F counts the number of fermionic creation operators used to construct the state the operator acts upon. Using this operator we can construct projection operators

$$\mathscr{P}_{\pm} = \tfrac{1}{2}\left(1 \pm (-1)^F\right) . \tag{2.88}$$

In the NS sector we want to get rid of all states constructed from the tachyonic ground state with an even number of fermionic operators, including the tachyonic ground state itself. In the NS sector we therefore work on the states with the projection operator

$$\mathscr{P}_{NS} = \tfrac{1}{2}\left(1 - (-1)^F_{NS}\right) , \tag{2.89}$$

with the fermion number operator $(-1)^F_{NS}$ given by

$$(-1)^F_{NS} = (-1)^{\sum_{r=1/2}^{\infty} b^{\mu}_{-r} b_{r\mu}} . \tag{2.90}$$

Clearly (2.89) projects out the tachyonic ground state and all higher level states constructed from the ground state with an even number of fermionic operators b^{μ}_{r}. The new ground state is now the massless vector constructed by $b^{\mu}_{-1/2}$ on the, projected out, tachyonic ground state.

In the R sector, basically we want to reduce the number of fermionic physical degrees of freedom. This can be done by introducing the chiral operator, projecting on a subspace of Weyl spinors having a definite chirality[9]. The chiral operator Γ_{11} in $D=10$ is defined as

$$\Gamma_{11} = \Gamma^0\Gamma^1 \ldots \Gamma^9 \,. \tag{2.91}$$

This operator squares to one and anticommutes with the other Γ–matrices. We can use it to define the GSO projection operator in the R–sector

$$\mathscr{P}_R = \tfrac{1}{2}\left(1+\Gamma_{11}(-1)^F_R\right) , \tag{2.92}$$

with the fermion number operator defined as

$$(-1)^F_R = (-1)^{\sum_{n=1}^{\infty} d^{\mu}_{-n} d_{n\mu}} \,. \tag{2.93}$$

Notice that we do not include d^{μ}_0 in the sum of course. Clearly this reduces the number of massless fermionic degrees of freedom by half, we project on positive helicity Weyl spinors which have 8 physical degrees of freedom. We will not show that this operator (2.92) also correctly reduces the number of massive fermionic degrees of freedom to match up with the massive bosonic degrees of freedom.

The resulting theory after GSO–projection is target spacetime supersymmetric at every mass level, although we are not going to prove that here. The resulting theory is also consistent when introducing perturbative spacetime string interactions. In fact the GSO–projection is forced upon us when introducing string interactions. After having discussed a consistent projection on the open superstring spectrum, we are now ready to discuss all the different spacetime supersymmetric, tachyonic free, superstring theories. We will mainly focus on the massless spectrum of these theories and we refer to [7, 8, 10] for a more thorough discussion.

2.2.2 Closed Type IIA and Type IIB superstrings

Basically we can now take two copies of the open superstring spectrum, one for the left–movers and one for the right–movers. We should keep in mind that we need a level–matching condition, so the total mass coming from the right–movers equals that of the right–movers. There is one subtlety in this prescription which has to do with the GSO–projection in the open superstring R–sector. In that case

[9] This can only be done consistently in $D=2$ mod 8 target spacetime dimensions.

we were free to choose a particular sign in the projection operator (2.92), we either project onto positive helicity states, or on negative helicity states. In the closed string we can now project onto different or the same helicity states in the left– and right–moving sector of the theory. The overall sign does not matter, but the relative sign does. We either get two massless spacetime fermions with the same chirality called type IIB superstring theory, or we get two massless spacetime fermions with opposite chirality called Type IIA superstring theory.

Let us discuss the four different sectors in both closed superstring theories. Between brackets we mention the number of physical degrees of freedom of a massless state.

- NS–NS sector.
  Clearly these states have to be bosons. At the massless groundstate of IIA and IIB closed superstrings we find a dilaton scalar Φ (1 d.o.f.), an antisymmetric 2–form tensor $B^{\mu\nu}$ (28 d.o.f.) and a symmetric 2–form tensor, the graviton $G^{\mu\nu}$ (35 d.o.f.). Basically these are the same states as found at the massless level in the bosonic string.

- R – NS and NS –R sector
  We can discuss both sectors at the same time, because they give the same states. Because one of the sectors is Ramond, these states have to be spacetime fermions. Again we find the same states for type IIA and type IIB superstrings. The massless groundstate consists of two spin $1/2$ Majorana–Weyl fermions ($2\times 8 = 16$ d.o.f.) and two higher spin $3/2$ fermions ($2\times 56 = 112$ d.o.f.) called gravitinos.

- R – R sector.
  In one R–sector acting with d_0^μ tells us the states have to be fermions. However when we act with d_0^μ and $\tilde{d}_0^\mu$, both being Γ–matrix representations, the states transform as bosons again. This is the sector where Type IIA and Type IIB theory differ from each other. Working out the representation theory is a little bit harder in this case, but let us just mention the results for the massless ground state

  – Type IIA
    The massless ground state consists of a vector A^μ (8 d.o.f.) and an antisymmetric 3–form tensor $A^{\mu\nu\rho}$ (56 d.o.f.).

  – Type IIB
    The massless ground state consists of a scalar λ (1 d.o.f.), an antisym-

metric 2–form tensor $A^{\mu\nu}$ (28 d.o.f.) and an antisymmetric 4–form tensor $A^{\mu\nu\rho\delta}$ (35 d.o.f.[10]).

So Type IIA contains so–called R–R gauge fields of odd rank and Type IIB contains R–R fields of even rank.

Adding the numbers between brackets we see that every sector contains 64 physical degrees of freedom. We find a total of 128 bosonic and 128 fermionic degrees of freedom, as we expect from supersymmetry. Because we obtain two spin $1/2$ Majorana–Weyl and two spin $3/2$ fermions we have $N=2$ supersymmetry in this case. In fact the left– and right–moving modes separately have $N=1$ spacetime supersymmetry, which enables us to introduce the closed Heterotic superstring.

2.2.3 The Heterotic superstring

Nothing prevents us from only introducing $N=1$ target spacetime supersymmetry by only including left–moving fermions on the superstring worldsheet. So the proposal is to introduce the worldsheet action[11]

$$S=\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\,\left[\partial_a X^\mu\partial^a X_\mu-2i\,\psi_+^\mu\partial_-\psi_{+\mu}\right]. \tag{2.94}$$

The complete left–moving side of this theory is consistent in $D=10$, while the theory on the right–moving side can only be consistent when we have 26 bosons X_- instead of the 10 X_- we have now. So we should add something to the right–moving side of this theory to make it consistent.

We cannot add 16 ordinary bosonic fields X, because that would ruin the consistency on the left–moving side. So we are going to add right–moving worldsheet fermions, but now without a spacetime vector index, because that would take us back to the standard superstring. Instead we will take target spacetime scalars with an internal symmetry index A and it turns out we need 32 of them for a consistent string theory. So we obtain

$$S=\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\,\left[\partial_a X^\mu\partial^a X_\mu-2i\,\psi_+^\mu\partial_-\psi_{+\mu}-2i\,\rho_-^A\partial_+\rho_-^A\right], \tag{2.95}$$

with $A=0,1,\ldots,31$.

[10]This gauge potential only has 35 degrees of freedom because its field strength in $D=10$ satisfies a self–duality constraint. What this means will become clear in the next chapter.

[11]Having $(1,0)$ worldsheet supersymmetry.

Again we should determine what boundary conditions to take on the ρ_-^A fermions. Consistency again plays an important role and it turns out there are only two possibilities.

- We impose the same boundary conditions, either periodic or anti–periodic on all of the right–moving ρ fields, giving us a theory with two sectors. These ρ fields are invariant under 32–dimensional rotations and we obtain a closed string theory with an internal $SO(32)$ symmetry group, the $SO(32)$ Heterotic string.

- We can also divide the 32 ρ fields into two groups of 16, on which we impose the same boundary conditions. So we now obtain 4 different sectors in our theory. The actual internal symmetry group is $E_8 \times E_8$ and not $SO(16) \times SO(16)$, as the reader might have expected. This is called the $E_8 \times E_8$ Heterotic string.

Both Heterotic superstring theories have the $D = 10$ $N = 1$ supergraviton multiplet as massless states, that is a graviton $G^{\mu\nu}$, a dilaton scalar Φ and an anti-symmetric two–form tensor $B^{\mu\nu}$. Besides that the internal symmetry groups of these Heterotic string theories manifest themselves as massless $SO(32)$ or $E_8 \times E_8$ gauge fields A_μ^a with a taking values in the gauge group. We now exhausted all consistent closed superstring possibilities and should move on to discuss open superstrings.

2.2.4 The Type I superstring

As we concluded from the section on the GSO–projection, the groundstate of this open superstring theory is massless. In the NS–sector the massless state is a vector and in the R–sector the massless state is a Majorana–Weyl spinor in $D = 10$. These combine into a vector multiplet of $N = 1$ supersymmetry in ten target spacetime dimensions. Inclusion of Chan–Paton factors will give a $U(N)$ gauge theory, at least as long as we are discussing oriented open strings (meaning that we can distinguish between the two endpoints of the string).

In the section on bosonic strings we concluded that open strings are not consistent by themselves. Through interactions they can form closed strings and therefore closed string states should be included. The same holds for the open Type I superstring in $D = 10$. To include closed superstring states consistently and keep $N = 1$ supersymmetry is not straightforward. It must be clear that we can not add Type IIA closed superstring states, because the open superstring is chiral, the fermions have a definite sign of chirality. We conclude that we should add Type IIB closed superstring states, but truncated to a $N = 1$ supersymmetric theory. The consistent truncation consists of identifying left– and right–movers, making the Type I superstring theory unoriented. Projecting Type IIB states onto states that are symmetric under parity reversal on the worldsheet is called an orientifold projection. Defining the parity reversal operation $\Omega : \sigma \to \pi - \sigma$ we can construct the orientifold projection operator

$$P_O = \frac{1}{2}(1 + \Omega) . \tag{2.96}$$

This breaks half of the supersymmetries as required by the open superstring. Let us summarize the closed superstring states that survive this orientifold projection (2.96).

- In the NS–NS sector only the graviton $G^{\mu\nu}$ and the dilaton scalar Φ survive.
- In the NS–R and R–NS sector only one linear combination of the Majorana–Weyl fermions and gravitinos survives.
- In the R–R sector only the antisymmetric 2–form $A^{\mu\nu}$ survives the orientifold projection.

These closed superstring states form a $N = 1$ supergravity multiplet and can be added consistently to the open superstring.

However, the Type I theory constructed in this way has a gravitational anomaly, i.e. we have non–conservation of the target spacetime energy momentum tensor in the quantum theory. We can cancel this anomaly by adding Chan–Paton factors to the open superstring endpoints and choosing the gauge group to be exactly $SO(32)$[12]. We refer to [7, 8, 10] for more details on this anomaly cancellation.

2.2.5 Superstring T–duality and D–branes

Just as for the bosonic string we expect to find a a symmetry between theories wrapped on a circle R and a dual circle $\frac{\alpha'}{R}$ exchanging winding and momentum modes. In the bosonic string we saw that the closed string is T–dual to itself.

[12]The reason why the gauge group is $SO(32)$ instead of $SU(32)$ is because of the fact that the superstring theory should now be unoriented, therefore it is not possible to distinguish between the two different endpoints of the string which constrains us to the orthogonal gauge groups.

That this is not true for the closed Type II superstrings follows from the observation that T–duality is a one–sided parity operation (2.55), working only on the right–moving modes. This will take the chirality operator, which we used in the GSO–projection on the R–sector, to minus itself for the right–moving modes. The conclusion therefore has to be that T–duality takes Type IIA to Type IIB superstring theory. This can in fact be proven order by order in Type IIA and IIB string perturbation theory. So the statement is that Type IIA superstring wrapped on a circle of radius R is indistinguishable from Type IIB superstring theory wrapped on a circle of radius $\frac{\alpha'}{R}$.

As long as we do not introduce Wilson lines breaking the gauge group, the Heterotic closed superstrings are T–dual to themselves. If we do introduce Wilson lines more possibilities arise. In fact, the $E_8 \times E_8$ Heterotic superstring is T–dual to the $SO(32)$ Heterotic superstring if we decide to break the gauge groups to $SO(16) \times SO(16)$ using Wilson lines.

Working out T–duality for the open Type I superstring will again lead us to D–branes, defined by Dirichlet boundary conditions in the T–dualized directions X^i on the open superstring. Let us consider these D–branes in an infinitely extended spacetime. The open superstring theory defined by the D–brane is clearly $N = 1$ supersymmetric. The open superstring can form closed superstrings that move off the D–brane and we saw that the consistent closed Type II superstrings have $N = 2$ supersymmetry. This means the D–brane breaks half of the supersymmetries and states with this property are usually called Bogomol'nyi–Prasad–Sommerfield (BPS) states.

BPS–states must carry conserved charges besides their mass as for example can be deduced from the supersymmetry algebra. The crucial observation by Polchinski [9] was that these D–branes are charged with respect to the R–R antisymmetric tensors in the Type II superstring theories. A Dp–brane naturally couples to an antisymmetric $p+1$–form R–R potential $A_{(p+1)}$, which has a $p+2$–form field strength $F_{(p+2)}$. Supergravity soliton solutions charged with respect to these antisymmetric R–R potentials show masses proportional to $1/g_s$, which is what we found for the D–branes (2.64) as well. This strongly suggests that we should identify D–branes and the R–R solitonic vacua of closed superstring theories.

Chapter 3

Superstring low energy limits and dualities

In this chapter we will introduce the low energy effective supersymmetric models of superstring theories and D–branes. We will start by discussing supersymmetry algebras and introduce the eleven–dimensional supersymmetry algebra as an example. We then explain the concept of Bogomol'nyi–Prasad–Sommerfield (BPS) states and show how to identify these states using the supersymmetry algebra. Using the eleven–dimensional supersymmetry algebra we will deduce all possible ten–dimensional supersymmetry algebras. After that we will introduce the corresponding eleven– and ten–dimensional supergravities, the concept of string dualities, and use the BPS states to test duality conjectures. We will introduce M–theory as the the eleven–dimensional non–perturbative theory which appears in the strong coupling limit of Type IIA superstring theory. There are many good texts on supersymmetry, supergravity, BPS states, dualities and M–theory. For introductory reviews covering most of the topics in this chapter we refer to [13, 14, 15, 16, 17].

3.1 Supersymmetry algebras and BPS states

In the previous chapter we concluded that the superstring states, after the GSO projection, belong to a supersymmetry multiplet. Supersymmetry, like any other symmetry, leads to conserved supersymmetry charges through Noether's theorem. The supersymmetry charges transform as spinors under the Lorentz group and are therefore anticommuting objects. This basically follows from the observation that supersymmetry transformations are governed by a spinorial parameter ε (2.75). By anticommuting the supersymmetry charges we obtain the supersymmetry algebra. Together with the generators of the Poincaré group, the translation generators P_M and Lorentz rotation generators M_{MN} which commute with the supercharges, they form the super–Poincaré algebra and supermultiplets of fields should transform in irreducible representations of this algebra.

By studying the possible irreducible representations of the super–Poincaré algebra it can be proven that eleven dimensions is the maximal dimension in which one can have a supergravity theory, by which we mean a theory with spins two (gravitons) and less. Higher dimensions introduce higher spin fields which can not be coupled consistently to the lower spin fields[1]. From the supersymmetry point of view eleven dimensions therefore seems to have a privileged role. We will use the eleven–dimensional supersymmetry algebra as a specific example to discuss the more general concepts of supersymmetry algebras and BPS states. Apart from minor details, our discussion can be readily generalized to include lower spacetime dimensions. The example of eleven dimensions will turn out to be useful for other reasons as well. First of all it will help us deduce the ten–dimensional supersymmetry algebras (and the corresponding supergravities) in which we are interested from the superstring point of view. Secondly, eleven dimensions turn out to play a prominent role in strongly coupled superstring theory, as we will argue when we discuss dualities and M–theory.

3.1.1 The $D = 11$ supersymmetry algebra

In eleven dimensions we introduce 32–component Majorana supercharges Q_α and generically these will satisfy ($M \in [0,10]$)

$$\{Q_\alpha, Q_\beta\} = (\Gamma^M C)_{\alpha\beta} P_M\,, \tag{3.1}$$

where $\Gamma^M_{\alpha\beta}$ are the Dirac matrices satisfying $\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}$ where we suppressed the spinor indices, $C_{\alpha\beta}$ is the charge conjugation matrix and P_M is the momentum vector. Supersymmetry charges in any dimension always lead to this term containing the momentum tensor. It tells us that two supersymmetry

[1]There are two (known) ways out of this problem. One is to introduce an infinite tower of massless higher spin fields and the other is to introduce extra timelike dimensions avoiding the problem of having to introduce higher spin fields. Both constructions have their own problems and will not be discussed in this thesis.

transformations combine to give a translation, explaining why local supersymmetric theories automatically give rise to General Relativity. Looking at (3.1) we note that the left–hand side is symmetric in its spinor indices and therefore has $\frac{1}{2} \times 32 \times 33 = 528$ components. The right–hand side has only 11 components and therefore we should expect that this supersymmetry algebra can be extended to include, what are called, central extensions. The unique central extension of the eleven–dimensional supersymmetry algebra having 528 components on the right–hand side is

$$\begin{aligned}\{Q_\alpha, Q_\beta\} = \quad & (\Gamma^M C)_{\alpha\beta} P_M + \frac{1}{2!}(\Gamma^{MN} C)_{\alpha\beta} Z_{MN} \\ & + \frac{1}{5!}(\Gamma^{MNPQR} C)_{\alpha\beta} Y_{MNPQR} \, . \end{aligned} \tag{3.2}$$

In this expression we used $\Gamma^{MN\ldots P} \equiv \Gamma^{[M}\Gamma^N \ldots \Gamma^{P]}$. The tensorial antisymmetric charges Z and Y are called central charges because they commute with the supersymmetry charges Q_α. Strictly speaking they are not central charges with respect to the full Poincaré algebra because they do not commute with the Lorentz rotation generators M_{MN}.

To discuss the interpretation of these central charges we first have to introduce some properties of Dirac matrices. In eleven dimensions there exist two representations of the Dirac matrices. They differ according to whether the product of all of them is $+1$ or -1. Because the choice is arbitrary we will choose

$$\Gamma^0 \Gamma^1 \ldots \Gamma^9 \Gamma^{||} = +1 \, , \tag{3.3}$$

where we used the symbol $||$ to denote the 10^{th} spatial direction , because we will use Γ^{10} to mean the product $\Gamma^1\Gamma^0$. From (3.3) and because $(\Gamma^{||})^2 = +1$ we deduce that

$$\Gamma^{||} = \Gamma^{01\ldots 9} \, . \tag{3.4}$$

Perhaps the reader remembers that this is the definition of the chirality operator Γ_{11} in $D = 10$ (2.91). We will use this when we reduce the eleven–dimensional supersymmetry algebra to ten dimensions. Using the Majorana representation of the Dirac matrices, Γ^0 has all the properties of the charge conjugation matrix $C_{\alpha\beta}$ and we can set $C_{\alpha\beta} \equiv \Gamma^0_{\alpha\beta}$.

In a quantum theory that realizes the algebra (3.2) as an asymptotic symmetry, the (supersymmetry preserving) vacuum state is annihilated by all the supersymmetry charges. Irreducible representations of the supersymmetry algebra (without central charges) are characterized by their mass and spin, and can be constructed using the supersymmetry charges as creation and annihilation operators, for more details on this construction we refer to [13, 15]. Let us now consider a (vacuum) state which preserves some amount, but not all, of the supersymmetries. Such a state will be annihilated by some combination of supersymmetry charges and the expectation value of $\{Q,Q\}$ will therefore have a number of zero eigenvalues. This means the determinant of that matrix will vanish. Let us first discuss what happens when all the central charges vanish and the momentum is the only term left in the supersymmetry algebra. Taking determinants we find

$$\det\{Q_\alpha, Q_\beta\} = \det \Gamma^M P_M = (P^2)^{16}\,. \tag{3.5}$$

Because $\{Q,Q\}$ is a positive operator it is guaranteed that $P^2 \geq 0$. This is an example of a Bogomol'nyi bound [18] and (3.5) only vanishes when $P^2 = m^2 = 0$. So the only state which can preserve some amount of supersymmetry is massless, it satisfies the Bogomol'nyi bound and is called a BPS state [19]. To determine the fraction of supersymmetry which is preserved by such a BPS state we choose a frame in which

$$P_M = \tfrac{1}{2}(-1,1,0,\ldots,0)\,. \tag{3.6}$$

This defines a massless particle moving in the X^1 direction. The algebra can then be rewritten (using $C = \Gamma^0$ and $(\Gamma^0)^2 = -1$)

$$\{Q_\alpha, Q_\beta\} = \tfrac{1}{2}(1-\Gamma_{01})_{\alpha\beta}\,. \tag{3.7}$$

Since Γ_{01} squares to one its eigenvalues are ± 1 and since it is also traceless half of the eigenvalues are $+1$ and half are -1. Therefore the eigenspinors of $\{Q,Q\}$ which have 16 zero eigenvalues satisfy

$$\Gamma_{01}\,\varepsilon = \pm\varepsilon \tag{3.8}$$

and we conclude that this BPS state breaks (and preserves) half of the number of supersymmetries. When we discuss supergravities we will present the half supersymmetry breaking classical background solution corresponding to this massless BPS state.

Looking for more supersymmetry breaking vacua involves turning on some central charges. As an example we will analyze the two–form central charge Z. Let us assume that the vacuum describing such a state is massive, i.e. $P^2 = m^2$, and by using Lorentz rotations we can go to a frame in which all spacelike momenta are zero, i.e. $P_0 = -m$. Further assume that just one of the components of the antisymmetric central charge Z_{MN} is non–zero, let us choose the spacelike components $Z_{12} = q_2$. The algebra (3.2) then reduces to

$$\{Q_\alpha, Q_\beta\} = (m - q_2 \Gamma_{012})_{\alpha\beta} \,. \tag{3.9}$$

Because Γ_{012} squares to one and is traceless, we again find that the eigenvalues of an eigenspinor ε of Γ_{012} are 16 times $+1$ and 16 times -1. The positiveness of the left–hand side assures us that $m \geq |q_2|$, which is the Bogomol'nyi bound. When this bound is satisfied, $m = |q_2|$, the determinant of (3.9) vanishes and we constructed a BPS state preserving half of the supersymmetries. The central charge Z_{12} suggests that such a state is extended in two spacelike directions X^1 and X^2. The classical supergravity background solution corresponding to this vacuum indeed describes a membrane, an object with two spacelike extended directions called the M2–brane.

The same analysis can be performed for the five–form central charge Y and leads to the construction of another $\frac{1}{2}$ supersymmetry preserving BPS state. The Bogomol'nyi bound $m = |q_5|$ is satisfied by an object having five spacelike extended directions called the M5–brane. Such a state can again be constructed as a solution of the $D = 11$ supergravity equations of motion.

At this point we conclude that any central charge of rank p appearing in the supersymmetry algebra corresponds to a BPS state with p spatial extended directions, breaking and preserving half of the number of supersymmetries. Let us now study what happens when we allow one of the spacetime indices of the central charges to be timelike[2]. Because of (3.3) any multiplication of n Dirac matrices is related to a multiplication of $11 - n$ Dirac matrices

$$\frac{1}{n!} \Gamma_{M_1 \cdots M_n}{}^{M_{n+1} \cdots M_{11}} \Gamma^{M_1 \cdots M_n} = \Gamma^{M_{n+1} \cdots M_{11}} \,, \tag{3.10}$$

where $n < 11$. In the same way the antisymmetric central charges can be dualized using the eleven–dimensional antisymmetric Levi–Civita tensor ε. In eleven dimensions this relates a rank $r < 11$ antisymmetric tensor to a rank $11 - r$ antisymmetric tensor in the following way

$$\frac{1}{r!} \varepsilon_{M_{r+1} \cdots M_{11}}{}^{M_1 \cdots M_r} Z_{M_1 \cdots M_r} = \tilde{Z}_{M_{r+1} \cdots M_{11}} \,. \tag{3.11}$$

[2]Central charges with no Lorentz indices, representing pointlike massive BPS objects, are necessarily excluded from this analysis.

When we choose a non–zero component of the central charges to be labeled by a timelike index, we can use the above relations to relate that component to a Hodge dual central charge with only spacelike indices. Let us again use eleven dimensions to illustrate this. The rank 2 central charge Z can be related to a rank 9 central charge $\tilde{Z}$. If we choose one of the indices on Z to be timelike, then $\tilde{Z}$ will only have spacelike indices. The latter central charge will have the usual interpretation as a BPS state with 9 extended spatial directions. The same holds for the rank 5 central charge Y with one timelike index. The corresponding rank 6 central charge $\tilde{Y}$ will have only spacelike indices and gives rise to a BPS state with 6 extended spatial directions. The 6–brane can again be constructed as a soliton solution in the corresponding supergravity theory. The 9–brane however, usually referred to as the M9–brane, is special. Although progress has been made in identifying and constructing this state in the supergravity model [20], its properties are still not completely understood.

All these states break half of the supersymmetry. Putting several BPS states together, letting them overlap or intersect, generically breaks all supersymmetries. Only when the branes are oriented in special ways with respect to each other we find BPS states preserving less than half of the supersymmetries [21, 22]. We will study these intersecting (or overlapping) configurations appearing in the eleven–dimensional supersymmetry algebra in the next chapter.

The massive irreducible representations belonging to these BPS states is shorter than the generic massive multiplet[3]. This follows from the fact that half of (combinations of) the supersymmetry charges annihilate the BPS state. Only the other half can be used as creation operators to construct representations and thus lead to fewer states. For a BPS multiplet to become a generic massive multiplet would require the sudden appearance of more states, which is impossible if we smoothly alter the parameters of the theory. Exactly for this reason we expect the number and type of BPS multiplets to be the same in the classical and quantum theories, regardless of the coupling constant of the quantum theory. This enables us to verify the existence of BPS states when the coupling is small and then conclude that these BPS states will still be present at strong coupling. Even better, in a sense these states are like massless particles (which are also described by short multiplets). Once a particle is massless it will stay massless, or put differently its mass will not be changed by quantum corrections. The same "no–renormalisation" theorem holds for massive BPS states. This enables us to

[3]The massless BPS state we constructed just corresponds to the generic massless short (supergraviton) multiplet.

determine the mass (and charge) of the BPS objects at weak coupling and extrapolate these expressions to strong coupling. These properties of BPS states play a very important role in checking duality conjectures relating strongly coupled theories to weakly coupled theories.

3.1.2 Supersymmetry algebras in ten dimensions

Reducing the eleven–dimensional supersymmetry algebra to ten dimensions by simply decomposing the $SO(10,1)$ representations into $SO(9,1)$ representations we obtain

$$\begin{aligned}\{Q_\alpha, Q_\beta\} = \quad & (\Gamma^\mu C)_{\alpha\beta} P_\mu + (\Gamma^{||} C)_{\alpha\beta} P_{||} \\ & + (\Gamma^\mu \Gamma^{||} C)_{\alpha\beta} Z_\mu + \frac{1}{2!} (\Gamma^{\mu\nu} C)_{\alpha\beta} Z_{\mu\nu} \\ & + \frac{1}{4!} (\Gamma^{\mu\nu\rho\sigma} \Gamma^{||} C)_{\alpha\beta} Y_{\mu\nu\rho\sigma} + \frac{1}{5!} (\Gamma^{\mu\nu\rho\sigma\delta} C)_{\alpha\beta} Y_{\mu\nu\rho\sigma\delta} .\end{aligned} \tag{3.12}$$

The first line in this expression is the straightforward decomposition of $M \in (\mu, ||)$ of the momentum P^M in eleven dimensions ($||$ represents the tenth spatial direction). The second and third line does the same for the rank 2 and 5 central charges in $D = 11$. Because the definition of the $D = 10$ chirality operator equals $\Gamma^{||}$, we can replace $\Gamma^{||}$ with Γ_{11}. One Majorana spinor in $D = 11$ should give rise to two Majorana–Weyl spinors in $D = 10$. Because the reduction will not project the spinors to one out of the two possible chiralities, we conclude that this supersymmetry algebra describes non–chiral $N = 2$ supersymmetry[4]. Therefore this algebra can only describe IIA superstrings, which is the only $N = 2$ spacetime supersymmetric theory which is non–chiral. Consequently the corresponding supergravities in $D = 10$ and $D = 11$ should also be related by dimensional reduction, which will have important consequences as we will see.

Looking at (3.12) we see that this algebra contains a scalar central charge $P_{||}$ related to the momentum in the internal 10^{th} direction. These are the Kaluza–Klein momentum modes already encountered in section 1.1.4, when we discussed them in the context of T–duality in string theory, and we now see that they are BPS states (these massive pointlike objects turn out to be the D0–branes of Type IIA string theory). Besides the scalar central charge we find rank 1, rank 2, rank 4 and rank 5 central charges. They would correspond, using the results of the previous

[4]Put differently, in $D = 10$ we can assemble two Majorana-Weyl supersymmetry charges of opposite chirality into a single non–chiral Majorana supercharge.

section, to BPS states having 1 and 9[5], 2 and 8, 4 and 6 and 5 extended spatial directions. The appearance of more central charges in the IIA algebra is of course due to the fact that, thinking in terms of the corresponding extended objects, one or none of the spatial legs of the extended object can be wrapped around the eleventh compact direction, giving rise to two distinct extended objects in $D = 10$. When we discuss IIA supergravity we will identify the corresponding BPS soliton solutions. For now let me just mention that all the even central charges in the Type IIA supersymmetry algebra in fact correspond to the D–brane BPS string solitons as discussed in section 1.2.5.

Let us continue and deduce the Type IIB supersymmetry algebra. To do this we rewrite the IIA supersymmetry algebra into a form in which the $D = 10$ Majorana supercharge Q is decomposed into the sum of two Majorana–Weyl supercharges $Q^{\pm}$ of opposite chirality

$$Q^{\pm} = \mathscr{P}^{\pm} Q \quad , \quad \mathscr{P}^{\pm} \equiv \tfrac{1}{2}(1 \pm \Gamma_{11})\,. \tag{3.13}$$

Using this decomposition the IIA supersymmetry algebra becomes

$$\begin{aligned}
\{Q^{+}_{\alpha}, Q^{+}_{\beta}\} = \quad & (\mathscr{P}^{+}\Gamma^{\mu} C)_{\alpha\beta}(P+Z)_{\mu} + \frac{1}{5!}(\Gamma^{\mu\nu\rho\sigma\delta} C)_{\alpha\beta} Y^{+}_{\mu\nu\rho\sigma\delta} \\
\{Q^{-}_{\alpha}, Q^{-}_{\beta}\} = \quad & (\mathscr{P}^{-}\Gamma^{\mu} C)_{\alpha\beta}(P-Z)_{\mu} + \frac{1}{5!}(\Gamma^{\mu\nu\rho\sigma\delta} C)_{\alpha\beta} Y^{-}_{\mu\nu\rho\sigma\delta} \\
\{Q^{+}_{\alpha}, Q^{-}_{\beta}\} = \quad & (\mathscr{P}^{+} C)_{\alpha\beta} P_{||} + \frac{1}{2!}(\mathscr{P}^{+}\Gamma^{\mu\nu} C)_{\alpha\beta} Z_{\mu\nu} \\
& + \frac{1}{4!}(\mathscr{P}^{+}\Gamma^{\mu\nu\rho\sigma} C)_{\alpha\beta} Y_{\mu\nu\rho\sigma}\,,
\end{aligned} \tag{3.14}$$

where the rank 5 $Y^{\pm}$ describe irreducible Hodge self–dual and anti self–dual charges[6] which are defined using the ten–dimensional Levi–Civita tensor

$$Y^{\pm}_{\mu_0\mu_1\cdots\mu_4} \equiv \pm\frac{1}{5!}\varepsilon_{\mu_0\mu_1\cdots\mu_4}{}^{\mu_5\mu_6\cdots\mu_9} Y^{\pm}_{\mu_5\mu_6\cdots\mu_9}\,. \tag{3.15}$$

This is just a special case of (3.11) where the tensors on both sides are of the same rank and therefore can be identified. The Y^{+} and Y^{-} can be used to construct two

[5]This would correspond to a spacetime filling brane which is special. It can be given a suitable interpretation and we refer to [23] for more details.

[6]The property of the Dirac matrices (3.3), together with the projection operator $\mathscr{P}^{\pm}$ is responsible for the fact that the rank 5 central charge Y decomposes into its self–dual and anti self–dual parts in the supersymmetry algebra.

independent charges by summing or subtracting them

$$Y = Y^+ + Y^- \quad , \quad \tilde{Y} = Y^+ - Y^- , \tag{3.16}$$

where we suppressed the Lorentz indices. The charges Y and $\tilde{Y}$ are related by the transformation (3.11), so symbolically we have $\tilde{Y} = \varepsilon_{(10)} Y$. These two charges represent different BPS objects because they will appear with different projection operators. This follows by using (3.10), giving the Hodge dual charge $\tilde{Y}$ an extra factor of Γ_{11}.

We can now obtain the ten–dimensional Type IIB supersymmetry algebra from the Type IIA algebra (3.14) by considering a T–duality transformation. We can interpret T–duality as a one–sided parity transformation changing the chirality in one sector of the theory. This is accomplished in the supersymmetry algebra by transforming Q^- into $\tilde{Q}^+\Gamma^9$, where X^9 is the compact direction. Because multiplication with Γ^9 changes the chirality, the supersymmetry charge $\tilde{Q}^+$ now has the same chirality as Q^+, so after this transformation the theory is indeed chiral. Explicitly performing the T–duality in the algebra requires a transformation of the IIA central charges as well, in order to be able to consider the ten–dimensional covariant limit $R_9 \to \infty$. For more details on this T–duality transformation we refer to [24].

The final $D = 10$ Type IIB algebra has two chiral supercharges which can be combined into a $SO(2)$ vector $Q^I = (Q^+, \tilde{Q}^+)$ and can be written in the form

$$\begin{aligned} \{Q^I_\alpha, Q^J_\beta\} \;=\; & (C\mathscr{P}^+\Gamma^\mu)_{\alpha\beta}\left(\delta^{IJ}P_\mu + \sigma_3^{IJ}Z_\mu + \sigma_1^{IJ}\tilde{Z}_\mu\right) \\ & + \frac{1}{3!}\sigma_2^{IJ}(C\mathscr{P}^+\Gamma^{\mu\nu\rho})_{\alpha\beta}W_{\mu\nu\rho} + \frac{1}{5!}\delta^{IJ}(C\mathscr{P}^+\Gamma^{\mu\nu\rho\sigma\lambda})_{\alpha\beta}K^+_{\mu\nu\rho\sigma\lambda} \\ & + \frac{1}{5!}(C\mathscr{P}^+\Gamma^{\mu\nu\rho\sigma\lambda})_{\alpha\beta}\left(\sigma_3^{IJ}V^+_{\mu\nu\rho\sigma\lambda} + \sigma_1^{IJ}\tilde{V}^+_{\mu\nu\rho\sigma\lambda}\right) , \end{aligned} \tag{3.17}$$

where the Pauli matrices σ set up the (standard) representation of the $SO(2)$ algebra. We note that all even rank central charges after T–duality become odd rank central charges. The resulting odd rank charges correspond to D–branes in IIB theory. This is in agreement with the results on T–duality in the open superstring. There we concluded that T–duality turns a Dp–brane into a D$(p-1)$– or a D$(p+1)$–brane depending on the direction in which we perform the T–duality. If IIA only contains even Dp–branes then the T–dual IIB theory will only contain odd Dp–branes, including a D5 brane representing the extended object responsible for the self–dual rank 5 central charge $\tilde{V}^+$. Similar observations can be made for the other central charges.

