ORBIFOLD COMPACTIFICATIONS OF STRING THEORY

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Abstract

The compactification of the heterotic string theory on a six-dimensional orbifold is attractive theoretically, since it permits the full determination of the emergent four-dimensional effective supergravity theory, including the gauge group and matter content, the superpotential and Kähler potential, as well as the gauge kinetic function. This review attempts to survey all of these calculations, covering the construction of orbifolds which yield (four-dimensional space–time) supersymmetry; orbifold model building, including Wilson lines, and the modular symmetries associated with orbifold compactifications; the calculation of the Yukawa couplings, and their connection with quark and lepton masses and mixing; the calculation of the Kähler potential and its string loop threshold corrections; and the determination of the non-perturbative effective potential for the moduli arising from hidden sector gaugino condensation, and its connection with supersymmetry breaking. We conclude with a brief discussion of the relevance of weakly coupled string theory in the light of recent developments on the strongly coupled theory.

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1. Orbifold constructions

1.1. Introduction

It is well known that the construction of a consistent quantum string theory is possible only for specific dimensionalities of the (target) space–time. For the bosonic string the required dimension is $D = 26$, while for the superstring dimension $D = 10$ is required. Thus from the outset we are forced to consider the "compactification" of the (spatial) dimensions which are surplus to the $d = 4$ dimensions of the world that we inhabit, if we are to have any chance of connecting the string theory with experimental (particle) physics. The string theory which is best placed to generate such a connection is the heterotic string [117], a theory of closed strings, in which the right-moving degrees of freedom of the superstring are adjoined to the twenty-six left-moving degrees of freedom of the bosonic string. To endow such a construction with a geometrical interpretation sixteen of the left-movers are compactified by associating them with a 16-dimensional torus, with radii of order the Planck length ($\ell_P \sim 10^{-35}$ m). Just as the compactification of one dimension onto a circle in the (original) five-dimensional Kaluza–Klein theory [135,141] generates a gauge boson, so here the compactification generates gauge fields, including some of a stringy origin which derive from the possibility of the string winding around the torus. In this way, the 16 left-movers generate an "internal" gauge symmetry with the (rank 16) gauge group $E_8 \times E_8$ being consistent with the cancellation of gauge and gravitational anomalies which is essential for a satisfactory *quantum* theory [113].

Although this scenario explains in a satisfying way how a gauge symmetry can emerge from string theory, there are serious problems which remain. Firstly, there is the fact that the symmetry group $E_8 \times E_8$ is far larger than the (rank 4) $SU(3) \times SU(2) \times U(1)$ gauge symmetry which we observe. Secondly, there remains a ten-dimensional space–time, six of whose dimensions must be compactified before we even contemplate questions like gauge symmetries and matter generations. The orbifolds [79,80], which are the subject of this review are one method of compactifying the unobserved six dimensions. An orbifold is obtained when a six-dimensional torus (T^6) is quotiented by a discrete ("point") group (P), as we shall see shortly. The identification of points on T^6 under the action of the point group generates a finite number of fixed points where the orbifold is singular. At all other points the orbifold is (Riemann) flat. It is for this reason that we are able to calculate rather easily all of the parameters and functions of the emergent supergravity theory: the gauge group and matter content; the Yukawa couplings and Kähler potential, which determine the quark and lepton masses and mixing angles; the gauge kinetic function, including string loop threshold corrections, which in turn determine the unification scale of the gauge coupling constants. We shall see also how modular invariance constrains the effective potential, and hence determines the actual value of the coupling constants at unification, as well as the nature of the supersymmetry breaking mechanism.

There are, of course, other methods of string compactification including Calabi–Yau manifolds [43,115,116], free fermion models [139,3], and $N = 2$ superconformal field theories [107,108,140], and (some) orbifold models are connected to some of these models [138,98,13,14,24]. However, none of the alternatives has so far been as fully worked out as the orbifold theories, and it is for this reason that we have focused upon them. If for no other reason, they illustrate the sort of predictive power which we should eventually like string theory to have (even if it should transpire that nature does not in fact utilize orbifolds!)

1.2. Toroidal compactifications

The construction of the ten-dimensional heterotic string has been fully described elsewhere (see, for example, [114,132,35]) and we need not review it here. As already noted, to have any chance of a realistic theory it is obviously essential that six of the (nine) spatial dimensions have to be compactified to a sufficiently small scale as to be unobservable at current accelerators. The simplest way to do this is to compactify on a torus. This ensures that the simple linear string (wave) equations of motion are unaffected, since the torus is flat. We work in the light-cone gauge. Then there are eight transverse bosonic degrees of freedom denoted by $X^i(\tau,\sigma)$ where $i = 1,2$ labels the two transverse four-dimensional space–time coordinates, and $X^k(\tau,\sigma)$ where $k = 3,\dots,8$ labels the remaining six spatial degrees of freedom. (τ, σ with $0 \leqslant \sigma \leqslant \pi$ are the world sheet parameters.) X^i and X^k are split into left and right moving components in the standard manner

$$X^{i(k)}(\tau,\sigma) = X_{\mathrm{R}}^{i(k)}(\tau-\sigma) + X_{\mathrm{L}}^{i(k)}(\tau+\sigma) . \tag{1.1}$$

In addition there are eight right-moving transverse fermionic degrees of freedom $\Psi^i_{\mathrm{R}}(\tau-\sigma)$, $\Psi^k_{\mathrm{R}}(\tau-\sigma)$, and the 16 (internal) left-moving bosonic degrees of freedom $X^I_{\mathrm{L}}(\tau+\sigma)$ ($I = 1,\dots,16$) which generate the $E_8 \times E_8$ gauge group of the ten-dimensional heterotic string. The (toroidal) compactification of the six spatial coordinates $X^k(\tau,\sigma)$ ($k = 3,\dots,8$) does not affect the mode expansions of $X^i(\tau,\sigma)$, $\Psi^i_{\mathrm{R}}(\tau-\sigma)$, $\Psi^k_{\mathrm{R}}(\tau-\sigma)$ or $X^I_{\mathrm{L}}(\tau+\sigma)$, so

$$X^i(\tau,\sigma) = x^i + p^i\tau + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\left[\frac{1}{n}\alpha^i_n \mathrm{e}^{-2\mathrm{i}n(\tau-\sigma)} + \frac{1}{n}\tilde{\alpha}^i_n \mathrm{e}^{-2n(\tau+\sigma)}\right] , \tag{1.2}$$

$$\Psi^{i(k)}_{\mathrm{R}}(\tau-\sigma) = \sum_{r} d^{i(k)}_n \mathrm{e}^{-2\mathrm{i}n(\tau-\sigma)} \quad (\mathrm{R}) \tag{1.3}$$

or

$$= \sum_{r\in Z+1/2} b^{i(k)}_r \mathrm{e}^{-2\mathrm{i}r(\tau-\sigma)} \quad (\mathrm{NS}) \tag{1.4}$$

depending on whether the world-sheet fermion field obeys periodic (Ramond, R) or anti-periodic (Neveu–Schwarz, NS) boundary conditions

$$\begin{aligned} \psi_{\mathrm{R}}(\tau-\sigma-\pi) &= +\psi_{\mathrm{R}}(\tau-\sigma) \quad (\mathrm{R}) ,\\ \psi_{\mathrm{R}}(\tau-\sigma-\pi) &= -\psi_{\mathrm{R}}(\tau-\sigma) \quad (\mathrm{NS}) . \end{aligned} \tag{1.5}$$

The mode expansion of the gauge degrees of freedom is

$$X^I_{\mathrm{L}}(\tau+\sigma) = x^I_{\mathrm{L}} + p^I_{\mathrm{L}}(\tau+\sigma) + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{\tilde{\alpha}^I_n}{n}\mathrm{e}^{-2\mathrm{i}n(\tau+\sigma)} \tag{1.6}$$

with the momenta p^I_{L} lying on the $E_8 \times E_8$ root lattice.

In an orthonormal basis, vectors on the E_8 root lattice the form

$$(n_1,n_2,\dots,n_8) \quad \text{or} \quad (n_1+\tfrac{1}{2},n_2+\tfrac{1}{2},\dots,n_8+\tfrac{1}{2}) \tag{1.7}$$

with n_i integers and

$$\sum_{i=1}^{8} n_i = 0 \bmod 2 \ . \tag{1.8}$$

There is an alternative formulation of these internal degrees of freedom which replaces the 16 bosonic left movers (X^I) compactified on the $E_8 \times E_8$ lattice by 32 real fermionic left-movers ($\lambda^A, \bar{\lambda}^A (A = 1, \ldots, 16)$) where $\lambda^A, \bar{\lambda}^A$ may separately have either periodic (R) or antiperiodic (NS) boundary conditions. Then

$$\lambda^A = \sum_n \lambda_n^A \mathrm{e}^{-2in(\tau+\sigma)} \quad (\mathrm{R})$$

$$= \sum_{r \in Z+1/2} \lambda_r^A \mathrm{e}^{-2ir(\tau+\sigma)} \quad (\mathrm{NS}) \ , \tag{1.9}$$

and similarly for the second set $\bar{\lambda}^A$. $(\lambda^A, \bar{\lambda}^A)$ transform as the $(\mathbf{16},\mathbf{1}) + (\mathbf{1},\mathbf{16})$ representation of the maximal subgroup $O(16) \times 0(16) \subset E_8 \times E_8$. The compactification of X^k entails the identification of the corresponding centre-of-mass coordinates x^k with points which are separated by a lattice vector of the torus. Thus

$$x^k \equiv x^k + 2\pi L^k \ , \tag{1.10}$$

where the factor 2π is for convenience and the vector $\boldsymbol{L}$ with coordinates L^k belongs to a six-dimensional lattice Λ

$$\Lambda \equiv \left\{ \sum_{t=3}^{8} \boldsymbol{r}_t \boldsymbol{e}_t \,|\, \boldsymbol{r}_t \in \boldsymbol{Z} \right\} \ , \tag{1.11}$$

where $\boldsymbol{e}_t$ $(t = 3, \ldots, 8)$ are the basis vectors of the lattice. Then the closed string boundary conditions for the coordinates X^k may also be satisfied when

$$X^k(\tau, \pi) = X^k(\tau, 0) + 2\pi L^k \tag{1.12}$$

corresponding to the string winding around the torus. The compactification also requires the quantization of the eigenvalues of the corresponding momentum operators p^k. The eigenfunctions $\exp(i \sum_k p^k x^k)$ are single-valued only if

$$\sum_{k=3}^{8} p^k L^k \in Z \ . \tag{1.13}$$

Thus, the momenta are quantized on the lattice Λ^* which is dual to Λ

$$\Lambda^* = \left\{ \sum_{t=3}^{8} m_t \boldsymbol{e}_t^* \,|\, \boldsymbol{m}_t \in \boldsymbol{Z} \right\} \ , \tag{1.14}$$

where the basis vectors $\boldsymbol{e}_t^*$ of Λ^* satisfy $\sum_{k=3}^{8} e_t^{*k} e_u^k \equiv \boldsymbol{e}_t^* \cdot \boldsymbol{e}_u = \delta_{tu}$.

Then the generalized mode expansions are

$$X^k_{\mathrm{R}}(\tau-\sigma) = x^k_{\mathrm{R}} + p^K_{\mathrm{R}}(\tau-\sigma) + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\alpha^k_n \mathrm{e}^{-2\mathrm{i}n(\tau-\sigma)} , \tag{1.15}$$

$$X^k_{\mathrm{L}}(\tau+\sigma) = x^k_{\mathrm{L}} + p^k_{\mathrm{L}}(\tau+\sigma) + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^k_n \mathrm{e}^{-2\mathrm{i}n(\tau+\sigma)} \tag{1.16}$$

with

$$p^k_{\mathrm{R}} \equiv \tfrac{1}{2}(p^k - 2L^k) , \tag{1.17}$$

$$p^k_{\mathrm{L}} \equiv \tfrac{1}{2}(p^k + 2L^k) , \tag{1.18}$$

$$x^k = x^k_{\mathrm{R}} + x^k_{\mathrm{L}} ,$$

where $\boldsymbol{p} \in \Lambda^*$ and $\boldsymbol{L} \in \boldsymbol{\Lambda}$.

The mass formula for the right movers in ten-dimensional heterotic string theory, which derives from the constraint equations, yields the four-dimensional mass formula

$$\tfrac{1}{4}m^2_{\mathrm{R}} = N(\beta) + \tfrac{1}{2}p^k_{\mathrm{R}}p^k_{\mathrm{R}} - a(\beta) , \tag{1.19}$$

where $\beta = \mathrm{R}$, NS labels the boundary conditions of the fermionic right-movers, and the number operators $N(\beta)$ is given by

$$N(\beta) = N_B + N_f(\beta) , \tag{1.20}$$

with

$$N_B = \sum_{n=1}^{\infty} (\alpha^i_{-n}\alpha^i_n + \alpha^k_{-n}\alpha^k_n) , \tag{1.21}$$

$$N_F(\mathrm{R}) = \sum_{n=1}^{\infty} (nd^i_{-n}d^i_n + nd^k_{-n}d^k_n) , \tag{1.22}$$

$$N_F(\mathrm{NS}) = \sum_{r=1/2}^{\infty} (rb^i_{-r}b^i_r + rb^k_{-r}b^k_r) . \tag{1.23}$$

$a(\beta)$ arises from the normal ordering of the operator L_0 in the Virasoro algebra and has the values

$$a(R) = 0 , \tag{1.24}$$

$$a(\mathrm{NS}) = \tfrac{1}{2} . \tag{1.25}$$

(Sums over $i = 1,2$ and $k = 3,\dots,8$ are implied by the repeated suffixes.) Similarly, the four-dimensional mass formula for the left movers is

$$\tfrac{1}{4}m^2_{\mathrm{L}} = \tilde{N} + \tfrac{1}{2}p^k_{\mathrm{L}}p^k_{\mathrm{L}} + \tfrac{1}{2}p^I_{\mathrm{L}}p^I_{\mathrm{L}} - 1 , \tag{1.26}$$

where a sum over $I = 1,\dots,16$ is also implied and

$$\tilde{N} = \sum_{n=1}^{\infty} (\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + \tilde{\alpha}^k_{-n}\tilde{\alpha}^k_n + \tilde{\alpha}^I_{-n}\tilde{\alpha}^I_n) . \tag{1.27}$$

If the fermionic formulation of the (left-moving) internal degrees of freedom is used the mass formula becomes

$$\tfrac{1}{4}m_{\mathrm{L}}^2(\beta,\gamma) = \tilde{N}(\beta,\gamma) + \tfrac{1}{2}p_{\mathrm{L}}^k p_{\mathrm{L}}^k - \tilde{a}(\beta,\gamma) \tag{1.28}$$

when $\beta,\gamma = \mathrm{R,NS}$ labels the (independent) boundary conditions for the two sets of real fermions $\lambda^A,\bar{\lambda}^A$, and

$$\tilde{N}(\beta,\gamma) = \tilde{N}_B + \tilde{N}_F(\beta) + \tilde{N}_F(\gamma)\ , \tag{1.29}$$

where

$$\tilde{N}_B = \sum_{n=1}^{\infty} (\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + \tilde{\alpha}^k_{-n}\tilde{\alpha}^k_n)\ , \tag{1.30}$$

$$\tilde{N}_F(R) = \sum_{n=1}^{\infty} n(\lambda^A_{-n}\lambda^A_n + \bar{\lambda}^A_{-n}\bar{\lambda}^A_n)\ , \tag{1.31}$$

$$\tilde{N}_F(NS) = \sum_{r=1/2}^{\infty} r(\lambda^A_{-r}\lambda^A_r + \bar{\lambda}^A_{-r}\bar{\lambda}^A_r)\ . \tag{1.32}$$

Similarly the normal ordering constant

$$\tilde{a}(\beta,\gamma) = \tilde{a}_B + \tilde{a}_F(\beta) + \tilde{a}_F(\gamma)\ , \tag{1.33}$$

where

$$\tilde{a}_B = \tfrac{1}{3}, \qquad \tilde{a}_F(\mathrm{R}) = -\tfrac{2}{3}, \qquad \tilde{a}_F(\mathrm{NS}) = \tfrac{1}{3}\ . \tag{1.34}$$

The mass formulae (1.19), (1.26) and (1.28) all include contributions from momenta $p_{\mathrm{R}}^k,p_{\mathrm{L}}^k$ in the compactified manifold, which, as we have shown in Eqs. (1.17) and (1.18), are quantized. As we shall see, the lattice Λ and hence its dual Λ^* generically have some arbitrary scale factors R_t, the lengths of the basis vectors $\boldsymbol{e}_t$, and angles between basis vectors. So, except for certain isolated values of these parameters, massless states, in particular, only arise when momenta and winding numbers on the compact manifold are zero

$$p_{\mathrm{R}}^k = 0 = p_{\mathrm{L}}^k\ . \tag{1.35}$$

In fact, the particles we observe in nature must all derive from massless string states, since otherwise their masses would be of the order of the string scale (10^{17} GeV).

We may now see why the simple toroidal compactification under consideration is unacceptable for phenomenological reasons. Let us consider a massless state, so

$$m_{\mathrm{L}}^2 = 0 = m_{\mathrm{R}}^2\ . \tag{1.36}$$

Suppose we fix the (massless) left-mover state; for example, we may use one of the $\tilde{\alpha}_{-1}$ operators on the left-movers' ground state $|0\rangle_{\mathrm{L}}$, or use momentum p_{L}^I on the $E_8 \times E_8'$ lattice with $p_{\mathrm{L}}^I p_{\mathrm{L}}^I = 2$. To each such left-moving state we may attach a massless right-moving state $b^i_{-1/2}|0\rangle_{\mathrm{R}}$ ($i = 1,2$) utilizing the NS fermionic oscillators. Since $i = 1,2$ corresponds to the two transverse space–time dimensions, the overall string state transforms as a space–time vector or a space–time tensor, the latter case arising only if the left-moving state is $\tilde{\alpha}^j_{-1}|0\rangle_{\mathrm{L}}$ ($j = 1,2$). Alternatively, we may attach the (massless) Ramond groundstate $|0\rangle_R$ to the fixed left-moving state. This transforms as an eight-component SO(8) *chiral* spinor, the opposite chirality spinor having been deleted by the GSO projection used in the superstring construction. This eight-component SO(8) chiral spinor may be decomposed into representations of $SO(2) \times SO(6) \subset SO(8)$, the SO(2) corresponding to the two transverse space–time coordinates, and the SO(6) to the six compactified coordinates. Then

$$\mathbf{8}_L = (+\tfrac{1}{2}) \times \mathbf{4} + (-\tfrac{1}{2})\bar{\mathbf{4}} \tag{1.37}$$

and it is clear that there are *four* space–time spinor particles of each chirality. Thus, if the (bosonic) string state constructed first was a vector particle, the fermionic state we have just constructed is four gauginos whereas if the bosonic state first constructed was a space–time tensor, the graviton, the fermionic state is four gravitinos. Evidently the toroidal compactification under consideration leads inevitably to $N = 4$ space–time supersymmetry, and hence to a non-chiral gauge symmetry. The observed cancellation of the gauge chiral anomaly within each generation of fermions strongly suggests (but does not conclusively prove) that the gauge symmetry is chiral, and hence that there can be at most $N = 1$ space–time supersymmetry; $N \geqslant 2$ supersymmetries automatically cancel chiral anomalies within each supermultiplet.

1.3. Point groups and space groups

In the previous section we considered the compactification of the ten-dimensional heterotic string in which the six left-movers and six right-movers X^k_R, X^k_L $(k = 3, \ldots, 8)$ are compactified onto the (same) torus T^6 generated by the lattice Λ, with the 16 left-movers X^I_L compactified on the (self-dual) $E_8 \times E_8$ torus $T^{E_8 \times E_8}$. This latter torus is generated by the root lattice of the group $E_8 \times E_8$. A torus is defined by identifying points of the underlying space which differ by a lattice vector $\boldsymbol{l} \in \boldsymbol{\Gamma} = 2\pi\boldsymbol{\Lambda}$

$$\boldsymbol{x} \equiv \boldsymbol{x} + \boldsymbol{l} \ . \tag{1.38}$$

This identification is called "modding" and in the six-dimensional toroidal case we write

$$T^6 = R^6/\Gamma \ . \tag{1.39}$$

We may generalize this process by identifying points on the torus which are related by the action of an isometry θ. To be well-defined on the torus θ must be an automorphism of the lattice, i.e. $\theta\boldsymbol{l} \in 2\pi\boldsymbol{\Lambda}$ if $\boldsymbol{l} \in 2\pi\boldsymbol{\Lambda}$ and preserve the scalar products

$$\theta\boldsymbol{e}_t \cdot \theta\boldsymbol{e}_u = \boldsymbol{e}_t \cdot \boldsymbol{e}_u \ . \tag{1.40}$$

The isometry group is called the point group (P) and an orbifold Ω is defined as

$$\Omega = T^6/P \times T^{E_8 \times E_8}/G \ , \tag{1.41}$$

where G is the embedding of P in the gauge group $E_8 \times E_8$. P and therefore G are discrete groups. Evidently the six-dimensional orbifold T^6/P may be obtained by identifying points of the underlying space (R^6) which are related by the action of the point group, up to a lattice vector $\boldsymbol{l}$

$$\boldsymbol{x} \equiv \theta\boldsymbol{x} + \boldsymbol{l} \ . \tag{1.42}$$

We may regard the right-hand side as the action of the pair $(\theta,\boldsymbol{l})$ upon the point $\boldsymbol{x}$, and the set of all such pairs

$$S \equiv \{(\theta,\boldsymbol{l})\,|\,\theta \in P, \boldsymbol{l} \in \mathbf{2}\pi\Lambda\} , \tag{1.43}$$

defines a group S, the space group, with the product defined in the obvious way by

$$[(\theta_1,\boldsymbol{l}_1)(\theta_2,\boldsymbol{l}_2)]\boldsymbol{x} \equiv (\theta_1,\boldsymbol{l}_1)[(\theta_2,\boldsymbol{l}_2)\boldsymbol{x}] . \tag{1.44}$$

Thus we may also write

$$T^6/P = R^6/S . \tag{1.45}$$

The solution of the string equations propagating on an orbifold are almost as straight forward as for a toroidal compactification, since the orbifold is flat almost everywhere. The exceptions are the points of the torus which are left fixed by the point group. Modding out the point group identifies different lines on the torus passing through the fixed points, so that a conical singularity occurs and the orbifold is not locally isomorphic to R^6 at such points. It follows from Eq. (1.42) that the fixed points satisfy

$$\boldsymbol{x}_f = \theta \boldsymbol{x}_f + \boldsymbol{l} \tag{1.46}$$

so if $1-\theta$ is singular there are fixed lines or tori, rather than isolated fixed points.

The full definition of an orbifold compactification requires the specification of T^6 or equivalently the lattice Γ, the discrete point group P, and its embedding G in the gauge degrees of freedom. The elements $\theta \in P$ act upon the bonsonic coordinates $X^k(\tau,\sigma)$ $(k = 3,\dots,8)$ of the string as SO(6) rotations. Possible choices of P are further restricted by the phenomenological requirement to obtain an $N = 1$ space–time supersymmetric spectrum; no supersymmetry $(N = 0)$ might also be acceptable, but the conventional wisdom is that $N = 1$ supersymmetry is preferred because of the solution to the technical hierarchy problem which it affords. To get $N = 1$ supersymmetry the point group P must be a subgroup of SU(3) [43]

$$P \subset \mathrm{SU}(3) . \tag{1.47}$$

This may be seen by recalling that SO(6) is isomorphic to SU(4), so if P satisfies Eq. (1.47) there is a covariantly constant spinor on the six-dimensional orbifold, and it is this extra symmetry which generates the required supersymmetry.

For the present we restrict our attention to the cases when the point group P is abelian. Then it must belong to the Cartan subalgebra of SO(6) associated with X^k $(k = 3,\dots,8)$. We denote the generators of this subalgebra by M_{34},M_{56},M_{78}. Then in the complex basis defined by

$$Z^1 \equiv (1/\sqrt{2})(X^3 + \mathrm{i}X^4) , \tag{1.48}$$

$$Z^2 \equiv (1/\sqrt{2})(X^5 + \mathrm{i}X^6) , \tag{1.49}$$

$$Z^3 \equiv (1/\sqrt{2})(X^7 + \mathrm{i}X^8) \tag{1.50}$$

the point group element θ acts diagonally and may be written

$$\theta = \exp[2\pi\mathrm{i}(v_1 M^{34} + v_2 M^{56} + v_3 M^{78})] \tag{1.51}$$

with $0 \leqslant |v_i| < 1$ ($i = 1,2,3$). The condition that Eq. (1.47) is satisfied then gives

$$\pm v_1 \pm v_2 \pm v_3 = 0 \tag{1.52}$$

for some choice of signs; this may be seen by noting that the eigenvalues of θ acting on a spinor are $e^{i\pi(\pm v_1 \pm v_2 \pm v_3)}$.

The requirement that θ acts crystallographically on the lattice Γ plus the condition (1.52) then leads to the conclusion [79,80] that P must either be Z_N with $N = 3,4,6,7,8,12$ or $Z_M \times Z_N$ with N a multiple of M and $N = 2,3,4,6$. Some of the point groups have two (inequivalent) embeddings in SO(6), i.e. they are realized by the inequivalent sets of v_1,v_2,v_3. The complete list is given in Tables 1 and 2. These results are the six-dimensional analogue of the famous result that crystals in three dimensions have only $N = 2,3,4,6$-fold rotational symmetries, (augmented by the $N = 1$ space–time supersymmetry requirement (1.52)). In all cases it is possible to find a lattice upon which P acts crystallographically, and in many cases there are several lattices for a given P. Often the massless spectrum and gauge group of the orbifold are independent of the choice of lattice, and are determined solely by P. However, we shall see in Section 2 that when the full space group, not just

Table 1
Point group generators for $Z_N \subset$ SU(3) orbifolds $\theta = \exp 2\pi i(v_1 M^{34} + v_2 M^{56} + v_3 M^{78})$

Point group	(v_1, v_2, v_3)
Z_3	$\frac{1}{3}(1,1,-2)$
Z_4	$\frac{1}{4}(1,1,-2)$
$Z_6 - I$	$\frac{1}{6}(1,1,-2)$
$Z_6 - II$	$\frac{1}{6}(1,2,-3)$
Z_7	$\frac{1}{7}(1,2,-3)$
$Z_8 - I$	$\frac{1}{8}(1,2,-3)$
$Z_8 - II$	$\frac{1}{8}(1,3,-4)$
$Z_{12} - I$	$\frac{1}{12}(1,4,-5)$
$Z_{12} - II$	$\frac{1}{12}(1,5,-6)$

Table 2
Point group generators for $Z_M \times Z_N \subset$ SU(3) orbifolds $\theta = \exp 2\pi i(v_1 M^{34} + v_2 M^{56} + v_3 M^{78})$; $\varphi = \exp 2\pi i(w_1 M^{34} + w_2 M^{56} + w_3 M^{78})$

Point group	(v_1,v_2,v_3)	(w_1,w_2,w_3)
$Z_2 \times Z_2$	$\frac{1}{2}(1,0,-1)$	$\frac{1}{2}(0,1,-1)$
$Z_3 \times Z_3$	$\frac{1}{3}(1,0,-1)$	$\frac{1}{3}(0,1,-1)$
$Z_2 \times Z_4$	$\frac{1}{2}(1,0,-1)$	$\frac{1}{4}(0,1,-1)$
$Z_4 \times Z_4$	$\frac{1}{4}(1,0,-1)$	$\frac{1}{4}(0,1,-1)$
$Z_2 \times Z_6 - I$	$\frac{1}{2}(1,0,-1)$	$\frac{1}{6}(0,1,-1)$
$Z_2 \times Z_6 - II$	$\frac{1}{2}(1,0,-1)$	$\frac{1}{6}(1,1,-2)$
$Z_3 \times Z_6$	$\frac{1}{3}(1,0,-1)$	$\frac{1}{6}(0,1,-1)$
$Z_6 \times Z_6$	$\frac{1}{6}(1,0,-1)$	$\frac{1}{6}(0,1,-1)$

the point group, is embedded in the $E_8 \times E_8$ group then the orbifold properties, not surprisingly, do depend upon the lattice Λ.

1.4. Orbifold compactifications

The existence of the point group P means that there are additional ways, over and above the toroidal conditions (1.12), in which the closed string boundary conditions may be satisfied. Let the Z_N point group be generated by an element θ, so that the general element is θ^n $(0 \leqslant n \leqslant N-1)$. (The generalization to $Z_M \times Z_N$ generated by θ,φ is trivial.) Then the identification (1.42) means that the closed string boundary conditions for the coordinates X^k $(k = 3,\dots,8)$ may also be satisfied when

$$\boldsymbol{X}(\tau,\pi) = (\theta^n,\boldsymbol{l})\boldsymbol{X}(\tau,0) = \theta^n \boldsymbol{X}(\tau,0) + \boldsymbol{l} \ . \tag{1.53}$$

Evidently the "untwisted" sector ($n = 0$) corresponds to the toroidal compactification discussed in the previous section. However, there are additional "twisted" sectors, satisfying Eq. (1.53), with $n \neq 0$, and these generate new string states which were not present in the toroidal compactification. Before considering these new states, however, an immediate question arises: what feature of the orbifold removes the unwanted gaugino and gravitino states which we showed are a generic feature of toroidal compactifications, and which are present in the untwisted sector of the orbifold compactification? We have explained that the definition of an orbifold requires the specification of a discrete group G comprising the space group S and its embedding in the gauge degrees of freedom. Thus to each element of $g \in G$ there corresponds an operator $\bar{g}$ which implements the action of g on the Hilbert space. Because the orbifold is defined by modding out the action of G, it follows that physical states must be invariant under G. That is to say, they are eigenvectors of $\bar{g}$ with eigenvalue unity. Now consider the four gravitino states in the untwisted sector

$$|0\rangle_{\mathrm{R}}\tilde{\alpha}^j_{-1}|0\rangle_{\mathrm{L}} \quad (j = 1,2) \ . \tag{1.54}$$

Since $j = 1,2$ corresponds to the transverse space–time coordinates which are unaffected by the point group transformations, it is clear that g acts trivially on the left-moving piece of the state. The right moving piece is the Ramond sector ground state, which is an SO(8) chiral spinor. The decomposition (1.37) is given explicitly by

$$8_{\mathrm{R}} = (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},\underline{\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}}) + (-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}),(-\tfrac{1}{2},\underline{-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}}) \ , \tag{1.55}$$

where the underlining indicates that all (three) permutations are included, and the individual entries are the eigenvalues of $M^{12}, M^{34}, M^{56}, M^{78}$ respectively. The point group generator θ is given by Eq. (1.51), and we see that acting on the first four states its eigenvalues are

$$\bar{\theta} = \exp[\mathrm{i}\pi(v_1 + v_2 + v_3)],\exp\mathrm{i}\pi(v_1 - v_2 - v_3),\exp[\mathrm{i}\pi(v_2 - v_3 - v_1)],\exp[\mathrm{i}\pi(v_3 - v_1 - v_2)] \tag{1.56}$$

with the second four states having complex conjugate eigenvalues. Condition (1.52) ensures that at least one of these states have $\bar{\theta} = 1$. Suppose, for example, that

$$v_1 + v_2 + v_3 = 0 \ . \tag{1.57}$$

(Similar arguments are easily constructed for the other possibilities.) Then the eigenvalues of the above four states are

$$\bar{\theta} = 1, \exp(2\pi i v_1), \exp(2\pi i v_2), \exp(2\pi i v_3) \; . \tag{1.58}$$

So provided that v_1, v_2 and v_3 are all non-zero, the last three states all have $\bar{\theta} \neq 1$. It follows that they are not invariant under the action of the point group, and are therefore not space-group invariant either. Thus three of the four gravitinos are deleted, as required if we are to obtain an $N = 1$ space–time supersymmetric theory. On the other hand, if one of $v_{1,2,3}$ is zero only two of the four gravitinos are deleted and we have at least $N = 2$ supersymmetries surviving. It is for, this reason that Table 1 lists only point the nine group elements with $v_{1,2,3}$ all non-zero. Similarly in Table 2 we list point group elements of the $Z_M \times Z_N$ orbifolds which for $\alpha = 1,2,3$ have v_α and w_α not both zero.

The twisted sectors of the orbifold string theory are defined by Eq. (1.53) with $n \neq 0$. Let us consider the case of a Z_N orbifold and the $n = 1$ twisted sector. The extension to $n > 1$ and $Z_N \times Z_M$ is easily done. The first thing to note is that the modified boundary conditions lead to a different form of the various mode expansions. In this complex basis defined in (1.48)–(1.50), the mode expansion of the string world sheet is

$$Z^\alpha = z_f^\alpha + \frac{i}{2} \sum_{n \neq 0} \left[\frac{1}{n + v_\alpha} \beta^\alpha_{n+v_\alpha} e^{-2i(n+v_\alpha)(\tau-\sigma)} + \frac{1}{n - v_\alpha} \tilde{\beta}^\alpha_{n-v_\alpha} e^{-2i(n-v_\alpha)(\tau+\sigma)} \right] \tag{1.59}$$

where $\alpha = 1,2,3$ labels the three *complex* planes. The fractional modings are needed to supply the phase factors $\exp(2\pi i v_\alpha)$ acquired by Z^α under the action of the point group. z_f^α is a complex fixed point, constructed from the real fixed points (1.46) analogously to (1.48)–(1.50). Evidently the full specification of a twisted sector requires not only the point group element (θ in this case) but also the particular fixed point (or torus) which appears in the zero mode part of the world sheet. Note too that the boundary conditions require that the momentum is zero, since θ acts non-trivially in all planes; this is not necessarily the case in all twisted sectors of non-prime orbifolds. For example it is clear from Table 1 that in the θ^2-sectors of the Z_4-orbifold the mode expansion of Z^3 *will* have non-zero, but quantized, momentum.

The complex conjugate mode expansion is

$$\bar{Z}_\alpha = \bar{z}_f^\alpha + \frac{i}{2} \sum_{n \neq 0} \left[\frac{1}{n - v_\alpha} \bar{\beta}^\alpha_{n-v_\alpha} e^{-2i(n-v_\alpha)(\tau-\sigma)} + \frac{1}{n + v_\alpha} \bar{\tilde{\beta}}^\alpha_{n+v_\alpha} e^{-2i(n+v_\alpha)(\tau+\sigma)} \right] \tag{1.60}$$

and operators $\beta^\alpha_{n+v_\alpha}$, $\tilde{\beta}^\alpha_{n-v_\alpha}$, $\bar{\beta}^\alpha_{n-v_\alpha}$, $\bar{\tilde{\beta}}_{n+v_\alpha}$which appear in Z^α and $\bar{Z}^\alpha$ obey the commutation relations

$$[\beta^\alpha_{n+v_\alpha}, \bar{\beta}^\gamma_{m-v_\gamma}] = \delta^{\alpha\gamma}(n + v_\alpha)\delta_{m+n,0} \; ,$$

$$[\tilde{\beta}^\alpha_{n-v_\alpha}, \bar{\tilde{\beta}}^\gamma_{m+v_\gamma}] = \delta^{\alpha\gamma}(n - v_\alpha)\delta_{m+n,0} \; .$$

Thus the β_{n+v} with $n + v > 0$ are (proportional to) annihilation operators and the $\bar{\beta}_{-n-v}$ the associated creation operators. Likewise the β_{n+v} with $n + v < 0$ are creation operators and the $\bar{\beta}_{-n-v}$ the associated annihilation operators. Similarly for $\tilde{\beta}_{n-v}$ and $\bar{\tilde{\beta}}_{n+v}$.

The point group also acts upon the right-mover fermionic degrees of freedom, so that in the θ-twisted sector the boundary conditions are modified:

$$\psi^\alpha_R(\tau - \sigma - \pi) = e^{2\pi i v_\alpha} \psi^\alpha_R(\tau - \sigma) \quad (R) \; ,$$

$$\psi^\alpha_R(\tau - \sigma - \pi) = -e^{2\pi i v_\alpha} \psi^\alpha_R(\tau - \sigma) \quad (NS) \; , \tag{1.61}$$

where the complex ψ_R^α ($\alpha = 1,2,3$) are constructed from the ψ_R^k just as the Z^α are defined in terms of the X^k ($k = 3,4,\dots,8$) in (1.48)–(1.50). Thus the modified mode expansions are

$$\psi_R^\alpha(\tau-\sigma) = \sum_n e^\alpha_{n+v_\alpha} e^{-2i(n+v_\alpha)(\tau-\sigma)} \quad (\mathrm{R})$$

$$= \sum_r c^\alpha_{r+v_\alpha} e^{-2i(r+v_\alpha)(\tau-\sigma)} \quad (\mathrm{NS}) \tag{1.62}$$

and

$$\bar\psi_R^\alpha(\tau-\sigma) = \sum_r \bar e^\alpha_{n-v_\alpha} e^{-2i(n-v_\alpha)(\tau-\sigma)} \quad (\mathrm{R})$$

$$= \sum_r \bar c^\alpha_{r-v_\alpha} e^{-2i(r-v_a)(\tau-\sigma)} \quad (\mathrm{NS})\ , \tag{1.63}$$

where

$$\{e^\alpha_{n+v_\alpha}, \bar e^\beta_{m-v_\beta}\} = \delta^{\alpha\beta}\delta_{m+n,0}\ ,$$
$$\{c^\alpha_{r+v_\alpha}, \bar c^\beta_{s-v\beta}\} = \delta^{\alpha\beta}\delta_{r+s,0}\ . \tag{1.64}$$

The space group may also be embedded in the gauge degrees of freedom, and in general, it must be, as we shall see. The element $(\theta,\boldsymbol{l})$ of the space group is generally mapped on to $(\Theta,\boldsymbol{V})$ where Θ is an automorphism of the $E_8\times E_8$ lattice and $\boldsymbol{V}$ is a shift on the lattice. In this section we only address the (compulsory) embedding of the point group elements $(\theta,\mathbf{0})$ in the gauge group. The (optional) embedding of the lattice elements $(1,\boldsymbol{l})$, Wilson lines, is discussed in Section 2.2.

It is easiest to consider first the embedding using the fermionic formulation of the gauge degrees of freedom. The 16 real fermions λ^A transform as the vector representation of $O(16)\subset E_8$. The simplest non-trivial embedding is achieved by picking an $O(6)$ subgroup of $O(16)$, in which the vector representation decomposes into a (six-dimensional) vector representation of SO(6) plus (ten) SO(6) singlets. We next form 3 complex fermions from the 6 real fermions, precisely as we did for the right-moving fermions ψ_R^k ($k = 3,\dots,8$), and then take the action of the point group on these 3 complex fermions to be precisely what it is on the three complex right-moving fermions ψ_R^α; the other ten-fermions are untransformed. This is called the *standard embedding* [80]. Evidently the mode expansions of these three complex gauge fermions will be modified precisely as are those of the complex fermionic right-movers. The second set of fermions ($\bar\lambda^A$) are left completely untransformed.

This embedding amounts to a shift on the $E_8\times E_8$ lattice when we use the bosonic formulation. To see why we need the relationship

$$\psi^I(\tau+\sigma) = :\exp(2iX_L^I): \tag{1.65}$$

between the bosonic toroidal coordinates X_R^I and the *complex* fermions. Then multiplying ψ by a phase factor $\exp(2\pi i V)$ amounts to adding πV^I to the bosonic coordinates X_L^I. Thus the embedding of $(\theta,0)$ on the $E_8\times E_8$ lattice is realized as $(1,\pi V^I)$, and the θ-twisted sector boundary conditions for the X_L^I become

$$X_L^I(\tau+\sigma+\pi) = X_L^I(\tau+\sigma) + \pi V^I \tag{1.66}$$

up to (π times an $E_8\times E_8$ root lattice vector, and the mode expansion satisfying this is

$$X^I_{\mathrm{L}} = x^I_{\mathrm{L}} + (p^I_{\mathrm{L}} + V^I)(\tau+\sigma) + \frac{\mathrm{i}}{2}\sum \frac{1}{n}\tilde{\alpha}^I_n \mathrm{e}^{-2\mathrm{i}n(\tau+\sigma)} . \tag{1.67}$$

Evidently the net effect of the twist θ is to shift the momentum p^I_{L} by V^I. In the standard embedding, which we have so far discussed,

$$V^I = (v_1, v_2, v_3, 0^5)(0^8) , \tag{1.68}$$

where v_α ($\alpha = 1,2,3$) are the twists of the 3 complex compactified coordinates.

However, we may also entertain the possibility of more general (non-standard) embeddings. Then (so far) the only constraint on the shift V^I is that for a Z_N orbifold $\mathrm{N}V^I$ is on the $E_8\times E_8$ root lattice (so that in the $\theta^N = 1$ sector the momenta $p^I_{\mathrm{L}} + \mathrm{N}V^I$ are on the same lattice as the p^I_{L} are):

$$\mathrm{N}V^I \in \Lambda_{E_8\times E_8} . \tag{1.69}$$

The requirement (1.52) on the v_a ($\alpha = 1,2,3$) ensures that the above constraint is always satisfied by the standard embedding.

In the absence of Wilson lines, the embedding of $(\theta,0)$ can always be realized as a shift $(1,\pi V^I)$ on the $E_8\times E_8$ lattice, and sometimes this shift is also realizable on an automorphism Θ of the lattice.

The changes in the mode expansions which we have described feed through into the calculations of the generators $L_m, \tilde{L}_n$ of the Virasoro algebra, and in particular to changes in the expressions for $L_0, \tilde{L}_0$ which lead to the mass formulae.. These now involve *fractional* number operators associated with the fractional-modings. The fractional modings also affect the calculations of the normal ordering constants. The general results are that a complex bosonic coordinate with moding shifted by v ($|v|<1$) contributes

$$a_B(v) = \tfrac{1}{12} - \tfrac{1}{2}|v|(1-|v|) \tag{1.70}$$

to the subtraction constant, while a complex Ramond fermion with moding shifted by v contributes

$$a_F(v) = -\tfrac{1}{12} + \tfrac{1}{2}|v|(1-|v|) . \tag{1.71}$$

The standard Neveu–Schwarz fermion may for these purposes be regarded as a Ramond fermion with shift $v=\frac{1}{2}$. Then a complex Neveu–Schwarz fermion with moding shifted by v contributes

$$a_{\mathrm{NS}}(v) = a_{\mathrm{F}}(|v+\tfrac{1}{2}|), \quad -1 < v < \tfrac{1}{2} \tag{1.72}$$

for

$$= a_{\mathrm{F}}(|v-\tfrac{1}{2}|), \quad \tfrac{1}{2} < v < 1 \tag{1.73}$$

The upshot of these changes is that the mass formula for the right movers in the θ-twisted sector has the general structure

$$\tfrac{1}{4}M^2_{\mathrm{R}} = N_B + N_F(\beta) - a_B - a_F(\beta) , \tag{1.74}$$

where, as in Eq. (1.20), $\beta = \mathrm{R}$, NS labels the (shifted) boundary conditions satisfied by the fermionic right movers,

$$N_B = \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + \sum_{\alpha, nn+v_\alpha>0} \bar{\beta}^\alpha_{-n-v_\alpha}\beta^\alpha_{n+v_\alpha} + \sum_{\alpha, nn-v_\alpha>0} \beta^\alpha_{-n+v_\alpha}\bar{\beta}^\alpha_{n-v_\alpha} , \tag{1.75}$$

$$N_F(\mathrm{R}) = \sum_{n=1}^{\infty} nd^i_{-n}d^i_n + \sum_{n,\alpha n+v_\alpha>0} (n+v_\alpha)\bar{e}^\alpha_{-n-v_\alpha}e^\alpha_{n+v_\alpha} + \sum_{\alpha,nn-v_\alpha>0} (n-v_\alpha)e^\alpha_{-n+v_\alpha}\bar{e}^\alpha_{n-v_\alpha}\ , \tag{1.76}$$

$$N_F(\mathrm{NS}) = \sum_{r\in Z+1/2r>0} rb^i_{-r}b^i_r + \sum_{r,\alpha\bar{r}v_\alpha>0} (r+v_\alpha)\bar{c}^\alpha_{-r-v_\alpha}c^\alpha_{r+v_\alpha} + \sum_{r,\alpha r-v_\alpha>0} (r-v_\alpha)c^\alpha_{-r+v_\alpha}\bar{c}^\alpha_{r-v_\alpha} \tag{1.77}$$

and

$$a_B = \frac{1}{3} - \frac{1}{2}\sum_{\alpha=1}^{3} |v_\alpha|(1-|v_\alpha|)\ , \tag{1.78}$$

$$a_F(\mathrm{R}) = -\frac{1}{3} + \frac{1}{2}\sum_{\alpha=1}^{3} |v_\alpha|(1-|v_\alpha|)\ ,$$

$$a_F(\mathrm{NS}) = \frac{-5}{24} + \frac{1}{2}\sum_{\alpha=1}^{3} \left|v_\alpha + \frac{1}{2}\right|\left(1 - \left|v_\alpha + \frac{1}{2}\right|\right). \tag{1.79}$$

(The form of $a_F(\mathrm{NS})$ assumes that $-1 < v_\alpha < \frac{1}{2}$ for all α, with the obvious change (1.73) to be made for any v_a satisfying $\frac{1}{2} < v_\alpha < 1$.) Note that there is no momentum contribution to m^2_{R}, since, as already observed, p_{R} is zero in a twisted sector (when all $v_a \neq 0$).

The mass formula for the left movers in the θ-twisted sector is

$$\tfrac{1}{4}m^2_{\mathrm{L}} = \tilde{N} + \tfrac{1}{2}(p^I_{\mathrm{L}} + V^I)^2 - \tilde{a}\ ,$$

where $\tilde{N}$ has the same form as N_B in Eq. (1.75) but with all operators replaced by their left-moving analogues. The subtraction constant $\tilde{a}$

$$\tilde{a} = 1 - \frac{1}{2}\sum_\alpha |v_\alpha|(1-|v_\alpha|)\ . \tag{1.80}$$

(The extra $\frac{2}{3}$ compared with a_B derives from the 16 internal bosonic left-movers.) There is a corresponding formula for m^2_{L} when the fermionic formulation of the gauge degrees of freedom is used. However we shall not quote it.

We may now see why the embedding of the point group in the gauge group is compulsory. First note that the mass formula (1.74) shows that the Ramond sector ground state $|0\rangle_{\mathrm{R}}$ is a (twisted sector) massless right-moving state, since

$$a_B + a_F(R) = 0 \tag{1.81}$$

and, by definition, no oscillators are utilized. Level matching then requires that there is a massless left-moving state. Now, since $\tilde{N}$ involves fractionally moded operators, it is easy to see that its eigenvalues are also fractional. For a Z_N orbifold

$$N\tilde{N} \in Z \tag{1.82}$$

so to obtain a massless left moving state, it follows from the mass formula (1.4) that

$$N(V^2 - v^2) \in 2Z \tag{1.83}$$

using Eq. (1.52) and the fact that p_L^I and NV^I are on the $E_8 \times E_8$ root lattice. This constraint is trivially satisfied by the standard embedding (1.68), but is not in general satisfied by the trivial embedding ($V = 0$). In fact, it follows from Tables 1 and 2 that of the Z_N orbifolds only the Z_3 and Z_7 orbifolds allow the trivial embedding; none of the $Z_M \times Z_N$ orbifolds do. It is in this sense that we say that the embedding of the twist in the gauge degrees of freedom is generally compulsory.

