Collective Quantum Fields

in Plasmas, Superconductors, and Superfluid ^{3}He,

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Preface

Under certain circumstances, many-body systems behave approximatly like a gas of weakly interacting collective excitations. Once this happens it is desirable to replace the original action involving the fundamental fields (electrons, nucleons, ^{3}He, ^{4}He atoms, quarks etc.) by another one in which all these excitations appear as explicit *independent* quantum fields. It will turn out that such replacements can be performed in many different ways without changing the physical content of the theory. Sometimes, there esists a choice of fields associated with *dominant* collective excitations displaying weak residual interactions which can be treated perturbatively. Then the collective field language greatly simplifies the description of the physical system.

It is the purpose of this book to discuss a simple technique via Feynman path integral formulas in which the transformation to collective fields amounts to mere changes of integration variables in functional integrals. After the transformation, the path formulation will again be discarded. The resulting field theory is quantized in the standard fashion and the fundamental quanta directly describe the collective excitations.

For systems showing plasma type of excitations, a real field depending on one space and time variable is most convenient to describe all physics. For the opposite situation in which dominant bound states are formed, a complex field depending on two space and one or two time coordinates will render th emore ecnonomic description. Such fields will be called bilocal. If the potential becomes extremely short range, the bilocal field degenerates into a local field. In the latter case a classical approximation to the action of a superconducting electron system has been known for some time: the Ginzburg-Landau equation. The complete *bilocal* theory has been studied in elementary-particle physics where it plays a role in the transition from inobservable quark to observable hadron fields.

The change of integration variables in path integrals will be shown to correspond to an exact resummation of the perturbation series thereby accounting for phenomena which cannot be described perturbatively. The path formulation has the great advantage of translating all quantum effects among the fundamental particles completely into the field language of collective excitations. All radiative corrections may be computed using only propagators and interaction vertices of the collective fields. The method presented here is particularly powerful when a system is in a region where several collective effects becomes simultaneously important. An example is the electron gas at lower density where ladder graphs gain increasing importance with respect to ring graphs thus mixing plasma and pair effects. ^{3}He pair effects are dominant but plasma effects provide strong corrections.

In Part I of the book we shall illustrate the functional approach by discussing first conventional systems such as electron gas and superconductors. Also, we investigate a simple soluble model to understand precisely the mechanism of the functional field transformations as well as the relation between the Hilbert spaces generated once from fundamental and once from collective quantum fields.
In Part II we apply the same techniques to superfluid ^{3}He.
In Part III, finally, we illustrate the working of the functional techniques by applying it to some simple solvabel models.

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Contents

List of Figures

List of Tables

Part I

Functional Integral Techniques

Introduction

In this book we shall study certain classes of phenomena which occur in systems of many fermions interacting with each other via two-body forces. These forces are caused by exchange processes of some more fundamental particles such as photons or phonons, but this will be of no concern here—the forces will be described by potentials. Depending on these potentials, the fermion systems exhibit two types of collective behavior: plasma oscillations and pair condensation. The first type is found if exchanged fundamental particles generating the potential couple strongly to virtual fermion-hole states. Examples are plasmons in a degenerate electron gas. The second type of behavior is found if the forces favor the formation of bound states between pairs of particles. This is possible only below a certain *critical* temperature T_c. Examples are excitons in a semiconductor or Cooper pairs in a superconductor. In the language of Feynman diagrams, the first type of behavior prevails if ring diagrams yield dominant contributions. The second type of behavior is generated by ladder diagrams.

For systems showing plasma type of excitations, real fields depending on space and time are most convenient to describe the physical phenomena. In the case of bound states, complex fields containing the two spatial arguments of the constituents and their common time coordinate render the most ecnonomic description. Such fields will be called *bilocal*. In relativistic systems, also the time coordinates may be different. If the potential has a sufficienly short range, the bilocal field degenerates into a local field. The most important example for a the latter case is the collective pair-field theory of superconducting electrons called the Ginzburg-Landau theory.

A bilocal theory has been studied in elementary-particle physics where it plays a role in the transition from inobservable quark to observable hadron fields. The new basic field quanta of the converted theory are no longer the fundamental particles but the set of all quark-antiquark meson bound states which are obtained by solving a Bethe-Salpeter bound-state equation in the ladder approximation. They are called *bare mesons*. The theory of mesons has its own Feynman graphs, in which every line represents an entire ladder of fermion pairs. Such a formulation can also be given to quantum electrodynamics of electrons and positrons, where the bare mesons are positronium atoms [9].

1

Nonrelativistic Fields

1.1 Free Fields

Consider free nonrelativistic particles, whose energy ε depends on the momentum $\mathbf{p}$ in by some function $\varepsilon(\mathbf{p})$. In free space, this has the form $\varepsilon(\mathbf{p}) = \mathbf{p}^2/2m$. For a particle moving in a periodic solid, the momentum dependence is usually more complicated. For many purposes it can, however, be approximated by the same quadratic behavior if the mass is exchanged by a mass parameter called the *effective mass*. The action of a free nonrelativistic field describing an ensemble of these particles reads

$$\mathcal{A}_0 = \int d^3x dt \psi^*(\mathbf{x},t)\left[i\partial_t - \epsilon(-i\boldsymbol{\nabla})\right]\psi(\mathbf{x},t) \tag{1.1}$$

By extremizing this, we find the equation of motion

$$\frac{\delta\mathcal{A}_0}{\delta\psi^*(\mathbf{x},t)} = \left[i\partial_t - \epsilon(-i\boldsymbol{\nabla})\right]\psi(\mathbf{x},t) = 0, \tag{1.2}$$

which coincides with the Schrödinger equation for a single free particle.

By changing the fields in this action into operators and postulating them to satisfy the harmonic-oscillator commutation rules at each point and equal times

$$[\hat{\psi}(\mathbf{x},t), \hat{\psi}(\mathbf{x}')] = 0, \tag{1.3}$$
$$[\hat{\psi}^\dagger(\mathbf{x},t), \hat{\psi}^\dagger(\mathbf{x}')] = 0, \tag{1.4}$$
$$[\hat{\psi}(\mathbf{x},t), \hat{\psi}^\dagger(\mathbf{x}')] = \delta^{(3)}(\mathbf{x}-\mathbf{x}'), \tag{1.5}$$

the single-particle theory changes into a theory of arbitrarily many identical particles. There exists a vacuum state $|0\rangle$, defined by the condition $\hat{\psi}(\mathbf{x},t=0)|0\rangle = 0$. Applying a field operator $\hat{\psi}^\dagger(\mathbf{x},t=0)$ to the vacuum state creates the state of a single-particle $|\mathbf{x}\rangle \equiv \hat{\psi}(\mathbf{x},t=0)|0\rangle = 0$ localized at the point $\mathbf{x}$. By applying any number of such field operators, we can generate a state with any number of particles at any place. This Hilbert space is called the *Fock space*, and the procedure of field quantization is called *second quantization*. The usual quantization is ensured by the correspondence rule $\mathbf{p} \to -i\boldsymbol{\partial}$ in the single-particle Schrödinder equation and the action .

Equivalent to the second quantization is another technique in which the thermodynamic partition of the above system is expressed as a functional integral over all possible fluctuating fields [1, 2]:

$$Z = N \int \mathcal{D}\psi^*(\mathbf{x},t)\mathcal{D}\psi(\mathbf{x},t) \exp\{i\mathcal{A}[\psi^*,\psi]\}, \tag{1.6}$$

where N is some constant which plays no role in all subsequent discussions. The functional formulation was found by Feynman by observating that the amplitudes of diffraction phenomena of light are obtained by summing over the individual amplitudes for all paths the light could possible have taken, each of them being a pure phase depending only on the action of the light particle along the path. In the general field system (1.33), this principle leads to the alternative formula (1.6).

The functional integral may conveniently be defined the space-time into finer and finer cubic lattices of size δ with corners at $(x,y,z,t) = (i_1,i_2,i_3,i_4)\,\delta$, introducing fields at each such points

$$\psi_{i_1 i_2 i_3 i_4} \equiv \psi\left(x_{i_1}, y_{i_2}, z_{i_3}, t_{i_4}\right) \sqrt{\delta}^{\,4}, \tag{1.7}$$

and performing the product of all the integrals at each lattice point, i.e.,

$$\int \mathcal{D}\psi^*(\mathbf{x},t)\mathcal{D}\psi(\mathbf{x},t) \equiv \prod_{\substack{i_1 i_2 i_3 i_4 \\ i'_1 i'_2 i'_3 i'_4}} \int\int \frac{d\psi^\dagger_{i_1 i_2 i_3 i_4} d\psi_{i'_1 i'_2 i'_3 i'_4}}{\sqrt{2\pi i}\sqrt{2\pi i}}. \tag{1.8}$$

The double integral over complex variables $\int\int d\psi^* d\psi$ symbolizes the real integrals

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\left(\frac{\psi+\psi^*}{\sqrt{2}}\right) d\left(\frac{\psi-\psi^*}{\sqrt{2}i}\right). \tag{1.9}$$

This naive definition of path integration is straightforward for Bose fields. If we want to use the functional technique to describe also the statistical properties of fermions, some modifications are necessary. Then the fields must be taken to be anticommuting c-numbers. In mathematics, such objects form a Grassmann algebra G. If ξ, ξ' are real elements of G, then

$$\xi\xi' = -\xi'\xi. \tag{1.10}$$

A trivial consequence of this condition is that the square of each Grassmann element vanishes, i.e., $\xi^2 = 0$. If $\xi = \xi_1 + i\xi_2$ is a complex element of G, then $\xi^2 = -\xi^*\xi = -2i\xi_1\xi_2$ is nonzero, but $(\xi^*\xi)^2 = (\xi\xi)^2 = 0$.

All results to be derived later will make use of only one simple class of integrals which are a generalization of the elementary Gaussian (or Fresnel) formula for $A > 0$ [3]:

$$\int_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi i}} \exp\left(\frac{i}{2}\xi A \xi\right) = A^{-1/2}. \tag{1.11}$$

First, one considers a multidimensional real space $(\xi_1, \ldots, \xi_k \ldots)$, in which clearly

$$\prod_k \left[\int_{-\infty}^{\infty} \frac{d\xi_k}{\sqrt{2\pi i}}\right] \exp\left(\frac{i}{2}\sum_k \xi_k A_k \xi_k\right) = \left[\prod_k A_k\right]^{-1/2}. \tag{1.12}$$

Now if A_{kt} is an arbitrary symmetric positive matrix, and the exponent has the form $(i/2)\sum_{k,l}\xi_k A_{kl}\xi_l$, an orthogonal transformation can be used to bring A_{kl} to diagonal form without changing the measure of integration. Thus an equation like (1.12) is still valid with the right-hand side denoting the product of eigenvalues of A_{kl}. This can also be written as

$$\prod_m \left[\int_{-\infty}^{\infty} \frac{d\xi_m}{\sqrt{2\pi i}}\right] \exp\left(\frac{i}{2}\sum_{k,l} \xi_k A_{kl} \xi_l\right) = [\det A]^{-1/2}. \tag{1.13}$$

If, more generally, ξ is complex and A hermitian and positive, the result (1.13) follows separately for the real and for the imaginary part yielding

$$\prod_m \left[\int \frac{d\xi_m^* d\xi_m}{\sqrt{2\pi i}\sqrt{2\pi i}}\right] \exp\left(i\sum_{k,l} \xi_k^* A_{kl} \xi_l\right) = [\det A]^{-1}. \tag{1.14}$$

If the integrals are performed over anticommuting real or complex variables ξ or $\xi^*\xi$, the right-hand sides of formulas (1.13) and (1.14(te-2.14)) appear in inverse, i.e., as $[\det A]^{1/2}$, $[\det A]^1$ respectively. This is immediately seen in the complex case. After bringing the matrix A_{kl} to diagonal form via a unitary transformation, the integral reads

$$\int \prod_m \left[\frac{d\xi_m^* d\xi_m}{\sqrt{2\pi i}\sqrt{2\pi i}}\right] \exp\left(i\sum_n \xi_n^* A_n \xi_n\right)$$
$$= \prod_m \int \frac{d\xi_m^* d\xi_m}{\sqrt{2\pi i}\sqrt{2\pi i}} \exp\left(i\xi_m + A_m\xi_m\right). \tag{1.15}$$

Expanding the exponentials into a power series leaves only the first two terms since $(\xi_m + \xi_m)^2 = 0$ thus the integral becomes

$$\prod_m \int \frac{d\xi_m^* d\xi_m}{\sqrt{2\pi i}\sqrt{2\pi i}} (1 + i\xi_m + A_m\xi_m). \tag{1.16}$$

But each of these integrals can immediately be performed using the very simple integration rules of Grassmann algebras

$$\int \frac{d\xi}{\sqrt{2\pi i}} = 0, \quad \int \frac{d\xi}{\sqrt{2\pi i}}\xi = 1, \quad \int \frac{d\xi}{\sqrt{2\pi i}}\xi^n = 0, \qquad n > 1 \tag{1.17}$$

for real ξ, from which we derive

$$\int \frac{d\xi}{\sqrt{2\pi i}} = 0, \quad \int \frac{d\xi^*}{\sqrt{2\pi i}}\frac{d\xi}{\sqrt{2\pi i}} i\xi^*\xi = 1,$$
$$\int \frac{d\xi^*}{\sqrt{2\pi i}}\frac{d\xi}{\sqrt{2\pi i}} (\xi^*\xi)^n = 0, \qquad n > 1, \tag{1.18}$$

for complex ξ^*, ξ.

Note that these integration rules imply that under a linear change of a Grassmann integration variable, the integral multiplies by the inverse of the usual Jacobian. If $\xi' = a\xi$, then since ξ' is another Grassmann variable, its integrals have the properties (1.28):

$$\int \frac{d\xi'}{\sqrt{2\pi i}} = 0, \quad \int \frac{d\xi'}{\sqrt{2\pi i}}\xi' = 1, \quad \int \frac{d\xi'}{\sqrt{2\pi i}}\xi'^n = 0, \; n > 1. \tag{1.19}$$

Hence, the measure changes as follows:

$$\int \frac{d\xi'}{\sqrt{2\pi i}} = \frac{1}{a}\int \frac{d\xi}{\sqrt{2\pi i}}\xi', \tag{1.20}$$

in contrast to ordinary integrals where the factor on the right-hand side would be a.

Note that these rules make the linear operation of integration in (1.28) coincide with the linear operation of *differentiation*. A function $F(\xi)$ of a real Grassmann variable ξ, is determined by only two parameters: the zeroth- and the first-order Taylor coefficients. Indeed, due to the property $\xi^2 = 0$, the Taylor series has only two terms $F(\xi) = F_0 + F'\xi$, where $F_0 = F(0)$ and

$$F' \equiv dF(\xi)/d\xi. \tag{1.21}$$

But according to (1.28), also the integral yields F':

$$\int \frac{d\xi}{\sqrt{2\pi i}} F(\xi) = F'. \tag{1.22}$$

As a consequence of the rules (1.28), the right-hand side of (1.16) becomes the product of eigenvalues A_m (apart from an irrelevant factor)

$$\prod_m A_m = \det A \tag{1.23}$$

which is exactly the inverse of the boson result (1.14).

The case of real Fermi fields is slightly more involved since now the hermitian matrix A_{kl} can no longer be diagonalized by a unitary transformation [i.e., without changing the measure of integration $\prod_m (d\xi_m/\sqrt{2\pi i})$]. However, the integral can be done after observing that A_{kl} may always be assumed to be antisymmetric. For if there were any symmetric part, it would cancel in the quadratic form $\sum_{kl} \xi_k A_{kl} \xi_l$ due to the anticommutativity of the Grassmann variables. Now, an antisymmetric hermititan matrix can always be written as $A = -iA_R$ where A_R is real antisymmetric. Such a matrix is a standard metric in symplectic spaces and can be brought to a canonical form $\mathbf{C}$ which is zero except for 2×2 matrices

$$c = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.24}$$

along the diagonal. Here σ^2 is the second Pauli matrix. Then A can be written as

$$A = -iT^T\mathbf{C}T \tag{1.25}$$

where the hermitian matrix $-i\mathbf{C}$ contains only σ^2-matrices along the diagonal. This matrix $\mathbf{\Sigma}^2$ has a unit determinant so that $\det T = \det^{1/2}(A)$. Thus, under a linear transformation of Grassman variables $\xi'_k \equiv T_k\xi_l$, the measure of integration changes according to

$$\prod_k d\xi_k = (\det T)\prod_k d\xi'_k. \tag{1.26}$$

This is a direct consequence of the rule (1.20). With the help of the integration rules (1.28), the Grassmann version of the functional integral (1.13) can now be evaluated as follows:

$$\begin{aligned}
&\prod_m \left[\int \frac{d\xi_m}{\sqrt{2\pi i}}\right] \exp\left(i\sum_{k,l}\xi_k A_{kl}\xi_l\right) \\
&\quad = (\det T)\prod_m \left[\int \frac{d\xi'_m}{\sqrt{2\pi i}}\right] \exp\left(-\sum_{kl}\xi'_k C_{kl}\xi'_l\right) \\
&\quad = (\det A)^{1/2}\prod_n^\infty \left[\int \frac{d\xi'_{2n}}{\sqrt{2\pi i}}\right]\frac{d\xi'_{2n+1}}{\sqrt{2\pi i}}\left(1+\xi'_{2n+1}\xi'_{2n}\right) = (\det A)^{1/2}.
\end{aligned} \tag{1.27}$$

The right-hand side is the inverse of the boson result (1.13).

In order to apply these formulas to fields $\psi(\mathbf{x}, t)$ defined on a continuous spacetime, both formulas have to be written in such a way that the limit of infintiely fine lattice grating $\delta \to 0$ can be performed with no problem. For this we recall the useful matrix identity

$$[\det A]^{\mp 1} = \exp[i(\pm i\mathrm{Tr}\log A)] \tag{1.28}$$

where $\log A$ may be expanded in the standard fashion as

$$\log A = \log\left(1 + (A-1)\right) = -\sum_{n=1}^\infty \left[-(A-1)\right]^n \frac{1}{n}. \tag{1.29}$$

This formula reduces the calculation of the determinant to a series of matrix multiplications. But in each of these, the limit $\delta \to 0$ is straight-forward. One simply replaces all sums over lattice indices by integrals over d^3xdt, for instance

$$\mathrm{tr}A^2 = \sum_{kl} A_{kl}A_{lk} \;\longrightarrow\; \mathrm{Tr}A^2 = \int d^3xdtd^3x'dt' A(\mathbf{x}, t; \mathbf{x}', t')A(\mathbf{x}', t', \mathbf{x}, t). \tag{1.30}$$

With this in mind, the field versions of (1.13) and (1.14(te-2.14)) amount to the following functional formulas:

$$\int \mathcal{D}\varphi(\mathbf{x}, t) \exp\left[\frac{i}{2}\int d^3xdtd^3x'dt'\varphi(\mathbf{x}, t)A(\mathbf{x}, t; \mathbf{x}', t')\varphi(\mathbf{x}', t')\right]$$

$$= \exp\left[i\left(\pm\frac{i}{2}\mathrm{Tr}\log\left\{\begin{array}{c}1\\i\end{array}\right\}A\right)\right] \tag{1.31}$$

$$\int \mathcal{D}\psi^*(\mathbf{x},t)\mathcal{D}\psi(\mathbf{x},t)\exp\left[i\int d^3xdtd^3x'dt'\psi^*(\mathbf{x},t)A(\mathbf{x},t;x',t')\psi(\mathbf{x}',t')\right]$$
$$= \exp\left[i(\pm i\mathrm{Tr}\log A)\right]. \tag{1.32}$$

Here φ, ψ are arbitrary real and complex fields, with the upper sign holding for bosons, the lower for fermions. The same result is of course true if φ and ψ have several components (describing for example spin) and A is a matrix in the corresponding space.

1.2 Interactions

Consider now a many-fermion system described by an action

$$\begin{aligned}\mathcal{A} \equiv \mathcal{A}_0 + \mathcal{A}_{\mathrm{int}} &= \int d^3xdt\psi^*(\mathbf{x},t)\left[i\partial_t - \epsilon(-i\boldsymbol{\nabla})\right]\psi(\mathbf{x},t) \qquad (1.33)\\ &-\frac{1}{2}\int d^3xdtd^3x'dt'\psi^*(\mathbf{x}',t')\psi^*(\mathbf{x},t)V(\mathbf{x},t;\mathbf{x}'t')\psi(\mathbf{x},t)\psi(\mathbf{x}',t')\end{aligned}$$

with a translationally invariant two-body potential

$$V(\mathbf{x},t;\mathbf{x}',t') = V(\mathbf{x}-\mathbf{x}',t-t'). \tag{1.34}$$

In the systems to be treated in this text we shall be concerned with the potential is, in addition, instantaneous in time

$$V(\mathbf{x},t;\mathbf{x}',t') = \delta(t-t')V(\mathbf{x}-\mathbf{x}'). \tag{1.35}$$

This property will greatly simplify the discussion.

The fundamental field $\psi(x)$ may describe bosons or fermions. The complete information on the the physical properties of the system resides in the Green functions. In the operator Heisenberg picture, these are given by the expectation values of the time-ordered products of the field operators

$$\begin{aligned}&G\left(\mathbf{x}_1,t_1,\ldots,\mathbf{x}_n,t_n;\mathbf{x}_{n'},t_{n'},\ldots,\mathbf{x}_{1'},t_{1'}\right) \qquad (1.36)\\ &\quad = \langle 0|\hat{T}\left(\hat{\psi}_H(\mathbf{x}_1,t_1)\cdots\hat{\psi}_H(\mathbf{x}_n,t_n)\hat{\psi}^\dagger_H(\mathbf{x}_{n'},t_{n'})\cdots\hat{\psi}^\dagger_H(\mathbf{x}_{1'},t_{1'})\right)|0\rangle\end{aligned}$$

The time-ordering operator $\hat{T}$ changes the position of the operators behind it in such a way that earlier times stand to the right of later times. To achieve the final ordering, a number of field transmutations are necessary. If F denotes the number of transmutations of Fermi fields, the final product receibes a sign factor $(-1)^F$.

It is convenient to view all Green functions (1.36) as derivatives of the generating functional

$$Z[\eta^*,\eta] = \langle 0|\hat{T}\exp\left\{i\int d^3xdt\left[\hat{\psi}^\dagger_H(\mathbf{x},t)\eta(\mathbf{x},t) + \eta^*(\mathbf{x},t)\hat{\psi}_H(\mathbf{x},t)\right]\right\}|0\rangle \tag{1.37}$$

namely

$$G\left(\mathbf{x}_1, t_1, \ldots, \mathbf{x}_n, t_n; \mathbf{x}_{n'}, t_{n'}, \ldots, \mathbf{x}_{1'}, t_{1'}\right) \qquad (1.38)$$
$$= (-i)^{n+n'} \left.\frac{\delta^{n+n'} Z[\eta^*, \eta]}{\delta\eta^*(\mathbf{x}_1, t_1) \cdots \delta\eta^*(\mathbf{x}_n, t_n) \delta\eta(\mathbf{x}_{n'}, t_{n'}) \cdots \delta\eta(\mathbf{x}_{1'}, t_{n'})}\right|_{\eta=\eta^*\equiv 0}.$$

Physically, the generating functional describes the amplitude that the vacuum remains a vacuum in spite of the presence of external perturbations.

The calculation of these Green functional is usually performed in the interaction picture which can be summarized by the operator expression for Z:

$$Z[\eta^*, \eta] = N\langle 0|T \exp\left\{i\mathcal{A}_{\rm int}[\psi^\dagger, \psi] + i\int d^3x dt \left[\psi^\dagger(\mathbf{x}, t)\eta(\mathbf{x}, t) + \text{h.c.}\right]\right\}|0\rangle. \quad (1.39)$$

In the interaction picture, the fields $\psi(\mathbf{x}, t)$ possess free-field propagators and the normalization constant N is determined by the condition [which is trivially true for (1.37)]:

$$Z[0, 0] = 1. \qquad (1.40)$$

The standard perturbation theory is obtained by expanding $\exp\{i\mathcal{A}_{\rm int}\}$ in (1.39) in a power series and bringing the resulting expression to normal order via Wick's expansion technique. The perturbation expansion of (1.39) often serves conveniently to *define* an interacting theory. Every term can be pictured graphically and has a physical interpretation as a virtual process.

Unfortunately, the perturbation series up to a certain order in the coupling constant is unable to describe many important physical phenomena, for example bound states in the vacuum and collective excitations in many-body systems. Those require the summation of infinite subsets of diagrams to all orders. In many situations it is well-known which subsets have to be taken in order to account approximately for specific effects. What is not so clear is how such lowest approximations can be improved in a systematic manner. The point is that as soon as a selective summation is performed, the original coupling constant has lost its meaning as an organizer of the expansion and there is need for a new systematics of diagrams. Such a systematics will be presented in what follows.

As soon as bound states or collective excitations are formed, it is very suggestive to use *them* as new quantum fields rather than the original fundamental particles ψ. The goal would then to be rewrite the expression (1.39) for $Z[\eta^*, \eta]$ in terms of new fields whose unperturbed propagator has the free energy spectrum of the *bound states* or collective excitations and whose $\mathcal{A}_{\rm int}$ describes *their* mutual interactions. In the operator form (1.39), however, such changes of fields are hard to conceive.

The ideal theoretical framework for describing a system in terms of the new quantum fields $Z[\eta^*, \eta]$ is offered by the above-intoduced functional integral techniques [1, 2, 3]. In these, changes of fields amount to changes of integration variables, as we shall see in the sequel.

1.3 Functional Formulation

In the functional integral approach, the generating functional (1.37) is given by

$$Z[\eta^*,\eta] = N\int \mathcal{D}\psi^*(\mathbf{x},t)\mathcal{D}\psi(\mathbf{x},t) \times \exp\left\{i\mathcal{A}[\psi^*,\psi] + i\int d^3x dt\,[\psi^*(\mathbf{x},t)\eta(\mathbf{x},t) + \text{c.c.}]\right\}. \tag{1.41}$$

It is worth emphasizing that the field $\psi(\mathbf{x},t)$ in the path integral formulation is a complex number and *not* an operator. All quantum effects are accounted for by fluctuations; the path integral includes not only the classical field configurations but also all classically forbidden ones, i.e., all those which do not run through the valley of extremal action in the exponent.

Finally, we may include an external source for the fields φ, ψ into the integral and solve by quadratic completion. If this is done in the elementary expressions (1.12) and (1.14(te-2.14)), we obtain for both bosons and fermions (dropping product and summation symbols)

$$\int_{-\infty}^{\infty}\frac{d\xi}{\sqrt{2\pi i}}\exp\left(\frac{1}{2}\xi A\xi + ij\xi\right) = \int_{-\infty}^{\infty}\frac{d\xi}{\sqrt{2\pi i}}\exp\left[\frac{i}{2}\left(\xi + jA^{-1}\right)A\left(\xi + A^{-1}j\right) - \frac{i}{2}jA^{-1}j\right] \tag{1.42}$$

$$\int\frac{d\xi^* d\xi}{\sqrt{2\pi i}\sqrt{2\pi i}}\exp(i\xi^* A\xi + ij^*\xi + i\xi^* j) = \int\frac{d\xi^* d\xi}{\sqrt{2\pi i}\sqrt{2\pi i}}\exp\left[i\left(\xi^* + j^*A^{-1}\right)A\left(\xi + A^{-1}j\right) - ij^*A^{-1}j\right]. \tag{1.43}$$

The shift in the integral $\xi \to \xi + A^{-1}\xi$ gives no change due to the infinite range of integration. Hence

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{d\xi}{\sqrt{2\pi i}}\exp\left(\frac{i}{2}\xi A\xi + ij\xi\right) &= \left\{\begin{array}{c}1\\ i^{1/2}\end{array}\right\}A^{\mp 1/2}\exp\left(-\frac{i}{2}jA^{-1}j\right)\\ \int_{-\infty}^{\infty}\frac{d\xi^* d\xi}{\sqrt{2\pi i}\sqrt{2\pi i}}\exp(i\xi^* A\xi + ij^*\xi + i\xi^* j) &= A^{\mp 1}\exp(-ij^*A^{-1}j).\end{aligned} \tag{1.44}$$

A corresponding operation on the functional formulas (1.31) and (1.32) leads to

$$\int \mathcal{D}\varphi(\mathbf{x},t)e^{\frac{i}{2}\int d^3x dt d^3x' dt'\left[\varphi(\mathbf{x},t)A(\mathbf{x},t;\mathbf{x}',t')\varphi(\mathbf{x}',t') + 2j(\mathbf{x},t)\delta^3(\mathbf{x}-\mathbf{x}',t)\delta(t-t')\right]} = e^{i\left(\pm\frac{i}{2}\mathrm{Tr}\log\left\{\begin{smallmatrix}1\\ i\end{smallmatrix}\right\}A\right) - \frac{i}{2}\int d^3x dt d^3x' dt' j(\mathbf{x},t)A^{-1}(\mathbf{x},t;\mathbf{x}',t')j(\mathbf{x}',t')} \tag{1.45}$$

$$\int \mathcal{D}\psi^*(\mathbf{x},t)\mathcal{D}\psi(\mathbf{x},t)e^{i\int d^3x dt d^3x' dt'\left\{\psi^*(\mathbf{x},t)A(\mathbf{x},t;\mathbf{x}',t')\psi(\mathbf{x},t') + [\eta^*(\mathbf{x},t)\psi(\mathbf{x})\delta^3(\mathbf{x}-\mathbf{x}')\delta(t-t') + \text{c.c.}]\right\}} = e^{i(\pm i\mathrm{Tr}\log A) - i\int d^3x dt d^3x' dt \eta^*(\mathbf{x},t)A^{-1}(\mathbf{x},t;\mathbf{x}',t')\eta(\mathbf{x}',t')}. \tag{1.46}$$

These integration formulas will be needed repeatedly in the remainder of this text. They are the basis for the treatment of any interacting quatum field theory.

1.4 Equivalence of Functional and Operator Methods

As an exercise we shall apply (1.45) and (1.46(te-2.24b)) to present a simple proof of the equivalence of Feynman's path integral formula (1.41) and the operator version (1.39(te-2.6)). First we notice that the interaction can be taken outside the integral or the vacuum expectation value in either formula as

$$Z[\eta^*,\eta] = \exp\left\{ i\mathcal{A}_{\rm int}\left[\frac{1}{i}\frac{\delta}{\delta\eta},\frac{1}{i}\frac{\delta}{\delta\eta^*}\right]\right\} Z_0[\eta^*,\eta], \tag{1.47}$$

where Z_0 is the generating functional for the free fields. Thus in Eq. (1.41) there is only $\mathcal{A}_0$ of (1.33(te-2.1)) in the exponent. Since

$$\mathcal{A}_0[\psi^*,\psi] = \int dxdt\psi^*(\mathbf{x},t)\left[i\partial_t - \epsilon(-i\boldsymbol{\nabla})\right]\psi(\mathbf{x},t) \tag{1.48}$$

the functional integral is of the type (1.46) with a matrix

$$A(\mathbf{x},t;\mathbf{x}',t') = \left[i\partial_t - \epsilon(-i\boldsymbol{\nabla})\right]\delta^{(3)}(\mathbf{x}-\mathbf{x}')\delta(t-t'). \tag{1.49}$$

This matrix is the inverse of the free propagator

$$A(\mathbf{x},t;\mathbf{x}',t') = iG_0^{-1}(\mathbf{x},t;\mathbf{x}',t') \tag{1.50}$$

where

$$G_0(\mathbf{x},t;\mathbf{x}',t') = \int\frac{dE}{2\pi}\int\frac{d^3p}{(2\pi)^4}e^{-i[E(t-t')-\mathbf{p}(\mathbf{x}-\mathbf{x}')]}\frac{i}{E-\epsilon(\mathbf{p})+i\eta}. \tag{1.51}$$

Inserting this into (1.46), we see that

$$Z_0[\eta^*,\eta] = N\exp\left[i\left(\pm i\mathrm{Tr}\log iG_0^{-1}\right) - \int d^3xdtd^3x'dt'\eta^*(\mathbf{x},t)G_0(\mathbf{x}',t')\eta(\mathbf{x}',t')\right].$$

We now fix N in accordance with the normalization (1.40) to

$$N = \exp\left[i\left(\pm i\mathrm{Tr}\log iG_0\right)\right] \tag{1.52}$$

and arrive at

$$Z_0[\eta^*,\eta] = \exp\left[-\int d^3xdtd^3x'dt'\eta^*(\mathbf{x},t)G_0(\mathbf{x},t;\mathbf{x}',t')\eta(\mathbf{x}',t')\right]. \tag{1.53}$$

This coincides exactly with what would have been obtained from the operator expression (1.39) for $Z_0[\eta^*,\eta]$ (i.e., without $\mathcal{A}_{\rm int}$).

Indeed, according to Wick's theorem [2, 3, 4], any time ordered product can be expanded as a sum of normal products with all possible contractions taken via Feynman propagators. The formula for an arbitrary functional of free fields ψ,ψ^* is

$$TF[\psi^*,\psi] \;=\; e^{\int d^3xdtd^3x'dt'\frac{\delta}{\delta\psi(\mathbf{x},t)}G_0(\mathbf{x},t;\mathbf{x}',t')\frac{\delta}{\delta\psi^*(\mathbf{x},t')}} : F[\psi^*,\psi] : . \tag{1.54}$$

Applying this to

$$\langle 0|TF[\psi^*,\psi]|0\rangle = \langle 0|T\exp\left[i\int dxdt(\psi^*\eta+\eta^*\psi)\right]|0\rangle \tag{1.55}$$

one finds:

$$\begin{aligned} Z_0[\eta^*,\eta] &= \exp\left[-\int dxdtdx'dt'\eta^*(\mathbf{x},t)G_0(\mathbf{x},t;\mathbf{x}',t')\eta(\mathbf{x}',t')\right] \\ &\quad \times\langle 0| : \exp\left[i\int dxdt(\psi^*\eta+\eta^*\psi)\right] : |0\rangle. \end{aligned} \tag{1.56}$$

The second factor is equal to unity thus proving the equality of this Z_0 with the path integral result (1.53) {which holds for the full $Z[\eta^*,\eta]$ because of (1.47)}.

1.5 Grand-Canonical Ensembles at Zero Temperature

All these results are easily generalized from vacuum expectation values to thermodynamic averages at fixed temperatures T and chemical potential μ. The change at $T=0$ is trivial: The single particle energies in the action (1.33) have to be replaced by

$$\xi(-i\boldsymbol{\nabla}) = \epsilon(-i\boldsymbol{\nabla}) - \mu \tag{1.57}$$

and new boundary conditions have to be imposed upon all Green functions via an appropriate $i\epsilon$ prescription in $G_0(\mathbf{x},t;\mathbf{x}',t')$ of (1.51) [see [2, 5]]:

$${}^{T=0}G_0(\mathbf{x},t;\mathbf{x}',t') = \int\frac{dEd^3p}{(2\pi)^4}e^{-iE(t-t')+i\mathbf{p}(\mathbf{x}-\mathbf{x}')}\frac{i}{E-\xi(\mathbf{p})+i\eta\,\mathrm{sgn}\,\xi(\mathbf{p})} \tag{1.58}$$

Note that, as a consequence of the chemical potential, fermions with $\xi<0$ inside the Fermi sea propagate backwards in time. Bosons, on the other hand, have in general $\xi>0$ and, hence, always propagate forward in time.

In order to simplify the notation we shall often use four-vectors $p=(p^0,\mathbf{p})$ and write the measure of integration in (1.58) as

$$\int\frac{dEd^3p}{(2\pi)^4} = \int\frac{d^4p}{(2\pi)^4}. \tag{1.59}$$

Note that in a solid, the momentum integration is really restricted to a Brioullin zone. If the solid has a finite volume V, the integral over spacial momenta becomes a sum over momentum vectors,

$$\int\frac{d^3p}{(2\pi)^3} = \frac{1}{V}\sum_{\mathbf{p}}, \tag{1.60}$$

and the Green function (1.58) reads

$${}^{T=0}G_0(\mathbf{x},t;\mathbf{x}',t') \equiv \int\frac{dE}{2\pi}\frac{1}{V}\sum_{\mathbf{p}}e^{-ip(x-x')}\frac{i}{p^0-\xi(\mathbf{p})+i\eta\,\mathrm{sgn}\,\xi(\mathbf{p})}. \tag{1.61}$$

The resulting formulas for ${}^{T=0}Z[\eta^*,\eta]$ can be brought to conventional form by performing a Wick rotation in the complex energy plane in all energy integrals (1.58) implied by formulas (1.56(te-2.32)) and (1.39(te-2.6)). For this, one sets $E = p^0 \equiv i\omega$ and replaces

$$\int_{-\infty}^{\infty} \frac{dE}{2\pi} \rightarrow i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}. \tag{1.62}$$

Then the Green function (1.58) becomes

$${}^{T=0}G_0(\mathbf{x},t;\mathbf{x}',t') = -\int \frac{d\omega}{2\pi}\frac{d^3p}{(2\pi)^3} e^{\omega(t-t')+i\mathbf{p}(\mathbf{x}-\mathbf{x}')}\frac{1}{i\omega-\xi(\mathbf{p})}. \tag{1.63}$$

Note that with formulas (1.53) and (1.47(te-2.25)), the generating functional ${}^{T=0}Z[\eta^*,\eta]$ is the grand-canonical partition function in the presence of sources [5].

Finally, we have to introduce arbitrary temperatures T. According to the standard rules of quantum field theory (for an elementary introduction see Chapter 2 in Ref. [2]), we must continue all times to imaginary values $t = i\tau$, restrict the imaginary time interval to the inverse temperature[1] $\beta \equiv 1/T$, and impose periodic or antiperiodic boundary conditions upon the fields $\psi(\mathbf{x},-i\tau)$ of bosons and fermions, respectively [2, 5]:

$$\psi(\mathbf{x},-i\tau) = \pm\psi(\mathbf{x},-i(\tau+1/T)). \tag{1.64}$$

When there is no danger of confusion, we shall usually drop the factor $-i$ in front of the imaginary times in the field arguments, for brevity. The same thing will be done in the Green functions.

By virtue of (1.47) and (1.53(te-2.30)), also the Green functions satisfy these boundary conditions. With the above notation:

$${}^{T}G_0\,(\mathbf{x},\tau+1/T;\mathbf{x}',\tau') \equiv \pm{}^{T}G_0(\mathbf{x},-i\tau;\mathbf{x}',-i\tau'). \tag{1.65}$$

This property is enforced automatically by replacing the energy integrations $\int_{-\infty}^{\infty} d\omega/2\pi$ in (1.63) by a summation over the discrete Matsubara frequencies [in analogy to the momentum sum (1.60), the temporal "volume" being $\beta = 1/T$]

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \rightarrow T\sum_{\omega_n} \tag{1.66}$$

which are even or odd multiples of πT

$$\omega_n = \left\{ \begin{array}{c} 2n \\ 2n+1 \end{array} \right\} \pi T \;\; \text{for} \;\; \left\{ \begin{array}{c} \text{bosons} \\ \text{fermions} \end{array} \right\}. \tag{1.67}$$

The prefactor T of the sum over the discrete Matsubara frequencies accounts for the density of these frequencies yielding the correct $T \rightarrow 0$-limit.

[1]Throughout these lectures we shall use natural units so that $k_B = 1, \hbar = 1$.

Thus we obtain for the imaginary-time Green function of a free nonrelativistic field at finite temperature (the so-called *free thermal Green function*) the following expression:

$$ {}^{T}G_0(\mathbf{x},\tau,\mathbf{x}',\tau') = -\,T\sum_{\omega_n}\int \frac{d^3p}{(2\pi)^3} e^{-i\omega_n(\tau-\tau')+i\mathbf{p}(\mathbf{x}-\mathbf{x}')}\frac{1}{i\omega_n-\xi(\mathbf{p})}. \tag{1.68}$$

Incorporating the Wick rotation in the sum notation we may write

$$ T\sum_{p_0} = -iT\sum_{\omega_n} = -iT\sum_{p_4}. \tag{1.69}$$

where $p_4 = -ip_0 = \omega$. If both temperatures, and volume are finite, the Green function will be written as

$$ {}^{T}G_0(\mathbf{x},\tau,\mathbf{x}',\tau') = -\,\frac{T}{V}\sum_{p_0}\sum_{\mathbf{p}} e^{-i\omega_n(\tau-\tau')+i\mathbf{p}(\mathbf{x}-\mathbf{x}')}\frac{1}{i\omega_n-\xi(\mathbf{p})}. \tag{1.70}$$

At equal space points and equal imaginary times, the sum can easily be evaluated. One must, however, specify the order in which $\tau \to \tau'$. Let η denote an infinitesimal positive number and consider the case $\tau' = \tau+\eta$, i.e., the Green function

$$ {}^{T}G_0(\mathbf{x},\tau,\mathbf{x},\tau+\eta) = -\,T\sum_{\omega_n}\int \frac{d^3p}{(2\pi)^3} e^{i\omega_n\eta}\frac{1}{i\omega_n-\xi(\mathbf{p})}.$$

The sum is now found by changing it into a contour integral

$$ T\sum_{\omega_n} e^{i\omega_n\eta}\frac{1}{i\omega_n-\xi(\mathbf{p})} = \frac{T}{2\pi i}\int_C dz\frac{e^{\eta z}}{e^{z/T}\mp 1}\frac{1}{z-\xi}. \tag{1.71}$$

The upper sign holds for bosons, the lower for fermions. The contour of integration C encircles the imaginary z axis in the positive sense, thereby enclosing all integer or half-integer valued poles of the integrand at the Matsubara frequencies $z = i\omega_m$ (see Fig. 1.1). The factor $e^{\eta z}$ ensures that the contour in the left half-plane does not contribute.

By deforming the contour C into C' and contracting C' to zero we pick up the pole at $z=\xi$ and find

$$ T\sum_{\omega_n} e^{i\omega_n\eta}\frac{1}{i\omega_n-\xi(\mathrm{p})} = \mp\frac{1}{e^{\xi(\mathbf{p})/T}\mp 1} = \mp\frac{1}{e^{\xi(\mathbf{p})/T}\mp 1} = \mp n(\xi(\mathbf{p})). \tag{1.72}$$

The phase $e^{\eta z}$ ensures that the contour in the left half-plane does not contribute. The function on the right is known as the *Bose or Fermi distribution function.*

By subtracting from (1.72) the sum with ξ replaced by $-\xi$, we obtain the important sum formula

$$ T\sum_{\omega_n}\frac{1}{\omega_n^2+\xi^2(\mathrm{p})} = \frac{1}{2\xi(\mathbf{p})}\coth^{\pm 1}\frac{\xi(\mathbf{p})}{T}. \tag{1.73}$$

FIGURE 1.1 Contour C in the complex z-plane for evaluating the Matsubara sum (1.72)

In the opposite limit $\tau' = \tau - \eta$, the phase factor in the sum would be $e^{-i\omega_m\eta}$ and would be converted into a contour integral

$$-k_BT\sum_{\omega_m} e^{i\omega_m\eta}\frac{1}{i\omega_m - \xi(\mathbf{p})} = \pm\frac{k_BT}{2\pi i}\int_C dz\frac{e^{-\eta z}}{e^{-z/k_BT}\mp 1}\frac{1}{z-\xi}, \tag{1.74}$$

yielding $1 \pm n_{\xi(\mathbf{p})}$.

In the operator language, these limits correspond to the expectation values

$$\begin{aligned}
{}^TG\left(\mathbf{x},\tau;\mathbf{x},\tau+\eta\right) &= \langle 0|\hat{T}\left(\hat{\psi}_H(\mathbf{x},\tau)\hat{\psi}_H^\dagger(\mathbf{x},\tau+\eta)\right)|0\rangle = \pm\langle 0|\hat{\psi}_H^\dagger(\mathbf{x},\tau)\hat{\psi}_H(\mathbf{x},\tau)|0\rangle \\
{}^TG\left(\mathbf{x},\tau;\mathbf{x},\tau-\eta\right) &= \langle 0|\hat{T}\left(\hat{\psi}_H(\mathbf{x},\tau)\hat{\psi}_H^\dagger(\mathbf{x},\tau-\eta)\right)|0\rangle = \langle 0|\hat{\psi}_H(\mathbf{x},\tau)\hat{\psi}_H^\dagger(\mathbf{x},\tau)|0\rangle \\
&= 1 \pm \langle 0|\hat{\psi}_H^\dagger(\mathbf{x},\tau)\hat{\psi}_H(\mathbf{x},\tau\mp\eta)|0\rangle
\end{aligned}$$

The function $n(\xi(\mathbf{p}))$ is the thermal expectation value of the number operator

$$\hat{N} = \hat{\psi}_H^\dagger(\mathbf{x},\tau)\hat{\psi}_H(\mathbf{x},\tau). \tag{1.75}$$

Also in the case of $T \neq 0$ ensembles, it is useful to employ a four-vector notation. The four-vector

$$p_E \equiv (p_4, \mathbf{p}) = (\omega, \mathbf{p}) \tag{1.76}$$

is called the *euclidean four-momentum*. Correspondingly, we define the *euclidean spacetime coordinate*

$$x_E \equiv (-\tau, \mathbf{x}). \tag{1.77}$$

The the exponential in (1.68) can be written as

$$p_E x_E = -\omega\tau + \mathbf{p}\mathbf{x}. \tag{1.78}$$

Collecting integral and sum in a single four-summation symbol, we shall write (1.68) as

$$ {}^{T}G_0(x_E - x') \equiv -\frac{T}{V}\sum_{p_E} \exp\left[-ip_E(x_E - x'_E)\right] \frac{1}{ip_4 - \xi(\mathbf{p})}. \tag{1.79} $$

It is quite straightforward to derive the general $T \neq 0$ Green function from a path integral formulation analogous to (1.41). For this we consider classical fields $\psi(\mathbf{x}, \tau)$ with the periodicity or anti-periodicity

$$ \psi(\mathbf{x}, \tau) = \pm\psi\left(\mathbf{x}, \tau + 1/T\right). \tag{1.80} $$

They can be Fourier-decomposed as

$$ \psi(\mathbf{x}, \tau) = \frac{T}{V}\sum_{\omega_n}\sum_{\mathbf{p}} e^{-i\omega_n\tau + i\mathbf{px}} a(\omega_n, \mathbf{p}) \equiv \frac{T}{V}\sum_{p_E} e^{-ip_E x_E} a(p_E) \tag{1.81} $$

with a sum over even or odd Matsubara frequencies ω_n. If now a free action is defined as

$$ \mathcal{A}_0[\psi^*, \psi] = -i\int_{-1/2T}^{1/2T} d\tau \int d^3x\psi^*(\mathbf{x}, \tau)\left[-\partial_\tau - \xi\left(-i\boldsymbol{\nabla}\right)\right]\psi(\mathbf{x}, \tau) \tag{1.82} $$

formula (1.46) renders [1, 6]

$$ {}^{T}Z_0[\eta^*, \eta] = e^{\mp\mathrm{Tr}\log A + \int\int_{-1/2T}^{1/2T} d\tau d\tau' \int d^3x d^3x' \eta^*(\mathbf{x},\tau)A^{-1}(\mathbf{x},\tau,\mathbf{x}',\tau')\eta(\mathbf{x}',\tau')} \tag{1.83} $$

with

$$ A(\mathbf{x}, \tau; \mathbf{x}', \tau') = \left[\partial_\tau + \xi\left(-i\boldsymbol{\nabla}\right)\right]\delta^{(3)}(\mathbf{x} - \mathbf{x}')\delta(\tau - \tau'), \tag{1.84} $$

and henceforth A^{-1} equal to the propagator (1.68), the Matsubara frequencies arising due to the finite τ interval of Euclidean space together with the periodic boundary condition (1.80).

Again, interactions are taken care of by multiplying ${}^{T}Z_0[\eta^*\eta]$ with the factor (1.47). In terms of the fields $\psi(\mathbf{x}, \tau)$, the exponent has the form:

$$ \begin{aligned} \mathcal{A}_{\mathrm{int}} &= \frac{1}{2}\int\int_{-1/2T}^{1/2T} d\tau d\tau' \\ &\quad\times\int d^3x d^3x' \psi^*(\mathbf{x}, \tau)\psi^*(\mathbf{x}', \tau')\psi(\mathbf{x}', \tau')\psi(\mathbf{x}, \tau)V(\mathbf{x}, -i\tau; \mathbf{x}', -i\tau'). \end{aligned} \tag{1.85} $$

In the case of an instantaneous potential (1.34), the potential becomes instantaneous in τ:

$$ V(\mathbf{x}, -i\tau; \mathbf{x}', -i\tau') = V(\mathbf{x} - \mathbf{x}')\; i\delta(\tau - \tau'). \tag{1.86} $$

In this case $\mathcal{A}_{\rm int}$ can be written in terms of the interaction Hamiltonian as

$$\mathcal{A}_{\rm int} = i\int_{-1/2T}^{1/2T} d\tau H_{\rm int}(\tau). \qquad (1.87)$$

Thus the grand canoncial partition function in the presence of external sources may be calculated from the path integral [6]:

$${}^{T}Z[\eta^*,\eta] = \int \mathcal{D}\psi^*(\mathbf{x},\tau)\mathcal{D}\psi(\mathbf{x},\tau)e^{i\,{}^{T}\!\mathcal{A}+\int_{-1/2T}^{1/2T}d\tau\int d^3x[\psi^*(\mathbf{x},\tau)\eta(\mathbf{x},\tau)+{\rm c.c.}]} \qquad (1.88)$$

where the grand-canonical action is

$$i\,{}^{T}\!\mathcal{A}[\psi^*,\psi] = -\int_{-1/2T}^{1/2T} d\tau \int d^3x\psi^*(\mathbf{x},\tau)\left[\partial_\tau + \xi(-i\boldsymbol{\nabla})\right]\psi(\mathbf{x},\tau) \qquad (1.89)$$
$$+\frac{i}{2}\int_{-1/2T}^{1/2T} d\tau d\tau' \int d^3xd^3x'\psi^*(\mathbf{x},\tau)\psi^*(\mathbf{x}',\tau')\psi(\mathbf{x},\tau')\psi(\mathbf{x},\tau)V(\mathbf{x},-i\tau;\mathbf{x},-i\tau').$$

$$G\left(\mathbf{x}_1,\tau_1,\dots,\mathbf{x}_n,\tau_n;\mathbf{x}_{n'},\tau_{n'},\dots,\mathbf{x}_{1'},\tau_{1'}\right) \qquad (1.90)$$
$$= (-i)^{n+n'}\left.\frac{\delta^{n+n'}Z[\eta^*,\eta]}{\delta\eta^*(\mathbf{x}_1,\tau_1)\cdots\delta\eta^*(\mathbf{x}_n,\tau_n)\delta\eta(\mathbf{x}_{n'},\tau_{n'})\cdots\delta\eta(\mathbf{x}_{1'},\tau_{n'})}\right|_{\eta=\eta^*\equiv 0}.$$

The right-hand side consists of the functional integrals

$$N\int \mathcal{D}\psi^*(\mathbf{x},t)\mathcal{D}\psi(\mathbf{x},t)\,\hat\psi(\mathbf{x}_1,\tau_1)\cdots\hat\psi(\mathbf{x}_n,\tau_n)\hat\psi^*(\mathbf{x}_{n'},\tau_{n'})\cdots\hat\psi^*(\mathbf{x}_{1'},\tau_{1'})e^{i\mathcal{A}[\psi^*,\psi]} \qquad (1.91)$$

In the sequel, we shall always assume the normalization factor to be chosen in such a way that $Z[0,0]$ is normalized to unity. Then the functional integrals (1.91) are obviously the *correlation function* of the the fields commonly written in the form

$$\langle\hat\psi(\mathbf{x}_1,\tau_1)\cdots\hat\psi(\mathbf{x}_n,\tau_n)\hat\psi^*(\mathbf{x}_{n'},\tau_{n'})\cdots\hat\psi^*(\mathbf{x}_{1'},\tau_{1'})\rangle$$

In contrast to Section 1.2, the bra and ket symbols denote now a thermal average of the classical fields.

The functional integral expression (1.88) for the generating functional offers the advantageous flexibility with respect to changes in the field variables.

Summarizing we have seen that the functional (1.88) defines the most general type of theory involving two-body forces. It contains all information on the physical system in the vacuum as well as in thermodynamic ensembles. The vacuum theory is obtained by setting $T = 0$, $\mu = 0$, and continuing the result back from T to physical times. Conversely, the functional (1.41) in the vacuum can be generalized to ensembles in the straight-forward manner by first continuing the times t to imaginary values $-i\tau$ via a Wick rotation in all energy integrals and then going to periodic functions in τ.

There is a complete correspondence between the real-time generating functional (1.41) and the thermodynamic imaginary-time expression (1.88(te-2.53)). For this reason it will be sufficient to exhibit all techniques only in one version for which we shall choose (1.41). Note, however, that due to the singular nature of the propagators (1.51) in real energy-momentum, the thermodynamic formulation specifies the way to specifies how to avoid singularities.

2

Relativistic Fields

We shall also study collective pheomena in relativistic fermion systems. For this we shall need fields describing relativistic particles of spin zero, 1/2, and 1. Their properties will now be briefly reviewed.

2.1 Lorentz and Poincaré Invariance

For relativistic particles, the relation between the physical laws in two coordinate frames which move with a constant velocity with resect to each other are different from the nonrelativistic case. Suppose a frame moves with velocity v into the $-z$-direction of another fixed frame. Then in the moving frame, the z momentum of the particle will appear increased. The particle appears *boosted* in the z direction with respect to the original observer. The momenta in x and y directions are unaffected. Now, the total four momentum still satisfies the energy momentum relation

$$E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + M^2}. \tag{2.1}$$

Introducing the four-vector notation

$$p^\mu \equiv (p^0, p^i) \quad \text{with} \quad p0 \equiv (\mathbf{p})/c, \tag{2.2}$$

we see that the four-vector satisfies the mass shell condition

$${p^0}^2 - \mathbf{p}^2 = M^2. \tag{2.3}$$

For the particle moving in z-direction, the combination ${p^0}^2 - {p^3}^2$ remains invariant. This implies that there must be a hyperbolic transformation mixing p^0 and p^3 which may be parametrized by a hyperbolic angle ζ, called *rapidity*:

$$\begin{aligned} p'^0 &= \cosh\zeta\, p^0 + \sinh\zeta\, p^3, \\ p'^3 &= \sinh\zeta\, p^0 + \cosh\zeta\, p^3. \end{aligned} \tag{2.4}$$

This is called a pure Lorentz transformation. We may write this transformation in a 4×4 matrix form as

$$p'^\mu = \begin{pmatrix} \cosh\zeta & 0 & 0 & \sinh\zeta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\zeta & 0 & 0 & \cosh\zeta \end{pmatrix}^\mu{}_\nu p^\nu \equiv B_3(\zeta)^\mu{}_\nu p^\nu. \tag{2.5}$$

The subscript 3 of B_3 indicates that the particle is boosted into the z-direction. A similar matrix can be written down for x and y-directions. In an arbitrary direction $\hat{\mathbf{p}}$, the the matrix elements are

$$B_{\hat{\mathbf{p}}}(\zeta) = \left(\begin{array}{c|c} \cosh\zeta & \hat{p}^i \sinh\zeta \\ \hline \hat{p}^i \sinh\zeta & \delta^{ij} + \hat{p}^i \hat{p}^j (\cosh\zeta - 1) \end{array} \right). \tag{2.6}$$

By combining rotations and boosts one obtains a 6-parameter manifold of matrices

$$\Lambda = B_{\hat{\mathbf{p}}}(\zeta) R_{\hat{\boldsymbol{\varphi}}}(\varphi), \tag{2.7}$$

called *proper Lorentz transformations*. For all these

$$p'^{0^2} - \mathbf{p}'^2 = p^{0^2} - \mathbf{p}^2 = M^2 c^2 \tag{2.8}$$

is an invariant. These matrices form a group, the *proper Lorentz group*. We can easily see that the Lorentz group allows reaching *every* momentum p^μ on the mass shell by applying an appropriate group element to some *fixed* reference momentum p_R^μ. For example, if the particle has a mass M we may choose for p_R^μ the so-called *rest momentum*

$$p_R^\mu = (M, 0, 0, 0), \tag{2.9}$$

and apply the boost in the $\hat{\mathbf{p}}$ direction

$$\Lambda = B_{\hat{\mathbf{p}}}(\zeta), \tag{2.10}$$

with the rapidity given by

$$\cosh\zeta = \frac{p^0}{M}, \quad \sinh\zeta = \frac{|\mathbf{p}|}{M}. \tag{2.11}$$

But we also may choose $\Lambda(p) = B_{\hat{\mathbf{p}}}(\zeta) R_{\hat{\boldsymbol{\varphi}}}(\varphi)$ where R is an arbitrary rotation, since these leave the rest momentum p_R^μ invariant. In fact, the rotations form the largest subgroup of the group of all proper Lorentz transformations which leaves the rest momentum p_R^μ invariant. It is referred to as the *little group* or *Wigner group* of a massive particle. It has an important physical significance since it serves to specify the intrinsic rotational degrees of freedom of the particle. If the particle is at rest it carries no orbital angular momentum. If it happens that its quantum mechanical state remains completely invariant under the little group R, the particle must also have zero intrinsic angular momentum or zero *spin*. Besides this trivial representation, the little group being a rotation group can have representations of any angular momentum $s = \frac{1}{2}, 1, \frac{3}{2}, \ldots$. In these cases, the state at rest has $2s+1$ components which are mixed with each other upon rotations.

The situation is quite different in the case of massless particles. They move with the speed of light and p^μ cannot be brought by a Lorentz transformation from the light cone to a rest frame. There is, however, another standard reference momentum from which one can generate all other momenta on the light cone. It is given by is

$$p_R^\mu = (1, 0, 0, 1)p. \tag{2.12}$$

It remains invariant under a different little group, which is again a three-parameter subgroup of the Lorentz group. This will be discussed later.

It is useful to write the invariant expression (2.8) as a square of a four vector p^μ formed with the metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \tag{2.13}$$

namely

$$p^2 = g_{\mu\nu} p^\mu p^\nu. \tag{2.14}$$

In general, we define a scalar product between any two vectors as

$$pp' \equiv g_{\mu\nu} p^\mu p'^\nu = p^0 p'^0 - \mathbf{p}\mathbf{p}'. \tag{2.15}$$

A space with this scalar product is called *Minkowski space*. It is useful to introduce the covariant components of any vector v^μ as

$$v_\mu \equiv g_{\mu\nu} v^\nu. \tag{2.16}$$

Then the scalar product can also be written as

$$pp' = p_\mu p'^\mu. \tag{2.17}$$

With this notation the mass shell condition for a particle before and after a Lorentz transformation reads simply

$$p'^2 = p^2 = M^2 c^2. \tag{2.18}$$

Note that, apart from the minus signs in the metric (2.13), the mass shell condition $p^2 = p^{0^2} - p^{1^2} - p^{2^2} - p^{3^2} = M^2c^2$, left invariant by the Lorentz group, is completely analogous to the spherical condition $p^{4^2} + p^{1^2} + p^{2^2} + p^{3^2} = M^2c^2$ which is left invariant by the rotation group in a four-dimensional euclidean space. Both groups are parametrized by six parameters which are associated with linear transformations in the six planes $12, 23, 31; 10, 20, 30$ or $12, 23, 31; 14, 24, 34$, respectively. In the case of the four-dimensional euclidean space these are all rotations which form the group of *special orthogonal matrices* called $SO(4)$. The letter S indicates the property *special* . A group is called *special* if all its transformation matrices have a unit determinant. In the case of the proper Lorentz group one uses by analogy the notation $SO(1,3)$. The numbers indicate the fact that in the metric (2.13), one diagonal element is equal to $+1$ and three are equal to -1.

The fact that all group elements are "special" follows from a direct calculation of the determinant of (2.6), (2.7(te-4.10)).

