DUALITIES OF STRING AND BRANES

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Para Fátima, con todo cariño.

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Chapter 1

Introduction

Twentieth century theoretical physics has been dominated by two major achievements which both revolutionized the way of thinking in physics: quantum mechanics and the theory of relativity.

The *theory of relativity*, written down by Einstein between 1905 and 1916, states that the laws of physics should be the same for all observers in the universe and must therefore be formulated in a covariant (observer independent) way. The theory of relativity consists of two parts: the theory of special relativity, which reformulates and corrects Newtonian mechanics at relativistic velocities (near the speed of light) and the theory of general relativity, which describes gravity in an observer independent way by introducing the concept of curved spaces.

Quantum mechanics, formulated in the nineteen twenties and thirties, is the theory that describes the behaviour of particles at (sub)-atomic scales, and is therefore the theory to be used if one is dealing with elementary particles. The main point of quantum mechanics is that some quantities in Nature do not have the continuous behaviour as described in classical mechanics, but turn out to be quantized, i.e. they can only take some discrete values. Furthermore there exist fundamental uncertainty relations: physical quantities can no longer be determined with the same accuracy as in classical mechanics, but due to quantum fluctuations the theory should be formulated in terms of probabilities.

Both theories, relativity and quantum mechanics, have become the pillars of modern physics and (general) covariance and quantum behaviour should be the basic ingredients of every fundamental theory. However, each of the two theories is only valid in its own specific range: relativity does not incorporate quantum effects needed to describe elementary particles at relativistic velocities, nor has quantum mechanics the necessary covariant formulation in order to be observer independent. It is therefore logical to look for a better formulation, a new theory which incorporates these two properties and of which both relativity and quantum mechanics are special limits.

Quantum field theory (formulated between the thirties and the sixties) was a first attempt to formulate a relativistic description for elementary particles. It made use of the concept of *gauge invariance*, which was inherited from classical electromagnetism. Gauge invariance is a symmetry which states that there are more degrees of freedom (fields) in the theory than physically relevant variables: the physically relevant variables are built from the degrees of freedom, the so-called gauge fields, but different gauge fields give rise to the same expression for the physical variables. Gauge transformations relate the gauge fields that build up the same physical variable, thereby dividing them into equivalence classes of identical physics. The physical variables, and therefore the theory, are (by construction) invariant under the gauge transformations. A gauge invariant formulation is thus in a sense a kind of over-description of the theory, but it can be used as a tool to calculate the physical quantities: at every point in space one can choose the form of the gauge field that is most convenient to solve a particular problem.

In the nineteen seventies, the Standard Model took form as the generally accepted way to describe elementary particles and their interactions: quarks and leptons were identified as the basic constituents of matter and photons, gluons and vector bosons as gauge particles that transmit the strong and electro-weak interactions. These interactions are governed by the gauge group $SU(3) \times SU(2) \times U(1)$, which at low energies is spontaneously broken to $SU(3) \times U(1)$ via the Higgs mechanism. The Higgs boson, of which the potential is responsible for the spontaneous symmetry breaking, accounts for the masses of the particles in the Standard Model.

The Standard Model is a very successful model, for various reasons. It gives an elegant and powerful description of the strong and the electro-weak interactions, making use of the principle of gauge invariance. Furthermore it agrees to a very high accuracy with experimental results, and made some predictions which were later verified in experiments.

In spite of this success, there are reasons to believe that the Standard Model is not the end of the story. These are not experimental reasons, since the Standard Model agrees very well with experiments, but theoretical reasons to believe the theory is not complete. A first indication comes from the theory itself: if the Standard Model is really the final and fundamental theory of particles and interactions, how come that there are still so many free parameters left? The masses of the particles, the mixing angles and the coupling constants of the interactions, all play an important role in the Model, but are not predicted by it. Their exact values are inserted by hand in order to agree with experiment. A fundamental theory would be more convincing if it could explain why all these parameters have the values we measure.

A second reason is more fundamental: although the Standard Model is the quantum field theory of the strong and electro-weak interactions, and therefore incorporates both a quantum and a (special) relativistic description of these interactions, it does not take the gravitational interactions into account. In other words, the Standard Model is a successful unification of special relativity and quantum mechanics, making use of gauge theories, but not of general relativity. There still does not exist a good theory which deals with the quantum aspects of gravity, or vice versa, gives a good description of the gravitational interaction between two elementary particles. This problem, the search for the theory of quantum gravity, has become an important challenge in modern high energy physics.

From the point of view of experiment, there is no direct problem: the gravitational interaction is much weaker than the strong or the electro-weak interaction so that in any realistic experiment it can be ignored completely. This is the reason why the Standard Model agrees so well with experiment: gravitational effects on elementary particles can simply not be measured with the present technology.

Yet, on theoretical grounds, one can argue that at a certain point the present theory will no longer hold: at higher energy scales, the gravitational interaction becomes increasingly important, and near the Planck-mass M_P it can no longer be ignored. The Planck-mass is the energy scale at which the Schwarzschild radius $R_S = 2mG_N/c^2$ of a particle becomes equal to its Compton wave length $\lambda_C = h/mc$:

$$M_P \;=\; \sqrt{\frac{hc}{2G_N}} \;\sim 10^{19}\ \mathrm{GeV}/c^2\ . \tag{1.1}$$

The Schwarzschild radius of an object of mass m is the limit beyond which the object has to be compressed in order to become a black hole and the Compton wave length is a measure for the quantum uncertainty in the position of a particle. So at the Planck-scale the structure of space-time gets interwoven with quantum uncertainties and a theory of quantum gravity is needed. Note the the Planck-mass can be expressed in terms of three fundamental constants of Nature: Planck's constant h, the speed of light c and Newton's gravitational constant G_N.

The energy scales corresponding to the Planck-mass are many orders of magnitude beyond the reach of present accelerators (the LHC, being built in Geneva, will be able to perform experiments at energies around 1.5×10^4 GeV), so in constructing a theory of quantum gravity, one will have to rely strongly on theoretical arguments and intuition, instead of following experimental indications.

The reason why it is so difficult to construct a quantum theory of gravity is that gravity is *not renormalizable*. Newton's gravitational constant, which is the coupling constant of gravity, has dimensions of (mass)$^{-2}$ (in units where $c = \hbar = 1$), such that the effective, dimensionless coupling constant $G_N E^2$ is proportional to the square of the energy of a given process. For higher and higher energies, the coupling will grow arbitrarily large and lead to divergences in perturbation theory that become larger in every order and make the theory difficult to handle at high energies. Effectively this means that something goes wrong in the short distance (= high energy) behaviour of the theory and that there is a cut-off beyond which the theory is no longer valid.

A parallel can be drawn between the non-renormalizability of gravity and of the four-fermi theory for the weak interaction. Both theories suffer from the same kind of problems due to a dimensionful coupling constant. In the case of the weak interactions the problem was solved by replacing the four-fermi theory by the $SU(2) \times U(1)$ gauge theory of electro-weak interactions, where the divergences were smoothed by the introduction of gauge bosons that spread out the interaction and weaken the short distance behaviour.

A logical attempt therefore would be to write gravity as a gauge theory for a new kind of symmetry, namely *supersymmetry.* This is a symmetry that relates bosons and fermions and associates to each particle a new particle of the opposite type. There is still no experimental evidence for the existence of supersymmetry: none of the extra particles predicted by supersymmetry has ever been observed. Therefore, if it exists, it has to be broken at low energies.

In spite of the lack of experimental evidence, many physicists believe that supersymmetry is an important ingredient for a description of quantum gravity: a local version of supersymmetry induces invariance under general coordinate transformations and thus leads to a theory with dynamical gravitational fields. In other words, a locally supersymmetric quantum field theory is a supersymmetric version of general relativity. Field theories with local supersymmetry are generally called *supergravity theories.*

Furthermore it was observed that supersymmetry softens the divergences in a quantum field theory. Since fermionic contributions to perturbative loop calculations have opposite signs compared to bosonic contributions, it was hoped that in this way the different divergences might cancel each other and give a finite result. However this turned out much more difficult to show than was first thought and people have more or less abandoned the idea that divergence cancellation might work in quantum field theory.

Since local supersymmetry alone is not sufficient to remove the divergences in gravity, a bigger step is needed. This is done by *string theory*, a theory that has as a starting point the idea that all elementary particles are not point-like, as we intuitively used to think, but one-dimensional objects, strings with a certain spatial extension. The theory has a natural cut-off built in at short distances, since the interactions are now spread out over the length of the string. In this way the short-distance behaviour is softened. The different oscillation modes of the string should correspond to the various elementary particles that we know from the Standard Model, that are predicted by supergravity and many more.

Introducing supersymmetry in string theory, one obtains the so-called *superstring.* The reason for introducing supersymmetry is that the simplest model, the bosonic string, which has only bosonic degrees of freedom, contains "unphysical" states in its spectrum. These are called tachyons and have the strange property that their mass squared is negative. However, this undesirable feature can be eliminated by introducing supersymmetry. Indeed, the superstring does not suffer from this problem, and at the same time it contains fermionic degrees of freedom, which the bosonic string did not have.

The short distance behaviour of superstring theory is better than that of most quantum field theories: it can be shown [32] that the superstring scattering amplitudes are ultraviolet finite. Whereas in quantum field theory perturbative calculations are done by computing Feynman diagrams, the perturbation expansion in string theory is a sum over the topologies of the two-dimensional world sheet which the string sweeps out in space. This means that in every order there is only one "diagram" to be considered (in a theory of closed strings), this in contrast to quantum field theory, where the number of diagrams increases rapidly with the order.

Important progress was made when it was realized [166, 136] that the superstring theory has massless states of spin two, which could be identified as the gravitons, the gauge particles of gravity. The identification of the graviton in the string spectrum also sets the length scale of the string: the typical size of a string should be of the order of the Planck-length L_P, the Compton wave length of a particle with mass equal to the Planck-mass:

$$L_P = \sqrt{\frac{G_N h}{c^3}} \ \sim \ 10^{-35} \text{ m}. \tag{1.2}$$

Note that present accelerators can probe distances down to about 10^{-18} m, so this explains why nothing has been noted of the stringy extendedness of elementary particles.

After what became known as the first superstring revolution (1984-1985), string theory really began to be considered as a serious candidate for a unifying theory. It turned out that there exist (only) five consistent versions, called the Type I string, the Type IIA, Type IIB, Heterotic $E_8 \times E_8$ and Heterotic $SO(32)$ string, which all have well-defined perturbation expansions and differ in their field content and the amount of space-time supersymmetry. Consistency in the quantization requires each of the five string theories to live in a ten-dimensional space-time.

The fact that the space-time is required to have ten dimensions, and not four, as we are used from general relativity or quantum field theory, is not such a big problem as it might seem. The explanation is that six of the ten dimensions are compact and very small (in fact of the order of the Planck-length [99]), so they cannot be detected at low energies. A technique, called *dimensional reduction*, is known to rewrite the ten-dimensional theory as an effectively four-dimensional one in order to make contact with our experimentally observable world.

Depending on how this dimensional reduction is performed, all kind of gauge symmetry groups can appear, some of which resemble the Standard Model at low energies. But there are many different reductions possible, leading to many low energy effective theories and many different vacua, and it is not at all clear why the universe as we see it has precisely four dimensions (and not any other number smaller than ten) and why precisely one particular reduction scheme should be preferred to others. A fundamental theory like string theory should be able to give a natural answer to these questions.

Another problem of string theory (which is maybe related to the previous ones) is that little more of it is known than a perturbative description. Glances into the non-perturbative regime have only recently become possible, since what is called the second superstring revolution, which started in the mid nineties. Then a new concept was introduced in string theory, namely the *duality symmetries*. In fact dualities might be one of the fundamental principles to understand string theory.

Dualities are symmetry transformations that relate different compactifications of a theory, different regimes and even different string theories to each other. There are many different types of duality transformations, but the ones we treat in this thesis are the most important ones: T-duality and S-duality. The other types of dualities can mostly be related to combinations of these two.

T-duality stands for target space duality, the duality on the space-time through which

the string moves. It relates small volumes to large ones and therefore physics of small scales to physics of large scales. Suppose one of the dimensions of the target space is rolled up into itself (for example as in a process of dimensional reduction) and forms a circle of radius R. A string running around in this compact dimension will have discrete momentum and energy states. In particular, the smaller the radius of the circle, the higher the energy of the string states. On the other hand, the string can also wind a number of times around this compact dimension. Since the energy of the string is also proportional to its length, the string winding states become more and more energetic as the radius of the circle becomes bigger and the string itself longer.

Now it turns out that energy levels of a string moving around on a circle with small radius correspond exactly to the energy levels of a string wound around a large circle and vice versa. In general, a string which is moving with momentum m and is wound n times around a circle of radius R, is equivalent to another string, moving with momentum n and wound m times around another circle of radius $\tilde{R} = \alpha'/R$, where α' is a constant related to the length of the string.

The duality transformation that relates these two descriptions is called T-duality and the two backgrounds (one with a compact dimension of radius R and the other of radius α'/R) are called T-dual. The string (and hence the observer) doesn't see whether it is in the first or in the second case, so it seems that there exists a kind of symmetry $R \rightarrow \alpha'/R$ between large and small scales. If the size of a compact dimension shrinks beyond a certain size ($R = \alpha'$), the theory behaves essentially as if in a dual description the dimension would be increasing again. This is an indication that the space-time at the Planck-scale may be very different from what we are intuitively used to.

It is clear that in this way many different compactifications can be related. If we perform a dimensional reduction over d dimensions of radii R_a ($a = 1, ..., d$), the obtained vacuum is physically equivalent to a dimensional reduction over coordinates of radii $\tilde{R}_a$, if the radii R_a and $\tilde{R}_a$ are related via T-duality and permutations in the index a. In this way T-duality divides the different vacua into equivalence classes and, although it does not say which vacuum is preferred to others, at least it reduces the problem significantly.

Not only can different compactifications of a specific theory become equivalent via T-duality, also the different theories themselves can be related via this procedure. As we will show later on in this thesis, the dimensionally reduced version of one theory compactified over a circle of radius R can be mapped onto the dimensionally reduced version of a different theory, which has been compactified over a circle of radius α'/R. In this way the Type IIA and the Type IIB theory and the Heterotic $E_8 \times E_8$ and $SO(32)$ can be related to each other: one theory compactified on a small volume is equivalent to the other theory compactified on a large volume.

Another duality that has been conjectured to exist, is the strong/weak coupling duality or *S-duality*. In perturbation theory only the weak-coupling regime of string theory can be explored but as the coupling grows too strong perturbative calculations break down and trustworthy results are hard to obtain. S-duality might give insight in the strong-coupling regime since it relates the strong and weak coupling regions of theories to each other. If the S-duality conjecture holds, a string theory A with fields ϕ_A and coupling constant g_A can be rewritten in terms of another string theory B with dual fields $\tilde{\phi}_B$ and coupling constant $g_B = 1/g_A$. In this way, non-perturbative calculations in one theory can be translated into perturbative calculations in the other theory. There are strong reasons to believe that in this way the strong coupling limit of the Type I theory corresponds to the weak coupling limit of Heterotic $SO(32)$ theory (and vice versa) and that the Type IIB theory is S-self dual, i.e. S-duality relates the strong and the weak coupling limits within the same theory.

The strong coupling limits of Type IIA and Heterotic $E_8 \times E_8$ are even more surprising: although these two theories are both ten-dimensional, as are all other string theories, in their strong coupling limit an extra, eleventh dimension appears. This is possible because this extra eleventh dimension is compact and its size is related to the ten-dimensional coupling constant. So at weak coupling the eleventh dimension is very small and in fact invisible, but as we let the coupling grow this extra dimension unfolds.

This discovery drew the attention back to eleven-dimensional supergravity, a theory which was known already from the times before string theory, when people still thought supergravity might lead to the theory of quantum gravity. However, eleven-dimensional supergravity was always neglected because of its possible non-renormalizability and the fact that one cannot obtain a chiral spectrum as in the Standard Model, where left and right handed components of fields behave differently under symmetry transformations. With the rise of string theory, it was considered an irrelevant curiosity, since all string theories live in ten dimensions and no connection to eleven-dimensional supergravity was found. Now that it turns out that some string theories have an eleven-dimensional limit, $D = 11$ supergravity gains importance as a possible low-energy effective theory for this strong-coupling limit.

We see that the duality symmetries weave a web of duality transformations between the different string theories and even eleven-dimensional supergravity: they are all interconnected via T or S-duality. This feeds the idea that the various string theories are in fact not the really fundamental theories, but rather different perturbation expansions around different vacua of one and the same underlying theory. This is an attractive and elegant idea, that explains both the wide variety of duality relations between the different theories, as well as the fact why we find no less then five versions of what we thought was the unifying, fundamental theory.

However, the other side of the picture is that it is not clear at all what this underlying theory looks like. It is usually referred to as *M-theory* (where the M can stand for many things, such as Membrane, Mother, Matrix, ...), but little more of it is known then that it has eleven-dimensional supergravity as its low-energy effective theory and that it is supposed to be the strong coupling limit of Type IIA and Heterotic $E_8 \times E_8$ theory. A lot of work is done nowadays to get a better picture of what M-theory actually is.

An important role in checking the M-theory conjecture and the duality relations between the different theories is played by the solutions of the equations of motion of the theory. In general, they appear as extended objects, objects with one or more spatial extension and are referred to as *p-branes*, where p stands for the dimensionality of the object: $p = 0$ is a particle, $p = 1$ is a string, $p = 2$ a membrane, ... Many of these

p-branes occur as solitons in the theory, i.e., not as solutions of perturbative calculations, but as topological defects which are very heavy and strongly interacting at weak coupling.

An example of such a brane is the solitonic five-brane, an object that has five spatial directions and carries a magnetic charge. Isolated magnetic charges have never been observed but occur typically in solitonic objects. This is in contrast to electrically charged objects which are considered to be the fundamental objects of the theory, since they appear in perturbation theory: they are light and weakly interacting at small coupling.

In an early version of S-duality a conjecture was made stating that, for a theory of electrically and magnetically charged particles, a dual formulation exists where the role of fundamental and solitonic particles is reversed: in the dual formulation the fundamental particles are the ones with magnetic charge, while the solitons are electrically charged. Furthermore, since the Dirac quantisation condition states that electric charge e and magnetic charge q are related via their inverses $q \sim 1/e$, the strong coupling limit of one theory corresponds to the weak coupling limit of the dual theory and vice versa: strongly interacting solitons in the fundamental theory can be viewed as weakly interacting fundamental particles in the dual theory.

In string theory, the fundamental, electrically charged object that interacts weakly at small coupling is the fundamental string, while the heavy, strongly interacting magnetic object is the solitonic five-brane. The string theory version of the electric/magnetic duality conjecture is the string/five-brane duality, which states that the strongly interacting string is dual to the weakly interacting five-brane. Instead of starting off with a theory for strings, we could have written down a theory for elementary five-branes that has string-like solitons (however, the problem with this dual formulations is that it is not clear how to quantize such an elementary five-brane). More generally, every p-brane in D dimensions has a dual $(D-p-4)$-brane of the opposite charge (electric vs. magnetic) and coupling (strong vs. weak).

Another type of extended objects that appear in string theory is the so-called Dirichlet-brane, or short D-brane. D-branes arise in the T-dual formulation of open strings: it turns out that open strings can also be described as strings whose end-points are attached to these D-branes. All D-branes are related to each other via T-duality and the strings attached to them make it possible to study their dynamics using familiar string perturbation theory. Furthermore, upon dimensional reduction to lower dimensions, D-branes might give insight into the microscopic description of the quantum states of black holes.

It was also realised that if Type IIA theory is really a compactified version of an eleven-dimensional theory, all solutions of Type IIA should be interpretable as reductions of eleven-dimensional objects. Indeed, it turns out that the fundamental object of $D = 11$ supergravity is a membrane, rather than a string, but upon dimensional reduction over one of the directions in which this membrane is oriented, a string-like object is found, which can be identified with the fundamental string solution of the Type IIA theory. Also the other Type IIA solutions (the solitonic five-brane, the D-branes, etc.) can be obtained from eleven-dimensional objects after dimensional reduction.

The fact that all these types of p-branes turn up in string theory has given rise to questions concerning the very nature of the theory: why call it string theory if there exist dual, equivalent formulations in terms of (for example) five-branes and if one of the theories entering in the duality web, $D = 11$ supergravity, does not even have a string-like solution, but a membrane? Why should in such a variety of objects, strings be more fundamental then other ones? Terms such as "p-brane democracy" and "Is string theory a theory of strings?" have become common amongst string theoreticians. It is hoped that M-theory will deal with these questions, but one of the reasons why it is so hard to formulate it, is that we do not know in terms of which objects the description is best given.

In this thesis, some of the aspects of the duality symmetries within string theory are discussed. This is done by looking at three main parts: the target space theory, the solutions and the world volume theory.

The *target space action* is the low-energy effective action of string theory, as seen from the space-time in which the theory lives, if one integrates out the massive modes. The action one obtains is one of supergravity theories, so that supergravity can be seen as a low-energy approximation of string theory. We will work often with these target space actions. They have many symmetries, which help us to understand the full string theory, even if these symmetries may not be completely conserved up to the level of the full theory. Also indications of the existence of the duality transformations between the various string theories are already present in the low-energy effective actions.

The equations of motion of the low-energy effective action give rise to the *solutions*. The duality transformations between these solutions, discussed above, are manifestations of the duality relations between the different string theories. Using the dualities on the solitonic solutions, one can get insight in the non-perturbative regime of the theory, while on the other hand looking at the duality relations between the solutions one can perform tests to check the conjectured dualities between the theories.

The dynamics of these solutions is described by the *world volume actions*. So in order to get a good understanding of the solutions it is necessary to look at their world volume actions. Also here there exist all kinds of duality relations between the world volume actions, much as they exist between the solutions themselves.

This thesis is organized as follows: in Chapter 2 we give a general introduction to string theory, the world volume theory and the dynamics, the different types of string theories, the target space action and the different solutions. In Chapter 3 we present T- and S-duality and show how they act at the level of the world volume, the target space actions and the solutions. We determine the strong coupling behaviour of the different theories and sketch the duality web between the actions and the solutions. We also explain how dimensional reduction is performed.

After these two introductory chapters, we will look in more detail at the different aspects of string theory. In Chapter 4 we study the target space actions, their symmetry groups and the duality relations, both in ten, nine, six and five dimensions. Chapter

5 is about the solutions of the target space actions and more specifically intersections of two or more p-branes. First a kind of stability condition is determined for an intersection of two such branes, and then this condition is used to construct and classify intersections consisting of more then two intersecting branes. Dimensional reductions of these intersections lead to new solutions in lower dimensions. The world volume theory is studied in Chapter 6. An overview is given of the world volume actions of the different solutions and the duality relations between them will be demonstrated at the level of these world volume actions. At the end of this chapter, the world volume action of one particular solution, namely the Kaluza-Klein monopole, is constructed, making use of the duality relations between the monopole solutions of the solitonic five-brane solution.

Chapter 2

String Theory

In this chapter we will give a general introduction to various aspects of string theory. We review in section 2.1 the basic string dynamics, introducing the sigma models of the classical bosonic string and the superstring. In section 2.2 we will look at the low energy effective actions of the various types of superstring theories, and in section 2.3 attention will be paid to the different solutions that arise in these theories.

A general introduction to the different aspects of string theory can be found in [78, 95, 105, 114, 127], for a review on string solutions and p-branes we refer to [61, 151].

2.1 World Volume Theory

Let us consider a classical bosonic string, moving in a D-dimensional Minkowski space, represented by the coordinates X^μ and the flat metric $\eta_{\mu\nu} = \mathrm{diag}[1, -1, -1, ..., -1]$.

While moving through space, the string sweeps out a two-dimensional surface Σ which we call the world sheet of the string, and which can be parametrised by the two-tuple $\sigma^i = (\tau, \sigma)$, where τ is a time-like parameter of the string and σ parametrises the length.

In analogy with the point particle, we can write down an action which describes the dynamics of the string, that is proportional to the surface of the world sheet:

$$S = -T \int_\Sigma d^2\sigma \sqrt{|\det(\partial_i X^\mu(\sigma^k)\partial_j X^\nu(\sigma^k)\eta_{\mu\nu})|}. \tag{2.1}$$

The action (2.1) is called the Nambu-Goto action for the bosonic string.

The constant T is the string tension and has the dimension of $(\mathrm{mass})^2$. Note that the X^μ are functions of τ and σ, and give the embedding of the string in the D-dimensional space-time. They are described by a two-dimensional field theory on the world sheet. They induce a metric g_{ij} on Σ via the expression $g_{ij} = \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu}$, so we see that (2.1) is indeed proportional to the surface of Σ.

There exists also another action which is, at least classically, equivalent to (2.1), but does not have the non-linearity caused by the square root:

$$S = -\frac{T}{2}\int d^2\sigma \sqrt{|\gamma|}\ \gamma^{ij}\partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu}. \tag{2.2}$$

This action, called the Polyakov action [131] (though first introduced in [55, 35]), makes use of the metric γ_{ij} on the world sheet as an independent but non-dynamical variable. We will see later that it can be gauged away completely. Its equation of motion defines the energy-momentum tensor

$$T_{ij} = -\frac{1}{T}\frac{1}{\sqrt{|\gamma|}}\ \frac{\delta S}{\delta \gamma^{ij}} = \tfrac{1}{2}\partial_i X^\mu \partial_j X_\mu - \tfrac{1}{4}\gamma_{ij}\gamma^{kl}\partial_k X^\mu \partial_l X_\mu = 0. \tag{2.3}$$

Taking the determinant of the matrix equation $T_{ij} = 0$ and taking the square root, we find

$$\sqrt{|\det(\partial_i X^\mu \partial_j X_\mu)|} = \tfrac{1}{2}\sqrt{|\gamma|}\gamma^{kl}\partial_k X^\mu \partial_l X_\mu, \tag{2.4}$$

which gives the relation between the Nambu-Goto and the Polyakov action. Let us now discuss the symmetries of the Polyakov action. First of all, Eqn (2.2) is, just as (2.1), invariant under reparametrisations of the world sheet $(\tau,\sigma) \to (f_1(\tau,\sigma), f_2(\tau,\sigma))$, as it should be. Since parametrisations (τ,σ) of the world sheet do not have a physical meaning and are in principle arbitrary, no physical result can depend on them. Furthermore, Eqn (2.2) has an extra symmetry which is intrinsically related to the fact that we are dealing with strings, one dimensional objects: the Weyl-rescaling. Only on a two-dimensional world sheet, is $\sqrt{|\gamma|}\ \gamma^{ij}$ invariant under

$$\gamma_{ij} \to \Lambda(\sigma)\ \gamma_{ij}. \tag{2.5}$$

We can use these local symmetries to gauge away the world sheet metric and write (2.2) in a simpler form. Making use of the reparametrisation invariance, we can write locally $\gamma_{ij} = \Omega(\sigma)\ \eta_{ij}$, the flat world sheet metric times a conformal factor, and scale away this conformal factor via the Weyl invariance. We then end up with the action of the free bosonic string.

$$S = -\frac{T}{2}\int d^2\sigma\ \eta^{ij}\ \partial_i X^\mu \partial_j X_\mu, \tag{2.6}$$

for which we can easily calculate the equation of motion of X^μ. This turns out to be the two-dimensional free wave equation

$$(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0, \tag{2.7}$$

with the well-known solution

$$X^\mu(\tau,\sigma) = X_+^\mu(\tau+\sigma) + X_-^\mu(\tau-\sigma), \tag{2.8}$$

$X_+^\mu(\tau+\sigma)$ and $X_-^\mu(\tau-\sigma)$ being arbitrary functions for the left and right moving modes on the string.

We still have to impose boundary conditions on Eqn (2.7). At this point, we have to distinguish between two topologically different types of strings: the open string, which is a string with free endpoints, and the closed string, which has no ends[1]. For closed strings we impose periodic boundary conditions $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi)$. The Fourier expansion of Eqn. (2.8) for the closed string, satisfying these periodic boundary conditions, is then given by

$$\begin{aligned} X^\mu_-(\tau-\sigma) &= \tfrac{1}{2}x^\mu + \tfrac{1}{2\pi T}\, p^\mu\ (\tau-\sigma) + \tfrac{i}{2}\tfrac{1}{\sqrt{\pi T}} \sum_{n\neq 0} \tfrac{1}{n}\ a^\mu_n\ e^{-in(\tau-\sigma)}, \\ X^\mu_+(\tau+\sigma) &= \tfrac{1}{2}x^\mu + \tfrac{1}{2\pi T}\, p^\mu\ (\tau+\sigma) + \tfrac{i}{2}\tfrac{1}{\sqrt{\pi T}} \sum_{n\neq 0} \tfrac{1}{n}\ \tilde{a}^\mu_n\ e^{-in(\tau+\sigma)}. \end{aligned} \tag{2.9}$$

x^μ and p^μ are the position and momentum of the center of mass and the a^μ_n and $\tilde{a}^\mu_n$ the Fourier coefficients of the oscillation modes of the string. Reality of X^μ requires that $(a^\mu_n)^\dagger = a^\mu_{-n}$ and $(\tilde{a}^\mu_n)^\dagger = \tilde{a}^\mu_{-n}$. The oscillation modes provide the string with extra dynamical degrees of freedom which distinguish the string from a point particle.

For the open string the boundary conditions come from the surface term in the variation of (2.6) between τ_i and τ_f (where we took $\delta X^\mu(\tau_i) = \delta X^\mu(\tau_f) = 0$):

$$-T\int d\tau\ \delta X^\mu \partial_\sigma X_\mu\ \Big|_{\sigma=0}^{\sigma=\pi} = 0. \tag{2.10}$$

This condition can be satisfied in two ways. The most obvious one is the Neumann boundary condition

$$\text{Neumann}: \quad \partial_\sigma X^\mu\Big|_{\sigma=0}^{\sigma=\pi} = 0, \tag{2.11}$$

because of its $SO(D-1,1)$ Poincaré invariance. Its physical meaning is that there is no momentum flow out of the string at both endpoints.

The Dirichlet boundary condition

$$\text{Dirichlet}: \quad \delta X^\mu\Big|_{\sigma=0}^{\sigma=\pi} = 0 \iff X^\mu\Big|_{\sigma=0}^{\sigma=\pi} = C^\mu, \tag{2.12}$$

with C^μ a constant vector, looks a bit strange at first sight, since it implies that the endpoints of the open string are fixed in space. However it will turn out that this is indeed a physically relevant boundary condition.

Suppose an open string satisfies Neumann boundary conditions in all but one direction, and Dirichlet boundary conditions in one direction X^1. This means that there is a $(D-2)$-dimensional hyperplane $X^1 = C$ in the Minkowski space to which the endpoints of the string are attached. This hyperplane is called a "Dirichlet-brane" or D-brane, because of the Dirichlet boundary conditions on the string. The interactions with open strings make the D-brane a dynamical object that, as we will see later, will play an important role in non-perturbative string theory.

[1] A string theory with open strings also contains closed strings, since the joining and splitting of open strings can lead to closed ones. The reverse is not true. For the open string we will choose the parametrisation $\sigma = [0,\pi]$, while for closed strings $\sigma = [0, 2\pi]$.

The Fourier expansion of the open string solution to (2.7), satisfying Neumann or Dirichlet conditions is given by

$$X^\mu_N(\tau,\sigma) \;=\; x^\mu + \tfrac{1}{2\pi T}\; p^\mu\; \tau + \tfrac{i}{2}\tfrac{1}{\sqrt{\pi T}}\sum_{n\neq 0}\tfrac{1}{n}\; a^\mu_n\; e^{-in\tau}\cos n\sigma, \tag{2.13}$$

$$X^\mu_D(\tau,\sigma) \;=\; x^\mu + \tfrac{1}{2\pi T}\; p^\mu\; \sigma + \tfrac{i}{2}\tfrac{1}{\sqrt{\pi T}}\sum_{n\neq 0}\tfrac{1}{n}\; \tilde{a}^\mu_n\; e^{-in\tau}\sin n\sigma, \tag{2.14}$$

where X^μ_N satisfies the Neumann conditions and X^μ_D the Dirichlet conditions.

At this point it would be logical to go beyond the purely classical analysis and try to quantize the bosonic string. Making use of techniques as conformal invariance and BRST-quantisation, one can compute the physical spectrum of this string theory and do string scattering amplitude calculations. However, these calculations go beyond the aim of this introduction. For a discussion of conformal symmetry and the BRST-formalism to compute string spectra, we refer to [74, 84]. Let us make some remarks though, which are worth mentioning because of their later relevance or because they complete the general picture.

First of all, a calculation of the spectrum of the bosonic string reveals that this string theory can only consistently be quantised in a 26-dimensional space-time. $D = 26$ is therefore called the critical dimension for the bosonic string and strings that live in other then the critical dimension are called non-critical strings. The fact that the dimensionality of the space-time is not a free parameter, but given by the theory is one of the nice surprises of string theory. Since string theory pretends to be the final, unifying theory, it also should be able to determine the precise value of quantities that entered as free parameters in other theories. It might be worrisome, however, that the number of dimensions, predicted by the bosonic string, differs so much from our "real", four-dimensional world. We will see that for other types of string theories, the number of dimensions will be lower, and that there exist techniques to make contact with the familiar $D = 4$ world.

A more worrying problem is the fact that in the spectrum of the bosonic string a tachyon appears, a particle with an imaginary mass, that moves faster than the speed of light. This will mess up the causality structure of the theory and is therefore an undesired feature. The problem is due to the fact that we are dealing with the bosonic string. Introducing the fermions in the right way will eliminate the tachyon from the spectrum.

Let us therefore make our string model a bit more realistic by also introducing fermions in the theory. We do this by allowing fermionic fields in the two-dimensional field theory on the world sheet, which will get the interpretation of "fermionic modes" of the string. As it turns out, these fermionic modes give rise to fermion fields in space-time. Let us consider the action:

$$S = -\frac{T}{2}\int d^2\sigma\Big[\partial_i X^\mu \partial^i X_\mu + i\bar{\psi}^\mu \rho^i \partial_i \psi_\mu\Big]. \tag{2.15}$$

Here, ψ^μ is a Majorana spinor on the world sheet that transforms as a vector under the $SO(D-1,1)$-Lorentz group of the Minkowski space. The ρ^i are the two-dimensional Dirac matrices.

The action (2.15) is invariant under a symmetry transformation that interchanges the bosonic and fermionic fields in the theory, the supersymmetry transformations

$$\begin{aligned}\delta X^\mu &= i\bar{\epsilon}\psi^\mu, \\ \delta\psi^\mu &= \rho^i \partial_i X^\mu \epsilon,\end{aligned} \tag{2.16}$$

where ϵ is a constant spinor. Because of the invariance under these supersymmetry transformations, the string model we are considering is called the superstring.

Note that we wrote the action (2.15) in the so-called conformal gauge, where the world sheet metric is already gauged away (compare with (2.6)). Therefore the fields in (2.15) have to obey certain constraints, such as the vanishing of the energy momentum tensor (as in (2.3)) and the conserved supersymmetry current. Though important in the general formulation of superstring theory, these constraints do not enter in the rest of our discussion, so we will not consider them.

The equations of motion and the dynamics of the bosonic part of (2.15) are the same as for the bosonic string. So let us concentrate on the fermionic part. Varying (2.15) with respect to $\bar{\psi}^\mu$ gives the equations of motion

$$\rho^i \partial_i \psi^\mu = 0, \tag{2.17}$$

and the boundary conditions

$$\bar{\psi}^\mu \rho^1 \delta\psi_\mu \Big|_{\sigma=0}^{\sigma=\pi} = 0. \tag{2.18}$$

In order to solve these equations it is convenient to choose a basis in which the Dirac matrices ρ^i are real:

$$\rho^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.19}$$

and to decompose ψ^μ into two real valued components

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix}. \tag{2.20}$$

ψ^μ_+ and ψ^μ_- are the left and right moving fermionic modes on the world sheet. The equations of motion can then be rewritten as:

$$\begin{aligned}(\partial_\tau - \partial_\sigma)\psi^\mu_+ = 0 \ , \\ (\partial_\tau + \partial_\sigma)\psi^\mu_- = 0 \ .\end{aligned} \tag{2.21}$$

Let us first look at the solution of these equations for the case of the open string. We see that the boundary condition

$$\psi^\mu_- \ \delta\psi_{-\mu} - \psi^\mu_+ \ \delta\psi_{+\mu} \Big|_0^\pi = 0 \tag{2.22}$$

is satisfied if $\psi^\mu_+ = \pm\psi^\mu_-$ and $\delta\psi^\mu_+ = \pm\delta\psi^\mu_-$ at $\sigma = 0, \pi$. Since an overall sign in the boundary conditions in irrelevant, we can set without loss of generality $\psi^\mu_+(0) = \psi^\mu_-(0)$. What remains to be fixed is the boundary condition at $\sigma = \pi$. There are two possibilities:

1. Ramond (R) boundary conditions: $\psi_+^\mu(\pi) = \psi_-^\mu(\pi)$. The solution of (2.21, 2.22) then yields

$$\psi_\pm^\mu = \frac{1}{\sqrt{2\pi T}} \sum_n b_n^\mu \, e^{-in(\tau \pm \sigma)}, \qquad n \in \mathbb{Z}. \tag{2.23}$$

2. Neveu-Schwarz (NS) boundary conditions: $\psi_+^\mu(\pi) = -\psi_-^\mu(\pi)$. Eqn (2.21, 2.22) is then solved by

$$\psi_\pm^\mu = \frac{1}{\sqrt{2\pi T}} \sum_r c_r^\mu \, e^{-ir(\tau \pm \sigma)}, \qquad r + \tfrac{1}{2} \in \mathbb{Z}. \tag{2.24}$$

String excitations coming from world sheet fields satisfying R-boundary conditions, will manifest themselves as fermionic fields from the space-time point of view, while excitations of fields satisfying the NS-boundary condition will appear as bosonic fields.

For closed strings we can impose either periodic or anti-periodic boundary conditions on each component ψ_+^μ and ψ_-^μ separately:

1. Periodic boundary conditions (R) $\psi_\pm^\mu(0) = \psi_\pm^\mu(\pi)$:

$$\psi_\pm^\mu = \frac{1}{\sqrt{2\pi T}} \sum_n d_n^\mu \, e^{-in(\tau \pm \sigma)}, \qquad n \in \mathbb{Z} \tag{2.25}$$

2. Anti-periodic boundary conditions (NS) $\psi_\pm^\mu(0) = -\psi_\pm^\mu(\pi)$:

$$\psi_\pm^\mu = \frac{1}{\sqrt{2\pi T}} \sum_r f_r^\mu \, e^{-ir(\tau \pm \sigma)}, \qquad r + \tfrac{1}{2} \in \mathbb{Z}. \tag{2.26}$$

So in total there are four possible combinations of left and right movers, each satisfying either one of the above boundary conditions: NS-NS, NS-R, R-NS and R-R. Excitations of the ψ^μ for which the different components satisfy NS-NS or R-R conditions, appear in the space-time as bosonic fields, whereas the ones that have NS-R or R-NS conditions manifest themselves as fermions.

The supersymmetry on the world sheet also induces supersymmetry transformations between the fermion and the boson fields in the space-time. For open strings this is $N = 1$ (so supersymmetry with one space-time supersymmetry generator) and for closed strings $N = 2$ supersymmetry (except for some special cases, as we will see in the next section).

The supersymmetry transformations (2.16) enable us to remove the tachyon we found in the spectrum of the bosonic string. Furthermore the number of space-time dimensions for the superstring is reduced to $D = 10$. From a phenomenological point of view, this is still a very high dimensional space, but as we will see in the section 3.1.2, there exist techniques to compactify over a number of dimensions to make contact with our $D = 4$ world.

Until now we have only considered strings moving in a Minkowski space, but in the end we are interested in strings moving in spaces with more general background fields, for example some curved space-time characterized by a metric $g_{\mu\nu}$. In section 2.2 we will give the most general covariant two-derivative action. These more general backgrounds complicate considerably the theory.

To perform string calculations one often uses perturbation expansions. One such is an expansion in α', a parameter with dimension of $(\text{length})^2$, which is related to the string tension via $\alpha' = \frac{1}{2\pi T}$. It introduces a fundamental length scale $\sqrt{\alpha'}$, which is the string scale, where stringy effects become important. Most of the time, we will work in the so-called "zero-slope limit"[2] $\alpha' \to 0$, unless mentioned differently. This corresponds to the string tension $T \to \infty$, so the size of the string shrinks to zero and it can be approximated by a point particle.

A second perturbation expansion is the expansion in the string coupling constant (given by the expectation value of the dilaton field e^{ϕ}, which we will introduce in the next section). This expansion counts the number of loops in string scattering processes, and thus the genus of the world sheet Σ. In fact this is the string generalisation of the Feynman diagrams in quantum field theory.

2.2 Target Space Action

Let us now for a moment go back to the bosonic string and try to write down a string moving in a more general space-time than the Minkowski space we have considered in the previous section. The most general covariant action we can write down with two world sheet derivatives is the non-linear sigma model action

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \Big\{ \Big(\sqrt{|\gamma|}\gamma^{ij} g_{\mu\nu}(X) - \varepsilon^{ij} B_{\mu\nu}(X) \Big) \partial_i X^\mu \partial_j X^\nu - \alpha' \sqrt{|\gamma|}\ \phi(X)\ \mathcal{R}^{(2)} \Big\}. \tag{2.27}$$

This is the action of a string moving through a background characterized by a metric $g_{\mu\nu}$, an antisymmetric tensor $B_{\mu\nu}$, called the axion, and a scalar field ϕ called the dilaton. $\mathcal{R}^{(2)}$ is the Ricci scalar of the two-dimensional world sheet metric γ_{ij} and ε^{ij} the fully antisymmetric tensor in two dimensions.

For a constant mode of the dilaton ϕ_0, the last term in (2.27) is a topological term which is proportional to the Euler characteristic

$$\chi(\Sigma) = \frac{1}{4\pi} \int_\Sigma d^2\sigma \sqrt{|\gamma|}\ \mathcal{R}^{(2)} = 2 - 2g, \tag{2.28}$$

where g is the genus (number of holes) of the surface Σ. In other words, the last term in (2.27) counts the number of loops in the string scattering diagrams. A g-loop diagram in the (Euclidean) path integral gets weighted by a factor $(e^{\phi})^{2-2g}$ and the string coupling constant can be identified with the expectation value of e^{ϕ}.

[2]The name zero-slope limit comes from the fact that α' is the proportionality constant between the angular momentum J of a rotating string with energy E and the square of the energy, so the slope of the plot $J(E^2)$.

The difference between the actions (2.2) and (2.27) is that the latter does not turn into the action (2.6) in the conformal gauge $\gamma_{ij} = \Omega(\sigma)\ \eta_{ij}$, which makes it a non-trivial two-dimensional field theory and forces us to a perturbation expansion in α', if we want to do quantum calculations.

The first two terms of (2.27) are invariant under Weyl rescaling (2.5) at the classical level, but the demand that Weyl invariance should hold at the quantum level forces the β-functions of the fields to vanish. This is because the β-functions give the scale dependence of the couplings of the various fields, so Weyl invariance (and therefore scale invariance) implies $\beta = 0$.

The conditions for Weyl invariance to hold are then, at first non-trivial order in α' and at tree level in the loop expansion [38]:

$$\begin{aligned}
\beta^g_{\mu\nu} &= R_{\mu\nu} - 2\nabla_\mu\partial_\nu\phi + \tfrac{9}{4}H_{\mu\rho\lambda}H_\nu{}^{\rho\lambda} + O(\alpha') = 0, \\
\beta^B_{\mu\nu} &= \nabla_\rho H^\rho{}_{\mu\nu} - 2H^\rho{}_{\mu\nu}\partial_\rho\phi\ + O(\alpha')\ = 0, \\
\tfrac{1}{\alpha'}\beta^\phi &= \tfrac{1}{\alpha'}(D-26) + 3\Big(R + 4(\partial\phi)^2 - 4\nabla^2\phi + \tfrac{3}{4}H_{\mu\nu\rho}H^{\mu\nu\rho}\Big)\ + O(\alpha')\ = 0.
\end{aligned} \tag{2.29}$$

Here $R_{\mu\nu}$ and R are the Ricci tensor and Ricci scalar for the background metric $g_{\mu\nu}$ and ∇_μ the covariant derivative on the space-time. $H_{\mu\nu\rho}$ is the rank three field strength tensor of $B_{\mu\nu}$:

$$H_{\mu\nu\rho}\ =\ \tfrac{1}{3}(\partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu})\ =\ \partial_{[\mu}B_{\nu\rho]}, \tag{2.30}$$

and is invariant under the gauge transformations $\delta B_{\mu\nu} = \partial_{[\mu}\Sigma_{\nu]}$.

The physical interpretation of these constraints is that they can be seen as the equations of motion of the action

$$S = \tfrac{1}{2}\int d^Dx\sqrt{|g|}e^{-2\phi}\left\{-\frac{(D-26)}{3\ \alpha'} - R + 4(\partial\phi)^2 - \tfrac{3}{4}H_{\mu\nu\rho}H^{\mu\nu\rho}\right\} + O(\alpha'), \tag{2.31}$$

This action is called the low-energy effective action or target space action, because it describes the massless modes of slowly varying X^μ's, as fields in the target space, the space in which the string moves. It can therefore be seen as a low energy approximation of string theory. For strings living in their critical dimensional space, the $(D-26)$-term in the third equation of (2.29) and in the action (2.31) drops out. From now on we will suppose that this is always the case.

The fact that the space-time metric $g_{\mu\nu}$ appears as a dynamical field, via the Ricci tensor, is the first indication we meet that gravity is contained in string theory. In fact (2.31) is the action for 26-dimensional gravity coupled to tensor and scalar fields. Higher orders in α' or string loop expansion will give rise to more terms in (2.31), and therefore predict corrections to general relativity. For a deeper analysis to higher order corrections, particularly for the Heterotic string, we refer to [155] and references therein.

The same procedure for computing the low-energy effective action can also be done for the supersymmetric string (2.15). It turns out that the low-energy description for the superstring is 10-dimensional supergravity, a locally supersymmetric quantum field theory. As already mentioned in the previous section, the $N=1$ world sheet supersymmetry induces $N=2$ space-time supersymmetry, i.e. a supersymmetry transformation with two space-time supersymmetry generators. The different ways these space-time supersymmetries can be introduced in the theory give rise to different types of superstring theories and different low energy effective actions:

- **Type I:** This is a theory of open strings. Closed strings however are also included in this theory because two interacting open strings can join and form a closed one. The boundary conditions for the open string eliminate one of the supersymmetries and break the original $N=2$ to $N=1$ supersymmetry. At the endpoints of the string charges can be attached, inducing a Yang-Mills gauge group in the theory. Consistency at the quantum level only allows $SO(32)$ as Yang-Mills group.