Condition (1.83) is sufficient to ensure level matching in the Neveu–Schwarz sector, and at general higher levels [183,105]. In fact, it is necessary and sufficient to ensure the modular invariance of the theory, as we shall see in Section 2.3.4; modular invariance means that the one-loop toroidal amplitude does not depend on the choice of the (two) basis vectors which generate the lattice defining the torus.

We have mentioned already that the (essential) primary virtue of orbifold models over toroidal compactifications is that the unwanted gravitinos (in the untwisted sector) are removed by the requirement of point group invariance. This point group invariance also reduces the gauge symmetry when the point group is embedded in the gauge degrees of freedom, as it has to be, in general. Precisely what gauge symmetry survives depends upon the details of the particular orbifold. However, we can make a general statement when the standard embedding is adopted. Then the constraint (1.47) ensures that the point group is embedded is an SU(3) subgroup of one of the E_8 groups. Since

$$E_8 \supset E_6 \times \mathrm{SU}(3) \tag{1.84}$$

it is clear that the surviving gauge symmetry will always include $E_6 \times E_8$. Further, the rank of the gauge group is unaffected by the embedding since the gauge bosons associated with the Cartan sub-algebra are all invariant under the action of the point group: They are given by

$$b^i_{-1/2}|0\rangle_R \tilde{\alpha}^I_{-1}|0\rangle_L \ . \tag{1.85}$$

We have already observed that the right-moving state is invariant under the action of P, and its embedding as a *shift* V on the $E^8 \times E^8$ lattice means that the oscillators $\tilde{\alpha}^I_n$ are also untransformed. Thus the standard embedding gives a gauge group of at least $E_6 \times U(1)^2 \times E_8$. The "charged" gauge bosons of $E_8 \times E_8$ are given by

$$b^i_{-1/2}|0\rangle_R |p^I_L\rangle \tag{1.86}$$

with $(p^I_L)^2 = 2$, and we shall show in Section 2.7 that, when the point group is embedded as a shift V^I on the lattice, the surviving gauge bosons satisfy

$$p^I_L V^I = 0 \bmod 1 \ . \tag{1.87}$$

Then, with the standard embedding, only the Z_3 and Z_4 orbifolds have more gauge symmetry. Z_3 has $E_6 \times \mathrm{SU}(3) \times E_8$ and Z_4 has $E_6 \times \mathrm{SU}(2) \times U(1) \times E_8$.

Non-standard embeddings, which embed non-trivially in both E_8 factors, may also be considered. They are constrained by Eqs. (1.69) and (1.83). Then, besides the trivial embeddings ($V = 0$) for the Z_3 and Z_7 orbifolds, the number of independent new embeddings ranges from three, for the Z_3-orbifolds, to 602 for the Z_{12}-*II* orbifold. Full details may be found in [124,125,102,104, 106,49,50,137]. As we have seen, the standard embedding breaks one of the E_8 symmetries to a smaller group with the same rank, while leaving the other E_8 unbroken. This affords the prospect of achieving further symmetry breaking, by Wilson lines, for example, leaving a realistic gauge symmetry. For this reason the broken E_8 is called the "observable" gauge group, and the unbroken E_8 the "hidden" gauge group.

1.5. Matter content of orbifold models

We have seen that the gauge symmetry in orbifold models is determined entirely by the point group P and its embedding in the gauge degrees of freedom. In particular the six-dimensional lattice T^6 on which the orbifold is compactified does not affect these results, so long as we do not embed the T^6 lattice vectors in the $E_8 \times E_8$ lattice. The same is true of the matter content of orbifold models: one just constructs massless, space group invariant, $N = 1$ chiral supermultiplets in all sectors using the fractionally moded creation operators and shifted momenta appropriate to the point-group twist. It might be thought that the lattice enters via the fixed points, which we have emphasized label the different twisted sectors. However, the *number* of fixed points (n_{fp}) under an SO(6) automorphism (θ) depends only upon the automorphism, and *not* in the specific lattice. In fact, n_{fp} may be calculated using the Lefschetz fixed point theorem which gives

$$n_{\mathrm{fp}} = \chi(\theta) = \det(1 - \theta) , \tag{1.88}$$

where $\chi(\theta)$ is the Euler character and θ is given in the vector representation of SO(6). The matter with which we shall be primarily concerned consists of chiral supermultiplets transforming non-trivially with respect to the observable gauge group. We have seen that the standard embedding breaks the E_8 symmetry to at least E_6, so the matter transforms as some representations of this group. It is easy to see the only representations which occur are the **27** and $\overline{\mathbf{27}}$. First note that we can construct (scalar) E_8 matter analogously to the gauge bosons:

$$b^k_{-1/2}|0\rangle_{\mathrm{R}}|p^I_{\mathrm{L}}\rangle \quad (k = 3, \dots ,8) \tag{1.89}$$

using the compactified untwisted oscillators $b^k_{-1/2}$, rather than the transverse space–time oscillators $b^i_{-1/2}$. However, since the right movers transform non-trivially under the action of the point group, the left-movers must too. Under the decomposition

$$E_8 \supset E_6 \times \mathrm{SU}(3) \tag{1.90}$$

the adjoint 248-dimensional representation of E_8 decomposes as

$$\mathbf{248} = (\mathbf{78},\mathbf{1}) + (\mathbf{1},\mathbf{8}) + (\mathbf{27},\mathbf{3}) + (\overline{\mathbf{27}},\overline{\mathbf{3}}) . \tag{1.91}$$

Thus the only matter which transforms non-trivially with respect to E_6 *and* with respect to $P \subset \mathrm{SU}(3)$ is the $(\mathbf{27},\mathbf{3})$ and $(\overline{\mathbf{27}},\overline{\mathbf{3}})$. Each **27** can accommodate one generation of fermions, together with some extra matter. This can be seen using the decomposition

$$E_6 \supset \mathrm{SO}(10) \tag{1.92}$$

in which

$$\mathbf{27} = \mathbf{16} + \mathbf{10} + \mathbf{1} . \tag{1.93}$$

Then the **16** accommodates the observed 15 chiral states together with an $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ singlet, presumably the right-chiral neutrino state.

For the standard embedding the net number of chiral generations is given by the formula [43,79,187]

$$n_G \equiv n(\mathbf{27}) - n(\overline{\mathbf{27}}) = \tfrac{1}{2}\chi$$
$$= \frac{1}{2|P|} \sum_{[h,g]=0} \chi(h,g) \;, \tag{1.94}$$

where $|P|$ is the order of the point group P, and $\chi(h,g)$ is the number of fixed points common to the elements $g,h \in P$. As we have seen, this last quantity does not depend on the lattice, and may easily be calculated using Eq. (1.88). This calculation is especially easy for the prime order orbifolds Z_3, Z_7, since the fixed points of the generator θ are fixed points of all θ^n $(1 \leqslant n \leqslant N-1)$, and then

$$\chi = (1/N)(N^2-1)\det(1-\theta) \;. \tag{1.95}$$

Remarkably in all abelian orbifolds n_G is a multiple of 12.

The orbifolds of even order all have fixed tori in some sectors. For example the Z_4 orbifold of Table 1 has a fixed torus (the third complex plane) in the θ^2 sector. In such sectors we effectively have $N=2$ supersymmetries and there are *two* invariant space-time spinors with *opposite* helicity. Equivalently such sectors contribute $\mathbf{27}+\overline{\mathbf{27}}$ pairs to the matter content. The full determination of the matter content of Z_N orbifolds may be found in [137] for Z_N orbifolds and in [98,143] for $Z_M \times Z_N$ orbifolds. It is clear that as they stand none of them has a realistic gauge group and/or matter content, and it is for this reason that in Section 2 we are led to study the embedding of the full space group S, not just P, in the gauge degrees of freedom.

1.6. Lattices

The complete specification of an orbifold requires the choice of a lattice T^6 upon which the point group P acts as an automorphism. In general there are several lattices for any given point group, but, as we saw in Section 1.4, many properties of the orbifold-compactified string theory do not depend on the choice of the lattice. However, when we embed the lattice in the gauge degrees of freedom non-trivially, as we do in Section 2, then the resulting theory manifestly depends upon T^6.

We consider the lattices of semi-simple Lie groups of rank 6. Inner automorphisms of such lattices are provided by the Weyl group of the algebra. It is generated by elements s_a whose action upon a vector x is to reflect it in the simple root e_a:

$$s_a(\mathbf{x}) = \mathbf{x} - 2(\mathbf{x}\cdot\mathbf{e}_a)\mathbf{e}_a/(\mathbf{e}_a\cdot\mathbf{e}_a) \;. \tag{1.96}$$

Such reflections are not SU(3) transformations, so the Weyl group is not contained in SU(3) and therefore cannot be the point group of any of our orbifolds. However it has some subgroups which *are* contained in SU(3). In particular, there is the cyclic subgroup generated by the Coxeter element [161,137,143]

$$C \equiv s_1 s_2 s_3 s_4 s_5 s_6 \tag{1.97}$$

which satisfies

$$C^N = 1 \;, \tag{1.98}$$

where the order N of the cyclic group is the Coxeter number. For a simple Lie algebra the Coxeter number is given by

$$N = \frac{\text{number of non-zero roots}}{\text{rank of Lie algebra}} . \tag{1.99}$$

It is these "Coxeter" orbifolds which we shall describe. We include in this class also the cyclic subgroups of SU(3) generated by the generalized Coxeter element(s), in which one (or more) of the Weyl reflections is replaced by an outer automorphism of the Dynkin diagram. Let us consider the rank 4 Lie algebra SO(8). The Coxeter element is

$$C_{\mathrm{SO}(8)} = s_1 s_2 s_3 s_4 \quad \text{with } N = 6 . \tag{1.100}$$

The Dynkin diagram has two automorphisms: (i) s_{34}, in which two of the (outer) roots, say $\boldsymbol{e}_3 \leftrightarrow \boldsymbol{e}_4$ are interchanged, and (ii) s_{134}, in which the outer roots are cyclically permuted $\boldsymbol{e}_1 \to \boldsymbol{e}_3 \to \boldsymbol{e}_4 \to \boldsymbol{e}_1$. ($\boldsymbol{e}_2$ is the central root.) Then

$$\begin{aligned} s_{34}(\boldsymbol{x}) &= \boldsymbol{x} - [\boldsymbol{x}\cdot(\boldsymbol{e}_3 - \boldsymbol{e}_4)](\boldsymbol{e}_3 - \boldsymbol{e}_4)/(\boldsymbol{e}_3\cdot\boldsymbol{e}_3) , \\ s_{134}(\boldsymbol{x}) &= \boldsymbol{x} - [(\boldsymbol{x}\cdot\boldsymbol{e}_1)(\boldsymbol{e}_1 - \boldsymbol{e}_3) + (\boldsymbol{x}\cdot\boldsymbol{e}_3)(\boldsymbol{e}_3 - \boldsymbol{e}_4) + (\boldsymbol{x}\cdot\boldsymbol{e}_4)(\boldsymbol{e}_4 - \boldsymbol{e}_1)]/(\boldsymbol{e}_3\cdot\boldsymbol{e}_3) \end{aligned} \tag{1.101}$$

s_{34} is of order 2, and s_{134} of order 3. Thus there are two generalized Coxeter elements associated with the SO(8) algebra:

$$\begin{aligned} C_{\mathrm{SO}(8)^{[2]}} &= s_1 s_2 s_3 s_{34} \quad \text{with } N = 8 , \\ C_{\mathrm{SO}(8)^{[3]}} &= s_1 s_2 s_{134} \quad \text{with } N = 12 , \end{aligned} \tag{1.102}$$

where the numbers in square brackets give the order of the outer automorphism used in the generalized Coxeter element. By considering products of such lattices, with Lie algebra having rank less than or equal to six we can find all Coxeter orbifolds. The results for the Z_N orbifolds are given in Table 3.

Even though we have specified the lattices upon which the various point groups act, it is important to recognize that there remain a number of "deformation parameters" which are not fixed. Generically there remain some undetermined scale factors, characterizing the size of the orbifold, as well as some undetermined angles between basis vectors, the complex structure of the lattice. Under the action of the point group θ a lattice vector $\boldsymbol{e}$ is transformed as

$$\boldsymbol{e}_t \to \boldsymbol{e}'_t = \theta_{ut}\boldsymbol{e}_u, \quad \theta_{ut} \in \boldsymbol{Z} . \tag{1.103}$$

Since θ is an isometry we require

$$(\theta\boldsymbol{e}_t \cdot \theta\boldsymbol{e}_u) = (\boldsymbol{e}_t \cdot \boldsymbol{e}_u) \tag{1.104}$$

so that

$$G = \theta^{\mathrm{T}} G \theta \tag{1.105}$$

where

$$G_{tu} \equiv \boldsymbol{e}_t \cdot \boldsymbol{e}_u \tag{1.106}$$

is the metric on the lattice.

Table 3
Z_N Coxeter orbifolds

Point group	Lattice
Z_3	$(\mathrm{SU}(3))^3$
Z_4	$(\mathrm{SU}(4))^2$
Z_4	$\mathrm{SO}(5) \times \mathrm{SU}(4) \times \mathrm{SU}(2)$
Z_4	$(\mathrm{SO}(5))^2 \times (\mathrm{SU}(2))^2$
$Z_6 - I$	$(\mathrm{G}_2)^2 \times \mathrm{SU}(3)$
$Z_6 - I$	$(\mathrm{SU}(3)^{[2]})^2 \times \mathrm{SU}(3)$
$Z_6 - II$	$\mathrm{SU}(6) \times \mathrm{SU}(2)$
$Z_6 - II$	$\mathrm{SO}(8) \times \mathrm{SU}(3)$
$Z_6 - II$	$\mathrm{SO}(7) \times \mathrm{SU}(3) \times \mathrm{SU}(2)$
$Z_6 - II$	$\mathrm{G}_2 \times \mathrm{SU}(3) \times (\mathrm{SU}(2))^2$
$Z_6 - II$	$\mathrm{SU}(3)^{[2]} \times \mathrm{SU}(3) \times (\mathrm{SU}(2))^2$
$Z_6 - II$	$\mathrm{SU}(4)^{[2]} \times \mathrm{SU}(3) \times \mathrm{SU}(2)$
Z_7	$\mathrm{SU}(7)$
$Z_8 - I$	$\mathrm{SO}(9) \times \mathrm{SO}(5)$
$Z_8 - I$	$\mathrm{SO}(8)^{[2]} \times \mathrm{SO}(5)$
$Z_8 - II$	$\mathrm{SO}(8)^{[2]} \times (\mathrm{SU}(2))^2$
$Z_8 - II$	$\mathrm{SO}(10) \times \mathrm{SU}(2)$
$Z_8 - II$	$\mathrm{SO}(9) \times (\mathrm{SU}(2))^2$
$Z_{12} - I$	E_6
$Z_{12} - I$	$F_4 \times \mathrm{SU}(3)$
$Z_{12} - I$	$\mathrm{SO}(8)^{[3]} \times \mathrm{SU}(3)$
$Z_{12} - II$	$\mathrm{SO}(4) \times F_4$
$Z_{12} - II$	$\mathrm{SO}(8)^{[3]} \times (\mathrm{SU}(2))^2$

We have seen that the specification of an orbifold includes the identification of the (six-dimensional) metric of the compactified space. We have also seen that besides the (symmetric) graviton and dilaton states the 10-dimensional spectrum also includes anti-symmetric tensor particles. Thus we may consider a more general situation than that which we have considered hitherto, in which there is an antisymmetric background field (B) besides the symmetric background metric fields (G). The possibility of doing this may also be seen by considering a generalization of the original Polyakov action

$$S_{\mathrm{p}} = -\frac{T}{2}\int \mathrm{d}^2\sigma(-h)^{1/2}[h^{\alpha\beta}G_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu + \varepsilon^{\alpha\beta}B_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu + \cdots] , \tag{1.107}$$

where σ^α ($\alpha = 1,2$) are the world sheet coordinates τ and σ, $h_{\alpha\beta}$ is the world sheet metric, $\varepsilon_{\alpha\beta}$ the anti-symmetric two-dimensional, tensor, and $G_{\mu\nu}$, $B_{\mu\nu}$ are the (constant) target space metric and antisymmetric tensor field. The unexhibited terms include Wilson line contributions (A^I_μ) linking the (ten-dimensional) string world sheet to the (16-dimensional) left-moving gauge degrees of freedom. These will be discussed in Section 2. The background field $B_{\mu\nu}$ is taken to be non-zero only in compactified dimensions. Then the new term is easily seen to be a total divergence, so the field equations and mode expansions are unaltered. Nevertheless, its presence affects the compactification because the field conjugate to X^μ becomes

$$\Pi_\mu = -T(G_{\mu\nu}\dot{X}^\nu + B_{\mu\nu}X'^\nu)\ , \tag{1.108}$$

where

$$\dot{X}^\nu \equiv \partial_0 X^\nu, \quad X'^\nu \equiv \partial_1 X^\nu\ . \tag{1.109}$$

Using the standard mode expansion for X^ν yields the momentum operator conjugate to x^k as

$$\tilde{p}_k = p_k + 2B_{k\ell}L^\ell$$

and it is $\tilde{p}$, rather than p, which has eigenvalues on the lattice Λ^* dual to Λ. The upshot is that the left and right mover mode expansions still have the form (1.15), (1.16), but *now* p_R,p_L are given by

$$p^k_\mathrm{R} = \tfrac{1}{2}\tilde{p}^k - L^k - B^{k\ell}L_\ell\ , \tag{1.110}$$

$$p^k_\mathrm{L} = \tfrac{1}{2}\tilde{p}^k + L^k - B^{k\ell}L_\ell\ . \tag{1.111}$$

with $\tilde{\boldsymbol{p}} \in \boldsymbol{\Lambda}^*$ and $\boldsymbol{L} \in \boldsymbol{\Lambda}$.

The full six-dimensional compactified space is evidently associated with 36 quantities, 21 associated with the (symmetric) metric parameters and 15 with the antisymmetric background field. In most applications far fewer parameters are non-zero, since the lattice is defined in terms of lower dimensional constructions. Many of these use two-dimensional lattices, which are specified by just four quantities G_{11},G_{12},G_{22},B_{12}. It is customary to combine these into two complex quantities T, U defined as follows. The metric quantities G_{ij}, are defined by two basis vectors whose relative size and orientation may be characterized by the complex number U which specifies the end point of the vector e_2, in the complex plane when e_1 is normalized to the unit vectors lying along the real axis of the Argand diagram.

Then U is given by

$$\mathrm{i}U = (1/G_{11})(G_{12} + \mathrm{i}\sqrt{\det G}) \tag{1.112}$$

and is called the "complex structure". As it involves only ratios of terms in the metric it carries no information about the overall size of the (two-dimensional) torus. This information is supplied by the complex numbers

$$\mathrm{i}T \equiv 2(B_{12} + \mathrm{i}\sqrt{\det G}) \tag{1.113}$$

so that

$$\det(G \pm B) = |T|^2/4$$

and the (square of the) imaginary part of T gives the area of the fundamental torus.

1.6.1. Example: Z_3 orbifold with standard embedding [79,80]

We illustrate the foregoing generalities by applying them to the Z_3-orbifold, the simplest of the (symmetric) abelian orbifolds. The point group generator θ satisfies

$$\theta^3 = 1 \tag{1.114}$$

and its action on the compactified dimensions is given by Eq. (1.51) with

$$(v_1,v_2,v_3)=\tfrac{1}{3}(1,1,-2)\ . \tag{1.115}$$

We have already noted that the gauge bosons arise in the untwisted sector, and are given by the states

$$b^{i}_{-1/2}|0\rangle_{\mathrm{R}}\tilde{\alpha}^{I}_{-1}|0\rangle_{\mathrm{L}} \tag{1.116}$$

corresponding to the Cartan sub-algebra, and the states

$$b^{i}_{-1/2}|0\rangle_{\mathrm{R}}|p^{I}_{\mathrm{L}}\rangle$$

with $(p^I_{\mathrm{L}})^2=2$ and $p^I_{\mathrm{L}}V^I\in Z$ corresponding to the charged state of $\mathrm{SU}(3)\times E_6\times E_8$. V^I is the standard embedding of the point group in the gauge group and is given by Eq. (1.68).

Similarly the chiral gauge non-singlet matter is given by the states

$$\bar{c}^{\alpha}_{-1/2}|0\rangle_{\mathrm{R}}|p^{I}_{\mathrm{L}}\rangle\ , \tag{1.117}$$

where the $\bar{c}^{\alpha}_{-1/2}$ ($\alpha=1,2,3$) are the untwisted fermionic oscillators in the complex basis (1.48)–(1.50). The right-movers are eigenstates of the operator $\bar{\theta}$, which implements the action of θ on the Hilbert space, with eigenvalue

$$\bar{\theta}=\mathrm{e}^{-2\pi\mathrm{i}/3}\ . \tag{1.118}$$

Then the corresponding left-mover momentum states $|p^I_{\mathrm{L}}\rangle$ are those with

$$(p^I_{\mathrm{L}})^2=2,\qquad p^I_{\mathrm{L}}V^I=\tfrac{1}{3}\,\mathrm{mod}\,1 \tag{1.119}$$

and it is easy to see that such states transform as the (**27**,**3**) representation of $E_6\times\mathrm{SU}(3)$. (The anti-particles have $p^I_{\mathrm{L}}V^I=\tfrac{2}{3}\,\mathrm{mod}\,1$.) Thus the untwisted sector generates a total of *nine* chiral matter generations.

The Z_3 point group is realized on the lattice Λ which comprises three copies of the root SU(3) lattice. The SU(3) lattice has two basis vectors $\boldsymbol{e}_1,\boldsymbol{e}_2$ satisfies

$$\boldsymbol{e}_1\cdot\boldsymbol{e}_1=\boldsymbol{e}_2\cdot\boldsymbol{e}_2=-2\boldsymbol{e}_1\cdot\boldsymbol{e}_2 \tag{1.120}$$

Its Coxeter element is

$$C=s_1s_2 \tag{1.121}$$

where s_1,s_2 are defined in Eq. (1.96). Then

$$\begin{aligned}C\boldsymbol{e}_1&=\boldsymbol{e}_2\ ,\\ C\boldsymbol{e}_2&=-\boldsymbol{e}_1-\boldsymbol{e}_2\end{aligned} \tag{1.122}$$

and

$$C^3=1 \tag{1.123}$$

as required. In this basis the matrix representing C is

$$C=\begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix} \tag{1.124}$$

so from Eq. (1.88) the number of fixed points in each plane is

$$\det(1 - C) = 3 \ . \tag{1.125}$$

It is easy to see that, up to a lattice vector, these fixed points of C are given by

$$\boldsymbol{x}_f = \tfrac{1}{3} n_1 (\boldsymbol{e}_1 + 2\boldsymbol{e}_2) \quad (n_1 = 0,1,2) \ . \tag{1.126}$$

The (six-dimensional) point group generator θ is defined as the product of the Coxeter elements associated with each of the (three) SU(3) lattices, so θ has a total of 27 fixed points

$$\boldsymbol{x}_f = \tfrac{1}{3} n_1 (\boldsymbol{e}_1 + 2\boldsymbol{e}_2) + \tfrac{1}{3} n_2 (\boldsymbol{e}_3 + 2\boldsymbol{e}_4) + \tfrac{1}{3} n_3 (\boldsymbol{e}_5 + 2\boldsymbol{e}_6) \tag{1.127}$$

with $n_i = 0,1,2$ for each $i = 1,2,3$.

For the twists (1.115) a_B and a_F (NS), given in Eqs. (1.78) and (1.79), both vanish

$$a_B = 0 = a_F \quad (\mathrm{NS}) \tag{1.128}$$

so the right movers' *twisted* ground state $|0\rangle_{\mathrm{R}}$ has

$$m_{\mathrm{R}}^2 = 0 \ . \tag{1.129}$$

Similarly, from Eq. (1.80), we find

$$\tilde{a} = \tfrac{2}{3} \tag{1.130}$$

so far a massless left-moving twisted state we require

$$\tfrac{1}{4} M_{\mathrm{L}}^2 = \tilde{N} + \tfrac{1}{2}(p_{\mathrm{L}}^I + V^I)^2 - \tfrac{2}{3} = 0 \ . \tag{1.131}$$

The only solutions with $\tilde{N} = 0$ have

$$p_{\mathrm{L}}^I + V^I = \left(\left(\frac{1}{3} \right)^3 \underline{\pm 10^4} \right) \tag{1.132}$$

$$= \left(\left(-\frac{1}{6} \right)^3 (\pm \frac{1}{2})^5 \right) \tag{1.133}$$

$$\left(\left(-\frac{2}{3} \right)^3 0^5 \right) , \tag{1.134}$$

where the underlining signifies that all (five) permutations are to be taken, and in Eq. (1.133) an odd number of $+\frac{1}{2}$ entries is required. Evidently, the above solutions constitute **10**, **16** and **1** representations of SO(10), and are all singlet representations of SU(3). Thus the twisted matter states

$$|0\rangle_{\mathrm{R}} |p_{\mathrm{L}}^I + V^I\rangle \tag{1.135}$$

with

$$(p_{\mathrm{L}}^I + V^I)^2 = \tfrac{4}{3} \tag{1.136}$$

transform as the (**27**, **1**) representation of the $E_6 \times \mathrm{SU}(3)$ gauge group, and in fact there is one such representation associated with each of the 27 fixed points. (The antiparticles, which transform as ($\overline{\mathbf{27}}$,**1**) representations, occur in the θ^2-twisted sector associated with the same fixed point. In this respect the Z_3 orbifold is atypical, since in general chiral matter in **27** representations may arise in any sector.)

Including the 9 chiral generations from the untwisted sector, we get a total matter content of 36 generations in the Z_3 orbifold. (This of course agrees with the general formula (1.95).)

It is clear from the definition (1.120) that the SU(3) lattice has a fixed U modulus

$$\mathrm{i}U = -\tfrac{1}{2} + \mathrm{i}\sqrt{3}/2 = \mathrm{e}^{2\pi \mathrm{i}/3} \tag{1.137}$$

while the T modulus specifying the overall size of the orbifold is arbitrary. Thus in the Z_3 orbifold all three U moduli have the common fixed value given above, and all three T moduli are unconstrained. As we have already said, these moduli are derived from the background field values associated with the (10-dimensional) graviton, dilaton and antisymmetric field. In the untwisted sector these (gauge singlet) particles are given by

$$\bar{c}^{\alpha}_{-1/2}|0\rangle_{\mathrm{R}}\tilde{\beta}^{\gamma}_{-1}|0\rangle_{\mathrm{L}} \quad (\alpha,\gamma = 1,2,3)\ , \tag{1.138}$$

where $\tilde{\beta}^{\gamma}$ ($\gamma = 1,2,3$) are the untwisted left-moving oscillators in the complex basis (1.48)–(1.50). Evidently the T_{α},U_{α} ($\alpha = 1,2,3$) moduli fields are associated with the diagonal ($\alpha = \gamma$) gauge singlet particles.

There are also (massless) gauge singlet states in the twisted sector. They are

$$|0\rangle_{\mathrm{R}}\tilde{\beta}^{\gamma}_{-2/3}|0\rangle_{\mathrm{L}}\ ,$$
$$|0\rangle_{\mathrm{R}}\bar{\tilde{\beta}}^{\gamma}_{-1/3}\bar{\tilde{\beta}}^{\delta}_{-1/3}|0\rangle_{\mathrm{L}} \quad (\gamma,\delta = 1,2,3)\ ,$$

and are associated with the so called "blowing up modes" (BUMs). When the background fields associated with the BUMs are taken to infinity, the conical orbifold singularities are "blown up", repaired, and we are left with a Calabi–Yau manifold [80].

1.7. Asymmetric orbifolds [162,163]

The treatment of orbifolds which we have presented so far rests on the geometrical notion of compactifying six spatial coordinates on a torus and then modding out an automorphism of the associated lattice. The mode expansions for the compactified left and right movers then follow from this geometrical construction. The action of the point group on the (left-moving) gauge degrees of freedom is then specified, consistent with modular invariance.

This symmetric treatment of the six compactified spatial coordinates contrasts with the asymmetric construction of the original heterotic string. In this we first consider the torus $\frac{1}{2}(\Gamma^8 + \Gamma^8)$, one Γ^8 for each E_8 group, turn on an appropriate anti-symmetric B-field, and then the left and right momenta are given by $(P_{\mathrm{L}},P_{\mathrm{R}})$, where P_{L} and P_{R} each belong to the $E_8 \times E_8$ root lattice. The standard heterotic string is then obtained by restricting to momenta of the form $(P_{\mathrm{L}},0)$ and using only left-moving oscillators to construct the states of the Hilbert space. It is natural to wonder whether this asymmetry *has* to be restricted to the gauge degrees of freedom or whether it can be continued further into the ten-dimensional space-time coordinates.

The work of Narain [164] and collaborators [165] has shown that the *combined* left-right momentum gives rise to an even self-dual lattice with a Lorentzian metric. For a six-dimensional toroidal compactification the signature is

$$[(+1)^{16+6},(-1)^{6}]\ . \tag{1.139}$$

The combined momenta have the form

$$P=(P_{\mathrm{L}},P_{\mathrm{R}})\ , \tag{1.140}$$

where P_{L} is a 22-dimensional vector, P_{R} is six-dimensional, and P belongs to a lattice $\Gamma^{22,6}$. We construct an orbifold by considering automorphisms of this lattice which do not necessarily treat the left- and right-moving components symmetrically. In doing this it is essential that the right and left-moving Hilbert spaces are not mixed. Then a general element g of the space group may be defined to act on the momentum degrees of freedom as follows:

$$g|P_{\mathrm{L}};P_{\mathrm{R}}\rangle=\exp[2\pi i(P_{\mathrm{L}}\cdot a_{\mathrm{L}}-P_{\mathrm{R}}\cdot a_{\mathrm{R}})]|\theta_{\mathrm{L}}P_{\mathrm{L}};\theta_{\mathrm{R}}P_{\mathrm{R}}\rangle\ , \tag{1.141}$$

where θ_{L} and θ_{R} are 22-dimensional and six-dimensional rotations, and a_{L} and a_{R} are 22-dimensional and six-dimensional shifts. The action of g on the bosonic oscillators is then simply their rotation by the matrices θ_{L} or θ_{R}. Similarly this action on the (right-moving) NSR world sheet fermions is also given by the θ_{R} rotation. Note that the action on the gauge degrees of freedom is already specified, as these are a part of the $\Gamma^{22,6}$ lattice.

The principal difficulty in constructing asymmetric orbifolds arises from the twisted sectors, i.e. string states which close only up to the action of the space group. Since the action of g is defined on momentum states, it does not give a sensible action on the configuration space (x) coordinates. In particular, the fixed points of the symmetric orbifold, defined in Eq. (1.46), have no immediate generalization to the asymmetric case because the action of the space-group may have a different number of fixed points for left- and right-moving degrees of freedom.

However, we may use the requirement of modular invariance (see Section 2) to obtain information about the twisted sector before constructing it. Then it can be shown that the generalization of the Lefschetz fixed point result (1.88) is

$$n_{\mathrm{fp}}^{\mathrm{asymm}}=\sqrt{\frac{\det(1-\theta_{\mathrm{L}})\det(1-\theta_{\mathrm{R}})}{|I^*/I|}}=\sqrt{\frac{\det(1-\theta)}{|I^*/I|}}\ , \tag{1.142}$$

where the determinant is over eigenvalues of $\theta=(\theta_{\mathrm{L}},\theta_{\mathrm{R}})$ which are not equal to unity; I is the subspace of lattice vectors in $\Gamma^{22,6}$ which are invariant under the action of θ, and I^* is its dual. $|I^*/I|$ denotes the index of I in I^*. It is far from obvious, but nevertheless true, that the formula (1.142) ensures that n_{fp} is an integer. The number of fixed points is of course the degeneracy of the twisted sector ground state, and the formula suggests that we should first consider a symmetric orbifold, and somehow take the square root of the number of fixed points. To do this we first consider the lattice $\Gamma^{22,6}$ but with a *euclidean* signature $[(+1)^{22},(+1)^{6}]$, and denote it by $\tilde{\Gamma}^{22,6}$ to avoid confusion. Now we consider a *symmetric* orbifold withwindings allowed on $\frac{1}{2}\tilde{\Gamma}^{22,6}$ and momenta on its dual. Although $\Gamma^{22,6}$ is self-dual with the Lorentzian signature, $\tilde{\Gamma}^{22,6}$ is *not* self-dual because of its Euclidean signature. However, it is easy to see that if

$$(p_1,p_2)\in\tilde{\Gamma}^{22,6}\ , \tag{1.143}$$

then

$$(p_1,-p_2)\in\tilde{\Gamma}^{22,6*}\ . \tag{1.144}$$

Thus we consider a lattice $\Gamma^{28;28}$ with momenta $(P_L;P_R)$ having the general form (1.110),(1.111):

$$P_R = \tfrac{1}{2}\tilde{P} - L - BL \,, \tag{1.145}$$

$$P_L = \tfrac{1}{2}\tilde{P} + L - BL \,, \tag{1.146}$$

where the windings L are on $\frac{1}{2}\tilde{\Gamma}^{22,6}$

$$L = \tfrac{1}{2}(p_1,p_2) \tag{1.147}$$

and the momenta $\tilde{P}$ are on its dual

$$\tilde{P} = 2(p_3 - p_4) \,. \tag{1.148}$$

The antisymmetric field $B_{\mu\nu}$ maybe chosen so that if the vectors e_i generate the lattice $\tilde{\Gamma}^{22,6}$

$$e_i \cdot Be_j = e_i . Ge_j \bmod 2 \,, \tag{1.149}$$

where G has the *Lorentzian* signature. Then the momentum vectors $(P_L;P_R)$ on the $\Gamma^{28,28}$ lattice have the form

$$(p_3,p_2 - p_4;p_3 - p_1, - p_4) \tag{1.150}$$

which is generated by vectors of the form

$$(k_1,0;0, - k_2),(0, - k_2;k_1,0) \tag{1.151}$$

with

$$(k_1,k_2) \in \tilde{\Gamma}^{22,6} \,. \tag{1.152}$$

Then, analogous to the $E_8 \times E_8$ compactification, we obtain the *untwisted* sector of the asymmetric orbifold by restricting to momenta of the form

$$(k_1 0;0, - k_2) \tag{1.153}$$

and using only the first 22 left-moving oscillators and the last 6 right-moving oscillators.

Now consider the twisted sector of the *symmetric* orbifold. As in Eq. (1.46), the fixed points $\boldsymbol{x}_f$ satisfy

$$(1 - \theta)\boldsymbol{x}_f = \boldsymbol{l} \tag{1.154}$$

so each fixed point is associated with a lattice vector $\boldsymbol{l} \in \tilde{\boldsymbol{\Gamma}}^{22,6}$. Of course, since we identify points which differ by a lattice vector

$$\boldsymbol{x}_f \equiv \boldsymbol{x}_f + \boldsymbol{l}_1 \quad \text{if } \boldsymbol{l}_1 \in \tilde{\Gamma}^{22,6} \tag{1.155}$$

$\boldsymbol{x}_f$ is also associated with $\boldsymbol{l} + (1 - \theta)\boldsymbol{l}_1$.

Let us denote by I the subspace of $\tilde{\Gamma}^{22,6}$ which is left invariance by θ

$$I = \{\boldsymbol{w} \in \tilde{\Gamma}^{22,6} \,|\, (1 - \theta)\boldsymbol{w} = \boldsymbol{0}\} \tag{1.156}$$

Evidently the lattice vector $\boldsymbol{l}$ associated with $\boldsymbol{x}_f$ is orthogonal to every vector in I. Thus $\boldsymbol{l}$ is in the subspace N of $\tilde{\Gamma}^{22,6}$ which is orthogonal to I. Clearly, $(1 - \theta)\tilde{\Gamma}^{22,6}$ is a subspace of N, and the number of *inequivalent* fixed points is given by the index of $(1 - \theta)\tilde{\Gamma}^{22,6}$ in N

$$n_{\mathrm{fp}}^{\mathrm{symm}} = |N/(1 - \theta)\tilde{\Gamma}^{22,6}| \tag{1.157}$$

and it can be shown [162] that this is precisely the square of $n_{\rm fp}^{\rm asymm}$ given in Eq. (1.142):

$$n_{\rm fp}^{\rm symm} = (n_{\rm fp}^{\rm asymm})^2 \ . \tag{1.158}$$

We can associate with each lattice vector $\boldsymbol{l} = (\ell_1, \ell_2) \in N$ an untwisted state of the asymmetric orbifold having momentum

$$P = (\ell_1, 0; 0, -\ell_2) \tag{1.159}$$

as in Eq. (1.153). Then the vertex operators for the emission of such states include matrices T^P which act upon the ground states of the twisted sector. The number of inequivalent $\boldsymbol{l} \in \tilde{\Gamma}^{22,6}$ is given by Eq. (1.157), and the matrices T^P constitute a representation of a group G with dimension

$$n_{\rm fp}^{\rm symm} = (n_{\rm fp}^{\rm asymm})^2 \ . \tag{1.160}$$

We could, of course, equally well have associated $\boldsymbol{l} \in N$ with the untwisted state of a (dual) asymmetric orbifold having momentum

$$\tilde{P} = (0, -\ell_2; \ell_1, 0) \ . \tag{1.161}$$

Then the matrices $T^{\tilde{P}}$ generate a group $\tilde{G}$ isomorphic to G. In fact the fixed point set constitutes an $(\boldsymbol{n}_{\rm fp}^{\rm asymm}, \bar{\boldsymbol{n}}_{\rm fp}^{\rm asymm})$ representation of $G \times \tilde{G}$, and for the symmetric orbifold (where we keep *both* P, $\tilde{P}$) we have a single irreducible representation. For the asymmetric orbifold generated by P we evidently have $n_{\rm fp}^{\rm asymm}$ copies of the $n_{\rm fp}^{\rm asymm}$-dimensional representation of G. Each of these copies gives rise to identical physics, and we retain only the $n_{\rm fp}^{\rm asymm}$ states in any single representation. This is what is meant by taking the "square root of the fixed point set".

We illustrate the foregoing ideas by constructing an asymmetric Z_3-orbifold which for the left-movers looks like a toroidal compactification and for the right-movers looks like a Z_3-orbifold. We take the even self-dual lattice $\Gamma^{22,6}$ to comprise

$$\Gamma^{22,6} = \Gamma^8 + \Gamma^8 + 3\Gamma^{2,2} \ , \tag{1.162}$$

where Γ^8 is the root lattice of E_8, and $\Gamma^{2,2}$ is defined by

$$\Gamma^{(2,2)} = \{(p_{\rm L}, p_{\rm R}) \,|\, p_{\rm L}, p_{\rm R} \in W,\ p_{\rm L} - p_{\rm R} \in R\} \tag{1.163}$$

where R is the (two-dimensional) root lattice of SU(3) and W is its (dual) weight-lattice. Then the 22-dimensional left momentum has the form

$$P_{\rm L} = (p_{\rm L}^I, p_{\rm L}'^I, p_{\rm L1}, p_{\rm L2}, p_{\rm L3}) \tag{1.164}$$

with p^I, p'^I $(I = 1, \ldots, 8)$ the E_8, E_8' momenta and $p_{{\rm L}\alpha}$ $(\alpha = 1,2,3)$ the left momenta on the $\Gamma^{2,2}$ lattices. Similarly the six-dimensional right moving momentum is

$$P_{\rm R} = (p_{\rm R1}, p_{\rm R2}, p_{\rm R3}) \tag{1.165}$$

Under the asymmetric Z_3-action the state $|P_{\rm L}, P_{\rm R}\rangle$ transforms as

$$|P_{\rm L}, P_{\rm R}\rangle \to {\rm e}^{2\pi i a.V} |P_{\rm L}, \theta P_{\rm R}\rangle \ , \tag{1.166}$$

where, as in the symmetric Z_3-orbifold, θ denotes a simultaneous rotation by $2\pi/3$ in all three tori, and V is the standard embedding (1.68) (with v_α given in Eq. (1.115)) of the twist in the gauge degrees of freedom by means of a shift.

The physical states in the untwisted sector are simply those in the toroidal compactification which are invariant under the action of the (asymmetric) Z_3 point group. As before, the graviton, antisymmetric tensor, and dilaton states and their $N=1$ space–time supersymmetric partners, are easily seen to survive, and again, as in Section 1.7, the gauge boson states

$$b^i_{-1/2}|0\rangle_{\mathrm{R}}\tilde{\alpha}^I_{-1}|0\rangle_{\mathrm{L}} , \tag{1.167}$$

$$b^i_{-1/2}|0\rangle_{\mathrm{R}}|p^I_{\mathrm{L}}\rangle_{\mathrm{L}} \tag{1.168}$$

with $(p^I_{\mathrm{L}})^2=2$ and $p^I_{\mathrm{L}}V^I\in Z$ corresponding to the gauge group $\mathrm{SU}(3)\times E_6\times E_8$ also survive. However, because the action of the point group on the left-movers is now toroidal, additional vectors states survive

$$b^i_{-1/2}|0\rangle_{\mathrm{R}}\tilde{\alpha}^a_{-1}|0\rangle_{\mathrm{L}} , \tag{1.169}$$

$$b^i_{-1/2}|0\rangle_{\mathrm{R}}|p_{\mathrm{L}1},p_{\mathrm{L}2},p_{\mathrm{L}3}\rangle , \tag{1.170}$$

with

$$p^2_{\mathrm{L}1}+p^2_{\mathrm{L}2}+p^2_{\mathrm{L}3}=2 \tag{1.171}$$

and these generate a further $\mathrm{SU}(3)^3$ gauge symmetry. The untwisted states (5.96) also survive and are of course singlets with respect to the (new) $\mathrm{SU}(3)^3$ gauge symmetry.

More interesting things happen in the twisted sector. First we construct the Euclidean lattice

$$\tilde{\Gamma}^{22,6}=\Gamma_8+\Gamma_8+3\tilde{\Gamma}^{2,2} . \tag{1.172}$$

There the invariant lattice I is given by

$$I=\Gamma_8+\Gamma_8+3(R,0) , \tag{1.173}$$

where as before R is the root lattice of SU(3), and in the same notation the normal lattice is

$$N=3(0,R) . \tag{1.174}$$

Since the action of the point group on the left-movers is toroidal

$$(1-\theta)\tilde{\Gamma}^{22,6}=N \tag{1.175}$$

it follows from Eqs. (1.157) and (1.158) that

$$n^{\mathrm{asymm}}_{\mathrm{fp}}=1 . \tag{1.176}$$

(For the symmetric Z_3 orbifold it will be recalled that there are 27 fixed points.) In fact [162], there is a single matter field in the $E_6\times\mathrm{SU}(3)^4$ representation

$$\begin{aligned}&(\overline{\mathbf{27}},\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1})+(\mathbf{27},\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1})+(\mathbf{27},\mathbf{1},\overline{\mathbf{3}},\mathbf{1},\mathbf{1})+[(\mathbf{1},\overline{\mathbf{3}},\mathbf{3},\mathbf{3},\mathbf{1})+(\mathbf{1},\overline{\mathbf{3}},\overline{\mathbf{3}},\mathbf{3},\mathbf{1})\\&\quad+(\mathbf{1},\overline{\mathbf{3}},\overline{\mathbf{3}},\overline{\mathbf{3}},\mathbf{1})+(\mathbf{1},\overline{\mathbf{3}},\mathbf{3},\overline{\mathbf{3}},\mathbf{1})+\text{perms}] ,\end{aligned} \tag{1.177}$$

where "perms" indicates the representations needed to make the last bracket symmetric with respect to the last three SU(3)s. As for the symmetric Z_3 orbifold, the θ^2 twisted sector gives the antiparticles of the θ twisted sector. Other examples of asymmetric orbifold compactification may be found in Refs. [110,180,85,146,147].

2. Orbifold model building

2.1. Introduction

As we have seen in Section 1, the observable gauge group of an orbifold compactified string theory is quite large e.g. $E_6 \times \mathrm{SU}(3)$ for the Z_3 orbifold with the standard embedding of the point group. It is therefore necessary to find mechanisms to break the gauge group to that of the standard model. The usual mechanism in an SU(5),SO(10) or E_6 grand unified theory is to employ Higgs bosons to spontaneously break the grand unified group. However, this requires the presence in the theory of massless scalar states in the adjoint representation or some larger representation of the gauge group. In a supersymmetric grand unified theory not derived from string theory, we can introduce any representations of the gauge group we require at will. On the other hand, in a grand unified theory derived from string theory, the spectrum of massless states is prescribed by the string theory for any specific compactification. Although by sifting through consistent orbifold compactifications we can find a range of possibilities for the massless spectrum, this range is not in general wide enough to permit the presence of adjoint or larger representations, as we now discuss.

The largest representations of the gauge group that can occur in a string theory are controlled by the level of the Kac–Moody algebra [111] (or current algebra) for the left movers, which is defined as follows. The vertex operator V^μ_a for a gauge boson is of the form

$$V^\mu_a = \psi^\mu(\bar{z})\mathrm{e}^{\mathrm{i}pX}J_a(z) \; , \tag{2.1}$$

where p is the momentum, ψ^μ is an NSR fermion right mover of conformal dimension $\frac{1}{2}$, and $J_a(z)$ is a left mover factor of conformal dimension 1, where

$$z = \mathrm{e}^{-2(\tau + \mathrm{i}\sigma)}, \qquad \bar{z} = \mathrm{e}^{-2(\tau - \mathrm{i}\sigma)} \tag{2.2}$$

with τ and σ the Wick rotated world sheet variables. In general, J_a satisfies the operator product expansion

$$J_a(z)J_b(w) \sim \bar{k}\delta_{ab}(z-w)^{-2} + \mathrm{i}f_{abc}J_c(z-w)^{-1} + \cdots \tag{2.3}$$

with f_{abc} the structure constants of the gauge group. The level k of the Kac–Moody algebra (or current algebra of the currents J_a) is a non-negative integer defined by

$$k = 2\bar{k}/\psi^2 \; , \tag{2.4}$$

where ψ is the highest root of the Lie algebra. In particular, for simply laced groups with normalisation $\psi^2 = 1$,

$$\bar{k} = \tfrac{1}{2}k \; . \tag{2.5}$$

The states of the string theory not only fall into representations of the Lie algebra of the gauge group but also into representations of the Kac–Moody algebra [111]. In practice, we are interested in unitary representations of the Lie algebra with a mass spectrum that is bounded below. For these representations, there is a bound on the highest weights of the representations of the Lie algebra that can occur, namely,

$$\sum_{i=1}^{\mathrm{rank}\, G} n_i m_i \leqslant k \; , \tag{2.6}$$

where n_i are the Dynkin labels of the highest weight of the representation, and m_i are positive integers that are fixed for a given Lie algebra G, and can be found tabulated in various places.