The IIB supersymmetry algebra (3.17) is invariant under the following two involutions

$$\begin{array}{rcl} Q^I & \to & \sigma_3^{IJ} Q^J \\ \tilde{Z} & \to & -\tilde{Z} \\ W & \to & -W \\ \tilde{V}^+ & \to & -\tilde{V}^+ \end{array} \qquad (3.18)$$

and

$$\begin{array}{rcl} Q^I & \to & \sigma_1^{IJ} Q^J \\ \tilde{Z} & \to & -Z \\ W & \to & -W \\ \tilde{V}^+ & \to & -V^+ . \end{array} \qquad (3.19)$$

Both symmetries can be used to truncate the IIB supersymmetry algebra (by modding out) to an $N = 1$ supersymmetry algebra. In fact we find that both supersymmetry algebras are isomorphic to each other. One can be obtained from the other by replacing Z^μ by $\tilde{Z}^\mu$ and the rank 5 Y^+ by $\tilde{V}^+$. The $N = 1$ algebra using the Z and Y^+ charges equals

$$\{Q^+_\alpha, Q^+_\beta\} = (\mathscr{P}^+\Gamma^\mu C)_{\alpha\beta}(P+Z)_\mu + \frac{1}{2\,5!}(\Gamma^{\mu\nu\rho\sigma\delta}C)_{\alpha\beta}(Y+\tilde{Y})_{\mu\nu\rho\sigma\delta}\,, \quad (3.20)$$

We already mentioned that the charges $\tilde{Z}$ and $\tilde{V}^+$ represent D–brane BPS solitons, whereas the charges Z and Y^+ represent a fundamental string winding mode (the F1) and a true solitonic[7] object (the NS5–brane). We note that this distinction can not be made by just comparing the $N = 1$ supersymmetry algebras. Because the Heterotic superstring theory does not contain D–branes, as opposed to Type I theory, we conclude that the Heterotic supersymmetry algebra (corresponding to either $SO(32)$ or the $E_8 \times E_8$ Heterotic string theory) has to be the one with the Z and Y^+ charges, which means the other isomorphic algebra has to belong to Type I theory. That the two $N = 1$ supersymmetry algebras are isomorphic is already an indication that the Heterotic superstrings and the Type I superstring are not as different as one might have thought.

Although much (kinetic) information is contained in the supersymmetry algebra, any dynamical information can only be studied by constructing the relevant

[7] By this we mean that the mass (and charge) of this object scales as $1/g_s^2$, which is the expected scaling of an ordinary soliton.

local or global supersymmetric field theories. When considering low energy limits of superstring theories (which contain gravitons), we need to construct a supergravity field theory. In the case of D–branes we need to construct a globally supersymmetric field theory. All the BPS states suggested by the supersymmetry algebra should appear as soliton solutions in the corresponding supergravities. The exact relation of these soliton solutions to the central charges appearing in the supersymmetry algebra will justify the identifications we made between central charges and BPS extended objects in this section. Of particular interest are the D–brane soliton solutions because of their microscopic description in terms of open superstrings.

3.2 Supergravities

From the analysis of the consistent superstring theories we deduce that there should exist five different effective low energy supergravities in $D=10$. Two of those should have $N=2$ (maximal) supersymmetry and three should have $N=1$ supersymmetry. Spacetime supersymmetry, field content, chirality properties and the gauge group in the $N=1$ case, uniquely determine the corresponding supergravity model.

Let us first define what we exactly mean by a low energy limit in string theory. At this point, before having discussed string dualities, this involves small string coupling g_s, because only in that regime is the interacting string theory well defined. By definition it involves considering low energy processes determined by a scale U, more precisely this means energies much smaller than the Planck mass. The Regge slope parameter α' is considered to be of the order of the Planck length squared, so in an equation the low energy condition reads

$$U^2\,\alpha' \ll 1\,. \tag{3.21}$$

As we saw in the previous chapter (2.86) the massive string states have masses which are of the order $M^2\alpha' \sim 1$. So this means we only have to consider massless asymptotic states in a low energy limit. In the small coupling and low energy limit we only need to consider string tree diagrams for these massless states which are well approximated by a classical supergravity field theory. This is an example of a correspondence principle. At small energies or at scales much larger than the string scale $l_s \equiv \sqrt{\alpha'}$, the string–like structure should become invisible and a particle theory should be adequate.

The above low energy limit is globally consistent when we scatter massless string states on a flat Minkowski background. This is because the background is exact (no stringy α' corrections) and because the string coupling $g_s = e^{\Phi}$ can be chosen to be small everywhere. For other backgrounds generically this is not true. First of all the background fields will receive stringy corrections (2.70) when the variations of the fields, basically measured by the curvatures, become large. More concretely, when the Ricci curvature $R\alpha' \sim 1$ the corrections become large and take us away from the supergravity limit. Another feature of generic backgrounds is that the string coupling $g_s = e^{\Phi}$ depends on where we are in the background and usually becomes large somewhere. String loop corrections will then become very important, again taking us away from the supergravity limit. This means that we can only trust supergravity backgrounds when the curvatures and the string coupling are small. Exactly in those regions we can use supergravity as a good approximation to superstring theory.

Constructing supergravities was a very active research subject in the 70's and 80's. In those days the hope was that supergravities by themselves were finite quantum gravity theories. Nowadays supergravity theories, although better behaved than ordinary General Relativity, are believed to be non–renormalizable as well. Our approach to constructing the $N = 1$ and $N = 2$ $D = 10$ supergravities will heavily rely on the information we obtained from the corresponding superstring theories. Although at first sight not related to superstring theories, we will start by constructing the eleven–dimensional supergravity and show that it appears in a strongly coupled limit of IIA superstring theory. We will not write down complete supergravity actions. We will only present the bosonic sector of the supergravities, keeping in mind that supersymmetry determines the fermionic sector. This will be all the information we need for our purposes in this thesis.

3.2.1 Supergravity in eleven dimensions

We cannot use the states found in a superstring theory to predict the bosonic massless states appearing in $D = 11$ supergravity. However (maximal) supersymmetry uniquely determines the the supergravity action in $D = 11$. We will denote all eleven–dimensional fields with hats, to distinguish them from ten–dimensional fields. It consists of the following massless bosonic fields (with $M, N, P \in [0, 10]$)

$$(\hat{G}^{MN},\hat{C}^{MNP})\,. \tag{3.22}$$

Only a metric tensor $\hat{G}^{MN}$ and a rank 3 antisymmetric tensor field $\hat{C}$ appear. The bosonic part of the action reads [25]

$$\begin{aligned} S_{D=11} = \frac{1}{\kappa_{11}^2}\int d^{11}x\sqrt{-\hat{G}} \quad \Big(\quad \hat{R} + \frac{1}{12}\hat{F}^2 \Big) \\ + \frac{2}{72^2}\int \hat{F}\wedge\hat{F}\wedge\hat{C}\,. \end{aligned} \tag{3.23}$$

In this action $\hat{F} = d\hat{C}$. The topological term in the action modifies the Bianchi identity of the Hodge dual field strength $\tilde{\hat{F}}$, which is a rank 7 tensor and which can be constructed using the $D = 11$ Levi–Civita tensor (as in (3.11)).

In the supersymmetry algebra discussion we noted that the IIA supersymmetry algebra is related to the eleven–dimensional supersymmetry algebra by a simple reduction. In that sense we used the eleven–dimensional supersymmetry algebra as a nice starting point from which all ten–dimensional supersymmetry algebras could be deduced (using superstring T–duality as well). The same should hold for the supergravity theories. However the precise relation between the fields in $D = 11$ supergravity and the fields in $D = 10$ Type IIA supergravity, suggests a much deeper connection.

Let us assume the existence of a compact spacelike direction giving rise to a $U(1)$ isometry with killing vector field k. This means that the Lie derivative with respect to k acting on all eleven–dimensional fields vanishes, i.e. $\mathscr{L}_k\hat{G} = 0$ and $\mathscr{L}_k\hat{F} = 0$. Splitting coordinates $x^M = (x^\mu, y)$ for which $k = \partial/\partial y$ we can decompose the $D = 11$ bosonic fields as follows

$$\begin{aligned} ds^2 &= e^{-\frac{2}{3}\Phi(x)}dx^\mu dx^\nu G_{\mu\nu}(x) + e^{\frac{4}{3}\Phi(x)}(dy - dx^\mu A_\mu(x))^2 \\ \hat{C} &= \frac{1}{6}dx^\mu\wedge dx^\nu\wedge dx^\rho C_{\mu\nu\rho}(x) + \tfrac{1}{2}dx^\mu\wedge dx^\nu\wedge dy B_{\mu\nu}(x)\,. \end{aligned} \tag{3.24}$$

The decomposition is done in such a way that the fields appearing all nicely transform in irreducible representation of the $SO(1,9)$ Poincaré algebra. This means we can just read off the fields that will appear in $D = 10$ from this decomposition. We find the dilaton Φ, the ten–dimensional metric $G^{\mu\nu}$ and the vector A_μ all appearing in the decomposition of the metric, and rank 2 and rank 3 antisymmetric tensors coming from the rank 3 antisymmetric tensor in $D = 11$. We recognize this as the massless field content of Type IIA superstrings and using this decomposition to rewrite the $D = 11$ action we will obtain the $D = 10$ Type IIA supergravity action, which we will write down explicitly in the next subsection.

Looking at the decomposition of the metric (3.24) it follows that the radius of the compact direction y, let us call it R_{11}, is measured by $e^{\frac{2}{3}\Phi}$. Because of the relation between the string coupling constant g_s and the dilaton we can make the following important identification (putting in the eleven–dimensional Planck length)

$$R_{11} = l_p^{(11)} g_s^{2/3}, \tag{3.25}$$

where g_s is the IIA superstring coupling constant. So the IIA superstring coupling constant measures the size of the eleventh dimension. To be more precise, this is the radius as measured by an observer using eleven–dimensional Planck units. By looking at (3.24) we see that the radius as measured by the ten–dimensional metric $G^{\mu\nu}$, which is measured using string length units $\sqrt{\alpha'}$, actually is proportional to g_s

$$R_{11} = \sqrt{\alpha'} e^{\Phi} \equiv \sqrt{\alpha'} g_s \,. \tag{3.26}$$

We note that this means that $l_p^{(11)} = \sqrt{\alpha'}\, g_s^{1/3}$. So when we increase the IIA coupling constant we blow up the eleventh dimension. The reason why nobody detected this hidden dimension in IIA superstring theory was because that theory is only defined at weak coupling $g_s \to 0$, which means the eleventh dimension is very small and therefore invisible in IIA superstring perturbation theory.

All this suggests the conjecture that eleven–dimensional supergravity is the low energy effective supergravity of strongly coupled ($g_s \to \infty$) IIA superstring theory. In order for this to be possible some states in the IIA superstring theory should become light at strong coupling, otherwise a low energy effective theory would not make sense. These states indeed exist in IIA superstring theory and we in fact already discussed them in the context of D–branes and in the context of the IIA supersymmetry algebra. In the IIA supersymmetry algebra (3.12) we found BPS particle states that from the eleven–dimensional point of view were Kaluza-Klein momentum modes. From the IIA superstring point of view these particle states are in fact D0–branes, point-like defects on which open strings can end. D–branes masses are proportional to $1/g_s$, so clearly when we increase the coupling the D0–branes become light. In the limit $g_s \to \infty$ these states become massless and at the same time we also decompactified to $D = 11$. This resulted in the conjecture that the effective low energy description of strongly coupled IIA superstring theory is $D = 11$ supergravity [26, 27] and it also suggests that the fundamental degrees of freedom in this eleven–dimensional theory are related to D0–branes.

3.2.2 Supergravities in ten dimensions

Let us now work our way through all the low energy effective supergravity theories of the different superstring theories. One way to proceed would be to write down the β–functions (2.70) of the corresponding superstring theories, but the uniqueness of the different supergravity theories in $D = 10$ allows us to use more direct field theoretical methods. For a more extensive discussion on the $D = 10$ supergravities and their original constructions we refer to [7] (part two) and the references therein. We will begin with the Type IIA supergravity action which is determined by the reduction of eleven–dimensional supergravity (3.24) as we already saw.

The following massless bosonic fields appear in IIA supergravity

$$(\Phi, G^{\mu\nu}, B^{\mu\nu}; A^{\mu}, A^{\mu\nu\rho})\,. \tag{3.27}$$

Where the first three fields are the Neveu–Schwarz fields combining to give the bosonic part of an $N = 1$ supergraviton multiplet. This basic Neveu–Schwarz supergravity multiplet will appear in all closed superstring low energy effective supergravity theories. The other two fields are Ramond–Ramond tensor fields giving rise to rank 2 and rank 4 field strengths (called F and G respectively) appearing in the supergravity action. We also find a topological term resulting in a modification of the Bianchi identity for the Ramond–Ramond field strength G giving $G = dA + 12B \wedge F$ (where B is the rank 2 Neveu–Schwarz field). The Type IIA supergravity action reads

$$\begin{aligned} S_{IIA} = \frac{1}{\kappa_{10}^2}\int \quad d^{10}x \quad \sqrt{-G}\,\{\, e^{-2\Phi}(R + 4(\partial\Phi)^2 - \frac{1}{3}H^2\,) \\ +F^2 + \frac{1}{12}G^2\,\} + \frac{1}{144}\int G\wedge G\wedge B\,. \end{aligned} \tag{3.28}$$

In this action κ_{10} is the ten–dimensional gravitational constant which can be related to α' by just counting mass dimensions, $\kappa_{10} \propto {\alpha'}^2$. Notice that all the fields of the Neveu–Schwarz $N = 1$ supergravity multiplet are multiplied by the same factor $e^{-2\Phi} \propto 1/g_s^2$. Extracting the powers of the string coupling constant g_s and redefining κ_{10} to include the powers of g_s we conclude that

$$\kappa_{10} \propto {\alpha'}^2 g_s\,. \tag{3.29}$$

The appearance of $e^{-2\Phi}$ is a result of the fact that this action can be deduced from tree level closed Type IIA superstring theory. The worldsheet at this level topologically is a sphere which has genus zero and Euler number $\chi = 2$. Therefore all diagrams are weighted with $e^{-2\Phi}$ (see 1.1.6), explaining the dilaton factor. This will be a common feature of all supergravity fields which are massless closed string states in the Neveu–Schwarz sector. Ramond–Ramond fields are special because they invalidate the conclusion that at tree level closed string states are weighted with $e^{-2\Phi}$. One could start with Ramond–Ramond fields C' weighted as usual, but it can be shown that when the dilaton background is non–trivial the Bianchi identity and the field equations pick up terms proportional to $d\Phi$. This means C' is no longer of the form dA. Defining $C \equiv e^{-\Phi}C'$ gets rid of the terms proportional to $d\Phi$ and restores the usual Bianchi identity (and field equations). This redefinition decouples the Ramond–Ramond fields from the dilaton and so they appear in the action without a factor of e^{Φ}. Again this will be a common feature of all supergravities containing Ramond–Ramond fields.

We should mention that IIA supergravity can be extended to include a cosmological constant consistent with supersymmetry. This cosmological constant, by using Hodge duality, can be related to a rank 10 field strength[8] which is of the Ramond–Ramond type which means that it does not couple to the dilaton. So the extension called massive IIA supergravity involves the addition of a cosmological constant in the Ramond–Ramond part of the action [28].

The action (3.28) is invariant under the dilation or global scale transformation

$$\begin{aligned} \Phi &\to \Phi + 4\lambda \\ G^{\mu\nu} &\to e^{-2\lambda}\, G^{\mu\nu} \\ B_{\mu\nu} &\to e^{2\lambda}\, B_{\mu\nu} \\ A_{\mu} &\to e^{-3\lambda}\, A_{\mu} \\ A_{\mu\nu\rho} &\to e^{-\lambda}\, A_{\mu\nu\rho}\,, \end{aligned} \tag{3.30}$$

explaining the name of the dilaton scalar. The vacuum in which Φ takes on a particular value is not invariant, so this symmetry is a spontaneously broken one for which the dilaton is the Nambu–Goldstone boson.

We could also consider local scale transformations with a function $e^{\lambda\Phi(x)}$ depending on the dilaton scalar. Such a transformation is called a conformal transformation and defines a new metric tensor $G_c^{\mu\nu}$

$$G_c^{\mu\nu} = e^{-\lambda\Phi} G^{\mu\nu}\,. \tag{3.31}$$

[8]Such a field strength can be consistently added without breaking supersymmetry because it does not describe local degrees of freedom. It only carries a global degree of freedom, consistent with the fact that it should be related to a cosmological constant.

This kind of transformation will transform the action (3.28) into

$$S^c_{IIA} = \frac{1}{\kappa^2_{10}} \int d^{10}x \sqrt{-G_c} \quad \Big(\quad e^{(-2+4\lambda)\Phi} \left[R_c + \delta\,(\partial\Phi)^2\right] \qquad (3.32)$$
$$-\frac{1}{3} e^{(-2+2\lambda)\Phi} H^2 + e^{3\lambda\Phi} F^2 + \frac{1}{12} e^{\lambda\Phi} G^2 \ \Big) \,,$$

with $\delta = 4 - 16\lambda^2\,(D-1)/(D-2)$. Every value of λ picks out a specific metric frame which determines the form of the supergravity action. Our starting point action (3.28) is written using the so–called string frame metric $G^{\mu\nu}_s$. This is the naturally preferred frame for strings probing a background solution, in the sense that the string action (2.1) in this frame does not depend on the dilaton scalar (it will depend on the dilaton scalar when performing a conformal transformation to another metric frame). Another convenient (and standard) frame is the Einstein frame, which is defined as the frame in which the Ricci curvature tensor is not multiplied with a dilaton factor. From (3.32) we see that this happens when we take $\lambda = \frac{1}{2}$, fixing the relation between the Einstein and string frame metric

$$G^{\mu\nu}_E = e^{-\frac{1}{2}\Phi} G^{\mu\nu}_s \,. \qquad (3.33)$$

The Einstein frame IIA supergravity action becomes

$$S^E_{IIA} = \frac{1}{\kappa^2_{10}} \int d^{10}x \sqrt{-G_E} \quad \Big(\quad \left[R_E - \tfrac{1}{2}(\partial\Phi)^2\right] \qquad (3.34)$$
$$-\frac{1}{3} e^{-\Phi} H^2 + e^{\frac{3}{2}\Phi} F^2 + \frac{1}{12} e^{\frac{1}{2}\Phi} G^2 \,.$$

Similar scaling properties exist for all the $D \leq 10$ supergravity actions which have a dilaton scalar. Although the actual physics should be independent of the frame we use, some properties of supergravity (or properties of solutions) are much easier detected using a specific frame. Different probes (appearing in the theory) have naturally preferred frames in which the probe worldvolume actions do not depend on the dilaton scalar.

Type IIB supergravity should be a chiral $N = 2$ supergravity which is uniquely determined by supersymmetry. From IIB superstring theory we concluded that at the massless level we find the following bosonic states

$$(\Phi, G^{\mu\nu}, B^{\mu\nu}; \lambda, A^{\mu\nu}, A^{\mu\nu\rho\sigma}_+) \,. \qquad (3.35)$$

The first three states are Neveu–Schwarz fields, which in the action should all be multiplied with a factor $e^{-2\Phi}$ because they arise at tree level in closed superstring theory. The other fields are Ramond–Ramond fields and we concluded that those should appear in the action without a factor of $e^{-2\Phi}$.

The rank 5 field strength $C^+ = dA^+$ constructed from the rank 4 Ramond–Ramond gauge field should be self–dual, i.e. symbolically $C^+ = \varepsilon C^+$. This follows from supersymmetry (otherwise the degrees of freedom do not match) but complicates the construction of an action for Type IIB supergravity because the self–duality condition does not follow from varying a (covariant) action. There are ways around that, but we will use a simpler procedure. The action will be constructed using a standard rank 5 field strength tensor and we will keep in mind that we have to add the self–duality constraint to the equations of motion following from the IIB action by hand (as an extra equation which does not follow from varying the action). Keeping this in mind, and omitting fermions, the Type IIB supergravity action is

$$\begin{aligned} S_{IIB} = \frac{1}{\kappa_{10}^2}\int d^{10}x\sqrt{-G} \quad \{ \quad & e^{-2\Phi}(R + 4(\partial\Phi)^2 - \frac{1}{3}H^2) \\ & -2(\partial\lambda)^2 - \frac{1}{3}(H' - \lambda H)^2 - \frac{1}{60}C^{+2} \} \\ & -\frac{1}{48}\int A^+ \wedge H \wedge H' . \end{aligned} \tag{3.36}$$

In this action $H = dB$ with B the rank 2 Neveu–Schwarz field and $H' = dA$ with A being the rank 2 Ramond–Ramond field. We also added a topological term which is responsible for a modification of the Bianchi identity of C^+ which adds Chern–Simons terms in the self–dual field strength C^+. We find that $dC^+ = H \wedge H'$. The pseudo–scalar λ is usually called the axion. Notice that H, H' and the axion are connected in the term $H - \lambda H'$ appearing in the action (3.36). The form of this term is determined by supersymmetry, and we will see that it is consistent with an $SL(2,R)$ symmetry of IIB supergravity.

It is also possible to deduce Type IIB supergravity from Type IIA supergravity using T–duality. As we concluded in 1.1.4, T–duality interchanges winding and Kaluza–Klein momentum modes and changes the chirality of one of the spacetime spinors. Winding and Kaluza–Klein modes need a compact direction and couple to abelian (massless) vectors in the lower dimensional (let us say 9–dimensional) theory, so interchanging these states from the supergravity point of view just means exchanging the winding and Kaluza–Klein vectors. This generalizes to all the other fields appearing in the IIA and IIB theories after reduction to $D = 9$. T–duality in this supergravity context then means that in the reduced $D = 9$ supergravity a simple relabeling of fields should relate the IIA reduction to the IIB reduction, telling us that these two different supergravity theories in $D = 10$ are equivalent in $D = 9$ [29].

Let us again write the supergravity theory using the Einstein frame metric as defined in (3.33). This results in the following Einstein frame action

$$\begin{aligned} S^E_{IIB} = \frac{1}{\kappa^2_{10}} \int d^{10}x\sqrt{-G} \quad \{ \quad & R - 2\,[\,(\partial\Phi)^2 + e^{2\Phi}(\partial\lambda)^2\,] - \frac{1}{60} C^{+2} \\ & - \frac{1}{3}\,[\,e^{-\Phi}H^2 + e^{\Phi}(H' - \lambda H)^2\,]\ \} \\ & - \frac{1}{48} \int A^+ \wedge H \wedge H' \,. \end{aligned} \tag{3.37}$$

Defining a complex scalar field $\tau = \lambda + ie^{-\Phi}$ we can rewrite the kinetic scalar terms as $-2e^{2\Phi}|\partial\tau|^2$. This kinetic term is invariant under fractional linear transformations working on the complex scalar τ as

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}\,, \tag{3.38}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})\,. \tag{3.39}$$

In order for this $SL(2,\mathbb{R})$ matrix to have unit determinant the parameters a, b, c and d have to satisfy $ad - bc = 1$. This symmetry extends to the full action (including fermions) provided that the rank 2 Neveu–Schwarz field B and the Ramond–Ramond field A transform as an $SL(2,\mathbb{R})$ doublet

$$\begin{pmatrix} B \\ -A \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} B \\ -A \end{pmatrix}\,. \tag{3.40}$$

Although we did not discuss it, an $SO(2)$ subgroup of this $SL(2,\mathbb{R})$ also appears in the IIB supersymmetry algebra.

Let us concentrate on a special $SL(2,\mathbb{R})$ matrix having $a = d = 0$ and $b = -c = 1$ and assume $\lambda = 0$. In this special case we conclude that IIB supergravity is invariant under a discrete subgroup taking $\Phi \rightarrow -\Phi$ if we at the same time interchange the rank 2 Ramond–Ramond and Neveu–Schwarz fields. From the superstring point of view this transforms strong string coupling to weak string coupling. This is called an S–duality transformation. As opposed to strongly coupled Type IIA superstring theory, which is described by $D = 11$ supergravity in a low energy limit, in Type IIB we find that S–duality maps strongly coupled IIB theory to the *same* weakly coupled IIB theory. Therefore IIB is called self–dual and the full non–perturbative symmetry group of IIB superstring theory is conjectured to be $SL(2,\mathbb{Z})$[9] [26, 30, 31, 32].

Let us now move on to the $N=1$ supergravity theories. In the Heterotic superstring theory we found four bosonic states at the massless level, which are

$$(\Phi, G^{\mu\nu}, B^{\mu\nu}; \mathscr{A}_H^{\mu}), \tag{3.41}$$

corresponding to a dilaton scalar, the metric tensor, a Neveu–Schwarz rank 2 gauge field and a Yang–Mills vector taking values in the Lie algebra of $SO(32)$ or $E_8 \times E_8$. Constructing a $N=1$ supergravity from these states again implies the appearance of the graviton supermultiplet, having one gravitino and one Majorana–Weyl fermion being the superpartners of the graviton, the Neveu–Schwarz tensor and the dilaton.

Besides that, $N=1$ supergravity can be coupled to a Yang–Mills supermultiplet consisting of a Yang–Mills vector and a Majorana–Weyl spinor. This coupling is non–trivial and will give rise to a modification of the field strength H of the rank 2 Neveu–Schwarz tensor B to include a so–called Yang–Mills Chern–Simons term. Schematically, using differential form notation, in the abelian case this gives $H = dB + \frac{1}{2}\mathscr{A}_H \wedge \mathscr{F}_H$ (where $\mathscr{F}_H = d\mathscr{A}_H$), which means $dH = \frac{1}{2}\mathscr{F}_H \wedge \mathscr{F}_H$ which is no longer the standard Bianchi identity. Canceling anomalies requires the gauge groups to be either $SO(32)$ or $E_8 \times E_8$. For simplicity we will consider the Heterotic supergravity action without the Chern–Simons terms (and we will do the same for Type I supergravity).

The bosonic sector of the action of $N=1$ Heterotic supergravity in the string frame is

$$S_{Het} = \frac{1}{\kappa_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\Phi} \left[R + 4(\partial\Phi)^2 - \frac{1}{3}H^2 - \alpha' \, Tr \, \mathscr{F}_H^2 \right]. \tag{3.42}$$

The symbol Tr means we trace over the gauge group indices which are either in $SO(32)$ or in $E_8 \times E_8$. We note that the Yang–Mills part of this action apparently is higher order in the α' expansion and that all the fields are multiplied by a common factor of $e^{-2\Phi}$ because this action can be deduced by considering (closed)

[9] The restriction to integers $\mathbb{Z}$ is because this symmetry interchanges charged objects (BPS states) in the theory. These charges are quantized and lie on an integer charge lattice, which only maps to itself if the symmetry group is restricted to $SL(2,\mathbb{Z})$.

Heterotic string states at tree level. Performing a conformal transformation to the Einstein frame action we find

$$S^E_{Het} = \frac{1}{\kappa^2_{10}} \int d^{10}x \sqrt{-G_E} \quad \Big(\quad \left[R_E - \tfrac{1}{2}(\partial\Phi)^2\right] \qquad\qquad (3.43)$$
$$ - \frac{1}{3} e^{-\Phi} H^2 - \alpha' e^{-\frac{1}{2}\Phi} Tr \mathscr{F}^2_H \Big) .$$

Although at this point there seems to be nothing especially interesting in writing the action this way, when we compare with the (Einstein frame) Type I supergravity it will become clear why this frame is useful.

As in the Heterotic superstring, in the Type I superstring theory we found four bosonic states at the massless level, which are

$$(\Phi, G^{\mu\nu}, A^{\mu\nu}; \mathscr{A}^{\mu}_I) , \qquad (3.44)$$

corresponding to a dilaton scalar, the metric tensor, a rank 2 Ramond–Ramond gauge field and a Yang–Mills vector taking values in the Lie algebra of $SO(32)$. Essentially these are the same fields as in the case of the Heterotic superstring and therefore the supergravity action should be similar, as we also saw in the corresponding supersymmetry algebras. Differences will arise from the fact that in Type I superstrings we are dealing with open and closed superstring states which are weighted differently with $e^{\Phi} \propto g_s$ at tree level. Also the rank 2 gauge field is a Ramond–Ramond field as opposed to the rank 2 Neveu–Schwarz gauge field in Heterotic supergravity.

The gauge fields arise from the open strings and therefore are weighted with $e^{-\Phi}$ whereas the metric and the dilaton are Neveu–Schwarz closed string states weighted by $e^{-2\Phi}$. Putting all these things together we find the following bosonic sector of the Type I supergravity action in the string frame

$$S_I = \frac{1}{\kappa^2_{10}} \int d^{10}x \sqrt{-G} \left(e^{-2\Phi} \left[R + 4(\partial\Phi)^2\right] - \frac{1}{3} C^2 - \alpha' e^{-\Phi} Tr \mathscr{F}^2_I \right) . \qquad (3.45)$$

In this action $C = dA$ and $\mathscr{F}_I = d\mathscr{A}_I$ are the rank 3 Ramond–Ramond field strength and the rank 2 Yang–Mills field strength respectively.

We find the following Einstein frame action for Type I

$$S^E_I = \frac{1}{\kappa^2_{10}} \int d^{10}x \sqrt{-G_E} \quad \Big(\quad \left[R_E - \tfrac{1}{2}(\partial\Phi)^2\right] \qquad\qquad (3.46)$$
$$ - \frac{1}{3} e^{+\Phi} C^2 - \alpha' e^{+\frac{1}{2}\Phi} Tr \mathscr{F}^2_I \Big) .$$

Except for the signs of the factors e^{Φ} this is exactly the same action as in the case of the Heterotic Einstein frame action (3.43). We can perform the following field transformations to go from the Type I to the Heterotic $SO(32)$ action

$$\begin{array}{rcl} C & \rightarrow & B \\ \mathscr{F}_I & \rightarrow & \mathscr{F}_H \\ e^{\Phi} & \rightarrow & e^{-\Phi} \end{array} \tag{3.47}$$

Because $e^{\Phi} \propto g_s$ this transformation takes strong coupling in Type I theory to weak coupling in the Heterotic theory and vice versa. It suggests that strongly coupled Type I superstrings can be described by weakly coupled Heterotic $SO(32)$ superstrings and vice versa [26, 31]. This sound peculiar because the Heterotic $SO(32)$ superstring theory is a theory of closed strings only, whereas the Type I superstring consists of closed and open superstrings. However lots of evidence has been gathered supporting this S–duality conjecture, most of them relying on supersymmetry and BPS states [33, 34].

3.3 Dualities, M–theory and *p*–branes

In this section we want to make the final connection between central charges, *p*–brane soliton solutions and superstring dualities. This will lead to the introduction of M–theory, first defined as the eleven–dimensional strong coupling limit of Type IIA superstring theory, but by now believed to reproduce all string theories as different perturbative corners in its moduli space[10]. The most important argument for believing in M–theory and the dualities following from it, is the existence of (extended) BPS soliton objects appearing in the low energy effective theories [35, 36, 37, 16].

3.3.1 BPS *p*–brane solutions

We want to construct solutions to the closed string low energy supergravity equations of motion preserving some amount of supersymmetry. The charge of such a solution we want to relate to the central charge appearing in the corresponding supersymmetry algebra. In general what we need to construct such a solution is General Relativity coupled to a dilaton scalar and an antisymmetric field strength

[10] A moduli space is the space of all free parameters and/or collective coordinates in a theory.

tensor. Because we want such a solution to preserve some amount of supersymmetry, it is sufficient to use the bosonic fields. Preservation of some amount of supersymmetry just means that a subset of supersymmetry transformations will leave the bosonic solution invariant, i.e. will not introduce fermions. For a more extensive review on BPS soliton solutions we refer to [16].

We will solve for BPS solutions to the equations of motion obtained from the following general (Einstein frame) action

$$S_D = \int \mathrm{d}^D x \, \sqrt{G} \Big[R - \frac{4}{D-2} (\partial\Phi)^2 - \frac{1}{2(p+2)!} e^{a\Phi} F^2_{p+2} \Big] \,, \tag{3.48}$$

For special values of a and p this is a (bosonic) truncation of a supergravity action in D dimensions. This will cover all $D = 11$ and $D = 10$ supergravity examples by choosing the parameters p and a (and D of course) appropriately. It also covers (truncated) possible compactifications of the $D = 11$ and $D = 10$ supergravities, either on tori or on more sophisticated compact manifolds like Calabi–Yau manifolds. Any supersymmetry preserving p–brane solution will therefore also have a natural place in any of the (compactified) superstring theories and/or M–theory. The advantage of this general action is of course that it enables us to discuss many different p–brane solutions in various dimensions at once.

A p–brane will naturally couple to a rank $p+1$ gauge potential which leads to a rank $p+2$ field strength, explaining the interpretation of the parameter p appearing in this action. Every field strength can be related to a rank $D-p-2$ field strength through Hodge duality of which we already saw some examples in the supersymmetry algebras, but in the presence of a dilaton scalar and a metric the relation (3.11) needs to be modified to become

$$\tilde{F}^{\mu_1 \cdots \mu_{D-p-2}} = \frac{(-1)^{D+p-1}}{\sqrt{G}\,(p+2)!} e^{a\Phi} \varepsilon^{\nu_1 \cdots \nu_{p+2} \mu_1 \cdots \mu_{D-p-2}} F_{\nu_1 \cdots \nu_{p+2}} \,. \tag{3.49}$$

Such a pair possibly gives rise to a p–brane and a $D-p-4$–brane. A special case occurs when the rank of the field strength equals the rank of the Hodge dual field strength. Then we can apply a self–duality condition identifying the two field strengths, as we did in the Type IIB action for the rank 5 field strength. The parameter a denotes the coupling of the dilaton to the field strength in the Einstein frame.