  The bosonic part of the low energy effective action of the Type I string is given by the bosonic part of $N=1, D=10$ supergravity [41, 19, 42]

$$S_I = \tfrac{1}{2}\int d^{10}x\sqrt{|g|}\,\Big[e^{-2\phi}\Big(-R+4(\partial\phi)^2\Big) - \tfrac{3}{4}H^2_{(3)} + \tfrac{1}{4}e^{-\phi}F^I_{(2)}F_{(2)I}\Big], \qquad (2.32)$$

  where we used the sub-index to indicate the rank of the field strength tensor. $F^I_{(2)}$ is the field strength of the vector field corresponding to the $SO(32)$-group and transforms under the adjoint representation of the group.

- **Type IIA:** This is a theory of closed strings only. The two space-time supersymmetries appear with opposite chirality, so the string itself is non-chiral and has $N=2$ supersymmetry. There is no freedom to introduce a Yang-Mills group, but in the bosonic field content we see, besides the metric, axion and dilaton of Type I, also a one-form $A_{(1)}$ and a three-from gauge field $C_{(3)}$ [93, 71, 40]:

$$\begin{aligned} S_{IIA} = \tfrac{1}{2}\int d^{10}x\sqrt{|g|}\,\Big\{&e^{-2\phi}\Big[-R+4(\partial\phi)^2 - \tfrac{3}{4}H^2_{(3)}\Big] \\ &+\tfrac{1}{4}F^2_{(2)} + \tfrac{3}{4}G^2_{(4)} + \tfrac{1}{64}\frac{\varepsilon_{(10)}}{\sqrt{|g|}}\partial C_{(3)}\partial C_{(3)}B_{(2)}\Big\}, \end{aligned} \qquad (2.33)$$

  with $F_{(2)}$ and $G_{(4)}$ the field strengths of the gauge fields $A_{(1)}$ and $C_{(3)}$ respectively and $\varepsilon_{(10)}$ the ten-dimensional fully anti-symmetric tensor. The NS-NS fields, satisfying double anti-periodic boundary conditions (2.26) on their world sheet fermions, appear differently in the above action as the R-R fields, satisfying double periodic boundary conditions (2.25). The fields of the NS-NS sector have an explicit dilaton coupling via the factor $e^{-2\phi}$, while the R-R fields are not multiplied by this factor. The R-R fields appear in the action (2.33) as the bosonic fields necessary to extend $N=1$ to $N=2$ supersymmetry. Their different dilaton coupling means that they correspond to a higher order in string coupling constant. As we will see later, the solutions that couple to these R-R fields do not belong to the perturbative spectrum.

- **Type IIB:** This is also a theory for closed strings with $N=2$ supersymmetry, though this time with two supersymmetries that have the same chirality, so the theory is chiral. Again it is impossible to introduce Yang-Mills groups and besides the NS-NS fields that appear in the same way as in Type IIA, the R-R sector consists of a scalar ℓ, a two-form gauge field $B^{(2)}_{\mu\nu}$ and a self-dual four-form gauge field $D^{+}_{\nu\mu\rho\lambda}$. Due to the self-duality condition of the four-form, it is impossible to write down a covariant low energy effective action for this theory [3]. The field equations of Type IIB supergravity can be found in [138]. In [17] an action is given in which the self-duality condition is not used, but is put in by hand as an extra equation of motion for the four-form:

$$\begin{aligned} S^{NSD}_{IIB} = \tfrac{1}{2}\int d^{10}x\sqrt{|g|}\,\Big\{ & e^{-2\phi}\Big[-R+4(\partial\phi)^2-\tfrac{3}{4}(\mathcal{H}^{(1)})^2\Big] \\ & -\tfrac{1}{2}(\partial\ell)^2-\tfrac{3}{4}\,(\mathcal{H}^{(2)}-\ell\mathcal{H}^{(1)})^2-\tfrac{5}{6}F^2_{(5)}(D) \\ & -\frac{1}{96\sqrt{|g|}}\varepsilon^{ab}\varepsilon^{(10)}D_{(4)}\mathcal{H}^{(a)}\mathcal{H}^{(b)}\Big\}, \end{aligned} \tag{2.34}$$

$$F(D^{+})_{\mu_1\dots\mu_5} = \frac{1}{5!\sqrt{|g|}}\,\varepsilon_{\mu_1\dots\mu_{10}}F(D^{+})^{\mu_6\dots\mu_{10}}. \tag{2.35}$$

$F_{\mu\nu\rho\lambda\sigma}$ and $\mathcal{H}^{(2)}_{\mu\nu\rho}$ are the field strengths of $D^{+}_{\mu\nu\rho\lambda}$ and $B^{(2)}_{\mu\nu}$.

- **Heterotic string:** This string theory makes use of the fact that for closed strings the left and the right moving sectors are independent. The left moving sector can be taken to be the left moving modes of the purely bosonic string, while for the right moving sector we take the modes from the superstring [79]. Since only one sector is supersymmetric, the Heterotic string has $N=1$ supersymmetry. This is however enough already to remove the tachyon from the bosonic spectrum. A Yang-Mills gauge group arises from the compactification of the bosonic sector on a 16-dimensional compact space, in order for the 26-dimensional bosonic string to match up with the superstring, living in 10 dimensions. Again quantum consistency restricts the gauge group to $SO(32)$ or $E_8\times E_8$.

  The bosonic part of the low energy effective action is given by

$$S_{Het} = \tfrac{1}{2}\int d^{10}x\sqrt{|g|}\;e^{-2\phi}\Big[-R+4(\partial\phi)^2-\tfrac{3}{4}H^2_{(3)}+\tfrac{1}{4}F^I_{(2)}F_{(2)I}\Big]. \tag{2.36}$$

Type I, Type IIA, Type IIB, Heterotic $SO(32)$ and Heterotic $E_8\times E_8$ are the only five consistent superstring theories in ten dimensions. Note that the metric, the dilaton and the axion appear in the same way in all string theories, except in Type I. We will therefore refer to this part of the action as the *common sector*.

Although the critical dimension for superstrings to live in is $D=10$, there does exist a supergravity theory in eleven dimensions. This has always been a mysterious subtlety, since on the one hand there seems to be an intimate relation between superstrings and supergravity theories, yet on the other hand this $D=11$ supergravity does not have a string theory counterpart of which it is the low energy effective action. We do mention

[3] However, see also [52]

it here though, because of the importance it has in a unifying description of the above string theories, as we will see in the next chapter.

- $D = 11$ **Supergravity:** Eleven dimensions is the highest number of dimensions for a supergravity theory to live in[4]. $D = 11$ supergravity turns out to be a unique theory with $N = 1$ supersymmetry. In its bosonic sector it has a field content consisting of a metric and a three-form gauge field $C_{\mu\nu\rho}$ and the action can be written as [47]

$$S_{D=11} = \tfrac{1}{2}\int d^{11}x\sqrt{|g|}\left\{-R + \tfrac{3}{4}G^2(C) + \tfrac{1}{384}\frac{1}{\sqrt{|g|}}\ \epsilon^{(11)}C\partial C\partial C\right\}. \qquad (2.37)$$

In Chapter 3 and Chapter 4 we will investigate the relations between these different supergravity actions and the symmetries they have. But let us first take a look at the solutions in string theory coming from these actions.

2.3 Solutions

Before we study in detail the solutions that appear in string theory, let us first focus on a special feature that occurs for field theories that have extended supersymmetry. We will see that then there exist states with special properties, namely states whose mass is related to their charge. The importance of these states is that they do not get any quantum corrections, so the semi-classical result is already exact.

The supersymmetry generators Q^I form an algebra which is typically of the form $\{Q, Q\} = \gamma^\mu P_\mu$, but for theories with more then two generators (so $I : 1, ..., N \geq 2$), in the presence of a soliton solution, a central charge term Z^{IJ} is present besides the usual momentum term P_μ,:

$$\{Q^I_\alpha, Q^J_\beta\} = \gamma^\mu_{\alpha\beta}P_\mu\delta^{IJ} + Z^{IJ}_{\alpha\beta} \qquad (2.38)$$

The central charge term arises as a boundary term in the supersymmetry algebra and has a non-zero value of solutions with non-trivial topological charges (solitons). It can therefore be thought of as the electric or magnetic charge of the soliton solution.

The presence of the central charge puts a bound on the mass of the particles. Because of the positivity of the supersymmetry algebra, the expectation value of (2.38) becomes (schematically)

$$\langle\psi|\ \{Q, Q\}\ |\psi\rangle\ =\ \langle\psi|\ H\ |\psi\rangle + \langle\psi|\ Z\ |\psi\rangle\ \geq\ 0, \qquad (2.39)$$

with H the Hamiltonian of the system. The first term on the right-hand side of (2.39) is then the energy (or the mass) of the state $|\psi\rangle$, and the second term its charge. So (2.39) actually states that the mass of a particle is bounded from below by its charge:

$$M \geq |Z|. \qquad (2.40)$$

[4]For supergravity theories in dimensions higher then eleven, fields with spin greater then two appear [118], and it is not clear how to deal with these higher spin fields in an adequate way.

This inequality is called the Bogomol'nyi bound or BPS-bound. It was first derived in the context of 't Hooft-Polyakov monopoles by Bogomol'nyi [33] and Prasad and Sommerfield [132], and later generalized to supersymmetric theories [165].

There exist particular states that saturate the above inequality (2.39), i.e. for states that have the minimal possible mass, the above inequality turns into an equality. This happens if a state $|\psi_0\rangle$ is annihilated by some of the supersymmetry generators, $Q^{I_0}|\psi_0\rangle = 0$. The mass of such a state is completely determined by its charge:

$$M = |Z|. \tag{2.41}$$

States that saturate the BPS-bound are called BPS-states. A special feature of these states, besides their mass formula, is that they form representations of the supersymmetry algebra which are shorter (lower-dimensional) than the usual representations. This can be understood from the fact that since they get annihilated by some of the generators, fewer different states appear in each multiplet. But this also implies that they are protected by supersymmetry from quantum corrections [165]: any quantum correction (perturbative or non-perturbative) would break up the mass-charge relation (2.41) and break the multiplet structure of the BPS-states. But since states always appear in multiplets and quantum corrections cannot change a short multiplet in a long (normal) one, BPS-states have to stay in their short multiplet representation and hence do not receive quantum corrections. Their relations and properties even hold if we let the coupling constant grow strong and perturbation theory no longer holds. Therefore BPS-states will turn out to be a very important tool to investigate the behaviour of theories at strong coupling (see Chapter 3).

Let us now have a look at solutions of the equations of motion of the actions (2.32) - (2.36). Amongst the various solutions of supergravity theories, there exists the class of spatially extended objects, called p-branes, where p refers to the dimensionality of the object ($p = 0$ would be a particle, $p = 1$ a string, $p = 2$ a membrane, ...). These extended objects appear because of the fact that in string theory the central charge of the supersymmetry algebra is in general a $(p+1)$-form antisymmetric tensor gauge field $Z^{IJ}_{\mu_1 \cdots \mu_{p+1}}$, rather then a Lorentz-scalar and the BPS-state carrying the $(p+1)$-form charge is typically a p-brane or, as we will see, a $(D-p-4)$-brane. For a detailed analysis of what kind of extended solutions correspond to each central extension of the supersymmetry algebra, we refer to [91].

We will discuss in the rest of this section some specific, "elementary" solutions that can be interpreted as the "fundamental" objects of string theory and supergravity. The fact that they can be interpreted as a single (fundamental) object is because they are all characterised by a single harmonic function $H(x)$, which determines their position in the target space. From now on we will restrict ourselves to the bosonic part only of the theories. In a first approach we will look at the solutions of the equations of motions of the common sector, since these will later reappear in the various theories. In a second step we will concentrate on solutions that occur in specific theories. For the general p-brane solution of the supergravity action, as a function of the spatial extension of the brane, the dimension of the space-time and the dilaton coupling of the gauge field, we refer to [16] and the references therein.

The variation of the action

$$S = \tfrac{1}{2} \int d^{10}x \sqrt{|g|}\; e^{-2\phi} \Big[-R + 4(\partial\phi)^2 - \tfrac{3}{4} H^2 \Big] \tag{2.42}$$

with respect to the different fields $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ gives

$$\begin{aligned}
[g_{\mu\nu}] \quad &: R_{\mu\nu} - 2\nabla_\mu \partial_\nu \phi + \tfrac{9}{4}\; H_{\mu\rho\lambda} H_\nu{}^{\rho\lambda} = 0, \\
[\phi] \quad &: R - 4\nabla_\mu \partial^\mu \phi + 4(\partial\phi)^2 + \tfrac{3}{4} H^2 = 0, \\
[B_{\mu\nu}] \quad &: \nabla_\rho (e^{-2\phi} H^{\rho\mu\nu}) = 0.
\end{aligned} \tag{2.43}$$

Since the action (2.42) is derived as a low energy effective action of a string moving in a curved space-time, it is not unreasonable to look for a string-like solution to Eqns (2.43), i.e. a solution that has an extension in one spatial and one time direction. Therefore it must have a two-dimensional Poincaré invariance times an eight-dimensional rotational symmetry: $P_2 \times SO(8)$. Such a solution, satisfying Eqns (2.43) is given in [50][5]:

$$F1 = \left\{ \begin{array}{l} ds^2 = H^{-1}(dt^2 - dx_1^2) - (dx_2^2 + ... + dx_9^2) \\ e^{-2\phi} = H \\ B_{01} = H^{-1} \end{array} \right. \tag{2.44}$$

The function H is a harmonic function of the coordinates $(x_2, ..., x_9)$:

$$H = 1 + \frac{c}{r^6}, \qquad\qquad r = \sqrt{x_2^2 + ... + x_9^2}. \tag{2.45}$$

In particular x_1 is an isometry direction and we can indeed interpret (2.44) as a string (a one-dimensional extended object) oriented in this x_1-direction. The solution (2.44) is generally referred to as the *fundamental string* ($F1$). The sub-space spanned by the coordinates $(x_2, ..., x_9)$ is called the transverse space of the string and the directions (t, x_1) the world volume directions.

A closer look at the solution (2.44) and the harmonic function $H = 1 + \frac{c}{r^6}$ reveals that the $F1$ is singular for $r \to 0$. Of course one always has to be very careful with singularities in particular coordinate systems, since they can be just an artifact of the chosen coordinates. But an analysis, done in [151], reveals that the fundamental string does indeed have a (time-like) singularity[6], which invites us to put a "material" string at the singularity by adding a delta-function source term to the supergravity action (2.42). Such a source term we already encountered, namely the non-linear sigma model (2.27), which describes the dynamics of the string. So we can say that the fundamental string solution (2.44) is a solution of the equations of motion of the combined "supergravity-matter" system

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{|g|}\; e^{-2\phi} \Big[-R + 4(\partial\phi)^2 - \tfrac{3}{4} H_{\mu\nu\rho} H^{\mu\nu\rho} \Big]$$

[5] A detailed derivation of this solution and the following ones and their supersymmetry can be found in [61].

[6] Though not in the coordinates given above. In order to see the singularity, one has to use an analytic extension of these coordinates. For a detailed analysis of the space-time structure of various p-branes and their Penrose diagrams, we refer to [151].

$$
\begin{aligned}
-\frac{T}{2}\int d^2\sigma\sqrt{|\gamma|}\ \gamma^{ij} g_{\mu\nu}(X)\ \partial_i X^\mu \partial_j X^\nu \\
+\ \frac{T}{2}\int d^2\sigma\ \varepsilon^{ij} B_{\mu\nu}(X)\ \partial_i X^\mu \partial_j X^\nu .
\end{aligned}
\tag{2.46}
$$

We can choose the parametrisation of the string source to be $(X^0, X^1, X^m) = (\tau, \sigma, \vec{0})$ and $\gamma_{ij} = \eta_{ij}$, so that all equations of motion reduce to

$$
\partial_n \partial_n H(x^m) = \ \kappa^2 T\ \delta(x^m). \tag{2.47}
$$

This gives us the relation between the constant c in the harmonic function $H(x^m)$, the string tension T and the coupling constant of general relativity κ^2:

$$
c = \frac{\kappa^2\ T}{3\ \Omega_7}, \tag{2.48}
$$

where Ω_7 is the volume of the unit 7-sphere around the string.

Although (2.44) is a purely bosonic configuration, it still preserves half of the supersymmetry of the theory. This can happen if not only the fermionic fields, but also their variations under supersymmetry transformations vanish for some Killing spinor ϵ. For the $N = 1$ case we have for the dilatino λ and the gravitino ψ_μ:

$$
\begin{aligned}
\delta\psi_\mu &= D_\mu \epsilon + \tfrac{3}{8}\ H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\ \ \epsilon \ = \ 0, \\
\delta\lambda &= \gamma^\mu \partial_\mu \phi\ \epsilon + \tfrac{1}{4}\ H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\ \ \epsilon \ = \ 0.
\end{aligned}
\tag{2.49}
$$

In particular, for the $F1$ this gives a condition for ϵ:

$$
(1 + \gamma^0 \gamma^1)\ \epsilon = 0. \tag{2.50}
$$

This condition defines in fact a projection operator on ϵ that breaks half of the supersymmetry and preserves the other half. This partial breaking of supersymmetry is due to the fact the the $F1$ is a BPS-state. This can be shown, comparing the mass per unit length, defined as the integral over the (00)-component of the energy-momentum tensor,

$$
M = \int_{V_8} T^{00}\ d^8x \ = \ 2\kappa^2 T \tag{2.51}
$$

to the electric charge conserved via the equations of motion of the two-form gauge field $B_{\mu\nu}$:

$$
e \ = \ \int_{V_8} \partial_m H^{01m} d^8x \ = \ \int_{S^7} H^{01i} dS_i \ = \ 2\kappa^2 T\ . \tag{2.52}
$$

There is also another way that the gauge field $B_{\mu\nu}$ can carry a conserved charge, but this time the charge is topologically conserved, not dynamically via the equations of motion.

$$
q = \int_{S_3} \varepsilon^{mnp} H_{mnp}\ d^3x. \tag{2.53}
$$

While (2.52) is the generalisation to higher dimensions and higher forms of the electric charge in Maxwell theory, (2.53) would correspond to the generalisation of the magnetic charge as it occurs in the Dirac monopole, a solitonic object in the context of electromagnetism. So also in the context of string theory, we expect the object that carries the magnetic charge as given in (2.53) to correspond to a solitonic object.

Indeed, a solution of the Eqns (2.43) carrying magnetic charge is given by [39, 63]

$$S5 = \begin{cases} ds^2 = dt^2 - dx_1^2 - ... - dx_5^2 - H(dx_6^2 + ... + dx_9^2) \\ e^{-2\phi} = H^{-1} \\ H_{mnp} = \varepsilon_{mnpr}\partial_r H \qquad\qquad (m,n,p,r: 6,...,9). \end{cases} \tag{2.54}$$

The harmonic function H depends this time on the coordinates $x_m = (x_6, ..., x_9)$, so we can interpret the solution as an object that has spatial extensions in the $(x_1, ..., x_5)$-directions, i.e. it has five plus one world volume directions and four transversal ones. We therefore refer to solution (2.54) as the *solitonic five-brane* ($S5$).

One can show [151] that there exist coordinate frames in which the $S5$ is completely singularity-free, so no source term is needed. The $S5$ is really a solitonic object in the sense that it corresponds to a topological defect with a large mass per unit volume, rather then with an elementary excitation of the vacuum. In fact one can show that the $S5$ is a BPS-state, so it conserves half of the supersymmetry and the Bogomol'nyi bound (2.41) between the mass and the magnetic charge is saturated.

Although the $S5$ is non-singular and a source term is not needed, we can still write down an effective action which describes the dynamics of the five-brane. Just as for the $F1$, the effective action of the $S5$ consists of two parts: a kinetic term, written in the form of a Born-Infeld (BI) term, which induces a metric on the five-brane, and a Wess-Zumino (WZ) term which gives the coupling to the gauge field. For the $N = 1$ five-brane this is:

$$\begin{aligned} S \;&=\; -\tfrac{T}{2}\int d^6\sigma\; e^{-2\phi}\sqrt{|\det(\partial_i X^\mu \partial_j X^\nu g_{\mu\nu})|} \\ &\quad +\tfrac{T}{6!}\int d^6\sigma\; \varepsilon^{i_1\cdots i_6}\; \partial_{i_1}X^{\mu_1}...\;\partial_{i_6}X^{\mu_6}C_{\mu_1\cdots\mu_6}. \end{aligned} \tag{2.55}$$

$C_{\mu_1\cdots\mu_6}$ is the dual (magnetic) potential of $B_{\mu\nu}$. More generally, every $(p+1)$-form potential can equivalently be written as a $(D-p-3)$-form, since their field strength tensors are related via Poincaré duality

$$F_{\mu_1\cdots\mu_{(p+2)}} = \frac{1}{(D-p-2)!}\frac{1}{\sqrt{|g|}}\;\varepsilon_{\mu_1\cdots\mu_{(p+2)}\mu_{(p+3)}\cdots\mu_D}F^{\mu_{(p+3)}\cdots\mu_D}. \tag{2.56}$$

The factor $e^{-2\phi}$ in the kinetic term of (2.55) states that we are dealing with a solitonic object, whose mass is inversely proportional to the square of the coupling constant:

$$M_{S5} \sim \frac{1}{g^2}. \tag{2.57}$$

This means that for weak coupling, so in the perturbative regime, the five-brane becomes very massive.

Let us now look at the solutions of the Type IIA/B theories (2.33) - (2.34). Again we encounter the fundamental string and the solitonic five-brane, because the common sector is contained in both Type II strings. However, due to the presence of the R-R gauge fields, there exists a entirely new class of solutions that are charged with respect to these fields: the so-called Dirichlet-branes or D-branes [128].

Dp-branes ($0 \leq p \leq 8$) arise as hyperplanes in space-time to which the endpoints of open fundamental strings can be attached. Such a string ending on a Dp-brane satisfies Dirichlet boundary conditions in $(9-p)$ directions, constraining it to live on the world volume of the D-brane [129]. The strings attached to the D-brane describe fluctuations on the surface of the brane and make the D-branes dynamical objects, rather then static hypersurfaces. The strings can interact with each other or with strings approaching the brane and then scatter off closed strings [82]. The D-branes appear as solutions of the equations of motion of both Type II theories in the form

$$Dp = \begin{cases} ds^2 = H^{-\frac{1}{2}}(dt^2 - dx_1^2 - ... - dx_p^2) - H^{\frac{1}{2}}(dx_{p+1}^2 + ... + dx_9^2) \\ e^{-2\phi} = H^{\frac{p-3}{2}} \\ F^{(R-R)}_{012...pm} = \partial_m H^{-1} \qquad\qquad (m : p+1, ..., 9). \end{cases} \tag{2.58}$$

Again H is a harmonic function that depends on the transverse coordinates $x_m = (x_{p+1}, ..., x_9)$. $F^{(R-R)}_{012...pm}$ is the field strength of the R-R p-form gauge field that carries the R-R charge of the brane. Note that for $p \geq 3$ we have used the equivalent expression for the field strength, in terms of the magnetic (dual) potential (2.56).

Dp-branes with even p ($D0, D2, D4, D6$) couple to odd-form gauge fields and therefore occur in Type IIA theory, while p-odd branes ($D1, D3, D5, D7$), coupling to even-form gauge-fields, occur in Type IIB.

From (2.56) we see that the Dp-branes with $p < 3$ carry an electric charge, and the Dp-branes with $p > 4$ a magnetic charge. The $D3$-brane is dyonic, i.e. it has both electric and magnetic charge, due to the self-duality condition of the $D^+_{\mu\nu\rho\lambda}$ in Type IIB. These charges can be calculated in the same way as for the $F1$ and $S5$ in (2.52)-(2.53). Again the Bogomol'nyi bound is saturated

$$M_{Dp} \sim \frac{1}{g} \sim Q^{\text{R-R}}. \tag{2.59}$$

The inverse coupling constant in the mass formula indicates that the D-branes also belong to the non-perturbative spectrum, though their solitonic character is not as strong as for the $S5$.

The dynamics of the D-brane are described by a sigma model type of action [109, 68], which also plays the role of source term for the equations of motion. The BI-term describes the coupling of the NS-NS fields with a world volume vector V_i and the WZ-term gives the coupling to the R-R gauge fields [68, 77]:

$$\begin{aligned} S &= -\tfrac{T}{2}\int d^{p+1}\sigma\ e^{-\phi}\sqrt{|det(g_{ij} + \mathcal{F}_{ij})|} \\ &\quad + \tfrac{T}{(p+1)!}\int d^{p+1}\sigma\ \varepsilon^{(p+1)}\Big[C_{(p+1)} + C_{(p-1)}\mathcal{F} + C_{(p-3)}\mathcal{F}^2 + ...\Big] \end{aligned} \tag{2.60}$$

where g_{ij} is the pull-back of the metric on the world volume and $\mathcal{F}_{ij}$ the field strength of the vector field V_i:

$$\begin{aligned} g_{ij} &= \partial_i X^\mu \partial_j X^\nu g_{\mu\nu}, \\ \mathcal{F}_{ij} &= \partial_i V_j - \partial_j V_i - \partial_i X^\mu \partial_j X^\nu B_{\mu\nu}. \end{aligned} \tag{2.61}$$

The $C_{(p+1)}$ are the different $(p+1)$-form R-R fields in a uniform notation. The interpretation of the world volume vector V_i is that of a $U(1)$-potential of a charged particle on the world volume of the D-brane. Charge conservation of the NS-NS two-form at the end of an open string ending on a D-brane is only maintained if there is an electric flux on the world volume coming out of the endpoint of the string. So the endpoints manifest themselves on the brane as charged particles, with a potential V_i associated to them [153].

Note that in the Type I action (2.32) the three-form field strength $H_{\mu\nu\rho}$ occurs in the same way as the R-R fields of Type IIA/B. The string and five-brane solutions of Type I should therefore be compared to the $D1$ and $D5$, rather than to the fundamental string or the solitonic five-brane.

The equations of motion of the $D=11$ supergravity action (2.37) do not contain an $F1$ or $S5$ solution (2.44, 2.54), but the three-form gauge field $C_{\mu\nu\rho}$ suggests that there has to be a two-brane and its eleven-dimensional magnetic dual, a five-brane, that couple to C. Indeed such an electrically charged membrane ($M2$) [65] and a magnetically charged five-brane ($M5$) [81] have been found[7]:

$$M2 \quad = \begin{cases} ds^2 = H^{-\frac{2}{3}}(dt^2 - dx_1^2 - dx_2^2) - H^{\frac{1}{3}}(dx_3^2 + ... + dx_{10}^2) \\ C_{012} = H^{-1} \end{cases} \tag{2.62}$$

$$M5 \quad = \begin{cases} ds^2 = H^{-\frac{1}{3}}(dt^2 - dx_1^2 - ... - dx_5^2) - H^{\frac{2}{3}}(dx_6^2 + ... + dx_{10}^2) \\ G(C)_{mnpr} = \varepsilon_{mnprs}\partial_s H \qquad (m,n,p,r,s: 6, ..., 10), \end{cases} \tag{2.63}$$

In many aspects these M-branes are much the same as their ten-dimensional counterparts (in fact in the next chapter we will see how they are related): the harmonic function H depends on the transversal coordinates x_m, they saturate the Bogomol'nyi bound and break half of the supersymmetry. The $M2$ is singular and needs a source term [30], while the $M5$ is a solitonic object that is very heavy at weak coupling.

Besides the above mentioned p-brane solutions, there exist two more solutions to both string theory and $D=11$ supergravity that are characterized by a single harmonic function and can therefore also be considered as fundamental objects of string theory and supergravity. We will encounter them often in the following chapters. They are special in the sense that they already occur as solutions of pure gravity, so they only consist of a non-trivial metric. Furthermore they do not have the typical two-block structure of world volume and transverse directions of p-branes. Therefore they can not be interpreted as "brane"-like solutions.

[7] The names $M2$ and $M5$ come from the fact that $D=11$ supergravity sometimes is called M-theory. Thus the p-branes that arise in M-theory are called M-branes.

dim	0	$\mathcal{W}_D$	1	2	3	4	5	6	$\mathcal{KK}_D$	7	8
$D = 11$		$\mathcal{W}_{11}$		$M2$			$M5$		$\mathcal{KK}_{11}$		
IIA	$D0$	$\mathcal{W}_{10}$	$F1$	$D2$		$D4$	$S5$	$D6$	$\mathcal{KK}_{10}$		$D8$
IIB		$\mathcal{W}_{10}$	$F1/D1$		$D3$		$S5/D5$		$\mathcal{KK}_{10}$	$D7$	
Het		$\mathcal{W}_{10}$	$F1$				$S5$		$\mathcal{KK}_{10}$		
I		$\mathcal{W}_{10}$	$D1$				$D5$		$\mathcal{KK}_{10}$		

Table 2.1: *The solutions of the various string theories and $D = 11$ supergravity.*

The first one is the D-dimensional gravitational wave or Brinkmann wave ($\mathcal{W}_D$) [36], propagating in the $z = x_1$ direction:

$$\mathcal{W}_D \ : \ ds^2 = (2 - H)dt^2 - Hdz^2 + 2(1 - H)dtdz - (dx_2^2 + ... + dx^2_{(D-1)}), \tag{2.64}$$

and the second the Kaluza-Klein monopole in D-dimensions ($\mathcal{KK}_D$) [150, 80]:

$$\mathcal{KK}_D \ : \ ds^2 = dt^2 - dx_1^2 - ... - dx^2_{(D-5)} - H^{-1}(dz + A_m dx_m)^2 - Hdx_m^2. \tag{2.65}$$

H is a harmonic function that depends in the case of the wave on the coordinates $t + z, x_2, ..., x_{D-1}$ and in the case of the monopole on x_m ($m = D-3, D-2, D-1$) and not on z. The z-direction is a compact isometry direction in order for the monopole to be non-singular. After a Kaluza-Klein compactification in this z-direction, one ends up with a $(D-5)$-brane, with a magnetic charge, which in the case of a five-dimensional monopole $\mathcal{KK}_5$ corresponds to a Dirac-monopole type particle. This explains its name.

Also A_i depends on x_m and the relation with H is given by:

$$F_{mn} = \partial_m A_n - \partial_n A_m = \varepsilon_{mnp}\partial_p H. \tag{2.66}$$

As mentioned above these solutions do not have a two-block structure due to off-diagonal terms in the metric, which makes it difficult to distinguish between world volume and transverse directions. We will choose, for later convenience, the z-directions in the case of the wave to be a world volume direction, but in the case of the Kaluza-Klein monopole a transverse direction.

Table 2.1 gives an overview of the different solutions we encountered in the various theories. In Chapter 3 we will see that these theories are related to each other via duality transformations. This means that there also must exist duality relations between the different solutions and the world volume actions that describe their dynamics. We will investigate in more detail these duality relations in Chapter 5 and see that under certain conditions different solutions can be superposed in a kind of "bound state". The relations between the world volume actions will be studied in Chapter 6.

Chapter 3

Duality

String theory is a very powerful tool in the attempt to find a unifying description of all interactions. However the theory, as it was known till the early nineties (i.e. as was briefly described in the previous chapter), has some problems. First of all, the theory is only defined at the perturbative level, as a Feynman "sum-over-histories" approach, without an understanding of the dynamical principles that form the theory and that allow one to go beyond perturbation theory. A second problem is the fact that, although techniques are known to come down from the ten-dimensional superstring world to our phenomenologically observable four-dimensional world, these techniques give rise to many degenerate ground states, parametrized by the scalars (moduli) that appear in these reductions. It is not at all clear which of these compactifications corresponds to a model that looks like something we know from experiments (the Standard Model) and why Nature chooses precisely this vacuum. But maybe the most annoying feature is that on the one hand string theory claims to be a unifying theory of gravity and quantum field theory, yet on the other hand five different versions of string theory are known: Type I, Type IIA, Type IIB, Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$. So there seem to exist five different unification candidates and five different ways to formulate a theory involving quantized gravity, which is not an appealing idea for a unification theory.

In the early nineties, the second "superstring revolution"[1] introduced the concept of "dualities", which indicated the possibility to solve many of the above problems at once: it was realised that a certain theory A, compactified on a large volume, could be equivalent to a theory B, compactified on a small volume, or that a theory C at weak coupling could be mapped to a theory D at strong coupling. In this way, it was possible to regard different vacua as being equivalent, find a more unifying description for the different string theories and to get insight into the physics beyond the perturbative level.

[1] The first superstring revolution was the one in the mid eighties, when it was realized that the above mentioned string theories are the only consistent ones and that these have a well defined perturbation expansion.

In this chapter we will give an overview of the different duality symmetries in string theory. In Section 3.1 we will present a duality that acts on the target space of the string, the Target Space Duality or T-duality. In section 3.2 we discuss the duality that relates the strong and weak coupling regime of the different theories, the so-called S-duality (Strong/Weak coupling duality). In section 3.3 we will present the unifying picture as it stands at this moment.

General references for string dualities are [141, 58, 158, 67, 162, 101, 104, 56, 148] .

3.1 Target Space Duality

Target Space duality, or for short T-duality, is a symmetry transformation that relates different string backgrounds to each other. It was first introduced at the level of the bosonic sigma model in the presence of an isometry as a $\mathbb{Z}_2$-symmetry that interchanges certain components of the metric with certain components of the axion [37]. The general T-duality transformations are intimately related with the idea of dimensional reduction via the appearance of the non-compact $O(d, d+n)$ groups. Their importance lies in the fact that T-duality gives a way to divide the many degenerate ground states in T-duality classes of equivalent physics. For extensive reviews about T-duality in string theory, we refer to [75, 3].

3.1.1 T-duality in World Volume Theory

The T-duality transformation rules can be derived from the non-linear sigma model (2.27):

$$\begin{aligned} S = &-\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{|\gamma|}\ \gamma^{ij}\partial_i X^{\hat{\mu}}\partial_j X^{\hat{\nu}} g_{\hat{\mu}\hat{\nu}} \\ &+\frac{1}{4\pi\alpha'}\int d^2\sigma\ \varepsilon^{ij}\ \partial_i X^{\hat{\mu}}\partial_j X^{\hat{\nu}} B_{\hat{\mu}\hat{\nu}}. \end{aligned} \tag{3.1}$$

Suppose that the background fields $g_{\hat{\mu}\hat{\nu}}$ and $B_{\hat{\mu}\hat{\nu}}$ are independent of one embedding coordinate X, so the D-dimensional indices $\hat{\mu}$ can be split into the index x of the isometry direction and the indices of the $(D-1)$ remaining directions: $\hat{\mu} = (x, \mu)$.

We can then consider the derivative of the isometry coordinate $\partial_i X$ to be an independent field V_i by adding a Lagrange multiplier $\tilde{X}$ and rewriting (3.1) as

$$\begin{aligned} S = &-\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{|\gamma|}\ \gamma^{ij}\Big[\partial_i X^{\mu}\partial_j X^{\nu} g_{\mu\nu} + 2\partial_i X^{\mu} V_j\ g_{\mu x} + V_i V_j\ g_{xx}\Big] \\ &+\frac{1}{4\pi\alpha'}\int d^2\sigma\ \varepsilon^{ij}\Big[\partial_i X^{\mu}\partial_j X^{\nu} B_{\mu\nu} + 2V_i\partial_j X^{\nu} B_{x\nu}\Big] \\ &-\frac{1}{4\pi\alpha'}\int d^2\sigma\ \varepsilon^{ij}\ \tilde{X}\partial_i V_j. \end{aligned} \tag{3.2}$$

The equation of motion of $\tilde{X}$ states that $V_i = \partial_i X$ and relates action (3.2) to action (3.1). On the other hand, solving the equation of motion of V_i and substituting in (3.2), we find the dual action, in terms of the dual coordinates $\tilde{X}^{\hat{\mu}} = (X^\mu, \tilde{X})$:

$$
\begin{aligned}
S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{|\gamma|}\ \gamma^{ij}\partial_i\tilde{X}^{\hat{\mu}}\partial_j\tilde{X}^{\hat{\nu}}\tilde{g}_{\hat{\mu}\hat{\nu}} \\
+\frac{1}{4\pi\alpha'} \int d^2\sigma\ \varepsilon^{ij}\ \partial_i\tilde{X}^{\hat{\mu}}\partial_j\tilde{X}^{\hat{\nu}}\tilde{B}_{\hat{\mu}\hat{\nu}}\ .
\end{aligned} \tag{3.3}
$$

This is again a non-linear sigma model action for a string moving in the dual background fields $\tilde{g}_{\mu\nu}$ and $\tilde{B}_{\mu\nu}$, where the relation between the original and the dual fields is given by the so-called T-duality rules [37]:

$$
\begin{aligned}
\tilde{g}_{\mu\nu} &= g_{\mu\nu} - (g_{x\mu}g_{x\nu} - B_{x\mu}B_{x\nu})/g_{xx}, \\
\tilde{B}_{\mu\nu} &= B_{\mu\nu} - (g_{x\mu}B_{x\nu} - g_{x\nu}B_{x\mu})/g_{xx}, \\
\tilde{g}_{x\mu} &= B_{x\mu}/g_{xx}, \\
\tilde{B}_{x\mu} &= g_{x\mu}/g_{xx}, \\
\tilde{g}_{xx} &= 1/g_{xx}\,.
\end{aligned} \tag{3.4}
$$

The transformation rule for the dilaton cannot be obtained via the equation of motion of V_i, but by demanding that the conformal invariance of (2.27) at order $(\alpha')^0$ can be found back in the dual action. The dilaton transforms as

$$
\tilde{\phi} = \phi - \tfrac{1}{2}\log|g_{xx}|. \tag{3.5}
$$

The T-duality rules relate two geometrically different, but dynamically equivalent sets of background fields: although the geometry of the space is altered, the physical properties of the model are unchanged under the duality transformation. Let us illustrate this with some simple examples for the closed and the open string in some simple backgrounds.

Suppose a closed string is moving in a flat space-time where one coordinate X is a circle of radius R. The metric is of the form $g_{\hat{\mu}\hat{\nu}} = \mathrm{diag}[1, -1, -1, ..., -R^2/\alpha']$, all other background fields are set equal to zero.

The boundary conditions on $X^{\hat{\mu}}$ are given by

$$
\begin{aligned}
X^\mu(\tau, \sigma + 2\pi) &= X^\mu(\tau, \sigma), \\
X(\tau, \sigma + 2\pi) &= X(\tau, \sigma) + 2\pi m R,
\end{aligned} \tag{3.6}
$$

where m is an integer that indicates how many times the string is wound around the compact direction X. The periodicity of X forces the momentum in this direction to be quantized: e^{iPX} should be single valued for X and $X + 2\pi R$, so $P = n/R$. The solution of the equation of motion (2.7) for the string, satisfying the above boundary conditions, is

$$
X^{\hat{\mu}}_\pm : \begin{cases} X^\mu_\pm & \text{as in (2.9)}, \\ X_\pm = \tfrac{1}{2}x + \sqrt{\tfrac{\alpha'}{2}}P_\pm\ (\tau \pm \sigma)\ +\ \text{oscillations}\ , \end{cases} \tag{3.7}
$$

where $X_+^{\hat{\mu}}$ and $X_-^{\hat{\mu}}$ are defined as in (2.8) and

$$P_{\pm} = \frac{1}{\sqrt{2}}\Big(\frac{\sqrt{\alpha'}}{R}\ n\ \pm\ \frac{R}{\sqrt{\alpha'}}\ m\Big). \tag{3.8}$$

We see that the expressions for $P_{\pm}$ are invariant under simultaneous interchange of $R \leftrightarrow \alpha'/R$ and $m \leftrightarrow n$ [103, 135], and since the on-shell mass condition is given by $M = (P^{\hat{\mu}}P_{\hat{\mu}} +$ oscillator terms$)$, also the spectrum is invariant under this interchange. The string does not see whether it is wound m times around a circle with small radius R while having a momentum n, or n times around a circle with large radius α'/R having momentum m.

It is not difficult to show that an $R \to \alpha'/R$ transformation is in fact a T-duality transformation where the compact direction X is dualized into the dual coordinate $\tilde{X} = X_+ - X_-$. The inversion of the radius seems to suggest that there exists a "minimal length" $R = \sqrt{\alpha'}$, at the string scale: going beyond this "minimal length" would give the same physics as at large length scales.

For an open string, freely moving in a flat space with one compact dimension $X = 2\pi R$ (i.e. satisfying Neumann boundary conditions $\partial_\sigma X^{\hat{\mu}} = 0$), we can rewrite (2.13) as ($X^{\hat{\mu}} = X_+^{\hat{\mu}} + X_-^{\hat{\mu}}$ and $P = n/R$):

$$\left\{ \begin{array}{lcl} X_{\pm}^{\mu} & & \text{as in (2.13)}, \\ X_{\pm} & = & \frac{1}{2}x \pm \frac{1}{2}C + \frac{1}{2}\alpha'\frac{n}{R}\ (\tau \pm \sigma)\ + \frac{1}{2}\sum_n \frac{1}{n}\ \tilde{a}_n^{\mu}\ e^{in(\tau\pm\sigma)}. \end{array} \right. \tag{3.9}$$

Again we can dualize X into $\tilde{X} = X_+ - X_-$ and we find [129]:

$$\tilde{X} = C + \alpha'\frac{n}{R}\ \sigma\ + \frac{1}{2}\sum_n \frac{1}{n}\ \tilde{a}_n^{\mu}\ e^{in\tau}\sin n\sigma. \tag{3.10}$$

This is the solution (2.14) for the equations of motion of a string satisfying Dirichlet boundary conditions. It turns out that T-duality has interchanged the Neumann boundary conditions $\partial_\sigma X|_{\sigma=0}^{\sigma=\pi} = 0$ for Dirichlet conditions $\partial_\tau \tilde{X}|_{\sigma=0}^{\sigma=\pi} = 0$ in the dualized direction: where for the freely moving string (2.13) the zero-modes of the string were independent of σ, here the zero-modes in the $\tilde{X}$ direction are independent of τ. This means that the endpoints of the string are fixed in the $\tilde{X}$-direction:

$$\left\{ \begin{array}{l} \tilde{X}(0) = C, \\ \tilde{X}(\pi) = C\ +\ 2n\pi\tilde{R}\ . \end{array} \right. \tag{3.11}$$

The string is attached to a $(D-2)$-dimensional hypersurface $\tilde{X} = C$, while it can wind n times around the the compact dimension $\tilde{X}$ of radius $\tilde{R} = \alpha'/R$. This hypersurface is in fact the D-brane we encountered as a solution of the equations of motion in Section 2.3.

3.1.2 Dimensional Reduction

Before we study the effect of T-duality on the low energy effective action of string theories, let us first make a small intermezzo about dimensional reduction and compactification.

In Chapter 2 we mentioned that superstring theory only can be quantized consistently if the string lives in a ten-dimensional space-time. However, if we want our theory to be "realistic", we have to be able to make contact with the phenomenologically observable world, which is four-dimensional: we have to find a way to hide away six dimensions and to rewrite high-dimensional results in terms of low-dimensional ones. This can be done through compactification. Suppose that six of the ten dimensions are compactified over a very small volume, with length scales of the order of the Planck-scale, such that they are invisible at low energies or at large length scales. The ten-dimensional manifold is a product of a four-dimensional space-time times a six-dimensional compact space: $\mathcal{M}^{10} = M^4 \times K^6$. We can translate the ten-dimensional theory to an effective theory in four dimensions, where the precise form of the effective theory depends on the geometry of the compact manifold. This idea is sometimes called Kaluza-Klein compactification, because Kaluza and Klein tried to write electromagnetism and general relativity in four dimensions as a single theory of pure gravity in five dimensions [97, 107].

There exist an infinite number of compact manifolds K over which we can compactify, but only a limited number of these give useful results for string theory[2]. The compactification we will study in this section and mostly use in the rest of this work, is the most simple case, namely the compactification over a d-dimensional torus T^d. Compactifications over more complicated manifolds, such as $K3$ or Calabi-Yau manifolds, may give phenomenologically more relevant results (chiral fermions, a Minimal Supersymmetric Standard Model, ...), but torus compactification will already be sufficient for the features we are interested in, namely the symmetry groups of compactified theories and the relations between different supergravity actions. Properties of a more complicated compactification will be discussed in Chapter 4, when we study the symmetries of Type IIA/B, compactified on $K3$.

If the fields of the uncompactified theory depend on the compact coordinates, then extra massive states appear in the lower-dimensional theory. This can be seen in a simple example: suppose a field $\hat{\Phi}(\hat{x}^{\hat{\mu}})$ in a flat D-dimensional space-time with one compact dimension x obeys the equations of motion[3]

$$\partial_{\hat{\mu}}\partial_{\hat{\mu}}\hat{\Phi}(\hat{x}^{\hat{\nu}}) \;=\; \partial_{\mu}\partial_{\mu}\hat{\Phi}(\hat{x}^{\hat{\nu}}) \;-\; \partial_x\partial_x\hat{\Phi}(\hat{x}^{\hat{\nu}}) = 0. \tag{3.12}$$

Since the field $\hat{\Phi}(\hat{x}^{\hat{\mu}})$ depends on all coordinates $\hat{x}^{\hat{\mu}}$, in particular also on the compact coordinate x, we can perform a separation of variables and do a Fourier expansion of field in the x-coordinate. This corresponds to an expansion of $\hat{\Phi}(\hat{x}^{\hat{\mu}})$ in modes of the quantised momentum in the compact direction:

$$\hat{\Phi}(\hat{x}^{\hat{\mu}}) = \sum_n \Phi_n(x^\mu)\; e^{\frac{inx}{R}}, \tag{3.13}$$

where R is the radius of the compact dimension. The equation of motion for the

[2]It turns out that only the compactifications that preserve some amount of supersymmetry are consistent. Compactifications that break all supersymmetry give rise to theories that do not have a well defined perturbation theory.

[3]From now on we will use the notation that hatted fields and indices are higher-dimensional ones and unhatted ones lower-dimensional. It should be clear from the context in which dimension each field (hatted or unhatted) lives.

coefficient $\Phi_n(x^\mu)$ of the n-th mode is now of the form

$$\partial_\mu \partial_\mu \ \Phi_n(x^\mu) \ + \ \tfrac{n^2}{R^2} \ \Phi_n(x^\mu) = 0, \tag{3.14}$$

which is the Klein-Gordon equation $(\Box + M_n^2)\Phi_n = 0$ for a field with mass $M_n = n/R$. So the different modes of the field $\hat{\Phi}$ manifest themselves in lower dimensions as an infinite tower of states with masses equal to the quantised momentum. The proportionality constant is the inverse radius of the compact direction $1/R$. These modes are called the Kaluza-Klein modes of $\hat{\Phi}$. If $R \to \infty$, so upon decompactification, the massive states become massless and form a continuous spectrum. For small R (comparable to the Planck-length) however, the states with $n \neq 0$ are very massive, with masses of the order of the Planck-mass.

At low energies, or equivalently at length scales much bigger then the size of the compact dimension, only the massless lowest mode can be detected. Since in the low energy effective actions of string theory the massive string modes have already been integrated out, it is therefore consistent to exclude also the massive Kaluza-Klein modes from the theory. This is the same as removing the dependence of the D-dimensional fields on the compactified coordinates. Throughout this section we will suppose that this is the case.