For level 1 ($k = 1$) the representations of the Lie algebra that can occur in string theory are very limited. In particular, for SO(10) or SU(5), the adjoint or larger representations do not occur [84]. This means that the usual spontaneous symmetry breaking mechanisms for breaking the symmetries in SO(10) or SU(5) grand unified theory can not be used in string theory with level 1 Kac–Moody algebras. It is possible [152] to use theories with Kac–Moody algebras with level greater than 1, but then a plethora of large exotic representations of the grand unified group occurs [99] for which it is difficult to generate large masses to remove them from a low energy theory. It is therefore attractive to stick with level 1 Kac–Moody algebras and to look instead for another mechanism to achieve some preliminary breaking of the gauge group, before spontaneous symmetry breaking, using the available smaller representations of the gauge group is applied. Such a mechanism exists in the form of Wilson lines, which we shall discuss in the next section.

2.2. Wilson lines

In Section 1, the point group was embedded in the gauge group in order to achieve some breaking of the gauge group [79], and, in the case of Z_N orbifolds other than Z_3 and Z_7 to ensure a modular invariant theory. Further breaking of the gauge group can be achieved (in a modular invariant way) by embedding the complete space group in the gauge group [80,123,15]. This means that not only should the point group element be embedded as a shift on the $E_8 \times E_8'$ bosonic degrees of freedom, but also the various basis vectors of the torus lattice underlying the orbifold should be embedded as such shifts. As we shall see, not only does this produce gauge symmetry breaking but it also modifies the matter field content, so that 3 generation models can be obtained [123,16,17].

Consider a twisted sector with boundary conditions twisted by the space group element (θ,ℓ), where θ is a point group element with ℓ as a lattice vector,

$$\ell = \sum \boldsymbol{r}_\rho \boldsymbol{e}_\rho \;, \tag{2.7}$$

where r_ρ are some integral coefficients and $\boldsymbol{e}_\rho$ are basis vectors of the 6 torus. To embed the space group in the gauge group, the point group element θ will be embedded as the shift πV^I, as before, and the lattice basis vector $\boldsymbol{e}_\rho$ as the shift πa_ρ^I. To ensure that we have an embedding we must check that we obtain a homomorphism. Thus, we must correctly image the product of two space group elements (θ_1,ℓ_1) and (θ_2,ℓ_2),

$$(\theta_1,\ell_1)(\theta_2,\ell_2) = (\theta_1\theta_2,\ell_1 + \theta_1\ell_2) \;. \tag{2.8}$$

For a Z_N point group generated by θ,

$$(\theta,\ell)^N = (I,\boldsymbol{o}) \;. \tag{2.9}$$

Consequently, we must require that

$$N(V^I + r_\rho a_\rho^I) \in \Lambda_{E_8 \times E_8'} \tag{2.10}$$

which implies that

$$N V^I \in \Lambda_{E_8 \times E_8'} \tag{2.11}$$

and

$$N \quad a^I_\rho \in \Lambda_{E_8 \times E'_8} \;, \tag{2.12}$$

so that

$$N \sum_{I \in E_8} V^I = 0 \bmod 2, \qquad N \sum_{I \in E'_8} V^I = 0 \bmod 2 \tag{2.13}$$

and

$$N \sum_{I \in E_8} a^I_\rho = 0 \bmod 2, \qquad N \sum_{I \in E'_8} a^I_\rho = 0 \bmod 2 \;. \tag{2.14}$$

In addition, the embedding of the space group must be chosen in such a way that the fundamental modular invariance property of the theory is preserved. The way to ensure a modular invariant theory is the subject of the next two sections.

2.3. *Modular invariance for toroidal compactification*

In the first instance, the evaluation of a string loop amplitude, such as Fig. 1, involves a path integral over world sheet metrics as well as over the bosonic and fermionic string degrees of freedom. The essential subtlety of the one loop string amplitudes for present purposes is contained in the toroidal world sheet of the vacuum to vacuum amplitude of Fig. 2. Infinities may arise in evaluating this amplitude (and other 1 loop amplitudes) unless we are careful to avoid including the contribution of equivalent world sheet tori infinitely many times. Tori may be characterised by the modular parameter $\tilde{\tau}$, which is defined as follows. First construct the complex variable

$$z = \sigma + \mathrm{i}\tau \tag{2.15}$$

from the world sheet coordinates σ and τ. Then a world sheet torus may be defined by making the identifications

$$z \equiv z + \pi(n_1\lambda_1 + n_2\lambda_2) \;, \tag{2.16}$$

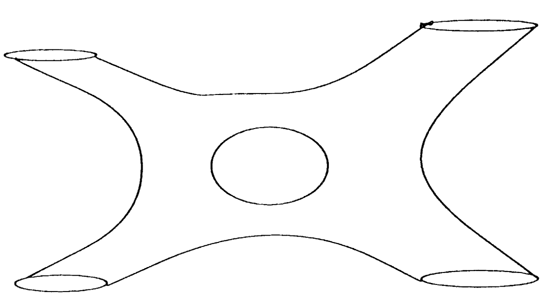


Fig. 1. The one-loop string amplitude.

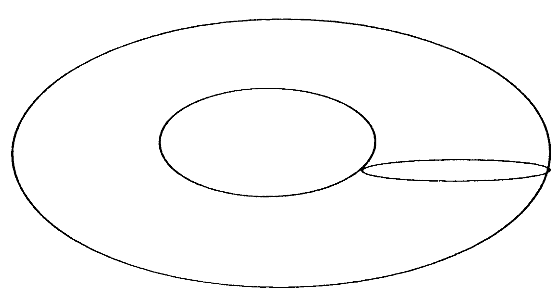


Fig. 2. The vacuum-to-vacuum string amplitude.

where λ_1 and λ_2 are two fixed complex numbers, and n_1 and n_2 are arbitrary integers. Points on the torus may be written as

$$z = \sigma_1 \lambda_1 + \sigma_{2\lambda_2}, \quad 0 \leqslant \sigma_1,\ \sigma_2 < \pi \ . \tag{2.17}$$

Because conformal invariance may be applied to rescale λ_1 to 1 if we wish, it is only the ratio

$$\tilde{\tau} = \lambda_2/\lambda_1 \tag{2.18}$$

that is relevant for characterising tori.

Not all values of $\tilde{\tau}$ specify inequivalent tori. If we consider the modular transformations

$$\begin{pmatrix} \lambda_2' \\ \lambda_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix}, \tag{2.19}$$

where a,b,c and d are integers satisfying

$$ad - bc = 1 \ , \tag{2.20}$$

then

$$n_1 \lambda_1 + n_2 \lambda_2 = n_1' \lambda_1' + n_2' \lambda_2' \ , \tag{2.21}$$

where n_1' and n_2' are also arbitrary integers. Thus, λ_1' and λ_2' define the same torus as λ_1 and λ_2 when the identification (2.16) is made. The corresponding transformations on $\tilde{\tau}$

$$\tilde{\tau}' = (a\tilde{\tau} + b)/(c\tilde{\tau} + d) \tag{2.22}$$

constitute the (world sheet) modular group SL(2,Z), and tori whose modular parameters are related by Eq. (2.22) are equivalent.

Infinities in the vacuum-to-vacuum amplitude (and other one-loop string amplitudes) may now be avoided by restricting the path integral over world sheet metrics to the range

$$-\tfrac{1}{2} \leqslant \mathrm{Re}\,\tilde{\tau} < \tfrac{1}{2}, \qquad \mathrm{Im}\,\tilde{\tau} \geqslant 0, \quad |\tilde{\tau}| \geqslant 1 \tag{2.23}$$

which ensures that inequivalent tori are counted only once. For this to work, it is necessary that the $\tilde{\tau}$ dependent path integral over the bosonic and fermionic string degrees of freedom for the vacuum-to-vacuum amplitude should be invariant under the modular transformations (2.22). This path integral is referred to as the partition function Z and after converting the Euclidean path integral to a determinant it is given by

$$Z = \mathrm{Tr}(q^{H_\mathrm{L}} \bar{q}^{H_\mathrm{R}}) \tag{2.24}$$

where the Hamiltonian has been written in terms of left and right mover contributions H_L and H_R as

$$H = H_\mathrm{L} + H_\mathrm{R} \tag{2.25}$$

and

$$q = \mathrm{e}^{\mathrm{i}\pi\tilde{\tau}}, \qquad \bar{q} = \mathrm{e}^{-\mathrm{i}\pi\bar{\tilde{\tau}}} \ . \tag{2.26}$$

2.4. Orbifold modular invariance

It will be convenient for the moment to use the fermionic formulation of the heterotic string to study the modular invariance of the orbifold partition function [80,183]. For the space group element (θ,ℓ), let the twists on the boundary conditions of the 3 complex right moving fermionic degrees of freedom associated with the compact manifold be $e^{2\pi i v^i}$, $i = 1,2,3$, and let the twists on the boundary conditions of the 16 complex left moving fermionic degrees of freedom associated with the $E_8 \times E_8'$ gauge group be $e^{2\pi i \tilde{v}^I}$, $I = 1,\dots,16$. These latter twists include the effect of embedding the lattice vectors $\boldsymbol{e}_\rho$ as well as the point group element θ, i.e. they include the Wilson lines. From Section 2.2, after switching from the bosonic to the fermionic formulation of the heterotic string we must have

$$\tilde{v}^I = V^I + r_\rho a_\rho^I \tag{2.27}$$

for a Z_N orbifold.

The orbifold partition function will be a sum over terms corresponding to the various choices of twisted boundary conditions in the σ_1 and σ_2 directions on the torus. For example, for a left-moving complex fermionic degree of freedom with boundary conditions twisted by $h = e^{2\pi i w}$ and $g = e^{2\pi i u}$ in the σ_2 and σ_1 directions, respectively, the generalisation of Eq. (2.24) to an orbifold is

$$Z_u^w = \mathrm{Tr}(q^{H_L(w)} e^{2\pi i(u-1/2)N_F(w)}) , \tag{2.28}$$

where $H_L(w)$ is the left-mover Hamiltonian for boundary conditions twisted by $e^{2\pi i w}$ and $N_F(w)$ is the fermion number (see, for example, Ref. [35, Section 11.2]). Evaluation of the trace gives

$$Z_u^w = e^{-\pi i u(1-w)} \theta\left(\begin{pmatrix} u \\ w \end{pmatrix}, \tilde{\tau}\right) , \tag{2.29}$$

where

$$\theta\left(\begin{pmatrix} u \\ w \end{pmatrix}, \tilde{\tau}\right) = q^{-w(1-w)+1/6} e^{\pi i u(1-w)} \prod_{n=1}^{\infty} (1 - q^{2(n-w)} e^{2\pi i u})(1 - q^{2(n+w-1)} e^{-2\pi i u}) . \tag{2.30}$$

For the purpose of studying the way in which partition function terms transform under modular transformations it is useful to note that the Jacobi θ function of Eq. (2.29) has the modular property

$$\theta\left(\alpha\begin{pmatrix} u \\ w \end{pmatrix}, \tilde{\tau}\right) = \varepsilon_\alpha \theta\left(\begin{pmatrix} u \\ w \end{pmatrix}, \alpha^{-1}\tilde{\tau}\right) , \tag{2.31}$$

where

$$\alpha : \tilde{\tau} \to (a\tilde{\tau} + b)/(c\tilde{\tau} + d) , \tag{2.32}$$

$$\alpha\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix} , \tag{2.33}$$

and ε_α is a 12th root of unity independent of u and w. Also useful are the shift properties

$$\theta\left(\left(\begin{smallmatrix} u+1 \\ w \end{smallmatrix}\right), \tilde{\tau}\right) = -e^{-\pi i w} \theta\left(\begin{pmatrix} u \\ w \end{pmatrix}, \tilde{\tau}\right) \tag{2.34}$$

and

$$\theta(({}^{u}_{w+1}),\tilde{\tau}) = -\,e^{\pi i u}\theta\left(\begin{pmatrix} u \\ w \end{pmatrix},\tilde{\tau}\right) . \tag{2.35}$$

A partition function term is a product of fermionic and bosonic factors for both right and left movers, but the phases arising from the modular transformation of the boundary conditions of the bosonic factors cancel between right and left movers. Consequently, only the fermionic factors need be considered for present purposes. A modular transformations (2.31) has the effect on the boundary conditions, $(h,g) \to (h',g')$ where

$$(h',g') = (h^d g^c, h^b g^a) . \tag{2.36}$$

If we consider a modular transformation that leaves the boundary conditions unaltered then, in order that the partition function can be uniquely defined, we must require that partition function terms, for given boundary conditions, transform into themselves without any modification, whereas, potentially, a phase factor could arise. In particular, if we consider the boundary conditions

$$(h,g) = (h,I) \tag{2.37}$$

and the modular transformation

$$\tilde{\tau} \to \tilde{\tau} + N , \tag{2.38}$$

where h is of order N, then (h',g') is the same as (h,g). The corresponding partition function factor for a left moving complex fermionic degree of freedom with boundary conditions twisted by $h = e^{2\pi i w}$ in the σ_2 direction undergoes the modular transformation

$$Z_0^w \to e^{\pi i N w(1-w)} Z_0^w . \tag{2.39}$$

Similarly, for a right mover partition function factor $\bar{Z}_0^w$ the corresponding transformation under the same modular transformation is

$$\bar{Z}_0^{\hat{w}} \to e^{-\pi i N \hat{w}(1-\hat{w})} \bar{Z}_0^w . \tag{2.40}$$

For a partition function term

$$Z = \prod_{i-1}^{3} \bar{Z}_0^{v^i} \prod_{I=1}^{16} Z_0^{\tilde{v}^I} , \tag{2.41}$$

where v^i are the twists on the right moving complex fermionic degrees of freedom, and $\tilde{v}^I$ are the twists on the left-moving $E_8 \times E_8'$ complex fermionic degrees of freedom, the transformation induced by the modular transformation (2.38) is

$$Z \to \exp\left[-\pi i N\left(\sum_{i=1}^{3} v^i(1-v^i) - \sum_{I=1}^{16} (1-\tilde{v}^I)\right)\right] Z . \tag{2.42}$$

Thus, to ensure that this partition function term transforms identically to itself, without any phase factor, we must require that

$$N\left(\sum_{i=1}^{3} v^i(1-v^i) - \sum_{I=1}^{16} \tilde{v}^I(1-\tilde{v}^I)\right) = 0 \bmod 2 . \tag{2.43}$$

The homomorphism condition (2.10), together with the requirement

$$N\sum_{i=1}^{3} v^i = 0 \bmod 2$$

for the action of the point group to be of order N acting on the spinor representation of SO(8), allow Eq. (2.43) to be simplified to

$$N\left(\sum_{i=1}^{3}(v^i)^2 - \sum_{I=1}^{16}(\tilde{v}^I)^2\right) = 0 \bmod 2\ , \tag{2.44}$$

with $\tilde{v}^I$ given by Eq. (2.27) for the θ twisted sector of the orbifold with Wilson lines. For the θ^n twisted sector,

$$N\left(n^2\sum_{i=1}^{3}(v^i)^2 - \sum_{I=1}^{16}(nV^I + r_\rho a_\rho^I)^2\right) = 0 \bmod 2 \tag{2.45}$$

with $n = 0,\ldots,N-1$ and $r_\rho = 0,\ldots,N-1$. In particular, embeddings of the point group in the gauge group consistent with modular invariance are required to satisfy

$$N\left(\sum_{I=1}^{16}(V^I)^2 - \sum_{i=1}^{3}(v^i)^2\right) \equiv N(V^2 - v^2) = 0 \bmod 2\ , \tag{2.46}$$

and Wilson lines consistent with modular invariance are required to satisfy

$$N\sum_{I=1}^{16}(a_\rho^I)^2 \equiv Na_\rho^2 = 0 \bmod 2\ , \tag{2.47}$$

$$N\sum_{I=1}^{16} a_\rho^I a_\sigma^I \equiv Na_\rho\cdot a_\sigma = 0 \bmod 1, \quad \rho \neq \sigma \tag{2.48}$$

and

$$N\sum_{I-1}^{16} V^I a_\rho^I \equiv NV\cdot a_\rho = 0 \bmod 1\ . \tag{2.49}$$

These results may be extended to $Z_M \times Z_N$ orbifolds.

2.5. GSO projections

As well as modular invariance imposing restrictions on the choice of point group embeddings and Wilson lines, it also imposes (generalised) GSO projections on the states [18,124,171]. For a Z_N orbifold with point group generated by θ, the complete partition function has the form

$$Z = \frac{1}{N}\sum_{m,n}\eta(m,n)Z_{(\theta^m,\theta^n)}\ , \tag{2.50}$$

where $Z_{(\theta^m,\theta^n)}$ is the partition function for twists θ^m and θ^n in the σ_1 and σ_2 directions, respectively, and $\eta(m,n)$ are phase factors fixed by modular invariance of the complete partition function Z, and determined by considering modular transformations that map one term in the sum (2.50) into another.

In the absence of Wilson lines, the contribution to (2.50) for boundary conditions twisted by θ^m in the σ_1 direction is

$$Z = \frac{1}{N} \sum_{n=1}^{N-1} \tilde{\chi}(\theta^m,\theta^n) \mathrm{Tr}(\Delta^n q^{H_{\mathrm{L}}(\theta^m)} \bar{q}^{H_{\mathrm{R}}(\theta^m)}) + \cdots \ , \tag{2.51}$$

where

$$\Delta = \mathrm{e}^{2\pi \mathrm{i} \delta} \tag{2.52}$$

with

$$\delta = (h + m\hat{v}) \cdot \hat{v} - (P + mV) \cdot V + (m/2)(V^2 - \hat{v}^2) + \varepsilon \ . \tag{2.53}$$

In Eq. (2.53), h is the so called H momentum for the bosonised right moving NSR fermionic degrees of freedom, P is the momentum on the $E_8 \times E_8'$ lattice,

$$\hat{v} \equiv (0,v^1,v^2,v^3) \tag{2.54}$$

describes the action of the point group on the compact manifold, and $\mathrm{e}^{2\pi \mathrm{i} \varepsilon}$ is the action of θ on the left mover oscillators involved in the construction of the state. (We shall only be interested in massless states, in which case these are no right mover oscillators.) The factor $\tilde{\chi}(\theta^m,\theta^n)$ as defined to be 1 for the untwisted sector $\theta^m = I$ and, otherwise, it is the number of simultaneous fixed points of θ^m and θ^n on the subspace rotated by θ^m. (This last remark is necessary to take account of the possibility of θ^m possessing fixed tori.)

The GSO projection deriving from Eq. (2.51) is particularly simple in the case of the θ twisted sector ($m = 1$) and in the absence of fixed tori in the θ twisted sector. Then,

$$\tilde{\chi}(\theta,\theta^n) = \chi(\theta) \quad \text{for all } n \ , \tag{2.55}$$

where $\chi(\theta)$ is the number of fixed points of θ, all of which must be fixed by θ^n. Thus, Eq. (2.49) simplifies to

$$Z = \frac{1}{N} \chi(\theta) \mathrm{Tr}\left(\sum_{n=0}^{N-1} \Delta^n q^{H_{\mathrm{L}}(\theta)} \bar{q}^{H_{\mathrm{R}}(\theta)} \right) \tag{2.56}$$

and states with $\Delta = 1$ survive the GSO projection. It turns out that all massless states in the θ sector have $\Delta = 1$, so that all massless states in this sector survive.

More generally, [124,142,100] the fixed points of θ^m and θ^n differ, and $\tilde{\chi}(\theta^m,\theta^n)$ does not have the same value for all n. This prevents us pulling out the $\tilde{\chi}(\theta^m,\theta^n)$ factor from the summation to leave a simple GSO projection. Instead, it is necessary to evaluate the degeneracy factor in the partition function.

$$D(\theta^m) = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{\chi}(\theta^m,\theta^n) \Delta^n \tag{2.57}$$

and states for which $D(\theta^m)$ is zero are projected out.

In the presence of Wilson lines, Eq. (2.51) still applies if Δ^n is replaced by $\tilde{\Delta}(n,m)$ where

$$\tilde{\Delta}(n,m) = \mathrm{e}^{2\pi \mathrm{i} \tilde{\delta}(n,m)} \tag{2.58}$$

with

$$\tilde{\delta}(n,m) = (h + m\hat{v})\cdot n\hat{v} - (P + mV + r_{\rho,m}a_\rho)\cdot(nV + r_{\rho,n}a_\rho) + \tfrac{1}{2}(mV + r_{\rho,m}a_\rho)\cdot(nV + r_{\rho,n}a_\rho)$$
$$= (mn/2)\hat{v}^2 + n\varepsilon \ . \tag{2.59}$$

In Eq. (2.59), the space group elements associated with fixed points in the θ^m and θ^n twisted sectors have been written as $(\theta^m, r_{\rho,m}\boldsymbol{e}_\rho + (\mathbf{1} - \theta^m)\boldsymbol{\Lambda})$ and $(\theta^n, r_{\rho,n}\boldsymbol{e}_\rho + (\mathbf{1} - \theta^n)\boldsymbol{\Lambda})$. The degeneracy factor is then

$$D(\theta^m) = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{r_{\rho,n}} \tilde{\chi}(\theta^m, \theta^n, r_{\rho,n})\tilde{\Delta}(n,m) \ . \tag{2.60}$$

For a given fixed point in the θ^m twisted sector (a choice of $r_{\rho,m}$) the degeneracy factor now has separate terms for each fixed point in the θ^n twisted sector (each choice of $r_{\rho,n}$). The factor $\tilde{\chi}(\theta^m, \theta^n, r_{\rho,n})$ now counts the number of simultaneous fixed points of θ^m and θ^n associated with a particular choice of $r_{\rho,n}$. For example, for the θ twisted sector of the Z_3 orbifold, with one Wilson line a_1, the 27 fixed points split into 3 sets of 9 associated with $V + r_1 a_1, r_1 = 0, \pm 1$. With 2 Wilson lines a_1 and a_3, the 27 fixed points split into 9 sets of 3 associated with $V + r_1 a_1 + r_3 a_3$, $r_1, r_3 = 0, \pm 1$.

2.6. Modular invariant Z_3 orbifold compactifications

The simplest case [80,123] in which to illustrate the way in which modular invariance restricts the consistent choices of point group embeddings and Wilson lines is the Z_3 orbifold. In that case, the inequivalent choices of the point group embedding V^I may be determined as follows. First write,

$$V = (V_1, V_2) \ , \tag{2.61}$$

where V_1 and V_2 are the components of V shifting the E_8 and E_8' lattices, respectively. Two shifts V_1 and V_1' that differ by an E_8 lattice vector are equivalent, as are two shifts that differ by a Weyl reflection of the E_8 lattice, and similarly for V_2. The homomorphism conditions are

$$3\sum_J V_1^J = 0 \bmod 2 \tag{2.62}$$

and

$$3\sum_K V_2^K = 0 \bmod 2 \ . \tag{2.63}$$

For the Z_3 orbifold,

$$v = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{2}{3}) \tag{2.64}$$

so that the modular invariance condition (2.46) is

$$3(V_1^2 + V_2^2) = 0 \bmod 2 \ . \tag{2.65}$$

Combining Eqs. (2.62)–(2.64), requires

$$V_1^2 = \tfrac{2}{9}q_1, \qquad V_2^2 = \tfrac{2}{9}q_2 , \tag{2.66}$$

where q_1 and q_2 are integers.

Any shift V_i, $i = 1,2$, is within a distance 1 from some lattice point on an E_8 lattice. Thus, by subtracting off an appropriate lattice vector we can always arrange that

$$V_1^2 \leqslant 1, \qquad V_2^2 \leqslant 1 . \tag{2.67}$$

Then, up to interchanging V_1 and V_2 the only inequivalent possibilities are

$$V_1^2 = V_2^2 = 0 , \tag{2.68}$$

$$V_1^2 = \tfrac{2}{3}, \qquad V_2^2 = 0 , \tag{2.69}$$

$$V_1^2 = \tfrac{2}{9}, \qquad V_2^2 = \tfrac{4}{9} , \tag{2.70}$$

$$V_1^2 = V_2^2 = \tfrac{2}{3} \tag{2.71}$$

and

$$V_1^2 = \tfrac{8}{9}, \qquad V_2^2 = \tfrac{4}{9} . \tag{2.72}$$

There is a large range of choices for the Wilson lines a_ρ. As for V we have the homomorphism conditions

$$3 \sum_{J \in E_8} a_\rho^J = 0 \bmod 2 \tag{2.73}$$

and

$$3 \sum_{K \in E'_8} a_\rho^K = 0 \bmod 2 . \tag{2.74}$$

Also, by subtracting off appropriate lattice vectors we can arrange that

$$\sum_{J \in E_8} (a_\rho^J)^2 \leqslant 1, \qquad \sum_{K \in E'_8} (a_\rho^K)^2 \leqslant 1 . \tag{2.75}$$

On the other hand, we can no longer use Weyl reflections to reduce the possibilities further because equivalent theories are connected by Weyl reflections on V and a_ρ *simultaneously*. However, not all Wilson lines satisfying (2.73)–(2.75) and the modular invariance conditions

$$3a_\rho^2 = 0 \bmod 2 , \tag{2.76}$$

and

$$3V \cdot a_\rho = 0 \bmod 2 \tag{2.77}$$

are independent. If the action of the point group element θ on the basis vectors for the compact manifold lattice is

$$\theta e_\rho = M_{\rho\sigma} e_\sigma , \tag{2.78}$$

then

$$a_\rho = M_{\rho\sigma} a_\sigma + \Lambda \ , \tag{2.79}$$

where Λ is an $E_8 \times E_8'$ lattice vector, reflecting the fact that these are inequivalent paths on the torus that are equivalent on the orbifold.

One approach [48] to writing down all possible models is to list all possible choices of $a_\rho^J, J \in E_8$, and then to use the modular invariance conditions on a_ρ to limit the possible choices of a_ρ^K, $K \in E_8'$, consistent with the choice of a_ρ^J. There are various other transformations on the Wilson lines, and the V^I and Wilson lines together, that give equivalent models. Phenomenologically promising models can then be selected by imposing requirements such as standard model gauge group, 3 generations and absence of extra colour triplets which may mediate rapid proton decay [125,49,50,102,103,100].

2.7. Untwisted sector massless states

Only initially massless states rather than states with masses on the string scale are directly relevant to the low energy world. It will be convenient to bosonise the NSR right mover fermionic degrees of freedom. Then, the 8 real fermions or 4 complex fermions become 4 real bosons with momentum on an SO(8) lattice. Denote the momentum components on the SO(8) lattice (the so-called H momentum) by h^i, $i = 0,1,2,3$. Then, the formulae for massless states of Section 1 become

$$M_\mathrm{R}^2 = M_\mathrm{L}^2 = 0 \ , \tag{2.80}$$

where

$$\frac{1}{4} M_\mathrm{R}^2 = N + \frac{1}{2} \sum_{i=0}^{3} (h^i) - \frac{1}{2} \tag{2.81}$$

and

$$\frac{1}{4} M_\mathrm{L}^2 = \tilde{N} - 1 + \frac{1}{2} \sum_{I=1}^{16} (P^I)^2 \ , \tag{2.82}$$

where P^I is the $E_8 \times E_8'$ lattice momentum. Now N contains only the contribution of transverse *bosonic* oscillators, and

$$\sum_{i=0}^{3} h^i = 1 \bmod 2 \tag{2.83}$$

because of the GSO projection.

As discussed in Section 1, the untwisted sector massless states include the gauge fields with NS sector right movers $b^i_{-1/2}|0\rangle_\mathrm{R}$, $i = 1,2$, created from the vacuum by space–time fermionic oscillators. In the case of the untwisted sector, the generalised GSO projections are equivalent to straightforward space group invariance without any phase factors. In the bosonic formulation, the space group element (θ,ℓ) with ℓ the linear combination of lattice basis vectors $\boldsymbol{e}_\rho$

$$\ell = \boldsymbol{r}_\rho \boldsymbol{e}_\rho \tag{2.84}$$

induces the translation on the $E_8 \times E_8'$ lattice $\pi(V^I + r_\rho a_\rho^I)$, so that the action on a state with momentum P^I is $\exp(2i\pi(V + r_\rho a_\rho)\cdot P)$. Since $b^i_{-1/2}$ does not transform under the space group, space-group invariance requires

$$P \cdot V = 0 \bmod 1 \tag{2.85}$$

and

$$P \cdot a_\rho = 0 \bmod 1 \quad \text{for all } \rho \tag{2.86}$$

for the gauge fields. These conditions result in breaking of the original $E_8 \times E_8'$ gauge group. For the Z_3 orbifold, a complete classification has been given with 1, 2 or 3 Wilson lines (the maximum independent number.) When 3 Wilson lines are deployed, examples of models with $\mathrm{SU}(3) \times \mathrm{SU}(2) \times U(1)^p$ gauge group can be obtained. (The breaking of the extra $U(1)$ factors not required by the standard model will be discussed later.)

Left chiral right movers for matter fields transform as $\bar{\mathbf{3}}$ of SU(3) contained in $\bar{\mathbf{4}} = \bar{\mathbf{3}} + \mathbf{1}$ when the SO(8) spinor Ramond sector ground state is decomposed as $\mathbf{4} + \bar{\mathbf{4}}$ of the compact manifold SO(6). Thus, the left chiral right movers transform with a phase factor $e^{-2\pi i/3}$ under θ. Consequently, the condition (2.85) for space group invariance is modified for

$$P \cdot V = \tfrac{1}{3} \bmod 1 \ , \tag{2.87}$$

for left chiral matter fields, and the condition (2.86) is unaltered. The surviving matter field content can be adjusted by adjusting the choice of Wilson lines.

2.8. Twisted sector massless states

In general, additional massless matter occurs in the twisted sectors of the orbifold. For the θ twisted sector of a Z_N orbifold, let the twists on the boundary conditions of the NSR fermions associated with the compact manifold be $e^{2\pi i v^i}$, $i = 1,2,3$. Then, in the θ^n twisted sector, the shift on the boundary conditions of the bosonised NSR fermions is $n\hat{v}$, with $\hat{v}$ as in Eq. (2.54), so that the H momentum is replaced by $h + n\hat{v}$. Also, the shift on the $E_8 \times E_8'$ degrees of freedom, including possible Wilson lines, is $\pi(nV^I + r_\rho a_\rho^I)$ so that the $E_8 \times E_8'$ lattice momentum P^I is replaced by $P^I + nV^I + r_\rho a_\rho^I$.

With the bosonic formulation of the heterotic string and the bosonised version of the NSR fermions, the only twisted boundary conditions are for the right and left-moving compact manifold bosonic degrees of freedom. Thus, the modification to the normal ordering constant is $\frac{1}{2}\sum_{i=1}^{3} v^i(1 - v^i)$, for both right and left movers. Then, the formulae for massless states become

$$M_\mathrm{R}^2 = M_\mathrm{L}^2 = 0 \ , \tag{2.88}$$

where

$$\frac{1}{4} M_\mathrm{R}^2 = N + \frac{1}{2} \sum_{i=0}^{3} (h^i + n\tilde{v}^i)^2 - a \tag{2.89}$$

and

$$\frac{1}{4} M_\mathrm{L}^2 = \tilde{N} + \frac{1}{2} \sum_{I=1}^{16} (P^I + nV^I + r_\rho a_\rho^I)^2 - \tilde{a} \tag{2.90}$$

with

$$a = \frac{1}{2} - \frac{1}{2}\sum_{i=1}^{3} v^i(1 - v^i) \tag{2.91}$$

and

$$\tilde{a} = 1 - \frac{1}{2}\sum_i v^i(1 - v^i) \; . \tag{2.92}$$

The oscillator terms N and $\tilde{N}$ now contain contributions from fractionally moded bosonic oscillators associated with the compact manifold.

For a $Z_M \times Z_N$ orbifold, generated by point group elements θ and ω, for the $\theta^k\omega^\ell$ twisted sector, $n\tilde{v}^i$ must be replaced by $k\tilde{v}^i + \ell\tilde{\omega}^i$ where $\tilde{v}^i$ and $\tilde{\omega}^i$ are the twists for θ and ω, respectively. Also $nV^I + r_\rho a_\rho^I$ must be replaced by $kV^I + \ell W^I + r_\rho a_\rho^I$, where V^I and W^I are the embeddings in the gauge group of θ and ω.

Naive space group invariance can not be applied in the twisted sectors, and the surviving states are those allowed by the generalised GSO projections [18,124,171]. In the case of the prime order orbifolds (Z_3 and Z_7) *all* twisted sector massless states survive these GSO projections, for arbitrary embeddings of the point group and arbitrary Wilson lines [18,124]. By a careful choice of Wilson lines, Z_3 orbifold models with 3 generations of quarks and leptons and $\mathrm{SU}(3) \times \mathrm{SU}(2) \times U^p(1)$ observable gauge group can be obtained [125,49,50,102,103,100]. This outcome depends crucially on the fact that Wilson lines differentiate the various fixed points [123,16,17], so that some fixed points have associated quark and lepton generations and others do not. Generically, additional massless matter with exotic gauge quantum numbers occurs [11]. This unwelcome matter will have to be removed from the observable low energy world, possibly by confinement due to non-trivial quantum numbers under a large non-abelian factor in the hidden sector gauge group [86]. Large non-abelian factors in the hidden sector gauge group have the additional virtue of providing the gaugino condensates necessary for non-perturbative supergravity, as will be discussed in Section 5.

A simple Z_3 orbifold example with a single Wilson line may be obtained by taking

$$V^I = \left(\frac{112}{333}0^5\right)(0^8)' \tag{2.93}$$

and

$$a_1^I = a_2^I = a_3^I = \left(0^5\frac{112}{333}\right)(0^8)' \; . \tag{2.94}$$

In this case, the observable sector gauge group is $[\mathrm{SU}(3)]^4$ where the first three SU(3) factors may be interpreted as $\mathrm{SU_C}(3) \times \mathrm{SU_L}(3) \times \mathrm{SU_R}(3)$. The untwisted sector massless matter fields are 9 copies of $(1,\mathbf{3},\bar{\mathbf{3}},\mathbf{1})$ of $[\mathrm{SU}(3)]^4$. The Wilson line differentiates the twisted sector fixed points so that the representations of $[\mathrm{SU}(3)]^4$ arising are 9 copies of $(1,\bar{\mathbf{3}},\mathbf{3},\mathbf{1}) + (\mathbf{3},\mathbf{3},\mathbf{1},\mathbf{1}) + (\bar{\mathbf{3}},\mathbf{1},\bar{\mathbf{3}},\mathbf{1})$ from the twisted sectors with $\sum_\rho r_\rho = 0 \bmod 3$, 9 copies of $(\mathbf{3},\bar{\mathbf{3}},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{3},\mathbf{1},\bar{\mathbf{3}}) + (\bar{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{3}) + 3(\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{1})$ from the twisted sectors with $\sum_\rho r_\rho = 1 \bmod 3$, and 9 copies of $(\bar{\mathbf{3}},\mathbf{1},\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{1},\bar{\mathbf{3}},\bar{\mathbf{3}}) + (\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{3}) + 3(\mathbf{1},\bar{\mathbf{3}},\mathbf{1},\mathbf{1})$ from the twisted sectors with $\sum_\rho r_\rho = 2 \bmod 3$.

With the definition of the electric charge

$$Q_{\mathrm{em}} = T_3^{\mathrm{L}} + T_3^{\mathrm{R}} + \tfrac{1}{2}Y_{\mathrm{L}} + \tfrac{1}{2}Y_{\mathrm{R}} \tag{2.95}$$

the twisted sectors with $\sum_\rho r_\rho = 0 \bmod 3$ contain 9 generations of quarks and leptons, together with associated states to make up 9 copies of the **27** of E_6. However, the other twisted sectors contain only exotic massless matter which can form fractionally charged colour singlet states. In this example, not all exotic matter can be confined by the extra SU(3) factor in the gauge group.

A complete classification of models in the absence of Wilson lines, their gauge groups and massless matter content, has been carried out for all Z_N orbifolds [137]. Potentially realistic models with Wilson lines producing standard model gauge group and 3 generations of quarks and leptons have been obtained in the cases of Z_3 as just discussed and Z_7 orbifolds [51] though a complete classification has not been carried out.

It is worth noting that there is never any need to adjust the theory to be free of gauge (and gravitational) anomalies due to chiral fermions. Freedom from such anomalies comes as an automatic consequence of the modular invariance of the string theory [172].

2.9. Anomalous $U(1)$ factors

In the first instance, model building leads to theories with $\mathrm{SU}(3)\times\mathrm{SU}(2)\times U^p(1)$ gauge group with $p > 1$. To reach the standard model, it is necessary for all but one of the $U(1)$ factors in the (observable) gauge group to be broken at a large scale. Frequently, one of the $U(1)$ factors is anomalous [76,12,75] with an anomaly arising from diagrams with 3 non-abelian gauge bosons, or one $U(1)$ gauge boson and two gravitons, as external legs. Then, at string one loop order a Fayet-Iliopoulos D-term is generated for this $U(1)$ factor, $U_A(1)$, and the corresponding D-term, D_A, in the Lagrangian takes the form

$$D_A = \frac{g}{192\pi^2}\sum_i q_A^i + \sum_i q_A^i|\phi_i|^2 , \tag{2.96}$$

whereas, for a non-anomalous $U(1)$, say $U_B(1)$,

$$D_B = \sum_i q_B^i|\phi_i|^2 , \tag{2.97}$$

where q_A^i and q_B^i are the corresponding $U(1)$ charges of the scalar fields ϕ_i. Since, in general, these ϕ_i carry not only the anomalous $U(1)$ charge but also other $U(1)$ charges, many $U(1)$ factors may be broken in this way [52,104].

As a consequence of selection rules on the Yukawa couplings and non-renormalizable couplings in an orbifold theory (which we shall discuss later) the effective potential often possesses F flat directions. Then, spontaneous symmetry breaking may occur along such a direction, with $U(1)$ factors in the gauge group being broken at a very large scale.

2.10. Continuous Wilson lines

The discussion of Wilson lines so far has assumed that the point group is embedded in the gauge group as a shift V^I on the bosonic $E_8 \times E_8'$ degrees of freedom. An alternative is to embed the point group in the gauge group as a discrete rotation [126,124] of the $E_8 \times E_8'$ lattice, still in the bosonic formulation of the heterotic string. Then, the space group element (θ,ℓ), with ℓ a lattice vector as in

Eq. (2.7), is embedded as (θ', a) where θ' is a rotation and

$$a^I = \pi \sum_\rho r_\rho a^I_\rho \tag{2.98}$$

is a shift, on the $E_8 \times E'_8$ degrees of freedom. For those components of a^I_ρ that are rotated by θ', no restriction is imposed on them by the homomorphism condition because, for these components

$$(\theta', a^I_\rho)^N = (I, \mathbf{0}) \ , \tag{2.99}$$

when θ is of order N. The components of a^I_ρ that are not rotated by θ' are restricted by Eq. (2.14), as usual. A priori, modular invariance might put conditions on the rotated components of a^I_ρ. However, this does not happen, and would not be expected to happen, because these components of the Wilson lines do not affect the mass operator for the twisted sector, and so do not affect level matching between left and right movers. Thus, the components of a^I_ρ that are rotated by θ' are continuously variable parameters (additional moduli) which are referred to as continuous Wilson lines.

Unlike the usual discrete Wilson lines, continuous Wilson lines are able to reduce the rank of the gauge group. The gauge fields associated with the Cartan subalgebra are of the form $b^i_{-1/2}|0\rangle_{\mathrm{R}} \tilde{\alpha}^I_{-1}|0\rangle_{\mathrm{L}}$, $I = 1, \dots, 16$, where i is a four dimensional space-time index. While θ acts trivally on $b^i_{-1/2}, \theta'$ has a non-trivial action on some of the left-mover bosonic oscillators $\tilde{\alpha}^I_{-1}$. As a consequence of point group invariance, some of the gauge fields of the Cartan subalgebra are projected out of the theory, leaving only that part of the Cartan subalgebra for while $\tilde{\alpha}^I_{-1}$ is unrotated by θ'. This is not the whole story because it is possible for there to be θ' invariant combinations of $E_8 \times E'_8$ momentum states $|P^I\rangle$, of the form $|P^I\rangle + |\theta' P^I\rangle + \cdots |(\theta')^{N-1} P^I\rangle$, in the Z_N case, which play the part of Cartan subalgebra states. However, the GSO projections due to Wilson lines generically project out some of the states, so that the rank of the gauge group is indeed reduced.

When the point group embedding in $E_8 \times E'_8$ is a rotation θ' rather than a shift, twisted sector states consistent with the boundary conditions will have to have centre-of-mass coordinates at a fixed point (or torus) of θ', as well as at a fixed point (or torus) of θ. If E^I_A are basis vectors for the $E_8 \times E'_8$ lattice, then the fixed points X^I_f will be of the form

$$X^I_f = [(I - \theta')^{-1}(a + t_A E_A)]^I \ , \tag{2.100}$$

where t_A are integers, and the form of a depends on the fixed point on the compact manifold according to Eqs. (2.98) and (2.7). Because X^I has only left-moving components, we are dealing here with an asymmetric orbifold. Consequently, the vacuum degeneracy for the twisted sector due to the $E_8 \times E'_8$ degrees of freedom is the square root of the number of fixed points determined in this way.

3. Yukawa couplings

3.1. Introduction

In this section, we shall discuss superpotential terms, focussing on the trilinear terms that give rise to Yukawa couplings. Before a Yukawa coupling is fully determined it is necessary to normalise correctly the fields involved. This requires a knowledge of the Kähler potential which will be deferred to Section 4.

Quite a lot can be learned about the Yukawa couplings in an orbifold compactified theory using the various selection rules which will be presented in the next few sections. In later sections, the detailed construction of Yukawa couplings will be discussed. The most important aspect of this is the leading exponential dependence of couplings amongst twisted sector states on the deformation parameters or moduli of the orbifold. This gives a possible starting point for understanding the hierarchy of quark and lepton masses, and the chapter closes with a discussion of progress to date in fitting the quark and lepton masses using orbifold compactifications.

3.2. Vertex operators for orbifold compactifications

The vertex operators for untwisted sector states will be described first before the modifications necessary for twisted sector vertex operators are given. So far as the right movers are concerned, the vertex operator V^{R}_{-1} in the -1 picture for emission of a scalar boson with four-dimensional space time momentum p is of the form

$$V^{\mathrm{R}}_{-1} = \mathrm{e}^{-\phi}\mathrm{e}^{\mathrm{i}p\cdot X_{\mathrm{R}}}\psi^{i}_{\mathrm{R}} \; . \tag{3.1}$$

This vertex operator is for a boson with right mover $b^{i}_{-1/2}|0\rangle_{\mathrm{R}}$ where $i = 1,2,3$ are complex basis indices for the three complex planes of the compact manifold, and ϕ is the phase of the bosonised superconformal ghost field. The corresponding 0 picture vertex operator V^{R}_{0}, which is required for superpotential terms with more than three chiral fields, is of the form

$$V^{\mathrm{R}}_{0} = \mathrm{e}^{\mathrm{i}p\cdot X_{\mathrm{R}}}\left(2\mathrm{i}z\frac{\partial X^{i}_{\mathrm{R}}}{\partial z} + \frac{p}{2}\psi_{\mathrm{R}}\psi^{i}_{\mathrm{R}}\right) \tag{3.2}$$

with z as in Eq. (2.2). The vertex operator $V^{\mathrm{R}}_{-1/2}$ in the $-1/2$ picture for the emission of a fermionic state is given by

$$V^{\mathrm{R}}_{-1/2} = \mathrm{e}^{-\phi/2}\mathrm{e}^{\mathrm{i}p\cdot X_{\mathrm{R}}}S \; , \tag{3.3}$$

where S is the spin field. It will often be convenient in what follows to bosonise the complex NSR world sheet fermionic degrees of freedom in the form $\mathrm{e}^{\mathrm{i}H_m}$, $m = 1, \ldots ,5$. In terms of the world sheet bosonic degrees of freedom H_m the vertex operator (3.3) takes the form

$$V^{\mathrm{R}}_{-1/2} = \mathrm{e}^{-\phi/2}\mathrm{e}^{\mathrm{i}p\cdot X_{\mathrm{R}}}\mathrm{e}^{\mathrm{i}h\cdot H} \; , \tag{3.4}$$

where

$$h = \alpha_{\mathrm{s}} = (\pm\tfrac{1}{2}, \ldots , \pm\tfrac{1}{2}) \tag{3.5}$$

with an even number of plus signs to satisfy the constraint on the ten-dimensional chirality of the state due to the GSO projection. The vertex operators for boson emission can also be recast in terms of H-momentum h by bosonising the NSR fermions so that

$$V^{\mathrm{R}}_{-1} = \mathrm{e}^{-\phi}\mathrm{e}^{\mathrm{i}p\cdot X_{\mathrm{R}}}\mathrm{e}^{\mathrm{i}h\cdot H} \; , \tag{3.6}$$

where, in this case, for a scalar boson

$$h = \alpha_{v} = (0,0\underline{\pm 1,0,0}) \; , \tag{3.7}$$

where underlining denotes permutations, and

$$V_0^{\mathrm{R}} = \mathrm{e}^{\mathrm{i}p\cdot X_{\mathrm{R}}}\left(2\mathrm{i}z\frac{\partial X_{\mathrm{R}}^{i}}{\partial z} + \frac{p}{2}\psi_{\mathrm{R}}\mathrm{e}^{\mathrm{i}\alpha_v\cdot H}\right) . \tag{3.8}$$

So far as the left movers are concerned, states with left movers of the type $\tilde{\alpha}^{j}_{-1}|0\rangle_{\mathrm{L}}$, where j is a compact manifold index in the complex basis, have vertex operators

$$V_{\mathrm{L}} = 2\mathrm{i}\bar{z}\frac{\partial}{\partial \bar{z}} X_{\mathrm{L}}^{j}\mathrm{e}^{\mathrm{i}p\cdot X_{\mathrm{L}}} \tag{3.9}$$

with $\bar{z}$ as in Eq. (2.2). This includes the moduli discussed in Section 1.6 Gauge fields in the Cartan subalgebra have left mover vertex operators of the same type, but with compact manifold index j replaced by an $E_8\times E_8'$ index I. Gauge fields not in the Cartan subalgebra, and also massless matter fields with non-trivial $E_8\times E_8'$ quantum numbers, have vertex operators associated with momentum P^I on the $E_8\times E_8'$ lattice (satisfying $\sum_I (P^I)^2 = 2$ for the untwisted sector)

$$V_{\mathrm{L}} = \mathrm{e}^{\mathrm{i}p\cdot X_{\mathrm{L}}}\mathrm{e}^{\mathrm{i}P^I X_{\mathrm{L}}^I} . \tag{3.10}$$

For twisted sector vertex operators some modifications are required. The vertex operator

$$V = V_{\mathrm{R}}V_{\mathrm{L}} \tag{3.11}$$

must now contain a product of twist fields that construct the twisted sector vacuum from the untwisted vacuum. The product of twist fields for the right moving NSR fermions is analogous to the spin field and is given by $\mathrm{e}^{\mathrm{i}n\hat{v}\cdot H}$ for the θ^n twisted sector, where $\hat{v}$ describes the action of the point group on the compact manifold, as in (2.54). Then the h momentum in Eqs. (3.4) and (3.6) is replaced by

$$h = \alpha_s + n\hat{v} \tag{3.12}$$

or

$$h = \alpha_v + n\hat{v} \tag{3.13}$$

respectively. The bosonic $E_8\times E_8'$ degrees of freedom are untwisted (except in models with continuous Wilson lines) but the momentum P^I is shifted by the embedding of the point group and the Wilson lines so that P^I is replaced in Eq. (3.10) by $P^I + nV^I + r_\rho a_\rho^I$ as in Section 2.8.