We will need to introduce an ansatz for the different fields to solve the equations of motion. This ansatz will be based on the fact that we expect to find a p–brane solution, which has $p+1$ isometries called worldvolume directions. For one thing, this means the solution can only depend on coordinates transverse to the worldvolume directions of the p–brane (this also means we will use a static gauge, i.e. the p–brane is not moving). Let us assume the p–brane is extended in the directions $x^{\mu} = 0, 1, \ldots p$ splitting up the D–dimensional coordinates into $x^{\mu} = (x^m, y^i)$ with $m \in [0, 1, \ldots, p]$ and $i \in [p+1, \ldots D-p-1]$, then we will use the following two–block ansatz

$$\begin{aligned} \mathrm{d}s_E^2 &= H(y)^{\alpha}\, \mathrm{d}x_{p+1}^2 + H(y)^{\beta}\, \mathrm{d}y_{D-p-1}^2\,, \\ e^{\Phi} &= H(y)^{\gamma}\,, \\ F_{01\ldots pi} &= \delta\, \partial_i H(y)^{\varepsilon}\,, \end{aligned} \tag{3.50}$$

where the function $H(y)$ is harmonic on the transverse space meaning

$$\partial_i \partial^i H(y) = 0\,. \tag{3.51}$$

This is a harmonic equation in $D-p-1$ dimensions which, when $D-p-1 \neq 2$ excluding $p = (D-3)$–branes, is solved by

$$H(y) = c + \left(\frac{r_0}{r}\right)^{D-p-3}\,, \tag{3.52}$$

with c an arbitrary integration constant which for all practical purposes can be set equal to one, and r is the radius in transverse space. Varying the action and solving the equations of motion[11] for the ansatz (3.50) we obtain the following expressions for the parameters α, β, γ, δ and ε [38, 39]

$$\begin{aligned} \alpha &= \frac{-4(D-p-3)}{(D-2)\Delta} \quad , \quad \beta = \frac{4(p+1)}{(D-2)\Delta} \\ \gamma &= \frac{(D-2)a}{4\Delta} \\ \delta^2 &= \frac{4}{\Delta} \quad , \quad \varepsilon = -1\,, \end{aligned} \tag{3.53}$$

where the special parameter Δ is

$$\Delta = \frac{(D-2)a^2}{8} + 2\frac{(p+1)(D-p-3)}{(D-2)}\,. \tag{3.54}$$

[11] When the field strength tensor also satisfies a self–duality condition, the p–brane solution is dyonic and the field strength solution F requires an extra factor $\frac{1}{2}\sqrt{2}$. For the moment we also assume that $p < D-2$. The cases with $p = D-2$ are called domain–walls and will be discussed in chapter 5.

This parameter is special because it is invariant under toroidal dimensional reduction of a specific solution in the Einstein frame [38]. This makes it easy to identify p–branes in different dimensions with their higher–dimensional "parent" p–brane solutions. Another special feature of Δ is that for supersymmetry preserving solutions it should equal

$$\Delta_{susy} = \frac{4}{n} \quad \text{with} \quad n \in \mathbb{Z}^{+} \,. \tag{3.55}$$

This picks out the values of a, D and p for which the action (3.48) is a truncation of a supergravity theory. In those cases the solutions can be shown to preserve some amount of supersymmetry by solving what are called the Killing spinor equations. Basically this means that for the p–brane solution there exist special supersymmetry transformation parameters η (the Killing spinors) for which the variation of the spin half fields appearing in the supergravity theory vanish $\delta\Psi = 0$. The (spinorial) dimension these Killing spinors span determines the number of supersymmetries preserved. Compare with breaking Lorentz symmetry by introducing an extended object in the theory. This will break some, but not all of the Lorentz symmetry. The theory will still be invariant under Lorentz transformations in the worldvolume directions of the extended object. In Table 3.1 we listed a variety of half supersymmetry preserving p–brane solutions in various dimensions covered by the solution (3.53), most notably the $D = 11$ M–branes and the $D = 10$ Dp–branes are listed in there as well.

All these *extreme* BPS solutions are limits of more general *black* p–brane solutions [36, 35] breaking all of the supersymmetry, but satisfying the Bogomol'nyi bound inequality $M > Q$. These non–extreme p–brane solutions are called black because they resemble black holes which have a true horizon, classically acting as a one–way gate. Semi–classically the black p–brane will Hawking radiate and lose energy until it reaches the extreme BPS p–brane ground state solution with $M = Q$.

The shape of an extended object should be allowed to fluctuate, or equivalently the p–brane is a dynamical object. It should be possible to describe the fluctuations of a brane by a worldvolume (effective) action. This worldvolume theory should be supersymmetric, precisely because the supergravity solution is BPS. A first guess for the coupling to the spacetime *string frame* metric would be a further generalization of the Nambu–Goto string action (2.1) to p–dimensional extended objects

$$S_p = T_p \int \mathrm{d}^{p+1}\xi \, e^{-k\Phi} \sqrt{\det \partial_i X^{\mu} \partial_j X^{\nu} g_{\mu\nu}} \tag{3.56}$$

with T_p representing the tension of the p–brane. In theories with $a \neq 0$ we have

Dimension	a	Δ	Name
11	0	4	M–branes
10	$\frac{3-p}{2}$	4	Dp–branes
	–1	4	NS1–brane
	1	4	NS5–brane
6	$1-p$	2	dp–branes
5	0	4/3	m–branes
4	$2\sqrt{3}$	4	black hole
	2	2	,,
	$2/\sqrt{3}$	4/3	,,
	0	1	RN black hole

Table 3.1: The Table indicates the values of a and Δ (in the Einstein frame) for a variety of branes in diverse dimensions.

to allow for a factor $e^{-k\Phi}$ where k is determined by adding S_p to the action (3.48) rewritten in the string frame and demanding that the total action transforms homogeneously under global scale transformations of the spacetime fields (3.30) [37]. Notice that a suitably chosen conformal transformation (3.31) can make the dilaton dependence of the worldvolume action (3.56) disappear. For every p–brane there exists a naturally preferred metric frame in which the worldvolume action does not depend on the dilaton. The dilaton dependence in the string frame will determine the scaling of the effective tension with $e^{-k\Phi} \propto g_s^{-k}$,

$$\tau_p = T_p g_s^{-k} \,. \tag{3.57}$$

Besides the coupling to the spacetime metric a p–brane couples to a $p+1$–form potential. In the worldvolume theory this is described by a term

$$S_{WZ} = \int_{\Sigma_{(p+1)}} A_{M_1 \ldots M_{p+1}} \, dX^{M_1} \wedge \ldots \wedge dX^{M_{p+1}} \,. \tag{3.58}$$

In this equation Σ_{p+1} represents the $p+1$ dimensional p–brane surface and the X^M are target spacetime coordinates. This is called a Wess–Zumino term and is a straightforward generalization of a point particle coupling to a gauge vector potential. This term will give rise to a rank $p+1$ spacetime current

$$J^{M_1 \ldots M_{p+1}}(X) = \int_{\Sigma_{(p+1)}} dX^{M_1} \wedge \ldots \wedge dX^{M_{p+1}} \, \delta(\Delta X) \,, \tag{3.59}$$

where $\Delta X = X - X(\sigma)$ in the δ–function, $X(\sigma)$ representing the position of the p–brane. For the static configuration we want to discuss, this expression reduces to a charge

$$Z_{i_1 \ldots i_p} \equiv J_{0\, i_1 \ldots i_p} = \tau_p \int_{\Sigma_{(p)}} dX^{i_1} \wedge \ldots \wedge dX^{i_p} \,, \tag{3.60}$$

where we integrated over worldvolume time and the δ–function and introduced a charge density $q_p \equiv \tau_p$. The i's are spatial indices and by definition (3.59) this tensor is completely antisymmetric. These are the central charges $Z_{i_1 \ldots i_p}$ appearing in the supersymmetry algebra. Looking at (3.60) this expression is a topological winding charge, proportional to the volume of the p–brane and vanishing for configurations which are not topologically stable. So we now established a precise connection between BPS soliton solutions and central charges in the supersymmetry algebra [40].

Let us as an example concentrate on the solutions appearing in $D = 11$ supergravity [35]. So we have to choose the appropriate parameters. Eleven–dimensional supergravity (3.23) does not have a dilaton scalar. Although that means (3.48) strictly speaking is no truncation of $D = 11$ supergravity, a solution having $\Phi = 0$ should also be solution of $D = 11$ supergravity. Taking $a = 0$ will do exactly that. The rank 4 field strength can give rise to a 2–brane solution and a $D - p - 4 = 5$–brane solution. This gives $\Delta = 4$ or $n = 1$ in both cases. The solutions are respectively called the M2– and M5–brane. For the M2–brane, extending in the 012 direction we find the solution

$$\begin{aligned} ds^2_{M2} &= {H_{M2}}^{-\frac{2}{3}}\, \mathrm{d}x_3^2 + {H_{M2}}^{\frac{1}{3}}\, \mathrm{d}y_8^2 \\ F_{012\,i} &= \partial_i {H_{M2}}^{-1} \,. \end{aligned} \tag{3.61}$$

For the M5–brane we find

$$\begin{aligned} ds^2_{M5} &= {H_{M5}}^{-\frac{1}{3}}\, \mathrm{d}x_6^2 + {H_{M5}}^{\frac{2}{3}}\, \mathrm{d}y_5^2 \\ {}^\star F_{01\ldots5\,i} &= \partial_i {H_{M5}}^{-1} \,. \end{aligned} \tag{3.62}$$

Although we will not prove this here, these solutions interpolate between different vacua of eleven–dimensional supergravity, one of them being flat Minkowski space at infinity. This is in fact a generic feature of most p–brane solutions [41, 42] and we will make this more precise in chapter 5. They only have one free physical parameter appearing in the harmonic function (3.52) which should represent both the mass density m and the charge density q of the p–brane[12]. The rela-

[12] Because the p–branes are infinitely extended, their total mass and charge must be infinite. Only the mass and charge density are finite.

tions between these charge densities satisfies the Bogomol'nyi bound we deduced from the eleven–dimensional supersymmetry algebra. For the M2– and M5–brane central charges we have

$$
\begin{aligned}
Z_{12} &= q_{M2} \int dX_1 \wedge dX_2 \\
Y_{1\ldots5} &= q_{M5} \int dX_1 \wedge \ldots \wedge dX_5 \,. \qquad (3.63)
\end{aligned}
$$

Let us be brief on the p–brane solutions in the $D = 10$ string low energy effective supergravities. Most of them can be found in Table 3.1. On plugging in the appropriate parameters which can be read off from the Einstein frame supergravity actions, it must be clear that 2 p–brane solutions exist for every field strength appearing in the supergravity action[13]. All solutions again have $\Delta = 4$. For the Heterotic supergravity (3.43) we find a Neveu–Schwarz string and a Neveu–Schwarz five–brane (we will not consider non–abelian solitons). In Type I supergravity we find a Ramond–Ramond string and a Ramond–Ramond five–brane. The Neveu–Schwarz string and five–brane also appear in Type IIA and Type IIB, the Neveu–Schwarz string being the BPS winding mode of a closed Heterotic, IIA or IIB superstring and the five–brane being the Hodge dual. In IIA we find all the even Ramond–Ramond Dp–branes and in IIB we find all the odd Ramond–Ramond Dp–branes. Masses and charges of Ramond–Ramond BPS solutions can be shown to scale as $1/g_s$ or $k = 1$ (3.57) (as they should if they are Dp–branes). For the Neveu–Schwarz five–branes we find a $1/(g_s)^2$ or $k = 2$ (3.57) behavior for the masses, which is what we expect for a true soliton solution. Because we already said that the Neveu–Schwarz strings correspond to the BPS winding modes of fundamental superstrings, it should be no surprise that the masses and charges of these object do not scale with g_s or $k = 0$ in (3.57).

We still seem to be missing some BPS soliton solutions, for example we did not find a massless BPS solution and in $D = 11$ we also did not find a $p = 6$–brane. These solutions do exist but they do not couple to a gauge potential and are described by purely gravitational solutions. Therefore they exist in any (super)gravity theory. The massless solution is nothing but a gravitational Brinkman wave. The other (massive and topologically stable) gravitational BPS solution is called a Kaluza–Klein monopole and is described by an off–diagonal Taub–Nut metric in only four spacelike dimensions (which we would consider transverse

[13]In the case of a non–vanishing dilaton coupling a the Hodge dual field strength will appear in the action with a dilaton coupling $-a$ because of (3.49).

to the Kaluza–Klein brane) of which one needs to be a special compact isometry direction[14]. For the wave we find the off–diagonal metric

$$\begin{aligned} ds_W^2 = \quad & (H(t+z,y)-2)\mathrm{d}t^2 + H(t+z,y)\mathrm{d}z^2 \\ & +2(H(t+z,y)-1)\mathrm{d}t\mathrm{d}z + \mathrm{d}y_{D-2}^2 \end{aligned} \tag{3.64}$$

and for the Kaluza–Klein monopole the off–diagonal metric is

$$ds_{KK}^2 = \mathrm{d}x_{D-4}^2 + H(y)^{-1}(\mathrm{d}z + A_i \mathrm{d}y_i)^2 + H(y)\mathrm{d}y_3^2 . \tag{3.65}$$

The wave corresponds to momentum traveling in the z direction and when z is a compact isometry direction it corresponds to a BPS Kaluza–Klein mode. The Kaluza–Klein monopole corresponds to a $p = D-5$–brane, because it clearly has p spatial isometry directions, the z isometry being special ensuring the topological stability of the solution. In $D = 11$ the Kaluza–Klein monopole therefore represents the 6–brane found in the $D = 11$ supersymmetry algebra [43]. The worldvolume theory of a Kaluza–Klein monopole is described by so–called gauged sigma–models, for more details we refer to [44].

3.3.2 Dualities and effective worldvolume theories

All the conjectured dualities between superstring theories [26, 5, 45, 46] can now be tested by studying the BPS soliton spectrum, which should be the same for two dual supersymmetric theories. Because we can trust the masses and charges of these BPS objects at weak and at strong coupling, we can analyze which of the states become light and can possibly be used as fundamental objects in a dual description allowing for a low energy effective supergravity approximation.

Let us first come back to the low energy effective worldvolume theories that should describe the small fluctuations of the p–brane solutions. Generically these theories are effective by definition because they are deduced from a solution to a low energy effective supergravity. We already mentioned that there is an exception. For N Dp–branes we do know the microscopic description in terms of open superstring theory with Dirichlet boundary conditions. As we saw in section 1.2.5, the states on N Dp–branes, at the massless level, are non–abelian $U(N)$ fotons and scalars whose fluctuations deform the shape of N Dp–branes. This means the low energy effective action can only be a supersymmetric $U(N)$ Yang–Mills

[14] This means 4 Euclidean dimensions is the smallest dimension in which such a solution exists and in that case it is called a Taub–Nut instanton.

theory, which has 16 real supersymmetries. This holds for all the Dp–branes and in fact all $p+1$–dimensional worldvolume theories are related to the $D=10$ supersymmetric $U(N)$ Yang–Mills theory by reduction. In $D=10$ the bosonic part is just

$$S_{SYM} = \frac{1}{g_{YM}^2} \int d^{10}x \, \mathrm{Tr} F^2 \, . \tag{3.66}$$

This theory would be describing the worldvolume theory of N D9–branes, which has no scalars denoting its position in the $D=10$ target spacetime because the brane is spacetime filling. If we reduce this theory on a torus T^n we obtain a theory with $U(N)$ vectors and n adjoint scalars with a commutator scalar potential, describing the degrees of freedom of N D$(9-n)$–branes moving in a ten–dimensional target spacetime. When $n=9$ we arrive at a supersymmetric quantum mechanics model describing the dynamics of N D0–branes in Type IIA superstring theory. We will say a lot more about that model in the next chapter.

Of course this is a lowest order in α' approximation. In fact we can do a lot better and the full action is known to be described by what is called a Dirac–Born–Infeld action [9]

$$S_{Dp} = T_{Dp} \int d^{p+1}x e^{-\Phi} \sqrt{\det(G_{\mu\nu} - \alpha' F_{\mu\nu})} \, . \tag{3.67}$$

Expanding the squareroot of the determinant we will arrive at the supersymmetric $U(1)$ Yang–Mills action. This action is hard to generalize to include non–abelian gauge groups. Proposals have been made, but so far it is unclear if those proposals are correct. Whenever we will need Dp–brane worldvolume theories in this thesis we will use the low energy effective supersymmetric Yang–Mills theory instead of the Born–Infeld theory.

Using all this information on BPS states in general and Dp–branes in particular we now want to re–analyze the conjectured dualities.

- Heterotic $SO(32)$ – Type I $SO(32)$ S–duality
  At strong coupling in Type I the BPS D1 string becomes light and in fact it can be shown to map perfectly to the fundamental closed superstring of Heterotic superstring theory [33]. All the other states also map perfectly giving strong evidence for this conjecture [34].

- Type IIB $SL(2,R)$ self–duality
  The same phenomenon occurs, the D1 string maps to the fundamental Type IIB superstring at strong coupling and vice versa. The $SL(2,R)$ symmetry

leads us to consider (p,q) strings in the theory representing a bound state of p Neveu–Schwarz strings and q Ramond-Ramond D1 strings [32]. The same mapping is at work for all the other BPS branes.

- Type IIA and M–theory
  At strong coupling of type IIA we concluded that an extra eleventh dimension is developed. All the BPS states of type IIA should have a counterpart in M–theory, where M–theory is defined as the eleven–dimensional fundamental theory giving Type IIA superstrings in $D = 10$ if the eleventh dimension is very small. Using the relation between eleven– dimensional supergravity and Type IIA supergravity we conclude that the M2–brane gives rise to the NS string and the D2–brane. The M5–brane reduces to the D4– and the NS5–brane. The $D = 11$ KK–monopole reduces to the $D = 10$ KK–monopole and to the D6–brane if reduced over the special compact isometry direction z in the $D = 11$ KK–monopole. Finally, but very importantly, the Kaluza–Klein momentum modes (or the gravitational waves having their momentum in the compact direction) reduce to the D0–branes[15] [47].

  One attempt to define M–theory made use of the fact that the theory contains M2–branes [27]. The idea was to try to quantize the supermembrane [48], just as we quantized the superstring. However, the standard quantized supermembrane has a continuous spectrum and no natural dimensionless coupling constant to set up a perturbative expansion, making a (standard) elementary particle interpretation problematic [49]. We will say a little bit more about this in the next chapter. The states becoming light at strong coupling are the D0–branes and therefore an obvious idea is to use D0–branes as the fundamental degrees of freedom describing M–theory. That theory is named Matrix theory [50] and is the subject of the next chapter.

- M–theory dualities
  Because Type IIB superstring theory can be obtained through Type IIA superstring theory using T–duality we can also obtain Type IIB theory from M–theory. T–duality requires another (besides the eleventh direction) compact direction and therefore should be related to M–theory on a two–torus T^2. Shrinking one cycle of the T^2 will give Type IIA superstring theory. Using T–duality, shrinking both cycles of the T^2 should give Type IIB superstring theory in $D = 10$ [6].

[15] The gravitational waves in $D = 10$ are just the gravitational waves in $D = 11$ not having their momentum in the compact direction.

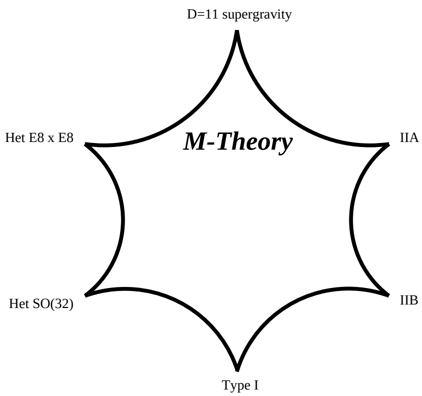

Figure 3.1: The moduli space of M–theory with its different perturbative corners.

We can also relate M–theory to the $N = 1$ superstring theories [51, 52]. This can be done by considering M–theory on an interval $S^1/\mathbb{Z}_2$ [52], this will break half of the supersymmetries giving a $N = 1$ theory in $D = 10$. This theory is related to the strong coupling limit of Heterotic $E_8 \times E_8$, where the coupling constant g_s is given by the length of the interval. Because Heterotic $E_8 \times E_8$ is T–dual to Heterotic $SO(32)$ (if we break the gauge group to $SO(16) \times SO(16)$), and Heterotic $SO(32)$ is S–dual to Type I superstring theory, we related all consistent superstring theories to M–theory. A situation pictorially displayed in Figure 3.1. If we consider other compact manifolds many more examples of these dualities can be found [26, 53, 54, 14]. One example is the IIA superstring theory compactified on a four–dimensional $K3$ Calabi–Yau manifold (breaking half of the supersymmetries), which is conjectured to be S–dual to Heterotic superstring theory on a four–dimensional torus T^4. Altogether this is significant evidence for the existence of one non–perturbative structure underlying all superstring theories called M–theory.

Chapter 4

Matrix theory

This chapter introduces Matrix theory as the candidate non–perturbative description of (eleven–dimensional) M–theory. After a discussion of the basic set up and the microscopic Matrix theory degrees of freedom, we will show how extended BPS objects can be constructed in Matrix theory. Using these extended BPS objects as our basic building blocks we will construct intersecting or overlapping BPS states which generically break more supersymmetries. A basic requirement for any theory claiming to describe M–theory would be that it can reproduce all BPS states appearing in the eleven–dimensional supersymmetry algebra. As a check of Matrix Theory we will therefore show how to identify Matrix theory BPS states with BPS states appearing in the eleven–dimensional supersymmetry algebra. This chapter reports on work published in [55]. The Matrix theory subject started with the paper by Banks, Fischler, Schenker and Susskind [50] and we also refer to the following review papers [56, 57, 58, 59].

4.1 The Matrix model conjecture

In the previous chapter we introduced M–theory as the eleven–dimensional non–perturbative theory underlying all consistent superstring theories. The low energy limit of this theory should give eleven–dimensional supergravity and upon compactification on a small circle M–theory should give the IIA superstring. Together with the eleven–dimensional supersymmetry algebra and its BPS spectrum, basically this is all we know of M–theory. At this point there are two ways to proceed in trying to define a microscopic description of M–theory (in a suitable limit). In the end both attempts turn out to be connected and will lead to the same M–theory description, the Matrix model.

The first approach is to use the fact that M–theory contains supermembranes [48], which can be shown to be related to the IIA superstring by a reduction over the compact circle of the 3–dimensional supermembrane worldvolume theory. We obtained the fundamental degrees of freedom of Type IIA theory by quantizing the IIA superstring worldvolume theory. The relation between supermembranes, M–theory and IIA superstrings suggests that we should try to quantize the supermembrane to obtain the fundamental degrees of freedom of M–theory.

Quantization of the supermembrane is however problematic. In one example of a supermembrane with the spatial topology of a sphere S^2 in a light–cone frame, a regularisation can be made based on the $SU(2)$ rotational symmetry of the S^2. The $SU(2)$ algebra can be used to associate coordinate functions on S^2 with matrices generating the $SU(2)$ algebra in the N–dimensional representation. This means integrals are replaced by traces and it can be verified that in the large N limit the matrix regularisation reproduces the continuum quantities. The matrix regularized Hamiltonian can be considered as a classical supersymmetric particle theory (all spatial integrals are replaced by traces over $N \times N$ matrices) with a finite number of degrees of freedom and the quantization of such a system is rather straightforward (solving the quantum theory is another issue). For the details of this construction we refer to [49]. This Hamiltonian turns out to be equivalent to the Hamiltonian describing N Type IIA D0–branes in a light–cone frame. This in fact is the other approach towards a description of M–theory, to which we will now turn our attention.

4.1.1 D0–branes and DLCQ M–theory

In section 2.2.5 we already hinted at a possible formulation of the microscopic M–theory degrees of freedom in terms of D0–branes. This was based on the fact that these are the states that become light at strong Type IIA coupling. Let us pursue this idea in a more precise manner. This discussion will mainly be based on [60, 61, 62] and the review paper [58].

Let us consider M–theory in a light–cone frame by choosing coordinates x_- and x_+ defined as

$$x_- = \frac{1}{\sqrt{2}}(x^{11} - t) \quad , \quad x_+ = \frac{1}{\sqrt{2}}(x^{11} + t)\,. \tag{4.1}$$

We choose the light–cone coordinate x_+ to play the role of time and consider the coordinate x_- to be compactified on a circle of radius R. So we identify

$$x_- \sim x_- + R \tag{4.2}$$

This compactification restricts the momentum P_+ to be quantized in units of N/R. Quantization of a theory in this special frame is called Discrete Light–Cone Quantization (DLCQ) and has some special features. Important for our purposes will be that the allowed values for the momentum P_+ will be strictly positive and that the theory, for any N, will have Galilean invariance[1] in the other transversal directions X^i.

Consider on the other hand M–theory compactified on a spatial circle R_s,

$$x^{11'} \sim x^{11'} + R_s\,. \tag{4.3}$$

This system can be related to the compactified light–cone coordinate x_- by performing a Lorentz boost

$$\begin{pmatrix} t' \\ x^{11'} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & \frac{\beta}{\sqrt{1-\beta^2}} \\ \frac{\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} \end{pmatrix} \begin{pmatrix} t \\ x^{11} \end{pmatrix}, \tag{4.4}$$

with

$$\beta = \frac{1}{\sqrt{1+\frac{2R_s^2}{R^2}}} \tag{4.5}$$

and x and t are identified as follows

$$\begin{pmatrix} -t \\ x^{11} \end{pmatrix} \sim \begin{pmatrix} -t + \frac{R}{\sqrt{2}} \\ x^{11} + \sqrt{\frac{R^2}{2} + R_s^2} \end{pmatrix}. \tag{4.6}$$

In the limit $R_s \to 0$, using the definition of the light–cone coordinates (4.1), we obtain x_+ and x_-, with x_- compactified on a circle of radius R. The conclusion is that we can interpret DLCQ M–theory with light–like radius R as a limit $R_s \to 0$ of Lorentz boosted M–theory compactified on a spacelike circle with radius R_s.

By definition we know that M–theory compactified on a spacelike circle is Type IIA superstring theory. That theory has two parameters which are of importance in this discussion, the string length $l_s \equiv \sqrt{\alpha'}$ and the string coupling g_s. Using (3.25) we can relate these quantities to $l_{11} \equiv l_p^{(11)}$ and the radius R_s

$$g_s = \left(\frac{R_s}{l_{11}}\right)^{3/2} \quad , \quad l_s^2 = \frac{l_{11}^3}{R_s}\,. \tag{4.7}$$

[1] These properties are general features of DLCQ theories. More details can be found in [60, 63].

Clearly the limit $R_s \to 0$ means that $g_s \to 0$, so higher genus corrections vanish in this limit. The string length tends to $l_s \to \infty$, which is a limit of vanishing string tension and at first sight does not allow us to use a low energy effective description. However, let us consider the energy of the states we are interested in when taking the limit $R_s \to 0$. Let us say we want to describe the behavior of a state with light–cone energy P_- and compact light–cone momentum $P_+ = N/R$. The momentum in the spacelike compactified theory is going to be $P' = N/R_s$, which means the total energy in the spacelike compactification is

$$E' = \frac{N}{R_s} + \Delta E \,. \tag{4.8}$$

The energy ΔE and P_- can be related by performing the boost (4.4) and we find (in the limit of small R_s)

$$\Delta E \approx \frac{R_s}{R} P_- \,. \tag{4.9}$$

This means that if we want to study states with finite light–cone energy P_-, we have to study states with vanishing ΔE in spacelike compactified M–theory in the limit $R_s \to 0$. In fact when we compute the ratio of ΔE and the string scale $m_s \equiv 1/l_s$ we find

$$\Delta E \, l_s = \frac{P_-}{R} R_s \, l_s = \frac{P_-}{R} \sqrt{R_s \, l_{11}^3} \,. \tag{4.10}$$

This ratio vanishes in the limit $R_s \to 0$ and therefore the energy scales of interest in the spacelike compactified M–theory are much smaller than the string mass scale m_s. In fact this equation is the square root of condition (3.21), telling us that we can safely use a low energy effective description. To make this more transparent it is useful to perform a change of units in the spacelike compactified theory, introducing a new Planck length $\tilde{l}_{11}$ defined by

$$\Delta E \, \tilde{l}_{11} = P_- \frac{R_s \, l_{11}^2}{R \, \tilde{l}_{11}} \,. \tag{4.11}$$

This is just (4.9) but with the units, which we now assume to be different for the DLCQ and spacelike compactified M–theory, explicitly inserted. All quantities of the DLCQ M–theory are measured with respect to the "old" l_{11} and all quantities of the spacelike compactified M–theory are measured with respect to the new $\tilde{l}_{11}$. We define the new Planck length $\tilde{l}_{11}$ in such a way that ΔE is independent of R_s and equal to P_-. Looking at (4.11) this means

$$\frac{R_s}{\tilde{l}_{11}^2} = \frac{R}{l_{11}^2} \,. \tag{4.12}$$

In the limit $R_s \to 0$ we also take $\tilde{l}_{11} \to 0$ to keep the ratio in (4.12) fixed. All the transverse coordinates scales as $\tilde{x}^i/\tilde{l}_{11} = x^i/l_{11}$. The new Planck length also gives a new string coupling $\tilde{g}_s$ and string length $\tilde{l}_s$ and we now find in the limit

$$\tilde{g}_s = \left(\frac{R_s}{\tilde{l}_{11}}\right)^{3/2} \to 0 \quad , \quad \tilde{l}_s^2 = \frac{\tilde{l}_{11}^3}{R_s} \to 0\,. \tag{4.13}$$

Using the new unit we succeeded in keeping the energies of interest fixed and the string theory parameters both tend to zero in the limit, enabling us to use a low energy effective description. The surprising thing is that this low energy effective theory can be used to describe DLCQ M–theory with finite parameters l_{11} and R.

The low energy effective theory was conjectured to be the one describing the dynamics of N D0–branes in the limit of the boosted spacelike circle. All other dynamics of the closed Type IIA superstring theory can be neglected for the following reason. The dynamics of the D0–branes is governed by the Yang–Mills coupling constant g_{YM}^2, which is related to the string theory parameters in the following way

$$g_{YM}^2 = c_p\, \tilde{g}_s\, \tilde{l}_s^{-3}\,, \tag{4.14}$$

where c_p is a fixed constant. We refer to chapter 5 on how to deduce this relation. Using the expressions for $\tilde{l}_s$ and $\tilde{g}_s$ in (4.13) it is not very hard to show that the Yang–Mills coupling constant is fixed in the limit (4.13), giving rise to finite D0–brane dynamics. The other dynamics in Type IIA superstring theory is governed by the gravitational constant (3.29), which in the limit (4.13) becomes $\kappa_{10} \propto \tilde{l}_s^4\, \tilde{g}_s \to 0$ and therefore can be neglected. Such decoupling limits will be discussed in more detail in the next chapter.

The Matrix theory conjecture then is that *DLCQ M–theory defined by the finite parameters N,R and l_{11} can be described by the low energy dynamics of N D0–branes with finite coupling parameter g_{YM}*. Uncompactified M–theory can be obtained by taking N and R to infinity keeping $P_+ = N/R$ fixed in DLCQ M–theory (it has to be accompanied by taking N to infinity as well because on a light–like circle the value of R can be changed by a boost). This clearly means that $P' = N/R_s$ goes to infinity even before taking the limit $R_s \to 0$, which is called the Infinite Momentum Frame (IMF). Initially this was the conjecture put forward by Banks, Fischler, Schenker and Susskind relating M–theory in the IMF to the quantum mechanical system describing the dynamics of $N \to \infty$ D0–branes [50].

The quantum mechanical model describing N D0–branes can be obtained by reducing the $D = 10$ Super Yang–Mills multiplet to $D = 0 + 1$ dimensions, as explained in section 2.3.2 . The limit $R_s \to 0$ ensures that we only need to consider the non–relativistic theory of N D0–branes (relativistic corrections are of order $\mathscr{O}(R_s/R)$ and therefore vanish in the limit $R_s \to 0$). The supersymmetric Lagrangian is (using static gauge)

$$\mathscr{L}_{IIA} = \frac{1}{2\tilde{g}_s \tilde{l}_s} \mathrm{Tr} \left\{ \dot{X}^a \dot{X}^a + \frac{1}{2\tilde{l}_s^4} \left[X^a, X^b \right]^2 + \frac{1}{\tilde{l}_s^2} \theta^T \left(i\dot{\theta} - \frac{1}{\tilde{l}_s^2} \Gamma_a [X^a, \theta] \right) \right\} . \quad (4.15)$$

In this Lagrangian the X^a are $N \times N$ Hermitian matrices with a running over the 9 spatial dimensions, the Hermitian $N \times N$ matrix θ is a 16–component Majorana spinor of $SO(9)$ and Γ_a are spatial Dirac matrices satisfying

$$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab} . \quad (4.16)$$

The dot represents a worldline time derivative and we have fixed the gauge to $A_0 = 0$. The matrices X have dimensions $[l]$ and the spinors have dimensions $[l]^{3/2}$. In order for the particle action $S = \int d\tau \mathscr{L}$ to be dimensionless the worldtime coordinate τ has dimension $[l]$. Before the BFSS conjecture the Lagrangian (4.15) was used to study N slowly moving D0–branes in Type IIA string theory [64, 65, 66].

We can replace all string theory parameters $\tilde{l}_s$ and $\tilde{g}_s$ by DLCQ M–theory parameters R and l_{11}, using (4.13) and (4.12). We also rescale the fields X and θ with $\tilde{l}_{11}/l_{11}$ according to the appropriate length dimension. We will be left with a quantum mechanics model with only finite M–theory parameters

$$\mathscr{L}_M = \tfrac{1}{2} \mathrm{Tr} \left\{ \frac{1}{R} \dot{X}^a \dot{X}^a + \frac{R}{2 l_{11}^6} \left[X^a, X^b \right]^2 + \frac{1}{l_{11}^3} \theta^T \left(i\dot{\theta} - \frac{R}{l_{11}^3} \Gamma_a [X^a, \theta] \right) \right\} . \quad (4.17)$$

It is also useful to construct the quantum mechanics Hamiltonian which is given by

$$\mathscr{H}_M = R \, \mathrm{Tr} \left\{ \tfrac{1}{2} P^2 - \frac{1}{4 l_{11}^6} \left[X^a, X^b \right]^2 + \frac{1}{2 l_{11}^6} \theta^T \Gamma_a [X^a, \theta] \right\} , \quad (4.18)$$

where we extracted an overall factor R and introduced the (standard) canonical momentum matrix

$$P^a \equiv \frac{d\mathscr{L}}{\partial \dot{X}^a} = \frac{\dot{X}^a}{R} . \quad (4.19)$$

The Matrix model is invariant under the following supersymmetry transformations

$$\begin{aligned} \delta X^a &= -2\bar{\varepsilon} \Gamma^a \theta , \\ \delta \theta &= \frac{1}{2} \left\{ \dot{X}^a \Gamma_a + \frac{i}{2 l_{11}^2} [X^a, X^b] \Gamma_{ab} \right\} \varepsilon + \tilde{\varepsilon} , \end{aligned} \quad (4.20)$$

where ε and $\tilde{\varepsilon}$ are two independent 16–dimensional supersymmetry parameters ($SO(9)$ spinors). As mentioned in 2.3.2 this model preserves 16 supersymmetries, which agrees with the number of preserved supersymmetries on the DLCQ M–theory side, where half of the initial supersymmetries are broken by the light–cone frame. The supersymmetry algebra can also be constructed and is given in [49, 67].

The most important feature of this quantum mechanics model is that its fields are $N \times N$ matrices which are invariant under $U(N)$ transformations. It is known as the Matrix model of M–theory. Looking at the potential in this model (4.17) we notice that there exist flat directions when the matrices X commute, which means the matrices only have diagonal entries. In that case the X can be given the usual interpretation of denoting the classical position of every D0 particle and there are no interactions. When two or more D0 particles approach each other, open strings will stretch between them giving rise to interactions. In the quantum mechanics model these interactions are represented by the quantum fluctuations of off-diagonal entries in the matrices X. These are suppressed by the potential, but when the D0 particles come very close together these fluctuations can become very large. In that case the matrix X can no longer be interpreted as representing the "classical geometric" position and we are in a non–commutative geometry regime.