The precise way the higher-dimensional fields reduce to lower dimensions is determined by gauge invariance: a general coordinate transformation in higher dimensions will manifest itself as a lower-dimensional general coordinate transformation and gauge symmetries. The reduction rules are given in [136, 46, 115]: let us derive them for some typical examples.

Suppose a D-dimensional metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ is independent of d coordinates. Coordinate transformations of the metric give:

$$\delta \hat{g}_{\hat{\mu}\hat{\nu}} = \hat{\xi}^{\hat{\lambda}} \ \partial_{\hat{\lambda}} \hat{g}_{\hat{\mu}\hat{\nu}} \ + \ \partial_{\hat{\mu}} \hat{\xi}^{\hat{\lambda}} \hat{g}_{\hat{\lambda}\hat{\nu}} \ + \ \partial_{\hat{\nu}} \hat{\xi}^{\hat{\lambda}} \hat{g}_{\hat{\mu}\hat{\lambda}}. \tag{3.15}$$

We can split the D-dimensional indices $\hat{\mu}$ in $\hat{\mu} = (\mu, a)$ with $0 \leq \mu \leq D-d-1$ and $1 \leq a \leq d$ and compactify over the coordinates x^a. The different components of $\hat{g}_{\hat{\mu}\hat{\nu}}$ then transform as:

$$\begin{array}{rl} \delta \hat{g}_{ab} = & \xi^\lambda \ \partial_\lambda \hat{g}_{ab} \ , \\ \delta \hat{g}_{\mu a} = & \xi^\lambda \ \partial_\lambda \hat{g}_{\mu a} \quad + \partial_\mu \xi^\lambda \hat{g}_{\lambda a} \qquad\qquad\qquad\quad + \partial_\mu \xi^b \hat{g}_{ba} \ , \\ \delta \hat{g}_{\mu\nu} = & \underbrace{\xi^\lambda \ \partial_\lambda \hat{g}_{\mu\nu} \quad + \partial_\mu \xi^\lambda \hat{g}_{\lambda\nu} \quad + \partial_\nu \xi^\lambda \hat{g}_{\mu\lambda}}_{\text{g.c.t in } (D-d)} \quad \underbrace{+ \partial_\mu \xi^a \hat{g}_{a\nu} \quad + \partial_\nu \xi^a \hat{g}_{\mu a}}_{\text{“internal”}} \ , \end{array} \tag{3.16}$$

where we also took the $\hat{\xi}^{\hat{\mu}}$ independent of x^a. The variations look like the transformation rules for a set of scalars, vectors and a metric under $(D-d)$-dimensional general coordinate transformations plus some extra variations coming from the internal components ξ^a.

In order to get rid of these extra terms, we define the $(D-d)$-dimensional quantities G_{ab}, A_μ^a and $g_{\mu\nu}$ as:

$$G_{ab} \ = \ \hat{g}_{ab},$$

$$
\begin{aligned}
A_\mu^a &= \hat{g}^{ab}\,\hat{g}_{\mu b}, \\
g_{\mu\nu} &= \hat{g}_{\mu\nu} - \hat{g}^{ab}\hat{g}_{\mu a}\hat{g}_{\nu b},
\end{aligned} \tag{3.17}
$$

where $\hat{g}^{ab}$ are the components of the inverse metric. It is easy to see that these fields transform in the correct way as a set of $\frac{1}{2}d(d+1)$ scalars, d vectors and one metric under the $(D-d)$-dimensional general coordinate transformations. Furthermore the D-dimensional transformations induce a $U(1)$-gauge transformation on the vector fields: $\delta A_\mu^a = \partial_\mu \xi^a$. The vectors A_μ^a are usually called Kaluza-Klein vectors, and the scalars G_{ab} the moduli of the compactification, since they parametrise the internal space.

The anti-symmetric tensor field $\hat{B}_{\hat{\mu}\hat{\nu}}$ transforms, besides under general coordinate transformations as in (3.15), also under the ten-dimensional gauge transformation $\delta\hat{B}_{\hat{\mu}\hat{\nu}} = \partial_{[\hat{\mu}}\hat{\Sigma}_{\hat{\nu}]}$. The variations of the different components yield:

$$
\begin{aligned}
\delta\hat{B}_{ab} &= \delta_L\hat{B}_{ab}, \\
\delta\hat{B}_{a\mu} &= \delta_L\hat{B}_{a\mu} + \partial_\mu\xi^b\hat{B}_{ab} - \partial_\mu\Sigma_a, \\
\delta\hat{B}_{\mu\nu} &= \delta_L\hat{B}_{\mu\nu} + \partial_{[\mu}\xi^a\hat{B}_{a\nu]} + \partial_{[\mu}\Sigma_{\nu]},
\end{aligned} \tag{3.18}
$$

where with δ_L we mean the variation under $(D-d)$-dimensional general coordinate transformations. In $(D-d)$ dimensions we therefore obtain a set of $\frac{1}{2}d(d-1)$ scalars B_{ab}, d vectors $B_{a\mu}$ and a rank-two anti-symmetric tensor $B_{\mu\nu}$, given as functions of the D-dimensional fields by:

$$
\begin{aligned}
B_{ab} &= \hat{B}_{ab}, \\
B_{a\mu} &= \hat{B}_{a\mu} - \hat{g}^{cb}\hat{g}_{c\mu}\hat{B}_{ab}, \\
B_{\mu\nu} &= \hat{B}_{\mu\nu} + \hat{g}^{ab}\hat{g}_{a[\mu}\hat{B}_{\nu]b} - 2\,\hat{g}^{ab}\hat{g}^{cd}\hat{g}_{a[\mu}\,\hat{B}_{bc}\,\hat{g}_{\nu]d}\ .
\end{aligned} \tag{3.19}
$$

Again all these fields transform in the proper way and $B_{a\mu}$ behaves like a $U(1)$-gauge field under the remnant gauge transformation of $\hat{B}_{\hat{\mu}\hat{\nu}}$: $\delta B_{a\mu} = \partial_\mu\Sigma_a$. $B_{a\mu}$ is usually called the winding vector, since one can show that it couples to string states that are wound a number of times around the compact dimension x^a. The scalars B_{ab} span, together with the G_{ab}, the moduli space of toroidal compactifications.

Let us now look at the reduction of the action of the common sector (2.42). For simplicity, we only reduce from ten to nine dimensions, since all typical and interesting features can already be found in this example. Later on we will study more extensively the reductions of the various superstring actions over more dimensions.

Suppose all fields in the ten-dimensional action (2.42)

$$
S = \tfrac{1}{2}\int d^{10}x\sqrt{|\hat{g}|}\;e^{-2\hat{\phi}}\Big[-\hat{R} + 4(\partial\hat{\phi})^2 - \tfrac{3}{4}\hat{H}^2\Big] \tag{3.20}
$$

are independent of the coordinate x, over which we are going the reduce.

At this point it is convenient to write the metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ locally as a flat metric:

$$
\hat{g}_{\hat{\mu}\hat{\nu}} \;=\; \hat{e}_{\hat{\mu}}{}^{\hat{\alpha}}\;\hat{e}_{\hat{\nu}}{}^{\hat{\beta}}\;\hat{\eta}_{\hat{\alpha}\hat{\beta}}, \tag{3.21}
$$

where $\hat{e}_{\hat{\mu}}{}^{\hat{\alpha}}$ is the ten-dimensional vielbein, which relates the curved indices $\hat{\mu}$ to the flat ones $\hat{\alpha}$. The vielbein transforms under Lorentz transformations, therefore we can choose a gauge in which the vielbein is of the form

$$\hat{e}_{\hat{\mu}}{}^{\hat{\alpha}} = \begin{pmatrix} e_\mu{}^\alpha & kA_\mu \\ 0 & k \end{pmatrix}, \tag{3.22}$$

where $e_\mu{}^\alpha$ is the nine-dimensional vielbein. This corresponds to a choice for the reduction rules

$$\begin{array}{rclcrcl} \hat{g}_{\mu\nu} & = & g_{\mu\nu} - k^2 A_\mu A_\nu\,, & \qquad & \hat{B}_{\mu\nu} & = & B_{\mu\nu} + A_{[\mu}B_{\nu]}\,, \\ \hat{g}_{x\mu} & = & -k^2 A_\mu\,, & & \hat{B}_{x\mu} & = & B_\mu\,, \\ \hat{g}_{xx} & = & -k^2\,. & & & & \end{array} \tag{3.23}$$

For the gauge choice (3.22), we have that

$$\sqrt{|\hat{g}|} = \det(\hat{e}_{\hat{\mu}}{}^{\hat{\alpha}}) = k\,\det(e_\mu{}^\alpha) = k\,\sqrt{|g|}. \tag{3.24}$$

So if we take for the reduction rule of the dilaton[4]

$$\hat{\phi} = \phi + \tfrac{1}{2}\log k, \tag{3.25}$$

we see that $\sqrt{|\hat{g}|}\,e^{-2\hat{\phi}} = \sqrt{|g|}\,e^{-2\phi}$. It can be shown that the first two terms of (3.20) reduce like

$$-\hat{R} \;+\; 4(\partial\hat{\phi})^2 = -R \;+\; 4(\partial\phi)^2 - (\partial\log k)^2 + \tfrac{1}{4}k^2 F_{\mu\nu}(A)F^{\mu\nu}(A), \tag{3.26}$$

while the axion field strength $\hat{H}_{\hat{\alpha}\hat{\beta}\hat{\gamma}} = \hat{e}_{\hat{\alpha}}{}^{\hat{\mu}}\;\hat{e}_{\hat{\beta}}{}^{\hat{\nu}}\;\hat{e}_{\hat{\gamma}}{}^{\hat{\rho}}\;\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}$ decomposes as:

$$\begin{array}{rcl} \hat{H}_{\alpha\beta x} & = & \frac{1}{3k}\;e_\alpha{}^\mu\;e_\beta{}^\nu F_{\mu\nu}(B)\;, \\ \hat{H}_{\alpha\beta\gamma} & = & e_\alpha{}^\mu\;e_\beta{}^\nu e_\gamma{}^\rho\;\left[\partial_{[\mu}B_{\nu\rho]} + \frac{1}{2}A_{[\mu}F_{\nu\rho]}(B) + \frac{1}{2}B_{[\mu}F_{\nu\rho]}(A)\right]\,, \\ & = & e_\alpha{}^\mu\;e_\beta{}^\nu e_\gamma{}^\rho\;H_{\mu\nu\rho} \;\;=\;\; H_{\alpha\beta\gamma}. \end{array} \tag{3.27}$$

So after dimensional reduction, (3.20) takes the form

$$\begin{aligned} S = \tfrac{1}{2}\int d^9x\;\sqrt{|g|}\;e^{-2\phi}\Big[-R \;+\; 4(\partial\phi)^2 -\; \tfrac{3}{4}H^2 -\; (\partial\log k)^2 \\ +\; \tfrac{1}{4}k^2F^2(A) \;+\; \tfrac{1}{4}k^{-2}F^2(B)\Big]\,. \end{aligned} \tag{3.28}$$

Note that the reduced action is invariant under nine-dimensional general coordinate transformations, as it should be, and under the $U(1)$-symmetries $\delta A_\mu = \partial_\mu\xi,\ \delta B_\mu = \partial_\mu\Sigma$, provided that the reduced axion transforms as:

$$\delta B_{\mu\nu} = \partial_{[\mu}\Sigma_{\nu]} + \partial_{[\mu}\xi B_{\nu]} - A_{[\mu}\partial_{\nu]}\Sigma. \tag{3.29}$$

Furthermore the action has an $O(1,1)$-symmetry, which is a direct product: $O(1,1) = SO^{\uparrow}(1,1)\times\mathbb{Z}_2^{(S)}\times\mathbb{Z}_2^{(T)}$. Just as in the case of the Lorentz group, this non-compact

[4]The higher dimensional analogue for this reduction rule is $\hat{\phi} = \phi + \frac{1}{2}\log|G_{ab}|$.

group consists of four disconnected parts, of which only the subgroup of proper, time-orientation preserving transformations $SO^{\uparrow}(1,1)$ is continuously connected to the identity. The other parts (the improper and/or non-orthochronous transformations) are connected via the mapping class group $\mathbb{Z}_2^{(S)} \times \mathbb{Z}_2^{(T)}$.

The continuous scale transformation $SO^{\uparrow}(1,1)$ scales the fields A_μ, B_μ and k with a factor $\Lambda > 0$ according to their weight under this scale transformation:

$$A_\mu \to \Lambda\ A_\mu, \qquad B_\mu \to \Lambda^{-1} B_\mu, \qquad k \to \Lambda^{-1} k. \tag{3.30}$$

The discrete subgroup $\mathbb{Z}_2^{(S)}$ flips the sign of the vector fields, while the $\mathbb{Z}_2^{(T)}$-symmetry is generated by an interchange of the vector fields and an inversion of k:

$$\tilde{A}_\mu = B_\mu, \qquad \tilde{B}_\mu = A_\mu, \qquad \tilde{k} = k^{-1}. \tag{3.31}$$

Using (the inverse of) the reduction rules (3.23), we can easily see that this symmetry (3.31) corresponds in ten dimensions to the T-duality transformation (3.4). The $O(1,1)$ is therefore called the T-duality group, which parametrises the moduli space of compactifications: the modulus k is directly related to the size of the compact dimension

$$k = \sqrt{|\hat{g}_{xx}|} = \frac{R}{\sqrt{\alpha'}}. \tag{3.32}$$

Different values of k label different compactifications, which are related via the $O(1,1)$ transformations. However T-duality ($\mathbb{Z}_2^{(T)}$) states that compactification over a radius R is equivalent to compactification over a radius $1/R$, so the points k and k^{-1} in moduli space are equivalent. Also the sign of the vector fields is irrelevant ($\mathbb{Z}_2^{(S)}$), so the moduli space of *inequivalent* compactifications is given by

$$\frac{O(1,1)}{\mathbb{Z}_2^{(T)} \times \mathbb{Z}_2^{(S)}} \;=\; SO^{\uparrow}(1,1). \tag{3.33}$$

In this simple example of compactification over one dimension, all generic features of toroidal compactification are present. In the next subsection we will study more general compactifications over d dimensions in the presence of n Abelian vector fields. This will give rise to bigger $O(d, d+n)$ groups and more complicated coset structures, but the same features will reappear. For an extensive study of the symmetry transformations of the dimensionally reduced action (3.28), we refer to [27].

3.1.3 T-duality in the Target Space Action

Let us now look at T-duality in the full low energy effective string theory actions. We will start with the Heterotic string theories. Their action is given by (2.36)

$$S = \tfrac{1}{2} \int d^{10}x \sqrt{|\hat{g}|}\ e^{-2\hat{\phi}} \Big[-\hat{R} + 4(\partial\hat{\phi})^2 - \tfrac{3}{4} \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} \hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}} + \tfrac{1}{4} \hat{F}^I_{\hat{\mu}\hat{\nu}} \hat{F}_I^{\hat{\mu}\hat{\nu}} \Big]. \tag{3.34}$$

The $\hat{F}^I_{\hat{\mu}\hat{\nu}}$ are the field strengths of the $SO(32)$ or $E_8 \times E_8$ gauge fields $\hat{V}^I_{\hat{\mu}}$; the axion field strength $\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}$ contains a Chern-Simons term:

$$\begin{aligned} \hat{F}^I_{\hat{\mu}\hat{\nu}} &= \partial_{\hat{\mu}}\hat{V}^I_{\hat{\nu}} - \partial_{\hat{\nu}}\hat{V}^I_{\hat{\mu}} - f_{KL}{}^I \hat{V}^K_{\hat{\mu}}\hat{V}^L_{\hat{\nu}}, \\ \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= \partial_{[\hat{\mu}}\hat{B}_{\hat{\nu}\hat{\rho}]} - \tfrac{1}{2}\Big[\hat{V}^I_{[\hat{\mu}}\hat{F}_{\hat{\nu}\hat{\rho}]I} + \tfrac{1}{3} f_{IKL}\hat{V}^I_{[\hat{\mu}}\hat{V}^K_{\hat{\nu}}\hat{V}^L_{\hat{\rho}]}\Big]. \end{aligned} \tag{3.35}$$

The gauge transformations of the Yang-Mills groups are given by:

$$\begin{aligned} \delta\hat{V}^I_{\hat{\mu}} &= \partial_{\hat{\mu}}\Lambda^I + f_{KL}{}^I\Lambda^K\hat{V}^L_{\hat{\mu}}, \\ \delta\hat{B}_{\hat{\mu}\hat{\nu}} &= \hat{V}^I_{[\hat{\mu}}\partial_{\hat{\nu}]}\Lambda_I. \end{aligned} \tag{3.36}$$

Dimensional reduction over T^d yields an action with a $(10-d)$-dimensional metric, axion and dilaton, d Kaluza-Klein vectors A^a_μ, d winding vectors $B_{\mu a}$, Yang-Mills vectors V^I_μ and moduli G_{ab}, B_{ab} and ℓ^I_a coming from the reduction of the metric, the axion and the Yang-Mills fields in 10 dimensions. The precise reduction rules will be given in (4.9).

In a generic point in the moduli space, the ℓ^I_a have a non-zero expectation value and via a Higgs mechanism they will give masses to the vector fields in the Yang-Mills group. Only the Abelian fields V^m_μ in the Cartan sub-algebra will remain massless after reduction. For both $SO(32)$ and $E_8 \times E_8$ this Cartan sub-algebra is 16-dimensional, so both groups break to $U(1)^{16}$. The low energy effective action therefore contains $(2d+16)$ Abelian vector fields, which form a $U(1)^{(2d+16)}$ gauge group. Furthermore these Abelian fields fit into a global $O(d,d+16)$-group representation such that the action can be written as [136, 115]:

$$\begin{aligned} S = \tfrac{1}{2}\int d^{10-d}x\sqrt{|g|}\; e^{-2\phi}\Big[&-R + 4(\partial\phi)^2 - \tfrac{3}{4}H_{\mu\nu\rho}H^{\mu\nu\rho} \\ &+ \tfrac{1}{8}\mathrm{Tr}(\partial_\mu M\partial^\mu M^{-1}) - \tfrac{1}{4}\mathcal{F}^i_{\mu\nu}M^{-1}_{ij}\mathcal{F}^{\mu\nu j}\Big], \end{aligned} \tag{3.37}$$

where

$$\begin{aligned} &\mathcal{F}^i_{\mu\nu}(\mathcal{A}) = \partial_\mu\mathcal{A}^i_\nu - \partial_\nu\mathcal{A}^i_\mu, \\ &H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]} + \tfrac{1}{2}\mathcal{A}^i_{[\mu}\mathcal{F}^j_{\nu\rho]}(\mathcal{A})L_{ij}, \\ &\mathcal{A}^i_\mu = \begin{pmatrix} A^a_\mu \\ B_{\mu a} \\ V^m_\mu \end{pmatrix}, \qquad L_{ij} = \begin{pmatrix} 0 & \mathbb{1}_d & 0 \\ \mathbb{1}_d & 0 & 0 \\ 0 & 0 & -\mathbb{1}_{16} \end{pmatrix}. \end{aligned} \tag{3.38}$$

The $d(d+16)$ moduli $G_{ab}, B_{ab}, \ell_a{}^m$ are combined into the symmetric $(2d+16)\times(2d+16)$ matrix M^{-1}, satisfying $M^{-1}LM^{-1} = L$, where L is the invariant metric on $O(d,d+16)$.

Different values of the moduli correspond to different radii of the torus and therefore to different compactifications. The moduli parametrise the $d(d+16)$-dimensional coset space $O(d,d+16)/(O(d)\times O(d+16))$ of different compactifications [119].

It is easy to see that (3.37) is invariant under general $O(d,d+16)$ transformations

$$\mathcal{A}'_\mu = \Omega\mathcal{A}_\mu, \qquad (M^{-1})' = \Omega M^{-1}\Omega^T, \qquad \Omega^T L\Omega = L. \tag{3.39}$$

This $O(d,d+16)$ is not a symmetry of the full string theory, since quantum corrections will break the group structure. An analysis at the level of the sigma model shows [119, 120] that the allowed $(2d+16)$ vector fields charges of the string states form a $(2d+16)$-dimensional, even self-dual lattice and the symmetry group of the full theory should leave this lattice invariant. The transformations that preserve this lattice form the discrete $O(d,d+16;\mathbb{Z})$-group, the sub-group of $O(d,d+16)$-transformations with integer parameters which is conjectured to be a symmetry of the full string theory. In fact this $O(d,d+16;\mathbb{Z})$ is the generalization of the T-duality transformations (3.31) and (3.4) and is usually called the T-duality group. It relates compactifications over different tori as equivalent ones. The moduli space of inequivalent toroidal compactifications is therefore given by the coset

$$\frac{O(d,d+16)}{O(d)\times O(d+16)}\Bigg/ O(d,d+16;\mathbb{Z}). \tag{3.40}$$

Note that this is the moduli space for both the $SO(32)$ as $E_8\times E_8$ theory. In fact the two theories correspond to two distinct points in this moduli space and can be continuously connected [73]. This means that they are two manifestations of one and the same Heterotic theory and can be mapped one into the other via T-duality.

A similar thing happens for the $N=2$ theories Type IIA and Type IIB: although the two theories look very different in ten dimensions, upon reduction over a circle the massless spectrum of the two theories precisely coincides: besides the NS-NS sector (3.28), they both have a scalar, a vector, a two-form and a three-form gauge field in their R-R sector[5]:

$$\begin{array}{ll} \text{Type IIA}: & \{\hat{A}^{(1)}_x, \quad \hat{A}^{(1)}_\mu, \quad \hat{C}_{\mu\nu x}, \quad \hat{C}_{\mu\nu\rho}\ \} \\ \text{Type IIB}: & \{\ \hat{\ell}, \quad\ \hat{B}^{(2)}_{x\mu}, \quad \hat{B}^{(2)}_{\mu\nu}, \quad \hat{D}^{+}_{\mu\nu\rho x}\}. \end{array} \tag{3.41}$$

Furthermore their low energy effective actions can be mapped on to one and the same Type II action in nine dimensions [26],

$$\begin{aligned} S = \tfrac{1}{2}\int d^9x\ \sqrt{|g|}\ \Bigg\{\ & e^{-2\phi}\Big[-R\ +\ 4(\partial\phi)^2 -\ \tfrac{3}{4}(H^{(1)})^2 -\ (\partial\log k)^2 \\ & \qquad +\ \tfrac{1}{4}k^2F^2(A)\ +\ \tfrac{1}{4}k^{-2}F^2(B)\Big] \\ & +\tfrac{1}{4}\Big(F^{(1)}+\ell F^{(2)}\Big)^2 - \tfrac{1}{2}\ k^{-1}(\partial\ell)^2 + \tfrac{3}{4}\ k\ G^2 \\ & -34\ k^{-1}\Big(H^{(1)}+\ell H^{(2)}\Big)^2\Bigg\} \\ -\tfrac{1}{64}\int d^9x\ \varepsilon_{(10)}\Big(& \partial C\partial CB + \partial C\partial B^{(a)}\partial B^{(b)}\varepsilon^{ab} + 2\ \partial CA^{(a)}\partial B^{(a)}B \\ & -\partial CA^{(a)}A^{(b)}\partial BB\varepsilon^{ab}\Big), \end{aligned} \tag{3.42}$$

[5]The four-form gauge field $\hat{D}^{+}_{\mu\nu\rho\lambda}$ is not an independent field in nine dimensions, but is completely determined by the self-duality condition (2.35) and can therefore be ignored.

provided that one uses two different reduction schemes for each theory. For the Type IIA theory the relation between the action (2.33) and the above action is given by the reduction rules

$$
\begin{array}{ll}
\hat{g}_{\mu\nu} = g_{\mu\nu} - k^2 A_\mu A_\nu, & \hat{B}_{\mu\nu} = B^{(1)}_{\mu\nu} + A_{[\mu}B_{\nu]}\,, \\
\hat{g}_{x\mu} = -k^2 A_\mu\,, & \hat{B}_{x\mu} = B_\mu, \\
\hat{g}_{xx} = -k^2, & \hat{\phi}^A = \phi + \tfrac{1}{2}\log k, \\
\hat{A}^{(1)}_x = \ell, & \hat{A}^{(1)}_\mu = A^{(1)}_\mu + \ell A_\mu, \\
\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}} = C_{\mu\nu\rho}, & \hat{C}_{\mu\nu x} = \tfrac{2}{3}(B^{(2)}_{\mu\nu} - A^{(1)}_{[\mu}B_{\nu]})\ ,
\end{array}
\tag{3.43}
$$

while for the Type IIB theory the relation between the ten and the nine-dimensional fields is given by

$$
\begin{array}{ll}
\hat{G}_{\mu\nu} = g_{\mu\nu} - k^{-2}B_\mu B_\nu, & \hat{\mathcal{B}}^{(1)}_{\mu\nu} = B^{(1)}_{\mu\nu} - A_{[\mu}B_{\nu]}, \\
\hat{G}_{x\mu} = -k^{-2}B_\mu\,, & \hat{\mathcal{B}}^{(1)}_{x\mu} = A_\mu, \\
\hat{G}_{xx} = -k^{-2}, & \hat{\phi}^B = \phi - \tfrac{1}{2}\log k, \\
\hat{\mathcal{B}}^{(2)}_{\mu\nu} = B^{(2)}_{\mu\nu} + A^{(1)}_{[\mu}B_{\nu]}, & \hat{\mathcal{B}}^{(2)}_{x\mu} = A^{(1)}_\mu, \\
\hat{D}_{\mu\nu\rho x} = \tfrac{3}{8}(C_{\mu\nu\rho} - A^{(a)}_{[\mu}B^{(a)}_{\nu\rho]} - \varepsilon^{ab}A^{(a)}_{[\mu}A^{(b)}_\nu B_{\rho]}), & \hat{\ell} = \ell.
\end{array}
\tag{3.44}
$$

The fact that Type IIA and Type IIB can be mapped on to the same Type II theory means that in ten dimensions they are different embeddings of one and the same theory which become equivalent after compactification on circles S^1_A and S^1_B, where the relation between the two compactification radii is given by:

$$
\frac{R_A}{\sqrt{\alpha'}} = \sqrt{|\hat{g}_{xx}|} = k = \frac{1}{\sqrt{|\hat{G}_{xx}|}} = \frac{\sqrt{\alpha'}}{R_B}. \tag{3.45}
$$

In other words the limits, $k \to \infty$ (Type IIA) and $k \to 0$ (Type IIB) are different limits in the moduli space of the Type II theory in nine dimensions. Furthermore, a careful analysis [57, 51] of the fermionic part of the action reveals a change in chirality of the fermions, which is necessary to relate the non-chiral Type IIA to the chiral Type IIB theory.

The relation between the fields of both theories in ten dimensions can be read off from (3.43) and (3.44):

$$
\begin{aligned}
\hat{C}_{x\mu\nu} &= \tfrac{2}{3}\left[\hat{\mathcal{B}}^{(2)}_{\mu\nu} + 2\hat{\mathcal{B}}^{(2)}_{x[\mu}\hat{G}_{\nu]x}/\hat{G}_{xx}\right], \\
\hat{C}_{\mu\nu\rho} &= \tfrac{8}{3}\hat{D}_{x\mu\nu\rho} + \varepsilon^{ab}\hat{\mathcal{B}}^{(a)}_{x[\mu}\hat{\mathcal{B}}^{(b)}_{\nu\rho]} + \varepsilon^{ab}\hat{\mathcal{B}}^{(a)}_{x[\mu}\hat{\mathcal{B}}^{(b)}_{|x|\nu}\hat{G}_{\rho]x}/\hat{G}_{xx}\,, \\
\hat{g}_{\mu\nu} &= \hat{G}_{\mu\nu} - \left(\hat{G}_{x\mu}\hat{G}_{x\nu} - \hat{\mathcal{B}}^{(1)}_{x\mu}\hat{\mathcal{B}}^{(1)}_{x\nu}\right)/\hat{G}_{xx}\,, \\
\hat{B}^{(1)}_{\mu\nu} &= \hat{\mathcal{B}}^{(1)}_{\mu\nu} + 2\hat{\mathcal{B}}^{(1)}_{x[\mu}\hat{G}_{\nu]x}/\hat{G}_{xx}\,,
\end{aligned}
$$

$$
\begin{array}{rclcrcl}
\hat{g}_{x\mu} & = & \hat{\mathcal{B}}^{(1)}_{x\mu}/\hat{G}_{xx}\,, & \qquad & \hat{B}^{(1)}_{x\mu} & = & \hat{G}_{x\mu}/\hat{G}_{xx} \\
\hat{A}^{(1)}_{\mu} & = & -\hat{\mathcal{B}}^{(2)}_{x\mu} + \hat{\ell}\hat{\mathcal{B}}^{(1)}_{x\mu}\,, & & \hat{g}_{xx} & = & 1/\hat{G}_{xx}\,, \\
\hat{\phi}^{A} & = & \hat{\phi}^{B} - \frac{1}{2}\log\left(-\hat{G}_{xx}\right), & & \hat{A}^{(1)}_{x} & = & \hat{\ell}\,.
\end{array}
\tag{3.46}
$$

These transformation rules look very similar to the T-duality rules (3.4), though this time the T-duality transformation is not a symmetry of the action, but a transformation that takes us from the Type IIB to the Type IIA action [26]. The inverse transformation from the Type IIA to the Type IIB action can easily be constructed in the same way.

We see that from the $O(1,1)$-symmetry group of the common sector (3.28), only the $SO^{\uparrow}(1,1) \times \mathbb{Z}_2^{(S)}$ survives as a symmetry of the action (3.42), while the $\mathbb{Z}_2^{(T)}$, which corresponds to (3.46), is a map from Type IIA to Type IIB and vice versa.

In a generalization to reduction over d dimensions, the T-duality group is $O(d,d;\mathbb{Z})$ and the moduli parametrise the coset $O(d,d)/(O(d) \times O(d))$. The moduli space of inequivalent compactifications is given by

$$
\frac{O(d,d)}{O(d)\times O(d)} \Big/ O(d,d;\mathbb{Z}). \tag{3.47}
$$

3.1.4 T-duality between Solutions

In the previous subsection we have seen that some of the string theories may be connected via T-duality, at least at the level of the string effective action. This implies that also T-duality transformations should exist between the solutions of these actions.

However, in the derivation of the T-duality rules (3.4) we intrinsically made use of the fact we were doing a duality transformation on a string-like solution: only on a two-dimensional world volume can a scalar X be dualized to another scalar $\tilde{X}$. From this procedure it is not clear how to generalize these rules to the extended objects we encountered in section 2.3.

Nevertheless there exists another, even more general way of deriving the T-duality rules, which in fact we already used, when we showed the T-duality between Type IIA and Type IIB theory: if we can map two actions (solutions) via different ways of dimensional reduction (one over a circle with radius R and the other over a circle with radius $1/R$) on to the same action (solution) one dimension lower, then we can say that the two actions (solutions) are connected via T-duality and the T-duality rules can be read off in the same way that we derived the Type II rules (3.46). In fact this procedure is more general, since not only do we find the transformation rules for all the participating fields (besides the NS-NS fields that enter in (3.4) also the rules for the R-R fields), but also this allows us to make T-duality transformations between objects of different spatial extension, while before we could in principle only go from string-like solutions to string-like solutions. The technique of performing T-duality via dimensional reduction will be studied more accurately in Chapter 6, where we will use it to prove the duality relations between the different world volume actions of solutions connected via T-duality.

Using the reduction rules (3.23) and (3.25), one easily sees that the reduction of the fundamental string solution (2.44)

$$F1 = \begin{cases} ds^2 = H^{-1}(dt^2 - dx_1^2) - (dx_2^2 + ... + dx_9^2) \\ e^{-2\phi} = H \\ B_{01} = H^{-1} \end{cases} \tag{3.48}$$

over the world volume direction x^1 gives rise to a nine-dimensional point particle solution

$$m0 = \begin{cases} ds^2 = H^{-1}dt^2 - (dx_2^2 + ... + dx_9^2) \\ e^{-2\phi} = H^{\frac{1}{2}} \\ k = H^{-\frac{1}{2}} \\ B_0 = -H^{-1} \\ B_{\mu\nu} = A_\mu = 0 \ , \end{cases} \tag{3.49}$$

while the reduction of the ten-dimensional gravitational wave (2.64)

$$\mathcal{W}_{10} \ : \ ds^2 = (2-H)dt^2 - Hdz^2 + 2(1-H)dtdz - (dx_2^2 + ... + dx_9^2) \tag{3.50}$$

over the propagation direction z of the wave gives

$$m\tilde{0} = \begin{cases} d\tilde{s}^2 = H^{-1}dt^2 - (dx_2^2 + ... + dx_9^2) \\ e^{-2\tilde{\phi}} = H^{\frac{1}{2}} \\ \tilde{A}_0 = -H^{-1} \\ \tilde{k} = H^{\frac{1}{2}} \\ \tilde{B}_{\mu\nu} = \tilde{B}_\mu = 0 \ . \end{cases} \tag{3.51}$$

We see that these two point particle solutions are actually the same if one identifies

$$\tilde{A}_\mu = B_\mu, \qquad \tilde{B}_\mu = A_\mu, \qquad \tilde{k} = 1/k. \tag{3.52}$$

Note that this is precisely the nine-dimensional T-duality transformation (3.31). Also direct application of the ten-dimensional rules (3.4) maps the $F1$ to the $\mathcal{W}_{10}$ and vice versa. Note that a T-duality transformation in a transverse direction leaves the string and the wave solution invariant.

The same procedure can be followed for the solitonic five-brane (2.54) and the Kaluza-Klein monopole (2.65): reduction of the $S5$ over a transverse direction gives a new five-brane solution in nine dimensions

$$mS5 = \begin{cases} ds^2 = dt^2 - dx_1^2 - ... - dx_5^2 - H(dx_6^2 + ... + dx_8^2) \\ e^{-2\phi} = H^{-\frac{1}{2}} \\ F_{mn}(B) = \varepsilon_{mnp} \ \partial_p H \\ k = H^{\frac{1}{2}} \\ B_{\mu\nu} = A_\mu = 0 \ , \end{cases} \tag{3.53}$$

while the reduction of the $\mathcal{KK}_{10}$ over the isometry direction z yields

$$m\tilde{S}5 = \begin{cases} d\tilde{s}^2 = dt^2 - dx_1^2 - ... - dx_5^2 - H(dx_6^2 + ... + dx_8^2) \\ e^{-2\tilde{\phi}} = H^{-\frac{1}{2}} \\ F_{mn}(\tilde{A}) = \varepsilon_{mnp}\ \partial_p H \\ \tilde{k} = H^{-\frac{1}{2}} \\ \tilde{B}_{\mu\nu} = \tilde{B}_\mu = 0\ . \end{cases} \tag{3.54}$$

Again the two solutions can be identified, using (3.52), which proves the T-duality between the $S5$ and the $\mathcal{KK}_{10}$. A T-duality transformation in a world volume direction leaves both solutions invariant.

It thus turns out that the solutions of the equations of motion of the common sector are related amongst each other via T-duality. This is not so strange, since we showed in Subsection 3.1.2 that the common sector (3.20) itself is invariant under T-duality. Let us now look at how the D-brane solutions, the solutions of Type IIA/B, transform under this duality.

T-duality is an important feature in the theory of D-branes: we already saw that a T-duality transformation on a freely moving open string changes the boundary conditions of the string and attaches it to a D-brane. But also the D-branes themselves are related [129]: applying T-duality on a string attached to a Dp-brane (so satisfying $(p+1)$ Neumann conditions and $(9-p)$ Dirichlet conditions) will change one of the Neumann conditions to a Dirichlet one or vice versa, so after the transformation the string will be attached to a $D(p\pm 1)$ brane. This should of course be visible at the level of the Dp-brane solutions (2.58) of the equations of motion [18].

Indeed, a straightforward application of the duality rules (3.4) on the D-brane solution $(p : 0, ..., 8)$

$$Dp = \begin{cases} ds^2 = H^{-\frac{1}{2}}(dt^2 - dx_1^2 - ... - dx_p^2) - H^{\frac{1}{2}}(dx_{p+1}^2 + ... + dx_9^2) \\ e^{-2\phi} = H^{\frac{p-3}{2}} \\ F^{(\text{R-R})}_{012...pm} = \partial_m H^{-1} \qquad\qquad (m : p+1, ..., 9), \end{cases} \tag{3.55}$$

inverts the metric component of the direction in which the T-duality is performed and changes a world volume direction into a transverse one and back. Also the dilaton and gauge field dependence change in the right way to obtain a $D(p\pm 1)$-brane. The exact form of the transformation rules for the R-R fields can be found in [18]. The fact that Dp-branes with p even (odd) get mapped to p-odd (even) branes corresponds to the fact that T-duality is a map from Type IIA(B) to Type IIB(A) theory.

3.2 Strong/Weak Coupling Duality

Another type of duality symmetry which has been found in string theory is the S-duality or Strong/Weak coupling duality, so called because it relates the strong and weak coupling limits of theories to each other. The importance of S-duality is that it gives a way to go beyond perturbation theory and to obtain a good picture of what string theory is like at strong coupling.

At small values of the coupling constant g, perturbative calculations give a reasonably good understanding of the theory: the weak coupling limit of the theory has a number of electrically charged, elementary states which can be handled in perturbation theory and some magnetically charged, solitonic states, which are very massive and strongly coupled (cfr. $F1$ and $S5$ in section 2.3). For large values of g these perturbative techniques break down and reliable results are much more difficult to obtain.

The idea of S-duality now is that in the large coupling limit the situation might be reversed: it is conjectured by Montonen and Olive [117] that when $g \to \infty$, the elementary, weakly coupled states are the magnetically charged ones and the strongly coupled, massive, solitonic states are electrically charged.

In other words, the Olive-Montonen conjecture states that at strong coupling the theory can be reformulated in terms of new, dual fields and a new coupling constant, such that it is again a weakly coupled theory in this dual formulation. This symmetry is believed to be exact for theories that have $N = 4$ supersymmetry [165], and to hold for some special cases with $N = 1, 2$ supersymmetry as well [143].

The interchange of electric and magnetic charge is very much connected to the interchange of strong and weak coupling through the Dirac quantization rule: the electric and magnetic charges of a state are a measure of how strongly the state interacts with other states and have therefore the role of coupling constants. Since due to the Dirac quantization rule magnetic charge is inversely proportional to electric charge, an electric/magnetic duality is equivalent to a strong/weak coupling duality.

In this section we will review some examples in string theory where S-duality is found and applied to get new results. We will start by looking at the S-duality symmetry in the Heterotic string, compactified on a six-torus. Then we will study the strong coupling limits of the different string theories and make contact with eleven dimensional supergravity.

3.2.1 The Heterotic String in Four Dimensions

The dimensional reduction over a six-torus T^6 of the low energy effective action (2.36) of the Heterotic string gives $N = 4$ supergravity coupled to Yang-Mills theory in four dimensions. The bosonic part of the four-dimensional action contains a metric, a dilaton and an axion, 28 Abelian vector fields and 132 scalars (3.37):

$$\begin{aligned} S = \tfrac{1}{2} \int d^4x \ \sqrt{|g|} \ e^{-2\phi} \Big[& -R + 4(\partial\phi)^2 - \tfrac{3}{4} H_{\mu\nu\rho} H^{\mu\nu\rho} \\ & + \tfrac{1}{8} \mathrm{Tr}(\partial_\mu M \partial^\mu M^{-1}) - \tfrac{1}{4} \mathcal{F}^i_{\mu\nu} M^{-1}_{ij} \mathcal{F}^{\mu\nu j} \Big], \end{aligned} \tag{3.56}$$

where $\mathcal{F}^i_{\mu\nu}, H_{\mu\nu\rho}$ and M^{-1}_{ij} are defined as in (3.38). As argued in Subsection 3.1.3, the vector and scalars transform under the $O(6, 22)$ group, which is a symmetry of the action.

There is yet another symmetry, which is a symmetry of the equations of motion, not of the action (3.56). This can be seen if we rewrite the above action by introducing a scalar field ψ and a rescaled metric $g^{\mathrm{E}}_{\mu\nu}$ via

$$
\begin{aligned}
H^{\mu\nu\rho} &= -\frac{1}{\sqrt{|g|}}\, e^{2\phi}\varepsilon^{\mu\nu\rho\lambda}\, \partial_\lambda \psi \,, && (3.57)\\
g^{\mathrm{E}}_{\mu\nu} &= e^{-2\phi}\, g_{\mu\nu}. && (3.58)
\end{aligned}
$$

The scalar ψ is the Poincaré dual of the anti-symmetric tensor $B_{\mu\nu}$, as in (2.56). The new metric $g^{\mathrm{E}}_{\mu\nu}$ is called the Einstein metric since this is the canonical metric that appears in the Einstein-Hilbert action. The metric $g_{\mu\nu}$ we have been using until now is usually called the string metric.

In terms of these new fields, the action (3.56) can be rewritten as

$$
\begin{aligned}
S = \tfrac{1}{2}\int d^4x\ \sqrt{|g^{\mathrm{E}}|}\ \Big[&-R_{\mathrm{E}} - \frac{1}{2(\lambda_2)^2}(\partial\lambda\ \partial\bar{\lambda}) + \tfrac{1}{8}\mathrm{Tr}(\partial M \partial M^{-1})\\
&-\tfrac{1}{4}\lambda_2 \mathcal{F}^i M^{-1}_{ij}\mathcal{F}^j - \tfrac{1}{16}\lambda_1 \mathcal{F}^i L_{ij}{}^*\mathcal{F}^j\Big]\,, \qquad (3.59)
\end{aligned}
$$

where we combined the scalars ψ and $e^{-2\phi}$ in one complex scalar

$$\lambda \;=\; \lambda_1 + i\lambda_2 \;=\; \psi + ie^{-2\phi}\,, \qquad (3.60)$$

and

$$^*\mathcal{F}^{\mu\nu i} = \tfrac{1}{\sqrt{|g^{\mathrm{E}}|}}\ \varepsilon^{\mu\nu\rho\sigma}\ \mathcal{F}^i_{\rho\sigma}. \qquad (3.61)$$

It can be shown that the equations of motion of the above action are invariant under the $SL(2,\mathbb{R})$ transformation [149, 146, 139]:

$$
\begin{aligned}
\lambda &\to \frac{a\lambda+b}{c\lambda+d}\,, \qquad ad-bc=1\\
\mathcal{F}^i_{\mu\nu} &\to (c\lambda_1+d)\mathcal{F}^i_{\mu\nu} + c\lambda_2 (ML)^i{}_j{}^*\mathcal{F}^j_{\mu\nu}. \qquad (3.62)
\end{aligned}
$$

More precisely, the equations of motion of the vector fields and their Bianchi identities can be written schematically as

$$
\begin{array}{ll}
\text{Eqns of motion:} & D_\mu[\lambda(ML\mathcal{F} + i^*\mathcal{F}) - \bar{\lambda}(ML\mathcal{F} - i^*\mathcal{F})] = 0\\
\text{Bianchi identity:} & D_\mu[(ML\mathcal{F} + i^*\mathcal{F}) - (ML\mathcal{F} - i^*\mathcal{F})] = 0
\end{array} \qquad (3.63)
$$

and it is straightforward to calculate that under an $SL(2;\mathbb{R})$ transformation (3.62) these two equations get mapped one into another. The equations of motion of all other fields are left invariant.

If we consider the particular $SL(2;\mathbb{R})$ transformation where the group parameters have the values $a=d=0$ and $b=-c=1$, we find the transformation

$$\lambda \to -\frac{1}{\lambda}\,, \qquad \mathcal{F}^i_{\mu\nu} \to -\lambda_1\mathcal{F}^i_{\mu\nu} - \lambda_2 (ML)^i{}_j{}^*\mathcal{F}^j_{\mu\nu}. \qquad (3.64)$$

For $\lambda_1 = 0$ this corresponds to a strong/weak coupling symmetry, where the electric fields of $\mathcal{F}^i_{\mu\nu}$ get interchanged with the magnetic ones of ${}^*\mathcal{F}^i_{\mu\nu}$, together with an inversion of the string coupling constant $e^{\phi} \to e^{-\phi}$.

So the low energy limit of the four-dimensional Heterotic string, compactified on T^6, has a symmetry which relates the strong coupling regime of the theory with its weak coupling regime. This is an example of the Olive-Montonen conjecture embedded in the context of string theory.

It is known that quantum effects break the $SL(2;\mathbb{R})$ symmetry to the discrete subgroup $SL(2;\mathbb{Z})$ [149, 145], the group of $SL(2;\mathbb{R})$ transformation with integer parameters and in analogy with the T-duality group $O(d, d+n;\mathbb{Z})$ of Narain, this $SL(2;\mathbb{Z})$ is conjectured to be a symmetry of the full string theory [66]. This is a very bold conjecture, since $SL(2;\mathbb{Z})$ is clearly a non-perturbative symmetry, as we can see already in (3.64).

However Sen was able to present indications that this is indeed the case [147] by showing that the charge spectrum of the theory and the BPS mass formula are invariant under $SL(2;\mathbb{Z})$ transformations. Furthermore he could identify elementary string excitations and known solitons as being $SL(2;\mathbb{Z})$ transforms of each other and therefore fitting in $SL(2;\mathbb{Z})$ multiplets.

In [142, 147] a low energy effective action was presented, which has a manifest $SL(2;\mathbb{R})$ symmetry with $O(6,22)$ as a symmetry of the equations of motion. This action is obtained by dimensional reduction of the ten-dimensional "dual" (six-form) action [63], where the $SL(2;\mathbb{Z})$ appears as the T-duality group of the reduced dual action. This hints at another type of duality, namely the string/five-brane duality [152, 64], which states that in ten dimensions string theory is equivalent to a theory of five-branes, that couple naturally to the six-form potential, which is the Poincaré dual (2.56) of the axion. In this duality the $O(6,22;\mathbb{Z})$ and the $SL(2;\mathbb{Z})$ appear on the same footing [142]: a symmetry of the action in one theory is a symmetry of the equations of motion in the other and vice versa. Their role gets interchanged and we can talk of a "duality of dualities".

3.2.2 Strong Coupling Limits of String Theories

Let us now look at the strong coupling limits of each of the string theories presented in section 2.2 and see whether S-duality can help us to find these limits.

From (2.32) and (2.36), we see that the Type I and the Heterotic $SO(32)$ string are quite similar: they have the same (bosonic) field content, the same gauge group $SO(32)$, and the same amount of supersymmetry. However the vector fields and the two-form anti-symmetric tensor are coupled in different ways to the dilaton in the two theories. The difference becomes more clear if we rescale the string metric to go to the Einstein frame:

$$g^{\mathrm{E}}_{\mu\nu} = e^{-\phi/2}\; g_{\mu\nu}. \tag{3.65}$$

The actions (2.32) and (2.36) in this frame yield

$$\begin{aligned} S_I &= \tfrac{1}{2}\int d^{10}x\sqrt{|g^{\mathrm{E}}|}\,\Big[-R_{\mathrm{E}} - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{3}{4}\,e^{\phi}\,H^2 + \tfrac{1}{4}\,e^{\phi/2}F^2\Big], \\ S_{Het} &= \tfrac{1}{2}\int d^{10}x\sqrt{|g^{\mathrm{E}}|}\,\Big[-R_{\mathrm{E}} - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{3}{4}\,e^{-\phi}\,H^2 + \tfrac{1}{4}\,e^{-\phi/2}F^2\Big]. \end{aligned} \tag{3.66}$$

We see that the difference between the Type I and the Heterotic $SO(32)$ low energy effective action is the sign of the dilaton: the transformation $\phi \to -\phi$ takes one action into the other. This seems to suggest that the strong coupling limit of the Heterotic string is the Type I string and vice versa [163].