It is difficult to give useful explicit expressions for the twist fields σ^i for the bosonic degrees of freedom, to be discussed later, but in practice what is usually sufficient is a knowledge of the operator product expansions involving these twist fields which will be given in Section 3.6. The $-\frac{1}{2}$ and -1 picture vertex operators then contain a factor $\sigma^1\sigma^2\sigma^3$ for the twist fields associated with the three complex planes of the compact manifold. The 0 picture vertex operator is more complicated and contains excited twist fields.

Tree level correlation functions (involving untwisted or twisted sector states) have to be constructed to cancel the ghost charge 2 of the vacuum (where $\mathrm{e}^{q\phi}$ has ghost charge q.) A Φ^3 term in the superpotential may be extracted from a Yukawa coupling of the form $\psi\,\psi\,\phi$, for which we need a 3-point function of the type $\langle V_{-1/2}(z_1,\bar{z}_1)V_{-1/2}(z_2,\bar{z}_2)V_{-1}(z_3,\bar{z}_3)\rangle$, and a Φ^{n+3} superpotential term may be extracted from a non-renormalisable coupling of the form $\psi\,\psi\,\phi^{n+1}$ for which we need an $n+3$ point function of the type

$$\langle V_{-1/2}(z_1,\bar{z}_1)V_{-1/2}(z_2,\bar{z}_2)V_{-1}(z_3,\bar{z}_3)V_0(z_4,\bar{z}_4)\cdots V_0(z_{n+3},\bar{z}_{n+3})\rangle\ .$$

3.3. Space group selection rules

For a non-zero correlation function, the product of space group elements associated with the twisted sector states involved should contain the identity element of the space group [78,120]. In particular, consider a 3-point function with the three states associated with the space group elements (α,ℓ_1),(β,ℓ_2) and (γ,ℓ_3) where α,β and γ are point group elements and

$$\begin{aligned}\ell_1 &= (I-\alpha)f_\alpha + (I-\alpha)\Lambda_\alpha\ ,\\ \ell_2 &= (I-\beta)f_\beta + (I-\beta)\Lambda_\beta\ ,\\ \ell_3 &= (I-\gamma)f_\gamma + (I-\gamma)\Lambda_\gamma\ ,\end{aligned} \tag{3.14}$$

where f_α, f_β and f_γ are fixed points in the α, β, and γ twisted sectors, respectively, and Λ_α, Λ_β and Λ_γ are arbitrary lattice vectors. Then,

$$(\alpha,\ell_1)(\beta,\ell_2)(\gamma,\ell_3) = (\alpha\beta\gamma, \ell_1 + \alpha\ell_2 + \alpha\beta\ell_3)\ . \tag{3.15}$$

For the identity element of the space group to be included, there is the requirement that

$$\alpha\beta\gamma = I \tag{3.16}$$

which is the point group selection rule, and the additional requirement (which we shall sometimes refer to as the space group selection rule) that

$$\ell_1 + \alpha\ell_2 + \alpha\beta\ell_3 = 0 \tag{3.17}$$

for some choice of Λ_α, Λ_β and Λ_γ. This can be simplified with the aid of the point group selection rule to

$$\ell_1 + \ell_2 + \ell_3 = 0 \tag{3.18}$$

for some choice of Λ_α,Λ_β and Λ_γ. In other words,

$$(I-\alpha)f_\alpha + (I-\beta)f_\beta + (I-\gamma)f_\gamma = 0 \tag{3.19}$$

up to the addition of $(I-\alpha)\Lambda_\alpha$, $(I-\beta)\Lambda_\beta$ or $(I-\gamma)\Lambda_\gamma$. This restricts the fixed points which can couple.

A simple example is provided by the Z_3 orbifold. As we saw earlier, the fixed points f for the θ twisted sector are given by

$$f = \frac{1}{3}\sum_{j=1}^{3} p_{2j-1}(2e_{2j-1} + e_{2j}) \tag{3.20}$$

for integers p_1,p_3 and p_5, with associated space group elements (θ,ℓ) where

$$\ell = (I-\theta)f + (I-\theta)\Lambda = p_1e_1 + p_3e_3 + p_5e_5 + (I-\theta)\Lambda\ . \tag{3.21}$$

If we consider a coupling of three states, each in the θ twisted sector, associated with fixed points f_1, f_2 and f_3 characterised by integers p^1_ρ, p^2_ρ and p^3_ρ, $\rho = 1,3,5$, respectively, then the point group selection rule is trivially satisfied and the space group selection rule gives

$$\sum_{J=1}^{3} p^J_\rho = 0 \bmod 3, \quad \rho = 1,3,5\ . \tag{3.22}$$

For non-prime order orbifolds [124,142,100,53], the discussion is a little more complicated. In this case, as we saw earlier, the fixed points of θ^m are not necessarily the same as those of θ^n, for $m \neq n$, and when constructing physical states we have to take linear combinations of fixed points to get an eigenstate of θ. If f_k is a fixed point of θ^k and n is the smallest integer such that $\theta^m f_k \sim f_k$ (up to a lattice vector) then we have to make the linear combinations

$$|p\rangle = \sum_{r=0}^{m-1} \mathrm{e}^{-\mathrm{i}\gamma r} |\theta^r f_k\rangle\ , \tag{3.23}$$

with

$$\gamma = \frac{2\pi p}{m}, \quad p = 0,1,\dots,m-1\ , \tag{3.24}$$

which have eigenvalues $\mathrm{e}^{\mathrm{i}\gamma}$ of θ. A subset of these survive the GSO projection. Then, a 3-point function will couple three states of the form (3.23). It can then be seen that, if the space group selection rule is satisfied by the states $|f_1\rangle, |f_2\rangle$ and $|f_3\rangle$, it is satisfied by these linear combinations.

3.4. H-momentum conservation

When the NSR right-moving fermionic degrees of freedom are bosonised, as discussed in Section 3.2, there is a conserved H-momentum associated with these bosonic degrees of freedom [78,120,65,104,142]. In the untwisted sector, spin 0 bosonic states in the NS sector have H-momentum h_v given by Eq. (3.7), and for superfield of a particular chirality we may fix attention on

$$h_v = (0\,0\ \underline{100})\ . \tag{3.25}$$

The 10-dimensional space–time $N = 1$ supercharge carries H-momentum

$$h^{\pm}_Q = (\pm 1, \mp 1,1,1,1)/2\ , \tag{3.26}$$

so that the superpartners of these bosonic states have H-momentum

$$h^{\pm}_s = h_v - h^{\pm}_Q = (\mp \tfrac{1}{2}, \pm \tfrac{1}{2}, \underline{\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}})\ , \tag{3.27}$$

where again the underlining denotes permutations. For a 3-point coupling of two fermions and one boson, conservation of H-momentum means that

$$h^1_v + h^{\pm(2)}_s + h^{\mp(3)}_s = 0 \tag{3.28}$$

or

$$\sum_{J=1}^{3} h^J_v = (1,1,1)\ , \tag{3.29}$$

where we are now displaying only the (non-zero) compact manifold components.

Untwisted sector bosons in chiral supermultiplets have right movers $b^{i}_{-1/2}|0\rangle_R$ where i refers to the complex basis and corresponding H-momenta (100), (010) and (001) as in (3.25). If we denote the bosons with i associated with the three complex planes by U_1, U_2 and U_3, then the only coupling allowed by H-momentum conservation is $U_1\ U_2\ U_3$.

For a Z_N orbifold, spin 0 bosons in the θ^n twisted sector have H-momentum of the form

$$h_v = (0,0\ \underline{1,0,0}) + n\hat{v} \tag{3.30}$$

for superfields of a particular chirality, with $\hat{v}$ as in Eq. (2.54). (For the $\theta^k\ \omega^\ell$ twisted sector of a $Z_M \times Z_N$ oribifold this becomes $k\hat{v} + \ell\hat{\omega}$.) The fermionic superpartners have H-momentum

$$h_s^{\pm} = h_v - h_Q^{\pm} \tag{3.31}$$

and the H-momentum conservation law for 3-point couplings remains (3.29) but with the modified form of h_v^J.

For $n+3$ point couplings all but three of the vertex operators are in the zero picture. The picture changing operation is implemented by the superconformal current for the compact manifold degrees of freedom

$$T_F = 2\mathrm{i}z(\partial_z X^i_R \bar{\psi}^i_R + \partial_z \bar{\psi}^i_\mathrm{R} \psi^i_R) \tag{3.32}$$

which adds H-momentum $(-\underline{100})$ when picture changing V^{R}_{-1} with compact manifold index $i = 1,2$ or 3 in the complex basis. If we write

$$\alpha_1 = (1,0,0), \quad \alpha_2 = (0,1,0), \quad \alpha_3 = (001)\ , \tag{3.33}$$

then the H-momentum conservation law for a vertex with two fermions and $n+1$ bosons as in Section 3.2 is

$$\sum_{J=1}^{n+3} h_v^J - \sum_{J=4}^{n+3} \alpha_i^J = (1,1,1)\ . \tag{3.34}$$

Table 4
H-momenta for massless spin 0 bosons for the twisted sectors of the $Z_3 \times Z_3$ orbifold

Twisted sector	Notation	h_v for massless states
θ	A	$\frac{1}{3}(1,0,2)$
ω	B	$\frac{1}{3}(0,1,2)$
$\theta\omega$	D	$\frac{1}{3}(1,1,1)$
θ^2	$\bar{A}$	$\frac{1}{3}(2,0,1)$
ω^2	$\bar{B}$	$\frac{1}{3}(0,2,1)$
$\theta\omega^2$	C	$\frac{1}{3}(1,2,0)$
$\theta^2\omega$	$\bar{C}$	$\frac{1}{3}(2,1,0)$
Untwisted	U_1	(1,0,0)
Untwisted	U_2	(0,1,0)
Untwisted	U_3	(0,0,1)

A simple illustration of the application of H momentum conservation to couplings involving twisted sector states is provided by the $Z_3 \times Z_3$ orbifold. In this case, the point group elements θ and ω are represented by

$$\hat{v} = \tfrac{1}{3}(1,0,-1) \tag{3.35}$$

and

$$\hat{\omega} = \tfrac{1}{3}(0,1,-1)\ , \tag{3.36}$$

respectively, and the H-momentum for spin 0 bosons in the $\theta^k \omega^\ell$ twisted sector as

$$h_v = (\underline{100}) + k\hat{v} + \ell\hat{\omega}\ . \tag{3.37}$$

The H-momenta for the massless states in the various twisted sectors are given in Table 4, where we continue to suppress the zero entries of the H-momentum. The Yukawa couplings consistent with the point group selection rule that are also allowed by H-momentum conservation are

$$U_1U_2U_3, A\bar{A}U_2, B\bar{B}U_1, C\bar{C}U_3, DDD, \bar{A}BC, A\bar{B}\bar{C}, ACD, B\bar{C}D, \bar{A}\bar{B}D\ . \tag{3.38}$$

3.5. Other selection rules

For a Z_N orbifold, the element θ generating the point group can be written in the complex space basis (corresponding to $(1/\sqrt{2})(X^1 + \mathrm{i}X^2)$, etc.) as $(e^{2\pi i v_1}, e^{2\pi i v_2}, e^{2\pi i v_3})$. These three-phase rotations of the complex coordinates for the compact manifold are automorphisms of the lattice and are thus symmetries of the 6-torus that are left unbroken by the construction of the orbifold [78,120,65,104]. We shall refer to this symmetry as point group invariance to distinguish it from the topological point group selection rule discussed in Section 3.3. If the action of the phase rotation on the ith complex plane is of order M the correlation functions involving

$$(\partial_z \bar{X}^i)^m (\partial_{\bar{w}} \bar{X}^i)^n (\partial_u X^i)^p (\partial_v \bar{X}^i)^q$$

are allowed by point group invariance only if

$$m + p - n - q = 0 \bmod M\ . \tag{3.39}$$

For a 3-point function, where the bosonic vertex operators are all in the -1 picture, the fact that there are no bosonic oscillators involved in the construction of massless right movers means that Eq. (3.39) simplifies to

$$m - n = 0 \bmod M\ . \tag{3.40}$$

This then restricts the allowed Yukawa couplings of massless twisted sector states for which bosonic oscillators act on the left mover ground state (excited twisted sector states.) For a 4 or more point function, even with no bosonic oscillators involved in the construction of the right mover state, derivatives of bosonic degrees of freedom can arise from the construction of zero picture vertex operators and the full expression (3.39) is required.

We shall discuss in Section 3 one further selection rule where derivatives of bosonic degrees of freedom at the same fixed point are involved.

3.6. 3-point functions from conformal field theory

The dependence of superpotential terms for Yukawa couplings upon the moduli or deformation parameters can be calculated by studying a 3-point function for vertex operators using conformal field theory methods [78,120]. However, the determination of the overall normalisation of the superpotential term requires the factorisation of a 4-point function into 3-point functions and this will be discussed in later sections. Of course, there is also the question of the correct normalisation of the fields involved using their kinetic terms. This requires a knowledge of the Kähler potential and discussion of this will be delayed to Section 4.

Schematically, we are interested in $\langle V^B V^F V^F \rangle$ where V^B denotes a bosonic vertex operator and V^F denotes a fermionic vertex operator. The factors involving $e^{ip\cdot X_R}$, $e^{ip\cdot X_L}$, $e^{iP\cdot X_L}$ and $e^{ih\cdot H}$ can be evaluated straightforwardly. The difficult part is the expectation value of the product of twist fields which, for twisted sector ground states, is a product of factors of the type

$$Z^i \equiv \langle \sigma^i_{k_i/N,f_a}(z_1,\bar{z}_1)\sigma^i_{\ell_i/N,f_b}(z_2,\bar{z}_2)\sigma^i_{-(k_i+\ell_i)/N,f_c}(z_3,\bar{z}_3)\rangle \,, \tag{3.41}$$

where $i = 1,2,3$ labels the three complex planes of the 6 torus for the compact manifold and σ^i is a twist field referring to that complex plane. (The case of twisted sector excited states will be discussed in Section 3.12.) The twists k_i/N, ℓ_i/N and $-(k_i+\ell_i)/N$ are for that complex plane and f_a, f_b and f_c are the fixed points involved. The twist fields are defined to construct the twisted sector ground state, denoted by $|\sigma_{k/N}\rangle$ from the untwisted ground state $|0\rangle$, so that

$$|\sigma_{k/N}\rangle = \sigma_{k/N}(0,0)|0\rangle \,. \tag{3.42}$$

The twisted sector boundary conditions for the ith complex plane are of the form

$$X^i(\tau,\sigma+\pi) = e^{-2\pi i k_i/N} X^i(\tau,\sigma) \tag{3.43}$$

and similarly for the other twists, or equivalently, after continuation to Euclidean space,

$$X^i(e^{2\pi i}z,\bar{e}^{2\pi i}\bar{z}) = e^{2\pi i k_i/N} X^i(z,\bar{z}) \,. \tag{3.44}$$

We shall mostly suppress display of the fixed points in what follows and shall often suppress the index i labelling the fixed plane.

The expectation value Z^i factors enter a quantum piece Z^i_{qu} and a classical piece Z^i_{cl}, so that

$$Z^i = Z^i_{qu} Z^i_{cl} \,, \tag{3.45}$$

where

$$Z^i_{cl} = \sum_{X^i_{cl}} \exp(-S^i_{cl}) \,, \tag{3.46}$$

where X^i_{cl} are the solutions for the classical field and the action S^i continued for Euclidean space is

$$S^i = \frac{1}{\pi}\int d^2z \left(\frac{\partial X^i}{\partial z}\frac{\partial \bar{X}^i}{\partial \bar{z}} + \frac{\partial X^i}{\partial \bar{z}}\frac{\partial \bar{X}^i}{\partial z}\right) \tag{3.47}$$

with z and $\bar{z}$ as in Eq. (2.2). The string equations of motion

$$\partial^2 X^i/\partial z\partial\bar{z} = 0 \tag{3.48}$$

require that $\partial X^i/\partial z$ and $\partial X^i/\partial \bar{z}$ are functions of z and $\bar{z}$ alone, respectively, which must be chosen to respect the boundary conditions implicit in the operator product expansions (OPEs) with the twist fields. These OPEs may be deduced from the mode expansions of the string degrees of freedom. If we write

$$X_{\mathrm{R}} = x_{\mathrm{R}} + \frac{\mathrm{i}}{2}\sum_{n=1}^{\infty}\frac{\beta_{n-k/N}}{(n-k/N)}z^{-(n-k/N)} - \frac{\mathrm{i}}{2}\sum_{n=0}^{\infty}\frac{\gamma^{\dagger}_{n+k/N}}{(n+k/N)}z^{n+k/n}\ , \tag{3.49}$$

where β and γ are oscillator modes, then for $z\to 0$,

$$\frac{\partial X}{\partial z} = \frac{\partial X_{\mathrm{R}}}{\partial z} \approx -\frac{\mathrm{i}}{2}\gamma^{\dagger}_{k/N}z^{k/N-1} \tag{3.50}$$

dropping the annihilation operator term, and so

$$\frac{\partial X}{\partial z}|\sigma_{k/N}\rangle = \frac{\partial X}{\partial z}\sigma_{k/N}(0,0)|0\rangle \sim -\frac{\mathrm{i}}{2}z^{k/N-1}\gamma^{\dagger}_{k/N}|\sigma_{k/N}\rangle\ . \tag{3.51}$$

Thus,

$$\partial_z X\sigma_{k/N}(0,0) \sim z^{-(1-k/N)}\tau_{k/N}(0,0)\ , \tag{3.52}$$

where $\tau_{k/N}$ (0,0) acting on the untwisted ground state creates an excited state of the twisted sector. Restoring the z and $\bar{z}$ dependence of the conformal fields, we conclude that the relevant OPEs are

$$\begin{aligned}
&\frac{\partial X}{\partial z}\sigma_{k/N}(w,\bar{w}) \sim (z-w)^{-(1-k/N)}\tau_{k/N}(w,\bar{w})\ ,\\
&\frac{\partial \bar{X}}{\partial z}\sigma_{k/N}(w,\bar{w}) \sim (z-w)^{-k/N}\tau'_{k/N}(w,\bar{w})\ ,\\
&\frac{\partial X}{\partial \bar{z}}\sigma_{k/N}(w,\bar{w}) \sim (\bar{z}-\bar{w})^{-k/N}\tilde{\tau}_{k/N}(w,\bar{w})\ ,\\
&\frac{\partial \bar{X}}{\partial \bar{z}}\sigma_{k/N}(w,\bar{w}) \sim (\bar{z}-\bar{w})^{-(1-k/N)}\tilde{\tau}'_{k/N}(w,\bar{w})\ ,
\end{aligned} \tag{3.53}$$

where $\tau,\tau',\tilde{\tau}$ and $\tilde{\tau}'$ are four varieties of excited twist fields. For $\sigma_{-k/N}$, k/N is replaced by $1-k/N$ in these expressions.

The classical solutions of the string equations of motion (3.48) with the correct boundary conditions in the presence of the twist fields as $z\to z_1$, z_2 and z_3 are of the form

$$\begin{aligned}
&\partial X_{\mathrm{cl}}/\partial z = a(z-z_1)^{-(1-k/N)}(z-z_2)^{-(1-\ell/N)}(z-z_3)^{-(k/N+(\ell/N))}\ ,\\
&\partial \bar{X}_{\mathrm{cl}}/\partial \bar{z} = \bar{a}(\bar{z}-\bar{z}_1)^{-(1-k/N)}(\bar{z}-\bar{z}_2)^{-(1-\ell/N)}(\bar{z}-\bar{z}_3)^{-(k/N+(\ell/N))}\ ,\\
&\partial X_{\mathrm{cl}}/\partial \bar{z} = b(\bar{z}-\bar{z}_1)^{-k/N}(\bar{z}-\bar{z}_2)^{-\ell/N}(\bar{z}-\bar{z}_3)^{-(1-k/N-(\ell/N))}\ ,\\
&\partial \bar{X}_{\mathrm{cl}}/\partial z = \bar{b}(z-z_1)^{-k/N}(z-z_2)^{-\ell/N}(z-z_3)^{-(1-k/N-(\ell/N))}\ ,
\end{aligned} \tag{3.54}$$

where a and b are constants to be determined, and we are continuing to suppress the index i labelling the complex plane. The constants a and b in Eq. (3.54) are determined by certain integrations round closed contours referred to as global monodromy conditions. In practice, only one of $\partial X_{\mathrm{cl}}/\partial z$ and $\partial X_{\mathrm{cl}}/\partial \bar{z}$, say $\partial X_{\mathrm{cl}}/\partial z$, is an acceptable classical solution, because the other gives a divergent classical action. Then, we may set b to zero. Integrating round a closed contour C such that X is shifted by an amount v but not rotated we have

$$\int_C \mathrm{d}z(\partial X_{\mathrm{cl}}/\partial z) = v \ . \tag{3.55}$$

For example, if we choose C to go ℓ times round z_1 counterclockwise and k times round z_2 clockwise, then X is unrotated. To find the shift v we have to multiply together the (powers of) space group elements associated with the fixed points. If we write α for the point group element with action $\mathrm{e}^{-2\pi \mathrm{i}k/N}$ in this complex plane and β for the point group element with action $\mathrm{e}^{-2\pi \mathrm{i}\ell/N}$, then what we need is

$$\begin{aligned}
&(\alpha,(I-\alpha)f_\alpha + (I-\alpha)\Lambda)^\ell(\beta^{-1},(I-\beta^{-1})f_\beta + (I-\beta^{-1})\Lambda)^k \\
&\quad = (I,(I-\alpha^\ell)(f_\alpha - f_\beta + \Lambda)) \\
&\quad = (I,v) \ ,
\end{aligned} \tag{3.56}$$

where, in each case, Λ is an arbitrary lattice vector. Strictly, if the complex plane in question is, for example, the $1+\mathrm{i}2$ plane, then we need the component of $(1-\alpha^\ell)(f_\alpha - f_\beta + \Lambda)$, on the $1+\mathrm{i}2$ direction to give v for $X^{1+\mathrm{i}2}$.

The integral (3.55) now determines the constant a as follows. For convenience, take

$$z_1 = 0, \qquad z_2 = 1, \qquad z_3 = z_\infty = \infty \ , \tag{3.57}$$

using SL(2,C) symmetry. Then, using the integral

$$\int_C \mathrm{d}z\, z^{-(1-(k/N)y)}(z-1)^{-(1-(\ell/N)y)} = -2\mathrm{i}\sin(k\ell\pi y)\frac{\Gamma(ky)\Gamma(\ell y)}{\Gamma((k+\ell)y)} \tag{3.58}$$

leads to

$$a = \frac{\mathrm{i}(-z_\infty)^{(k+\ell)/N}}{2}\frac{\Gamma((k+\ell)/N)v}{\sin(k\ell\pi/N)\Gamma(k/N)\Gamma(\ell/N)} \ . \tag{3.59}$$

Consequently, after performing the $\int \mathrm{d}^2 z$, using Appendix A of [118], Z_{cl} (with the index i suppressed) is given by Eq. (3.46) with

$$S_{\mathrm{cl}} = \frac{|v|^2}{4\pi\sin^2(k\ell\pi/N)}\frac{|\sin(k\pi/N)||\sin(\ell\pi/N)|}{|\sin((k+\ell)\pi/N)|} \tag{3.60}$$

with v as in Eq. (3.56) and the sum over X_{cl} reducing to a sum over the choices of v parameterised by the arbitrary lattice vector Λ.

When there are two independent twists k and ℓ involved [42] there are two distinct global monodromy conditions [177,88] which can be obtained by encircling the pairs of points z_1 and z_2 and z_1 and z_3 in turn. This leads to two different expressions for S_{cl}. Consistency between these two expansions has to be achieved by restricting the sum over the initially arbitrary lattice vectors Λ occurring in the expression for v. When the two twists k and ℓ are identical, there is only a single independent global monodromy condition and this subtlety does not arise.

Another subtlety is that if one of the twisted sectors involved has a fixed plane then the factor Z^i in the 3-point function corresponding to this complex plane reduces to a two twist field correction function and can be normalised to 1.

3.7. 3-point function for Z_3 orbifold

The ideas of the previous section can be illustrated and the result made explicit so far as the moduli and fixed point dependence is concerned by considering the coupling of three states each in the θ twisted sector of the Z_3 orbifold. This coupling is allowed by the point group selection rule and also by H-momentum conservation because it is analogous to the DDD coupling for the $Z_3 \times Z_3$ orbifold discussed earlier. The space group selection rule (3.22) specifies the fixed point with which the third θ twisted sector state must be associated given the fixed points for the first 2 states.

The twists for the three complex planes are

$$\frac{k_i}{3} = \frac{\ell_i}{3} = \frac{2}{3}, \quad i = 1,2,3 \; . \tag{3.61}$$

It then follows from Eq. (3.60) that

$$S^i_{\mathrm{cl}} = \frac{|v_i|^2}{2\pi\sqrt{3}}, \quad i = 1,2,3 \tag{3.62}$$

and thus

$$Z_{\mathrm{cl}} \sim \sum_v \exp\left(\frac{-1}{2\pi\sqrt{3}} \sum_i |v_i|^2\right) , \tag{3.63}$$

where v_i is the component of v in the ith complex plane (e.g. v_1 is the component of v in the $1 + \mathrm{i}2$ direction.) Also, from Eq. (3.56)

$$v = (I - \theta^2)(f_1 - f_2 + \Lambda) \; . \tag{3.64}$$

For the Z_3 orbifold, with f of the form given in Eq. (3.20), v takes the form

$$v = \sum_{j=1}^{3} d_{2j-1}(e_{2j-1} + e_{2j}) + (I - \theta^2)\Lambda \; , \tag{3.65}$$

where

$$d_{2j-1} = p^1_{2j-1} - p^2_{2j-1} \tag{3.66}$$

and the integers p^J_{2j-1}, $J = 1,2$, take the values

$$p^J_{2j-1} = 0, \pm 1 \; . \tag{3.67}$$

In addition, Λ is an arbitrary linear combination with integral coefficients of the basis vectors $e_\rho, \rho = 1,\dots,6$, and using the action of the point group on the basic vectors we can write

$$v = \sum_{j=1}^{3} [(d_{2j-1} + 2k_{2j-1} - k_{2j})e_{2j-1} + (d_{2j-1} + k_{2j-1} + k_{2j})e_{2j}] \tag{3.68}$$

with k_{2j-1} and k_{2j} arbitrary integers.

The orbifold possesses deformation parameters or moduli which are continuously variable quantities corresponding to radii and angles defining the underlying torus. These parameters can be absorbed into the definition of the basis vectors e_ρ. The most general lattice basis compatible with the point group is obtained by requiring that all scalar products $e_\rho \cdot e_\sigma$ are preserved by the point group action. Here, we restrict attention to the moduli

$$R^2_{2j-1} = |e_{2j-1}|^2 = |e_{2j}|^2 \; . \tag{3.69}$$

The angles $\theta_{2j-1,2j}$ defined in terms of the scalar products $e_{2j-1} \cdot e_{2j}$ are fixed at $2\pi/3$ for compatibility with the point group. The other six angles are also moduli and the dependence of the Yukawa couplings on these can be found elsewhere [53,144].

The orthonormal space basis g_r, $r = 1,\dots,6$, in which the point group element θ has the action

$$\theta(g_{2j-1} + \mathrm{i}g_{2j}) = \mathrm{e}^{2\pi \mathrm{i}/3}(g_{2j-1} + \mathrm{i}g_{2j}) \tag{3.70}$$

for $j = 1,2,3$, is related to the lattice basis e_ρ, $\rho = 1,\dots,6$ by

$$e_{2j-1} = R_{2j-1} g_{2j-1}, \qquad e_{2j} = R_{2j-1}(\cos\tfrac{2}{3}\pi g_{2j-1} + \mathrm{i}\sin\tfrac{2}{3}\pi g_{2j}) \; . \tag{3.71}$$

Consequently, the component of v in the direction $g_{2j-1} + \mathrm{i}g_{2j}$ corresponding to the jth complex plane is

$$\begin{aligned} v_j = {} & (d_{2j-1} + 2k_{2j-1} - k_{2j})R_{2j-1} + (d_{2j-1} + k_{2j-1} + k_{2j})R_{2j-1}\cos\tfrac{2}{3}\pi \\ & + i(d_{2j-1} + k_{2j-1} + k_{2j})R_{2j-1}\sin\tfrac{2}{3}\pi \; . \end{aligned} \tag{3.72}$$

Thus, in Eq. (3.63)

$$|v_i|^2 = [(d_{2i-1} + 2k_{2i-1} - k_{2i})^2 + (d_{2i-1} + k_{2i-1} + k_{2i})(2k_{2i} - k_{2i-1})]R^2_{2i-1} \; . \tag{3.73}$$

To find the leading term in the Yukawa for large values of R_{2i-1} we need to minimise the coefficient of R^2_{2i-1} in Eq. (3.73) with k_{2i} and k_{2i-1} arbitrary integers, and, as a consequence of Eq. (3.67)

$$d_{2i-1} = 0, \pm 1, \pm 2 \; . \tag{3.74}$$

The result is

$$\begin{aligned} |v_i|^2_{\mathrm{MIN}} &= 0 && \text{for } d_{2i-1} = 0 \\ &= R^2_{2i-1} && \text{for } d_{2i-1} = \pm 1, \pm 2 \; . \end{aligned} \tag{3.75}$$

In this approximation, the Yukawa coupling between three θ twisted sectors takes the form

$$Z_{\mathrm{cl}} \sim \exp\left(-\frac{1}{2\pi\sqrt{3}} \sum_i |v_i|^2_{\mathrm{MIN}} \right) \tag{3.76}$$

with the moduli and fixed-point-independent constant of proportionality to be fixed by Z_{qu}. It will be noticed that the size of the Yukawa coupling is controlled by the "distances" d_{2i-1} between the fixed points on the 3 complex planes. Such calculations have been carried out for all Z_N and $Z_M \times Z_N$ orbifolds [78,120,42,53,177,88,144,22,19,178,20,21].

3.8. B field backgrounds

If the components on the compact manifold of the anti-symmetric tensor field always present in heterotic string theory develop vacuum expectation values then this B field background affects the Yukawa couplings [89]. The B field background is included through the term in the action

$$S_B = -\frac{1}{2\pi}\int_0^\pi d\sigma \int d\tau\, \varepsilon^{\alpha\beta} B_{rS} \partial_\alpha X^r \partial_\beta X^s , \tag{3.77}$$

where the indices r and s refer to the real space basis. In the complex basis and after Wick rotation,

$$S = \frac{1}{\pi}\int d^2 z \left(\frac{\partial X^i}{\partial z}\frac{\partial \bar{X}^i}{\partial \bar{z}} + \frac{\partial X^i}{\partial \bar{z}}\frac{\partial \bar{X}^i}{\partial z}\right) - \frac{iB_{2i-1,2i}}{\pi}\int d^2 z \left(\frac{\partial X^i}{\partial z}\frac{\partial \bar{X}^i}{\partial \bar{z}} - \frac{\partial X^i}{\partial \bar{z}}\frac{\partial \bar{X}^i}{\partial z}\right), \tag{3.78}$$

where we have retained only the background B fields with both indices in a single complex plane. Assuming, as before, that $\partial X^i_{\text{cl}}/\partial \bar{z}$ gives a divergent classical action and should be dropped, we obtain

$$Z_{\text{cl}} = \sum_{X^i_{\text{cl}}} \exp(-S_{\text{cl}}) = \sum_{v_i} \exp\left(\frac{-(1 - iB_{2i-1,2i})|v_i|^2 |\sin(k_i\pi/N)||\sin(\ell_i\pi/N)|}{4\pi^2 \sin^2((k_i\ell_i\pi/N)) \qquad |\sin((k_{i+}\ell_i)\pi/N)|}\right) \tag{3.79}$$

as the generalisation of Eq. (3.60).

In the case of the Z_3 orbifold, R^2_{2i-1} is replaced in Eq. (3.73) by $(1 - iB_{2i-1,2i})\ R^2_{2i-1}$. This can be written in terms of the moduli T_i as follows. In general, for the ith complex plane,

$$iT_i = 2(b_{2i-1,2i} + i\sqrt{(\det g)_i})(2\pi)^{-2} , \tag{3.80}$$

where, in terms of the basis vectors e_a for the lattice of the 6 torus,

$$g_{ab} = e_a \cdot e_b , \tag{3.81}$$

$$b_{ab} = e^r_a B_{rs} e^s_b \tag{3.82}$$

and the determinant refers to the 2×2 matrix for the ith complex plane. For the Z_3 orbifold, the (deformed) lattice basis vectors are

$$e_1 = (1,0)R_1, \quad e_2 = (\cos\tfrac{2}{3}\pi, \sin\tfrac{2}{3}\pi)R_1 \tag{3.83}$$

and similarly for the other complex planes with R_2 and R_3 replacing R_1. Thus,

$$\sqrt{(\det g)_i} = (\sqrt{3}/2)R^2_{2i-1} , \tag{3.84}$$

$$b_{2i-1,2i} = (\sqrt{3}/2)R^2_{2i-1}B_{2i-1,2i} , \tag{3.85}$$

$$(1 - iB_{2i-1,2i})R^2_{2i-1} = (T_i/\sqrt{3})(2\pi)^2 . \tag{3.86}$$

The effect of the B field background on Z_{cl}, and so on the Yukawa coupling, is therefore to replace R_{2i-1}^2 by $T_i(2\pi)^2/\sqrt{3}$ so that the Yukawa couplings are functions of the T moduli.

3.9. Classical part of 4-point function from conformal field theory

The determination of the moduli independent normalisation of a Yukawa coupling requires the factorisation of a 4-point function into 3-point functions. The difficult part is the expectation value of a product of 4 twist fields and (for twisted sector ground states) we are interested in factors Z_4^i for the ith complex plane of the form

$$Z_4^i = \langle \sigma^i_{-k_i/N,f_1}(z_i,\bar{z}_1)\sigma^i_{k_i/N,f_2}(z_2,\bar{z}_2)\sigma^i_{-\ell_i/N,f_3}(z_3,\bar{z}_3)\times\sigma^i_{\ell_i/N,f_4}(z_4,\bar{z}_4)\rangle\ . \tag{3.87}$$

This can be written as a product of a classical and a quantum part as

$$Z_4^i = (Z_4^i)_{\mathrm{qu}}(Z_4^i)_{\mathrm{cl}}\ , \tag{3.88}$$

$$(Z_4^i)_{\mathrm{cl}} = \sum_{X^i_{\mathrm{cl}}} \exp(-S^i_{\mathrm{cl}})\ , \tag{3.89}$$

with

$$S^i_{\mathrm{cl}} = \frac{1}{\pi}\int \mathrm{d}^2 z\left(\frac{\partial X^i_{\mathrm{cl}}}{\partial z}\frac{\partial \bar{X}^i_{\mathrm{cl}}}{\partial \bar{z}} + \frac{\partial X^i_{\mathrm{cl}}}{\partial \bar{z}}\frac{\partial \bar{X}^i_{\mathrm{cl}}}{\partial z}\right) \tag{3.90}$$

and the classical solutions now in the presence of four twist fields. The solutions of the string equations of motion with correct boundary conditions as $z \to z_1,\ z_2,\ z_3$ and z_4 in the presence of the twist fields are of the form

$$\begin{aligned}
\partial X/\partial z &= a\omega_{k/N,\ell/N}(z)\ ,\\
\partial \bar{X}/\partial \bar{z} &= \bar{a}\bar{\omega}_{k/N,\ell/N}(\bar{z})\ ,\\
\partial X/\partial \bar{z} &= b\bar{\omega}_{1-k/N,1-\ell/N}(\bar{z})\ ,\\
\partial \bar{X}/\partial z &= \bar{b}\omega_{1-k/N,1-\ell/N}(z)\ ,
\end{aligned} \tag{3.91}$$

where

$$\omega_{k/N,\ell/N}(z) = (z-z_1)^{-k/N}(z-z_2)^{k/N-1}(z-z_3)^{-\ell/N}(z-z_4)^{\ell/N-1}\ , \tag{3.92}$$

where we have suppressed the index i.

For the case $k=\ell$ there are two independent contours for global monodromy conditions [177,88] to fix a and b which can be taken to be C_1 and C_2 of Fig. 3. For $k \neq \ell$, there are three independent contours and, much as for the 3-point function, this results in a restriction on the sum over initially arbitrary lattice vectors in the final expressions. We shall focus on $k=\ell$. If v_j is the shift in X_{cl} in going round the contour C_j then

$$v_j = \oint_{C_j} \frac{\partial X_{\mathrm{cl}}}{\partial z}\mathrm{d}z + \oint_{C_j}\frac{\partial X_{\mathrm{cl}}}{\partial \bar{z}}\mathrm{d}\bar{z}\ , \tag{3.93}$$

where

$$v_1 = (I-\theta^k)(f_2 - f_1 + \Lambda) \tag{3.94}$$

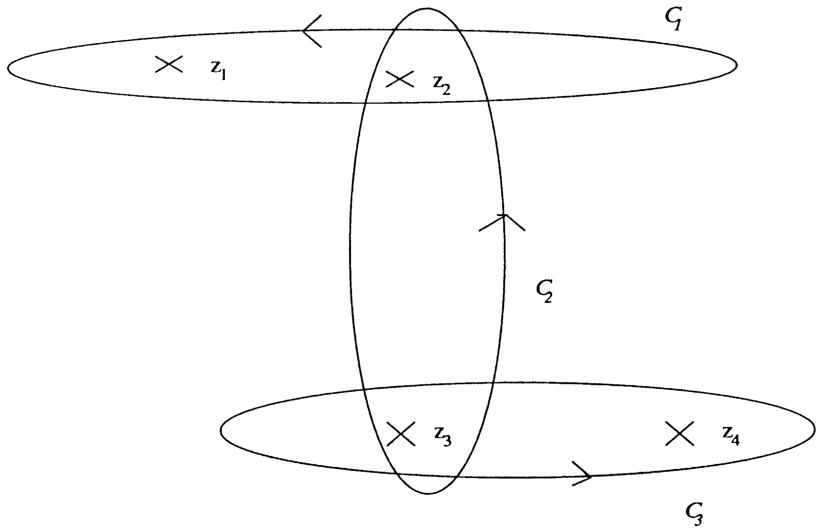


Fig. 3. Independent contours for global monodromy conditions for the 4-point function. In general, the points z_1, z_2 and z_3 may be encircled more than once.

and

$$v_2 = (I - \theta^k)(f_2 - f_3 + \Lambda) \tag{3.95}$$

with projection of the shifts onto the appropriate complex planes. Carrying out the contour integrals and solving for a and b leads to

$$\pi S_{\mathrm{cl}} = \frac{1}{4\tau_2 \sin(k\pi/N)}\{|v_2|^2 + |\tau|^2|v_1|^2 + \mathrm{i}\tau_1(\mathrm{e}^{\pi \mathrm{i}k/N} v_1\bar{v}_2 - \mathrm{e}^{-\pi \mathrm{i}k/N} v_2\bar{v}_1)\} , \tag{3.96}$$

where SL(2,C) symmetry has been used to set

$$z_1 = 0, \quad z_2 = x, \quad z_3 = 1, \quad z_4 = z_\infty = \infty , \tag{3.97}$$

the quantity τ is defined by

$$\tau = \tau_1 + \mathrm{i}\tau_2 = \frac{\mathrm{i}F(1-x)}{F(x)} \tag{3.98}$$

and $F(x)$ is the hypergeometric function

$$F(x) \equiv F\left(\frac{k}{N};1-\frac{k}{N};1;x\right) . \tag{3.99}$$

The real and imaginary parts of τ are thus given by

$$\tau_1 = \frac{\mathrm{i}}{2}\frac{(\bar{F}(\bar{x})F(1-x) - F(x)\bar{F}(1-\bar{x}))}{|F|^2} \tag{3.100}$$

and

$$\tau_2 = I(x,\bar{x})/2|F(x)|^2 \tag{3.101}$$

where

$$I(x,\bar{x}) \equiv F(x)\bar{F}(1-\bar{x}) + F(x)F(1-x)\ . \tag{3.102}$$

3.10. Quantum part of the 4-point function

The quantum part of Z_4 is determined with the aid of a differential equation for its dependence on the variable z_2 derived using the stress tensor method [78]. (The variables z_1, z_3 and z_4 can be fixed to constant values using SL(2,C) symmetry as in (3.97).) This method relies on the operator product expansion (OPE) of the stress tensor $T(z)$ with the twist field $\sigma_{k/N}$, namely,

$$T(z)\sigma_{k/N}(w,\bar{w}) \sim \frac{h_{k/N}\sigma_{k/N}(w,\bar{w})}{(z-w)^2} + \frac{\partial_w \sigma_{k/N}(w,\bar{w})}{(z-w)} + \cdots\ , \tag{3.103}$$

where

$$h_{k/N} = \frac{1}{2}\frac{k}{N}\left(1-\frac{k}{N}\right) \tag{3.104}$$

is the conformal dimension of the twist field. The stress tensor is the normal ordered product

$$T(z) = -\tfrac{1}{2}{:}\partial_z X \partial_z \bar{X}{:} \tag{3.105}$$

and $h_{k/N}$ can be identified by considering the expectation value of $T(z)$ between twisted sector ground states $|\sigma_{k/N}\rangle$ and $|\sigma_{-k/N}\rangle$. The method also relies on the OPE

$$-\frac{1}{2}\partial_z X(z)\partial_w \bar{X}(w) \sim T(z) + \frac{1}{(z-w)^2} + \cdots\ . \tag{3.106}$$

As usual, the index i labelling the particular complex plane of the 6 torus is being omitted.

The OPE (3.103) implies that

$$\partial_{z_2}\ln(Z_4)_{\mathrm{qu}} = \lim_{z\to z_2}\left(\frac{(z-z_2)}{Z_4}\langle T(z)\sigma_{-k/N}(z_1)\sigma_{k/N}(z_2)\sigma_{-\ell/N}(z_3)\sigma_{\ell/N}(z_4)\rangle - \frac{h_{k/N}}{(z-z_2)}\right)\ . \tag{3.107}$$

Moreover, the OPE (3.106) implies that

$$\langle T(z)\sigma_{-k/N}(z_1)\sigma_{k/N}(z_2)\sigma_{-\ell/N}(z_3)\sigma_{\ell/N}(z_4)\rangle/Z_4 = \lim_{z\to w}\left(g(z,w;z_j) - \frac{1}{(z-w)^2}\right)\ , \tag{3.108}$$

where

$$g(z,w;z_j) = \langle -\tfrac{1}{2}\partial_z X(z)\partial_w \bar{X}(w)\sigma_{-k/N}(z_1)\sigma_{k/N}(z_2)\sigma_{-\ell/N}(z_3)\sigma_{\ell/N}(z_4)\rangle/Z_4\ . \tag{3.109}$$

Thus, $\partial_{z_2}\ln(Z_4)_{\mathrm{qu}}$ can be derived if we can calculate $g(z,w;z_j)$. Because of the OPEs of $\partial_z X$ and $\partial_w \bar{X}$, with the twist fields and the OPE of $\partial_z X$ with $\partial_w \bar{X}$, g has the behaviour for $z,w \to z_1,\dots,z_4$ and $z \to w$,

$$\begin{aligned} g(z,w;z_i) &\sim (z-w)^{-2} + \text{finite}, & z &\to w \\ &\sim (z-z_1)^{-k/N}, & z &\to z_1 \\ &\sim (z-z_3)^{-\ell/N}, & z &\to z_3 \\ &\sim (z-z_2)^{-(1-k/N)}, & z &\to z_2 \\ &\sim (z-z_4)^{-(1-\ell/N)}, & z &\to z_4 \\ &\sim (w-z_1)^{-(1-k/N)}, & w &\to z_1 \\ &\sim (w-z_3)^{-(1-\ell/N)}, & w &\to z_3 \\ &\sim (w-z_2)^{-k/N}, & w &\to z_2 \\ &\sim (w-z_4)^{-\ell/N}, & w &\to z_4 \ . \end{aligned} \tag{3.110}$$

In terms of the holomorphic function defined in Eq. (3.92), g is fixed to be of the form

$$g(z,w;z_j) = \omega_{k/N,\ell/N}(z)\omega_{1-k/N,1-\ell/N}(w)\left[\frac{P(z,w)}{(z-w)^2} + A(z_j,\bar{z}_j)\right], \tag{3.111}$$

where $P(z,w)$ is a polynomial quadratic in z and w separately,

$$P(z,w) = \sum_{i,j=0}^{2} a_{ij} w^i z^j \ . \tag{3.112}$$

The coefficients a_{ij} are determined by requiring that there is *no* simple pole for $z \to w$ and that the numerator of the double pole is 1. This fixes all coefficients a_{ij} except for a_{20}, a_{02} and a_{11}, for which there are only two equations. This freedom corresponds to the freedom to absorb the constant part of $P(z,w)/(z-w)^2$ into A. It is convenient to fix all coefficients a_{ij} before calculating A, without loss of generality. Specialising to the case $k=\ell$, a convenient choice is

$$a_{20} = \frac{k}{N} z_1 z_3 + \left(1 - \frac{k}{N}\right) z_2 z_4 \ . \tag{3.113}$$

Then,

$$\begin{aligned} P(z,w) = {} & \frac{k}{N}(z-z_1)(z-z_3)(w-z_2)(w-z_4) \\ & + \left(1-\frac{k}{N}\right)(z-z_2)(z-z_4)(w-z_1)(w-z_3) \ . \end{aligned} \tag{3.114}$$

Using SL(2,C) symmetry to set

$$z_1 = 0, \quad z_2 = x, \quad z_3 = 1, \quad z_4 = z_\infty = \infty \ , \tag{3.115}$$

the expression (3.107) for the derivative of $\ln(Z_4)_{\mathrm{qu}}$ now reduces to

$$\partial_x \ln(Z_4)_{\mathrm{qu}} = -\frac{k}{N}\left(1-\frac{k}{N}\right)\left(\frac{1}{x}-\frac{1}{(1-x)}\right)-\frac{\tilde{A}(x,\bar{x})}{x(1-x)}\ , \tag{3.116}$$

where

$$\tilde{A}(x,\bar{x}) = \lim_{z_4\to\infty} (-z_4)^{-1} A(0,x,1,z_4)\ . \tag{3.117}$$

Before $(Z_4)_{\mathrm{qu}}$ can be calculated it remains to determine A.