How do we have to describe the quantum mechanics of a free relativistic particle in Minkowski space? The energy and momenta $p^0, \mathbf{p}$ must be related to the time and space derivatives of particle waves in the usual way

$$\begin{aligned} p^0 &= \frac{\epsilon}{c} = i\hbar \frac{\partial}{\partial ct} \equiv i\hbar \frac{\partial}{\partial x^0}, \\ p^i &= -i\hbar \frac{\partial}{\partial x^i}. \end{aligned} \tag{2.19}$$

They satisfy the canonical commutation rules

$$\begin{aligned}
[p^\mu, p^\nu] &= 0, \\
[x^\mu, x^\nu] &= 0, \\
[p^\mu, x^\nu] &= -i\hbar g^{\mu\nu}.
\end{aligned} \tag{2.20}$$

We expect that associated with the pure momentum state $\mathbf{p}$ there will be some wave function

$$f_{\mathbf{p}}(x) = e^{-i(p^0x^0 - p^ix^i)/\hbar} \equiv e^{-ipx/\hbar}. \tag{2.21}$$

At this point we do not yet know the proper scalar product necessary to extract physical information from such wave functions.

We have stated previously that permissible energy momentum states of a free particle can be realized by considering *one and the same* particle in different coordinate frames connected by the transformation $\Lambda(p)$. Suppose that we change the coordinates of the same space time point as follows:

$$x \to x' = \Lambda x. \tag{2.22}$$

Under this transformation the scalar product of any two vectors remains invariant:

$$x'y' = xy. \tag{2.23}$$

This holds also for scalar products between momentum and coordinate vectors

$$p'x' = px. \tag{2.24}$$

For the transformation matrix Λ this implies that

$$(\Lambda p)(\Lambda x) = px. \tag{2.25}$$

If the scalar products are written out explicitly in terms of the metric $g_{\mu\nu}$ this amounts to

$$g_{\mu\nu}\Lambda^\mu{}_\lambda p^\lambda \Lambda^\nu{}_\kappa x^\kappa = g_{\lambda\kappa} p^\lambda x^\kappa, \tag{2.26}$$

for all p, x. The Lorentz matrices Λ satisfy therefore the identity

$$g_{\mu\nu}\Lambda^\mu{}_\lambda \Lambda^\nu{}_\kappa = g_{\lambda\kappa}, \tag{2.27}$$

or, written without indices,

$$\Lambda^T g \Lambda = g. \tag{2.28}$$

If the metric were euclidean, this would be the definition of orthogonal matrices. In fact, In the notation of scalar products in which the metric is suppressed as in (2.29), there is no difference between the manipulation of orthogonal and Lorentz matrices, since

$$(\Lambda p)(\Lambda x) = p\Lambda^{-1}\Lambda x = px. \tag{2.29}$$

When changing the coordinates, the same particle wave in space will behave like

$$\begin{aligned} f_p(x) &= e^{-ip\Lambda^{-1}x'/\hbar} \\ &= e^{-i(\Lambda p)x'/\hbar} = f_{\Lambda p}(x') = f_{p'}(x'). \end{aligned} \tag{2.30}$$

This shows that in the new coordinates the same particle appears with a different momentum components

$$p' = \Lambda p. \tag{2.31}$$

Consider a wave $\psi(x)$ which is an arbitrary superposition of different momentum states. After a coordinate transformation it will still have the same value at the same space time point. Thus $\psi'(x')$, as seen in the new frame, must be equal to $\psi(x)$ in the old frame

$$\psi'(x') = \psi(x). \tag{2.32}$$

At this place one defines the *substantial change* under the Lorentz transformation Λ as the change at the same values of the coordinates x (which corresponds to a transformed point in space)

$$\psi(x) \xrightarrow{\Lambda} \psi'_\Lambda(x) = \psi(\Lambda^{-1}x). \tag{2.33}$$

We have marked the transformation under which $\psi'(x)$ arises as a subscript. Clearly, this transformation property is valid only if the particle does not possess any intrinsic orientational degree of freedom, i.e., no spin. A field with this properties is called a *scalar field* or, for historical reasons, a *Klein-Gordon field*.

If a particle has spin degrees of freedom the situation is quite different. Then the wave function has several components to account for the spin orientations. The transformation law must be such that the spin orientation in space remain the same at the same space point. This implies that the field components which specify the orientation with respect to the different coordinate axes will have to be transformed by certain matrices. How this goes is well-known in the case of electromagnetic and gravitational fields which have vector and tensor transformation properties. In the next sections these will be recalled. Afterwards it will be easy to generalize everything to the case of arbitrary spin.

Before coming to this, however, let us conclude this section by mentioning that there are other space transformations which leave the scalar products $p_\mu x^\mu$ invariant but which are not contained in the group $SO(1,3)$: These are the *space inversion*, also called *mirror reflection* or *parity transformation*

$$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \tag{2.34}$$

which reverses the direction of the spatial vectors, $\mathbf{x} \to -\mathbf{x}$, and the *time inversion*

$$T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \tag{2.35}$$

which changes the sign of x^0. If P and T are incorporated into the special Lorentz group $SO(1,3)$ one speaks of the *full Lorentz group*.

Note that the determinants of both (2.34) and (2.35(te-4.25)) is negative so that the full Lorentz group no longer deserves the letter S in its name. It is called $O(1,3)$.

2.2 Relativistic Free Scalar Fields

It is then obvious how the non-relativistic free field action

$$\mathcal{A} = \int dt dx \psi^*(\mathbf{x},t) \left[i\hbar\partial_t + \hbar^2 \frac{{\partial_\mathbf{x}}^2}{2M} \right] \psi(\mathbf{x},t) \tag{2.36}$$

must describe relativistic n-particle states. In order to accommodate the kinematic features discussed in the last section we require the action to be invariant under Lorentz transformations. Depending on the possible internal spin degrees of freedom there are different ways of making the action relativistic. These will now be discussed separately.

2.2.1 Scalar Fields

If the field $\psi(\mathbf{x},t)$ carries no spin degree of freedom which varies under space rotations, the spatial derivative $\partial_\mathbf{x}$ always has to appear squared in the action to guarantee rotational invariance. With the Lorentz symmetry between ∂_0 and $\partial_\mathbf{x}$ we are led to a classical action

$$\mathcal{A} = \int dx^0 L = \int dx^0 d^3x \psi^*(\mathbf{x},t) \left[c_1 \partial^\mu \partial_\mu + c_2 \right] \psi(\mathbf{x},t), \tag{2.37}$$

where c_1, c_2 are two arbitrary real constants. It is now easy to see that this action is indeed Lorentz invariant: Under the transformation (2.22), the four-volume element does not change

$$dx^0 d^3x \equiv d^4x \to d^4x' = d^4x. \tag{2.38}$$

If we therefore take the action in the new frame

$$\mathcal{A} = \int d^4x' \psi^{*\prime}(x') \left[c_1 \partial'^\mu \partial'_\mu + c_2 \right] \psi'(x'), \tag{2.39}$$

we can use (2.37) and (2.32(te-4.24)) to rewrite

$$\mathcal{A} = \int d^4x \psi^*(x) \left[c_1 \partial'^\mu \partial'_\mu + c_2 \right] \psi(x). \tag{2.40}$$

But since

$$\partial'_\mu = \Lambda_\mu{}^\nu \partial_\nu, \quad \partial^{\mu\prime} = \Lambda^\mu{}_\nu \partial^\nu \tag{2.41}$$

with $\Lambda_\mu{}^\nu \equiv g_{\mu\lambda} g^{\nu\kappa} \Lambda^\lambda{}_\kappa$, we see that

$$\partial'^2 \;=\; \partial^2, \tag{2.42}$$

and the transformed action becomes

$$\mathcal{A} = \int dx^0 d^3x \psi^*(\mathbf{x}, t) \left[c_1 \partial^\mu \partial_\mu + c_2\right] \psi(\mathbf{x}, t), \tag{2.43}$$

which is the same as (2.37).

It is useful to introduce the integrand of the action as the so called *Lagrangian density*

$$\mathcal{L}(\mathbf{x}, t) = \psi^*(\mathbf{x}, t) \left[c_1({\partial^0}^2 - {\partial_{\mathbf{x}}}^2) + c_2\right] \psi(\mathbf{x}, t). \tag{2.44}$$

Then the invariance of the action under Lorentz transformation is a direct consequence of the Lagrangian density being a scalar field, satisfying the transformation law (2.32),

$$\mathcal{L}'(x') = \mathcal{L}(x), \tag{2.45}$$

as implied by (2.39), (2.40(te-4.31)), and (2.46(te-4.33)).

The free-field equation of motion are derived from (2.37) as follows. We write

$$\mathcal{A} = \int dx^0 L = \int dx^0 \int d^3x \psi^*(\mathbf{x}, t) \left[c_1({\partial^0}^2 - {\partial_{\mathbf{x}}}^2) + c_2\right] \psi(\mathbf{x}, t), \tag{2.46}$$

and vary this with respect to the fields $\psi(x), \psi^*(x)$ independently. The independence of these variables is expressed by the functional differentiation rules

$$\begin{aligned} \frac{\delta\psi(x)}{\delta\psi(x')} &= \delta^{(4)}(x - x'), \quad \frac{\delta\psi^*(x)}{\delta\psi^*(x')} = \delta^{(4)}(x - x'), \\ \frac{\delta\psi(x)}{\delta\psi^*(x')} &= 0, \qquad\qquad\quad \frac{\delta\psi^*(x)}{\delta\psi(x')} = 0. \end{aligned} \tag{2.47}$$

Applying these rules to (2.46) we obtain directly

$$\begin{aligned} \frac{\delta\mathcal{A}}{\delta\psi^*(x)} &= \int d^4x' \delta^{(4)}(x' - x)(c_1\partial^2 + c_2)\psi(x) \\ &= (c_1\partial^2 + c_2)\psi(x) = 0. \end{aligned} \tag{2.48}$$

Similarly

$$\begin{aligned} \frac{\delta\mathcal{A}}{\delta\psi(x)} &= \int d^4x' \psi^*(x')(c_1\partial^2 + c_2)\delta(x' - x) \\ &= \psi^*(x)(c_1 \overleftarrow{\partial}^2 + c_2), \end{aligned} \tag{2.49}$$

where the arrow on top of the last derivative indicates it acts on the field to the left. The second equation is just the complex conjugate of the previous one. Then the functional derivative with respect to $\psi^*(x)$ is simply In terms of the Lagrangian density, the extremality condition can be expanded in terms of partial derivatives with respect to increasin partial derivatives of all fields in $\mathcal{L}$,

$$\frac{\delta\mathcal{A}}{\delta\psi(x)} = \frac{\partial\mathcal{L}(x)}{\partial\psi(x)} - \partial_\mu \frac{\partial\mathcal{L}(x)}{\partial\partial_\mu\psi(x)} + \partial_\mu\partial_\nu \frac{\partial\mathcal{L}(x)}{\partial\partial_\mu\partial_\nu\psi(x)} + \ldots\ , \tag{2.50}$$

with the same equation for $\psi^*(x)$. This follows directly from the defining relations (2.47). The field equation for $\psi(x)$ is particularly simple:

$$\frac{\delta \mathcal{A}}{\delta\psi^*(x)} = \frac{\partial\mathcal{L}(x)}{\partial\psi^*(x)}. \tag{2.51}$$

For $\psi^*(x)$, on the other hand, all derivatives written out in (2.50) have to be evaluated.

Both field equations (2.48) and (2.49(te-4.equm2)) are solved by the quantum mechanical plane wave (2.21)

$$f_p(x) = e^{-ipx/\hbar}, \tag{2.52}$$

if the momentum satisfies the condition

$$-c_1 p^\mu p_\mu + c_2 = 0. \tag{2.53}$$

This has precisely the form of the mass shell relation (2.18) if we choose

$$c_2\hbar^2/c_1 = M^2c^2. \tag{2.54}$$

It is customary to normalize c_1 to

$$c_1 = -\hbar^2. \tag{2.55}$$

The sign is necessary to have stable field fluctuations. The size can always be brought to this value by a multiplicative renormalizaton of the field. Then the mass shell condition fixes the free field action to the standard form

$$\mathcal{A} = \int dx^0 d^3x \psi^*(\mathbf{x},t)\left[-\hbar^2\partial^\mu\partial_\mu - M^2c^2\right]\psi(\mathbf{x},t), \tag{2.56}$$

The appearance of the constants $\hbar$ and c in all future formulas can be avoided if we agree to work from now on with new fundamental units l_0, m_0, t_0, E_0 different from the ordinary cgs units. They are chosen to give $\hbar$ and c have the values 1. Expressed in terms of the conventional length, time, mass, and energy, these new *natural units* are given by

$$l_0 = \frac{\hbar}{Mc} = \frac{\hbar}{E_0}c, \qquad t_0 = \frac{\hbar}{Mc^2}, \tag{2.57}$$

$$m_0 = M, \quad E_0 = Mc^2. \tag{2.58}$$

If, for example, the particle is a proton with mass m_p, these units are

$$l_0 = 2.103138 \times 10^{-11}\text{cm} = \text{Compton wavelength of proton}, \tag{2.59}$$

$$t_0 = l_0/c = 7.0153141 \times 10^{-22}\text{sec} = \text{time it takes light to cross the Compton wavelength}, \tag{2.60}$$

$$m_0 = m_p = 1.6726141 \times 10^{-24}\text{g}, \tag{2.61}$$

$$E_0 = 938.2592\,\text{MeV}. \tag{2.62}$$

For any other mass, they can easily be rescaled.

With these natural units we can drop c and $\hbar$ in all formulas and write the action simply as

$$\mathcal{A} = \int d^4x \mathcal{L}(x) = \int d^4x \psi^*(x)(-\partial^2 - M^2)\psi(x). \tag{2.63}$$

Actually, since we are dealing with relativistic particles there is no fundamental reason to assume $\psi(x)$ to be a complex field. In the non-relativistic theory this was necessary in order to construct a term linear in the time derivative

$$\int dt \psi^* i\partial_t \psi. \tag{2.64}$$

For a real field $\psi(x)$ this would have been a pure surface term and thus not influenced the dynamics of the system. For second-order time derivatives as in (2.63) this is no longer necessary.

Thus we shall also study the real scalar field with an action

$$\mathcal{A} = \int d^4x \mathcal{L}(x) = \frac{1}{2} \int d^4x \phi(x)(-\partial^2 - M^2)\phi(x). \tag{2.65}$$

In this case, a prefactor $\frac{1}{2}$ is the normalization convention for the field. Also, we have used the letter $\phi(x)$ to denote the real field, as is commonly done.

2.3 Electromagnetic Field

Electromagnetic fields move with light velocity and have no mass term.[1] The fields have two polarization degrees of freedom (right and left polarized) and are described by the usual electromagnetic action. Historically, this was the very first example of a relativistic classical field theory. Thus it could also have served as a guideline for the previous construction of the action of the scalar field $\phi(\mathbf{x})$.

The action may be given in terms of a real auxiliary four-vector potential $A_\mu(x)$ from which the physical electric and magnetic fields can be derived as follows

$$E^i = -(\partial^0 A^i - \partial^i A^0) = -\partial_t A^i - \partial_i A^0, \tag{2.66}$$

$$H^i = -\frac{1}{2}\epsilon_{ijk}(\partial^i A^k - \partial^k A^i) = \frac{1}{2}\epsilon_{ijk}(\partial_j A^k - \partial_k A^j). \tag{2.67}$$

Here ϵ_{ijk} is the completely antisymmetric *Levy-Cività* tensor with $\epsilon_{123} = 1$. It is useful to introduce the so-called *four-curl* of the vector potential

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.68}$$

[1] The best upper limit for the mass of the electromagnetic field M_γ deduced under terrestial conditions, from the shape of the earth's magnetic field, is $M_\gamma < 4 \cdot 10^{-48}$g corresponding to a Compton wavelength $\lambda_\gamma = \hbar/M_\gamma c > 10^{10}$cm (= larger than the diameter of the sun). Astrophysical considerations ("whisps" in the crab nebula) give $\lambda_\gamma > 10^{16}$cm. If metagalactic magnetic fields could be discovered, the Compton wavelength would be larger than $10^{24} - 10^{25}$cm, quite close to the ultimate limit set by the horizon of the universe$=c\times$ age of the universe $\sim 10^{28}$cm. See G.V. Chibisov, Sov. Phys. Usp. *19*, 624 (1976).

Its six components are directly the field strengths

$$E^i = -F^{0i} = F_{0i}, \quad H^i = -F^{jk} = -F_{jk}; \qquad ijk = \text{cyclic}. \tag{2.69}$$

For this reason $F_{\mu\nu}$ is also called the *field tensor* The electromagnetic action reads

$$\mathcal{A} = \int d^4x \mathcal{L}(x) = \int d^4x \frac{1}{2}(\mathbf{H}^2 - \mathbf{E}^2) = -\frac{1}{4}\int d^4x F_{\mu\nu}{}^2. \tag{2.70}$$

The four-curl $F_{\mu\nu}$ satisfies the so-called *Bianchi identity* for any smooth A_μ [which satisfies the Schwartz integrability condition $(\partial_\lambda\partial_\kappa - \partial_\kappa\partial_\lambda)A_\mu = 0$]

$$\partial_\mu \tilde{F}^{\mu\nu} = 0, \tag{2.71}$$

where

$$\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\lambda\kappa} F_{\lambda\kappa} \tag{2.72}$$

is the so called *dual field tensor*, with $\epsilon^{\mu\nu\lambda\kappa}$ being the four-dimensional Levy-Cività tensor with $\epsilon_{0123} = 1$.

The equations of motion which extremize the action are

$$\frac{\delta\mathcal{A}}{\delta A^\mu(x)} = -\partial_\mu \frac{\partial\mathcal{L}(x)}{\partial_\mu A_\nu(x)} = \partial_\mu F^{\mu\nu}(x) = 0. \tag{2.73}$$

Separating the equations (2.72) and (2.73(te-4.58)) into space and time components they are seen to coincide with Maxwell's equation in empty space

$$\begin{aligned} \partial_\mu \tilde{F}^{\mu\nu} = 0: &\quad \nabla\cdot\mathbf{B} = 0, \quad \nabla\times\mathbf{E} + \partial_t\mathbf{B} = 0, \\ \partial_\mu F^{\mu\nu} = 0: &\quad \nabla\cdot\mathbf{E} = 0, \quad \nabla\times\mathbf{B} - \partial_t\mathbf{E} = 0. \end{aligned} \tag{2.74}$$

The field tensor is invariant under local gauge transformations

$$A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu\Lambda(x), \tag{2.75}$$

where $\Lambda(x)$ is any smooth field which satisfies the integrability condition $(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\Lambda = 0$. In terms of the vector field A^μ, the action reads explicitly

$$\begin{aligned} \mathcal{A} = \int d^4x \mathcal{L}(x) &= -\frac{1}{2}\int d^4x [\partial^\mu A^\nu(x)\partial_\mu A_\nu(x) - \partial^\nu A_\nu(x)\partial^\mu A_\mu(x)] \\ &= \frac{1}{2}\int dx A_\mu(x)(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A_\nu(x). \end{aligned} \tag{2.76}$$

The latter form is very similar the scalar action (2.36). The first piece is the same as (2.37) for each of the spatial components A^1, A^2, A^3. The time component A^0, however, appears with an opposite sign. A field with this property is called a *ghost field*. When trying to quantize such a field the associated particle states turn out to have a negative norm. It order for the theory to be physically consistent it will be necessary to make sure that such states can never appear in any scattering process. The second piece in the action $\partial^\nu A_\nu \partial^\mu A_\mu$ is novel with respect to the scalar case. It exists here as an additional Lorentz invariant since A_μ is a vector field under Lorentz transformation.

In order to see the Lorentz transformation properties, let us remember that in electrodynamics the Lorentz forces on a moving particle carrying a charge and a classical magnetic pole are obtained from the field transformation

$$\begin{aligned} E_{||}{}' = E_{||}\ , \quad & E_{\perp}{}' = \gamma(E_{\perp} + \mathbf{v}\times\mathbf{B}), \\ B_{||}{}' = B_{||}\ , \quad & B_{\perp}{}' = \gamma(B_{\perp} - \mathbf{v}\times\mathbf{E}), \end{aligned} \tag{2.77}$$

with $\mathbf{v}$ being the velocity of the particle and $\gamma \equiv \sqrt{1-\mathbf{v}^2/c^2}$. Here $\mathbf{E}$, $\mathbf{B}$ are the fields in the laboratory and $\mathbf{E}'$, $\mathbf{B}'$ the fields in the frame of the moving particle. They exert electric and magnetic forces $e\mathbf{E}' + g\mathbf{B}'$. The subscripts $||$ and $\perp$ denote the components parallel and orthogonal to $\mathbf{v}$.

From this experimental fact we can derive the transformation law of the vector field A_μ under Lorentz transformations. The frame in which the moving particle is at rest is related to the laboratory frame by

$$x' = B_{\hat{\mathbf{v}}}(\zeta)x \tag{2.78}$$

where $B_{\hat{\mathbf{v}}}(\zeta)$ is a boost in $\mathbf{v}$ direction with the rapidity

$$\cosh\zeta = \gamma, \quad \sinh\zeta = \gamma\frac{v}{c}, \quad \tanh\zeta = \frac{v}{c}. \tag{2.79}$$

the transformation law (2.77) is equivalent to

$$A'^{\mu}(x') = B_{\hat{\mathbf{v}}}(\zeta)^{\mu}{}_{\nu}A^{\nu}(x). \tag{2.80}$$

An analogous transformation law holds for rotations so that we can write, in general,

$$A'^{\mu}(x') = \Lambda^{\mu}{}_{\nu}A^{\nu}(x). \tag{2.81}$$

This transformation law differs from that of a scalar field (2.32) in the way envisaged above for particles with non-zero intrinsic angular momentum. The field has several components. It points in the same spatial direction before and after the coordination change. This is ensured by its components changing in the same way as the coordination of the point x_μ. Notice that as a consequence, $\partial^\mu A_\mu(x)$ is a scalar field in the sense defined in (2.32). Indeed

$$\partial'^{\mu}A'_{\mu}(x') = (\Lambda^{\mu}{}_{\nu}\partial^{\nu})\Lambda_{\mu}{}^{\lambda}A_{\lambda}(x) = \partial^{\nu}A_{\nu}(x). \tag{2.82}$$

For this reason the second term in the action (2.92) is Lorentz invariant, just as the mass term in (2.65). The invariance of the first term is shown similarly

$$\begin{aligned} A'^{\nu}(x')\partial'^2 A'_{\nu}(x') &= \Lambda^{\nu}{}_{\lambda}A^{\nu}(x)\partial'^2\Lambda_{\nu}{}^{\kappa}A_{\kappa}(x') \\ &= A^{\nu}(x)\partial'^2 A_{\nu}(x) = A^{\nu}(x)\partial^2 A_{\nu}(x). \end{aligned} \tag{2.83}$$

Hence the action (2.92) does not change under Lorentz transformations, as it should.

Just as the scalar action, also the electromagnetic action (2.70) is invariant under the extensions of the Lorentz group by translations (the Poincaré group)

$$A'^{\mu}(x') = A^{\mu}(x) \tag{2.84}$$

where

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}. \tag{2.85}$$

Similarly, under parity

$$A^{\mu} \xrightarrow{P} A'^{\mu}_{P}(x) = \tilde{A}^{\mu}(\tilde{x}), \tag{2.86}$$

and under time reversal

$$A \xrightarrow{T} A'^{\mu}_{T}(x) = \tilde{A}^{\mu}(x_T) \tag{2.87}$$

where

$$\tilde{A}^{\mu} = (A^0, -A^i). \tag{2.88}$$

In principle, there would have been the possibility of a parity transformation

$$A^{\mu} \xrightarrow{P} A^{\mu}_{P}(x) = \eta_P \tilde{A}^{\mu}(\tilde{x}), \tag{2.89}$$

with $\eta_P = \pm 1$ and in the case $\eta_P = -1$ the field A^{μ} would have been called an *axial vector field*. The electromagnetic gauge field A^{μ}, however, is definitely a vector field. This follows from the vector nature of the electric and the axial vector nature of the magnetic field which are observed in the laboratory. Similarly, the phase under time reversal of A^{μ}, which in principle could have been

$$A^{\mu} \xrightarrow{T} A'^{\mu}_{T}(x) = \eta_T \tilde{A}^{\mu}(x_T) \tag{2.90}$$

with $\eta_T = \pm 1$, is given by (2.87). This is because under time reversal, all currents change their direction. This reverses the direction of the **B**-field but has no influence on the **E**-field.

It is also possible to perform the operation of charge conjugation by exchanging the sign of all charges without changing their direction of flow. Then **E** and **B** change directions. Hence

$$A^{\mu} \xrightarrow{C} A'^{\mu}_{C}(x) = -A^{\mu}(x). \tag{2.91}$$

In general, the vector field could have transformed as

$$A^{\mu} \xrightarrow{C} A'^{\mu}_{C}(x) = \eta_C A^{\mu}(x) \tag{2.92}$$

with $\eta_C = \pm 1$. The fact that $\eta_C = -1$ means that the electromagnetic field is odd under charge conjugation.

2.4 Relativistic Free Fermi Fields

For Fermi fields, the situation is technically more involved. Experimentally, fermions always have an even number of spin degrees of freedom. In order to describe these we give the field ψ a spin index α running through $(2s+1)$ components. Under rotations, these spin components are mixed with each other as observed experimentally in the *Stern-Gerlach experiment*. Lorentz transformations lead to certain well defined mixtures of different spin components.

The question arises whether we can construct a Lorentz invariant action involving $(2s+1)$ spinor field components. To see the basic construction principle we use the known transformation law (2.81) for the 4-vector field A^μ as a guide. For an arbitrary *spinor field* we postulate the transformation law

$$\psi(x)_\alpha \xrightarrow{\Lambda} \psi'_\alpha(x') = D_\alpha{}^\beta(\Lambda)\psi_\beta(x), \tag{2.93}$$

with an appropriate $(2s+1) \times (2s+1)$ spinor transformation matrix $D_\alpha{}^\beta(\Lambda)$ which we have to construct. This can be done by purely mathematical arguments. The construction is the subject of the so-called *group representation theory*. First of all, we perform two successive Lorentz transformations,

$$x'' = \Lambda x = \Lambda_2 x' = \Lambda_2\Lambda_1 x. \tag{2.94}$$

Since the Lorentz transformations Λ_1, Λ_2 are elements of a group, the product $\Lambda \equiv \Lambda_2\Lambda_1$ is again a Lorentz transformation. Under the individual factors Λ_2 and Λ_1, the field transform as

$$\begin{aligned} \Psi(x) \xrightarrow{\Lambda_1} \Psi'(x') &= D(\Lambda_1)\Psi(x), \\ \Psi'(x) \xrightarrow{\Lambda_2} \Psi''(x'') &= D(\Lambda_2)\Psi'(x'), \end{aligned} \tag{2.95}$$

so that under $\Lambda = \Lambda_2\Lambda_1$,

$$\Psi(x) \xrightarrow{\Lambda_2\Lambda_1} \Psi''(x'') = D(\Lambda_2)D(\Lambda_1)\Psi(x). \tag{2.96}$$

But for Λ itself, the transformation matrix is $D(\Lambda)$ and

$$\Psi''(x'') = D(\Lambda_2\Lambda_1)\Psi(x). \tag{2.97}$$

Comparison of this with (2.96) shows that the matrices $D(\Lambda)$ which mix the spinor field components under the Lorentz group must follow a group multiplication law which has to be conmpatible with that of the group itself. The mapping

$$\Lambda \longrightarrow D(\Lambda) \tag{2.98}$$

is a homomorphism and the $D(\Lambda)$'s form a *matrix representation* of the group.

Notice that the transformation law (2.81) for A_μ follows the same rule with

$$D(\Lambda) \equiv \Lambda \tag{2.99}$$

being the defining 4×4 representation of the Lorentz group.

The group laws for Λ and $D(\Lambda)$ are sufficiently stringent to allow* only for a countable set of fundamental[2] finite dimensional transformation laws $D(\Lambda)$. They are characterized by two quantum numbers, s_1 and s_2, with either one taking the possible half-integer or integer values $0, \frac{1}{2}, 1, \frac{3}{2}, \dots$.
A representation $D^{(s_1,s_2)}(\Lambda)$ will turn out to harbor particles of spin $|s_1 - s_2|$ to $s_1 + s_2$. Hence, particles with a single fixed spin s can only follow the $D^{(s,0)}(\Lambda)$ or $D^{(0,s)}(\Lambda)$ transformation laws.

For spin 1/2, the relativistic free-field which is invariant under parity has four components and is called the *Dirac field*. It is described by the action

$$\mathcal{A} = \int d^4x \mathcal{L}(x) = \int d^4x \bar{\psi}(x) \left(i\gamma^\mu \partial_\mu - M \right) \psi(x), \tag{2.100}$$

where M is the mass of the spin-1/2 -particles described by $\psi(x)$. The quantities γ^μ are the so-called *Dirac matrices*, defined by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}, \tag{2.101}$$

where σ^μ is a four vector formed from the the 2×2 Pauli matrices as follows:

$$\sigma^\mu \equiv (\mathbf{1}, \sigma^i), \tag{2.102}$$

and

$$\tilde{\sigma}^\mu \equiv (\mathbf{1}, -\sigma^i). \tag{2.103}$$

The symbol $\bar{\psi}(x)$ is short for

$$\bar{\psi} \equiv \psi^\dagger \gamma^0. \tag{2.104}$$

As a historical note we mention that Dirac found his by considering the naive relativistic time independent Schrödinger equation of an electron

$$\hat{H}\psi(\mathbf{x}) = \sqrt{\hat{\mathbf{p}}^2 + M^2}\psi(\mathbf{x}) = E\psi(\mathbf{x}). \tag{2.105}$$

He asked the question whether the square root could be found explicitly if the equation were considered as a matrix equation acting on several components of $\psi(\mathbf{x}, t)$ to represent the spin degrees of freedom of the electron. So he made the ansatz

$$\hat{H}_D\psi(\mathbf{x}) = (-i\alpha_i\hat{p}_i + \beta M)\psi(\mathbf{x}) = E\psi(\mathbf{x}), \tag{2.106}$$

with α_i, β being unknown matrices. Then he required that by applying $\hat{H}_D$ twice upon $\psi(\mathbf{x})$ should give $(\hat{\mathbf{p}}^2 + M^2)\psi(\mathbf{x}) = E^2\psi(\mathbf{x})$. This led him to the algebraic relations

$$\begin{aligned} \{\alpha_i \alpha_j\} &= \delta_{ij}, \\ \{\alpha_i, \beta\} &= 0, \\ \beta^2 &= 1. \end{aligned} \tag{2.107}$$

[2]Mathematically, “fundamental” means that the representation is irreducible. Any arbitrary representation is equivelent to a direct sum of irreducible ones.

By multiplying equ. (2.108) with β and going over to a time dependent equation by replacing E by $i\partial_{x^0}$, he obtained the *Dirac equation*

$$(i\gamma^\mu \hat{p}_\mu - M)\psi(x) = 0. \tag{2.108}$$

with the matrices

$$\gamma^0 \equiv \beta, \quad \gamma^i \equiv \beta\alpha_i. \tag{2.109}$$

These satisfy the anticommutation rules

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{2.110}$$

which are indeed solved by the Dirac matrices (2.101).

It has become customary to abbreviate the contraction of γ^μ with any vector v^μ by

$$\not{v} \equiv \gamma^\mu v_\mu, \tag{2.111}$$

and write the Dirac equation as

$$(\not{p} - M)\psi(x) = 0, \tag{2.112}$$

or

$$(i\not{\partial} - M)\psi(x) = 0. \tag{2.113}$$

2.5 Perturbation Theory of Relativistic Fields

If interactions are present, the Lagrangian consists of a sum

$$\mathcal{L}\left(\psi, \bar{\psi}, \varphi\right) = \mathcal{L}_0 + \mathcal{L}_{\text{int}}. \tag{2.114}$$

As in the case of nonrelativistic felds, all time ordered Green's functions can be obtained from the derivatives with respect to the external sources of the generating functional

$$Z[\eta, \bar{\eta}, j] = \text{const } \times \langle 0|Te^{i\int dx\left(\mathcal{L}_{\text{int}}+\bar{\eta}\psi+\bar{\psi}\eta+j\varphi\right)}|0\rangle. \tag{2.115}$$

The fields in the exponent follow free equations of motion and $|0\rangle$ is the free-field vacuum. The constant is conventionally chosen to make $Z[0,0,0] = 1$, i. e.

$$\text{const} = \left[\langle 0|Te^{i\int dx\mathcal{L}_{\text{int}}\left(\psi,\bar{\psi},\varphi\right)}|0\rangle\right]^{-1}. \tag{2.116}$$

This normalization may always be enforced at the very end of any calculation such that $Z[\eta, \bar{\eta}, j]$ is only interesting as far as its functional dependence is concerned, modulo the irrelevant constant in front.

It is then straight-forward to show that $Z[\eta, \bar{\eta}, j]$ can alternatively be computed via the Feynman path integral formula

$$Z[\eta, \bar{\eta}, j] = \text{const } \times \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\varphi e^{i\int dx\left[\mathcal{L}_0\left(\psi,\bar{\psi},\varphi\right)+\mathcal{L}_{\text{int}}+\bar{\eta}\psi+\bar{\psi}\eta+j\varphi\right]}. \tag{2.117}$$

Here the fields are no more operators but classical functions (with the mental reservation that classical Fermi fields are anticommuting objects). Notice that contrary to the operator formula (2.115) the *full* action appears in the exponent.

For simplicity, we demonstrate the equivalence only for one real scalar field $\varphi(x)$. The extension to other fields is immediate [7], [8]. First note that it is sufficient to give the proof for the free field case, i. e.

$$\begin{aligned} Z_0\left[j\right] &= \langle 0|Te^{i\int dx j(x)\varphi(x)}|0\rangle \\ &= \text{const} \times \int \mathcal{D}\varphi e^{i\int dx\left[\frac{1}{2}\varphi(x)\left(-\Box_x-\mu^2\right)\varphi(x)+j(x)\varphi(x)\right]}. \end{aligned} \tag{2.118}$$

For if it holds there, a simple multiplication on both sides of (2.118) by the differential operator

$$e^{i\int dx \mathcal{L}_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta j(x)}\right)} \tag{2.119}$$

would extend it to the interacting functionals (2.115) or (2.117(2.12)). But (2.118(2.13)) follows directly from Wick's theorem according to which any time ordered product of a free field can be expanded into a sum of normal products with all possible time ordered contractions. This statement can be summarized in operator form valid for any functional $F[\varphi]$ of a free field $\varphi(x)$:

$$TF[\varphi] = e^{\frac{1}{2}\int dxdy \frac{\delta}{\delta\varphi(x)} D(x-y) \frac{\delta}{\delta\varphi(y)}} : F[\varphi] : \tag{2.120}$$

where $D(x-y)$ is the free-field propagator

$$D(x-y) = \frac{i}{-\Box_x - \mu^2 + i\epsilon}\delta(x-y) = \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-y)} \frac{i}{q^2-\mu^2+i\epsilon}. \tag{2.121}$$

Applying this to (2.120) gives

$$\begin{aligned} Z_0 &= e^{\frac{1}{2}\int dxdy \frac{\delta}{\delta\hat\varphi(x)} D(x-y) \frac{\delta}{\delta\hat\varphi(y)}} \langle 0| : e^{i\int dx j(x)\hat\varphi(x)} : |0\rangle \\ &= e^{-\frac{1}{2}\int dxdy j(x)D(x-y)j(y)} \langle 0| : e^{i\int dx j(x)\hat\varphi(x)} : |0\rangle \\ &= e^{-\frac{1}{2}\int dxdy j(x)D(x-y)j(y)} \end{aligned} \tag{2.122}$$

The last part of the equation follows from the vanishing of all normal products of $\varphi(x)$ between vacuum states.

Exactly the same result is obtained by performing the functional integral in (2.118) and using the functional integral formula (1.45(te-2.24a)). The matrix A is equal to $A(x,y) = (-\Box_x - \mu^2)\,\delta(x-y)$, and its inverse yields the propagator $D(x-y)$:

$$A^{-1}(x,y) = \frac{1}{-\Box_x - \mu^2 + i\epsilon}\delta(x-y) = -iD(x-y) \tag{2.123}$$

yielding again (2.122).

For the generating functional of a free Dirac field theory

$$\begin{aligned} Z_0\left[\eta,\bar{\eta}\right] &= \langle 0|Te^{i\int(\bar{\eta}\hat{\psi}+\hat{\bar{\psi}}\eta)dx}|0\rangle \\ &= \text{const} \times \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{i\int dx\left[\mathcal{L}_0\left(\psi,\bar{\psi}\right)+\bar{\eta}\psi+\bar{\psi}\eta\right]}. \end{aligned} \tag{2.124}$$

with the free-field Lagrangian

$$\mathcal{L}_0(x) = \bar{\psi}(x)\left(i\gamma^\mu\partial_\mu - M\right)\psi(x), \tag{2.125}$$

we obtain, similarly,

$$\begin{aligned} Z_0[\bar{\eta},\eta] &= e^{\frac{1}{2}\int dxdy\frac{\delta}{\delta\psi(x)}G_0(x-y)\frac{\delta}{\delta\bar{\psi}(y)}}\langle 0| :e^{i\int dx(\bar{\eta}\hat{\psi}+\hat{\bar{\psi}})\eta} : |0\rangle \\ &= e^{-\frac{1}{2}\int dxdy\bar{\eta}(x)G_0(x-y)\eta(y)}\langle 0| :e^{i\int dx(\bar{\eta}\hat{\psi}+\hat{\bar{\psi}})\eta} : |0\rangle \\ &= e^{-\frac{1}{2}\int dxdy\bar{\eta}(x)G_0(x-y)\eta(y)}. \end{aligned} \tag{2.126}$$

Now,

$$A(x,y) = \left(i\gamma^\mu\partial_\mu - M\right)\delta(x-y), \tag{2.127}$$

and its inverse yields the fermion propagator $G_0(x-y)$:

$$A^{-1}(x,y) = \frac{1}{i\gamma^\mu\partial_\mu - M + i\epsilon}\delta(x-y) = -iG_0(x-y). \tag{2.128}$$

Note that it is Wick's expansion which supplies the free part of the Lagrangian when going from the operator form (2.120) to the functional version (2.117).

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Part II

Plasmas and Superconductors

1

Introduction

In this part we shall develop the theory of collective quantum fields by treating the collective phenomena in in two important many-electron systems. In the first, the electrons interact only via their long-range Coulomb forces, in the other they have an attractive short-range interaction. How this can happen in a solid will be discussed. The Coulomb forces give rise to collective modes called plasmons, the attractive short range forces lead to the formation of bound pairs which follow Bose statistics and are the physical origin for superconductivity.

2

Plasmas

Let us give a first application of the functional method by transforming the grand partition function (1.88) to plasmon coordinates.

For this, we make use of the Hubbard-Stratonovic transformation (1.45) and observe that a two-body interaction (1.33) in the generating functional can always be generated by an auxiliary field $\varphi(x)$ as follows.

$$\exp\left[-\frac{i}{2}\int dx dx' \psi^*(x)\psi^*(x')\psi(x)V(x,x')\right] \tag{2.1}$$
$$= \text{const}\times\int\mathcal{D}\varphi\left\{\frac{i}{2}\int dx dx'\left[\varphi(x)V^{-1}(x,x')\varphi(x')-2\varphi(x)\psi^*(x)\psi(x)\delta(x-x')\right]\right\}$$

To abbreviate the notation, we have used four-vector notation with

$$x\equiv(\mathbf{x},t),\quad dx\equiv d^3x dt,\quad \delta(x)\equiv\delta^3(\mathbf{x})\delta(t).$$

The symbol $V^{-1}(x,x')$ denotes the functional inverse of the matrix $V(x,x')$, i.e., the solution of the equation

$$\int dx' V^{-1}(x,x')V(x',x'')=\delta(x-x''). \tag{2.2}$$

The constant prefactor in (2.1) is $[\det V]^{-1/2}$. Absorbing this in the always omitted normalization factor N of the functional integral, the grand-canonical partition function $\Omega = Z$ becomes

$$Z[\eta^*,\eta]=\int\mathcal{D}\psi^*\mathcal{D}\psi D\varphi\exp\left[i\mathcal{A}+i\int dx\,(\eta^*(x)\psi(x)+\psi^*(x)\eta(x))\right] \tag{2.3}$$

where the new action is

$$\mathcal{A}[\psi^*,\psi,\varphi]=\int dx dx'\Big\{\psi^*(x)\left[i\partial_t-\xi(-i\boldsymbol{\nabla})-\varphi(x)\right]\delta(x-x')\psi(x') \tag{2.4}$$
$$+\frac{1}{2}\varphi(x)V^{-1}(x,x')\varphi(x')\Big\}.$$

Note that the effect of using formula (1.45) in the generating functional amounts to the addition of the complete square in φ in the exponent:

$$\frac{1}{2}\int dx dx'\left[\varphi(x)-\int dy V(x,y)\psi^*(y)\psi(y)\right]V^{-1}(x,x')$$
$$\times\left[\varphi(x')-\int dy' V(x',y')\psi^*(y')\psi(y')\right] \tag{2.5}$$

together with the additional integration over $\mathcal{D}\varphi$. This procedure of going from (1.33) to (2.4) is probably simpler mnemonically than formula (1.45). The fact that the functional Z remains unchanged by this addition follows, as before, since the integral $D\varphi$ produces only the irrelevant constant $[\det V]^{-1/2}$.

The physical significance of the new field $\varphi(x)$ is easy to understand: $\varphi(x)$ is directly related to the particle density. At the classical level this is seen immediately by extremizing the action (2.4) with respect to variations $\delta\varphi(x)$:

$$\frac{\delta\mathcal{A}}{\partial\varphi(x)} = \varphi(x) - \int dyV(x,y)\psi^*(y)\psi(y) = 0. \tag{2.6}$$

Quantum mechanically, there will be fluctuations around the field configuration $\varphi(x)$ determined by Eq. (2.6), making the Green functions of $\varphi(x)$ and of the composite operator $\int dyV(x,y)\psi^*(y)\psi(y)$ different. But due to the Gaussian nature of the $D\varphi$ integration, the fluctuations are quite simple. One can easily show that, for example, the propagators of either field differ only by the direct interaction, i.e.,

$$\begin{aligned}&\langle T\left(\varphi(x)\varphi(x')\right)\rangle \\ &\quad = V(x-x') + \langle T\left[\int dyV(x,y)\psi(y)\right]\left[\int dy'V(x',y')\psi^*(y')\psi(y')\right]\rangle.\end{aligned} \tag{2.7}$$

For the proof, the reader is referred to Appendix 3A. Note, that for a potential V which is dominantly caused by a single fundamental-particle exchange, the field $\varphi(x)$ coincides with the field of this particle: If, for example, $V(x,y)$ represents the Coulomb interaction

$$V(x,x') = \frac{e^2}{|\mathbf{x}-\mathbf{x}'|}\delta(t-t') \tag{2.8}$$

Eq. (2.6) amounts to

$$\varphi(\mathbf{x},t) = -\frac{4\pi e^2}{\boldsymbol{\nabla}^2}\psi^*(\mathbf{x},t)\psi(\mathbf{x},t) \tag{2.9}$$

revealing the auxiliary field as the electric potential.
If the particles $\psi(x)$ have spin indices, the potential will, in this example, be thought of a spin conserving at every vertex and Eq. (2.6) must be read as spin contracted: $\varphi(x) \equiv \int d^4yV(x,y)\psi^{*\alpha}(y)\psi_\alpha(y)$. This restricion is just for convenience and can easily be lifted later. Nothing in our procedure depends on this particular property of V and φ. In fact, V could arise from the exchange of many different fundamental particles and their multiparticle configurations (for example $\pi, \pi\pi, \sigma, \varphi$, etc. in nuclei) so that the spin dependence is the rule rather than the exception.

The important point is now that the auxiliary field $\rho(x)$ can be made the *only* field of the theory by integrating out ψ^*, ψ in Eq. (2.3), using formula (1.46). Thus one obtains

$$Z[\eta^*,\eta] \equiv \Omega[\eta^*,\eta] = Ne^{i\mathcal{A}} \tag{2.10}$$

where the new action is

$$\mathcal{A}[\varphi] \;=\; \pm\mathrm{Tr}\log\left(iG_\varphi^{-1}\right) + \frac{1}{2}\int dx dx' \eta^*(x) G_\varphi(x,x')\eta(x') \tag{2.11}$$

with G_φ being the Green function of the fundamental particles in an external classical field $\varphi(x)$:

$$\left[i\partial_t - \chi(-i\boldsymbol{\nabla}) - \varphi(x)\right] G_\varphi(x,x') = i\delta(x-x'). \tag{2.12}$$

The field $\varphi(x)$ is called a plasmon field. The new plasmon action can easily be interpreted graphically. For this, one expands G_φ in powers φ

$$G_\varphi(x,x') = G_0(x-x') - i\int dx_1 G_0(x-x_1)\varphi(x_1-x') + \ldots \tag{2.13}$$

Hence the couplings to the external currents η^*, η in (2.11) amount to radiating one, two, etc. φ fields from every external line of fundamental particles (see Fig. 2.1). An expansion of the Tr log expression in φ gives

FIGURE 2.1 This diagram displays the last, pure current, piece of the collective action (2.11). The original fundamental particle (fat line) can enter and leave the diagrams only via external currents, emitting an arbitrary number of plasmons (wiggly lines) on its way

$$\begin{aligned}\pm i\mathrm{Tr}\log(iG_\varphi^{-1}) &= \pm i\mathrm{Tr}\log(iG_0^{-1}) \pm i\mathrm{Tr}\log(1+iG_0\varphi)\\ &= \pm i\mathrm{Tr}\log(iG_0^{-1}) \mp i\mathrm{Tr}\sum_{n=1}^{\infty}(-iG_0\varphi)^n\frac{1}{n}. \end{aligned}\tag{2.14}$$

The first term leads to an irrelevant multiplicative factor in (2.10). The nth term corresponds to a loop of the original fundamental particle emitting $n\varphi$ lines (see Fig. 2.2).

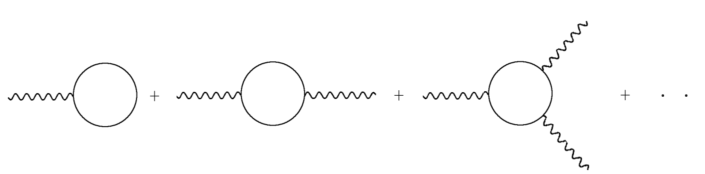

FIGURE 2.2 The non-polynomial self-interaction terms of the plasmons arising from the Tr log in (2.11) are equal to the single loop diagrams emitting n plasmons

Let us now use the action (2.11) to construct a quantum field theory of plasmons. For this we may include the quadratic term

$$\pm i\mathrm{Tr}(G_0\varphi)^2\frac{1}{2} \tag{2.15}$$

into the free part of φ in (2.11) and treat the remainder perturbatively. The free propagator of the plasmon becomes

$$\{0|T\varphi(x)\varphi(x')|0\} \equiv (2s+1)G_0(x',x). \tag{2.16}$$

This corresponds to an inclusion into the V propagator of all ring graphs (see Fig. 2.3). It is worth pointing out that the propagator in momentum space $G^{\mathrm{pl}}(k)$ con-

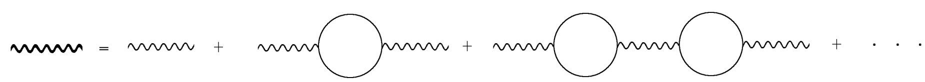

FIGURE 2.3 The free plasmon propagator containing an infinite sequence of single loop coorections ("bubblewise summation")

tains actually two important physical informations. From the derivation at fixed temperature it appears in the transformed action (2.11) as a function of discrete Euclidean frequencies $\nu_n = 2\pi nT$ only. In this way it serves for the time independet fixed T description of the system. The calculation (2.16), however, renders it as a function in the whole complex energy plane. It is this function which determines the *time dependent* collective phenomena for *real* times[1]

With the propagator (2.16) and the interactions given by (2.14), the original theory of fundamental fields ψ^*, ψ has been transformed into a theory of φ fields whose bare propagator accounts for the original potential which has absorbed ringwise an infinite sequence of fundamental loops.

This transformation is exact. Nothing in our procedure depends on the statistics of the fundamental particles nor on the shape of the potential. Such properties are important when it comes to *solving* the theory perturbatively. Only under appropriate physical circumstances will the field φ represent important collective excitations with weak residual interactions. Then the new formulation is of great use in understanding the dynamcis of the system. As an illustration consider a dilute fermion gas of very low temperature. Then the function $\xi(-i\boldsymbol{\nabla})$ is $\epsilon(-i\boldsymbol{\nabla})-\mu$ with $\epsilon(-i\boldsymbol{\nabla}) = -\boldsymbol{\nabla}^2/2m$.

Let the potential be translationally invariant and instantaneous

$$V(x,x') = \delta(t-t')V(\mathbf{x}-\mathbf{x}'). \tag{2.17}$$

[1]See the discussion in Ch. 9 of the last of [3] and G. Baym and N. D. Mermin, J. Math. Phys. **2**, 232 (1961).

Then plasmon propagator (2.16) reads in momentum space

$$G^{\rm pl}(\nu,\mathbf{k}) = -\frac{1}{[V(\mathbf{k})]^{-1} - \pi(\nu,\mathbf{k})} \tag{2.18}$$

where the single electron loop is[2]

$$\pi(\nu,\mathbf{k}) = 2\frac{T}{V}\sum_p V\sum_p \frac{1}{i\omega - \mathbf{p}^2/2m+\mu}\,\frac{1}{i(\omega+\nu)-(\mathbf{p}+\mathbf{k})^2/2m+\mu}. \tag{2.19}$$

The frequencies ω and ν are odd and even multiples of πT. The sum is calculated in the standard fashion by introducing a convergence factor $e^{i\omega\eta}$, rewriting

$$\begin{aligned}\pi(\nu,\mathbf{k}) &= 2\int\frac{d^3p}{(2\pi)^3}\frac{1}{\xi(\mathbf{p}+\mathbf{k})-\xi(\mathbf{p})-i\nu} \\ &\quad\times T\sum_{\omega_n} e^{i\omega_n\eta}\left[\frac{1}{i(\omega_n+\nu)-\xi(\mathbf{p}+\mathbf{k})} - \frac{1}{i\omega_n-\xi(\mathbf{p})}\right],\end{aligned} \tag{2.20}$$

and using the summation formula (1.72), this becomes

$$\pi(\nu,\mathbf{k}) = 2\int\frac{d^3p}{(2\pi)^3}\frac{n(\mathbf{p}+\mathbf{k})-n(\mathbf{p})}{\epsilon(\mathbf{p}+\mathbf{k})-\epsilon(\mathbf{p})-i\nu}. \tag{2.21}$$

If one performs a long wavelength, small-frequency expansion of the integrand, one finds for $T\approx 0$

$$\pi(\nu,\mathbf{k}) \approx -\frac{mp_F}{\pi^2}(1-\rho\,\mathrm{arctg}\,\rho^{-1}) \tag{2.22}$$

where p_F denotes the Fermi momentum and ρ the ratio[3]

$$\rho \equiv m\nu/p_F|\mathbf{k}| \tag{2.23}$$

The analytic continuation to physical energies $k_0 = i\nu$ yields, with $\tilde\rho \equiv mk_0/p_F|\mathbf{k}| = i\rho$:

$$\pi(k_0,\mathbf{k}) = -\frac{mp_F}{\pi^2}\left[1-\frac{\tilde\rho}{2}\log\left|\frac{\tilde\rho+1}{\tilde\rho-1}\right| - i\frac{\pi}{2}|\tilde\rho|\,\Theta(1-|\tilde\rho|)\right]. \tag{2.24}$$

The real poles of $G^{\rm pl}(\nu,\mathbf{k})$ determine the elementary excitations. Suppose $[V(\mathbf{k})]^{-1}$ has a long-wavelength expansion

$$[V(\mathbf{k})]^{-1} = [V(\mathbf{0})]^{-1} + a\mathbf{k}^2 + \dots \tag{2.25}$$

Then there are real poles at energies k_0 for which

$$[V(\mathbf{0})]^{-1} + a\mathbf{k}^2 + \dots = -\frac{mp_F}{\pi^2}\left(1-\frac{\tilde\rho}{2}\log\left|\frac{\tilde\rho+1}{\tilde\rho-1}\right|\right), \tag{2.26}$$

[2] The factor 2 stems from the trace over the electron spin.

[3] For a discussion of this expression, see a standard textbook, for example Ref. [3].

as long as $[V(\mathbf{0})]^{-1}$ is finite and positive, i.e., for a well behaved overall repulsive potential ($V(\mathbf{0}) = \int d^3x V(\mathbf{x}) > 0$). The value $\tilde{\rho}_0$ for which (2.1) is fulfilled at $\mathbf{k} = 0$ determines the zero-sound velocity c_0 according to

$$\tilde{\rho}_0 = \frac{m}{p_F} \frac{k_0}{|\mathbf{k}|} = \frac{1}{v_F} c_0. \tag{2.27}$$

In the neighbourhood of the pole the propagator has the form

$$G^{\mathrm{pl}}(k_0, k) \approx \text{ const } \times \frac{|\mathbf{k}|}{k_0 - c_0|\mathbf{k}|}. \tag{2.28}$$

The case of an electron gas has to be discussed separately since the potential is not well behaved:

$$V(x, x') = \delta(t - t') \frac{e^2}{|\mathbf{x} - \mathbf{x}'|} \tag{2.29}$$

so that

$$[V(\mathbf{k})]^{-1} = \frac{\mathbf{k}^2}{4\pi e^2}. \tag{2.30}$$

Hence, (2.1) has to be solved for $[V(\mathbf{0})]^{-1} = 0$ and $a = 1/4\pi e^2$. Obviously, $\tilde{\rho}$ has to go to infinity as $\mathbf{k} \to 0$. In this limit

$$\pi(k_0, \mathbf{k}) \to \frac{m p_F}{\pi^2} \frac{\tilde{\rho}^{-2}}{3} = \frac{p_F^3}{3\pi^2 m} \frac{\mathbf{k}^2}{k_0^2} \tag{2.31}$$

and there is a pole at energy[4]

$$k_0^2 = 4\pi e^2 \frac{p_F^2}{3\pi^2 m} = 4\pi e^2 \frac{n}{m} \tag{2.32}$$

which is the well-known plasmon frequency. Thus the long-range part of the propagator can be written as

$$G^{\mathrm{pl}}(k_0, \mathbf{k}) \approx 4\pi e^2 \frac{k_0^2}{k_0^2 - 4\pi e^2 n/m} \frac{i}{\mathbf{k}^2}. \tag{2.33}$$

Using the plasmon propagator (2.18) and the multi-plasmon interactions from (2.14) one can develop a fully fledged quantum field theory of plasmons.

Great simplications arise if the system is investigated only with respect to its long-range behaviour in space and time. Then expressions like (2.3) and (2.10) become good approximations to the propagator. Moreover, the higher terms in the expansion (2.14) become more and more irrelevant due to their increasing field dimensionality. Such discussions are standard and will not be repeated here [7].

[4] n is the number density: $n = 2\int d^3p_F/(2\pi)^3 = p_F^3/3\pi^2$.

3

Superconductors

3.1 General Formulation

There is a collective field complementary to the plasmon field which describes dominant collective excitations in many systems such as type II superconductors, ^3He, excitonic insulators, etc. A *pair field* $\Delta(\mathbf{x}t;\mathbf{x}'t')$ with two space and two time indices, called bilocal, is introduced into the generating functional by rewriting the exponential of the interaction (1.41) different from (2.1) as[1] [8]

$$\exp\left[-\frac{i}{2}\int dxdx'\psi^*(x)\psi^*(x')\psi(x')\psi(x)V(x,x')\right] = \text{const} \times\int \mathcal{D}\Delta(x,x')\mathcal{D}\Delta^*(x,x')$$
$$\times e^{\frac{i}{2}\int dxdx'\left\{|\Delta(x,x')|^2\frac{1}{V(x,x')}-\Delta^*(x,x')\psi(x)\psi(x')-\psi^*(x)\psi^*(x')\Delta(x,x')\right\}}. \tag{3.1}$$

Hence the grand-canonical potential becomes

$$Z[\eta,\eta^*] = \int \mathcal{D}\psi^*\mathcal{D}\psi\mathcal{D}\Delta^*\mathcal{D}\Delta e^{i\mathcal{A}[\psi^*,\psi,\Delta^*,\Delta]+i\int dx(\psi^*(x)\eta(x)+\text{c.c.})} \tag{3.2}$$

with the action

$$\mathcal{A}[\psi^*,\psi,\Delta^*,\Delta] = \int dxdx'\left\{\psi^*(x)\left[i\partial_t - \xi(-i\boldsymbol{\nabla})\right]\delta(x-x')\psi(x')\right.$$
$$\left.-\frac{1}{2}\Delta^*(x,x')\psi(x)\psi(x') - \frac{1}{2}\psi^*(x)\psi^*(x')\Delta(x,x') + \frac{1}{2}|\Delta(x,x')|^2\frac{1}{V(x,x')}\right\} \tag{3.3}$$

Note that this new action arises from the original one in (1.41) by adding to it the complete square

$$\frac{i}{2}\int dxdx'|\Delta(x,x) - V(x',x)\psi(x')\psi(x)|^2\frac{1}{V(x,x')}$$

which upon functional integration over $\int\mathcal{D}\Delta^*\mathcal{D}\Delta$ gives an irrelevant constant factor to the generating functional but which has the virtue of removing the quartic interaction term.

[1] In this expression, $1/V(x,x')$ is understood as numeric division, no matrix inversion being implied.

At the classical level, the field $\Delta(x, x')$ is nothing but a convenient abbreviation for the composite field $V(x, x')\psi(x)\psi(x')$. This follows from the equation of motion obtained by extremizing the new action with respect to $\delta\Delta^*(x, x')$ which gives

$$\frac{\delta\mathcal{A}}{\delta\Delta^*(x, x')} = \frac{1}{2V(x, x')}\left[\Delta(x, x') - V(x, x')\psi(x)\psi(x')\right] \equiv 0. \tag{3.4}$$

Quantum mechanically, there are Gaussian fluctuations around this solution which are discussed in detail in Appendix Appendix 3B. The expression (3.3) is quadratic in the fundamental fields ψ and can be rewritten in matrix form as

$$\begin{aligned}\frac{1}{2}f^*(x)A(x, x')f(x') \\ = \frac{1}{2}f^\dagger(x)\begin{pmatrix} [i\partial_t - \xi(-i\boldsymbol{\nabla})]\,\delta(x - x') & -\Delta(x, x') \\ -\Delta^*(x, x') & \mp[i\partial_t + \xi(i\boldsymbol{\nabla})]\,\delta(x - x')\end{pmatrix} f(x'),\end{aligned} \tag{3.5}$$

where $f(x)$ denotes the fundamental field doublet $f(x) = \begin{pmatrix}\psi(x) \\ \psi^*(x)\end{pmatrix}$ and $f^\dagger \equiv f^{*T}$, as usual. Now, $f^*(x)$ is not independent of $f(x)$. Indeed, $f^\dagger A f$ can also be written as

$$f^\dagger A f = f^T\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} A f.$$

Therefore, the real-field formula (1.45) can be used to derive the functional integral for the generating functional

$$Z[\eta^*, \eta] = \int \mathcal{D}\Delta^*\mathcal{D}\Delta e^{i\mathcal{A}[\Delta^*, \Delta] - \frac{1}{2}\int dx \int dx' j^\dagger(x) G_\Delta(x, x') j(x')} \tag{3.6}$$

where $j(x)$ collects the external source $\eta(x)$ and its complex conjugate, $j(x) \equiv \begin{pmatrix}\eta(x) \\ \eta^*(x)\end{pmatrix}$, and the collective action reads

$$\mathcal{A}[\Delta^*, \Delta] = \pm\frac{i}{2}\mathrm{Tr}\log\left[i\mathbf{G}_\Delta^{-1}(x, x')\right] + \frac{1}{2}\int dx dx' |\Delta(x, x')|^2 \frac{1}{V(x, x')}. \tag{3.7}$$

The 2×2 matrix $\mathbf{G}_\Delta$ denotes the propagator iA^{-1} which satisfies the functional equation

$$\begin{aligned}\int dx'' \begin{pmatrix} [i\partial_t - \xi(-i\boldsymbol{\nabla})]\,\delta(x - x'') & -\Delta(x, x'') \\ -\Delta^*(x, x'') & \mp[i\partial_t + \xi(i\boldsymbol{\nabla})]\,\delta(x - x'')\end{pmatrix} \\ \times\mathbf{G}_\Delta(x'', x') = i\delta(x - x').\end{aligned} \tag{3.8}$$

Writing $\mathbf{G}_\Delta$ as a matrix $\begin{pmatrix} G & F \\ F^\dagger & \tilde{G}\end{pmatrix}$ the mean-field equations associated with this action are precisely the equations used by Gorkov to study the behaviour of type II superconductors.[2] With $Z[\eta^*, \eta]$ being the *full* partition function of the system,

[2]See, for example, p. 444 of Ref. [3]

the fluctuations of the collective field $\Delta(x, x')$ can now be incorporated, at least in principle, thereby yielding corrections to these equations.