Although the basic constituents of this model are the N D0 particles, we could consider block diagonal matrices. Every block has $N_n \times N_n$ entries and $\sum N_n = N$. If the classical distance represented by the difference between the averages of the N_n diagonal elements in every block is large, these blocks will move independently of each other and in fact represent different (asymptotic) states. In this precise sense the model can be interpreted as a second quantized theory (remember that perturbative string theory only described the dynamics of a single string). It is conjectured that these block diagonal matrices represent supergravitons, each carrying an integer N_n times the minimal light–cone momentum $1/R$. That also means that for every N there exists a threshold BPS bound state in the theory, in order to match the number of excitations of such a state with that of a supergraviton. The existence of such a state was proven for $N = 2$ in [68], and more general evidence (including different gauge groups) for the existence of those states was presented in [69]. The scattering of 2 and 3 D0 particles indeed reproduces $D = 11$ supergraviton results, giving evidence for the conjecture [70, 71, 72, 73].

If the Matrix model describes DLCQ M–theory, the Matrix model on a compact space C should describe DLCQ M–theory on the same compact space C. Compactifying Matrix theory is not as straightforward as in ordinary field theory, because of the open string degrees of freedom which should be treated carefully on compact spaces [74]. In fact we already saw how to deal with toroidal compactifications of such open string models when we considered D–brane T–duality. Compactifying one of the transverse directions of a Dp–brane on a circle of radius R_c was equivalent to wrapping one leg of a D$(p+1)$–brane on a T–dual circle of radius $1/R_c$. In the limit $R_c \to 0$ we are left with the (unwrapped) theory of the D$(p+1)$–brane. Using this we conclude that the theory describing DLCQ M–theory on a circle R_c, which by definition is DLCQ IIA superstring theory, is the 2–dimensional $U(N)$ Yang–Mills worldvolume theory of N D1–branes [75]. When we consider higher–dimensional tori T_p we will obtain higher dimensional $U(N)$ Yang–Mills theories describing DLCQ M–theory on T_p[2]. Strangely enough DLCQ M–theory on higher–dimensional tori (thus being a *lower–dimensional* effective theory) acquires more degrees of freedom (the dimension of the corresponding Matrix theory *increases*).

4.2 BPS objects in Matrix theory

The Matrix model Lagrangian (4.17) is the same as the matrix regularized supermembrane model. This can be considered additional evidence for the correctness of the Matrix model. This connection also clarified the initial problems people had in interpreting the supermembrane spectrum, which could be shown to be continuous. The absence of a discrete spectrum was considered a serious drawback because the hope was that states in the supermembrane spectrum could be put in a one–to–one correspondence with elementary particle states (like in superstring theory), which clearly is impossible if the spectrum is continuous. We now understand that the supermembrane theory, or the Matrix model, should be understood as a second–quantized theory, describing multiple particle states explaining the appearance of a continuous spectrum. The connection with the regularized supermembrane model also suggests that a stable supermembrane should be considered a condensate of $N \to \infty$ D0–branes [47]. The same idea can be used to construct higher–dimensional solitons as we will see.

[2]For $p > 3$ we are in trouble because the corresponding field theories are non–renormalizable, meaning that new degrees of freedom appear at some scale. This means that the theories describing DLCQ M–theory on T_p for $p > 3$ can not be described by the $U(N)$ Yang–Mills field theory at all scales and new ingredients are needed [61, 62].

4.2.1 Basic Matrix theory extended objects

Extended objects in Matrix theory have to be condensates of D0–branes. These objects, like the M2–brane and the M5–brane, can only be stable when they are infinitely extended[3]. This means that we can only construct stable BPS extended objects in Matrix theory in the $N \to \infty$ limit. This is also clear from the connection with the supermembrane model, only in the limit $N \to \infty$ can we replace the matrices by smooth functions on a membrane surface of a particular topology. In fact every spatial topology has its own matrix regularisation, basically governed by the Poisson bracket of smooth functions which we replace by a matrix commutation relation [49].

How it is possible to construct membranes out of D0–branes can be most easily seen in the context of Type IIA superstring theory. In Type IIA theory this would correspond to the construction of a IIA D2–brane out of the degrees of freedom describing a system of N D0–branes. An important observation is that D0–branes can be identified with the magnetic flux of the gauge field living on a set of N D2–branes [76]. Consider N D2–branes wrapped on a 2–torus T^2 with k units of magnetic flux on its worldvolume

$$\int_{T^2} F = 2\pi k, \tag{4.21}$$

which can be identified with k D0–branes living on the worldvolume of N D2–branes. Performing a T–duality transformation in both directions of the torus T^2 will give us a (worldvolume) theory of N D0–branes. The magnetic flux condition on the worldvolume of the D2–brane (4.21) is translated into a commutator condition on the D0–brane matrices in the dual torus $\tilde{T}^2$ directions

$$\mathrm{Tr}[X^1, X^2] = \frac{iAk}{2\pi}. \tag{4.22}$$

In this expression A denotes the two–dimensional area of the T–dual torus. Of course the trace of a commutator of matrices can only be non–zero when the matrices are infinite and because of the compactness of the torus and the details of open string T–duality they are indeed infinite [74]. The parameter k should now be interpreted as the number of D2–brane charges. This again follows from T–duality, the k D0–brane charges on the worldvolume of N D2–branes under T–duality are transformed into k D2–brane charges. This shows that it is possible to create membrane charges in a theory of just D0–particles.

[3]Or they have to be wrapped onto a compact surface, in which case we should consider a compactification of the Matrix model (4.17).

So in the context of Matrix theory we should identify non–trivial commutators of ($N \to \infty$) matrices with membrane charges in DLCQ M-theory. There are some differences with the Type IIA situation described above. We would like to describe infinitely extended membranes, so the membrane is not wrapping any compact directions. The only way in which we can then make the matrices X infinite is by taking $N \to \infty$. These subtleties aside, a single infinitely extended Matrix theory M2–brane with one unit of charge density should correspond to the following commutator

$$\mathrm{Tr}\,[X^a, X^b] = iA_2\,, \tag{4.23}$$

where A_2 is the surface area of the Matrix M2–brane, which should be infinite in the limit $N \to \infty$. We will assume the area of a single M2–brane to scale linearly with the number N of D0–particles

$$A_2 = N\varepsilon_2\,, \tag{4.24}$$

which defines a minimal area ε_2. The commutator defines one unit of total membrane (central) charge Z^{ab} with the standard dimensions of mass through

$$Z^{ab} = -\frac{1}{l_{11}^3}\, i\,\mathrm{Tr}\,[X^a, X^b]\,. \tag{4.25}$$

The unit charge density q_{M2} with the appropriate dimensions obviously is

$$q_{M2} = \frac{1}{l_{11}^3}\,. \tag{4.26}$$

Although the equation (4.23) can only be satisfied when $N \to \infty$, a useful formal solution exists by taking matrices X satisfying

$$[X^a, X^b]_{ij} = i\,\varepsilon_2\,\delta_{ij}\,, \tag{4.27}$$

where we inserted the matrix indices and used the minimal area defined through (4.24). After taking the trace we will obtain a factor of N which nicely represents the scaling of the total area of the single M2–brane.

Higher dimensional (infinitely extended) branes can be constructed by considering tensor products of the basic non–trivial commutator. In this sense higher dimensional objects like M5–branes, Kaluza–Klein monopoles and hypothetical M9–branes are build out of all the lower dimensional brane charges, which ultimately can be related to the D0 particles. Every basic M2 commutator is accompanied by a scale factor $1/l_{11}^3$, which means that to relate the tensor products to total charges (which have dimension of mass) we need to multiply with another length scale which has to be the radius R of the DLCQ M–theory. This enables us to define the following higher–dimensional unit charges in Matrix theory

$$
\begin{aligned}
Z_{M5}^{a_1\cdots a_4} \equiv \frac{A_5}{l_{11}^6} &= \frac{-R}{l_{11}^6}\mathrm{Tr}X^{[a_1}X^{a_2}X^{a_3}X^{a_4]}\,, \\
Z_{KK}^{a_1\cdots a_6} \equiv \frac{A_6}{l_{11}^7} &= \frac{iR^2}{l_{11}^9}\mathrm{Tr}X^{[a_1}X^{a_2}\cdots X^{a_6]}\,, \\
Z_{M9}^{a_1\cdots a_8} \equiv \frac{A_7}{l_{11}^{10}} &= \frac{R^3}{l_{11}^{12}}\mathrm{Tr}X^{[a_1}X^{a_2}\cdots X^{a_8]}\,.
\end{aligned} \tag{4.28}
$$

It is not entirely obvious how these charges can be related to the higher–dimensional branes in M–theory. This is basically because the number of indices only refer to the “transverse” dimensions (from the light–cone perspective). We can however make the following identifications. The Z_{M5} is identified with the charge of an M5–brane wrapped around the compactified light–cone dimension of DLCQ M–theory. This also explains the appearance of the factor of R in the definition of Z_{M5}, the total charge (and tension) of this wrapped M5–brane has to be proportional to the radius R of the wrapped light–cone dimension. The Z_{KK} is identified with a Kaluza–Klein monopole with its special Taub–Nut direction in the compact light–cone direction. The Kaluza–Klein charge and tension have indeed been found to scale with the squared radius of the Taub–Nut direction [43, 44], which explains the factor of R^2. The Z_{M9} will be identified with the M9–brane wrapped around the compact light–cone direction. Although the M9–brane is not understood very well, it is known that the tension and charge of such an object scale with R^3 [20, 77], in agreement with the definition of Z_{M9} in (4.28).

Matrix configurations giving rise to these higher–dimensional charges can be constructed by considering tensor products of (4.27). For example, to construct a non–vanishing wrapped M5–brane charge extended in the directions $X^1 \dots X^4$ we can take

$$
\begin{aligned}
[X^1, X^2]_{ij} &= i\,\varepsilon_2\,\delta_{ij}\,, \\
[X^3, X^4]_{ij} &= i\,\varepsilon_2\,\delta_{ij}\,.
\end{aligned} \tag{4.29}
$$

Besides M5–brane charge this configuration will carry M2–brane charges in the directions X^1, X^2 and X^3, X^4 as well. The different charges in this configuration are

$$
Z^{12} = \frac{N\varepsilon_2}{l_{11}^3}
$$

$$\begin{aligned} Z^{34} &= \frac{N\varepsilon_2}{l_{11}^3} \\ Z_{M5}^{1234} &= \frac{RA_2^2}{l_{11}^6} \end{aligned} \tag{4.30}$$

We would like this configuration to carry one unit of wrapped M5–brane charge and the area $A_4 = A_2^2$ of a single M5–brane should scale linearly with N. This means the area of the M2–branes scales as $A_2 \propto \sqrt{N}$, leaving us with a factor of $\sqrt{N}$ in the membrane charges Z^{12} and Z^{34}. We interpret this by saying we have one unit of wrapped M5–brane charge which is build out of an infinite number $n = \sqrt{N}$ of stacked M2–branes in the X^1, X^2 and X^3, X^4 directions. Although the number of M2–branes in a single stack is infinite, the charge density in a single stack is equivalent to that of a single M2–brane and therefore finite (an intuitive picture would be to consider a single M2–brane and fold it in such a way that it can also be interpreted as a stack of different M2–branes, this will obviously not affect the charge density). This is a natural continuation of our story so far. We constructed an M2–brane out of an infinite number of Matrix theory quanta (or D0–particles) and we now conclude that we can construct a (wrapped) M5–brane out of infinite stacks of M2–branes.

Similarly we can construct matrix configurations giving rise to KK and M9–brane charges. We just repeat the structure as given in (4.29). So for obtaining a single non–zero KK–charge extended in the directions $X^1 \ldots X^6$ we can take

$$\begin{aligned} [X^1, X^2]_{ij} &= i\varepsilon_2 \delta_{ij}, \\ [X^3, X^4]_{ij} &= i\varepsilon_2 \delta_{ij}. \\ [X^5, X^6]_{ij} &= i\varepsilon_2 \delta_{ij}, \end{aligned} \tag{4.31}$$

giving rise to the following charges

$$\begin{aligned} Z^{ab} &= \frac{N^{2/3}A_2}{l_{11}^3} \quad ab = 12, 34, 56 \\ Z_{M5}^{abcd} &= \frac{RN^{1/3}A_4}{l_{11}^6} \quad abcd = 1234, 1256, 3456 \end{aligned} \tag{4.32}$$

$$Z_{KK}^{123456} = \frac{R^2 A_6}{l_{11}^9}. \tag{4.33}$$

The interpretation of this Matrix configuration should be clear. We have $3 \times n = 3N^{2/3}$ M2–brane charges, $3 \times m = 3N^{1/3}$ wrapped M5–brane charges and one

KK–charge in this Matrix configuration. We can repeat this procedure in constructing one M9–brane charge, using four copies of (4.27) and scaling the areas appropriately. This gives $4\times N^{3/4}$ M2–brane charges, $12\times N^{1/2}$ M5–brane charges, $8\times N^{1/4}$ KK–charges and one M9–brane charge. There is a structure in these constructions which can be summarized by the following expression

$$Z_{2n} = P_+^{1-n}\prod_{i=1}^{n} Z_{2,i} \qquad n=1,\ldots,4\,, \tag{4.34}$$

relating all higher–dimensional charges to the membrane charges appearing in the configuration by multiplying with a power of the light–cone momentum $P_+ = N/R$. The parameter n denotes the number of indices on a particular Matrix charge and it should be clear to what kind of Matrix brane this is related.

Looking at the Hamiltonian of the Matrix model (4.18) we find it straightforward to determine the energy of an extended object. Assuming vanishing momentum and fermions we obtain the (light–cone) energy expression

$$P_- = -\frac{R}{4l_{11}^6}\mathrm{Tr}[X^a,X^b]^2\,. \tag{4.35}$$

For all the Matrix configurations discussed so far (which all have membrane charges), the light–cone energy reduces to

$${P_-}^{(2n)} = \frac{1}{2P_+}\sum_n {Z_2}^2\,, \tag{4.36}$$

where the sum is over the membrane charges in the different stacks, so $n=1$ denotes the energy of a single M2–brane, $n=2$ denotes the energy of a wrapped M5–brane and so on.

We did not yet show that these objects are in fact BPS states. To show that these object indeed preserve half of the 32 maximal supersymmetries, we will use the supersymmetry transformation rules (4.20). If the Matrix configurations preserve some supersymmetry there should exist so–called Killing spinors which make the fermion supersymmetry variation vanish. Then the solution without fermions ($\theta=0$) will be invariant under those supersymmetry transformations. So we are only interested in the fermionic variation and we also assume the momentum P^a vanishes. Then we obtain the following condition for the vanishing of the supersymmetry transformation of θ

$$\delta_{ij}\tilde{\varepsilon} = -\frac{i}{4l_{11}^2}[X^a,X^b]_{ij}\Gamma_{ab}\,\varepsilon\,, \tag{4.37}$$

where we inserted the matrix indices i, j. So we need to cancel the supersymmetry transformation of $\tilde{\varepsilon}$ against the supersymmetry transformation of ε. This requires that the commutators of X^a are multiples of the unit matrix

$$[X^a, X^b] = i\mathscr{F}^{ab}\delta_{ij}\,. \tag{4.38}$$

In that case $\tilde{\varepsilon}$ is completely determined in terms of ε through (4.37) and no extra breaking of supersymmetry occurs and so 16 supersymmetries are preserved. As can be checked, all extended objects constructed so far satisfy this requirement and therefore break and preserve half of the supersymmetries.

The extended object constructed so far are not the only BPS states carrying one unit of charge density. For example, we expect a BPS state representing momentum, corresponding to the M–theory BPS wave. From (4.37) we see that we can consider the following BPS configuration, preserving half of the supersymmetries

$$(\dot{X}^a)_{ij} = R\,p^a\,\delta_{ij}\,, \tag{4.39}$$

representing Matrix theory momentum traveling in the transversal direction X^a. The factor R is included because of (4.19).

It turns out to be possible to define "pure" M5–brane, KK–monopole and M9–brane charges. By "pure" we mean configurations with just the charge corresponding to the brane we want to be describing. So far we constructed higher brane charges which consisted of lower brane charges as well, the exception being the M2–brane. Consider the following configuration

$$\begin{aligned}
[X^1, X^2]_{ij} &= i\,\varepsilon_2\,\mathrm{diag}(\mathbb{1}, -\mathbb{1})_{ij}\,, \\
[X^3, X^4]_{ij} &= i\,\varepsilon_2\,\mathrm{diag}(\mathbb{1}, -\mathbb{1})_{ij}\,,
\end{aligned} \tag{4.40}$$

The opposite signs in the matrix $\mathrm{diag}(\mathbb{1}, -\mathbb{1})_{ij}$ will ensure that the trace of each commutator separately vanishes, that means we do not have M2–brane charge. However the tensor product of these commutators will give the ordinary unit matrix, that means we still have M5–brane charge. We find the following two equations that have to be satisfied if we want this configuration to preserve some supersymmetry (4.37)

$$\begin{aligned}
\tilde{\varepsilon} &= (\varepsilon_2\Gamma^{12} + \varepsilon_2\Gamma^{34})\varepsilon \\
\tilde{\varepsilon} &= -(\varepsilon_2\Gamma^{12} + \varepsilon_2\Gamma^{34})\varepsilon\,.
\end{aligned} \tag{4.41}$$

Obviously this implies that $\tilde{\varepsilon} = 0$ and

$$(\Gamma^{12} + \Gamma^{34})\varepsilon = 0\,. \tag{4.42}$$

Multiplying with Γ^{12} we obtain the projection equation

$$(1-\Gamma^{1234})\varepsilon = 0\,. \tag{4.43}$$

Because Γ^{1234} squares to one and the its trace vanishes, its eigenvalues are $8\times(+1)$ and $8\times(-1)$. Only 8 supersymmetries are therefore preserved by an eigenspinor ε in this configuration. This is $1/4$ of the total number of supersymmetries, as opposed to the wrapped M5–brane we constructed before, which preserved $1/2$ of the supersymmetries. We will call this Matrix theory BPS state the *P*5–brane.

The same idea can be used to construct higher–dimensional pure branes. Consider for example

$$\begin{aligned}
[X^1,X^2]_{ij} &= i\,\varepsilon_2\,\mathrm{diag}(\mathbb{1},\mathbb{1},-\mathbb{1},-\mathbb{1})_{ij}\,,\\
[X^3,X^4]_{ij} &= i\,\varepsilon_2\,\mathrm{diag}(\mathbb{1},-\mathbb{1},\mathbb{1},-\mathbb{1})_{ij}\,,\\
[X^5,X^6]_{ij} &= i\,\varepsilon_2\,\mathrm{diag}(\mathbb{1},-\mathbb{1},-\mathbb{1},\mathbb{1})_{ij}\,.
\end{aligned} \tag{4.44}$$

All M2–brane charges obviously vanish after taking the trace. In this configuration the trace of all tensor products of two commutators vanishes, so M5–brane charges are absent as well. The trace of the tensor product of all three commutators instead gives the $N\times N$ unit matrix, meaning we do find a KK–charge. In a precise sense, looking at (4.44), this KK–charge can be thought of a being build out of 4 KK–monopoles with different signs for the internal brane charges in the different directions. "Summing" these 4 KK–monopoles all internal charges cancel except for the total KK–charge, which is normalized to one unit of charge density. However the equations governing the number of supersymmetries preserved by this configuration become more complicated. We again find that $\tilde{\varepsilon}=0$, which is a consequence of the fact that the sum of terms in any commutator is zero, and we are left with four conditions on ε

$$\begin{aligned}
(\Gamma^{12}+\Gamma^{34}+\Gamma^{56})\varepsilon &= 0\\
(\Gamma^{12}-\Gamma^{34}-\Gamma^{56})\varepsilon &= 0\\
(-\Gamma^{12}+\Gamma^{34}-\Gamma^{56})\varepsilon &= 0\\
(-\Gamma^{12}-\Gamma^{34}+\Gamma^{56})\varepsilon &= 0\,.
\end{aligned} \tag{4.45}$$

This system of equations has no Killing spinor solution and we therefore conclude that this configuration, which we call the P6, breaks all supersymmetries. The same thing happens if we want to construct a pure M9–brane, the P9 configuration.

By doubling the tensor structure in (4.44) all lower–dimensional charges will be absent, but again the supersymmetry Killing equations have no solution.

Copying the matrix configuration in (4.40) in directions $X^5 \ldots X^8$ we construct an 9–brane charge which, although it is not entirely pure since the constituent P5–charges do not vanish, has no M2 or KK–monopole charges. From the supersymmetry Killing equations we find the following equation for ε ($\tilde{\varepsilon} = 0$ of course)

$$(\Gamma^{12} + \Gamma^{34} + \Gamma^{56} + \Gamma^{78})\varepsilon = 0\,, \tag{4.46}$$

which can be rewritten as $(1 - P)\varepsilon = 0$ with

$$P = (\Gamma^{1234} + \Gamma^{1256} + \Gamma^{1278})\,. \tag{4.47}$$

The Dirac matrices in P all commute with each other and they all square to one. Also their trace, and the trace of their products, vanishes. These conditions determine the eigenvalues of P. We want to count the number of $+1$ eigenvalues of P. Necessarily the $+1$ eigenvalues of P are the sum of two $+1$ eigenvalues and one -1 eigenvalue of the three different Dirac matrices in P, which because they commute are separately diagonalizable. Because the traces of the products also vanish, the eigenvalues of the Dirac matrices have to be divided over the (16) available diagonal entries in the following way

$$\begin{array}{rl} \Gamma^{1234} = & (+ \;\; + \;\; + \;\; + \;\; - \;\; - \;\; - \;\; -) \\ \Gamma^{1256} = & (+ \;\; + \;\; - \;\; - \;\; - \;\; - \;\; + \;\; +) \\ \Gamma^{1278} = & (+ \;\; - \;\; - \;\; + \;\; - \;\; + \;\; + \;\; -) \end{array}\,, \tag{4.48}$$

where we only showed half of them for simplicity of presentation. Looking at this expression we find that three of the 8 combinations consist of two $+1$ and one -1 eigenvalue giving rise to an eigenvalue $+1$ for the operator P. This means $3/8 \times 1/2 = 3/16$ of the 32 supersymmetries are preserved by this configuration. This strange number is a result of the the different P5 constituents. This configuration we will call the $P9_5$. This ends our discussion on extended BPS objects in Matrix theory carrying one unit of charge (density).

As already mentioned, all these Matrix configurations only make sense in the limit $N \to \infty$, where they should describe the semiclassical branes of IMF M–theory if the Matrix theory conjecture is correct. We will only use these infinitely extended BPS objects to study their kinematical properties. Extended objects with different topology and properties have been discussed in the literature [78, 79, 80, 81] and also in a dynamical context in [82, 83, 84, 85, 86]. Although we introduced a wrapped (around the light–cone direction) M5–brane, we did not find a transverse M5–brane charge. If Matrix theory describes DLCQ M–theory, a priori one might think that such a state should exist. Although the absence of the transversal M5–brane is still not completely understood, it is believed to be the result of the special light–cone frame. In superstring theory in a light–cone formalism, the strings must have Neumann boundary conditions in both the light–cone directions X^+ and X^- which ensures that all Dp–branes have one leg in the light–cone direction. The same thing happens for supermembranes and because the M5–brane can be interpreted as a a topological defect on which open membranes can end, this means it would only be possible to see light–cone wrapped M5–branes [67]. More on transverse M5–branes can be found in [87, 88]. The same argument can in fact be used to explain the absence of a transverse M9–brane. We did not discuss light–cone wrapped M2–branes which gives rise to the fundamental string BPS state in Type IIA theory. Such a state can be shown to exist [67] and is sometimes called a matrix string, but we will not use it in this thesis.

More Matrix configurations can be constructed representing overlapping or intersecting branes breaking more, but not all, of the supersymmetries. We will construct these states and match those with states appearing in the eleven–dimensional supersymmetry algebra. In this way we will check if and how Matrix theory reproduces (part of) the BPS spectrum of the eleven–dimensional supersymmetry algebra, which can be considered a test of the Matrix theory conjecture.

4.2.2 Intersecting BPS objects in Matrix theory

In the preceding sections we always considered one unit of charge of the (highest–dimensional) BPS object. It should be obvious how to extend the discussion to include k charges. We just multiply the basic non–trivial commutator (4.27) with an integer q and demand that for an object with $2n$ (transversal) indices which we want to have k units of charge that $k \equiv \prod_n q_n$. We will use the freedom to add these factors in the following.

Using the basic objects in Matrix theory we are interested in considering intersecting or overlapping configurations. In eleven–dimensional supergravity much research was done in constructing intersecting M–brane configurations [21, 22, 89]. The basic phenomenon is that when two (or more) M–branes are oriented in a particular way with respect to each other, the system will preserve some supersymmetry [90] (which will be less than $1/2$) and is therefore stable. When considering intersections at threshold, we mean that these systems do not have any binding energy. The different constituents can be separated without a cost in energy. Real bound states (also called non–threshold intersections) of different branes also exist [91] and in fact most of the Matrix theory configurations considered so far correspond to non–threshold bound states in the $D = 11$ supersymmetry algebra, as we will show when we compare with BPS states in the supersymmetry algebra. For the moment we will be interested in constructing the basic pairwise threshold intersecting configurations in Matrix theory.

In Matrix theory we can break more supersymmetries by taking configurations with commutators which are not proportional to the unit matrix (4.38). This is also the mechanism we will use to introduce intersecting brane configurations. Let us discuss a specific example which is based on the expectation that two M2–branes can intersect over a point [21, 22]. Let us consider the following matrix configuration

$$\begin{aligned} [X^1, X^2]_{ij} &= i\frac{N k_1}{N_1}\varepsilon_2 \,\mathrm{diag}(\mathbb{1}_{N_1}, 0)_{ij}\,, \\ [X^3, X^4]_{ij} &= i\frac{N k_2}{N_2}\varepsilon_2 \,\mathrm{diag}(0, \mathbb{1}_{N_2})_{ij}\,, \end{aligned} \tag{4.49}$$

where the unit matrices are $N_1 \times N_1$ and $N_2 \times N_2$ dimensional and of course $N_1 + N_2 = N$. We interpret this configuration as representing $N k_1/N_1$ M2–branes in the X^1, X^2–plane and $N k_2/N_2$ M2–branes in the X^3, X^4–plane. The factors N/N_1 and N/N_2 are included because we can only compare the number of charges if the total charges, which requires a trace, scale in the same way with N. Obviously to obtain a finite number of charges, the limit $N \to \infty$ is taken in such a way that the ratio N_1/N_2 stays fixed. Notice also that this configuration does not carry M5–brane charges. Preservation of supersymmetry (4.37) implies

$$\begin{aligned} \tilde{\varepsilon} &= -\frac{N k_1}{N_1}\Gamma^{12}\varepsilon \\ \tilde{\varepsilon} &= -\frac{N k_2}{N_2}\Gamma^{34}\varepsilon\,, \end{aligned} \tag{4.50}$$

which reduces to the following equation for ε ($\tilde{\varepsilon}$ is determined by any of the above equations (4.50))

$$(1 - \frac{N_1 k_2}{N_2 k_1}\Gamma^{1234})\varepsilon = 0\,. \tag{4.51}$$

To solve this equation we need $N_1 k_2 = N_2 k_1$. Because the ratio N_1/N_2 is arbitrary (and finite), the number of charges are arbitrary as well. Because Γ^{1234} squares to one and is traceless, this Killing equation is then solved by an eigenspinor with 8 positive unit eigenvalues. The system therefore preserves $1/4$ of the maximal supersymmetry. The properties of this intersecting M2–brane system in Matrix theory, $1/4$ supersymmetry preservation and arbitrary charges, are in agreement with results from the $D = 11$ supersymmetry algebra and from the supergravity soliton solutions.

The idea should now be clear. We consider n separate independent extended objects in Matrix theory through commutators equal to N_n unit matrices with $\mathbb{1}_{N_1} \oplus \ldots \oplus \mathbb{1}_{N_n} = \mathbb{1}_N$. Intersections can then be described by taking non–vanishing commutators in different directions. Generically the supersymmetry Killing equations will then fall in the following class

$$(k_1 \Gamma^{12} + k_2 \Gamma^{34} + k_3 \Gamma^{56} + k_4 \Gamma^{78})\, \varepsilon = 0\,, \tag{4.52}$$

which we can rewrite as $(1-P)\varepsilon = 0$ with

$$P = \frac{1}{k_1}(k_2 \Gamma^{1234} + k_3 \Gamma^{1256} + k_4 \Gamma^{1278})\,. \tag{4.53}$$

To look for solutions it is useful to calculate P^2

$$P^2 = \frac{1}{k_1^2}(k_2^2 + k_3^2 + k_4^2 - 2k_2k_3\Gamma^{3456} - 2k_2k_4\Gamma^{3478} - 2k_3k_4\Gamma^{5678})\,. \tag{4.54}$$

The Dirac matrices in P^2 all commute, square to one and are traceless, meaning they can be diagonalized simultaneously and their eigenvalues are $8 \times (+1)$ and $8 \times (-1)$. To solve the equation $(1-P)\varepsilon = 0$ we need at least one constraint (to make $P^2 = 1$) on the parameters k_n and at most 3. More constraints means more supersymmetry is preserved, to be precise $2n$ of the 32 supersymmetries are preserved when we have $n = 1, 2, 3$ constraints. This means that in all the cases where the supersymmetry equations look like (4.52) the possible fractions of supersymmetry are $1/16, 1/8$ or $3/16$. We already saw an example of this last fraction in (4.46), where we took $k_1 = k_2 = k_3 = k_4$. As soon as we truncate by taking some k_n to be zero, the number of possible preserved supersymmetries increases to either $1/4$ or $1/2$. These are then the only fractions we will encounter in our analysis. Let us end this subsection by giving the results for the different pairwise intersections.

- Pairwise Matrix theory configurations with a wave. Matrix momentum (a gravitational wave) can only be added to the “non–pure” Matrix branes (M2,

Matrix configuration	SUSY
$(1\|W, M2)$	$\frac{1}{4}$
$(1\|W, M5)$	$\frac{1}{4}$
$(1\|W, M6)$	$\frac{1}{8}$
$(1\|W, M9)$	$\frac{1}{8}, \frac{1}{16}, \frac{3}{16}$

Table 4.1: Supersymmetric pair intersections involving W. The notation $(p|A,B)$ indicates that the objects A and B have p common spacelike worldvolume directions. The second column gives the amount of residual supersymmetry that can be obtained.

Configuration	SUSY	Configuration	SUSY
$(0\|M2, M2)$	$\frac{1}{4}$	$(4\|M5, M5)$	$\frac{1}{2}, \frac{1}{4}$
$(2\|M2, M2)$	$\frac{1}{2}$	$(2\|M5, M6)$	$\frac{1}{8}, \frac{1}{16}, \frac{3}{16}$
$(0\|M2, M5)$	$\frac{1}{8}$	$(4\|M5, M6)$	$\frac{1}{4}, \frac{1}{8}$
$(2\|M2, M5)$	$\frac{1}{4}$	$(4\|M5, M9)$	$\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{3}{16}$
$(0\|M2, M6)$	$\frac{1}{8}, \frac{1}{16}, \frac{3}{16}$	$(4\|M6, M6)$	$\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{3}{16}$
$(2\|M2, M6)$	$\frac{1}{4}, \frac{1}{8}$	$(6\|M6, M6)$	$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$
$(2\|M2, M9)$	$\frac{1}{8}. \frac{1}{16}, \frac{3}{16}$	$(6\|M6, M9)$	$\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{3}{16}$
$(0\|M5, M5)$	$\frac{1}{8}, \frac{1}{16}, \frac{3}{16}$	$(8\|M9, M9)$	$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{3}{16}$
$(2\|M5, M5)$	$\frac{1}{4}, \frac{1}{8}$		

Table 4.2: Pair intersections of $M2$, $M5$, $M6$, $M9$. Only branes are considered which are built up out of membranes in the 12, 34, 56 and 78 directions.

M5, KK–monopole and the M9) if we want to preserve some supersymmetry. The direction of the wave must necessarily be in a worldvolume direction of the brane. We summarized the possibilities and fractions of preserved supersymmetry in Table 4.1.

- Pairwise intersections of M2, M5, KK-monopole and M9. This is going to be a rather extensive list. There exist many possibilities and many fractions of supersymmetry can be preserved. We only considered Matrix branes which are build out of non–vanishing commutators in the X^1, X^2, X^3, X^4, X^5, X^6 and X^7, X^8 directions. We summarized the possibilities in Table 4.2.

4.2.3 Matrix BPS states and the M–theory algebra

Clearly we found a lot of BPS states in Matrix theory. We would now like to check if the same states appear in the M–theory supersymmetry algebra (3.2). The conjecture is that Matrix theory describes DLCQ M–theory, so we better compare with the the DLCQ supersymmetry algebra. The easiest way to compare is to note that the only thing the special (compactified) light–cone frame does from the supersymmetry algebra point of view, is it places momentum P_+ in the (compact) x_- direction. This can be interpreted as a BPS wave which has to added to any state we would like to consider. Let us first see to what extent it is possible to relate states by assuming this BPS wave to have momentum in a spatial direction X^{11} (instead of a light–like direction). As we will see this works fine in all situations. The reason for this is that we know the relation between a (limit of a) spatial compactification of M–theory and a light–like compactification, as was discussed in section 3.1.1.

Keeping this in mind, let us look for the appropriate BPS states in the supersymmetry algebra. First we are going to rewrite the supersymmetry algebra in the following form

$$\{Q,Q\} = P^0(\mathbb{1} + \bar{\Gamma}), \tag{4.55}$$

with the operator $\bar{\Gamma}$ equal to

$$\bar{\Gamma} = \frac{1}{P^0}\left(\Gamma^{0m}P_m + \frac{1}{2}\Gamma^{0MN}Z_{MN} + \frac{1}{5!}\Gamma^{0M_1\dots M_5}Y_{M_1\dots M_5}\right), \tag{4.56}$$

where the small Roman index m only runs over the spatial directions $m \in (11, i)$ where i is of course an $SO(9)$ index and the capital Roman index M runs over all directions $M \in (0, 11, i)$. Writing the algebra this way (4.55) makes it clear that in order to preserve some supersymmetry we need $\bar{\Gamma}^2 = 1$, meaning the eigenvalues of $\bar{\Gamma}$ are ± 1.