There is more evidence to support this idea: the fundamental string solution in the Heterotic theory, which couples to the axion, can be shown to coincide with the D-string of Type I theory, which couples to the R-R two-form. The same goes for the Heterotic $S5$ and the Type I $D5$ [49, 90]. Furthermore after compactification to nine dimensions the points in moduli space of the Heterotic string, for which an enhancement of the gauge symmetry occurs, correspond exactly to the points where the perturbative description of Type I theory breaks down [130].

Type IIB theory is manifestly $SL(2;\mathbb{R})$ invariant [92]. This can be seen best by rewriting the Type IIB action (2.34) in the Einstein-frame metric (3.65):

$$\begin{aligned} S_{IIB} = \tfrac{1}{2}\int d^{10}x\sqrt{|g^{\mathrm{E}}|}\,\Big[&-R_{\mathrm{E}} + \tfrac{1}{4}\mathrm{Tr}(\partial N\partial N^{-1}) - \tfrac{3}{4}\mathcal{H}^{(a)}N_{ab}\mathcal{H}^{(b)} \\ &-\tfrac{5}{6}F_{(5)}^2 - \tfrac{1}{96\sqrt{|g^{\mathrm{E}}|}}\varepsilon^{ab}\varepsilon^{(10)}D_{(4)}\mathcal{H}^{(a)}\mathcal{H}^{(b)}\Big], \end{aligned} \tag{3.67}$$

where N_{ab} is the $SL(2;\mathbb{R})$ matrix

$$N_{ab} = \frac{1}{\lambda_2}\begin{pmatrix} |\lambda|^2 & -\lambda_1 \\ \lambda_1 & 1 \end{pmatrix} \tag{3.68}$$

and $\lambda = \lambda_1 + \lambda_2 = \ell + ie^{-\phi}$.

In this form, the action is invariant under the $SL(2;\mathbb{R})$ transformation [26, 17]

$$\begin{aligned} \mathcal{H}^{(a)} &= \omega^a{}_b\,\mathcal{H}^{(b)}, \\ N'_{ab} &= \omega\,N\,\omega^{-1}, \\ \omega &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \\ ad &- bc = 1. \end{aligned} \tag{3.69}$$

The transformation rule for the complex scalar λ is as in (3.62)

$$\lambda \to \frac{a\lambda + b}{c\lambda + d}\,, \tag{3.70}$$

which implies again an S-duality symmetry $\lambda \to \lambda^{-1}$. For $a = d = \ell = 0$ and $b = -c = 1$ we have

$$e^{\phi} \to e^{-\phi}, \qquad \mathcal{B}^{(1)} \to \mathcal{B}^{(2)}, \qquad \mathcal{B}^{(2)} \to -\mathcal{B}^{(1)}. \tag{3.71}$$

Note however that this S-duality is different from the one given in (3.62), in the sense that here the transformation does not interchange the field strength $\mathcal{H}$ with its Poincaré dual $\mathcal{H} \rightarrow {}^*\mathcal{H}$, but mixes the NS-NS form with the R-R form and vice versa. The strong/weak coupling duality can be understood as the interchange of states from the perturbative (NS-NS) sector with states from the non-perturbative (R-R) sector. In particular, the Type IIB fundamental string gets mapped to the $D1$-brane and the solitonic five-brane to the $D5$.

The behaviour of the Type IIA theory at strong coupling is rather different from the way the Heterotic, Type I or Type IIB behave. Type IIA at strong coupling does not get related to a different, previously known string theory, but it turns out that there is an intimate relation with $D = 11$ supergravity and a not yet well formulated theory, called M-theory.

At the level of the low energy effective action, the connection between the Type IIA action (2.33) and the $D = 11$ supergravity action (2.37) is that the former is a simple dimensional reduction of the latter over a circle S^1 [93, 71, 40]. Using the conventions of [26], the reduction rules between ten and eleven dimensions are:

$$\begin{array}{ll} \hat{g}_{xx} = -e^{\frac{4}{3}\phi}, & \hat{C}_{\mu\nu x} = \frac{2}{3}B_{\mu\nu}, \\ \hat{g}_{\mu x} = -e^{\frac{4}{3}\phi}A^{(1)}_\mu, & \hat{C}_{\mu\nu\rho} = C_{\mu\nu\rho}, \\ \hat{g}_{\mu\nu} = e^{-\frac{2}{3}\phi}g_{\mu\nu} - e^{\frac{4}{3}\phi}A^{(1)}_\mu A^{(1)}_\nu . & \end{array} \tag{3.72}$$

We see that the ten-dimensional R-R vector $A^{(1)}_\mu$ is actually the Kaluza-Klein vector from the reduction and the Kaluza-Klein scalar, the measure of the compactification radius, is given by the ten-dimensional dilaton ϕ. But in ten dimensions the dilaton is associated with the coupling constant of the theory. We therefore see that the Type IIA (perturbation) theory is nothing other than an expansion around the zero-radius limit of eleven dimensions. On the other hand, in the strong coupling limit of Type IIA theory (thus for large values of the dilaton), an eleventh dimension unfolds, which previously in perturbation theory could not be seen [163]:

$$R_{11} \;=\; e^{\frac{2}{3}\phi} \;=\; g^{\frac{2}{3}}. \tag{3.73}$$

If the idea that Type IIA is really a dimensional reduction of something eleven dimensional holds also beyond the level of the low energy effective action, then this means that the (non-perturbative) spectrum of the Type IIA theory should contain all kinds of Kaluza-Klein modes coming from the wrapping of the eleven-dimensional solutions around the compact dimension. It was shown [163] that these modes would have masses inversely proportional to the coupling constant and therefore they could be identified with the Type IIA D-branes.

In fact the whole spectrum of Type IIA fundamental objects can be given an eleven-dimensional interpretation [156]: using the reduction rules (3.72), one sees that the Type IIA fundamental string can be understood as the eleven dimensional $M2$-brane wrapped around the compact dimension (double dimensional reduction), while the $D2$-brane is the direct reduction (reduction over a transverse direction) of the same $M2$-brane. The same goes for the $D4$ and the $S5$-brane in type IIA, which turn out to be

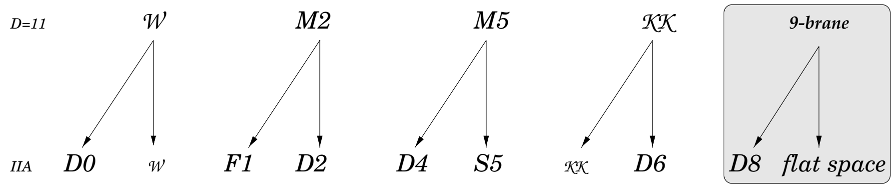

Figure 3.1: *The relation between $D = 10$ IIA and $D = 11$ solutions: Vertical lines imply direct dimensional reduction, diagonal lines double dimensional reduction. The shadowed area indicates the relationship between known ten-dimensional solutions and a conjectured 9-brane in $D = 11$.*

the double and direct reductions of the $M5$. The reduction of the eleven-dimensional gravitational wave $\mathcal{W}_{11}$ yields again a gravitational wave $\mathcal{W}_{10}$ in ten dimensions upon reduction over a transverse direction and a massive $D0$-brane if one reduces over the propagation direction of the wave. In the same way the Kaluza-Klein monopole $\mathcal{KK}_{11}$ in eleven dimensions gives rise to a ten dimensional monopole $\mathcal{KK}_{10}$ and a $D6$-brane upon reduction over a world-volume coordinate or the isometry direction z, respectively.

Only the interpretation of the Type IIA $D8$-brane is still mysterious: it is believed to be related to the equally mysterious eleven-dimensional 9-brane upon double reduction of the latter. Direct reduction of the 9-brane would give ten-dimensional Minkowski space[6]. In Figure 3.1 the relations between the various ten and eleven-dimensional solutions is summarized.

Also the strong coupling limit of Heterotic $E_8 \times E_8$ theory (2.36) is believed to be $D = 11$ supergravity [85], though this time the Heterotic theory turns out to be a compactification of $D = 11$ supergravity on a interval with length L, or equivalently on a circle sector $S^1/\mathbb{Z}_2$. The eleven-dimensional space-time consists of two nine-dimensional hyperplanes, separated by the interval of length L. On the two boundaries, gauge fields of E_8 live and in the limit $L \to 0$, a ten-dimensional theory with $E_8 \times E_8$ gauge symmetry is recovered. As in the case of Type IIA theory, the ten-dimensional coupling constant is related to the compact dimension by $L = g^{2/3}$.

If Type IIA supergravity (2.33) and Heterotic $E_8 \times E_8$ theory (2.36) are the weak coupling limits of $D = 11$ supergravity and the low energy limit of their respective string theories, we could ask the question: "What is the strong coupling limit of Type IIA (Het $E_8 \times E_8$) string theory?" or equivalently, "Of which theory is $D = 11$ supergravity the low energy limit?". This is conjectured to be M-theory, a non-perturbative, fundamental theory, which is believed to unify the various known string theories in one picture, although little more is known about it than that it has $D = 11$ supergravity as its low energy effective theory.

In the next section we will discuss the unifying picture and the role M-theory is believed to play.

[6]Comments on the conjectured 9-brane and the relation with the $D8$ have been given in [130, 22, 86, 125, 129, 61, 21].

3.3 General Picture

In the previous sections we have encountered two kinds of duality transformations: the T-duality which relates different string compactifications with each other and the S-duality that maps the strong coupling limit of a theory to the weak coupling limit of another (or, in the case of Type IIB and $D = 4,\ N = 4$ Heterotic theory the same) theory.

These duality symmetries shed new light on the problems that arose in string theory up to the beginning of the nineteen nineties:

- The wide variety of possible compactification manifolds and the different degenerate string vacua that follow from them. It is not clear which of all these vacua corresponds to our phenomenologically observable $D = 4$ world and why precisely this vacuum is the preferred one to be picked out.
- The difficulties to extend the known string theories beyond the perturbative level at which they are formulated. Little was known about a non-perturbative formulation or the basic dynamical principles that lie at the basis of string theory.
- The fact that five different versions exist of the theory which claims to be the "final" unification of gravity and all other fundamental interactions in Nature. It was believed (hoped) that sooner or later some of these theories would turn out to be inconsistent and/or equivalent to other ones, so that in the end one final version of string theory would be left over.

The surprising fact of the duality symmetries is that they were able to solve many (though certainly not all) of these problems, or at least to make some remarkable progress.

T-duality showed that different compactifications in string theory can be considered to be equivalent: upon dimensional reduction on a d dimensional torus T^d, for example, the T-duality group $O(d, d+n;\mathbb{Z})$ maps a given point in the moduli space (i.e., a given string vacuum) to a different point in moduli space with equivalent dynamics and equivalent physics as the first one. All vacua can thus be classified in T-duality classes and the moduli space of inequivalent compactifications is given by the coset

$$\mathcal{M} = \frac{O(d, d+n)}{O(d) \times O(d+n) \times O(d, d+n; \mathbb{Z})}\ . \tag{3.74}$$

Non-toroidal compactifications will give rise to other T-duality groups and other moduli spaces, but the main principles will be the same as in the easier case of toroidal compactification.

S-duality gives insight into the strong coupling regimes of theories: the S-duality group $SL(2;\mathbb{Z})$ is intrinsically a non-perturbative symmetry, since it acts non-trivially on the coupling constant of the theory. Under this symmetry the strong coupling regime of a theory gets mapped to the weak coupling regime of another theory and vice versa.

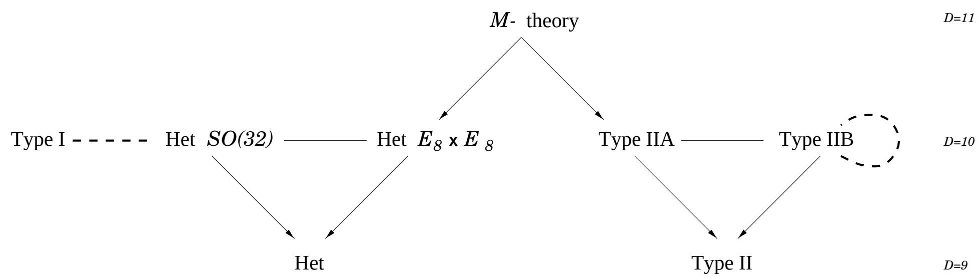

Figure 3.2: *Duality relations between the various string theories in ten dimensions and M-theory in eleven dimensions: the arrows indicate dimensional reduction from D to $D-1$ dimensions, the dotted lines represent an S-duality and a straight lines T-duality.*

Strong and weak coupling regimes therefore turn out to be different, but equivalent formulations of the same underlying theory. This yields a simple and elegant way to go beyond the level of perturbation theory: non-perturbative results in one theory can be computed in the other theory by simple perturbative calculations.

But perhaps the most striking issues of the concept of dualities is that the five, previously known string theories all turn out to be equivalent and in a rather surprising way interconnected via these dualities: T-duality relates the Type IIA and Type IIB theories in the presence of an isometry: one theory compactified on a circle of radius R gives exactly the same physics as the other theory compactified on a circle of radius $1/R$. The two theories are just different limits in moduli space of the same underlying theory. The same goes for Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$ theory. Furthermore the strong coupling limit of Heterotic $SO(32)$ coincides with the weak coupling limit of Type I strings (and vice versa), while the strong coupling of Type IIA and Heterotic $E_8 \times E_8$ both are conjectured to give a new theory, called M-theory, that has eleven dimensional supergravity as its low energy limit. Type IIB theory is believed to be S–self-dual, in the sense that its strong coupling limit is again the same Type IIB theory. A schematique picture of the relations between these theories can be seen in Figure 3.2.

The relations between the various string theories also imply connections between the solutions of their low energy effective actions: in Figure 3.1 we already showed how the Type IIA solutions were connected to the solutions of $D = 11$ supergravity, but also within ten dimensions the various solutions are related via dualities: T-duality connects all D-branes of Type IIA and Type IIB with each other, the wave with the fundamental string and the solitonic five-brane with the Kaluza-Klein monopole. S-duality connects the $F1$ with the $D1$ and the $S5$ with the $D5$ of theories that are each other's S-dual. Furthermore Poincaré duality (2.56) relates p-branes with a $(6-p)$-brane, i.e. the Dp-brane with the $D(6-p)$-brane and the $F1$ and the $S5$. These relations can be seen in Figure 3.3.

The fact that all these theories are related has led to the idea that they are not the fundamental theories we are looking for, but that all five string theories and $D = 11$ supergravity are different limits of one and the same underlying theory, called M-theory.

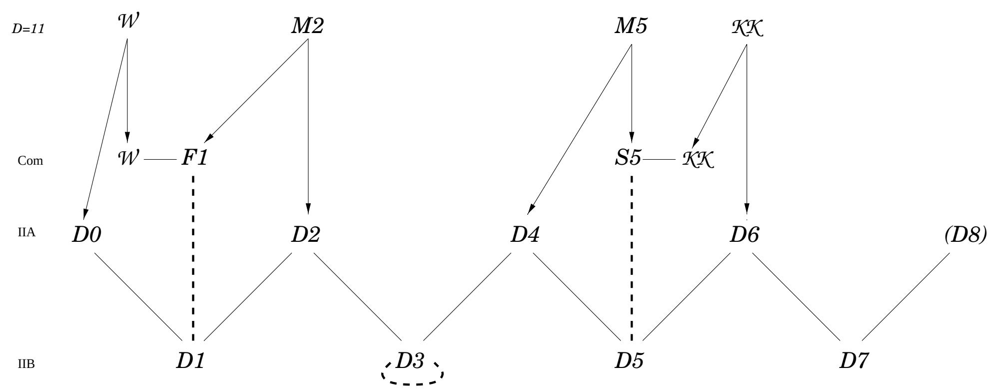

Figure 3.3: *Duality relations between the different solutions of string theory in ten dimensions and M-theory in eleven dimensions: the arrows indicate dimensional reduction from D to $D-1$ dimensions, the dotted lines represent an S-duality and a straight lines T-duality*

The different string theories and eleven dimensional supergravity can then be thought off as different perturbation expansions in different points of the moduli space of M-theory, characterized by the value of the coupling constant and the size of the compact dimensions. A full picture of what M-theory itself looks like is not yet known, though serious attempts are being made using techniques of Matrix-theory [12]. It is believed to have membrane and five-brane solutions, to be non-perturbative,... In fact, the idea of M-theory being the fundamental, underlying theory even has brought the name string theory in question, since strings no longer play a preferred role in this picture.

In the following Chapters we will apply the techniques of duality symmetries and duality transformations on the various aspects of string theory and supergravity: in Chapter 4 we will study the symmetries of the target space actions of string theories in more detail and find duality relations between them in dimensions lower than ten. In Chapter 5 we will look at the solutions of the supergravity actions and in particular the bound states they can form, and in Chapter 6 we study the duality transformations between the effective actions of the solutions, finding that also these are related in the same way as the solutions themselves.

Chapter 4

Target Space Actions

In Section 3.1 we already discussed briefly the global symmetries of the low energy effective actions of the common sector, the Heterotic and Type IIA/B theories. In this chapter we will study in more detail these discrete and non-compact symmetries. We will look in Section 4.1 at the symmetries of theories after compactification over one dimension, since here most of the properties of (toroidally) compactified theories are present in a simple form. In Section 4.2 we will find the same properties in the duality relations between the six-dimensional Heterotic, Type IIA and Type IIB theory at the level of the target space actions.
This chapter contains results presented in [27] and [14].

4.1 Duality Symmetries in Ten and Nine Dimensions

In this section we look in detail to the symmetry properties that arise from the dimensional reduction of the low energy effective string action from ten to nine dimensions. First we discuss the symmetry group of the common sector and explain the origin of the different symmetry transformations. Then we look at how the group structure gets enhanced, c.q. broken, in the presence of (Abelian) vector fields in the case of the Heterotic string or R-R fields in the case of the Type IIA/B theory.

4.1.1 Symmetries of the Common Sector

As discussed in Section 3.1, dimensional reduction over a circle already shows many of the interesting features of toroidal compactification. Let us therefore look in more detail at the dimensionally reduced $D = 9$ common sector action (3.28):

$$S^{(9)} = \tfrac{1}{2}\int d^9x\ \sqrt{|g|}\ e^{-2\phi}\Big[-R\ +\ 4(\partial\phi)^2 -\ \tfrac{3}{4}H^2 -\ (\partial\log k)^2$$

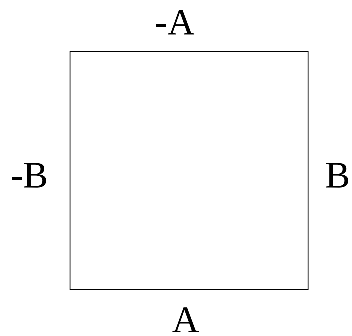

Figure 4.1: *Each discrete symmetry of the square corresponds to a symmetry acting on the two vectors A and B. The four sides of the square correspond to the pairs $(A, -A)$ and $(B, -B)$.*

$$
\begin{aligned}
& + \tfrac{1}{4}k^2 F^2(A) \ + \ \tfrac{1}{4}k^{-2}F^2(B)\Big] \ , \qquad (4.1)\\
H_{\mu\nu\rho} &= \partial_{[\mu}B_{\nu\rho]} \ + \ \tfrac{1}{2}A_{[\mu}F_{\nu\rho]}(B) \ + \ \tfrac{1}{2}B_{[\mu}F_{\nu\rho]}(A).
\end{aligned}
$$

As mentioned earlier, this action has a manifest global $O(1,1)$-symmetry, coming from the dimensional reduction, which decomposes in a subgroup of proper $O(1,1)$ transformation and a mapping class group:

$$O(1,1) = SO^{\uparrow}(1,1)_x \times \mathbb{Z}_2^{(S)} \times \mathbb{Z}_2^{(T)}. \qquad (4.2)$$

The different subgroups act on the Kaluza-Klein scalar k and the vector fields A_μ and B_μ as

$$
\begin{array}{llll}
SO^{\uparrow}(1,1)_x : & k \to \Lambda^{-1}k, & A_\mu \to \Lambda \ A_\mu, & B_\mu \to \Lambda^{-1}B_\mu, \\
\mathbb{Z}_2^{(S)} : & k \to k & A_\mu \to - \ A_\mu, & B_\mu \to - \ B_\mu, \\
\mathbb{Z}_2^{(T)} : & k \to k^{-1}. & A_\mu \to B_\mu, & B_\mu \to A_\mu
\end{array} \qquad (4.3)
$$

The $SO^{\uparrow}(1,1)_x \times \mathbb{Z}_2^{(S)}$-symmetry comes from the fact that the action (4.1) was obtained via a dimensional reduction over x of the ten-dimensional action (3.20), and is therefore invariant under reflections and rescalings in the compact x-direction:

$$
\begin{array}{ll}
\mathbb{Z}_2^{(S)} : & x' = -x, \\
SO^{\uparrow}(1,1)_x : & x' = \Lambda \ x.
\end{array} \qquad (4.4)
$$

It is not difficult to see that these ten-dimensional transformations act on the nine-dimensional fields as in (4.3).

The appearance of the $\mathbb{Z}_2^{(T)}$, the T-duality transformation (3.4), cannot be explained from the point of view of dimensional reduction, and is difficult to interpret, ignoring the stringy character of the action (4.1). We refer to the discussion in Section 3.1.

In addition to the two discrete groups given above, there exists yet another $\mathbb{Z}_2$ transformation, which we call $\mathbb{Z}_2^{(A)}$ because of the fact that it acts on the axion and the winding

Name	$g_{\mu\nu}$	$B_{\mu\nu}$	A_μ	B_μ	e^ϕ	k	$S^{(9)}$
$SO^\uparrow(1,1)_x$	0	0	1	-1	0	-1	0
$SO^\uparrow(1,1)_y$	-1	-1	0	-1	$-\frac{5}{4}$	$\frac{1}{2}$	-1
$\mathbb{R}_\phi$	0	0	0	0	1	0	-2

Table 4.1: *Weights of the common sector fields under the two $SO^\uparrow(1,1)$ symmetries of the action $S^{(9)}$ and the $\mathbb{R}_\phi$ which scales the action in nine dimensions.*

vector:

$$\mathbb{Z}_2^{(A)}: \qquad B'_{\mu\nu} = -B_{\mu\nu}, \qquad B'_\mu = -B_\mu. \tag{4.5}$$

This $\mathbb{Z}_2^{(A)}$ does not commute with $\mathbb{Z}_2^{(T)}$ and therefore the three $\mathbb{Z}_2$'s together combine into the non-Abelian dihedral group D_4, the group of symmetry transformations of a square with undirected sides: every D_4-transformation on the vectors A_μ and B_μ corresponds to a transformation that leaves a square with sides $(A,-A)$ and $(B,-B)$ invariant (see Figure 4.1). The only D_4-transformation that acts non-trivially on the scalar k is $\mathbb{Z}_2^{(T)}$.

Furthermore there are two more non-compact symmetries, $SO^\uparrow(1,1)_y$ and $\mathbb{R}_\phi$[1], which scale the action but leave the equations of motion invariant. Their interpretation will become clear later on in this section, in the context of the symmetries of the Type II theory. The weights of the fields under the various scale transformations is given in Table 4.1. The full group of symmetries the equations of motion is then given by

$$SO^\uparrow(1,1)_x \times\ SO^\uparrow(1,1)_y \times\ \mathbb{R}_\phi \times\ D_4\,. \tag{4.6}$$

In the presence of fields that do not belong to the common sector, such as vector fields in the case of the Heterotic theory, or R-R fields in the case of Type IIA/B theory, part of the symmetry gets broken. How much the symmetry gets broken depends on the situation. Let us therefore discuss each of the two cases separately.

4.1.2 Symmetries of the Heterotic Theory

In the presence of an (Abelian) vector field $\hat{V}_\mu$, the situation changes in two ways: the extra Chern-Simons term in the axion field strength tensor will break the $\mathbb{Z}_2^{(A)}$ symmetry (and thus the D_4), while on the other hand the Abelian vector field combines with the A_μ and B_μ into the bigger reduction group $SO^\uparrow(1,2)_x$.

[1]With $\mathbb{R}$ we denote the additive group of real numbers, which is isomorphic to $SO^\uparrow(1,1)$. However we reserve the notation $SO^\uparrow(1,1)$ for groups that can combine with their mapping class group into a full $O(1,1)$.

We start our analysis from the action of the ten-dimensional Heterotic string in the presence of one Abelian vector field:

$$S = \tfrac{1}{2}\int d^{10}x\sqrt{|\hat{g}|}\; e^{-2\hat{\phi}}\left[-\hat{R} + 4(\partial\hat{\phi})^2 - \tfrac{3}{4}\hat{H}^2 + \tfrac{1}{4}\hat{F}_{\hat{\mu}\hat{\nu}}(\hat{V})\hat{F}^{\hat{\mu}\hat{\nu}}(\hat{V})\right]\,, \tag{4.7}$$

where the three-form field strength is of the form

$$\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} = \partial_{[\hat{\mu}}\hat{B}_{\hat{\nu}\hat{\rho}]} - \tfrac{1}{2}\hat{V}_{[\hat{\mu}}\hat{F}_{\hat{\nu}\hat{\rho}]}(\hat{V})\,. \tag{4.8}$$

In principle, there is an ambiguity in the relative sign between $\partial\hat{B}$ and the Yang–Mills Chern–Simons term $\hat{V}\hat{F}(\hat{V})$. In fact, there are two theories whose only difference is this relative sign and which are related by the change of sign of $\hat{B}_{\hat{\mu}\hat{\nu}}$ ($\mathbb{Z}_2^{(A)}$), which is no longer a symmetry of each separate theory. Therefore, the group D_4 is broken to $\mathbb{Z}_2^{(T)} \times\ \mathbb{Z}_2^{(S)}$ in each theory. In fact $\mathbb{Z}_2^{(A)}$ is a duality transformation that brings us from one theory to the other, exactly as happens in the Type II duality (3.46) (see also [26]). From the sigma-model point of view, these theories are related by a change of the sign of $\hat{B}_{\hat{\mu}\hat{\nu}}$ and the simultaneous interchange of left- and right-movers. For the sake of definiteness, we will work with the above choice of relative sign.

Following the standard rules for dimensional reduction in the presence of vector fields [115]

$$\begin{array}{rclrcl}
\hat{g}_{\mu\nu} & = & g_{\mu\nu} - k^2 A_\mu A_\nu\,, & \hat{B}_{\mu\nu} & = & B_{\mu\nu} + A_{[\mu}B_{\nu]} + \ell A_{[\mu}V_{\nu]}\,,\\
\hat{g}_{x\mu} & = & -k^2 A_\mu\,, & \hat{B}_{x\mu} & = & B_\mu + \tfrac{1}{2}\,\ell\, V_\mu\,,\\
\hat{g}_{xx} & = & -k^2\,, & \hat{\phi} & = & \phi + \tfrac{1}{2}\log k\,,\\
\hat{V}_x & = & \ell\,, & \hat{V}_\mu & = & V_\mu +\ \ell A_\mu\,,
\end{array} \tag{4.9}$$

we obtain the nine-dimensional action, which is the generalisation of (4.1):

$$\begin{aligned}
S = \tfrac{1}{2}\int d^9x\ \sqrt{|g|}\ e^{-2\phi}\ \Big\{ & -R + 4(\partial\phi)^2 - \tfrac{3}{4}H^2 - \left[(\partial\log k)^2 + \frac{1}{2k^2}(\partial\ell)^2\right]\\
& + \tfrac{1}{4}\left[\frac{(2k^2+\ell^2)^2}{4k^2}F^2(A) + k^{-2}F^2(B) + \frac{\ell^2}{k^2}F(A)F(B)\right]\\
& + F(V)\left[\left(\frac{2k^2\ell+\ell^3}{4k^2}\right)F(A) + \frac{\ell}{2k^2}F(B)\right]\\
& + \tfrac{1}{4}\left(\frac{k^2+\ell^2}{k^2}\right)\ F^2(V)\Big\}\,.
\end{aligned} \tag{4.10}$$

This can be written in a manifestly $O(1,2)$ invariant notation (3.37) [115]:

$$\begin{aligned}
S\ =\ & \tfrac{1}{2}\int d^9x\ \sqrt{|g|}\ e^{-2\phi}\Big\{-R + 4(\partial\phi)^2 - \tfrac{3}{4}H^2\\
& + \tfrac{1}{8}\mathrm{Tr}\ (\partial_\mu M^{-1}\partial^\mu M) - \tfrac{1}{4}\mathcal{F}^i_{\mu\nu}(\mathcal{A})M_{ij}\mathcal{F}^{\mu\nu j}(\mathcal{A})\Big\}\,,
\end{aligned} \tag{4.11}$$

Name	$g_{\mu\nu}$	$B_{\mu\nu}$	A_μ	B_μ	e^ϕ	k	ℓ	V_μ	$S^{(9)}$
$SO^\uparrow(1,1)_y$	-1	-1	0	-1	$-\frac{7}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0
$\mathbb{R}_\phi$	0	0	0	0	1	0	0	0	-2
$SO^\uparrow(1,1)_\alpha$	0	0	1	-1	0	-1	-1	0	0

Table 4.2: *Weights of the fields under the two $SO^\uparrow(1,1)$ duality symmetries of the action of the nine-dimensional Heterotic string and the $\mathbb{R}_\phi$ which scales it.*

where $H_{\mu\nu\rho}$, $\mathcal{F}^i_{\mu\nu}$ and $\mathcal{A}^i_\mu$ are as in (3.38) and M^{-1} is the $O(1,2)$-matrix

$$M^{-1}_{ij} = \begin{pmatrix} -(2k^2+\ell^2)^2/4k^2 & -\ell^2/2k^2 & -(2k^2\ell+\ell^3)/2k^2 \\ -\ell^2/2k^2 & -1/k^2 & -\ell/k^2 \\ -(2k^2\ell+\ell^3)/2k^2 & -\ell/k^2 & -(k^2+\ell^2)/k^2 \end{pmatrix} . \tag{4.12}$$

Let us now analyse in detail the different symmetries of this theory. First of all, the $SO^\uparrow(1,1)_y \times \mathbb{R}_\phi$ of (4.6) can be extended straightforwardly to the action (4.10). The weights of the fields are given in Table 4.2.

The dihedral group D_4 gets broken to the mapping class group of $O(1,2)$, namely $\mathbb{Z}_2^{(S)} \times \mathbb{Z}_2^{(T)}$, which now, due to the presence of the vector field is of the form:

$$\begin{array}{lcl} \mathbb{Z}_2^{(S)} & : & \left\{ \begin{array}{ll} A'_\mu = -A_\mu\,, & B'_\mu = -B_\mu\,, \\ (k^2)' = (k^2), & \ell' = -\ell\,, \end{array} \right. \\ \mathbb{Z}_2^{(T)} & : & \left\{ \begin{array}{ll} \tilde{A}_\mu = B_\mu\,, & \tilde{B}_\mu = A_\mu\,, \\ \tilde{k}^2 = \frac{4k^2}{(\ell^2+2k^2)^2}\,, & \tilde{\ell} = \frac{2\ell}{\ell^2+2k^2}\,. \end{array} \right. \end{array} \tag{4.13}$$

The interpretation of these $\mathbb{Z}_2$ transformations is the same as in (4.3): $\mathbb{Z}_2^{(S)}$ corresponds to a change of sign of the compact direction and $\mathbb{Z}_2^{(T)}$ corresponds to the T-duality transformations, which now in ten dimensions appear as a generalization of (3.4):

$$\begin{array}{rcl} \tilde{\hat{g}}_{\mu\nu} & = & \hat{g}_{\mu\nu} + \left(\hat{g}_{xx}\hat{G}_{x\mu}\hat{G}_{x\nu} - 2\hat{G}_{xx}\hat{G}_{x(\mu}\hat{g}_{\nu)x}\right)/\hat{G}^2_{xx}\,, \\ \tilde{\hat{B}}_{\hat{\mu}\hat{\nu}} & = & \hat{B}_{\hat{\mu}\hat{\nu}} - \hat{G}_{x[\mu}\hat{G}_{\nu]x}/\hat{G}_{xx}\,, \\ \tilde{\hat{g}}_{x\mu} & = & (-\hat{g}_{x\mu}\hat{G}_{xx} + \hat{g}_{xx}\hat{G}_{x\mu})/\hat{G}^2_{xx}\,, \\ \tilde{\hat{B}}_{x\mu} & = & (\hat{G}_{x\mu} - \hat{B}_{x\mu})/\hat{G}_{xx}\,, \end{array}$$

$$
\begin{aligned}
\tilde{\hat{g}}_{xx} &= \hat{g}_{xx}/\hat{G}_{xx}^2\,, \\
\tilde{\hat{\phi}} &= \hat{\phi} - \tfrac{1}{2}\log|\hat{G}_{xx}|\,, \\
\tilde{\hat{V}}_x &= -\hat{V}_x/\hat{G}_{xx}\,, \\
\tilde{\hat{V}}_\mu &= \hat{V}_\mu - \hat{V}_x\hat{G}_{x\mu}/\hat{G}_{xx}\,,
\end{aligned}
\tag{4.14}
$$

where $\hat{G}_{\hat{\mu}\hat{\nu}}$ is an "effective metric"

$$
\hat{G}_{\hat{\mu}\hat{\nu}} = \hat{g}_{\hat{\mu}\hat{\nu}} + \hat{B}_{\hat{\mu}\hat{\nu}} - \tfrac{1}{2}\hat{V}_{\hat{\mu}}\hat{V}_{\hat{\nu}}\,, \tag{4.15}
$$

which transforms under $\mathbb{Z}_2^{(T)}$ in the following particularly simple form:

$$
\begin{array}{ll}
\tilde{\hat{G}}_{\mu\nu} = \hat{G}_{\mu\nu} - \hat{G}_{x\mu}\hat{G}_{\nu x}/\hat{G}_{xx}\,, & \tilde{\hat{G}}_{xx} = 1/\hat{G}_{xx}\,, \\
\tilde{\hat{G}}_{x\mu} = \hat{G}_{x\mu}/\hat{G}_{xx}\,, & \tilde{\hat{G}}_{\mu x} = -\hat{G}_{\mu x}/\hat{G}_{xx}\,.
\end{array}
\tag{4.16}
$$

Note that for $\hat{V}_{\hat{\mu}} = V_\mu = \ell = 0$, (4.13) and (4.14) reduce to the known T-duality transformations (3.4) and (4.3).

We next consider the continuous $SO^\uparrow(1,2)_x$ transformations. It is convenient to first consider the $so(1,2)$ Lie algebra with generators $J_3,\ J_+$ and J_-:

$$
[J_3, J_+] = J_+\,, \qquad [J_3, J_-] = -J_-\,, \qquad [J_+, J_-] = J_3\,. \tag{4.17}
$$

The generators $J_3,\ J_+$ and J_- can be represented by 3×3 matrices

$$
J_+ = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

The exponentiation of $J_3,\ J_+$ and J_- leads to the following $SO^\uparrow(1,2)$ group elements:

$$
\begin{aligned}
\exp\alpha J_3 &= \begin{pmatrix} e^{\alpha} & 0 & 0 \\ 0 & e^{-\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
\exp\beta J_- &= \begin{pmatrix} 1 & 0 & 0 \\ \tfrac{1}{2}\beta^2 & 1 & -\beta \\ -\beta & 0 & 1 \end{pmatrix}, \\
\exp\gamma J_+ &= \begin{pmatrix} 1 & \tfrac{1}{2}\gamma^2 & -\gamma \\ 0 & 1 & 0 \\ 0 & -\gamma & 1 \end{pmatrix}.
\end{aligned}
\tag{4.18}
$$

An arbitrary $SO^\uparrow(1,2)$ group element Ω can be written as the product of these basis elements. Using the fact that the vectors $A_\mu,\ B_\mu$ and V_μ transform in the fundamental representation of $SO^\uparrow(1,2)$ and the scalars as $(M^{-1})' = \Omega M^{-1}\Omega^T$ (see (3.39)), one can verify that the transformations in the basis above induce three transformations in nine dimensions.

First of all, the transformation generated by J_3 in nine dimensions is just the scale transformation $SO^\uparrow(1,1)_x$ of previous sections and corresponds to a general coordinate transformation (g.c.t.) $x \to e^\alpha x$. The weights of the various fields under this transformation are given in Table 4.2.

We next consider the transformation generated by J_-. The nine-dimensional rules are given by

$$
\begin{aligned}
&A'_\mu = A_\mu\,, && (k^2)' = k^2\,,\\
&B'_\mu = B_\mu - \beta V_\mu + \tfrac{1}{2}\beta^2 A_\mu\,, && \ell' = \ell + \beta\,,\\
&V'_\mu = V_\mu - \beta A_\mu\,.
\end{aligned}
\tag{4.19}
$$

The corresponding transformation of the ten-dimensional fields is

$$
\begin{aligned}
\hat{V}'_x &= \hat{V}_x + \beta\,,\\
\hat{B}'_{x\mu} &= \hat{B}_{x\mu} - \tfrac{1}{2}\beta\hat{V}_\mu\,.
\end{aligned}
\tag{4.20}
$$

All other fields are invariant. It turns out that this transformation is a particular finite $U(1)$ gauge transformation, under which also the axion transforms (cfr: (3.35)):

$$
\begin{aligned}
\hat{V}'_{\hat\mu} &= \hat{V}_{\hat\mu} + \partial_{\hat\mu}\Lambda\,,\\
\hat{B}'_{\hat\mu\hat\nu} &= \hat{B}_{\hat\mu\hat\nu} + \hat{V}_{[\hat\mu}\partial_{\hat\nu]}\Lambda\,,
\end{aligned}
\tag{4.21}
$$

with the parameter Λ given by $\Lambda = \beta x$.

Finally, we consider the transformation generated by J_+. The transformation rules in nine dimensions are given by:

$$
\begin{aligned}
A'_\mu &= A_\mu + \tfrac{1}{2}\gamma^2 B_\mu - \gamma V_\mu\,, & (k^2)' &= \left(\frac{4k}{4+4\gamma\ell+(\ell^2+2k^2)\gamma^2}\right)^2\,,\\
B'_\mu &= B_\mu\,, & \ell' &= \frac{4\ell+2(\ell^2+2k^2)\gamma}{4+4\gamma\ell+(\ell^2+2k^2)\gamma^2}\,,\\
V'_\mu &= -\gamma B_\mu + V_\mu\,.
\end{aligned}
\tag{4.22}
$$

For the (complicated) expression for these transformations in ten dimensions we refer to [27]. This transformation cannot be interpreted as a g.c.t. or a gauge transformation in ten dimensions. Together with the T-duality transformation $\mathbb{Z}_2^{(T)}$, it forms the $O(1,2)$-subgroup $\mathbb{Z}_2^{(T)} \times O(2)$ of solution generating transformations [144, 83]. Note that the subgroup $\mathbb{Z}_2^{(T)} \times O(2) = O(1) \times O(2)$ corresponds exactly to the subgroup that is factored out in the Narain coset $O(1,2)/(O(1)\times O(2))$ of inequivalent compactifications [119] we encountered in Section 3.1. Therefore, the $\mathbb{Z}_2^{(T)} \times O(2)$ transformations parametrise the elements within each class of equivalent compactifications of (4.7). Acting with these transformations on a given solution, generates all other solution within the same equivalence class.

More generally, one can show [144, 83] that for a dimensionally reduced action with an $O(d,d+n)$ symmetry, the transformations belonging to the $O(d)\times O(d+n)$ subgroup are non-trivial solution generating transformations, while the coset $O(d,d+n)/(O(d)\times O(d+n))$ corresponds to the coset of gauge transformations.

Name	$C_{\mu\nu\rho}$	$g_{\mu\nu}$	$B^{(1)}_{\mu\nu}$	$B^{(2)}_{\mu\nu}$	$A^{(1)}_{\mu}$	A_μ	B_μ	k	ℓ	e^ϕ	S
$\mathbb{R}_{\text{brane}}$	1	1	1	1	0	0	1	$\frac{1}{2}$	0	$\frac{1}{4}$	3
$SO^\uparrow(1,1)_{x-y}$	0	1	1	-1	-1	1	0	$-\frac{1}{2}$	-2	$\frac{7}{4}$	0
$SO^\uparrow(1,1)_{x+y}$	0	1	1	1	-1	-1	2	$\frac{3}{2}$	0	$\frac{3}{4}$	2
$\mathbb{Z}_2^{(S)}$	$-$	$+$	$-$	$+$	$+$	$-$	$+$	$+$	$-$	$+$	$+$

Table 4.3: *Weights of the $D = 9$ Type II supergravity fields and action under $SL(2,\mathbb{R}) \times SO^\uparrow(1,1)_{x+y} \times\ \mathbb{R}_{\text{brane}} \times\ \mathbb{Z}_2^{(S)}$.*

4.1.3 Symmetries of Type IIA/B

As we have seen in section 3.1, Type IIA and Type IIB theory in the presence of an isometry are related via the Type II T-duality rules (3.46) [26]. Therefore they also have the same symmetry group [27]:

$$SL(2;\mathbb{R}) \times SO^\uparrow(1,1)_{x+y} \times \mathbb{R}_{\text{brane}} \times \mathbb{Z}_2^{(S)}. \tag{4.23}$$

The $SL(2,\mathbb{R})$ group is a symmetry of the action. From the Type IIB point of view it is the manifest $SL(2,\mathbb{R})$ symmetry (3.69)-(3.70) of the original theory [92, 26, 17], while from the point of view of the Type IIA it is a part of the symmetry group coming from the dimensional reduction of the eleven dimensional supergravity theory: the group of two-dimensional general coordinate transformations $GL(2,\mathbb{R}) = SL(2,\mathbb{R}) \times SO^\uparrow(1,1) \times \mathbb{Z}_2$.[2]

The $SL(2,\mathbb{R})$ contains one particular subgroup of scalings: $SO^\uparrow(1,1)_{x-y}$, corresponding to the eleven-dimensional g.c.t. $x \to e^\alpha x\,, y \to e^{-\alpha} y$. This is of course the particular combination of the scaling symmetries $SO^\uparrow(1,1)_x$ and $SO^\uparrow(1,1)_y$ of the previous sections. Another (linearly independent) combination is the $SO^\uparrow(1,1)_{x+y}$, which scales the fields and the action and corresponds to the eleven-dimensional g.c.t. $x \to e^\alpha x\,, y \to e^\alpha y$. The $\mathbb{R}_{\text{brane}}$ is a symmetry that can already be found back in eleven dimensions and that scales the action, giving each field a weight according to its mass dimension [27]. Finally, $\mathbb{Z}_2^{(S)}$ corresponds to improper g.c.t.s in the internal space, for instance $x \to -x$ (up to $SL(2,\mathbb{R})$ rotations) [85]. The weights of the different nine-dimensional fields are summarized in Table 4.3.

The discrete $\mathbb{Z}_2^{(S)}$ is the only part that remains from the dihedral group D_4 in (4.6). The $\mathbb{Z}_2^{(A)}$-symmetry is broken by the topological term in (3.42), and the $\mathbb{Z}_2^{(T)}$ is the Type II T-duality (3.46), which is not a symmetry of the nine-dimensional Type II

[2] Schwarz [140] and Aspinwall [7] interpreted the presence of the $SL(2,\mathbb{Z})$ symmetry of the Type IIB string as the $SL(2,\mathbb{Z})$ modular invariance of the torus on which the $D = 11$ supergravity is compactified in order to relate this to the Type IIB theory.

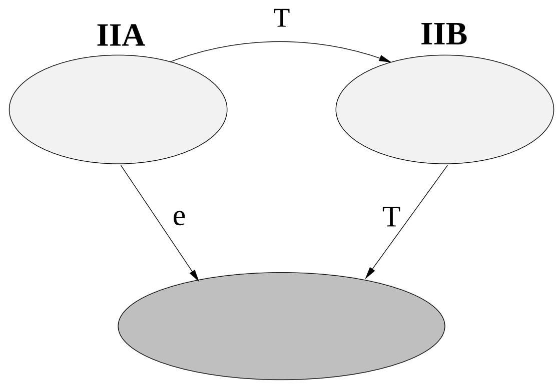

Figure 4.2: *The Type II T-duality in ten dimensions describes a map between the Type IIA and Type IIB theory. The reduction to $D = 9$ of the Type IIA (Type IIB) is indicated with* **e** (**T**)

action (3.42), but relates the ten-dimensional Type IIA and Type IIB with each other. Instead of being a symmetry of a single theory, it is a map between two different theories, which can be constructed relating the two different reduction schemes (3.43) and (3.44) to each other (see Figure 4.2)

We will call these two reduction schemes **e** and **T** respectively, the reason for this being that the reduction scheme **T** is the T-dual formulation (3.31) of the reduction scheme **e**, when restricted to the common sector[3].

An advantage of this notation is that one can easily see the $\mathbb{Z}_2$ group structure:

$$\begin{aligned} T(IIB \to IIA) \times T(IIA \to IIB) &= \mathbb{1}(IIA \to IIA)\,, \\ T(IIA \to IIB) \times T(IIB \to IIA) &= \mathbb{1}(IIB \to IIB)\,. \end{aligned} \tag{4.24}$$

This is due to our notation of the reduction formulae, which is such that, when restricted to the common sector, each reduction scheme (and its inverse) is in one-to-one correspondence with a specific $\mathbb{Z}_2$–symmetry of the action (4.1).

The above analysis can also be repeated for the more complicated case of $D = 5, 6$. The six-dimensional Type IIA/B theories compactified on $K3$ are in the same way T-dual to each other upon reduction to five dimensions. Furthermore they can be related to the Heterotic theory compactified on a four-torus T^4, which will give rise to bigger discrete duality groups.

[3]Upon truncation of the R-R fields, the Type II theories reduce to the common sector (4.1) and $\mathbb{Z}_2^{(T)}$ becomes a symmetry of the action.

4.2 Duality Symmetries in Six and Five Dimensions

In this section we will discuss the duality symmetries between the Heterotic, Type IIA and Type IIB theory in six and five dimensions. We will see that all three are related to each other via a string/string/string triality structure [62, 100]. Just as in the previous section we will start with the symmetries of the common sector, then present the six-dimensional form of each of the theories and reduce them to the same five-dimensional theory. In the end we will construct duality maps between the various theories.