Let us set the sources in the generating functional $Z[\eta^*, \eta]$ equal to zero and investigate the behaviour of the collective quantum field Δ. In particular, we want to develop Feynman rules for a perturbative treatment of the fluctuations of $\Delta(x, x')$. As a first step we expand the Green function $\mathbf{G}_{\boldsymbol{\Delta}}$ in powers of Δ as

$$\mathbf{G}_\Delta = \mathbf{G}_0 - i\mathbf{G}_0 \begin{pmatrix} 0 & \Delta \\ \Delta^* & 0 \end{pmatrix} \mathbf{G}_0 - \dots \tag{3.9}$$

with

$$\mathbf{G}_0(x, x') = \begin{pmatrix} \dfrac{i}{i\partial_t - \xi(-i\boldsymbol{\nabla})}\delta(x - x') & 0 \\ 0 & \mp\dfrac{i}{i\partial_t + \xi(i\boldsymbol{\nabla})}\delta(x - x') \end{pmatrix}. \tag{3.10}$$

We shall see later that this assumption is justified only in a very limited range of thermodynamic parameters, namely close to the critical temperature T_c. With such an expansion, the source term in (3.6) can be interpreted graphically by the absorption and emission of lines $\Delta(k)$ and $\Delta^*(k)$, respectively, from virtual zig-zag configurations of the underlying particles ψ, ψ^* (see Fig. 3.1)

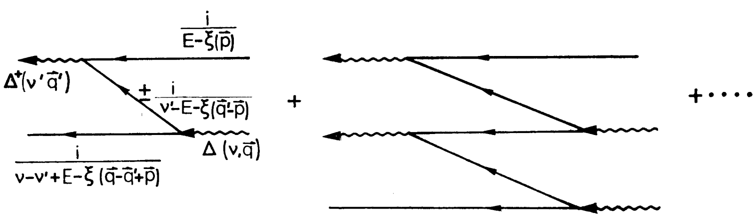

FIGURE 3.1 The fundamental particles (fat lines) entering any diagram only via the external currents in the last term of (3.6), absorbing n pairs from the right (the past) and emitting the same number of the left (the future).

The functional submatrices in $\mathbf{G}_0$ have the Fourier representation

$$G_0(x, x') = \frac{T}{V}\sum_p \frac{i}{p^0 - \xi(\mathbf{p})} e^{-i\left(p^0 t - \mathbf{px}\right)}, \tag{3.11}$$

$$\tilde{G}_0(x, x') = \pm\frac{T}{V}\sum_p \frac{i}{-p^0 - \xi(-\mathbf{p})} e^{-i\left(p^0 t - \mathbf{px}\right)}. \tag{3.12}$$

The first matrix coincides with the operator Green function

$$G_0(x - x') = \langle 0|T\psi(x)\psi^\dagger(x')|0\rangle. \tag{3.13}$$

The second one corresponds to

$$\begin{aligned} \tilde{G}_0(x - x') &= \langle 0|T\psi^\dagger(x)\psi(x')|0\rangle = \pm\langle 0|T\left(\psi(x')\psi^\dagger(x)\right)|0\rangle \\ &= \pm G_0(x' - x) \equiv \pm[G_0(x, x')]^T \end{aligned} \tag{3.14}$$

where T denotes the transposition in the functional sense (i.e., x and x' are interchanged). After a Wick rotation of the energy integration contour, the Fourier components of the Green functions at fixed energy read

$$G_0(\mathbf{x}-\mathbf{x}',\omega) = -\sum_{\mathbf{p}} \frac{1}{i\omega - \xi(\mathbf{p})} e^{i\mathbf{p}(\mathbf{x}-\mathbf{x}')} \tag{3.15}$$

$$\tilde{G}_0(\mathbf{x}-\mathbf{x}',\omega) = \mp\sum_{\mathbf{p}} \frac{1}{-i\omega - \xi(-\mathbf{p})} e^{i\mathbf{p}(\mathbf{x}-\mathbf{x}')} = \mp G_0(\mathbf{x}'-\mathbf{x},-\omega). \tag{3.16}$$

The Tr log term in Eq. (3.7) can be interpreted graphically just as easily by expanding according to (3.9):

$$\pm\frac{i}{2}\mathrm{Tr}\log\left(i\mathbf{G}_0^{-1}\right) \mp \frac{i}{2}\mathrm{Tr}\left[-i\mathbf{G}_0\begin{pmatrix} 0 & \Delta \\ \Delta^* & 0\end{pmatrix}\Delta^*\right]^n\frac{1}{n}. \tag{3.17}$$

The first term only changes the irrelevant normalization N of Z. To the remaining sum only even powers can contribute so that we can rewrite

$$\mathcal{A}[\Delta^*,\Delta] = \mp i\sum_{n=1}^{\infty}\frac{(-)^n}{2n}\mathrm{Tr}\left[\left(\frac{i}{i\partial_t - \xi(-i\boldsymbol{\nabla})}\delta\right)\Delta\left(\frac{\mp i}{i\partial_t + \xi(i\boldsymbol{\nabla})}\delta\right)\Delta^*\right]^n$$
$$+\frac{1}{2}\int dxdx'|\Delta(x,x')|^2\frac{1}{V(x,x')} \tag{3.18}$$
$$= \sum_{n=1}^{\infty}\mathcal{A}_n[\Delta^*,\Delta] + \frac{1}{2}\int dxdx'|\Delta(x,x')|^2\frac{1}{V(x,x')}. \tag{3.19}$$

This form of the action allows for an immediate quantization of the collective field Δ. The graphical rules are slightly more involved technically than in the plasmon case since the pair field is bilocal. Consider at first the *free* quanta which can be obtained from the quadratic part of the action:

$$\mathcal{A}_2[\Delta^*,\Delta] = -\frac{i}{2}\mathrm{Tr}\left[\left(\frac{i}{i\partial_t - \xi(-i\boldsymbol{\nabla})}\delta\right)\Delta\left(\frac{i}{i\partial_t + \xi(i\boldsymbol{\nabla})}\delta\right)\Delta^*\right]. \tag{3.20}$$

Variation with respect to Δ displays the equations of motion

$$\Delta(x,x') = iV(x,x')\left[\left(\frac{i}{i\partial_t - \xi(-i\boldsymbol{\nabla})}\delta\right)\Delta\left(\frac{i}{i\partial_t + \xi(i\boldsymbol{\nabla})}\delta\right)\right]. \tag{3.21}$$

This equation coincides exactly with the Bethe-Salpeter equation [9], in ladder approximation, for two-body bound-state vertex functions usually denoted in momentum space by

$$\Gamma(p,p') = \int dxdx'\exp[i(px+p'x')]\Delta(x,x').$$

Thus the free quanta of the field $\Delta(x,x')$ consist of bound pairs of the original fundamental particles. The field $\Delta(x,x')$ will consequently be called pair field. If we introduce total and relative momenta q and $P = (p - p')/2$, then (3.21) can be written as[3]

$$\Gamma(P|q) = -i \int \frac{d^4P'}{(2\pi)^4} V(P-P') \frac{i}{q_0/2 + P_0' - \xi(\mathbf{q}/2 + \mathbf{P'}) + i\eta \,\mathrm{sgn}\, \xi} \times \Gamma(P'|q) \frac{i}{q_0/2 - P_0' - \xi(\mathbf{q}/2 - \mathbf{P'}) + i\eta \,\mathrm{sgn}\, \xi}. \tag{3.22}$$

Graphically this formula can be represented as follows: The Bethe-Salpeter *wave*

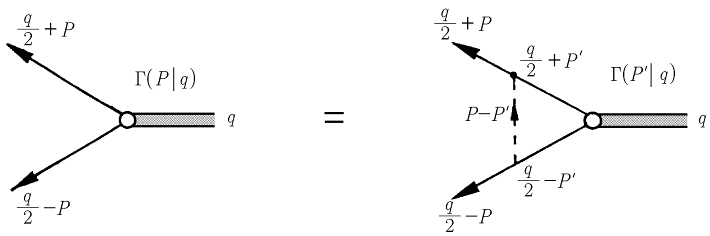

FIGURE 3.2 The free pair field following the Bethe-Salpeter equation as pictured in this diagram

function is related to the vertex $\Gamma(P|q)$ by

$$\Phi(P|q) = N \frac{i}{q_0/2 + P_0 - \xi(\mathbf{q}/2 + \mathbf{P}) + i\eta \,\mathrm{sgn}\, \xi} \times \frac{i}{q_0/2 + P_0 - \xi(\mathbf{q}/2 + \mathbf{P}) + i\eta \,\mathrm{sgn}\, \xi} \Gamma(P|q). \tag{3.23}$$

It satisfies

$$G_0(q/2 + P) G_0(q/2 - P) \Phi(P|q) = -i \int \frac{dP'}{(2\pi)^4} V(P, P') \Phi(P'|q) \tag{3.24}$$

thus coinciding, up to a normalization, with the Fourier transform of the two-body state wave functions

$$\psi(\mathbf{x}, t; \mathbf{x}', t') = \langle 0 | T(\psi(\mathbf{x}, t)\psi(\mathbf{x}', t')) | B(q) \rangle. \tag{3.25}$$

If the potential is instantaneous, then (3.21) shows $\Delta(x, x')$ to be factorizable according to

$$\Delta(x, x') = \delta(t - t') \Delta(\mathbf{x}, \mathbf{x}'; t) \tag{3.26}$$

so that $\Gamma(P|q)$ becomes independent of P_0.

[3]Here q stands short for $q_0 = E$ and $\mathbf{q}$.

Consider now the system at $T = 0$ in the vacuum. Then $\mu = 0$ and $\xi(-i\boldsymbol{\nabla}) = \epsilon(-i\boldsymbol{\nabla}) > 0$. One can perform the P_0 integral in (3.22) with the result

$$\Gamma(\mathbf{P}|q) = \int \frac{d^3P'}{(2\pi)^4} V(\mathbf{P}-\mathbf{P}') \frac{1}{q_0 - \epsilon\left(\mathbf{q}/2+\mathbf{P}'\right) - \epsilon\left(\mathbf{q}/2-\mathbf{P}'\right) + i\eta} \Gamma(\mathbf{P}'|q). \quad (3.27)$$

Now the equal-time Bethe-Salpeter wave function

$$\begin{aligned} \psi(\mathbf{x},\mathbf{x}';t) \quad &\equiv \quad N \int \frac{d^3\mathbf{P}\,dq_0\,d^3\mathbf{q}}{(2\pi)^7} \exp\left[-i\left(q_0 t - \mathbf{q}\cdot\frac{\mathbf{x}+\mathbf{x}'}{2} - \mathbf{P}\cdot(\mathbf{x}-\mathbf{x}')\right)\right] \\ &\qquad \times \frac{1}{q_0 - \epsilon\left(\mathbf{q}/2+\mathbf{P}\right) - \epsilon\left(\mathbf{q}/2-\mathbf{P}\right) + i\eta} \end{aligned} \quad (3.28)$$

satisfies

$$\left[i\partial_t - \epsilon(-i\boldsymbol{\nabla}) - \epsilon(-i\boldsymbol{\nabla}')\right]\psi(\mathbf{x},\mathbf{x}';t) = V(\mathbf{x}-\mathbf{x})\psi(\mathbf{x},\mathbf{x}';t) \quad (3.29)$$

which is simply the Schrödinger equation. Thus, in the instantaneous case, the free collective excitations in $\Delta(x,x')$ are the bound states as they follow from the Schrödinger equation.

In a thermodynamic ensemble the energies in (3.22) have to be summed over Matsubara frequencies only. First, we write the Schrödinger equation as

$$\Gamma(\mathbf{P}|q) = -\int \frac{d^3\mathbf{P}'}{(2\pi)^3} V(\mathbf{P}-\mathbf{P}')\, l(\mathbf{P}'|q)\,\Gamma(\mathbf{P}'|q) \quad (3.30)$$

with

$$\begin{aligned} l(\mathbf{P}|q) &= -i\sum_{P_0} G_0\left(q/2+P\right)\tilde{G}_0\left(P-q/2\right) \qquad (3.31) \\ &= -i\sum_{P_0} \frac{i}{q_0/2+P_0-\xi\left(\mathbf{q}/2+\mathbf{P}\right)+i\eta\,\mathrm{sgn}\,\xi}\;\frac{i}{q_0/2-P_0-\xi\left(\mathbf{q}/2-\mathbf{P}\right)+i\eta\,\mathrm{sgn}\,\xi}. \end{aligned}$$

After a Wick rotation and setting $q_0 \equiv i\nu$, the replacement of the energy integration by a Matsubara sum leads to

$$\begin{aligned} l(\mathbf{P}|q) \quad &= \quad -T\sum_{\omega_n} \frac{1}{i\left(\omega_n+\nu/2\right) - \xi\left(\mathbf{q}/2+\mathbf{P}\right)}\;\frac{1}{i\left(\omega_n-\nu/2\right) + \xi\left(\mathbf{q}/2-\mathbf{P}\right)} \\ &= \quad T\sum_{\omega_n} \frac{1}{i\nu - \xi\left(\mathbf{q}/2+\mathbf{P}\right) - \xi\left(\mathbf{q}/2-\mathbf{P}\right)} \\ &\qquad \times\left[\frac{1}{i(\omega_n+\nu/2) - \xi\left(\mathbf{q}/2+\mathbf{P}\right)} - \frac{1}{i(\omega_n-\nu/2) + \xi\left(\mathbf{q}/2-\mathbf{P}\right)}\right] \\ &= \quad -\frac{\pm\left[n\left(\mathbf{q}/2+\mathbf{P}\right) + n\left(\mathbf{q}/2-\mathbf{P}\right)\right]}{i\nu - \xi\left(\mathbf{q}/2+\mathbf{P}\right) - \xi\left(\mathbf{q}/2-\mathbf{P}\right)}. \end{aligned} \quad (3.32)$$

Here we have used the frequency sum [see (1.72)]

$$T\sum_{\omega_n} \frac{1}{i\omega_n - \xi(\mathbf{p})} = \mp\frac{1}{e^{\xi(\mathbf{p})/T} \mp 1} \equiv \mp n(\mathbf{p}) \quad (3.33)$$

with $n(\mathbf{p})$ being the acupation number of the state of energy $\xi(\mathbf{p})$. The expression in brackets is antisymmetric if both $\xi \to -\xi$ since under this substitution $n \to \mp 1 - n$. In fact, one can write it in the form $-N(\mathbf{P}, \mathbf{q})$ with

$$\begin{aligned} N(\mathbf{P}|q) &\equiv 1 \pm [n(\mathbf{q}/2 + \mathbf{P}) + n(\mathbf{q}/2 - \mathbf{P})] \\ &= \frac{1}{2}\left[\tanh^{\mp 1}\frac{\xi(\mathbf{q}/2 + \mathbf{P})}{2T} + \tanh^{\mp 1}\frac{\xi(\mathbf{q}/2 - \mathbf{P})}{2T}\right] \end{aligned} \tag{3.34}$$

so that

$$l(\mathbf{P}|q) = -\frac{N(\mathbf{P}|\mathbf{q})}{i\nu - \xi(\mathbf{q}/2 + \mathbf{P}) - \xi(\mathbf{q}/2 - \mathbf{P})}. \tag{3.35}$$

Defining again a Schrödinger type wave function as in (3.28), the bound-state problem can be brought to the form (3.27) but with a momentum dependent potential $V(\mathbf{P} - \mathbf{P}') \times N(\mathbf{P}'|\mathbf{q})$. We are now ready to construct the propagator of the pair field $\Delta(x, x')$ for $T = 0$. In many cases, this is most simply done by considering Eq. (3.22) with a potential $\lambda V(P, P')$ rather than V and asking for all eigenvalues λ_n at *fixed* q. Let $\Gamma_n(P|q)$ be a complete set of vertex functions for this q. Then one can write the propagator as

$$\dot{\Delta}(P|q)\dot{\Delta}^*(P'|q') = -i\sum \left.\frac{\Gamma_n(P|q)\Gamma_n^*(P'|q)}{\lambda - \lambda_n(q)}\right|_{\lambda=1} (2\pi)^4\delta^{(4)}(q - q') \tag{3.36}$$

where a common dot denotes, as usual, the Wick contraction of the fields. Obviously the vertex functions have to be normalized in a specific way, as discussed in Appendix 3A.

Expansion in powers of $[\lambda/\lambda_n(q)]^n$ displays the propagator of Δ as a ladder sum of exchanges (see Appendix 3A). In the instantaneous case either side is independent

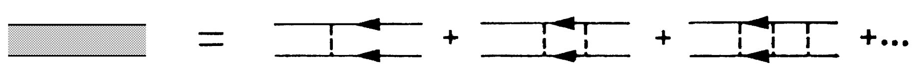

FIGURE 3.3 The free pair propagator, amounting to a sum of all ladders of fundamental potential exchanges. This is revealed explicitly by the expansion of (3.36) in powers $(\lambda/\lambda_n(q))$

of P_0, P_0'. Then the propagator can be shown to coincide directly with the scattering matrix T of the Schrödinger equation (3.29) and the associated integral equation in momentum space (3.27) [see Eq. (3A.13)].

$$\dot{\Delta}\dot{\Delta}^* = iT \equiv iV + iV\frac{1}{E - H}V. \tag{3.37}$$

Consider now the higher interactions $\mathcal{A}_n, n \geq 3$ of Eq. (3.19). They correspond to zig-zag loops shown in Fig. 3.4. These have to be calculated with every possible

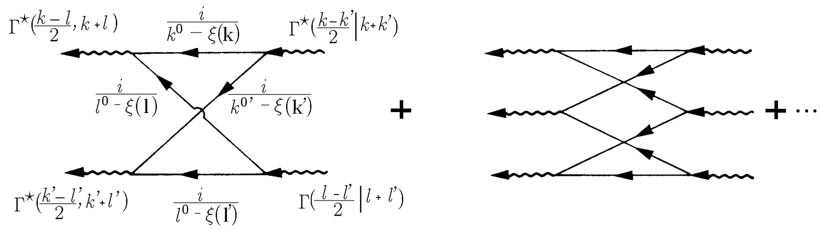

FIGURE 3.4 The self-interaction terms of the non-polynomial pair Lagrangian amounting to the calculation of al single zig-zag loop diagrams absorbing and emitting n pair fields

$\Gamma_n(P|q), \Gamma_m^*(P|q)$ entering or leaving, respectively.

Due to the P dependence at every vertex, the loop integrals become very involved. A slight simplification arises for instantaneous potential in that at least the frequency sums can be performed immediately. Only in the special case of a completely local action the full P-dependence disappears and the integrals can be calculated at least approximately. This will be done in the following section.

3.2 Local Interaction and Ginzburg-Landau Equations

Let us study the case of a completely local potential in detail. For the electrons in a crystal, this is only an approximation which, however, happens to be quite good. In a crystal, the interaction between the electrons is mediated by phonon exchange. An electron moving through the lattice attracts the positive ions in its neighborhood and thus creates a cloud of positive charge around its path. This cloud, in turn, attracts other electrons and this is the the origin of pair formation. The size of the cloud is of the order of the lattice spacing, i.e., a few Å. Although this can hardly be called local, it is effectively so as far as the formation of bound states is concerned. The reason is that the strength of the interaction is quite small. This leads to a rather wide bound-state wave function – it radius will be seen to be of the order of 100 Å, i.e. many lattice spacings. Thus, as far as the bound-states are concerned, the potential may just as well be considered as local. This is what justifies the following considerations.

We therefore assume the fundamental interaction to be of the form

$$\mathcal{A}_{\rm int} = \frac{g}{2} \sum_{\alpha,\beta} \int d^3x dt \psi^*_\alpha(\mathbf{x},t)\psi^*_\beta(\mathbf{x},t)\psi_\beta(\mathbf{x},t)\psi_\alpha(\mathbf{x},t). \tag{3.38}$$

Following the general arguments leading to (3.1) we can rewrite the exponential of this interaction as[4]

$$\exp\left[\frac{i}{2} g \sum_{\alpha,\beta} \int d^3x\, dt\, \psi^*_\alpha(\mathbf{x},t)\psi^*_\beta(\mathbf{x},t)\psi_\beta(\mathbf{x},t)\psi_\alpha(\mathbf{x},t)\right] = \text{const} \times \int \mathcal{D}\Delta(\mathbf{x},t)\mathcal{D}\Delta^*(\mathbf{x},t)$$

[4]In analogy to [10], the hermitian adjoint $\Delta^*_{\alpha\beta}(x)$ comprises transposition in the spin indices, i.e., $\Delta^*_{\alpha\beta}(x) = [\Delta_{\beta\alpha}(x)]^*$.

$$\times \exp\left[-\frac{i}{2}\int d^3x dt \sum_{\alpha\beta}\left(|\Delta_{\alpha\beta}|^2\frac{1}{g} - \psi_\beta \Delta^*_{\beta\alpha}\psi_\alpha - \psi^*_\alpha \Delta_{\alpha\beta}\psi^*_\beta\right)\right] \quad (3.39)$$

where the new auxiliary field is a $(2s+1)\times(2s+1)$ non-hermitian matrix which satisfies the equation of constraint:

$$\Delta_{\alpha\beta}(\mathbf{x},t) = g\psi_\alpha(\mathbf{x},t)\psi_\beta(\mathbf{x},t). \quad (3.40)$$

Observe the hermiticity property

$$\Delta_{\alpha\beta}(\mathbf{x},t)^* = \Delta^*_{\beta\alpha}(\mathbf{x},t). \quad (3.41)$$

Thus the matrix A in (3.5) reads

$$A(\mathbf{x},t;\mathbf{x}',t') = \begin{pmatrix} [i\partial_t - \xi(-i\boldsymbol{\nabla})]\,\delta(x-x')\delta_{\alpha\beta} & -\Delta_{\alpha\beta}(x)\delta(x-x') \\ -\Delta^*_{\alpha\beta}(x)\delta(x-x') & \mp[i\partial_t + \xi(i\boldsymbol{\nabla})]\,\delta(x-x')\delta_{\alpha\beta} \end{pmatrix} \quad (3.42)$$

and the action (3.19) becomes

$$\mathcal{A}[\Delta^*,\Delta] = \mp i\sum_{n=1}^{\infty}\frac{(-)^n}{2n}\mathrm{Tr}\,\mathrm{tr}_{\mathrm{spin}}\left[\left(\frac{i}{i\partial_t - \xi(-i\boldsymbol{\nabla})}\delta\right)(\Delta\,\delta)\left(\frac{\mp i}{i\partial_t + \xi(i\boldsymbol{\nabla})}\delta\right)(\Delta^*\,\delta)\right]^n, \quad (3.43)$$

where Tr acts on the functional matrix space while $\mathrm{tr}_{\mathrm{spin}}$ is restricted to the spin indices.

Consider now fermions of spin 1/2 close to a critical region, i.e., $T \approx T_c$ in which long-range properties of the system dominate. As far as such questions are concerned, the expansion

$$\mathcal{A}[\Delta^*,\Delta] = \sum_2^\infty \mathcal{A}_n[\Delta^*,\Delta] \quad (3.44)$$

may be truncated after the fourth term without much loss of information (the dimensions of the neglected terms are so high that they become invisible at long distances [7]. The free part of the action $\mathcal{A}_2[\Delta^*,\Delta]$ is given by

$$\begin{aligned}\mathcal{A}_2[\Delta^*,\Delta] &= \pm i\mathrm{Tr}\,\mathrm{tr}_{\mathrm{spin}}\left[\left(\frac{i}{i\partial_t - \xi(-i\boldsymbol{\nabla})}\delta\right)(\Delta\delta)\left(\frac{\mp i}{i\partial_t + \xi(i\boldsymbol{\nabla})}\delta\right)(\Delta^*\delta)\right] \\ &\quad -\frac{1}{2}\mathrm{tr}_{\mathrm{spin}}\int dx \Delta^*(x)\Delta(x)\frac{1}{g}. \end{aligned} \quad (3.45)$$

The spin traces can be performed by noticing that due to Fermi statistics and remembering the constraint Eq. (3.4), (3.40), there is really only one independent pair field:

$$\Delta(x) \equiv \Delta_{\downarrow\uparrow}(x) = g\psi_\downarrow(x)\psi_\uparrow(x) = -\Delta_{\uparrow\downarrow}(x). \quad (3.46)$$

Thus $\mathcal{A}_2$ becomes:

$$\mathcal{A}_2[\Delta^*\Delta] = -i\int dxdx' G_0(x,x')\tilde{G}_0(x',x)\Delta^*(x)\Delta(x') - \frac{1}{g}\int dx|\Delta(x)|^2. \tag{3.47}$$

In momentum space, this can be rewritten as

$$\mathcal{A}_2[\Delta^*,\Delta] = \frac{T}{V}\sum_k \Delta^*(k)L(k)\Delta(k) \tag{3.48}$$

with

$$\begin{aligned} L(k) &\equiv -i\frac{T}{V}\sum_p \frac{i}{p^0+k^0-\xi(\mathbf{p}+\mathbf{k})+i\eta\,\mathrm{sgn}\,\xi(\mathbf{p}+\mathbf{k})}\frac{i}{p^0+\xi(\mathbf{p})-i\eta\mathrm{sgn}\,\xi(\mathbf{p})} - \frac{1}{g} \\ &= \frac{T}{V}\sum_p l(\mathbf{p}|k) - \frac{1}{g} \end{aligned} \tag{3.49}$$

as pictured by the diagram Fig. 3.5. The expression $l(\mathbf{p}|k)$ was discussed before in

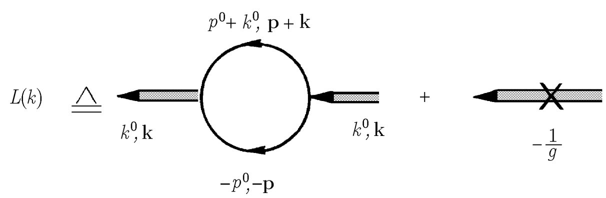

FIGURE 3.5 The free part of the pair field Δ Lagrangian containing the direct term plus the one loop diagram. As a consequence, the free Δ propagator sums up an infinite sequence of such loops

general and brought to the form (3.32). In the present case of Fermi statistics this leads to

$$L(\nu,\mathbf{k}) = \frac{1}{2}\frac{1}{V}\sum_{\mathbf{p}}\frac{1}{\xi(\mathbf{p}+\mathbf{k})+\xi(\mathbf{p})-i\nu}\left[\tanh(\xi(\mathbf{p}+\mathbf{k})/2T)+\tanh(\xi(\mathbf{p})/2T)\right]-\frac{1}{g} \tag{3.50}$$

At $k=0$ one has

$$\begin{aligned} L(0) &= \frac{1}{2}\frac{1}{V}\sum_{\mathbf{p}}\frac{\tanh\xi(\mathbf{p})/2T}{\xi(\mathbf{p})} - \frac{1}{g} \\ &\approx \mathcal{N}(0)\int_0^\infty \frac{d\xi}{\xi}\tanh\frac{\xi}{2T} - \frac{1}{g}. \end{aligned} \tag{3.51}$$

In going from the first to the second line we have used the approximation

$$\frac{1}{V}\sum_{\mathbf{p}} \equiv \int\frac{d^3p}{(2\pi)^3} = \frac{1}{(2\pi)^3}\int p^2\frac{dp}{d\xi}d\xi \approx \mathcal{N}(0)\int d\xi \tag{3.52}$$

with

$$\mathcal{N}(0) = \frac{mp_F}{2\pi^2} = \frac{3n}{4\mu}. \tag{3.53}$$

Note that the Fermi temperature $T_F = mp_F^2/2mk_B$ lies in most material around 10000 K. The ξ-integral is logarithmically divergent. This, however, is an unphysical feature of the local approximation of the assumed interaction between the electrons. As we argued above, the attraction between electrons is caused by phonon exchange. Phonons, however, have frequencies which are at most of the order of the *Debeye frequency* ω_D which may be used as a cutoff in all energy integrals $\int d\xi$, resticting them to the interval $\xi \in (0, \omega_D)$, caused by the lattice structure of the system. The temperature $T_D = \hbar\omega_D/k_B$ is of the order of 1000 K and thus quite large compared to the characteristic temperature where superconductivity sets in. But since it is an order of magnitude smaller than T_F, the attraction between electrons is active only between states within a thin layer in the neighborhood of surface of the Fermi sphere. With the cutoff at ω_D, Eq. (3.51) yields

$$\begin{aligned} L(0) &\approx \mathcal{N}(0) \int_0^{\omega_D} \frac{d\xi}{\xi} \tanh \frac{\xi}{2T} - \frac{1}{g} \\ &= \mathcal{N}(0) \log\left(\frac{\omega_D}{T}\frac{2e^\gamma}{\pi}\right) - \frac{1}{g}, \end{aligned} \tag{3.54}$$

where γ is Euler's constant

$$\gamma = -\Gamma'(1)/\Gamma(1) \approx 0.577, \tag{3.55}$$

so that $e^\gamma/\pi \approx 1.13$.

The integral in (3.54) is done as follows: It is integrated by part to

$$\int_0^{\omega_D} \frac{d\xi}{\xi} \tanh \frac{\xi}{2T} = \log \frac{\xi}{T} \tanh \frac{\xi}{2T}\bigg|_0^{\omega_D/T} - \frac{1}{2} \int_0^\infty d\frac{\xi}{T} \log \frac{\xi}{T} \frac{1}{\cosh^2 \dfrac{\xi}{2T}}. \tag{3.56}$$

Since $\omega_D/\pi T >> 1$, the first term is given by $\log(\omega_D/2T))$ with exponentially small corrections which can be ignored. In the second integral, we have taken the upper limit of integration to infinity since it converges. We may use the integral formula[5]

$$\int_0^\infty dx \frac{x^{\mu-1}}{\cosh^2(ax)} = \frac{4}{(2a)^\mu}\left(1 - 2^{2-\mu}\right)\Gamma(\mu)\zeta(\mu - 1), \tag{3.57}$$

set $\mu = 1 + \delta$, expand the formula to order δ, and insert the special values

$$\Gamma'(1) = -\gamma, \quad \zeta'(0) = -\frac{1}{2}\log(2\pi)\log(4e^\gamma/\pi) \tag{3.58}$$

[5]See, for instance, I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York, 1980, Formula 3.527.3.

to find from the linear terms in δ:

$$\int_0^\infty dx \frac{\log x}{\cosh^2(x/2)} = -2\log(2e^\gamma/\pi), \tag{3.59}$$

so that we obtain

$$\int_0^{\omega_D} \frac{d\xi}{\xi} \tanh \frac{\xi}{2T} = \log\left(\frac{\omega_D}{T}\frac{2e^\gamma}{\pi}\right). \tag{3.60}$$

Expression $L(0)$ in (3.54) vanishes at a critical temperature determined by

$$T_c \equiv \frac{2e^\gamma}{\pi}\omega_D e^{-1/\mathcal{N}(0)g}. \tag{3.61}$$

In terms of T_c, Eq. (3.54) can be rewritten as

$$L(0) = \mathcal{N}(0)\log\frac{T_c}{T} \approx \mathcal{N}(0)\left(1 - \frac{T}{T_c}\right). \tag{3.62}$$

The constant $L(0)$ obviously plays the role of the chemical potential of the pair field. Its vanishing at $T = T_c$ implies that at that temperature the field propagates over long range (with a power law) in the system. Critical phenomena are observed [7]. For $T < T_c$ the chemical becomes positive indicating the appearance of a Bose condenstate. If $\nu \neq 0$ but $\mathbf{k} = 0$ one can write (3.47) as in the subtracted form

$$L(\nu,\mathbf{0}) - L(0,\mathbf{0}) = \frac{T}{V}\sum_p \left[\frac{1}{2\xi(\mathbf{p}) - i\nu} - \frac{1}{2\xi(\mathbf{p})}\right]\tanh\frac{\xi(\mathbf{p})}{2T} \tag{3.63}$$

$$\approx i\nu\mathcal{N}(0)\int_{-\omega_D}^{\omega_D} \frac{d\xi}{2\xi - i\nu}\frac{1}{2\xi}\tanh\frac{\xi}{2T}. \tag{3.64}$$

Since the integral converges fast it can be performed over the whole ξ axis with the small error $T/\omega_D \ll 1$. For $\nu < 0$, the contour may be closed above picking up poles exactly at the Matsubara frequencies $\xi = i(2n+1)\pi T = i\omega_n$. Hence

$$L(\nu,\mathbf{0}) - L(0,\mathbf{0}) \approx \nu\mathcal{N}(0)\pi T \sum_{\omega_n>0} \frac{1}{\omega_n - \nu/2}\frac{1}{\omega_n}. \tag{3.65}$$

The sum can be expressed in terms of digamma functions: For $|\nu| \ll T$ one expands

$$\sum_{\omega_n>0}\left[\frac{1}{\omega_n^2} + \frac{\nu}{2}\frac{1}{\omega_n^3} + \frac{\nu^2}{4}\frac{1}{\omega_n^4} + \dots\right] \tag{3.66}$$

and applies the formula

$$\sum_{\omega_n>0}\frac{1}{\omega_n^k} = \frac{1}{\pi^k T^k}[1 - 2^{-k}]\zeta(k), \tag{3.67}$$

where

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z} \tag{3.68}$$

is the Rieman zeta function. Some of its values are

$$\begin{aligned} \zeta(2) &= \frac{\pi^2}{6}, \quad \zeta(3) = 1.202057, \\ \zeta(4) &= \frac{\pi^4}{90}, \quad \zeta(5) = 1.036928, \\ &\vdots \; . \end{aligned} \tag{3.69}$$

Thus:

$$\sum_{\omega_n>0} \frac{1}{\omega_n^2} = \frac{1}{\pi^2 T^2} \frac{3}{4} \frac{\pi^2}{6} = \frac{1}{8T^2}, \tag{3.70}$$

$$\sum_{\omega_n>0} \frac{1}{\omega_n^3} = \frac{1}{\pi^3 T^3} \frac{7}{8} \zeta(3), \tag{3.71}$$

$$\sum_{\omega_n>0} \frac{1}{\omega_n^4} = \frac{1}{\pi^4 T^4} \frac{15}{16} \frac{\pi^4}{90} = \frac{1}{96 T^4}. \tag{3.72}$$

$$\vdots \tag{3.73}$$

Using the power series for the digamma function

$$\psi(1-x) = -\gamma - \sum_{k=2}^{\infty} \zeta(k) x^{k-1} \tag{3.74}$$

the sum is

$$-\frac{2}{\nu\pi T}\left[\psi\left(1-\frac{\nu}{2\pi T}\right) - \psi\left(1-\frac{\nu}{4\pi T}\right)/2 + \gamma/2\right] = \frac{1}{\nu\pi T}\left[\psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2}-\frac{\nu}{4\pi T}\right)\right]$$

with the first term

$$\frac{1}{8T^2} + \nu \frac{1}{2\pi^3 T^3} \frac{7}{8} \zeta(3) + \frac{\nu^2}{4 \cdot 96 T^4} + \dots \tag{3.75}$$

For $\nu > 0$ the integration contour is closed below and the same result is obtained with ν replaced by $-\nu$. Thus one finds

$$\begin{aligned} L(\nu, \mathbf{0}) - L(0, \mathbf{0}) &= \mathcal{N}(0)\left[\psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2} + \frac{|\nu|}{4\pi T}\right)\right] \\ &\approx -\mathcal{N}(0)\left[\frac{\pi}{8T}|\nu| - \nu^2 \frac{1}{2\pi^2 T^2} \frac{7}{8} \zeta(3) + \dots\right]. \end{aligned} \tag{3.76}$$

The $\mathbf{k}$-dependence at $\nu = 0$ is obtained by expanding directly

$$\begin{aligned} L(0, \mathbf{k}) &= \frac{T}{V} \sum_{\omega, \mathbf{p}} \frac{1}{i\omega - \xi(\mathbf{p}+\mathbf{k})} \frac{1}{-i\omega - \xi(\mathbf{p})} - \frac{1}{g} \\ &= T \sum_{n=0}^{\infty} \frac{1}{V} \sum_{\omega, \mathbf{p}} \frac{1}{[i\omega - \xi(\mathbf{p})]^{n+1}} \left(\frac{\mathbf{pk}}{m} + \frac{\mathbf{k}^2}{2m}\right)^n \frac{1}{-i\omega - \xi(\mathbf{p})} - \frac{1}{g}. \end{aligned} \tag{3.77}$$

The sum over $\mathbf{p}$ may be split into radial and angular integrals as

$$\frac{1}{V}\sum_{\mathbf{p}}\int\frac{d^3p}{(2\pi)^3}\approx\mathcal{N}(0)\int d\xi\int\frac{d\hat{\mathbf{p}}}{4\pi}, \tag{3.78}$$

where

$$\int\frac{d\hat{\mathbf{p}}}{4\pi}=\int_{-\pi}^{\pi}\frac{d\phi}{2\pi}\int_{-1}^{1}\frac{d\cos\theta}{2} \tag{3.79}$$

is the integral over all directional angles of the momentum $\mathbf{p}$. The denominators are strongly peaked at $\xi\approx 0$ so that only the narrow region $|\xi|\leq T$ contributes. Hence, the momentum p may be replaced by the Fermi momentum p_F with only a small error $\mathcal{O}(T/\mu)\approx 10^{-3}$. Introducing now the Fermi velocity $v_F=p_F/m$, for convenience, and performing the ξ-integrals

$$\int d\xi\frac{1}{(i\omega-\xi)^{n+1}}\frac{1}{-i\omega-\xi}=(-i\,\mathrm{sgn}\,\omega)^n\frac{\pi}{2^n|\omega|^{n+1}}, \tag{3.80}$$

one finds

$$L(0,\mathbf{k})\approx 2\mathcal{N}(0)\mathrm{Re}\sum_{n=0}^{\infty}(-i)^n\frac{\pi}{2^n|\omega|^{n+1}}\int\frac{d\hat{\mathbf{p}}}{4\pi}\left(v_F\hat{\mathbf{p}}\mathbf{k}+\frac{\mathbf{k}^2}{2m}\right)^n-\frac{1}{g}. \tag{3.81}$$

For $\mathbf{k}=0$ we recover the logarithmically divergent sum

$$L(0,\mathbf{0})=\mathcal{N}(0)\sum_{\omega}\frac{\pi}{|\omega|}-\frac{1}{g}. \tag{3.82}$$

This is just another representation of the energy integral (3.54), and can therefore be made finite by the same cutoff procedure.

The higher powers can be summed via formula (3.67) with the result

$$\begin{aligned}L(0,\mathbf{k})&=L(0,\mathbf{k})+2\mathcal{N}(0)\mathrm{Re}\sum_{n=1}^{\infty}\frac{(-i)^n}{2^n\pi^nT^n}(1-2^{-(n+1)})\zeta(n+1)\int\frac{d\hat{\mathbf{p}}}{4\pi}\left(v_F\hat{\mathbf{p}}\mathbf{k}+\frac{\mathbf{k}^2}{2m}\right)^n\\&=L(0,\mathbf{0})+\mathcal{N}(0)\mathrm{Re}\int\frac{d\hat{\mathbf{p}}}{4\pi}\left[\psi\left(\frac{1}{2}\right)-\psi\left(\frac{1}{2}-i\left(v_F\hat{\mathbf{p}}\mathbf{k}+\frac{\mathbf{k}^2}{2m}\right)\frac{1}{4\pi T}\right)\right].\end{aligned} \tag{3.83}$$

Comparing this with Eq. (3.76) one sees that the full $\mathbf{k}$- and ν-dependence is obtained by adding $|\nu|/4\pi T$ to the arguments of the second digamma function. This can also be checked by a direct calculation. In the long-wave length limit in which $kv_F/T\ll 1$ one has also

$$\frac{k^2/2m}{T}\approx\frac{k}{p_F}\frac{kv_F}{T}\ll\frac{kv_F}{T},$$

and one may truncate the sum after the quadratic term as follows:

$$L(0,\mathbf{k})=L(0,\mathbf{0})+\sum\Lambda_{kl}(0)k_kk_l \tag{3.84}$$

where

$$\Lambda_{kl}(0) = -2\mathcal{N}(0)\frac{1}{4\pi^2T^2}\frac{7}{8}\zeta(3)v_F^2\int\frac{d\hat{\mathbf{p}}}{4\pi}\hat{p}_k\hat{p}_l. \tag{3.85}$$

The angular integration yields

$$\int\frac{d\hat{\mathbf{p}}}{4\pi}\hat{p}_k\hat{p}_l = \frac{1}{3}\delta_{kl}. \tag{3.86}$$

Hence, the lowest terms in the expansion of $L(\nu,\mathbf{k})$ for $kv_F \ll T$ and $\nu \ll T$ are

$$L(\nu,\mathbf{k}) \approx L(0,\mathbf{0}) - \mathcal{N}(0)\left[\frac{\pi}{8T}|\nu| + \frac{1}{6\pi^2T^2}\frac{7}{8}\zeta(3)v_F^2\mathbf{k}^2\right]. \tag{3.87}$$

The term (3.85) may also conveniently be calculated in x space, since for large $x(\gg 1/p_F)$,

$$G_0(\mathbf{x},\omega) \approx -\frac{m}{2\pi|\mathbf{x}|}\exp\left[ip_F|\mathbf{x}|\operatorname{sgn}\omega - \frac{|\omega|}{v_F}|\mathbf{x}|\right] \tag{3.88}$$

so that the second spatial derivative contributes to (3.43):

$$\int dx\left[\frac{1}{2}\int d^3x'T\sum_{\omega_n}G_0(\mathbf{x}-\mathbf{x}',\omega_n)G_0(\mathbf{x}-\mathbf{x}',-\omega_n)(x-x')_i(x-x')_j\right] \times\Delta^*(x)\boldsymbol{\nabla}_i\boldsymbol{\nabla}_j\Delta(x). \tag{3.89}$$

The parenthesis becomes

$$\begin{aligned}&\frac{1}{2}\int d^3zT\sum_{\omega_n}\left(\frac{m}{2\pi|z|}\right)^2\exp\left(-2\frac{|\omega_n|}{v_F}|z|\right)z_iz_j\\ &\quad= \frac{1}{24}\delta_{ij}T\int d^3z\frac{1}{\sinh 2\pi|z|T/v_F} = \delta_{ij}\frac{7\zeta(3)}{48}\mathcal{N}(0)\frac{v_F^2}{\pi^2T^2}\end{aligned} \tag{3.90}$$

making (3.87) coincide with (3.84).

In many formulas to come it is useful to introduce the characteristic length parameter

$$\xi_0 \equiv \sqrt{\frac{7\zeta(3)}{48}}\frac{v_F}{\pi T_c}. \tag{3.91}$$

Inserting $\zeta(3) \approx 1.202057$ this becomes

$$\xi_0 \approx 0.4187 \times \frac{v_F}{\pi T_c}. \tag{3.92}$$

Using $T_F \equiv \mu \equiv p_F^2/2m$, the right-hand side of (3.91) can also be written as

$$\xi_0 = \sqrt{\frac{7\zeta(3)}{48}}\frac{2T_Fp_F^{-1}}{\pi T_c} \approx 0.25 \times \frac{T_F}{T_c}p_F^{-1}. \tag{3.93}$$

Since in most superconductors, T_c is of the order of one degree Kelvin, about 1/1000 of the Fermi temperature T_F, the length parameter ξ_0 is of the order of 1000 Å. Then, in the action (3.48), the low-frequency and long-wavelength result (3.87) corresponds to [6]

$$\mathcal{A}_2[\Delta^*,\Delta] \approx -i\mathcal{N}(0)T \sum_{\nu\ll T,\mathbf{k}} \Delta^*(\nu,\mathbf{k}) \left\{ \left(1-\frac{T}{T_c}\right) - \xi_0^2\mathbf{k}^2 - \frac{\pi}{8T}|\nu| \right\} \Delta(\nu,\mathbf{k}). \quad (3.94)$$

For $T \le T_c$, the field Δ has therefore a propagator

$$\dot{\Delta}^*(\nu_n,\mathbf{k})\dot{\Delta}(\nu_m,\mathbf{k}') = -(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\frac{1}{T}\delta_{nm}\frac{1}{\mathcal{N}(0)} \times \left[-\frac{\pi}{8T}|\nu_n| + \left(1-\frac{T}{T_c}\right) - \xi_0^2\mathbf{k}^2 \right]^{-1} \quad (3.95)$$

The spectrum of collective excitations can be read off from this expression by continuing the energy back to real values from the upper half of the complex plane:

$$k_0 = -i\frac{8}{\pi}(T-T_c) - i\frac{8T}{\pi}\xi_0^2\mathbf{k}^2. \quad (3.96)$$

These excitations are purely dissipative.

If the system is close enough to the critical temperature all interaction terms except $\mathcal{A}_4[\Delta^*,\Delta]$ become irrelevant because of their high dimensions [7]. And in $\mathcal{A}_4$ only the momentum independent contribution is of interest, again because it has the lowest dimension.

Its calculation is standard:

$$\begin{aligned}
\mathcal{A}_4[\Delta^*\Delta] &= \frac{i}{4}\mathrm{Tr}\,\mathrm{Tr}_{\mathrm{spin}}\left[\left(\frac{i}{i\partial_t - \xi(-i\boldsymbol{\nabla})}\Delta\delta\right)\left(\frac{i}{i\partial_t + \xi(i\boldsymbol{\nabla})}\delta\right)\Delta^*\delta\right]^2 \\
&= -\frac{i}{2}\int dx_1dx_2dx_3dx_4 G_0(x_1x_2)\tilde{G}_0(x_2x_3)G_0(x_3x_4)\tilde{G}_0(x_4x_1)\Delta^*(x_1)\Delta(x_2)\Delta^*(x_3)\Delta(x_4) \\
&\approx -\frac{1}{2}\int dx|\Delta(x)|^4 \int d^3x_2d^3x_3d^3x_4 \\
&\quad \times T\sum_{\omega_n}[G_0(\mathbf{x}-\mathbf{x}_2,\omega_n)G_0(\mathbf{x}_3-\mathbf{x}_2,-\omega_n)G_0(\mathbf{x}_3-\mathbf{x}_4,\omega_n)G_0(\mathbf{x}-\mathbf{x}_4,-\omega_n)] \\
&\equiv -\frac{\beta}{2}\int dx|\Delta(x)|^4. \qquad (3.97)
\end{aligned}$$

The coefficient can be computed as usual

$$\begin{aligned}
\beta &= T\sum_{\omega_n\mathbf{p}}\frac{1}{(\omega_n^2+\xi^2(\mathbf{p}))^2} \approx \mathcal{N}(0)T\sum_{\omega_n}\int d\xi\frac{1}{(\omega_n^2+\xi^2)^2} \\
&= \mathcal{N}(0)\frac{\pi}{2}T\sum_{\omega_n}\frac{1}{|\omega_n|^3} = \mathcal{N}(0)\frac{7\zeta(3)}{8(\pi T_c)^2} = 6\mathcal{N}(0)\frac{\xi_0^2}{v_F^2} \approx 10^{-3}\frac{p_F^3}{T_FT_c^2}. \qquad (3.98)
\end{aligned}$$

---

[6] Note that only Matsubara frequency $\nu_0 = 0$ satisfies the condition $\nu \ll T$. The neighbourhood of $\nu_0 = 0$ with the linear behaviour $|\nu|$ becomes visible only after analytic continuation of (3.95) to the retarded Green function which amounts to replacing $|\nu_n| \to -ik_0$.

The time independent part of this action at the classical level has been derived a long time ago by Gorkov on the basis of Green function techniques [3, 11]. Certainly, his technical manipulations are exactly the same as presented here. The difference lies in the theoretical foundation [4, 5, 6, 7] and the ensuing prescriptions on how to improve upon the approximations. Our action of (3.7) is the *exact* translation of the fundamental theory into pair fields. These fields are made quantum fields in the standard fashion by leaving the functional formalism and going to the operator language. The result is a perturbation theory of Δ-quanta with (3.95) as a free propagator and $\mathcal{A}_n, n > 2$ treated as perturbations. The higher terms $\mathcal{A}_6, \mathcal{A}_8, \ldots$ are very weak residual interactions as far as long distance questions are concerned. In fact, for the calculation of the critical indices, $\mathcal{A}_2$ and $\mathcal{A}_4$ contain *all* information about the system.

3.3 Inclusion of Electromagnetic Fields into the Pair Field Theory

The original action $\mathcal{A}$ of (1.33) can be made invariant under general spacetime dependent gauge transformations

$$\psi(\mathbf{x}, T) \to \exp[-i\Lambda(\mathbf{x}, t)]\psi(\mathbf{x}, t) \tag{3.99}$$

if an electromagnetic potential

$$A = (\varphi, \mathbf{A}) \tag{3.100}$$

is present, capable of absorbing the generated derivative terms via

$$\begin{aligned} \varphi &\to \varphi - \frac{1}{e}\partial_t\Lambda \\ A_i &\to A_i + \frac{c}{e}\boldsymbol{\nabla}_i\Lambda. \end{aligned} \tag{3.101}$$

The complete action including electromagnetism in the Coulomb gauge, $\boldsymbol{\nabla}A = 0$, becomes:

$$\begin{aligned} \mathcal{A}_{\rm compl} = \mathcal{A}[\psi^*, \psi] \Big|_{i\partial_t \to i\partial_t + \frac{1}{e}\varphi, -\boldsymbol{\nabla}_i \to -i\boldsymbol{\nabla}_i + \frac{c}{e}A_i} \\ + \frac{1}{8\pi}\int dx \left(-\varphi\boldsymbol{\nabla}^2\varphi + \frac{1}{c^2}A^2 + A\boldsymbol{\nabla}^2 A\right) \end{aligned} \tag{3.102}$$

where the arrows denote the gauge invariant substitutions in the action (1.33). Since the final pair action (3.7) describes the same system as the initial action (1.33), it certainly has to posses the same invariance after inclusion of electromagnetism. But from the constraint equation (3.4) we see

$$\Delta(x, x') \to \exp\{-i[\Lambda(x) + \Lambda(x')]\}\,\Delta(x, x'). \tag{3.103}$$

For the local pair field appearing in (3.39) this gives

$$\Delta(x) \to \exp[-2i\Lambda(x)]\,\Delta(x). \tag{3.104}$$

Hence the final action (3.94) with $\mathcal{A}_4$ from (3.97) added is gauge invariant after replacing

$$i\partial_t \to i\partial_t + 2e\varphi, \quad -i\boldsymbol{\nabla}_i \to -i\boldsymbol{\nabla}_i + 2\frac{e}{c}A_i,$$
$$k_0 \to k_0 + 2e\varphi, \quad k_i \to k_i + 2\frac{e}{c}A_i. \tag{3.105}$$

This leads to the full time dependent Lagrangian close to the critical point

$$\begin{aligned}\mathcal{L} &= \frac{\mathcal{N}(0)\pi}{8T}\Delta^*(x)(-\partial_t + 2ie\varphi)\Delta(x) + \mathcal{N}(0)\left(1+\frac{T}{T_c}\right)\Delta^*\Delta \\ &-\mathcal{N}(0)\xi_0^2\left(\boldsymbol{\nabla}_i - 2i\frac{e}{c}A_i\right)\Delta^*(x)\left(\boldsymbol{\nabla}_i + 2i\frac{e}{c}A_i\right)\Delta(x) \\ &-3\mathcal{N}(0)\frac{\xi_0^2}{v_F^2}|\Delta(x)|^4 + \frac{1}{8\pi}\left(-\varphi\boldsymbol{\nabla}^2\varphi + \frac{1}{c^2}\dot{A}^2 + A\boldsymbol{\nabla}^2 A\right).\end{aligned} \tag{3.106}$$

The discussion of this Lagrangian is standard. At the classical level there are, above T_c, doubly charged pair states of chemical potential

$$\mu_{\text{pair}} = L(0) = \mathcal{N}(0)\left(1-\frac{T}{T_c}\right) < 0; T > T_c. \tag{3.107}$$

Below T_c the chemical potential becomes positive causing an instability which settles, due to the stabilizing quartic term, at a nonzero field value, the "gap":

$$\Delta_0(T) = \sqrt{\frac{\mu_{\text{pair}}}{\beta}} = \sqrt{\frac{8}{7\zeta(3)}}\pi T_c\left(1-\frac{T}{T_c}\right)^{1/2}. \tag{3.108}$$

Inserting $\zeta(3) \approx 1.202057$ this is approximately

$$\Delta_0(T) \approx 3.063 \times T_c\left(1-\frac{T}{T_c}\right)^{1/2}. \tag{3.109}$$

The new vacuum obviously breaks gauge invariance spontaneously: the field Δ will now oscillate radially with a chemical potential

$$\mu_{\text{pair}} = -2\mathcal{N}(0)\left(1-\frac{T}{T_c}\right) < 0; T < T_c. \tag{3.110}$$

Due to this, spatial changes of the field $|\Delta|$ can take place over a length scale, defined as coherence length [3, 11]

$$\xi_c(T) \equiv \sqrt{\frac{\text{coefficient of } |\boldsymbol{\nabla}\Delta|^2}{|\mu_{\text{pair}}|}} = \xi_0\left(1-\frac{T}{T_c}\right)^{-1/2}. \tag{3.111}$$

The azimuthal oscillations experience a different fate in the absence of electromagnetism; they have a vanishing chemical potential due to the invariance of $\mathcal{L}$ under phase rotations. As an electromagnetic field is turned on, the new center of oscillations (3.108) is seen in (3.106) to generate a mass term $1/8\pi\mu_A^2 A^2$ for the photon. The vector potential aquires a mass

$$\mu_A^2 = 8\pi \text{ coefficient of } A^2 \text{ in } |\boldsymbol{\nabla}\Delta|^2 - \text{term } = 8\omega\frac{4e^2}{c^2}\mathcal{N}(0)\xi_0^2\Delta_0^2. \tag{3.112}$$

This mass limits the penetration of magnetic field into a superconductor. The penetration depth is defined as [3, 11]

$$\begin{aligned} \lambda(T) \equiv \mu_A^{-1} &= \sqrt{\frac{3}{\pi\mathcal{N}(0)}}\frac{c}{4ev_F}\left(1-\frac{T}{T_c}\right)^{-1/2} \\ &= \sqrt{\frac{3\pi}{8}}\sqrt{\frac{c}{v_F\alpha}}p_F^{-1}\left(1-\frac{T}{T_c}\right)^{-1/2}. \end{aligned} \tag{3.113}$$

Here we have introduced the feinstructure constant

$$\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}. \tag{3.114}$$

The ratio

$$\kappa(T) \equiv \frac{\lambda(T)}{\xi(T)} = \sqrt{\frac{9\pi^3}{14\zeta(3)}}\sqrt{\frac{c}{v_F\alpha}}\frac{T_c}{T_F} \approx 4.1 \times \sqrt{\frac{c}{v_F\alpha}}\frac{T_c}{T_F} \tag{3.115}$$

is the Ginzburg-Landau parameter deciding whether it is energetically preferable for the superconductor to have flux lines invading it or not. For ($\kappa > 1/\sqrt{2}$ they do invade and the superconductor is said to be of type II, for $\kappa < 1/\sqrt{2}$ they don't and the superconductor is of type I.

3.4 Far below the Critical Temperature

We have seen in the last section that for T smaller than T_c the chemical potential of the pair field becomes positive, causing oscillations around a new minimum which is the gap value Δ_0 given by (3.108). That formula was based on the expansion (3.9) of the pair action and can be valid only as long as $\Delta \ll T_c$, i.e., $T \approx T_c$. If T drops far below T_c, one must account for Δ_0 non-perturbatively by inserting it as an open parameter into $\mathbf{G}_\Delta$ of (3.8). Everything has to be recalculated.

3.4.1 The Gap

In the general bilocal form collective action we separate

$$\Delta(x,x') = \Delta_0(x-x') + \Delta'(x,x') \tag{3.116}$$

and expands $\mathbf{G}_\Delta$ inpowers of $\Delta'(x,x')$ around

$$\mathbf{G}_{\Delta_0}(x,x') = i\begin{pmatrix} [i\partial_t - \xi(-i\boldsymbol{\nabla})]\delta & -\Delta_0 \\ -\Delta_0^\dagger & \mp i[\partial_t - \xi(i\boldsymbol{\nabla})]\delta \end{pmatrix}^{-1}(x,x') \tag{3.117}$$

instead of (3.10). This leads to the replacement $\mathbf{G}_0 \to \mathbf{G}_{\Delta_0}$ in every term of (3.17). Observe that in the underlying theory of fields ψ^*, ψ the matrix $\mathbf{G}_{\Delta_0}$ collects the bare one-particle Green functions:

$$\mathbf{G}_{\Delta_0}(x, x') = \begin{pmatrix} \dot{\psi}(x)\dot{\psi}^\dagger(x') & \dot{\psi}(x)\dot{\psi}(x') \\ \dot{\psi}^\dagger(x)\dot{\psi}^\dagger(x') & \dot{\psi}^\dagger(x)\dot{\psi}(x') \end{pmatrix} \tag{3.118}$$

Contrary to (3.9) and (3.13) the off-diagonal Green functions are nonvanishing, since at $T < T_c$ a condensate is present in the vacuum. The presence of Δ_0 causes a linear dependence of the action on $\Delta'(x, x')$

$$\begin{aligned} \mathcal{A}_1[\Delta'^*, \Delta'] = \pm \mathrm{Tr} \left[\mathbf{G}_{\Delta_0} \begin{pmatrix} 0 & \Delta' \\ \Delta'^* & 0 \end{pmatrix} \right] \\ + \frac{1}{2} \int dx dx' \left[\Delta_0^*(x - x') \Delta'(x, x') \frac{1}{V(x, x')} + \text{c.c.} \right]. \end{aligned} \tag{3.119}$$

The gap function may now be determined optimally by minimizing the action with respect to $\delta\Delta'$ at $\Delta' = 0$ which amounts to the elimination of $\mathcal{A}_1[\Delta'^*, \Delta']$. Taking the functional derivative of (3.119) gives the *gap equation*

$$\Delta_0(x - x') = \pm V(x - x') \,\mathrm{tr}_{2\times 2} \left[\mathbf{G}_{\Delta_0}(x, x') \frac{\tau^-}{2} \right] \tag{3.120}$$

where $\tau^-/2$ is the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ in the 2×2 dimensional matrix space of (3.8). If the potential is instantaneous, the gap has a factor $\delta(t - t')$, i.e.,

$$\Delta_0(x - x') \equiv \delta(t - t') \times \Delta_0(\mathbf{x} - \mathbf{x}')$$

and the Fourier transform of the spatial part satisfies

$$\Delta_0(\mathbf{p}) = \pm \frac{T}{V} \sum_{\omega, \mathbf{p}'} V(\mathbf{p} - \mathbf{p}') \,\mathrm{tr}_{2\times 2} \left[\mathbf{G}_{\Delta_0}(\omega, \mathbf{p}') \frac{\tau^-}{2} \right]. \tag{3.121}$$

Inverting (3.117) renders the propagator:

$$\begin{aligned} &\mathbf{G}_{\Delta_0}(\tau, x) = \\ &\mp \frac{T}{V} \sum_{\omega, \mathbf{p}} \exp\left[-i(\omega\tau - \mathbf{p}\mathbf{x})\right] \frac{1}{\omega^2 + \xi^2(\mathbf{p}) \mp |\Delta_0(\mathbf{p})|^2} \begin{pmatrix} \mp [i\omega + \xi(\mathbf{p})] & \Delta_0(\mathbf{p}) \\ \Delta_0^*(\mathbf{p}) & [i\omega - \xi(\mathbf{p})] \end{pmatrix}, \end{aligned} \tag{3.122}$$

so that the gap equation (3.121) takes the explicit form

$$\Delta_0(\mathbf{p}) = -\frac{T}{V} \sum_{\omega, \mathbf{p}'} V(\mathbf{p} - \mathbf{p}') \frac{\Delta_0(\mathbf{p}')}{\omega^2 + \xi^2(\mathbf{p}') \mp |\Delta_0(\mathbf{p}')|^2}. \tag{3.123}$$

Performing the frequency sum yields

$$\Delta_0(\mathbf{p}) = -\sum_{\mathbf{p}'} V(\mathbf{p}-\mathbf{p}')\frac{\Delta_0(\mathbf{p}')}{2E(\mathbf{p}')}\tanh^{\mp 1}\frac{E(\mathbf{p}')}{2T} \tag{3.124}$$

where

$$E(\mathbf{p}) = \sqrt{\xi^2(\mathbf{p}) \mp |\Delta_0(\mathbf{p})|^2}. \tag{3.125}$$

For the case of the superconductor with an attractive local potential

$$V(x-x') = -g\delta^{(3)}(\mathbf{x}-\mathbf{x}')\delta(t-t') \tag{3.126}$$

this becomes

$$\Delta_0 = g\frac{T}{V}\sum_{\omega,\mathbf{p}}\frac{\Delta_0}{\omega^2+\xi^2(\mathbf{p})+|\Delta_0|^2} = \left[g\frac{1}{V}\sum_{\mathbf{p}}\frac{1}{2E(\mathbf{p})}\tanh\frac{E(\mathbf{p})}{2T}\right]\Delta_0. \tag{3.127}$$

There is a nonzero gap if

$$g\frac{1}{V}\sum_{\mathbf{p}}\frac{1}{2E(\mathbf{p})}\tanh\frac{E(\mathbf{p})}{2T} = 1. \tag{3.128}$$

Let $T = T_c$ denote the critical temperature at which the gap vanishes. There, $E(\mathbf{p}) = \xi(\mathbf{p})$ so that Eq. (3.128) determines the same T_c as the previous Eqs. (3.54) (3.61) which were derived for $T \approx T_c$ in a different fashion. The result (3.128) holds for any temperature.