As a first example, let us consider the Matrix M5–brane, which besides M5–brane charge carries M2–brane charges in the appropriate directions as well. We also need to put one leg of the Matrix M5–brane in the X^{11} direction. We take the following central charge configuration

$$\begin{aligned} Z^{12} &= z_1 \quad , \quad Z^{34} = z_2 \\ Y^{-1234} &= y \end{aligned} \tag{4.57}$$

and we need momentum p in the direction X^{11}. Plugging these charges into (4.56) and calculating its square we find

$$\bar{\Gamma}^2 = \frac{1}{P^{0\,2}} \left(p^2 + z_1^2 + z_2^2 + y^2 + 2(py - z_1 z_2)\Gamma^{1234} \right) . \tag{4.58}$$

If we want this configuration to break only half of the supersymmetries, as it does in Matrix theory, the term with the Dirac matrix needs to vanish. This leads to a constraint on the charges

$$p\,y = z_1 z_2 \, , \tag{4.59}$$

which is exactly the relation between the charges we found in Matrix theory (4.34) if we replace p by P_+. This replacement can be made by performing the change of units to the DLCQ M–theory units, defined by (4.12). We can make $\bar{\Gamma}^2 = 1$ by choosing P^0 appropriately. From (4.58) and (4.59) we deduce that

$$P^0 = \sqrt{p^2 + z_1^2 + z_2^2 + \frac{z_1^2 z_2^2}{p^2}} \tag{4.60}$$

We note that this has to be a non–threshold bound state. The sum of the BPS energies of all the separate constituents is larger than P^0 in (4.60), so this system must have non–zero binding energy. An extensive study of non–threshold bound states in M–theory and their construction can be found in [91]. The momentum p in this expression equals N/R_s. To relate the energy P^0 to the DLCQ energy P_- we take the limit $R_s \to 0$, as discussed in section 3.1.1. This means we can safely neglect the last term in P^0, which is of order $1/p^2$ and vanishes in the limit. Extracting a factor of p and expanding the squareroot we find

$$P^0 \equiv \frac{N}{R_s} + \Delta E = \frac{N}{R_s} + \frac{1}{2}\frac{z_1^2 + z_2^2}{p} \, . \tag{4.61}$$

Now using the relation (4.9) we see that the light–cone energy P_- equals

$$P_- = \frac{1}{2P_+}(z_1^2 + z_2^2), \tag{4.62}$$

which is equivalent to the (more general) expression (4.36). So the supersymmetry algebra nicely reproduces the Matrix theory results.

Similar results can be obtained for the matrix KK–monopole and M9–brane. Both matrix branes have all lower–dimensional charges activated which should be incorporated in the supersymmetry analysis. Again we find that when we want to preserve $1/2$ of the supersymmetry we have to impose constraints on the charges. These constraints exactly reproduce the relations (4.34) found in Matrix theory. These are all non–threshold bound states and the light–cone energy as obtained from the supersymmetry algebra nicely reproduces (4.36).

We can also consider the P5, which does not carry any membrane charges so (4.58) reduces to

$$\bar{\Gamma}^2 = \frac{1}{{P^0}^2}\left(p^2 + y^2 + 2py\Gamma^{1234}\right). \tag{4.63}$$

Now we can not get rid of the Dirac matrix Γ^{1234}. However we can set 16 of the eigenvalues of $\bar{\Gamma}^2$ to one by choosing P^0 appropriately. This means we have 8 eigenvalues equal to -1 and $1/4$ of the supersymmetry is preserved. The energy P_0 equals

$$P^0 = p + y, \tag{4.64}$$

which therefore has to be a threshold bound state of a wave and a M5–brane, with the wave and one leg of the M5–brane in the eleventh direction. The light–cone energy P_- is exactly equal to y in this case. This is also what we find in the Matrix theory.

In Matrix theory we saw that pure P6 and P9 charges did not preserve any supersymmetry. This can in fact not be deduced from the supersymmetry algebra which suggests that the P6 is a non–threshold bound state of a KK–monopole and a wave breaking $1/2$ of the supersymmetry. The P9 is considered a threshold bound state with a wave, from the algebra point of view. This can be deduced using the same techniques as in the P5–brane case. The exact reason for this discrepancy is not clear. However probably the fact that both branes require a special compact direction that scales unusually with R_s (or R in the DLCQ frame) (4.28) has something to do with it. Remember that the momentum wave has to lie in this special direction. In Type IIA these configurations would correspond to D6– and D8–branes with D0–branes on their worldvolume. It is known that these bound states are special [92]. In [92] classically stable bound states of D0–branes and D6– or D8–branes are constructed which do not preserve any supersymmetry. These solutions are constructed out of four D6–branes and eight D8–branes, which is in agreement with our Matrix theory constructions of these states.

As an example of a state preserving 3/16 of the supersymmetry we analyze $P9_5$. There is one M9–brane charge, mixed with 6 M5–brane charges and momentum in the eleventh direction. The M9–brane charge corresponds to[4] $Z_{09} = q$. Including all charges we obtain

$$\begin{aligned} P^0\bar{\Gamma} &= \Gamma^{09}p + \Gamma^{01234||}y_1 + \Gamma^{01256||}y_2 + \Gamma^{01278||}y_3 \\ &\quad + \Gamma^{03456||}y_4 + \Gamma^{03478||}y_5 + \Gamma^{05678||}y_6 + \Gamma^9 q\,, \end{aligned} \tag{4.65}$$

In $(P^0\bar{\Gamma})^2$ there are three independent commuting Γ-matrices so that in the generic case this configuration will preserve 1/16 of the supersymmetry. This corresponds to a threshold bound state of six M5–branes, an M9–brane and a wave. We can also obtain configurations which preserve 1/8 and 3/16, by restricting the coefficients. If we set $y_2 = y_3 = y_4 = y_5 = y$, leaving y_1 and y_6 arbitrary, we find that $(P^0\bar{\Gamma})^2$ has the following eigenvalues: $(p - q \pm (y_1 - y_6))^2$ with multiplicity 8 for each choice of sign, $(p + q + y_1 + y_6)^2$ with multiplicity 8, and $(p + q - y_1 - y_6 \pm 4y)^2$ with multiplicity 4 for each sign. Therefore, by choosing P^0 appropriately, we preserve 1/8 supersymmetry, for each of the eigenvalues of multiplicity 8. If we also set $y_1 = y_6 = y$, the eigenvalues simplify further. There are then 12 eigenvalues equal to $(p + q + 2y)^2$, leading to 3/16 of the maximal supersymmetry, which is the case related to the Matrix theory $P9_5$–brane we constructed. This can also be interpreted as a threshold bound state of one M9–brane with 2 M5–branes (suggested by the factor 2 in the energy expression) which intersect at an angle. Indeed a system of two M5–branes at angles intersecting over a string, which can be related to two D4–branes at angles in Type IIA superstring theory, was shown in [93] to preserve $3/16$ of the supersymmetry. A more general discussion of M–branes at angles can be found in [94, 95].

Except for the P6 and P9 charges all basic Matrix theory charges correspond to BPS states in the M–theory supersymmetry algebra. Intersections of these basic branes can be considered in the supersymmetry algebra as well and are in agreement with the obtained results in Matrix theory. We also hinted at an explanation of why the P6– and P9–branes are special. Although we did not explicitly check every Matrix theory state, we do believe that any Matrix theory BPS state has an

[4]Remember that the symbol $||$ indicates the eleventh direction and $\Gamma^9 = \Gamma^{012345678||}$.

analogue in the M–theory supersymmetry algebra. Obviously this is not true the other way around. Not all BPS states in the M–theory supersymmetry algebra can be found in the Matrix theory. This is mainly due to the special DLCQ frame in which we should analyze the Matrix theory BPS states (except for the P6 and P9). The exotic fraction $3/16$ of supersymmetry preservation which can occur in Matrix theory, has also been found in the analysis of intersecting branes at angles. We expect that the Matrix theory states can in fact be identified with those states, as we saw in one explicit example.

4.3 The status of Matrix theory

A great deal of evidence has been gathered in favor of the Matrix theory conjecture. In this chapter we showed that extended objects appear naturally in the Matrix model and can be used to construct more involved intersecting BPS states, which also occur in the M–theory supersymmetry algebra. This can be considered (kinematical) evidence in favor of the conjecture. Besides that much dynamical evidence in favor of the conjecture has been obtained as well [73, 70, 71, 72, 86, 85, 84, 82] and also evidence based on M–theory dualities [96, 97, 98, 99, 100].

Although the model seems very attractive as a description of M–theory because of its simplicity and its non–perturbative nature, there are problems with the model as well. Most of them can be traced back to the broken covariance and background dependence of the Matrix model. Different backgrounds all lead to different Matrix models. Matrix theory on curved backgrounds is a subject on which some progress has been made [101, 102], but in general this is not yet well understood. There have also been attempts to look for covariant Matrix models [103], but so far no real progress has been made in that direction. It seems that although the Matrix model looks simple and nice, the (calculational) details of the conjecture turn out to be rather unclear and difficult (mainly because of the matrix structure and the large N limit). Also when compactifying Matrix theory on tori the theory gets more and more involved when the dimension of the torus grows. Matrix theory compactified on T^4 and higher requires new degrees of freedom at a particular scale. This also means that four–dimensional physics cannot be well understood using the Matrix theory.

If correct, Matrix theory does teach us a lot about the nature of quantum–gravity. It is very surprising, to say the least, that a $0+1$–dimensional quantum mechanics model can describe certain aspects of (eleven–dimensional) quantum gravity (like graviton scattering). That this is possible can be traced back to the supersymmetry and the matrix nature of the quantum mechanics model. Without supersymmetry bound states of the basic D0–particles would not have been possible and without the matrix structure the interactions could not possibly have described gravity. Matrix theory in fact realizes some interesting ideas about quantum gravity. First of all, spacetime is represented by matrices and therefore noncommutative [104] and secondly Matrix theory realizes the so–called Holographic principle, which is an idea first discussed by 't Hooft [105] and Susskind [106]. The Holographic principle tells us that the entropy in a theory of quantum gravity is not proportional to the volume of a part of spacetime (as it is in ordinary quantum field theory), but to the area surrounding that part of spacetime with an upper limit of one bit per Planck area. This idea is motivated by the Bekenstein–Hawking entropy formula for black holes. All the information in a particular volume of spacetime is stored, or can be mapped, holographically in the area surrounding it, hence the name for the principle. In the original conjecture [50] it was already suggested that Matrix theory is in correspondence with the Holographic principle. We will see very explicit manifestations of the Holographic principle in another, but related, context in the next chapter.

We conclude that, although many details of the Matrix theory conjecture are still poorly understood, there exists much evidence in favor of the conjecture that the Matrix model describes the DLCQ sector of M–theory. The model is a nice tool to potentially learn more about the nature of quantum–gravity and string theory. Matrix theory can be "deduced" as a certain limit (4.13) of Type IIA superstring theory with D0–branes. We are going to consider this limit, and a generalization of it to higher–dimensional branes, in more detail in the next chapter.

Chapter 5

String solitons and the field theory limit

In this chapter we will study the soliton solutions appearing in the supergravity actions more carefully. Most importantly this will involve putting in the string theory parameters and a study of the near–horizon region. We will introduce a special metric frame, called the *dual* frame, in which the special properties of the near–horizon geometry are most easily detected. After that we will study a string theory limit which will leave us with the decoupled soliton worldvolume field theory on the one side, and a (well behaved) near–horizon supergravity on the other side, which are conjectured to be dual descriptions of the same system. From the outset our analysis is valid in an arbitrary number of dimensions and for very generic brane solutions. To obtain well behaved near–horizon supergravities we will need a constraint on our parameters, leading us to consider mainly Dp–branes and their intersections. We will end by presenting some examples. This chapter is based on work done in [107], which generalizes work done in [108] and [109]. Dualities between (conformal) field theories and (Anti–de Sitter) near–horizon supergravities were first discussed in [110]. Many good review articles on the subject have appeared and we refer to [111] for a nice pedagogical introduction and to [112] for an extensive overview.

5.1 String soliton geometries

We want to take a closer look at the geometries of all kind of solitons appearing as solutions to the low energy effective actions of string and M–theory. This interest is motivated by the fact that the physics of p–branes can be described in two (different) ways. On the one hand, when the target spacetime supergravity (also called the *bulk* supergravity) is decoupled, the p–brane fluctuations can be described by a (effective) worldvolume field theory living on the worldvolume of the p–brane. In another, semiclassical, regime we are also allowed to describe the physics of the p–brane by probing the p–brane background with supergravity fields. A priori one might think these two descriptions are valid in completely different string theory regimes. However, in the previous chapter we encountered an example of a worldvolume field theory of N D0–branes which was able to describe gravitational physics. This suggests that there exist string theory regimes where both descriptions describe the same physics. This is also suggested by the string interactions of Dp–branes, which from one point of view describe exchanges of *closed* strings leading to bulk supergravity physics, or from another point of view describe vacuum diagrams of *open* strings which lead to worldvolume (quantum) field theory physics (see Figure 2.6).

We want to understand this phenomenon in more generality and detail. As a first step towards that understanding we will need to analyze the p–brane geometries again. We want to study a string theory limit in which the bulk supergravity decouples and which leaves us with a non–trivial worldvolume field theory. As we will see this limit takes us into the near–horizon region of the corresponding p–brane solution. Let us therefore first discuss p–brane near–horizon geometries.

5.1.1 Near–horizon geometries of p–branes

Our starting point will be a slightly different action than the one given in (3.48). We will replace the rank $p+2$ field strength in (3.48) by its rank $D-p-2$ Hodge dual and look for p–brane solutions which are magnetically charged with respect to the Hodge dual potential (so they are electrically charged with respect to the rank $p+1$ gauge potential). This will turn out to be useful as we go along. Besides that it will also be important to keep track of all factors of g_s appearing in the action. We refer to Appendix A for the details of how to obtain the appropriate scalings with g_s, but basically these can be read off from the exponential dilaton factors in the string frame action. Our action then is

$$S_D = \int \mathrm{d}^D x\, \sqrt{g}\frac{1}{(\sqrt{\alpha'})^{D-2}g_s^2}\Big[R - \frac{4}{D-2}(\partial\Phi)^2 - \frac{g_s^{(4-2k)}}{2(\tilde{d}+1)!}\left(\frac{e^{\Phi}}{g_s}\right)^{-a} F^2_{\tilde{d}+1}\Big]\,, \tag{5.1}$$

where we introduced a parameter $\tilde{d}$ and we will also introduce a parameter d, which are defined by

$$\begin{cases} d &= p+1 \quad \text{dimension of the worldvolume}\,, \\ \tilde{d} &= D-d-2 \quad \text{dimension of the dual brane worldvolume}\,. \end{cases} \tag{5.2}$$

We note that $d+\tilde{d}=D-2$. Also notice the change in sign in the dilaton coupling parameter a in (5.1), which is a result of performing the Hodge duality transformation (3.49). We also introduced a parameter k, which is related to a,d and D in the following way (see appendix A)

$$k = \frac{a}{2} + \frac{2d}{D-2} \tag{5.3}$$

and which determines the scaling with g_s of the rank $\tilde{d}+1$ field strength. Notice that when $k=1$ the overall scaling with g_s vanishes in front of the field strength. This is the appropriate scaling for a Ramond–Ramond field strength. For $k=2$ and $k=0$ we find the appropriate scaling of Neveu–Schwarz field strengths and their Hodge duals respectively.

We will consider the following class of "two–block" p–brane solutions of the action (5.1)

$$\begin{aligned} ds_E^2 &= H^{-\frac{4\tilde{d}}{(D-2)\Delta}} dx_d^2 + H^{\frac{4d}{(D-2)\Delta}} dx_{\tilde{d}+2}^2\,, \\ e^{\Phi} &= g_s H^{\frac{(D-2)a}{4\Delta}}\,, \\ g_s^{(2-k)} F &= \sqrt{\frac{4}{\Delta}}\, {}^*(dH \wedge dx_1 \wedge \cdots \wedge dx_d)\,, \end{aligned} \tag{5.4}$$

where * is the Hodge operator on D–dimensional spacetime. This is the magnetically charged analog of (3.50) where we now took care of the appropriate scalings with g_s. The parameter Δ is the same as in (3.54), which expressed in terms of d and $\tilde{d}$ equals

$$\Delta = \frac{(D-2)a^2}{8} + \frac{2d\tilde{d}}{(D-2)}\,. \tag{5.5}$$

The function H is harmonic on the $\tilde{d}+2$ transverse coordinates if $\tilde{d} \neq 0,-2$ and can be expressed using $\tilde{d}$ as

$$H(r) = 1 + \left(\frac{r_0}{r}\right)^{\tilde{d}}\,, \tag{5.6}$$

where $r_0^{\tilde{d}}$ is related to the charge (and mass) of the p–brane. Looking at (5.4) it follows that the charge should scale as $g_s^{(2-k)}$[1]. We note that codimension one objects, which have $\tilde{d} = -1$ and are usually called domain–walls, are included in (5.4) as opposed to codimension 2 objects. The magnetically charged field strength belonging to a domain–wall is a rank 0 object, a cosmological constant. In fact, the solution involving linear harmonic functions presented in (5.4) is not uniquely defined for domain–walls. It will be useful to discuss these objects separately in section 4.1.2.

The Dp–branes and NS–branes in $D = 10$ and M–branes in $D = 11$ are included in (5.4), but also two–block p–branes in dimensions $D < 10$. These can arise in string theory by considering string compactifications. We will mainly be interested in two–block BPS p–branes which can be obtained from an intersection of the basic BPS p–branes in $D = 10$ or $D = 11$. When the relative transverse directions of such an intersection are all wrapped on a torus T^r with r the number of relative transverse directions, the result will be a two–block p–brane in $D = 10 - r$ (or $D = 11 - r$). Supersymmetry preserving BPS p–brane solutions in any dimension are distinguished by having $\Delta = 4/n$ with n an integer (3.55) denoting the number of participating (higher–dimensional) branes [38].

To discuss the near–horizon geometry of these p–branes, we want to consider a limit in which the constant part in the harmonic function (5.6) is negligible, which means

$$\begin{aligned} r &\gg r_0 \quad \text{for } \tilde{d} = -1\,, \\ r &\ll r_0 \quad \text{all other cases}\,. \end{aligned} \tag{5.7}$$

The p–branes are positioned at $r = 0$ so this limit brings us close to the brane when $\tilde{d} > 0$. When $\tilde{d} = -1$, so for domain–walls, this limit actually takes us far away from the brane. We will still refer to this limit as a near–horizon limit. Strictly speaking, because we are considering extremal BPS p–branes, near–“horizon” is not good terminology even in those cases where $\tilde{d} \neq -1$. This is because the p–brane Einstein frame metric in (5.4) is singular at $r = 0$, except for some special cases where the dilaton is constant (e.g. the D3–brane in $D = 10$ Type IIB supergravity). Therefore it would perhaps be more suitable to call this a near–core limit. We will soon see however that this singularity in the metric at $r = 0$ can be removed by a conformal transformation (3.31) to a special frame called the dual frame, in which the hypersurface at $r = 0$ has become a non–singular horizon.

[1]To obtain natural units we still have to divide with the gravitational constant $\kappa_D \propto g_s^2$ (3.29), giving the expected scaling of the charges and tensions of the different p–branes $\propto g_s^{-k}$.

This can be understood by noting that $e^{\pm\Phi}$ is singular at $r=0$ as well and therefore we can perform a conformal transformation "canceling" the singularity in the metric. Of course we are still left with a singularity in $e^{\pm\Phi}$, but the limit (5.7) can in this sense be referred to as a near–horizon limit.

In the limit (5.7) the Einstein metric and the dilaton can be written as

$$\mathrm{d}s_E^2 = \left(\frac{r_0}{r}\right)^{-\frac{4\tilde{d}^2}{\Delta(D-2)}} \mathrm{d}x_d^2 + \left(\frac{r_0}{r}\right)^{\frac{4d\tilde{d}}{\Delta(D-2)}} \mathrm{d}x_{\tilde{d}+2}^2 \quad , \quad e^{\Phi} = g_s \left(\frac{r_0}{r}\right)^{\frac{(D-2)a\tilde{d}}{4\Delta}} . \tag{5.8}$$

Let us now introduce the conformal transformation which will factor off the singularities in the above metric. The following conformal transformation will do exactly that

$$g_D^{\mu\nu} = \left(\frac{e^{\Phi}}{g_s}\right)^{a/\tilde{d}} g_E^{\mu\nu} , \tag{5.9}$$

where we divided by g_s to not introduce (extra) g_s dependence in the metric. This conformal transformation will have the following effect on the action (5.1)

$$S_D = \int \mathrm{d}^D x\, \sqrt{g_D} \frac{1}{(\sqrt{\alpha'})^{D-2} g_s^2} \left(\frac{e^{\Phi}}{g_s}\right)^{\delta} \left[R_D + \gamma(\partial\Phi)^2 - \frac{g_s^{(4-2k)}}{2(\tilde{d}+1)!} F_{\tilde{d}+1}^2\right], \tag{5.10}$$

with

$$\delta = -\frac{(D-2)a}{2\tilde{d}}, \qquad \gamma = \frac{D-1}{D-2}\delta^2 - \frac{4}{D-2} . \tag{5.11}$$

So the dual frame can be characterized by saying that all fields in the action are multiplied with the same e^{Φ} factor. We note that this would not have been true if we had used electrically charged potentials. Another special feature of this frame, which explains the name *dual* frame, is that Hodge dual $(D-p-4)$–branes probing the p–brane background solution couple naturally to the dual frame metric without a dilaton term.

The regular dual frame metric is given by

$$\mathrm{d}s_D^2 = \left(\frac{r_0}{r}\right)^{2(1-\frac{2\tilde{d}}{\Delta})} \mathrm{d}x_d^2 + \left(\frac{r_0}{r}\right)^2 \mathrm{d}r^2 + r_0^2 \mathrm{d}\Omega_{\tilde{d}+1}^2 . \tag{5.12}$$

Notice that the size of the transverse sphere $S^{\tilde{d}+1}$ no longer depends on r, it has become constant with radius r_0. Because the charge can be calculated by integrating the flux over the transverse sphere, we conclude that the (dualized) field strength in (5.4) can no longer depend on r either. Therefore we will not consider the solution for the field strength and just consider the metric and the dilaton expression. The metric in (5.12) generically describes a $d+1$–dimensional Anti–de Sitter spacetime times a $\tilde{d}+1$–dimensional sphere. Let us next consider coordinate transformations to connect the metric (5.12) to more standard and familiar parameterizations of Anti–de Sitter spacetimes.

Consider the following coordinate transformation redefining the radius r as

$$\left(\frac{r_0}{r}\right) = e^{-\lambda/r_0}\,, \tag{5.13}$$

which transforms the metric and dilaton into

$$\begin{aligned} \mathrm{d}s_D^2 &= e^{-2(1-\frac{2\tilde{d}}{\Delta})\lambda/r_0}\mathrm{d}x_d^2 + \mathrm{d}\lambda^2 + r_0^2\mathrm{d}\Omega_{\tilde{d}+1}^2 \\ \Phi &= \ln(g_s) - \frac{(D-2)a\tilde{d}}{4\Delta r_0}\lambda. \end{aligned} \tag{5.14}$$

As already mentioned the metric in (5.14) (generically) is a parameterization of a $d+1$–dimensional Anti–de Sitter spacetime times a $\tilde{d}+1$–dimensional sphere, in shorthand notation $AdS_{d+1}\times S^{\tilde{d}+1}$. The full p–brane geometry can therefore be described as interpolating between an asymptotic flat Minkowski and a near–horizon curved $AdS_{d+1}\times S^{\tilde{d}+1}$ geometry, connected by a throat as shown in Figure 5.1. This interpolating property of two–block p–branes (5.4) was first discussed in [41, 42] and generalized in [113, 114]. The only exception occurs when $1-2\tilde{d}/\Delta = 0$, because then the metric describes a $d+1$–dimensional Minkowski spacetime times the same sphere. Another special case is when $a=0$, giving a constant dilaton background. A nice feature of the stereographic coordinates (5.13), as they are called, is that the dilaton depends linearly on the radial coordinate λ.

We can also use horospherical coordinates to parametrise the $AdS_{d+1}\times S^{\tilde{d}+1}$ spacetime, which are defined as

$$u\beta = \frac{r^\beta}{r_0^{\beta+1}}\,, \tag{5.15}$$

with the dimensionless parameter β given by

$$\beta = \frac{2\tilde{d}}{\Delta} - 1\,. \tag{5.16}$$

These coordinates clearly do not make sense when $\beta = 0$. Comparing with (5.14) we see that this is exactly when the near–horizon spacetime becomes Minkowski

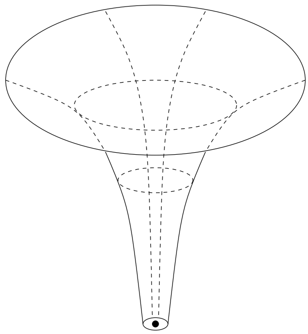

Figure 5.1: The p–brane solutions interpolating between Minkowski and Anti–de Sitter geometries.

and that is why the horospherical coordinates can only be used to describe *AdS* spacetimes.

We note that u carries dimensions of $[l]^{-1} = [m]$, which defines an energy scale. This will have interesting consequences. Rewriting the dual frame solution (5.12) using the horospherical coordinates we obtain

$$\begin{aligned} \mathrm{d}s_D^2 &= r_0^2 \left[(u\beta)^2 \mathrm{d}x_d^2 + \left(\frac{1}{u\beta}\right)^2 \mathrm{d}u^2 + \mathrm{d}\Omega^2_{\tilde{d}+1} \right] \\ e^{\Phi} &= g_s r_0^{-\frac{(D-2)a}{8}\left(\frac{\beta+1}{\beta}\right)} (u\beta)^{-\frac{(D-2)a}{8}\left(\frac{\beta+1}{\beta}\right)} \, . \end{aligned} \tag{5.17}$$

We will prefer these coordinates when analyzing the string limit in which the worldvolume field theory decouples. In the metric (5.17) the $u = 0$ hypersurface is a non–singular horizon and $u \to \infty$ corresponds to the boundary of *AdS*. We will say more about some of the properties of *AdS* spacetimes in section 4.1.2.

Summarizing, we showed that in the dual frame, defined by (5.9), all p–branes solutions in (5.4) generically have a $AdS_{d+1} \times S^{\tilde{d}+1}$ near–horizon geometry and a non–trivial dilaton. Special cases arise when $\beta = 0$ or when $a = 0$. The constant sphere allows for a reduction of the D–dimensional fields. All the indices of the non–trivial field strength lie on the sphere, so this will give rise to a cosmological constant in $d+1$ dimensions and so does the curvature of the sphere. Therefore the reduced solution (throwing away the sphere part), generically consisting of an *AdS* metric and a non–trivial dilaton, solves the equations of motion of an action with a cosmological constant. The reduced object indeed has $p = (d+1) - 2$

spatial extended directions and this is what we called a domain–wall. Before actually performing a (truncated) reduction [115], we would first like to discuss domain–wall solutions in general.

5.1.2 Domain–walls and Anti–de Sitter spacetimes

Domain–wall spacetimes [116] solve the equations of motion obtained by varying a (super)gravity action with a cosmological constant Λ and a dilaton. They correspond to p–branes with worldvolume dimension $d = p+1$ which is one less than the dimension D of the target spacetime they live in (this also means that $\tilde{d} = -1$). Although domain–wall solutions do appear in (5.4), it turns out that these are not the most general solutions one can write down. Because all p–brane near–horizon solutions are described by a domain–wall in $d+1$ dimensions (when reduced over the sphere), it will be useful to study domain–wall solutions more carefully to make a connection with p–brane near–horizon solutions.

Again performing a Hodge dualization, which replaces the cosmological constant Λ by a rank $d+1$ field strength F_{d+1} in (5.1), we can naturally discuss objects of codimension one coupling to a d-form potential, defining a domain–wall. In terms of the field strength F_{d+1} the action is now given by

$$S^E_{d+1} = \int \mathrm{d}^{d+1}x\, \sqrt{g}\frac{1}{(\sqrt{\alpha'})^{d-1}g_s^2}\left[R - \frac{4}{d-1}(\partial\Phi)^2 - \frac{g_s^{(4-2\tilde{k})}}{2(d+1)!}\left(\frac{e^{\Phi}}{g_s}\right)^b F^2_{d+1}\right]. \tag{5.18}$$

We introduced a different dilaton coupling parameter b to stress the difference with (5.1) and $\tilde{k}$ equals

$$\tilde{k} = \frac{-b}{2} + \frac{-2}{d-1}, \tag{5.19}$$

which is just the appropriate modification of (5.3) to this domain–wall case using an electrically charged potential. The equations of motion following from the action (5.18) can be solved using the general p–brane Ansatz (3.50) involving harmonic functions, but not uniquely. The solutions are

$$\begin{aligned} \mathrm{d}s^2_E &= H^{-\frac{4\varepsilon}{(d-1)\Delta_{DW}}}\mathrm{d}x_d^2 + H^{\frac{-4\varepsilon d}{(d-1)\Delta_{DW}}-2(\varepsilon+1)}\mathrm{d}y^2\,, \\ e^{\Phi} &= g_s H^{\frac{-(d-1)b\varepsilon}{4\Delta_{DW}}}\,, \\ g_s^{(2-\tilde{k})}F_{01\ldots d-1y} &= \sqrt{\frac{4}{\Delta_{DW}}}\,\partial_y H^{\varepsilon}\,, \end{aligned} \tag{5.20}$$

where ε is now an arbitrary parameter as opposed to $\varepsilon = -1$ for ordinary p–branes (when using electrically charged potentials). The parameter Δ_{DW} is defined by

$$\Delta_{DW} = \frac{(d-1)b^2}{8} - \frac{2d}{d-1}\,, \tag{5.21}$$

which is just (5.5) with $\tilde{d} = -1$ and $a \to b$.

The function H is harmonic on the 1-dimensional transverse space with coordinate y and equals

$$\begin{aligned} H(y) &= 1 + Q_+ y \quad \forall y > 0\,, \\ H(y) &= 1 + Q_- y \quad \forall y < 0\,, \end{aligned} \tag{5.22}$$

with $Q_\pm$ constants and we fixed an arbitrary integration constant c to equal 1. The equations of motion allow for a discontinuity and so Q_+ and Q_- do not have to be equal[2]. It is understood that the domain–wall is positioned at the discontinuity $y = 0$. The value of $Q_\pm$ on any side of the domain–wall can be expressed in terms of a mass parameter $m_\pm$ in the following way

$$Q_\pm \varepsilon = m_\pm\,, \tag{5.23}$$

where $m_\pm$ is related to the cosmological constant through the equation

$$\Lambda_\pm = \frac{-2m_\pm^2}{\Delta_{DW}}\,. \tag{5.24}$$

So a domain–wall is an object which interpolates between two different cosmological constant vacua. The charge $Q_\pm$ should not be associated with the physical charge or mass of the domain–wall because it cannot be measured, which follows from the dependence on the arbitrary parameter ε. The physical mass and charge of a domain–wall have to be proportional to the (discontinuous) change in the cosmological constant. This is the only way to detect such an object.

We saw that use of the Ansatz (3.50) allows for an undetermined parameter ε in the domain–wall solution. The origin of this parameter is the fact that there are coordinate transformations, labeled by ε, that keep the solution within the same Ansatz. The explicit form of these coordinate transformations is given in [20]. Another way of understanding this is that the Ansatz (3.50) is not a suitable one

[2] Strictly speaking this is only true when using the rank $d+1$ field strength formulation as we did.

in the domain–wall case because it does not uniquely specify the solution. This also means that it should be possible to consider coordinate transformations that get rid of the free parameter ε.

We will now focus on one side of the domain–wall, let us say $y > 0$ and define $Q \equiv Q_+$ (from here on we will also drop the $+$ subscript on all parameters related to Q). Let us also assume that we are far away from the domain wall discontinuity[3]. This is what we called a "near–horizon" limit in the previous section and allows us to neglect the constant 1 in the harmonic function (5.22). We can get rid of the free parameter ε by making the following $y \to \lambda$ coordinate transformation

$$Qy = e^{-Q\lambda}\,. \tag{5.25}$$

The domain-wall solution in the new stereographic λ coordinate reads

$$\begin{aligned} \mathrm{d}s_E^2 &= e^{m\lambda\left(\frac{(d-1)b^2}{4\Delta_{DW}}\right)}\left(e^{-2m\lambda\left(\frac{2+\Delta_{DW}}{\Delta_{DW}}\right)}\mathrm{d}x_d^2 + \mathrm{d}\lambda^2\right) \\ \Phi &= \ln(g_s) + \frac{(d-1)bm}{4\Delta_{DW}}\lambda\,. \end{aligned} \tag{5.26}$$

This is a solution of the action (5.18) with Λ given by (5.24). The overall term in the metric can be removed by performing a conformal transformation to the dual frame, which is now defined as

$$g_D^{\mu\nu} = \left(\frac{e^{\Phi}}{g_s}\right)^{-b} g_E^{\mu\nu}\,, \tag{5.27}$$

which is just (5.9) with $\tilde{d} = -1$ and $a \to b$. The solution in the dual frame becomes

$$\begin{aligned} \mathrm{d}s_D^2 &= e^{-2m\lambda\left(\frac{2+\Delta_{DW}}{\Delta_{DW}}\right)}\mathrm{d}x_d^2 + \mathrm{d}\lambda^2 \\ \Phi &= \ln(g_s) + \frac{(d-1)bm}{4\Delta_{DW}}\lambda\,. \end{aligned} \tag{5.28}$$

This is just (5.14) with $\tilde{d} = -1$, $m = 1/r_0$ and $a \to b$ and generically the dual frame domain–wall metric describes an AdS_{d+1} spacetime [117]. When $\Delta_{DW} = -2$ the metric becomes flat Minkowski spacetime, which is equivalent to taking $\beta = 0$, $\tilde{d} = -1$ and $a \to b$ in (5.14).

[3] We could also perform a shift coordinate transformation, but this would also shift the position of the discontinuity changing the range of the transversal coordinate y.

Near–horizon spacetimes of p–branes should fall in this category of domain–wall solutions after the reduction over the sphere. To make this connection we have to relate the original parameters of the p–brane solution $a, d, \tilde{d}$ and r_0 to the parameters of the $d+1$–dimensional domain–wall which are just b, d and m. Reducing just the fields participating in the solution (5.4) in the dual frame will only replace the field strength by a cosmological constant whose value is determined by the Ricci curvature of the sphere and the charge of the original p–brane

$$S^R_{d+1} = \int \mathrm{d}^{d+1}x\,\sqrt{g}\frac{1}{(\sqrt{\alpha'})^{d-1}g_s^2}\left(\frac{e^{\Phi}}{g_s}\right)^{\delta}\left[R+\gamma(\partial\Phi)^2+g_s^{(4-2k)}\Lambda\right]. \qquad (5.29)$$

This action should be a truncation of a gauged supergravity action which presumably can be obtained by reducing the complete higher–dimensional supergravity action on a sphere [115]. To compare with (5.18) we have to perform a conformal transformation to the Einstein frame and rescale the dilaton $\Phi \to \Phi/c$ to obtain the standard normalization of the dilaton kinetic term. The scale factor c equals

$$c^2 = \frac{2\tilde{d}^2}{\Delta(\tilde{d}+1)-2\tilde{d}}. \qquad (5.30)$$

We can then read off the domain–wall dilaton coupling parameter b

$$b = a\,\frac{(d+\tilde{d})}{(d-1)\tilde{d}}c. \qquad (5.31)$$

This is all we need to express Δ_{DW} (5.21) in terms of the parameters of the original p–brane[4]. We find

$$\Delta_{DW} = \frac{-2\tilde{d}\Delta}{\Delta(\tilde{d}+1)-2\tilde{d}}. \qquad (5.32)$$

Comparing the reduced p–brane solution in the Einstein frame with (5.26) we conclude that m and r_0 are related as follows

$$m = \frac{-\tilde{d}}{r_0}. \qquad (5.33)$$

As a consistency check we should find $c^2 = 1, a = b, \Delta_{DW} = \Delta$ and $m = 1/r_0$ when $\tilde{d} = -1$, in which case the original p–brane is already a domain–wall. The relations (5.30), (5.31), (5.32) and (5.33) indeed satisfy this requirement.