4.2.1 The Common Sector

The common sector of the Heterotic, Type IIA and Type IIB theory in six dimensions is given by

$$S^{(6)} = \tfrac{1}{2}\int d^6x\sqrt{|\hat{g}|}\; e^{-2\hat{\phi}}\Big[-\hat{R} + 4(\partial\hat{\phi})^2 - \tfrac{3}{4}\hat{H}^2\Big]\,. \tag{4.25}$$

The special thing about six dimensions is that the equations of motion corresponding to the common sector are invariant under so-called string/string duality transformations [60, 61, 163]. These transformations are easiest formulated in the (6-dimensional) Einstein-frame metric

$$\hat{g}^{\mathrm{E}}_{\mu\nu} = e^{-\hat{\phi}}\hat{g}_{\mu\nu}\,, \tag{4.26}$$

which is invariant under the string/string duality transformations. The action for the common sector in the Einstein-frame metric is given by:

$$S^{(6)} = \tfrac{1}{2}\int d^6x\sqrt{|\hat{g}^{\mathrm{E}}|}\;\Big[-\hat{R} - (\partial\hat{\phi})^2 - \tfrac{3}{4}e^{-2\hat{\phi}}\hat{H}^2\Big]\,. \tag{4.27}$$

It is not difficult to see that the equations of motion of the above action are invariant under:

$$\hat{\phi}' = -\hat{\phi}\,, \qquad\qquad \hat{H}' = e^{-2\hat{\phi}}\;{}^*\hat{H}\,, \tag{4.28}$$

where ${}^*\hat{H}$ is the Poincaré dual (2.56) of the axion field strength $\hat{H}$:

$${}^*\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} \equiv \frac{1}{3!\sqrt{|\hat{g}|}}\;\varepsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}\hat{\sigma}\hat{\tau}}\hat{H}^{\hat{\lambda}\hat{\sigma}\hat{\tau}}\,. \tag{4.29}$$

This string/string duality is the six-dimensional analogue of the strong/weak coupling duality (3.64), or equivalently the string/five-brane duality in ten dimensions. It states that in the strong coupling limit of the six-dimensional common sector the fundamental string gets related to the solitonic string, the direct reduction of the solitonic five-brane.

We now discuss the reduction to five dimensions, assuming there is an isometry in the x-direction. Using both in $D=6$ as well as $D=5$ the string–frame metric, the 6-dimensional fields are expressed in terms of the five-dimensional ones as follows:

$$\begin{aligned}
\hat{g}_{xx} &= -e^{-4\sigma/\sqrt{3}}\,,\\
\hat{g}_{x\mu} &= -e^{-4\sigma/\sqrt{3}}A_\mu\,,
\end{aligned}$$

$$
\begin{aligned}
\hat{g}_{\mu\nu} &= g_{\mu\nu} - e^{-4\sigma/\sqrt{3}} A_\mu A_\nu \,, \\
\hat{B}_{\mu\nu} &= B_{\mu\nu} + A_{[\mu} B_{\nu]} \,, \\
\hat{B}_{x\mu} &= B_\mu \,, \\
\hat{\phi} &= \phi - \tfrac{1}{\sqrt{3}} \sigma \,.
\end{aligned} \tag{4.30}
$$

Note that for later convenience we have renamed the Kaluza-Klein scalar $k^2 = e^{-4\sigma/\sqrt{3}}$. The reduced action in the (five-dimensional) string frame metric is given by

$$
\begin{aligned}
S^{(5)} = \tfrac{1}{2} \int d^5x \sqrt{|g|}\; e^{-2\phi} \Big[&-R + 4(\partial\phi)^2 - \tfrac{3}{4} H^2 - \tfrac{4}{3}(\partial\sigma)^2 \\
&+ e^{-4\sigma/\sqrt{3}} F(A)^2 + e^{4\sigma/\sqrt{3}} F(B)^2 \Big] \,,
\end{aligned} \tag{4.31}
$$

with $H_{\mu\nu\rho}$ as in (4.1).

We next use the fact that five dimensions is special in the sense that in this dimension the antisymmetric tensor $B_{\mu\nu}$ is Poincaré dual to a vector C_μ [163, 164]:

$$
H_{\mu\nu\rho} \equiv \frac{1}{3\sqrt{|g|}}\; e^{2\phi}\; \varepsilon_{\mu\nu\rho\lambda\sigma} F(C)^{\sigma\tau} \tag{4.32}
$$

In terms of this vector C_μ the action is given by:

$$
\begin{aligned}
S^{(5)} = \tfrac{1}{2} \int d^5x \sqrt{|g|}\; e^{-2\phi}\; \Big[&-R + 4(\partial\phi)^2 - \tfrac{4}{3}(\partial\sigma)^2 + e^{4\phi} F(C)^2 \\
&+ e^{-4\sigma/\sqrt{3}} F(A)^2 + e^{4\sigma/\sqrt{3}} F(B)^2 \Big] \\
&- \tfrac{1}{2} \int d^5x\; \varepsilon_{(5)}\; A\; \partial B\; \partial C \,.
\end{aligned} \tag{4.33}
$$

To study the symmetries of the dimensionally reduced action it is convenient to use the (five-dimensional) Einstein frame metric

$$
g^{\mathrm{E}}_{\mu\nu} = e^{-4\phi/3} g_{\mu\nu} \,, \tag{4.34}
$$

so that the action becomes

$$
\begin{aligned}
S^{(5)} = \tfrac{1}{2} \int d^5x \sqrt{|g^{\mathrm{E}}|} \Big[&-R - \tfrac{4}{3}(\partial\phi)^2 - \tfrac{4}{3}(\partial\sigma)^2 + e^{-4\vec{Q}_C \cdot \vec{\Phi}/3} F(C)^2 \\
&+ e^{-4\vec{Q}_A \cdot \vec{\Phi}/3} F(A)^2 + e^{-4\vec{Q}_B \cdot \vec{\Phi}/3} F(B)^2 \Big] \\
&- \tfrac{1}{2} \int d^5x\; \varepsilon_{(5)} A\; \partial B\; \partial C \,,
\end{aligned} \tag{4.35}
$$

where $\vec{\Phi} = (\sigma, \phi)$ and

$$
\begin{aligned}
\vec{Q}_A &= (\sqrt{3}, 1) \,, \\
\vec{Q}_B &= (-\sqrt{3}, 1) \,, \\
\vec{Q}_C &= (0, -2) \,.
\end{aligned} \tag{4.36}
$$

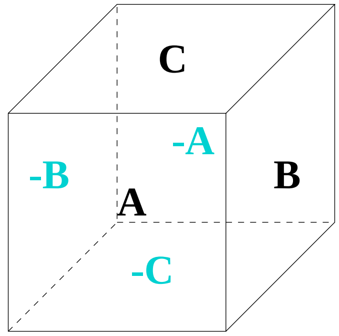

Figure 4.3: *Each proper discrete symmetry of the cube corresponds to a symmetry acting on the three vectors. The six faces of the cube correspond to the pairs* $(A,-A),(B,-B)$ *and* $(C,-C)$.

Given the above form of the dimensionally reduced action, it is not difficult to analyse its discrete duality symmetries. It turns out that on the 3 vectors one can realize the 24-element finite group $\mathcal{C}/\mathbb{Z}_2$ where $\mathcal{C}$ is the so-called cubic group. The easiest way to see how this group is realized is to write a cube, like in Figure 4.3, with faces $(A,-A),(B,-B)$ and $(C,-C)$.

The reason that we only consider the 24 proper symmetries and not the full 48-element cubic group is that only the proper elements leave the last (topological) term in the action (4.35) invariant. The proper cubic group has elements of order 2 and 3.[4] An example of a 2-order element is the reflection around the diagonal vertical plane that connects the right-front of the cube to the left-back of the cube. An example of a 3-order element is given by a (counter-clockwise) rotation of 120 degrees with axis the line going from the upper right corner at the front to the lower left corner at the back of the cube. Each of the 24 proper discrete symmetries of the cube naturally leads to a discrete symmetry acting on the 3 vectors. For instance, the 2– and 3–order elements given above induce the following discrete symmetries acting on the vectors, respectively:

$$\begin{aligned} A' &= B\,, \quad B' = A\,, \quad C' = C\,, \\ A' &= B\,, \quad B' = C\,, \quad C' = A\,. \end{aligned} \tag{4.37}$$

To see which discrete group is realized on the 2 scalars, it is easiest to write the 3 vectors $\vec{Q}_A, \vec{Q}_B$ and $\vec{Q}_C$ as the corners of an equilateral triangle, like in Figure 4.4. It was pointed out by Kaloper [96] that on the scalars one can realize the 6-element dihedral group

$$D_3 = \mathcal{C}/\big(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\big)\,, \tag{4.38}$$

i.e., to every 4 symmetries acting on the vectors one relates a single symmetry acting on the scalars. The action of the 6 elements of D_3 on the scalars is given by:

$$\mathbf{e}: \qquad \sigma' = \sigma\,,$$

[4] The full cubic group also has elements of order 4. An example of such a 4-order element is a rotation of 90 degrees with axis the line going from the center of the lower face to the center of the upper face of the cube.

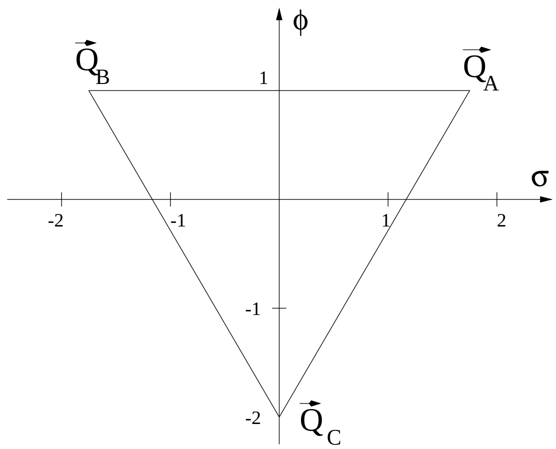

Figure 4.4: *Each symmetry of the equilateral triangle corresponds to a symmetry acting on the two scalars. The three corners of the triangle are given by the three vectors* $\vec{Q}_A, \vec{Q}_B$ *and* $\vec{Q}_C$ *defined in eq. (4.36).*

$$
\begin{array}{rrcl}
 & \phi' & = & \phi\,, \\
\mathbf{T}: & \sigma' & = & -\sigma\,, \\
 & \phi' & = & \phi\,, \\
\mathbf{S}: & \sigma' & = & \frac{1}{2}\sigma + \frac{1}{2}\sqrt{3}\phi\,, \\
 & \phi' & = & \frac{1}{2}\sqrt{3}\sigma - \frac{1}{2}\phi\,, \\
\mathbf{TS}: & \sigma' & = & -\frac{1}{2}\sigma + \frac{1}{2}\sqrt{3}\phi\,, \\
 & \phi' & = & -\frac{1}{2}\sqrt{3}\sigma - \frac{1}{2}\phi\,, \\
\mathbf{ST}: & \sigma' & = & -\frac{1}{2}\sigma - \frac{1}{2}\sqrt{3}\phi\,, \\
 & \phi' & = & +\frac{1}{2}\sqrt{3}\sigma - \frac{1}{2}\phi\,, \\
\mathbf{TST}: & \sigma' & = & \frac{1}{2}\sigma - \frac{1}{2}\sqrt{3}\phi\,, \\
 & \phi' & = & -\frac{1}{2}\sqrt{3}\sigma - \frac{1}{2}\phi\,.
\end{array}
\tag{4.39}
$$

Note that all D_3-transformations can be obtained as products of two elements, **T** and **S**, where the **T**-element corresponds to the usual T-duality transformation (3.31) and the **S**-element corresponds to the string/string duality (4.28). By **TS** we mean the symmetry that is obtained by a composition of **T** and **S** as follows:

$$
\sigma'' = \tfrac{1}{2}\sigma' + \tfrac{1}{2}\sqrt{3}\phi' = -\tfrac{1}{2}\sigma + \tfrac{1}{2}\sqrt{3}\phi\,. \tag{4.40}
$$

To every element of D_3 corresponds 4 elements of $\mathcal{C}/\mathbb{Z}_2$ acting on the 3 vectors. The

	e	T	S	ST	TS	TST
e	e	T	S	ST	TS	TST
T	T	e	TS	TST	S	ST
S	S	ST	e	T	TST	TS
ST	ST	S	TST	TS	e	T
TS	TS	TST	T	e	ST	S
TST	TST	TS	ST	S	T	e

Table 4.4: *Group multiplication table of the 6-element dihedral group D_3.*

specific transformations of the vectors are given by[5]

$$
\begin{aligned}
\mathbf{e}: \quad & A' = A\,, \quad B' = B\,, \quad C' = C\,, \\
\mathbf{T}: \quad & A' = B\,, \quad B' = A\,, \quad C' = C\,, \\
\mathbf{S}: \quad & A' = A\,, \quad B' = C\,, \quad C' = B\,, \\
\mathbf{TS}: \quad & A' = B\,, \quad B' = C\,, \quad C' = A\,, \\
\mathbf{ST}: \quad & A' = C\,, \quad B' = A\,, \quad C' = B\,, \\
\mathbf{TST}: \quad & A' = C\,, \quad B' = B\,, \quad C' = A\,.
\end{aligned}
\tag{4.41}
$$

Finally, for the sake of completeness we give the complete group multiplication table of D_3 in Table 4.4.

4.2.2 $D = 6$ Heterotic, Type IIA and Type IIB Theory

In this Subsection we describe the actions and symmetries of the six-dimensional Heterotic compactified on T^4 and Type IIA and Type IIB theory compactified on $K3$.

The toroidally compactified Heterotic theory was already discussed in subsection 3.1.3: its field content consists of the usual metric, axion and dilaton, 24 Abelian vector fields and 80 scalars parametrising an $O(4,20)/(O(4) \times O(20))$ coset. They can be combined into a $O(4,20)$-matrix $\hat{M}^{-1}$, satisfying $\hat{M}^{-1} L \hat{M}^{-1} = L$, where L is the $O(4,20)$-metric (3.38). The Heterotic action has six-dimensional $N = 2$ supersymmetry and can be written in the string-frame in an manifest $O(4,20)$-invariant way:

$$
\begin{aligned}
S_{\text{Het}} = \tfrac{1}{2} \int d^6x \sqrt{|\hat{g}|}\; e^{-2\hat{\phi}} \Big[& -\hat{R} + 4(\partial\hat{\phi})^2 - \tfrac{3}{4} \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} \hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}} \\
& + \tfrac{1}{8} \mathrm{Tr}\,(\partial_{\hat{\mu}} \hat{M} \partial^{\hat{\mu}} \hat{M}^{-1}) - \hat{F}(\hat{V})^{i}_{\hat{\mu}\hat{\nu}} \hat{M}^{-1}_{ij} \hat{F}(\hat{V})^{\hat{\mu}\hat{\nu} j} \Big]\,,
\end{aligned}
\tag{4.42}
$$

where $\hat{H}$ is defined as in (3.38).

[5] We only give 6 elements of $\mathcal{C}/\mathbb{Z}_2$. To every element below one can associate 3 more elements by changing (in 3 possible ways) *two* signs in the given transformation rules.

In order to rewrite this theory in $D=5$, we make the following Ansatz for the reduction scheme:

$$
\mathbf{e}: \left\{ \begin{array}{rcl}
\hat{g}_{xx} & = & -e^{-4\sigma/\sqrt{3}}\,, \\
\hat{g}_{x\mu} & = & -e^{-4\sigma/\sqrt{3}} A_\mu\,, \\
\hat{g}_{\mu\nu} & = & g_{\mu\nu} - e^{-4\sigma/\sqrt{3}} A_\mu A_\nu\,, \\
\hat{B}_{\mu\nu} & = & B_{\mu\nu} + A_{[\mu} B_{\nu]} + \ell^i V^j_{[\mu} A_{\nu]} L_{ij}\,, \\
\hat{B}_{x\mu} & = & B_\mu - \frac{1}{2}\ell^i V^j_\mu L_{ij}\,, \\
\hat{\phi} & = & \phi - \frac{1}{\sqrt{3}}\sigma\,, \\
\hat{V}^i_\mu & = & V^i_\mu + \ell^i A_\mu\,, \\
\hat{V}^i_x & = & \ell^i\,, \\
\hat{M} & = & M\,.
\end{array} \right. \tag{4.43}
$$

Just as in the previous subsection the axion $B_{\mu\nu}$ can be dualized to a vector C_μ via the formula:

$$
H^{\mu\nu\rho} = \frac{1}{3\sqrt{|g|}}\; e^{2\phi} \varepsilon^{\mu\nu\rho\lambda\sigma} F(C)_{\lambda\sigma}\,. \tag{4.44}
$$

The dimensionally reduced action in the (five-dimensional) string-frame is given by

$$
\begin{aligned}
S = \frac{1}{2}\int d^5x\sqrt{|g|}\; e^{-2\phi}\Big[& -R + 4(\partial\phi)^2 + \frac{1}{8}\mathrm{Tr}\;(\partial_\mu \mathcal{M}\partial^\mu \mathcal{M}^{-1}) \\
& + e^{4\phi}F(C)^2 - F(\mathcal{A})^I_{\mu\nu}\mathcal{M}^{-1}_{ij} F(\mathcal{A})^{\mu\nu J}\Big] \\
& -\frac{1}{4}\int d^5x\; \varepsilon_{(5)} C\; \partial\mathcal{A}^I\; \partial\mathcal{A}^J \mathcal{L}_{IJ}\,,
\end{aligned} \tag{4.45}
$$

where $\mathcal{L}$ is the invariant metric on $O(5,21)$. The $O(5,21)$-vectors $\mathcal{A}^I$ $\;(I=1,\cdots,26)$ are given by

$$
\mathcal{A}^I_\mu = \begin{pmatrix} A_\mu \\ B_\mu \\ V^i_\mu \end{pmatrix}\,. \tag{4.46}
$$

The explicit expression of the $O(5,21)$-matrix $\mathcal{M}$ in terms of the 105 scalars σ, ℓ^i and the 80 scalars contained in the $O(4,20)$ matrix M is given by

$$
\mathcal{M}^{-1} = \begin{pmatrix}
-e^{-4\sigma/\sqrt{3}} + \ell^i\ell^j M^{-1}_{ij} - \frac{1}{4}e^{4\sigma/\sqrt{3}}\ell^4 & \frac{1}{2}e^{4\sigma/\sqrt{3}}\ell^2 & \ell^i M^{-1}_{ij} - \frac{1}{2}e^{4\sigma/\sqrt{3}}\ell^2\ell_j \\
\frac{1}{2}e^{4\sigma/\sqrt{3}}\ell^2 & -e^{4\sigma/\sqrt{3}} & e^{4\sigma/\sqrt{3}}\ell_j \\
\ell^i M^{-1}_{ij} - \frac{1}{2}e^{4\sigma/\sqrt{3}}\ell^2\ell_j & e^{4\sigma/\sqrt{3}}\ell_j & M^{-1}_{ij} - e^{4\sigma/\sqrt{3}}\ell_i\ell_j
\end{pmatrix}
$$

where $\ell^2 \equiv \ell^i\ell^j L_{ij}$ and $\ell_i \equiv \ell^j L_{ij}$. These scalars parametrise the coset $O(5,21)/\big(O(5)\times O(21)\big)$.

The action (4.45) defines the Type II theory in 5 dimensions. It clearly contains the common sector given in (4.33). This may be seen by imposing the following constraints:

$$
\ell^i = V^i_\mu = 0\,, \qquad M^{-1}_{ij} = \delta_{ij}\,. \tag{4.47}
$$

Now, we will compare this result to the actions of the Type IIA/B theories, compactified on $K3$. $K3$ is a four-dimensional manifold that can be best seen as an orbifold of the four-torus T^4: it can be obtained from the T^4 after identification of the points on the torus that are mapped to each other under the $\mathbb{Z}_2$ transformation $x^a \to -x^a$ on the coordinates. It has 16 fix-points (points that under the $\mathbb{Z}_2$ are mapped to themselves) and an 80-dimensional moduli space of inequivalent string compactifications

$$\frac{SO(4,20)}{SO(4)\times SO(20)\times SO(4,20;\mathbb{Z})}. \tag{4.48}$$

Reduction over $K3$ breaks exactly half of the supersymmetry the theory would have if it were compactified on T^4.

The field content of the Type IIA theory reduced over $K3$ to 6 dimensions is identical to the Heterotic theory. Furthermore the reduction over $K3$ breaks half of the supersymmetry, such that we find also here six-dimensional $N=2$. The action, however, is different. Instead of a Chern-Simons term inside $\hat{H}$, the action contains an additional topological term as compared to the Heterotic action. We thus have

$$\begin{aligned} S_{\mathrm{IIA}} = \tfrac{1}{2}\int d^6x\sqrt{|\hat{g}|}\; e^{-2\hat{\phi}}\Big[&-\hat{R} + (\partial\hat{\phi})^2 - \tfrac{3}{4}\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}} \\ &+\tfrac{1}{8}\mathrm{Tr}\,\big(\partial_{\hat{\mu}}\hat{M}\partial^{\hat{\mu}}\hat{M}^{-1}\big) - e^{2\hat{\phi}}\hat{F}(\hat{V})^i_{\hat{\mu}\hat{\nu}}\hat{M}^{-1}_{ij}\hat{F}(\hat{V})^{\hat{\mu}\hat{\nu}j}\Big] \\ &-\tfrac{1}{8}\int d^6x\;\varepsilon_{(6)}\;\hat{B}\;\partial\hat{V}^i\;\partial\hat{V}^j L_{ij}\,. \end{aligned} \tag{4.49}$$

Just as in the ten-to-nine reduction of Type IIA/B in the previous section, the six-dimensional Type IIA action can be mapped onto the same five-dimensional Type II action (4.45) as the Heterotic theory, provided we use a different reduction scheme for the Type IIA theory:

$$\mathbf{S}:\left\{\begin{array}{rcl} \hat{g}_{xx} &=& -e^{-2\phi-2\sigma/\sqrt{3}}\,, \\ \hat{g}_{x\mu} &=& -e^{-2\phi-2\sigma/\sqrt{3}}A_\mu\,, \\ \hat{g}_{\mu\nu} &=& e^{-2\phi+2\sigma/\sqrt{3}}g_{\mu\nu} - e^{-2\phi-2\sigma/\sqrt{3}}A_\mu A_\nu\,, \\ \hat{B}_{\mu\nu} &=& B_{\mu\nu} + A_{[\mu}C_{\nu]}\,, \\ \hat{B}_{x\mu} &=& C_\mu\,, \\ \hat{\phi} &=& -\phi + \frac{1}{\sqrt{3}}\sigma\,, \\ \hat{V}^i_\mu &=& V^i_\mu + \ell^i A_\mu\,, \\ \hat{V}^i_x &=& \ell^i\,, \\ \hat{M} &=& M\,. \end{array}\right. \tag{4.50}$$

The five-dimensional antisymmetric tensor $B_{\mu\nu}$ is dualized to a vector B_μ via the relation

$$H^{\mu\nu\rho} = \frac{1}{3\sqrt{|g|}}\;e^{2\phi+4\sigma/\sqrt{3}}\;\varepsilon^{\mu\nu\rho\lambda\sigma}\Big[F(B)_{\lambda\sigma} + \ell^i F(V)^j_{\lambda\sigma}L_{ij} + \ell^2 F(A)_{\lambda\sigma}\Big]\,. \tag{4.51}$$

The field content of the Type IIB theory on $K3$ is given by a metric, 5 self-dual anti-symmetric tensors, 21 anti-self-dual anti-symmetric tensors and 105 scalars. The 105 scalars parametrize an $O(5,21)/\big(O(5)\times O(21)\big)$ coset and are combined into the symmetric 26×26 dimensional matrix $\hat{\mathcal{M}}$ satisfying the condition $\hat{\mathcal{M}}^{-1}\mathcal{L}\hat{\mathcal{M}}^{-1}=\mathcal{L}$ where $\mathcal{L}$ is the invariant metric on $O(5,21)$. The theory has $N=2$ supersymmetry.

Due to the (anti-)self-duality of the tensor fields, a covariant action is hard to write down[6], but omitting the self-duality constraint, a non-self-dual action can be constructed. We find that in the Einstein-frame the non-self-dual Type IIB action is given by

$$S_{\text{IIB}}=\tfrac{1}{2}\int d^6x\sqrt{|\hat{g}^{\text{E}}|}\,\Big[-\hat{R}+\tfrac{1}{8}\text{Tr}\,\big(\partial_{\hat{\mu}}\hat{\mathcal{M}}\partial^{\hat{\mu}}\hat{\mathcal{M}}^{-1}\big)+\tfrac{3}{8}\hat{H}^{I}_{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{\mathcal{M}}^{-1}_{IJ}\hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}J}\Big]\,, \tag{4.52}$$

where $\hat{H}^{I}_{\hat{\mu}\hat{\nu}\hat{\rho}}=\partial_{[\hat{\mu}}\hat{B}^{I}_{\hat{\nu}\hat{\rho}]}$. The field equations corresponding to this action lead to the correct Type IIB field equations, provided that we substitute by hand the following (anti-)self-duality conditions for the antisymmetric tensors $\hat{B}^I$ $(I=1,\cdots,26)$:

$$\hat{H}^I=\mathcal{L}^{IJ}\hat{\mathcal{M}}^{-1}_{JK}\;{}^*\hat{H}^K\,. \tag{4.53}$$

In order to extract the common sector out of the Type IIB theory, it is necessary to use a particular parametrisation of the matrix $\hat{\mathcal{M}}^{-1}$ in terms of the 105 scalars, thereby identifying a particular scalar as the Type IIB dilaton $\hat{\phi}$. This dilaton may then be used to define a string-frame metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ via (4.26). We use the following parametrisation:

$$\hat{\mathcal{M}}^{-1}=\begin{pmatrix} -e^{-2\hat{\phi}}+\hat{\ell}^i\hat{\ell}^j\hat{M}^{-1}_{ij}-\frac{1}{4}e^{2\hat{\phi}}\hat{\ell}^4 & \frac{1}{2}e^{2\hat{\phi}}\hat{\ell}^2 & \hat{\ell}^i\hat{M}^{-1}_{ij}-\frac{1}{2}e^{2\hat{\phi}}\hat{\ell}^2\hat{\ell}_j \\ \frac{1}{2}e^{2\hat{\phi}}\hat{\ell}^2 & -e^{2\hat{\phi}} & e^{2\hat{\phi}}\hat{\ell}_j \\ \hat{\ell}^i\hat{M}^{-1}_{ij}-\frac{1}{2}e^{2\hat{\phi}}\hat{\ell}^2\hat{\ell}_j & e^{2\hat{\phi}}\hat{\ell}_j & \hat{M}^{-1}_{ij}-e^{2\hat{\phi}}\hat{\ell}_i\hat{\ell}_j \end{pmatrix}\,, \tag{4.54}$$

where 80 scalars are contained in the $O(4,20)$ matrix $\hat{M}^{-1}$, 24 scalars are described by the $\hat{\ell}^a$ and where $\hat{\phi}$ is identified as the Type IIB dilaton.

The common sector is then obtained by imposing the constraints:

$$\hat{B}^i_{\hat{\mu}\hat{\nu}}=0\,,\quad (i=3,\cdots,26)\,,\quad \hat{\ell}^i=0\,,\quad \hat{M}^{-1}_{ij}=\delta_{ij}\,. \tag{4.55}$$

The (anti-)self-duality conditions (4.53) reduce to

$$\hat{H}^{(2)}=-e^{-2\hat{\phi}}\;{}^*\hat{H}^{(1)}\,. \tag{4.56}$$

Substituting the constraints (4.55) and the constraint (4.56) back into the Type IIB action (4.52) one obtains the standard form of the action for the common sector in the Einstein metric as given in (4.27). Having identified the Type IIB dilaton it is straightforward to convert this result to the string-frame metric as given in (4.25).

The above discussion for the Type IIA theory also applies to the Type IIB theory. We find that the dimensional reduction of the Type IIB theory leads to the same $D=5$ Type II theory (4.45) as the dimensional reduction of the Heterotic and Type IIA theory

[6]However, see [52].

provided we use the following dimensional reduction formulae for the Type IIB fields:

$$\mathbf{ST}: \left\{ \begin{array}{rcl} \hat{g}_{xx} &=& -e^{2\phi+2\sigma/\sqrt{3}}\,, \\ \hat{g}_{x\mu} &=& -e^{2\phi+2\sigma/\sqrt{3}}C_\mu\,, \\ \hat{g}_{\mu\nu} &=& e^{-2\phi+2\sigma/\sqrt{3}}g_{\mu\nu} - e^{2\phi+2\sigma/\sqrt{3}}C_\mu C_\nu\,, \\ \hat{B}^I_{\mu\nu} &=& B^I_{\mu\nu} + C_{[\mu}\mathcal{A}^I_{\nu]}\,, \\ \hat{B}^I_{x\mu} &=& \mathcal{A}^I_\mu\,, \\ \hat{\phi} &=& \frac{2}{\sqrt{3}}\sigma\,, \\ \hat{\ell}^i &=& \ell^i\,, \\ \hat{M} &=& M\,. \end{array} \right. \tag{4.57}$$

Note that due to the (anti-)self-duality relations (4.53) both $\hat{B}^I_{\mu\nu}$ as well as $\hat{B}^I_{x\mu}$ get related to the 5-dimensional vector fields $\mathcal{A}^I_\mu$. The dimensionally reduced expression for the (anti-)self-duality condition (4.53) states that the 26 anti-symmetric tensors $B^I_{\mu\nu}$ and the 26 vector $\mathcal{A}^I_\mu$ are not independent degrees of freedom, but each other's Poincaré dual:

$$H^{\rho\sigma\lambda K} = -\frac{1}{3\sqrt{|g|}}\; e^{-2\phi}\; \varepsilon^{\mu\nu\rho\sigma\lambda}\mathcal{M}^{KI}\mathcal{L}_{IJ}F(\mathcal{A})^J_{\mu\nu}\,. \tag{4.58}$$

Now that we are able to map the three different six-dimensional theories, Heterotic, Type IIA and Type IIB, onto one and the same five-dimensional Type II theory, we can use these reduction formulae to construct discrete duality transformations between the different theories in six dimensions, as an analogue of the Type IIA/B T-duality in ten dimensions. This will be done in the next subsection.

First we should make a remark about the symmetries of the Type II action (4.45): the action is clearly invariant under the group

$$O(5,21) = SO^\uparrow(5,21)\times\mathbb{Z}_2^{(S)}\times\mathbb{Z}_2^{(T)}\,, \tag{4.59}$$

where the mapping class group $\mathbb{Z}_2^{(S)}\times\mathbb{Z}_2^{(T)}$ is the straightforward generalization of the nine-dimensional Heterotic case (4.13):

$$\begin{array}{ll} \mathbb{Z}_2^{(S)}: & \mathcal{A}'^I_\mu = -\mathcal{A}^I_\mu \\ & \\ \mathbb{Z}_2^{(T)}: & \left\{ \begin{array}{l} A'_\mu = B_\mu \\ B'_\mu = A_\mu \\ V'^i_\mu = L^i{}_j V^j_\mu \\ e^{-4\sigma'/\sqrt{3}} = N^{-1}e^{-4\sigma/\sqrt{3}} \\ \ell'^i = N^{-1}(e^{-4\sigma/\sqrt{3}}\ell^j M^{-1}_{jk}L^{ki} - \frac{1}{2}\ell^2\ell^i) \end{array} \right. \end{array} \tag{4.60}$$

with $N = (e^{-8\sigma/\sqrt{3}} - e^{-4\sigma/\sqrt{3}}\ell^j\ell^k M^{-1}_{jk} + \frac{1}{4}\ell^4)$. All other fields remain invariant.

The breaking of the symmetry group of the common sector $\mathcal{C}/\mathbb{Z}_2$ to the above mapping class group is the five-dimensional analogue of the breaking of the dihedral group D_4 in $D=9$ to $\mathbb{Z}_2^{(S)}\times\mathbb{Z}_2^{(T)}$ in the presence of vector fields. If we restrict ourselves to transformations that act non-trivially on the scalars, we see that the D_3-group of the common sector gets broken to $\mathbb{Z}_2^{(T)}$.

4.2.3 Type II Dualities

The fact that it is possible to compactify the Heterotic, Type IIA and Type IIB action onto the same Type II action in five dimensions, means that the three theories in six dimensions are intimately related. On one hand, we have the string/string duality between Heterotic and Type IIA theory [60, 163], while on the other hand the T-duality between Type IIA and Type IIB theory on $K3$ can be constructed in the same way as in nine dimensions [26]. Together they form a web of string/string/string "triality" transformations [62]. These transformations can now be constructed via the different reduction schemes that map the three theories onto the same one in five dimensions.

The presence of the T-duality symmetry (4.60) in five dimensions means that to each reduction formula given above we can associate a T-dual version, in the way that the Type IIB reduction scheme (3.44) was the T-dual of the Type IIA reduction scheme (3.43). Its explicit form is obtained by replacing in the original reduction formula each five-dimensional fields by its T-dual expression. The T-dual reduction formula so obtained should lead to the same action in five dimensions. This is guaranteed by the fact that the five-dimensional action is invariant under T-duality. We will indicate the T-dual versions of the reduction formulae constructed in the previous section as follows:

$$\mathbf{e} \rightarrow \mathbf{T}\,, \qquad \mathbf{S} \rightarrow \mathbf{TS}\,, \qquad \mathbf{ST} \rightarrow \mathbf{TST}\,. \tag{4.61}$$

Again we have named the different reduction schemes by the group elements of D_3, since they are each other's D_3-transforms (4.39)-(4.41) when restricted to the common sector.

We thus obtain six different reduction formulae which correspond to the three down-pointing arrows in Figure 4.5. Similarly, there are six inverse reduction (decompactification) formulae which go opposite the vertical arrows in Figure 4.5. These decompactification formulae will be indicated by the inverse group elements ($\mathbf{e}^{-1}, \mathbf{T}^{-1}, \mathbf{S}^{-1}$, ...) and can be constructed easily from the reduction formulae. The claim is now that, using these six reduction and decompactification formulae only, one is able to construct in a simple way all the discrete dualities that act within and between the Heterotic, Type IIA and Type IIB theories that are indicated in Figure 4.5.

Each discrete duality symmetry has been given a name which corresponds to the proper combination of reduction and decompactification schemes and, when restricted to the common sector, the duality becomes the corresponding D_3 duality symmetry that acts in the common sector.

To show how the dualities of Figure 4.5 may be constructed, starting from the different reduction and decompactification formulae, it is instructive to give a few examples.

1. The T-duality that acts within the Heterotic theory is obtained by first reducing the theory using the $\mathbf{e}$ reduction formulae given in (4.43) and then using the

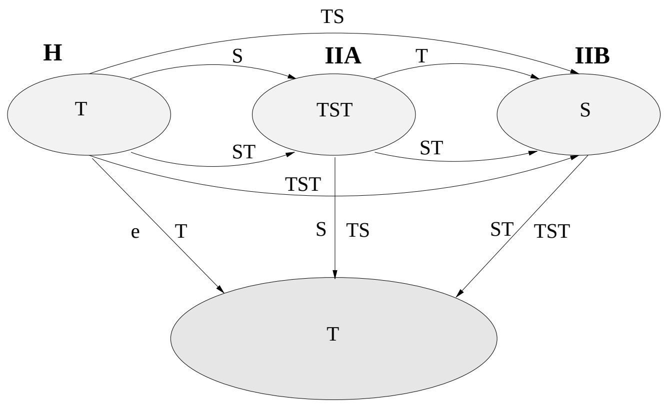

Figure 4.5: *The 3 down-pointing arrows indicate the six possible ways to map the three $D=6$ theories (Heterotic, Type IIA, Type IIB) onto the same $D=5$ Type II theory. Each reduction formula is indicated by a (boldface) element of D_3. As explained in the text these six reduction formula and their inverses may be used to construct the explicit form of all the discrete $D=6$ dualities that are indicated in the figure.*

T-dual decompactification formulae, defined in (4.61), i.e.

$$T(H \to H) = \mathbf{T^{-1}} \times \mathbf{e} = \mathbf{T}\,. \tag{4.62}$$

The duality rules are the uplifted form of (4.60).

2. The S-duality that maps the Heterotic onto the Type IIA theory is obtained by first reducing the Heterotic theory via $\mathbf{e}$ and next decompactifying the $D=5$ theory via $\mathbf{S^{-1}}$. As Figure 4.5 shows there are three other possibilities, one of them gives the same answer while the other two are related to the ST map indicated in Figure 4.5:

$$\begin{array}{rcl} S(H \to IIA) &=& \mathbf{S^{-1}} \times \mathbf{e} = \mathbf{S^{-1}} = \mathbf{S}\,, \\ S(H \to IIA) &=& (\mathbf{TS})^{-1} \times \mathbf{T} = \mathbf{ST} \times \mathbf{T} = \mathbf{S}\,, \\ (ST)(H \to IIA) &=& \mathbf{S^{-1}} \times \mathbf{T} = \mathbf{S} \times \mathbf{T} = \mathbf{ST}\,, \\ (ST)(H \to IIA) &=& (\mathbf{TS})^{-1} \times \mathbf{e} = \mathbf{ST} \times \mathbf{e} = \mathbf{ST}\,. \end{array} \tag{4.63}$$

The S-duality map corresponds to the known $D=6$ string/string duality rule [60, 163]. We find that the S-duality rules are given by (using the string-frame metric):

$$\begin{array}{rcl} \hat{G}_{\hat{\mu}\hat{\nu}} &=& e^{-2\hat{\phi}}\hat{g}_{\hat{\mu}\hat{\nu}}\,, \\ \hat{\Phi} &=& -\hat{\phi}\,, \\ \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} &=& e^{-2\hat{\phi}}\,{}^*\hat{h}_{\hat{\mu}\hat{\nu}\hat{\rho}}\,, \end{array} \tag{4.64}$$

where the other fields are invariant. The capital fields are Type IIA and the small-script fields Heterotic. To derive this string/string duality rule one must also use the two dualization formulae (4.44) and (4.51). Note that one may only derive a string/string duality rule for $\hat{H}$ and not $\hat{B}$. This is of course related to the fact that from the six-dimensional point of view the string/string duality is a symmetry of the equations of motion only.

3. The S-duality that acts within the Type IIB theory is obtained by first reducing the Type IIB theory with $\mathbf{ST}$ and then decompactifying with $(\mathbf{TST})^{-1}$. The other way round gives the same answer:

$$\begin{aligned} S(IIB \to IIB) &= (\mathbf{TST})^{-1} \times \mathbf{ST} = \mathbf{TST} \times \mathbf{ST} \\ &= \mathbf{STS} \times \mathbf{ST} = \mathbf{S}\,, \\ S(IIB \to IIB) &= (\mathbf{ST})^{-1} \times \mathbf{TST} = \mathbf{TS} \times \mathbf{TST} \\ &= \mathbf{TS} \times \mathbf{STS} = \mathbf{S}\,, \end{aligned} \tag{4.65}$$

where we have used the multiplication properties of the group D_3.

The S-duality rules can be written covariantly in a six-dimensional way, i.e., in terms of the $\hat{\mu}$–indices, in contrast to the T-duality, whose presence requires the existence of a special isometry direction. We find that the S-duality is given by a particular $O(5,21)$-transformation with parameter Ω given by $\Omega = \mathcal{L}$, where $\mathcal{L}$ is the flat $O(5,21)$ metric given in (3.38). In components, its action on the antisymmetric tensors and scalars is given by

$$\begin{aligned} \hat{H}'^{(1)}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= \hat{H}^{(2)}_{\hat{\mu}\hat{\nu}\hat{\rho}}\,, \\ \hat{H}'^{(2)}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= \hat{H}^{(1)}_{\hat{\mu}\hat{\nu}\hat{\rho}}\,, \\ \hat{H}'^{a}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= L^a{}_b \hat{H}^{b}_{\hat{\mu}\hat{\nu}\hat{\rho}}\,, \\ e^{-2\hat{\phi}'} &= e^{-2\hat{\phi}}\left(e^{-4\hat{\phi}} - e^{-2\hat{\phi}'}\hat{\ell}^a\hat{\ell}^b\hat{M}^{-1}_{ab} + \tfrac{1}{4}\ell^4\right)^{-1}\,, \\ \hat{\ell}'^a &= \frac{e^{-2\hat{\phi}}\hat{\ell}^c\hat{M}^{-1}_{cd}L^{da} - \frac{1}{2}\hat{\ell}^2\hat{\ell}^a}{e^{-4\hat{\phi}} - e^{-2\hat{\phi}}\hat{\ell}^a\hat{\ell}^b\hat{M}^{-1}_{ab} + \frac{1}{4}\hat{\ell}^4}\,, \\ \hat{M} &= \hat{M}^{-1}\,. \end{aligned} \tag{4.66}$$

Note that, when restricted to the common sector, this duality transformation indeed reduces to the standard S-duality rule given in (4.39)-(4.41).

4. We deduce from Figure 4.5 that there is not only a T-duality that acts within the Heterotic theory but also a T-duality that maps the Type IIA theory onto the Type IIB theory. This is then the analogue of the Type IIA/B T-duality in ten dimensions [26]. It may be obtained in the following two ways from the reduction/decompactification formulae:

$$\begin{aligned} T(IIA \to IIB) &= (\mathbf{ST})^{-1} \times \mathbf{S} = \mathbf{TS} \times \mathbf{S} = \mathbf{T}\,, \\ T(IIA \to IIB) &= (\mathbf{TST})^{-1} \times \mathbf{TS} = \mathbf{TST} \times \mathbf{TS} = \mathbf{T}\,. \end{aligned} \tag{4.67}$$

Following our method described above we find the following expression for this duality transformation:

$$
\begin{aligned}
\hat{\Phi} &= \hat{\phi} - \tfrac{1}{2}\log(-\hat{g}_{xx})\,, \\
\hat{G}_{xx} &= 1/\hat{g}_{xx}\,, \\
\hat{G}_{x\mu} &= \hat{b}_{x\mu}/\hat{g}_{xx}\,, \\
\hat{G}_{\mu\nu} &= \hat{g}_{\mu\nu} - \Big(\hat{g}_{x\mu}\hat{g}_{x\nu} - \hat{b}_{x\mu}\hat{b}_{x\nu}\Big)/\hat{g}_{xx}\,, \\
\hat{B}^{(1)}_{x\mu} &= \hat{g}_{x\mu}/\hat{g}_{xx}\,, \\
\hat{B}^{(1)}_{\mu\nu} &= \hat{b}_{\mu\nu} - \Big(\hat{g}_{x\mu}\hat{b}_{x\nu} - \hat{g}_{x\nu}\hat{b}_{x\mu}\Big)/\hat{g}_{xx}\,, \\
\hat{B}^{i}_{x\mu} &= \hat{v}^{i}_{\mu} - \hat{v}^{i}_{x}\hat{g}_{x\mu}/\hat{g}_{xx}\,, \\
\hat{\ell}^{i} &= \hat{v}^{i}_{x}\,, \\
\hat{M}_{ij} &= \hat{m}_{ij}\,,
\end{aligned}
\tag{4.68}
$$

where the capital fields are Type IIB and the small-script fields are Type IIA fields, respectively. Note that the duality transformations of $\hat{B}^{(2)}_{\hat{\mu}\hat{\nu}}$ and $\hat{B}^{i}_{\mu\nu}$ are not given. Their transformation rules follow from the ones given above via the the self-duality conditions (4.53).

5. Finally, we observe that ST is a 3-order element of D_3. This means that starting with the Heterotic theory and applying the ST-duality three times we should get back the Heterotic theory. In the diagram of Figure 4.5 this is seen as follows: The first ST duality brings us to the Type IIA theory, the second one brings us from the Type IIA to the type IIB theory. Finally, to perform the last ST-duality we observe that $ST = (TS)^{-1}$, i.e., this duality brings us back from the Type IIB theory to the Heterotic theory via the opposite direction of the oriented arrow at the top of the diagram.

Clearly, the above given examples are not all the D_3 string/string/string triality transformations. The other transformations can be constructed in the same way as the transformations above.

These six-dimensional duality relations are, just as the duality map in ten dimensions, an indication that the various string theories are different manifestations of the underlying M-theory (section 3.3). Different compactifications of different limits become equivalent and can be related via duality transformations. Figure 4.5 can therefore be compared to Figure 3.2 in section 3.3.

Chapter 5

Solutions

In this chapter we will study the extended object solutions of the ten- and eleven-dimensional low energy effective action, presented in Section 2.3, and more particularly, solutions that consist of more then one object: the so-called intersections or superpositions of various extended objects.

The interest in these intersections lays in the fact that, after dimensional reduction, they give rise to various new single-brane solutions in lower dimensions. As we will see later on in this chapter, these lower-dimensional single-brane solutions differ in the way they are coupled to the dilaton, which is determined by the number of branes in the original ten- or eleven-dimensional intersection. So in order to have an overview of the lower-dimensional p-branes, it is necessary to have a classification of the p-brane intersections that reduce to these. A special interest has risen recently in those intersections that reduce to black holes, because the number of micro-states of a black hole (which is a measure for its entropy) is determined by the number of intersections that reduce to this black hole [154].

In Section 5.1 we study the conditions two objects should satisfy in order to form a stable configuration, we classify the different intersection classes and compute the amount of supersymmetry of the intersections. In Section 5.2 we use the conditions for stable two-object intersections to construct multiple intersections, consisting of more then two objects per configuration. Again we will give a classification of the intersection classes for different numbers of objects involved, and determine the maximum number of objects in a configuration. In Section 5.3 we construct new, lower-dimensional solutions from the obtained ten and eleven-dimensional intersections.

The results presented in this chapter are a summary of [15, 20, 21].

5.1 Pair Intersections of Extended Objects

In this section we will study the conditions for two fundamental objects to combine into a two-object intersection. We start with the intersection of two D-branes in Subsection 5.1.1, and generalize the results to any two fundamental objects in Subsection 5.1.2.

5.1.1 D-brane Pair Intersections

The elementary Dirichlet p–brane solutions in ten dimensions are characterized by a single function H that depends on the $(9-p)$ transverse coordinates and is harmonic with respect to these variables. In the string-frame metric, the solution with p $(0 \leq p < 9)$ is given by (2.58):

$$Dp = \begin{cases} ds^2 = H^{-\frac{1}{2}}(dt^2 - dx_1^2 - ... - dx_p^2) - H^{\frac{1}{2}}(dx_{p+1}^2 + ... + dx_9^2) \\ e^{-2\phi} = H^{\frac{p-3}{2}} \\ F^{(R-R)}_{012...pm} = \partial_m H^{-1} \qquad\qquad (m : p+1, ..., 9). \end{cases} \tag{5.1}$$

For even (odd) p this metric corresponds to a solution of IIA (IIB) supergravity.

We have seen in Section 3.1 that T-duality relates the various D-branes to each other. If one assumes an isometry direction x, the only non-trivial T-duality rule involving the metric is given by (3.46)[1]:

$$\tilde{g}_{xx} = 1/g_{xx}\,. \tag{5.2}$$

Clearly, under this duality transformation the metric of a Dirichlet p-brane becomes that of a $(p+1)$-brane if the duality is performed over one of the transverse directions of the p-brane. In other words, one of the transverse directions of the p-brane has become a world volume direction of the $(p+1)$-brane. It is of course also possible to perform T-duality in an orthogonal direction and change a world volume coordinate into a transverse one. However, in this case one has be careful, since then we have to suppose that the harmonic function after dualization depends on the direction in which we have dualized and it is not guaranteed that this is the case.