The full temperature dependence of the gap cannot be obtained in closed form from (3.128). For $T \approx T_c$ one may expand directly (3.127) in powers of Δ_0:

$$1 = g\frac{T}{V}\sum_{\omega,\mathbf{p}}\left\{\frac{1}{\omega^2+\xi^2(\mathbf{p})} - \Delta_0^2\frac{1}{[\omega^2+\xi^2(\mathbf{p})]^2}+\dots\right\} \tag{3.129}$$

The first sum on the right-hand side yields the same integral as in (3.54), and we obtain

$$\begin{aligned} 1 &= g\mathcal{N}(0)\left[\log\frac{\omega_D}{T}2\frac{e^\gamma}{\pi} - \Delta_0^2\frac{7\zeta(3)}{8\pi^2T^2}+\dots\right] \\ &= 1+\mathcal{N}(0)\left[\left(1-\frac{T}{T_c}\right) - \Delta_0^2\frac{7\zeta(3)}{8\pi^2T^2}+\dots\right]. \end{aligned} \tag{3.130}$$

and finds

$$\Delta_0^2(T) \approx \frac{8}{7\zeta(3)}\pi^2T_c^2\left(1-\frac{T}{T_c}\right) \tag{3.131}$$

in agreement with (3.108). For very small temperatures, on the other hand, Eq. (3.127) can be written as

$$\begin{aligned} 1 &= g\mathcal{N}(0)\int_0^{\omega_D}\frac{d\xi}{\sqrt{\xi^2+\Delta_0^2}}\left[1-2\exp(-\sqrt{\xi^2+\Delta_0^2}/T)-\dots\right] \\ &= g\mathcal{N}(0)\left[\log\frac{2\omega_D}{\Delta_0}-2K_0(\Delta_0/T)\right]+\dots \end{aligned} \tag{3.132}$$

For small T, K_0 vanishes exponentially fast:

$$2K_0\left(\frac{\Delta_0}{T}\right)\rightarrow\frac{1}{\Delta_0}\sqrt{2\pi T\Delta_0}e^{-\Delta_0/T}. \tag{3.133}$$

Hence one finds at $T=0$ the gap

$$\Delta_0(0)=2\omega_D e^{-1/g\mathcal{N}(0)} \tag{3.134}$$

or, from (3.61),

$$\Delta_0(0)=\pi e^{-\gamma}T_c\approx 1.76\times T_c. \tag{3.135}$$

This value is approached exponentially as $T\rightarrow 0$ since from (3.132)

$$\log\frac{\Delta_0(T)}{\Delta_0(0)}\approx\frac{\Delta_0(T)}{\Delta_0(0)}-1\approx-\frac{1}{\Delta_0(0)}\sqrt{2\pi T\Delta_0(0)}e^{-\Delta_0(0)/T}. \tag{3.136}$$

For arbitrary T, the calculation of (3.128) is conveniently done by using the expansion into Matsubara frequencies Eq. (ge)

$$\frac{1}{2E}\tanh\frac{E}{2T}=\frac{1}{2E}T\sum_{\omega_n}\left(\frac{1}{i\omega_n+E}-\frac{1}{i\omega_n-E}\right)=T\sum_{\omega_n}\frac{1}{\omega_n^2+\xi^2+\Delta_0^2}. \tag{3.137}$$

This can be integrated over ξ and we find for the gap equation (3.128): Eq. (9.24)

$$\log\frac{T}{T_c}=2\pi T\sum_{\omega_n>0}\left(\frac{1}{\sqrt{\omega_n^2+\Delta_0^2}}-\frac{1}{\omega_n}\right). \tag{3.138}$$

It is now convenient to introduce the auxiliary dimensionless quantity Eq. (9.25)

$$\delta=\frac{\Delta_0}{\pi T} \tag{3.139}$$

and a reduced version of the Matsubara frequencies: Eq. (9.26)

$$x_n\equiv(2n+1)/\delta. \tag{3.140}$$

Then the gap equation (3.138) takes the form Eq. (9.27b)

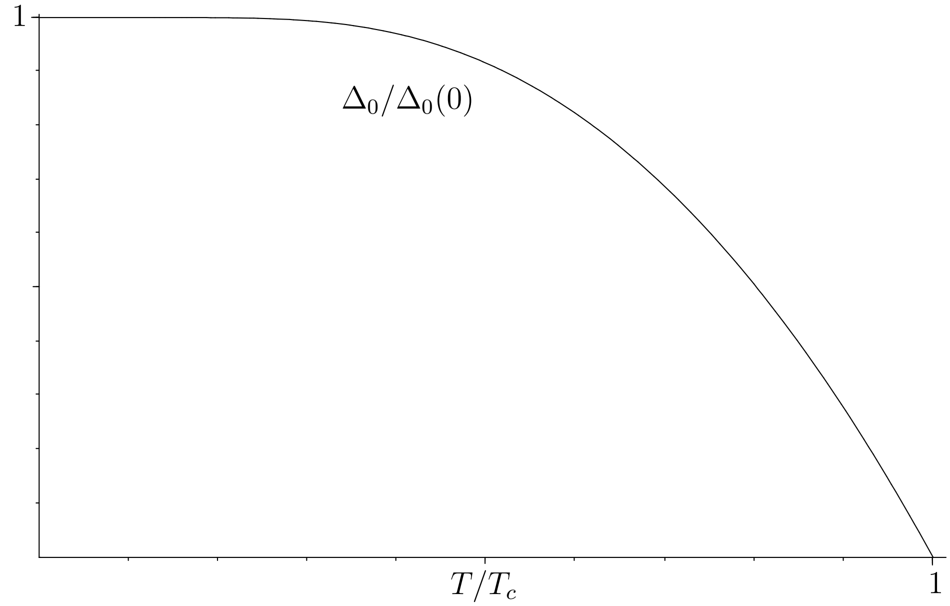

FIGURE 3.6 Energy gap of superconductor as a function of temperature.

$$\log \frac{T}{T_c} = \frac{2}{\delta} \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{x_n^2+1}} - \frac{1}{x_n} \right). \tag{3.141}$$

The temperature dependence of Δ_0 is plotted in Fig. 3.6. The behavior in the vicity of the critical temperature T_c can be extracted from Eq. (3.141) by expanding the sum under the assumption of large x_n. The leading term gives Eq. (9.29)

$$-\log \frac{T}{T_c} \approx \frac{2}{\delta} \sum_{n=0}^{\infty} \frac{1}{2x_n^3} = \delta^2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \delta^2 \frac{7}{8} \zeta(3) \tag{3.142}$$

so that

$$\delta^2 \approx \frac{8}{7\zeta(3)} \left(1 - \frac{T}{T_c}\right) \tag{3.143}$$

Eq. (9.30) and

$$\Delta_0/T_c = \pi\delta = \pi \sqrt{\frac{8}{7\zeta(3)}} \left(1 - \frac{T}{T_c}\right)^{1/2} \approx 3.063 \times \left(1 - \frac{T}{T_c}\right)^{1/2}, \tag{3.144}$$

as before in (3.109).

3.4.2 The Free Pair field

The action quadratic in the pair fields Δ' reads

$$\mathcal{A}_2[\Delta'^*, \Delta'] = \pm \frac{i}{4} \mathrm{Tr} \left[\mathbf{G}_{\Delta_0} \begin{pmatrix} 0 & \Delta' \\ \Delta'^* & 0 \end{pmatrix} \mathbf{G}_{\Delta_0} \begin{pmatrix} 0 & \Delta' \\ \Delta'^* & 0 \end{pmatrix} \right]$$
$$+ \frac{1}{2} \int dx dx' |\Delta(x,x')|^2 \frac{1}{V(x,x')} \tag{3.145}$$

with an equation of motion

$$\begin{pmatrix} \Delta'(x,x') \\ \Delta'^*(x,x') \end{pmatrix} = \mp\frac{i}{2}V(x,x')\mathrm{tr}_{2\times 2}\left[\mathbf{G}_{\Delta_0}\begin{pmatrix} 0 & \Delta' \\ \Delta'^* & 0 \end{pmatrix}\mathbf{G}_{\Delta_0}\begin{pmatrix} 0 & \Delta' \\ \Delta'^* & 0 \end{pmatrix}\frac{\tau^\pm}{2}\right](x,x') \tag{3.146}$$

rather than (3.21). Inserting the momentum space representation (3.122) of $\mathbf{G}_{\Delta_0}$, this renders the two equations

$$\begin{aligned} \Delta'(P|q) &= -\frac{T}{V}\sum_{p'} V(P-P')\left[l_{11}(P'|q)\Delta'(P'|q) + l_{12}(P'|q)\Delta'^*(P'|q)\right] \\ \Delta'^*(P|q) &= -\frac{T}{V}\sum_{p'} V(P-P')\left[l_{11}(P'|q)\Delta'^*(P'|q) + l_{12}(P'|q)\Delta'(P'|q)\right], \end{aligned} \tag{3.147}$$

where (with $P_0 \equiv i\omega$)

$$\begin{aligned} l_{11}(P|q) &= \frac{\omega^2 - \nu^2/4 + \xi(\mathbf{q}/2+\mathbf{P})\,\xi(\mathbf{q}/2-\mathbf{P})}{\left[(\omega+\nu/2)^2 + E^2(\mathbf{q}/2+\mathbf{P})\right]\left[(\omega-\nu/2)^2 + E^2(\mathbf{q}/2-\mathbf{P})\right]} \\ l_{12}(P|q) &= \pm\frac{\Delta_0^2(\mathbf{q}/2+\mathbf{P})}{\left[(\omega+\nu/2)^2 + E^2(\mathbf{q}/2+\mathbf{P})\right]\left[(\omega-\nu/2)^2 + E^2(\mathbf{q}/2-\mathbf{P})\right]}. \end{aligned} \tag{3.148}$$

Thus for $T \ll T_c$ the simple bound-state problem (3.30) takes quite a different form due to the presence of the off-diagonal terms in the propagator (3.122).

Note that the parenthesis on the right-hand side Eqs. (3.147) contain precisely the Bethe-Salpeter wave function of the bound state (compare (3.23), (3.25) in the gapless case)

$$\begin{aligned} \psi(P|q) &\equiv \pm\frac{i}{2}\mathrm{tr}_{2\times 2}\left[\mathbf{G}_{\Delta_0}\left(\frac{q}{2}+P\right)\begin{pmatrix} 0 & \Delta'(P|q) \\ \Delta'^*(P|q) & 0 \end{pmatrix}\mathbf{G}_{\Delta_0}\left(P-\frac{q}{2}\right)\frac{\tau^*}{2}\right] \\ &= l_{11}(P|q)\Delta'(P|q)l_{12}(P'|q)\Delta'^*(P|q). \end{aligned} \tag{3.149}$$

Not much is known on the general properties of solutions of equations (3.147). Even for the simple case of a $\delta^{(4)}(x-x')$ function potential, only the long wavelength spectrum has been studied. There is, however, one important solution which always occurs for $T < T_c$ due to symmetry considerations: The original action (1.33) is symmetric under phase transformations

$$\psi \to e^{-i\alpha}\psi \tag{3.150}$$

guaranteeing the conservation of particle number. If the pair fields oscillate around a nonzero value $\Delta_0(x-x')$, this symmetry is spontaneously broken (since the complex c-number does not take part in such a phase transformation). As a consequence, there must now be an excitation of the system related to the infinitesimal symmetry transformation. This is known as *Goldstone's Theorem*. If the whole system is transformed at once this corresponds to $\mathbf{q}=0$. The symmetry ensures that this corresponds to energy $q_0 = 0$. Indeed, suppose the gap equation did have a non-trivial solution $\Delta_0(P) \equiv 0$. Then we can easily see that

$$\Delta'(P|q=0) \equiv i\Delta_0(P) \tag{3.151}$$

is a solution of the bound-state equations (3.147) at $q=0$. Take

$$l_{11}(P|q=0) = \frac{\omega^2+\xi^2(\mathbf{P})}{\omega^2+E^2(\mathbf{P})} \tag{3.152}$$

and insert (3.151) into (3.147). This gives

$$\begin{aligned}\Delta_0(P) &= \frac{T}{V}\sum_{P'} V(P-P')\left\{\frac{1}{[\omega'^2+E^2(\mathbf{P'})]^2}\left[\omega^2+\xi^2(\mathbf{P'})\mp|\Delta_0(P')|^2\right]\right\}\\ &= -\frac{T}{V}\sum_{P'} V(P-P')\frac{1}{\omega'^2+E^2(\mathbf{P'})}\Delta_0(P'),\end{aligned} \tag{3.153}$$

i.e., the bound-state equation at $q=0$ reduces to the gap equation. Moreover, due to (3.149), the expression

$$\psi_0(P|q=0) \equiv \frac{1}{\omega^2+E^2(\mathbf{P})}\Delta_0(P) \tag{3.154}$$

is the Bethe-Salpeter wave function of the $q=0$ bound state. If the potential is instantaneous, it is possible to calculate the *equal-time amplitude* $\psi_0(\mathbf{x}-\mathbf{x}',\tau) \equiv \psi(\mathbf{x},\tau;\mathbf{x}'\tau)$. Doing the sum over ω in (3.153) we find

$$\begin{aligned}\psi_0(\mathbf{x}-\mathbf{x}',\tau) &= \int\frac{d^3P}{(2\pi)^3}e^{i\mathbf{P}(\mathbf{x}-\mathbf{x}')}T\sum_\omega \psi_0(\mathbf{P}|q=0)\\ &= \int\frac{d^3P}{(2\pi)^3}e^{i\mathbf{P}(\mathbf{x}-\mathbf{x}')}\tanh^{\mp1}\frac{E(\mathbf{P})}{2T}\frac{\Delta_0(\mathbf{P})}{2E(\mathbf{P})}.\end{aligned} \tag{3.155}$$

Note that the time dependence of this amplitude happens to be trivial since the bound state has no energy. The $q=0$ bound state described by $\psi_0(\mathbf{x}-\mathbf{x}')$ is called the *Cooper pair*.

In configuration space (3.153) amounts to a Schrödinger type of equation.

$$-2E(-i\boldsymbol{\nabla})\psi_0(\mathbf{x}) = V(\mathbf{x})\psi(\mathbf{x}). \tag{3.156}$$

This may be interpreted as the $q=0$ bound state of two quasi-particles whose energies are

$$E(\mathbf{P}) = \sqrt{\xi^2(\mathbf{P})\mp|\Delta_0(\mathbf{P})|^2}. \tag{3.157}$$

The equation (3.156) is, however, non-linear since $\Delta_0(\mathbf{P})$ in $E(\mathbf{P})$ depends itself on $\psi_0(x)$. In order to establish contact with the standard discussion of pairing effects via canonical transformations (see Rev. [3]) a few comments may be useful. Let us restrict the discussion to instantaneous potentials. From equation (3.122) one sees that the propagator $\mathbf{G}_\Delta$ can be diagonalized by means of an ω-independent Bogoljubov transformation

$$B(\mathbf{p}) = \begin{pmatrix} u^*_{\mathbf{p}} & \mp v^*_{\mathbf{p}} \\ -v_{\mathbf{p}} & u_{\mathbf{p}} \end{pmatrix}, \tag{3.158}$$

where

$$|u_{\mathbf{p}}|^2 = \frac{1}{2}\left[1 + \frac{\xi(\mathbf{p})}{E(\mathbf{p})}\right], \quad |v_{\mathbf{p}}|^2 = \mp\frac{1}{2}\left[1 - \frac{\xi(\mathbf{p})}{E(\mathbf{p})}\right], \quad 2u_{\mathbf{p}}v^*_{\mathbf{p}} = \frac{\Delta_0(\mathbf{p})}{E(\mathbf{p})}. \tag{3.159}$$

Since

$$|u_{\mathbf{p}}|^2 \mp |v_{\mathbf{p}}|^2 = 1. \tag{3.160}$$

one finds

$$B^{-1}(\mathbf{p}) = \begin{pmatrix} u_{\mathbf{p}} & \pm v^*_{\mathbf{p}} \\ v_{\mathbf{p}} & u^*_{\mathbf{p}} \end{pmatrix} = \left\{ \begin{array}{c} \sigma_3 B^*(\mathbf{p})\sigma_3 \\ B^*(\mathbf{p}) \end{array} \right\}. \tag{3.161}$$

Thus $B(\mathbf{p})$ is a unitary spin rotation in the Fermi case whereas for bosons it is a non-unitary element of the non-compact group $SU(1,1)$ [12].

$$\begin{aligned} \mathbf{G}^d_{\Delta_0}(\omega,\mathbf{p}) &= B(\mathbf{p})\mathbf{G}_{\Delta_0}(\omega,\mathbf{p})B(\mathbf{p})^* \\ &= -\begin{pmatrix} [i\omega - E(\mathbf{p})]^{-1} & \\ & \pm[i\omega + E(\mathbf{p})]^{-1} \end{pmatrix} \end{aligned} \tag{3.162}$$

may be interpreted as describing free quasi-particles of energy

$$E(\mathbf{p}) = \sqrt{\xi^2(\mathbf{p}) \mp |\Delta_0(\mathbf{p})|^2}. \tag{3.163}$$

In fact, if one would introduce new creation and annihilation operators

$$\begin{pmatrix} \alpha(\mathbf{p},\tau) \\ \beta^*(-\mathbf{p},\tau) \end{pmatrix} = B(\mathbf{p}) \begin{pmatrix} a(\mathbf{p},\tau) \\ a^*(-\mathbf{p},\tau) \end{pmatrix} \tag{3.164}$$

their propagators would be

$$\begin{aligned} \mathbf{G}^d_{\Delta_0}(\tau-\tau',\mathbf{p}) &\equiv \begin{pmatrix} \dot\alpha(\mathbf{p},\tau)\dot\alpha^*(\mathbf{p}\tau') & \dot\alpha(\mathbf{p},\tau)\dot\beta(-\mathbf{p},\tau') \\ \dot\beta^*(-\mathbf{p},\tau)\dot\alpha^*(\mathbf{p},\tau') & \dot\beta^*(-\mathbf{p},\tau)\dot\beta(-\mathbf{p},\tau') \end{pmatrix} \\ &= T\sum_\omega e^{-i\omega(\tau-\tau')}\mathbf{G}^d_{\Delta_0}(\omega,\mathbf{p}). \end{aligned} \tag{3.165}$$

At equal "times", $\tau' = \tau + \epsilon$, the frequency sums may be performed with the result

$$\sum_\omega \mathbf{G}^d_{\Delta_0}(\omega,\mathbf{p}) = \begin{pmatrix} \pm n^{\mathrm{qu}}(\mathbf{p}) & 0 \\ 0 & \pm 1 + n^{\mathrm{qu}}(\mathbf{p}) \end{pmatrix}, \tag{3.166}$$

where $n^{\rm qu}(\mathbf{p})$ are the usual Bose and Fermi occupation factors for the quasi-particle energy (3.163):

$$n^{\rm qu}(\mathbf{p}) = \frac{1}{e^{E(\mathbf{p})/T} \mp 1}. \tag{3.167}$$

The corresponding frequency sum for the original propagator becomes

$$T\sum_\omega \mathbf{G}_{\Delta_0}(\omega,\mathbf{p}) = T\sum_\omega B^{-1}(\mathbf{p})\mathbf{G}^d_{\Delta_0}(\omega,\mathbf{p})B^{-1}(\mathbf{p})^* \tag{3.168}$$

$$= \begin{pmatrix} \pm|v_{\mathbf{p}}|^2 \tanh^{\mp 1}\frac{E(\mathbf{p})}{2T} \pm n(\mathbf{p}) & u_{\mathbf{p}}v^*_{\mathbf{p}}\tanh^{\mp 1}\frac{E(\mathbf{p})}{2T} \\ u^*_{\mathbf{p}}v_{\mathbf{p}}\tanh^{\pm 1}\frac{E(\mathbf{p})}{2T} & \pm|u_{\mathbf{p}}|^2\tanh^{\mp 1}\frac{E(\mathbf{p})}{2T} - n(\mathbf{p}) \end{pmatrix}. \tag{3.169}$$

The off-diagonal elements of $\mathbf{G}_{\Delta_0}$ describe, according to Eq. (3.118), the vacuum expectation values of $\langle\psi(x)\psi(x')\rangle$,i.e.,

$$\begin{aligned} \langle\psi(\mathbf{x},\tau)\psi(\mathbf{x},\tau)\rangle &= \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{px}} u_{\mathbf{p}}v^*_{\mathbf{p}} \tanh^{\mp 1}\frac{E(\mathbf{p})}{2T} \\ &= \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{px}} \tanh^{\mp 1}\frac{E(\mathbf{p})}{2T}\frac{\Delta_0(\mathbf{p})}{2E(\mathbf{p})}. \end{aligned}$$

But from Eq. (3.155) this coincides with the Schrödinger type of wave function of the bound state $\langle\psi(\mathbf{x},\tau)\psi(\mathbf{x},\tau)|B(q)$ at $q=0$. After this general discussion let us now return to the superconductor. The action quadratic in the pair fields Δ' [instead of (3.45)]

$$\mathcal{A}_2[\Delta'^*,\Delta'] = -\frac{i}{2}\mathrm{Tr}\left[\mathbf{G}_{\Delta_0}\begin{pmatrix} 0 & \Delta' \\ \Delta'^* & 0 \end{pmatrix}\mathbf{G}_{\Delta_0}\begin{pmatrix} 0 & \Delta' \\ \Delta'^* & 0 \end{pmatrix}\right] - \frac{1}{g}\int dx|\Delta'(x)|^2, \tag{3.170}$$

where the spin traces have been taken. This action can be written in momentum space as

$$\begin{aligned} \mathcal{A}_2[\Delta'^*,\Delta'] &= \frac{1}{2}\frac{T}{V}\sum_k [\Delta'^*(k)L_{11}(k)\Delta'(k) + \Delta'(-k)L_{22}(k)\Delta'(-k) \\ &\qquad + \Delta'^*(k)L_{12}(k)\Delta'^*(-k) + \Delta'(-k)L_{21}(k)\Delta'(k)]\,. \end{aligned} \tag{3.171}$$

The Lagrangian matrix elements $L_{ij}(k)$ are obtained by inserting the Fermi form of the propagator (3.122) into (3.170) [compare (3.147), (3.148)]. Setting $\nu = ik_0$ one has:

$$\mathcal{A}_2[\Delta'^*,\Delta'] = -\frac{1}{2}\frac{T}{V}\sum_{\omega,\mathbf{p}} \frac{1}{\left(\omega+\frac{\nu}{2}\right)^2 + E^2\left(\mathbf{p}+\frac{\mathbf{k}}{2}\right)}\,\frac{1}{\left(\omega-\frac{\nu}{2}\right)^2 + E^2\left(\mathbf{p}-\frac{\mathbf{k}}{2}\right)}$$

$$\times\,\mathrm{Tr}\left[\begin{pmatrix} i\left(\omega+\frac{\nu}{2}\right)+\xi\left(\mathbf{p}+\frac{\mathbf{k}}{2}\right) & \Delta_0 \\ \Delta_0^* & i\left(\omega+\frac{\nu}{2}\right)-\xi\left(\mathbf{p}+\frac{\mathbf{k}}{2}\right) \end{pmatrix}\begin{pmatrix} 0 & \Delta'(k) \\ \Delta'^*(k) & 0 \end{pmatrix}\right.$$

$$\times \begin{pmatrix} i\left(\omega - \frac{\nu}{2}\right) + \xi\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right) & \Delta_0 \\ \Delta_0^* & i\left(\omega - \frac{\nu}{2}\right) - \xi\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right) \end{pmatrix} \begin{pmatrix} 0 & \Delta'(-k) \\ \Delta'^*(k) & 0 \end{pmatrix} \Bigg] \\ -\frac{1}{g}\sum_k \Delta'^*(k)\Delta'(k) \tag{3.172}$$

which is equal to

$$\mathcal{A}_2[\Delta'^*, \Delta'] = \frac{1}{2}\frac{T}{V}\sum_{\omega,\mathbf{p}} \left\{\left[\left(\omega + \frac{\nu}{2}\right)^2 + E^2\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right)\right]\left[\left(\omega - \frac{\nu}{2}\right)^2 + E^2\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right)\right]\right\}^{-1} \\ \times \left\{\left[\omega^2 - \frac{\nu^2}{4} + \xi\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right)\xi\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right)\right][\Delta'^*(k)\Delta'(k) + \Delta'(-k)\Delta'^*(-k)] \right. \\ \left. - |\Delta_0|^2[\Delta'^*(k)\Delta'^*(-k) + \Delta'(k)\Delta'(-k)]\right\} - \frac{1}{g}\sum_k \Delta'^*(k)\Delta'(k). \tag{3.173}$$

From this we read off: Eq. (9.32)

$$\begin{aligned} L_{11}(k) &= L_{22}(k) = \frac{T}{V}\sum_{\omega,\mathbf{p}} l_{11}(p|k) \\ &= -\int \frac{d^3p}{(2\pi)^3} T \sum_{\omega_n} \frac{\omega_n^2 - \nu^2/4 + \xi_+\xi_-}{\left[\left(\omega_n + \frac{\nu}{2}\right)^2 + E_+^2\right]\left[\left(\omega_n - \frac{\nu}{2}\right)^2 + E_-^2\right]} - \frac{\delta_{ij}}{g}. \end{aligned} \tag{3.174}$$

and Eq. (9.33)

$$\begin{aligned} L_{12}(k) &= [L_{21}(k)]^* = \frac{T}{V}\sum_{\omega,\mathbf{p}} l_{12}(p|k) \\ &= \int \frac{d^3p}{(2\pi)^3}|\Delta_0|^2 T \sum_{\omega_n} \frac{1}{\left[\left(\omega_n + \frac{\nu}{2}\right)^2 + E_+^2\right]\left[\left(\omega_n - \frac{\nu}{2}\right)^2 + E_-^2\right]}, \end{aligned} \tag{3.175}$$

with the notation $k_0 = -i\nu$ and Eq. (9.34)

$$\begin{aligned} \left\{\begin{array}{c} \xi_+ \\ \xi_- \end{array}\right\} &\equiv \frac{(\mathbf{p} \pm \mathbf{k}/2)^2}{2m} = \frac{\mathbf{p}^2}{2m} \pm \frac{1}{2}\frac{\mathbf{p}}{m}\mathbf{k} + \frac{\mathbf{k}^2}{8m} \approx \xi \pm \frac{1}{2}\mathbf{v}\mathbf{k} + \dots\,, \\ \left\{\begin{array}{c} E_+ \\ E_- \end{array}\right\} &= \sqrt{\left\{\begin{array}{c} \xi_+^2 \\ \xi_-^2 \end{array}\right\} + \Delta^2} \approx E \pm \frac{1}{2}(\mathbf{v}\mathbf{k})\frac{\xi}{E} + \frac{1}{8}(\mathbf{v}\mathbf{k})^2\frac{\Delta^2}{E^3} + \dots\,, \end{aligned} \tag{3.176}$$

where ξ and $E = \sqrt{\xi^2 + \Delta^2}$ are now the average values of ξ_+, ξ_- and E_+, E_-, respectively. As usual, the integral over d^3p can be split into size and directional integral according to (3.78) and we can set $\mathbf{v} \equiv \mathbf{p}/m \approx v_F\hat{\mathbf{p}}$.

We now rearrange the terms in the sum in such a way that we obtain combinations of single sums of the type

$$T\sum_{\omega_n}\frac{1}{i\omega_n - E_+} \tag{3.177}$$

which lead to the Fermi distribution function Eq. (9.dis)

$$T\sum_{\omega_n}\frac{1}{i\omega_n - E} = n(E) \equiv \frac{1}{e^{E/T}+1} \tag{3.178}$$

with the property

$$n(E) = 1 - n(-E). \tag{3.179}$$

If we drop the subscripts n and introduce the notation $\omega_\pm \equiv \omega \pm \nu/2$, the first term
Eq. (9.dec) in the sum (3.175) for $L_{12}(k)$ can be decomposed as follows:

$$\begin{aligned}
&\frac{1}{[\omega_+^2+E_+^2]\,[\omega_-^2+E_-^2]}\\
&\quad= \frac{1}{4E_+E_-}\left(\frac{1}{i\omega_+ + E_+} - \frac{1}{i\omega_+ - E_+}\right)\left(\frac{1}{i\omega_- + E_-} - \frac{1}{i\omega_- - E_-}\right)\\
&\quad= \frac{1}{4E_+E_-}\left\{-\frac{1}{E_+ + E_- - i\nu}\left(\frac{1}{i\omega_+ - E_+} - \frac{1}{i\omega_- - E_-}\right)\right.\\
&\qquad\qquad +\frac{1}{E_+ + E_- + i\nu}\left(\frac{1}{i\omega_+ + E_+} - \frac{1}{i\omega_- + E_-}\right)\\
&\qquad\qquad -\frac{1}{E_+ - E_- + i\nu}\left(\frac{1}{i\omega_+ + E_+} - \frac{1}{i\omega_- + E_-}\right)\\
&\qquad\qquad \left.+\frac{1}{E_+ - E_- - i\nu}\left(\frac{1}{i\omega_+ - E_+} - \frac{1}{i\omega_- - E_-}\right)\right\}
\end{aligned} \tag{3.180}$$

We now use the summation formula (1.72) and the fact that the frequency shifts ν in $\omega_\pm$ do not appear in the final result since they amount to a mere discrete translation
Eq. (9.33b) in the infinite sum (3.178). Collecting the different terms we find

$$\begin{aligned}
L_{12}(k) = [L_{21}(k)]^* = &-\int\frac{d^3p}{(2\pi)^3}|\Delta_0|^2\frac{1}{2E_-E_+} \\
&\times\left\{\frac{E_+ + E_-}{(E_+ + E_-)^2+\nu^2}\left[1-n(E_+)-n(E_-)\right] + \frac{E_+ - E_-}{(E_+ - E_-)^2+\nu^2}\left[n(E_+)-n(E_-)\right]\right\}.
\end{aligned} \tag{3.181}$$

Eq. () In the first expression we decompose

$$\begin{aligned}
\frac{\omega_n^2 - \nu^2/4 + \xi_+\xi_-}{[\omega_+^2+E_+^2]\,[\omega_-^2+E_-^2]} = \frac{1}{2}&\left\{\frac{1}{\omega_+^2+E_+^2} + \frac{1}{\omega_-^2+E_-^2}\right.\\
&\left. -(E_+^2+E_-^2+\nu^2-2\xi_+\xi_-)\frac{1}{(\omega_+^2+E_+^2)\,(\omega_-^2+E_-^2)}\right\}.
\end{aligned} \tag{3.182}$$

When summing the first two terms we use the formula

$$T\sum_{\omega}\frac{1}{\omega^2+E^2}=\frac{1}{2E}[n(-E)-n(E)]=\frac{1}{2E}\tanh\frac{E}{2T}. \tag{3.183}$$

In the last term, the right-hand factor was treated before. Replacing the factor $E_-^2+E_+^2+\nu^2$ once by $(E_-+E_+)^2+\nu^2-2E_-E_+$ and once by $(E_--E_+)^2+\nu^2+2E_-E_+$ we obtain immediately Eq. (9.32b)

$$\begin{aligned}L_{11}(k)=L_{22}(k)=&\\ \int\frac{d^3p}{(2\pi)^3}\tilde p_i\tilde p_j&\left\{\frac{E_+E_-+\xi_+\xi_-}{2E_+E_-}\frac{E_++E_-}{(E_++E_-)^2+\nu^2}\left[1-n(E_+)-n(E_-)\right]\right.\\ &\left.-\frac{E_+E_--\xi_+\xi_-}{2E_+E_-}\frac{E_+-E_-}{(E_+-E_-)^2+\nu^2}\left[n(E_+)-n(E_-)\right]\right\}-\frac{\delta_{ij}}{g}\end{aligned} \tag{3.184}$$

Let us study in more detail the static case and consider only the long-wavelength limit of small $\mathbf{k}$. Hence, we shall take $k_0=0$ and study the lowest orders in k only. At $\mathbf{k}=0$ we find from (3.184) and (3.181) Eq. (9.35)

$$L_{11}(0)=\mathcal{N}(0)\int d\xi\left\{\frac{E^2+\xi^2}{4E^3}\left[\tanh\frac{E}{2T}+2f'(E)\right]-\frac{1}{g}\right\}. \tag{3.185}$$

and Eq. (9.36)

$$L_{12}(0)=-\frac{1}{2}\mathcal{N}(0)\phi(\Delta_0) \tag{3.186}$$

where we have introduced the so-called *Yoshida function* Eq. (9.37)

$$\phi(\Delta_0)\equiv\Delta_0^2\left[\int_0^\infty d\xi\frac{1}{E^3}\tanh\frac{E}{2T}+2\int_0^\infty d\xi\frac{1}{E^2}f'(E)\right]. \tag{3.187}$$

We now observe that due to the gap equation (3.123), $L_{11}(k)$ can also be expressed in terms this function as of Eq. (9.37a)

$$L_{11}(0)=-\frac{1}{2}\mathcal{N}(0)\phi(\Delta_0). \tag{3.188}$$

The first integral in Eq. (3.187) can be done in parts and brought to the more convenient form Eq. (9.38)

$$\phi(\Delta_0)=1-\frac{1}{2T}\int_0^\infty d\xi\frac{1}{\cosh^2(E/2T)}. \tag{3.189}$$

For $T\approx 0$, this function approaches zero exponentially. The full temperature behaviour is best calculated by using the Matsubara sum of (3.175) to write Eq. (9.38)

$$\begin{aligned}\phi(\Delta_0)&=2T\sum_{\omega_n}\int d\xi\frac{\Delta_0^2}{(\omega_n^2+E^2)^2}=-2\Delta_0^2T\sum_{\omega_n}\frac{\partial}{\partial\omega_n^2}\int d\xi\frac{1}{\omega_n^2+\xi^2+\Delta_0^2}\\ &=-2\Delta_0^2T\sum_{\omega_n}\frac{\partial}{\partial\omega_n^2}\frac{\pi}{\sqrt{\omega_n^2+\Delta_0^2}}=2T\pi\sum_{\omega_n>0}\frac{1}{\sqrt{\omega_n^2+\Delta_0^2}^{\,3}}.\end{aligned} \tag{3.190}$$

Using again the variables δ and x_n from (3.139) and (3.140), this becomes Eq. (9.39)

$$\phi(\Delta_0) = \frac{2}{\delta} \sum_{n=0}^{\infty} \frac{1}{\sqrt{x_n^2+1}^3}. \tag{3.191}$$

Eq. () For $T \approx T_c$, $\delta \to 0$ and

$$\phi(\Delta_0) \approx 2\delta^2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} = 2\delta^2 \frac{7\zeta(3)}{8} \approx 2\left(1-\frac{T}{T_c}\right). \tag{3.192}$$

In the limit $T \to 0$, the sum turns into an integral. Using the formula

$$\int_0^{\infty} dx \frac{1}{(x^1+1)^{\nu}} = \frac{1}{2} B(\mu/2, \nu-\mu/2) \tag{3.193}$$

with $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ we see that

$$\phi(\Delta_0)|_{T=0} = 1. \tag{3.194}$$

Eq. (9.42) Note that we can write $L_{11}(0)$ also as

$$L_{11}(0) = -\frac{3}{4m^2 v_F^2} \rho_s \tag{3.195}$$

Eq. (9.45s) with

$$\rho_s \equiv \rho\phi(\Delta_0) \tag{3.196}$$

This function has an important physical meaning. Let us calculate the bending energies of the collective field $\Delta(x)$. For this, we expand $L_{11}(k)$ and $L_{12}(k)$ at $\nu = 0$ into powers of the momentum $\mathbf{k}$ up to $\mathbf{k}^2$. Let us denote the zero-frequency parts of $L_{11}(k)$ and $L_{12}(k)$ by $L_{11}(\mathbf{k})$ and $L_{11}(\mathbf{k})$, respectively, with the explicit form

Eq. (C1)

$$\begin{aligned} L_{11}(\mathbf{k}) &= \frac{T}{V} \sum_{\omega_n, \mathbf{p}} \frac{\omega^2 + \xi_+\xi_-}{(\omega^2+E_+^2)(\omega^2+E_-^2)} - \frac{1}{g} \\ L_{12}(\mathbf{k}) &= -\frac{T}{V} \sum_{\omega_n, \mathbf{p}} \frac{\Delta_0^2}{(\omega^2+E_+^2)(\omega^2+E_-^2)}. \end{aligned} \tag{3.197}$$

Eq. (C2) Inserting the expansions

$$\begin{aligned} \xi_+\xi_- &= \xi^2 - \frac{1}{4}(\mathbf{vk})^2 + \dots \\ \left\{ \begin{array}{c} E_+^2 \\ E_-^2 \end{array} \right\} &= E^2 \pm \xi \mathbf{vk} + \frac{1}{2}(\mathbf{vk})^2 + \dots \end{aligned} \tag{3.198}$$

Eq. (C3) we have

$$L_{11}(\mathbf{k}) - L_{12}(\mathbf{k}) \approx \int \frac{d^3p}{(2\pi)^3} T \sum_{\omega} \frac{\omega^2 + \Delta_0^2 + \xi^2 - \frac{1}{4}(\mathbf{vk})^2}{(\omega^2 + E^2)^2 \left[1 + \frac{1}{2}(\mathbf{vk})^2 \frac{\omega^2 - \xi^2 + \Delta_0^2}{(\omega^2+E^2)^2}\right]} - \frac{1}{g} + \dots$$
$$= \int \frac{d^3p}{(2\pi)^3} \Bigg\{ \left(T \sum_{\omega} \frac{1}{\omega^2 + E^2} - \frac{1}{g} \right) + T \sum_{\omega_n} \left[\frac{1}{4} \frac{1}{(\omega^2 + E^2)^2} - \frac{\omega^2 + \Delta_0^2}{(\omega^2 + E^2)^3} \right] (\mathbf{vk})^2 \Bigg\} + \dots . \quad (3.199)$$

Due to the gap equation the first parenthesis vanishes so that we are left with Eq. (C4)

$$L_{11}(\mathbf{k}) - L_{12}(\mathbf{k}) \approx \mathcal{N}(0)(\mathbf{vk})^2 \times \int \frac{d\hat{\mathbf{p}}}{4\pi} \int_{-\infty}^{\infty} d\xi \left[\frac{1}{4} \frac{1}{(\omega^2 + \xi + \Delta_0^2)^2} - \frac{\omega^2 + \Delta_0^2}{(\omega^2 + \xi^2 + \Delta_0^2)^3} \right] \quad (3.200)$$

Similarly we obtain Eq. (C5)

$$L_{12}(\mathbf{k}) \approx -\mathcal{N}(0) \int \frac{d\hat{\mathbf{p}}}{4\pi} \int_{-\infty}^{\infty} d\xi \Bigg\{ \frac{\Delta_0^2}{(\omega^2 + \xi^2 + \Delta_0^2)^2} + (\mathbf{vk})^2 \left[\frac{1}{2} \frac{1}{(\omega^2 + \xi^2 + \Delta_0^2)^3} - \frac{\omega^2 + \Delta_0^2}{(\omega^2 + \xi^2 + \Delta_0^2)^4} \right] \Bigg\} . \quad (3.201)$$

Using the integrals Eq. (C6)

$$\int_{-\infty}^{\infty} d\xi \frac{1}{(\omega^2 + \xi^2 + \Delta_0^2)^{2,3,4}} = \left(\frac{1}{2}, \frac{3}{8}, \frac{5}{16} \right) \frac{\pi}{\sqrt{\omega^2 + \Delta_0^2}^{3,5,7}} \quad (3.202)$$

we find Eq. (C7)

$$L_{11}(\mathbf{k}) - L_{12}(\mathbf{k}) \approx -\frac{\mathcal{N}(0)}{4} \frac{(\mathbf{vk})^2}{\Delta_0^2} \int \frac{d\hat{\mathbf{p}}}{4\pi} \phi(\Delta_0) \quad (3.203)$$
$$L_{12}(\mathbf{k}) \approx -\frac{\mathcal{N}(0)}{2} \phi(\Delta_0) + \frac{\mathcal{N}(0)}{8} (\mathbf{vk})^2 \int \frac{d\hat{\mathbf{p}}}{4\pi} \bar{\phi}(\Delta_0) \quad (3.204)$$

where $\phi(\Delta_0)$ is the Yoshida function (3.191), while $\bar{\phi}(\Delta_0)$ is a further gap function: Eq. (C9)

$$\bar{\phi}(\Delta_0) = \Delta_0^4 \pi T \sum_{\omega_n > 0} \frac{1}{\sqrt{\omega_n^2 + \Delta_0^2}^5} \equiv \frac{1}{\delta} \sum_{n=0}^{\infty} \frac{1}{\sqrt{x_n^2 + 1}^5} . \quad (3.205)$$

For $T \approx T_c$ this behaves like

$$\bar{\phi}(\Delta_0) \approx \delta^4 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^5} = \delta^4 \frac{31\zeta(5)}{32}, \quad (3.206)$$

and thus, by (3.143),

$$\bar{\phi}(\Delta_0) \approx \delta^4 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^5} = \delta^4 \frac{31\zeta(5)}{32} \approx 0.9072 \times \left(1 - \frac{T}{T_c}\right)^2 \quad (3.207)$$

while for $T \to 0$ the sum turns into an integral whose value is, by formula (3.193),

$$\bar{\phi}(\Delta_0^2)\Delta_0^2 \,|_{T=0} = \frac{1}{3}. \tag{3.208}$$

Eq. (9.63) Thus we find for the energy density the gradient terms

$$e(x) = \frac{1}{4m^2}\left[\rho_{ij}^{11}\partial_i\Delta^*(x)\partial_j\Delta(x)/\Delta_0^2 + \operatorname{Re}\rho_{ij}^{12}\partial_i\Delta^*(x)\partial_j\Delta^*(x)/\Delta_0^2\right]. \tag{3.209}$$

Here we have dropped in the primes on the fields since in the presence of the derivatives the additional constant Δ_0 does not matter. The first coefficient is given by

Eq. (9.64)

$$\rho_{ij}^{11} = \frac{3\rho}{2}\int\frac{d\hat{\mathbf{p}}}{4\pi}\hat{p}_i\hat{p}_j\left[\phi(\Delta_0) - \frac{1}{2}\bar{\phi}(\Delta_0)\right] \tag{3.210}$$

Eq. (9.64b) while the second is

$$\rho_{ij}^{12} = -\frac{3\rho}{2}\int\frac{d\hat{\mathbf{p}}}{4\pi}\hat{p}_i\hat{p}_j\frac{1}{2}\bar{\phi}(\Delta_0) \tag{3.211}$$

Performing the angular integral gives

$$\rho_{ij}^{11} = \frac{1}{2}\rho\left[\phi(\Delta_0^2) - \frac{1}{2}\bar{\phi}(\Delta_0)\right]\delta_{ij} \tag{3.212}$$

$$\rho_{ij}^{12} = -\frac{1}{4}\rho\bar{\phi}(\Delta_0)\delta_{ij}. \tag{3.213}$$

Decomposing the collective field $\Delta(x)$ into size $|\Delta(x)|$ and and phase $\varphi(x)$,

$$\Delta(x) = |\Delta(x)|e^{i\varphi(x)} \tag{3.214}$$

Eq. (9.63c) the energy density reads

$$e(x) = \frac{1}{4m^2}\left\{(\rho^{11} - \rho^{12})(\partial\phi)^2 + (\rho^{11} + \rho^{12})(\partial|\Delta(x)|)^2/\Delta_0^2\right\}. \tag{3.215}$$

The first coefficient is seen to coincide with the function (3.196) encountered earlier. Introducing, in addition, the notation

$$\bar{\rho}_s \equiv \rho\bar{\phi}(\Delta) \tag{3.216}$$

and adding to the energy density the earlier $\mathbf{k} = 0$ result we find the total quadratic
Eq. (9.63d) fluctuation energy density

$$e(x) = \rho_s(\partial\varphi)^2 + (\rho_s - \bar{\rho}_s)(\partial|\Delta(x)|)^2/\Delta_0^2 + 6\rho_s(\delta|\Delta(x)|)^2)/v_F^2. \tag{3.217}$$

The behaviour of ρ_s and $\bar{\rho}_s$ for all $T \le T_c$ is shown in Fig. (3.7).

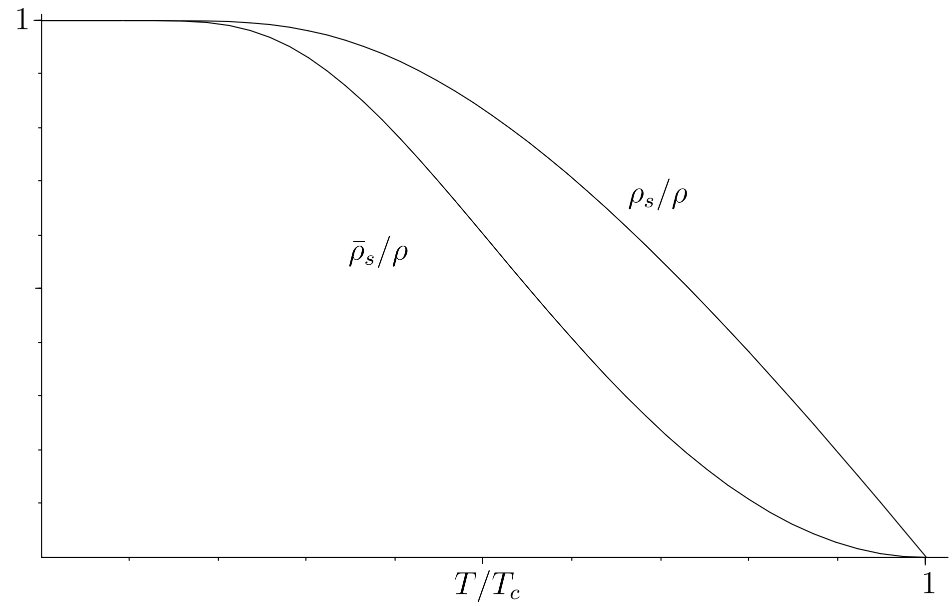

FIGURE 3.7 Temperature behaviour of superfluid density ρ_s/ρ (Yoshida function) and the gap function $\bar{\rho}_s/\rho$.

The phase fluctuations are of infinite range, the size fluctuations have a finite range characterized by the temperature-dependent *coherence length*

$$\xi(T) = \sqrt{\frac{v_F^2}{6\Delta^2}\frac{\rho_s - \bar{\rho}_s}{\rho_s}}. \tag{3.218}$$

For T close to T_c, the second ratio tends towards one while Δ^2 goes to zero according to Eq. (3.108). Thus we recover the previous result (3.93) for the coherence length:

$$\xi(T) \approx \xi_0 \left(1 - \frac{T}{T_c}\right) \tag{3.219}$$

with

$$\xi_0 = \sqrt{\frac{7\zeta(3)}{48}}\frac{v_F}{\pi T_c} \approx 0.419 \times \frac{v_F}{\pi T_c}. \tag{3.220}$$

For $T \to 0$, $\xi(T)$ tends exponentially fast against

$$\xi(0) = \frac{e^\gamma}{3}\frac{v_F}{\pi T} \approx 0.591 \times \frac{v_F}{\pi T} \approx 1.4179 \times \xi_0. \tag{3.221}$$

The behaviour of $\xi_0^2/\xi^2(T)$ is displayed in Fig. (3.8).

At low temperatures we can ignore the size fluctuations of the collective field parameter $\Delta(x)$. This is called the *hydrodynamic limit* or *London limit*. Thus we approximate

$$\Delta(x) \approx \Delta_0 e^{i\phi(x)}, \tag{3.222}$$

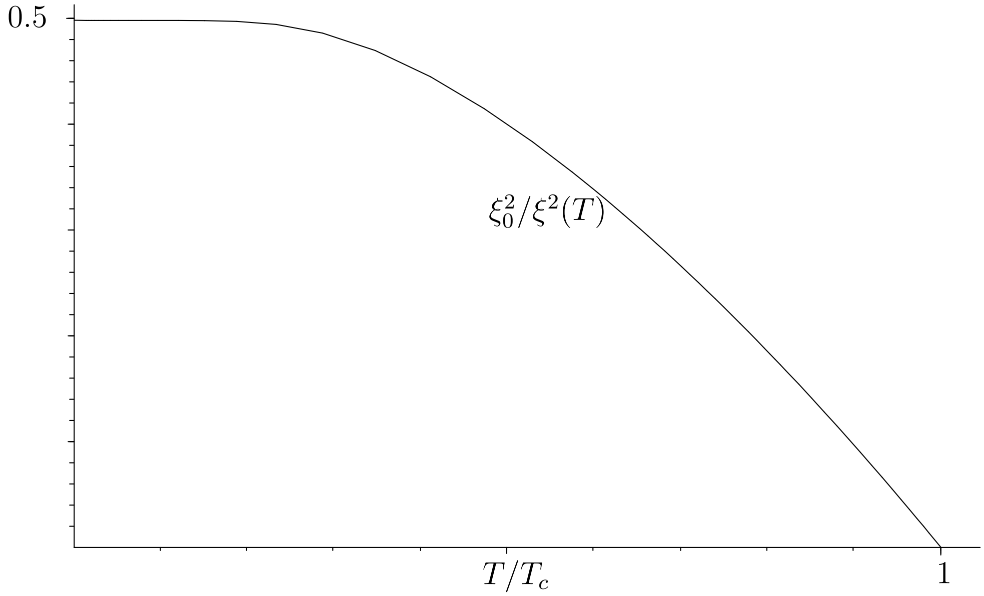

FIGURE 3.8 Temperature behaviour of the inverse square coherence length $\xi^{-2}(T)$. The dashed line shows the Ginzburg-Landau limit.

In this limit, the bending energy is simply Eq. (9.63d)

$$e(x) = \frac{1}{4m^2}\rho_s[\partial_i\phi(x)]^2 \tag{3.223}$$

By studying the behaviour of this expression under Galilei transformations we identify the superfluid velocity of the condenstate

$$\mathbf{v}_s = \frac{1}{2m}\boldsymbol{\nabla}\phi. \tag{3.224}$$

In terms of this the energy density takes the form

$$e(x) = \frac{1}{2}\rho_s\mathbf{v}_s^2 \tag{3.225}$$

This shows that ρ_s is the *superfluid density* of the condensate.

For temperatures close to zero, the sum over Matsubara frequencies $T\sum_\omega$ may also be performed as an integral $\int d\omega/2\pi$, and the result is [recall (3.197)]

$$\begin{aligned} L_{11}(k) &= L_{22}(k) = \frac{T}{V}\sum_{\omega,\mathbf{p}} l_{11}(p|k) - \frac{1}{g} = \frac{1}{V}\sum_{\mathbf{p}}\frac{E_+E_- + \xi_+\xi_-}{2E_+E_-}\frac{E_+ + E_-}{(E_+ + E_-)^2+\nu^2} - \frac{1}{g}, \\ L_{12}(k) &= L_{21}(k) = \frac{T}{V}\sum_{\omega,\mathbf{p}} l_{12}(p|k) = -|\Delta_0|^2\frac{1}{V}\sum_{\mathbf{p}}\frac{1}{2E_+E_-}\frac{E_+ + E_-}{(E_+ + E_-)+\nu^2}. \end{aligned} \tag{3.226}$$

Because of the gap equation (3.128) at $T=0$, we insert

$$\frac{1}{g} = \sum_{\mathbf{p}}\frac{1}{2E(\mathbf{p})}, \tag{3.227}$$

so that the last term in $L_{11}(k)$ provides us with a subtraction of the sum. The energies of fundamental excitations are obtained by diagonalizing the action $\mathcal{A}_2[\Delta'^*, \Delta']$ and searching for zero eigenvalues of the matrix $L(k)$ via

$$L_{11}(k)L_{22}(k) - L_{12}^2(k) = 0. \tag{3.228}$$

Since $L_{11}(k) = L_{22}(k)$ this amounts to the two equations

$$L_{11}(k) = \pm L_{12}(k). \tag{3.229}$$

These equations can be solved for small k. Expanding to forth order in ν and k [13] and using the gap equation (3.128) at $T = 0$

$$\frac{1}{g} = \sum_{\mathbf{p}} \frac{1}{2E(\mathbf{p})} \tag{3.230}$$

one obtains

$$\begin{aligned} L_{11}(k) &= -\frac{m^2 v_F}{4\pi^2}\left(1 + \frac{\nu^2}{3\Delta_0^2} + \frac{v_F^2\mathbf{k}^2}{9\Delta_0^2} - \frac{v_F^2\nu^2\mathbf{k}^2}{30\Delta_0^4} - \frac{\nu^4}{20\Delta_0^4} - \frac{v_F^4\mathbf{k}^4}{100\Delta_0^4}\right) + \dots \\ L_{12}(k) &= -\frac{m^2 v_F}{4\pi^2}\left(1 - \frac{\nu^2}{6\Delta_0^2} + \frac{v_F^2\mathbf{k}^2}{18\Delta_0^2} - \frac{v_F^2\nu^2\mathbf{k}^2}{45\Delta_0^4} - \frac{\nu^4}{30\Delta_0^4} - \frac{v_F^4\mathbf{k}^4}{150\Delta_0^4}\right) + \dots \end{aligned} \tag{3.231}$$

so that the first of Eq. (3.229) has the small $k_0, \mathbf{k}$ solution ($k_0 = -i\nu$)

$$k_0 = \pm c|\mathbf{k}|(1 - \gamma\mathbf{k}^2), \quad c \equiv \frac{v_F}{\sqrt{3}}, \quad \gamma = \frac{v_F^2}{45\Delta_0^2}. \tag{3.232}$$

The other Eq. (3.229) can be solved for small $\mathbf{k}$ and $i\nu$ directly. Using (3.226) and (3.230) one can write $-L_{11}(k) - L_{12}(k) = 0$ as[7]

$$\frac{1}{V}\sum_{\mathbf{p}}\left[\frac{1}{2E} + \frac{(\Delta_0^2 - EE' - \xi\xi')(E + E')}{2EE'\left[(E + E')^2 + \nu^2\right]}\right] = 0. \tag{3.233}$$

For small $\mathbf{k}$ this leads to the energies [13]

$$k_0^{(n)} = 2\Delta_0 + \Delta_0\left(\frac{v_F\mathbf{k}}{2\Delta_0}\right)^2 z_n \tag{3.234}$$

with z_n being the solutions of the integral equation

$$\int_{-1}^{1} dx \int_{-\infty}^{\infty} dy \frac{x^2 - z}{x^2 + y^2 - z} = 0. \tag{3.235}$$

[7]For $T \neq 0$ each result appears with a factor $\frac{1}{2}\left(\tanh\frac{E}{2T} + \tanh\frac{E'}{2T}\right)$ to which one has to add once more the whole expression with E' replaced by $-E'$.

Setting $e^t = \left(\sqrt{1-z}+1\right)/\left(\sqrt{1-z}-1\right)$ this is equivalent to

$$\sinh t + t = 0 \tag{3.236}$$

which has infinitely many solutions t_n starting with

$$t_1 = 2.251 + i4.212 \tag{3.237}$$

and tending asymptotically to

$$t_n \approx \log[\pi(4n-1)] + i\left(2\pi n - \frac{\pi}{2}\right). \tag{3.238}$$

The excitation energies are

$$k_0^{(n)} = 2\Delta_0 - \frac{v_F^2}{4\Delta_0}\mathbf{k}^2 \frac{1}{\sinh^2 t_n/2}. \tag{3.239}$$

Of these only the first one at $k_0^{(1)} \approx 2\Delta_0 + (.24 - .30i)v_F^2/4\Delta_0^2\mathbf{k}^2$ lies on the second sheet and may have observable consequences while the others are hiding under lower and lower sheets of the two-particle branch cut from $2\Delta_0$ to ∞ (which is logarithmic due to the dimensionality of the surface of the Fermi sea at $T = 0$).

3.5 Ground State Properties

The superfluid densities do not only characterize the hydrodynamic bending energies. They also appear in the description of the thermodynamic quantities of the ground state.

3.5.1 Free Energy

Since the ground state field Δ_0 is constant in space and time the lowest-order terms in Eq. (3.7) reduces to

$$\mathcal{A}[\Delta_0^*, \Delta] = \frac{i}{2}\mathrm{Tr}\log\left[i\mathbf{G}_{\Delta_0}^{-1}(x,x')\right] - \frac{1}{g}\int dx|\Delta_0|^2 \tag{3.240}$$

and can be calculated explicitly. In energy momentum space the matrix inside the
Eq. (9.85) trace log is diagonal

$$\begin{pmatrix} \epsilon - \xi(\mathbf{p}) & \Delta_0 \\ \Delta_0^* & \epsilon + \xi(\mathbf{p}) \end{pmatrix} \tag{3.241}$$

in the functional indices $\epsilon, \mathbf{p}$. In the 4×4 matrix space this can be diagonalized via
Eq. (9.86) a Bogoljubov transformation with the result

$$\begin{pmatrix} [\epsilon - E(\mathbf{p})]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & [\epsilon + E(\mathbf{p})]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \tag{3.242}$$

where $E(\mathbf{p})$ are the quasi-particle energies (3.163). Thus the first trace log term in the expression (3.240) can be written as Eq. (9.87)

$$-i(t_b-t_a)V\int\frac{d\epsilon}{2\pi}\frac{d^3p}{(2\pi)^3}\log\left[\epsilon-E(\mathbf{p})\right]\left[\epsilon+E(\mathbf{p})\right]. \qquad (3.243)$$

The second term contributes simply Eq. ()

$$-\frac{1}{g}\Delta^2(t_b-t_a)V. \qquad (3.244)$$

After a Wick rotation, this action corresponds to the free energy density[8] Eq. (9.98)

$$f=-\sum_{\omega_n}\sum_{\mathbf{p}}\log\{[i\omega_n-E(\mathbf{p})][i\omega_n+E(\mathbf{p})]\}+\frac{1}{g}|\Delta_0|^2+\text{const}\ . \qquad (3.245)$$

The constant accounts for the unspecified normalization of the functional integration. It is removed by substracting the free fermion system with $\Delta=0,\ g=0$ (note that $\Delta^2\sim e^{-1/g\mathcal{N}(0)}\to 0$ for $g\to 0$). Since the energy of the free fermion system is well-known Eq. (9.90)

$$f_0=-2T\sum_{\mathbf{p}}\log\left(1-e^{\xi(\mathbf{p})/T}\right) \qquad (3.246)$$

it is sufficient to study only Eq. (9.91)

$$\Delta f=f-f_0=-T\sum_{\omega_n,\mathbf{p}}\log\frac{i\omega_n-E(p)}{i\omega_n-\xi(p)}+(E\to -E,\ \xi\to-\xi)+\frac{1}{g}|\Delta_0|^2. \qquad (3.247)$$

This energy difference is the condensation energy associated with the transition into the superfluid phase.