The global monodromy conditions for the quantum part of X using the same two contours C_i (Fig. 3) as were used for the classical part of X give

$$\Delta_{C_i} X_{\mathrm{qu}} = \oint_{C_i} \frac{\partial X_{\mathrm{qu}}}{\partial z}\mathrm{d}z + \oint_{C_i} \frac{\partial X_{\mathrm{qu}}}{\partial \bar{z}}\mathrm{d}\bar{z} = 0 \tag{3.118}$$

and consequently

$$\oint_{C_i} \mathrm{d}z\, g(z,w) + \oint_{C_i} \mathrm{d}\bar{z}\, h(\bar{z},w) = 0\ , \tag{3.119}$$

where an auxiliary correlation function $h(\bar{z},w;z_i)$ is defined by

$$h(\bar{z},w;z_i) = \langle -\tfrac{1}{2}\partial_{\bar{z}}X(z,\bar{z})\partial_w\bar{X}(w,\bar{w})\sigma_{-k/N}(z_1)\sigma_{k/N}(z_2)\sigma_{-\ell/N}(z_3)\sigma_{\ell/N}(z_4)\rangle/Z_4\ . \tag{3.120}$$

The most general form of h consistent with the OPEs of $\partial_{\bar{z}}X$ and $\partial_w\bar{X}$ with the twist fields and the non-singular OPE of $\partial_{\bar{z}}X$ with $\partial_w\bar{X}$ is

$$h(\bar{z},w;z_i) = \bar{\omega}_{N-k/N,N-\ell/N}(\bar{z})\omega_{N-k/N,N-\ell/N}(w)B(z_i,\bar{z}_i)\ . \tag{3.121}$$

Specialising to $k=\ell$, dividing through by $\omega_{N-k/N,N-k/N}(w)$, choosing $z_1,\dots,z_4$ as in Eq. (3.115), and taking the limit $w\to\infty$, gives a pair of equations for A and B which can be solved to give

$$\tilde{A}(x,\bar{x}) = x(1-x)\partial_x \ln I(x,\bar{x})\ , \tag{3.122}$$

where $I(x,\bar{x})$ was defined in Eq. (3.102).

Now that $\tilde{A}(x,\bar{x})$ is known, Eq. (3.116) can be integrated to give

$$(Z_4)_{\mathrm{qu}} = \tilde{c}|x(1-x)|^{-2k/N(1-k/N)}[I(x,\bar{x})]^{-1}\ , \tag{3.123}$$

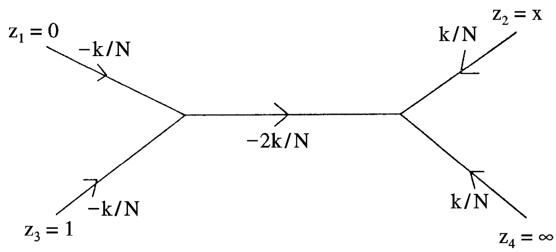
where $\tilde{c}$ is a constant. Multiplying together Eqs. (3.89) and (3.123) to obtain Z_4 gives

$$Z_4 = \frac{c|x(1-x)|^{-2k/N(1-k/N)}}{\tau_2|F(x)|^2}\sum_{v_1,v_2} \mathrm{e}^{-S_{\mathrm{cl}}(v_1,v_2)}\ , \tag{3.124}$$

where we have used Eq. (3.101), the constant c is $\tilde{c}/2$, and $S_{\mathrm{cl}}(v_1,v_2)$ is given by Eq. (3.96).

3.11. Factorisation of the 4-point function to 3-point functions

To derive the Yukawa couplings in which we are interested we now have to factorise the 4-point function by writing it as a sum of terms which are products of 3-point functions [78,177,88]. We



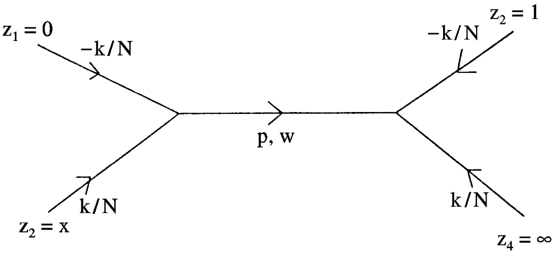


Fig. 4. u channel factorisation of 4-point function.

Fig. 5. s channel factorisation of 4-point function.

first factorise in the u channel (Fig. 4) to derive the required Yukawa coupling up to a moduli and fixed point independent normalisation factor and then factorise in the s channel (Fig. 5) to establish the normalisation.

To study the u channel factorisation it is necessary to take the limit $x \to \infty$. Using the asymptotic form for $F(x)$, we then obtain

$$S_{\mathrm{cl}} \approx \frac{\pm 1}{4\pi \sin(2k\pi/N)}(|\tilde{v}_1|^2 + |\tilde{v}_2|^2) \quad \text{for } x \to \infty \ , \tag{3.125}$$

where

$$\tilde{v}_1 = v_1 - v_2, \quad \tilde{v}_2 = \mathrm{e}^{2\pi \mathrm{i}k/N} v_1 + v_2 \tag{3.126}$$

and the plus sign and minus sign correspond to $k/N < 1 - k/N$ and $k/N > 1 - k/N$, respectively. The factorisation of the 4-point function into a sum of terms that are products of 3 point functions depends on the OPE of two twist fields. In general, for conformal fields A_i and A_j of conformal weights $h_i, \bar{h}_i$ and $h_j, \bar{h}_j$ the OPE can be written in the form

$$A_i(z,\bar{z})A_j(w,\bar{w}) \sim \sum_k \frac{C_{ijk} A_k(w,\bar{w})}{(z-w)^{h_i+h_j-h_k}(\bar{z}-\bar{w})^{\bar{h}_i+\bar{h}_j-\bar{h}_k}} \tag{3.127}$$

for $z \to w$, where C_{ijk} are some coefficients. For twist fields, we can write

$$\sigma_{k/N,f_2}(x,\bar{x})\sigma_{k/N,f_4}(z_\infty,\bar{z}_\infty) \sim \sum_f Y^{k/n}_{f_2 f_4 f}\sigma_{2k/N,f}(z_\infty \bar{z}_\infty)|x - z_\infty|^{-2(2h_{k/N}-h_{2k/N})} \tag{3.128}$$

for $x \to z_\infty$, where the sum is over fixed points, the coefficients Y can be interpreted as Yukawa couplings, as will be seen shortly, and the conformal weights are given by

$$h_{k/N} = \bar{h}_{k/N} = \frac{1}{2}\frac{k}{N}\left(1 - \frac{k}{N}\right) \tag{3.129}$$

as in Eq. (3.104). The 4-point function can then be written in the form valid for $x \to z_\infty$,

$$\begin{aligned} Z_4 &= \langle \sigma_{-k/N,f_1}(0)\sigma_{k/N,f_2}(x)\sigma_{-k/N,f_3}(1)\sigma_{k/N,f_4}(z_\infty)\rangle \\ &\sim \sum_f Y^{k/n}_{f_2 f_4 f}\langle \sigma_{-k/N,f_1}(0)\sigma_{-k/N,f_3}(1)\sigma_{2k/N,f}(z_\infty)\rangle \times |x - z_\infty|^{-2(2h_k/N - h_{2k}/N)} \ . \end{aligned} \tag{3.130}$$

Moreover, for conformal fields A_i,A_j and A_k,

$$\langle A_i(z_1,\bar{z}_1)A_j(z_2,\bar{z}_2)A_k(z_3,\bar{z}_3)\rangle = C_{ijk}\prod_{i<j}(z_i - z_j)^{-h_{ij}}(\bar{z}_i - \bar{z}_j)^{-\bar{h}_{ij}} \tag{3.131}$$

with

$$h_{ij} = h_i + h_j - h_k \tag{3.132}$$

and similarly for $\bar{h}_{ij}$. In the case of twist fields,

$$\langle \sigma_{-k/N,f_1}(0)\sigma_{-k/N,f_3}(1)\sigma_{2k/N,f}(z_\infty)\rangle = Y^{*k/N}_{f_1 f_3 f}|z_\infty|^{-4h_{2k/N}} \tag{3.133}$$

Consequently, Z_4 takes the form for $x \to \infty$,

$$Z_4 \approx |x|^{-2(2h_{k/N}-h_{2k/N})}|z_\infty|^{-4h_{2k/N}}\sum_f Y^{k/N}_{f_2 f_4 f}Y^{*k/N}_{f_1 f_3 f}\ . \tag{3.134}$$

To complete the factorisation, we have to use the requirement that the u channel fixed points f summed over must be consistent with the space group selection rule, which takes the form

$$(1-\theta^{2k})(f+\Lambda) = \theta^k(1-\theta^k)(f_3+\Lambda) + (1-\theta^k)(f_1+\Lambda)\ , \tag{3.135}$$

where the action of the point group element in this complex plane is

$$\theta = \mathrm{e}^{2\pi \mathrm{i}/N}\ . \tag{3.136}$$

The relation (3.135) is also correct if we interchange f_1 and f_3, and there are similar relations with f_2 and f_4 replacing f_1 and f_3. Aided by the space group selection rule we can show that

$$\tilde{v}_1 = \theta^{-k}(1-\theta^{2k})(f-f_1+\Lambda) \tag{3.137}$$

and

$$\tilde{v}_2 = -(1-\theta^{2k})(f-f_2+\Lambda)\ . \tag{3.138}$$

This allows Z_4 to be written in the factorised form for $x \to \infty$, and $k/N < 1-k/N$,

$$Z_4 \approx c\frac{|x|^{2k/N(2k/N-1)}(\Gamma(1-k/N))^4}{\cos(k\pi/N)(\Gamma(1-2k/N))^2}\sum_f\left(\sum_{\tilde{v}_1}\mathrm{e}^{-\tilde{S}(\tilde{v}_1)}\right)\left(\sum_{\tilde{v}_2}\mathrm{e}^{-\tilde{S}(\tilde{v}_2)}\right) \tag{3.139}$$

with

$$\tilde{S}(\tilde{v}) = \frac{|\tilde{v}|^2}{4\pi\sin(2k\pi/N)}\ . \tag{3.140}$$

It remains to fix the normalisation constant in Eq. (3.139). This can be done by considering the s channel factorisation (Fig. 5) which gives the coupling for the annihilation of two twisted states into an untwisted state. To study these s channel couplings we need to Poisson resum $\sum_{v_1,v_2}\mathrm{e}^{-S_{\mathrm{cl}}}$ so as to write Z_4 in terms of momenta on the dual Λ^* of the lattice Λ corresponding to the momenta of untwisted S channel states. Because the sum over v_2 is over the coset $(1-\theta^k)(f_2-f_3+\Lambda)$ rather than Λ, it is necessary first to arrange for a sum over Λ by writing

$$v_2 = -2\mathrm{i}\mathrm{e}^{\mathrm{i}\pi k/N}\sin\left(\frac{\pi k}{N}\right)(f_2-f_3+q) \tag{3.141}$$

where $q \in \Lambda$. Writing $S_{\rm cl}$ in terms of q and using the Poisson resummation identity

$$\frac{1}{V_\Lambda \sqrt{\det A}} \sum_{q\in\Lambda} \exp(-\pi)(q+\phi)^{\rm T} A^{-1}(q+\phi) - 2\pi i \delta^{\rm T}(q+\phi)$$

$$= \sum_{p\in\Lambda^*} \exp(-\pi(p+\delta)^{\rm T} A(p+\delta) + 2\pi i p^{\rm T}\phi) \tag{3.142}$$

with V_Λ the volume of the unit cell of the lattice, leads to

$$Z_4 = 2c \frac{|x(1-x)|^{-2k/N(1-k/N)}}{V_\Lambda \sin(k\pi/N)|F(x)|^2} \sum_{p\in\Lambda^*, v\in\Lambda_C} \exp(2\pi i p\cdot(f_2 - f_3)) W^{(p+v/2)^2/2} \bar{W}^{(p-v/2)^2/2} , \tag{3.143}$$

where $F(x)$ is as in Eq. (3.99), Λ_C is the set $(1-\theta^k)(f_2 - f_1 + \Lambda)$ corresponding to the v_1 summation,

$$W = e^{i\pi\tau/\sin(k\pi/N)} \tag{3.144}$$

with τ as in Eq. (3.98), and we have used

$$\sqrt{\det A} = \frac{2\tau_2}{\sin(k\pi/N)} . \tag{3.145}$$

To carry out the s channel factorisation it is now necessary to consider the limit $x \to 0$. The relevant OPE of twist fields is

$$\sigma_{-k/N,f_1}(0,0)\sigma_{k/N,f_2}(x,\bar{x}) \sim \sum_{p,w} (-x)^{h-2h_{k/N}}(-\bar{x})^{\bar{h}-2h_{k/N}} C^{k/N}_{f_1,f_2,p,w} V_{p,w}(x,\bar{x}) , \tag{3.146}$$

where $V_{p,w}$ is the twist invariant vertex operator for the emission of an untwisted sector state with $p \in \Lambda^*$ and winding number $w \in (1-\theta)(f_2 - f_1 + \Lambda)$, and h and $\bar{h}$ are the conformal dimensions of the corresponding untwisted state. Then, for $x \to 0$,

$$Z \approx \sum_{p,w} C^{k/N}_{f_1,f_2,h,w}(-x)^{h-2h_{k/N}}(-\bar{x})^{\bar{h}-2h_{k/N}} \langle V_{p,w}(x)\sigma_{-k/N,f_3}(1)\sigma_{k/N,f_4}(z_\infty)\rangle . \tag{3.147}$$

Moreover, using Eq. (3.131), for $x \to 0$,

$$\langle V_{p,w}(x)\sigma_{-k/N,f_3}(1)\sigma_{k/N,f_4}(z_\infty)\rangle \approx C^{k/N}_{p,w,f_3,f_4}|z_\infty|^{-4h_{k/N}}(-1)^{h+\bar{h}} \tag{3.148}$$

which results in

$$Z_4 \approx \sum_{p,w} C^{k/N}_{f_1,f_2,p,w} C^{k/N}_{p,w,f_3,f_4} x^{h-2h_{k/N}}\, \bar{x}^{\bar{h}-2h_{k/N}}|z_\infty|^{-4h_{k/N}} . \tag{3.149}$$

The 4-point function can now be normalised by considering the contribution to the sum over untwisted states from I, which is the untwisted state with $p = w = 0$ and $h = \bar{h} = 0$. Taking $f_1 = f_2$ and $f_3 = f_4$, in Eq. (3.148),

$$C_{0,0,f_3,f_3} = |z_\infty|^{4h_{k/N}} \langle I\sigma_{-k/N,f_3}(1)\sigma_{k/N,f_3}(z_\infty)\rangle = 1 , \tag{3.150}$$

where the 2-point function for two twist fields is normalised (consistently with Eq. (3.127)) by

$$\langle \sigma_{-k/N,f_3}(1)\sigma_{k/N,f_3}(z_3)\rangle = |z_\infty|^{-4h_{k/N}} . \tag{3.151}$$

Thus, the contribution from I on Eq. (3.149) can be written as

$$Z_4 \approx |x|^{-4h_{k/N}}|z_\infty|^{-4h_{k/N}} . \tag{3.152}$$

Comparing with the term with $p = v = 0$ in Eq. (3.143) the corresponding contribution is

$$Z_4 \approx c|x|^{-4h_{k/N}}/V_\Lambda \sin(k\pi/N) \tag{3.153}$$

so that

$$c = V_\Lambda \sin\frac{k\pi}{N}|z_\infty|^{-4h_{k/N}} . \tag{3.154}$$

Returning to Eq. (3.139) with constant of proportionality now fixed, and comparing with Eq. (3.134), gives the result for the 3-point function

$$Y^{k/N}_{f_2 f_4 f} = \sqrt{V_\Lambda \tan\frac{k\pi}{N}\frac{\Gamma^2(1-k/N)}{\Gamma(1-2k/N)}}\sum_{\tilde{v}_2} \mathrm{e}^{-\tilde{S}(\tilde{v}_2)} , \tag{3.155}$$

where

$$\tilde{S}(\tilde{v}_2) = \frac{1}{4\pi \sin 2k\pi/N}|\tilde{v}_2|^2 \tag{3.156}$$

and $\tilde{v}_2$ is given by Eq. (3.138). If there are any complex planes that are unrotated by one of the three twists involved, the normalisation factor should be restricted to the rotated complex planes, because, for the unrotated plane, the 3-point function reduces to a 2-point function that can be normalised to 1.

3.12. Yukawa couplings involving excited twisted sector states

The vertex operators for excited states, i.e. states with oscillators acting on the ground state, involve derivatives of string degrees of freedom as well as twist fields. For correlation functions involving excited states there is then the selection rule [78,120,65,102–104] that a correlation function for which the product of vertex operators contains the factor $(\partial_{\bar{z}} X^i)^p (\partial_{\bar{z}} \bar{X}^i)^q$ must have

$$p - q = 0 \bmod N \tag{3.157}$$

if the action of a point group element in the ith complex plane is of order N.

To calculate the moduli dependence [78,21,22] consider for simplicity the situation where there are two excited twisted sector states involved each of which is created from the vacuum by a single bosonic left mover oscillator. The description of excited twisted sector states requires the excited twisted fields $\tilde{\tau}_{k/N}$ and $\tilde{\tau}'_{k/N}$ that occur in the OPEs (3.53). Thus, the non-trivial part of the 3-point function we wish to consider if of the form

$$(Z_3)_{\text{excited}} = \langle \tilde{\tau}'_{k/N}(z_1,\bar{z}_1)\tilde{\tau}_{\ell/N}(z_2,\bar{z}_2)\sigma_{-(k+\ell)/N}(z_3,\bar{z}_3)\rangle , \tag{3.158}$$

where the index i referring to the complex plane and the fixed point dependence have been suppressed. Consideration of the 2-point function $\langle \tilde{\tau}'_{k/N}(z_1,\bar{z}_1)\tilde{\tau}_{-k/N}(z_2,\bar{z}_2)\rangle$ and the twisted sector mode expansions shows that the excited twist fields that create normalised states are $(2k/N)^{-1/2}\tilde{\tau}'_{k/N}$

and $(2(1-k/N))^{-1/2}\tilde{\tau}_{-k/N}$. For an acceptable solution with convergent classical action, $(Z_3)_{\text{excited}}$ is found to have the same moduli dependence as the 3-point function with unexcited twist fields. However, the overall normalisation of the 3-point function changes. This normalisation, which depends on the twisted sectors involved, can be fixed by considering the 4-point function

$$(Z_4)_{\text{excited}} = \langle \tilde{\tau}'_{-k/N}(z_1,\bar{z}_1)\sigma_{k/N}(z_2,\bar{z}_2)\tilde{\tau}_{-\ell/N}(z_3,\bar{z}_3)\sigma_{\ell/N}(z_4,\bar{z}_4)\rangle \; . \tag{3.159}$$

With the aid of the OPEs (3.53) this can be written as

$$\begin{aligned}(Z_4)_{\text{excited}} &= \lim_{w\to z_1, z\to z_3} (\bar{w}-\bar{z}_1)^{k/N}(\bar{z}-\bar{z}_3)^{1-\ell/N} \\ &\quad \times \langle \partial_{\bar{z}}X\partial_{\bar{w}}\bar{X}\sigma_{-k/N}(z_1,\bar{z}_1)\sigma_{k/N}(z_2,\bar{z}_2)\sigma_{-\ell/N}(z_3,\bar{z}_3)\sigma_{\ell/N}(z_4,\bar{z}_4)\rangle \\ &= \sum_{X_{\text{cl}}} \mathrm{e}^{-S_{s\ell}}\langle \partial_{\bar{z}}X_{\text{qu}}\partial_{\bar{w}}\bar{X}_{\text{qu}}\rangle_{4\ \text{twists}} + \sum_{X_{\text{cl}}} \mathrm{e}^{-S_{s\ell}}\partial_{\bar{z}}X_{s\ell}\partial_{\bar{w}}\bar{X}_{s\ell}(Z_4)_{\text{qu}} \; .\end{aligned} \tag{3.160}$$

In Eq. (3.160), X has been separated into a classical and a quantum part, $\langle \partial_{\bar{z}}X_{\text{qu}}\partial_{\bar{w}}X_{\text{qu}}\rangle_{4\ \text{twists}}$ is the expectation value in the presence of the four twist fields, and $(Z_4)_{\text{qu}}$ is given by

$$(Z_4)_{\text{qu}} = \int DX_{\text{qu}}\mathrm{e}^{-S_{\text{qu}}}\sigma_{-k/N}(z_1,\bar{z}_1)\sigma_{k/N}(z_2,\bar{z}_2)\sigma_{-\ell/N}(z_3,\bar{z}_3)\sigma_{\ell/N}(z_4,\bar{z}_4) \; . \tag{3.161}$$

Using operator product expansion methods, setting

$$z_1 = 0, \quad z_2 = x, \quad z_3 = 1 \quad \text{and} \quad z_4 = z_\infty \tag{3.162}$$

using SL(2,C) invariance, and taking the limit $x \to z_\infty$ to achieve u channel factorisation, leads to

$$\begin{aligned}\frac{\langle \tilde{\tau}'_{-k/N}(0)\tilde{\tau}_{-\ell/N}(1)\sigma_{(k+\ell)/N}(z_\infty)\rangle}{\langle \sigma_{-k/N}(0)\sigma_{-\ell/N}(1)\sigma_{(k+\ell)/N}(z_\infty)\rangle} &= 2\tfrac{\ell}{N}(-1)^{2(k/N)-\ell/N} \quad \text{for } \tfrac{k}{N} < 1-\tfrac{\ell}{N} \; , \\ \frac{\langle \tilde{\tau}'_{-k/N}(0)\tilde{\tau}_{-\ell/N}(1)\sigma_{(k+\ell)/N}(z_\infty)\rangle}{\langle \sigma_{-k/N}(0)\sigma_{-\ell/N}(1)\sigma_{(k+\ell)/N}(z_\infty)\rangle} &= 2(1-\tfrac{k}{N})(-1)^{2(k/N)-\ell/N} \quad \text{for } \tfrac{k}{N} > 1-\tfrac{\ell}{N} \; .\end{aligned} \tag{3.163}$$

Taking account of the normalisation of the excited twist fields discussed above and powers of -1 from the conformal field theory of the 3-point function, the Yukawa coupling Y^E involving excited states should be defined by

$$Y^E_{-k/N,-\ell/N,(k+\ell)/N} = \frac{1}{2(N)}\frac{\ell}{}^{-1/2}(1-\tfrac{k}{N})^{-1/2}(-1)^{(\ell/N)-2k/N}\langle \tilde{\tau}'_{-k/N}(0)\tilde{\tau}_{-\ell/N}(1)\sigma_{(k+\ell)/N}(z_\infty)\rangle \tag{3.164}$$

and consequently

$$\begin{aligned}\frac{Y^E_{-k/N,-\ell/N,(k+\ell)/N}}{\langle \sigma_{-k/N}(0)\sigma_{-\ell/N}(1)\sigma_{(k+\ell)/N}(z_\infty)\rangle} &= \sqrt{\frac{\ell/N}{1-k/N}} \quad \text{for } \tfrac{k}{N} < 1-\tfrac{\ell}{N} \; , \\ \frac{Y^E_{-k/N,-\ell/N,(k+\ell)/N}}{\langle \sigma_{-k/N}(0)\sigma_{-\ell/N}(1)\sigma_{(k+\ell)/N}(z_\infty)\rangle} &= \sqrt{\frac{1-k/N}{\ell/N}} \quad \text{for } \tfrac{k}{N} > 1-\tfrac{\ell}{N} \; .\end{aligned} \tag{3.165}$$

There are thus twist-dependent suppression factors arising in the excited twisted sector Yukawa couplings relative to the Yukawa couplings amongst twisted sector ground states [21,22].

3.13. Quark and lepton masses and mixing angles

The exponential suppressions [78,120,112] due to the moduli dependence of twisted sector Yukawa couplings can lead to a hierarchial quark and lepton mass matrix [127,54,55]. By utilising all the possible embeddings of the point group and all possible choices of Wilson lines a huge number of models can be obtained for each Z_N or $Z_M \times Z_N$ orbifold. The strategy that has been adopted [55] in exploring the possibilities for the quark and lepton masses (and weak mixing angles) has been to allow the quarks and leptons and Higgses to be assigned to arbitrary twisted sectors and arbitrary fixed points.

In general, the Lagrangian terms L_q for the quark masses take the form

$$L_q = (\bar{d}_0,\bar{s}_0,\bar{b}_0)_{\mathrm{L}} M_d \begin{pmatrix} d_0 \\ s_0 \\ b_0 \end{pmatrix}_{\mathrm{R}} + (\bar{u}_0,\bar{c}_0,\bar{t}_0)_{\mathrm{L}} M_u \begin{pmatrix} u_0 \\ c_0 \\ t_0 \end{pmatrix}_{\mathrm{R}} + \text{h.c.} , \tag{3.166}$$

where M_d and M_u are matrices deriving from couplings to Higgses H_1 and H_2. In Eq. (3.166), $(u_0)_{\mathrm{L}}$,$(d_0)_{\mathrm{L}}$,$(c_0)_{\mathrm{L}}$,$(s_0)_{\mathrm{L}}$,$(t_0)_{\mathrm{L}}$ and $(b_0)_{\mathrm{L}}$ are the $[\mathrm{SU}(2)]_{\mathrm{L}}$ doublet quark fields, in terms of which the weak current J^{μ}_{+} coupled to the W boson takes the form

$$J^{\mu}_{+} = (\bar{u}_0,\bar{c}_0,\bar{t}_0)_{\mathrm{L}} \gamma^{\mu} I \begin{pmatrix} d_0 \\ s_0 \\ b_0 \end{pmatrix}_{\mathrm{L}} . \tag{3.167}$$

On the other hand, in terms of the states u,d,c,s,t and b that diagonalise the quark mass matrix the weak current J^{μ}_{+} takes the form

$$J^{\mu}_{+} = (\bar{u},\bar{c},\bar{t})_{\mathrm{L}} \gamma^{\mu} V \begin{pmatrix} d \\ s \\ b \end{pmatrix}_{\mathrm{L}} , \tag{3.168}$$

where the matrix V is the usual Kobayashi–Maskawa matrix

$$V = \begin{pmatrix} C_1 & C_3 S_1 & S_1 S_3 \\ -C_2 S_1 & C_1 C_2 C_3 - S_2 S_3 e^{i\delta} & C_1 C_2 S_3 + C_3 S_2 e^{i\delta} \\ S_1 S_2 & -C_1 C_3 S_2 - C_2 S_3 e^{i\delta} & -C_1 S_2 S_3 + C_2 C_3 e^{i\delta} \end{pmatrix}, \tag{3.169}$$

where

$$C_i = \cos\theta_i, \qquad S_i = \sin\theta_i . \tag{3.170}$$

For massless neutrinos, the Lagragian terms L_ℓ for the lepton mass take the form

$$L_\ell = (\bar{e},\bar{\mu},\bar{\tau})_{\mathrm{L}} M_e \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}_{\mathrm{R}} \tag{3.171}$$

and diagonalisation of the lepton mass matrix should not be required.

To reproduce the Kobayashi–Maskawa matrix it is necessary for the quark mass matrices M_d and M_u to have off-diagonal entries. Whether this is possible depends on the space group selection rules. For the prime order orbifolds Z_3 and Z_7, all the Yukawa couplings are diagonal in the sense that any 2 quark or lepton or Higgs states can only couple to a unique third state. This derives from 2 twisted sectors coupling to a unique third twisted sector because of the point group selection rule, from two fixed points coupling to a unique third fixed point because of the space group selection rule, and from there being only one state of given gauge quantum numbers associated with a particular fixed point. Then the (renormalisable) Yukawa couplings can not reproduce the Kobayashi–Maskawa matrix. Moreover the (diagonal) elements of the mass matrix do not have any observable phases because they can be observed into a redefinition of the right handed quark states.

However, non-renormalisable superpotential terms occur in general and can give rise to effective Yukawa couplings amongst quarks, leptons and Higgses when some gauge singlet scalars in the non-renormalisable coupling acquire expectation values. This gives the scope to obtain off-diagonal entries and phase factors in the quark mass matrices. In general, we can obtain quark and lepton mass matrices M_d, M_u and M_e of the form

$$M = \begin{pmatrix} \varepsilon & a & b \\ \tilde{a} & A & c \\ \tilde{b} & \tilde{c} & B \end{pmatrix}, \tag{3.172}$$

where $a,b,c,\tilde{a},\tilde{b} \ll \varepsilon,A,B$ because they are induced by non-renormalisable terms in the superpotential. It is also natural to assume that $\varepsilon \ll A,B$ because of the smallness of the first generation quark and lepton masses, and so to assume that ε also derives from a non-renormalisable term.

Things are more complicated for non-prime-order orbifolds. However, it is still unlikely that a suitable Kobayashi–Maskawa matrix can arise from non-renormalisable terms, and it therefore still appropriate to look for matrices M of the same form.

The strategy that has been adopted [55] has been to try to fit the second and third generation quarks and lepton masses with A and B in M given in terms of all the moduli (deformation parameters) of the orbifold, under the assumption that the smaller first generation masses are induced by non-renormalisable terms. Then the relevant Yukawa couplings L_{Y} are

$$L_{\mathrm{Y}} = h_c Q_c c^c H_2 + h_S Q_c S^c H_1 + h_t Q_t t^c H_2 + h_b Q_t b^c H_1 + h_\mu L_\mu \mu^c H_1 + h_\tau L_\tau \tau^c H_1 \;, \tag{3.173}$$

where Q and L denote quark and lepton doublets. At this time, the expectation values of H_1 and H_2 are additional parameters constrained by

$$\langle H_1 \mathrm{e}^2 + \langle H_2 \mathrm{e}^2 = 2\left(\frac{m_W}{g_2}\right)^2 . \tag{3.174}$$

The masses obtained at the string scale have to be run to 1 GeV using renormalisation group equations to make contact with the point at which quark and lepton masses are usually given. A subtlety is that the Yukawa couplings have to be run from the string scale of about 0.5×10^{18} GeV whereas, because we know that gauge coupling constants unify at about 10^{16} GeV (perhaps because of string loop threshold corrections), the gauge coupling constants should only be run from about 2×10^{16} GeV.

Of the Z_n orbifolds, only Z_3, Z_4, $Z_6 - I$ and Z_7 are able [55] to fit the quark and lepton masses. However, a number of possible effects have been neglected in these calculations. The effect of the tree level moduli dependent Kahler potential in normalising the matter states has not been included, nor have the twist dependent suppression factors if the Yukawa couplings are between excited twisted sector states, nor have the string loop threshold corrections to the Yukawa couplings from the one-loop Kahler potential.

In the absence of a definite model for the entries of the mass matrix deriving from the non-renormalisable superpotential terms, the Kobayashi–Maskawa mixing angles and phases cannot be determined. However, a simple model for M_d and M_u with vanishing (11), (13) and (31) entries and opposite phases for the Eqs. (23) and (32) entries can give mixing angles consistent with experiment together with an approximately maximal weak CP violating angle $\delta \approx 95°$.

4. Kähler potentials and string loop threshold corrections to gauge coupling constants

4.1. Introduction

A supergravity theory is specified by the superpotential, the Kähler potential and the gauge kinetic function. The light shed by orbifold compactifications of superstring theory on the form of the superpotential (especially the renormalisable terms) was the subject of the previous section. The Kähler potential and the gauge kinetic function, which yields the gauge coupling constants of the theory, will be studied in this section. A knowledge of the Kähler potential allows the normalisation of the states of the theory to be carried out and is also necessary for the construction of the effective potential. On the other hand, a knowledge of the gauge kinetic function is necessary to determine the values of the string loop corrected gauge coupling constants at the string scale, which, with the aid of the renormalisation group equations, can be compared with the measured low energy values. Like the Yukawa coupling, the Kähler potential and the gauge kinetic function both depend on the moduli of the orbifold discussed in Sections 3.7 and 3.8 and the values of the moduli are required before conclusions can be drawn. The determination of the moduli from the effective potential will be one of the topics discussed in the next section.

4.1.1. Modular properties of the Kähler potential

Associated with the ith complex plane of the underlying 6-torus of the orbifold, all abelian orbifolds have a modulus T_i defined in Eq. (3.80) by

$$\mathrm{i}T_i = 2(b_{2i-1,2i} + \mathrm{i}\sqrt{\{(\det g)_i\}}), \quad i = 1,2,3 , \tag{4.1}$$

where the matrices g and b are the metric and anit-symmetric tensor in the lattice basis, as in (3.81)–(3.82), and the determinant refers to the 2×2 matrix for the ith complex plane. When the point group acts as Z_2 in the ith complex plane there is also a U modulus, U_i, defined by

$$\mathrm{i}U_i = \frac{1}{g_{2i-1,2i-1}}(g_{2i,2i} + \mathrm{i}\sqrt{(\det g)_i}) . \tag{4.2}$$

On the other hand, when the point group acts as Z_N with $N \neq 2$ in the ith complex plane the modulus U_i is forced to take a fixed value and only T_i survives as a continuous modulus. Specific orbifolds possess additional T moduli but in what follows we shall focus on the moduli defined above. The T moduli may be thought of as continuously variable quantities corresponding to deformations of the underlying torus. Moduli may also be thought of as expectation values of scalar fields in the corresponding supergravity theory for which the effective potential is flat to all orders. Looked at this way, the existence of T and U moduli is equivalent to the existence of untwisted sector states of the string theory of the type $b^i_{-1/2}|0\rangle_R \tilde{\alpha}^{\bar{j}}_{-1}|0\rangle_L$ or $b^i_{-1/2}|0\rangle_R \tilde{\alpha}^{j}_{-1}|0\rangle_L$. In general, depending on the point group, the first type of state can exist for $i = j$ and for some choices of $i \neq j$. The second type of state is only permitted by point group invariance for $i = j$ and then only if the ith complex plane is a Z_2 plane (a plane in which the point group acts as Z_2). The states in Eqs. (4.1) and (4.2) are the states with $i = j$.

Orbifold compactifications of string theory are known to possess certain modular symmetries to all orders in string perturbation theory. Generically, these symmetries are transformations of the form

$$T_i \to (a_i\, T_i - \mathrm{i}b_i)/(\mathrm{i}c_i T_i + d_i) \tag{4.3}$$

and

$$U_i \to (a'_i U_i - \mathrm{i}b'_i)/(\mathrm{i}c'_i U_i + d'_i) \tag{4.4}$$

where a_i, b_i, c_i, d_i, a'_i, b'_i, c'_i and d'_i are integers,

$$a_i d_i - b_i c_i = 1 \tag{4.5}$$

and

$$a'_i d'_i - b'_i c'_i = 1\ . \tag{4.6}$$

These symmetries are thus PSL(2,Z) modular groups, referred to as target space modular symmetries if these is a need to distinguish them for the world sheet modular symmetries discussed in Section 2. In some cases, string loop corrections can restrict the symmetries to subgroups of PSL(2,Z), or, equivalently can restrict the allowed range of values of these integers, as we shall see later. Further subtleties are that beyond string tree level the dilaton field S can participate in the modular transformations, and that, if Wilson line moduli are present, these may also enter the modular symmetries.

We shall see in subsequent sections that at string tree level the Kähler potential takes the form

$$K = \hat{K} + \sum_\alpha |\phi_a|^2 \prod_{i=1}^{3} (T_i + \bar{T}_i)^{n^i_\alpha} + \cdots\ , \tag{4.7}$$

where

$$\hat{K} = -\ln(S + \bar{S}) - \sum_{i=1}^{3} \ln(T_i + \bar{T}_i)\ . \tag{4.8}$$

Here, only the diagonal T moduli, T_i, $i = 1,2,3$, have been retained, the U moduli have not been displayed, and K has been taken to quadratic order in the matter fields ϕ_α. The powers n^i_α are referred to as the modular weights of the matter fields.

In the absence of matter fields, the transformation of the Kähler potential under a modular transformation on T_i is

$$K \to K + \ln|\mathrm{i}c_i T_i + d_i|^2 , \tag{4.9}$$

which is a specific Kähler transformation. For

$$G = K + \ln|W|^2 , \tag{4.10}$$

where W is the superpotential, to be invariant under modular transformations, W must transform with modular weight -1, by which is meant

$$W \to W(\mathrm{i}c_i T_i + d_i)^{-1} . \tag{4.11}$$

Because of non-renormalisation theorems this must be true to arbitrary orders in perturbation theory. If the matter fields are now introduced, then, to retain the modular invariance of G, the matter fields must transform with modular weights n^i_α, by which is meant

$$\phi_\alpha \to \phi_\alpha(\mathrm{i}c_i T_i + d_i)^{n^i_\alpha} . \tag{4.12}$$

The modular properties of a Yukawa coupling

$$W = h_{\alpha\beta\gamma}(T_i)\phi_\alpha\phi_\beta\phi_\gamma \tag{4.13}$$

in the superpotential may then be deduced. For W to have modular weight -1 we must have

$$h_{\alpha\beta\gamma}(T_i) \to h_{\alpha\beta\gamma}(T_i)(\mathrm{i}c_i T_i + d_i)^{-(1+n^i_\alpha+n^i_\beta+n^i_\gamma)} . \tag{4.14}$$

4.2. Kähler potentials for moduli

There are several different approaches to deriving Kähler potentials from orbifold compactifications of string theory, including truncation of the corresponding 10-dimensional supergravity theory to four dimensions [188,92,23,93,94] identification of accidental symmetries of the string action which can then be applied to the supergravity action [66,67,58,59,69], and comparison of amplitudes calculated in the string theory and in the supergravity theory with the aid of the $N = 2$ superconformal algebra [81,25,26]. In this section, we shall present the second of these methods, and, very briefly, in a discussion of the dilaton Kähler potential, the first of these methods. In the next section, the last of these three methods will be used in a discussion of the matter field contribution to the Kähler potential. Any of these methods may be used to discuss the moduli and matter field Kähler potentials but it is useful here to present a different method in each section to illuminate different aspects of the origin of the form of Kähler potentials.

Employing the accidental symmetry approach [66,67], let consider first a complex plane of the underlying 6-torus for which the action of the point group is Z_N with $N \neq 2$. Then, there is associated with this complex plane only a T modulus ((1,1) modulus) and no U modulus ((1,2) modulus.) The background field term in the string action for this T modulus may be written as

$$S = \frac{1}{\pi}\int \mathrm{d}^2 z(T\partial_z X \partial_{\bar z}\bar X + \text{h.c.}) , \tag{4.15}$$

where the index i referring to the complex plane has been suppressed. This action possesses the "accidental" symmetries

$$X \to \Lambda X + C \;, \tag{4.16}$$

$$T \to T\Lambda^{-1}\bar{\Lambda}^{-1} \;, \tag{4.17}$$

where Λ and C are arbitrary complex numbers, and

$$T \to T + \mathrm{i}D \;, \tag{4.18}$$

where D is an arbitrary real number. These symmetries of the world sheet action must appear as symmetries of the low-energy effective action for the moduli. The most general Lagrangian compatible with these symmetries is

$$\mathscr{L} = k(T + \bar{T})^{-2}\partial_\mu T \partial^\mu \bar{T} \;, \tag{4.19}$$

where k is a constant. The constant may be fixed by comparing the four moduli amplitude calculated at tree level in the supergravity theory and the string theory with the result that

$$\mathscr{L} = (T + \bar{T})^{-2}\partial_\mu T \partial^\mu \bar{T} \tag{4.20}$$

which derives from the Kähler potential

$$K = -\ln(T + \bar{T}) \;. \tag{4.21}$$

If instead we consider a complex plane for which the action of the point group is Z_2, then there is both an associated T modulus and an associated U modulus. It is then convenient to introduce the metric $g_{\rho\sigma}$ and anti-symmetric tensor $b_{\rho\sigma}$ background fields on the (real) lattice basis. The corresponding background field term in the string action is

$$S = \frac{1}{\pi}\int \mathrm{d}^2 z \, F_{\rho\sigma}\partial_z \hat{X}^\rho \partial_{\bar{z}} \hat{X}^\sigma \;, \tag{4.22}$$

where $\rho,\sigma = 1,2$, $\hat{X}^\rho$ and the string degrees of freedom in the lattice basis, defined by

$$\hat{X}^\rho = e^\rho_r X^r \;, \tag{4.23}$$

where r refers to the real space basis,

$$e^\rho_r \equiv e^{*r}_\rho \tag{4.24}$$

are basis vectors of the dual lattice, and

$$F_{\rho\sigma} = g_{\rho\sigma} + b_{\rho\sigma} \;. \tag{4.25}$$

The T and U moduli for this complex plane are then defined by

$$T = T_1 + \mathrm{i}T_2 = 2(\sqrt{\det g} - \mathrm{i}b_{12}) \tag{4.26}$$

and

$$U = U_1 + \mathrm{i}U_2 = \frac{1}{g_{11}}(\sqrt{\det g} - \mathrm{i}g_{12}) \;, \tag{4.27}$$

or, consequently, the matrices g and b are given by

$$g = \frac{1}{2}\frac{\operatorname{Re} T}{\operatorname{Re} U}\begin{pmatrix} 1 & -\operatorname{Im} U \\ -\operatorname{Im} U & |U|^2 \end{pmatrix} \tag{4.28}$$

and

$$b = \frac{1}{2}\begin{pmatrix} 0 & -\operatorname{Im} T \\ \operatorname{Im} T & 0 \end{pmatrix}. \tag{4.29}$$

The string (world sheet) action has the "accidental" symmetries

$$\hat{X}^\rho \to M_{\rho\sigma}\hat{X}^\sigma + C^\rho , \tag{4.30}$$

$$F_{\rho\sigma} \to F_{\lambda\tau}(M^{-1})_{\lambda\rho}(M^{-1})_{\tau\sigma} , \tag{4.31}$$

and

$$F_{\rho\sigma} \to F_{\rho\sigma} + D_{\rho\sigma} , \tag{4.32}$$

where M is a real non-singular matrix, C^ρ are real constants and D is a real anti-symmetric matrix. Applying these symmetries to the low energy supergravity effective action for the moduli, the most general consistent form of Lagrangian is

$$\begin{aligned} \mathscr{L} &= -\tfrac{1}{2}\mathrm{Tr}((F + F^{\mathrm{T}})^{-1}\partial_\mu F^{\mathrm{T}}(F + F^{\mathrm{T}})^{-1}\partial^\mu F) \\ &= -\tfrac{1}{8}\mathrm{Tr}(g^{-1}\partial_\mu g g^{-1}\partial^\mu g - g^{-1}\partial_\mu b g^{-1}\partial^\mu b) , \end{aligned} \tag{4.33}$$

where the overall multiplication constant has been fixed by comparing the $ggbb$ amplitude in the low-energy supergravity theory and the string theory using the vertex operators coupled to the background fields g and b. Substituting for g and b in terms of the T and U moduli gives

$$\mathscr{L} = (T + \bar{T})^{-2}\partial_\mu T\partial^\mu \bar{T} + (U + \bar{U})^{-2}\partial_\mu U\partial^\mu \bar{U} \tag{4.34}$$

which derives from the Kähler potential

$$K = -\ln(T + \bar{T}) - \ln(U + \bar{U}) . \tag{4.35}$$

Another modulus, in the sense of a field with flat effective potential to all orders in the corresponding supergravity theory, is the dilation S. A simple way of deriving the Kähler potential for the dilaton field is by truncation to 4 dimensions of the 10-dimensional supergravity that is the effective field theory below the string scale. The supergravity multiplet of supergravity in 10-dimensions contains bosonic states which are the symmetric metric tensor g_{AB}, the antisymmetric tensor b_{AB} and the 10-dimensional dilaton scalar ϕ, where A and B range over the 10-dimensional space. The dilaton S for the 4-dimensional reduction of the 10-dimensional supergravity is constructed from the degrees of freedom of the 10-dimensional theory as

$$S = \sqrt{\Delta}\mathrm{e}^{\phi} + 3\sqrt{2}\mathrm{i}D \tag{4.36}$$

where

$$\Delta = \det g_{ij} \tag{4.37}$$

with i and j referring to the compact six-dimensional manifold and D being the dual of the $b_{\mu\nu}$ field, where μ and ν refer to four-dimensional space–time. The field D is given in terms of the field strength $h_{\mu\nu\rho}$ for $b_{\mu\nu}$ as

$$e^{2\phi}\Delta h_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma}\partial^{\sigma}D \ . \tag{4.38}$$

The kinetic term for S in the dimensionally reduced Lagrangian derives from the Kähler potential term

$$K = -\ln(S + \bar{S}) \ . \tag{4.39}$$

This is present not just in toroidal compactifications but also in the untwisted sector of any orbifold compactification, constructed in this approach by truncating the dimensionally reduced theory. This is done by retaining only singlets under the action of some finite subgroup of the rotation group SO(6) on the compact manifold designed to leave only an $N = 1$ supergravity in four dimensions [188,23,92–94].

4.3. Kähler potentials for untwisted matter fields

The method described in this section, which can be found in greater detail in the original literature, [81] relies on the fact that the $N = 2$ super Virasoro algebra for the left movers for a string theory with $N = 1$ space–time supersymmetry, relates the left mover vertex operators $\Psi^{\pm}$ for **27** and $\overline{\mathbf{27}}$ matter fields in the **10** of the SO(10) subgroup of E_6 (apart from $E_8 \times E'_8$ factors in the vertex operator) to other left mover vertex operators $\Phi^{\pm}$ in the same $N = 2$ chiral multiplets of this algebra. In general, the left mover vertex operators for moduli fields M associated with matter fields ϕ can be written as linear combinations of the vertex operators $\Phi^{\pm}$. Thus, the vertex operators for the (1,1) moduli denoted by M^a with associated **27**'s denoted by ϕ_α can be identified by

$$M^a \leftrightarrow U^{\alpha}_{a}\Phi^{+}_{\alpha} \tag{4.40}$$

for some coefficients U^{α}_{a}, and the vertex operators for the (1,2) moduli denoted by M^m with associated **27**'s denoted by ϕ_μ can be identified by

$$M^m \leftrightarrow U^{\mu}_{m}\Phi^{-}_{\mu} \tag{4.41}$$

for some coefficients U^{μ}_{m}. Let us define the Kähler metrics $g_{a\bar{b}}$ and $G_{\alpha\bar{\beta}}$ for (1,1) moduli and **27** matter fields by

$$g_{a\bar{b}} \equiv \partial^2 G/\partial M^a \partial \bar{M}^b = \partial^2 K/\partial M^a \partial \bar{M}^b \tag{4.42}$$

and

$$G_{\alpha\bar{\beta}} \equiv \partial^2 G/\partial\phi_\alpha \partial\bar{\phi}_\beta = \partial^2 K/\partial\phi_\alpha\partial\bar{\phi}_\beta \tag{4.43}$$

and similarly for the (1,2) moduli and $\overline{\mathbf{27}}$ matter fields. The two point functions for vertex operators $\Psi^{\pm}$ can be related to 2-point functions for vertex operators $\Phi^{\pm}$ with the result that the Kähler metrics are related by

$$g_{a\bar{b}} = U^{\alpha}_{a} G_{\alpha\bar{\beta}} \bar{U}^{\beta}_{b} \tag{4.44}$$

and similarly for (1,2) moduli. Thus, if the matrices U^{α}_{a} can be calculated relations can be found between moduli and matter field Kähler metrics.