It is convenient to represent every coordinate that corresponds to a world volume direction by $\times$ and every direction transverse to the brane by $-$. We thus obtain the following representation of the metric of a Dp-brane solution:

$$ds^2 = \underbrace{\times|\ \times\ ...\ \times}_{p+1}\ \overbrace{-\ -\ ...\ -}^{9-p}. \tag{5.3}$$

Note that the first $\times$ on the left hand side represents the time direction, which is necessarily a world volume direction. It is easy to see that acting with T-duality on this metric, a $-$ changes into a $\times$ or vice versa. This representation will turn out to be very useful in the study of intersection solutions.

[1]The T-duality rules for the R-R fields can be found in [18].

We will study a special type of intersections: the so-called *orthogonally intersecting threshold BPS bound states*. These are intersections where each participating brane corresponds to an independent harmonic function H_i in the solution. Furthermore, the branes intersect each other orthogonally and the forces between the different branes vanish [161], so that there is no potential energy. The total energy of the intersection is the sum of the energy of each brane separately. The precise form of such a solution is given by the *harmonic function rule* [160], which prescribes how products of powers of the harmonic functions H_i of the intersecting branes must occur in the composite solution. In particular, it implies that if one removes one of the N branes of the configuration (i.e., one of the H_i is set equal to one), a solution with $(N-1)$ intersecting branes is obtained. Solutions satisfying these intersection conditions have been studied extensively in the literature [124, 160, 106, 15, 70, 102, 125, 43, 161, 4, 123, 20, 6, 5, 21]. We will not consider non-threshold bound states, branes at angles, rotating branes or transversely boosted branes. For this we refer to [134, 48, 121, 34, 44, 159].

Let us now study in detail the pair intersections and derive the conditions necessary to form a stable solution of the equation of motion. An Ansatz describing the (string frame) metric of a $D(p+r)$-brane intersecting a $D(p+s)$-brane over p coordinates, is given by [160]:

$$\begin{aligned} ds^2 = (H_1 H_2)^{-1/2}\ ds^2_{p+1} - \Big(\frac{H_1}{H_2}\Big)^{1/2}\ dx_s^2 \\ -\Big(\frac{H_2}{H_1}\Big)^{1/2}\ dx_r^2 - (H_1 H_2)^{1/2}\ dx_m^2\ . \end{aligned} \tag{5.4}$$

The harmonic function H_1 describes the $(p+r)$-brane, while H_2 describes the $(p+s)$-brane. It is easy to see that this Ansatz satisfies the harmonic function rule: the metric of a single D-brane is recovered upon setting the other harmonic function equal to one. We will denote this solution of a $D(p+r)$-brane and a $D(p+s)$-brane intersecting over p coordinates as

$$p\,|D(p+r),\ D(p+s)) \tag{5.5}$$

We see that the coordinates naturally split into three parts: (1) the overall world volume coordinates $x_i, (i = 0, ..., p)$, which are common to the two branes, (2) the overall transverse coordinates x_m, with $m = 1, ..., 9-p-r-s$, which are orthogonal to both branes and (3) the other coordinates x_a with $a : 1, ..., n = r+s$ which are called relative transverse coordinates and are transverse to one brane but parallel to the other one. Using the notation of (5.3), we can write an intersection of the type (5.4) as:

$$ds^2 = \left\{ \begin{array}{c|ccccccccc} \times & \times & \times & \times & \times & \times & - & - & - & - \\ \times & \times & - & - & - & - & - & - & - & - \end{array} \right. \begin{array}{l} : H_1 \\ : H_2\ . \end{array} \qquad \underbrace{\quad}_{x_i}\ \underbrace{\qquad\quad}_{x_a}\ \underbrace{\qquad\quad}_{x_m} \tag{5.6}$$

Every column represents a direction x_μ, which can be either common world volume (x_i), relative transverse (x_a) or overall transverse (x_m).

The labels $p,\ r$ and s in the configuration (5.4) have to fulfill certain conditions: first of all $p+r+s \leq 9$ for the obvious reason that we only have 9 spatial dimensions to fill.

$n=2$	
(0|0,2)	(0|1,1)
(1|1,3)	(1|2,2)
(2|2,4)	(2|3,3)
(3|3,5)	(3|4,4)
(4|4,6)	(4|5,5)
(5|5,7)	(5|6,6)
(6|6,8)	(6|7,7)
(7|7,9)	(7|8,8)

$n=6$			
(0|0,6)	(0|1,5)	(0|2,4)	(0|3,3)
(1|1,7)	(1|2,6)	(1|3,5)	(1|4,4)
(2|2,8)	(2|3,7)	(2|4,6)	(2|5,5)
(3|3,9)	(3|4,8)	(3|5,7)	(3|6,6)

$n=4$		
(0|0,4)	(0|1,3)	(0|2,2)
(1|1,5)	(1|2,4)	(1|3,3)
(2|2,6)	(2|3,5)	(2|4,4)
(3|3,7)	(3|4,6)	(3|5,5)
(4|4,8)	(4|5,7)	(4|6,6)
(5|5,9)	(5|6,8)	(5|7,7)

$n=8$				
(0|0,8)	(0|1,7)	(0|2,6)	(0|3,5)	(0|4,4)
(1|1,9)	(1|2,8)	(1|3,7)	(1|4,6)	(1|5,5)

Table 5.1: *The tables of the T-duality classes of intersecting configurations of two D-branes in ten dimensions. Via T-duality one can move horizontally and vertically within a table. Only the tables with $n=4$ and $n=8$ correspond to threshold solutions to the equations of motion.*

Furthermore we only want to combine objects which come from the same theory (Type IIA or Type IIB), so r and s have to be both odd or both even. In other words $r+s$ has to be an even number n.

A T-duality transformation on a configuration (5.6) acts in a certain direction, changing in the column corresponding to that direction every $\times$ for a $-$ and vice versa. In general, a $(p\ |p+r,\ p+s)$-configuration can transform under T-duality in two ways: either the T-duality is performed in a relative transverse direction

$$\Big(p \,\Big|\, p+r,\ p+s\Big) \rightarrow \Big(p \,\Big|\, p+(r\pm 1),\ p+(s\mp 1)\Big)\,, \tag{5.7}$$

and the duality interchanges a relative transverse direction of one object with a relative transverse direction of the other object. The second possibility is that the T-duality is applied to an overall transverse direction or a common world volume direction

$$\Big(p \,\Big|\, p+r,\ p+s\Big) \rightarrow \Big(p\pm 1 \,\Big|\, (p\pm 1)+r,\ (p\pm 1)+s\Big)\,. \tag{5.8}$$

In either case (5.7) or (5.8) the number $r+s=n$ remains constant, so that n can be used to label the four different classes, as given in Table 5.1 [15, 70]. Within each class we can move horizontally or vertically via the T-duality transformations: horizontal movements correspond to a T-duality transformation in a relative transverse direction (5.7), while vertical movements are generated by T-duality transformations of the type (5.8).

For the Ansatz for the dilaton we take the product of the dilaton expressions for each brane separately:

$$e^{-2\phi} = (H_1)^{\frac{p+r-3}{2}}\ (H_2)^{\frac{p+s-3}{2}}\ . \tag{5.9}$$

In this way the harmonic function rule is satisfied in a straightforward way. Furthermore it is guaranteed that (5.9) transforms correctly under the T-duality rule for the dilaton (3.5) to give the right dilaton expression of the T-dual intersection (5.7) or (5.8).

The expression for the R-R gauge fields can easily be obtained by the requirement that, if one of the harmonic functions is set equal to one, the intersecting configuration should reduce to one of the D-brane solution (5.1). The explicit form of the R-R gauge fields is most easily given by using a formulation where the magnetic configurations are described by magnetic (dual) potentials. This leads us to consider the Lagrangian

$$\mathcal{L} = \sqrt{|g|}\left\{e^{-2\phi}\Big[R - 4(\partial\phi)^2\Big] + \frac{(-)^{p+r+1}}{2(p+r+2)!}F^2_{(p+r+2)} + \frac{(-)^{p+s+1}}{2(p+s+2)!}F^2_{(p+s+2)}\right\}\ , \tag{5.10}$$

where it is understood that in the field equations one imposes the constraint that $F_{(8-p)}$ is the dual of $F_{(p+2)}$. In particular, $F_{(5)}$ is self-dual. Pseudo-Lagrangians of this form have been discussed in [17]. It is also understood that the two kinetic terms for the gauge fields become identical if $r = s$.

We next distinguish three different cases:

- **Case 1:** Both harmonic functions depend on the overall transverse directions x_m. The R-R gauge fields are given by

$$F^{(1)}_{0\cdots p1\cdots rm} = \partial_m H_1^{-1}\ , \qquad\qquad F^{(2)}_{0\cdots p1\cdots sm} = \partial_m H_2^{-1}\ . \tag{5.11}$$

- **Case 2:** The function H_1 depends on the overall transverse directions x_m, whereas H_2 depends on its relative transverse directions x_a. The R-R gauge fields are given by

$$F^{(1)}_{0\cdots p1\cdots rm} = H_2^{\alpha}\partial_m H_1^{-1}\ , \qquad\qquad F^{(2)}_{0\cdots p1\cdots sa} = \partial_a H_2^{-1}\ . \tag{5.12}$$

- **Case 3:** Both harmonic functions depend on their (own) relative transverse directions x_a and x_b. The R-R gauge fields are given by

$$F^{(1)}_{0\cdots p1\cdots rb} = H_2^{\alpha}\partial_b H_1^{-1}\ , \qquad\qquad F^{(2)}_{0\cdots p1\cdots sa} = H_1^{\beta}\partial_a H_2^{-1}\ . \tag{5.13}$$

The α in Case 2 and the α, β in Case 3 are arbitrary (real) parameters that cannot be fixed by the Bianchi identities. We will determine them via the equations of motion.

So far, we have only applied T-duality to generate the Ansatz (5.4), (5.9), (5.11–5.13) for intersecting D-brane configurations, without really knowing whether they correspond to solutions to the equations of motion. Our next task is to determine which of these configurations corresponds to a (supersymmetric) solution of the Lagrangian (5.10). Substituting our Ansatz into the vector field and dilaton equation[2], we see that [14]:

- Case 1 can only be a solution for $n = 4$,
- Case 2 for $n = 4$ and $\alpha = 0$
- Case 3 requires that $n = 8$ and $\alpha = \beta = 1$.

Configurations in our Ansatz with $n = 2, 6$ relative transverse directions do not appear as solutions of the equations of motion. Non-threshold bound states with 2 relative transverse directions have been argued to exist [45], but it is not clear whether these solutions are of the form given above.

It turns out that the Cases 1 and 2 can naturally be combined into a more general configuration where H_1 only depends on the overall transverse directions, as before, but where H_2 is given by the sum of two harmonics $H_2^{(a)}, H_2^{(b)}$, which depend on the overall and relative transverse directions, respectively, i.e.

$$H_2(x_m, x_b) = H_2^{(a)}(x_m) + H_2^{(b)}(x_b)\,. \tag{5.14}$$

We will now investigate the supersymmetry of these solutions. For a single D-brane the supersymmetry condition is $\delta\lambda = \delta\psi_\mu = 0$, where λ is the dilatino and ψ_μ the gravitino in the IIA/IIB supergravity multiplet. Their variations (in the string frame) are given by (compare to (2.49)):

$$\begin{aligned}
\delta\psi_\mu &= \partial_\mu\epsilon - \tfrac{1}{4}\omega_\mu^{ab}\gamma_{ab}\epsilon + \frac{(-)^p}{8(p+2)!}\ e^\phi\ F_{\mu_1\ldots\mu_{p+2}}\ \gamma^{\mu_1\cdots\mu_{p+2}}\gamma_\mu\ \epsilon'_{(p)} = 0\,,\\
\delta\lambda &= \gamma^\mu(\partial_\mu\phi)\epsilon + \frac{3-p}{4(p+2)!}\ e^\phi F_{\mu_1\cdots\mu_{p+2}}\ \gamma^{\mu_1\cdots\mu_{p+2}}\ \epsilon'_{(p)} = 0\,,
\end{aligned} \tag{5.15}$$

where $\epsilon'_{(p)} = \epsilon$ for $p = 0, 4, 8$; $\epsilon'_{(p)} = \gamma_{11}\epsilon$ for $p = 2, 6$; $\epsilon'_{(p)} = i\epsilon$ for $p = 7$ and $\epsilon'_{(p)} = i\epsilon^\star$ for $p = 1, 5$. Substituting the single D-brane solution into the above equation leads to the condition

$$\epsilon + \gamma_{01\cdots p}\ \epsilon'_{(p)} = 0\,, \tag{5.16}$$

which defines a projection operator on ϵ that breaks half of the supersymmetry.

Now consider the intersection of a $(p+r)$-brane with a $(p+s)$-brane. Then the two supersymmetry conditions corresponding to the $(p+r)$-brane and $(p+s)$-brane are given by

$$\begin{aligned}
\epsilon + \gamma_{01\cdots p+r}\epsilon'_{(p+r)} &= 0\,,\\
\epsilon + \gamma_{01\cdots p+s}\epsilon'_{(p+s)} &= 0\,,
\end{aligned} \tag{5.17}$$

[2]The case that only intersecting 3-branes are involved is special since for this case the dilaton equation is trivially satisfied. By applying T–duality one can relate this case to the other cases and show that the same restrictions as given below apply.

respectively. Each one breaks half of the supersymmetry. Combining the two supersymmetry conditions we get

$$\epsilon'_{(p+r)} = (-)^{\frac{1}{2}r(r+1)}\gamma_{r+s}\epsilon'_{(p+s)}\,. \tag{5.18}$$

We now distinguish four cases in which the two spinors in the above equation are given by $(\epsilon,\epsilon), (\epsilon,\gamma_{11}\epsilon), (i\epsilon, i\epsilon)$ or $(i\epsilon, i\epsilon^\star)$, respectively. All four cases lead to the consistency condition that $\gamma^2_{r+s} = 1$, or

$$n = 4 \quad \text{or} \quad 8\,. \tag{5.19}$$

This reproduces the result of [129], where it is stated that the only supersymmetric (1/4 of the supersymmetry is unbroken) pair intersections are the ones with $r+s = 0\ mod\ 4$.

We next extend this analysis and consider the Killing spinor equation that follows from $\delta\lambda = 0$ for the case that we substitute the complete intersecting configuration and not only the separate D-brane configurations. In the string–frame we obtain the following equation from $\delta\lambda = 0$:

$$\begin{aligned}\gamma^\mu(\partial_\mu\phi)\;\epsilon \quad &+ \quad \tfrac{1}{4}(3-p-r)\; e^\phi F^{(1)}_{0\cdots p+r\mu}\gamma^{0\cdots p+r\mu}\;\epsilon'_{(p+r)}\\ &+ \quad \tfrac{1}{4}(3-p-s)\; e^\phi F^{(2)}_{0\cdots p+s\mu}\gamma^{0\cdots p+s\mu}\;\epsilon'_{(p+s)} = 0\,.\end{aligned} \tag{5.20}$$

Substituting the explicit form of the general intersecting configuration (5.4), (5.9), (5.11-5.13) into the above Killing spinor equation leads, for case 1 to $n = 4$, for Case 2 to $n = 4, \alpha = 0$ and for Case 3 to $n = 8, \alpha = \beta = 1$ [14]. This nicely agrees with our earlier finding that only these configurations can be solutions to the equations of motion.

Summarizing, we come to the following conclusions: there exist three types of D-brane pair intersections in ten dimensions satisfying the Ansatz (5.4), (5.9), (5.11-5.13), each conserving one quarter of the original supersymmetry. The three types of intersections differ in the dependence of the harmonic function on the coordinates and in the number n of relative transverse directions in the intersection, which labels the T-duality classes of intersections:

1. both harmonic functions depend on the overall transverse coordinates x_m. The only allowed intersections are the ones that have $n = 4$ relative transverse directions. The gauge fields are of the form (5.11).

2. one of the harmonics depends on the overall transverse coordinates, while the other depends on its relative transverse directions. Also here the only allowed intersections are the ones in the $n = 4$ class. The gauge fields are of the form (5.12) with $\alpha = 0$.

3. both harmonic functions depend on their relative transverse directions. Now the intersections must have $n = 8$ relative transverse coordinates and the gauge fields are of the form (5.13) with $\alpha = \beta = 1$.

In the next subsection we will try to generalize these results to M-brane intersections in eleven dimensions and ten-dimensional intersections that also involve other objects than D-branes.

	common wv.	relative trv.	overall trv.
$(0\|M2,M2)$	$-$	$(0\|F1,D2)$	$(0\|D2,D2)$
$(1\|M2,M5)$	$(0\|F1,D4)$	$(1\|F1,S5)$ $(1\|D2,D4)$	$(1\|D2,S5)$
$(3\|M5,M5)$	$(2\|D4,D4)$	$(3\|D4,S5)$	$(3\|S5,S5)$
$(1\|M2,\mathcal{W})$	$(0\|F1,D0)$	$(1\|F1,W)$	$(1\|D2,W)$
$(1\|M5,\mathcal{W})$	$(0\|D4,D0)$	$(1\|D4,W)$	$(1\|S5,W)$
$(2\|M2,\mathcal{KK})$	$(1\|F1,KK)$	$(2\|D2,KK)$	$(2\|D2,D6)$
$(5\|M5,\mathcal{KK})$	$(4\|D4,KK)$	$(5\|S5,KK)$	$(5\|S5,D6)$
$(0\|M2,\mathcal{KK})$	$-$	$(0\|F1,D6)$ $(0\|D2,KK)$	$(0\|D2,D6)^*$
$(3\|M5,\mathcal{KK})$	$(2\|D4,KK)$	$(3\|D4,D6)$ $(3\|S5,KK)$	$(3\|S5,D6)^*$
$(1\|\mathcal{W},\mathcal{KK})$	$(0\|D0,KK)$	$(1\|W,KK)$	$(1\|W,D6)$
$(4\|\mathcal{KK},\mathcal{KK})^a$	$(3\|KK,KK)^a$	$(4\|D6,KK)^*$	$(4\|D6,D6)$
$(4\|\mathcal{KK},\mathcal{KK})^b$	$(3\|KK,KK)^b$	$(4\|D6,KK)$	$(4\|D6,D6)^*$

Table 5.2: *Pair intersections in $D = 11$ and their reductions to $D = 10$ with dependence on overall transverse coordinates: the first column represents the pair intersections in $D = 11$, reductions to non-trivial solutions in $D = 10$, obtained by compactification in different directions (common world volume, relative transverse and overall transverse) with respect to the branes, are indicated in the remaining columns. The $D = 10$ solutions marked with * are not of the usual harmonic form.*

5.1.2 General Pair Intersections

The results of subsection 5.1.1 can be easily be uplifted to eleven dimensions, since the relations between the ten-dimensional Type IIA D-brane solutions and the solutions of $D = 11$ supergravity are known [156] (see Figure 3.1). On the other hand, dimensional reduction of the intersections in $D = 11$ yields new ten-dimensional intersections that do not only contain D-branes, but also fundamental strings, solitonic five-branes, waves and monopoles.

In Tables 5.2 and Table 5.3 we summarize the results on the pair intersections [21]. The two independent harmonic functions of the pairs in Table 5.2 depend on the overall transverse coordinates[3]. For the pairs in Table 5.3 both harmonic functions must depend on the relative transverse coordinates.

In the first three rows of Table 5.2 we list the intersections of $M2$- and $M5$-branes

[3] Here we will not consider the case where one of the harmonic functions in the intersections depends on the relative transverse directions. Their intersections are the same as for the case where the two harmonic functions depend on the overall transverse coordinates.

[125, 160] and their reductions to ten dimensions:

$$(0|M2,M2) = \left\{ \begin{array}{c|cccccccccc} \times & \times & \times & - & - & - & - & - & - & - & - \\ \times & - & - & \times & \times & - & - & - & - & - & - \end{array} \right. \tag{5.21}$$

$$(1|M2,M5) = \left\{ \begin{array}{c|cccccccccc} \times & \times & \times & - & - & - & - & - & - & - & - \\ \times & \times & - & \times & \times & \times & \times & - & - & - & - \end{array} \right. \tag{5.22}$$

$$(3|M5,M5) = \left\{ \begin{array}{c|cccccccccc} \times & \times & \times & \times & \times & \times & - & - & - & - & - \\ \times & \times & \times & \times & - & - & \times & \times & - & - & - \end{array} \right. \tag{5.23}$$

As an example, we will discuss the $(1|M2,M5)$ configuration and its different compactifications to ten dimensions. Reduction over x_1 gives $(0|F1,D4)$ in ten dimensions. For the relative transverse directions the possibilities are: either reduction over x_2, giving $(1|F1,S5)$, or reduction over one of the directions $x_3,\ldots,x_6$, giving $(1|D2,D4)$. Finally, one can impose an isometry in one of the overall transverse directions by restricting the dependence of the harmonic functions to three coordinates. Reduction over such a direction gives $(1|D2,S5)$.

The next two rows represent the addition of a wave (2.64) to the $D=11$ M-branes. The z-direction of the wave must be placed in the world volume of the M-brane. The dependence of the harmonic functions is only on the directions transverse to the M-brane, so that the wave does not propagate. The metric for these two $D=11$ pairs can be represented by[4]:

$$(1|M2,\mathcal{W}) = \left\{ \begin{array}{c|cccccccccc} \times & \times & \times & - & - & - & - & - & - & - & - \\ \times & z & - & - & - & - & - & - & - & - & - \end{array} \right. \tag{5.24}$$

$$(1|M5,\mathcal{W}) = \left\{ \begin{array}{c|cccccccccc} \times & \times & \times & \times & \times & \times & - & - & - & - & - \\ \times & z & - & - & - & - & - & - & - & - & - \end{array} \right. \tag{5.25}$$

The next four rows in Table 5.2 denote the pairs involving one M-brane and one Kaluza-Klein monopole (2.65). The metric for these four cases takes on the form

$$(2|M2,\mathcal{KK}) = \left\{ \begin{array}{c|cccccccccc} \times & - & - & - & - & \times & \times & - & - & - & - \\ \times & A_1 & A_2 & A_3 & z & \times & \times & \times & \times & \times & \times \end{array} \right. \tag{5.26}$$

$$(5|M5,\mathcal{KK}) = \left\{ \begin{array}{c|cccccccccc} \times & - & - & - & - & \times & \times & \times & \times & \times & - \\ \times & A_1 & A_2 & A_3 & z & \times & \times & \times & \times & \times & \times \end{array} \right. \tag{5.27}$$

$$(0|M2,\mathcal{KK}) = \left\{ \begin{array}{c|cccccccccc} \times & \times & - & - & \times & - & - & - & - & - & - \\ \times & A_1 & A_2 & A_3 & z & \times & \times & \times & \times & \times & \times \end{array} \right. \tag{5.28}$$

$$(3|M5,\mathcal{KK}) = \left\{ \begin{array}{c|cccccccccc} \times & \times & - & - & \times & \times & \times & \times & - & - & - \\ \times & A_1 & A_2 & A_3 & z & \times & \times & \times & \times & \times & \times \end{array} \right. \tag{5.29}$$

As we see, there are two possibilities. The z-direction of the Kaluza-Klein monopole can be placed either in a direction transverse to ($(2|M2,\mathcal{KK})$ and $(5|M5,\mathcal{KK})$) or in the world volume of the M-brane ($(0|M2,\mathcal{KK})$ and $(3|M5,\mathcal{KK})$). The solutions (5.26) and (5.27) have also been given in [160, 43]. For these, the reduction to $D=10$ is straightforward. Note that the reduction over an overall transverse direction can be

[4]Note that we extend the notation $(p|p+r;p+s)$ to include waves and monopoles with the understanding that the world volume directions of the "$\mathcal{W}$-brane" are given by t,z (see (2.64)), and the transverse directions of the "$\mathcal{KK}$-brane" are given by the isometry direction z and the coordinates in which the Kaluza-Klein vector is oriented. These directions (called x_m in (2.65)) will be denoted by A_m.

either over a direction indicated by z, or, by imposing an additional isometry, in the direction of a component of the vector field.

In the solutions (5.28) and (5.29) the harmonic functions depend only on the two overall transverse coordinates, so that the Kaluza-Klein monopole has one additional isometry direction (indicated by A_1). In both of these solutions the reduction over the relative transverse A_1 and z directions yields, after a coordinate transformation, the same result[5].

The last three rows of Table 5.2 correspond to intersections of Kaluza-Klein monopoles and waves. The possibilities are shown in (5.30-5.32)[6]. Note that there are two ways to intersect two Kaluza-Klein monopoles, both with a five-dimensional common world volume. In solution (5.31) the two harmonic functions depend on a single coordinate (x^1), in (5.32) on two coordinates (x^1, x^2).

$$(1|\mathcal{W},\mathcal{KK}) = \left\{ \begin{array}{c|cccccccccc} \times & - & - & - & - & z_1 & - & - & - & - & - \\ \times & A_1 & A_2 & A_3 & z_2 & \times & \times & \times & \times & \times & \times \end{array} \right. \tag{5.30}$$

$$(4|\mathcal{KK},\mathcal{KK})^a = \left\{ \begin{array}{c|cccccccccc} \times & A_1 & A_2 & A_3 & z & \times & \times & \times & \times & \times & \times \\ \times & B_1 & \times & \times & z & B_5 & B_6 & \times & \times & \times & \times \end{array} \right. \tag{5.31}$$

$$(4|\mathcal{KK},\mathcal{KK})^b = \left\{ \begin{array}{c|cccccccccc} \times & A_1 & A_2 & A_3 & z_1 & \times & \times & \times & \times & \times & \times \\ \times & B_1 & B_2 & \times & \times & B_5 & z_2 & \times & \times & \times & \times \end{array} \right. \tag{5.32}$$

The solution (5.31) solves the equations of motion, since it is the known ten-dimensional solution $(4|D6, D6)$ lifted up to $D = 11$. The configuration (5.32) must be a solution because, after reduction over a common world volume direction, it can be related to a known solution involving two solitonic five-branes via the following T-duality chain in $D = 10$:

$$(3|S5, S5) \rightarrow (3|S5, KK) \rightarrow (3|KK, KK)^b \,. \tag{5.33}$$

Similarly, the intersection of a wave and a Kaluza-Klein monopole can be obtained from ten dimensions by first constructing an intersection in $D = 10$ of a $D1$-brane with the solitonic five-brane and performing a T-duality in the direction of the string:

$$(0|D1, S5) \rightarrow (0|D0, KK) \,, \tag{5.34}$$

and by lifting this to eleven dimensions.

In Table 5.3 we consider intersections in which the two harmonic functions depend on the relative coordinates. There is one pair involving only $M5$ [70], and five pairs involving Kaluza-Klein monopoles. Some of these configurations and their generalization to non-orthogonal intersections were discussed in [69].

Below we present the metric of these pairs in the usual, short-hand way. The pairs involving Kaluza-Klein monopoles are each related to known solutions through $D = 10$, so that we can be sure that they solve the equations of motion. For example, $(2|\mathcal{KK},\mathcal{KK})$

[5]For a more detailed discussion of the possible dependences of the harmonic function of the monopole on one or two coordinates only, we refer to [21]. In general one can say that upon reduction of the monopole solution $\mathcal{KK}_D$ over any transverse direction z or x_m one always finds a magnetic $(D-5)$-brane.

[6]Solution (5.30) was presented in [160].

	common wv.	relative trv.	overall trv.
$(1\|M5, M5)$	$(0\|D4, D4)$	$(1\|D4, S5)$	$(1\|S5, S5)$
$(0\|M2, \mathcal{KK})$	$-$	$(0\|D2, KK)$ $(0\|F1, D6)^*$	$(0\|D2, D6)$
$(1\|M5, \mathcal{KK})$	$(0\|D4, KK)$	$(1\|S5, KK)$ $(1\|D4, D6)$	$-$
$(3\|M5, \mathcal{KK})$	$(2\|D4, KK)$	$(3\|S5, KK)$ $(3\|D4, D6)^*$	$(3\|S5, D6)$
$(2\|\mathcal{KK}, \mathcal{KK})$	$(1\|KK, KK)$	$(2\|D6, KK)$	$-$
$(4\|\mathcal{KK}, \mathcal{KK})$	$(4\|KK, KK)$	$(4\|D6, KK)$ $(4\|D6, KK)^*$	$(4\|D6, D6)^*$

Table 5.3: *Pair intersections in $D = 11$ and their reductions to $D = 10$ with dependence on relative transverse coordinates. The reductions indicated by a * are not expressed in a standard way in terms of harmonic functions.*

can be reduced to $(1|KK, KK)$ in ten dimensions and applying T-duality twice, in the directions z_1 and z_2, we find

$$(1|KK, KK) \to (1|S5, KK) \to (1|S5, S5)\,, \tag{5.35}$$

and this can be lifted up to $(1|M5, M5)$, which is a known solution. The intersections of Table 5.3 are of the form:

$$(1|M5, M5) \quad = \quad \left\{ \begin{array}{c|cccccccccc} \times & \times & \times & \times & \times & \times & - & - & - & - & - \\ \times & \times & - & - & - & - & \times & \times & \times & \times & - \end{array} \right. \tag{5.36}$$

$$(0|M2, \mathcal{KK}) \quad = \quad \left\{ \begin{array}{c|cccccccccc} \times & \times & \times & - & - & - & - & - & - & - & - \\ \times & A_1 & A_2 & A_3 & z & \times & \times & \times & \times & \times & \times \end{array} \right. \tag{5.37}$$

$$(1|M5, \mathcal{KK}) \quad = \quad \left\{ \begin{array}{c|cccccccccc} \times & \times & \times & \times & \times & \times & - & - & - & - & - \\ \times & A_1 & A_2 & A_3 & z & \times & \times & \times & \times & \times & \times \end{array} \right. \tag{5.38}$$

$$(3|M5, \mathcal{KK}) \quad = \quad \left\{ \begin{array}{c|cccccccccc} \times & \times & \times & - & - & - & - & - & \times & \times & \times \\ \times & A_1 & A_2 & A_3 & z & \times & \times & \times & \times & \times & \times \end{array} \right. \tag{5.39}$$

$$(2|\mathcal{KK}, \mathcal{KK}) \quad = \quad \left\{ \begin{array}{c|cccccccccc} \times & A_1 & A_2 & A_3 & z_1 & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & z_2 & B_6 & B_7 & B_8 & \times & \times \end{array} \right. \tag{5.40}$$

$$(4|\mathcal{KK}, \mathcal{KK}) \quad = \quad \left\{ \begin{array}{c|cccccccccc} \times & A_1 & A_2 & A_3 & z_1 & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & B_3 & B_4 & B_5 & z_2 & \times & \times & \times & \times \end{array} \right. \tag{5.41}$$

Let us finally make a remark about the ten-dimensional intersections that are not characterized by the usual harmonic functions. They come from regular eleven-dimensional intersections, but fall out of the usual Ansatz by the way we have reduced to $D = 10$. They are indicated in Table 5.2 and Table 5.3 by a *.

As an example, consider the reduction of (5.31). The harmonic functions depend on x_1, the gauge field components can be chosen to be all zero except A_2 and B_5, which then depend on x_3 and x_6, respectively. It is not difficult to see that this satisfies the condition (2.66) on the off-diagonal components of the metric.

Reduction over z gives $(4|D6,D6)$, but also reduction over x_2 is possible. This gives a $D=10$ configuration which has the properties of $(4|D6,KK)$, but the fields do not have the standard harmonic form. It is given by [21]:

$$\begin{aligned} ds^2 &= \varphi^{-1/2}(dt^2 - dx^2_{(7-10)} - H_2 dx^2_{(5-6)}) \\ &\quad -H_2^{-1}\varphi^{1/2}\ \big(dz + B_5 dx_5)^2 + (H_1^2 H_2 + A_2^2)(dx_3^2 + H_2 dx_1^2)\big)\ , \\ e^{2\phi} &= \varphi^{-3/2}\,, \\ C_z &= \frac{\varphi A_2}{H_1 H_2}\,, \qquad C_5 = \frac{\varphi A_2 B_5}{H_1 H_2}\,, \end{aligned} \tag{5.42}$$

where

$$\varphi = H_1 H_2/(A_2^2 + H_1^2 H_2)\,. \tag{5.43}$$

The nonzero components of the R-R-vector field in $D=10$ are denoted by C_μ. Note that φ is indeed not harmonic in x_1, x_3. If $H_2 = 1$ and $B_5 = 0$, φ does become harmonic, and we obtain a standard $D6$ solution, after the coordinate transformation

$$d(u+iv) = (H + iA_2)\, d(x_1 + ix_3). \tag{5.44}$$

Conversely, for $H_1 = 1$, $A_2 = 0$ a standard Kaluza-Klein monopole is obtained in $D=10$.

5.2 Multiple Intersections of Extended Objects

In this section we will generalize the results we found in the previous sections to intersections consisting of more then two extended objects. The conditions for pair intersections will form the basis to construct the multiple intersections. We will follow the same strategy as before, namely first we will construct the multiple D-brane and multiple M-brane intersections, and then see where we can add other objects.

For simplicity we will limit ourselves from now on to intersections of the first type, namely intersections that have $n=4$ relative transverse dimensions and where the harmonic functions depend on the overall transverse directions. For multiple intersections of the other types we refer to [20, 69].

5.2.1 Multiple D-brane Intersections

The construction of multiple D-brane intersections is completely determined by the harmonic function rule and the conditions for pair intersections of D-branes. Since our Ansatz describes threshold BPS bound states, which do not exert forces on each other, one can always remove all but two D-branes in the multiple intersection to infinity, without cost of energy. The remaining pair intersection should of course satisfy the conditions found in the previous section.

One can therefore follow an iterative procedure by adding to a given configuration an extra brane, such that is has $n=4$ relative transverse directions with all other branes.

In the language of (5.6), this means that we add an extra row with $\times$'s and $-$'s, such that the new row has four different entries of $\times$ and $-$ with every other row.

To streamline the construction, it is useful to characterize an intersection by the contents of the columns (components of the metric) corresponding to the relative transverse coordinates. These columns will be the building blocks of the intersections. For an N-brane intersection, a certain column will consist of k $\times$'s and $(N-k)$ $-$'s. Since T-duality replaces in a column all $\times$'s for $-$, it is not difficult to construct a T-duality invariant quantity: we define n_k as the number of columns with k $\times$'s or k $-$, where $k \leq [N/2]$. The square brackets indicate the integer part of $N/2$.

In an N-brane intersection ($N \geq 2$) there are $\frac{1}{2}N(N-1)$ pairs of intersecting branes. The total number of differences between $\times$ and $-$ in the N-brane intersection is therefore four times the number of pairs, or $2N(N-1)$. On the other hand, a column with k $\times$'s contributes $k(N-k)$ differences. Then we must have

$$\sum_{k=1}^{[N/2]} k(N-k)n_k = 2N(N-1)\,, \tag{5.45}$$

with $\sum_k n_k < 9$. Given N, this is an equation for the n_k.

Let us give a few examples. For $N=2$ there is only one type of building block, namely $k=1$. Equation (5.45) for this case reduces to the equation $n_1 = 4$, which is the condition for a stable threshold BPS bound state found in the previous section. For $N=3$ there is again only one type of building block ($k=1$) and we find $n_1 = 6$. For $N=4$, there are two types of building blocks, with $k=1$ and with $k=2$. Thus (5.45) reduces to $3n_1 + 4n_2 = 24$ which has 3 solutions namely $(n_1, n_2) = (8,0), (4,3)$ and $(0,6)$. For $N=5$ there are again two types of building blocks with $k=1,2$ and we find $4n_1 + 6n_2 = 40$ leading to 2 solutions given by $(n_1, n_2) = (4,4)$ and $(1,6)$.

Clearly, (5.45) is only a necessary condition for the existence of a solution. Given a set of n_k allowed by (5.45), it is not clear that one can actually realize such a solution in terms of the available building blocks and consistent with condition for pair intersections. This is because (5.45) is just an expression for the total number of differences in $\times$ and $-$, but does not contain information about how the configurations should be realized in terms of the no-force condition. Indeed it turns out that there exist solutions of (5.45) that do not correspond to intersecting configurations. However, (5.45) remains useful as a tool in the classification of multiple D-brane intersections.

Note that the numbers $(n_1, n_2, ...n_{[N/2]})$ form a good label for the classification: by construction the n_k's are invariant under T-duality and the set $(n_1, n_2, ...n_{[N/2]})$ labels a unique D-brane configuration,[7] up to T-duality and interchanges of rows and columns. The latter are in fact nothing else then a relabeling of the space-time coordinates and the harmonic functions.

The construction of multiple intersections of D-branes is now straightforward: start adding branes in all possible ways to a known intersection, such that the harmonic

[7] Although this has not been proved rigorously, the uniqueness can be seen in a case by case analysis.

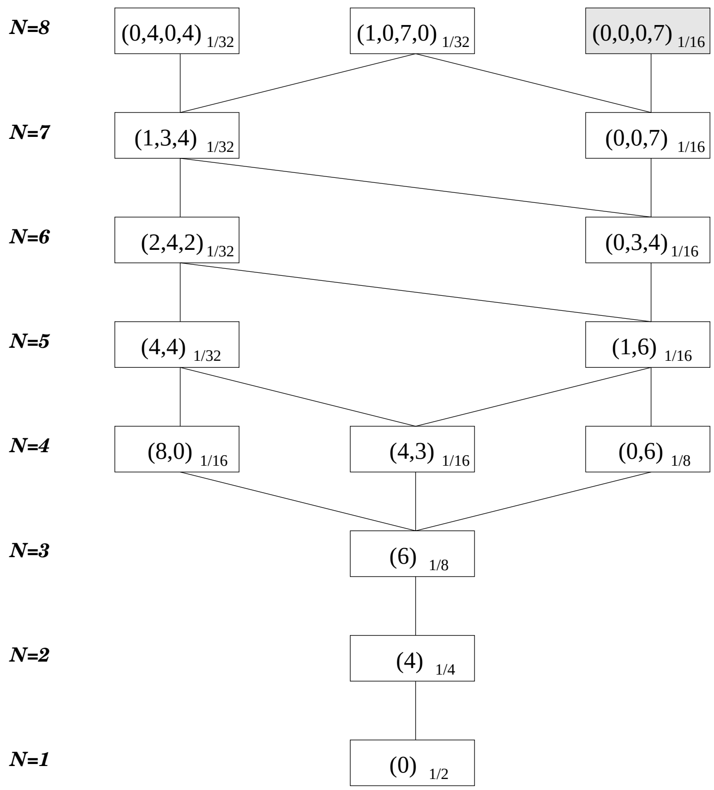

Figure 5.1: *D-brane intersections with $n = 4$ in 10 dimensions: the numbers $(n_1, n_2, \ldots)$ label the number of times a building block with $(1, 2, \ldots)$ world volume directions is used. The subscript in the Figure indicates the amount of supersymmetry preserved in each solution. The number N indicates the number of independent harmonics. The lines between solutions indicate how one configuration follows from another by adding (or truncating) a harmonic function. The configuration (0,0,0,7) cannot be extended to 11 dimensions in terms of $M2$- and $M5$-branes only.*

function rule and equation (5.45) are satisfied. The label $(n_1, n_2, ...n_{[N/2]})$ will tell to which T-duality class the new intersection belongs.

We can repeat this analysis till $N = 8$, for which we find three different (T-inequivalent) configurations. At this point the procedure stops. Although (5.45) has solutions for $N = 9$, it turns out to be impossible to add a ninth brane such that it has $n = 4$ relative transverse directions with the eight other branes. An overview of the different intersection classes and their relations is given in Figure 5.1, the three $N = 8$ configurations are given by (all other configurations with $N < 8$ can be obtained via truncation of harmonic function in the above configurations) [20]:

$$(0,4,0,4): \quad \left\{ \begin{array}{c|cc|cc|cc|cc|c}
\times & - & - & - & - & - & - & - & - & - \\
\times & \times & \times & \times & \times & - & - & - & - & - \\
\times & \times & \times & - & - & \times & \times & - & - & - \\
\times & - & - & \times & \times & \times & \times & - & - & - \\
\times & \times & - & \times & - & \times & - & \times & - & - \\
\times & - & \times & \times & - & \times & - & - & \times & - \\
\times & \times & - & \times & - & - & \times & - & \times & - \\
\times & - & \times & \times & - & - & \times & \times & - & -
\end{array} \right. \tag{5.46}$$

$$(1,0,7,0): \quad \left\{ \begin{array}{c|cc|cc|cc|cc|c}
\times & - & - & - & - & - & - & - & - & - \\
\times & \times & \times & \times & \times & - & - & - & - & - \\
\times & \times & \times & - & - & \times & \times & - & - & - \\
\times & - & - & \times & \times & \times & \times & - & - & - \\
\times & \times & - & \times & - & \times & - & \times & - & - \\
\times & - & \times & \times & - & \times & - & - & \times & - \\
\times & - & \times & - & \times & \times & - & \times & - & - \\
\times & - & \times & \times & - & - & \times & \times & - & -
\end{array} \right. \tag{5.47}$$

$$(0,0,0,7): \quad \left\{ \begin{array}{c|cc|cc|cc|cc|c}
\times & - & - & - & - & - & - & - & - & - \\
\times & \times & \times & \times & \times & - & - & - & - & - \\
\times & \times & \times & - & - & \times & \times & - & - & - \\
\times & - & - & \times & \times & \times & \times & - & - & - \\
\times & \times & - & \times & - & \times & - & \times & - & - \\
\times & \times & - & - & \times & - & \times & \times & - & - \\
\times & - & \times & \times & - & - & \times & \times & - & - \\
\times & - & \times & - & \times & \times & - & \times & - & -
\end{array} \right. \tag{5.48}$$

At this stage one should still check whether the above configurations satisfy the Einstein equation and the dilaton equations of motion. This can be done for the three $N = 8$ configurations, using the computer. This implies that the intersections with $N \geq 5$ are also solutions. For lower N the number of overall transverse coordinates increases, so that the harmonic functions can depend on more coordinates. One can check that the equations of motion indeed allow this.

Let us now consider the supersymmetry of the solutions. Just as for the pair intersections, the solution is supersymmetric if $\delta\lambda = \delta\psi_\mu = 0$ (5.15). Each brane contributes a projection operator (5.16) on ϵ, and each time we add a new projection operator, half of the remaining supersymmetry gets broken. However, sometimes it is possible to add a D-brane in such a way that its projection operator is not independent, but given by a product of previous operators [76, 106, 70]. In that case no additional supersymmetry generator is broken. In Figure 5.1 we see this happen for example in the $N = 4$ intersection. For $N = 3$ we have one 0-brane and two 4-branes which preserve 1/8th of the supersymmetry because of the three independent projection conditions

$$\begin{aligned}
&(1 + \gamma_0)\epsilon = 0\,, \\
&(1 + \gamma_{01234})\epsilon = 0\,, \\
&(1 + \gamma_{01256})\epsilon = 0\,.
\end{aligned} \tag{5.49}$$

From Figure 5.1 we see that there are three different ways to add a fourth brane. Two of them break an extra half of the remaining supersymmetry (configurations (8,0) and (4,3)), since in these cases the new brane introduces an independent projection operator. The third way (corresponding to configuration (0,6)) is by adding a 4-brane oriented in such a way that its projection operator

$$(1 + \gamma_{03456})\epsilon = 0 \tag{5.50}$$

is exactly the product of the previous three operators (5.49). In this way no extra conditions on the Killing spinor arise and no more supersymmetry gets broken.

The construction of projection operators for supersymmetry is another way of building up Figure 5.1. Apparently supersymmetry and the equations of motion go hand in hand: supersymmetry protects the stability of a configuration and vice versa, all stable solutions are supersymmetric. For a more systematic approach on how supersymmetry can be used to obtain intersections, we refer to [54]. The amount of unbroken supersymmetry of each configuration can be found in Figure 5.1.

By using T-duality one can express all intersections in terms of $D2$- and $D4$-branes, except the $N = 8$ $(0,0,0,7)$ solution. Writing an intersection in terms of $D2$- and $D4$-branes has the advantage that an uplifting to eleven dimensions is straightforward in terms of $M2$- and $M5$-branes. As we will see in the next subsection, the uplifting of the $N = 8$ $(0,0,0,7)$ solution is a little more involved, since it requires the presence of a eleven-dimensional gravitational wave. This solution is indicated by a grey box in Figure 5.1.

5.2.2 Multiple Intersections in Eleven Dimensions

Intersections consisting of $D2$- and $D4$-branes can be rewritten straightforwardly in eleven dimensions in terms of $M2$- and $M5$-branes. However, as we have seen in the previous subsection, not all intersection classes can be written in as a purely $D2$-$D4$ intersection. From the point of view of the relation between $D = 11$ supergravity and Type IIA theory, we would like to have to have an eleven dimensional interpretation for these solutions as well.

In general, if there is really a one to one map between eleven-dimensional supergravity and Type IIA theory, then we expect all intersections that involve $D0$- and $D6$-branes to be directly related to an eleven-dimensional solution, and not indirectly via a T-duality transformation to a $D2$-$D4$ intersection.

In this subsection we will give a classification of the eleven-dimensional intersections that reduce to intersections of D-branes in ten dimensions with the harmonic functions depending on the overall transverse directions.. We first give a classification of $M2$-$M5$ intersections and relate them to the $D2$-$D4$ intersection of the previous subsection. Then we will see how we can add wave and monopole solutions, in order to give a $D = 11$ interpretation for the other D-brane intersections.

The M-brane pair intersections that satisfy the eleven-dimensional equivalent of the $n = 4$ condition, are the ones we found in (5.21-5.23): $(0|M2, M2), (1|M2, M5)$ and $(3|M5, M5)$. Next, we add further $M2$-branes and/or $M5$-branes, always satisfying this intersection condition for each pair. Like in $D = 10$, we find that this procedure stops at $N = 8$. We will not present the details of our constructive procedure but instead present the results below. In this way we recover the $M2$-$M5$ intersections which are the direct uplifting of the D-brane intersections found in the previous subsection, but also some extra one, which cannot be reduced to pure D-brane intersections in ten dimensions. One can go from M-branes in $D = 11$ to D-branes in $D = 10$ only if there is one specific direction, such that all $M2$-branes are reduced to $D2$-branes, and all $M5$-branes to $D4$-branes. This will not be true in general: some configurations (which have $N \geq 4$) in $D = 11$ will only reduce to $D = 10$ intersections that involve NS-NS branes. Although these intersections do not have direct relevance for our original motivation (construct the eleven-dimensional version of the D-brane intersections), we will list them here anyway for the sake of completeness. In this way we can give a complete classification of intersecting M-branes with overall transverse dependence of the harmonic functions.