The sum over Matsubara frequency can be performed as usual by using Cauchy's formula: Eq. (9.92)

$$T\sum_{\omega_n}\log\left(1-\frac{E}{i\omega_n}\right)=-\frac{1}{2\pi i}\int\frac{dz}{e^{z/T}+1}\log\left(1-\frac{E}{z}\right) \qquad (3.248)$$

where the contour C encircles all poles along the imaginary axis at $z=i\omega_n$ in the positive sense but passes the logarithmic cut from $z=0$. By deforming the contour C into C' and by contracting C' to zero one picks up the pole at $z=E$ and finds (see Fig. 1.1). Eq. (9.93)

$$-\int_0^E\frac{dz}{e^{z/T}+1}=\int_0^E dE\ n(E). \qquad (3.249)$$

Since Eq. (9.94)

$$\frac{\partial n(E)}{\partial E}=-n(1-n)/T \qquad (3.250)$$

this can be calculated as Eq. (9.95)

$$-\int_0^E dEn(E)=T\int_{1/2}^n dn'\frac{1}{1-n'}=-T\log 2(1-n(E)). \qquad (3.251)$$

The expression (3.247) becomes therefore Eq. (9.96)

$$\Delta f = T \sum_{\mathbf{p}} \left[\log(1-n)n - \log(1-n_0)n_0\right] + \frac{1}{g}\Delta^2 \tag{3.252}$$

Eq. (9.97) where n_0 denotes the free-fermion distribution. Alternatively, one may write

$$\Delta f = 2T \sum_{\mathbf{p}} \{\log(1-n) - (E-\xi)\} + \frac{1}{g}|\Delta_0|^2 - 2T\sum_{\mathbf{p}} \log(1-n_0) \tag{3.253}$$

The last term is recognized as minus the energy of the free system so that the first line gives the full energy of the superfluid ground state.

The explicit calculation can conveniently be done by studying Δf of (3.253) at fixed T as a function of g. At $g=0$, $\Delta_0 = 0$ and $\Delta f = 0$. As g is increased to its physical value, the gap increases to Δ_0. Now, since Δf is extremal in changes of Δ
Eq. (9.98) at fixed g and T, all g-dependence comes from the variation of the factor $1/g$, i.e.,

$$\left.\frac{\partial \Delta f}{\partial g}\right|_T = |\Delta_0|^2. \tag{3.254}$$

Eq. () We can therefore calculate Δf by simply performing the integral

$$\Delta f = -\int_{1/g}^{\infty} d\left(1/g'\right)|\Delta_0|^2\left(1/g'\right) \tag{3.255}$$

Eq. () The $1/g$-dependence of the gap is obtained directly from (3.132), (3.141) as

$$\frac{1}{g\mathcal{N}(0)} - \log\left(2\frac{e^\gamma}{\pi}\frac{\omega_c}{T}\right) = \frac{1}{\delta}\sum_{n=0}^{\infty}\left[\frac{1}{\sqrt{x_n^2+1}} - \frac{1}{x_n}\right].$$

Eq. (9.101) From this we find

$$\frac{\partial}{\partial \delta^2}\left.\left(\frac{1}{g\mathcal{N}(0)}\right)\right|_T = -\frac{1}{2\delta^2}\phi(\delta^2) = -\frac{1}{2\delta^2}\frac{\rho_s}{\rho}, \tag{3.256}$$

where $\phi(\delta^2)$ is the Yoshida function $\phi(\Delta)$ in which we have here emphasized the fact that it is a pure function of δ^2, as we can see from (3.187). In the last line we have expressed it in terms of the superfluid density ρ_s . Using this we can change
Eq. (9.102) variables in the integration and write

$$\Delta f = \mathcal{N}(0)\pi^2 T^2 \frac{1}{2}\int_0^{\delta^2} d\delta'^2 \phi(\delta'^2). \tag{3.257}$$

Inserting ϕ from the upper part of equation (3.191) we can perform the integral with
Eq. (9.103) the result:

$$\frac{1}{\delta^2}\int_0^{\delta^2} d\delta'^2 \phi(\delta'^2) = \frac{4}{\delta}\sum_{n=0}^{\infty}\left[-\frac{1}{\sqrt{x_n^2+1}} + 2\left(\sqrt{x_n^2+1} - x_n\right)\right] \tag{3.258}$$

Eq. () In analogy to $\phi = \rho_s/\rho$ we shall denote this new function as $\tilde{\phi} \equiv \tilde{\rho}_s/\rho$, i.e.,

$$\frac{\tilde{\rho}_s}{\rho} \equiv \tilde{\phi} = \frac{4}{\delta}\sum_{n=0}^{\infty}\left[-\frac{1}{x_n} + 2\left(\sqrt{x_n^2+1} - x_n\right)\right]. \tag{3.259}$$

When plotted against temperature, this starts out as $(1 - T/T_c)$ for $T \sim T_c$ and goes to unity for $T \to 0$. The approach to unity is found by means of the Euler-McLaurin expansion applied to (3.259): Eq. (9.122)

$$\begin{aligned}\sum_{n=0}^{\infty} f(x_n) &= \frac{\delta}{2}\int dx f(x) - \frac{1}{2!\cdot 3\delta^2}\left[f'(\infty) - f'(0)\right] \\ &+ \left[\left(\frac{1}{2!\cdot 3}\right)^2 - \frac{1}{4!\cdot 5}\right]\frac{1}{\delta^4}\left[f'''(\infty) - f'''(0)\right] + \dots\,.\end{aligned} \tag{3.260}$$

For $\tilde{\rho}_s$ this implies Eq. (9.123)

$$\begin{aligned}\left.\frac{\tilde{\rho}_s}{\rho}\right|_{\delta^2\to 0} &= 2\int_0^{\infty} dx\left[-\frac{1}{\sqrt{x^2+1}} + 2\left(\sqrt{x^2+1} - x\right)\right] - \frac{2}{3\delta^2} + \dots \\ &= 1 - \frac{2}{3\delta^2} + \dots\,.\end{aligned} \tag{3.261}$$

The full temperature behaviour is plotted in Fig. 3.9. The condensation energy can therefore be written in the simple form Eq. (9.105)

$$\Delta f = -\mathcal{N}(0)\pi^2 T^2 \frac{1}{2}\frac{\tilde{\rho}_s}{\rho}\delta^2. \tag{3.262}$$

For $T \to T_c$, the function $\tilde{\rho}_s$ this goes to zero linearly, just like ρ_s, and we find Eq. (9.106)

$$\Delta f \approx -\mathcal{N}\pi^2 T^2 \frac{1}{2}\left(1 - \frac{T}{T_c}\right)^2 \frac{8}{7\zeta(3)}. \tag{3.263}$$

in agreement with our previous calculation (3.20) in the Ginzburg-Landau regime for $T \sim T_c$.

For $T \to 0$, on the other hand, where $\delta^2\pi^2T^2 \to 3.111 \times T_c^2$, the condensation energy becomes Eq. ()

$$\Delta f|_{T=0} \approx -0.236 \times c_n(T_c), \tag{3.264}$$

where we have normalized the right-hand part of the equation by the specific heat of the liquid just above the critical temperature. Eq. (9.109)

$$c_n(T_c) = -\frac{2}{3}\pi^2\mathcal{N}(0)T_c \tag{3.265}$$

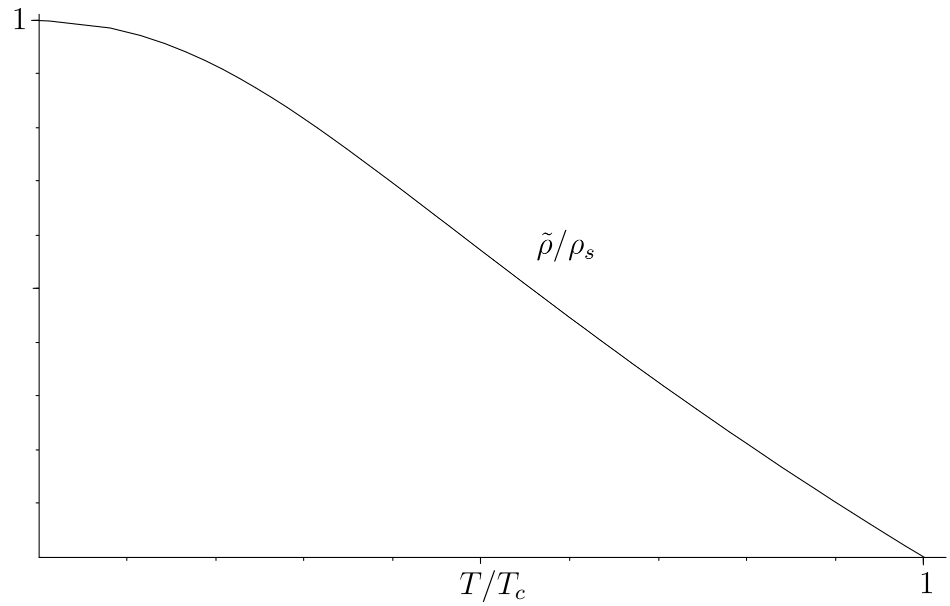

FIGURE 3.9 The gap function $\tilde{\rho}_s$ appearing in the condensation energy of a superconductor as a function of temperature

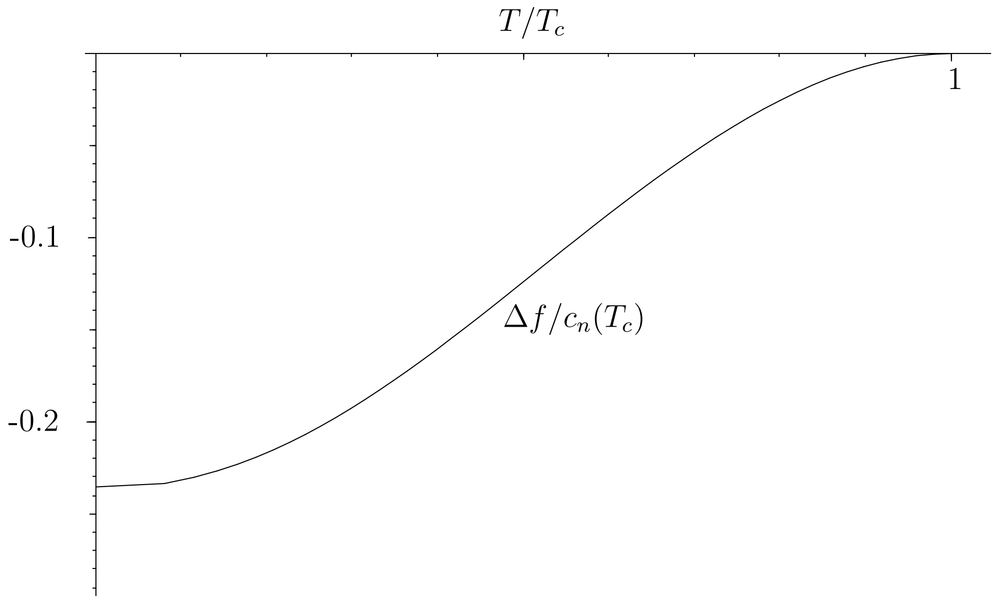

FIGURE 3.10 Condensation energy of a superconductor as a function of temperature.

The full temperature dependence of Δf can be seen in Fig 3.10.

A remark is in order concerning the application of the Euler-McLaurin expansion (3.260). First, it cannot be used to find an exponential approach to the zero-

[8]The relation between the ground-state action and the free energy F is $\mathcal{A} = iF/T,\ t_b - t_a = -i/T,\ \int_{-\infty}^{\infty} d\epsilon = iT\sum_{\omega_n}$

temperature limit (of the type $e^{-\delta}$).
Second, it works only if the integral over the function $f(x)$ has no singularity at $x = 0$. Consider, for example, the $T \to 0$ limit of (3.141). The sum on the right-hand side requires the following more careful limiting procedure: Eq. (sc-9.127)

$$\begin{aligned} \sum_{n=0}^{N} \frac{1}{x_n} &= \delta \sum_{n=1}^{N} \frac{1}{2n+1} = \delta \left(\sum_{n=1}^{2(N+1)} \frac{1}{n} - \frac{1}{2} \sum_{n=1}^{N+1} \frac{1}{n} \right) \\ &\underset{N \text{ large}}{\approx} \delta \left\{ \log 2(N+1+\gamma - \frac{1}{2} [\log(N+1)+\gamma] \right\} \\ &= \frac{\delta}{2} \left\{ \int_{1/\delta}^{x_N} \frac{dx}{x} + \log(2e^{\gamma}) \right\} . \end{aligned} \tag{3.266}$$

Thus one would obtain Eq. (9.127a)

$$\begin{aligned} \log \frac{T}{T_c} &\underset{T\to 0}{\longrightarrow} \int_0^{\infty} dx \frac{1}{\sqrt{x^2+1}} - \int_{1/\delta}^{\infty} \frac{1}{x} \\ &= -\log\left(\delta e^{\gamma}\right) . \end{aligned} \tag{3.267}$$

in agreement with (3.144). The following rule is useful: If $\sum_{n=0}^{\infty} 1/x_n$ appears in a sum, it can be made finite by subtracting $\delta/2$ times

$$\frac{2}{\delta} \sum_{n=0}^{\infty} = \frac{1}{x_n} \int_{-\omega_c}^{\omega_c} \frac{d\xi}{2\xi} \tan \frac{\xi}{2T} = \log \left(\frac{2\omega_c e^{\gamma}}{T\pi} \right) .$$

In a sloppy procedure, one may replace the left-hand side for $T \to 0$ by the integral Eq. ()

$$\frac{\delta}{2} \int_0^x \frac{dx'}{x'} \to \frac{\delta}{2} (\log x - \log 0), \tag{3.268}$$

if one substitutes, at the lower limit, Eq. (9.128)

$$\log 0 \quad \to \quad -\log\left(2\delta e^{\gamma}\right) - \log \frac{2\Delta_0 e^{\gamma}/\pi}{T} = -\log \frac{2\Delta_0}{\Delta_0(0)} \frac{T_c}{T} \tag{3.269}$$

with the zero-temperature gap Eq. (9.128a)

$$\Delta_0(0) = \pi e^{-\gamma} T_c \sim 1.764 \times T_c \,. \tag{3.270}$$

3.5.2 Entropy

Let us now calculate the entropy. For this it is useful to note that at fixed T and $1/g$ the energy is extremal with respect to small changes in Δ. It is this condition which previously lead to the gap equation [see for instance Eq. (3.138)]. Thus when forming Eq. (9.110)

$$s = -\frac{\partial f}{\partial T} \tag{3.271}$$

we do not have to take into account the fact that Δ^2 varies with temperature. Therefore we find Eq. (9.111)

$$\Delta s = -\frac{\partial \Delta f}{\partial T} = -2\sum_{\mathbf{p}}\left[\log(1-n(\mathbf{p})) - \frac{T}{n(1-n)}\frac{\partial n}{\partial T}\right]. \tag{3.272}$$

But the derivative is Eq. (9.112)

$$\frac{\partial n}{\partial T} = n(1-n)\frac{E}{T^2} \tag{3.273}$$

Eq. () so that the entropy becomes

$$\Delta s = -2\sum_{\mathbf{p}}\left[\log\left(1-n(\mathbf{p})\right) - n\frac{E(\mathbf{p})}{T}\right] \tag{3.274}$$

Eq. (9.114) which can be rewritten in the more familiar form

$$\Delta s = -2\sum_{\mathbf{p}}\left[(1-n)\log(1-n) + n\log n\right] \tag{3.275}$$

Eq. (9.115) after having inserted the identity

$$\frac{E}{T} = \log\frac{1-n}{n} \tag{3.276}$$

For the explicit calculation we differentiate (3.257) with respect to the temperature
Eq. (9.116) and find

$$\Delta s = -\frac{\partial \Delta f}{\partial T} \;=\; \mathcal{N}(0)\pi^2 T\int_0^{\delta^2} d\delta'^2\phi(\delta'^2) + \mathcal{N}(0)\pi^2 T^2\frac{1}{2}\phi\frac{\partial \delta^2}{\partial T}. \tag{3.277}$$

Eq. (9.117) From Eq. (3.141) we know $\log(T/T_c)$ as a function of δ^2. Differentiation yields

$$\frac{1}{T}\frac{dT}{d\delta^2} = -\frac{1}{2\delta^2}\phi^{B,A} \tag{3.278}$$

Eq. (9.118) so that the condensation entropy is simply

$$\Delta s = -\mathcal{N}(0)\pi^2 T\int_0^{\delta^2} d\delta'^2\left[1-\phi^{B,A}(\delta'^2)\right] \tag{3.279}$$

Eq. (9.120) If we normalize this again with the help of $c_n(T_c)$ this can be written as

$$\frac{s}{c_n(T_c)} = -\frac{2}{3}\left(1-\tilde{\rho}_s/\rho\right)\delta^2. \tag{3.280}$$

Eq. (9.121) For $T \to T_c$ this behaves like

$$\frac{s}{c_n(T_c)} \underset{T\approx T_c}{\approx} -\frac{3}{2}\left(1-\frac{T}{T_c}\right)\frac{8}{7\zeta(3)}. \tag{3.281}$$

Eq. (9.125) The $T \to 0$ limit of the condensation entropy density is therefore

$$\Delta s|_{T=0} = -\frac{2}{3}\mathcal{N}(0)\pi^2 T, \tag{3.282}$$

Eq. (9.126) thereby cancelling exactly with the normal entropy

$$s_n = \frac{2}{3}\mathcal{N}(0)\pi^2 T \tag{3.283}$$

so that the total entropy vanishes, as it should. The approach to zero is parabolic
Fig. XXXIV due to (3.261). The full temperature behaviour is plotted in Fig. 3.11.

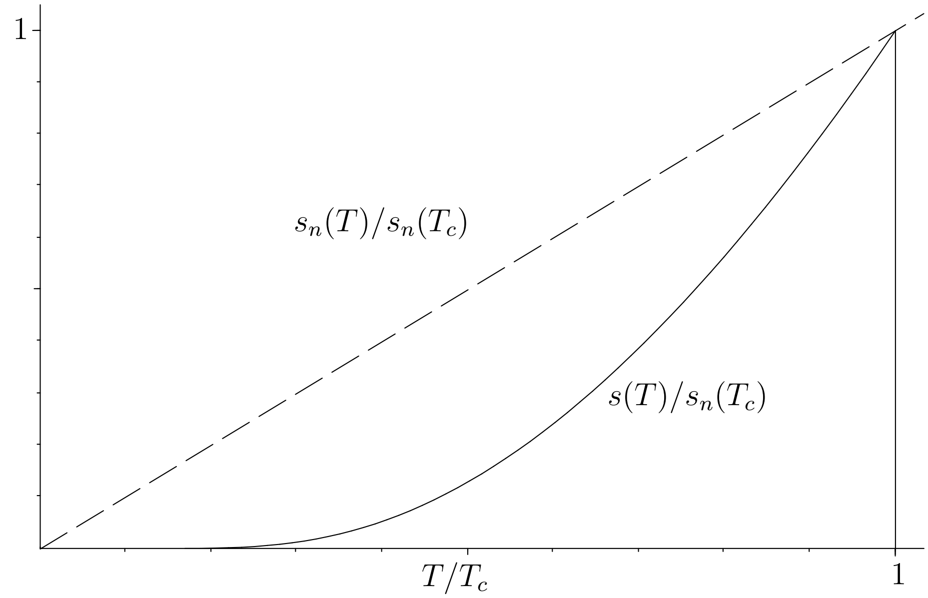

FIGURE 3.11 The temperature behaviour of the condensation entropy of a superconductor.

3.5.3 Specific Heat

By a further differentiation with respect to the temperature we immediately obtain the specific heat Eq. (9.129)

$$\begin{aligned}\Delta c &= T\frac{\partial \Delta s}{\partial T} = \mathcal{N}(0)T^2\Delta s - \mathcal{N}(0)\pi^2 T\left[1-\phi(\delta^2)\right]T\frac{\partial\delta^2}{\partial T} \\ &= -\mathcal{N}(0)\pi^2\left[\Delta s - 2T\frac{1-\phi(\delta^2)}{\phi(\delta^2)}\delta^2\right].\end{aligned} \tag{3.284}$$

This can be rewritten in terms of the superfluid density as Eq. (9.130)

$$\frac{\Delta c}{c_n(T_c)} = \frac{T}{T_c}\left[-\frac{3}{2}\left(1-\tilde{\rho}_s/\rho\right)+3(\rho/\rho_s-1)\right]\delta^2. \tag{3.285}$$

At $T = T_c$ there is a finite descontinuity Eq. (9.131)

$$\frac{\Delta c}{c_n(T_c)} = \frac{3}{2}\frac{8}{7\zeta(3)} = 1.4261, \tag{3.286}$$

which could have been found from the Ginzburg-Landau treatment in Section 2.2. For the full specific heat one has to add the normal contribution of the normal Fermi liquid in (3.246), which is simply equal to T/T_c. The result is shown in Fig. 3.12.

For $T \to 0$ we use the results (3.263), (3.281) to find Eq. ()

$$\frac{\Delta c}{c_n(T_c)} = -T/T_c. \tag{3.287}$$

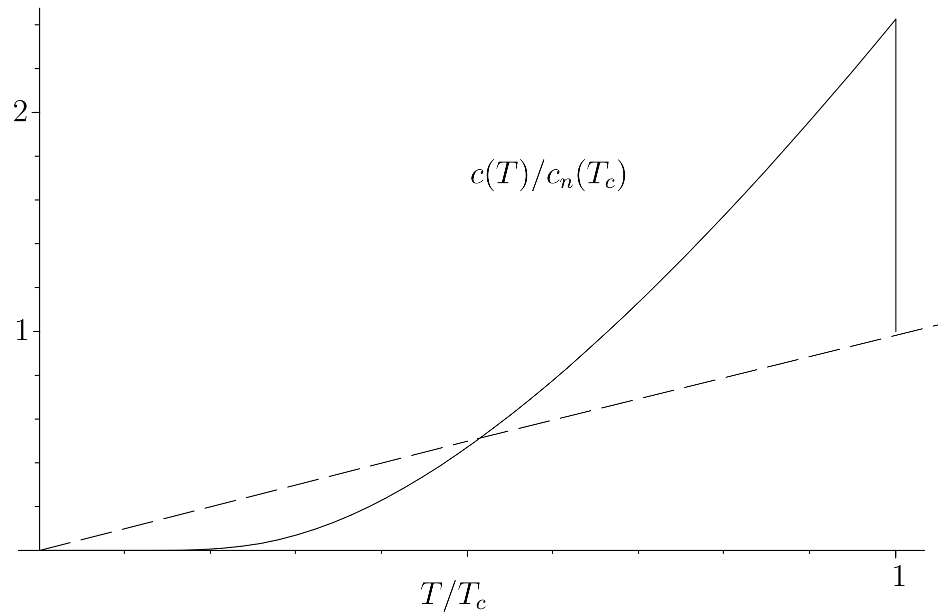

FIGURE 3.12 Total specific heat (normal plus condensate part) of a superconductor as a function of temperature.

This exactly the opposite of the specific heat of the normal liquid so that the curve for the total $c/c_n(T_c)$ starts out exponentially flat at the origin (exponentially due to the nonzero gap amounting to finite activation energy).

3.6 Plasmons versus Pairs

Very often the two-body potential $V(x,x')$ will consist of several pieces favouring different collective excitations

$$V(x,x') = \sum_i V_i(x,x'). \tag{3.288}$$

Thus V may have a long-range part supporting plasma oscillations and, in addition, a strong short-range contribuiton giving rise to tightly bound pairs. It is obvious that in such situations it is convenient to eliminate each potential V_i separately by the introduction of different collective fields. Only then has a perturbation expansion a chance of showing fast convergence.

Also, for one fundamental potential there may be different domains in T, μ, V with different collective phenomena being dominant. Thus a system of electrons will, at lower density, not be governed by plasmons due to ring graphs but corrections of the type
will become increasingly important. The path integral formalism has no formal

problem in incorporating such effects. One simply performs, in the grand-canonical action, an artificial splitting

$$V(x,x') = V_1(x,x') + V_2(x,x') \tag{3.289}$$

with an arbitrary $V_1(x,x')$ which may depend on μ, T, V and defines

$$V_2(x,x') \equiv V(x,x') - V_1(x,x'). \tag{3.290}$$

Then V_1 may be turned into plasmons, V_2 into pairs. The full final answer should not depend on the parameters characterizing the splitting (3.289). But at every given order in the collective perturbation theory there will be an optimal set of these parameters minimizing the free energy.

Certainly, physical intuition and experience has to guide the selection of V_1, and general rules have yet to be worked out.

Appendix 3A Propagator of the Bilocal Pair Field

Consider the Bethe-Salpeter equation (3.22) with a potential λV instead of V

$$\Gamma = -i\lambda V G_0 G_0 \Gamma. \tag{3A.1}$$

Take this as an eigenvalue problem in λ at fixed energy-momentum $q = (q^0, \mathbf{q})= (E, \mathbf{q})$ of the bound states. Let $\Gamma_n(P|q)$ be all solutions, with eigenvalues $\lambda_n(q)$. Then the convenient normalization of Γ_n is:

$$-i \int \frac{d^4P}{(2\pi)^4} \Gamma_n^\dagger (P|q)\, G_0\left(\frac{q}{2}+P\right) G_0\left(\frac{q}{2}-P\right) \Gamma_{n'}(P|q) = \delta_{nn'}. \tag{3A.2}$$

If all solutions are known, there is a corresponding completeness relation (the sum may comprise an integral over a continuous part of the spectrum)

$$-i \sum_n G_0\left(\frac{q}{2}+P\right) G_0\left(\frac{q}{2}-P\right) \Gamma_n(P|q)\Gamma_n^\dagger(P'|q) = (2\pi)^4 \delta^{(4)}(P-P'). \tag{3A.3}$$

This completeness relation makes the object given in (3.36) the correct propagator of Δ. In order to see this write the free Δ action $\mathcal{A}_2[\Delta^\dagger\Delta]$ as

$$\mathcal{A}_2 = \frac{1}{2}\Delta^\dagger \left(\frac{1}{\lambda V} + iG_0 \times G_0\right)\Delta \tag{3A.4}$$

where we have used λV instead of V. The propagator of Δ would have to satisfy

$$\left(\frac{1}{\lambda V} + iG_0 \times G_0\right) \dot{\Delta}\dot{\Delta}^\dagger = i. \tag{3A.5}$$

Performing this calculation on (3.28) one has, indeed, by virtue of (3A.1) for Γ_n, λ_n:

$$\left(\frac{1}{\lambda V}+iG_0\times G_0\right)\times\left\{-i\lambda\sum_n\frac{\Gamma_n\Gamma_n^\dagger}{\lambda-\lambda_n(q)}\right\}$$
$$=-i\lambda\sum_n\frac{\frac{1}{\lambda V}\Gamma_n\Gamma_n^\dagger+iG_0\times G_0\Gamma_n\Gamma_n^\dagger}{\lambda-\lambda_n(q)}$$
$$=i\lambda\sum_n\frac{-\frac{\lambda_n(q)}{\lambda}+1}{\lambda-\lambda_n(q)}(-iG_0\times G_0\Gamma_n\Gamma_n^\dagger)$$
$$=i\left(-i\sum_i G_0\times G_0\Gamma_n\Gamma_n^\dagger\right)=i. \tag{3A.6}$$

Note that the expansion of the propagator in powers of λ

$$\dot\Delta\dot\Delta^\dagger=i\sum_k\left(\sum_n\left(\frac{\lambda}{\lambda_n(q)}\right)^k\Gamma_n\Gamma_n^\dagger\right) \tag{3A.7}$$

corresponds to the graphical sum over one, two, three, etc. exchanges of the potential λV. For $n=1$ this is immediately obvious due to (3A.1):

$$i\sum_n\frac{\lambda}{\lambda_n(q)}\Gamma_n\Gamma_n^\dagger=\sum\frac{\lambda}{\lambda_n(q)}\lambda_n(q)VG_0\times G_0\Gamma_n\Gamma_n^\dagger=i\lambda V. \tag{3A.8}$$

For $n=2$ one can rewrite, using the orthogonality relation,

$$i\sum_n\left(\frac{\lambda}{\lambda_n(q)}\right)^2\Gamma_n\Gamma_n^\dagger=\sum_{nn'}\frac{\lambda}{\lambda_n(q)}\Gamma_n\Gamma_n^\dagger G_0\times G_0\Gamma_{n'}\Gamma_{n'}^\dagger\frac{\lambda}{\lambda_{n'}(q)}=\lambda VG_0\times G_0\lambda V \tag{3A.9}$$

which displays the exchange of two λV terms with particles propagating in between. The same procedure applies at any order in λ. Thus the propagator has the expansion

$$\dot\Delta\dot\Delta^\dagger=i\lambda V-i\lambda VG_0\times G_0 i\lambda V+\dots. \tag{3A.10}$$

If the potential is instantaneous, the intermediate $\int dP_0/2\pi$ can be performed replacing

$$G_0\times G_0\to i\frac{1}{E-E_0(\mathbf{P}|q)} \tag{3A.11}$$

where

$$E_0(\mathbf{P}|q)=\xi\left(\frac{\mathbf{q}}{2}+\mathbf{P}\right)+\xi\left(\frac{\mathbf{q}}{2}-\mathbf{P}\right)$$

is the free particle energy which may be considered as the eigenvalue of an operator H_0. In this case the expansion (3A.10) reads

$$\dot\Delta\dot\Delta^\dagger=i\left(\lambda V+\lambda V\frac{1}{E-H_0}\lambda V+\dots\right)=i\lambda V\frac{E-H_0}{E-H_0-\lambda V}. \tag{3A.12}$$

We see it related to the resolvent of the complete Hamiltonian as

$$\dot{\Delta}\dot{\Delta}^{\dagger} = i\lambda V(R\lambda V + 1) \tag{3A.13}$$

where

$$R \equiv \frac{1}{E - H_0 - \lambda V} = \sum_n \frac{\psi_n \psi_n^{\dagger}}{E - E_n} \tag{3A.14}$$

with ψ_n being the Schrödinger amplitudes in standard normalization. We can now easily determine the normalization factor N in the connection between Γ_n and the Schrödinger amplitude ψ_n. Eq. (3A.2) gives in the instantaneous case

$$\int \frac{d^3P}{(2\pi)^3} \Gamma_n^{\dagger}(\mathbf{P}|q) \frac{1}{E - H_0} \Gamma_{n'}(\mathbf{P}|q) = \delta_{nn'} \tag{3A.15}$$

Inserting ψ from (3.28) renders

$$\frac{1}{N^2} \int \frac{d^3P}{(2\pi)} \psi_n^{\dagger}(E - H_0)\psi_{n'}(\mathbf{P}|q) = \delta_{nn'}. \tag{3A.16}$$

But since

$$(E - H_0)\psi = \qquad \lambda V \psi \tag{3A.17}$$

this is also

$$\frac{1}{N^2} \int \frac{d^3P}{(2\pi)^3} \psi_n^{\dagger}\,(\mathbf{P}|q)\,\lambda V \psi_{n'}\,(\mathbf{P}|q) = \delta_{nn'}. \tag{3A.18}$$

For ψ_n wave functions in standard normalization the integral expresses the differential

$$\lambda \frac{dE}{d\lambda}.$$

For a typical calculation of a resolvent, the reader is referred to Schwinger's treatment [45] of the Coulomb problem. His result may directly be used for a propagator of electron hole pairs bound to excitons.

Appendix 3B Fluctuations around the Composite Field

Here we show that the quantum mechanical fluctuations around the *classical* equations of motion (2.6)

$$\varphi(x) = \int dy V(x, y) \psi^{\dagger}(y) \psi(y), \tag{3B.1}$$

or (3.4)

$$\Delta(x, y) = V(x - y)\psi(x)(y), \tag{3B.2}$$

are quite simple to calculate. For this let us compare the Green functions of $\varphi(x)$ or $\Delta(x,y)$ with those of the composite operators on the right-hand side of Eqs. (3B.1) or (3B.2). The Green functions or φ or Δ are generated by adding external currents $\int dx\varphi(x)I(x)$ or $1/2\int dxdy(\Delta(y,x)I^\dagger(x,y)+h.c.)$ to the final actions (2.11) or (3.7), respectively, and by forming functional derivatives $\delta/\delta I$. The Green functions of the composite operators, on the other hand, are obtained by adding

$$\int dx\left(\int dyV(x,y)\psi^\dagger(y)\psi(y)\right)K(x)$$

$$\frac{1}{2}dxdyV(x-y)\psi(x)\psi(y)K^\dagger(x,y)+h.c.$$

to the original actions (2.4) or (3.3), respectively, and by forming functional derivatives $\delta/\delta K$. It is obvious that the sources K can be included in the final actions (2.11) and (3.7) by simply replacing

$$\varphi(x)\to\varphi'(x)=\varphi(x)-\int dx'K(x')V(x',x)$$

or

$$\Delta(x,y)\to\Delta'(x,y)=\Delta(x,y)-K(x,y).$$

If one now shifts the functional integrations to these new translated variables and drops the irrelevant superscript "prime", the actions can be rewritten as

$$\begin{aligned}\mathcal{A}[\varphi] &= \pm i\mathrm{Tr}\log(iG_\varphi^{-1})+\frac{1}{2}\int dxdx'\varphi(x)V^{-1}(x,x')\varphi(x')+i\int dxdx'\eta^\dagger(x)G_\varphi(x,x')\eta(x)\\ &\quad+\int dx\varphi(x)\left[I(x)+K(x)\right]+\frac{1}{2}\int dxdx'K(x)V(x,x')K(x') \qquad (3B.3)\end{aligned}$$

or

$$\begin{aligned}\mathcal{A}[\Delta] &= \pm\frac{i}{2}\mathrm{Tr}\log\left(i\mathbf{G}_\Delta^{-1}\right)+\frac{1}{2}\int dxdx'|\Delta(x,x')|^2\frac{1}{V(x,x')}\\ &\quad+\frac{i}{2}\int dxdx'j^\dagger(x)\mathbf{G}_\Delta(x,x')\frac{1}{V(x,x')}\\ &\quad+\frac{1}{2}\int dxdx'\left\{\Delta(y,x)\left[I^\dagger(x,y)+K^\dagger(x,y)\right]+\mathrm{h.c.}\right\}\\ &\quad+\frac{1}{2}\int dxdx'|K(x,x')|^2V(x,x'). \qquad (3B.4)\end{aligned}$$

In this form the actions display clearly the fact that derivatives with respect to the sources K or I coincide exactly, except for all possible insertions of the direct interaction V. For example, the propagators of the plasmon field $\varphi(x)$ and of the composite operator $\int dyV(x,y)\psi^\dagger(y)\psi(y)$ are related by

$$\begin{aligned}\dot\varphi(x)\dot\varphi(x') &= -\frac{\delta^{(2)}Z}{\delta I(x)\delta I(x')}=V^{-1}(x,x')-\frac{\delta^{(2)}Z}{\delta K(x)\delta K(x')} \qquad (3B.5)\\ &=V^{-1}(x,x')+\langle 0|(\int dyV(x,y)\psi^\dagger(y)\psi(\varphi))(\int dy'V(x'y')\psi^\dagger(y')\psi^\dagger(y')\psi(y'))|0\rangle\end{aligned}$$

in agreement with (2.7). Similarly, one finds for the pair fields:

$$\begin{aligned}\dot{\Delta}(x,x')\dot{\Delta}(y,y')^\dagger &= \delta(x-y)\delta(x'-y')iV(x-x') \\ +\langle 0|(V(x',x)\psi(x')\psi(x))&(V(y',y)\psi^\dagger(y)\psi^\dagger(y'))|0\rangle.\end{aligned} \tag{3B.6}$$

Note that the latter relation is manifestly displayed in the representation (3A.10) of the propagator Δ. Since

$$\dot{\Delta}\dot{\Delta}^\dagger = iVG^{(4)}V$$

one has from (3B.6)

$$\langle 0|V(\psi\psi)(\psi^\dagger\psi^\dagger V)|0\rangle = V\,G^{(4)}V \tag{3B.7}$$

which is correct, remembering that $G^{(4)}$ is the full four-point Green function. In the equal-time situation at instantaneous potential, $G^{(4)}$ is replaced by the resolvent R.

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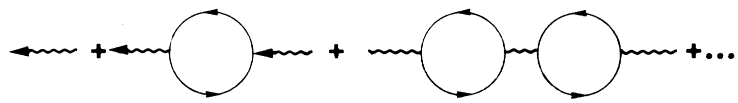

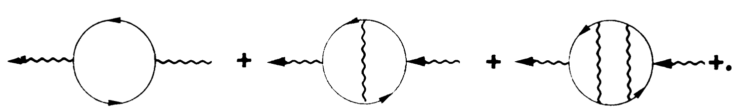

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Part III

Superfluid ^{3}He

1

Introduction

In 1958 the theory by Bardeen, Cooper, and Schrieffer succeeded in explaining superconductivity via the formation and subsequent condensation of bosonic s-wave Cooper pairs in a fermionic electron gas [1]. Immediately afterwards a search was started for similar phenomena in other Fermi systems such as nuclei [2] and liquid ^{3}He [3]. While nuclear forces did, in fact, allow for a direct application of the BCS formalism [4], it was soon noticed [5] that in ^{3} He the strong repulsive core of the interatomic potential would not permit exactly the same type of pair formation as in superconductors. If we take a look at the shape of the potential shown in Fig. 1.1, we see that the hard core starts at a radius of about $r \approx 2.5$ Å. At $r \approx 3$ Å there is a minimum of roughly -10 K. Beyond this, the potential approaches zero with the van-der Waals behavior r^{-6}. It is now obvious that there can be no formation of s-wave bound states: In the Fermi liquid, only the atoms moving close to the surface of the Fermi sphere are capable of substantial interactions. But they move with a momentum $p \approx p_F \approx 8 \times 10^{-19}$ g cm/sec. Then for each angular momentum

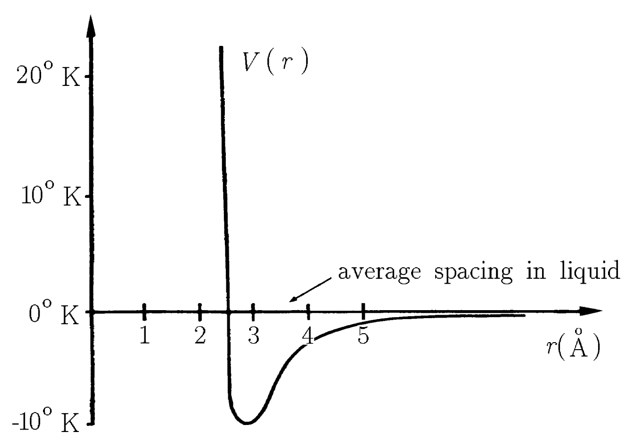


FIGURE 1.1 Interatomic potential between ^{3}He atoms as a function of the distance r. There is a hard core at about 2.35 Å and a minimum at 3 Å at which the potential is about -10 K. The mean distance between the atoms in the liquid lies at ≈ 3.5 Å, i.e., well within the range of attraction. Beyond this, the potential approaches zero with the van-der Waals behavior r^{-6}.

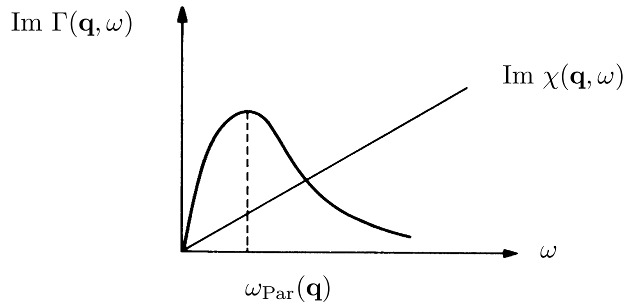


FIGURE 1.2 Imaginary part of the susceptibility caused by repeated exchange of spin fluctuations, as a function of energy ω. There is a pronounced peak whose sharpness increases with decreasing q. Thus, for small q, there are long-lived excitations in the system which are called paramagnons. The straight line shows the inginary part of the susceptibility for for a free Fermi system.

$l = 0, \hbar, 2\hbar, 3\hbar$ the impact parameter, i.e., the distance, art which the particles pass one another is of the order of $l/p_F \approx 0$, 1.25Å, 2.5 Å, 3.75 Å, ... With the repulsive core rising at $r \leq 2.5$ Å it appears as if the first partial wave which has a chance to bind is the d-wave. As a matter of fact, in the first quantitative analyses, d-wave pairs were argued to make up the superfluid condensate and the first careful extension of BCS formalism was done for this case [6].

The situation is not, however, that simple. There are strong many-body effects which have been neglected in such a simple-minded consideration. They lead to a screening of the fundamental interatomic potential so that the partial wave estimates have to be modified. Moreover, the hard core together with the Pauli exclusion principle generate strong spin-spin correlations. As a consequence there is a pronounced resonance in the dynamic susceptibility (see Fig. 1.2) which is usually referred to as a paramagnon excitation. The exchange of these particle-like states between two atoms gives rise to an additional attraction between parallel spins and therefore enhances the bound states of odd annular momenta.

It would be desirable to calculate these effects quantitatively from first principles, i.e., an n-body Hamiltonian of ^{3}He atoms with the fundamental interaction $V(r)$ shown in Fig. 1.1. Unfortunately the strength of this interaction has made this an impossible task until today. Therefore we have to take the evidence from experiment that the Cooper pairs form at a lower angular home than expected, namely at $l = 1$. Apparently, the screening effects do somewhat weaken the hard core and the paramagnons provide sufficient additional attraction between parallel spins so that binding can occur in this p-wave state of low impact parameter. By statistics this state has to be symmetric in the spin wave functions, so that its total spin has to be $S = 1$ (spin triplet).

With the difficulties of calculating which orbital wave would bind, also the estimates for the transition temperature considerable variations. Early estimates were as high as 0.1 K. They has to be lowered successively when experiments had reached such low temperatures without seeing the phase transition.

The transition was finally discovered experimentally by cooling liquid ^{3}He down along the melting curve at 2.7 mK, with another transition at 2.1 mK. The reason why measurements were first performed along the melting curve lies in the simplicity of the cooling techniques and temperature control via the so-called *Pomeranchuk effect*. It is useful to keep in mind how temperatures in the milli-Kelvin range can be reached and maintained: First, the system can easily be pre-cooled to roughly 77 K by working inside a Dewar container filled with liquid nitrogen. Embedded in this is another container filled with liquid ^{4}He which maintains, at atmospheric pressure, a temperature of 4 K. Enclosed in this lies a dilution refrigerator. It is constructed on the basis of the following physical principle: Liquid ^{3}He, when brought into contact with ^{4}He, forms a well defined interface. Across it, diffusion takes place just in the same way as in the evaporation process at a water surface. This lowers the temperature. The process can be made cyclic just like in an ordinary evaporation refrigerator. Nowadays, temperature of a few mK can be reached in this way. In the beginning, the dilution cooling was used only down to roughly 100 mK. From there on, the Pomeranchuk effect can be exploited. This is based on the observation that according to the *Clausius-Clapeyron equation*, Eq. (he-1.3)

$$\frac{dP}{dT} = \frac{S_{\text{liquid}} - S_{\text{solid}}}{V_{\text{liquid}} - V_{\text{solid}}}, \tag{1.1}$$

the temperature goes down with increasing pressure since the entropy of the liquid becomes smaller than that of the solid in spite of its larger volume. Thus, in order to cool, all one has to do is to compress the system.

If one wants to measure the phase diagram away from the melting curve adiabatic demagnetization may be used in addition to the Pomeranchuk effect. The best magnetic materials for this purpose are either CMN (cereous magnesium nitrate) or copper. In the first material it is the magnetic moments of the electrons, in the second that of the nuclei which is demagnetized. With copper, temperatures of 35 mK have meanwhile be reached and maintained for hours (below 2mK for 3 days).

With such techniques the phase diagram has meanwhile been measured for very low pressures (see Fig. 1.3). The two phases discovered originally along the melting curve are now called A and B. For large magnetic fields, there is another phase, called A_1, which forms between A phase and normal liquid. In order to improve visibility in the Figure we have exaggerated the corresponding temperature interval by a factor.

Many properties of these three phases have meanwhile been investigated experimentally and they all are in complete agreement with the theoretical description via p-wave spin triplet Cooper pairs.

As hard as it was to give the correct prediction for the transition temperature, the final observation of the critical temperature $T_c = 2.7$ mK is in perfect scale when compared with normal superconductors (see Table 1.1 [10]. Tab. masses

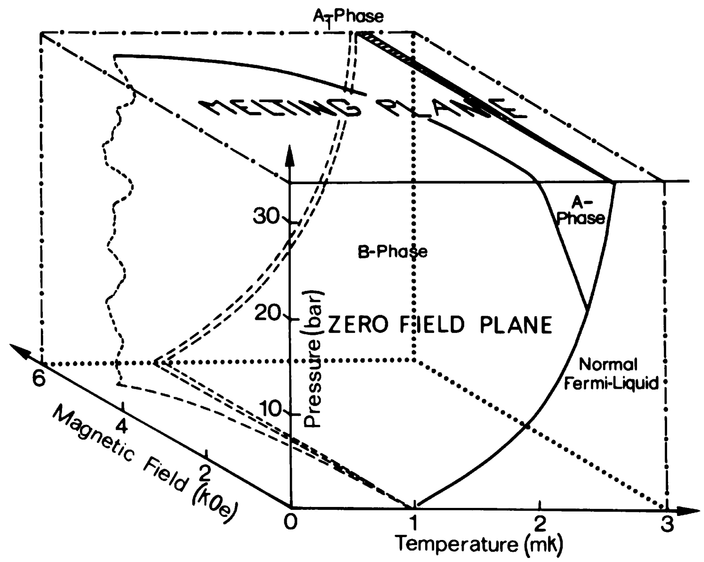


FIGURE 1.3 Phase diagram of ^{3}He plotted against temperature, pressure, and magnetic field. As $H = 0$ there are two phases, A and B. For strong magnetic fields, an additional phase A_1 develops.

	T_F	mass	T_c
Superconductor	1000 K	1 $\mathrm{m_{electron}}$	2.7 K
^{3}He	1 K	1000 $\mathrm{m_{electron}}$	2.7 mK

TABLE 1.1 There is a factor of roughly 1000 between the characteristic quantities of superconductors and ^{3}He

2

Preparation of Functional Integral

2.1 The Action of the System

It will turn out to be convenient to consider grand canonical enesembles in which the particle number can fluctuate and only the average is fixed. Then, instead of the Hamiltonian H, we shall work with $H - \mu N$ where N counts the particle number

$$N = \int d^3x \psi^*(\mathbf{x}, t)\psi(\mathbf{x}, t) \tag{2.1}$$

and μ is an appropriate Lagrangian multiplyer acting as a chemical potential. Then the action can be written as [1] Eq. (he-3.2)

$$\mathcal{A} = \int d^4x \psi^*(x) i\partial_t \psi(x) - \int dt (H - \mu N) \tag{2.2}$$

with Eq. (he-3.3)

$$\begin{aligned} H - \mu N &= \int d^3x \psi^\dagger(x) \left(-\hbar^2 \frac{\nabla^2}{2m} - \mu\right) \psi(x) \\ &+ \frac{1}{2} \int d^3x d^3x' \psi^*(x) \psi^*(x') V(x', x) \psi(x') \psi(x) \end{aligned} \tag{2.3}$$

The potential V may be taken as instantaneous and time independent Eq. (he-3.4)

$$V(x', x) = \delta(t' - t) V(\mathbf{x}', \mathbf{x}) \tag{2.4}$$

The dominant part of $V(\mathbf{x}', \mathbf{x})$ consists in the van der Waals molecular potential $V(\mathbf{x}' - \mathbf{x})$ which was displayed in Fig. 1.1.

Since the ^{3}He atoms are electrically neutral, there are no Coulomb forces at atomic distances. There is, however, a weak nuclear magnetic moment $\gamma \approx 2.04 \times 10^4$ (gauss sec)$^{-1}$ causing an additional small spin-spin dipole interaction Eq. (he-3.5)

$$\begin{aligned} \mathcal{A}_\mathrm{d} = -\int dt H_\mathrm{d} &= \int dt \int d^3x' d^3x \left[\delta_{ab} - 3\frac{(x' - x)_a (x - x)_b}{|\mathbf{x}' - \mathbf{x}|^2}\right] \\ &\times \psi^*(\mathbf{x}', t) \frac{\sigma_a}{2} \psi(\mathbf{x}', t) \psi^*(\mathbf{x}, t) \frac{\sigma_b}{2} \psi(\mathbf{x}, t). \end{aligned} \tag{2.5}$$

[1] Here and later we shall use four-vector notation where the symbol x has time and space components $x^\mu \equiv (x^0, \mathbf{x}) = (t, \mathbf{x})$ with $\mu = 0, 1, 2, 3$ and $d^4x \equiv dt d^3\mathbf{x}$, for brevity.

Due to its smallness, the interaction is negligible in the normal Fermiliquid. In the sensitive superfluid phase, however, it will have interesting consequences causing a variety of domain structures.

As shown in the last chapter, the thermodynamic action relevant at the statistical
Eq. (he-3.6) level is obtained by analytic continuation to imaginary time τ:

$$-i\mathcal{A} \to \mathcal{A}^T = \int_{-1/2T}^{1/2T} d\tau d^3x \psi^*(x)\partial_\tau\psi(x) + \int_{1/2T}^{1/2T} d\tau \left(H - \mu N + H_{\rm d}\right) \tag{2.6}$$

with $\psi(x)$ in such euclidean expressions standing for $\psi(\mathbf{x}, \tau)$. In the partition function, the path integral extends over all fields $\psi(x) = \psi(\mathbf{x}, \tau)$ antiperiodic for
Eq. (he-3.7) $\tau \to \tau + 1/T$:

$$\psi(\mathbf{x}, \tau) = -\psi\left(\mathbf{x}, \tau + 1/T\right). \tag{2.7}$$

Confronted with the action (3.6) it appears, at first sight, quite hopeless to attempt any perturbative treatment. First of all, the potential $V(r)$ has an essentially infinite repulsive core. Moreover, from the experimental density we can estimate the average distance between the atoms in the liquid to be about 3.5 Å where the potential is still of considerable strength. The salvation from this difficulty is provided by Landau's observation that many features of this strongly interacting Fermi liquid till obey the same laws observed in a free Fermi system:

(a) The specific heat behaves like $C_V \sim T$.

(b) The susceptibility behaves like $\chi \sim$ const .

(c) The compressibility behaves like $\kappa \sim$ const for small T (but in the normal liquid).

In fact, all free Fermi liquid laws for these quantities are valid provided one substitutes an effective mass $m_{eff} \approx 3 - 6m_{^3\rm He}$ instead of the true mass $m_{^3\rm He}$, depending on whether one works close to zero or melting pressure (35 bar). In addition, there is a simple multiplicative renormalization by a factor which can be attributed to molecular field effects, similar as in Weiss' theory of ferromagnetism. Landau's interpretation of this phenomenon is the following: By restricting one's attention to low-energy and momentum properties of a system the strong-interaction problems simplify considerably. The rapid fluctuations cause an almost instantaneous readjustment of the particle distribution. For this reason, if slow and long-wavelength disturbances are applied to the system, several ^{3}He atoms which are in their mutual range of interaction will respond simultaneously as a cluster, called quasiparticle, with an effective mass larger than the atomic mass. The residual interaction between these quasiparticles is very smooth and weak since any potential hole, which could appear as a result of a small displacement in the liquid, is immediately filled up and screened away by a rapid redistribution of the atoms. It is this screening effect mentioned in the introduction which makes quantitative calculations at least at the level of quasiparticles. Apparently the fast fluctuations generate a new effective action of approximately the same form as (3.2) except that ψ has to be read as quasiparticle field, m as effective mass, and V as the residual effective potential between the quasiparticles. The energy range of integrations in the Fourier decomposition of the fields is, however, limited to some cutoff frequency ω_{cutoff} beyond which the effective action becomes invalid.

Using the path integral formulation of the partition function it will be quite easy to formulate this transition from the fundamental expression $\mathcal{A}$ to a quasiparticle action at least in principle.

2.2 From Particles to Quasiparticles

It was argued that fluctuations cause a significant screening of the potential. The screened lumps of particles move almost freely but with a larger effective mass. In order to formulate this situation we need first a precise distinction between fast and slow fluctuations. For this we expand the field in a Fourier series Eq. (he-)

$$\psi(\mathbf{x},t) = \frac{1}{\sqrt{(t_b-t_a)V}} \sum_{\omega_n,\mathbf{k}} e^{-i\omega_n t + i\mathbf{k}\mathbf{x}} \tag{2.8}$$

where V is the spatial volume of the system and ω_n are the Matsubara frequencies Eq. (he-)

$$\omega_n \equiv \frac{2\pi\,(n+1/2)}{(t_b-t_a)} \tag{2.9}$$

which enforce the anti-periodic boundary condition (3.7). Apparently, there are natural energy and momentum scales ω_0 and k_0 so that a separation of the field into slow and long-wavelength and fast and short-wavelength, Eq. (he-3.9)

$$\begin{aligned} \psi(\mathbf{x},t) &\equiv \psi_s(\mathbf{x},t) + \psi_h(\mathbf{x},t) \qquad (2.10)\\ &= \frac{1}{\sqrt{(t_b-t_a)V}} \left(\sum_{\substack{|\omega_n|<\omega_0\\ |\mathbf{k}|<k_0}} e^{-i\omega_n t + i\mathbf{k}x} \psi(\omega_n,\mathbf{k}) + \sum_{\substack{|\omega_n|\geq\omega_0\\ |\mathbf{k}|>k_0}} e^{-i\omega_n t + i\mathbf{k}x} \psi(\omega_n,\mathbf{k}) \right) \end{aligned}$$

can be used to simplify the path integral. The two pieces will be referred to as *soft* and *hard components* of the field ψ. When written in energy momentum space, the functional integral measure may be separated accordingly: Eq. (he-3.10)

$$\begin{aligned} &\int \mathcal{D}\psi\mathcal{D}\psi^* = \int \mathcal{D}\psi_s\mathcal{D}\psi_s^* \int \mathcal{D}\psi_h\mathcal{D}\psi_h^* \equiv \\ &\prod_{|\omega|<\omega_0,|\mathbf{k}<k_0} \frac{d\psi(\omega_n,\mathbf{k})d\psi^*(\omega_n k)}{2\pi i} \prod_{|\omega|\geq\omega_0,|\mathbf{k}\geq k_0} \frac{d\psi(\omega_n,\mathbf{k})d\psi^*(\omega_n k)}{2\pi i}. \end{aligned} \tag{2.11}$$

If we now perform the path integral over the hard components we remain with a partition function Eq. (he-3.11)

$$Z = \int \mathcal{D}\psi_s\mathcal{D}\psi_s^* e^{i\mathcal{A}_s[\psi_s^*,\psi_s]} \tag{2.12}$$

where $\mathcal{A}_s[\psi_s^*, \psi_s]$ is a functional of only the soft components. The point of Landau's argument is now that due to the high quality of the free Fermi gas laws there seems to exist an optimal choice for ω_0, K_0 so that the action looks like the action of the initial ^{3}He particles except that the new fields ψ_s have a larger effective mass m^* and that the interactions are much weaker than in the original fundamental form (3.2).

Certainly, the execution of the path integral over the fast components is extremely difficult due to the strength of the interactions. We shall therefore accept Landau's argument purely on phenomenological grounds and see its justification in the successful derivation of the physical properties of the liquid.

At first sight, the precise choice of ω_0 and k_0 seems to be a rather ad hoc matter and one might fear that all results derived from the partition function (3.12) depend strongly on which values are taken. It is gratifying to note, however, that this is not really true. Only the predictions as to the size of the transition temperature T_c varies strongly with ω_0, k_0. But in all final results ω_0, k_0 can be eliminated in favor of the observable temperature T_c in this way the arbitrariness is removed. This is completely analogous to the independence of all physical amplitudes on the cutoff in renormalizable field theory.

For the phenomena of superfluidity, the optimal choice of ω_0, k_0 will be so that ω_0 is about 10 times larger than the transition temperature T_c while k_0 comprises 10 approximately atomic distances (i.e., $k_0 \approx 2\pi/10\text{Å}$). In this way quasiparticle fields are well enough localized in space and time to describe excitations with frequencies in the range of $T_c \sim MHZ$ and wavelength of about 100 Å.

2.3 The Approximate Quasiparticle Action

We are thus confronted with a simplified problem of calculating the partition function over soft field components ψ_s. For brevity, the subscript's will be dropped. The soft field *quanta* are precisely what Landau introduced as quasiparticles. Since we are not able to calculate $\mathcal{A}_s$ explicitly, we have to deduce its structure from experimental facts. As argued above,the action must account for the free particle like behavior of specific heat, susceptibility transition temperature T except for simple renormalization factors.

Let us briefly take a look at the experimental situation: For a free Fermi gas it is easy to calculate the three quantities (the standard derivation is omitted here, in
Eq. (he-3.13) order to proceed with the argument. The derivation will appear later. [2]

$$C_V = \frac{mp_F}{3\hbar^3} {k_B}^2 T \tag{2.13}$$

$$\chi_N = \frac{\gamma^2}{4} \frac{mp_F}{\pi^2} \frac{1}{\hbar} \tag{2.14}$$

$$\kappa_T = \frac{m}{\rho^2} \frac{p_F}{\pi^2 \hbar^3} \tag{2.15}$$

[2]For explicitness, we keep $\hbar$ and k_B in these particular formulas.

with Eq. (he-)

$$\frac{\rho}{m} = \frac{N}{V} \tag{2.16}$$

being the particle density whereas Eq. (he-3.16)

$$p_F = \left(3\pi^2\right)^{1/3} \left(\frac{N}{V}\right)^{1/3} \hbar \approx g \times 10^{-20}\text{g cm/sec} \tag{2.17}$$

is the Fermi momentum which is only slightly pressure dependent. The associated Fermi velocity $v_F \equiv p_F/m$ varies from $5.5 \cdot 10^3$ cm/sec at zero pressure to about $3 \cdot 10^3$ cm/sec at melting pressure (see Table 2.1).

Experimentally one finds the linear behavior for C_V below 20 mK for the absolute size is enhanced by a factor 6 to 14 for pressures ranging from atmospheric to 35 bar (melting pressure). This enhancement may be attributed to a change in the effective mass from m to m^*. It is customary to introduce the parameter F_1^s defined by Eq. (he-3.17)

$$\frac{m^*}{m} = 1 + \frac{F_1^s}{3}. \tag{2.18}$$

The precise values of F_1^s can be seen on Table 2.1.

The spin susceptibility is found to be independent of temperature below 40 mK. If one, however, inserts the effective mass m^* into formula (3.17) one finds a value

TABLE 2.1 Pressure dependence of Landau parameters F_0^a, F_0^s, and F_1^s in ^{3}He together with the molar volume and the effective mass ratio m^*/m. The values of V, m^*/m and F_1^s are taken from Greywall (1986), whereas F_0^a, F_0^s are from Wheatley (1975) except for corrections using more recent values of m^*/m. At $P = 34.39$ bar this was done on the basis of Wheatley's values at $P = 34.36$ bar.