A rather messy, but easily solved, differential equation involving the moduli metric matrix g and the matrix of coefficients U can be derived by first using the $N = 2$ super-Virasoro algebra to relate pure moduli amplitudes of the type $MM \to \bar{M}\bar{M}$ to pure matter field amplitudes of the type $\phi\phi \to \bar{\phi}\bar{\phi}$ and also to relate amplitudes of the type $M\phi \to \bar{M}\bar{\phi}$ to amplitudes of the type $\phi\phi \to \bar{\phi}\bar{\phi}$. In each case, because it is the vertex operators $\Phi^{\pm}$ and $\Psi^{\pm}$ that belong to $N = 2$ supermultiplets, the matrices U occur. In the second stage of the derivation, the various amplitudes are calculated from the corresponding supergravity theory as follows [81]. In the case of $MM \to \bar{M}\bar{M}$ and $M\phi \to \bar{M}\bar{\phi}$ amplitudes, there are contributions from sigma model interactions due to the non-minimal Kähler potential and from graviton exchange. In the case of $\phi\phi \to \bar{\phi}\bar{\phi}$ amplitudes, at leading order in the momenta, there are contributions from 4 scalar F terms, from gauge boson exchange and from corresponding D terms. The reason that the calculation will be able to determine the matrix U and so to determine the combinations of gauge singlet scalars that are moduli fields, is that the (defining) flatness of the effective potential with respect to moduli to all orders has been used to drop all moduli–moduli interactions other than sigma model interactions. The details of the calculation depend on the gauge group assumed. We shall assume for the moment that the gauge group is simply $E_6 \times E'_8$. When the gauge group is instead $E_6 \times E'_8 \times U^p(1)$ or $E_6 \times E'_8 \times \mathrm{SU}(3) \times U^p(1)$, there are extra gauge boson exchanges and corresponding D term contributions as well as F terms modified by modified Yukawa couplings that affect the $\phi\phi \to \bar{\phi}\bar{\phi}$ amplitudes.

Finally, the expressions at leading order in the momenta for the amplitudes derived from the supergravity theory are inserted in the string relations between amplitudes. In this way, after elimination of the terms containing the matter field Yukawa coupling coefficients between equations, a matrix equation involving the matrices g and U is arrived at, namely,

$$\partial_a(U^{\dagger}g^{-1}U\partial_{\bar{d}}(U^{-1}g(U^{\dagger})^{-1}))_{\alpha\gamma} = (U^{\dagger}\partial_a(g^{-1}\partial_{\bar{d}}g)(U^{\dagger})^{-1})_{a\gamma} + \frac{\kappa^2}{3}\partial_a\partial_{\bar{d}}(K_2 - K_1)\delta_{\alpha\gamma} \tag{4.45}$$

and a similar equation with a,d replaced by m,n, where ∂_a and $\partial_{\bar{a}}$ denote $\partial/\partial M^a$ and $\partial/\partial \bar{M}^a$, and the pure moduli term $\hat{K}$ in the Kähler potential has been decomposed in the form (proved possible in Ref. [81])

$$\hat{K} = K_1 + K_2 \tag{4.46}$$

with K_1 depending only on M^a and $\bar{M}^a$ and K_2 depending only on M^m and $\bar{M}^m$. Eq. (4.45) has the solution

$$U^{\alpha}_{a} = V^{\alpha}_{a}(M)\exp\tfrac{1}{6}\kappa^2(K_1 - K_2) \tag{4.47}$$

and

$$U^{\mu}_{m} = V^{\mu}_{m}(M)\exp\tfrac{1}{6}\kappa^2(K_2 - K_1)\ , \tag{4.48}$$

where $V^{\alpha}_{a}(M)$ and $V^{\mu}_{m}(M)$ are arbitrary holomorphic functions of the moduli. The occurrence of these arbitrary functions corresponds to the freedom to redefine the matter fields by taking linear combinations of the various **27**'s and linear combinations of the various $\overline{\mathbf{27}}$'s with coefficients that are functions only of the moduli but not their conjugates, in order to preserve the Kähler geometry.

The matter fields may be chosen in such a way as to replace the V matrices by identity matrices so that

$$U^{\alpha}_{a} = \delta^{\alpha}_{a} \exp \tfrac{1}{6}\kappa^2(K_1 - K_2) \tag{4.49}$$

and

$$U^{\mu}_{m} = \delta^{\mu}_{m} \exp \tfrac{1}{6}\kappa^2(K_2 - K_1)\ . \tag{4.50}$$

Knowing U, the connection between matter field and moduli Kähler metrics following from Eq. (4.44) is

$$G_{a\bar{b}} = g_{a\bar{b}} \exp \tfrac{1}{3}\kappa^2(K_2 - K_1) \tag{4.51}$$

and

$$G_{m\bar{n}} = g_{m\bar{n}} \exp \tfrac{1}{3}\kappa^2(K_1 - K_2)\ . \tag{4.52}$$

Returning to the equations derived from the string relations between amplitudes *before* elimination of the matter field Yukawa coupling coefficients between equations and utilising Eqs. (4.49)–(4.52) yields equations that relate the Kähler metric for the moduli to the matter field Yukawa couplings. Once these Yukawa couplings have been specified, the Kähler metric can be solved for in specific cases [81].

A more realistic case is obtained [81,25] if the gauge group is $E_6 \times U^p(1) \times E'_8$ or $E_6 \times \mathrm{SU}(3) \times U^p(1) \times E'_8$. If, for example, [25] we take the gauge group to be $E_6 \times \mathrm{SU}(3) \times E'_8$ then this corresponds to the Z_3 orbifold with standard embedding of the point group in the gauge degrees of freedom. In that case, the matter fields are in (**27**,**3**) of $E_6 \times \mathrm{SU}(3)$ and singlet under E'_8 and we denote the vertex operators for matter fields $\phi_{\alpha i}$ in the **10** of the SO(10) subgroup of E_6 by $\Psi_{\alpha i}$, where α is a global index labelling the various copies of (**27**,**3**) and $i = 1,2,3$ is an SU(3) index labelling the basis states of **3** of SU(3). (The free fermion factor in the vertex operator carrying the SO(10) quantum numbers is not displayed.) Associated with these matter fields are the $E_6 \times \mathrm{SU}(3)$ singlet scalars which are members of the same $N = 2$ chiral multiplets and whose vertex operators we denote by $\Phi_{\alpha i}$. In this case, there are only (1,1) moduli fields, denoted by M_{AI}, where A and I are both global indices. It is convenient to decompose the global index on the modulus field in this way to mirror the decomposition of the index αi on the corresponding matter field into a global index α and an SU(3) index i. The vertex operators for the (1,1) moduli are in general linear combinations which can be identified by

$$M_{AI} \leftrightarrow U^{\alpha i}_{AI} \gamma \Phi_{\alpha i}\ , \tag{4.53}$$

where the $U^{\alpha i}_{AI}$ are functions of the moduli and their conjugates. There is some arbitrariness in the definition of U because we can make a redefinition of the matter fields by taking a linear combination of the various (**27**,**3**)'s or by making a change of basis in the SU(3) space. Thus, new matter field vertex operators $\Psi'_{\alpha' i'}$ may be defined by

$$\Psi_{\alpha i} = R^{\alpha'}_{\alpha}(M) S^{i'}_{i}(M) \Psi'_{\alpha' i'} \tag{4.54}$$

where R and S are functions of the moduli but not their conjugates, in order to preserve the Kähler geometry, and S is unitary. Consequently, there is the freedom to replace $U^{\alpha i}_{AI}$ by $\tilde{U}^{\alpha' i'}_{AI}$ where

$$\tilde{U}^{\alpha' i'}_{AI} = U^{\alpha i}_{AI} R^{\alpha'}_{\alpha}(M) S^{i'}_{i}(M)\ . \tag{4.55}$$

The $N=2$ superconformal algebra now relates the moduli metric $g_{AI,\bar{B}\bar{J}}$ to the matter field metric $G_{\alpha i,\bar{\beta}\bar{j}}$ through

$$g_{AI,\bar{B}\bar{J}} = U^{\alpha i}_{AI} G_{\alpha\beta} \bar{U}^{\bar{\beta}\bar{i}}_{\bar{B}\bar{J}} , \tag{4.56}$$

where the unbroken SU(3) gauge symmetry has been used to block diagonalise G in the form

$$G_{\alpha i,\bar{\beta}\bar{j}} = G_{\alpha\bar{\beta}} \delta_{i\bar{j}} . \tag{4.57}$$

Equations involving g,G,U and the Yukawa coupling coefficients for matter fields may again be derived [25] by studying amplitudes for moduli and matter field with the following slight differences. Yukawa couplings have to be modified to take account of the SU(3) indices so that the corresponding superpotential terms are

$$W = \tfrac{1}{3} W_{\alpha\beta\gamma}(M) \varepsilon_{ijk} \phi_{\alpha i} \phi_{\beta j} \phi_{\gamma k} + \cdots . \tag{4.58}$$

The four scalar vertex contribution to the $\phi\phi \to \bar{\phi}\bar{\phi}$ amplitude is then modified by the modification of the F terms in the effective potential. In addition, the $\phi\phi \to \bar{\phi}\bar{\phi}$ amplitude is affected by SU(3) gauge boson exchanges and corresponding D terms. After elimination of the matter field Yukawa coupling coefficients between equations a matrix equation involving g and U is arrived at, which now takes the form

$$\begin{aligned} \partial_{AI}[U^{\dagger} g^{-1} U \partial_{\bar{D}\bar{L}} (U^{-1} g (U^{\dagger})^{-1})]_{\alpha i,\gamma k} &= [U^{\dagger} \partial_{AI} (g^{-1} \partial_{\bar{D}\bar{L}} g)(U^{\dagger})^{-1}]_{\alpha i,\gamma k} - \frac{\kappa^2}{3} g_{AI,\bar{D}\bar{L}} \delta_{\alpha\gamma} \delta_{ik} \\ &\quad - \frac{\kappa^2}{6} (\bar{U}\lambda_\rho U)^{\bar{\delta}\varepsilon}_{\bar{D}\bar{L},AI} (U^{-1})^{EM}_{\varepsilon\ell} g_{EM,\bar{F}\bar{N}} (\bar{U}^{-1})^{\bar{F}\bar{N}}_{\bar{\delta}\bar{\ell}} \delta_{\alpha\gamma} (\lambda_\rho)_{ki} , \end{aligned} \tag{4.59}$$

where the λ_ρ are the Gell–Mann matrices for SU(3),

$$(\bar{U}\lambda_\rho U)^{\bar{\delta}\varepsilon}_{\bar{D}\bar{L},AI} \equiv \bar{U}^{\bar{\delta}\bar{\ell}}_{\bar{D}\bar{L}} (\lambda_\rho)_{\bar{\ell}\,\bar{i}} U^{\varepsilon i}_{AI} , \tag{4.60}$$

and ∂_{AI} and $\partial_{\bar{A}\bar{I}}$ denote $\partial/\partial M_{AI}$ and $\partial/\partial \bar{M}_{AI}$. To obtain the solution, we also require one of the 2 original equations derived from the matter field and moduli amplitudes using the $N=2$ superconformal algebra for left movers, which may be taken to be

$$\begin{aligned} \kappa^{-2} R_{AI,\bar{C}\bar{K},BJ,\bar{D}\bar{L}} &= g_{AI,\bar{D}\bar{L}} g_{BJ,\bar{C}\bar{K}} + g_{AI,\bar{C}\bar{K}} g_{BJ,\bar{D}\bar{L}} - \tfrac{1}{3} \exp(\kappa^2 \hat{K}) g^{\bar{F}\bar{N},EM} (WUUU)^{ijg}_{AI,BJ,EM} \\ &\quad \times (\bar{W}\bar{U}\bar{U}\bar{U})_{\bar{C}\bar{K},\bar{D}\bar{L},\bar{F}\bar{N}} \varepsilon_{ijh} \varepsilon_{k\ell h} , \end{aligned} \tag{4.61}$$

where

$$(WUUU)^{ijg}_{AI,BJ,EM} \equiv W_{\alpha\beta\varepsilon} U^{\alpha i}_{AI} U^{\beta j}_{BJ} U^{\varepsilon g}_{EM} \tag{4.62}$$

and the Riemann tensor of the Kähler geometry is given by

$$R_{AI,\bar{C}\bar{K},BJ,\bar{D}\bar{L}} = \partial_{AI} \partial_{\bar{D}\bar{L}} g_{BJ,\bar{C}\bar{K}} - \partial_{AI} g_{BJ,\bar{E}\bar{M}} g^{\bar{E}\bar{M},FN} \partial_{\bar{D}\bar{L}} g_{FN,\bar{C}\bar{K}} . \tag{4.63}$$

Eqs. (4.59) and (4.63) have the solution

$$\hat{K} = -\kappa^{-2} \ln \det B , \tag{4.64}$$

where $\hat{K}$ is the pure moduli term in the Kähler potential,

$$B_{AI} = M_{AI} + \bar{M}_{IA} \tag{4.65}$$

or, as a matrix,

$$B = M + M^{\dagger} \tag{4.66}$$

and

$$U^{\alpha i}_{AI} = X_{A\alpha} Y_{Ii} \tag{4.67}$$

with

$$Y = B^{-1/2} V \ , \tag{4.68}$$

where V is an arbitrary matrix,

$$W_{\alpha\beta\gamma} = w \varepsilon_{\alpha\beta\gamma} \tag{4.69}$$

and

$$\det(XX^{\dagger}) = \frac{1}{2|w|^2} \tag{4.70}$$

with X a function of M but not of $\bar{M}$. The degree of arbitrariness occuring in the solution is consistent with Eq. (4.54). If we make the choice

$$\sqrt{2}X = V = I \ , \tag{4.71}$$

then the solution for U simplifies to

$$U^{\alpha i}_{AI} = (1/\sqrt{2})\delta_{A\alpha}(B^{-1/2})_{Ii} \ . \tag{4.72}$$

It follows that the moduli and matter field metrics are

$$g_{AI,\bar{B}\bar{J}} = \kappa^2 B^{-1}_{I\bar{J}} B^{-1}_{\bar{B}A} \tag{4.73}$$

and

$$G_{AI,\bar{B}\bar{J}} = 2\kappa^2 B^{-1}_{\bar{B}A} \delta_{I\bar{J}} \ . \tag{4.74}$$

If we retain only the diagonal moduli of Section 4.3, "switch off" the other moduli and write

$$T_i \equiv M_{ii}, \quad i = 1,2,3 \ , \tag{4.75}$$

then the moduli and corresponding matter field metrics simplify to

$$g_{i\bar{i}} = 2\kappa^2 (T_i + \bar{T}_i)^{-2} \tag{4.76}$$

and

$$G_{i\bar{i}} = 2\kappa^2 (T_i + \bar{T}_i)^{-1} \ . \tag{4.77}$$

In terms of redefined matter fields

$$\hat{\phi}_i \equiv \sqrt{2}\phi_{ii} \tag{4.78}$$

the Kähler potential K to quadratic order in the matter fields is

$$K = -\kappa^2 \sum_i \ln(T_i + \bar{T}_i) + \kappa^2 \sum_i (T_i + \bar{T}_i)^{-1} |\hat{\phi}_i|^2 + \cdots \; . \tag{4.79}$$

For an orbifold possessing a complex plane where the point group acts as Z_2, so that there is both a T modulus and a U modulus associated with this complex plane, the situation is slightly more complicated. If the Z_2 complex plane is the jth complex plane then the corresponding contribution to the Kähler potential takes the form

$$K = -\ln[(T_j + \bar{T}_j)(U_j + \bar{U}_j) - (B_j + \bar{C}_j)(\bar{B}_j + C_j)] \; , \tag{4.80}$$

where B_j and C_j are two complex matter fields.

All of the above discussion assumes that there are no Wilson lines breaking the gauge symmetry whereas in practice this will be necessary if the gauge group is to be reduced to a subgroup of $E_6 \times \mathrm{SU}(3)$ as a suitable starting point for spontaneous symmetry breaking to the standard model. However, the Kähler potential of the moduli and certain of the matter fields in the theory with Wilson lines can be calculated from the corresponding terms in the Kähler potential in the underlying theory without Wilson lines as a consequence of two observations [81]. First, the amplitudes are the same for the states which survive the GSO projections as in the original theory, and, second, the relationships amongst vertices that follow from the $N = 2$ superconformal algebra are also unmodified. This means that the Kähler potential in the theory with Wilson lines may be derived by calculating in the theory without Wilson lines the Kähler potential of the moduli and matter fields associated with moduli that survive the GSO projections due to the Wilson lines.

4.4. *Kähler potentials for twisted sector matter fields*

In the previous section, the Kähler potential was derived for the moduli and the untwisted sector matter fields related to the moduli by the superconformal algebra. However, such methods can not be employed when, as occurs for matter fields in twisted sectors with Wilson lines, the matter fields are not related to moduli. Other methods are then required [129,27,28]. One approach [27,28] is to make a direct comparison of amplitudes in the string theory with amplitudes in the supergravity theory without the benefit of the superconformal algebra (in the spirit of the earliest papers [119,150] on the derivation of low energy supergravity from string theory.)

First notice that holomorphic redefinitions of the fields allow the moduli and matter field Kähler potentials and metrics to be written in a variety of forms [81]. For example, the Kähler potential

$$K = -\ln(T + \bar{T} - |\phi|^2) \; , \tag{4.81}$$

where ϕ is a matter field, may be written in the form

$$K = -\ln(1 - |\tilde{T}|^2 - |\tilde{\phi}|^2) \tag{4.82}$$

by the redefinition

$$T = \frac{1 - \tilde{T}}{1 + \tilde{T}}, \qquad \phi = \frac{\sqrt{2}\tilde{\phi}}{1 + \tilde{T}} \; . \tag{4.83}$$

These are equivalent Kähler potentials because they differ only by $h + \bar{h}$ with

$$h = \ln\left(\frac{1 + \tilde{T}}{\sqrt{2}}\right). \tag{4.84}$$

In this new form, the Kähler potential may be expanded in powers of the moduli as well as in powers of the matter field. It is this form of Kähler potential that arises naturally in calculations of string amplitudes. In consequence, the Kähler potential at quadratic order in the matter fields (4.7) and (4.8) arises in the form

$$K = -\sum_{i-l}^{3} \ln(1 - |\tilde{T}_i|^2) + \sum_{\alpha} |\tilde{\phi}_\alpha|^2 \prod_i (1 - |\tilde{T}_i|^2)^{n_\alpha^i} \tag{4.85}$$

so that the matter field Kähler metric is

$$G_{\alpha\bar{\beta}}(\tilde{T}_i, \bar{\tilde{T}}_i) = \delta_{\alpha\bar{\beta}} \prod_i (1 - |\tilde{T}_i|^2)^{n_\alpha^i}\ . \tag{4.86}$$

Information about the matter field metric may be derived from the two moduli – two matter field amplitude of Fig. 6. With zero moduli expectation values and to quadratic order in the momenta the supergravity amplitude is given by

$$A(\tilde{T}_i, \tilde{\phi}_\alpha, \bar{\tilde{\phi}}_\beta, \bar{\tilde{T}}_j) = \tfrac{1}{32}\left(\tfrac{us}{t}\delta_{i\bar{j}}\delta_{\alpha\bar{\beta}} + sG_{\alpha\bar{\beta},i\bar{j}}(0,0)\right), \tag{4.87}$$

where the indices i and $\bar{j}$ on $G_{\alpha\bar{\beta},i\bar{j}}$ denote derivatives with respect to $\tilde{T}_i$ and $\bar{\tilde{T}}_j$, and s, t and u are the usual Mandelstam variables.

$$s = -(k_1 + k_2)^2, \qquad t = -(k_1 + k_4)^2, \qquad u = -(k_1 + k_3)^2\ . \tag{4.88}$$

A string theory calculation of this amplitude determines the matter field metric to quadratic order in the moduli (expectation values).

$$G_{\alpha\bar{\beta}}(\tilde{T}_i, \bar{\tilde{T}}_i) = \delta_{\alpha\bar{\beta}} + G_{\alpha\bar{\beta},i\bar{j}}(0,0)\tilde{T}_i \bar{\tilde{T}} j + \cdots\ . \tag{4.89}$$

Once this is known to quadratic order the values of the modular weights n_α^i are obtained by comparison with Eq. (4.86). Explicit expressions for the modular weights in terms of the powers of oscillators involved in the construction of the twisted sector matter states may be found in Refs. [129,28].

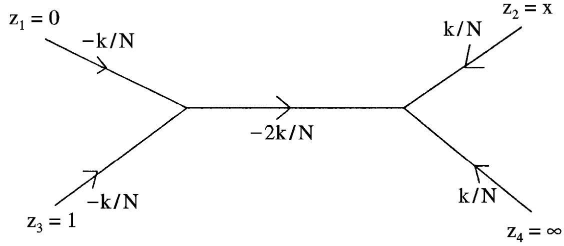


Fig. 6. Two moduli – two twisted matter field scattering amplitude.

These expressions allow all possible values of matter field modular weights for a specific orbifold (with arbitrary choices of point group embedding and Wilson lines) to be determined. In general, for a massless left mover the oscillator number N_{L} is given by

$$N_{\mathrm{L}} = a_{\mathrm{L}} - h_{\mathrm{KM}} \,, \tag{4.90}$$

where a_{L} is the normal ordering constant for the particular orbifold twisted sector and h_{KM} is the contribution to the conformal weight of the state from the $E_8 \times E_8'$ algebra. For level 1 gauge group factors it is given by

$$h_{\mathrm{KM}} = \sum_a \frac{\dim G_a}{\dim R_a} \frac{T(R_a)}{(C(G_a) + 1)} \,, \tag{4.91}$$

where $C(G_a)$ is the quadratic Casimir for the adjoint representation of G_a and $T(R_a)$ is the quadratic Casimir for the representation R_a of G_a to which the state belongs

$$T(R_a) = \mathrm{Tr}\, Q_a^2 \,, \tag{4.92}$$

where Q_a is any generator of G_a in the representation R_a. For a specific gauge group e.g. $\mathrm{SU}(3) \times \mathrm{SU}(2) \times U(1)$ of the standard model, flipped $\mathrm{SU}(5) \times U(1)$, $[\mathrm{SU}(3)]^3$ or $\mathrm{SO}(6) \times \mathrm{SO}(4)$ and chosen representations for the matter fields, we should use Eq. (4.91) to set a lower bound on h_{KM} for each matter field to allow for the possibility of additional contributions to h_{KM} from any extra $U(1)$ factors in the gauge group which are spontaneously broken along flat directions at a large energy scale, as frequently happens in orbifold theories. In this way, it is possible to derive the allowed range of modular weights [129,28] for the various twisted sectors of the Z_N and $Z_M \times Z_N$ orbifolds for a specific gauge group and matter field representations. This knowledge is useful in studying string loop threshold corrections to gauge coupling constants, as we see later.

4.5. String loop threshold corrections to gauge coupling constants

It is possible to derive effective low energy theories by integrating out the fields with masses above a chosen scale to leave a theory containing only fields with masses below this scale which can be employed in low energy calculations. [185]. So far as gauge coupling constants are concerned this means that renormalisation group equations may be run from the chosen scale to any lower energy with the coefficients in the renormalisation group equations calculated using only the light states provided a threshold correction is made to the gauge coupling constants at the chosen scale which contains the contributions from the heavy states. In the case of heterotic string theory, the gauge coupling constant $g_a(\mu)$ at energy scale μ is related to the string scale coupling constant g_{STRING} by

$$16\pi^2 g_a^{-2}(\mu) = 16\pi^2 k_a g_{\mathrm{STRING}}^{-2} + b_a \ln\left(\frac{M_{\mathrm{STRING}}^2}{\mu^2}\right) + \Delta_a \,, \tag{4.93}$$

where k_a is the level of the gauge group factor G_a,

$$M_{\mathrm{STRING}} \approx 0.53 g_{\mathrm{STRING}} \times 10^{18}\,\mathrm{GeV} \tag{4.94}$$

and

$$g_{\mathrm{STRING}} \approx 0.7 \tag{4.95}$$

is the common value of the gauge coupling constants [109,136] at the string tree level unification scale M_{STRING}. We shall usually assume that all gauge group factors have level 1 (with the $U(1)$ factors suitably normalised.) The threshold correction Δ_a has been derived in terms of the complete spectrum of states for any four-dimensional heterotic string theory that is tachyon free [136]. It is given by

$$\Delta_a = \int_\Gamma \frac{d^2\tau}{\tau_2}(B_a(\tau,\bar{\tau}) - b_a) , \tag{4.96}$$

where, for convenience, we are denoting the modular parameter $\tilde{\tau}$ of Section 2 by τ,

$$\tau = \tau_1 + i\tau_2 \tag{4.97}$$

and Γ as the fundamental domain,

$$\Gamma\colon -\tfrac{1}{2} \le \tau_1 \le \tfrac{1}{2}, \quad \tau_2 \ge 0, \ |\tau| \ge 1. \tag{4.98}$$

In Eq. (4.96),

$$B_a(\tau,\bar{\tau}) = |\eta(-i\tau)|^{-4} \sum_{(s_1,s_2)\neq(1,1)} \frac{(-1)^{s_1+s_2}}{2\pi i}\frac{dZ_\psi(s_1 s_2,\bar{\tau})}{d\bar{\tau}} \mathrm{Tr}_{s_1}(Q_a^2(-1)^{s_2 N_F} q^{H_L}\bar{q}^{H_R}) , \tag{4.99}$$

where q and $\bar{q}$ are as in Eq. (2.26), $\eta(\tau)$ is the Dedekind η function,

$$\eta(-i\tau) = q^{1/12} \prod_{n=1}^{\infty} (1 - q^{2n}) \tag{4.100}$$

and Z_ψ is the light cone gauge partition function for a single complex free fermion with s_1 and s_2 taking the values 0 and 1 for NS or R boundary conditions for the two directions on the world sheet torus; the trace is over the internal degrees of freedom i.e. all degrees of freedom other than those of four-dimensional space–time. The charge Q_a is any generator of the factor G_a of the gauge group, and N_F is the "fermion number". Explicitly,

$$\begin{aligned} Z_\psi(s_1,s_2,\bar{\tau}) &= \bar{q}^{-1/12} \prod_{n=1}^{\infty} (1 + \bar{q}^{2n-1})^2, \quad (s_1,s_2) = (0,0) \\ &= \bar{q}^{-1/12} \prod_{n=1}^{\infty} (1 - \bar{q}^{2n-1})^2, \quad (s_1,s_2) = (0,1) \\ &= 2\bar{q}^{1/6} \prod_{n=1}^{\infty} (1 + \bar{q}^{2n})^2, \quad (s_1,s_2) = (1,0) \\ &= 0, \quad (s_1 s_2) = (1,1). \end{aligned} \tag{4.101}$$

Specialising to the case of an abelian orbifold, with point group G [82], the trace can be written as a sum over twisted sectors (h,g). Then, the trace factor in Eq. (4.99) is

$$\mathrm{Tr}_{s_1}(Q_a^2(-1)^{s_2 N_F} q^{H_L}\bar{q}^{H_R}) = \frac{1}{|G|}\sum_{h,g\varepsilon G} \mathrm{Tr}_{(h,s_1)}(Q_a^2 g(-1)^{s_2 N_F(h)} q^{H_L(h)}\bar{q}^{H_R(h)}) \tag{4.102}$$

which is just the orbifold partition function with the insertion of Q_a^2.

For an orbifold theory with $N = 1$ space–time supersymmetry the point group must be a finite subgroup of the SU(3) group which is a subgroup of the SO(6) acting on the compact manifold degrees of freedom [80]. Any element of such a group either rotates all but one of the three complex planes for the compact manifold, rotates all 3 complex planes or rotates none of the complex planes. The corresponding twisted sectors are then referred to as $N = 2$, $N = 1$ or $N = 4$ sectors, respectively.

For an $N = 1$ sector the boundary conditions do not allow any momentum or winding number associated with the compact manifold. As we shall see later, the moduli enter the Hamiltonian through the left and right mover momenta (or, equivalently, through the momenta and winding numbers) for the compact manifold, and so there can be no dependence of the threshold correction on the moduli for an $N = 1$ sector.

When there is at least one complex plane unrotated by h, in general there is a moduli dependent threshold correction for the h twisted sected. We must then ask what is the effect of g on the pair of boundary conditions (h,g) for the world sheet torus. The answer is that g must leave the same complex plane unrotated as h does if there is to be moduli dependence because the trace projects out states with non-trivial winding numbers or momenta if g rotates the complex plane. In the special case when h is the identity (the $N = 4$ sector), if g is also the identity then there is no contribution to the threshold correction. This is because the $(h = I,\ g = I)$ sector is a self-contained $N = 4$ supersymmetric theory and both the renormalisation group coefficients and the 1 loop threshold corrections vanish in such a theory.

Thus, the moduli dependent threshold corrections come from (h,g) twisted sectors where h leaves a single complex plane unrotated (the $N = 2$ sectors) and in addition g leaves the same complex plane unrotated [82]. Moduli dependence in threshold corrections is important because, as we shall see later, it provides a possible mechanism to move the unification scale for gauge coupling constants down from the tree level string scale to the lower scale "observed" empirically [2,90].

4.6. Evaluation of string loop threshold corrections

The first step in evaluating the moduli dependent part of the threshold correction Δ_a is the observation that the contribution to the threshold correction from a twisted sector with a fixed plane (an $N = 2$ sector) can be factorised in the form [82]

$$B_a(\tau,\bar{\tau}) = Z_{\mathrm{TORUS}}(\tau,\bar{\tau})C_a(\tau)\ , \tag{4.103}$$

where Z_{TORUS} is the zero-mode partition function for the 2-dimensional torroidal compactification corresponding to the fixed plane and the holomorphic function $C_a(\tau)$ is the contribution from all other string degrees of freedom. Because $\tau_2 Z_{\mathrm{TORUS}}$ is modular invariant and also $\tau_2((B_a(\tau,\bar{\tau})/k_a) - B_b(\tau,\bar{\tau})/k_b)$ for two different factors G_a and G_b of the gauge group is also known to be modular invariant, it may be concluded that $C_a(\tau)/k_a - (C_b(\tau)/k_b)$ is also modular invariant. The theory of modular forms then requires this function to be a constant which must equal $b_a/k_a - (b_b/k_b)$ by taking the limit of Z_{TORUS} and B_a, B_b for $\tau \to \mathrm{i}\,\infty$, and noting that

$$\lim_{\tau\to i\infty} Z_{\mathrm{TORUS}} = 1 \tag{4.104}$$

and, as shown in Ref. [51], that

$$\lim_{\tau \to i\infty} B_a(\tau,\bar{\tau}) = b_a \, . \tag{4.105}$$

Thus, we are able to conclude that

$$\frac{B_a(\tau,\bar{\tau})}{k_a} - \frac{B_b(\tau,\bar{\tau})}{k_b} = Z_{\mathrm{TORUS}}\left(\frac{b_a}{k_a} - \frac{b_b}{k_b}\right) \tag{4.106}$$

so that

$$\Delta_a = b_a \int_\Gamma \frac{d^2\tau}{\tau_2}(Z_{\mathrm{TORUS}}(\tau,\bar{\tau}) - 1) \tag{4.107}$$

with the understanding that the formula is only to be applied to the difference $\Delta_a/k_a - \Delta_a/k_b$.

The problem of evaluating the contribution to Δ_a from a particular $N = 2$ twisted sector thus reduces to the evaluation of Z_{TORUS} in the fixed plane for this sector [82]. To calculate this quantity it is necessary to express the left and right mover Hamiltonians H_{L} and H_{R} in terms of windings and momenta in this fixed plane, to which the two-dimensional toroidal compactification corresponds. The windings and momenta enter the right and left mover mode expansions through

$$X^r_{\mathrm{R}}(t - \sigma) = x^r_{\mathrm{R}} + p^r_{\mathrm{R}}(t - \sigma) + \text{oscillator terms} \, , \tag{4.108}$$

and

$$X^r_{\mathrm{L}}(t + \sigma) = x^r_{\mathrm{L}} + p^r_{\mathrm{L}}(t + \sigma) + \text{oscillator terms} \, , \tag{4.109}$$

where

$$p^r_{\mathrm{R}} = \tfrac{1}{2}(p^r - 2L^r), \qquad p^r_{\mathrm{L}} = \tfrac{1}{2}(p^r + 2L^r) \tag{4.110}$$

with p^r and L^r the momenta and winding numbers, respectively, and r a real index for the space basis. (The world sheet variables are being denoted by (t,σ) rather than (τ,σ) to avoid confusion with the modular parameter τ.) In the conventions being used here

$$X^r = X^r_{\mathrm{R}}(t - \sigma) + X^r_{\mathrm{L}}(t + \sigma) \, . \tag{4.111}$$

In terms of the basis vectors e^r_ρ, $\rho = 1, \dots, 6$, of the lattice for the 6 torus and with the "radii" absorbed into the definition of the basis vectors,

$$L^r = \sum_\rho m^\rho e^r_\rho \tag{4.112}$$

where m^ρ are integers. For convenience, we are using basis vectors here that are smaller by than those used in Section 3 by a factor of 2π.

Symmetric and anti-symmetric background fields G_{rs} and B_{rs} are introduced in the world sheet action S through the term

$$S = -\frac{1}{2\pi}\int_0^\pi d\sigma \int dt[G_{rs}\partial_\alpha X^r \partial_\alpha X^s + \varepsilon^{\alpha\beta} B_{rs} \partial_\alpha X^r \partial_\beta X^s] \, . \tag{4.113}$$

In the presence of the background fields, the conjugate momentum which is quantized on the dual lattice with basis vectors

$$e^{*r}_{\rho} \equiv e^{\rho}_{r} \tag{4.114}$$

is

$$\tilde{p}_r = G_{rs}p^s + 2B_{rs}L^s = \sum_{\rho} n_{\rho}e^{\rho}_{r} \tag{4.115}$$

where n_ρ are integers. In terms of $\tilde{p}_r$ and the windings L^r, the momentum p^r is given by

$$p^r = G^{rs}\tilde{p}_s - 2G^{rs}B_{st}L^t \ , \tag{4.116}$$

where G^{rs} is the inverse of G_{rs}.

It will be convenient to write all quantities in the lattice basis in which the string degrees of freedom are

$$\hat{X}^{\rho} \equiv e^{\rho}_{r}X^r \ . \tag{4.117}$$

Then, we define

$$b_{\rho\sigma} \equiv e^{r}_{\rho}B_{rs}e^{s}_{\sigma} \ , \tag{4.118}$$

$$g_{\rho\sigma} \equiv e^{r}_{\rho}G_{rs}e^{s}_{\sigma} \ , \tag{4.119}$$

$$p_{\rho R} \equiv g_{\rho\sigma}p^{\sigma}_{\mathrm{R}} \tag{4.120}$$

and

$$p_{\rho \mathrm{L}} \equiv g_{\rho\sigma}p^{\sigma}_{\mathrm{L}} \ , \tag{4.121}$$

where p^{ρ}_{R} and p^{ρ}_{L} are the coefficient of $t-\sigma$ and $t+\sigma$ in $\hat{X}^{\rho}_{\mathrm{R}}$ and $\hat{X}^{\rho}_{\mathrm{L}}$, respectively. In terms of the background fields

$$p_{\rho \mathrm{R}} = \tfrac{1}{2}n_{\rho} - g_{\rho\sigma}m^{\sigma} - b_{\rho\sigma}m^{\sigma} \tag{4.122}$$

and

$$p_{\rho \mathrm{L}} = \tfrac{1}{2}n_{\rho} + g_{\rho\sigma}m^{\sigma} - b_{\rho\sigma}m^{\sigma} \tag{4.123}$$

which may be written succinctly as

$$p_{\mathrm{R}} = \tfrac{1}{2}n - (g+b)m \tag{4.124}$$

and

$$p_{\mathrm{L}} = \tfrac{1}{2}n = (g-b)m \ . \tag{4.125}$$

The Hamiltonian is

$$H = H_{\mathrm{R}} + H_{\mathrm{L}} \tag{4.126}$$

with

$$H_{\mathrm{R}} = \tfrac{1}{2}p_{\rho \mathrm{R}}g^{\rho\sigma}p_{\sigma \mathrm{R}} \equiv \tfrac{1}{2}p^{\mathrm{T}}_{\mathrm{R}}g^{-1}p_{\mathrm{R}} \tag{4.127}$$

and

$$H_{\mathrm{L}} = \tfrac{1}{2} p_{\rho \mathrm{L}} g^{\rho\sigma} p_{\sigma \mathrm{L}} \equiv \tfrac{1}{2} p_{\mathrm{L}}^{\mathrm{T}} g^{-1} p_{\mathrm{L}} \tag{4.128}$$

and the world sheet momentum is

$$P = H_{\mathrm{L}} - H_{\mathrm{R}} \ . \tag{4.129}$$

The zero-mode partition function Z for the 6 torus

$$Z = \sum_{p_{\mathrm{R}}, p_{\mathrm{L}}} q^{H_{\mathrm{L}}} \bar{q}^{H_{\mathrm{R}}} \tag{4.130}$$

with

$$q = \mathrm{e}^{\mathrm{i}\pi\tau}, \qquad \bar{q} = \mathrm{e}^{-\mathrm{i}\pi\bar{\tau}} \tag{4.131}$$

may now be written in the form

$$Z = \sum_{n,m} \mathrm{e}^{2\pi \mathrm{i} \tau n^{\mathrm{T}} m} \times \exp(-\pi\tau_2[\tfrac{1}{2} n^{\mathrm{T}} g^{-1} n - 2n^{\mathrm{T}} g^{-1} b m + 2m^{\mathrm{T}}(g - b g^{-1} b) m - 2n^{\mathrm{T}} m]) \tag{4.132}$$

with

$$\tau = \tau_1 + \mathrm{i}\tau_2 \ . \tag{4.133}$$

What we require is the partition function Z_{TORUS} for the 2-dimensional toroidol compactification corresponding to the fixed plane of an $N = 2$ twisted sector. Choosing the labelling of the complex planes such that it is the first complex plane that is the fixed plane, we should then take m and n of the form

$$m = \begin{pmatrix} m^1 \\ m^2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad n = \begin{pmatrix} n_1 \\ n_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.134}$$

The zero-mode partition function Z_{TORUS} may then be cast in terms of m^1,m^2,n_1,n_2,b_{12},g_{11},g_{12} and g_{22} as

$$Z_{\mathrm{TORUS}} = \sum_{n,m} \mathrm{e}^{2\pi \mathrm{i}\tau(m^1 n_1 + m^2 n_2)} \exp\left(\frac{-\pi\tau_2}{T_1 U_1} | - T U m^2 + \mathrm{i} T m^1 - \mathrm{i} U n_1 + n_2|^2 \right), \tag{4.135}$$

where the moduli T and U associated with the $N = 2$ complex plane are defined as in Eqs. (4.26) and (4.27).

Returning to Eq. (4.107) and performing the τ integration, as described in detail in Ref. [82], gives the contribution to the threshold correction from this $N = 2$ twisted sector

$$\Delta_a = -b_a[\ln((T + \bar{T})|\eta(T)|^4) + \ln((U + \bar{U})|\eta(U)|^4)] + (\text{moduli independent constant}) \ , \tag{4.136}$$

where the Dedekind eta function is

$$\eta(T) = \mathrm{e}^{-\pi T/12} \prod_{n=1}^{\infty} (1 - \mathrm{e}^{-2\pi n T}) \,. \tag{4.137}$$

The string loop threshold correction (4.136) can be seen to be invariant under the (target space) modular transformation

$$T \to (aT - \mathrm{i}b)/(\mathrm{i}cT + d) \tag{4.138}$$

corresponding to Eq. (4.3) by observing that under this transformation

$$T + \bar{T} \to (T + \bar{T})/|\mathrm{i}cT + d|^2 \tag{4.139}$$

and

$$\eta(T) \to (\mathrm{i}cT + d)^{1/2}\eta(T) \tag{4.140}$$

and similarly for the U dependent term.

The complete threshold correction to the gauge coupling constant may be obtained as follows [82]. If the ith complex plane is left unrotated by a subgroup G_i of the point group G, then T^6/G_i is an orbifold with $N = 2$ space–time supersymmetry. The threshold correction Δ_a for the original orbifold T^6/G may be written as

$$\Delta_a = -\sum_i \frac{(b_a^{N=2})^i |G_i|}{|G|} \begin{pmatrix} \ln((T_i + \bar{T}_i)|\eta(T_i)|^4) \\ + \ln((U_i + \bar{U}_i)|\eta(U_i)|^4) \end{pmatrix} + (\text{moduli independent terms}) \,, \tag{4.141}$$

where the sum over i is over the $N = 2$ complex planes i.e. the complex planes left unrotated in at least one twisted sector of the original orbifold and the moduli independent part of the threshold corrections contains the contribution of the $N = 1$ complex planes. Here, $(b_a^{N=2})^i$ is the renormalisation group coefficient for the $N = 2$ orbifold T^6/G_i. If the ith complex plane is a Z_M plane with $M \neq 2$ then U_i is not a (continuously variable) modulus but takes a fixed value, so that the U_i dependent term in Eq. (4.141) is just an additional constant term. To arrive at Eq. (4.141) it should be noticed that the complete set of $N = 2$ twisted sectors of the original orbifold T^6/G for which the ith complex plane is unrotated constitutes the twisted sectors of the $N = 2$ orbifold T^6/G_i and that the $N = 4$ untwisted sector does not contribute to the threshold correction nor to b_a. Thus, combining the contributions of all these $N = 2$ sectors of the original orbifold yields a coefficient which is the renomalisation group coefficient $(b_a^{N=2})^i$.

Although the derivation of the string loop threshold correction presented in this section is a one string loop order calculation it has been shown in an alternative approach using integrability conditions that there are no additional contributions from higher orders in string-perturbation theory [4,5].

4.7. Modular anomaly cancellation and threshold corrections to gauge coupling constants

The form of the moduli-dependent threshold corrections to gauge coupling constants can be partly understood in terms of cancellation of (target space) modular anomalies [71]. This approach also gives an alternative form for the numerical coefficient in the threshold correction which involves the modular weights of the light states and is often more useful in practice.

In the following discussion, we shall focus attention on the three moduli T_i with modular transformation as in Eq. (4.3). The transformation induced on the Kähler potential is a particular Kähler transformation as in Eq. (4.9) which we may write as

$$K \to K + h_i(T_i) + \bar{h}_i(\bar{T}_i) \tag{4.142}$$

with

$$h_i(T_i) = \ln(\mathrm{i}c_i T_i + d_i) \tag{4.143}$$

and rewriting Eq. (4.12) the transformation on the scalar matter fields is

$$\phi_\alpha \to \phi_\alpha \mathrm{e}^{n^i_\alpha h_i} , \tag{4.144}$$

where n^i_a is the modular weight of ϕ_α, with a corresponding transformation on the fermionic partners ψ_α of the scalar matter fields [44,71] and on the gauginos λ_a chosen to maintain modular invariance of the supergravity Lagrangian at classical level. However, this classical symmetry acting on chiral fermions is potentially broken at quantum level by anomalies due to triangle diagrams [44,71] with two gauge bosons plus a number of moduli as external legs and massless fermionic matter fields and gauginos as internal lines. The one-loop anomaly for the gauge group factor G_a is a variation of the Lagrangian of the form

$$\delta\mathscr{L} = \tfrac{1}{8}(\bar{C}_a)_i(h_i - \bar{h}_i)F^b_{\mu\nu}\tilde{F}^{\mu\nu}_b , \tag{4.145}$$

where $\tilde{F}^{\mu\nu}_b$ is the dual field strength and the real constants $(\tilde{C}_a)_i$ will be given shortly. This derives from the variation of a supersymmetric Lagrangian term of the form

$$\delta\mathscr{L}_{\mathrm{ANOMALOUS}} = \int \mathrm{d}^2\theta((\tilde{C}_a)_i h_i W^\alpha_b W_{b\alpha} + \text{h.c.}) , \tag{4.146}$$

where W^α_b is the field strength (spinor) superfield.

The coefficients $(\tilde{C}_a)_a$ are calculated from the interaction terms in the low energy supergravity theory that contribute to the anomaly triangle diagrams and these interaction terms are controlled by the Kähler potential. For the gauge group factor G_a, the resulting coefficient is

$$(\tilde{C}_a)^i = (b'_a)^i/8\pi^2 \tag{4.147}$$

with

$$(b'_a)^i = -C(G_a) + \sum_\alpha T(R^\alpha_a)(1 + 2n^i_\alpha) , \tag{4.148}$$

where $C(G_a)$ is the quadratic Casimir for the adjoint representation of G_a and

$$T(R^\alpha_a) = \mathrm{Tr}\, Q^2_a , \tag{4.149}$$

where Q_a is any generator of G_a in the representation R^α_a to which the matter field ϕ_α belongs.

In general, there can be two different contributions to the cancellation of the modular anomaly to restore modular invariance at the quantum level. The first of these contributions is generated by a Green–Schwarz-type mechanism which involves allowing the dilaton field S, which does not transform under modular transformations at tree level, to undergo a transformation of the form

$$S \to S - \sum_i \frac{\delta^i_{\mathrm{GS}}}{8\pi^2} h_i \tag{4.150}$$

at one string loop level for some real coefficients δ^i_{GS}. The tree-level gauge kinetic term

$$\mathscr{L}_{\mathrm{GK}} = \int d^2\theta (f_{bc} W^\alpha_b W_{c\alpha} + \mathrm{h.c.}) \tag{4.151}$$

with

$$f_{bc} = S\delta_{bc} \tag{4.152}$$

then transforms into $\mathscr{L}_{\mathrm{GK}} + \delta\mathscr{L}_{\mathrm{GK}}$ with

$$\delta\mathscr{L}_{\mathrm{GK}} = -\frac{\delta^i_{\mathrm{GS}}}{8\pi^2} \int d^2\theta (h_i W^\alpha_b W_{b\alpha} + \mathrm{h.c.}) \tag{4.153}$$

and this cancels a part of the anomaly that is the same for each factor of the gauge group. To maintain modular invariance of the Kähler potential, $\hat{K}$ of Eq. (4.8) must be modified at one string loop level to

$$\hat{K} = -\ln Y - \sum_i \ln(T_i + \bar{T}_i) \tag{4.154}$$

with

$$Y = S + \bar{S} - \sum_i \frac{\delta^i_{\mathrm{GS}}}{8\pi^2} \ln(T_i + \bar{T}_i) . \tag{4.155}$$

The remainder of the anomaly, which is not universal for all factors of the gauge group, will have to be cancelled by massive string mode contributions. Thus, the massive string mode contribution will have to cancel the variation

$$\delta\mathscr{L}_{\text{MASSLESS MODES}} = \frac{((b'_a)^i - \delta^i_{\mathrm{GS}})}{8\pi^2} \int d^2\theta (h_i W^\alpha_b W_{b\alpha} + \mathrm{h.c.}) . \tag{4.156}$$

At this point, the knowledge gained in the previous section (in particular Eqs. (4.141) and (4.140)) suggests that the appropriate Lagrangian terms whose variation cancels the remainder of the anomaly is

$$\mathscr{L}_{\text{MASSIVE MODES}} = -\frac{((b'_a)^i - \delta^i_{\mathrm{GS}})}{8\pi^2} \int d^2\theta (\ln(\eta(T_i))^2 W^\alpha_b W_{b\alpha} + \mathrm{h.c.}) . \tag{4.157}$$

This is a holomorphic term as would be expected for a term obtained from integrating out massive modes.