To characterize the configurations, we use again the contents of the columns in the representation of the metric. For an N-intersection each column can have $1, \ldots, N$ $\times$'s, indicating world volume directions. The numbers of columns with k world volume directions label the solutions, in the notation $\{n_1, \ldots, n_N\}$ (using curly brackets). It is convenient to classify, in a first stage, the eleven-dimensional intersections up to T-duality. T-duality works as follows in $D = 11$ [26]: two $D = 11$ solutions are called T-dual if, upon reduction to $D = 10$ dimensions they lead to T-dual D-brane configurations. These T-dual $D = 11$ solutions can be represented by the labels $(n_1, \ldots, n_{[N/2]})$ (using round brackets) which were used in the previous section to label T-dual D-brane configurations. Of course, this notation can only be used for $D = 11$ intersections that can be reduced to D-branes only. For the other classes we will stick to the curly bracket notation.

The results we find in $D = 11$ can be represented in three different ways [20]. First of all, in Figure 5.2 we present the solutions up to T-duality in $D = 11$. For those M-brane intersections that reduce to one of the D-brane intersections given in Figure 5.1, we use the same notation $(n_1, \cdots, n_{[N/2]})$ as in the previous Section. The gray rectangles indicate the solutions which necessarily contain NS-NS-branes in $D = 10$, and for those the $D = 11$ notation $\{n_1, \cdots, n_N\}$ is used. Secondly, in Table 5.4 more details are given about the contents of Figure 5.2 by showing all $D = 11$ solutions that correspond to the same $D = 10$ D-brane intersection. Finally, we give the $N = 8$ intersections explicitly:

$$\{0,4,0,5,0,0,0,0\}_{1/32}: \quad \left\{ \begin{array}{c|cc|cc|cc|cc|cc}
\times & \times & \times & - & - & - & - & - & - & - & - \\
\times & - & - & \times & \times & - & - & - & - & - & - \\
\times & - & - & - & - & \times & \times & - & - & - & - \\
\times & - & - & - & - & - & - & \times & \times & - & - \\
\times & - & \times & \times & - & \times & - & \times & - & \times & - \\
\times & \times & - & - & \times & \times & - & \times & - & \times & - \\
\times & \times & - & \times & - & - & \times & \times & - & \times & - \\
\times & \times & - & \times & - & \times & - & - & \times & \times & -
\end{array} \right. \tag{5.51}$$

$$\{1,0,6,1,1,0,0,0\}_{1/32}: \quad \left\{ \begin{array}{c|cc|cc|cc|cc|cc}
\times & \times & \times & - & - & - & - & - & - & - & - \\
\times & - & - & \times & \times & - & - & - & - & - & - \\
\times & - & - & - & - & \times & \times & - & - & - & - \\
\times & - & - & - & - & - & - & \times & \times & - & - \\
\times & - & \times & - & \times & \times & - & - & \times & \times & - \\
\times & \times & - & - & \times & \times & - & \times & - & \times & - \\
\times & - & \times & \times & - & \times & - & \times & - & \times & - \\
\times & \times & - & \times & - & \times & - & - & \times & \times & -
\end{array} \right. \tag{5.52}$$

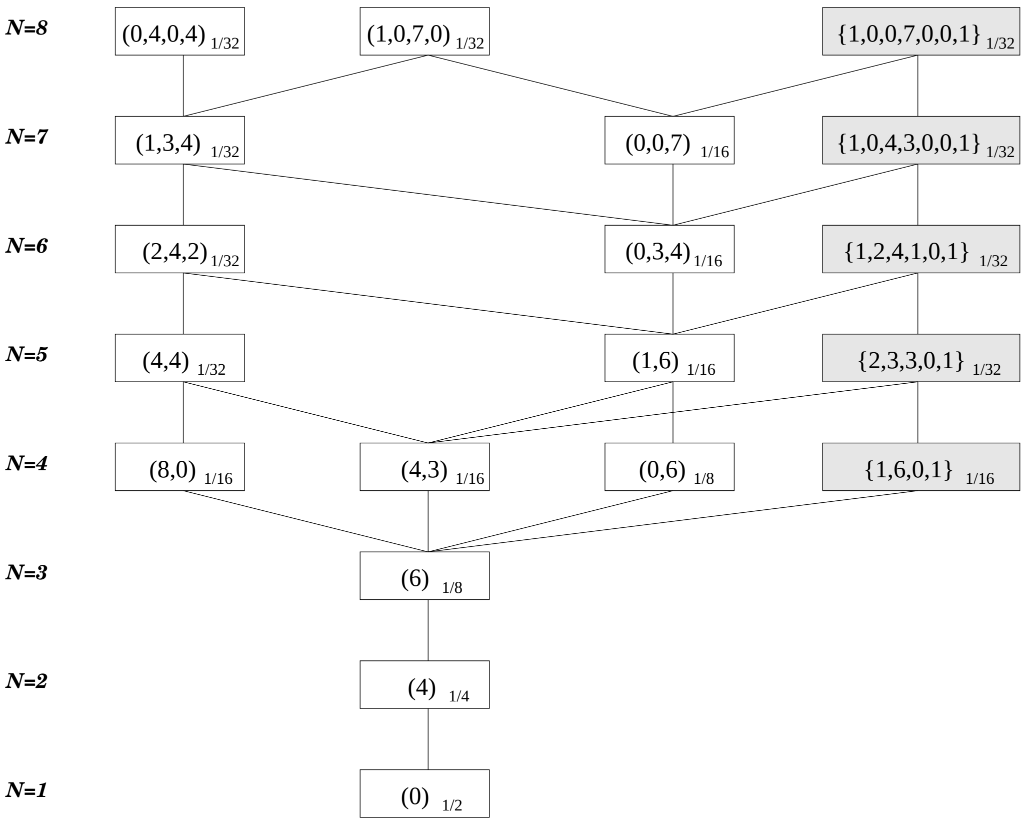

Figure 5.2: *M-brane intersections with $n = 4, 5$ in 11 dimensions: the numbers $(n_1, \cdots, n_{[N/2]})$ are the same labels used in $D = 10$, and indicate to which D-brane intersection the $D = 11$ solution reduces. The configurations in gray rectangles only reduce to $D = 10$ intersections involving NS-NS branes. For these configurations we use the eleven-dimensional notation $\{n_1, \cdots, n_N\}$ explained in the text. The subscripts indicate the amount of residual supersymmetry.*

$$\{1,0,0,7,0,0,0,1\}_{1/32} : \quad \left\{ \begin{array}{c|cc|cc|cc|cc|cc} \times & \times & \times & - & - & - & - & - & - & - & - \\ \times & \times & - & \times & \times & \times & \times & - & - & - & - \\ \times & \times & - & \times & \times & - & - & \times & \times & - & - \\ \times & \times & - & - & - & \times & \times & \times & \times & - & - \\ \times & \times & - & \times & - & \times & - & \times & - & \times & - \\ \times & \times & - & - & \times & - & \times & \times & - & \times & - \\ \times & \times & - & - & \times & \times & - & - & \times & \times & - \\ \times & \times & - & \times & - & - & \times & - & \times & \times & - \end{array} \right. \tag{5.53}$$

Again the explicit form of all other intersections with $N < 8$ can be obtained via truncation of these configurations. It can be checked that these intersections indeed solve the equations of motion.

As in $D = 10$, the complete structure of the $D = 11$ intersections can be recovered

N=8	**(0,4,0,4)**	**(1,0,7,0)**	$\mathbf{\{1,0,0,7,0,0,0,1\}}$	
	$[2^4,5^4]\{0,4,0,5,0,0,0,0\}$	$[2^4,5^4]\{1,0,6,1,1,0,0,0\}$	$[2^1,5^7]\{1,0,0,7,0,0,0,1\}$	
N=7	**(1,3,4)**	**(0,0,7)**	$\mathbf{\{1,0,4,3,0,0,1\}}$	
	$[5^7]\{1,0,4,0,3,0,1\}$ $[5^7]\ \{0,3,0,4,0,1,1\}$ $[2^3,5^4]\{1,2,4,1,1,0,0\}$ $[2^3,5^4]\{1,3,1,4,0,0,0\}$ $[2^4,5^3]\{1,3,4,1,0,0,0\}$	$[5^7]\{0,0,7,0,0,0,2\}$ $[5^7]\{0,0,0,7,0,0,1\}$ $[2^3,5^4]\{0,0,6,2,0,0,0\}$	$[2^1,5^6]\{1,0,4,3,0,0,1\}$	
N=6	**(2,4,2)**	**(0,3,4)**	$\mathbf{\{1,2,4,1,0,1\}}$	
	$[5^6]\{1,2,2,2,1,1\}$ $[2^2,5^4]\{1,4,2,1,1,0\}$ $[2^2,5^4]\{2,2,2,3,0,0\}$ $[2^3,5^3]\{2,3,3,1,0,0\}$ $[2^4,5^2]\{2,5,2,0,0,0\}$	$[5^6]\{0,0,4,3,0,1\}$ $[5^6]\{0,3,4,0,0,2\}$ $[2^2,5^4]\{0,2,4,2,0,0\}$ $[2^3,5^3]\{0,3,5,0,0,0\}$	$[2^1,5^5]\{1,2,4,1,0,1\}$	
N=5	**(4,4)**	**(1,6)**	$\mathbf{\{2,3,3,0,1\}}$	
	$[5^5]\{2,2,2,2,1\}$ $[2^1,5^4]\{3,1,3,2,0\}$ $[2^2,5^3]\{3,3,2,1,0\}$ $[2^3,5^2]\{4,3,2,0,0\}$ $[2^4,5^1]\{5,4,0,0,0\}$	$[5^5]\{1,4,2,0,2\}$ $[5^5]\{0,2,4,1,1\}$ $[2^1,5^4]\{0,4,2,2,0\}$ $[2^1,5^4]\{1,6,0,1,1\}$ $[2^2,5^3]\{1,3,4,0,0\}$ $[2^3,5^2]\{1,6,1,0,0\}$	$[2^1,5^4]\{2,3,3,0,1\}$	
N=4	**(8,0)**	**(4,3)**	**(0,6)**	$\mathbf{\{1,6,0,1\}}$
	$[2^2,5^2]\{6,1,2,0\}$ $[2^4]\{8,0,0,0\}$ $[5^4]\{4,0,4,1\}$	$[5^4]\{3,3,1,2\}$ $[5^4]\{1,3,3,1\}$ $[2^1,5^3]\{4,3,1,1\}$ $[2^1,5^3]\{2,3,3,0\}$ $[2^2,5^2]\{3,4,1,0\}$ $[2^3,5^1]\{5,3,0,0\}$	$[2^2,5^2]\{0,7,0,0\}$ $[5^4]\{0,6,0,2\}$	$[2^1,5^3]\{1,6,0,1\}$
N=3	**(6)**			
	$[5^3]\{6,0,3\}$ $[5^3]\{0,6,1\}$	$[5^3]\{3,3,2\}$ $[2^1,5^2]\{5,2,1\}$	$[2^1,5^2]\{2,5,0\}$ $[2^2,5^1]\{5,2,0\}$ $[2^3]\{6,0,0\}$	
N=2	**(4)**			
	$[5^2]\{4,3\}$	$[2^1,5^1]\{5,1\}$	$[2^2]\{4,0\}$	

Table 5.4: *Table of M-brane intersections in D=11: the number N indicates the number of independent harmonics. The boldface labels* $(n_1,\ldots,n_{[N/2]})$ *correspond to the* $D=10$ *D-brane intersection to which the* $D=11$ *solutions reduce (when applicable). The numbers between square brackets indicate the number of* $M2$*-branes and* $M5$*-branes involved in the intersection. The labels* $\{n_1,\ldots,n_N\}$ *specify the structure of the* $D=11$ *metric as explained in the text.*

by the requirement of partially unbroken supersymmetry [54]. Since the procedure is identical to the one used in $D = 10$ we will not give the details. The amount of unbroken supersymmetry for the different solutions is indicated in Figure 5.2.

Having determined the "no-force" condition between the basic eleven-dimensional solutions in Subsection 5.1.2, we next consider multiple intersections that also involve gravitational waves and Kaluza-Klein monopoles. We will again restrict ourselves to the configurations that can be reduced to intersections with only D-branes in $D = 10$. Looking back at Table 5.2, we see that all pairs involving monopoles should then be of the form $(2|M2, \mathcal{KK})$, $(3|M5, \mathcal{KK})$ or $(4|\mathcal{KK}, \mathcal{KK})^a$, and that with a wave only $(1|M5, \mathcal{W})$ may be used.

Our strategy will be to take Table 5.4 as our starting point and then to consider to which M-brane intersections waves and/or monopoles can be added. The rule for adding a wave is known [160, 134]: to each intersection involving at least a common string a wave can be added in such a way that the z-isometry direction of the wave lies in the space-like common string direction. Furthermore, at most one wave can be added to any given intersection.

From the intersection (5.26) we see that the world volume of the $M2$-brane must lie in the world volume directions of the monopole. Two intersecting $M2$-branes have distinct (space-like) world volume directions and since the monopole has six (space-like) world volume directions we conclude that monopoles may be added to configurations that contain at most three $M2$-branes [43]:

$$\left\{ \begin{array}{c|cccccccccc} \times & \times & \times & - & - & - & - & - & - & - & - \\ \times & - & - & \times & \times & - & - & - & - & - & - \\ \times & - & - & - & - & \times & \times & - & - & - & - \\ \times & \times & \times & \times & \times & \times & \times & z & A_8 & A_9 & A_{10} \end{array} \right. \tag{5.54}$$

We next consider the $M5$-branes. Using only the pair $(3|M5, \mathcal{KK})$ we see that the z-isometry direction of the monopole should lie in a common world volume direction of the $M5$-branes. One finds that to a single monopole one can add at most four $M5$-branes. An example of such a configuration is:

$$\left\{ \begin{array}{c|cccccccccc} \times & - & \times & \times & - & \times & - & \times & - & \times & - \\ \times & \times & - & - & \times & \times & - & \times & - & \times & - \\ \times & \times & - & \times & - & - & \times & \times & - & \times & - \\ \times & \times & - & \times & - & \times & - & - & \times & \times & - \\ \times & \times & \times & \times & \times & \times & \times & A_7 & A_8 & z & A_{10} \end{array} \right. \tag{5.55}$$

The harmonic functions depend only on the coordinate x_{10}. However, one may add more than one monopole to the four five-branes. From (5.55) it is clear that the monopole could also have been placed with two components of the vector field in the $(x_1, x_2), (x_3, x_4)$ or (x_5, x_6) directions. In fact, in this way one can combine four monopoles with the four $M5$-branes:

$$\left\{ \begin{array}{c|cccccccccc} \times & - & \times & \times & - & \times & - & \times & - & \times & - \\ \times & \times & - & - & \times & \times & - & \times & - & \times & - \\ \times & \times & - & \times & - & - & \times & \times & - & \times & - \\ \times & \times & - & \times & - & \times & - & - & \times & \times & - \\ \times & \times & \times & \times & \times & \times & \times & A_7 & A_8 & z & A_{10} \\ \times & \times & \times & \times & \times & B_5 & B_6 & \times & \times & z & B_{10} \\ \times & \times & \times & C_3 & C_4 & \times & \times & \times & \times & z & C_{10} \\ \times & D_1 & D_2 & \times & \times & \times & \times & \times & \times & z & D_{10} \end{array} \right. \tag{5.56}$$

N=8	**(0,4,0,4)**$_{SUSY=1/32}$	**(1,0,7,0)**$_{SUSY=1/32}$	**(0,0,0,7)**$_{SUSY=1/16}$
	$[2^4,5^4]\{\mathit{0,4,0,5,0,0,0}\}$	$[2^4,5^4]\{\mathit{1,0,6,1,1,0,0,0}\}$	$[2^3,5^4]\{\mathit{0,0,6,2,0,0,0}\}+\mathcal{KK}$
	$[2^3,5^4]\{\mathit{1,2,4,1,1,0,0}\}+\mathcal{KK}$	$[2^3,5^4]\{\mathit{1,3,1,4,0,0,0}\}+\mathcal{KK}$	$[2^1,5^4]\{\mathit{1,6,0,1,1}\}+3\mathcal{KK}$
	$[2^2,5^4]\{\mathit{2,2,2,3,0,0}\}+2\mathcal{KK}$	$[2^2,5^4]\{\mathit{1,4,2,1,1,0,0}\}+2\mathcal{KK}$	$[5^7]\{\mathit{0,0,0,7,0,0,1}\}+\mathcal{W}$
	$[2^1,5^4]\{\mathit{0,4,2,2,0}\}+3\mathcal{KK}$	$[2^2,5^4]\{\mathit{0,2,4,2,0,0}\}+2\mathcal{KK}$	
	$[5^4]\{\mathit{4,0,4,1}\}+4\mathcal{KK}$	$[2^1,5^4]\{\mathit{3,1,3,2,0}\}+3\mathcal{KK}$	
	$[5^7]\{\mathit{0,3,0,4,0,1,1}\}+\mathcal{W}$	$[5^4]\{\mathit{0,6,0,2}\}+4\mathcal{KK}$	
		$[5^7]\{\mathit{0,0,7,0,0,0,2}\}+\mathcal{W}$	
		$[5^7]\{\mathit{1,0,4,0,3,0,1}\}+\mathcal{W}$	

Table 5.5: *N=8 intersections that reduce to pure D-brane intersections: The boldface numbers indicate the ten dimensional T-duality class. The notation* $[2^k, 5^l] + n\mathcal{KK}$ *indicates that the intersections contain k M2-branes, l M5-branes and n monopoles. An additional wave is indicated by* $+\mathcal{W}$.

One may verify that this intersection is consistent with the $M5-\mathcal{KK}$ intersection rule (5.29) and the $\mathcal{KK}-\mathcal{KK}$ rule (5.31).

Having established the rule of how to add waves and monopoles to an intersection of $M2$-branes and $M5$-branes or a mixture thereof, we are able to list all intersections involving $M2$-branes, $M5$-branes, waves and monopoles. It is enough to give only the intersection with the largest number of independent harmonics. All other intersections can be obtained from these by setting one or more of the harmonic functions equal to one.

The result is given in Table 5.5 [21]. The maximum number of intersecting objects N equals eight if we restrict ourselves to configurations which can be reduced to pure D-brane intersections in $D=10$. This is not surprising, since the maximum number of intersecting D-branes is also $N=8$. To label the different configurations we use the M-brane notation for the intersecting brane part and indicate with $+n\mathcal{KK}$ and $+\mathcal{W}$ the waves and monopoles added to the solution. Furthermore we have divided the different solutions in classes, corresponding to the T-duality classes of the D-branes in six dimensions. In Table 5.5 we have also indicated the unbroken supersymmetry which directly follows from the unbroken supersymmetry of the corresponding D-brane intersection. Note the the solution $[5^7]\{\mathit{0,0,7,0,0,0,2}\}+\mathcal{W}$ correspond to the uplifting of the $N=8$ D-brane intersection (5.48), the one that could not be described in eleven dimensions by $M2$ and $M5$-branes only.

It is instructive to consider also the pair $(1|M2,\mathcal{W})$. The reduction to $D=10$ will then necessarily include also NS-NS branes and will therefore go beyond our original motivation to find the intersections that reduce to strictly D-branes. However, this extra pair will allow us the complete the classes that are indicated by the grey colour in table 5.2 in terms of waves and monopoles. It turns out that there are three such maximum intersections. All other intersections follow by truncation of these ones. We find one intersection with $N=8$ and two intersections with $N=9$ independent harmonics:

$$
\begin{aligned}
N=8: &\qquad [2^1,5^6]\{\mathit{1,0,4,3,0,0,1}\}+\mathcal{W}\,,\\
N=9: &\qquad [2^1,5^7]\{\mathit{1,0,0,7,0,0,0,1}\}+\mathcal{W}\,,\\
&\qquad [2^1,5^4]\{\mathit{1,6,0,1,1}\}+3\mathcal{KK}+\mathcal{W}\,.
\end{aligned}
\tag{5.57}
$$

All three solutions have 1/32 unbroken supersymmetry. Interestingly enough we find intersections with nine independent harmonics. The two intersections with $N = 9$ are extensions of $N = 8$ intersections with 1/16 supersymmetry in Table 5.5.

The remaining intersection of the class with 1/16 supersymmetry in Table 5.5, $[2^3, 5^4] + \mathcal{KK}$, can also be extended to $N = 9$, 1/32 supersymmetric solutions but this necessarily requires the use of a pair from Table 5.3. For example, an additional five-brane can be added, giving 1/32 supersymmetry.

5.3 Dimensional Reductions of Intersections

A natural application of our results is the reduction of the M-brane and D-brane intersections we found in the previous two sections to p-branes in lower dimensions. This will lead to dilatonic p-brane solutions which can be understood as D- and/or M-brane bound states in $D = 10, 11$. The interpretation of lower-dimensional solutions in terms of bound states of D- and/or M-branes in $D = 10, 11$ is a useful tool for understanding the properties of these lower dimensional solutions, especially in the case of (extremal) black holes where it has opened up the possibility for a microscopic explanation of the Bekenstein-Hawking entropy in terms of p-branes and p-brane bound states [154].

The (Einstein frame) form of our reduced action (upon truncating the scalars coming from the reduction and identifying many of the gauge fields) for $D > 2$ will always be in the class of Lagrangians of the form [8]

$$\mathcal{L}_D = \sqrt{|g^{\mathrm{E}}|}\,\Big[R_{\mathrm{E}} + \tfrac{1}{2}(\partial\phi)^2 + \frac{(-)^{p+1}}{2(p+2)!}e^{a\phi}F^2_{(p+2)}\Big]. \tag{5.58}$$

With the Ansatz

$$\begin{aligned} ds^2_{\mathrm{E},D} &= H^{\alpha}ds^2_{p+1} - H^{\beta}ds^2_{d-p-1}, \\ e^{2\phi} &= H^{\gamma}, \\ F_{0..pi} &= \delta\;\partial_i H^{-1}, \end{aligned} \tag{5.59}$$

one finds the general p-brane solution ($D > 2$) [112]:

$$\begin{aligned} \alpha &= -\frac{4(D-p-3)}{\Delta(D-2)} \quad , \qquad \beta = \frac{4(p+1)}{\Delta(D-2)}, \\ \gamma &= \frac{4a}{\Delta} \quad , \qquad \delta^2 = \frac{4}{\Delta}, \end{aligned} \tag{5.60}$$

with

$$\Delta = a^2 + 2\frac{(p+1)(D-p-3)}{D-2}\,. \tag{5.61}$$

[8] For $D = 2$ there does not exist a transformation to go from the string frame to the Einstein frame. Therefore the calculations should be done in the string frame. For the details and the precise form of the solutions, we refer to [20].

The lower dimensional p-brane solutions which follow from the reduced D-brane and M-brane intersections (now containing only one independent harmonic function) must fall inside this class of solutions. A property of supersymmetric solutions is that [112, 111]:

$$\Delta = 4/N\,, \tag{5.62}$$

where N is an integer labeling the number of participating field strengths, or equivalently, the number of intersecting branes.

Any toroidal Kaluza-Klein reduction of the $D = 10, 11$ intersections will be a supersymmetry preserving p-brane solution in a lower dimension. Because the number of participating field strengths is equal to the number of intersecting branes we can immediately read off the dilatonic p-brane solution from (5.60) and (5.61).

For example, combining (5.62) and (5.61), we find that for the $D = 4$ black hole ($p = 0$) the possible dilaton couplings are [124, 106]

$$a = \sqrt{4/N - 1}. \tag{5.63}$$

We find four types of $D = 4$ (extremal) dilaton black holes preserving half of the supersymmetry with different values for a. These can therefore be interpreted as bound states of D-branes (M-branes) compactified on a six-torus (seven-torus) [124, 160, 106, 70, 9, 13]:

1. $a = \sqrt{3}$: compactification of a single D-brane
2. $a = 1$: compactification of two intersecting D-branes
3. $a = 1/\sqrt{3}$: compactification of three intersecting D-branes
4. $a = 0$: compactification of four intersecting D-branes

More precisely this corresponds to the compactification of the $N = 4$ (0,6) class of solutions (see Figure 5.1), upon identifying the different harmonic functions, and its truncations to intersections with lower N.

As another illustration, consider the $N = 8$ D-brane intersections (see Figure 5.1). We see that one of them, labeled by (0,0,0,7), can be naturally reduced to 0-branes in $D = 3$ by reducing over all relative transverse directions. Every truncation of this solution can of course also be reduced to 0-branes, giving rise to 8 different supersymmetry preserving solutions in $D = 3$. Doing the explicit Kaluza-Klein reduction we find that the different values of a representing the different solutions (the explicit solution can be determined using (5.60)) are given by

$$a = \sqrt{4/N}, \tag{5.64}$$

which is just (5.61) with $p = 0$, $D = 3$ and N running from 1 to 8. So we find eight supersymmetry preserving 0-branes in $D = 3$ (in contrast to the four 0-branes in $D = 4$) with the dilaton coupling given by (5.64) [113].

To see how many a-values correspond to a particular p-brane solution in D dimensions, one has to find the highest N intersection in the D- or M-brane intersections that

	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$
$D=6$	2	2	2	2	2
$D=5$	3	3	4	3	–
$D=4$	4	7	7	–	–
$D=3$	8	8	–	–	–
$D=2$	9	–	–	–	–

Table 5.6: *Bound state interpretation of dilatonic p-branes in $D \leq 6$ dimensions: the numbers in the table give the number of dilatonic p-brane solutions in D dimensions with different values for the dilaton coupling, coming from different intersections in higher dimensions.*

can be reduced to a single p-brane in a lower dimension. The p-brane solutions in the lower dimension are given by (5.60) and (5.61) with $\Delta = 4/N$. Note that N is the only parameter, and that therefore different configurations of intersecting D- or M-branes with the same N, will all reduce to the same p-brane in lower dimensions upon identification of the harmonic functions (even if the $D = 10, 11$ intersecting solutions preserve different amounts of supersymmetry).

All p-brane solutions in lower dimensions preserve half of the maximal (lower-dimensional) supersymmetry in contrast to the intersecting D- or M-intersections in $D = 10, 11$. This gain in supersymmetry is a result of the identification of the different harmonics (equal charges). For an overview of the number of dilatonic p-brane solutions in lower dimensions ($D \leq 6$) with an interpretation as bound states of Table 5.5 or their truncations, we refer to Table 5.6. Many of the solutions that arise in the reduction and are listed in Table 5.6 were given in [112, 111, 113, 124, 160, 106, 24, 108, 21].

Chapter 6

World Volume Actions

In this chapter we will study the world volume theory, and more in particular the world volume actions of the extended objects we encountered in the previous chapters. These effective actions describe the dynamics of the objects and their energy-momentum tensor occurs as a source term in the equations of motion of the solutions. Since there exist all kinds of duality relations between the different solutions, the same relations should connect the various world volume actions to each other.
The aim of this chapter is to see how some of these dualities are realized. In Section 6.1, we derive via dimensional reduction the form of the world volume actions of the fundamental string, the $D2$-brane, the solitonic five-brane and the $D4$-brane from the world volume actions of the $M2$ and $M5$-brane. In Section 6.2 we investigate the T-duality map between the world volume actions of the ten-dimensional gravitational wave and the fundamental string and in Section 6.3 we use the T-duality between the solitonic five-brane and the Kaluza-Klein monopole to construct the world volume action of the monopole.

Part of the work presented here can also be found in [94].

6.1 Type IIA Branes from $D = 11$

In section 3.2 we have seen that there exists a direct relation between $D = 11$ supergravity (2.37) and Type IIA theory (2.33): the latter can be obtained from the former via a dimensional reduction over a circle S^1. The different extended objects that appear as solutions of the Type IIA theory can be interpreted as direct and double dimensionally reduced objects from eleven-dimensional supergravity [156], as shown in Figure 3.1.

This implies of course also that the world volume actions of the Type IIA extended objects, presented in Section 2.3, should be related to the world volume actions of the $D = 11$ supergravity solutions. In this section, we will show that the world volume actions for the fundamental string (2.46) and for the $D2$-brane (2.60) can be obtained from the $M2$-action and the $S5$ and $D4$-action from the action of the $M5$.

We will not discuss in this section the world volume actions of the gravitational wave or the Kaluza-Klein monopole and the ten-dimensional objects they reduce to. The ten-dimensional gravitational wave and monopole will be discussed in Section 6.2 and Section 6.3 respectively, where they are constructed making use of the T-duality relations with the fundamental string and the solitonic five-brane.

6.1.1 The Membrane Action

Let us consider the bosonic part of the $M2$-brane action of eleven-dimensional supergravity, given by [30]:

$$\begin{aligned} S_{M2} = -\tfrac{1}{2}\int d^3\sigma\ \sqrt{|\det(\partial_{\hat\imath}\hat X^{\hat\mu}\partial_{\hat\jmath}\hat X^{\hat\nu}\hat g_{\hat\mu\hat\nu})|} \\ + \tfrac{1}{6}\int d^3\sigma\ \varepsilon^{\hat\imath\hat\jmath\hat k}\partial_{\hat\imath}\hat X^{\hat\mu}\partial_{\hat\jmath}\hat X^{\hat\nu}\partial_{\hat k}\hat X^{\hat\rho}\hat C_{\hat\mu\hat\nu\hat\rho}\ . \end{aligned} \tag{6.1}$$

The $\hat X^{\hat\mu}$ ($\hat\mu = 0, 1, ..., 10$) are the target space embedding coordinates and the $\sigma^{\hat\imath}$ ($\hat\imath = 0, 1, 2$) the world volume coordinates on the brane. The $D = 11$ supergravity background fields induce a metric and a three-form gauge field on the world volume:

$$\begin{aligned} \hat g_{\hat\imath\hat\jmath} &= \partial_{\hat\imath}\hat X^{\hat\mu}\partial_{\hat\jmath}\hat X^{\hat\nu}\hat g_{\hat\mu\hat\nu}\ , \\ \hat C_{\hat\imath\hat\jmath\hat k} &= \partial_{\hat\imath}\hat X^{\hat\mu}\partial_{\hat\jmath}\hat X^{\hat\nu}\partial_{\hat k}\hat X^{\hat\rho}\hat C_{\hat\mu\hat\nu\hat\rho}\ . \end{aligned} \tag{6.2}$$

The world sheet action for the fundamental string is obtained via dimensional reduction of (6.1) over a world volume direction [59]. Therefore we identify one of the embedding coordinates with a world volume direction

$$\sigma^{\hat\imath} = (\sigma^i, \sigma), \qquad \hat X^{\hat\mu} = (X^\mu, \sigma), \tag{6.3}$$

where the world volume indices now run over $i = 0, 1$ and the target space indices $\mu = 0, ...9$. Using the reduction rules (3.72) between ten and eleven dimensions, the induced metric and gauge field can be expressed in terms of the ten-dimensional fields as:

$$\begin{aligned} \hat g_{\sigma\sigma} &= (-e^{4\phi/3})\ , \\ \hat g_{i\sigma} &= \partial_i X^\mu(-e^{4\phi/3}A^{(1)}_\mu)\ , \\ \hat g_{ij} &= \partial_i X^\mu\partial_j X^\nu(e^{-2\phi/3}g_{\mu\nu} - e^{4\phi/3}A^{(1)}_\mu A^{(1)}_\nu)\ , \\ \varepsilon^{\hat\imath\hat\jmath\hat k}\ &\partial_{\hat\imath}\hat X^{\hat\mu}\partial_{\hat\jmath}\hat X^{\hat\nu}\partial_{\hat k}\hat X^{\hat\rho}\ \hat C_{\hat\mu\hat\nu\hat\rho} = 3\ \varepsilon^{ij}\ \partial_i X^\mu\partial_j X^\nu\ B_{\mu\nu}\ . \end{aligned} \tag{6.4}$$

Making use of the formula (3.24), it is easy to see that the square root of the determinant in (6.1) reduces as

$$\sqrt{|\det\hat g_{\hat\imath\hat\jmath}|} = \sqrt{|\det g_{ij}|}, \tag{6.5}$$

so that the reduced action is the world sheet action for a fundamental string (2.27):

$$S_{F1} = -\tfrac{1}{2}\int d^2\sigma \, \sqrt{|\det(\partial_i X^\mu \partial_j X^\nu g_{\mu\nu})|} + \tfrac{1}{2}\int d^2\sigma \, \varepsilon^{ij}\partial_i X^\mu \partial_j X^\nu B_{\mu\nu} \, . \tag{6.6}$$

The action for the $D2$-brane can be derived from the action (6.1) via direct reduction over a direction transverse to the brane [137, 157]. To show this we start from the Howe-Tucker formulation [55, 89] of the $M2$ action, which makes use of an auxiliary world volume metric $\hat{\gamma}_{\hat{\imath}\hat{\jmath}}$ and is (classically) equivalent to (6.1) [137]:

$$\begin{aligned} S_{M2} = -\tfrac{1}{2}\int d^3\sigma \, \sqrt{|\hat{\gamma}|} \, \left\{\hat{\gamma}^{\hat{\imath}\hat{\jmath}}\partial_{\hat{\imath}}\hat{X}^{\hat{\mu}}\partial_{\hat{\jmath}}\hat{X}^{\hat{\nu}}\hat{g}_{\hat{\mu}\hat{\nu}} - 1\right\} \\ + \tfrac{1}{6}\int d^3\sigma \, \varepsilon^{\hat{\imath}\hat{\jmath}\hat{k}}\partial_{\hat{\imath}}\hat{X}^{\hat{\mu}}\partial_{\hat{\jmath}}\hat{X}^{\hat{\nu}}\partial_{\hat{k}}\hat{X}^{\hat{\rho}}\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}} \, . \end{aligned} \tag{6.7}$$

Since we are reducing over a direction orthogonal to the brane, the world volume metric on the brane does not change: $\hat{\gamma}_{\hat{\imath}\hat{\jmath}} = \gamma_{\hat{\imath}\hat{\jmath}}$. Splitting the embedding coordinates $\hat{X}^{\hat{\mu}}$ in the ten-dimensional embedding coordinates X^μ and a world volume scalar S, we find for the reduced action:

$$\begin{aligned} S_1 = -\tfrac{1}{2}\int d^3\sigma\sqrt{|\gamma|} \, \left\{e^{-\frac{2}{3}\phi}\gamma^{\hat{\imath}\hat{\jmath}}g_{\hat{\imath}\hat{\jmath}} - e^{\frac{4}{3}\phi}\gamma^{\hat{\imath}\hat{\jmath}}F_{\hat{\imath}}F_{\hat{\jmath}} - 1\right\} \\ + \tfrac{1}{6}\int d^3\sigma \, \varepsilon^{\hat{\imath}\hat{\jmath}\hat{k}}\left\{\tfrac{3}{2}C_{\hat{\imath}\hat{\jmath}\hat{k}} + 3B_{\hat{\imath}\hat{\jmath}}F_{\hat{k}} - 3B_{\hat{\imath}\hat{\jmath}}A^{(1)}_{\hat{k}}\right\} \, , \end{aligned} \tag{6.8}$$

where $B_{\hat{\imath}\hat{\jmath}}$ is the pull-back of $B_{\mu\nu}$ and $F_{\hat{\imath}}$ is the gauge invariant field strength of the world volume scalar S:

$$F_{\hat{\imath}} = \partial_{\hat{\imath}}S + \partial_{\hat{\imath}}X^\mu A^{(1)}_\mu . \tag{6.9}$$

In order to relate the action (6.8) to the action of the $D2$-brane, we have to replace $F_{\hat{\imath}}$ by its world volume Poincaré dual, which is done by considering $F_{\hat{\imath}}$ as an independent field and imposing its Bianchi identity via the Lagrange multiplier term

$$S_2 = \int d^3\sigma \, \varepsilon^{\hat{\imath}\hat{\jmath}\hat{k}} \, V_{\hat{\imath}} \, \left\{\partial_{\hat{\jmath}}F_{\hat{k}} - \partial_{\hat{\jmath}}A^{(1)}_{\hat{k}}\right\} \, . \tag{6.10}$$

The equation of motion for $F_{\hat{\imath}}$

$$F_{\hat{\imath}} \quad = \quad \frac{1}{\sqrt{|\gamma|}} \, e^{-\frac{3}{4}\phi}\gamma_{\hat{\imath}\hat{\jmath}} \, \varepsilon^{\hat{\jmath}\hat{k}\hat{l}} \, (\partial_{\hat{k}}V_{\hat{l}} - \partial_{\hat{l}}V_{\hat{k}} - B_{\hat{k}\hat{l}}) \tag{6.11}$$

expresses $F_{\hat{\imath}}$ in terms of its Poincaré dual $\mathcal{F}_{\hat{\imath}\hat{\jmath}}$

$$\mathcal{F}_{\hat{\imath}\hat{\jmath}} \quad = \quad \partial_{\hat{\imath}}V_{\hat{\jmath}} - \partial_{\hat{\jmath}}V_{\hat{\imath}} - B_{\hat{\imath}\hat{\jmath}}. \tag{6.12}$$

Substituting (6.9) and (6.11) in the action and redefining the world volume metric $\gamma_{\hat{\imath}\hat{\jmath}} \to e^{-2\phi/3}\gamma_{\hat{\imath}\hat{\jmath}}$, we find for the dual action in terms of the world volume vector $V_{\hat{\imath}}$:

$$\begin{aligned} S_{D2} = -\tfrac{1}{2}\int d^3\sigma \, e^{-\phi}\sqrt{|\gamma|} \, \left\{\gamma^{\hat{\imath}\hat{\jmath}}g_{\hat{\imath}\hat{\jmath}} + \tfrac{1}{2}\mathcal{F}_{\hat{\imath}\hat{\jmath}}\mathcal{F}^{\hat{\imath}\hat{\jmath}} - 1\right\} \\ + \tfrac{1}{4}\int d^3\sigma \, \varepsilon^{\hat{\imath}\hat{\jmath}\hat{k}}\left\{C_{\hat{\imath}\hat{\jmath}\hat{k}} + 2\mathcal{F}_{\hat{\imath}\hat{\jmath}}A_{\hat{k}}\right\} \, , \end{aligned} \tag{6.13}$$

which is the Howe-Tucker form of the world volume action for the $D2$-brane [157]. The world volume vector $V_{\hat{i}}$ is of course the Born-Infeld vector and the factor $e^{-\phi}$ indicates the property of D-branes that their mass is proportional to the inverse of the coupling constant $g = e^{<\phi>}$.

6.1.2 The Five-brane Action

The construction of the world volume actions for the $M5$ and the Type IIA $S5$ is a more subtle problem, due to the non-linearity of the kinetic term and the presence of a self-dual world volume two-form $\hat{W}^{+}_{\hat{i}\hat{j}}$ [72, 98]. The equations of motion and a fully covariant action, involving an auxiliary scalar field can be found in [87, 126, 10, 1, 88, 11]. In this section we will restrict ourselves to the non-self-dual action, up to quadratic order in $\hat{W}^{+}_{\hat{i}\hat{j}}$, presented in [23]:

$$\begin{aligned} S_{M5} = \int d^6\sigma \sqrt{|\det \hat{g}_{\hat{i}\hat{j}}|}\ \left(1 + \tfrac{1}{2}\hat{\mathcal{H}}_{\hat{i}\hat{j}\hat{k}}\hat{\mathcal{H}}^{\hat{i}\hat{j}\hat{k}}\right) \\ + \int d^6\sigma\ \varepsilon^{\hat{i}_1\ldots\hat{i}_6}\left(\tfrac{1}{70}\hat{C}_{\hat{i}_1\ldots\hat{i}_6} + \tfrac{3}{4}\partial_{\hat{i}_1}\hat{W}^{+}_{\hat{i}_2\hat{i}_3}\hat{C}_{\hat{i}_4\hat{i}_5\hat{i}_6}\right)\ . \end{aligned} \tag{6.14}$$

The six-form gauge field $\hat{C}_{\hat{\mu}_1\ldots\hat{\mu}_6}$ is the Poincaré dual of $\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}}$ in the dual $D = 11$ supergravity [1]. The tensor $\hat{\mathcal{H}}_{\hat{i}\hat{j}\hat{k}}$ is the field strength of the self-dual field $\hat{W}_{\hat{i}\hat{j}}$:

$$\hat{\mathcal{H}}_{\hat{i}\hat{j}\hat{k}} \quad = \quad \partial_{[\hat{i}}\hat{W}_{\hat{j}\hat{k}]} - \tfrac{1}{2}\hat{C}_{\hat{i}\hat{j}\hat{k}}\ . \tag{6.15}$$

The self-duality condition $\hat{\mathcal{H}} = {}^*\hat{\mathcal{H}}$ does not follow from (6.14), but has to be put in by hand as an extra equation of motion. Note that this procedure is analogous to the one used to write down an action for the Type IIB supergravity theory in Section 2.2 [17].

Double dimensional reduction, using (3.72), gives for the induced metric the same reduction rules as (6.4), only now $\sqrt{\hat{g}_{\hat{i}\hat{j}}} = e^{-\phi}\sqrt{g_{ij}}$. The components of the gauge fields reduce as

$$\hat{W}_{ij} = W_{ij}, \qquad \hat{W}_{i\sigma} = V_i, \qquad \hat{C}_{\mu_1\ldots\mu_5\sigma} = \tfrac{7}{6}\ C_{\mu_1\ldots\mu_5} \tag{6.16}$$

so that in ten dimensions the field strength tensors are of the form

$$\begin{aligned} \mathcal{H}_{ijk} &= 3\ (\partial_{[i}W_{jk]} - \tfrac{1}{2}C_{ijk} - A^{(1)}_{[i}\mathcal{F}_{jk]}), \\ \mathcal{F}_{ij} &= \partial_i V_j - \partial_j V_i - B_{ij}. \end{aligned} \tag{6.17}$$

The action (6.14) reduces then to [23]

$$S = \int d^5\sigma\ \sqrt{|\det g_{ij}|}\ \left\{e^{-\phi} + \tfrac{1}{2}e^{\phi}\mathcal{H}_{ijk}\mathcal{H}^{ijk} - \tfrac{3}{2}e^{-\phi}\mathcal{F}_{ij}\mathcal{F}^{ij}\right\}$$

[1] The construction of a dual formulation of $D = 11$ supergravity in terms of a six-form gauge field is a notorious problem: since the action cannot be written in terms of the field strength tensors only, but contains terms in which the gauge field $\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}}$ occurs explicitly, a dual formulations in terms of the dual gauge field $\hat{C}_{\hat{\mu}_1\ldots\hat{\mu}_6}$ has not been found yet. One could avoid the problem by considering the dual theory only on-shell, such that $\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}}$ can be eliminated via its equations of motion [2], or try to formulate the dual theory making use of an auxiliary field [29]. For our purposes it is sufficient to know that the dual field exists and a dual formulation can be written down.

$$+\int d^5\sigma\ \varepsilon^{i_1\dots i_5}\Big\{\tfrac{1}{10}C_{i_1\dots i_5}+\tfrac{3}{2}(\partial_{i_1}W_{i_2i_3}B_{i_4i_5}-\partial_{i_1}V_{i_2}C_{i_3i_4i_5})\Big\}\ . \tag{6.18}$$

The self-duality condition has reduced to a duality relation $\mathcal{H}=e^{-\phi}\ {}^*\mathcal{F}$ between the world volume one- and two-form, which can be used to consistently eliminate W_{ij} from the equations of motion of (6.18). It can be shown [23] that the then obtained equations follow from the action

$$\begin{aligned}S=&\int d^5\sigma\ e^{-\phi}\ \sqrt{|\det g_{ij}|}\ \Big\{1-\ 3\ \mathcal{F}^2\Big\}\\&+\int d^5\sigma\ \varepsilon^{(5)}\ \Big\{\tfrac{1}{10}C_{(5)}-3\partial VC_{(3)}+\tfrac{3}{4}BC_{(3)}-\tfrac{3}{2}A^{(1)}\mathcal{F}\mathcal{F}\Big\}\ ,\end{aligned} \tag{6.19}$$

which is precisely the action of the $D4$-brane [77] up to quadratic order in the kinetic term.

The action of the solitonic five-brane, obtained via direct reduction of the action of the $M5$, has the same subtleties as the $M5$ action since after dimensional reduction the world volume field $W_{\hat{\imath}\hat{\jmath}}$ still satisfies the self-duality condition $\mathcal{H}_{\hat{\imath}\hat{\jmath}\hat{k}}={}^*\mathcal{H}_{\hat{\imath}\hat{\jmath}\hat{k}}$. The reduction of (6.14) is straightforward: the induced metric and the world volume field strength reduce as

$$\begin{aligned}\hat{g}_{\hat{\imath}\hat{\jmath}}&=e^{-\frac{2}{3}\phi}g_{\hat{\imath}\hat{\jmath}}\ -\ e^{\frac{4}{3}\phi}F_{\hat{\imath}}F_{\hat{\jmath}}\ ,\\\hat{\mathcal{H}}_{\hat{\imath}\hat{\jmath}\hat{k}}&=3\ \partial_{[\hat{\imath}}W^+_{\hat{\jmath}\hat{k}]}+\tfrac{1}{2}C_{\hat{\imath}\hat{\jmath}\hat{k}}+3\ B_{[\hat{\imath}\hat{\jmath}}\partial_{\hat{k}]}S=\mathcal{H}_{\hat{\imath}\hat{\jmath}\hat{k}}\ ,\end{aligned} \tag{6.20}$$

with $F_{\hat{\imath}}$ as in (6.9). The action of the solitonic five-brane, to quadratic order, is of the form [29]:

$$\begin{aligned}S_{S5}=&\int d^6\sigma\ e^{-2\phi}\sqrt{|\det(g_{\hat{\imath}\hat{\jmath}}-e^{2\phi}F_{\hat{\imath}}F_{\hat{\jmath}})|}\ \Big\{1+e^{2\phi}\ \mathcal{H}^2\Big\}\\&+\int d^6\sigma\ \varepsilon^{(6)}\ \Big\{\tfrac{1}{70}C_{(6)}+\tfrac{1}{10}C_{(5)}\partial S+\tfrac{3}{4}\partial W(C_{(3)}+B\partial S)\Big\}\ .\end{aligned} \tag{6.21}$$

Note that the dilaton factor in front of the kinetic term indicates that the mass of the $S5$ is proportional to the inverse coupling constant squared, the typical behaviour of a solitonic object.

6.2 Wave/String Duality

In Section 3.1 we showed that the gravitational wave ($\mathcal{W}$) and the fundamental string solution ($F1$) were related via a T-duality transformation in the propagation direction of the wave or, the other way round, in the world volume direction of the string. To derive this T-duality we had to use a different procedure then in the original derivation [37], presented in Subsection 3.1.1, since the latter relates only fundamental string backgrounds to other fundamental string backgrounds. In order to relate other than fundamental string solutions to each other, we used the idea of T-duality via dimensional reduction: the gravitational wave and the fundamental string are dual to each other in ten dimensions because they can be mapped onto the same nine-dimensional solution, using two different (T-dual) reduction schemes. The ten-dimensional T-duality rules, obtained by relating the two reduction schemes, map one solution to the other. The dualized coordinate corresponds to the direction over which we have reduced and decompactified. We used the same procedure in Chapter 4 to relate the Type IIA and Type IIB and Heterotic actions in ten and six dimensions.