P(bar)	V(cm^3)	m^*/m	F_1^s	F_0^s	F_0^a
0	36.84	2.80	5.39	9.30	-0.695
3	33.95	3.16	6.49	15.99	-0.723
6	32.03	3.48	7.45	22.49	-0.733
9	30.71	3.77	8.32	29.00	-0.742
12	29.71	4.03	9.09	35.42	-0.747
15	28.89	4.28	9.85	41.73	-0.753
18	28.18	4.53	10.60	48.46	-0.757
21	27.55	4.78	11.34	55.20	-0.755
24	27.01	5.02	12.07	62.16	-0.756
27	26.56	5.26	12.79	69.43	-0.755
30	26.17	5.50	13.50	77.02	-0.754
33	25.75	5.74	14.21	84.79	-0.755
34.39	25.50	5.85	14.56	88.47	-0.753

about four times too small. This is attributed to molecular field effects. if the atomic magnetic moments are partially oriented, the magnetic field seen by an individual atom consists of the external field plus that of the other moments in the liquid. The
Eq. (he-) enhancement factor is usually denoted as

$$\frac{1}{1+F_0^a} \equiv \frac{1}{1+Z_0/4} \tag{2.19}$$

with $F_0^a \equiv Z_0/4$ being roughly -3 up to melting pressure 35 bar (see Table 2.1).
Eq. (he-3.19) The compressibility, finally is measured via the velocity of sound

$$c = \frac{1}{\sqrt{\rho \kappa_T}} = \frac{v_F}{\sqrt{3}} \left(1+\frac{F_1^s}{3}\right)^{1/2}. \tag{2.20}$$

Here

$$v_F = \frac{p_F}{m^*} \tag{2.21}$$

is the Fermi velocity for the effective mass m^* which ranges from 5 to 3×10^3 cm/sec (see Table 2.1).

Experimentally, formula (3.22) turns out to be wrong by a factor 3 to 10. This is again attributed to molecular field effects for the density field and one multiplies
Eq. (he-3.21) the compressibility with a correction factor ${F_0}^s \equiv 1/(1+F_0^s)$ so that

$$c = \frac{v_F}{\sqrt{3}} \left[\left(1+\frac{F_1^s}{3}\right)(1+F_0^s)\right]^{1/2} \tag{2.22}$$

with F_0^s and ranging from 10 to 100 (see Table 2.1).

2.4 The Effective Interaction

Which action do we have to take in order to explain these simple features of liquid ^{3}He in a wide range above the superfluid transition temperature? It appears simple to include the effective mass. All we have to do is choose a free-particle Hamiltonian
Eq. (he-3.22)

$$H_0 = \int d^3x \psi^*(x) \left(i\partial_t + \frac{\nabla^2}{2m^*}\right) \psi(x). \tag{2.23}$$

This naive approximation would immediately lead to the specific heat (3.15) with the mass m replaced by m^*, provided the number of quasiparticles is taken to be equal to the true particle number (so that also the Fermi momentum p_F which depends only on the particle density N/V is the same).
Eq. (he-3.23) If one would set the system into motion by displacing all particle velocities

$$v = \frac{p}{m^*} \tag{2.24}$$

Eq. (he-3.24) by a certain amount Δv, the total momentum P of the system would change by

$$\Delta P = \Delta v N m^* \tag{2.25}$$

Eq. (he-3.25) rather than what it must be physically:

$$\Delta P = \Delta v N m. \tag{2.26}$$

This can only be corrected by introducing an additional interaction which, however, must not modify the previous calculation of the specific heat. Such interactions are well known in molecular field theories. We simply add to the free Hamiltonian a current interaction Eq. (he-3.26)

$$H_{\text{curr-curr}} = \frac{1}{2\rho^*}\frac{F_1^s}{3}\int d^3x \psi^*(x)\frac{i}{2}\overleftrightarrow{\nabla}\psi(x)\psi^*(x)\frac{i}{2}\overleftrightarrow{\nabla}\psi(x) \tag{2.27}$$

where $\overleftrightarrow{\nabla}$ is the right-minus-left derivative $\overrightarrow{\nabla} - \overleftarrow{\nabla}$, the constant F_a^s denotes the coupling strength, and

$$\rho^* = \frac{m^* N}{Y}$$

is the mass density of quasi-particles. Then the kinematic properties of single quasi-particle states are automatically correct: For such a state we see the energy to be Eq. (he-3.27)

$$E = \frac{p^2}{2m^*} + \frac{{F_1}^s/3}{2m^*}p^2 = \frac{p^2}{2m^*}\left(1 + {F_1}^s/3\right) \tag{2.28}$$

so that the velocity is Eq. (he-3.28)

$$v = \frac{\partial E}{\partial p} = \frac{p}{m^*}\left(1 + {F_1}^s/3\right) \tag{2.29}$$

and the momentum changes upon a shift in velocity by Eq. (he-3.29)

$$\Delta P = \frac{\Delta v N m^*}{\left(1 + \frac{F_1^s}{3}\right)} \tag{2.30}$$

For an N-particle state, this leads to the correct result (2.26) if Eq. (he-)

$$m^* = m\left(1 + \frac{F_1^s}{3}\right) \tag{2.31}$$

which is just the relation introduced before in (3.21). That the interaction (2.27) does leave the specific heat in the form (3.15) with only m replaced by m^* is not so easy to see and will be shown later.

The renormalization factors for susceptibility and compressibility have to be inferred in a similar manner.

For this we realize that when going from one Galilean frame of reference to another, which moves with velocity v, the energy changes by Eq. (he-3.31)

$$\Delta H_v = -\int d^3x \psi^*(x)\frac{\overleftrightarrow{\nabla}}{2}\psi(x)\ \Delta v. \tag{2.32}$$

When turning on a magnetic field, the energy changes by Eq. (he-3.32)

$$\Delta H_H = \int d^3x \psi^*(x) \frac{\sigma^a}{2} \psi(x)\ \gamma H_a \tag{2.33}$$

due to the coupling to the spin magnetic moments.
Finally, if chemical potential is introduced by contact with a particle reservoir, the
Eq. (he-3.33) energy has to be modified by

$$\Delta H_\mu = -\int d^3x \psi^*(x)\psi(x)\ \mu. \tag{2.34}$$

Thus, current density

$$j_i \equiv \frac{1}{2}\psi^* \overset{\leftrightarrow}{\nabla}_i \psi,$$

spin density

$$s^a \equiv \psi^* \frac{\sigma^a}{2}\psi,$$

and particle density

$$n \equiv \psi^* \psi$$

appear on exactly the same footing.

We have seen that the quadratic current density coupling brings changes in the
Eq. (he-3.34) kinetic energy to the correct form

$$\frac{1}{m^*} pdp \to \frac{1}{m^*}\left(1 + \frac{F_1^s}{3}\right) pdp = p\frac{dp}{m}. \tag{2.35}$$

Eq. (he-3.35) Thus we expect quadratic spin density and particle density couplings

$$\begin{aligned} H_{sd} &= \frac{1}{2}\frac{{F_0}^s}{\rho^*}\int d^3x \psi^*(x)\frac{\sigma^a}{2}\psi(x)\psi^*(x)\frac{\sigma^a}{2}\psi(x) \\ H_{\rm d} &= \frac{1}{2}\frac{F_0}{\rho^*}\int d^3x \psi^*(x)\psi(x)\psi^*(x)\psi(x) \end{aligned} \tag{2.36}$$

to produce the correction factors for changes in the magnetic and chemical energy
Eq. (he-3.36) density

$$\chi H dH \to \chi\left(1 + F_0^a\right) H dH \tag{2.37}$$

$$\kappa \mu d\mu \to \kappa\left(1 + F_0^s\right) \mu d\mu \tag{2.38}$$

which were needed to obtain agreement of these quantities with experiment. Certainly, the couplings introduced are just a specially important selection of a more
Eq. (he-3.38) complete expansion

$$\begin{aligned} H_{\rm int} &= \frac{1}{2\rho^*}\sum_{l=0}^{\infty}\int d^3x \frac{F_l^s}{2l+1}\psi^*(x)\partial_{lm}\psi(x)\psi^*(x)\partial_{lm}\psi(x) \\ &+ \frac{1}{2\rho^*}\sum_{l=0}^{\infty}\int d^3x \frac{F_l^a}{2l+1}\psi^*(x)\frac{\sigma^a}{2}\partial_{lm}\psi^*(x)\frac{\sigma^a}{\sqrt{2}}\partial_{lm}\psi(x) \end{aligned} \tag{2.39}$$

in which each parameter F can depend also on the momentum transfer, i.e., the momentum of $\psi^*\partial_{lm}\psi$ composite field. Such a dependence is referred to as a form factor. The symbol ∂_{lm} is a short notation for the product of l derivatives which are chosen traceless in order to project out definite angular momenta, i.e., for instance

Eq. (he-3.39)

$$\begin{aligned}\partial_{2m} &\propto \partial_i\partial_j - \frac{1}{3}\delta_{ij}\partial^2 \\ \partial_{3m} &\propto \partial_i\partial_j\partial_k - \frac{1}{5}\left(\delta_{ij}\partial^2\partial_k + 2 \text{ cyclic permutations}\right). \end{aligned} \quad (2.40)$$

The proportionality factor is chosen to comply with the following definition in terms of spherical harmonics:

Eq. (he-3.40)

$$\partial_{lm} = \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\hat{\partial})|\partial^l|. \quad (2.41)$$

Here the components m refer to a spherical basis so that one must distinguish ∂_{lm} and ∂^*_{lm} and contract them accordingly in (2.40).

It will turn out that many phenomena depend only on the values of F_l at zero momentum transfer. Moreover, only the three parameters discussed explicitly before are easily accessible to experimental measurement.

2.5 Pairing Interaction

With the couplings introduced until now the properties of the degenerate Fermi liquid can be explained within very simple approximations as long as the temperature is above the critical value T_c. As explained in the introduction the superfluid properties below T_c require the formation of p-wave spin triplet Cooper pairs. This can only happen due to an additional attractive interaction which must consist of one screened version of the original potential V. Its accurate shape is unknown. This, however, turns out to be no handicap. The reason is the following: The attractive force is extremely weak. Therefore the Cooper pairs are only barely bound (as manifested by the critical temperature T_c being much smaller than the characteristic temperature unit of the system which is $T_F = p_F^2/2m$ (= the Fermi energy of the system $\approx$ 1 K). Therefore the radius of the bound-state wave functions is much larger than $1/p_F \approx 1$ Å. It will turn out to be a few hundred Å. Therefore, whatever the detailed shape of V, since it can be nonzero only at distances of the order of a few Å, it is seen by the bound state only as a completely local attractive interaction. Since this must bind in p-wave spin triplet state, we may directly write

Eq. (he-3.41)

$$H_{\text{pair}} = -\frac{3g}{4p_F^2}\int d^3x \psi^*(x)\frac{\sigma^a}{2}c^\dagger \overleftrightarrow{\nabla} \psi^*(x)\psi(x)c\frac{\sigma^a}{2} \overleftrightarrow{\nabla} \psi(x). \quad (2.42)$$

The matrix c is

Eq. (he-3.42)

$$c = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (2.43)$$

and makes sure that $\psi c\sigma_a\psi$ transforms in the same way as $\psi^*\sigma^a\psi$, i.e., like a vector (since the 2×2 rotation matrix U and its complex conjugate are equivalent by $cUc^{-1} = U^*$.

2.6 Dipole Interaction

The Hamiltonian (3.27) with interactions $H_2, H_{sd}, H_{\rm d}, H_{\rm pair}$ will be sufficient to explain quantitatively most of the properties of the normal and superfluid ^{3}He. As stated in the beginning of this chapter, the condensate of Cooper pairs is a very sensitive system. We shall see that many of its interesting phenomena are a direct manifestation of the very small dipole coupling (even though this is properly a hyperfine interaction).

3

Transformation from Fundamental to Collective Fields

While fundamental fields provide the theortically most satisfactory way of *defining* the action of a theory, they are quite ineconomic as far as the description of low-energy and long-wavelength phenomena of systems like ^{3}He and superconductors is concerned. The reason is basically the following: Below the transition temperature T_c to the superfluid phase the binding of the fundamental particles in Cooper pairs results in an anergy gap Δ of the single particle spectrum which becomes Eq. (he-3.1)

$$E(\mathbf{p}) = \sqrt{\xi^2(\mathbf{p}) + \Delta^2} \tag{3.1}$$

For ^{3}He the size of the gap is of the order of mK while for most superconductors Δ lies in the K range. As a consequence, the propagator Eq. (he-3.2)

$$\langle 0| T\left(\psi(x)\psi^*(y)\right)|0\rangle \tag{3.2}$$

has no singularities in the energy plane below $E = \Delta$. A description of the rich set of physical phenomena with energies much smaller than Δ[1] such as zero-sound waves, spin waves etc. in terms of ψ is therefore quite complicated: An infinite set of Feynman graphs is necessary even for a lowest order understanding of these phenomena. On the other hand, there are Green functions which directly display excitations of this type in the complex energy plane, for example those of the composite field operators Eq. (he-3.3)

$$\langle 0| T\left(\psi^*(x)\psi(x)\psi^*(y)\psi(y)\right)|0\rangle \tag{3.3}$$

Eq. (he-3.4)

$$\langle 0| T\left(\psi^*(x)\frac{\sigma_a}{2}\psi(x)\psi^*(y)\frac{\sigma_b}{2}\psi(y)\right)|0\rangle. \tag{3.4}$$

Singularities which appear in such composite Green functions but not in (3.2) are called collective excitations. One may expect that the most economic description of the associated physical phenomena can be obtained by transforming the full theory first to the appropriate composite fields. Such transformations have, in fact, been studies a long time ago in many-body theory at the quasiclassical level. For

[1] These will often be called "infrared" phenomena, for brevity.

superconductors [13] and ^{3}He the result is the so-called Ginzburg-Landau equation [14]. This equation has been extremely successful in explaining many low-energy properties of the system. The approximate methods leading to this equation have been describes in Part I and applied to plasmons and superconductors in Part II. As before, we add to the action (3.27) (2.42(he-3.41)) a complete square involving an
Eq. (he-3.5) auxiliary field A_{ai}

$$\Delta\mathcal{A} = -\frac{1}{3g^2}\int d^3x |A_{ai} - \frac{3g}{2p_F}\psi_i \overleftrightarrow{\nabla}_i c\frac{\sigma_a}{2}\psi|^2. \tag{3.5}$$

This does not change the theory, for A_{ai} is obviously a dependent field (no $\partial_t A_{ai}$ appears) and can be eliminated by solving the "equations of motion" $\delta\mathcal{A}/\delta A_{ai}(x) = 0$
Eq. (he-3.6) which are

$$A_{ai}(x) = \frac{3g}{2p_F}\psi(x) i\overleftrightarrow{\nabla}_i c\frac{\sigma_a}{2}\psi(x). \tag{3.6}$$

Hence, A_{ai} coincides, at the classical level, with the composite field of a pair of ^{3}He atoms in a p-wave spin triplet configuration. Since it will serve to describe the collective phenomena it will, from now on, be called the *collective pair field*. Reinserting (3.6) into (3.5(he-3.5)) gives $\Delta\mathcal{A} = 0$ so that, at the classical level, the addition of $\delta\mathcal{A}$ really leaves the action unchanged. That this remains true at the full
Eq. (he-3.7) quantum level can be seen by considering the generating functional of the theory

$$Z = \int \mathcal{D}\psi^*\mathcal{D}\psi\mathcal{D}A^*_a\mathcal{D}A_{ai} e^{i(\mathcal{A}_0+\mathcal{A}_{\rm int}+\Delta\mathcal{A})} \tag{3.7}$$

The integral over the auxiliary field $\mathcal{D}A_{ai}$ is of the gaussian type peaks for each spacetime point x at $\frac{3g}{2p_F}\psi(x)i\overleftrightarrow{\nabla}_i\frac{\sigma_a}{2}\psi(x)$. Since the integral runs at each point from $-\infty$ to $+\infty$, the finite shift is irrelevant and the integral renders the same irrelevant constant for each x. The particular choice (3.5) of $\Delta\mathcal{A}$ consists in its eliminating
Eq. (he-3.8) the quartic term in the action in (3.7) so that the remainder

$$\begin{aligned}\mathcal{A} \;=\; & \int d^4x \,\{\psi^*(x)\,(i\partial_t - \xi(-i\boldsymbol{\nabla}))\,\psi(x) \\ & + \left(A^*_{ai}(x)\psi i\tilde{\nabla} c\frac{\sigma_a}{2}\psi + c.c.\right) - \frac{1}{3g}A^*_{ai}A_{ai}\right\}\end{aligned} \tag{3.8}$$

is quadratic in the fields ψ. For this reason, the integral over $\mathcal{D}\psi^*\mathcal{D}\psi$ can be performed in Z leaving a quantum field theory formulated only in terms of the pair field A_{ai}. In (3.8) we have gone over to a dimensionless right-minus-left derivative

$$\tilde{\nabla} \equiv \frac{1}{2p_F}\overleftrightarrow{\nabla}_i, \tag{3.9}$$

for convenience.

In order to bring the fermion integration to a standard form we introduce a four-component field which assembles the $\psi(x)$ and $\psi^*(x)$ components into a single
Eq. (he-3.f) "bispinor"

$$f = \begin{pmatrix} \psi \\ c\psi^* \end{pmatrix}. \tag{3.10}$$

Then (3.8) can be rewritten as [2] Eq. (he-3.9)

$$\mathcal{A} = \int d^4x \left[\frac{1}{2} f^*(x) \begin{pmatrix} i\partial_t - \xi(-i\boldsymbol{\nabla}) & i\tilde{\nabla}_i \sigma_a A_{ai} \\ i\tilde{\nabla}_i \sigma_a A^*_{ai} & i\partial_t + \xi(i\boldsymbol{\nabla}) \end{pmatrix} f(x) - \frac{1}{3g} A^*_{ai} A_{ai} \right]. \tag{3.11}$$

Performing now $\int \mathcal{D}f^* \mathcal{D}f$ results in [3] Eq. (he-3.10)

$$Z = \int \mathcal{D}A^*_{ai} \mathcal{D}A_{ai} e^{i\mathcal{A}_{\rm coll}[A^*,\ A]} \tag{3.12}$$

with the collective action Eq. (he-3.11)

$$\begin{aligned} &\mathcal{A}_{\rm coll}\left[A^*,\ A\right] = \\ &\quad -\frac{i}{2} \mathrm{Tr} \log \begin{pmatrix} i\partial_t - \xi(-i\nabla) & i\tilde{\nabla}_i \sigma_a A_{ai} \\ i\tilde{\nabla}_i \sigma_a A^*_{ai} & i\partial_t + \xi(i\nabla_i) \end{pmatrix} - \frac{1}{3g} \int d^4x A^*_{ai}(x) A_{ai}(x) \end{aligned} \tag{3.13}$$

The trace log part can also be written as Eq. (he-3.12)

$$\begin{aligned} &-\frac{i}{2} \mathrm{Tr} \log \begin{pmatrix} i\partial_t - \xi(-i\boldsymbol{\nabla}) & 0 \\ 0 & i\partial_t + \xi(i\boldsymbol{\nabla}) \end{pmatrix} \\ &-\frac{i}{2} \mathrm{Tr} \log \left[1 - i \begin{pmatrix} \frac{i}{i\partial_t - \xi(-i\boldsymbol{\nabla})} & \\ 0 & \frac{i}{i\partial_t + \xi(i\boldsymbol{\nabla})} \end{pmatrix} \begin{pmatrix} 0 & i\tilde{\nabla}_i \sigma_a A_{ai} \\ i\tilde{\nabla}_i \sigma_a A^*_{ai} & 0 \end{pmatrix} \right]. \end{aligned} \tag{3.14}$$

The first term is an irrelevant constant.[4] The second term can be expanded in powers of A_{ai} as follows: Eq. (he-3.13)

$$i \sum_{n=1}^{\infty} \frac{(-i)^{2n}}{2n} \mathrm{Tr} \left(\frac{i}{i\partial_t - \xi(-i\boldsymbol{\nabla})} i\overset{\leftrightarrow}{\nabla}_i \sigma_a A_{ai} \frac{i}{i\partial_t + \xi(i\boldsymbol{\nabla})} i\overset{\leftrightarrow}{\nabla}_j \sigma_a A^*_{bj} \right)^n. \tag{3.15}$$

The lowest terms of this expansion correspond to the loop diagrams shown in Fig. 3.4. The free part of the collective action is given by the $n = 1$ term of (3.15) and the last term in (3.8). Performing the single-loop integral with fixed temperature Green functions we find for the Fourier-transformed fields defined by

$$A_{ai}(x) = \frac{1}{\sqrt{V/T}} \sum_k A_{ai}(k) e^{-ikx} \equiv \frac{1}{\sqrt{V/T}} \sum_{\omega_n, \mathbf{k}} A_{ai}(\omega_n, \mathbf{k}) e^{-i(\omega_n \tau - \mathbf{kx})} \tag{3.16}$$

with no time dependence [i.e., only $A(\omega_0, \mathbf{k})$ is nonzero] the action Eq. (he-3.14)

[2] The derivatives $\tilde{\nabla}_i$ do not act on the collective field A_{ai} but only on $f^*,\ f$.

[3] Here we have used $\int \mathcal{D}f^* \mathcal{D}f e^{i\frac{1}{2}f^*Mf} = (\mathrm{Det}\, M)^{\frac{1}{2}} = e^{\frac{1}{2}ßmboxTr\ \log M}$.

[4] This can be integrated as $\mathrm{Tr} \log\left((i\partial_t - \xi)\delta_{\alpha\beta}\right) = 2\int \frac{d^3p}{(2\pi)^3} \log\left(1 + e^{-\xi(\mathbf{p})/T}\right) \equiv -E_0/T$ with E_0 = energy of a free-fermion system.

$$\mathcal{A}_0[A^*, A] \approx \frac{\mathcal{N}(0)}{3} \int dt \sum_{\mathbf{k}} A^*_{ai}(\mathbf{k}) \left[\left(1 - \frac{T}{T_c}\right) \delta^{ij} - \frac{3}{5}\xi_0^2 \left(\mathbf{k}^2 \delta^{ij} + 2k^i k^i\right) \right] A_{aj}(\mathbf{k}) \tag{3.17}$$

where

$$\xi_0 = \sqrt{\frac{7\zeta(3)}{48\pi^2}} \frac{v_F}{T_c} \approx 120\text{Å} \tag{3.18}$$

is the basic *coherence length* of the superfluid[5] and T_c is the critical temperature given by the solution of the gap equation

$$\begin{aligned} 0 &= \int \frac{d^3p}{(2\pi)^3} \frac{\tanh \xi(\mathbf{p})/2T}{2\xi(\mathbf{p})} - \frac{1}{g} \\ &\approx \mathcal{N}(0) \int_{-\omega_{\text{cutoff}}}^{\omega_{\text{cutoff}}} \frac{d\xi}{2\xi} \tanh \frac{\xi}{2T} - \frac{1}{g} \\ &= \mathcal{N}(0) \log \left(2 \frac{2^\gamma}{\pi} \frac{\omega_{\text{cutoff}}}{T} \right) - \frac{1}{g} \end{aligned} \tag{3.19}$$

Eq. (he-3.16) in which $\mathcal{N}(0)$ denotes the density of states at the surface of the Fermi sea:

$$\mathcal{N}(0) = \frac{mp_F}{2\pi^2} = \frac{3}{4} \frac{\rho}{mT_F} = \frac{3}{2} \frac{\rho}{m^2} \frac{1}{v_F^2} \tag{3.20}$$

Eq. (he-3.17) Thus, one has

$$T_c \equiv \omega_{\text{cutoff}}\ 2\frac{e^\gamma}{\pi} e^{-1/g\mathcal{N}(0)}. \tag{3.21}$$

Close to T_c, the right-hand side of Eq. (3.19) can be approximated as

$$\mathcal{N}(0) \left(1 - \frac{T}{T_c}\right)$$

which is exactly the first term in (3.17). The lowest order collective interaction is quartic in the A fields and becomes in the static case with the long-wavelength limit
Eq. (he-3.18) taken

$$\begin{aligned} \mathcal{A}_{\text{int}}[A^*, A] = - \int d^4x \Big[&\beta_1 A^*_{ai} A_{bj} A^*_{ai} A_{bj} + \beta_2 \left(A^*_{ai} A_{ai}\right)^2 \\ &+ \beta_3 A^*_{ai} A_{aj} A^*_{bi} A_{bj} + \beta_4 A^*_{ai} A_{bi} A^*_{bj} A_{aj} + \beta_5 A^*_{ai} A_{bi} A^*_{aj} A_{bj} \Big]. \end{aligned} \tag{3.22}$$

The coefficients β_i are found from the loop integral for $n = 2$ in the same way as in
Eq. (he-) the case of the superconductor:

$$-2\beta_1 = \beta_2 = \beta_3 = \beta_4 = -\beta_5 = \frac{2}{5} \mathcal{N}(0) \frac{{\xi_0}^2}{{v_F}^2} \tag{3.23}$$

The full interaction contains infinite powers in the collective field A. If one restricts the consideration to temperatures close to the critical point ($T \approx T_c$),

[5]The constant is $\zeta(3) \equiv \sum_{n=1}^{\infty} 1/n^3 \approx 1.202$.

however, the fields A becomes massless and fluctuations take place over long range. As far as long-wavelength properties are concerned, higher and higher powers in A become more and more irrelevant (since the dimension of A goes as length [1]). This type of discussion is familiar from the renormalization group treatment of critical phenomena [17].

In x-space, the free part of the action can be written as Eq. (he-3.20)

$$\mathcal{A}_0[A^*, A] = \int d^4x \,(\mu_A A^*_{ai} A_{ai} - \frac{K_1}{2}\partial_i A^*_{aj}\partial_i A_{aj} - \frac{K_2}{2}\partial_i A^*_{aj}\partial_j A_{ai} - \frac{K_3}{2}\partial_i A^*_{ai}\partial_j A_{aj}\Big), \tag{3.24}$$

with the temperature dependence residing all in Eq. (he-3.21)

$$\mu_A = \frac{1}{3}\mathcal{N}(0)\left(1 - \frac{T}{T_c}\right) \tag{3.25}$$

and the stiffness constants Eq. (he-3.21)

$$K_1 = \frac{2}{5}\mathcal{N}(0)\xi_0{}^2, \quad K_2 + K_3 = 2K_1. \tag{3.26}$$

The static long wavelength action (3.22) and (3.24(he-3.20)) for $T \approx T_c$ when considered classically is usually referred to a s the "weak-coupling" Ginzburg-Landau action of ^{3}He. Notice, however, that its validity is not confined to the classical regime since the path integral renders the collective action as a fully fledged quantum field theory.

There is not enough space and time here to discuss all the corrections which become necessary when including the other parts of the interaction potential which were omitted in this derivation of the collective action. [16] Let us just mention that the general form (3.22) and (3.24) remains as it is sufficiently close to the critical temperature if one stays in the long-wavelength limit. Only the numerical values of the coefficients change and will, in general, no longer satisfy the many rotations (2.20), (3.26(he-3.21)) which were obtained in the weak-coupling limit.

That this must be true can be seen on the grounds of the symmetry properties of the action (3.2). It is obvious that the action is invariant under separate rotations in spin and orbital space and under phase transformations $\psi \to e^{i\alpha}\psi$. The collective action as a direct translation has to display the same invariance. In the static long-wavelength limit with $T \approx T_c$ this leaves only the form (3.22) and (3.24(he-3.20)). On the same symmetry grounds it is obvious that the dipole action (3.5) *cannot* be included by a mere change of the coefficients: The action contracts spatial with spin indices and is no longer invariant under separate spin and orbital rotations. It can be shown [14, 16] that the collective form of the dipole action gives rise to an additional mass term for the A field: Eq. (he-3.22)

$$\mathcal{A}_{\rm d} = g_{\rm d} \int d^4x \left(A^*_{aa}A_{bb} + A^*_{ab}A_{ba} - \frac{2}{3}A^*_{ab}A_{ab}\right). \tag{3.27}$$

The coupling of spatial and spin degrees will result in the most interesting observable phenomena of the superfluid phase.

At the mean-field level, the integrand of the collective action yields the Ginzburg-Landau free energy to be used in the sequel:

$$
\begin{aligned}
f &= f_0 + f_{\rm int} + f_{\rm d} \\
&= \mu_A A^*_{ai} A_{ai} - \frac{K_1}{2} \partial_i A^*_{aj} \partial_i A_{aj} - \frac{K_2}{2} \partial_i A^*_{aj} \partial_j A_{ai} - \frac{K_3}{2} \partial_i A^*_{ai} \partial_j A_{aj} \\
&\quad -\beta_1 A^*_{ai} A_{bj} A^*_{ai} A_{bj} + \beta_2 \left(A^*_{ai} A_{ai}\right)^2 \\
&\quad +\beta_3 A^*_{ai} A_{aj} A^*_{bi} A_{bj} + \beta_4 A^*_{ai} A_{bi} A^*_{bj} A_{aj} + \beta_5 A^*_{ai} A_{bi} A^*_{aj} A_{bj}. \\
&\quad +g_{\rm d} \left(A^*_{aa} A_{bb} + A^*_{ab} A_{ba} - \frac{2}{3} A^*_{ab} A_{ab} \right).
\end{aligned}
\tag{3.28}
$$

4

General Properties of Collective Action

The static action (3.22) with (3.24) describes the ^{3}He liquid in terms of a complex 3×3 matrix, i.e., an 18-component field called the *order field*. If the dipole interaction is left out, the action is invariant under global $SU(2)\times SU(2)\times U(1)$ transformations:

Eq. (he-4.1)

$$\begin{aligned} A_{ai} &\rightarrow R_{ab}(\boldsymbol{\varphi}^s)R_{ij}(\boldsymbol{\varphi}^o)e^{-2i\varphi} \\ &= \left(e^{-\boldsymbol{\varphi}^s\boldsymbol{\epsilon}}\right)_{ab}\left(e^{-2\boldsymbol{\varphi}^o\boldsymbol{\epsilon}}\right)_{ij}e^{-2i\varphi}A_{bj} \end{aligned} \tag{4.1}$$

where $(\epsilon_a)_{bc}\equiv\epsilon_{abc}$ are the 3×3 generators of the three-dimensional rotation group and $\boldsymbol{\varphi}^s$, $\boldsymbol{\varphi}^o$ denote the associated rotation parameters. Remembering the classical equality

Eq. (he-4.2)

$$A_{ai}=\frac{3g}{2p_F}\psi(x)i\overleftrightarrow{\nabla}_i c\frac{\sigma_a}{2}\psi(x) \tag{4.2}$$

we see that the first transformation corresponds to pure spin, the second to pure orbital rotations to the original field ψ. The last phase is associated with particle number conservation and is doubled because of the two fields occurring in (4.2).

Accordingly, there are three conserved Noether currents which are obtained by functional derivatives with respect to infinitesimal *x-dependent* symmetry transformations:
First there is the particle current:

Eq. (he-4.3)

$$\begin{aligned} j_i\equiv\frac{\delta\mathcal{A}}{\delta\partial_i\varphi}=i\Big\{&K_1A^*_{aj}\overleftrightarrow{\nabla}_iA_{aj}+K_2\left(A^*_{aj}\overleftrightarrow{\nabla}_jA_{ai}-A^*_{ai}\overleftrightarrow{\nabla}_jA_{aj}\right) \\ &+K_3\left(A^*_{ai}\overleftrightarrow{\nabla}_jA_{aj}-A^*_{ai}\overleftrightarrow{\nabla}_jA_{ai}\right)\Big\}. \end{aligned}$$

This current j_i coincides also with the $(1/m)T^{0i}$ components of the energy momentum tensor. Under an infinitesimal Galilei transformation

Eq. (he-)

$$\psi\rightarrow e^{-im\mathbf{v}\cdot\mathbf{x}}\psi \tag{4.3}$$

the collective field changes as follows:

$$A^{ai}\rightarrow e^{-2im\mathbf{v}\cdot\mathbf{x}}A_{ai} \tag{4.4}$$

so that the energy transforms as Eq. (he-4.4)

$$\delta E = m \int d^3x \; \mathbf{j} \cdot \mathbf{v}_s. \tag{4.5}$$

Eq. (he-4.5) From infinitesimal spin rotations one derives the conserved spin current:

$$\begin{aligned} j^{spin}_{ai} \;\; &\equiv \;\; \frac{\delta \mathcal{A}}{\delta \partial_i \varphi^s_a} = \epsilon_{abc} \left[K_1 \left(A^*_{bj} \overleftrightarrow{\nabla}_i A_{cj} + A^*_{cj} \overleftrightarrow{\nabla}_i A_{bj} \right) \right. \\ &\left. + K_2 \left(A^*_{bj} \overleftrightarrow{\nabla}_j A_{cj} + A^*_{cj} \overleftrightarrow{\nabla}_j A_{bj} \right) + K_3 \left(A^*_{bi} \overleftrightarrow{\nabla}_j A_{cj} + A^*_{cj} \overleftrightarrow{\nabla}_j A_{bi} \right) \right]. \end{aligned} \tag{4.6}$$

Eq. (he-) The orbital current can be written as

$$m j^{orb}_i = \epsilon_{ijk} \left(x^j T^{0k} - x^k T^{0i} \right) \tag{4.7}$$

Eq. (he-4.6) i.e.,

$$\mathbf{j}^{orb} \equiv \mathbf{x} \times \mathbf{j} \tag{4.8}$$

displaying the fact that the angular momentum density is the vector product of $\mathbf{x}$ with the momentum density $\mathbf{j}m$. The conservation laws Eq. (he-)

$$\nabla \cdot \mathbf{j} = 0 \tag{4.9}$$

Eq. (he-4.6a) of the first two currents follow from Noether's relation

$$\partial_i \frac{\partial \mathcal{L}}{\partial \partial_i \boldsymbol{\varphi}(x)} = \frac{\partial \mathcal{L}}{\partial \boldsymbol{\varphi}(x)}, \tag{4.10}$$

Eq. (he-4.6b)

$$\partial_i \frac{\partial \mathcal{L}}{\partial \partial_i \boldsymbol{\varphi}(x)} = \frac{\partial \mathcal{L}}{\partial \boldsymbol{\varphi}(x)}, \tag{4.11}$$

and the invariance under *spatially independent* symmetry transformations as expressed by the equation

$$\partial \mathcal{L} / \partial \boldsymbol{\varphi} = 0.$$

Similarly, one finds that the integral over (4.8), which is the total angular momentum

Eq. (he-)

$$\mathbf{L} = \int d^3x \; \mathbf{x} \times \mathbf{j}, \tag{4.12}$$

is a time-independent quantity. Since the invariance of the collective action under (4.1) is a direct consequence of the original fundamental action being invariant under separate phase, spin, and orbital rotations defined as Eq. (he-4.7)

$$\begin{aligned} \psi \;\; &\rightarrow \;\; e^{-i\varphi} \psi \\ \psi \;\; &\rightarrow \;\; e^{-i\boldsymbol{\varphi}^s \boldsymbol{\sigma}} \psi \\ \psi \;\; &\rightarrow \;\; e^{-\boldsymbol{\varphi}^o \boldsymbol{\epsilon}_{jk} (x_i \partial_j - x_j \partial_i)} \psi \end{aligned} \tag{4.13}$$

the currents (4.2), (4.5(he-4.4)), and (4.6(he-4.5)) are simply the collective versions of the fundamental Noether currents following from (4.13) Eq. (he-4.8)

$$\begin{aligned} j_i &\equiv \frac{1}{2mi}\psi^* \overleftrightarrow{\nabla}_i \psi, \\ j_{ai} &\equiv \frac{1}{2mi}\psi^* \sigma_a \overleftrightarrow{\nabla}_i \psi, \\ \mathbf{j}^{orb} &\equiv \mathbf{x}\times\mathbf{j}. \end{aligned} \tag{4.14}$$

Because of the invariance (4.1) and the quartic form of the collective action, the theory at hand is what is called a $3+1$ dimensional $SU(2)\times SU(2)\times U(1)$ -symmetric linear σ of model related to the Landau model of ferromagnetism (see Appendix 8A for a comparison).

When confronted with such a model, the discussion usually starts with the question for stable vacua. For this one examines small oscillations in the field A. Since, in the static case, the action $\mathcal{A}$ can be expressed in terms of the energy as Eq. (he-4.8a)

$$\mathcal{A} = -\int dtE = -\int dtd^3xe \tag{4.15}$$

small oscillations of A_{ai} around zero will be stable as long as Eq. (he-)

$$\mu_A = \frac{\mathcal{N}(0)}{3}\left(1-\frac{T}{T_c}\right) < 0, \qquad \text{i.e., } T > T_c. \tag{4.16}$$

As the temperature drops below the critical value T_c, the quadratic potential becomes unstable and the quartic term is needed to control the fluctuations. The field A_{ai} will settle at some new minimum away from zero. Unfortunately, no full mathematical analysis is available on the minima for all possible configurations of the coefficients β_i. Among the many minima discussed in the literature [24] (see Appendix 4A), there are three which apparently have been found in the laboratory associated with the phases which were shown in Fig. 1.3. Each of these is non-unique due to a residual degeneracy and can be parametrized as follows:

A **phase**

Eq. (he-4.10)

$$A^0_{ai} = \Delta_A d_a \left(\phi^{(1)} + i\phi^{(2)}\right)_i. \tag{4.17}$$

Here, $\mathbf{d}$, $\boldsymbol{\phi}^1$, $\boldsymbol{\phi}_i$ are arbitrary real unit vectors with $\boldsymbol{\phi}^{(1)}\perp\boldsymbol{\phi}^{(2)}$.

B **phase**

Eq. (he-4.11)

$$A^0_{ai} = \Delta_B R_{ai}(\hat{\mathbf{n}},\theta)e^{i\varphi}. \tag{4.18}$$

Here, R_{ai} is an arbitrary rotation around an axis $\hat{\mathbf{n}}$ by an angle θ with φ being some phase.

A_1 **phase**

Eq. (he-4.12)

$$A_{ai} = \Delta_{A_i}\left(d^{(1)} + id^{(2)}\right)_a \left(\phi^{(1)} + i\phi^{(2)}\right)_i \tag{4.19}$$

Here, $\mathbf{d}^{(1)}, \mathbf{d}^{(2)}$; $\boldsymbol{\phi}^{(1)}, \boldsymbol{\phi}^{(2)}$ are unit vectors with $\mathbf{d}^{(1)}\perp\mathbf{d}^{(2)}$ and $\boldsymbol{\phi}^{(1)}\perp\boldsymbol{\phi}^{(2)}$. The magnitudes of Δ are controlled by the free energy. This becomes in the three cases at hand: Eq. (he-4.13)

$$\begin{aligned} f_A &= -2\mu_A\Delta_A^2 + (\beta_2+\beta_4+\beta_5)\,4\Delta_A^4,\\ f_B &= -3\mu_A\Delta_B^2 + (\beta_1+\beta_2)\,g\Delta_B^4 + (\beta_3+\beta_4+\beta_5)\,3\Delta_B^4,\\ f_{A_1} &= -4\mu_A\Delta_{A_1}^2 + 16\,(\beta_2+\beta_4)\,\Delta_{A_1}^4, \end{aligned} \tag{4.20}$$

Eq. (he-4.14) and is minimized for $\mu_A < 0$ at the nonzero values

$$\begin{aligned} \Delta_A &= \sqrt{\frac{\mu_A}{4\beta_{245}}} = \pi_c\sqrt{\frac{10}{7\zeta(3)}}\sqrt{1-\frac{T}{T_c}},\\ \Delta_B &= \sqrt{\frac{\mu_A}{6\beta_{12}+2\beta_{345}}} = \pi T_c\sqrt{\frac{8}{7\zeta(3)}}\sqrt{1-\frac{T}{T_c}},\\ \Delta_{A_1} &= \sqrt{\frac{\mu_A}{8\beta_{24}}} = \pi T_c\sqrt{\frac{10}{7\zeta(3)}}\sqrt{1-\frac{T}{T_c}}, \end{aligned} \tag{4.21}$$

Eq. (he-4.15) where β_{ij}, $\beta_{ijk}, \dots$ are short for $\beta_i+\beta_j$, $\beta_i+\beta_j+\beta_k$, $\dots$. The minimal values are

$$\begin{aligned} f_A^{\min} &= \mu_A\Delta_A^2 = -\rho\left(1-\frac{T}{T_c}\right)\frac{1}{2\xi_0^2}\cdot\frac{5}{24},\\ f_B^{\min} &= -\frac{3}{2}\mu_A\Delta_A^2 = -\rho\left(1-\frac{T}{T_c}\right)\frac{1}{2\xi_0^2}\cdot\frac{1}{4},\\ f_{A_1}^{\min} &= -2\mu_A\Delta_{A_1}^2 = -\rho\left(1-\frac{T}{T_c}\right)\frac{1}{2\xi_0^2}\cdot\frac{5}{48}, \end{aligned} \tag{4.22}$$

respectively and the different combinations of β determine which minimum is the most stable depending on pressure and temperature (see Fig. 1.3).

The fluctuations around the new minima can be separated according to massive and massless ones. The massive oscillations all occur with a mass of the same order of magnitude as is found for the oscillations of Δ at the new minimum. This can be
Eq. (he-4.16) calculated as follows: Introducing

$$\Delta \to \Delta + \Delta' \tag{4.23}$$

Eq. (he-4.17) one finds for Δ' oscillations

$$f = f^{\min} + \delta^2 f \tag{4.24}$$

Eq. (he-4.18) with a mass term twice the opposite of that in (4.20):

$$\delta^2 f = 4\mu_A\Delta'^2 \tag{4.25}$$

The massive oscillations in directions other than Δ have the same type of mass term except that it is accompanied by a numerical factor (Clebsch-Gordon type of coefficient. The massless oscillations arise for small displacements of the *direction* vectors $\mathbf{d}$ and $\boldsymbol{\phi}$ and the phase φ characterizing the minima. They are called Goldstone bosons.

Group-theoretically, the following considerations are useful. The action is invariant and $SU(2)\times SU(2)\times U(1)$. The infinitesimal transformations of this group consist of those which change the directions of the minima and a subgroup leaving them invariant. The first ones coincide with the long-wavelength limit of Goldstone bosons oscillating around the new minimum. The mass of these oscillations is zero, since the action is invariant in the limit of infinite wavelength in which the small displacements become uniform rotations of $\mathbf{d}$, $\boldsymbol{\phi}$, φ. The subgroup of symmetry transformations which leave the directions at the minima invariant, on the other hand, will mix the Goldstone modes with each other. These transformations describe the residual symmetry left for the physics of the Goldstone modes.

The collective field A_{ai} has 18 parameters while A_{ai} has only 6, 5, or 7 parameters in A, B, and A_1 phase, respectively. The parametrization of the vacuum, therefore, does not allow to describe *all* massive oscillations (only those Δ are included).

In field theoretic considerations a particular direction di^0 is usually chosen as a vacuum of the theory. The freedom of taking an arbitrary direction corresponds to an infinite degeneracy of the possible vacua. In ^{3}He physics such a uniform choice is usually not possible since, as we shall see, boundary effects do not permit the ground state to settle in a uniform direction of the A^0_{ai} field. The "vacua" will be nontrivial. In addition to boundaries, also external fields[1] currents[2] and topology may serve to stabilize different non-uniform field configurations. The latter fact establishes links with present-day discussions of topologically interesting vacua in field theory.

As we have stressed repeatedly, we shall analyze the quantum liquid only with respect to those phenomena which take place at energies much smaller than the gap energy Δ. In this limit, all massive oscillations become unimportant (since their energy lies in the Δ regime). We can therefore assume Δ to be pinned down tightly at one of the degenerate extremal values (4A.1) and allow only for fluctuations of the *direction* of A^0_{ai}. This approximation, in which only the Goldstone modes are studied, is called the *hydrodynamic or London limit* of the theory. In σ-models of field theory, this corresponds to letting the mass of the σ-particle (the σ-oscillations) go to infinity. This limit leaves only the pion as a dynamical field in what is called a nonlinear σ-model. In the following, we shall restrict all our discussions to this hydrodynamic limit.

Appendix 4A Comparison with O(3)-Symmetric Linear σ-Model

For Comparison, we briefly describe the symmetry-breakdown in the simple $O(3)$-symmetric σ-model, also known as the *classical Heisenberg model* of ferromagnetism. There the free-energy density reads, for constant fields, Eq. (he-4.19)

$$f = \frac{\mu^2}{2}\left(\pi_1^2+\pi_2^2+\pi_3^2\right) + \frac{\lambda}{4}\left(\pi_1^2+\pi_2^2+\pi_3^2\right)^2 . \tag{4A.1}$$

[1]For a general discussion and references see H. Kleinert, Fortschr. Phys. *26*, Nr. 9/10, 1978.

[2]Non-trivial helix-like vacua in the presence of currents have been found in: H. Kleinert, Y.R. Lin-Liu and K. Maki; Paper presented at the 1978 Grenoble Conference on Low Temperature Physics, and USC Preprints, March 1978.

Eq. (he-4.20) For $\mu^2 < 0$, this has the following set of degenerate minima:

$$\pi_i^0 \equiv \Delta^0 d_i^0 \quad \text{with} \qquad \Delta^0 = \sqrt{-\frac{\mu^2}{\lambda}}, \tag{4A.2}$$

where d_i^0 is an arbitrary unit vector in three-space. The oscillations of $\pi_i \equiv \Delta d_i$
Eq. (he-4.21) around π_i^0 consist of massive radial oscillations in Δ controlled by

$$f = -\frac{\mu^4}{4\lambda} + (-\mu^2)(\Delta - \Delta^0)^2 \tag{4A.3}$$

and massless oscillations of d_i around the direction of d_i^0. If d_i^0 points along the
Eq. (he-) 3−axis, these oscillations can be parametrized as

$$d_i = \left(\frac{\pi_i'}{\Delta}, \sqrt{1 - \frac{\boldsymbol{\pi}'^2}{\Delta^2}} \right). \tag{4A.4}$$

The energy depends only on Δ. Rotations leaving d_i^0 invariant transform π_1', π_2' among each other and correspond to the residual $O(2)$ symmetry after spontaneous symmetry breakdown of the original $O(3)$. The situation here is simpler than for ^{3}He since the parametrization $\pi_i = \Delta d_i$ of the ground state can be used to cover the *entire* three-dimensional field space.

Appendix 4B Other Possible Phases

By factorizing the order parameter A_{ai} into a size Δ and a "direction" $\hat{A}_{ai}$,

$$A_{ai} = \Delta \hat{A}_{ai}, \tag{4B.1}$$

with

$$\mathrm{tr}\left(\hat{A}^\dagger \hat{A}\right) \equiv \sum_{a,i} \hat{A}_{ai}^* \hat{A}_{ai} = 1 \tag{4B.2}$$

the free-energy density associated with the action (3.22) is

$$f = -\mu_A \Delta^2 + \beta_0 \beta \Delta^4 \tag{4B.3}$$

where

$$\begin{aligned} \beta_0 \quad &\equiv \quad \frac{3}{5} \frac{\rho^*}{m^{*2}} \frac{\xi_0^2}{\upsilon_F^{*\,4}} \\ \beta \quad &\equiv \quad \sum_{i=1}^{5} \beta_i t_i \end{aligned} \tag{4B.4}$$

with t_i being the traces

$$
\begin{aligned}
t_1 &= |\mathrm{tr}(\hat{A}\hat{A}^T)|^2 \\
t_2 &= \left[\mathrm{tr}(\hat{A}\hat{A}^\dagger)\right]^2 \\
t_3 &= \mathrm{tr}\left[(\hat{A}^\dagger\hat{A})(\hat{A}^\dagger\hat{A})^*\right]^2 \\
t_4 &= \mathrm{tr}(\hat{A}\hat{A}^\dagger)^2 \\
t_5 &= \mathrm{tr}\left[(\hat{A}\hat{A}^\dagger)(\hat{A}\hat{A}^\dagger)^*\right].
\end{aligned}
\tag{4B.5}
$$

Minimizing e with respect to Δ we find

$$\Delta^2 = \frac{\mu_A}{2\beta_0\beta} \tag{4B.6}$$

so that

$$f_{\min} = -\frac{\mu_A^2}{4\beta_0\beta}. \tag{4B.7}$$

We now search for extremal points common to all t_i, $i = 1, \ldots, 5$. Their direction vectors $\hat{A}_{ai}$ will be independent of the size of β_i. The associated phases are called *inert*.

1) In t_1 we write

$$\hat{A}_{ai} = x_{ai} + iy_{ai} \tag{4B.8}$$

so that it becomes

$$t_1 = X^2 + Y^2 \tag{4B.9}$$

with

$$
\begin{aligned}
X &= \sum_{a,i} x_{ai}^2 - y_{ai}^2 \\
Y &= 2\sum_{a,i} x_{ai}y_{ai}.
\end{aligned}
\tag{4B.10}
$$

In terms of x_{ai}, y_{ai}, the constraint (4B.2) reads

$$\sum_{a,i}\left(x_{ai}^2 + y_{ai}^2\right) = 1 \tag{4B.11}$$

Introducing a Lagrange multiplyer σ the extremality conditions read

$$(X+\sigma)x_{ai} + Yy_{ai} = 0 \tag{4B.12}$$

$$Yx_{ai} + (-Xx\sigma)y'ai = 0 \tag{4B.13}$$

Since (4B.11) forces at least one x_{ai}, y_{ai} to be non-zero, the determinant must vanish:

$$\sigma^2 = X^2 + Y^2. \tag{4B.14}$$

Multiplying (4B.12) with x_{ai}, (4B.13) with y_{ai}, and subtracting the two equations from each other gives

$$(X+\sigma)x_{ai}^2 - (X+\sigma)y_{ai}^2 = 0. \tag{4B.15}$$

Summing over ai leads to

$$X(1+\sigma) = 0. \tag{4B.16}$$

A similar operation in opposite order produces

$$Y(1+\sigma) = 0 \tag{4B.17}$$

Thus one has the alternative $\sigma \neq -1$ and $X = Y = 0$ implying $t_1 = 0$, or $\sigma = -1$ and therefore, due to (4B.14), $t_1 = 1$. In conclusion, t_1 has two stationary points

$$t_1\big|_{\rm ext} = \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\}. \tag{4B.18}$$

Notice that $t_1 = 1$ is possible only if A is real up to an overall phase.

2) The second invariant t_2 is always equal to 1 due to the normalization condition (4B.11).

3) The invariant t_4 can be extremized as follows: An unitary transformation brings the hermitian matrix $\hat{A}^\dagger \hat{A}$ to the diagonal form

$$\hat{A}^\dagger \hat{A} = \Lambda \equiv \left(\begin{array}{ccc} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{array} \right) \tag{4B.19}$$

with non-negative eigenvalues

$$\lambda_i \geq 0. \tag{4B.20}$$

In terms of λ_i, the invariant t_4 reads

$$t_4 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2. \tag{4B.21}$$

It has to be extremized with respect to the constraint (4B.2):

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 \tag{4B.22}$$

This is a diagonal plane in three dimensions. The quantity t_4 is the square of the radius for each of the points on this plane. Its minimum is clearly the symmetric point where the smallest possible sphere touches the plane

$$\lambda_i = \frac{1}{3}, \qquad t_4 = \frac{1}{3} \tag{4B.23}$$

Its maximum is any of the end points (up to permutations)

$$\lambda_1 = 1, \quad \lambda_2 = \lambda_3 = 0, \qquad t_4 = 1. \tag{4B.24}$$

There is a saddle point when the sphere is tangent to the diagonal line in each of the coordinate planes (up to permutations)

$$\lambda_1 = \lambda_2 = \frac{1}{2}, \quad \lambda_3 = 0, \qquad t_4 = \frac{1}{2} \tag{4B.25}$$

Thus t_4 can have the extremal values

$$t_4\big|_{\text{ext}} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \\ 1 \end{pmatrix} \tag{4B.26}$$

4) The invariants t_3 and t_5 can be treated together since they differ only by exchanging $\hat{A}^\dagger \hat{A}$ by $\hat{A}\hat{A}^\dagger$ and none of the subsequent statements will depend on this. Consider

$$t_3 = \text{tr}\left[(\hat{A}^\dagger \hat{A})(\hat{A}^\dagger \hat{A})^*\right]. \tag{4B.27}$$

By a unitary transformation this takes the form

$$t_3 = \text{tr}\left[U\hat{A}^\dagger \hat{A}U^\dagger (U^*(\hat{A}^\dagger \hat{A})^* U^T)^*\right]. \tag{4B.28}$$

Suppose U is chosen to reach the diagonal form

$$U\hat{A}^\dagger \hat{A}U^\dagger = \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}. \tag{4B.29}$$

Then

$$t_3 = \text{tr}\left(\Lambda U U^T \Lambda (U U^T)^*\right) \tag{4B.30}$$

Let us denote the matrix UU^T by V, for brevity. Then

$$t_3 = \sum_{n,m} \lambda_n \lambda_m |V_{nm}|^2 \tag{4B.31}$$

to be extremized with the constraint (4B.22) and

$$\sum_{n,m} \sigma_{mn} \left(V_{mp} V^*_{mp} - \delta_{nm} \right) = 0 \tag{4B.32}$$

which guarantees unitarity of V. Using σ and σ_{nm} as Lagrange multiplyers this leads to the equations

$$\begin{aligned} \sum_m \lambda_m |V_{nm}|^2 + \sigma\lambda_n &= 0, &(4B.33)\\ \lambda_n\lambda_p V_{np} + \sum_m \sigma_{nm} V_{mp} &= 0, &(4B.34)\\ \lambda_n\lambda_p V^*_{np} + \sum_m \sigma_{mn} V^*_{mp} &= 0. &(4B.35)\end{aligned}$$

Comparing the last two equations we see that σ_{mn} must be a hermitian matrix,

$$\sigma_{mn} = \sigma^*_{nm}. \tag{4B.36}$$

We now take advantage of the unitarity of V by multiplying (4B.34) with V^*_{kp} and summing over p, yielding

$$\sum_p \lambda_k \lambda_p V_{np} V^*_{kp} + \sigma^*_{kn} = 0. \tag{4B.37}$$

Using (4B.30) we conclude

$$\sum_p (\lambda_n - \lambda_k) V_{np} \lambda_p (V^\dagger)_{pk} = 0 \tag{4B.38}$$

If the matrix elements λ_i are all different this implies that also $V\Lambda V^\dagger$ is diagonal. If two or more of the λ_i's are equal to each other we can choose a new basis on the degenerate subspace so that $V\Lambda V^\dagger$ is diagonal. Supposing that we have chosen such a basis then

$$U\hat{A}^\dagger\hat{A}U^\dagger = \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \tag{4B.39}$$

and

$$U(\hat{A}^\dagger\hat{A})^* U^\dagger = \hat{\Lambda} = \begin{pmatrix} \tilde{\lambda}_1 & & \\ & \tilde{\lambda}_2 & \\ & & \tilde{\lambda}_3 \end{pmatrix}. \tag{4B.40}$$

Since $\hat{A}^\dagger\hat{A}$ is real, the eigenvalues λ_i , and $\tilde{\lambda}_i$ can only differ by permutations

$$\tilde{\lambda}_i = \lambda_{p(i)}. \tag{4B.41}$$

In principle, this leaves 6 choices but out of these two can be ruled out: If v_i are the complex eigenvectors of $\hat{A}^\dagger \hat{A}$, i.e.,

$$\hat{A}^\dagger \hat{A}\, v_i = \lambda_i\, v_i, \tag{4B.42}$$

then v_i are eigenvectors of $(\hat{A}^\dagger \hat{A})^*$. Suppose now that $(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3) = (\lambda_2, \lambda_3, \lambda_1)$ or $(\lambda_3, \lambda_1, \lambda_2)$ with all λ_i's being different from each other. In the first case

$$v_1^* = v_2, \quad v_2^* = v_3, \quad v_3^* = v_1 \tag{4B.43}$$

which leads to the contradiction

$$v_1 = v_3. \tag{4B.44}$$

In the second case

$$v_1^* = v_3, \quad v_2^* = v_1, \quad v_3^* = v_2 \tag{4B.45}$$

and this is ruled out by

$$v_1 = v_2. \tag{4B.46}$$

Thus only four possibilities remain:

$$(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3) = (\lambda_1, \lambda_2, \lambda_3) \tag{4B.47}$$

or

$$(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3) = (\lambda_2, \lambda_1, \lambda_3)\,, \tag{4B.48}$$

each with 2 possible cyclic permutations. The first case gives

$$t_3 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \tag{4B.49}$$

and thus leads to the same extrema as for t_4:

$$t_3 = \frac{1}{3}, \frac{1}{2}, 1. \tag{4B.50}$$

In the second case one has to extremize

$$t_3 = 2\lambda_1 \lambda_2 + \lambda_3^2 \tag{4B.51}$$

on the diagonal plane (4B.22). Within this plane this gives

$$\lambda_i + \sigma = 0$$

and therefore

$$\lambda_i = \frac{1}{3}, \qquad t_4 = \frac{1}{3}. \tag{4B.52}$$

This represents no new extremal value. On the diagonal boundary lines in the coordinate planes, say λ_1 (or $\lambda_2 = 0$) one has

$$t_3 = \lambda_3^2, \tag{4B.53}$$

whose smallest value is $t_3 = 0$ while the largest value is 1, already obtained earlier. On the boundary line $\lambda_3 = 0$ one finds once more $t_3 = 0$.

Thus, altogether t_3 has four possible extrema

$$t_3|_{\rm ext} = \begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{2} \\ 1 \end{pmatrix}. \tag{4B.54}$$

The same result holds for t_5:

$$t_5|_{\rm ext} = \begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{2} \\ 1 \end{pmatrix}. \tag{4B.55}$$

The question which of these extrema can be realized by a single matrix $\hat{A}_{ai}$ can be decided as follows. First of all it is certainly true that

$$t_4 \geq t_5 \tag{4B.56}$$

since in the above diagonalization

$$\begin{aligned} t_4 &= \sum_i \lambda_i^2 \\ t_5 &= \sum_i \lambda_i \lambda_{p(i)}. \end{aligned} \tag{4B.57}$$

Second, by going through the same arguments as for t_4 we can see that the traces

$$\begin{aligned} t_{X^2} &= {\rm tr}(X^2) \\ t_{Y^2} &= {\rm tr}(Y^2) \end{aligned} \tag{4B.58}$$

of the two auxiliary matrices

$$\begin{aligned} X &= [\hat{A}\hat{A} + (\hat{A}\hat{A})^*]/2 \\ Y &= [\hat{A}^\dagger \hat{A} + (\hat{A}^\dagger \hat{A})^*]/2 \end{aligned} \tag{4B.59}$$

have the same extremely values

$$t_{X^2}|_{\rm ext} = t_{Y^2}|_{\rm ext} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \\ 1 \end{pmatrix}. \tag{4B.60}$$

But it is easy to check that

$$\begin{aligned} t_{X^2} &= (t_4 + t_5)/2, \\ t_{Y^2} &= (t_4 + t_3)/2. \end{aligned} \tag{4B.61}$$

Third, the stationary value $t_1 = 1$ is reached only if all $y_{ai} = 0$, i.e., if $\hat{A}_{ai}$ is real up to a phase and therefore $t_3 = t_4 = t_5$.