[4]The parameter Δ is only invariant under toroidal reductions and not under reductions on spheres. This explains why $\Delta_{DW} \neq \Delta$.

Using these relations we can express the value of the cosmological constant Λ in terms of the original p–brane parameters. We just use (5.24) and plug in (5.33) and (5.32). This gives

$$g_s^{4-2k}\Lambda = \frac{\tilde{d}}{2r_0^2}\left[2(\tilde{d}+1) - \frac{4\tilde{d}}{\Delta}\right]. \tag{5.34}$$

The first term in this expression originated in the reduction of the D–dimensional Ricci scalar and the second term comes from the reduction of the magnetically charged rank $\tilde{d}+1$ field strength curvature. Analyzing this expression we find that all p–brane near-horizon geometries give a $\Lambda > 0$ (with $1 \leq \tilde{d} \leq (D-3)$) except for the domain-walls, which have $\tilde{d} = -1$ and a sign change occurs, giving $\Lambda < 0$. We note that this is not in contradiction with the fact that all p–branes (including the domain–walls) have AdS geometries, which are defined by having $\Lambda > 0$, in the near–horizon limit. The dilaton kinetic term in (5.18) will contribute to an *effective* cosmological constant which is always positive, as can be most easily seen using stereographic coordinates when the dilaton is a linear function of λ (5.28)[5].

So we have now related all near–horizon geometries of p–branes in (5.4) to domain–wall solutions. In the dual frame the generic domain–wall metric described an AdS_{d+1} spacetime, with one Minkowski spacetime exception when $\Delta_{DW} = -2$. Let us now discuss some of the special properties of AdS spacetimes, for a more extensive discussion we refer to [118].

Anti–de Sitter metrics describe spacetimes of constant negative curvatures. By considering a $d+1$–dimensional Einstein–Hilbert action with a cosmological constant term we find

$$\begin{aligned} R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R &= \Lambda g_{\mu\nu} \Rightarrow \\ R &= -\frac{d+1}{d-1}\Lambda \Rightarrow \\ R_{\mu\nu} &= -\frac{\Lambda}{d-1}g_{\mu\nu}\,. \end{aligned} \tag{5.35}$$

So these spaces have the property that the Ricci tensor is proportional to the metric tensor, which is the definition of Einstein spacetimes. When $\Lambda > 0$ and $d > 1$ the solutions describe spacetimes of constant negative curvature and to obtain AdS we

[5]As one might expect one obtains a flat Minkowski near–horizon geometry when the *effective* cosmological constant vanishes.

need maximal symmetry implied by demanding

$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d+1)}(g_{\nu\sigma}g_{\mu\rho} - g_{\nu\rho}g_{\mu\sigma})\,. \tag{5.36}$$

It is possible to embed AdS_{d+1} in a $d+2$–dimensional flat space. The metric of this $d+2$–dimensional flat space is

$$\eta_{ab} = \mathrm{diag}(-,+,+,\ldots,+,-)\,. \tag{5.37}$$

The $d+2$–dimensional spacetime therefore has two times, or signature $(2,d)$. The invariant distance or length (positive for timelike worldlines) is defined as

$$-l^2 \equiv \sum_{i=1}^{i=d}(y^i)^2 - (y^0)^2 - (y^{d+2})^2\,. \tag{5.38}$$

We note that this length is preserved by a generalization of the Lorentz group rotations into $SO(2,d)$. An AdS_{d+1} embedded surface is then defined as a hyperboloid with $l^2 = \mathscr{R}^2 =$ constant

$$-\mathscr{R}^2 = \sum_{i=1}^{i=d}(y^i)^2 - (y^0)^2 - (y^{d+2})^2\,. \tag{5.39}$$

The length scale $\mathscr{R}$ can be interpreted as the embedding radius of the AdS_{d+1} surface. Through this embedding equation (5.39) the isometry group of an AdS_{d+1} spacetime obviously is $SO(2,d)$, which has $\frac{1}{2}(d+1)(d+2)$ generators[6]. Quantum theories on AdS_{d+1} should therefore have an $SO(2,d)$ invariance. We note that the group of conformal transformations in d dimensions is also $SO(2,d)$. The AdS_{d+1} embedding equation (5.39) also implies the existence of closed timelike curves in the embedded surface. This can be avoided by considering the universal cover of the AdS_{d+1} geometry (which means we introduce an infinite set of AdS_{d+1} geometries allowing timelike curves to pass through different AdS_{d+1} geometries avoiding closed timelike curves).

We can now choose suitable coordinates on AdS_{d+1} satisfying the embedding constraint (5.39) and defining an induced AdS_{d+1} metric. For example let us define

$$u' \equiv y^{d+2} + y^d \quad , \quad v \equiv y^{d+2} - y^d \tag{5.40}$$

[6]Notice that the number of generators is the same as the Poincaré group in $d+1$ dimensions.

where we picked out y^d as one of the spacelike coordinates in (5.39). Also define the left–over coordinates as

$$y^{\mu} \equiv \frac{u'}{\mathscr{R}} x^{\mu} \qquad \mu \in [0, 1, \ldots, d-1] \tag{5.41}$$

and introduce a flat d–dimensional metric $\eta_{\mu\nu}$ with usual Minkowski signature to lower the Greek indices on the x coordinates. The induced (mostly plus) AdS_{d+1} metric is just

$$ds^2 = dy^{\mu} dy_{\mu} - du' dv\,. \tag{5.42}$$

Working out the differentials and expressing v in terms of u and x^{μ} through the embedding equation (5.39) we obtain

$$ds^2 = \left(\frac{u'}{\mathscr{R}}\right)^2 dx^{\mu} dx_{\mu} + \left(\frac{\mathscr{R}}{u'}\right)^2 du'^2\,. \tag{5.43}$$

This can be recognized as the horospherical parameterization of AdS_{d+1} (5.17) if we identify $u' \equiv u r_0^2$ and

$$\mathscr{R} \equiv \frac{r_0}{\beta} = \frac{\Delta r_0}{2\tilde{d} - \Delta}\,. \tag{5.44}$$

An important property of AdS_{d+1} is that it has a “projective boundary”. This has the effect that in many (physical) situations AdS_{d+1} spacetime acts as a finite volume box. Lightlike trajectories can reach this AdS boundary in finite time as opposed to timelike trajectories. Considering the embedding (5.39) and defining new coordinates Ry' with R very large, the boundary can be parametrized (approximately) as

$$-\left(\frac{\mathscr{R}}{R}\right)^2 \to 0 = \sum_{i=1}^{i=d} (y'^i)^2 - (y'^0)^2 - (y'^{d+2})^2\,. \tag{5.45}$$

Since tR with $t \in \mathbb{R}$ is just as good as R, we should consider the boundary as a projective equivalence class defined as

$$\begin{aligned} 0 &= \sum_{i=1}^{i=d} (y'^i)^2 - (y'^0)^2 - (y'^{d+2})^2 \\ y &\sim ty\,. \end{aligned} \tag{5.46}$$

We can use the scaling equivalence to fix one of the coordinates and this means the boundary is a d–dimensional surface, as it should be. For example we can fix

$y'^{d+2} \equiv 1$. In that case we find that

$$1 = \sum_{i=1}^{i=d} (y'^i)^2 - (y'^0)^2 , \tag{5.47}$$

which means the topology of the boundary is $S^1 \times S^{d-1}$ [7]. Considering the universal cover of AdS_{d+1} decompactifies the S^1, avoiding closed timelike curves. An important property of the boundary of AdS_{d+1} is that the isometry group $SO(2,d)$ acts precisely as the conformal group on Minkowski space. The conformal group consists of the usual Poincaré group together with the following conformal transformations

- Dilations or scale transformations acting as

$$x^\mu \to \lambda x^\mu , \quad \lambda \in \mathbb{R} . \tag{5.48}$$

- Special conformal transformations acting as

$$\begin{aligned} x^\mu &\to x'^\mu , \quad \text{such that} \\ \frac{x'^\mu}{x'^2} &= \frac{x^\mu}{x^2} + a^\mu . \end{aligned} \tag{5.49}$$

Together with the Poincaré group these make up the group $SO(2,d)$. It is not very hard to see that some infinitesimal $SO(2,d)$ isometries, namely infinitesimal translations $u' \to u' + a$ (using horospherical coordinates (5.40)), indeed reproduce dilations on the boundary, which follows from the equivalence class condition (5.46) of coordinates on the boundary. For a more extensive discussion on this point we refer to [111]. We conclude that $SO(2,d)$ AdS isometries can be identified with conformal transformations from the boundary point of view.

We found that generic p–brane near–horizon geometries have AdS metrics, but the complete generic solution is also described by a non–trivial dilaton. A non–trivial dilaton breaks the $SO(2,d)$ isometries of the complete solution. For example infinitesimal $SO(2,d)$ translations will not leave the value of e^Φ invariant, breaking the symmetry. From the boundary point of view this has to correspond to broken conformal or scale invariance. Only when the dilaton background is

[7] This result only refers to the topology of the boundary and does not mean that the boundary is a curved geometry. Rather the boundary is a flat Minkowski geometry which can be thought of as the infinite radius limit of a sphere.

constant do we expect the complete background solution to be invariant under the $SO(2,d)$ isometries of AdS_{d+1}. In that case we also find supersymmetry enhancement in the near–horizon limit [41, 42, 119], meaning that a pure AdS background solution preserves all of the supersymmetries, just like flat Minkowski space.

Let us end this section by discussing the two special near–horizon cases in the dual frame.

- Flat Minkowski near–horizon spacetime. This requires

$$\beta = 0 \quad \text{or} \quad 2\tilde{d} = \Delta . \tag{5.50}$$

  We will only consider supersymmetry preserving cases, which means $\Delta = 4/n$ with n an integer. It is important to note that the parameter $\tilde{d}$ is invariant under double dimensional reductions. This means that once we found a p–brane satisfying the constraint (5.50), p–branes with r legs compactified on a T^r giving a $p-r$–brane in $D-r$ dimensions, will also satisfy the constraint (5.50). So we find families of solutions. Because $\tilde{d}$ has to be an integer solutions can only be found for $n=1$ and $n=2$. Relating our results to existing branes in string– or M–theory we find the 10–dimensional $p=5$–branes for $n=1$. When $n=2$ we find $p=5$–branes in $D=9$, which can be obtained from reduction of $D=10$ Kaluza–Klein monopoles in the $N=1$ supergravity theories.

- Pure Anti–de Sitter backgrounds. This requires

$$a = 0 \quad \text{or} \quad \frac{2d\tilde{d}}{\Delta} = (d+\tilde{d}) . \tag{5.51}$$

  This condition can be satisfied for the cases where we preserve some supersymmetry or equivalently $\Delta = 4/n$. We summarized the results [41, 42, 120, 119] in Table 5.1.

  The p–brane listed in Table 5.1 with $\Delta = 2$ in $D=6$ can be traced back to an intersection of 2 Dp–branes in $D=10$, hence the terminology. The same holds for the p–branes in $D=5$ with $\Delta = 4/3$, which are related to intersections of 3 M–branes. Finally, the 0–brane or extreme Reissner–Nordström black hole in $D=4$ is related to an intersection of 4 Dp–branes in $D=10$. Notice that only three possible values of β occur: $\frac{1}{2}, 1$ and 2. Remember that β defines the ratio of the radius r_0 of the transverse sphere $S^{\tilde{d}+1}$ to the radius r_0/β of the embedded AdS_{d+1}. Considering $D=10$ Dp–branes

D	β	$\Delta_{\rm DW}$	Δ	Name
11	$\frac{1}{2}$	–12/5	4	M5–brane
	2	–3	4	M2–brane
10	1	–8/3	4	D3–brane
6	1	–4	2	d1–brane
5	$\frac{1}{2}$	–4	4/3	m1–brane
	2	∞	4/3	m0–brane
4	1	∞	1	RN black hole

Table 5.1: The Table indicates the values of β, $\Delta_{\rm DW}$ and Δ for all p–branes that have a pure *AdS* near–horizon background.

or their reduced intersections we always find this ratio to be 1. Considering $D = 11$ M2–branes or M5–branes and their reduced intersections we always find 2 and $\frac{1}{2}$ respectively.

Finally notice that $p = 0$ is special because Δ_{DW} blows up and using (5.35) we find that $\Lambda \equiv 0$. However we can still consider maximally symmetric Einstein spaces of constant negative curvature and the AdS_2 metric satisfies these requirements.

This finishes our discussion on p–brane near–horizon geometries and their relation to domain–wall and Anti–de Sitter spaces. We will next introduce a string theory low energy limit which leads us into the p–brane near–horizon region, decoupled from the asymptotic Minkowski supergravity. From the p–brane worldvolume theory point of view the same low energy limit also decouples bulk Minkowski supergravity and leaves us with the non–trivial field theory living on the p–brane. This will lead to the surprising and interesting conjecture that p–brane field theories can be mapped to closed superstring theories on (p–brane near–horizon) domain–wall backgrounds. By now the best understood and checked example is that of N D3–branes in Type IIB string theory, which was first discussed by Maldacena, together with the other pure *AdS* backgrounds, in [110]. This was generalized to the other $D = 10$ Dp–branes in [108] and somewhat later in [109] using the dual frame metric.

5.2 The field theory limit

In this section we will set up the limit taking us into the near–horizon region, fixing the worldvolume field theory coupling constant and energy scale. We will work out this limit for the general class of p–branes described by (5.4). To obtain well–behaved near–horizon background solutions we need a constraint on our p–brane parameters. The result will be that dualities relating domain–wall supergravities having $a \neq 0$ to large N worldvolume field theories are only well–behaved for Dp–branes and their reduced intersections. We will first try to be as general as possible, only excluding the flat Minkowski near–horizon spacetimes, which will not be treated in this thesis.

5.2.1 The general setup

A string low energy[8] limit will always involve

$$u^2 \alpha' \to 0 \,, \tag{5.52}$$

as explained in section 2.2, where we substituted u for U to denote the natural energy scale. There are two ways to interpret this limit (5.52). One usually considers $u \to 0$ and keeps α' fixed. However one can equivalently consider fixed energies u and consider the limit $\alpha' \to 0$. We will use the last option and therefore consider the limit

$$\alpha' \to 0 \,, \tag{5.53}$$

keeping fixed a natural energy scale u. We will also assume that from the outset g_s is small, in order for the p–brane soliton solution to make sense in a string theory low energy limit.

In one regime the system we want to analyze consists of N p–branes described by a worldvolume theory, coupled perturbatively to a Minkowski bulk supergravity theory (so we neglect the back reaction of the N p–branes on the spacetime geometry). The dynamics of the (effective) field theory on the corresponding p–brane should be non–trivial. This means that at least one field theory coupling constant should be fixed in the limit (5.53). The field content of the p–brane worldvolume field theory determines the dimensions of the different coupling constants (scalars have dimensions different from vectors), which can be easily read off from the different kinetic terms. The dependence on g_s is fixed through

[8] It should be clear that when we consider M–theory we just replace the string length scale by the Planck length scale and we lose the string theory coupling constant.

the scaling of the effective tension of the p–brane under consideration, which is denoted by the parameter k (5.3). A p–brane soliton solution must be thought of as a stack of N microscopic single p–branes. We will fix (generalized) ’t Hooft coupling constants[9], which involve this integer N. In general we will assume the following structure of a p–brane worldvolume (’t Hooft) coupling constant

$$g_f^2 = c_p N g_s^k (\sqrt{\alpha'})^x, \tag{5.54}$$

with c_p some (dimensionless) constant. When considering M–theory branes, the g_s dependence is of course absent[10]. We introduced the parameter x to denote the dimension of the coupling constant, which is unconstrained (for now). Both c_p and x depend on the specific p–brane worldvolume theory fields under consideration. When considering Yang–Mills coupling constants of Dp–branes, x is equal to $p-3$. On the other hand, when considering scalar coupling constants $x = p-1$. We want the coupling constant (5.54) to stay fixed in the limit (5.53). Depending on the sign of x this has the following consequences

$$\begin{aligned} x < 0 \quad &\rightarrow \quad N g_s^k \sim \sqrt{\alpha'}^{-x} \rightarrow 0 \\ x > 0 \quad &\rightarrow \quad N g_s^k \sim \sqrt{\alpha'}^{-x} \rightarrow \infty . \end{aligned} \tag{5.55}$$

We will only be considering p–branes with $k > 0$. This is reasonable because otherwise the effective tension would scale with a positive power of g_s, saying that in a weak string coupling limit the tension of such an object would vanish. This implies the absence of solitonic solutions to the string effective equations of motion which are only defined in a weak coupling limit, and so we arrive at a contradiction because we do want to consider the existence of p–brane soliton solutions. Positive k and $g_s \ll 1$ imply that in order to keep g_f fixed we need $g_s \rightarrow 0$ when $x < 0$ and $N \rightarrow \infty$ when $x > 0$.

The non–trivial worldvolume field theory will be decoupled from the bulk Minkowski supergravity theory when the gravitational coupling constant vanishes in the limit (5.53). The gravitational coupling constant in D spacetime dimensions is proportional to

$$G_D \propto (\sqrt{\alpha'})^{D-2} g_s^2, \tag{5.56}$$

[9]These are called ’t Hooft coupling constants after ’t Hooft’s idea to treat $U(N)$ Yang–Mills theories in a $1/N$ expansion. The suggestion was that these theories might simplify when the number of colors N is large, because only planar Feynman diagrams contribute in a large N limit. If one could solve the theory with $N = \infty$ exactly, the hope was that one could analyze $SU(3)$ QCD beyond the perturbative weak coupling expansion by doing an expansion in $1/N = 1/3 < 1$.

[10]In fact the α' dependence also disappears in that case because the worldvolume field theories are scale invariant.

which clearly vanishes in the limit (5.53) and u and g_f fixed, as long as we do not consider taking $g_s \to \infty$. We conclude that in the limit described above, which we will refer to as the field theory limit, we end up with a bulk Minkowski supergravity theory decoupled from a non–trivial p–brane low energy effective worldvolume theory.

Now consider the p–brane supergravity soliton solution (5.4). We have to define a natural energy scale with respect to the p–brane solution which coincides with the one in the worldvolume field theory and which should be kept fixed in the limit (5.53). We already encountered an energy scale when we discussed the AdS_{d+1} horospherical coordinates, where we defined a parameter u with the dimensions of mass (5.15). This can be shown to be the natural energy scale associated to a massless supergravity field probing the p–brane near–horizon geometry [121]. That energy scale could only be defined when $\beta \neq 0$ (5.16), so we will only be discussing p–branes with AdS_{d+1} near–horizon geometries (in the dual frame). The cases with $\beta = 0$, resulting in a dual frame flat Minkowski near–horizon geometry, can be treated but we refer to [108, 109] and [122] to learn more about the holographic duality conjectures in these special cases.

We want to replace all quantities appearing in the harmonic function by the fixed parameters in the field theory limit. All r dependence will be replaced by u and we also need to express r_0 in terms of the appropriate string theory parameters. In Appendix A we deduce that

$$r_0^{\tilde{d}} = (d_p N g_s^{2-k}) \sqrt{\alpha'}^{\tilde{d}} , \tag{5.57}$$

where we introduced a dimensionless constant d_p. We can now rewrite the harmonic function $H(r)$ in terms of the fixed quantities u and g_f^2, extracting powers of α' and g_s which are left–over. This gives

$$H = 1 + (\sqrt{\alpha'})^{\frac{x-\tilde{d}}{\beta}} (g_s)^{\frac{2(k-1)}{\beta}} \left[g_f^2 (u\beta)^{\tilde{d}} \left(\frac{d_p}{c_p} \right) \right]^{-1/\beta} . \tag{5.58}$$

The field theory limit will take us into the p–brane near–horizon region when

$$\frac{x-\tilde{d}}{\beta} < 0 . \tag{5.59}$$

The power of g_s could still spoil this behavior, but we will soon constrain our parameters in such a way that this possibility is excluded.

Let us for the moment assume that the constraint (5.59) is fulfilled and the field theory limit takes us into the p–brane near–horizon geometry. We will use the dual frame metric solution written in terms of horospherical coordinates and we will not give the expression for the field strength, which is of less importance. Expressing the p–brane near–horizon solution in terms of the fixed quantities, removing all the α' dependence in the dilaton, we find

$$\begin{aligned} \mathrm{d}s^2 &= (d_p N g_s^{2-k})^{2/\tilde{d}}\,\alpha' \left[(u\beta)^2 \mathrm{d}x_d^2 + \left(\frac{1}{u\beta}\right)^2 \mathrm{d}u^2 + \mathrm{d}\Omega^2_{\tilde{d}+1} \right] \\ e^{\Phi} &= g_s^{1+\frac{(D-2)a}{2\Delta\beta}(k-1)} \left(N g_s^k\right)^{\frac{a(D-2)(\tilde{d}-x)}{4\Delta\beta x}} \left[(g_f^2)^{1/x} (u\beta) \left(\frac{d_p^{1/\tilde{d}}}{c_p^{1/x}} \right) \right]^{\frac{-(D-2)a}{8}\left(\frac{\beta+1}{\beta}\right)} \end{aligned} \tag{5.60}$$

We can rescale the metric to lose the factors $(d_p N g_s^{2-k})^{2/\tilde{d}}$ and α'. This will introduce (extra) α' and $(d_p N g_s^{2-k})^{2/\tilde{d}}$ dependence in the dual frame action (5.10). Collecting all α' dependence, we find the important result that all α''s drop out. All g_s dependence nicely combines into the $e^{\delta\Phi}$ in front of the dual frame action. After the rescaling our action (5.10) becomes

$$S_D = \int \mathrm{d}^D x \,\sqrt{g_D} (d_p N)^{(D-2)/\tilde{d}} e^{\delta\Phi} \left[R_D + \gamma(\partial\Phi)^2 - \frac{1}{2(d_p N)^2 (\tilde{d}+1)!} F^2_{\tilde{d}+1} \right]. \tag{5.61}$$

This means that if the dilaton expression is non–singular in the field theory limit, we are left with a near–horizon supergravity theory with a finite Planck length and a finite string coupling constant defined by e^{Φ}! This strongly suggests that in the field theory limit on the supergravity soliton side, we end up with a superstring theory on the $AdS_{d+1} \times S^{\tilde{d}+1}$ p–brane near–horizon background with new (finite) parameters $\tilde{\alpha}'$ and $\tilde{g}_s$. The field theory limit decouples this near–horizon superstring theory from the asymptotic Minkowski superstring theory we started with.

The special cases $a = 0$ imply a constant dilaton and the supergravity background carries an unbroken $SO(2,d)$ isometry. This should match with a conformal symmetry group in the worldvolume field theory description, meaning the fixed coupling constant (5.54) should be dimensionless or $x = 0$. Those cases (which include the M–branes) do not require a parameter restriction and will be discussed separately in 4.2.2. When $a \neq 0$ non–singular dilaton expressions in the field theory limit require a restriction on our p–brane parameters. We will now constrain our p–brane parameters such that the near–horizon limit (5.59) is guaranteed and the new string coupling constant e^{Φ} is finite or independent of the old string coupling constant g_s.

In the analysis of the effect of the field theory limit on the supergravity soliton, we defined a fixed energy scale u. A priori there is no reason for this fixed energy scale to be the same as the natural energy scale in the worldvolume field theory. To be able to compare both descriptions we need related or better, equivalent fixed energy scales. To determine these relations we need to probe the system under consideration in both descriptions by the same objects or fields. Suppose we are dealing with N Dp–branes probed by a single Dp–brane. Then we know how to relate a natural energy scale in the Dp–brane Yang–Mills worldvolume field theory to the length of stretched strings, defining a distance scale in the bulk. Open strings stretching from the probe brane to the system of N Dp–branes correspond to energy scales equal to

$$U = \frac{r}{\alpha'}\,, \tag{5.62}$$

which is just the distance between the probe and the system of N Dp–branes times the open string tension (see 1.1.5). This is obviously not the same as the definition of the energy scale u (5.17), which we kept fixed when considering the field theory limit in the supergravity soliton description. The two energy scales are related in the following way

$$u\beta = \alpha'^{\frac{x+\tilde{d}-\Delta}{\Delta}}\, g_s^{\frac{4(k-1)}{\Delta}} \left(\frac{d_p}{c_p} g_f^2\right)^{\frac{-2}{\Delta}} U^{\beta}\,. \tag{5.63}$$

We would like u and U to be related through fixed quantities only. Otherwise fixed energy scales U in the worldvolume field theory would correspond to diverging energy scales u in the supergravity soliton description and vice versa. Looking at (5.63) this is only possible when

$$k = 1 \quad , \quad x = \Delta - \tilde{d}\,. \tag{5.64}$$

The first constraint $k = 1$ just confirms our restriction to Dp–branes in any dimension. The second constraint is interesting, because it tells us which coupling constant in the worldvolume theory we should keep fixed. Until now we kept x as a free parameter, but now we see we have to fix it in order to connect the Dp–brane supergravity soliton and Dp–brane field theory energy scales. When $D = 10$ and $\Delta = 4$ we find $x = p - 3$, telling us that we should keep fixed the 't Hooft Yang–Mills coupling constant (5.54). The connection between the two energy scales was discussed extensively in [121]. There it was observed that u is the natural energy scale for supergravity probes (instead of Dp–brane probes), which could also be obtained in the worldvolume field theory by considering the self–energy of a point charge. The point charge is the interpretation of the stretched string from the worldvolume gauge theory point of view, which has energy U. However, the self–energy is also proportional to the effective strength of the Coulomb interaction and this will reproduce (5.63). To get the correct holographic relation between the number of degrees of freedom on both sides of the duality [123], the energy scale u should be used and therefore this parameter is also called the holographic energy scale. This also means that from the holographic point of view, the supergravity fields are the natural holographic probes of the AdS geometry. In [109] it was noted for the first time that the holographic energy scale u is the natural energy scale coordinate for AdS spacetimes obtained as the near–horizon geometries of Dp–branes in $D=10$ in the dual frame. We have seen that this phenomenon extends to p–branes in arbitrary dimensions.

Using the restriction (5.64) first of all drops all g_s dependence in (5.58), and the power of α' (5.59) becomes $-\Delta$. Because we only consider $\Delta > 0$, it is guaranteed that the field theory limit takes us into the near–horizon region. Importantly, using the constraints (5.64) we find the following expression for e^{Φ} in the field theory limit

$$e^{\Phi} = \frac{1}{N}\left[(g_f^2)^{1/x}(u\beta)\left(\frac{d_p^{1/\tilde{d}}}{c_p^{1/x}}\right)\right]^{\frac{-(D-2)a}{8}\left(\frac{\beta+1}{\beta}\right)} . \tag{5.65}$$

Remember that $x = \Delta - \tilde{d}$ in this expression. This is finite (at least when we do not consider $N \to \infty$) and defines a new string coupling constant independent of the old Minkowski string coupling constant g_s. From (5.65) we conclude that this new coupling constant is proportional to $1/N$. So in a large N limit we obtain a weakly coupled string theory and string theory quantum corrections are $1/N$ effects.

We are now ready to state the conjecture naturally following from the above analysis, restricting the parameters as in (5.64) when $a \neq 0$. Fixing an energy scale u and a 't Hooft coupling constant g_f, the low energy limit $\alpha' \to 0$ decouples Minkowski supergravity in both descriptions. We are left with a p–brane worldvolume field theory on one side and a well defined domain–wall supergravity solution on the other side. Choosing a fixed energy scale and 't Hooft coupling constant we can now either describe the system of p–branes by a closed superstring theory on a domain–wall background, or by the p–brane worldvolume field theory and both descriptions are conjectured to give the same results.

This automatically leads to the conjecture that *closed superstring theory on a $DW_{d+1} \times S^{\tilde{d}+1}$ p–brane near–horizon background is dual to the d–dimensional p–brane worldvolume field theory*. These kind of relations we will very often call domain–wall/quantum field theory dualities (or in short DW/QFT dualities). The suggestion to go beyond the supergravity approximation is based on the fact that we found finite Planck length and string coupling in the supergravity analysis. Of course when considering M–branes we loose the dilaton (and thus the string coupling) and we should replace "*closed superstring theory on a $DW_{d+1} \times S^{\tilde{d}+1}$ p–brane near–horizon background*" by "*M–theory on a pure $AdS_{d+1} \times S^{\tilde{d}+1}$ M–brane near–horizon background*".

Let us explain what we mean when we say that the two descriptions are *dual* to each other. This will become clear when we start analyzing the regions of the fixed quantities u and g_f (and usually N) where the different descriptions are in their perturbative, calculable, regime. We will first analyze the perturbative regime of the worldvolume field theory giving a restriction on the quantities u and g_f and after that deduce the restriction needed on the quantities u and g_f (and N) to be in a perturbative closed string regime.

Although the field theory description is in principle defined non–perturbatively, in practice we (almost always) need a perturbative expansion which is only defined for small effective dimensionless coupling constant. This effective dimensionless coupling constant in the worldvolume field theory can be constructed from the energy scale u and the coupling constant g_f^2 and should be much smaller than one, giving

$$g_{\text{eff}}^2 = g_f^2 u^x \ll 1\,. \tag{5.66}$$

Depending on the sign of x the perturbative field theory description will either be valid when $u \ll 1$ or $u \gg 1$. This effective coupling constant determines the (classical) scaling of the supersymmetric d–dimensional quantum field theory under consideration. We note that this is the combination of u and g_f that appears in the dilaton background (5.65).

The situation is different for the string theory (as explained in 1.1.6), which first of all is only *defined* for weak string coupling. Also the domain–wall background solution on which the string theory is defined can only be trusted as long as the spacetime curvatures are small and finite size corrections can be neglected. We can use a supergravity approximation (neglecting string loop diagrams) when the string coupling and the curvature of the background are both small. Let us investigate the regions in the p–brane near–horizon background solution where we can trust this supergravity approximation.

Small curvature (as seen by closed strings) can be translated in demanding that the effective tension in the dual frame times the characteristic spacetime length is large. The characteristic spacetime length is determined by the dual frame $AdS_{d+1} \times S^{\tilde{d}+1}$ metric and is of order one (in $\tilde{\alpha}'$ units). Calculating the effective string tension in the dual frame (using (5.9) and (2.1) we find small curvature when

$$\tau_s = \left(d_p N e^{(2-k)\Phi}\right)^{2/\tilde{d}} \gg 1\,. \tag{5.67}$$

We note that we did not yet use the constraint $k = 1$ in the above expression.

Small string coupling, which is defined by the dilaton expression (5.65) after the constraint (5.64), can be translated into the constraint

$$e^{\Phi} = \frac{1}{N}\left[g_{\mathrm{eff}}^2 \left(\frac{\beta^x d_p^{1/\tilde{d}}}{c_p}\right)\right]^{\frac{-(D-2)a}{8}\left(\frac{\beta+1}{\beta x}\right)} \ll 1\,. \tag{5.68}$$

This last constraint can always be satisfied for generic u (except for the special points $u \to 0$ or $u \to \infty$) by taking N very large. This is a general feature of DW/QFT dualities, a supergravity approximation at least requires a large N limit. We note that when we take $k = 1$ in (5.67) the overall N dependence will drop out. This means that generically we can only expect supergravity to be a good approximation within a finite region of the complete background [108, 109]. Quantum–gravitational corrections can be included by taking into account string loop diagrams (which as we already mentioned can be identified with $1/N$ effects) and this string loop expansion will be restricted to the same region on the domain–wall background.

As we will see, in most cases the region where supergravity is a valid description is not overlapping with the region in which the perturbative field theory is a good description. This means that at a particular scale u and coupling constant g_f there only exists one perturbatively well–defined theory which, through the conjecture, can be mapped to a non–perturbative regime in the other theory. So a strongly coupled theory can be mapped to a weakly coupled (different) theory and it is precisely in this sense that the theories are said to be dual to each other. In fact we could have expected that, because sure enough we know that perturbative quantum field theory is very different from any (super)gravity theory.

This new kind of duality between large N SYM field theories and closed superstring theories (including quantum gravity) can be used to study non–perturbative physics on either side. First of all, although we did not show this here, in the procedure leading to the conjecture we can also introduce a small non–extremality parameter (giving rise to some excitation energy) in the background solution breaking the supersymmetry. This will correspond to a finite temperature configuration on the dual field theory side. Such a background solution has a true horizon and will Hawking radiate (a quantum process) towards thermal equilibrium[11]. Using the conjectured duality we can map this semiclassical gravity result to a process in a finite temperature large N SYM field theory, which means that Hawking radiation can be described by a unitary process associated with $1/N$ effects in the large N quantum field theory.

On the other hand we could also use the duality map to learn about large N quantum field theory, using (semiclassical) supergravity. Let us also remark that this duality is an explicit manifestation of the holographic principle [105, 106]. It was shown in [123] that the number of degrees of freedom in AdS space satisfy the holographic bound of one per surrounding Planck area. The way the two theories are related is by identifying the coordinate u on the domain–wall supergravity side with the scale parameter in the quantum field theory. Moving from $u = 0$ to the boundary at $u = \infty$ in the AdS spacetime should be interpreted as going from the infrared (IR) to the ultraviolet (UV) in the quantum field theory. This also means that large distances (IR) in the quantum field theory correspond to small distances in the domain–wall supergravity (near the center) and vice versa. The special properties of AdS metrics allow for this so–called UV–IR connection. Let us next discuss more specific examples present in our general setup. Along the way we will give some more details of the duality map.

5.2.2 The AdS/CFT examples

In this subsection we will take a better look at the pure AdS examples, which require a constant dilaton. This means that the dilaton coupling parameter $a = 0$. Remember that we do not need the constraint (5.64) now. Also remember that we want $a = 0$ to cover M–branes as well, in which case the string coupling constant is non–existent and we should replace the string length scale by the appropriate M–theory Planck length scale.

The complete background will have unbroken $SO(2,d)$ isometries, which will correspond to invariance under the conformal group in the dual field theory. The

[11]For a black p–brane in Minkowski spacetime this means that the p–brane will evolve into the extremal BPS p–brane. In AdS spacetimes, which act as finite volume boxes, the equilibrium situation will be the one where the temperature of the black p–brane equals the temperature of the surrounding gas of emitted Hawking radiation.

dual field theory therefore has to be a superconformal field theory, which means the parameter x should vanish (otherwise the coupling would classically run). For Δ and β we find in this case

$$\beta = \frac{\tilde{d}}{d} \quad , \quad \Delta = \frac{2d\tilde{d}}{d+\tilde{d}} . \tag{5.69}$$

Using these results we find that the harmonic function in the field theory limit can be written as (5.58)

$$H = 1 + (l_f)^{-d} \left[g_f^2 \, (u\beta)^{\tilde{d}} \left(\frac{d_p}{c_p} \right) \right]^{-1/\beta} , \tag{5.70}$$

where we replaced the $\sqrt{\alpha'}$ by the appropriate fundamental length scale l_f under consideration (which is the Planck length in M–theory and the string length in string theory). Because $d > 0$ always, the field theory limit is guaranteed to take us into the near–horizon region of the background solution. All the branes listed in Table 5.1 are covered by this analysis, which were studied in the paper by Maldacena [110].