In general, the (non-holomorphic) anomalous massless mode contribution $\delta\mathscr{L}_{\text{ANOMALOUS}}$ is generated by a non-local Lagrangian term. However, if we focus on the $F^b_{\mu\nu}F^{\mu\nu}_b$ term, with a view to obtaining the string loop correction to the gauge coupling constant, then (for covariantly constant moduli) this part of $L_{\text{ANOMALOUS}}$ is the local Lagrangian term

$$\mathscr{L}_{\text{ANOMALOUS}} = \frac{(b'_a)^i}{64\pi^2}\ln(T_i + \bar{T}_i)F^b_{\mu\nu}F^{\mu\nu}_b + \cdots \ . \tag{4.158}$$

Noticing that, for any function ϕ,

$$\int d^2\theta(\phi W^\alpha_b W_{b\alpha} + \text{h.c}) = -\frac{(\phi+\bar{\phi})}{8}F^b_{\mu\nu}F^{\mu\nu}_b + \frac{(\phi-\bar{\phi})}{8}F^b_{\mu\nu}(\tilde{F}^{\mu\nu}_b) + \cdots \ . \tag{4.159}$$

and combining the $F^b_{\mu\nu}F^{\mu\nu}_b$ terms from Eqs. (4.151),(4.157) and (4.158), yields the string loop corrected gauge coupling constant g_a given by [71]

$$g_a^{-2} = \frac{Y}{2} - \sum_i \frac{((b'_a)^i - \delta^i_{\text{GS}})}{16\pi^2}\ln((T_i + \bar{T}_i)|\eta(T_i)|^4) \ , \tag{4.160}$$

where the running of the gauge coupling constants has been ignored. Including the field theoretic one loop running of the gauge coupling constants $g_a(\mu)$ at energy scale μ,

$$16\pi^2 g_a^{-2}(\mu) = 16\pi^2 g^{-2}_{\text{STRING}} + b_a\ln\left(\frac{M^2_{\text{STRING}}}{\mu^2}\right) + \Delta_a \tag{4.161}$$

for level 1 gauge group factors, where

$$g^{-2}_{\text{STRING}} = \tfrac{1}{2}Y \tag{4.162}$$

gives the (redefined) gauge coupling constant at the string scale excluding the threshold correction Δ_a, and

$$\Delta_a = -\sum_i ((b'_a)^i - \delta^i_{\text{GS}})\ln((T_i + \bar{T}_i)|\eta(T_i)|^4) \ . \tag{4.163}$$

The Green–Schwarz coefficients δ^i_{GS} may be determined by comparing the threshold correction (4.163) in the approach of this section with the threshold correction (4.141) in the approach of the previous section. We then see that

$$\delta^i_{\text{GS}} = (b'_a)^i - (b^{N=2}_a)^i|G_i|/|G| \ . \tag{4.164}$$

In general, the $N=2$ renormalisation group coefficient is given by

$$(b^{N=2}_a)^i = -2C(G_a) + 2\sum_i T(R^i_a) \ , \tag{4.165}$$

where the sum over i is a sum over matter $N=2$ hypermultipletes in representations R^i_a for the $N=2$ orbifold T^6/G_i. A special case is when there is a pure gauge hidden sector. Then,

$$(b'_a)^i = -C(G_a) = \tfrac{1}{3}b_a \tag{4.166}$$

and

$$\delta^i_{\text{GS}} = \tfrac{1}{3}b_a(1 - 2|G_i|/|G|) \ . \tag{4.167}$$

Table 5
Non-T^2+T^4 Coexter Z_N orbifolds. For the point group generator θ we display (ξ_1,ξ_2,ξ_3) such that the action of θ in the complex orthogonal space basis is $(e^{2\pi i\xi_1},e^{2\pi i\xi_2},e^{2\pi i\xi_3})$

Orbifold	Point group generator θ	Lattice
Z_4-a	$(1,1,-2)/4$	$SU(4)\times SU(4)$
Z_4-b	$(1,1,-2)/4$	$SU(4)\times SO(5)\times SU(2)$
Z_6-II-a	$(2,1,-3)/6$	$SU(6)\times SU(2)$
Z_6-II-b	$(2,1,-3)/6$	$SU(3)\times SO(8)$
Z_6-II-c	$(2,1,-3)/6$	$SU(3)\times SO(7)\times SU(2)$
Z_8-II-a	$(1,3,-4)/8$	$SU(2)\times SO(10)$
$Z_{12}-I-a$	$(1,-5,4)/12$	E_6

In particular, if the ith complex plane is a Z_2 plane, i.e. a plane where the point group acts as Z_2, then δ^i_{GS} is zero.

4.8. Threshold corrections with reduced modular symmetry

In Section 4.6, the assumption was made in the derivation of the threshold corrections to the gauge coupling constants that, whenever a twisted sector has a fixed plane, a decomposition of the 6-torus $T^6=T^2+T^4$ can be made with the fixed plane lying in T^2. When this assumption is not correct, which we shall refer to as the case of non T^2+T^4 orbifolds, the discussion can be generalised as follows [158,29,30]. We shall see that the resulting threshold corrections have modular symmetries that are subgroups of PSL(2,Z). Non T^2+T^4 Coxeter Z_N orbifolds are tabulated in Table 5.

Analogously to Eq. (4.107) we start from

$$\Delta_a=\sum_{(h,g)} b_a^{(h,g)}\int_\Gamma \frac{d^2\tau}{\tau_2} Z^{\mathrm{TORUS}}_{(h,g)}(\tau,\bar\tau)-b_a^{N=2}\int_\Gamma \frac{d^2\tau}{\tau_2}\,, \tag{4.168}$$

where only the twisted sectors (h,g) for which there is a complex plane of the 6 torus T^6 fixed by both h and g contribute i.e. sectors which are twisted sectors of an $N=2$ space–time supersymmetric theory. In Δ_a, $Z^{\mathrm{TORUS}}_{(h,g)}$ is the moduli-dependent part of the zero mode partition function for the 2 dimensional toroidal compactification corresponding to the fixed plane of the (h,g) twisted sector, $b_a^{(h,g)}$ is the contribution of the massless states in the (h,g) sector to the one-loop renormalisation group equation coefficient and $b_a^{N=2}$ is the contribution of all $N=2$ twisted sectors. Unlike the T^2+T^4 case, $b_a^{N=2}$ no longer factors out from the first term in Eq. (4.168) because $Z^{\mathrm{TORUS}}_{(h,g)}$ now depends on the particular twisted sector. It is convenient to write Δ_a in terms of a subset (h_0,g_0) of $N=2$ twisted sectors (referred to as the fundamental elements) with the integration over an enlarged region $\tilde\Gamma$ depending on (h_0,g_0). Then,

$$\Delta_a=\sum_{(h_0,g_0)} b_a^{(h_0,g_0)}\int_{\tilde\Gamma} \frac{d^2\tau}{\tau_2} Z^{\mathrm{TORUS}}_{(h_0,g_0)}(\tau,\bar\tau)-b_a^{N=2}\int_{\tilde\Gamma} \frac{d^2\tau}{\tau^2}\,. \tag{4.169}$$

Here, the single twisted sector (h_0,g_0) replaces a set of twisted sectors which can be obtained from it by applying those PSL(2,Z) transformations that generate the fundamental region $\tilde{\Gamma}$ of the world sheet modular symmetry group of $Z^{\text{TORUS}}_{(h_0,g_0)}$ from the fundamental region of PSL(2,Z). In general, $Z^{\text{TORUS}}_{(h_0,g_0)}$ is invariant under a congruence subgroup of PSL(2,Z) obtained by restricting the parameters a,b,c,d in the $PSL(2,Z)$ transformation

$$\tau \to (a\tau + b)/(c\tau + d) \ . \tag{4.170}$$

If we denote such groups by $\Gamma_0(n)$ defined by

$$c = 0 \ (\text{mod},n) \tag{4.171}$$

and $\Gamma^0(n)$ defined by

$$b = 0 \, (\text{mod},n) \tag{4.172}$$

then, for example, for $\Gamma_0(3)$,

$$\tilde{\Gamma} = \{I,S,ST,ST^2\}\Gamma \ , \tag{4.173}$$

where S and T are the PSL(2,Z) transformations

$$S{:}\tau \to 1/\tau \tag{4.174}$$

and

$$T{:}\tau \to \tau + 1 \tag{4.175}$$

To calculate $Z^{\text{TORUS}}_{(h_0,g_0)}$, for an orbifold with point group generated by θ we first write the action of θ on the basis vectors e^r_ρ of the lattice of the 6-torus as

$$\theta{:}e^r_\rho \to e^r_\sigma Q_{\sigma\rho} \ . \tag{4.176}$$

Then the action of θ on m and n of Eqs. (4.112) and (4.115) is

$$\theta{:}m \to m' = Qm \tag{4.177}$$

and

$$\theta{:}n \to n' = (Q^{\text{T}})^{-1}n \ . \tag{4.178}$$

For the θ^k twisted sector, the fixed plane assocated with $g_0 = \theta^k$ is determined by

$$Q^k m = m, \qquad ((Q^{\text{T}})^{-1})^k n = n \tag{4.179}$$

and m and n in the fixed plane are then parameterised by two integers. Using this form for m and n in Eq. (4.132), and introducing a metric $g_\perp$ and an anti-symmetric tensor $b_\perp$ for the two-dimensional sublattice of the fixed plane with the moduli T and U defined in terms of $g_\perp$ and $b_\perp$, the τ integrations may be performed to obtain an expression for $Z_{(I,g_0)}$. For all non-$T^2 + T^4$ orbifolds, it is found that all fundamental sectors can be generated from fundamental sectors of the form (I,g_0)

by applying world sheet modular transformations. The final result for the threshold correction is always of the form [158,29,30]

$$\Delta_a = -\sum_i ((b'_a)^i - \delta^i_{\mathrm{GS}})\left(\ln(T_i + \bar{T}_i) + \sum_m \frac{C_{im}}{2}\ln\left|\eta\left(\frac{T_i}{\ell_{im}}\right)\right|^4\right)$$
$$-\sum_i ((d'_a)^i - \tilde{\delta}^i_{\mathrm{GS}})\left(\ln(U_i + \bar{U}_i) + \sum_m \frac{\tilde{C}_{im}}{2}\ln\left|\eta\left(\frac{U_i}{\tilde{\ell}_{im}}\right)\right|^4\right), \tag{4.180}$$

where the sum over i is restricted to complex planes which are unrotated in at least one twisted sector ($N = 2$ complex planes), and for the U moduli is further restricted to complex planes for which the point group acts as Z_2. The coefficients $(d'_a)^i$ are defined analogously to Eq. (4.148) with the modular weights with respect to U moduli replacing modular weights with respect to T moduli, and the $\tilde{\delta}^i_{\mathrm{GS}}$ are the Green Schwarz parameters for the U_i modular transformations. The values of C_{im}, ℓ_{im}, $\tilde{C}_{im}$ and $\tilde{\ell}_{im}$ are given in Table 6 for the various non-$T^2 + T^4$ Coxeter Z_N orbifolds. In the case of $Z_6 - II - b$, the modulus U_3 is understood to be replaced by $U_3 - 2i$. The range over which m runs depends on the value of i but always

$$\sum_m C_{im} = \sum_m \tilde{C}_{im} = 2\ . \tag{4.181}$$

In Eq. (4.180), the coefficients $(b_a^{N=2})^i$ have been identified using Eq. (4.164).

The threshold correction Δ_a now has (target space) modular symmetries that are subgroups of PSL(2,Z), e.g. for the $Z_6 - II - a$ orbifold, the part of the threshold correction involving T_3 and U_3 has the form

$$\Delta_a = -((b'_a)^3 - \delta^3_{\mathrm{GS}})(\ln(T_3 + \bar{T}_3)|\eta(T_3)|^4(U_3 + \bar{U}_3)|\eta(U_3)|^4)$$
$$+\ln\left((T_3 + \bar{T}_3)\left|\eta\left(\frac{T_3}{3}\right)\right|^4(U_3 + \bar{U}_3)|\eta(3U_3)|^4\right), \tag{4.182}$$

Table 6
Values of C_{im}, ℓ_{im}, $\tilde{C}_{im}$ and $\tilde{\ell}_{im}$ for non-$T^2 + T^4$ Coxeter Z_N orbifolds

Orbifold	C_{im},$\tilde{C}_{im}$	ℓ_{im}, $\tilde{\ell}_{im}$
$Z_4 - a$	$C_{31} = 2$ $\tilde{C}_{31} = 2$	$\ell_{31} = 2$ $\tilde{\ell}_{31} = 1$
$Z_4 - b$	$C_{31} = C_{32} = 1$ $\tilde{C}_{31} = \tilde{C}_{32} = 1$	$\ell_{31} = 1$, $\ell_{32} = 2$ $\tilde{\ell}_{31} = 1$, $\tilde{\ell}_{32} = \frac{1}{2}$
$Z_6 - IIa$	$C_{11} = 2$, $C_{31} = C_{32} = 1$ $\tilde{C}_{31} = \tilde{C}_{32} = 1$	$\ell_{11} = 2$, $\ell_{31} = 1$, $\ell_{32} = 3$ $\tilde{\ell}_{31} = 1$, $\tilde{\ell}_{32} = \frac{1}{3}$
$Z_6 - IIb$	$C_{11} = 2$, $C_{31} = C_{32} = 1$ $\tilde{C}_{31} = \tilde{C}_{32} = 1$	$\ell_{11} = \ell_{31} = 1$, $\ell_{32} = 3$ $\tilde{\ell}_{31} = 3$, $\tilde{\ell}_{32} = 1$
$Z_6 - IIc$	$C_{11} = 2$, $C_{31} = C_{32} = 1$ $\tilde{C}_{31} = \tilde{C}_{32} = 1$	$\ell_{11} = \ell_{31} = 1$, $\ell_{32} = 3$ $\tilde{\ell}_{31} = 1$, $\tilde{\ell}_{32} = \frac{1}{3}$
$Z_8 - IIa$	$C_{31} = C_{32} = 1$ $\tilde{C}_{31} = \tilde{C}_{32} = 1$	$\ell_{31} = 1$, $\ell_{32} = 2$ $\tilde{\ell}_{31} = 1$, $\tilde{\ell}_{32} = \frac{1}{2}$
$Z_{12} - Ia$	$C_{31} = 2$	$\ell_{31} = 2$

which for modular transformations on T_3 is invariant under $\Gamma^0(3)$ and for modular transformations on U_3 is invariant under $\Gamma_0(3)$ with $\Gamma^0(3)$ and $\Gamma_0(3)$defined by imposing the conditions (4.171) or (4.172) in Eq. (4.3) or (4.4).

The modular symmetries of the threshold corrections for the non-T^2+T^4 case may also be determined without explicit calculation of the threshold corrections [31] by using a method [151,174,175] which explores the modular group that leaves invariant the spectrum of the twisted sectors. In the presence of discrete Wilson lines, knowledge of the explicit threshold corrections [160] is limited to the case without moduli but the modular symmetries may be determined by a generalisation of the above approach [87,174,31,153] both for the T^2+T^4 case and for the non-T^2+T^4 case. On the other hand, explicit calculations of the effect of Wilson line moduli on the threshold corrections are available [6,45,159].

4.9. Unification of gauge coupling constants

From Eq. (4.161), the running of the gauge coupling constants (assume level 1 gauge group factors) is given by

$$16\pi^2 g_a^{-2}(\mu) = 16\pi^2 g_{\mathrm{STRING}}^{-2} + b_a \ln\left(\frac{M_{\mathrm{STRING}}^2}{\mu^2}\right) + \Delta_a \tag{4.183}$$

with Δ_a given by Eq. (4.163) for T^2+T^4 orbifolds and by Eq. (4.180) to include non-T^2+T^4 orbifolds, and with

$$g_{\mathrm{STRING}} \approx 0.7 \tag{4.184}$$

and

$$M_{\mathrm{STRING}} \approx 0.53\ g_{\mathrm{STRING}} \times 10^{18}\,\mathrm{GeV}\ , \tag{4.185}$$

where g_{STRING} is the common value of the gauge coupling constants at the string tree level unification scale M_{STRING}.

If there are no additional states, over and above the (minimal supersymmetric) standard model states, with masses interemediate between the electroweak scale and the string scale, [7,32] then it may be necessary to explain the difference between the "observed" unification of gauge coupling constants at

$$M_X \approx 2\times 10^{16}\,\mathrm{GeV} \tag{4.186}$$

and tree-level unficiation scale M_{STRING} by the occurrence [130,129,33] of suitable moduli dependent threshold corrections Δ_a. (The moduli-independent part of the threshold correction is small [136,8,133].) If g_a and g_b are gauge coupling constants for two factors of the $\mathrm{SU}(3)\times\mathrm{SU}(2)\times U(1)$ standard model gauge group then

$$\frac{M_{\mathrm{STRING}}^2}{M_X^2} = \prod_i \alpha_i^{((b_a')^i-(b_b')^i)/b_a-b_b} \prod_j \tilde{\alpha}_j^{((d_a')^j-(d_b')^j)/b_a-b_b}\ , \tag{4.187}$$

where the product over i is over $N=2$ complex planes and the product over j is over $N=2$ complex planes for which the point group acts as Z_2,

$$\alpha_i = (T_i + \bar{T}_i)\prod_m \left|\eta\left(\frac{T_i}{\ell_{im}}\right)\right|^{2C_{im}} \tag{4.188}$$

and

$$\tilde{\alpha}_j = (U_j + \bar{U}_j)\prod_m \left|\eta\left(\frac{U_j}{\tilde{\ell}_{jm}}\right)\right|^{2\tilde{C}_{jm}} . \tag{4.189}$$

The coefficients $(b'_a)^i$ and $(d'_a)^j$ may be written in terms of the modular weights of 3 generations of quarks and lepton and the electroweak Higgses h and $\bar{h}$ in the supersymmetric standard model as

$$(b'_3)^i = 3 + \sum_{g=1}^{3} (2n^i_{Q(g)} + n^i_{u(g)} + n^i_{d(g)}) , \tag{4.190}$$

$$(b'_2)^i = 5 + n^i_h + n^i_{\bar{h}} + \sum_{g=1}^{3} (3n^i_{Q(g)} + n^i_{\mathrm{L}(g)}) \tag{4.191}$$

and

$$(b'_1)^i = \frac{33}{5} + \frac{3}{5}(n^i_h + n^i_{\bar{h}}) + \frac{1}{5}\sum_{g=1}^{3} (n^i_{Q(g)} + 8n^i_{u(g)} + 2n^i_{d(g)} + 3n^i_{\mathrm{L}(g)} + 6n^i_{e(g)}) \tag{4.192}$$

with similar expressions for $(d'_a)^j$ with modular weights with respect to U_j replacing modular weights with respect to T_i, where g labels the generations and $L(g)$ and $Q(g)$ are lepton and quark $\mathrm{SU_L}(2)$ doublets.

It is possible to generate all possible modular weights of the massless matter with quark, lepton and Higgs quantum numbers in the twisted sectors of an arbitrary Z_N or $Z_M \times Z_N$ orbifold with $\mathrm{SU}(3)\times\mathrm{SU}(2)\times U^p(1)$ gauge group (allowing for extra $U(1)$ factors to be spontaneously broken along flat directions at a high-energy scale), as discussed in Section 4.5. Then, under the simplifying assumption that a single T modulus is dominating the threshold corrections (either one of the T_i or $T = T_1 = T_2 = T_3$) the Z_N and $Z_M \times Z_N$ orbifolds that permit a unification solution with $M_X < M_{\mathrm{STRING}}$ can be identified [129]. For any particular choice of modular weights that permits such a solution the value of the dominant modulus to achieve unifications at 2×10^{16} GeV can then be calculated. In general, this results in values of the dominant modulus T_d that are unnaturally large in Planck scale units. (Re $T_d \sim 20$ is typical). Smaller values can be obtained in a variety of ways e.g. by including Wilson line moduli as well as T and U moduli in the threshold correction [169] or by assuming that unification at M_X occurs to a gauge group larger than the standard model [34,36] with the massless states of the supersummetric standard model below that scale. In this latter case, the renormalisation group coefficients b_a are those of the supersymmetric standard model but the threshold corrections, which are determined by the massless states at the string scale, are modified. For a review of other options see Dienes [72]. A more radical possibility is that the difficulty lies not in the running of the gauge coupling constants but in the gravitational constant which is modified by the appearance of a fifth dimension above a certain energy in M-theory [190].

5. The effective potential and supersymmetry breaking

5.1. Introduction

We have seen in earlier sections that the moduli dependence of the Yukawa couplings, the gauge kinetic functions, and the Kähler potential can, in principle, account for many otherwise puzzling features of low-energy phenomenology, such as the hierarchy of fermion masses and the "precocious" unification of the observed gauge couplings at an energy scale a factor of 20 or so below the string scale. Our foregoing discussion, however, does not address the question of why and whether the moduli have the particular values needed to solve these problems. Nor does it explain why the $N=1$ space-time supersymmetry is broken at a hierarchically low energy scale compared with the string scale; this is required phenomenologically, both in order to protect the TeV scale of electroweak symmetry breaking from string scale corrections, and in order to achieve the "observed" unification of the gauge coupling strengths.

We shall see in this section how these two shortcomings are related, and remedied. The obvious approach to the first problem, the stabilisation of the moduli, is to calculate the effective potential for the relevant fields and determine which values of the moduli minimise it. However, the moduli potential is flat to all orders in string perturbation theory, when space–time supersymmetry is unbroken [74,189]. This follows from a non-renormalisation theorem in string theory directly analogous to the familiar non-renormalisation theorems in supersymmetric field theories. The point is that each (scalar) moduli field is a component of a chiral supermultiplet which necessarily contains a pseudoscalar partner of the scalar mode. In the case of the T-modulus defined in (Section 4.1), the real part, associated with the overall size of the torus, is (the expectation value of) a scalar field while the imaginary part is (the expectation value of) a pseudoscalar field. The vertex operator for the pseudoscalar field is

$$V_B \propto \int \mathrm{d}^2 z\, B\partial_z X(\partial_{\bar z}\bar X + \tfrac{1}{2}k\cdot\Psi\cdot\bar\Psi)\mathrm{e}^{-\mathrm{i}k\cdot X}\ , \tag{5.1}$$

where k is the four-momentum, and X is the *complex* world sheet made from the two compactified dimensions under consideration. At zero momentum only the first (bosonic) term survivies, and this vanishes because it is a total derivative. Thus the zero momentum mode decouples, and the theory is invariant under the Peccei-Quinn axionic symmetry (see Eq. (4.18)).

$$B \to B + \text{const}\ , \tag{5.2}$$

as noted in Eq. (4.18). As a result, the superpotential is independent of the pseudoscalar field B. However, because of supersymmetry, B can only appear in the combination T given in Eq. (4.1). So the superpotential is independent of T and the moduli effective potential is therefore flat to all orders in string perburbation theory.

It follows that the moduli, and in particular the size of the compactified dimensions, have their values fixed by non-perturbative effects and/or supersymmetry breaking. A similar argument applies to the dilaton moduli field S. The non-perturbative mechanism which has attracted most attention, and upon which we shall concentrate in this section, is hidden sector gaugino condensation [91,70,73]. Because of asymptotic freedom gauge coupling strengths increase as the energy scale M is reduced from the string scale (m_{string}). The quantitative relationship is given by the renormalisation group equation which to one loop order gives

$$M e^{8\pi^2/bg^2(M)} = m_{\text{string}} e^{8\pi^2/bg^2(m_{\text{string}})} ,$$

where

$$b = -3c(G) + \sum_\alpha T(R^\alpha) \tag{5.3}$$

determines the leading term of the beta function

$$\beta(g) = \frac{b}{16\pi^2} g^3 + \frac{b_2}{(16\pi^2)^2} g^5 + \cdots \tag{5.4}$$

$c(G)$ is the quadratic Casimir for the adjoint representation of the (simple) gauge group G and $T(R^\alpha)$ the usual Casimir for chiral supermultiplets:

$$T(R^\alpha) = \mathrm{Tr}(Q^{\alpha 2}) \tag{5.5}$$

where Q^α is the matrix representing any generator of G in the representation R^α to which the chiral matter belong. The gauge coupling becomes large at a scale Λ where $\exp(8\pi^2/bg^2(\Lambda))$ is of order unity, and is given by

$$\Lambda \simeq m_{\text{string}} e^{8\pi^2/bg^2(m_{\text{string}})} \tag{5.6}$$

which is exponentially suppressed relative to the string scale. When this occurs we entertain the possibility of gaugino condensation in which the quantity $\lambda_a \lambda_b$, bilinear in the gaugino fields, acquires a non-zero vacuum expectation value (VEV) with

$$|\langle \lambda_a \lambda_b \rangle_0| \sim \Lambda^3 . \tag{5.7}$$

In this regime the use of a field theoretic description in terms of gauge and gaugino fields alone is inadequate.

In a globally supersymmetric theory the supersymmetry can only be broken by the F-term of a chiral supermultiplet acquiring a non-zero VEV. The gaugino bilinear $\lambda_a \lambda_b$ is (proportional to) the lowest component of the (composite) chiral superfield $W^\alpha_a W_{b\alpha}$, where W^α_a is the usual field strength chiral superfield, and is therefore *not* an F-term. Thus gaugino condensation does not break global supersymmetry, and this is confirmed by explicit claculations [184]; this also agrees with conclusions following from Witten's index theorem [186,187].

However, in a locally supersymmetric theory, a supergravity theory, things are different [70,73,179]. Under a *local* supersymmetry transformation of the spinor component ψ of a chiral supermultiplet, this gaugino bilinear $\lambda_a \lambda_b$ *does* appear in $\delta\psi$. So $\langle 0|\delta\psi|0\rangle \neq 0$ if a gaugino condensate occurs, and supersymmetry is broken; for this to happen the gauge kinetic function must be non-minimal. This breaking of local supersymmetry in the hidden sector provides a seed for supersymmetry breaking in the observable sector, which is coupled to the hidden sector only by gravitational interactions.

5.2. Non-perturbative superpotential due to gaugino condensate(s)

In the strongly interacting regime we need more than just the usual gauge kinetic piece of a globally supersymmetric Lagrangian:

$$\mathscr{L}_{\mathrm{GK}} = \int \mathrm{d}^2\theta f_{bc}(\Phi) W_b^{\alpha} W_{c\alpha} + \text{h.c.}\ , \tag{5.8}$$

where $f_{bc}(\Phi)$ is the gauge kinetic function, dependent on the gauge singlet chiral superfields Φ, including the moduli superfields; $W_{c\alpha}$ is the standard (spinor-valued, chiral) gauge field strength superfield whose lowest dimension component is the gaugino field $\frac{1}{2}\lambda_{c\alpha}$. In addition, we need an effective Lagrangian to describe the interactions of the (bound states and) possible gaugino condensate, which arise as consequences of the strong gauge interactions. We therefore construct a composite supermultiplet U to describe the lightest of the non-perturbative states. This is assumed to be a gauge singlet chiral superfield

$$U \equiv 4 W_b^{\alpha} W_{b\alpha} \tag{5.9}$$

which has the (singlet) gaugino, bilinear combination $\lambda_b\lambda_b$ as its lowest dimension $[M^3]$ component. Then U develops a vacuum expectation value if the gaugino condensate forms. To determine if it does we need an effective Lagrangian for the composite field U. Because of its non-canonical dimensions the kinetic term for U is [184]

$$\mathscr{L}_{\mathrm{K}} = \frac{9}{\gamma}\int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, (U\bar{U})^{1/3}\ , \tag{5.10}$$

where γ is a dimensionless constant. The inclusion of this term in the effective theory means that the Kähler potential K, discussed in the previous section is modified by these non-perturbative effects. If we now denote by $\tilde{K}$ the Kähler potential in the absence of a condensate then the complete Kähler potential is given by

$$K = \tilde{K} + K^{\mathrm{np}}\ , \tag{5.11}$$

where

$$K^{\mathrm{np}} = -3\ln\left[1 + \frac{9}{\gamma}\mathrm{e}^{\tilde{K}/3}(U\bar{U})^{1/3}(S_0\bar{S}_0)^{-1}\right] \tag{5.12}$$

with S_0 a “chiral compensator” superfield with scaling dimension unity [63,148]. The choice of S_0 determines the normalization of the gravitational action

$$\mathscr{L}_{\mathrm{grav}} \propto \mathrm{e}^{-K/3} S_0\bar{S}_0|_{\theta=\bar{\theta}=0}\,\mathscr{R} \tag{5.13}$$

so

$$\mathrm{e}^{-K/3} S_0\bar{S}_0|_{\theta=\bar{\theta}=0} = 1/16\pi G_N \propto m_{\mathrm{p}}^2 \tag{5.14}$$

and we see that the condensate contribution to the Kähler potential is suppressed by the square of the Planck mass m_{p}^2.

The Lagrangian must also be augmented by a term which reproduces the anomalies of the underlying theory [148,131]. The anomalies in question are the chiral anomaly, the scaling (energy-momentum trace) anomaly, and the supersymmetry current γ-trace anomaly, and all are proportional to (different) components of the composite superfield U. The chiral anomaly, for example, is given by

$$\partial^\mu J^5_\mu = -(\beta(g)/2g^3)F_{a\mu\nu}\tilde{F}^{\mu\nu}_a \tag{5.15}$$

where $\beta(g)$ is given in Eq. (5.4); $F_{a\mu\nu}$ is the usual non-Abelian field strength, and $\tilde{F}_{a\mu\nu}$ is its dual. In the same notation the anomaly of the energy-momentum tensor is

$$\theta^\mu_\mu = (\beta(g)/2g^3)F_{a\mu\nu}F^{\mu\nu}_a \tag{5.16}$$

and the supercurrent trace anomaly is

$$\gamma^\mu S_\mu = (\beta(g)/g^3)F_{a\mu\nu}\sigma^{\mu\nu}\lambda_a\ . \tag{5.17}$$

In order to ensure that the anomalous Ward identities are satisfied to tree order we add a term [182]

$$\mathscr{L}_{\text{anom}} = -\frac{\beta(g)}{6g^3}\int \mathrm{d}^2\theta\, U\ln(cU/S_0^3) + \text{h.c.} \tag{5.18}$$

to the Lagrangian (c is an unknown constant). $\mathscr{L}_{\text{anom}}$ is chosen so that under chiral transformations

$$U(x,\theta,\bar{\theta}) \to \mathrm{e}^{3\mathrm{i}\alpha}U(x,\theta\,\mathrm{e}^{-3\mathrm{i}\alpha/2},\bar{\theta}\,\mathrm{e}^{3\mathrm{i}\alpha/2}) \tag{5.19}$$

and under scale transformations

$$U(x,\theta,\bar{\theta}) \to \mathrm{e}^{3\gamma}U(x\,\mathrm{e}^{\gamma},\theta\,\mathrm{e}^{\gamma/2},\bar{\theta}\,\mathrm{e}^{\gamma/2}) \tag{5.20}$$

the variation of the action $\int \mathrm{d}^4x\, \mathscr{L}_{\text{anom}}$ gives precisely the required chiral, scaling and superconformal anomalies.

It is easy to see how this works. The F term of $U\ln U$ clearly includes, among others, the term $F_U \ln u$, where u is the scalar component of U and F_U is the F-part. Under the transformations (5.19) and (5.20) above, F_U transforms covariantly whereas

$$\ln u \to \ln u + 3\mathrm{i}\alpha \tag{5.21}$$

and

$$\ln u \to \ln u + 3\gamma\ , \tag{5.22}$$

respectively. Then $\mathscr{L}_{\text{anom}}$ generates the anomalous terms $\mathrm{i}\alpha(\beta(g)/2g^3)(F_U - F_U^\dagger)$ and $\gamma(\beta(g)/2g^3)\times(F_U + F_U^\dagger)$ which are just the required anomalies.

Taking the (hidden sector) gauge kinetic function in Eq. (5.8) to be

$$f_{bc}(\Phi) = f_G(\Phi)\delta_{bc} \tag{5.23}$$

we see that we may combine $\mathscr{L}_{\text{GK}}$ and $\mathscr{L}_{\text{anom}}$ to yield the non-perturbative superpotential

$$\tilde{W}^{\text{np}} = \tfrac{1}{4}f_G(\Phi)U - (\beta(g)/6g^3)U\ln(cU/S_0^3)\ . \tag{5.24}$$

Although we have taken proper account of the scaling and chiral anomalies, we must also ensure that the effective theory is invariant under the target space modular transformations [95,101,41,166] defined in Eq. (4.3):

$$T_i \rightarrow \frac{a_i T_i - \mathrm{i} b_i}{\mathrm{i} c_i T_i + d_i} \quad (i = 1,2,3) \tag{5.25}$$

with a_i, b_i, c_i, d_i integers satisfying

$$a_i d_i - b_i c_i = 1 \ . \tag{5.26}$$

Since the Kähler potential $\tilde{K}$ in the absence of the condensate(s) already satisfies Eq. (4.9)

$$\tilde{K} \rightarrow \tilde{K} + \ln|\mathrm{i} c_i T_i + d_i|^2 \ , \tag{5.27}$$

we require that the additional piece K^{np} arising from the condensate is modular invariant. Then from Eq. (5.12) we infer that U/S_0^3 has modular weight -1

$$(U/S_0^3) \rightarrow (U/S_0^3)(\mathrm{i} c_i T_i + d_i)^{-1} \ . \tag{5.28}$$

The modular invariance of

$$G \equiv K + \ln|W/S_0^3|^2 \tag{5.29}$$

requires that W/S_0^3 has modular weight -1, as noted in Eq. (4.11). In general, for this to be satisfied by the non-perturbative contribution W^{np} given in Eq. (5.24), we have to include some further T_i -dependence in W^{np}. It follows from Eqs. (4.161), (4.162) and (4.163) that (the holomorphic part of) the gauge kinetic function is

$$f_G(\Phi) = S - \frac{1}{8\pi^2} \sum_i (b'^i_G - \delta^i_{GS}) \ln \eta^2(T_i) \ , \tag{5.30}$$

where the second term derives from string loop threshold corrections to the (hidden sector) gauge group coupling constant, and the δ^i_{GS} are to cancel anomalies under the target space duality transformations. We may write

$$f_G(\Phi) = \Sigma - \frac{1}{8\pi^2} \sum_i b^{i\prime}_G \ln \eta^2(T_i) \ , \tag{5.31}$$

where

$$\Sigma \equiv S + \frac{1}{8\pi^2} \sum_i \delta^i_{GS} \ln \eta^2(T_i) \ . \tag{5.32}$$

Then under the duality transformations, it follows from Eqs. (4.150) and (4.140) that Σ is invariant and

$$f_G(\Phi) \rightarrow f_G(\Phi) - \frac{1}{8\pi^2} b'^i_G \ln(\mathrm{i} c_i T_i + d_i) \ . \tag{5.33}$$

To ensure that W^{np} has the required modular weight we replace $\tilde{W}^{\mathrm{np}}$ in Eq. (5.24) by

$$W^{\mathrm{np}} = \tfrac{1}{4} f_G(\Phi) U - \frac{\beta(g)}{6g^3} U \ln[U v(T_i)/S_0^3] ,$$

where $v(T_i)$ has modular weight n_i. Then W^{np}/S_0^3 has weight -1 provided

$$n_i = 1 - \frac{3g^3}{16\pi^2 \beta(g)} b_G^{i'} = 1 - 3b_G^{ir}/b_G \tag{5.34}$$

keeping only the first term of $\beta(g)$ given in Eq. (5.4). Now

$$v(T_i) \propto \prod_i \eta(T_i)^{2n_i} \tag{5.35}$$

has weight n_i, and Ferrara et al. [95–97] have argued that this is the *unique* T_i dependence which does not lead to unphysical zeros or poles in the upper-half of the $\mathrm{i}T_i$ complex plane. Thus finally we obtain the superpotential

$$\begin{aligned} W^{\mathrm{np}} &= \frac{1}{4} f_G(\Phi) U - \frac{\beta(g)}{6g^3} U \ln\left[cU \prod_i \eta(T_i)^{2n_i}/S_0^3 \right] \\ &= \frac{1}{4} U\Sigma - \frac{b_G}{96\pi^2} U \ln\left[cU \prod_i \eta(T_i)^2/S_0^3 \right]. \end{aligned} \tag{5.36}$$

The above treatment is easily generalised to the formation of several gaugino condensates, associated with hidden sector (non-abelian simple) gauge groups G_n $(n = 1, \ldots, p)$. There are then p composite chiral superfields U_n, and the non perturbative superpotential is

$$W^{\mathrm{np}} = \sum_{n=1}^{p} \left\{ \frac{1}{4} U_n \Sigma - \frac{b_n}{96\pi^2} \ln\left[c_n U_n \prod_i \eta(T_i)^2/S_0^3 \right] \right\} \tag{5.37}$$

with b_n determining the leading term of the beta function $\beta_n(g_n)$ of G_n, and the c_n unknown constants.

To determine whether gaugino condensation, and hence supersymmetry breaking, actually occurs we need to calculate the effective potential deriving from the supergravity theory we have obtained, and to see whether the scalar component(s) u_n of U_n have non-zero values at the minimum. This is the calculation to which we now turn.

5.3. *Effective potential*

The effective potential in any supergravity theory is given by

$$V_{\mathrm{eff}} = \mathrm{e}^G [G_A (G^{-1})^A_B G^B - 3] , \tag{5.38}$$

where

$$G \equiv K + \ln|W|^2 \tag{5.39}$$

with K the Kähler potential and W the superpotential, and we are keeping only the scalar components φ_A and φ^*_A of the chiral superfields Φ_A and $\Phi^\dagger_A$ in terms of which K, W and $\bar{W}$ are defined. The derivatives of G are written as

$$G^A \equiv \partial G/\partial\varphi_A, \qquad G_A \equiv \partial G/\partial\varphi^*_A \tag{5.40}$$

and

$$G^A_B \equiv \partial^2 G/\partial\varphi_A\partial\varphi^*_B \ . \tag{5.41}$$

Then $(G^{-1})^A_B$ is the inverse of the matrix G^A_B.

In the case under consideration the chiral superfields involved are those whose scalar components are the dilation field S, defined in Eq. (4.36); the orbifold moduli fields T_i, U_i, defined in Eqs. (4.1) and (4.2), some of which are fixed by the point group; the condensates u_n; and other matter fields φ_α, including Higgs fields H_1 and H_2. Evidently the calculation and minimization of V_{eff} in full generality is a formidable calculation when several moduli and gauge condensates are active. The calculation of V_{eff} in the case of a single (overall) modulus T, and when the dilation field S is modular invariant ($\delta^i_{GS} = 0$), but with several gaugino condensates, has been done by Taylor [181]. He notes the existence of a zero-energy local, but not global, minimum, which corresponds to the weak coupling (i.e. $\mathrm{Re}\, S \to \infty$) limit. In this limit $V_{\mathrm{eff}} = W = 0$, corresponding to a supersymmetric vacuum. This supersymmetry is not surprising. The weak coupling limit corresponds to infinite Planck mass, since as we have seen in Section 4 the Kähler potential has a leading term

$$K \sim -\ln(\mathrm{Re}\, S) \quad \text{as } \mathrm{Re}\, S \to \infty \tag{5.42}$$

and then from Eq (5.14) we see that

$$m_{\mathrm{p}} \to \infty \quad \text{as } \mathrm{Re}\, S \to \infty \ . \tag{5.43}$$

In this limit only global supersymmetry survives, and we have already noted that a gaugino condensate cannot break global supersymmetry.

In this weak coupling limit the potential is minimised when

$$\partial W/\partial U_n = 0 \ , \tag{5.44}$$

i.e. the global F-terms vanish. Using Eq. (5.37) we find that the condensate is then given by

$$u_n(S, T_i) = \frac{\mu^3}{c_n e} \mathrm{e}^{24\pi^2\Sigma/b_n} \prod_i \eta(T_i)^{-2} \ . \tag{5.45}$$

Substituting Eq. (5.45) into W^{np} eliminates the dependence upon the condensate and we obtain the "truncated" superpotential

$$W^{\mathrm{np}}_{\mathrm{trunc}} = \sum_n \frac{b_n}{96\pi^2} u_n(\Sigma, T_i) \tag{5.46}$$

entirely in terms of the moduli fields. As we have said, this is a good approximation provided that Re S is stabilised at a "large" value at the minimum of the effective potential. Phenomenologically we require

$$(4\pi \mathrm{Re}\ S)^{-1} = \frac{g^2(m_{\mathrm{string}})}{4\pi} \sim \frac{1}{24} \tag{5.47}$$

for coupling constant unification, so $\langle \mathrm{Re}\, S \rangle \sim 2$ which is *not* particularly large. Further, we shall see that for a single condensate at least, the effective potential does not have a local minimum at a finite value of S [73,101,95]. So we shall assume that some other mechanism is responsible for stabilising the dilation VEV.

In view of the complexity of minimizing the full effective potential, it is desirable to find a more economical procedure, and the one which has received considerable attention consists of using the truncated superpotential (5.46), in which the condensates are assumed to have the form (5.45), rather than the full non-perturbative superpotential (5.37). The justification for doing this is first to note that the form (5.46)

$$W^{\mathrm{np}}_{\mathrm{trunc}} = \Omega(\Sigma) \Big/ \prod_i \eta^2(T_i) , \tag{5.48}$$

where

$$\Omega(\Sigma) = \sum_n d_n \mathrm{e}^{24\pi^2 \Sigma / b_n} \tag{5.49}$$

with

$$d_n = b_n \mu^3 / 96\pi^2 c_n e = \text{constant} , \tag{5.50}$$

is essentially required by the fact that W^{np} must have modular weight -1. It has further been noted [46,156,56] in the case of a single overall T modulus and $\delta^i_{GS} = 0$, that for *small* values of $\sum_n |u_n|^2 / |\mu|^6$, the form (5.45) for the condensate can be *deduced* from the extremum conditions on the full effective potential with the assumption of modular covariance. Then for

$$\mathrm{Re}\, S > -b_n / 24\pi^2 , \tag{5.51}$$

$\sum_n |u_n|^2 \ll |\mu|^6$, and it follows that the full effective potential is well approximated by the truncated effective potential obtained using $W^{\mathrm{np}}_{\mathrm{trunc}}$ and the original (condensate-independent) Kähler potential $\tilde{K}$. The above condition is satisfied for a wide range of values of $\mathrm{Re}\, S$ including the realistic case where $\mathrm{Re}\, S \sim 2$.

5.3.1. Pure gauge hidden sector

For the remainder of this section we shall therefore use the truncated superpotential (5.48) and the effective potential which derives from it using the Kähler potential K.

The simplest case is when the hidden sector is a pure gauge Yang–Mills theory, i.e. there is no hidden sector matter. Then the Kähler potential is given in (4.154)

$$\tilde{K} = -\ln Y - \sum_i \ln(T_i + \bar{T}_i) \tag{5.52}$$

and Y, given in Eq. (4.155), can be written

$$Y = \Sigma + \bar{\Sigma} - \frac{1}{8\pi^2} \sum_i \delta^i_{GS} \ln(T_i + \bar{T}_i) |\eta(T_i)|^4 . \tag{5.53}$$

The effective potential is calculated using Eqs. (5.38) and (5.39):

$$V_{\mathrm{eff}} = Y^{-1}\prod_i (T_i+\bar{T}_i)^{-1}|\eta(T_i)|^{-4}\Bigg\{|\Omega - Y\Omega_\Sigma|^2 - 3|\Omega|^2 + \sum_i \frac{Y}{Y-(1/8\pi^2)\delta^i_{GS}}\left|\Omega - \frac{1}{8\pi^2}\delta^i_{GS}\Omega_\Sigma\right|^2 (T_i+\bar{T}_i)^2|\hat{G}^i|^2\Bigg\}\,, \tag{5.54}$$

where

$$\hat{G}^i \equiv (T_i+\bar{T}_i)^{-1} + 2\eta(T_i)^{-1}\frac{\mathrm{d}\eta}{\mathrm{d}T_i} \tag{5.55}$$

and

$$\Omega_\Sigma \equiv \mathrm{d}\Omega/\mathrm{d}\Sigma\ . \tag{5.56}$$

The hope is that this potential has a minimum at finite values for the moduli T_i and Σ, and that the consequent value of Y corresponds to a realistic values $2g^{-2}_{\mathrm{string}}$. Unfortunately, this does not happen generically. For a reasonable value of Σ (and hence Y) the potential does develop a minimum at finite values of T_i. If Σ is fixed, then for reasonable values and the case of a single overall modulus $T \equiv T_1 = T_2 = T_3$, there is always a minimum [101,68] with $T \sim 1.23$. However, as mentioned previously, the potential does not obviously have a minimum at a finite value of Σ: in fact for a single condensate the only stationary point of V_{eff} at finite Σ is a maximum [47]. The condition for a stationary point is Σ gives

$$(\Omega - Y\Omega_\Sigma)\left[2\bar{\Omega} - \sum_i \frac{Y^2}{(Y-d_i)^2}(T_i+\bar{T}_i)^2|\hat{G}^i|^2(\bar{\Omega} - d_i\bar{\Omega}_{\bar{\Sigma}})\right] - Y^2\Omega_{\Sigma\Sigma}(\bar{\Omega} - Y\bar{\Omega}_{\bar{\Sigma}}) = Y^2\Omega_{\Sigma\Sigma}\sum_i \frac{d_i}{Y-di}(T_i+\bar{T}_i)^2|\hat{G}^i|^2(\bar{\Omega} - d_i\bar{\Omega}_{\bar{\Sigma}})\ , \tag{5.57}$$

where

$$d_i \equiv \delta^i_{\mathrm{GS}}/8\pi^2\ , \tag{5.58}$$

$$\Omega_\Sigma \equiv \partial\Omega/\partial\Sigma,\ \text{etc}\ . \tag{5.59}$$

In the case that $\delta^i_{GS} = 0$, so $Y = 2\mathrm{Re}\,\Sigma = 2\mathrm{Re}\,S$, the above condition reduces to

$$(2-\tilde{G})(\Omega - Y\Omega_\Sigma)\bar{\Omega} = Y^2\Omega_{\Sigma\Sigma}(\bar{\Omega} - Y\bar{\Omega}_{\bar{\Sigma}})\ , \tag{5.60}$$

where

$$\tilde{G} \equiv \sum_i (T_i+\bar{T}_i)^2|\hat{G}^i|^2\ ,$$

which may be satisfied trivially, when

$$\Omega - Y\Omega_\Sigma = 0\ , \tag{5.61}$$

or non-trivially. De Carlos et al. [47] have shown that if the trivial solution gives a reasonable value of Y, then it will always correspond to a minimum of V_{eff}, whereas the non-trivial solution is never a minimum. Further, in the trivial case, we see by inspection that the minimum of V_{eff} occurs at any zero of the modified Eisenstein function $\hat{G}^i$, and in particular at the fixed points $T = 1$ and $e^{\mathrm{i}\pi/6}$ of the modular group.