Thus, for $t_1 = 1$ we remain with the following three possibilities for simultaneous extrema of $(t_1, t_2, t_3, t_4, t_5)$:

1) B phase: $(1, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
2) Planar phase: $(1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
3) Polar phase: $(1, 1, 1, 1, 1)$

For $t_1 = 0$ there are five more possibilities:
4) α phase: $(0, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
5) Bipolar phase: $(0, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
6) A(xial) phase: $(0, 1, 0, 1,)$
7) β phase: $(0, 1, 1, 1, 0)$
8) γ phase: $(0, 1, 0, 1, 0)$

Actually there is one more combination

$$(0, 1, 1, 1, 1)$$

which is compatible with the above conditions but can be ruled out on the following grounds: If $t_3 = t_5 = 0$ the matrix $\hat{A}^\dagger \hat{A}$ can be chosen to have the form

$$(\hat{A}\hat{A}^\dagger) = \left(\hat{A}^\dagger \hat{A}\right) = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}. \tag{4B.62}$$

The only matrix $\hat{A}$ compatible with this is

$$\hat{A}_{ai} = e^{i\phi} \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}_{ai} \tag{4B.63}$$

which has $t_1 = 1,\ t_2 = 1$ corresponding again to the polar phase. For the Ginzburg-Landau values

$$-2\beta_1 = \beta_2 = \beta_3 = \beta_4 = -\beta_5 = 1 \tag{4B.64}$$

the extrema have the free energy

$$f = -f_c \, \frac{1}{\beta} \tag{4B.65}$$

where

$$f_c \equiv -\frac{\mu_A^2}{4\beta_0}. \tag{4B.66}$$

The factor $1/\beta$ becomes, in the different phases,

1) B: $\beta_B^{-1} = \left(\beta_{12} + \frac{1}{3}\beta_{345}\right)^{-1} = \frac{6}{5}$

2) P: $\beta_P^{-1} = \left(\beta_{12} + \frac{1}{2}\beta_{345}\right)^{-1} = 1$

3) Pol: $\beta_{Pol} = (\beta_{12} + \beta_{345})^{-1} = \frac{2}{3}$

4) α : $\beta_\alpha^{-1} = \left(\beta_2 + \frac{1}{3}\beta_{345}\right)^{-1} = \frac{3}{4}$

5) Bip: $\beta_{Bip}^{-1} = \left(\beta_2 + \frac{1}{2}\beta_{345}\right)^{-1} = \frac{2}{3}$

6) A: $\beta_A^{-1} = \beta_\beta^{-1} = (\beta_2 + \beta_{34})^{-1} = \frac{1}{3}$

7) β : $\beta_\beta^{-1} = (\beta_2 + \beta_{34})^{-1} = \frac{1}{3}$

8) γ : $\beta_\gamma^{-1} = \beta_{24}^{-1} = \frac{1}{2}$

Obviously, only the B phase is absolutely stable. The phases A and P are the next higher ones.

We are now going to find convenient parametrizations for the direction "vectors" $\hat{A}_{ai}$:

1) In the B phase, $\hat{A}\hat{A}^\dagger$ can be brought to the form

$$\hat{A}\hat{A}^\dagger = \frac{1}{3}\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = \hat{A}^\dagger\hat{A}. \tag{4B.67}$$

The most general matrix which allows for this is

$$\hat{A}_{ai} = e^{i\phi}R_{ai}(\hat{\mathbf{n}}\theta) \tag{4B.68}$$

where $R(\hat{\mathbf{n}}\theta)$ is a rotation by an angle θ around the direction $\hat{\mathbf{n}}$ and $e^{i\phi}$ is some phase. A convenient special form is

$$\hat{A}_{ai} = \frac{1}{3}\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \tag{4B.69}$$

2) In the planar phase, $t_1 = 1$ so that A=real (up to an overall phase). Since $AA^\dagger$ and $A^\dagger A$ have both the eigenvalues $\frac{1}{2}, \frac{1}{2}, 0$, the most general representation of $\hat{A}_{ai}$ is

$$\hat{A}_{ai} = \frac{1}{\sqrt{2}}e^{i\phi}(d_a^{(1)}\phi_i^{(1)} + d_a^{(2)}\phi_i^{(2)}) \tag{4B.70}$$

where $\mathbf{d}^{(1)} \cdot \mathbf{d}^{(2)} = 0$, $\boldsymbol{\phi}^{(1)} \cdot \boldsymbol{\phi}^{(2)} = 0$ are real orthogonal unit vectors. For example

$$\hat{A}_{ai} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4B.71}$$

3) In the polar phase A must be real up to a phase with $\hat{A}^\dagger \hat{A}$, $\hat{A}\hat{A}^\dagger$ having the eigenvalues $(1, 0, 0)$. This leaves

$$\hat{A}_{ai} = e^{i\phi} d_a \phi_i \tag{4B.72}$$

with two unit vectors $\mathbf{d}$ and $\boldsymbol{\phi}$. For example

$$\hat{A}_{ai} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4B.73}$$

4) In the α phase one can show that a typical direction is

$$\hat{A}_{ai} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & \\ & e^{i\pi/3} & \\ & & e^{-i\pi/3} \end{pmatrix} \tag{4B.74}$$

which can be transformed to other forms by phase $e^{i\phi}$ and by two different spin and orbital rotations of a and i, respectively.

5) In the bipolar phase one can show that $\hat{A}_{ai}$ has the general form

$$\hat{A}_{ai} = \frac{1}{\sqrt{2}} \left(d_a^{(1)} \mathbf{\Phi}_i^{(1)} + d_a^{(2)} \mathbf{\Phi}_i^{(2)} \right) \tag{4B.75}$$

where $\mathbf{d}$ are real and $\mathbf{\Phi}_i^{(1)}$, $\mathbf{\Phi}_i^{(2)}$ are complex orthogonal unit vectors. For example

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4B.76}$$

6) In the A phase, the matrices $\hat{A}^\dagger \hat{A}$ and $\hat{A}\hat{A}^\dagger$ have eigenvalues $(1, 0, 0)$. The most general $\hat{A}_{ai}$ with this property is

$$\hat{A}_{ai} = a_a b_i \tag{4B.77}$$

where $\mathbf{a}$ and $\mathbf{b}$ are two complex unit vectors. Then

$$\begin{aligned} (\hat{A}\hat{A}^\dagger)_{ab} &= a_a a_b^* \\ \left(A^\dagger A \right)_{ij} &= b_i^* b_j \end{aligned} \tag{4B.78}$$

and we find

$$\begin{aligned} t_3 &= b_i^* b_j b_j b_i^* = |\mathbf{b}^2|^2 \\ t_5 &= a_a a_b^* a_b^* a_a = |\mathbf{a}^2|^2. \end{aligned} \tag{4B.79}$$

This implies that[3]

$$t_1 = \left| \sum_{a,i} a_a b_i b_i a_a \right|^2 = |\mathbf{a}^2|^2 |\mathbf{b}^2|^2 = t_3 t_5. \tag{4B.80}$$

Now we observe that since $|\mathbf{a}|^2$ and $t_5 = 1$, $\mathbf{a}$ has to be a real unit vector for which we use again the symbol $\mathbf{d}$. Similarly, since $t_3 = 0$, b_i have to be complex with

$$\mathbf{b}^2 = 0.$$

Hence we may express $\mathbf{b}$ in terms of two real unit vectors

$$b_i = \left(\phi_i^{(1)} + i\phi_i^{(2)} \right) / \sqrt{2}$$

which are orthogonal to each other, $\boldsymbol{\phi}^{(1)} \cdot \boldsymbol{\phi}^{(2)} = 0$. Thus

$$\hat{A}_{ai} = \frac{1}{\sqrt{2}} d_a \left(\phi_i^{(1)} + i\phi_i^{(2)} \right). \tag{4B.81}$$

A particular example is

$$\hat{A}_{ai} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4B.82}$$

1) The β phase has t_3 and t_5 and thus the role of spin and orbital indices interchanged, i.e.,

$$\hat{A}_{ai} = \frac{1}{\sqrt{2}} \left(d^{(1)} + i d^{(2)} \right)_a \phi_i \tag{4B.83}$$

where $\mathbf{d}^{(1)}, \mathbf{d}^{(2)}$ are two orthogonal real unit vectors.

If the condition of simultaneous stationarity is relaxed the minimization problem has not yet been solved. If one allows the matrix A_{ai} to take restricted generic forms with some elements identical to zero and requires that a small deviation of these elements from zero does not cause linear changes in any of the invariants t_i only two forms remain as candidates

$$\hat{A}_{ai} = \begin{pmatrix} a & 0 & d \\ 0 & b & 0 \\ 0 & c & 0 \end{pmatrix} \left(\text{or equivalently} \begin{pmatrix} 0 & a & d \\ b & 0 & 0 \\ c & 0 & 0 \end{pmatrix} \right) \tag{4B.84}$$

or

$$\hat{A}_{ai} = \begin{pmatrix} a & 0 & d \\ 0 & b & 0 \\ e & 0 & c \end{pmatrix}. \tag{4B.85}$$

[3]Exactly for this reason the combination $(0, 1, 1, 1, 1)$ is outruled.

By calculating f_c we find in the first case the following set of new phases

9) Axi-planar phase

$$\hat{A} = \begin{pmatrix} \cos\theta & 0 & -i\sin\theta\cos\phi \\ 0 & \sin\theta\sin\phi & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4B.86}$$

Its name is due to the fact that it interpolates between the axial phase at $\phi = 0, \theta = \pi/4$ and the planar phase at $\phi = \pi/2,\ \theta = \pi/4$. The extremum lies at

$$\tan^2\phi = 2\beta_{13}\beta_{45}/\beta_{345}\,(2\beta_1 + \beta_3)\,, \tag{4B.87}$$

$$\cos^2\theta = \frac{1}{2}\beta_3\,(2\beta_1 + \beta_{345})\,/\,[\beta_3(2\beta_1 + \beta_{345}) + \beta_{45}/2\beta_1 + \beta_3)]\,. \tag{4B.88}$$

The energy has a value f_c of (4B.7) with

$$\beta_{AP}^{-1} = \frac{\beta_{34} + S}{\beta_2\beta_{34} + S\beta_{234}}, \qquad S = \beta_{345} + \frac{\beta_1\beta_3}{\beta_{13}}. \tag{4B.89}$$

In the Ginzburg-Landau regime, $S = 0$ and

$$\beta_{AP}^{-1} = 1 \tag{4B.90}$$

implying that this phase is degenerate with axial and planar phases. This is, in fact true for any choice of the angle ϕ, θ which manifest itself in both ratios (4B.88) being undefined [equal to (0/0)].

The extremum is minimal with respect to variations in ϕ, θ if

$$\beta_{45} > 0,\ S > 0, \quad \beta_1\beta_{45} < \beta_{13}S \tag{4B.91}$$

10) β-Planar Phase
By interchanging the rows and columns (i.e., the spin and orbital indices), the matrix (4B.81) interpolates between the planar and the β phase. The algebras the same except for β_3 and β_5 interchanged. In the Ginzburg-Landau regime, the energy is determined by

$$\beta_{\beta P}^{-1} = \frac{1}{3} \tag{4B.92}$$

and thus lies higher than A and B phase. The angles θ and θ are fixed to zero and $\pi/2$.

11) ϵ-Phase
The matrix

$$\hat{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sin\theta & 0 & i\sin\theta \\ 0 & \cos\theta & 0 \\ 0 & i\cos\theta & 0 \end{pmatrix} \tag{4B.93}$$

is an extremum for

$$\tan^2\theta = \beta_{34}/\beta_{45} \tag{4B.94}$$

with

$$\beta_\epsilon^{-1} = \frac{\beta_{34}+\beta_{45}}{\beta_2\left(\beta_{34}+\beta_{45}\right)+\beta_{34}\beta_{45}}. \tag{4B.95}$$

In the Ginzburg-Landau regime, $\theta = \pi/2$ and this phase coincides with the A phase.

These are all solutions of the generic form a). A similarly complete list for the form b) does not exist. One additional extremum is known, the

12) ζ-phase

$$\hat{A}_{ai} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sin\theta\cos\phi & 0 & -i\sin\theta\sin\phi \\ 0 & \sqrt{2}\cos\phi & 0 \\ -i\sin\phi\sin\phi & 0 & \sin\theta\cos\phi \end{pmatrix}. \tag{4B.96}$$

The angles have to be taken as

$$\begin{aligned} \tan^2\theta &= \left(T\beta_1+\beta_{1345}\right)/\beta_4, \\ T &\equiv 2\beta_1\left(\beta_4-\beta_{135}-\beta_1\right), \\ \cos 2\phi &= T/\tan^2\theta, \end{aligned} \tag{4B.97}$$

and the extremal energy is determined by

$$\beta_\zeta^{-1} = \frac{T\beta_1+\beta_{1345}+\beta_4}{\left(T\beta_1+\beta_{1345}\right)\beta_4} + \beta_2. \tag{4B.98}$$

The extremum is minimal if

$$\beta_4 > 0, \quad T\beta_1+\beta_{1345} > 0, \; |T\beta_1\beta_4| \le |\left(T\beta_1+\beta_{1345}\right)\beta_1|. \tag{4B.99}$$

5

Hydrodynamic Limit Close to T_c

In the hydrodynamic limit the only degrees of freedom of the liquid consist in ground state configurations A^0_{ai} with slow spatial variations of the directional vectors. In the A-phase, in which Eq. (he-5.1)

$$\begin{aligned} A^0_{ai} &= \Delta_A d_a \phi_i \\ \phi_i &\equiv \phi_i^{(1)} + i\phi_i^{(2)}, \end{aligned} \tag{5.1}$$

where $\phi_i^{(1)}$ and $\phi_i^{(2)}$ are orthogonal unit vectors, the magnitude Δ_A is pinned down at the potential minimum (4.22) with a value (4.21(he-4.14)). The unit vectors d_a and $\phi_i^{(1)}$, $\phi_i^{(2)}$ vary in space. It is useful to visualize the physical meaning of these directions. Remembering the relation (4.2) expressing the collective field A_{ai} in terms of the pair of fundamental fields, the vector d_a indicates the direction along which the spin has the wave function $\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$, i.e., along which its third component vanishes.[1] The plane in which the Cooper pair moves is given by the plane spanned by the unit vectors $\boldsymbol{\phi}^{(1)}$ and $\boldsymbol{\phi}^{(2)}$. It has become customary to introduce a vector Eq. (he-5.2)

$$\mathbf{l} \equiv \boldsymbol{\phi}^{(1)} \times \boldsymbol{\phi}^{(2)} \tag{5.2}$$

which denotes the direction of the intrinsic orbital angular momentum of the Cooper pairs in the condensate. For the completeness of the description one has to specify, in addition, the azimuthal angle α of $\phi^{(1)}$ in the plane orthogonal to 1. This specification can be made unique, for example, by the following choice of parametrization:

Eq. (he-5.3)

$$\begin{aligned} \boldsymbol{\phi} &\equiv \boldsymbol{\phi}^{(1)} + i\boldsymbol{\phi}^{(2)} && (5.3)\\ &= e^{-i\alpha}\left\{(-\sin\gamma,\ \cos\gamma, 0) + i(-\cos\beta\cos\gamma,\ -\cos\beta\sin\gamma, \sin\beta)\right\}, \\ \mathbf{l} &\equiv (\sin\beta\cos\gamma,\ \sin\beta\sin\gamma, \cos\beta)\,. && (5.4) \end{aligned}$$

Consider now the energy density of (3.22) and (3.24) when expressed in terms of the parametrization of the hydrodynamic limit, (5.1). The derivative terms in (3.24) yield Eq. (he-5.5)

[1] To verify this, let $d_a = (0,0,1)_a$ then $d_a(c\sigma^a)_{\alpha\beta} = -(\uparrow_\alpha\downarrow_\beta + \downarrow_\alpha\uparrow_\beta)$ where $\uparrow_\alpha,\ \downarrow_\beta$ are the spin $\frac{1}{2}$ two-spinors with spin up and down, respectively.

$$e = \frac{1}{2}\Delta_A^2 \left\{ K_1|\nabla_i\phi_j|^2 + K_2\nabla_i\left[\phi_j^\dagger\nabla_j\phi_i - \phi_i^\dagger\nabla_j\phi_j\right] + K_{23}|\nabla_i\phi_i|^2 + K_{23}|\boldsymbol{\phi}\cdot\boldsymbol{\nabla}d_a|^2 + 2K_1(\nabla_i d_a)^2 \right\}, \tag{5.5}$$

with the notation $K_{12} \equiv K_2 + K_3$. The last term is a pure divergence and can be neglected in most discussions. Since the magnitude of all directional vectors is unity, the mass term in (3.24) and the quartic term (3.22(he-3.18)) add up to the minimal values given in (4.22). Since Δ is tightly pinned down at that minimal value, any deviation of the energy from this minimum is completely determined by the derivative terms of (5.5). These vanish for uniform field configurations and increase with increasing bending of the field lines. For this reason, the energy (5.5) is often referred to as *bending energy*.

The factor $\frac{1}{2}\Delta_A^2$ in front can be brought to a physically more transparent form: Using (2.20) and (4.21), we find in the weak-coupling limit:

Eq. (he-5.6)

$$\begin{aligned} \frac{1}{2}\Delta_A^2 &= \frac{1}{2}\frac{\mu_A}{4\beta_{245}} = \frac{1}{2}\frac{1}{3}\mathcal{N}(0)\frac{1}{\frac{8}{5}\mathcal{N}(0)\xi_0^2/v_F^2}\left(1-\frac{T}{T_c}\right) = \frac{5}{48}\frac{v_F^2}{\xi_0^2}\left(1-\frac{T}{T_c}\right) \\ &= \frac{1}{16m^2}\rho\left(1-\frac{T}{T_c}\right)\frac{1}{K_1} \end{aligned} \tag{5.6}$$

where ρ is the mass density of ^{3}He particles per unit volume. Now, if a collective excitation of wave vector $\mathbf{k}$ runs through the liquid, its energy density per particle is of the order $(\mathbf{k}^2/2m)\,(1 - T/T_c)$. It grows with decreasing temperature due to the increasing condensation energy.

If one uses, instead of the complex vector $\boldsymbol{\phi}$, the more physical vector $\mathbf{l}$ of (5.4), one can express the energy density in a somewhat more intuitive fashion. For this we define a gradient vector called the *macroscopic superfluid velocity*:

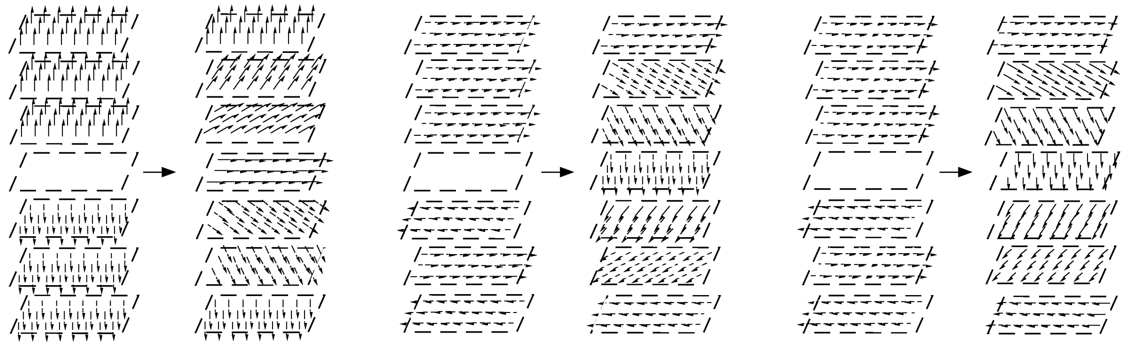
$$v_{si} = \frac{1}{2m}\boldsymbol{\phi}^{(1)}\nabla_i\boldsymbol{\phi}^{(2)} = \frac{i}{4m}\boldsymbol{\phi}^\dagger\nabla_i\boldsymbol{\phi} \tag{5.7}$$

Then e takes a form (see Appendix 5A)

$$\begin{aligned} f &= \frac{1}{2}\rho_s\mathbf{v}_s^2 - \frac{1}{2}\rho_0(\mathbf{l}\cdot\mathbf{v}_s)^2 + c\mathbf{v}_s\cdot(\boldsymbol{\nabla}\times\mathbf{l}) - c_0(\mathbf{l}\cdot\mathbf{v}_s)\left[\mathbf{l}\cdot(\boldsymbol{\nabla}\times\mathbf{l})\right] \\ &+\frac{1}{2}K_s(\boldsymbol{\nabla}\cdot\mathbf{l})^2 + \frac{1}{2}K_t\left[\mathbf{l}\cdot(\boldsymbol{\nabla}\times\mathbf{l})\right]^2 + \frac{1}{2}K_b\left[\mathbf{l}\times(\boldsymbol{\nabla}\times\mathbf{l})\right]^2 \\ &+\frac{1}{2}K_1^d(\nabla_i d_a)^2 - \frac{1}{2}K_2^d\,(\mathbf{l}\cdot\nabla d_a)^2 \end{aligned} \tag{5.8}$$

with coefficients

$$\begin{aligned} \rho_s &= \Delta_A^2(K_1 + \tfrac{1}{2}K_{23})\,4m^2, \\ \rho_0 &= \Delta_A^2 K_{23}\,4m^2, \\ c &= \Delta_A^2 K_3 2m, \\ c_0 &= \rho_0/2m, \end{aligned}$$



(a) (b) (c)

FIGURE 5.1 Three fundamental planar textures: splay (a), bend (b), and twist (c). The left-hand side of the figure shows field configurations with a singular plane where the fields reverse direction. Since the superfluid would have to be normal in this plane, it prefers the right-hand configuration in which the direction changes smoothly through a domain wall of finite size. The thickness $\xi_{\rm d}$ is determined by the competition of the dipole and bending energy.

$$\begin{aligned} K_s &= K_t = \Delta_A^2 K_1, \\ K_b &= \Delta_A^2 (K_a + K_{23}), \\ K_1^d &= \rho_s/4m^2, \\ K_2^d &= \rho_0/4m^2. \end{aligned} \tag{5.9}$$

With the weak-coupling results (3.26), (4.21), these expressions simplify to Eq. (he-5.10)

$$\begin{aligned} \frac{\rho_s}{2} &= \rho_0 = 2mc_0 = 4mc \qquad (5.10) \\ &= (2m)^2 2K_s = (2m)^2 2K_t = (2m)^2 \frac{2}{3} K_b = (2m)^2 \frac{1}{2} K_1^d = (2m^2) K_2^d \end{aligned}$$

The indices on K_s, b, t stand for splay, bend and twist. This nomenclature is taken from the theory of liquid crystals [25]. In that theory, a vector field which bends in the way shown in Figs. 5.1a-c) is called a splay, bend, and twist texture, respectively. Fig. V
It is easily seen that in these cases the terms with K_s, K_b, and K_t give, indeed, the dominant contributions. For, if the spatial changes of the **1** field take place only in z directions one can write (with (5.8)) Eq. (he-5.11)

$$\begin{aligned} e &\approx \frac{1}{2} K_s (\boldsymbol{\nabla} \cdot \mathbf{l})^2 + \frac{1}{2} K_t [\mathbf{l} \cdot (\nabla \times \mathbf{l})]^2 + \frac{1}{2} K_b [\mathbf{l} \times (\boldsymbol{\nabla} \times \mathbf{l})]^2 \qquad (5.11) \\ &= \frac{1}{2} K_s \sin^2 \beta \; \beta_z^2 + \frac{1}{2} K_t \sin^4 \beta \; \gamma_z^2 + \frac{1}{2} K_b \left[\cos^2 \beta \; (\sin^2 \beta \; \gamma_z^2 + \beta_z^2) \right] . \end{aligned}$$

In the twist texture, **1** changes in the xy plane from x to $-x$ direction. Hence, $\beta \equiv \pi/2$ and only γ_z contributes Eq. (he-5.12)

$$f = \frac{1}{2} K_z \gamma_z^2. \tag{5.12}$$

Eq. (he-5.13) The other two textures are not that cleanly separated: In both $\gamma = 0$ so that

$$f = \frac{1}{2} K_s \sin^2 \beta \; \beta_z^2 + \frac{1}{2} K_b \cos^2 \beta \; \beta_z^2. \tag{5.13}$$

In the splay case, **l** turns in the xz plane from z to $-z$ direction. In the middle of the texture, i.e., the place of largest β_z, the angle β is $\frac{\pi}{2}$ and therefore the first term dominates. In the bend case, **l** turns in the xz plane from x to $-x$ direction. Thus, for the largest β_z, $\beta \sim \pi$ and the second term dominates.

The currents can now be calculated by inserting (5.1) into (4.2) and (4.3(he-4.3)).

Eq. (he-5.14) For the mass current we find

$$j_i = \rho_{sij} v_{sj} + c_{ij} \left(\nabla \times \mathbf{l} \right)_j \tag{5.14}$$

Eq. (he-5.14a) with the matrices

$$\begin{aligned} \rho_{sij} &\equiv \rho_s \delta_{ij} - \rho_0 l_i l_j \\ c_{ij} &= c \delta_{ij} - c_0 l_i l_j . \end{aligned} \tag{5.15}$$

Notice, that this result also follows directly from an infinitesimal Galilean transformation. If one multiplies A by $e^{2im\mathbf{vx}}$, this leaves **l** invariant while changing

$$\mathbf{v}_{si} = \frac{1}{2mi} \phi^\dagger \nabla_i \phi$$

Eq. (he-5.15) as follows

$$\mathbf{v}_s \to \mathbf{v}_s + \mathbf{v} \tag{5.16}$$

showing that $\mathbf{v}_s$ transforms like a velocity (thus justifying its name) Using this transformation together with (4.5) on (5.8(he-5.8)) yields again the current (5.14). This current describes the superflow of Cooper pairs in the rest frame of the normal liquid. The superfluid density is a tensor with a component longitudinal to **l**, $\rho_s^\parallel = \rho_s - \rho_0$, and a transverse one, $\rho_{s\perp} = \rho_s$.

We now turn to the "orbital current". It describes the collective motion of the atoms *within* the Cooper pairs. It is similar to the current $\nabla \times \mathbf{M}$ which appears in magnetostatics in the presence of magnetizable matter [26] in the Maxwell equation

Eq. (he-5.16)

$$\boldsymbol{\nabla} \times \mathbf{B} = 4\pi \left(\mathbf{j} + \boldsymbol{\nabla} \times \mathbf{M} \right). \tag{5.17}$$

The second current term is the electronic current flowing within the molecular orbits of the matter. In complete analogy to this, there is a local matter current associated with the rotation of ^{3}He atoms inside the Cooper pairs. This current contributes to the total superflow.

The spin current can be derived similarly to the matter current via the appropriate symmetry transformation which brings $A \rightarrow e^{-2\boldsymbol{\varphi}^s\boldsymbol{\epsilon}}A$ and $d_a \rightarrow d_a + \delta d_a$ with

Eq. (he-5.16)

$$\delta d_b = -2\varphi_a^s \epsilon_{abc} d_c. \tag{5.18}$$

Since the spin current is defined by $j_{ai}^{\rm spin} \equiv -\partial e/\partial_i \varphi_a^s$ we find from the hydrodynamic energy directly Eq. (he-5.16a)

$$j_{ai} = 2\left(K_1^d \delta_{ij} - K_2^d l_i l_j\right) \epsilon_{abc} d_b \nabla_j d_c. \tag{5.19}$$

In order to keep as much analogy as possible with the superfluid velocity we may define a superspin velocity Eq. (he-5.17)

$$v_{sai} \equiv \frac{1}{2m} \epsilon_{abc} d_b \nabla_i d_c \tag{5.20}$$

in terms of which the current becomes Eq. (he-5.18)

$$j_{ai} = 4m\left(K_1^d \delta_{ij} - K_2^d l_i l_j\right) v_{saj} \tag{5.21}$$

where, again, there is a longitudinal piece proportional to $K_1^d - K_2^d$ and a transverse one with a factor K_1^d.
Under a spin rotation (5.18), the velocity transforms according to Eq. (he-5.19)

$$v_{sai} \rightarrow -2\varphi_a^s \epsilon_{abc} v_{sci} + \nabla_i \varphi^s/m. \tag{5.22}$$

The orbital angular momentum current is obtained from (5.14) by forming the vector product with x.

The action is still incomplete since, until now, we have left out the dipole force. Inserting the parametrization (5.1) and (5.3(he-5.3)) into (3.27), we find Eq. (he-5.19a)

$$f_{\rm d} = -2_{gd} \Delta_A^2 \left[(\mathbf{d}\cdot\mathbf{l})^2 - \frac{1}{3}\right]. \tag{5.23}$$

Thus, the dipole force tends to align **d** and **l** parallel or antiparallel. This can physically be understood as follows: Let the atoms orbit around each other, say, in the xy plane. If the spin points in z direction the two nuclear moments have equal poles all the time adjacent to each other. In the $S_z = 0$ configuration they are, on the other hand, aligned so that opposite poles face each other for half the orbit. This is energetically favored. The latter case corresponds to $d^{\|}1$.

A comparison of the strength of the dipole energy with the main piece (5.2) of the bending energy is possible if we write Eq. (he-5.20)

$$f_{\rm d} = -{\Delta_A}^2 K_{23} \frac{1}{\xi_a^{\perp 2}} (\mathbf{d}\cdot\mathbf{l})^2 + \text{const} . \tag{5.24}$$

Then, the dipole length Eq. (he-5.21)

$$\xi_{\rm d}^{\perp} = \sqrt{K_{23}/2g_{\rm d}} \tag{5.25}$$

measures the length scale over which the direction of field lines has to vary appreciably in e of (5.2) in order that the bending energy is of comparable size with the dipole energy. The microscopic calculation yields $\xi_{\rm d}^{\perp} \approx 10^{-3}$ cm $(1 - T/T_c)$ which is two orders of magnitude larger than the coherence length ≈ 1000Å$\,(1 - T/T_c)$.

The dipole energy (5.22) in the σ-model of ^{3}He plays a very similar role as the small PCAC violation present in σ-models of particle physics. Before $f_{\rm d}$ is turned on, all Goldstone modes are massless. With (5.22) the oscillations in which the relative angle between **d** and **l** vibrates acquire a small mass. The experimental resonance frequency is $\Omega_A \approx 50$ kHz corresponding, energetically, to the temperature $T_A \approx 5 \times 10^{-7}$ K. It is, therefore, much smaller than the gap energy ($\approx m \cdot K$).

While **l** and $\mathbf{v}_s$-vectors have physically the most transparent meaning, they are dynamically not independent (since $v_{si} = \frac{1}{2m}\boldsymbol{\phi}^{(1)}\nabla_i\boldsymbol{\phi}^{(2)}$ is itself a derivative). In
Eq. (he-5.22) fact, the curl of $\mathbf{v}_s$ is related to the **l** field as follows

$$\boldsymbol{\nabla} \times \mathbf{v}_s = \frac{1}{4m}\epsilon_{ijk}\, \mathbf{l} \cdot (\nabla_j \mathbf{l} \times \nabla_k \mathbf{l})\,. \tag{5.26}$$

Eq. (he-5.22a) For the proof, one forms the derivative of (5.7)

$$\begin{aligned}(\boldsymbol{\nabla} \times \mathbf{v}_s)_i &= \frac{1}{2m}\epsilon_{ijk}\nabla_j\left(\boldsymbol{\phi}^{(1)}\nabla_k\boldsymbol{\phi}^{(2)}\right)\\ &= \frac{1}{2m}\epsilon_{ijk}\left(\nabla_j\boldsymbol{\phi}^{(1)}\nabla_k\boldsymbol{\phi}^{(2)}\right).\end{aligned} \tag{5.27}$$

Since $\boldsymbol{\phi}^{(1)}\nabla_k\boldsymbol{\phi}^{(1)} = 0$ (due to $\boldsymbol{\phi}^{(1)2} = 1$), $\nabla_k\boldsymbol{\phi}^{(1,2)}$ has only a component along **l** and
Eq. (he-5.23) $\boldsymbol{\phi}^{(2,1)}$. Thus

$$\nabla_j\boldsymbol{\phi}^{(1)}\nabla_k\boldsymbol{\phi}^{(2)} = \left(\mathbf{l}\cdot\nabla_j\boldsymbol{\phi}^{(1)}\right)\left(\mathbf{l}\cdot\nabla_k\boldsymbol{\phi}^{(2)}\right). \tag{5.28}$$

Eq. (he-5.24) But $\mathbf{l}\cdot\nabla_j\boldsymbol{\phi}^{(1,2)} = -\boldsymbol{\phi}^{(1,2)}\nabla_j\mathbf{l}$ (due to $\boldsymbol{\phi}^{(2)}\mathbf{l} = 0$) so that we can write

$$(\nabla \times \mathbf{v}_s)_i = \frac{1}{4m}\epsilon_{ijk}\left[\left(\boldsymbol{\phi}^{(1)}\nabla_j\mathbf{l}\right)\left(\boldsymbol{\phi}^{(2)}\cdot\nabla_k\mathbf{l}\right) - \left(\boldsymbol{\phi}^{(1)}\nabla_k\mathbf{l}\right)\left(\boldsymbol{\phi}^{(2)}\nabla_j\mathbf{l}\right)\right]. \tag{5.29}$$

From this, Eq. (5.25) follows directly since $\boldsymbol{\phi}^{(1)}$, $\boldsymbol{\phi}^{(2)}$, **l** are an orthonormal triplet. The relation (5.25) will be powerful in relating the flow vortices to the geometric properties of the container of the liquid. For, if one takes the scalar product of (5.25)
Eq. (he-5.25) with **l** one finds [28]

$$2m\mathbf{l}\cdot(\nabla \times \mathbf{v}_s) = \frac{1}{2}\epsilon_{ijk}l_i\;\left[\mathbf{l}\cdot(\nabla_j\mathbf{l} \times \nabla_k\mathbf{l})\right] = K. \tag{5.30}$$

The right-hand side is the gaussian curvature of the surface cutting normal to the **l** field. If there is a closed normal surface anywhere inside the liquid, the integral over k gives 2π times the Euler invariant characteristic E of a closed surface. This
Eq. (he-5.26) characteristic is

$$E = 2(1 - m) \tag{5.31}$$

Fig. VI for a surface equivalent to a sphere with m handles (see (5.2). Performing the same integral over the left-hand side renders 2π times the number of singular vortex lines which have to enter the closed surface at some place. For, consider a closed contour on top of the closed surface (see Fig. 5.3).

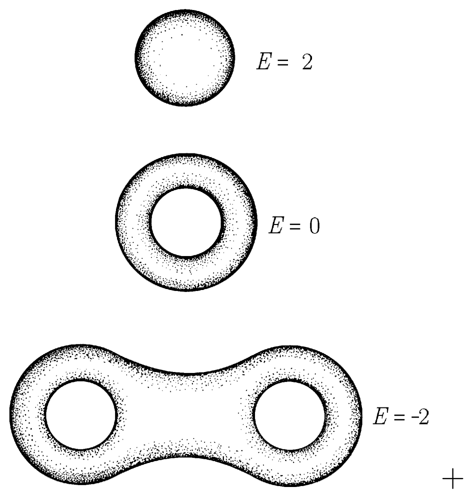


FIGURE 5.2 Sphere with no, one, or two handles, and the respective Euler characteristics $E = 2, 0$, or -2.

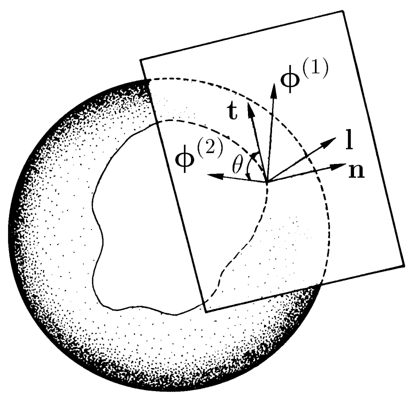


FIGURE 5.3 Local tangential coordinate system $\mathbf{n}, \mathbf{t},\ \mathbf{i}$ for an arbitrary curve on the surface of a sphere.

Let $\mathbf{t}$ be the tangent vector and $\mathbf{n} = \mathbf{l} \times \mathbf{t}$ be the normal to the contour inside the surface. Since $\boldsymbol{\phi}^{(1)},\ \boldsymbol{\phi}^{(2)}$ lie in the tangent plane they can be spanned as follows:

Eq. (he-5.27)

$$\begin{aligned} \boldsymbol{\phi}^{(1)} &= \cos\theta\ \mathbf{n} + \sin\theta\ \mathbf{t} \\ \boldsymbol{\phi}^{(2)} &= -\sin\theta\ \mathbf{n} + \cos\theta\ \mathbf{t}. \end{aligned} \tag{5.32}$$

As one proceeds a little way along the surface the tangential component of $\mathbf{v}_s$ is Eq. (he-5.28)

$$\begin{aligned} 2m\mathbf{v}_s \cdot \mathbf{t} &= ds\boldsymbol{\phi}^{(1)} \times \frac{d}{ds}\boldsymbol{\phi}^{(2)} \\ &= ds\left[\frac{d\theta}{ds} + \mathbf{t} \cdot \left(\mathbf{l} \times \frac{d}{ds}\mathbf{t}\right)\right]. \end{aligned} \tag{5.33}$$

Eq. (he-) The second piece

$$\gamma = \mathbf{t} \cdot \left(\mathbf{l} \times \frac{d}{ds} \mathbf{t} \right) \tag{5.34}$$

is called the geodesic curvature since it describes the rate of change of **t** away from the **t** direction (it is zero on the equator of a sphere). If we now convert the integral over the left-hand side of (5.30) into a contour integral and increase the contour
Eq. (he-5.30) throughout the surface leaving out all singular points, the result will be

$$\sum_i \oint ds \left[\frac{d\theta}{ds} + \gamma \right] \tag{5.35}$$

with the sum over all enclosed singularities. If the circles are infinitesimal, the surface can be considered as a plane and the integral over the geodesic curvature
Eq. (he-5.31) renders

$$\oint ds\, \gamma = 2\pi. \tag{5.36}$$

The integral over $d\theta/ds$, on the other hand, depends on the vertex strength N_i at
Eq. (he-5.32) the point i as

$$2\pi(N_i - 1). \tag{5.37}$$

For, if there is no vortex, the vectors $\phi^{(1)}$, $\phi^{(2)}$ stay fixed in space along the contour. Thus, the intrinsic coordinate θ of (5.32) changes by -2π. If, on the other hand, there is vortex with $\phi^{(1)}, \phi^{(2)}$ rotating N_i times around **l** in the positive sense when going around the contour, there will be an additional change of $N_i \cdot 2\pi$. Thus the Euler characteristic determines the number of vortex lines passing through any closed surface normal to the **l** field inside the liquid. This theorem will be useful for the discussion to follow.

Eq. (he-5.33) In the B phase, in which

$$A^0_{ai} = \Delta_B R_{ai}(\theta)\, e^{i\varphi} \tag{5.38}$$

the magnitude of Δ_B is pinned down at the potential minimum (4.22) with a value (4.21) and only θ and φ are allowed to vary. The gradient energy becomes due to
Eq. (he-5.34) (3.24)

$$\begin{aligned} f &= \frac{1}{2}\Delta_B^2 \left[K_1 \delta_{ij}\delta_{kl} + \frac{1}{2} K_{23} \left(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl} \right) \right] \\ &\quad \times \nabla_k \left(R_{ai}\ (\theta) e^{-i\varphi} \right) \nabla_l \left(R_{aj}(\theta) e^{i\varphi} \right). \end{aligned} \tag{5.39}$$

Eq. (he-5.35) The derivative terms can be written as

$$\nabla_k \varphi \nabla_l \varphi + \nabla_k R_{ai} \nabla_l R_{aj} + \dots \tag{5.40}$$

with mixed terms $\nabla R \nabla \varphi$ vanishing in the contraction with the tensor (5.35) due to
Eq. (he-5.36) symmetry. If we parametrize small oscillations in θ as

$$R_{ai}(\theta) = R_{aj}(\theta_0) R_{ji}(\theta) \tag{5.41}$$

Eq. (he-5.37) the energy becomes

$$\begin{aligned} f &= \frac{1}{2}\Delta_B^2 \left\{ K_1 \left[3(\boldsymbol{\nabla}\varphi)^2 + 2(\nabla_i \tilde{\theta}_j)^2 \right] \right. \\ &\quad \left. + K_{23} \left[(\boldsymbol{\nabla}\varphi)^2 + (\nabla_i \tilde{\theta}_j)^2 - \frac{1}{2}(\boldsymbol{\nabla}\tilde{\theta})^2 - \frac{1}{2}(\nabla_i \tilde{\theta}_j \nabla_j \tilde{\theta}_i) \right] \right\}. \end{aligned} \tag{5.42}$$

Using the result of Eq. (4.21) together with (5.6), we have Eq. (he-5.38)

$$\frac{1}{2}\Delta_B^2 = \frac{8}{10}\frac{1}{2}\Delta_A^2 = \frac{8}{10}\frac{1}{16m^2}\rho\left(1 - \frac{T}{T_c}\right)\frac{1}{K_1} \tag{5.43}$$

which can be used to bring the energy to the form Eq. (he-5.39)

$$\begin{aligned} f &= \frac{1}{4m^2}\frac{1}{2}\rho_s^B \left\{ \frac{3}{5}(\boldsymbol{\nabla}\varphi)^2 + \frac{2}{5}(\nabla_i \tilde{\theta}_j)^2 \right. \\ &\quad \left. + \frac{K_{23}}{2K_1}\left[\frac{2}{5}(\boldsymbol{\nabla}\varphi)^2 + \frac{2}{5}(\nabla_i \tilde{\theta}_j)^2 - \frac{1}{5}(\boldsymbol{\nabla}\tilde{\theta})^2 - \frac{1}{5}\nabla_i \tilde{\theta}_j \nabla_j \hat{\theta}_i \right] \right\}. \end{aligned} \tag{5.44}$$

Here, we have introduced Eq. (he-5.40)

$$\rho_s^B \equiv 2\rho\left(1 - \frac{T}{T_c}\right) \tag{5.45}$$

which is defined for $T \approx T_c$ and is called the superfluid density of the B phase. The current density can be obtained either by inserting (5.41) into (4.3) or by performing $\varphi \to \varphi + 2\delta\varphi$ in (5.44) Eq. (he-5.41)

$$j_i = \rho_s^B \frac{1}{2m^2}\frac{1}{5}(3 + K_{23}/K_1)\nabla_i \varphi. \tag{5.46}$$

The spin current may be obtained by inserting (5.41) into (4.6) from which we find Eq. (he-5.42)

$$\begin{aligned} j_{ai} &= R_{aa'}(\boldsymbol{\theta}_0)\tilde{j}_{a'i} \\ \tilde{j}_{ai} &= -\frac{1}{2m^2}\rho_s \left\{ \frac{2}{5}\left(1 + \frac{K_{23}}{2K_1}\right)\nabla_i \hat{\theta}_a - \frac{1}{5}\frac{K_{23}}{2K_1}\nabla_a \tilde{\theta}_i + \delta_{ia}\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{\theta}} \right\} \end{aligned} \tag{5.47}$$

Appendix 5A Hydrodynamic Coefficients for $T \approx T_c$

Here we give a brief derivation of the hydrodynamic energy (5.8) as it follows from the original form (5.5) which we rewrite as Eq. (he-A.1)

$$e = \frac{1}{2}\Delta_A^2 \left\{ K_1 |\partial_i \phi_j|^2 + K_2 \left(\partial_i \phi_j^* \partial_j \phi_i\right) + K_3 |\boldsymbol{\partial}\boldsymbol{\phi}|^2 + K_{23}|\boldsymbol{\phi}\boldsymbol{\partial} d_a|^2 + 2K_1 \left(\partial_i d_a\right)^2 \right\}. \tag{5A.1}$$

with the notation $K_{12} \equiv K_2 + K_3$. First we process the pure ϕ parts. The first term is decomposed as follows: Eq. (he-A.2)

$$|\partial_i \phi_j|^2 = (\partial_i \boldsymbol{\phi}^{(1)})^2 + (\partial_i \boldsymbol{\phi}^{(2)})^2. \tag{5A.2}$$

Observing that the vector $\partial_i \boldsymbol{\phi}^{(1)}$ has only an $\mathbf{l}$ and a $\boldsymbol{\phi}^{(2)}$ component, due to the trivial orthogonality relation $\boldsymbol{\phi}^{(1)} \partial_i \boldsymbol{\phi}^{(1)} = 0$, we write Eq. (he-A.3)

$$\partial_i \boldsymbol{\phi}^{(1)} = \left(\mathbf{l} \partial_i \boldsymbol{\phi}^{(1)}\right) \mathbf{l} + \left(\boldsymbol{\phi}^{(2)} \partial_i \boldsymbol{\phi}^{(1)}\right) \boldsymbol{\phi}^{(2)}. \tag{5A.3}$$

Eq. (he-A.4) In terms of the superfluid velocity

$$v_{si} = \frac{1}{2m} \boldsymbol{\phi}^{(1)} \partial_i \boldsymbol{\phi}^{(2)} \tag{5A.4}$$

and using the further orthogonality relation $\mathbf{l}\partial_i \boldsymbol{\phi}^{(1,2)} = -(\partial_i \mathbf{l})\, \boldsymbol{\phi}^{(1,2)}$ which follows
Eq. (he-A.5) from the orthogonality between $\mathbf{l}$ and $\boldsymbol{\phi}^{(1,2)}$, we have

$$\partial_i \boldsymbol{\phi}^{(1,2)} = -(\boldsymbol{\phi}^{(1,2)} \partial_i \mathbf{l}) \mathbf{l} \mp 2m v_{si} \boldsymbol{\phi}^{(2,1)}. \tag{5A.5}$$

Eq. (he-A.6) Squaring this gives

$$(\partial_i \boldsymbol{\phi}^{(1)})^2 = (\boldsymbol{\phi}^{(1)} \partial_i \mathbf{l})^2 + 4m^2 \mathbf{v}_s^2. \tag{5A.6}$$

Eq. (he-A.7) Adding once more the same term with $\boldsymbol{\phi}^{(1)}$ and $\boldsymbol{\phi}^{(2)}$ interchanged we obtain

$$\begin{aligned} |\partial_i \boldsymbol{\phi}|^2 &= (\boldsymbol{\phi}^{(1)} \partial_i \mathbf{l})^2 + (\boldsymbol{\phi}^{(2)} \partial_i \mathbf{l})^2 + 8m^2 \mathbf{v} s^2 \\ &= (\partial_i \mathbf{l})^2 + 8m^2 \mathbf{v}_s^2 \end{aligned} \tag{5A.7}$$

having dropped a trivially vanishing term $-(\mathbf{l}\partial_i \mathbf{l})^2$. The first term can be decom-
Eq. (he-A.8) posed into splay, twist, and bend terms as

$$(\partial_i \mathbf{l})^2 = (\boldsymbol{\nabla} \cdot \mathbf{l})^2 + [\mathbf{l} \cdot (\mathbf{v} \times \mathbf{l})]^2 + [\mathbf{l} \times (\mathbf{v} \times \mathbf{l})]^2 \tag{5A.8}$$

Eq. (he-A.9) so that we find the final form

$$|\partial_i \boldsymbol{\phi}|^2 = (\boldsymbol{\nabla} \cdot \mathbf{l})^2 + [\mathbf{l} \cdot (\boldsymbol{\nabla} \times \mathbf{l})]^2 + [\mathbf{l} \times (\boldsymbol{\nabla} \times \mathbf{l})]^2 + 8m^2 \mathbf{v}_s^2. \tag{5A.9}$$

Eq. (he-A.10) The third derivative term ϕ is treated as follows:

$$\begin{aligned} |\boldsymbol{\partial}\boldsymbol{\phi}|^2 &= (\boldsymbol{\partial}\boldsymbol{\phi}^{(1)})^2 + (\boldsymbol{\partial}\boldsymbol{\phi}^{(2)})^2 \\ &= \left[(\mathbf{l}\partial_i \boldsymbol{\phi}^{(1)}) l_i + (\boldsymbol{\phi}^{(2)} \partial_i \boldsymbol{\phi}^{(1)}) \phi_i^{(2)}\right]^2 + (1 \leftrightarrow 2) \\ &= [-(\boldsymbol{\phi}^{(1)} \partial_i \mathbf{l}) l_i - 2m v_{si} \phi_i^{(2)}]^2 + (1 \leftrightarrow 2, v_s \to -v_s) \\ &= (\mathbf{l}\boldsymbol{\partial} l_j)^2 + 4m v_{sk} [\phi_k^{(2)} \phi_j^{(1)} - (1 \leftrightarrow 2)]\, (\partial_i l_j)\, l_i + 4m^2 \left[\mathbf{v}_s^2 - (\mathbf{l} \cdot \mathbf{v}_s]^2\right). \end{aligned} \tag{5A.10}$$

Eq. (he-A.11) Here the first term is of the pure bend form

$$[\mathbf{l}\boldsymbol{\partial} l_j]^2 = [\mathbf{l} \times (\boldsymbol{\nabla} \times \mathbf{l})]^2. \tag{5A.11}$$

Eq. (he-A.12) The second term can be rewritten using

$$\phi_k^{(2)}\phi_j^{(1)} - \phi_k^{(1)}\phi_j^{(2)} = -\epsilon_{kjm} l_m \tag{5A.12}$$

Eq. (he-A.13) as

$$- m v_{sk} \epsilon_{kjm} l_m (\partial_i l_j) l_i . \tag{5A.13}$$

Eq. (he-A.14) With the formula

$$l_i \epsilon_{kjm} = l_k \epsilon_{ijm} + l_j \epsilon_{kim} + l_m \epsilon_{kji} \tag{5A.14}$$

it becomes

$$- 4m \left(\mathbf{v}_s \cdot \mathbf{l}\right) \left[\mathbf{l} \cdot (\boldsymbol{\nabla} \times \mathbf{l})\right] + 4m \mathbf{v}_s \cdot (\boldsymbol{\nabla} \times \mathbf{l}). \tag{5A.15}$$

The second gradient term in (5A.1). finally becomes, by a similar treatment: Eq. (he-A.15)

$$\partial_i \phi_j^* \partial_j \phi_i = [\mathbf{l} \times (\nabla \times \mathbf{l})]^2 + 4m^2 [\mathbf{v}_s^2 - (\mathbf{v}_s \cdot \mathbf{l})^2] - 4m (\mathbf{v}_s \cdot \mathbf{l}) [\mathbf{l} \cdot (\nabla \times \mathbf{l})]. \tag{5A.16}$$

Hence, the pure ϕ part of the gradient energy is Eq. (he-A.16)

$$\begin{aligned} e^{\phi} &= \frac{1}{2} \Delta_A^2 \Big\{ 4m^2 (2K_1 + K_{23}) \mathbf{v}_s^2 - 4m^2 K_{23} \left(\mathbf{l} \cdot \mathbf{v}_s\right)^2 \\ &\quad + 4m K_3 \mathbf{v}_s \cdot (\boldsymbol{\nabla} \times \mathbf{l}) - 4m K_{23} \left(\mathbf{v}_s \cdot \mathbf{l}\right) \left[\mathbf{l} \cdot (\boldsymbol{\nabla} \times \mathbf{l})\right] \\ &\quad + K_1 \left(\boldsymbol{\nabla} \cdot \mathbf{l}\right)^2 + K_2 \left[\mathbf{l} \cdot (\boldsymbol{\nabla} \times \mathbf{l})\right]^2 + (K_1 + K_{23}) \left[\mathbf{l} \times (\boldsymbol{\nabla} \times \mathbf{l})\right]^2 \Big\} . \end{aligned} \tag{5A.17}$$

If the **d** bending energies are neglected, we find the hydrodynamic energy (5.8) with the coefficients Eq. (he-A.17)

$$\begin{aligned} \rho_s &= \Delta_A (2K_1 + K_{23})\ 4m^2, && (5A.18) \\ \rho_0 &= 2mc_0 = \Delta_A^2 K_{23}\ 4m^2, && (5A.19) \\ c &= \Delta_A^2 K_3\ 2m, && (5A.20) \\ c_0 &= \Delta_A^2 K_{23}\ 2m, && (5A.21) \\ K_s &= K_t = \Delta_A^2 K_1, && (5A.22) \\ K_b &= \Delta_A^2 (K_1 + K_{23}). && (5A.23) \end{aligned}$$

Inserting the weak-coupling results (3.26) for $K_1, 2, 3$, one has Eq. (he-A.20)

$$\rho_s = 2\rho \left(1 - \frac{T}{T_c}\right) \tag{5A.24}$$

and the relations Eq. (he-A.21)

$$\rho_0 = \frac{1}{2}\, \rho_s = c_0\ 2m = 2c\ 2m, \tag{5A.25}$$

Eq. (he-A.22)

$$K_s = K_t = \frac{1}{4m^2} \frac{1}{4} \rho_s; \quad K_b = \frac{1}{4m^2} \frac{3}{4} \rho_s. \tag{5A.26}$$

The terms containing the **d**-vectors can processed similarly. With Eq. (he-A.23)

$$
\begin{aligned}
|\boldsymbol{\phi}\boldsymbol{\partial} d_a|^2 &= (\boldsymbol{\phi}^{(1)}\boldsymbol{\partial} d_a)^2 + (\boldsymbol{\phi}^{(2)}\boldsymbol{\partial} d_a)^2 \\
&= (\partial_i d_a)^2 - (\mathbf{l}\boldsymbol{\partial} d_a)^2
\end{aligned}
\tag{5A.27}
$$

we obtain Eq. (he-A.24)

$$
e^d = \frac{1}{2}\Delta_A^2\left\{(2K_1+K_{23})(\partial_i d_a)^2 - K_{23}(\mathbf{l}\boldsymbol{\partial} d_a)^2\right\} \tag{5A.28}
$$

amounting to the bending constants

$$
K_1^d = \Delta_A^2(2K_1+K_{23}), \qquad K_2^d = \Delta_A^2\, K_{23}. \tag{5A.29}
$$

In the case that **d** and **l** are locked to each other by the dipole energy, the general
Eq. (he-A.24a) bending energy of the d_a field

$$
e^d = \frac{1}{2}\left\{K_1^d(\partial_i d_a)^2 - K_2^d(\mathbf{l}\boldsymbol{\partial} d_a)^2\right\} \tag{5A.30}
$$

Eq. (he-A.25) contributes to the l field an energy

$$
\begin{aligned}
f^d_{\text{locked}} &= \frac{1}{2}(K_1^d\left\{(\boldsymbol{\nabla}\cdot\mathbf{l})^2 + [\mathbf{l}\cdot(\boldsymbol{\nabla}\times\mathbf{l})]^2 + [\mathbf{l}\times(\boldsymbol{\nabla}\times\mathbf{l})]^2\right\} \\
&\qquad -K_2^d\,[\mathbf{l}\times(\boldsymbol{\nabla}\times\mathbf{l})]^2\,).
\end{aligned}
\tag{5A.31}
$$

Adding this to (5.8) we obtain again the general form (5.8(he-5.8)), now with the coefficients

$$
K_t^l = K_t + K_1^d, \quad K_s^l = K_s + K_1^d, \quad K_b = K_b + K_1^d - K_2^d. \tag{5A.32}
$$

Eq. (he-A.2) For the present case with the coefficients (5A.17) and (5A.28(he-A.24)) this gives

$$
K_s = K_t = K_b = \Delta_A^2(3K_1+K_{23}). \tag{5A.33}
$$

Eq. (he-A.27) In the weak-coupling limit these are related to the superfluid density by

$$
K_{s,t,b} = \frac{1}{4m^2}\frac{5}{4}\rho_s. \tag{5A.34}
$$

6

Bending the Superfluid ^{3}He-A

The experimental interest lies in the possibility of preparing many nontrivial field configurations by gaining control over the directions of **l**- and **d**-vectors. Their presence can be detected by magnetic and sonic resonances. The principal means of enforcing certain field directions are the following:

1. External Magnetic Fields
   These try to enforce $\mathbf{d} \perp \mathbf{H}$ with a strength comparable to the dipole energy if $\mathbf{H} \approx 35$ Oe. The energy is proportional to $(\mathbf{d} \cdot \mathbf{H})^2$. The microscopic reason for this collective effect is clear. **H** becomes the quantization axis so that the direction **d** which is defined by the magnetic quantum number in that direction being zero, $S_3 = 0$, is orthogonal to **H**.

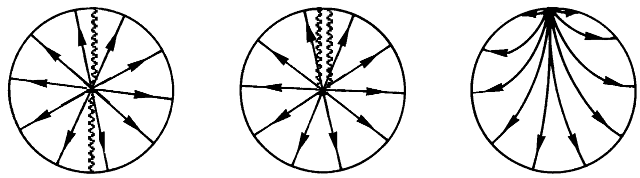
2. *Walls*
   Since **l** denotes the direction of the orbital angular momentum of the Cooper pairs one expects **l** to stand orthogonal to the walls of the container since a plane of orbital motion parallel to the walls should energetically be favored over the orthogonal configuration. This expectation is borne out by calculations.

Apart from this, currents and probably also electric fields act as directional agents upon **l**.

Let us now discuss what is called an open system. It is defined by a liquid in a container which is large compared with the dipole length $\xi_{\rm d}$ (i.e., much bigger than 10^3 cm) and with no magnetic field being present. In order to avoid the pile-up of dipole energy, **d** and **l**-vectors will stay aligned over most of the volume. Only in the neighborhood with a radius $\xi_{\rm d}$ around singularities where bending energies become comparable with a dipole energy may alignment be destroyed. Such singularities will be present in any sample prepared carelessly. Moreover, even with the most delicate cooling into the superfluid phase, the geometry of most containers will enforce the existence of some singularities:

6.1 Monopoles

If a sphere is cooled smoothly through the transition region, the l field lines will be planted uniformly orthogonal to the walls and develop towards the inside like



a) b) c)

FIGURE 6.1 The $\mathbf{l} \parallel \mathbf{d}$ field lines in a spherical container. There are necessarily two flux quanta associated with a singular point either in form of two vortex lines (a) or of one with double circulation (b). Since vortex lines store condensation energy, they act approximately like a rubber band and draw the point to the wall (c), thereby generating a flower-like texture called boojum [27].

the spines of a hedgehog. At some place there has to be a point-like singularity. Moreover, since the Euler characteristic of the sphere is $E = 2$, any closed surface orthogonal to the $\mathbf{l}$ field inside the liquid has to be passed by two vortex quanta. Possible field configurations are shown in Fig. 6.1. In the first case, two separate vortex lines of strength one emerge from the singularity, one running to the north, the other to the south pole. In the second case, there is, instead, one single line of vortex strength two at the north pole. In the third case the singularity has settled at the boundary forming a flower-like structure, a texture called *boojum* [28]. The last case is apparently favored energetically since there is considerable condensation energy stored in the vortex line inside of which the liquid is normal. The vortex line acts like a rubber band (compare the next section on vortex lines) pulling the singularity to the boundary. The first situation corresponds to the field lines of $\phi^{(1)}$ and $\phi^{(2)}$ running along the lines of equal longitude or latitude like on a globe, the $\mathbf{l}$-vector pointing, of course, radially outward, north and south poles are singularities. The two other situations correspond to a parametrization of the globe with only one
Fig. IX singularity at the north pole (see Fig. 6.2).

Eq. (he-6.1) In order to estimate the energies let us parametrize the field lines as[1]

$$\mathbf{l} = \mathbf{e}_r, \quad \boldsymbol{\phi} = (\mathbf{e}_\theta + i\mathbf{e}_\varphi)\, e^{i\chi}. \tag{6.1}$$

Eq. (he-6.2) Then the superfluid velocity is:

$$\mathbf{v}_s = \frac{1}{2m}\left(\boldsymbol{\nabla}\chi - \frac{\cot\theta}{r}\mathbf{e}_\varphi\right) \tag{6.2}$$

Eq. (he-6.3) with a vorticity

$$2m\,(\boldsymbol{\nabla} \times \mathbf{v}_s) = \frac{1}{r^2}\mathbf{e}_r. \tag{6.3}$$

Integrating this over a spherical closed surface gives $4\pi = 2 \times 2\pi$ corresponding to the passing of two vortex units. Choosing $\chi \equiv 0$ we see $\mathbf{v}_s$ to be singular at $\theta = 0$

[1]We neglect, for simplicity, all energy terms involving the $\mathbf{d}$-vector.