When embedded in a string theory we always find $\beta = 1$, which also means $d = \tilde{d}$ and $\Delta = d$ through (5.69). Using (5.3) we can also conclude that $k = 1$, so the branes under consideration have to be D–branes. This also means the constraint (5.64) is satisfied and using (5.63) we find that $u \propto U$, so in this case the two energy scales are essentially equivalent. The fixed conformal field theory coupling constant is proportional to

$$g_f^2 \propto N g_s . \tag{5.71}$$

This coupling constant has to be small if we want to obtain perturbative conformal field theory results. In theories without supersymmetry classical conformal invariance is usually broken by quantum effects, as is represented by the Wilsonian renormalisation group equations. In supersymmetric theories however the conformal invariance can be maintained at the quantum level, which is necessary for the AdS/CFT duality. Looking at the condition for small AdS curvature (5.67) we find

$$\tau_s \propto (g_s N)^{2/d} = (g_f^2)^{2/d} \gg 1 . \tag{5.72}$$

The new string coupling constant can be written as $g_s = g_f^2/N$, which has to be small in a supergravity approximation. Combining these two requirements we conclude that we need large $N \gg g_f^2 \gg 1$ for the supergravity approximation to be a valid description. String quantum corrections are governed by the string coupling constant and are therefore $1/N$ corrections, as we concluded before. The size of the sphere $S^{\tilde{d}+1}$ and the radius of the AdS_{d+1} embedding are both equal to r_0 and therefore proportional (in $\tilde{\alpha}'$ units) to

$$r_0 \propto (g_s N)^{1/d} = (g_f^2)^{1/d} . \tag{5.73}$$

We can use (5.34) to translate this into the value of the cosmological constant

$$\Lambda \propto \frac{d(d-1)}{(g_f^2)^{2/d}} . \tag{5.74}$$

We clearly obtain small $\Lambda \ll 1$ or large $r_0 \gg 1$ in the supergravity approximation.

Similar equations hold for M–branes. We loose the constraint of small string coupling, g_s disappears altogether and the fixed conformal field theory coupling constant is just proportional to N. The only requirement for a supergravity approximation is now small curvature and this will involve taking a large N limit. For M–branes the parameter β can take on two values. For M2–branes (or intersections) we find $\beta = 2$ and for M5–branes (or intersections) we find $\beta = \frac{1}{2}$. The energy parameter u (5.15) can be written as

$$u\beta = \frac{r^{\beta}}{(d_p N)^{(\beta+1)/\tilde{d}} l_{11}^{(\beta+1)}} . \tag{5.75}$$

When $\beta = \frac{1}{2}$ this energy scale can be interpreted as the squareroot of the distance scale between M5–branes times the tension of a membrane $T_{\mathrm{M2}} \propto 1/l_{11}^3$. So in the field theory limit we keep the length of stretched membranes fixed. This is of course very similar to Dp–branes in string theory and confirms the interpretation of M5–branes as topological defects on which open supermembranes can end. For $\beta = 2$ we do not find such a nice interpretation.

Because of exact conformal invariance these dualities are not restricted to particular regimes in the AdS background. The coupling constants are independent of the scale parameter u, and so the supergravity approximation constraints can be satisfied on the complete $AdS_{d+1} \times S^{\tilde{d}+1}$ background by just considering a large N limit. Conformal invariance also makes it easier to perform tests of these dualities, basically because some observables (like 2– and 3–point functions) satisfy very stringent constraints because of conformal invariance. This ensures that some perturbative properties on the conformal field theory side can be extended to strong ’t Hooft coupling. These results can then be compared with (semiclassical) AdS

supergravity results to test the conjecture. Of course we need a specific relation between the fields on both sides of the conjecture to be able to compare results. The general AdS/CFT duality map was first constructed by Witten in [124] and was suggested earlier for D3–branes in [125]. These maps position the conformal field theory on the boundary of AdS_{d+1} and couple boundary values of AdS supergravity fields to conformal operators living on the boundary. The choice for the boundary of AdS is a natural one because the AdS isometries indeed act on the boundary as ordinary conformal field theory transformations, as we discussed in 4.1.2.

A first requirement for the duality to be possible is that the symmetries match on both sides. These can be checked case by case and should include the isometries from the sphere $S^{\tilde{d}+1}$, giving a $SO(\tilde{d}+2)$ global symmetry group on the supergravity side. These global symmetries are indeed also found as the so–called $\mathscr{R}$–symmetry group on the superconformal field theory side. Conformal symmetry and the number of supersymmetries also match up in all cases. Using the duality maps many checks were made of the AdS/CFT conjecture, mainly for the D3–brane and the self–dual string (a D1–D5 intersection) in $D=6$. Especially the D3–brane case, where the dual field theory is a $D=4$ superconformal Yang–Mills theory, attracted a lot of attention. All these checks so far confirmed the AdS/CFT conjecture and for more details we refer to the review paper [112] and references therein.

The examples which are not yet understood are the five– and four–dimensional Reissner–Nordström extremal black holes. Those cases are conjectured to give rise to an $AdS_2 \times S^2$ and $AdS_2 \times S^3/CFT_1$ duality respectively. The field theories in these cases should reduce to supersymmetric conformal quantum mechanics models [126, 127], which are difficult to construct. An immediate problem that arises is that, as opposed to higher dimensional field theories, a (conformal) quantum mechanics model does not describe any internal dynamics if we have to assume that all the BPS particles lie on top of each other. The same problem occurs in the conjectured dual AdS_2 supergravity description which displays a mass gap, telling us that small excitations of the AdS_2 supergravity fields can not exist. The duality conjecture therefore reduces to a map between two non–dynamical theories, which does not seem very interesting. Another basic problem refers to the fact that AdS_2 has two boundaries and the question then arises on which boundary the dual conformal quantum mechanics model should live [128]. Another problem was discussed in [129], where it was observed that the AdS_2 spacetime is not a stable background vacuum solution, but instead can fragment into multiple AdS_2

spacetimes.

Although potentially these AdS_2/CFT_1 dualities could teach us a lot about (extremal) black hole dynamics, as advocated in in [130, 131], the situation at this moment is not well understood. What has become clear is that the AdS_2/CFT_1 case requires an understanding of what the duality conjecture means which is different from all the higher–dimensional cases. More recent progress in constructing supersymmetric conformal quantum mechanics models and the interpretation of the duality can be found in [132, 133, 134, 135] and in the review paper [136].

5.2.3 Non–trivial dilaton Dp–branes in $D = 10$

Let us first discuss all ten–dimensional Dp–branes, except for the already covered D3–brane. These were first discussed in [108, 109], except for the special case of the D8–brane. The only difference with the pure AdS cases is the appearance of a non–trivial dilaton, forcing us to use the constraint (5.64) and signalling the breaking of the conformal isometry group of the complete background solution. The non–trivial dilaton, turning the background solution into a generic domain–wall, is also responsible for the breaking of supersymmetry. The number of broken supersymmetries is obviously the same as the number of broken supersymmetries in the original Dp–brane solution, which is $\frac{1}{2}$ of the maximum number 32. Remember that the pure AdS vacuum solution preserved all of the supersymmetries and with respect to the original p–brane soliton solution this represented supersymmetry enhancement in the near–horizon geometry.

The string coupling constant will depend on the (radial) AdS energy scale u and therefore naturally represents a (classically) running coupling constant in the dual field theory. This also means that when we want to use a supergravity approximation we will be forced into a region (an energy scale u interval) of the complete domain–wall supergravity background where the string coupling is small. Similarly the spacetime curvature of the background, when probed with strings, is no longer constant because the dilaton scalar will now contribute as well. Therefore it is non–trivial to find energy regions where supergravity will be a good approximation.

The constraint (5.64) for $D = 10$ and $\Delta = 4$ gives $k = 1$ and more importantly $x = p - 3$. The fixed ’t Hooft coupling constant (5.54) therefore equals

$$g_f^2 = c_p N g_s \sqrt{\alpha'}^{(p-3)} = N g_{YM}^2 \,, \tag{5.76}$$

which is the appropriate scaling for a coupling constant of a $p+1$–dimensional Yang–Mills gauge theory. We will keep this quantity fixed, which for $p > 3$ means we have to take $N \to \infty$ if we want to be sure that the bulk supergravity decouples. From the Yang–Mills theory point of view this is the 't Hooft limit of an infinite number of colors. The supersymmetric Yang–Mills theory under consideration will of course be the one obtained as a low–energy limit of Dp–branes, which as we discussed in section 2.3.2 can all be obtained from a toroidal reduction of the $D = 10$ SYM theory. These include scalars which for $p \geq 1$ will be frozen into their vacuum expectation values[12] and will play no role in the actual dynamics. Remember that even p D–branes are string solitons of IIA superstrings, whereas odd p D–branes are part of IIB superstring theory.

Other $D = 10$ Dp–brane parameters we will need are

$$\beta = \tfrac{1}{2}(5-p) \quad , \quad a = \tfrac{1}{2}(3-p)\,. \tag{5.77}$$

For $p = 5$ we see that $\beta = 0$ and therefore the near–horizon geometry becomes $d+1$–dimensional flat space in the dual frame metric. This excludes the D5–branes from our discussion. Also remember that the IIB D7–brane is excluded because $\tilde{d} = 0$. The relation between the Dp–brane energy scale U and the holographic energy scale u now becomes

$$U^{5-p} = \tfrac{1}{4}(5-p)^2 \left(\frac{d_p}{c_p}\right) g_f^2 u^2\,. \tag{5.78}$$

Plugging in the different $D = 10$ Dp–brane parameters the dilaton background (5.65) becomes

$$e^{\Phi} = \frac{1}{N}\left[\tfrac{1}{2}|5-p|(g_f^2)^{\frac{1}{p-3}}\, u \left(\frac{d_p^{1/(7-p)}}{c_p^{1/(p-3)}}\right)\right]^{\frac{(p-3)(7-p)}{2(5-p)}}\,. \tag{5.79}$$

Let us analyze for each Dp–brane where we can trust a supergravity approximation on the one hand and a perturbative supersymmetric Yang–Mills gauge theory approximation on the other hand.

The effective dimensionless coupling constant (5.66) in the perturbative Yang–Mills theory equals

$$g_{eff}^2 = g_f^2\, u^{(p-3)}\,. \tag{5.80}$$

This represents the classical scaling of the coupling constant which because of supersymmetry should not be affected by quantum effects in the field theory. Necessarily a perturbative field theory analysis can only be trusted in the following

[12]Put differently, for $p \geq 1$ the positions of the N Dp–branes are fixed and determine the gauge theory vacuum.

energy regimes

$$\begin{aligned} u &\gg (g_f^2)^{\frac{-1}{(p-3)}} \quad \forall\, p<3 \\ u &\ll (g_f^2)^{\frac{-1}{(p-3)}} \quad \forall\, p>3\,. \end{aligned} \tag{5.81}$$

In a supergravity approximation we need small curvature (5.67) and at the same time small string coupling e^{Φ} (5.79). Small curvature means[13]

$$\tau_s \propto \left(g_{eff}^2\right)^{\frac{1}{(5-p)}} \gg 1\,. \tag{5.82}$$

Similarly small string coupling (5.79), when rewritten using g_{eff}, gives the condition

$$e^{\Phi} \propto \frac{1}{N}\left(g_{eff}^2\right)^{\frac{(7-p)}{2(5-p)}} \ll 1\,. \tag{5.83}$$

Translating the conditions (5.82) and (5.83) into conditions on the energy scale u we find

$$\begin{aligned} u &\ll (g_f^2)^{\frac{-1}{(p-3)}} \quad \forall\, p<3 \\ u &\gg (g_f^2)^{\frac{-1}{(p-3)}} \quad \forall\, 3<p<5 \\ u &\ll (g_f^2)^{\frac{-1}{(p-3)}} \quad \forall\, p>5\,, \end{aligned} \tag{5.84}$$

for small curvature. Notice that these regimes are opposite to the perturbative field theory regimes (5.81) for $p<5$. For $p>5$ these regimes overlap. Small string coupling (5.83) translates into

$$\begin{aligned} u &\gg (g_f^2)^{\frac{-1}{(p-3)}}\, N^{\frac{(p-3)(7-p)}{2(5-p)}} \quad \forall\, p<3 \\ u &\ll (g_f^2)^{\frac{-1}{(p-3)}}\, N^{\frac{(p-3)(7-p)}{2(5-p)}} \quad \forall\, 3<p<5 \\ u &\gg (g_f^2)^{\frac{-1}{(p-3)}}\, N^{\frac{(p-3)(7-p)}{2(5-p)}} \quad \forall\, 5<p<7 \\ u &\ll (g_f^2)^{\frac{-1}{(p-3)}}\, N^{\frac{(p-3)(7-p)}{2(5-p)}} \quad \forall\, p>7\,. \end{aligned} \tag{5.85}$$

Combining the conditions (5.84) and (5.85) to find the regions where supergravity is a good approximation, will necessarily give a condition on N to be able to satisfy both conditions at the same time (except for the D8–brane). We find

$$N^{\left|\frac{(p-3)(7-p)}{2(5-p)}\right|} \gg 1 \quad \forall\, p<7\,. \tag{5.86}$$

[13]In our analysis of small curvature and string coupling we suppress all constants of order 1, e.g. p, c_p and d_p.

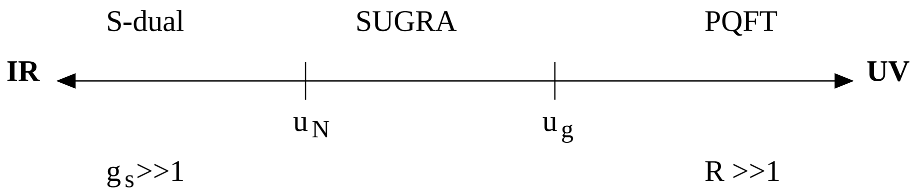

Figure 5.2: Different regimes in the energy plot for Dp–branes with $0 \leq p < 3$ and $N \gg 1$. The terminology should be self–explanatory and was discussed in the main text.

A supergravity approximation therefore always involves a large N limit. The only exception being the D8–brane, but remember that in that case we had to take $N \to \infty$ anyhow to keep g_f^2 fixed and to decouple gravity. The D8–brane was first discussed in [107].

The $D = 10$ Dp–branes can be divided into four types of qualitatively different behavior [108, 109, 107].

- Dp–branes with $0 \leq p < 3$. Summarizing the previous discussion we find that the supergravity approximation is valid in the IR when

$$u_N = (g_f^2)^{\frac{-1}{(p-3)}} N^{\frac{(p-3)(7-p)}{2(5-p)}} \ll u \ll u_g = (g_f^2)^{\frac{-1}{(p-3)}}, \qquad (5.87)$$

which implies large N. The perturbative quantum field theory description applies in the UV, when

$$u \gg u_g . \qquad (5.88)$$

We end up in a strongly coupled string theory regime when

$$u \ll u_N . \qquad (5.89)$$

In the inequality (5.87) we defined the two critical points u_N and u_g, where the subscript refers to their (different) dependence on the fixed quantities. The critical point u_g describes the crossover from small to large curvature ($R \gg 1$) in the string theory description and the transition from perturbative (PQFT) to non–perturbative in the field theory description. The critical point u_N describes the crossover from weak to strong string coupling. When $N = 1$ the two critical points become equal and the separation between u_g and u_N increases with N, which motivates the subscript N on u_N. We plotted the different regimes in Figure 5.2.

Depending on whether we are in Type IIA ($p = 0$ and $p = 2$) or Type IIB superstring theory ($p = 1$) we can invoke M–theory or S–dual Type IIB theory respectively to try to obtain another weakly coupled description of this system of Dp–branes in the far IR. The Dp–brane background itself should then also be transformed. For the D2–brane the solution transforms into the M–theory M2–brane, which we already discussed in the AdS/CFT subsection. It follows that the $(2+1)$–dimensional field theory flows to a conformal fixed point in the IR, which has a dual description as the $AdS_4 \times S^7$ M2–brane near–horizon supergravity in the large N limit [108].

The S–dual background for the Type IIB D1–brane is the fundamental ($k = 0$) Type IIB F1–brane, which has a curvature singularity at $u = 0$ and therefore a supergravity approximation is not possible in the far IR. However the $(1+1)$–dimensional Yang–Mills gauge theory flows to a conformal fixed point in the IR [75]. This suggests that the curvature singularity in the S–dual F1 background is an artifact and is resolved by a description as a conformal fixed point in the $1+1$–dimensional gauge theory [108].

The S–dual D0–brane background is the M–wave, which just represents $D = 11$ momentum. Wrapping the M–wave on a light–like compact direction with N units of momentum will give the N D0–branes near–horizon solution [137]. The S–dual gravitational theory in the IR is therefore M–theory on a compact light–like direction with N units of momentum. This is the strongly coupled region of the dual N D0–branes quantum mechanics model. This quantum mechanics model is nothing but the Matrix model and we now conclude that only the strongly coupled IR limit of the Matrix model will describe DLCQ M–theory. We can now also understand why we reached a different conclusion in section 3.1.1. When we "deduced" the Matrix model we neglected the gravitational backreaction of the N units of M–theory momentum. The agreement between *perturbative* Matrix model results and DLCQ M–theory (in a supergravity limit) should probably be understood as a consequence of non–renormalisation theorems due to supersymmetry. The Matrix theory conjecture has now been reduced to a special example of the more general DW/QFT correspondence [137, 138]. Although the above sketched scenario seems plausible, a remaining problem is that we do not understand the duality map between quantum mechanics models and DW_2 backgrounds very well. These problems were already explained in section 4.2.2 where we discussed the AdS_2/CFT_1 examples.

The established relation between the (uncompactified) Matrix model and

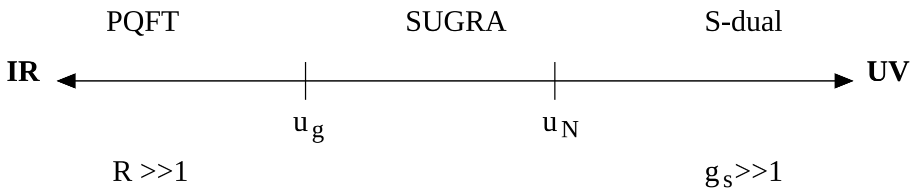

Figure 5.3: Different regimes in the energy plot for the D4–brane with $N \gg 1$.

the more general DW_2/QFT_1 correspondence seems to break down when we consider Matrix model compactifications. According to the Matrix model conjecture $p+1$–dimensional Yang–Mills theory defined on a torus T_p will describe DLCQ M–theory on the T–dual torus. At first sight these Matrix model conjectures can not be obtained as special examples of the DW/QFT correspondence if $p > 0$. Supposedly a relation should exist [138], but a detailed understanding seems to be missing so far.

- The D4–brane. The supergravity regime is defined in the UV by

$$u_g = (g_f^2)^{-1} \ll u \ll u_N = (g_f^2)^{-1} N^{\frac{3}{2}} . \tag{5.90}$$

The $D = 5$ perturbative quantum field theory description applies in the IR, when

$$u \ll u_g . \tag{5.91}$$

Strong string coupling is encountered in the far UV when

$$u \gg u_N . \tag{5.92}$$

We plotted the different regimes in Figure 5.3.

In the strong string coupling regime we can try to go to the S–dual description, which would be the M–theory M5–brane. In the UV regime we therefore obtain the AdS/CFT duality between $AdS_7 \times S^4$ supergravity and a $(5+1)$–dimensional conformal field theory. This also means that the $(4+1)$–dimensional gauge theory of the D4–branes should flow in the UV to a $(5+1)$–dimensional conformal fixed point theory. This was also suggested earlier in studies involving Matrix theory on T^4 and T^5 [61, 62] . To decouple gravity and to fix the 't Hooft coupling constant we mentioned that we had to take $N \to \infty$ and small g_s. However in this case we can

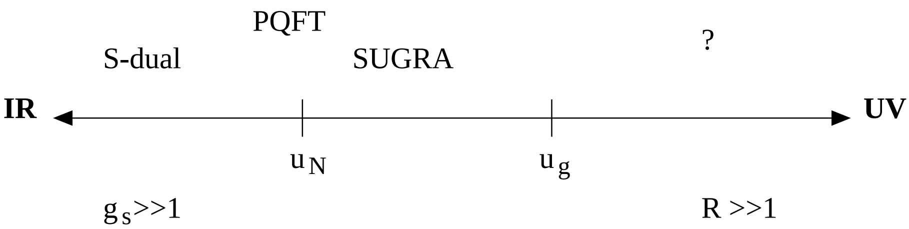

Figure 5.4: Different regimes in the energy plot for the D6–brane with $N \gg 1$. The question mark represents our lack of knowledge for both the QFT and the string theory in the UV. When $N \to \infty$, $u_N \to 0$ and the S–dual string theory region will shrink to zero size.

take $g_s \to \infty$ and finite N to fix the coupling constant and decouple *eleven–dimensional* bulk gravity to leave us with the (conformal) M5–brane worldvolume field theory. Classically this is a self–dual rank 2 gauge theory, whose quantum version is not yet understood.

- The D6–brane. In the IR there exists a valid supergravity regime bounded by

$$u_N = (g_f^2)^{\frac{-1}{3}} N^{\frac{-3}{2}} \ll u \ll u_g = (g_f^2)^{\frac{-1}{3}}\,, \tag{5.93}$$

which partially overlaps with the perturbative field theory regime, defined by

$$u \ll u_g\,. \tag{5.94}$$

At the same IR end of the energy spectrum, when

$$u \ll u_N\,, \tag{5.95}$$

we end up in a region of strong string coupling. We plotted the different regimes in Figure 5.4.

The strong string coupling regime can perhaps be resolved by going to eleven–dimensional M–theory. This will amount to considering Kaluza–Klein monopoles, their near–horizon region and their worldvolume field theory description. This time it is impossible to consider a limit $g_s \to \infty$ and finite N to fix the worldvolume field theory coupling constant and decouple *eleven–dimensional* gravity. So the best we can do is consider the limit $N \to \infty$. Essentially this excludes the appearance of a strong string

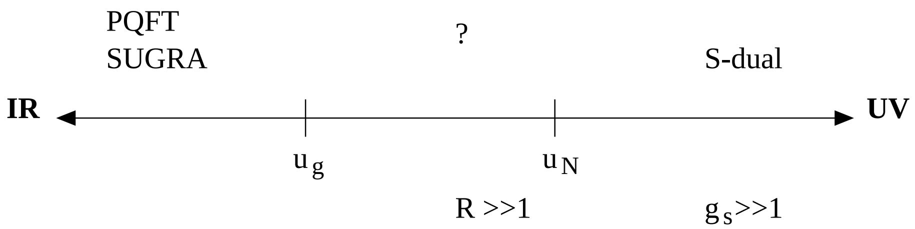

Figure 5.5: Different regimes in the energy plot for the D8–brane with $N \gg 1$. The question mark represents our lack of knowledge for both the QFT and the string theory. When $N \to \infty$, $u_N \to \infty$ and the S–dual string theory region will shrink to zero size.

coupling regime and describes a free string theory limit. The *DW/QFT* duality then suggests that in the IR a perturbative 't Hooft limit ($N \to \infty$) of the $(6+1)$–dimensional (non–renormalizable) gauge theory *equals* a free string theory on the domain–wall background. Different interpretations of the correspondence in this D6–branes example, although investigated in a slightly different context, were pointed out in [108, 121].

- The D8–brane. This case is rather special, although somewhat similar to the D6–brane discussion. As we already pointed out the supergravity approximation does not need to involve a large N limit and is valid in the energy range

$$u \ll u_g = (g_f^2)^{\frac{-1}{5}}\,, \tag{5.96}$$

which becomes larger in the limit of large N. The perturbative gauge theory regime is bounded by

$$u \ll u_g\,, \tag{5.97}$$

which is the same energy range as in the supergravity approximation. Large string coupling is encountered in the UV when

$$u \gg u_N = (g_f^2)^{\frac{-1}{5}} N^{\frac{5}{6}}\,, \tag{5.98}$$

which gets larger when N is increasing. We plotted the different regimes in Figure 5.5.

Again we would like to go to M–theory to resolve this regime. We need M9–branes for that but these solutions will still have large curvature in the UV, so a supergravity approximation does not exist in that regime. Also the worldvolume theory of M9–branes is not yet well understood [20, 23]. As in the case of the D6–brane it is again impossible to take the limit $g_s \to \infty$ and finite N to fix the worldvolume coupling constant and decouple *eleven–dimensional* gravity. So we have to consider the limit $N \to \infty$, which is a free string theory limit. The analysis above suggests that in the IR a 't Hooft limit of the $(8+1)$–dimensional (non–renormalizable) gauge theory can also be described by a free string theory on the domain–wall background.

This ends our discussion on $D = 10$ Dp–branes. We will now move on to their six–dimensional analogues.

5.2.4 Non–trivial dilaton dp–branes in $D = 6$

The examples in $D = 6$ were first discussed in [107]. The p–brane solutions in $D = 6$ we will treat all have $\Delta = n = 2$, where n is the integer counting the (higher–dimensional) constituents which make up the $D = 6$ p–brane solution. This can be understood as follows. In ten dimensions one can consider intersecting or overlapping Dp–branes, let us say we consider a Dp–brane intersecting with a Dq–brane with p' common worldvolume directions. These solutions are stable and BPS, breaking $1/4$ of the 32 supersymmetries, only when the number of relative transverse directions[14] equals 4 [139, 140]. Reducing all 4 relative transverse directions on a torus T^4 we end up in $D = 6$ with a p'–brane solution. Reducing on the T^4 relative transverse space also means the constituent branes are *delocalized*, the solution can not depend on the relative transverse directions anymore. All information of the $D = 10$ intersection is hidden in the 4 small compact directions, from the $D = 6$ point of view this can not be distinguished from an ordinary p'–brane solution. To obtain a solution with just one harmonic functions we will decide to identify the $D = 10$ Dp– and Dq–brane charges. This means we identify the number N_1 of Dp–branes and the number N_2 of Dq–branes to equal the number N of p'–branes. The resulting $D = 6$ p'–brane can be found in our general solution (5.4) if we take $D = 6$ and $\Delta = 2$ and inherits many properties of the $D = 10$ Dp–brane parents, so we decide to call them dp'–branes. For example their tension again scales as $1/g_s$, which is customary for D–branes.

It is also possible to give these string solitons an interpretation without referring to $D = 10$ Dp–brane intersections. Instead we can relate them directly to $D = 10$ Dp–branes by considering $K3$ compactifications. A $K3$–manifold is

[14] These are transverse directions of one of the constituent branes and worldvolume directions of the other brane.

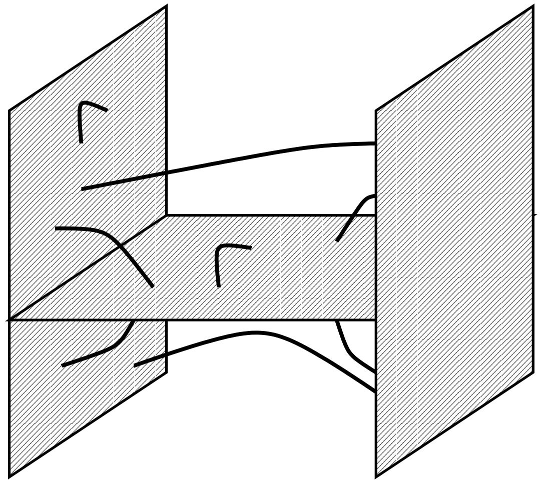

Figure 5.6: Intersecting D–branes and some different open string excitations.

a 4–dimensional Calabi–Yau manifold breaking $1/2$ of the $D = 10$ supersymmetries. Compactification of Type IIA or Type IIB superstrings on a $K3$–manifold gives a corresponding $D = 6$ superstring theory with 16 supersymmetries. String solitons in this $D = 6$ theory will break another half and therefore correspond to supergravity soliton solutions preserving 8 supersymmetries. When we consider Dp–branes in Type IIA or Type IIB superstring theory and compactify the theory on a $K3$–manifold, the $D = 6$ dp'–branes with $\Delta = 2$ will arise naturally.

The worldvolume theory of these dp'–branes can not be the same as their Dp–brane parents. For one thing they should preserve a smaller number of supersymmetries. To construct them the easiest approach is to use their interpretation as intersections of Dp–branes in $D = 10$. Open strings, which will determine the worldvolume field theory fluctuations, now have the possibility to stretch from the Dp–branes to the Dq–branes, see Figure 5.6. These will give rise to extra states in the worldvolume field theory. Extra fermions and scalars denoting the relative position in the intersection space (which is the space of relative transverse directions), called a supersymmetry hypermultiplet, will appear on both the Dp–brane and Dq–brane worldvolume theory. These scalars and fermions will transform in the fundamental representation of the (different) gauge groups. The appearance of the hypermultiplet will break the supersymmetry of the system to one preserving only $1/4$ of the maximum of 32 supersymmetries. Most important for our discussion will be that the $D = 6$ dp'–brane worldvolume field theory will have extra hypermultiplet scalars which after identification of the number of branes transform in a single fundamental representation of the $U(2N)$ gauge group.

We already encountered one example of a $D=6$ dp–brane in our $AdS/\,CFT$ discussion, where we described the $D=6$ self–dual string ($p=1$) as the pure $AdS_3\times S^3$ near–horizon geometry example which is dual to a 2–dimensional conformal field theory. The d2–brane is the special flat Minkowski spacetime near–horizon geometry example, the d3–brane has $\tilde{d}=0$, so we will discuss the d0–brane and the (domain–wall) d4–brane as new examples of DW/QFT dualities in $D=6$.

Interestingly enough the constraint (5.64) tells us that in this case

$$x = p-1\,, \tag{5.99}$$

which means that the dimension of the fixed coupling constant (5.54) is that of a scalar field theory[15]. This is a very important difference as compared with the $D=10$ Dp–branes. It means that the Yang–Mills coupling constant $g_{YM}\to\infty$ in the field theory limit. Generically the dp–brane worldvolume field theory will be in a vacuum with non–zero vacuum expectation values for the hypermultiplet scalars, which is called the Higgs branch[16]. This will give rise to a Higgs effect giving mass to all the vector bosons on the worldvolume proportional to the Yang–Mills coupling constant. Because the Yang–Mills coupling constant diverges in the field theory limit, all vector bosons will become infinitely massive and decouple. The effective field theory which is left–over is a worldvolume scalar field theory (with the necessary fermions of course to make it supersymmetric). These kinds of limits were discussed earlier in the context of Matrix theories [141, 142] and in the context of the $D=6$ self–dual string AdS/CFT duality in [110].

In principle the dynamics of the positions of the dp–branes, represented by the vector multiplet scalars (as opposed to the hypermultiplet scalars) is now included as well, which is called the Coulomb branch when the hypermultiplet scalars have vanishing expectation values. In [141, 142] it is argued that the Coulomb branch decouples and one should only consider the Higgs branch scalar field theory. We will assume this conclusion to be correct and will only consider, when needed, the Higgs branch of the corresponding dp–brane worldvolume field theory.

Let us now analyze the $D=6$ supergravity near–horizon backgrounds. We are mainly interested in the dilaton–expression which governs the analysis of where

[15] This is of course consistent with the fact that the $p=1$ case should be a conformal field theory.

[16] Strictly speaking, this is only called the Higgs branch if the scalars in the vector multiplet have vanishing expectation values, which means that all the dp–branes are on top of each other.

we will be able to use a supergravity approximation. We are going to repeat the analysis done for the $D = 10$ Dp–branes, just replacing the $D = 10$ parameters by the appropriate six–dimensional ones. Using that $\Delta = 2$ and $D = 6$ we find

$$\beta = 2 - p \quad , \quad a = 1 - p \,. \tag{5.100}$$

The D–brane constraint (5.64) gives the following relation between the D–brane energy scale U and the holographic energy scale u

$$U^{2-p} = |2 - p| \left(\frac{d_p}{c_p} \right) g_f^2 u \,. \tag{5.101}$$

The singular case $p = 2$ is the one with the flat Minkowski near–horizon region, as the $p = 5$ case in $D = 10$. The effective dimensionless coupling constant equals

$$g_{eff}^2 = g_f^2 u^{p-1} \,. \tag{5.102}$$

In terms of this effective coupling constant the dilaton background (5.65) can be expressed as

$$e^{\Phi} = \frac{1}{d_p N} |2 - p|^{\frac{(p-1)(3-p)}{2(2-p)}} \left(\frac{d_p g_{eff}^2}{c_p} \right)^{\frac{(3-p)}{2(2-p)}} \,. \tag{5.103}$$

This is of course very similar to the expression in (5.83).

From (5.102) it follows that a perturbative field theory analysis is valid when

$$\begin{aligned} u &\gg (g_f^2)^{\frac{1}{(1-p)}} \quad \forall\, p < 1 \\ u &\ll (g_f^2)^{\frac{1}{(1-p)}} \quad \forall\, p > 1 \,. \end{aligned} \tag{5.104}$$

The supergravity approximation requires us to satisfy the following two conditions simultaneously (we neglect constants of order 1, e.g. p, c_p and d_p)

$$\begin{aligned} \tau_s &\propto \left(g_{eff}^2\right)^{\frac{1}{(2-p)}} \gg 1 \\ e^{\Phi} &\propto \frac{1}{N} \left(g_{eff}^2\right)^{\frac{(3-p)}{2(2-p)}} \ll 1 \,. \end{aligned} \tag{5.105}$$

It should be clear that the first condition represents small curvature and the second condition represents small string coupling. Translating these conditions into conditions on the energy u will give us the energy regimes where supergravity

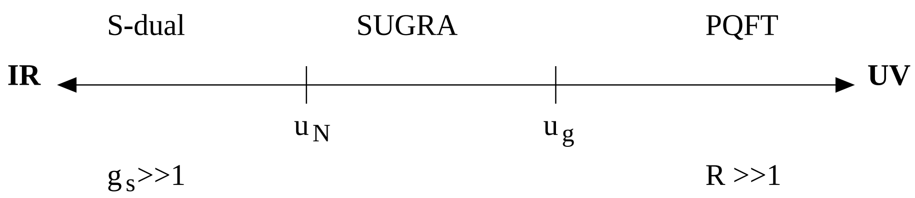

Figure 5.7: Different regimes in the energy plot for d0–branes with $N \gg 1$. The terminology should be self–explanatory and was discussed in section 4.2.3.

is a good description. For $p = 0$ these conditions can again only be satisfied simultaneously by taking a large N limit, whereas for $p = 4$ a large N limit is not implied by (5.105). However, we need to take $N \to \infty$ in the d4–brane case to fix the coupling constant (5.54) and to decouple the worldvolume field theory from gravity.

Below we discuss the two $D = 6$ DW/QFT examples in more detail.

- The $D = 6$ d0–brane. A supergravity approximation can be used in the following IR energy regime

$$u_N = g_f^2 N^{\frac{-4}{3}} \ll u \ll u_g = g_f^2 \,, \qquad (5.106)$$

which can only be satisfied for large N. In the UV we can use perturbative field theory

$$u \gg u_g \,, \qquad (5.107)$$

which in this case reduces to a quantum mechanics model. We therefore find the typical DW/QFT behavior that the supergravity regime and the perturbative field theory regime do not overlap, avoiding inconsistencies. In the far IR, when

$$u \ll u_N \,, \qquad (5.108)$$

the string coupling becomes large and we could try to use an S–dual description. The different regimes are plotted in Figure 5.7.