These statements are easily verified for the case of a single condensate

$$\Omega(\Sigma) = d\mathrm{e}^{-\alpha\Sigma} \tag{5.62}$$

with

$$\alpha = -24\pi^2/b > 0\ . \tag{5.63}$$

Eq. (5.61) gives

$$Y = -1/\alpha < 0 \tag{5.64}$$

an unphysical value. The non-trivial solution with $Y > 0$ is

$$Y = \sqrt{2 - \tilde{G}}/\alpha \tag{5.65}$$

which is clearly a maximum of

$$V_{\mathrm{eff}} \propto \frac{\mathrm{e}^{-\alpha Y}}{Y}[(1 + \alpha Y)^2 - 3 + \tilde{G}]\ . \tag{5.66}$$

The situation is not much better when we have two or more condensates. For realistic values of Y

$$24\pi^2\ \mathrm{Re}\ \Sigma/|b_n| \gg 1 \tag{5.67}$$

as already noted. Then the trivial (minimum) condition (5.61) reduces to

$$\Omega_\Sigma = 0 \tag{5.68}$$

or

$$\sum_n c_n^{-1}\,\mathrm{e}^{24\pi^2\Sigma/b_n} = 0\ . \tag{5.69}$$

So for two condensates we get

$$\frac{1}{2}Y = \mathrm{Re}\,\Sigma = \frac{1}{24\pi^2}\left(\frac{1}{b_1} - \frac{1}{b_2}\right)^{-1}\ln\left|\frac{c_1}{c_2}\right|\ , \tag{5.70}$$

and for the unknown constants c_n of order unity this is typically small, and therefore unrealistic. Similar conclusions are reached for three or more condensates.

The foregoing conclusion is largely unaffected by consideration of the more realistic case with $\delta^i_{GS} \neq 0$, although the complexity of Eq. (5.57) necessitates a numerical treatment. In essence, the parameter d_i, in which the δ^i_{GS} appears in Eq. (5.57), is generically small, so the effects may be calculated perturbatively in d_i. In any case, it is important to note that $\delta^i_{GS} \neq 0$ severely constrains the formation of multiple pure gauge condensates. The reason is that any complex plane i which is *not* an $N=2$ plane does not contribute to the threshold corrections to the gauge coupling constants, and consequently not to the gauge kinetic function either. So for this particular plane

$$b_n^{i'} = \delta_{GS}^i \tag{5.71}$$

for each gauge group G_n. However, for a pure gauge condensate we see from Eq. (4.166) that

$$b_n^{i'} = \tfrac{1}{3} b_n \ , \tag{5.72}$$

so

$$b_n = b_m \equiv b \tag{5.73}$$

for any two of the hidden sector gauge group G_n and G_m, since the right-hand side of Eq. (5.71) is independent of n. Thus each of the condensates has the *same* exponential $\exp(24\pi\Sigma/b)$ and the system effectively has just one condensate. This eliminates all Z_N orbifolds from consideration, since each of them has at least one non $N=2$ complex plane, as is apparent from Table 1. The $Z_M \times Z_N$ models are, however, unaffected.

5.3.2. *Hidden sector with matter*

In view of the difficulty in stabilizing the dilaton field Y at an acceptable value with a pure gauge hidden sector, the natural recourse is to study the effects of hidden matter [156,182,56,46,157,9,134] Then, besides the field strength supermultiplets $W^{\alpha a}$, with a labelling the generators of the gauge group G, we have chiral matter multiplets Q^i_m, with $m = 1, \dots, M$ labelling the multiplets, and i labelling the components of the representation of G to which Q_m belong. We assume that for each multiplet Q^i_m there is a chiral supermultiplet $\bar{Q}_{mi}$ belonging to the complex conjugate representation of G to which Q^i_m belongs. Then, in the strong coupling regime discussed in Section 5.1, besides the formation of a gaugino condensate, we entertain the possible formation of chiral matter condensates $\sum_i \langle \bar{q}_{mi} q^i_m \rangle_0 \neq 0$ and bound states, just as in QCD we get mesons from quark anti-quark bound states; in a supersymmetric theory we have also the possibility of bound squark–antisquark states. We assume too that the charged matter fields Q^i_m and $\bar{Q}_{mi}$ are coupled to gauge singlet superfields A_m by trilinear terms in the perturbative superpotential

$$W^{\mathrm{pert}} = \sum_{m,i} h_m(T_i) A_m Q^i_m \bar{Q}_{mi} \tag{5.74}$$

such that the "quarks" develop non-zero masses

$$m_m = h_m(T_i) \langle A_m \rangle_0 \tag{5.75}$$

when the gauge singlet fields develop non-zero VEVs. The trilinear terms give a contribution

$$\mathscr{L}_{\mathrm{tr}} = \int \mathrm{d}^2\theta \sum_{m,i} h_m(T_i) A_m Q^i_m \bar{Q}_{mi}$$

to the Lagrangian.

To describe the bound states we define the M gauge singlet composite chiral superfields

$$V_m \equiv \sum_i Q^i_m \bar{Q}_{mi} \quad (m = 1, \dots, M) \tag{5.76}$$

which contain the squark–antisquark bilinear $\sum_i \tilde{q}_{mi} \tilde{\bar{q}}^i_m$ as the lowest dimension $[M^2]$ components.

In the absence of the mass terms the global symmetry of the (supersymmetric hidden sector gauge) theory is

$$\mathrm{SU}(M)_{\mathrm{L}} \times \mathrm{SU}(M)_{\mathrm{R}} \times U(1)_{\mathrm{V}} \times U(1)_{A} \times U(1)_{\mathrm{R}} \tag{5.77}$$

if the M "quark" superfields all belong to the same representation R of G. In any case there is an extra $U(1)$ symmetry compared with the non-supersymmetric case which relates to the gaugino field. The chiral $U(1)_A$ acts on the gaugino composite superfield as in Eq. (5.19) and on the matter composite superfields V_m as

$$V_m(x,\theta,\theta) \rightarrow \mathrm{e}^{2\mathrm{i}\alpha} V_m(x,\theta \mathrm{e}^{-3\mathrm{i}\alpha/2},\bar{\theta}\mathrm{e}^{3\mathrm{i}\alpha/2}) , \tag{5.78}$$

while the $U(1)_{\mathrm{R}}$ symmetry acts only on the matter superfields so

$$U(x,\theta,\bar{\theta}) \rightarrow U(x,\theta,\bar{\theta}) , \tag{5.79}$$

$$V_m(x,\theta,\bar{\theta}) \rightarrow \mathrm{e}^{2\mathrm{i}\beta} V_m(x,\theta,\bar{\theta}) .$$

Both of the above $U(1)$ symmetries are broken at the quantum level by the Adler–Bell–Jackiw anomaly. Under the chiral $U(1)_A$ we get

$$\delta \mathscr{L}_A = -\alpha(b/32\pi^2)F\tilde{F} \tag{5.80}$$

with b defined in Eq. (5.3), so

$$b = -3c(g) + 2MT(R) \tag{5.81}$$

in the case that the M "quark" superfields are all in the representation R of G. Under the $U(1)_{\mathrm{R}}$ transformation

$$\delta \mathscr{L}_{\mathrm{R}} = \beta(2c/32\pi^2)F\tilde{F} \tag{5.82}$$

where

$$c = 2\sum_m T(R_m) = 2MT(R) . \tag{5.83}$$

As before, we need an effective Lagrangian expressed in terms of the composite superfields, which reproduces these anomalies, and which yields an effective non-perturbative superpotential with the correct modular weight (-1); the modular weight of the gaugino composite field is -1, as before, and it is easy to see that the matter composite fields V_m/S_0^2 have modular weight $-2/3$. Then proceeding as before we obtain the full non-perturbative superpotential to be

$$W^{\mathrm{np}} = \frac{1}{4}U\Sigma - \frac{b}{96\pi^2}U \ln\left[cU^{1+2c/b}\prod_m V_m^{-6T(R)/b}\prod_i \eta^2(T_i)/S_0^3\right] - \sum_m h_m(T_i)A_m V_m . \tag{5.84}$$

Also as before, we shall instead use the "truncated" superpotential which is obtained by eliminating the composit superfields U,V_m using

$$\partial W/\partial U = 0 = \partial W/\partial V_m . \tag{5.85}$$

(It is unclear whether this enjoys the same numerical justification as in the pure gauge case.) This gives

$$V_m = \frac{2T(R)}{32\pi^2}\frac{U}{h_m(T_i)A_m}\ , \tag{5.86}$$

$$\frac{2T(R)}{32\pi^2}U = \mu^3\left(\frac{32\pi^2 e}{2T(R)}\right)^{(b+2c)/c-b} \mathrm{e}^{24\pi^2\Sigma/(b-c)}\left\{c\prod_i \eta^2(T_i)\prod_m [h_m(T_i)A_m]^{6T(R)/b}\right\}^{-b/(b-c)}$$

and

$$W^{\mathrm{np}}_{\mathrm{trunc}} = \frac{b-c}{96\pi^2}U\ . \tag{5.87}$$

The form of the trilinear coupling (5.74) may be generalised to the form

$$W^{\mathrm{pert}} = \sum_{m,n,\alpha} h_{\alpha mn}(T_i)A_\alpha Q^i_m \bar{Q}_{ni}\ , \tag{5.88}$$

where there are arbitrary number of gauge singlet superfields A_α with more general couplings. The effect is the replacement in $W^{\mathrm{np}}_{\mathrm{trunc}}$

$$\prod_m h_m(T_i)\varphi_m \to \det M\ , \tag{5.89}$$

where

$$M_{mn} \equiv \sum_\alpha h_{\alpha mn}(T_i)A_\alpha \tag{5.90}$$

is the "quark" mass matrix. The dependence of the Yukawa couplings h_{amn} on the moduli T_i is well-understood, as we saw in Section 3. Non-trivial dependence arises only when all three of the coupled fields are (point group) twisted sector states.

It is also easy enough to generalize to the case when the "quark" composite fields V_m belong to different representations R_m of G. However, the multi-gaugino condensate is typically difficult to handle. The reason is that in general the quark field Q_m belongs to non-trivial representations R_{mn} of several gauge groups G_n, just as the quark fields in the standard model belong to non-trvial representations of SU(3) and SU(2). Thus the different gaugino condensates are coupled to each other unless, for each m, R_{mn} is non-trivial for precisely one n. In that case the quark condensate is proportional to a single gaugino condensate, just as in the single condensate case already discussed.

To procede further we need the Kähler potential K_m for the matter fields A_α. At tree graph level we have seen in Eq. (4.79) that for untwisted matter, and for orbifolds whose point group does not act as Z_2 in any complex plane (so the U moduli are fixed) the matter contribution to the Kähler potential is

$$K_m = \sum_i (T_i + \bar{T}_i)^{-1}|\varphi_i|^2 \tag{5.91}$$

so the fields φ_i have modular weight -1. For Z_2 planes the situation is slightly more complicated, and for twisted matter we have seen in Section 4.5 that

$$K_m = \sum_\alpha \prod_i (T_i + \bar{T}_i)^{n_{\alpha i}} |\varphi_\alpha|^2 \equiv \sum_\alpha K_m^{(\alpha)} |\varphi_\alpha|^2 \ , \tag{5.92}$$

where the modular weights $n_{\alpha i}$ are model-dependent and calculable.

The effect of the matter condensate has been studied [156] in the simplified case that there is a single overall T modulus $T_1 = T_2 = T_3 = T$ and a single untwisted gauge singlet field A having modular weight -1, and $\delta_{GS}^i = 0$. Then

$$K = -\ln(S+\bar{S}) - 3\ln(T + \bar{T} - |A|^2) \tag{5.93}$$

and the effective potential is given by

$$V_{\text{eff}} = (S+\bar{S})^{-1}(T+\bar{T}-|A|^2)^{-3}\left\{ |(S+\bar{S})W_S - W|^2 + \frac{1}{3}(T+\bar{T}-|A|^2)|W_A + \bar{A}W_T|^2 \right.$$
$$\left. + \frac{1}{3}(T+\bar{T}-|A|^2)^2 \left| W_T - 3\frac{W}{T+\bar{T}-|A|^2} \right|^2 - 3|W|^2 \right\} , \tag{5.94}$$

where

$$W_S \equiv \partial W / \partial S \text{,etc.} \tag{5.95}$$

In this case the truncated superpotential reduces to

$$W_{\text{trunc}}^{\text{np}} \propto \left[\frac{e^{24\pi^2 \Sigma}}{\eta(T)^{6b} A^{3c}} \right]^{1/(b-c)} \tag{5.96}$$

and then the effective potential is

$$V_{\text{eff}} = \frac{1}{3}|W|^2 (S+\bar{S})^{-1}(T+\bar{T})^{-3}(1-|\tilde{A}|^2)^{-2} \left\{ 3(1-|\tilde{A}|^2)^{-1}|f_S|^2 + \left(\frac{3c}{b-c}\right)^2 |\tilde{A}|^{-2} \right.$$
$$\left. + \left(\frac{3b}{b-c}\right)^2 [|(T+\bar{T})\hat{G}|^2 - 1] \right\} , \tag{5.97}$$

where

$$|\tilde{A}|^2 \equiv |A|^2/(T+\bar{T}) \tag{5.98}$$

is duality unvariant,

$$f_S \equiv 1 - (S+\bar{S})\frac{W_S}{W} \tag{5.99}$$

and $\hat{G}$ is defined in Eq. (5.55).

As before, for the single gaugino condensate under consideration V_{eff} has no stable minimum for the dilaton at a finite value, so Lüst and Taylor [156] take S and f_S as free parameters whose value is fixed by some other mechanism. The modular invariance of V_{eff} means that the self-dual points $T = 1$ and $T = e^{i\pi/6}$ are stationary points, but they may be maxima, minima or saddle points depending on the parameters b,c and f_S. In the case $b+2c<0$, V_{eff} always has a non-trivial minimum with $\tilde{A} \neq 0$, so non-vanishing "quark" masses are dynamically generated, and local supersymmetry is spontaneously broken. Further, the parameter f_S can be fine-tuned so that V_{eff} is zero at this minimum; in other words the cosmological constant vanishes. Simultaneously the compactification scale is determined to be of order the Planck mass. In the case $b+2c>0$, however, there is always a zero energy minimum of V_{eff} at $\tilde{A}=0$, the condensates are zero, and supersymmetry is unbroken; so there is no dynamical mass generation and the compactification radius is undetermined.

The continuing difficulty of stablizing the dilaton has led de Carlos, Casas and Munõz [46] to study multiple gaugino condensate, with the (tacit) assumption that the matter fields transform non-trivially with respect to only one of the gauge groups. Again they take a single overall modulus T, a single gauge singlet field A, and $\delta^i_{GS}=0$. Their numerical analysis indicates that V_{eff} does not have a true minima even for two condensates. This is understood by noting that the VEV of $\tilde{A}$, defined in Eq. (5.98), is expected to be small, since it vanishes perturbatively. Then, since the superpotential has a power dependence of $\tilde{A}$, see Eq. (5.96), the dominant contribution to V_{eff} in Eq. (5.94) comes from the term proportional to $|W_A|^2$, except for a small region where $W_A=0$. Thus

$$V_{\mathrm{eff}} \sim \tfrac{1}{3}(S+\bar{S})^{-1}(T+\bar{T})^{-2}|W_A|^2 \tag{5.100}$$

which has an absolute minimum at

$$W_A = 0 \; . \tag{5.101}$$

However, it is clear that this *cannot* be satisfied for a single condensate of the form (5.96). The authors note that this deficiency can be remedied if the superpotential is augmented by a perturbative contribution

$$W^{\mathrm{pert}} = A^3 \tag{5.102}$$

which models the generic cubic self-interaction of the gauge singlet fields A_α:

$$W^{\mathrm{pert}} = \sum_{\alpha,\beta,\gamma} \hat{h}_{\alpha\beta\gamma}(T_i) A_\alpha A_\beta A_\gamma \; . \tag{5.103}$$

Then Eq. (5.101) gives

$$A^3 = \frac{c}{b-c} W^{\mathrm{np}} = \frac{c}{96\pi^2} U \; . \tag{5.104}$$

If we now substitute back into W^{np} we get an effective superpotential as a function of S and T alone. Not surprisingly it has the form (5.48) previously derived from the requirements of modular invariance and the consideration of anomalies

$$W^{\mathrm{np}} \propto \mathrm{e}^{24\pi^2\Sigma/b} \eta(T)^{-6} \tag{5.105}$$

although the value of b is now includes contributions from the matter as well as the pure gauge contributions. Again, of course, the dilaton cannot be stablised with a single condensate gauge group.

However it is now rather easy to do so with two gauge groups [56,181,145] with the unknown constants taking values of order unity. As before, the minimum occurs at a value $T \sim 1.23$ in all cases [101,47], but the value of Y depends upon the exact gauge groups and the representations occupied by the hidden matter. There is no difficulty in obtaining physically reasonable values of Y [47].

5.4. Supersymmetry breaking

Spontaneous breakdown of local supersymmetry occurs when the Goldstone fermion is "eaten" by the gravitino, thereby giving it the extra degrees of freedom needed for a massive spin 3/2 particle. The supergravity Lagrangian contains a four fermion term

$$\mathscr{L}_{4F} = \tfrac{1}{2} f^A_{bc} \psi_{AL} \sigma^{\mu\nu} \lambda_{b\mathrm{L}} \psi_{\nu\mathrm{L}} \gamma_\mu \lambda_{c\mathrm{R}} + \text{h.c.} \,, \tag{5.106}$$

where λ_b are the gaugino fields of the (hidden) gauge group G, ψ_ν is the gravitino field, ψ_A are the fermionic components of the chiral superfields Φ_A, $f_{ab}(\Phi)$ is the (non-minimal) gauge kinetic function and

$$f^A_{bc} \equiv \partial f_{bc} / \partial \varphi_A \,. \tag{5.107}$$

Evidently, if there is a gaugino condensate, the above term mixes the Goldstone fermion field

$$\eta = f^A_{bc} \langle \lambda_{b\mathrm{L}} \lambda_{c\mathrm{R}} \rangle \psi_A \tag{5.108}$$

with the gravitino field. Thus, provided that the gaugino condensate and f^A_{bc} are non-zero at the minimum of the effective potential we have been examining, the local supersymmetry is broken, and the gravitino acquires a non-zero mass

$$m_{3/2} = \mathrm{e}^{G_0/2} m_\mathrm{p} \,, \tag{5.109}$$

where G_0 is the value of

$$G = K + \ln |W|^2 \tag{5.110}$$

at the minimum.

This conclusion is in accord with the general result that for spontaneous supersymmetry breaking to occur the variation of at least one of the fields in the theory must have a non-zero VEV. The variation $\delta\psi_A$ of ψ_A under a local supersymmetry transformation contains the terms

$$\delta\psi_A = -\sqrt{2} \mathrm{e}^{G/2} (G^{-1})^B_A G_B \xi - \tfrac{1}{8} f_{bcB} (G^{-1})^B_A \lambda_b \lambda_c + \cdots \,, \tag{5.111}$$

where

$$f_{bcB} \equiv \partial f_{bc} / \partial \varphi^{B*} \tag{5.112}$$

so again a non-zero condensate and non-minimal gauge kinetic function with f_{bcB} is non-zero, indicate a breakdown of local superstring.

This breaking of supersymmetry by the hidden sector gaugino condensate leads to soft supersymmetry breaking in the observable sector. In particular, it is easy to see that (all) gauginos acquire non-zero masses, while the corresponding gauge fields remain massless. The mass terms derive from the following two-fermion term in the supergravity Lagrangian

$$\mathscr{L}_{2F} = \tfrac{1}{4}\mathrm{e}^{G/2}\hat{f}_{bcA}(G^{-1})^A_B G^B \hat{\lambda}_b \hat{\lambda}_c \ , \tag{5.113}$$

where we $\hat{f}_{bc}$ is an *observable* sector gauge kinetic function and $\hat{\lambda}_b$ are observable sector gaugino fields. Thus, with a diagonal gauge kinetic function, the mass of the canonically normalised gaugino $\hat{\lambda}_{\text{phys}} = (\mathrm{Re}\hat{f})^{1/2}\hat{\lambda}$ is

$$M = \tfrac{1}{2}m_{3/2}(\mathrm{Re}\hat{f}_0)^{-1}\hat{f}_{0A}(G_0^{-1})^A_B G_0^B \ , \tag{5.114}$$

where the suffix '0' indicates that the quantity is evaluated at the minimum of the effective potential. Formula (5.114) gives the gaugino mass at the string scale where

$$(\mathrm{Re}\hat{f})^{-1} = \hat{g}^2(m_{\text{string}}) = 4\pi\hat{\alpha}(m_{\text{string}}) \ . \tag{5.115}$$

We may use the renormalization group equation

$$M(\mu)/\alpha(\mu) = M(m_{\text{string}})/\alpha(m_{\text{string}}) \tag{5.116}$$

to determine the gaugino mass $\hat{M}(\mu)$ at the scale μ. If we also use the form

$$W^{\text{np}} = \Omega(\Sigma) \Big/ \prod_i \eta^2(T_i) \tag{5.117}$$

for the effective non-perturbative potential, as discussed in the previous sections, then

$$\hat{M}(\mu) = 2\pi\hat{\alpha}(\mu)m_{3/2}\left\{ - Yf_S - \frac{1}{8\pi^2}\sum_i (\hat{b}'^i_G - \delta^i_{GS})\frac{1}{Y-d_i}[(1-f_S)d_i - Y](T_i + \bar{T}_i)^2|\hat{G}^i|^2 \right\}, \tag{5.118}$$

where

$$f_S \equiv 1 - Y\Omega_\Sigma/\Omega \ ,$$
$$d_i \equiv \delta^i_{GS}/8\pi^2. \tag{5.119}$$

$\hat{G}^i$ is the (modular covariant) Eisenstein function, defined in Eq. (5.55), and b'^i_G and defined in Eq. (4.148). In deriving Eq. (5.118) it is necessary to augment the form (5.114) in order to obtain a modular invariant expression for the gaugino mass; in particular the term $2b'^i_G\eta(T_i)^{-1}\mathrm{d}\eta/\mathrm{d}T_i$ which arises from $\hat{f}_{0T_i}$ is replaced by $2b'^i_G\hat{G}^i$.

The supersymmetry breaking also generates non-zero masses for the matter scalar fields φ_α. With the form (5.92) for the Kähler potential, valid for small values of φ_α, we may expand the effective potential to quadratic order in φ_α and read off the scalar masses. This gives

$$m^2_{\varphi_\alpha} = V_0 + m^2_{3/2}\left[1 + \sum_i \frac{|d_i(1-f_S) - Y|^2}{(Y-d_i)^2}(T_i + \bar{T}_i)^2|\hat{G}^i|^2 n^i_\alpha\right], \tag{5.120}$$

where V_0 is the ground state energy, the cosmological constant, given by

$$V_0 = m^2_{3/2}\left[|f_S|^2 - 3 + \sum_i Y^{-1}(Y-d_i)^{-1}|d_i(1-f_S) - Y|^2(T_i + \bar{T}_i)^2|\hat{G}^i|^2\right]. \tag{5.121}$$

We have already noted that $V_{\rm eff}$ is minimized with values of T close to the self-dual points at which $\hat{G}^i$ is zero, and because of this the last term of both V_0 and $m^2_{\varphi_\alpha}$ is generally small. Further, f_S is zero at the minimum in the case that δ^i_{GS} is zero, and in general f_S too is small. It follows that

$$V_0 \sim -3m^2_{3/2} \tag{5.122}$$

and that the scalar masses squared are generally negative

$$m^2_{\varphi_\alpha} \sim -2m^2_{3/2} \ , \tag{5.123}$$

completely unacceptable predictions, which cast doubt on either the validity or the relevance of the whole gaugino condensate mechanism for supersymmetry breaking.

The scalar mass problem would be solved if the cosmological constant were small, and indeed the observed flatness of the universe on large scales supports the view that V_0 is zero, or very small. It is worth noting, however, that in principle the cosmological constant is not necessarily the same as the particle physics vacuum energy. The observed flatness on large scales may be an average value of highly curved values at very small scales [40]. Nevertheless, we shall take the economical view that V_0 is zero, and that we must therefore seek mechanisms to achieve this. In particular we regard the philosophy of setting $V_0 = 0$ in contradiction to the prediction (5.122) as being unacceptable.

5.5. Cosmological constant

The vanishing of the cosmological constant V_0 evidently requires the existence of additional matter whose contribution cancels those discussed hitherto, although we shall not attempt to explain why this should be so when supersymmetry is broken. We assume that this extra matter arises only in an additional term K_1 of the Kähler potential. Then the new Kähler potential is

$$K^{\rm new} = \tilde{K} + K_1(X,\bar{X}) \tag{5.124}$$

with $\tilde{K}$ as in Eq. (4.154) and

$$G^{\rm new} = K^{\rm new} + \ln|W|^2 \ . \tag{5.125}$$

The consequence is that the effective potential becomes

$$V_{\rm eff} = {\rm e}^{G^{\rm new}} v \ , \tag{5.126}$$

where

$$v = |f_\Sigma|^2 - 3 + \sum_i \frac{1}{Y(Y-d_i)} |Y - d_i(1-f_\Sigma)|^2 (T_i + \bar{T}_i)^2 |\hat{G}^i|^2 + K_{1X}^{-1X} |K_{1X}|^2 \tag{5.127}$$

and, as before

$$f_\Sigma = 1 - Y\Omega_\Sigma/\Omega \ , \tag{5.128}$$

$$K_{1X} = \partial K_1/\partial X \quad \text{etc} \ . \tag{5.129}$$

Evidently, we can tune the X-dependent terms to ensure that the cosmological constant vanishes by arranging that

$$v_0 = 0 \,, \tag{5.130}$$

where v_0 is the value of v at the minimum of V_{eff}. With this requirement the minimum of V_{eff} is obtained by minimising v. So for the case $d_i = 0$ we find that the T_i are near the fixed point value at which $\hat{G}^i$ is zero and $f_\Sigma = 0$.

The additional contribution to G means that the gravitino mass is now given by

$$m_{3/2} = \mathrm{e}^{G_0^{\mathrm{new}}/2} m_{\mathrm{p}} \,, \tag{5.131}$$

but otherwise has no effect on the formula (5.118) for the observable sector gaugino masses. Similarly the scalar masses are still given by (5.120) but with the cosmological constant now tuned to zero. So

$$m^2_{\varphi_\alpha} \sim m^2_{3/2} \tag{5.132}$$

which is quite acceptable, in principle.

5.6. A-terms and B-terms

The generic cubic term (4.14) in the perturbative superpotential

$$W_3^{\mathrm{p}} = h_{\alpha\beta\gamma}(T_i)\Phi_\alpha\Phi_\beta\Phi_\gamma \,, \tag{5.133}$$

where $\Phi_{\alpha,\beta,\gamma}$ are chiral superfields, generates Yukawa couplings and quartic scalar couplings in the supersymmetric field theory. In the presence of supersymmetry breaking effects, such as we are considering, it also generates (soft), trilinear couplings of the scalar fields $\varphi_{\alpha,\beta,\gamma}$ of the form

$$\mathscr{L}_3 = A_{\alpha\beta\gamma}\hat{h}_{\alpha\beta\gamma}\varphi_\alpha\varphi_\beta\varphi_\gamma \tag{5.134}$$

and it is straightforward to calculate these; including the contribution W^{p} to the superpotential we merely expand V_{eff} to third order in the scalar fields. Then

$$\begin{aligned} A_{\alpha\beta\gamma} m_{3/2}^{-1} = {} & -\bar{f}_\Sigma + \sum_i \frac{Y - d_i(1-\bar{f}_\Sigma)}{Y - d_i}(T_i + \bar{T}_i)\hat{\bar{G}}^i \\ & \times\left[-(1 + n^i_\alpha + n^i_\beta + n^i_\gamma) + (T_i + \bar{T}_i)\frac{\partial \ln h_{\alpha\beta\gamma}}{\partial T_i}\right] , \end{aligned} \tag{5.135}$$

where

$$\hat{h}_{\alpha\beta\gamma} = \mathrm{e}^{K/2} \prod_{\rho=\alpha,\beta,\gamma} (K_m^{(\rho)})^{-1/2} h_{\alpha\beta\gamma} \tag{5.136}$$

with $K_m^{(\rho)}$ the Kähler potential for the matter field φ_ρ, and n^i_ρ its modular weight, see Eq. (5.92). It is well-known that to avoid axions, and to break the observable sector electroweak symmetry successfully, it is necessary to include a "μ-term"

$$W_2^{\mathrm{p}} = \mu_W H_1 H_2$$

bilinear in the two Higgs fields $H_{1,2}$ into the perturbative superpotential. Then, as for the trilinear terms, there are corresponding soft terms, bilinear in the scalar fields, induced by the supersymmetry-breaking hidden sector

$$\mathscr{L}_2 = -B_W\hat{\mu}h_1h_2\ , \tag{5.137}$$

where

$$B_{\text{cond}}m_{3/2}^{-1} = 1 - \bar{f}_\Sigma(1 - Y\mu_{W\Sigma}) + \sum_i \frac{Y - d_i(1-\bar{f}_\Sigma)}{Y - d_i}(T_i + \bar{T}_i)\hat{G}^i\left[(T_i + \bar{T}_i)\frac{\partial \ln \mu_W}{\partial T_i} + d_i\frac{\partial \ln \mu_W}{\partial \Sigma} - (1 + n_1^i + n_2^i)\right] \tag{5.138}$$

and

$$\hat{\mu} = \mathrm{e}^{K/2}\prod_{\rho=1,2}(K_m^{(\rho)})^{-1/2}\mu_W \tag{5.139}$$

μ_W can be calculated [6], and for the $Z_6 - IIb$ orbifold we have

$$\mu_W \propto W^{\text{np}}\frac{\partial \ln \eta(T_3)\eta(T_3/3)}{\partial T_3}\frac{\partial \ln \eta(U_3)\eta(U_3/3)}{\partial U_3} \tag{5.140}$$

and

$$n_1^3 = n_2^3 = -1, \qquad n_1^i = n_2^i = 0, \quad i \neq 3\ . \tag{5.141}$$

There is also a term [6] in the Kähler potential

$$K_Z = ZH_1H_2 + \text{h.c.}\ , \tag{5.142}$$

where

$$Z = (T_3 + \bar{T}_3)^{-1}(U_3 + \bar{U}_3)^{-1} \tag{5.143}$$

which generates a soft scalar bilinear term

$$\mathscr{L}'_2 = -B_Z\mu_Z^{\text{eff}}h_1h_2 + \text{h.c.}\ , \tag{5.144}$$

where

$$\mu_Z^{\text{eff}} = |W^{\text{np}}|Z[1 - (T_3 + \bar{T}_3)\hat{G}(T_3,\bar{T}_3) - (U_3 + \bar{U}_3)\hat{G}(U_3,\bar{U}_3)] \tag{5.145}$$

is the coefficient of the higgsino bilinear term in the Lagrangian and

$$\begin{aligned} -m_{3/2}^{-1}B_Z\mu_Z^{\text{eff}} = W^{\text{np}}Z\Big\{ &-1 + (T_3 + \bar{T}_3)(\hat{G}(T_3,\bar{T}_3) + \text{h.c.}) + (U_3 + \bar{U}_3)(\hat{G}(U_3,\bar{U}_3) + \text{h.c.}) \\ &+ (T_3 + \bar{T}_3)(U_3 + \bar{U}_3)\hat{G}(T_3,\bar{T}_3)\hat{G}(U_3,\bar{U}_3) + |f_\Sigma|^2 \\ &- \sum_i \frac{(T_i + \bar{T}_i)^2}{Y(Y - d_i)}|Y - d_i(1 - f_\Sigma)|^2|\hat{G}(T_i,\bar{T}_i)|^2\Big\}\ . \end{aligned} \tag{5.146}$$

The calculations of de Carlos et al. [47] show that a gravitino mass $m_{3/2}$ in the range $10^2\,\text{GeV} < m_{3/2} < 10^4\,\text{GeV}$ is easily obtained in models with hidden sector matter. There is therefore every reason to suppose that the incorporation of these supersymmetry breaking terms into the renormalization group equations will yield a sparticle spectrum on the same scale.

5.7. Further considerations

The stabilization of the dilaton was achieved by using two or more gaugino condensates with suitably chosen hidden sector matter content [56,181,145]. An alternative, which requires only a single condensate has recently been proposed [57,39,173]. This utilises the observation that there are good reasons to believe that there are sizeable stringy non-perturbative corrections to the Kähler potential. The effect is to replace the $\ln Y$ term in Eq. (4.154) by a so far unknown function $P(Y)$. Then Casas [57] has shown in several examples how $P(Y)$ can be chosen so that the dilaton is stabilized with just a single condensate. However, it has not so far been possible to do this while simultaneously achieving a zero cosmological constant. It is straightforward to generalize the foregoing calculations of the supersymmetry breaking to this case [37,38].

We saw in Section 5.2 how the requirement that the non-perturbative physics preserves the modular invariance severely constrains the form of the non-perturbative superpotential. It was observed [68] that superpotentials involving the modular invariant function $j(T)$ may in principle arise in orbifold theories with gauge non-singlet states which become massless at special values of the moduli, although examples are lacking. $j(T)$ must appear in a function

$$H(T) = (j - 1728)^{m/2} j^{n/3} P(j) \tag{5.147}$$

multiplying W^{np} (m,n are integers and P is a polynomial)

$$W^{\mathrm{np}} \to W^{\mathrm{np}} H(j) \tag{5.148}$$

in order to avoid singularities in the fundamental domain

$$\mathscr{F} = \{T \colon |T| \geq 1,\ 0 \leq \mathrm{Im}\, T \leq 1\} \ . \tag{5.149}$$

This observation has been given added force recently [83,64] by the discovery that F-theory constructions of W^{np} are indeed modular forms, in fact E_8 theta functions. Although the appearance of $H(j)$ does not affect the stabilization of the dilaton when there is a single condensate, it clearly does affect the values of the T_i moduli at the minimum of V_{eff}. One interesting feature is that mimima arise in the interior of the fundamental domain [37,38] $\mathscr{F}$, whereas previously they were on the boundary [68].

It is natural to wonder whether the minimization of V_{eff} at complex values of the moduli might induce CP-violation via the moduli dependence of the soft supersymmetry breaking terms [128,40,1] calculated in the previous sections, although it has been argued [77,60] that there is no explicit CP-violation in string theory, perturbative or non-perturbative. Indeed the CP-violating phases of the soft supersymmetry breaking A and B terms are constrained to be less than $O(10^{-3})$ by the current limit on the electric dipole moment of the neutron [40]. Thus if CP-violation does arise in this way the challenge to string theory is to explain why these phases are so small. It is found [37,38] that the phases are either zero or well below the experimental bounds, unless *both* a non-minimal Kähler potential, as discussed above, and the modular invariant function $j(T)$ is present via the appearance of $H(j)$ multiplying W^{np}. In those circumstances CP-violation comparable to the current upper bounds does occur.

6. Conclusions and outlook

The "observed" unification [2] of the $SU(3)\times SU(2)\times U(1)$ gauge coupling strengths of the minimal supersymmetric extension of the standard model (MSSM) is to date the best evidence that the low energy world really is supersymmetric. Compactified string theory naturally generates an effective four-dimensional supergravity – Yang Mills theory and, as we have seen in Eq. (4.155), it requires coupling constant unification at a value

$$\alpha_{\text{string}} \equiv g^2_{\text{string}}/4\pi = (4\pi\,\mathrm{Re}\,S)^{-1} \tag{6.1}$$

determined by the dilaton S, ignoring contributions Δ_a from the string loop threshold corrections and the Green–Schwarz anomaly cancelling coefficients δ^i_{GS} for the present. If/when we understand the non-perturbative physics which stabilizes the dilaton field at a value with

$$\langle \mathrm{Re}\,S\rangle \sim 2 \tag{6.2}$$

the observed unification with $\alpha \sim 1/25$ would also be evidence for an underlying string theory. However, to date we have *no* a priori convincing theory which leads to this result. In addition (and unlike a grand unified theory, which also requires unification), string theory predicts the energy scale at which unification is achieved to be

$$m_{\text{string}} \simeq 4\times 10^{17}\,\text{GeV} \tag{6.3}$$

as follows from Eq. (4.185) using the "observed" value of g_{string}, which is a factor of 20 or so higher than the "observed" unification scale (4.186). In Section 4.9 we discussed the feasibility of bridging this gap using calculations of the string loop threshold corrections Δ_a calculated in various orbifold compactifications. Our conclusion is that it *is* possible that these can remove the discrepancy, but that large values of the T modulus

$$\langle \mathrm{Re}\,T\rangle \sim 20 \tag{6.4}$$

are required to do so. However, we saw in Section 5.3 that when the T modulus is stabilized by hidden sector gaugino condensation, its value is generically of order unity, so again we have no a priori convincing theory as to how such a large value might arise. Of course, as we noted, the assumption that the only the matter content is that of the MSSM might be wrong, but here too we have no a priori convincing reason for including the extra matter needed to remove the discrepancy. Thus, although not conclusive, at face value the "observed" unification is also the best evidence to date that (perturbative) string theory is wrong.

We can see this another way. With six dimensions compactified on a space of volume V, the 10-dimensional effective supergravity theory arising from heterotic string theory relates the four-dimensional gravitational coupling G_N and the unified gauge coupling strength α_{string} to the string tension α' and the dilaton field φ as follows:

$$G_N = (\alpha')^4 e^{2\varphi}/64\pi V\ , \tag{6.5}$$

$$\alpha_{\text{string}} = (\alpha')^3 e^{2\varphi}/16\pi V\ . \tag{6.6}$$

Since $e^{2\varphi}$ and V enter in the same combination in both expressions we can eliminate both and relate the string tension, and hence the string energy scale, to the coupling strength and the Planck mass $m_P \equiv G_N^{-1/2}$

$$m_{\mathrm{string}} \equiv (\alpha')^{-1/2} = \tfrac{1}{2}\alpha_{\mathrm{string}}^{1/2} m_P \\ \sim \tfrac{1}{10} m_P \tag{6.7}$$

if we use the "observed" value of the unified gauge coupling. Defining $m_{\mathrm{GUT}} = aV^{-1/6}$, with $1 \le a \le 2\pi$ expected, it follows that

$$G_N m_{\mathrm{GUT}}^2 = \left(\frac{m_{\mathrm{GUT}}}{m_P}\right)^2 = a^2 e^{-\varphi/3}\left(\frac{\pi}{4}\right)^{1/3} \alpha_{\mathrm{string}}^{4/3} \\ \sim \frac{a^2}{79} \tag{6.8}$$

if we require that $e^{2\varphi} < 1$, so that a perturbative treatment is justified. In contrast the "observed" unification scale is

$$m_{\mathrm{GUT}} \simeq 3 \times 10^{16}\,\mathrm{GeV} \tag{6.9}$$

and $m_P = 1.22 \times 10^{19}\,\mathrm{GeV}$, so the observed ratio is far smaller than the perturbatively predicted lower bound. In fact to get the observed value requires

$$e^{2\varphi} \sim 10^{10} \tag{6.10}$$

way beyond any perturbative validity.

One possibility, therefore, is that in the real world string theory is strongly coupled, and that the perturbative treatment underlying this review is irrelevant to particle phenomenology. Developments in the past few years have shown that what were formerly regarded as different string vacua may all be related using a web of duality transformations. (Two theories A, compactified on a space X, and B, compactified on Y, are "dual" to each other if the physics in the common uncompactified space M is identical [10].) In particular, it has been established that the (10-dimensional) strongly coupled $E_8 \times E_8$ heterotic string theory, compactified on a Calabi Yao threefold X, is dual to a new 11-dimensional M-theory [190,121,122], compactified on $X \times S^1/Z_2$. In the field theory limit M-theory reduces to an 11-dimensional supergravity theory with two E_8 super-Yang Mills theories on each of the (two) 10-dimensional hyperplanes corresponding to the fixed points of the S^1/Z_2 orbifold. It is beyond the scope of this review to give much detail of this. Suffice it to say that in this case, when the theory is compactified on a Calabi Yao space of volume V, the theory relates the four-dimensional gravitational coupling G_N and the unified gauge coupling strength α_{string} to the 11-dimensional gravitational coupling κ and the length $R_{11} = \pi\rho$ of the orbifold interval as follows:

$$G_N = \kappa^2/8\pi R_{11} V \ , \tag{6.11}$$

$$\alpha_{\mathrm{string}} = (4\pi\kappa^2)^{2/3}/V \ . \tag{6.12}$$

Thus defining $m_{\mathrm{GUT}} = aV^{-1/6}$, as before, with $1 \le a \le 2\pi$ expected

$$m_{11} \equiv \kappa^{-2/9} = (4\pi\alpha_{\mathrm{string}}^{-3/2})^{1/9} m_{\mathrm{GUT}}/a \tag{6.13}$$

$$\simeq \frac{2.266}{a} m_{\mathrm{GUT}} \tag{6.14}$$

using the observed value of α, and

$$R_{11}^{-1} = 32\pi^2 \alpha_{\mathrm{string}}^{-3/2} \left(\frac{m_{\mathrm{GUT}}}{m_{\mathrm{P}}}\right)^2 a^{-3} m_{\mathrm{GUT}} \tag{6.15}$$

$$\simeq \frac{0.238}{a^3} m_{\mathrm{GUT}} \tag{6.16}$$

if we use the observed unification scale. So the length scale associated with the GUT is of the same order as, or a bit larger than, the fundamental scale $m_{11}^{-1}$of the 11-dimensional theory at which unification of the GUT and gravitational forces presumably occurs, and the orbifold length scale R_{11} is

$$R_{11} \sim 9.5 a^2 m_{11}^{-1} \tag{6.17}$$

an order of magnitude larger than the fundamental scale. In this picture, at low energies the world is four-dimensional with gauge couplings evolving logarithmically and power law evolution of the gravitational coupling. Around R_{11}^{-1} a fifth dimension opens up, and the power law evolution of the gravitational coupling changes; the logarithmic evolution of the gauge couplings is unaffected since the gauge fields are confined to the walls at the fixed points of the extra dimension. Finally, at m_{GUT} the gauge couplings unify and six further dimensions open up; the theory is now 11-dimensional and has (sixth) power evolution of the couplings. Although weakly coupled at this scale, the gauge and gravitational couplings unify at m_{11} with a value $\alpha \sim 1$. Thus, unlike the weakly coupled heterotic string theory, analysed above, M-theory allows a consistent incorporation of the parameters associated with "observed" unification.

However, there are several points which should be borne in mind. One is that M-theory does not *explain* the parameters, any more than perturbative string theory did. As in the weakly coupled heterotic string, the effective supergravity theory emerging from the compacified M-theory has two model independent moduli with

$$\langle \mathrm{Re}\, S \rangle \equiv (1/4\pi)(4\pi\kappa^2)^{-2/3} V \,, \tag{6.18}$$

$$\langle \mathrm{Re}\, T \rangle \equiv 6^{1/3} (4\pi\kappa^2)^{-1/3} R_{11} V^{1/3} \,. \tag{6.19}$$

Using the previous formulae (6.13) and (6.15), we find [62]

$$\langle \mathrm{Re}\, S \rangle = 1/g_{\mathrm{string}}^2 \sim 2 \tag{6.20}$$

and

$$\langle \mathrm{Re}\, T \rangle = \frac{6^{1/3}\alpha_{\mathrm{string}}}{32\pi^2} \left(\frac{a m_{\mathrm{P}}}{m_{\mathrm{GUT}}}\right)^2 \sim 39 a^2 \tag{6.21}$$

and, as before, we have no a priori convincing theory of why the moduli should have these values. Further, it is amusing to observe that the large value required for $\langle \mathrm{Re}\, T\rangle$ would suffice to bridge the previously noted unification gap of the weakly coupled theory, thereby dispensing with the need for a strongly coupled theory!

Physically, the most important feature distinguishing between M-theory and the weakly coupled string theory is that the gravitational fields propagate in the bulk (compactified) 11-dimensional world, while the gauge and matter fields are confined to the (compactified) 10-dimensional hyperplanes. One effect of this is that because of the variation with the extra coordinate, the effective (four-dimensional) supergravities differ at the two ends. In particular, the gauge kinetic function of the (observable sector) E_6 gauge fields is

$$f_6 = S + \alpha T \tag{6.22}$$

with α an integer determined by the Hodge numbers of the Calabi–Yao threefold X upon which the theory is compactified. (In the "standard" embedding the gauge connection of one of the E_8 theories is set equal to the spin connection of X, and this breaks the gauge symmetry (in the observable sector) to E_6.) The (hidden sector) E_8 has gauge kinetic function

$$f_8 = S - \alpha T \; . \tag{6.23}$$

These expressions have a striking similarity to those

$$f_{6,8} = S \pm \varepsilon T \tag{6.24}$$

which occur when the weakly coupled heterotic string is compactified, with the εT terms arising from the string loop threshold corrections (in the large T limit) and ε determined by the anomaly. The other quantities needed to specify the effective supergravity theory have also been calculated [61,167,168,154], and these may be applied straightforwardly to determine the soft supersymmetry breaking terms. So the second point to note is that since, as we have previously observed, it is not yet unambiguously determined that we *are* in the strongly coupled regime, it is important to have calculations for both the weakly coupled case and the strongly coupled case in order to decide the matter phenomenologically. In any case, it is already known that some features of the weakly coupled regime (e.g. the Kähler potential) carry across to the strongly coupled case with little or no modification, so some results from the former have a wider validity than their parentage might indicate.

At the time of writing, the form of M-theory compactified on a Calabi Yao threefold with standard [121,122,190], and non-standard [149,155] embeddings, is known, and there are some results for orbifold compactifications [170,176], with the concomitant modular symmetry groups. For this, and all of the previously given reasons, we hope that this review of weakly coupled orbifold compactifications of heterotic string theory will also be of relevance to the exciting new developments that are now occurring.

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