Concretely, the fundamental string solution (2.44) and the gravitational wave solution (2.64) can be reduced both onto the same nine-dimensional massive 0-brane solution (3.49)

$$m0 = \begin{cases} ds^2 = H^{-1}dt^2 - (dx_2^2 + \ldots + dx_9^2) \\ e^{-2\phi} = H^{\frac{1}{2}} \\ k = H^{-\frac{1}{2}} \\ B_0 = -H^{-1} \\ B_{\mu\nu} = A_\mu = 0\ , \end{cases} \tag{6.22}$$

if we take for the reduction rules for the $F1$

$$F1: \quad \begin{cases} \hat{g}_{\mu\nu} = g_{\mu\nu} - k^2 A_\mu A_\nu\ , & \hat{B}_{\mu\nu} = B_{\mu\nu} + A_{[\mu}B_{\nu]}\ , \\ \hat{g}_{x\mu} = -k^2 A_\mu\ , & \hat{B}_{x\mu} = B_\mu\ , \\ \hat{g}_{xx} = -k^2\ , & \hat{\phi} = \phi + \frac{1}{2}\log k, \end{cases} \tag{6.23}$$

while for the reduction scheme of the $\mathcal{W}$ we use the T-dual version

$$\mathcal{W}: \quad \begin{cases} \hat{g}_{\mu\nu} = g_{\mu\nu} - k^{-2} B_\mu B_\nu\ , & \hat{B}_{\mu\nu} = B_{\mu\nu} + B_{[\mu}A_{\nu]}\ , \\ \hat{g}_{x\mu} = -k^{-2} B_\mu\ , & \hat{B}_{x\mu} = A_\mu\ , \\ \hat{g}_{xx} = -k^{-2}\ , & \hat{\phi} = \phi - \frac{1}{2}\log k. \end{cases} \tag{6.24}$$

In the case of the $F1$ the x-direction is the world volume direction of the string, while for the $\mathcal{W}$ it corresponds to the propagation direction of the wave. It is easy to verify that the combination of the two reduction schemes gives the well-known T-duality rules (3.4).

The T-duality between the wave/string solutions suggests that a same type of duality exists between the world volume actions that describe the dynamics of these solutions. We will use the procedure of T-duality via direct and double dimensional reduction to construct the duality map between the actions.

We start from the Nambu-Goto form for the kinetic term for the fundamental string action together with a Wess-Zumino term (6.6)

$$\mathcal{L}_{F1} \;=\; \sqrt{|\det(\partial_i \hat{X}^{\hat{\mu}} \partial_j \hat{X}^{\hat{\nu}} \hat{g}_{\hat{\mu}\hat{\nu}})|} \;+\; \tfrac{1}{2}\,\varepsilon^{ij}\partial_i \hat{X}^{\hat{\mu}} \partial_j \hat{X}^{\hat{\nu}} \hat{B}_{\hat{\mu}\hat{\nu}} \tag{6.25}$$

The indices $i,\ j$ are the world volume indices $\tau,\ \sigma$. If we assume that the direction in which the string is oriented is compact and the string is wound m times around this direction, we can make a split in the target space coordinates as follows:

$$\hat{X}^{\hat{\mu}}(\tau,\sigma) = X^\mu(\tau),\ m\sigma\big). \tag{6.26}$$

This means that we have identified the world volume direction of the string with the space-time direction in which the string is oriented. Using this split in the coordinates and the reduction rules given in (6.23), we can rewrite the action (6.25) after double dimensional reduction as

$$
\begin{aligned}
\mathcal{L}_{F1} &= \left| \begin{array}{cc} \partial X^\mu \partial X^\nu\,(g_{\mu\nu} - k^2 A_\mu A_\nu) & m\partial X^\mu(-k^2 A_\mu) \\ m\partial X^\mu(-k^2 A_\mu) & m^2(-k^2) \end{array} \right|^{\frac{1}{2}} + \; m\,\varepsilon^{\sigma\tau}\partial X^\mu B_\mu \\
&= mk\,\sqrt{|\det(\partial X^\mu \partial X^\nu g_{\mu\nu})|} \; - \; m\,\partial X^\mu B_\mu, \qquad (6.27)
\end{aligned}
$$

which is the action for a massive particle with mass m. With ∂X^μ we mean the partial derivative of X^μ with respect to τ.

The world volume action of a gravitational wave is given by

$$\mathcal{L}_{\mathcal{W}} = \tfrac{1}{2}\sqrt{|\gamma|}\;\gamma^{-1}\;\partial\hat{X}^{\hat{\mu}}\;\partial\hat{X}^{\hat{\nu}}\;\hat{g}_{\hat{\mu}\hat{\nu}}. \qquad (6.28)$$

Direct dimensional reduction via the reduction scheme (6.24) gives an action of the form

$$\mathcal{L} = \tfrac{1}{2}\sqrt{|\gamma|}\;\gamma^{-1}\Big[\partial X^\mu \partial X^\nu g_{\mu\nu} \; - \; k^{-2}(\partial S + B_\mu \partial X^\mu)^2\Big], \qquad (6.29)$$

where S the world sheet scalar coming from the compact dimension: $S = \hat{X}^x$. It can be eliminated via its equations of motion

$$\partial\,\Big[\sqrt{|\gamma|}\gamma^{-1}\;k^{-2}\;(\partial S + B_\mu \partial X^\mu)\Big] = 0, \qquad (6.30)$$

or equivalently

$$\partial S = \sqrt{|\gamma|}\;\mu\;k^2 - B_\mu \partial X^\mu, \qquad (6.31)$$

where μ is a constant that corresponds to the momentum in the compactified direction. Before we can substitute this expression directly in (6.28) we have to verify whether this is consistent with the other equations of motion. It turns out that the substitution can be done if we add to the Lagrangian (6.28) a total derivative term [8, 31], giving

$$\mathcal{L} = \tfrac{1}{2}\sqrt{|\gamma|}\gamma^{-1}\partial X^\mu \partial X^\nu g_{\mu\nu} \; + \; \tfrac{1}{2}\sqrt{|\gamma|}\;\mu^2 k^2 \; - \; \mu\;B_\mu \partial X^\mu. \qquad (6.32)$$

Again we can go to the Nambu-Goto formulation eliminating the world line metric γ via its equation of motion

$$\gamma^{-1} = \frac{\mu^2 k^2}{\partial X^\mu \partial X^\nu g_{\mu\nu}}. \qquad (6.33)$$

This yields the nine-dimensional action for a massive particle with mass μ

$$\mathcal{L} = \mu\;k\;\sqrt{|\det(\partial X^\mu \partial X^\nu g_{\mu\nu})|} \; - \; \mu\;\partial X^\mu B_\mu, \qquad (6.34)$$

which is exactly the same Lagrangian as was obtained via T-dual double dimensional reduction from the fundamental string, provided that we identify the constants m and μ. Physically this means that the T-duality interchanges the winding number of the string with the momentum of the wave in the dualized direction.

Note that the world volume action (6.28) of the gravitational wave coincides with the world volume action for a massless particle. As a matter of fact the massless particle is the source of the gravitational wave solution: a massless particle, moving at the speed of light, drags along a gravitational wave as it moves through space. This can be seen best if we rewrite the gravitational wave solution (2.64) in light cone coordinates

$$u = \tfrac{1}{\sqrt{2}}(t+z), \qquad v = \tfrac{1}{\sqrt{2}}(t-z). \tag{6.35}$$

The solution (2.64) then takes the form

$$ds^2 = 2dudv + 2(1-H)du^2 - dx_m^2. \tag{6.36}$$

Consider now the action of a massless particle coupled to gravity

$$S = -\tfrac{1}{\kappa}\int d^{10}x\sqrt{|g|}\;R \quad -\tfrac{T}{2}\int d\sigma\sqrt{|\gamma|}\gamma^{-1}\partial X^\mu \partial X^\nu g_{\mu\nu}, \tag{6.37}$$

and the equation of motion of $g_{\mu\nu}$

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -\tfrac{\kappa T}{2\sqrt{|g|}}\int d\sigma\sqrt{|\gamma|}\gamma^{-1}\partial X_\mu \partial X_\nu\;\delta(X^\mu - x^\mu). \tag{6.38}$$

The gravitational wave solution (6.36) satisfies this equation of motion, if we choose the following parametrisation for the embedding coordinates:

$$U = 0, \qquad V = \tau, \qquad X^m = 0, \tag{6.39}$$

which are the embedding coordinates of a massless particle moving at the speed of light. The equation of motion then reduces to

$$\partial_m \partial^m H = -\tfrac{\kappa T}{2}\;\delta(u)\;\delta(x^m) \tag{6.40}$$

which has as a solution

$$H(u, x^m) = -\frac{\kappa T}{2}\;\frac{\delta(u)}{\sqrt{|x^m x_m|}^6}\,. \tag{6.41}$$

6.3 The Five-brane/Monopole Duality

Let us now make use of the known T-duality between the solitonic five-brane ($S5$) and the Kaluza-Klein monopole ($\mathcal{KK}_{10}$) in ten dimensions in order to construct a world volume action for the monopole. We will present the bosonic part of the world volume action for the Heterotic ($N=1$ supersymmetric) monopole. For the kinetic part of the eleven-dimensional and ten-dimensional Type IIA monopole action we refer to [28].

Our strategy will be similar to the one in the previous chapter: because of the T-duality we know that a correct action for the monopole is one that, upon a T-dual compactification, reduces to the nine-dimensional form of the $S5$. The reduction of the monopole must be performed over the isometry direction z, since the monopole solution (2.65) transforms into the five-brane solution (2.54) after dualization in this direction.

The reduction of the $N = 1$ five-brane world volume action is straightforward. Starting from the action (2.55)

$$\begin{aligned} S_{(S5)} &= -\tfrac{T}{2}\int d^6\sigma\ e^{-2\hat{\phi}}\sqrt{|\det(\partial_i \hat{X}^{\hat{\mu}}\partial_j \hat{X}^{\hat{\nu}}\hat{g}_{\hat{\mu}\hat{\nu}})|} \\ &\quad +\tfrac{T}{6!}\int d^6\sigma\ \varepsilon^{i_1\ldots i_6}\ \partial_{i_1}\hat{X}^{\hat{\mu}_1}\ldots\ \partial_{i_6}\hat{X}^{\hat{\mu}_6}\hat{C}_{\hat{\mu}_1\ldots\hat{\mu}_6}, \end{aligned} \tag{6.42}$$

and using the reduction rules

$$\begin{array}{ll} \hat{X}^{\mu} = X^{\mu}\,, & \hat{X}^{x} = S \\ \hat{g}_{\mu\nu} = g_{\mu\nu} - k^2 A_\mu A_\nu\,, & \hat{C}_{x\mu_1\ldots\mu_5} = D_{\mu_1\ldots\mu_5},, \\ \hat{g}_{x\mu} = -k^2 A_\mu\,, & \hat{C}_{\mu_1\ldots\mu_6} = C_{\mu_1\ldots\mu_6} + 6A_{[\mu_1}D_{\mu_2\ldots\mu_6]}\,, \\ \hat{g}_{xx} = -k^2\,, & \hat{\phi} = \phi + \frac{1}{2}\log k, \end{array} \tag{6.43}$$

we find for the reduced five-brane action

$$\begin{aligned} S &= -\tfrac{T}{2}\int d^6\sigma\ e^{-2\phi}k^{-1}\sqrt{|\det(g_{ij} - k^2F_iF_j)|} \\ &\quad +\tfrac{T}{6!}\int d^6\sigma\ \varepsilon^{i_1\ldots i_6}\ \Big[C_{i_1\ldots i_6} - 6\ D_{i_1..i_5}F_{i_6}\Big], \end{aligned} \tag{6.44}$$

where

$$\begin{aligned} g_{ij} &= \partial_i X^\mu \partial_j X^\nu g_{\mu\nu}, \\ F_i &= \partial_i S + A_\mu \partial_i X^\mu. \end{aligned} \tag{6.45}$$

Our task is now to find an action for the monopole that, upon the reduction (6.43) gives an action that is T-dual to (6.44), or equivalently, gives the same action upon T-dual reduction.

However, due to the presence of the isometry direction z in the monopole solution, a subtlety occurs in the counting of the degrees of freedom: it turns out that this z-direction can not be interpreted as a world volume direction [91]. We are therefore dealing with a five-brane, (i.e. with a six-dimensional world volume), that has an extra isometry in its transverse space. Its degrees of freedom are then, just as in the case of the solitonic five-brane, given by the scalar multiplet of an $N = 1$ supersymmetric field theory in six dimensions, consisting of four scalars. Naively, one could think that these four scalars again, as for the five-brane, correspond to the four transversal coordinates $(\hat{X}^i, Z)$, the collective coordinates for the position of the monopole. However, since Z is an isometry direction, it does not correspond to a degree of freedom (being an isometry, the position of the monopole in the Z-direction is not determined) and therefore it should not be taken in account the counting.

So on the one hand, we have to find a way to get rid of this extra degree of freedom Z in a proper way (note that this cannot be done by a simple extra gauge fixing of a world volume diffeomorphisms, since this would turn the monopole into a six-brane), yet on the other hand we have to introduce a new scalar in order to obtain the $N=1, D=6$ scalar multiplet. We will do this via a *gauged sigma model*, where we will gauge the isometry direction, eliminating the Z degree of freedom and introducing a scalar S to get the counting right [28].

Our proposal for the kinetic term of the monopole action is

$$S_{\mathcal{KK}} = \tfrac{T}{2}\int d^6\sigma\ \hat{k}^2\ e^{-2\hat{\phi}}\ \sqrt{\left|\det\Big(\partial_i\hat{X}^{\hat{\mu}}\partial_j\hat{X}^{\hat{\nu}}\ \hat{\mathcal{G}}_{\hat{\mu}\hat{\nu}} - \hat{k}^{-2}\hat{\mathcal{F}}_i\hat{\mathcal{F}}_j\Big)\right|}. \tag{6.46}$$

The vector $\hat{k}^{\hat{\mu}}$ is a Killing vector associated with the isometry direction z, and

$$\hat{k}^2 = -\hat{k}^{\hat{\mu}}\hat{k}^{\hat{\nu}}\hat{g}_{\hat{\mu}\hat{\nu}}. \tag{6.47}$$

In coordinates adapted to the isometry direction, $\hat{k}^{\hat{\mu}}$ will be of the form $\hat{k}^{\hat{\mu}} = \delta^{\hat{\mu}z}$. Furthermore we introduced a "metric" $\hat{\mathcal{G}}_{\hat{\mu}\hat{\nu}}$ and a scalar $\hat{S}$ via

$$\begin{aligned} \hat{\mathcal{G}}_{\hat{\mu}\hat{\nu}} &= \hat{g}_{\hat{\mu}\hat{\nu}}\ +\ \hat{k}^{-2}\ \hat{k}_{\hat{\mu}}\ \hat{k}_{\nu}, \\ \hat{\mathcal{F}}_i &= \partial_i\hat{S}\ +\ \partial_i\hat{X}^{\hat{\mu}}\ \hat{k}^{\hat{\nu}}\ \hat{B}_{\hat{\mu}\hat{\nu}}. \end{aligned} \tag{6.48}$$

The action (6.46) is a gauged sigma model, because of the symmetries

$$\delta\hat{X}^{\hat{\mu}} = \Lambda\ \hat{k}^{\hat{\mu}}, \qquad \delta\hat{S} = -\hat{k}^{\hat{\mu}}\Sigma_{\hat{\mu}}, \qquad \delta\hat{B}_{\hat{\mu}\hat{\nu}} = \partial_{[\hat{\mu}}\hat{\Sigma}_{\hat{\nu}]}, \tag{6.49}$$

under which the two terms in (6.46) are separately invariant. This symmetry occurs because of the presence of the Killing vector $\hat{k}^{\hat{\mu}}$, which effectively projects the z-direction and the corresponding field $Z(\sigma)$ out of the action. This can be seen in the contraction of the $\hat{k}^{\hat{\mu}}$ with the "metric" $\hat{\mathcal{G}}_{\hat{\mu}\hat{\nu}}$:

$$\begin{aligned} \hat{k}^{\hat{\mu}}\ \hat{\mathcal{G}}_{\hat{\mu}\hat{\nu}} &= \hat{k}^{\hat{\mu}}\hat{g}_{\hat{\mu}\hat{\nu}} + \hat{k}^{-2}\hat{k}^{\hat{\mu}}\hat{k}_{\hat{\mu}}\hat{k}_{\hat{\nu}} \\ &= \hat{k}^{\hat{\mu}}\hat{g}_{\hat{\mu}\hat{\nu}} - \hat{k}^{-2}\hat{k}^2\ \hat{k}^{\hat{\rho}}\hat{g}_{\hat{\rho}\hat{\nu}}\ = 0. \end{aligned} \tag{6.50}$$

The "metric" $\hat{\mathcal{G}}_{\hat{\mu}\hat{\nu}}$ is therefore effectively a nine-dimensional metric, written in a ten-dimensional covariant form. Reducing the action (6.46) over the isometry direction, using the reduction rules (6.23), we find for the reduced monopole action

$$S = -\tfrac{T}{2}\int d^6\sigma\ e^{-2\phi}k\ \sqrt{\left|\det\ g_{ij} - k^{-2}\tilde{\mathcal{F}}_i\tilde{\mathcal{F}}_j)\right|}, \tag{6.51}$$

where

$$\tilde{\mathcal{F}}_i = \partial_i S + B_\mu\partial_i X^\mu. \tag{6.52}$$

This is indeed precisely the action (6.44) of the reduced five-brane up to a T-duality transformation (3.31)

$$\tilde{A}_\mu = B_\mu, \qquad \tilde{B}_\mu = A_\mu, \qquad \tilde{k} = k^{-1}. \tag{6.53}$$

In order to construct the Wess-Zumino term of the monopole action, we first have to get a closer look at the Wess-Zumino term of the reduced five-brane action (6.44) and to see what is the origin of each field. As we know, the ten-dimensional six-form gauge field $\hat{C}_{\hat{\mu}_1\ldots\hat{\mu}_6}$ is the Poincaré dual of the axion $\hat{B}_{\hat{\mu}\hat{\nu}}$ and reduces in nine dimensions to a six-form and a five-form field, which are the Poincaré duals of the nine-dimensional winding vector B_μ and axion $B_{\mu\nu}$ respectively. Since under T-duality the Kaluza-Klein vector A_μ gets interchanged with the winding vector B_μ, we expect that under the same transformations also the Poincaré dual of the winding vector, the six-form $C_{\mu_1\ldots\mu_6}$, will get interchanged with some six-form $A_{\mu_1\ldots\mu_6}$, being the Poincaré dual of the Kaluza-Klein vector. The axion does not transform under T-duality, so it is logical to suppose that also its dual, the five-form $D_{\mu_1\ldots\mu_5}$ will be invariant.

Taking this in account, we make the following Ansatz for the ten-dimensional Wess-Zumino term:

$$S = \frac{T}{6!}\int d^6\sigma\varepsilon^{i_1\ldots i_6}\hat{k}^{\hat{\mu}_1}\ \partial_{i_1}\hat{X}^{\hat{\mu}_2}\ldots\partial_{i_6}\hat{X}^{\hat{\mu}_7}\Big[\hat{A}_{\hat{\mu}_1\ldots\hat{\mu}_7} - 6\hat{D}_{\hat{\mu}_1\ldots\hat{\mu}_6}(\partial_{\hat{\mu}_7}\hat{S} - \hat{k}^{\hat{\nu}}\hat{B}_{\hat{\mu}_7\hat{\nu}})\Big]\,, \quad (6.54)$$

where

$$\begin{aligned}
\partial_{[\hat{\mu}_1}\hat{A}_{\hat{\mu}_2\ldots\hat{\mu}_8]} &= \tfrac{1}{2!7!\sqrt{|\hat{g}|}}\hat{k}^2 e^{-2\hat{\phi}}\varepsilon_{\hat{\mu}_1\ldots\hat{\mu}_{10}}[\hat{k}^2\hat{F}^{\hat{\mu}_9\hat{\mu}_{10}}(\hat{A}) + 2\hat{k}^{\hat{\nu}}\hat{B}_{\hat{\nu}\hat{\rho}}\hat{H}^{\hat{\rho}\hat{\mu}_9\hat{\mu}_{10}}],\\
\partial_{[\hat{\mu}_1}\hat{D}_{\hat{\mu}_2\ldots\hat{\mu}_7]} &= \tfrac{1}{8!\sqrt{|\hat{g}|}}e^{-2\hat{\phi}}\varepsilon_{\hat{\mu}_1\ldots\hat{\mu}_{10}}\hat{H}^{\hat{\mu}_8\hat{\mu}_9\hat{\mu}_{10}}, \qquad (6.55)\\
\hat{A}_{\hat{\mu}} &= \hat{k}^{-2}\hat{k}^{\hat{\nu}}\hat{g}_{\hat{\mu}\hat{\nu}}\,.
\end{aligned}$$

The vector $\hat{A}_{\hat{\mu}}$ is the uplifting of the Kaluza-Klein vector A_μ, which can be written in a ten-dimensional form via the Killing vector $\hat{k}^{\hat{\mu}}$. The seven-form $\hat{A}_{\hat{\mu}_1\ldots\hat{\mu}_7}$ and the six-form $\hat{D}_{\hat{\mu}_1\ldots\hat{\mu}_6}$ are the Poincaré dual of $\hat{A}_{\hat{\mu}}$ and $\hat{B}_{\hat{\mu}\hat{\nu}}$ and the uplifting of $A_{\mu_1\ldots\mu_6}$ and $D_{\mu_1\ldots\mu_5}$ respectively . It is not difficult to show, using the definitions (6.55), that the latter are the only non-zero components of $\hat{A}_{\hat{\mu}_1\ldots\hat{\mu}_7}$ and $\hat{D}_{\hat{\mu}_1\ldots\hat{\mu}_6}$.

The Wess-Zumino term (6.54) transforms as a total derivative under the gauge transformations

$$\begin{aligned}
\delta\hat{A}_{\hat{\mu}_1\ldots\hat{\mu}_7} &= \partial_{[\hat{\mu}_1}\Lambda_{\hat{\mu}_2\ldots\hat{\mu}_7]} + \partial_{[\hat{\mu}_1}\,,\ _{\hat{\mu}_2\ldots\hat{\mu}_6}\hat{k}^{\hat{\nu}}\hat{B}_{|\hat{\nu}|\hat{\mu}_7]}\,,\\
\delta\hat{D}_{\hat{\mu}_1\ldots\hat{\mu}_6} &= \partial_{[\hat{\mu}_1}\,,\ _{\hat{\mu}_2\ldots\hat{\mu}_6]},\\
\delta\hat{S} &= -\hat{k}^{\hat{\mu}}\Sigma_{\hat{\mu}}, \qquad (6.56)\\
\delta\hat{B}_{\hat{\mu}\hat{\nu}} &= \partial_{[\hat{\mu}}\Sigma_{\hat{\nu}]}.
\end{aligned}$$

Again the contractions with the Killing vector $\hat{k}^{\hat{\mu}}$ take care of the fact that the z-direction is projected out of the action (6.54). Reduction over the isometry direction z gives

$$S = \tfrac{T}{6!}\int d^6\sigma\ \varepsilon^{i_1\ldots i_6}\ \partial_{i_1}X^{\mu_1}\ldots\partial_{i_6}X^{\mu_6}\Big[A_{\mu_1\ldots\mu_6} - 6\ D_{\mu_1\ldots\mu_5}(\partial_{\mu_6}S + B_{\mu_6})\Big], \qquad (6.57)$$

which can be mapped onto the Wess-Zumino term of the $S5$-brane, via the T-duality transformation

$$\tilde{B}_\mu = A_\mu, \qquad \tilde{A}_{\mu_1\ldots\mu_6} = C_{\mu_1\ldots\mu_6}. \qquad (6.58)$$

Besides the fact that the world volume action (6.46)-(6.54) is T-dual to the $S5$ action, we will give another evidence in favour of this action: the constructed action also serves as a source term for the monopole solution (2.65)[2]. For simplicity we only look at the purely gravitational part. This leads to the following action[3]

$$S = -\tfrac{1}{\kappa}\int d^{10}x\sqrt{|g|}\ R - \tfrac{T}{2}\int d^6\sigma\sqrt{|\gamma|}\gamma^{ij}\partial_i X^\mu \partial_j X^\nu k^2 \mathcal{G}_{\mu\nu} \tag{6.59}$$

Varying this action with respect to $g^{\mu\nu}$ (and taking care of all the metric factors hidden in $\mathcal{G}_{\mu\nu}$ and k^2), we find for the equation of motion:

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = \tfrac{\kappa T}{2\sqrt{|g|}}\int d^6\sigma\sqrt{|\gamma|}\gamma^{ij}\partial_i X^\rho \partial_j X^\lambda \delta(X^\mu - x^\mu)\times \tag{6.60}$$

$$\times\Big[-k_\mu k_\nu g_{\rho\lambda} - k^2 g_{\mu\rho}g_{\nu\lambda} + k_\nu k_\lambda g_{\mu\rho} + k_\rho k_\nu g_{\mu\lambda}\Big] \tag{6.61}$$

For the monopole solution (2.65) and the parametrisation

$$X^i = \sigma^i, \qquad Z = X^m = 0 \tag{6.62}$$

the equation (6.61) reduces to

$$\partial^2 H = \frac{\kappa T}{2}\delta(x^m). \tag{6.63}$$

The solution to this equation is given by

$$H(x^m) = \frac{\kappa T}{2}\frac{1}{\sqrt{|x^m x_m|}}. \tag{6.64}$$

We therefore can conclude that the (gravitational part) of the world volume action for the monopole is indeed a source for the ten-dimensional monopole solution.

Gauged sigma models have been used lately [110, 122, 29] to give an eleven-dimensional interpretation to the world volume actions of p-brane solutions in the background of massive Type IIA supergravity [133]. The relation between massive Type IIA theory and eleven-dimensional supergravity is a notorious problem, but it turns out that a massive version of $D = 11$ supergravity can be formulated if one assumes an isometry direction, characterized by a Killing vector $\hat{k}^{\hat{\mu}}$. The world volume action of the p-brane in massive Type IIA theory can be described in terms of the massive $D = 11$ supergravity background fields, if one gauges the isometry direction, using gauge transformations that involve a mass parameter m. Reduction over the direction associated to the isometry gives rise to the world volume actions for the p-branes in a massive ten-dimensional background.

An interesting question now is what does the world volume action for a massive eleven-dimensional Kaluza-Klein monopole look like, if the massless monopole is already described by a gauged sigma model? It turns out that there are two possibilities [25]: one

[2] Strictly speaking source terms are only needed for singular objects. For non-singular objects, such as the Kaluza-Klein monopole, a source term can be introduced if in a certain coordinate frame (non-physical) coordinate singularities appear. This is the case considered here.

[3] At this point we omit the hatted notation for ten-dimensional fields.

can extend the gauging (6.49) of the action (6.46)-(6.54) to the massive gauge transformations of [29] and reduce over the isometry direction in order to obtain the massive $D6$-brane action, or one can assume an extra isometry direction with a new Killing vector $\hat{h}^{\hat{\mu}}$. The massive Type IIA monopole action is then obtained by a massive gauging of and reduction over the new isometry direction. The massless limit can be taken consistently by setting the mass parameter $m = 0$.

Knowing the world volume action of the (massive) Type IIA monopole, it would be interesting to perform a (massive) T-duality transformation and see if one can obtain a world volume action for the Type IIB solitonic five-brane.

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Samenvatting

Elementaire deeltjesfysica, of hoge-energiefysica, is de tak van de natuurkunde die zich bezighoudt met het bestuderen van de elementaire deeltjes en hun wisselwerkingen. Aangezien de materie rondom ons is opgebouwd uit deze deeltjes, geeft de studie van elementaire deeltjes ons een idee van hoe de natuur in elkaar zit. In die zin behoort hoge-energiefysica tot het fundamenteel onderzoek: men zoekt niet direct naar bruikbare toepassingen, maar wil inzicht krijgen in hoe de wereld om ons heen functioneert en waarom hij zo functioneert.

In het algemeen is het zo: hoe kleiner de beschouwde structuren zijn waarnaar we kijken, hoe hogere energieën we nodig hebben om die structuren te zien. Dit komt doordat we, om het gedrag en de samenstelling van elementaire deeltjes te zien, andere (test)deeltjes erop moeten afschieten en kijken hoe die twee deeltjes onderling wisselwerken. Als we de testdeeltjes met steeds hogere energie afschieten (dit wil ruwweg zeggen: met steeds hogere snelheid), dringen ze steeds dieper door in het te bestuderen object en kunnen eventuele substructuren zichtbaar worden gemaakt.

Zo weten we bijvoorbeeld, dat atomen (waarvan men vroeger dacht dat het de elementaire bouwstenen van de materie waren) in feite bestaan uit een elektronenwolk die een hele kleine kern omringt. Die kern blijkt dan weer te zijn opgebouwd uit twee soorten deeltjes, die protonen en neutronen genoemd worden, en die op hun beurt weer opgebouwd zijn uit quarks. Voor zover we nu weten, hebben quarks en elektronen geen verdere substructuur meer en kunnen ze daarom “elementair” genoemd worden. Maar zoals inmiddels blijkt, is de term “elementair” een tijdsafhankelijk begrip en zal hij misschien in de toekomst niet meer van toepassing zijn op datgene wat er tegenwoordig mee wordt aangeduid.

Er zijn twee manieren om het gedrag van deze elementaire deeltjes te bestuderen: men kan ofwel experimenten doen waarbij de deeltjes met hoge energie op elkaar worden geschoten, vervolgens kijkt men dan hoe de deeltjes zich gedragen en probeert daaruit een theorie op te stellen die dit gedrag verklaart. Of men kan uitgaan van een centraal idee, hierrond een goede theorie bouwen en dan kijken of deze theorie experimentele toetsen kan doorstaan. In dit proefschrift houden wij ons bezig met de laatste werkwijze: de theoretische hoge-energiefysica.

Zo’n centraal idee kan bijvoorbeeld symmetrie zijn. Symmetrie is een eigenschap van een theorie die zegt dat de vorm van de theorie niet verandert als we er een bepaalde operatie, een symmetrietransformatie, op uitvoeren. Zo verandert een theorie die translatiesymmetrie heeft, niet als we het coördinatenstelsel verschuiven van één punt in de ruimte naar een ander punt. Fysisch betekent dit dat de wetten van de natuurkunde in de hele ruimte dezelfde zijn. Theorieën met veel symmetrie zijn ook gemakkelijker om mee te werken, omdat de symmetrie het probleem vereenvoudigt. In een theorie met translatiesymmetrie bijvoorbeeld hoef je een resultaat maar voor één punt te berekenen, de resultaten in de rest van de ruimte zijn dezelfde. Ook mogen er geen expliciete positie-afhankelijkheden in de formulering van de theorie voorkomen, omdat anders de translatiesymmetrie gebroken wordt. De symmetrie beperkt dus ook de mogelijke formuleringen van de theorie.

De op dit moment algemeen aanvaarde theorie voor elementaire deeltjes is het zogenaamde Standaardmodel, dat zowel een classificatie van de deeltjes geeft als een beschrijving van hun wisselwerkingen onder drie fundamentele interacties: de sterke, de zwakke en de elektromagnetische interactie. Het Standaardmodel gaat uit van het centrale idee van ijkinvariantie. IJkinvariantie is een interne symmetrie die stelt dat deeltjes in families (multipletten) verdeeld kunnen worden en dat de verschillende leden van zo'n multiplet zich symmetrisch moeten gedragen. Verder worden de interacties verklaard door de uitwisseling van zogenaamde ijkdeeltjes, die ook weer in multipletten voorkomen. Dit alles beperkt de mogelijke interacties heel sterk. Experimenteel gezien is het Standaardmodel een erg succesvolle theorie: de resultaten stemmen tot op heel hoge precisie overeen met de experimenten en de theorie was in staat om voorspellingen te doen die later experimenteel bevestigd werden.

Er is echter nog een vierde fundamentele interactie in de natuur waarmee het Standaardmodel helemaal geen rekening houdt, namelijk de zwaartekracht. De zwaartekracht wordt verklaard in een andere theorie: de algemene relativiteitstheorie (ART). Deze gaat uit van het principe dat de natuurwetten dezelfde moeten zijn voor alle waarnemers en beschrijft zwaartekracht als een vervorming van de ruimte door de aanwezige materie. Ook de ART is vanuit experimenteel oogpunt een succesvolle theorie: zij kan fenomenen beter verklaren dan de traditionele theorie van Newton en meerdere voorspellingen zijn experimenteel bevestigd.

Het feit dat deze twee theorieën, elk op hun eigen gebied, het zo goed doen, komt omdat de invloed van het ene verschijnsel op het andere erg klein is: in versnellerexperimenten is de zwaartekracht tussen de elementaire deeltjes veel te zwak om er iets van te merken, terwijl men de ART meestal gebruikt voor de bewegingen van hemellichamen, waarbij krachten tussen elementaire deeltjes niet van belang zijn.

Toch blijft het vreemd dat we twee totaal onafhankelijke theorieën hebben, die allebei dezelfde natuur proberen te beschrijven. Als beide theorieën uitgaan van zulke centrale principes, zou het dan niet meer voor de hand liggen als deze principes in beide theorieën zouden voorkomen? Met andere woorden: zou het niet logischer zijn als we een geïntegreerde theorie hadden, die zowel het gedrag van elementaire deeltjes als de effecten van zwaartekracht beschrijft?

Zoals gezegd, op experimentele gronden is er niets aan de hand, omdat beide theorieën binnen hun eigen bereik erg succesvol zijn. Maar op theoretische gronden kan men zien dat er vroeg of laat problemen ontstaan. Immers, als we in botsingsexperimenten de energie maar blijven opdrijven, zou de zwaartekracht alsmaar sterker worden, aangezien deze net zozeer aan massa als aan energie koppelt. Vanaf een bepaalde schaal (de Planck-energie of de Planckmassa) zou de invloed van de zwaartekracht zelfs net zo groot worden als die van andere krachten in het Standaardmodel, en zou geen van de twee theorieën meer een goede beschrijving kunnen geven van de experimenten. Deze energieschalen liggen weliswaar ver buiten het bereik van de huidige deeltjesversnellers, maar het (theoretische) probleem is gesteld.

Het vinden van een geünificeerde beschrijving van elementaire deeltjesfysica en zwaartekracht is één van de grote uitdagingen van de moderne hoge-energiefysica. Dit is zo moeilijk omdat men bij pogingen om de typische quantum-effecten van elementaire deeltjesfysica te incorporeren in ART (die niet-quantummechanisch is), steeds tegen grote wiskundige problemen oploopt.

Eén van de veelbelovende kandidaten voor een theorie van quantum-zwaartekracht is de stringtheorie. Deze gaat uit van het idee dat de elementaire deeltjes niet puntvormig zijn, zoals we ons die intuïtief voorstellen, maar ééndimensionale objecten, snaartjes (Engels: strings). Deze snaren kunnen, net als de snaren van een gitaar, trillen en de verschillende trillingswijzen (die bij een gitaar overeenkomen met verschillende toonhoogtes) corresponderen hier met verschillende deeltjes, onder andere ook die van het Standaardmodel. Het feit dat er nog niets gemerkt is van de snaarstructuur van deeltjes, komt volgens de theorie doordat de afmeting van de snaartjes zo klein is, dat ze vanaf "grote" afstanden (afstanden te vergelijken met structuren zoals in het Standaardmodel) puntvormig lijken. Maar dit houdt tegelijkertijd in, dat om deze snaarstructuur te detecteren zulke hoge energieën nodig zijn, dat de eerstvolgende generaties versnellers nog volkomen ontoereikend zullen zijn.

Ondanks het grote gebrek aan experimentele gegevens blijkt de stringtheorie toch erg interessant te zijn. Immers, één van de deeltjes die in de stringtheorie voorkomen, blijkt het graviton te zijn, het ijkdeeltje van zwaartekracht. Dit betekent dat zwaartekracht al automatisch in de stringtheorie ingebouwd zit en dat we ART als speciale limiet kunnen terugvinden. Verder vinden we nog allerlei symmetriestructuren, die lijken op de ijksymmetrieën van het Standaardmodel, zodat dit inderdaad wijst in de goede richting.

Toch is er nog veel werk aan de winkel: de stringtheorie is namelijk nog alles behalve volledig. De tot voor kort gebruikte formulering is gebaseerd op storingstheorie, waarbij wordt uitgegaan van de eenvoudige situatie dat de snaren onderling geen wisselwerking hebben: de interacties worden dan gezien als een storing op deze ideale situatie. Zo'n benadering is relatief goed zolang de storing (de koppeling tussen de snaren) klein blijft, want zolang kunnen technieken (storingsrekening) gebruikt worden om resultaten te berekenen. In principe kan men dan weer storingen op deze storingen gaan berekenen, enz., maar in de praktijk houdt het snel op, omdat de berekeningen al snel te groot worden. Bovendien is de benadering alleen maar houdbaar als de storingen klein blijven. Bij sterke interacties tussen de snaren (meestal juist de interessantste situatie) is deze methode echter niet meer te betrouwen: men heeft op deze manier namelijk geen idee van wat er gebeurt in zogenaamde niet-storingseffecten.

Ten tweede is het problematisch om het Standaardmodel terug te vinden als lage-energie-benadering van de stringtheorie. Uit consistentie-eisen blijkt dat de snaren zich in een tiendimensionale ruimte-tijd moeten voortbewegen, terwijl de ons bekende wereld van ART en van het Standaardmodel vierdimensionaal is (de drie ruimte-dimensies en de tijd). De verklaring daarvoor is dat zes van die tien dimensies zo klein zijn (van de orde van de snaar zelf) dat ze bij lage energieën niet te zien zijn en er bestaat een techniek (dimensionele reductie) die resultaten van de tiendimensionale ruimte kan vertalen naar een effectieve vierdimensionale ruimte. Afhankelijk van hoe de dimensionele reductie gedaan wordt, kunnen allerlei symmetriegroepen verschijnen, waarvan sommige zelf erg veel op die van het Standaardmodel lijken. Maar er zijn nog vele andere mogelijkheden en er is niets dat wijst op een goede reden waarom nu precies die ene reductie verkozen moet worden boven alle andere.

Een derde probleem, misschien wel het meest vervelende, is dat er niet één versie van de stringtheorie bekend is, maar vijf verschillende, met elk hun eigen deeltjes en storingsbenadering. Dit is natuurlijk geen aantrekkelijk idee voor een unificerende theorie. Tot voor kort meende men dan ook dat vroeg of laat sommige van die theorieën inconsistent en/of equivalent zouden blijken te zijn en dat er (hopelijk) uiteindelijk één zou overblijven.

In de laatste paar jaren is er echter in korte tijd veel vooruitgang geboekt. Zoveel zelfs dat er sprake is van een heuse stringrevolutie. Deze ontwikkelingen hebben het idee van dualiteiten geïntroduceerd, een soort symmetrie die zegt dat verschillende formuleringen van een theorie aan elkaar gerelateerd kunnen worden (duaal zijn), hetgeen hen in feite equivalent maakt. Resultaten van de ene formulering kunnen via de dualiteitstransformaties vertaald worden naar de andere formulering. Binnen de groep van stringtheoretici wordt veel verwacht van deze dualiteiten; het zou zelfs één van de fundamentele principes kunnen zijn, nodig om de theorie te begrijpen.

Zo blijkt bijvoorbeeld dat een theorie gereduceerd op een groot volume duaal is aan dezelfde theorie gereduceerd op een klein volume. De dualiteitstransformatie die daar voor zorgt, heet T-dualiteit en relateert dus op een verrassende manier grote en kleine lengteschalen. Maar het verschil tussen formuleringen op een groot en een klein volume is precies het verschil tussen twee manieren om een theorie dimensioneel te reduceren (van tien naar vier dimensies bijvoorbeeld). T-dualiteit stelt dus dat verschillende reducties equivalent kunnen zijn en verdeelt de verschillende mogelijke compactificaties in equivalentieklassen. Hoewel we daarmee nog niet het probleem hebben opgelost welke reductie verkozen moet worden boven andere, is het tenminste significant gereduceerd.

S-dualiteit is een dualiteit die inzicht kan geven in het gebied dat verder ligt dan de storingstheorie. Zoals gezegd is het erg moeilijk om betrouwbare resultaten te krijgen als de koppeling (de interacties tussen de snaren) erg sterk wordt, omdat storingsrekening dan niet meer toereikend is. S-dualiteit, ook wel sterke/zwakke-koppelingsdualiteit genoemd, stelt dat als de koppeling erg groot wordt, er een duale formulering gevonden kan worden waarin de snaren weer zwak gekoppeld zijn. In deze duale formulering kunnen we dan weer gewoon storingsrekening gebruiken. En omgekeerd: het sterke-koppelingsgebied van de duale formulering komt weer overeen met het zwakke-koppelingsgebied van de originele theorie. Op die manier zijn niet-storingsresultaten toch vrij gemakkelijk te verkrijgen uit storingsrekening.

Niet alleen binnen eenzelfde theorie blijken verschillende formuleringen duaal te zijn, maar ook hele theorieën kunnen via dualiteiten aan elkaar gerelateerd worden. Zo kan één theorie op een klein volume precies dezelfde blijken te zijn als een andere theorie op een groot volume, of kan het sterke-koppelingsgebied van de ene overeenkomen met het zwakke-koppelingsgebied van de andere. T- en S-dualiteit spannen dus een heel net op van dualiteitsrelaties tussen de vijf stringtheorieën. Deze zijn dan misschien op het eerste gezicht (in het storingsgebied) erg verschillend, maar, met niet-storingseffecten in rekening gebracht, in feite equivalent.

Samen met de ontdekking van dit dualiteitennet is ook het idee ontstaan dat de vijf stringtheorieën misschien niet de finale theorieën zijn waarnaar we op zoek zijn, maar eigenlijk verschillende benaderingen van een nog niet ontdekte, onderliggende theorie. De dualiteitstransformaties relateren dan deze benaderingen met elkaar. Die onderliggende theorie, wordt meestal M-theorie genoemd, al weet niemand precies waarvoor die "M" staat (Moeder, Membraan, Mysterie, ...). Ook is het nog helemaal niet duidelijk hoe M-theorie eruit ziet en of snaren uiteindelijk nog wel de elementaire bouwstenen van de theorie zijn.

Immers, een andere bijdrage van de dualiteiten aan de stringtheorie is dat, naast snaren, ook andere objecten voorkomen: puntdeeltjes, membranen, drie-, vier-, en nog hogerdimensionale objecten. Deze worden p-branen genoemd, (in analogie met membranen) waar p het aantal dimensies is van het object. De dualiteitstransformaties spannen een web tussen al deze objecten, net zoals ze dat doen tussen de theorieën: sommige p-branen kunnen via T-dualiteit gerelateerd worden aan $(p+1)$- of $(p-1)$-branen, dus aan objecten die net één dimensie groter of kleiner zijn. Sterke/zwakke-koppelingsdualiteit zegt dan bijvoorbeeld weer dat een sterk gekoppelde snaar equivalent is aan een zwak gekoppelde 5-braan. We hadden dus net zo goed van een "5-branentheorie" kunnen uitgaan als van een stringtheorie. In feite is een string net zo veel of zo weinig fundamenteel als elke andere p-braan. En dit maakt het er natuurlijk niet gemakkelijker op om een goede formulering van M-theorie te vinden.

We zien dus dat dualiteiten en M-theorie een heel ander beeld geven van (wat vroeger bekend stond als) de stringtheorie dan datgene wat we tot voor kort hadden. En ondanks het feit dat het centrale deel van het nieuwe beeld nog erg vaag blijft, kunnen toch al heel wat implicaties ervan getest worden. Immers, een essentiële veronderstelling in het geheel is dat de dualiteiten tussen de vijf stringtheorieën correct zijn. Door deze te testen, toetsen we ook indirect het M-theorie-beeld. Bovendien leveren de dualiteitstransformaties een beter begrip op van hoe de stringtheorie zelf in elkaar zit (denk maar aan de niet-storingseffecten en S-dualiteit).

Als twee theorieën duaal aan elkaar zijn, moet die dualiteit natuurlijk in alle sectoren terug te vinden zijn: zowel in de lage-energielimiet, in de oplossingen, als in de dynamica van de theorie. Omgekeerd, gegeven een dualiteit tussen één theorie en een andere, kunnen we ons beeld van de ene theorie vervolledigen dankzij de kennis van de andere: nieuwe oplossingen construeren uit oude bijvoorbeeld, of de dynamica afleiden uit die van een duale theorie.

In dit proefschrift hebben we beide dingen gedaan: zowel het testen van de dualiteiten op de verschillende niveaus van de theorie, als het gebruiken van de dualiteiten om resultaten mee af te leiden. Na een drietal inleidende hoofdstukken, waarin we iets dieper ingaan op de probleemstelling en een introductie geven in de basisbegrippen van de stringtheorie en dualiteiten, onderzoeken we in hoofdstuk 4 uitvoerig de symmetrieën van de lage-energielimieten van de verschillende theorieën. We schrijven de dualiteitsregels op die deze lage-energielimieten onderling relateren en vinden dat deze overeenkomen met de dualiteitsrelaties tussen de stringtheorieën waarvan de lage-energielimieten beschouwden.

In hoofdstuk 5 kijken we naar de p-braan-oplossingen van de stringtheorie, en meer in het bijzonder naar de manieren waarop meerdere van zulke p-branen samen kunnen voorkomen en elkaar kunnen snijden. Dergelijke intersecties zijn interessant omdat, ze na dimensionele reductie oplossingen opleveren in lagere dimensies. Een classificatie van de mogelijke intersecties in de stringtheorie geeft een overzicht van de verschillende oplossingen in bijvoorbeeld een vierdimensionale ruimte-tijd als de onze.

Hoofdstuk 6 gaat over de dynamica van de oplossingen. Die wordt gegeven door een wiskundige uitdrukking, die men de effectieve actie van de oplossing noemt. Zoals gezegd moeten de dualiteitsrelaties die tussen de oplossingen bestaan, ook tussen de effectieve acties van deze oplossingen terug te vinden zijn. We geven een overzicht van deze effectieve acties, tonen de dualiteiten aan en construeren aan de hand daarvan de effectieve actie van een bepaalde oplossing (de Kaluza-Klein monopool).

De laatste jaren zijn in een snel tempo nieuwe vorderingen gemaakt in de stringtheorie, en dat zal nog wel even blijven duren. In de tijd die er nodig was om dit proefschrift te schrijven, zijn er weer ontwikkelingen geweest, die veelbelovend lijken. Eén ding is duidelijk: of er nu al dan niet snel een goede formulering van M-theorie gevonden wordt, zelfs of M-theorie nu wel of niet de uiteindelijke, langgezochte “theorie van alles” blijkt te zijn, dualiteiten hebben zich een blijvende, belangrijke plaats weten te verwerven in de stringtheorie en meer algemeen, in de hoge-energiefysica.

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