The d0–brane is related to the D4–brane by considering a IIA compactification on a $K3$–manifold. It is conjectured (and by now well established) that Type IIA superstring theory on a $K3$–manifold is S–dual to Heterotic superstring theory on a T^4 [26]. On the Heterotic side the S–dual soliton solution would be a fundamental state ($k = 0$) and has a curvature singularity

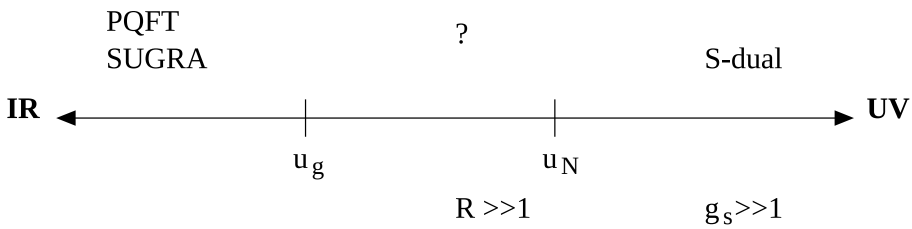

Figure 5.8: Different regimes in the energy plot for the d4–brane with $N \gg 1$. The question mark represents our lack of knowledge for both the QFT and the string theory. When $N \to \infty$, $u_N \to \infty$ and the S–dual string theory region will shrink to zero size.

at $u = 0$, so a supergravity approximation will not make sense. The situation resembles the D1–brane case in $D = 10$ Type IIB theory. There the curvature singularity in the S–dual F1–brane solution was resolved by the strong coupling conformal fixed point of the $(1+1)$–dimensional gauge theory. It is suggestive to propose the occurrence of a similar phenomenon in this case. It would therefore be interesting to determine the strongly coupled IR limit of the corresponding quantum mechanics model. As mentioned in section 4.2.2 and 4.2.3, we will have to deal with the problems involving the interpretation of the DW_2/QFT_1 correspondence if we want to understand this d0–branes example in all its detail.

- The d4–brane. Just like the $D = 10$ D8–brane this is a special case, because it is a $D = 6$ domain–wall solution. The supergravity regime is bounded from above by

$$u \ll u_g = (g_f^2)^{\frac{-1}{3}} \tag{5.109}$$

and we do not need large N. The perturbative field theory is valid in

$$u \ll u_g \,, \tag{5.110}$$

so supergravity and perturbative field theory are valid in the same regime, which seems implausible. Large string coupling is encountered in the UV when

$$u \gg u_N = (g_f^2)^{\frac{-1}{3}} N^{\frac{4}{3}} \,. \tag{5.111}$$

We plotted the different regimes in Figure 5.8.

As in the D8–brane case however, we should remember that we had to take $N \to \infty$ to fix the coupling constant and to decouple gravity. So the statement should be that the IR 't Hooft limit of the $(4+1)$–dimensional (non–renormalizable) field theory can equivalently be described by a *free* string theory on the corresponding domain–wall background. Again we can not consider a limit in which we take $g_s \to \infty$ and keep N finite to decouple an S–dual gravity theory (like we could for the Type IIA D4–brane). In the UV, after taking the limit $N \to \infty$ which shrinks the S–dual region to zero size, a well–defined description is unknown. This is also suggested by the non–renormalizability of the worldvolume field theory.

At the end of this subsection let us make the following remarks. We did not present a detailed investigation of the dp–brane worldvolume field theories. Our discussion was focussed on the dp–brane geometry in the field theory limit and the search for well–defined supergravity regions. We showed that the near–horizon geometries of the $D=6$ dp–branes indeed have regions where a supergravity approximation seems valid and the analysis is strikingly similar to that of the $D=10$ Dp–branes. We did make some general remarks on the nature of the field theory, which is governed by scalar dynamics, presumably in the Higgs branch of the $p+1$–dimensional gauge theory, consisting of supersymmetry vector multiplets and hypermultiplets.

We should point out that other work was done on *localized* Dp–brane intersections and the field theory limit [143, 144, 145]. In these investigations a limit is considered taking one into the near–horizon geometry of the lower–dimensional D–brane in the intersection and the dual field theory should then also be the one living on the lower–dimensional D–brane. Although the field theory limit in that case fixes the Yang–Mills coupling constant, there could be a connection with the results presented here in the sense that both investigations start off with the same intersecting D–brane system.

The status of these DW/QFT dualities, in $D=10$ as well as in $D=6$, is not entirely clear at this moment because they are hard to check explicitly. Basically this is because the supergravity approximation and the perturbative field theory are generically valid in opposite energy regimes, making it very hard to perform explicit checks of the duality conjecture. What can be checked of course are the symmetries and it is not very hard to show that these match in all examples presented. However, the very general mechanism leading to these proposed dualities, the explicit checks of the AdS/CFT duality and the string theory interpretation of Dp–branes as discussed in 1.1.5, can all be considered strong circumstantial evidence for the correctness of the DW/QFT duality conjecture.

This ends the chapter on string solitons and the field theory limit. In the next concluding chapter we will summarize our results presented here and in the previous chapter and try to establish a common understanding of these results, apparently teaching us that gravity and gauge field theories, as limits of an underlying string theory, are connected in a very interesting way.

Chapter 6

Emerging structure and discussion

To conclude it is useful to summarize and discuss the results presented in chapter 4 and 5 and, if possible, try to establish a common understanding. One of the goals of this thesis was to try to convey the idea that string theory, regarded as a natural extension of point particle theories, can be used as a tool to find interesting relationships between point particle theories which were previously thought to be very distinct. The correspondence between supergravities and large N supersymmetric gauge theories was the example we concentrated on in this thesis. In chapter 4 we introduced Matrix theory as an example of a large N quantum mechanics model capable of describing (certain aspects of) M–theory and, in a low energy limit, $D = 11$ supergravity. In chapter 5 we discussed a limit which incorporated the p–brane geometries in an important way and which led to the conjectured DW/QFT correspondence.

First of all let us summarize our contributions. In the context of Matrix theory we constructed additional (kinematical) evidence in favor of the conjecture. We constructed intersecting BPS states in the Matrix model and concluded that all of these BPS states can be mapped to corresponding intersecting BPS states in M–theory. We also presented some arguments as to why some BPS states are missing in Matrix theory, which is perhaps a consequence of the light–cone frame. Besides the additional evidence in favor of the Matrix theory conjecture, the method of constructing intersecting BPS states is interesting by itself and can be applied to other cases as well (e.g. higher dimensional supersymmetric Yang–Mills theories).

In the chapter on string solitons and the field theory limit we showed that dualities between field theories on the one hand and domain–wall superstring theories on the other hand can be derived systematically. This involved introducing the dual frame metric and also the restriction to D–branes and their intersections (reduced over *all* relative transverse directions) when a non–trivial dilaton background was involved. This led to new conjectures of DW/QFT correspondences in 6 uncompactified spacetime dimensions, which are very similar to the 10–dimensional cases. We also presented some results of the correspondence when considering $D-2$–branes. We argued that the situation in those cases not necessarily leads to contradictions because we explicitly have to take $N \to \infty$ in that case. In the case of the AdS/CFT correspondences many tests have been performed which are all in support of the conjecture and in some cases one can even go beyond the supergravity approximation, considering string theory on AdS spacetimes[1].

We have to say that explicit tests of these DW/QFT correspondences (with non–trivial dilaton) have not (yet) been made, mainly because generically the dual descriptions are supposed to be valid in different regimes. We should also point out that the DW/QFT correspondence is not well understood when the dual field theory is a supersymmetric (conformal) quantum mechanics model. Although we did describe flat near–horizon geometries, we did not discuss a limit taking us into the flat near–horizon region and leading to non–trivial worldvolume dynamics. These limits are described in [108, 109] and a more thorough investigation on (the meaning of) the correspondence in those cases can be found in [122].

We noticed that the Matrix theory conjecture can be understood as embedded in the more general DW/QFT correspondence. This has some consequences for the Matrix theory conjecture(s). Although we introduced the Matrix theory conjecture as a limit of IIA superstring theory, in retrospect we have to conclude that we neglected the effect of the limit on the (non–trivial) background of the N units of momentum in DLCQ M–theory (or D0–branes in Type IIA superstring theory). If we would not have neglected the background we could have observed that the radius of the compact (light–like) direction is not a constant, but instead depends on the radial parameter of the background solution. In the context of the DW/QFT correspondence this then led to the understanding that Matrix theory, or the quantum mechanics of N D0–branes, is only describing DLCQ M–theory in the far IR, where the quantum mechanics model is strongly coupled. Remember that we do seem to understand how to construct DLCQ M–theory supergraviton states in the Matrix theory, whereas we do not know the explicit map in the

[1] In general this is very hard because the background involves non–trivial Ramond–Ramond flux, which is hard to treat in a quantized closed string σ–model approach. Progress has been made in the case of $AdS_3 \times S^3$ in which case the R–R rank 2 potential can be replaced by a NS–NS rank 2 potential [146, 147, 148].

DW_2/QM correspondence. Perhaps the Matrix theory conjecture can teach us how we should construct the DW_2/QM map. Presumably similar considerations hold for the tori compactified Matrix theory conjectures. We note that in the DW/QFT correspondence we did not consider wrapping the string solitons on tori, so we can not immediately compare with the Matrix theory conjectures.

The overall topic in this thesis were the relations between supersymmetric (large N Yang–Mills) field theories and closed superstring theories, containing gravity, on a particular BPS background. Although string theory was used to establish these relations, it is not impossible that these relations could have been discovered without string theory. After all, in a large N limit where string loop corrections can be neglected, we are left with two particle theories on both sides. In that case one would not have had any understanding as to the origin of these relations between gravity and field theory. In that sense string theory naturally explains the appearance of these relations, understood by the covariance property of the (stretched) string worldsheet relating closed and open strings in the presence of D–brane solitons. This could be another strong theoretical argument in favor of string theory, assuming these dualities and/or correspondences are actually correct.

Perhaps these relations are very dependent on supersymmetry. In fact a lot of results and techniques were based on the fact that we were dealing with BPS states. Recently there has been a lot of activity in trying to get results beyond the BPS states. Also in the context of the AdS/CFT correspondence one is trying to make contact with our real (non–supersymmetric) world, for example trying to understand confinement in QCD from a dual string theory on a particular background [149, 150]. Another interesting approach is making use of the idea that the DW/QFT correspondence tells us that one can think of the renormalization group scale in field theories as a (special) coordinate in a background solution of a gravitational theory living in one dimension higher (the renormalization group flow equations can then be deduced from the classical gravitational equations of motion). In a cosmological context this can perhaps explain the smallness of the cosmological constant in our $D = 4$ world [151, 152, 153]. It is satisfying to notice that these new techniques and ideas in string theory can be more directly applied to $D = 4$ (theoretical) physics problems.

From our point of view string theory is an interesting theoretical structure in which new ideas about space, time and quantum mechanics seem to be realized in a very concrete and tractable way. The concepts of dualities, non–commutative geometry and the holographic principle were all mentioned somewhere in this thesis. This is just one among many other motivations to continue to investigate string theory and its consequences.

Appendix A

String theory units and charge conventions

In this Appendix we specify our units and charge conventions. In $D = 10$ and lower we use the convention $l_s^2 = \alpha'$ which is understood to be replaced by the Planck length l_{11} in eleven dimensions.

In a low energy approximation of string theory we want to make a clear distinction between the field e^{Φ} and the string coupling constant g_s in order to determine the dependence of the low energy effective action on g_s. Remember that g_s is defined as the constant part of e^{Φ}. This means that in a low energy, tree level approximation of string theory, which naturally involves the string frame metric, all g_s dependence can be extracted from the field e^{Φ}. We note that we do not want these dependences to change when we perform a conformal transformation. Our starting point for deducing the dependence of the action on g_s will be the general Einstein frame action (3.48), but with the rank $d+1$ field strength replaced by its rank $\tilde{d}+1$ Hodge dual. Next we perform a conformal transformation to the string frame using

$$g_{\mu\nu}^{(E)} = e^{\frac{-4\Phi}{D-2}} g_{\mu\nu}^{(S)} . \tag{A.1}$$

This gives the following string frame action

$$S_D = \frac{1}{\kappa_D^2} \int \mathrm{d}^D x \sqrt{g_S}\, e^{-2\Phi} \left[R_S + 4(\partial\Phi)^2 - \frac{e^{\left[4-\left(a+\frac{4d}{D-2}\right)\right]\Phi}}{2(p+2)!} F_{\tilde{d}+1}^2 \right] , \tag{A.2}$$

where we extracted an overall factor of $e^{-2\Phi}$. We can now read off the g_s dependence by just extracting g_s's from e^Φ and writing the string frame action as

$$S_D = \frac{1}{\alpha'^{\frac{1}{2}(D-2)} g_s^2} \int \mathrm{d}^D x \sqrt{g_S} \quad \left(\frac{e^\Phi}{g_s}\right)^{-2} \Big[R_S + 4(\partial\Phi)^2 - \frac{g_s^{4-2k}}{2(p+2)!} \left(\frac{e^\Phi}{g_s}\right)^{4-2k} F^2_{\tilde{d}+1} \Big] \,, \qquad \text{(A.3)}$$

where we used that $\kappa_D^2 \propto \alpha'^{\frac{1}{2}(D-2)}$ and defined the parameter k

$$k = \frac{a}{2} + \frac{2d}{D-2} \,. \qquad \text{(A.4)}$$

This parameter is useful because it is directly related to the scaling of the effective tension of the corresponding p–brane, which is equal to g_s^{-k} and that can be deduced by adding a p–brane source term to the supergravity action and using a scaling argument [37]. So for D–branes $k = 1$, for Neveu–Schwarz solitons $k = 2$ and for fundamental branes $k = 0$, which are the only types of branes we will encounter in string theory.

At this point we want this g_s dependence to stay fixed under conformal transformations, which means we should always perform conformal transformations with the g_s independent field $\frac{e^\Phi}{g_s}$

$$g^{(C)}_{\mu\nu} = \left(\frac{e^\Phi}{g_s}\right)^{\alpha} g^{(S)}_{\mu\nu} \,. \qquad \text{(A.5)}$$

The mass τ_p per unit p–volume of a p–brane is given by the ADM–formula

$$\begin{aligned} \tau_p &= \frac{1}{2\kappa_D^2} \int_{\partial M^{\tilde{d}+2}} d^{\tilde{d}+1}\Sigma^m \left(\partial^n h_{mn} - \partial_m h^b{}_b \right) \\ &= \frac{2\tilde{d}}{\Delta \kappa_D^2} r_0^{\tilde{d}} \Omega_{\tilde{d}+1} \,, \end{aligned} \qquad \text{(A.6)}$$

where we plugged in the solution given in (5.4), in which r_0 is an integration constant with the dimension of length. On the other hand, the charge μ_p per unit p–volume is given, in terms of the same integration constant r_0, by the Gauss–law formula

$$\begin{aligned} \mu_p &= \frac{1}{2\kappa_D^2} \int (d^{\tilde{d}+1}\Sigma)^{m_1 \cdots m_{\tilde{d}+1}} g_s^{(2-k)} F_{m_1 \cdots m_{\tilde{d}+1}} \\ &= \pm \sqrt{\frac{\Delta}{4}} \tau_p \,. \end{aligned} \qquad \text{(A.7)}$$

Hence, the p–brane solution satisfies the Bogomol'nyi bound

$$\tau_p = \sqrt{\frac{4}{\Delta}}\,|\mu_p|\,. \tag{A.8}$$

Because the effective tension (A.6) should scale as g_s^{-k} we can now derive an expression for r_0 in terms of the string parameters l_s and g_s

$$\left(\frac{r_0}{l_s}\right)^{\tilde{d}} = d_p N g_s^{2-k}\,, \tag{A.9}$$

where d_p is just a number and N counts the number of p–branes.

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Samenvatting

Dacht men aan het einde van de 19de eeuw dat atomen de kleinste bouwstenen waren, in de loop van de 20ste eeuw bleek al snel dat een atoom is opgebouwd uit een atoomkern en daaromheen cirkelende elektronen. De atoomkern is opgebouwd uit protonen en neutronen en de protonen en neutronen zijn opgebouwd uit de zogenaamde quarks, waarvan men nu aanneemt dat ze ondeelbaar zijn. Uit bovenstaande blijkt dat wat men elementaire of kleinste bouwstenen noemt onderhevig kan zijn aan veranderingen in de loop der tijd. De theoretische elementaire–deeltjesfysica is die tak van de natuurkunde die zich bezighoudt met het opstellen en bestuderen van wiskundige modellen die de eigenschappen en wisselwerkingen beschrijven van de kleinste bouwstenen der natuur.

De bewegingen van deze deeltjes in ruimte en tijd worden bepaald door de eigenschappen van de verschillende deeltjes en hun onderlinge wisselwerkingen. Zeer kleine en snelle objecten blijken zich op een fundamenteel andere wijze te gedragen dan we gewend zijn van macroscopische objecten, waarvan het gedrag nauwkeurig kan worden beschreven met behulp van de wetten van Newton. Bij afstanden die de grootte van een atoom benaderen ($\sim 10^{-10}$ meter) of bij snelheden in de buurt van de lichtsnelheid (3.10^8 meter per seconde) dienen de wetten van Newton aangepast te worden.

Microscopische objecten worden beschreven door de kwantummechanica. De natuur laat zich volgens deze theorie bij zeer kleine afstanden niet meer precies voorspellen. In plaats daarvan beschrijven de uitkomsten van kwantummechanische berekeningen slechts de kans dat een bepaald proces zich voordoet. Een ander specifiek gevolg van de kwantummechanica is dat in veel omstandigheden allerlei meetbare grootheden, zoals bijvoorbeeld energie, slechts discrete waarden aannemen. De toestanden die overeenkomen met deze discrete waarden worden vaak aangeduid als kwantumtoestanden. Hoe kleiner de afstanden, des te doeltreffender wordt de kwantummechanische beschrijving. Hoe groter de afstanden, des te meer komt de kwantummechanische beschrijving overeen met de klassieke mechanica van Newton[1].

Wanneer de snelheden van objecten de lichtsnelheid benaderen dient de mechanica van Newton te worden vervangen door de speciale relativiteitstheorie van Einstein. De speciale relativiteitstheorie veronderstelt dat de lichtsnelheid gemeten door verschillende waarnemers die ten opzichte van elkaar met constante snelheid bewegen, altijd dezelfde is. Gebruikmakend van deze symmetrie kunnen de relativistische bewegingswetten worden opgesteld[2], waaruit bijvoorbeeld blijkt dat de lichtsnelheid bovendien de maximaal bereikbare snelheid is. In de limiet van relatief kleine snelheden vinden we de klassieke mechanica van Newton weer terug.

De kwantummechanica en de speciale relativiteitstheorie zijn toepasbaar op een groter gebied van afstanden en snelheden dan de mechanica van Newton en in die zin beschouwen we deze theorieën als fundamentelere beschrijvingen. In de huidige elementaire–deeltjesfysica wil men processen beschrijven die zowel op kleine schaal als bij zeer hoge snelheden plaatsvinden. Dit vereist een verdere aanpassing van de wetten der mechanica, namelijk één die de kwantummechanica en de speciale relativiteitstheorie verenigt. Het resultaat wordt kwantumveldentheorie genoemd en deze theorie is in staat gebleken, althans formeel, vele eigenschappen van elementaire deeltjes en hun wisselwerkingen met grote precisie te beschrijven. Zoals de kwantummechanica en de speciale relativiteitstheorie afzonderlijk fundamentelere beschrijvingen zijn vergeleken met de mechanica van Newton, zo is kwantumveldentheorie wederom een fundamentelere beschrijving van de natuur, aangezien ze een nog groter bereik heeft.

De combinatie van kwantumveldentheorieën die alle waargenomen deeltjes en hun wisselwerkingen beschrijven wordt het Standaard Model genoemd. In het Standaard Model kan men onderscheid maken tussen materiedeeltjes (elektronen, quarks en neutrinos, ook wel fermionen genoemd), waaruit alle macroscopische objecten zijn opgebouwd, en krachten uitwisselende deeltjes (bosonen), die uitgewisseld worden wanneer er een interactie plaatsvindt tussen materiedeeltjes. Er bestaan drie soorten krachten uitwisselende deeltjes in het Standaard Model. De fotonen zijn verantwoordelijk voor de elektromagnetische wisselwerking, de massieve vectorbosonen beschrijven de zwakke wisselwerking en de sterke wisselwerking komt tot stand door uitwisseling van gluondeeltjes. De kwantumvelden-

[1] In sommige gevallen kan dit aangetoond worden, in andere gevallen is het zeer moeilijk (wiskundig) te bewijzen. In dat opzicht is de kwantummechanica nog geen gesloten (en zeker niet begrepen) boek.

[2] Met als absoluut meest bekende resultaat de equivalentie tussen massa en energie, uitgedrukt in de vergelijking $E = mc^2$.

theorieën in het Standaard Model die deze deeltjes en hun interacties beschrijven hebben de belangrijke eigenschap dat ze onveranderd blijven onder zogenaamde ijktransformaties en worden kwantumijkveldentheorieën genoemd.

Al meer dan twintig jaar beschrijft het Standaard Model met (veelal ongekende) precisie allerlei botsingsexperimenten tussen elementaire deeltjes in de verscheidene experimentele versnellerinstituten, zoals CERN in Genève (Zwitserland), SLAC in Stanford en Fermilab in Chicago (beide Verenigde Staten). In dat opzicht is het Standaard Model een van de succesvolste natuurkundige theorieën ooit en is er op dit moment zeker geen experimentele noodzaak om theorieën te ontwikkelen (of te bestuderen) die beter of fundamenteler, in de zin zoals eerder uitgelegd, zouden zijn dan het Standaard Model[3]. En toch zijn er velen die denken dat er iets aan het Standaard Model mankeert.

Voor een fundamentele theorie die de kleinste bouwstenen van de materie beschrijft, bevat het Standaard Model veel vrije parameters die bepaald moeten worden door een experiment te doen. Velen binnen de theoretische elementaire–deeltjesfysica verwachten (of hopen) dat een fundamentele theorie van de natuur weinig mogelijkheden voor variatie toelaat, en dus een minimum aan vrije parameters bevat. Op die manier zou het heelal zijn zoals het is omdat er gewoon geen andere mogelijkheden zijn, wat voor veel theoretisch natuurkundigen een afdoende verklaring is. Aangezien dit voor het Standaard Model niet geldt, wordt vaak verondersteld dat het Standaard Model slechts een effectieve beschrijving is, geldig voor processen beneden een bepaalde energie. Voor processen boven die energie (die ook wel de Grand Unified Theory (GUT)–energie wordt genoemd) zou een fundamentelere theorie geldig zijn die minder vrije parameters bevat. Dit idee wordt ondersteund door experimentele data die laten zien dat de drie verschillende koppelingconstanten, die de sterkte van de drie verschillende wisselwerkingen tussen de elementaire deeltjes weergeven, als functie van de energie naar elkaar toe bewegen bij hogere energie. Dit betekent dat het niet onmogelijk is dat de drie wisselwerkingen hun oorsprong vinden in een theorie met slechts één koppelingconstante, een Grand Unified Theory. Omdat de GUT–energie ver boven de bereikbare experimentele energieën ligt, kan dit idee niet direct getoetst worden en is een veelgehoorde kritiek op de bovenstaande argumenten dat ze gebaseerd zijn op esthetische principes die a priori helemaal niets met de werkelijkheid te maken hoeven hebben. Er zijn echter meer redenen om aan te nemen

[3]Hier dient te worden opgemerkt dat recentelijk experimenteel is aangetoond dat niet alle neutrino deeltjes massaloos zijn, zoals wordt verondersteld in het Standaard Model. De wijziging van het Standaard Model die dit echter tot gevolg heeft zouden wij in de context van dit proefschrift niet willen betitelen als een fundamentele verbetering van het Standaard Model.

dat er iets mis is met het Standaard Model.

De opmerkzame lezer zal het opgevallen zijn dat in het lijstje van wisselwerkingen of krachten die opgenomen zijn in het Standaard Model de zwaartekracht ontbreekt. De zwaartekracht wordt beschreven door de algemene relativiteitstheorie die een extensie is van de speciale relativiteitstheorie, eveneens ontwikkeld door Einstein. Het blijkt dat wanneer we steeds grotere massa's (of energieën) beschouwen, de speciale relativiteitstheorie niet langer van toepassing kan zijn en we over moeten stappen naar de algemene relativiteitstheorie. Speciale relativiteitstheorie moet dus worden opgevat als een redelijke benadering van algemene relativiteitstheorie wanneer de energieën of massa's niet te groot zijn. De algemene relativiteitstheorie beschrijft met zeer grote precisie de processen die met zwaartekracht te maken hebben, zoals bijvoorbeeld de banen van de planeten om de zon, de afbuiging van licht door sterren en de evolutie van het heelal. De algemene relativiteitstheorie en het Standaard Model vormen samen de hedendaagse pijlers van de theoretische hoge–energiefysica en hebben, ieder in hun eigen gebied van geldigheid, vele experimentele toetsen met glans doorstaan.

Voor elementaire deeltjes speelt de zwaartekracht geen rol omdat de massa's van deze deeltjes te klein zijn. Alle andere interacties zijn veel sterker en de zwaartekracht is dus verwaarloosbaar. Maar omdat volgens de speciale relativiteitstheorie massa en energie equivalent zijn, zal op een bepaald moment wanneer we de energieën van de elementaire deeltjes steeds groter maken de zwaartekracht wel degelijk een rol gaan spelen. Hoewel deze energie, ook wel de Planck energie genoemd, ver buiten het huidige experimentele bereik ligt moeten de energieën een fractie na het ontstaan van het heelal (ongeveer vijftien miljard jaar geleden als gevolg van een enorme explosie, de Big Bang) van vergelijkbare grootte zijn geweest. Het is dus van (theoretisch) belang het effect van de algemene relativiteitstheorie op de elementaire deeltjes in het Standaard Model te bepalen. Dit betekent dat gezocht moet worden naar een theorie die de algemene relativiteitstheorie en de kwantummechanica verenigt. Op algemene gronden kan men aannemen dat die theorie een graviton dient te bevatten, het deeltje dat verantwoordelijk is voor de gravitationele wisselwerking.

Er bestaan standaard (technische) procedures om een theorie te verenigen met de kwantummechanica. Alhoewel dit in veel gevallen gepaard gaat met allerlei technische problemen, is het in het geval van het Standaard Model gelukt om een betekenisvolle theorie (dat wil zeggen dat de theorie uitkomsten geeft die tussen de 0 en 100 procent liggen) op te zetten[4]. Past men deze kwantisatiemethode

[4]Voor het oplossen van deze technische problemen in het Standaard Model kregen de Neder-

echter toe op de algemene relativiteitstheorie dan is het resultaat een onzinnige kwantumtheorie (dat wil zeggen dat men oneindigheden berekent). Hoewel dit misschien slechts een technisch probleem lijkt, zijn er ook fundamentelere problemen bij het kwantiseren van de algemene relativiteitstheorie. De algemene relativiteitstheorie beschrijft zwaartekracht als een gevolg van de kromming van de ruimte en tijd. Kwantisatie van de algemene relativiteitstheorie is daarom hetzelfde als kwantisatie van de ruimte en tijd. Of aan dat laatste een betekenisvolle interpretatie kan worden gegeven is maar de vraag. De heersende opvatting is dat een betekenisvolle constructie van een kwantumtheorie van de zwaartekracht radicaal nieuwe natuurkunde zal bevatten.

In het Standaard Model worden de elementaire deeltjes beschreven als puntdeeltjes, objecten die geen vorm of grootte hebben. Al in de jaren ’70 speelde men met het idee om deeltjes als de verschillende kwantumtoestanden van een snaar te beschrijven. Deze resonantiemodellen, zoals ze genoemd werden, waren bedoeld om een onderdeel van het Standaard Model, namelijk de sterke wisselwerking waarvan men op dat moment nog weinig wist, beter te begrijpen. Deze modellen werden al snel aan de kant geschoven omdat bleek dat bepaalde toestanden van de snaar niet voorkwamen in de theorie van de sterke wisselwerking. Na een korte stilte traden de snaarmodellen weer op de voorgrond, maar nu met een volstrekt ander doel. Men was er namelijk achter gekomen dat een van de toestanden van de snaar alle eigenschappen had van een graviton, het deeltje dat verantwoordelijk is voor de gravitationele wisselwerking. De snaartheorie bleek een consistente beschrijving van de kwantum–zwaartekracht te kunnen geven in een benadering waarin de zwaartekracht niet te sterk mag zijn.

In een snaartheorie worden alle elementaire–materiedeeltjes en de deeltjes verantwoordelijk voor de uitwisseling van krachten beschouwd als de verschillende (harmonische) toestanden van één enkele snaar. Voor de opzet van een betekenisvolle snaartheorie is een aantal theoretische concepten vereist. Zo is het noodzakelijk om supersymmetrie in te voeren. Supersymmetrie is een symmetrie die elementaire–materiedeeltjes (fermionen) en krachten uitwisselende deeltjes (bosonen) met elkaar verbindt. In een supersymmetrische theorie heeft elk bosondeeltje een fermion superpartner deeltje en omgekeerd. Om contact te maken met het Standaard Model moet worden aangenomen dat supersymmetrie een gebroken symmetrie is omdat superpartners vooralsnog niet zijn waargenomen.

Alle consistente supersnaartheorieën zijn gedefinieerd in tien ruimte– en tijddimensies. Om contact te maken met onze vierdimensionale ruimte en tijd, wordt

landers ’t Hooft en Veltman in 1999 de Nobelprijs!

aangenomen dat zes dimensies opgerold en zeer klein zijn, zodat we ze niet kunnen waarnemen[5]. Gebruikmakend van supersymmetrie en extra dimensies is tijdens de zogenaamde eerste supersnaar revolutie (1984–1985) aangetoond dat er vijf verschillende supersnaartheorieën in tien ruimte– en tijddimensies bestaan. Al deze supersnaartheorieën bevatten een betekenisvolle beschrijving van kwantumgravitatie en zijn daarnaast *in principe* in staat om de verschillende kwantumijkveldentheorieën in het Standaard Model te beschrijven. Snaartheorie lijkt daarom uitermate geschikt als kandidaat voor een geunificeerde elementaire–deeltjestheorie.

Het grote probleem in supersnaartheorie is dat deze slechts bij benadering gedefinieerd is. Dat wil zeggen alleen wanneer de snaarkoppelingconstante klein is, wat equivalent is aan niet al te grote zwaartekracht. Zeer waarschijnlijk verhindert deze restrictie ons ook om direct contact te maken met het succesvolle vierdimensionale Standaard Model. Reproduktie van het Standaard Model is natuurlijk van uitermate groot belang voor snaartheorie als serieuze kandidaat voor een geunificeerde elementaire–deeltjestheorie. Het huidige onderzoek in snaartheorie is dan ook bijna volledig geconcentreerd op het begrijpen van het gedrag van snaren wanneer de snaarkoppelingconstante groot is (het niet–perturbatieve regime). De laatste tijd is er enorme vooruitgang geboekt op dit gebied door de ontdekking van dualiteiten in snaartheorie.

Supersymmetrie zorgt ervoor dat sommige objecten, BPS–objecten genoemd, massa's en ladingen dragen die geen correcties krijgen wanneer de koppelingconstante groot wordt. Deze BPS–gegevens zijn dus volkomen onafhankelijk van de perturbatieve definitie van snaartheorie. Het blijkt dat BPS–objecten veel voorkomen in snaartheorie. Bovendien ontdekte men dat sommige van deze objecten microscopisch beschreven konden worden met behulp van open–snaartheorie. Door aan te nemen dat de eindpunten van een open snaar eindigen op een p–dimensionaal vlak, bleek men in staat een BPS–Dp–braan[6] te beschrijven. Gebruikmakend van de BPS–eigenschap van deze objecten kwam men erachter dat er zeer waarschijnlijk allerlei verbanden bestaan tussen de vijf verschillende snaartheorieën en deze verbanden worden dualiteiten genoemd.

Het meest opvallend zijn die dualiteiten die een sterk gekoppelde snaartheorie relateren aan een zwak gekoppelde (niet noodzakelijk andere) snaartheorie. Tijdens de zogenaamde tweede supersnaar revolutie (1994–1996) ontdekte men dat

[5]Denk aan een tuinslang. Van dichtbij is deze tweedimensionaal, maar wanneer je er van een flinke afstand naar kijkt begint de tuinslang steeds meer op een ééndimensionale lijn te lijken.

[6]Deze randcondities op een open string worden Dirichlet randcondities genoemd, vandaar de naam D–branen.

alle vijf de snaartheorieën verbonden zijn met elkaar en de verschillende perturbatieve (zwakke koppeling) limieten beschrijven van een unieke elfdimensionale theorie die M–theorie wordt genoemd. Hoewel dit een enorme vooruitgang is[7], ontbreekt er tot op de dag van vandaag een microscopische formulering van M–theorie. In dit proefschrift bestuderen we een voorstel voor een microscopische definitie van M–theorie, Matrix theorie genoemd, en laten we zien dat Matrix theorie in staat is het spectrum van BPS–objecten van M–theorie grotendeels te reproduceren.

Als vervolg op deze ontwikkelingen, gebruikmakend van dezelfde technieken, werden allerlei onverwachte relaties (wederom dualiteiten genoemd) ontdekt tussen kwantumijkveldentheorieën die, zoals reeds vermeld, ook onderdeel zijn van het Standaard Model, en snaartheorieën gedefinieerd op Anti–de Sitter–ruimtes (dit zijn oplossingen van de bewegingsvergelijkingen van algemene relativiteitstheorie met een negatieve kosmologische constante). In een bepaalde limiet leiden deze dualiteiten tot de verrassende conclusie dat sommige kwantumijkveldentheorieën tevens beschreven kunnen worden door supergravitaties gedefinieerd op Anti–de Sitter–ruimtes. A priori hadden deze dualiteiten ook ontdekt kunnen worden zonder enige kennis van snaartheorie, maar in dat geval lijkt het onwaarschijnlijk dat dit zou hebben geleid tot het huidige begrip van dit verschijnsel. In dit proefschrift laten we bovendien zien dat snaartheorie het mogelijk maakt deze relaties tussen gravitatie en kwantumveldentheorie systematisch af te leiden en presenteren we enkele nieuwe gevallen.

De laatste tien jaar bleek snaartheorie in staat ons allerlei diepe inzichten te geven in de eigenschappen van kwantumgravitatie en kwantumveldentheorieën. Het lijdt nauwelijks twijfel dat deze resultaten ook hun invloed zullen hebben buiten de snaartheorie. Snaartheorie lijkt namelijk bij uitstek geschikt om ons allerlei concrete en handelbare voorbeelden te geven van nieuwe ideeën over ruimte en tijd en kwantumveldentheorie, die bijvoorbeeld door theoretisch natuurkundigen niet werkend aan snaartheorie geopperd zijn[8]. Het valt te hopen en te verwachten dat er de komende jaren nog vele nieuwe ontwikkelingen zullen volgen.

[7] Voorheen waren er vijf ongerelateerde snaartheorieën en bestond er dus geen unieke geunificeerde theorie.

[8] Voorbeelden daarvan zijn het Holografische principe van ’t Hooft en de niet–commutatieve meetkunde van Connes.

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