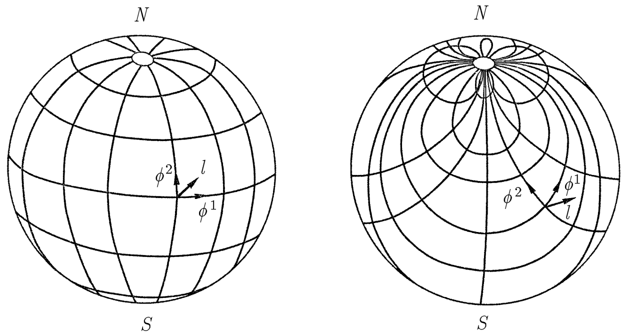


FIGURE 6.2 Two possible parametrization of a sphere, either with two singularities, as the standard geographic coordinate frame on the globe, or with one singularity, as shown in the lower figure. The geographic parametrization corresponds to liquid ^{3}He having two singular vortex lines, one emerging at the north pole and one at the south pole. The vectors $\boldsymbol{\phi}^{(i)} \times \boldsymbol{\phi}^{(2)}$ are tangential to the coordinates. The vector $\mathbf{l} = \boldsymbol{\phi}^{(1)} \times \boldsymbol{\phi}^{(2)}$ points radially outwards. The lower parametrization corresponds to one vortex line with double circulation emerging at the north pole. The south pole is a regular point.

and π, so that two vortices of one quantum each run from the center upwards or downwards [see Fig. 6.1a)], respectively. If χ is chosen to be $\chi = \varphi$, the singularity on the north (south) pole is cancelled with the other one being doubled [see Fig. 6.1(b)]. Inserting these configurations into the energy (5.9) with $2K, \sim K_{23} = 2K$ one has [29] $[\rho_s \approx 2\rho\,(1 - T/T_c)]$ Eq. (he-6.4)

$$E = \frac{\rho_s}{m^2}\frac{\pi}{4}R\log\left(\frac{2R}{\xi} - \frac{5}{2}\right) \tag{6.4}$$

in the first case. The energy of the second case is obtained by replacing $\log\,(2p/\xi - 5/2)$ by the larger value $2\log\,(2R/\xi - 7/4)$. The volume integration has to be cut off at the coherence distance $\xi = \xi_0/\sqrt{1 - T/T_c}$ away from the singularity. This is physically the correct procedure closer than ξ, the liquid cannot support the large bending energies concentrated in the directional change of $\phi^{(1)}$, $\phi^{(2)}$ around the $\mathbf{v}_s$ vortex line and escapes by Δ leaving the valley of minimal action and returning to the normal liquid point $\Delta = 0$. At that point, $\mathbf{d}$ and $\boldsymbol{\phi}$ in (5.1) lose their meaning and the singularity is avoided. Since the energy is proportional to Δ^2 and Δ^4, it vanishes in the normal region so that the integration can be cut off there. (Remember, though, that the complete energy consists of the sum of the bending energy e of (5.2) and the negative condensation energy $f_{\min}$ of (4.6)). When comparing this structure with monopole-like solutions in gauge theories coupled with Higgs fields [30] there is an essential difference: The energy increases with the radius of the sphere. The energy of a monopole, on the other hand, is constant. The reason for this is simple: In a σ-type of model, a field configuration which is radial asymptotically has a bending energy Eq. (he-6.5)

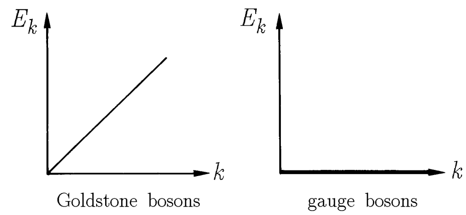


FIGURE 6.3 Spectra of Goldstone bosons versus gauge bosons. Goldstone bosons have energies going to zero with increasing wavelength due to an underlying symmetry. Gauge bosons have no energy for any vector case since their fields correspond to *local* symmetry transformations under which a gauge theory is invariant.

$$\left(\nabla_i \frac{x_j}{r}\right)^2 = \left(\frac{\delta_{ij} - x_i x_j / r^2}{r}\right)^2 \sim \frac{1}{r^2}. \tag{6.5}$$

Hence, the integral diverges with R. In a gauge field theory, on the contrary, the vector potential is oriented radially but the bending energy measures only the gauge invariant derivative $F_{\mu\nu}^2 = (\nabla_\mu A_\nu - \nabla_\nu A_\mu)^2$. This vanishes asymptotically very fast and all the energy is concentrated around the origin.

The situation can be described also in the particle language. In the σ-model, the nontrivial vacuum consists of a coherent superposition of static off-shell Goldstone bosons with many **k**-vectors. Their energy increases with k^2 and even in the asymptotic region there is a considerable amount of energy. In the gauge theory, the asymptotic region contains only longitudinal gauge particles which, by gauge invariance, correspond to Goldstone bosons with energies which vanish identically
Fig. X for all **k**-vectors (see Fig. 6.3).

Therefore, the asymptotic region is free of energy. Since it is the curvature of the container walls which enforces asymptotic bending energy (or the presence of Goldstone bosons close to the walls) the growth of energy with the radius of a sphere cannot be avoided, even if one patches together the field of a monopole with that of another monopole and forms what may be called a monopolium [29]. In order to study the situation, the point singularities sit at $(0, 0, C)$ and $(0, 0, -C)$. Then an
Eq. (he-6.6) Ansatz

$$\mathbf{l} = \frac{r^2 - C^2}{\lambda} \cos\theta \; \mathbf{e}_r + \frac{r^2 + C^2}{\lambda} \sin\theta \; \mathbf{e}_\theta \tag{6.6}$$

Eq. (he-6.7) with

$$\lambda \equiv \left[\left(r^2 - c^2\right)^2 + 4C^2 r^2 \sin^2\theta\right] \tag{6.7}$$

Eq. (he-6.8) can be used to construct a vector field ϕ so that the superfluid velocity is

$$\mathbf{v}_s = \frac{1}{2M} \left\{ \boldsymbol{\nabla}\chi - \frac{r^2 \cos^2\theta - C^2}{\lambda r \sin\theta} \mathbf{e}_\theta \right\}. \tag{6.8}$$

Eq. (he-6.9) The energy becomes

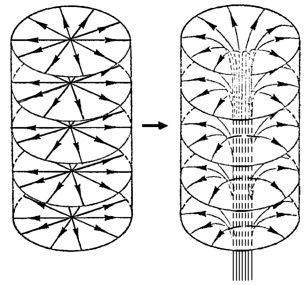


FIGURE 6.4 Cylindrical container with the $\mathbf{l} \parallel \mathbf{d}$ field lines spreading outwards when moving upwards. The line singularity on the left stores condensation energy. The curved configuration on the right contains bending energy. In a large container, this is preferable. Small containers and magnetic fields give preference to the singular line.

$$E = \frac{56}{24}\pi\frac{\rho_s}{m^2}R + \frac{\pi}{2}\frac{\rho_s}{m^2}C\left[\ln\frac{2c}{\xi} - \left(\frac{9}{4} + \frac{3\pi^2}{32}\right)\right] \tag{6.9}$$

within a sphere of radius R. Thus, in addition to the energy proportional to R enforced by the spherical container, there is a linear binding energy with a logarithmic correction which stems from the bending energy in the neighborhood of the vortex line. The vortex line pulls the point singularities together according to an almost constant force.

Notice, that apart from the first piece in the energy caused by geometry, there is an essential difference of this σ-model result with what one expects for string like solutions of pure gauge theories. There, color is supposed to be screened completely in the vacuum so that the color field does not leave the vortex line. This is the reason why the force is purely linear! The monopolium state can be stabilized by placing ions of equal charge at both ends.

6.2 Line Singularities

If a cylindrical container is cooled, the $\mathbf{l}$ lines will develop radially inwards. One therefore expects a singular line along the axis. At this line, the liquid would have to be in its normal state since the $\mathbf{l}$-vectors are undefined. This amounts to the accumulation of a large condensation which is completely superfluid and contains only bending energies. This can, indeed, be achieved by the $\mathbf{l}$ lines flaring upwards like in a chimney [29] (see Fig. 6.4). Fig. XI

Quantitatively, the energy can be minimized by an $\mathbf{l}$ field Eq. (he-6.10)

$$\mathbf{l} = \mathbf{e}_z\cos\beta + \mathbf{e}_\rho\sin\beta \tag{6.10}$$

with Eq. (he-6.11)

$$\beta(\rho = 0) = 0, \quad \beta(\rho = R) = \frac{\pi}{2}. \tag{6.11}$$

There are many complex vectors ϕ which can be constructed with this $\mathbf{l}$, for example:

Eq. (he-6.12)

$$\phi = e^{im\varphi}\left[-\sin\beta\,\mathbf{e}_z + \cos\beta\,\mathbf{e}_\rho + i\mathbf{e}_\varphi\right]. \tag{6.12}$$

Eq. (he-6.13) They lead to a superfluid velocity

$$\mathbf{v}_s = \frac{1}{2M\rho}\left(m - \cos\beta\right)\mathbf{e}_\rho. \tag{6.13}$$

At $m = 1$, there is no vortex line along the axis. This situation is favored energetically. Inserting $\mathbf{v}_s$ and $\mathbf{l}$ into the energy (5.8) and extremizing with respect to $\delta\beta(\rho)$,

Eq. (he-6.14) one finds the solution:

$$\frac{\rho}{R} = \exp\left(\int_{\beta(\rho)}^{T/2}\left\{\frac{K_s\cos^2\beta + K_b\sin^2\beta}{K_s\sin^2\beta + \frac{\rho_s}{4m^2}\left(1-\cos\beta\right)^2}\right]^{1/2} d\beta\right\}. \tag{6.14}$$

Eq. (he-6.15) The total energy of this configuration is

$$E \approx 1.145\pi\frac{\rho_s}{m^2}L \tag{6.15}$$

where L is the length of the cylinder. Here, the weak coupling equalities (5.3) have been used.

Notice that from (6.13) there is an azimuthal current flowing in this field configuration which therefore may have a nonvanishing orbital angular momentum. In

Eq. (he-6.16) order to calculate this, consider the second, convective part of the current

$$\boldsymbol{\nabla}\times\mathbf{l} = (\mathbf{l}\cdot\boldsymbol{\nabla})\,\beta\mathbf{e}_\varphi = -(\cos\beta)'\mathbf{e}_\varphi. \tag{6.16}$$

This also circulates around the axis but with a different radial dependence. The

Eq. (he-6.17) total angular momentum is then, due to (4.8),

$$\mathbf{L} = \int d^3x\,(\mathbf{x}\times\mathbf{j}) \tag{6.17}$$

Eq. (he-6.18) directed along the z-axis with a value

$$\begin{aligned} L_z &= 2\pi\int dz d\rho\rho\left[\frac{\rho_s}{2M\rho}(1-\cos\beta) - c\frac{\nabla}{\nabla\rho}(cos\beta)\right] \\ &= 2\pi\int dz d\rho\left[\frac{\rho_s}{2M}(1-\cos\beta) + c\cos\beta\right] \\ &\approx R^2\frac{\rho_s}{2m}\left(1 - \frac{1}{\pi} + \frac{2}{\pi^2}\right)L. \end{aligned} \tag{6.18}$$

For the last equation, we have again inserted the weak- coupling result $c = \rho_s/4$. During the phase transition, the angular momentum must manifest itself in a recoil imparted to the container. It would be interesting to detect this effect experimentally. [2]

There is also a way to prepare the singular vortex line. For this, a magnetic field has to be turned on along the z axis which drives the **d**-vectors into the xy-plane. This enforces a singularity in the **d** field lines along the axis causing the liquid to be normal there. Once the condensation energy is spent, the weak dipole force is sufficient to pull also the **l** field into the radial direction. [31]

[2] Also, the boojum in a sphere has an angular momentum which would set the sphere into rotation when cooling through the transition point.

6.3 Solitons

Let us now turn to planar textures in an open geometry. [29]. A direction may be defined by magnetic field pointing, say, along the z axis. Then, the **d** vectors will be forced to lie in the xy plane: Eq. (he-)

$$\mathbf{d} = \sin\psi + \hat{\mathbf{x}} + \cos\psi + \hat{\mathbf{y}}. \tag{6.19}$$

The bending energy is minimized by a constant ψ in space. The dipole force pulls the **l** vector in the same or in the opposite direction. Since this force is very weak, there will be some regions where **l** is parallel and others where **l** is anti-parallel to d. The wall separating the different domains is stabilized by the competition between bending and dipole energy. If the thickness of the domain wall, a, shrinks, the bending energy density grows as $\frac{\rho_s}{m^2}\frac{1}{a^2} + a$ while the corresponding dipole term drops as $\frac{\rho_s}{m^2}\frac{1}{\xi d^2} \times a$. Conversely, a large domain accumulates an overwhelming dipole energy. Equilibrium is reached at $a \approx \xi d^2$. If one studies, for simplicity, only configurations with a pure z-dependence and with **l** in the xy plane Eq. (he-6.20)

$$\mathbf{l} = \sin\chi\hat{x} + \cos\chi\hat{y} \tag{6.20}$$

the most general ϕ-vector is Eq. (he-6.21)

$$\phi = e^{i\varphi}\left(-\cos\chi\hat{x} + \sin\chi\hat{y} + iz\right) \tag{6.21}$$

and the bending energy becomes with the weak-coupling values of the parameters Eq. (he-6.22)

$$f_{\text{bend}} = \frac{\rho_s^{\|}}{8m^2}\left\{2(\nabla\psi)^2 + 2(\nabla\varphi)^2 + \frac{1}{2}(\nabla\chi)^2\right\}. \tag{6.22}$$

The dipole energy contributes Eq. (he-6.23)

$$f_{\text{dip}} = \frac{\rho_s^{\|}}{8m^2}\frac{2}{\xi_{\text{d}}^{\perp 2}}\sin^2\left(\chi - \psi\right). \tag{6.23}$$

The phase φ occurs only in the bending energy and is uniform, $\varphi = \text{const}$. at equilibrium. The remaining dependence on the fields χ, ψ can be diagonalized by setting Eq. (he-6.24)

$$\begin{aligned} v &\equiv \chi - \psi \\ u &\equiv \chi + 4\psi. \end{aligned} \tag{6.24}$$

Then, the energy takes the sine-Gordon form Eq. (he-6.25)

$$f = f_{\text{bend}} + f_{\text{dip}} = \frac{\rho_s^{\|}}{4m^2}\left(\frac{1}{20}u_z^2 + \frac{1}{5}v_z^2 + \frac{1}{\xi_{\text{d}}^{\perp 2}}\sin^2 v\right). \tag{6.25}$$

This is minimized by a constant u and a soliton in the variable v: Eq. (he-6.26)

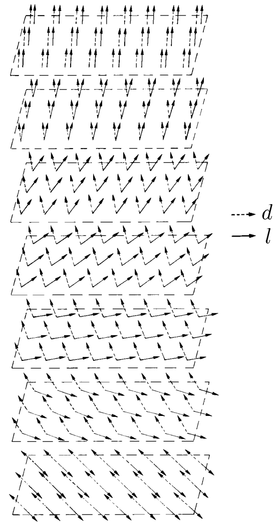


FIGURE 6.5 Field vectors in a composite soliton. At $z = +\infty$ and $\mathbf{l}$, $\mathbf{d}$ are parallel, at $z = -\infty$ they are antiparallel. Inside the domain wall of size $\xi_{\rm d}$ the vectors change direction, $\mathbf{l}$ four times as much as $\mathbf{d}$.

$$\sin v_{\rm sol} = \cosh^{1}\left(z/\xi_{\rm sol}\right), \quad \tan g\frac{v_{\rm sol}}{2} = e^{\pm z/\xi_{\rm sol}} \tag{6.26}$$

where the width of the soliton is of the order of the dipole length Eq. (he-6.27)

$$\xi_{\rm sol} \equiv \frac{1}{\sqrt{5}}\xi_s^{\perp}, \tag{6.27}$$

as expected. The energy per unit area of the domain wall is Eq. (he-6.28)

$$\frac{E}{\sigma} = \frac{\rho_s^{\parallel}}{4m^2}\frac{2}{\xi_d^2}\int_{-\infty}^{\infty} dz \cosh^4\left(\frac{z}{\xi_{\rm sol}}\right) = \frac{\rho_s^{\parallel}}{m^2}\frac{\xi_{\rm sol}}{\xi_{\rm d}^2} = \frac{\rho_s}{m^2}\frac{1}{\sqrt{5}}\frac{1}{\xi_{\rm d}}. \tag{6.28}$$

The soliton corresponds to $\mathbf{d}$- and $\mathbf{l}$-vectors twisting in opposite direction inside the
Fig. XII domain wall with $\mathbf{l}$ moving four times as far as $\mathbf{d}$ (see Fig. 6.5).

The presence of such a domain wall can be detected in the laboratory via a nuclear magnetic resonance experiment (NMR). Suppose a vibrating field is turned on along the z-axis (in addition to the static orienting field H^{ext}). This is called a longitudinal resonance experiment. The vector $\mathbf{l}$ associated with the spin starts
Eq. (he-6.29) oscillating around the z-axis (see Ref. 16), say as

$$\psi = \psi_{\rm sol} + \delta \tag{6.29}$$

Eq. (he-6.30) and consequently

$$\begin{aligned} u &= u_{\rm sol} + 4\delta \\ v &= v_{\rm sol} - \delta. \end{aligned} \tag{6.30}$$

Eq. (he-6.31) This gives an additional vibrational energy

$$\delta^2 f = \frac{\rho_s^{\parallel}}{4m^2} \times \left[\delta_z^2 + \frac{1}{\xi_{\rm d}^{\perp 2}}\left(1 - \frac{2}{\cosh^2(z/\xi_{\rm sol})}\right)\delta^2\right]. \tag{6.31}$$

The extrema of this energy correspond to the bound states of the Schrödinger equation [3] Eq. (he-6.32)

$$\left[-\nabla_z^2 + \frac{1}{\xi_{\rm d}^{\perp 2}}\left(1 - \frac{2}{\cosh^2(z/\xi_{\rm sol})}\right)\right]\delta(z) = \lambda\delta(z). \tag{6.32}$$

This is a standard soluble problem (see the textbook on quantum mechanics by Landau-Lifshitz, ch. 23). The ground state is Eq. (he-6.33)

$$\delta(z) \propto \frac{1}{\cosh^2(z/\xi_{\rm sol})} \tag{6.33}$$

with Eq. (he-6.34)

$$s \equiv \frac{1}{2}\left[-1 + \sqrt{1 + 4\frac{2}{\xi^{\perp 2} d}\xi_{\rm sol}^2}\right] = \frac{1}{2}\left[-1 + \sqrt{1 + 4\frac{2}{5}}\right] \approx .306. \tag{6.34}$$

Since $s \leq 1$ there is only one bound state (if s were ¿ 1,there would be more, for $n = 0, 1, 2, \ldots, s$). This bound state has an energy Eq. (he-6.35)

$$\lambda = \frac{1}{2}\left(\sqrt{65} - 7\right)\frac{1}{\xi_{\rm d}^{\perp 2}}. \tag{6.35}$$

The continuum has a spectrum Eq. (he-6.36)

$$\lambda = k^2 + \frac{1}{\xi_{\rm d}^{\perp 2}}. \tag{6.36}$$

Experimentally, the vibrating field is homogeneous so that in the continuum only the $\mathbf{k} = 0$ value is excited. This leads to the main NMR resonance absorption line. If now soliton was present this would be the only signal observed. The bound state trapped by the soliton has the effect of causing at a frequency which lies by factor $\frac{1}{2}\left(\sqrt{65} - 7\right) \approx (.728)^2$ lower than the main line. Such a "satellite" frequency has indeed been observed experimentally (see Fig. 6.6). Fig. XIII

Notice that the satellite line provides a good test for the weak-coupling values of the coefficients $K_{1,2,3}$ in the bending energies. If κ denotes the ratio Eq. (he-6.36a)

$$\kappa \equiv 2K_1/(K_2 + K_3), \tag{6.37}$$

the frequency should be found at Eq. (he-6.36b)

$$\frac{1}{2\kappa}\left[\sqrt{(3\kappa + 2)(\kappa + 2)} - (5\kappa + 2)\right] \tag{6.38}$$

instead of $(.728)^2$. The experimental value $(.74)^2$ limits κ close to 1 in agreement with the weak coupling result.

[3] The time driving term can be shown to go as $\frac{1}{2}\dot{\delta}^2$ so that λ corresponds to a frequency square.

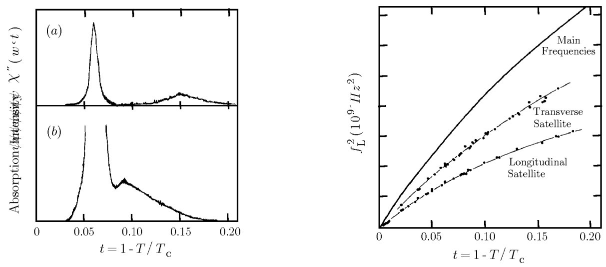


FIGURE 6.6 Nuclear magnetic resonance frequencies in a superfluid ^{3}He-A sample in an external magnetic field. The large peak corresponds to the main absorption line, the small peak to the right is a satellite frequency line attributed to the trapping of spin waves in planar domain walls. The lower part of the figure shows the position of these lines for different external frequencies of the longitudinal applied magnetic field. The ratio of satellite frequency to main frequency agrees with the theoretical calculation.

6.4 Localized Lumps

We have argued before, that the energies of point and line singularities are necessarily not localized. In a hedgehog like field structure, the σ-model bending energy goes asymptotically as $\frac{1}{R^2}$ so that the spherical (or cylindrical) integral diverges linearly (or logarithmically). The energy can be confined to a small region only for a field configuration which is asymptotically flat but contains some knots, say, close to the origin. Topologically, one has to find a nontrivial mapping of the whole three-dimensional space into the parameter space of the liquid with everything, except a small neighborhood of the origin, mapped into one point. In the A-phase of ^3He there exists, in fact, such a mapping with $\mathbf{l}$ and $\mathbf{d}$-vectors aligned [36] (see Fig. 6.7). For, the covering space of the parameter space SO(3) is $SU(2)$ which

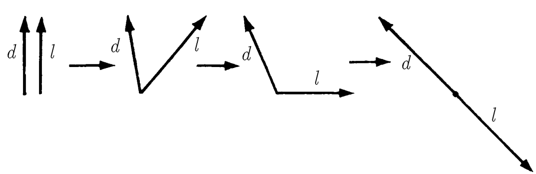


FIGURE 6.7 Vectors of orbital and spin orientation in The A-phase of superfluid ^{3}He

is equivalent to S_3, the surface of a sphere in four dimensions. Since the ordinary space corresponds to S_3, which is the space S_3 with the north pole removed, one has a nontrivial mapping $S_3 \to S_3'$ with a large neighborhood of the north pole of S^3 mapped into one point of S_3, accounting for asymptotic uniformity. This corresponds to a diffuse smoke ring type of configuration which moves through the liquid with a velocity $\mathbf{v} = \hbar/mR$, a momentum $P \approx \hbar\rho_s R^2/m$ and an energy $E \approx \hbar^2\rho_s R/m^2$, respectively. Notice that the velocity is inversely proportional to R and the energy of the smoke-ring. Actually, the topological stability does not prevent this object from having only a small lifetime. While it moves through the liquid, orbital friction eats up the energy and decreases the size. Once the object has shrunk to the size of the order of the dipole length $\xi_{\rm d}$, the locking and **d** and **l**-vectors will be overcome, the parameter space increases to $\tilde{S}^2 \times \tilde{S}O_3(z_2$ and the topological stability is lost. The knot unwinds and disappears.

6.5 Use of Topology in the A-Phase

As in gauge theories, topological considerations are helpful in classifying the different stable field configurations. In the superfluid, topological stability means that there is no continuous deformation to a lower energy state within the hydrodynamic limit. since this limit is an approximation, the stability is not perfect. the size of the order parameter Δ which in the hydrodynamic limit, is assumed to be pinned down at the value of minimal energy does, in fact, fluctuate and may arrive on rare occasions at the point $\Delta = 0$ where the liquid becomes normal. This process is called nucleation. For example, there is topological stability in a superconductor contained in a torus with the phase of the order parameter changing by $e^{in\varphi}$ when going once around the circle. There is no continuous way to relax the ensuing supercurrent in the hydrodynamic limit. But the supercurrent *does* decay within years. The reason is that at some place at the inner boundary the size of the order parameter may, by fluctuations, climb up from the valley of lowest energy into the normal phase with $\Delta = 0$. There the phase φ looses its meaning and can unwind by one unit of 2π. This point may lie at the inside of the torus and can develop into a thin flux tube. This tube can carry one unit of electric flux away from the supercurrent. such a process is facilitated by putting together two superconductors in a Josephson junction where the diffusion of such units can be observed in the clearest fashion. Thus topological stability in the hydrodynamic regime really amounts to metastability with extremely long life times. For practically purposes such life times can be assumed to be infinite and topological classification provides for good quantum numbers of field configurations. is the connection between two field configurations of the same topological class? they can be deformed into each other by continuous changes only of the directions of the fields with the magnitude being fixed. If both, initial and final state, are dynamically stable, there certainly is an energy barrier to be crossed during such a deformation. Its energy density is only due to the bending of the field lines and, therefore, extremely small as compared with the condensation energy which enforces topological stability.

Consider now the topology in the parameter space of ${}^3He$ [33].

In the A phase, the vacuum is determined by the product of the vectors d_a and ϕ_i. The vector d_a covers the surface of the unit sphere in 3 dimensions, S^2, the complex vector $\boldsymbol{\phi} = \boldsymbol{\phi}^{(1)} + i\boldsymbol{\phi}^{(2)}$ is a three parameter space equivalent to the space SO(3), i.e., a sphere of radius π with diametrally opposite points at the surface identified. Every point is determined by the direction of the vector $\mathbf{l} = \boldsymbol{\phi}^{(1)} \times \boldsymbol{\phi}^{(2)}$ and the length which characterizes the azimuthal angle of $\boldsymbol{\phi}^{(1)}$ in the plane orthogonal to $\mathbf{l}$. Due to the product $d_a\phi_i$ a sign change of d_a can be absorbed in ϕ_i so that the total parameter space is Eq. (he-7.1)

$$R = S^2 \times \mathrm{SO}(3)/Z_2. \tag{6.39}$$

Stable singular points exist if the homotopy group $\pi_2(R)$ of mappings of the sphere S^2 in three dimensions into this parameter space is nontrivial. But it is well-known [34] that for the above space $\pi_2(R) = Z$, the group of integer numbers. Thus all point singularities are characterized by an integer. There can be infinitely many different stable monopole type of classical field configurations. This purely topological argument is based on the independence of the vectors **d** and **l**. We know, however, that the dipole force tries to align the **d** and **l**-vectors. For this reason, as soon as the size of the container exceeds the dipole length $\xi_{\rm d} \approx 10^{-3}$ cm, **d** and **l** will stay
Eq. (he-7.2) parallel asymptotically thereby reducing the parameter space to

$$R = \mathrm{SO}(3) \quad (\mathbf{d} \parallel \mathbf{l}). \tag{6.40}$$

Then the homotopy group is $\pi_2(R) = 0$ and there are no monopoles.

Thus monopoles could be created only in very small regions ($r << 10^{-3}$ cm) of the liquid. Their **d** and **l** field lines are non-aligned. As a consequence, their neighborhood contains considerable dipole energy. If the volume of the neighborhood becomes much larger than the dipole length, fluctuations in the liquid cause nucleation to the normal phase with the monopole vanishing in favor of a $\mathbf{d}^{\parallel}\mathbf{l}$ alignment and no dipole energy. Quantitatively, it is the competition of the small dipole energy density $f_{\rm d} \sim \frac{e}{m^2}\frac{1}{\xi d^2}$ stored in a finite volume with the large condensation energy density $f_c \sim \frac{\rho}{m^2}\frac{1}{\xi^2}$ stored in the immediate neighborhood ξ^3 of the singularity of size $\xi(\approx 1000$ Å) which determines the transition point to the $\mathbf{d}^{\parallel}\mathbf{l}$ configuration.
Eq. (he-7.3) The relaxation occurs at

$$\frac{R^3}{\xi d^2} > \frac{\xi^3}{\xi^2} \tag{6.41}$$

Eq. (he-7.4) or

$$R > \sqrt[3]{\xi\xi d^2} \approx 10^{-4} cm. \tag{6.42}$$

For line singularities we have to consider $\pi, (R)$. For **d** and **l** independent this
Eq. (he-7.5) homotopy group is

$$\pi_1(R) = Z_4. \tag{6.43}$$

Hence there are four types of line singularities which can be labelled by their vortex
Eq. (he-7.6) strengths $\delta = \pm\frac{1}{2}, \pm 1$. Examples:

$$\begin{array}{llcl} \pm\frac{1}{2}: & \boldsymbol{\phi} & = & e^{\pm i\xi/2}(\mathbf{e}_x + i\mathbf{e}_y) \quad , \quad \mathbf{d}_a \; = \; \mathbf{e}_x \cos\frac{\gamma}{2} \mp \mathbf{e}_y \sin\frac{\gamma}{2} \\ \mathbf{l} & & = & \mathbf{l}_z \\ \pm 1: & \boldsymbol{\phi} & = & (\mathbf{e}_z + i\mathbf{e}_\varphi) \qquad\quad , \quad d_a \; = \; f_\rho \\ \mathbf{l} & & = & \mathbf{e}_\rho. \end{array} \tag{6.44}$$

As the volume increases, $R >> \xi_{\rm d}$, the dipole force leads again to alignment of **d**
Eq. (he-7.7) and **l** reducing the parameter space to

$$\pi_1(R) = Z_2 \quad (\mathbf{l} \parallel \mathbf{d}). \tag{6.45}$$

Thus only two types of singular lines survive and one sees from (6.44) that it is the ± 1 vortex lines which survive.

6.6 Topology in the B-Phase

The discussion of the hydrodynamic limit can be extended to the B-Phase. Consider the parametrization (4.18) of the degenerate ground state Eq. (he-8.1)

$$A_{ai} = \Delta_B R^{ai}(\mathbf{n}, \theta)\, e^{i\varphi} \tag{6.46}$$

with Δ_B pinned at the point (4.21) of minimal energy density (4.21). The matrix R may be written explicitly as Eq. (he-8.2)

$$R_{ai}(\mathbf{n}, \theta) = \cos\theta\delta_{ai} + (1-\cos\theta)\, n_a n_i + \sin\theta\epsilon_{aik} n_k. \tag{6.47}$$

Inserting this into the collective action (3.22), (3.24(he-3.20)) the energy becomes the sum of bending energies involving gradients of $\theta, \mathbf{n}$ and φ.

The parameter space of (6.46) consists of the direct product of a phase (which is isomorphic to the circle S^1) and the group space SO(3). As the dipole force is turned on, the angle θ is pined at the value $\theta \approx 104^0$ and the space SO(3) is narrowed down to the different *directions* of $\mathbf{n}$ only, covering the surface of a sphere S^2. The point and lien singularities are classified by considering the homotopy groups $\pi_2(R)$ and $\pi,\ (R)$ of the parameter spaces $R = S^2\times$ SO(3) for small configurations, $r << \xi_{\rm d}$, and $R = S^2 \times S^2$ for large ones.

In the first case one has Eq. (he-8.3)

$$\pi_2(R) = 0; \quad \pi_1(R) = Z + Z_2. \tag{6.48}$$

Thus, there are no topologically stable point singularities while there are two types of vortex lines: One set has its origin in the pure phase $e^{i\phi}$ of the parametrization and is characterized by an arbitrary integer N. These vortex lines are of exactly the same type as those of superfluid ^4He. In addition, there are singular lines in the $\mathbf{n}$, θ parameter space, two of which can annihilate each other (due to Z_2). For large samples, with $\theta = 104^0$ the homotopy groups are Eq. (he-8.4)

$$\pi_2(R) = Z. \quad \pi_1(R) = Z. \tag{6.49}$$

Thus, there are stable point like solutions of arbitrary integer charge, the simplest being a hedgehog with the $\mathbf{n}$-vector pointing radially. The line singularities are all due to the phase $e^{i\varphi}$ and therefore again of the same nature as in $HeII$.

There are interesting planar structures in the B-phase. In order to classify them one has to map the line $z\epsilon(-\infty, \infty)$ into the parameter space SO(3). In an open geometry any such mapping can be deformed into the identity. Stable configurations arise if a magnetic field is turned on along the z-direction which aligns the $\mathbf{n}$ vector parallel or anti-parallel. Notice, however, that, contrary to the A-phase, the directional energy of the magnetic field is quite weak: Since the B-phase corresponds to $J = 0$ Cooper pairs, it is only the small distortion of the wave function caused by the dipole coupling which leads to a net magnetic energy of the order of Eq. (he-8.5)

$$f_{\rm mg}(H) \sim g_{\rm d}\left(\frac{\gamma}{\Delta}\right)^2 (\mathbf{n}\cdot\mathbf{H})^2 \tag{6.50}$$

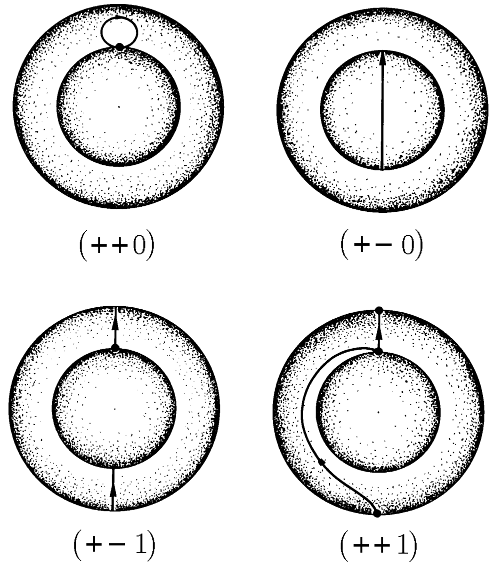


FIGURE 6.8 Parameter space of ^{3}He-B containing that of the rotation group. Thus points which lie diametrally opposite on the surface are identical. The dipole energy (hyperfine coupling between the spins) favors a spherical shell within this sphere. It corresponds to rotations around any axis by 104^0. Planar textures (solitons) have to start and end asymptotically on this shell. The figure shows the four topologically distinct classes of paths starting and ending on this spherical shell.

Thus, the characteristic length over which bending and magnetic energies are comparable is much larger than ξ_d, namely Eq. (he-8.6)

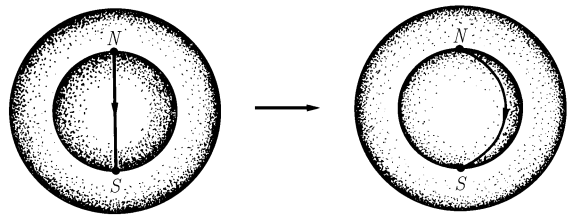
$$\xi_\mathrm{mg}(H) \sim \frac{\Delta}{\gamma H}\xi_\mathrm{d}. \tag{6.51}$$

With $\Delta \sim |mg$ and $\gamma H \sim .156 \times 10^3 \frac{mg-\lambda}{\mathrm{gauss}}$ this is, at 100 gauss, of the order of one mm. At large distances, however, this weak-coupling does result in the **n**-vector lying parallel or anti-parallel to . By the same token, also the dipole force is active and the angle θ settles at the value $\theta \approx 104^0$.

We can visualize this asymptotic situation by drawing a sphere of radius π and specifying, within this, the surface of fixed radius $\theta \approx 104^0$. Then, any planar field configuration corresponds to aline starting and ending at the north or south pole of the $\theta \approx 104^0$ surface. Thus, the asymptotic space is Z_2. There are eight classes of mappings, four of which are the mirror images of the others. They are shown in
Fig. XIV Fig. 6.8.

The first class $(+\,+\,0)$ is trivial and can be deformed continuously into the uniform field configuration. The second $(+-0)$ can be described as a pure θ soliton
Eq. (he-8.7) by setting

$$\mathbf{n} = f_z\,\mathrm{sgn}(z), \quad \theta = 2\mathrm{arctan}\left(\sqrt{\frac{5}{3}}\tan\frac{|z|}{\xi_\mathrm{d}}\right). \tag{6.52}$$



(+−0)

FIGURE 6.9 Possible path followed by the order parameter in a planar texture (soliton) when going from $z = -\infty$ to $z = +\infty$. The path tries to stay all the way on the spherical shell preferred by the dipole energy.

The third soliton, $[+-1)$, has the angle θ run from 104^0 to π and come back from the identical point at the south pole into 104^0 with $\mathbf{n}$ pointing in the opposite direction. An explicit parametrization is Eq. (he-)

$$\mathbf{n} = l_z \operatorname{sgn}(z), \quad \theta = 2 \arctan\left(\sqrt{\frac{5}{3}} \tan \frac{|z|}{\xi_{\mathrm{d}}}\right). \tag{6.53}$$

The last class, $(++1)$, is topologically equivalent to the sum of the two described before and can, in fact decay into them.

In order to imagine the different energies of these field configurations remember that the dipole force makes the radial shells have constant dipole energy with a minimal valley at the shell $\theta \approx 104^0$. The magnetic force, on the other hand, draws $\mathbf{n}$ into z direction thus creating a potential valley running through the sphere from north to south. Since the magnetic force is much weaker than the dipole force, however, this valley is extremely flat. Let us now follow the movement of the field configuration as z runs from $+\infty$ to $-\infty$. Clearly, the liquid likes to stay for the largest possible portion of the z axis close to the north and south poles. The crossing over to the other side will take place on a small piece only. The dipole energy is the strongest effect at hand, the value of θ will stay fixed at 104^0. Thus, the curve representing the field moves as shown in Fig. 6.9. Fig. XV

While crossing to the other side it moves through the valley $\theta \approx 104^0$ and has to overcome only the magnetic energy. Correspondingly, the soliton $(+-0)$ has the size determined by ξ_{mg} which is quite large.

The soliton $(+-1)$, on the other hand, always has to cross the dipole barrier and has therefore the much smaller size ξ_{d}.

Finally, the last configuration $(++1)$ can lower its energy by deforming the line as shown in Fig. 6.10. Fig. XVI

As is obvious by inspecting this figure, such a soliton can decay into the previous two, one with dipole and one with the much lower magnetic energy.

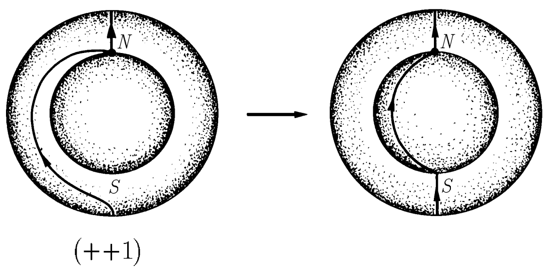


FIGURE 6.10 Another possible class of solitons has an order parameter which starts at the north pole (say) of the spherical shell, goes to the surface of the sphere, re-emerges at the diametrally opposite (identical) point and ends up at the same point it started out. Along the way it tries to stay as much as possible on the spherical shell preferred by the dipole energy.

7

Hydrodynamic Limit at any Temperature $T < T_c$

Until now, the discussion has been limited to temperatures in the vicinities of T_c. Only then can the expansion of the collective action (3.11) in A_{ai} converge. For $T << T_c$ the field A_{ai} can no longer be assumed to be small since it oscillates around a nonzero average value A^0_{ai} whose size increases as the temperature drops. For $T \approx T_c$, such a behavior is shown by formula (4.21). In order to obtain an approximation to the collective action (3.11) valid for all temperatures we have to expand in fluctuations around the finite average value A^0_{ai} by setting Eq. (he-9.1)

$$A_{ai} = A^0_{ai} + A'_{ai}. \tag{7.1}$$

Then the collective action becomes a functional of A'_{ai} only Eq. (he-9.2)

$$\begin{aligned}\mathcal{A}_{\rm coll}[A'] = &-\frac{i}{2}\mathrm{Tr}\log\begin{pmatrix} i\partial_t - \xi(-i\boldsymbol{\nabla}) & i\tilde{\nabla}_i\sigma_a(A^0_{ai}+A'_{ai}) \\ i\tilde{\nabla}_i\sigma_a(A^{0*}_{ai}+A'_{ai}{}^*) & i\partial_t+\xi(i\nabla)\end{pmatrix}\\ &-\frac{1}{3g}\int d^4x A^{0*}_{ai}A^0_{ai} - \frac{1}{3g}\int d^4x\left(A^{0*}_{ai}A'_{ai}+c.c.\right) - \frac{1}{3g}\int d^4x A'_{ai}{}^*A'_{ai}.\end{aligned} \tag{7.2}$$

Here we have used the dimensionless derivative Eq. (he-)

$$\tilde{\nabla}_i \equiv \frac{1}{2p_F}\overleftrightarrow{\nabla}_i. \tag{7.3}$$

The trace log part can be rewritten in analogy with (3.14) as Eq. (he-9.3)

$$\begin{aligned}\mathcal{A}_{\rm coll}[A'] = &-\frac{i}{2}\mathrm{Tr}\log\begin{pmatrix} i\partial_t - \xi(-i\boldsymbol{\nabla}) & i\tilde{\nabla}_i\sigma_a A^0_{ai} \\ i\tilde{\nabla}_i\sigma_a A^{0*}_{ai} & i\partial_t+\xi\end{pmatrix} - \frac{1}{3g}\int d^4x A^{0*}_{ai}A^0_{ai}\\ &-\frac{i}{2}\mathrm{Tr}\log\left\{1 - iG_{A^0}\begin{pmatrix} 0 & i\tilde{\nabla}_i\sigma_a A'_{ai} \\ i\tilde{\nabla}_i\sigma_a A'_{ai}{}^* & 0\end{pmatrix}\right\}\\ &-\frac{1}{3g}\int d^4x\,(A^*_{ai}A'_{ai}+c.c.) - \frac{1}{3g}\int d^4x|A'_{ai}|^2\end{aligned} \tag{7.4}$$

where Eq. (he-9.4)

$$G_{A^0} \equiv i\begin{pmatrix} i\partial_t - \xi(-i\boldsymbol{\nabla}) & i\tilde{\nabla}_i\sigma_a A^0_{ai} \\ i\tilde{\nabla}_i\sigma_a A^{0*}_{ai} & i\partial_t+\xi(i\boldsymbol{\nabla})\end{pmatrix}^{-1} \tag{7.5}$$

is the propagator in the presence of the constant A^0 field.

The first two terms can be dropped since they are an irrelevant constant due to their lack of depending on the fluctuating field A'. Expanding in powers of A', we
Eq. (he-9.5) have

$$\mathcal{A}_{\rm coll}[A'] = \sum_{n=1}^{\infty} \mathcal{A}_n[A'] \tag{7.6}$$

Eq. (he-9.6) with a linear term

$$\mathcal{A}_1[A'] = \frac{1}{2}\mathrm{Tr}\left(G_{A^0} i\tilde{\nabla}_i \sigma_a A'_{ai}\frac{\tau^+}{2}\right) - \frac{1}{3g}\int d^4x A^{0*}_{ai} A'_{ai} + c.c. \tag{7.7}$$

where $\tau^+/2$ is the same 2×2-matrix as $\sigma^+/2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ but acting on the two field
Eq. (he-9.7) components of (3.10). The quadratic term is

$$\mathcal{A}_2[A'] = \frac{i}{4}\mathrm{Tr}\left[G_{A^0}\begin{pmatrix} 0 & i\tilde{\nabla}_i\sigma_a A'_{ai} \\ i\tilde{\nabla}_i\sigma_a {A'}^*_{ai} & 0 \end{pmatrix}\right]^2 . \tag{7.8}$$

The linear term is eliminated by the requirement that $\mathcal{A}$ be stationary under fluc-
Eq. (he-gap) tuations in A'_{ai}. This condition yields the *gap equation*:

$$A^0_{ai} = \frac{3g}{2}\mathrm{Tr}\left(\sigma_a i\tilde{\nabla}_i G_{A^0}(x,y)\frac{\tau^-}{2}\right)\Bigg|_{x=y-\epsilon} . \tag{7.9}$$

The propagator (7.5) can be calculated most easily for the case of a unitary
Eq. (he-9.10) matrix

$$\Delta_{\alpha\beta}(\tilde{p}) \;\;\equiv\;\; \tilde{p}_i(\sigma_a)_{\alpha\beta}A_{ai}, \tag{7.10}$$

Eq. (he-) where $\tilde{\mathbf{p}}$ denotes the dimensionless vector $\mathbf{p}/p_F$. Then

$$\Delta_{\alpha\beta}\Delta^\dagger_{\beta\gamma} = \frac{1}{2}\mathrm{Tr}(\Delta\Delta^\dagger)\delta_{\alpha\gamma}, \tag{7.11}$$

The condition is satisfied if A^0_{ai} has the form (4.17) of the A-phase or (4.18) of the B-phase [not, however, for the A_1 phase (4.19)]. In A- and B-phases the right hand
Eq. (he-9.10a) side becomes

$$\left\{ \begin{array}{c} \Delta_A^2 \sin^2\theta \\ \Delta_B^2 \end{array} \right\} \tilde{p}^2 \equiv \Delta^2 \tilde{p}^2 \tag{7.12}$$

where θ is the angle between $\mathbf{l}$ and the momentum vector $\tilde{p}$. In momentum space,
Eq. (he-9.11) the propagator is

$$G_{A^0}(\omega, p) = \frac{1}{\omega^2 + \xi^2(p) + \Delta_A^2\sin^2\theta\ \tilde{p}^2}\begin{pmatrix} (i\omega + \xi(\mathbf{p}))\delta_{\alpha\beta} & -\Delta_{\alpha\beta}(\tilde{p}) \\ -\Delta^\dagger_{\alpha\beta}(\tilde{p}) & (i\omega - \xi(\mathbf{p}))\delta_{\alpha\beta} \end{pmatrix} \tag{7.13}$$

for the A-phase, with $\Delta_A^2\sin^2\theta$ replaced by Δ_B^2 in the B-phase. This matrix can be diagonalized via a so-called *Bogoljubov transformation* with the diagonal values
Eq. (he-9.12) displaying pure propagators of energy

$$E(\mathbf{p}) = \pm\sqrt{\xi(\mathbf{p})^2 + \left\{ \begin{array}{c} \Delta_A^2 \sin^2\theta \\ \Delta_B^2 \end{array} \right\}}. \tag{7.14}$$

The energies show a gap $\Delta_A^2 \sin^2\theta$ or Δ_B^2. In the B-phase, the gap is isotropic just as in a superconductor. In the A-phase, on the other hand, there is an unisotropy along the $\mathbf{l}$ axis with the gap vanishing for momenta along $\mathbf{l}$.

The size of the gap is found by solving the gap equation (7.9). Inserting (7.13), this takes the form Eq. (he-9.13a)

$$\begin{aligned} A^0_{ai} &= 3g\sum_j T \sum_{\omega_n,\mathbf{p}} \tilde{p}^i\tilde{p}^j \frac{1}{\omega_n^2 + E^2(\mathbf{p})} A^0_{aj} \\ &= 3g\sum_j \sum_{\mathbf{p}} \tilde{p}^i\tilde{p}^j \frac{1}{2E(\mathbf{p})} \tan\frac{E(\mathbf{p})}{2T} A^0_{aj}. \end{aligned} \tag{7.15}$$

or Eq. (he-9.13)

$$\begin{aligned} \frac{1}{g}\frac{1}{3}\delta_{ij} &= T\sum_{\omega_n,\mathbf{p}} \tilde{p}^i\tilde{p}^j \frac{1}{\omega_n^2 + E^2(\mathbf{p})} \\ &= \sum_{\mathbf{p}} \tilde{p}^i\tilde{p}^j \frac{1}{2E(\mathbf{p})} \tan\frac{E(\mathbf{p})}{2T}. \end{aligned} \tag{7.16}$$

the momentum integration can be split into size and direction Eq. (he-9.14)

$$\int \frac{d^3p}{(2\pi)^3} \approx \mathcal{N}(0) \int \frac{d\hat{\mathbf{p}}}{4\pi} \int d\xi \tag{7.17}$$

where $\mathcal{N}(0)$ is the density of states at the surface of the Fermi sea. Since the integration over $d\xi$ is cut off at a value $\omega_{\text{cutoff}} \approx \frac{1}{10}T_F$, the momenta stay sufficiently close to the Fermi sphere to make $\tilde{p}$ approximately to unit vectors: $\tilde{\mathbf{p}} \approx \hat{\mathbf{p}} \equiv \mathbf{p}/|\mathbf{p}|$. Then (7.16) becomes Eq. (he-9.15)

$$\frac{1}{g}\int \frac{d\hat{\mathbf{p}}}{4\pi}\hat{p}_i\hat{p}_j \approx \mathcal{N}(0)\sum_j \int \frac{d\hat{\mathbf{p}}}{4\pi}\hat{p}_i\hat{p}_j \left(\int_{-\omega_{\text{cutoff}}}^{\omega_{\text{cutoff}}} d\xi \frac{1}{2E} \tan\frac{E}{2T} \right). \tag{7.18}$$

We may eliminate the coupling constant g in favor of the critical temperature T_c by using (3.20). This gives Eq. (he-9.16)

$$\begin{aligned} &\int_{-\omega_{\text{cutoff}}}^{\omega_{\text{cutoff}}} d\xi \frac{1}{2\xi} \tan\frac{\xi}{2T_c} = \\ &\quad \frac{3}{4}\int_{-1}^{1} dz \left(1 - z^2\right) \int_{\omega_{\text{cutoff}}}^{\omega_{\text{cutoff}}} d\xi \frac{1}{2\sqrt{\xi^2 + \Delta^2}} \tan\left(\sqrt{\xi^2 + \Delta^2}/2T\right) \end{aligned} \tag{7.19}$$

In order to extract the finite content, one may subtract

$$\int_{\omega_{\text{cutoff}}}^{\omega_{\text{cutoff}}} d\xi \frac{1}{2\xi} \tan\frac{\xi}{2T}$$

on both sides. Then ξ-integral converges and we can remove the cutoff which leads to Eq. (he-9.17)

$$\log\frac{T}{T_c} = \frac{3}{4}\int_{-1}^{1} dz\left(1-z^2\right)\int_{-\infty}^{\infty} d\xi \times\left[\frac{1}{2\sqrt{\xi^2+\Delta^2}}\tan\left(\sqrt{\xi^2+\Delta^2}/2T\right) - \frac{1}{2\xi}\tan\frac{\xi}{2T}\right]. \tag{7.20}$$

From this we may calculate T/T_c as a function of Δ_{AB}/T_c as follows:

For small T, the integral increases as $\log T$ due to the small ξ behavior of the second term. The finite piece is determined by setting $\tan\left(\sqrt{\xi^2+\Delta^2}/2T\right) \approx 1$ (which is good to experimental accuracy $e^{-\Delta/T}$ except in the A phase for $z \approx \pm 1$)
Eq. (he-9.18) integrating the first and partially integrating the second term:

$$\begin{aligned}\log\frac{T}{T_c} &\approx \frac{3}{4}\int_{-1}^{1} dz(1-z^2)\left\{\left[\log\left(\sqrt{\xi^2+\Delta^2}+\xi\right)/2T - \log\frac{\xi}{2T}\tan\frac{\xi}{2T}\right] + \int_0^\infty d\mu\frac{\log\mu}{\cosh^2\mu}\right\}\\ &= \frac{3}{4}\int_{-1}^{1} dz(1-z^2)\left\{\log\frac{4T}{\Delta} - \log\frac{4e^\gamma}{\pi}\right\}\\ &= \log\frac{T}{\Delta_{\rm max}e^\gamma/\pi} - \frac{3}{8}\int_{-1}^{1} dz(1-z^2)\log\frac{\Delta^2}{\Delta^2_{\rm max}}\end{aligned} \tag{7.21}$$

Eq. (he-9.20) In the B phase, $\Delta \equiv \Delta_{\rm max} = \Delta_B$ and

$$\Delta_B/T_c = \pi e^{-\gamma} \approx 1.76; \quad T \approx 0. \tag{7.22}$$

Eq. (he-9.21) In the A phase, $\Delta = \Delta_A \sin\Theta$ and the integral becomes

$$-\frac{3}{8}\int_{-1}^{1} dz(1-z^2)\log(1-z^2) = \frac{5}{6} - \log 2 \approx \log 1.15 \tag{7.23}$$

Eq. (he-9.22) so that

$$\Delta_A/T_c = \pi e^{-\gamma}\frac{e^{5/6}}{2} \approx 2.03\,. \tag{7.24}$$

For small T, this value is approached exponentially $\sim e^{\Delta_B/T}$ for the B-phase and with a power law T^4 for the A-phase (due to the vanishing of $\Delta_A \sin g\theta$ along the unisotropy axis $\mathbf{1}$).

For arbitrary T, the calculation of (7.20) is done (as in the case of superconduc-
Eq. (he-ge) tivity in Part II) by using the expansion into Matsubara frequencies

$$\frac{1}{2E}\tan\frac{E}{2T} = \frac{1}{2E}T\sum_{\omega_n}\left(\frac{1}{i\omega_n+E} - \frac{1}{i\omega_n - E}\right) = T\sum_{\omega_n}\frac{1}{\omega^2+\xi^2+\Delta^2}. \tag{7.25}$$

Eq. (he-9.24) This can be integrated over ξ and we find for the gap equation (7.20):

$$\log \frac{T}{T_c} = 2\pi \; \frac{3}{4} \int_{-1}^{1} dz(1-z^2) \; T \sum_{\omega_n > 0} \left(\frac{1}{\sqrt{\omega_n^2 + \Delta^2}} - \frac{1}{\omega_n} \right). \tag{7.26}$$

Eq. (he-9.25) At this place one introduces the auxiliary dimensionless quantity

$$\delta = \frac{\Delta}{\pi T} \tag{7.27}$$

and a reduced version of the Matsubara frequencies: Eq. (he-9.26)

$$x_n \equiv (2n+1)/\delta. \tag{7.28}$$

Then, the gap equation (7.20) takes the form Eq. (he-9.27)

$$\log \frac{T}{T_c} = \frac{2}{\delta} \left[\frac{3}{4} \int_{-1}^{1} dz(1-z^2) \right] \sum_{n=0}^{\infty} \left(1 \middle/ \sqrt{x_n^2 + \left\{ \begin{array}{c} 1 \\ 1-z^2 \end{array} \right\}} - 1/x_n \right) \tag{7.29}$$

in the B and the A phase, respectively. In the B phase, the angular integral in the brackets gives a factor 1, so that Eq. (he-9.27b)

$$\text{B}: \quad \log \frac{T}{T_c} = \frac{2}{\delta} \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{x_n^2 + 1}} - \frac{1}{x_n} \right). \tag{7.30}$$

In the second case, it leads to Eq. (he-9.28)

$$\text{A}: \quad \log \frac{T}{T_c} = \frac{2}{8} \sum_{n=0}^{\infty} \left\{ \frac{3}{4} \left[(1-x_n^2) \arctan \frac{1}{x_n} + x_n \right] - \frac{1}{x_n} \right\}. \tag{7.31}$$

The curves $\Delta_{A,B}/T_c$ are plotted in Fig. 7.1. Fig. IXXX

The $T \approx T_c$ behavior can be extracted from (7.29) by expanding the sum for large x_n. The leading term is Eq. (he-9.29)

$$\left\{ \begin{array}{c} 1 \\ 1-z^2 \end{array} \right\} \sum_{n=0}^{\infty} \frac{1}{2x_n^3} = \left\{ \begin{array}{c} 1 \\ 1-z^2 \end{array} \right\} \frac{\delta^3}{2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \left\{ \begin{array}{c} 1 \\ 1-z^2 \end{array} \right\} \frac{\delta^3}{2} \frac{7}{8} \zeta(3) \tag{7.32}$$

so that Eq. (he-9.30)

$$\Delta_B/T_c = \pi\delta = \pi \sqrt{\frac{8}{7\zeta(3)}} \left(1 - \frac{T}{T_c} \right)^{1/2} \approx 3.063 \left(1 - \frac{T}{T_c} \right)^{1/2},$$
$$\Delta_A/T_c = \pi\delta = \pi \sqrt{\frac{10}{7\zeta(3)}} \left(1 - \frac{T}{T_c} \right)^{1/2} \tag{7.33}$$

in agreement with the determination (4.21).

Consider now the free-field part $\mathcal{A}_2[A']$ of the collective action, Eq. (7.8). In momentum space, it can be written in the form Eq. (he-9.31)

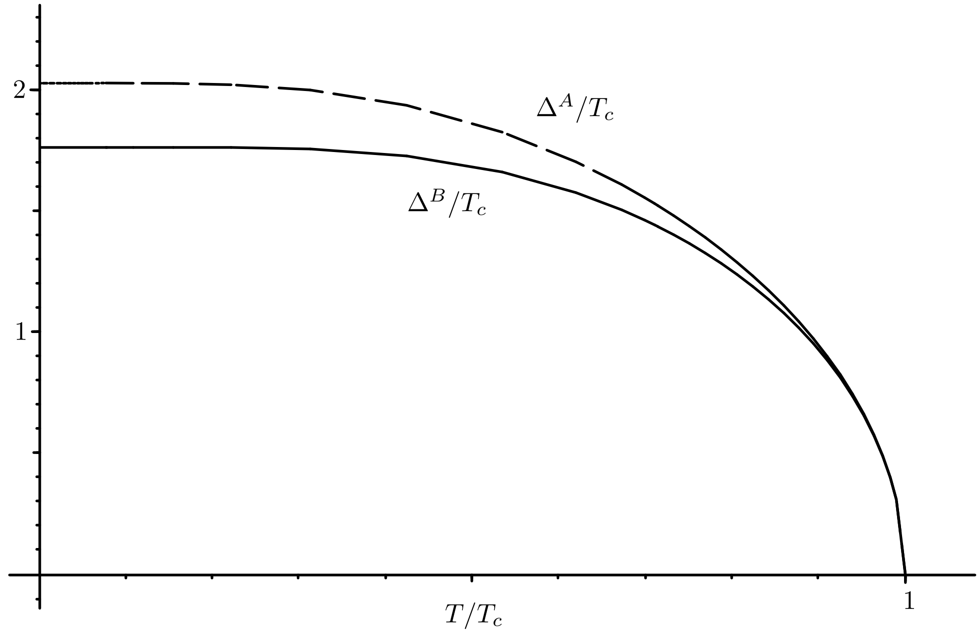


FIGURE 7.1 The fundamental quantities of superfluid ^{3}He-B and A are shown as a function of temperature. The superscript FL denotes the Fermi liquid corrected values.

$$\mathcal{A}_2 = \frac{1}{2}\sum_k \Big\{ A'_{ai}(k)^* L^{ij}_{11}(k) A'_{aj}(k) + A'_{ai}(-k) L^{ij}_{22}(k) A'_{aj}(-k)^* + A'_{ai}(k)^* L^{ij}_{12}(k) A'_{aj}(-k)^* + A'_{ai}(-k) L^{ij}_{21}(k) A'_{ai}(k) \Big\} \tag{7.34}$$

where, with the notation $k_0 = -i\nu$, Eq. (he-9.32)

$$L^{ij}_{11}(k) = L^{ij}_{22}(k) = \int \frac{d^3p}{(2\pi)^3} \tilde{p}_i \tilde{p}_j T \sum_{\omega_n} \frac{\omega_n^2 - \nu^2/4 + \xi_+\xi_-}{\left[\left(\omega_n - \frac{\nu}{2}\right)^2 + E_+^2\right]\left[\left(\omega_n + \frac{\nu}{2}\right)^2 + E_-^2\right]} - \frac{\delta_{ij}}{g}. \tag{7.35}$$

Eq. (he-9.33) and

$$L^{ij,ab}_{12}(k) = \left[L^{ij,ab}_{21}(k)\right]^* = -\int \frac{d^3p}{(2\pi)^3} \tilde{p}_i \tilde{p}_j \tilde{p}_{i'} \tilde{p}_{j'} A^0_{a'i'} A^{0*}_{b'j} t_{a'b',ab} T \sum_{\omega_n} \frac{1}{\left[\left(\omega_n - \frac{\nu}{2}\right)^2 + E_+^2\right]\left[\left(\omega_n + \frac{\nu}{2}\right)^2 + E_-^2\right]}. \tag{7.36}$$

Eq. (he-) with

$$t_{a'b'ab} \equiv \frac{1}{2}\mathrm{tr}(\sigma_{a'}\sigma_{b'}\sigma_a\sigma_b) = \delta_{a'a}\delta_{b'b} + \delta_{a'b}\delta_{b'a} - \delta_{a'b'}\delta_{ab} \tag{7.37}$$

Eq. (he-9.34) and

$$\begin{Bmatrix} \xi_+ \\ \xi_- \end{Bmatrix} \equiv \frac{(\mathbf{p}\pm\mathbf{k}/2)^2}{2m}+\ldots = \frac{\mathbf{p}^2}{2m}\pm\frac{1}{2m}\mathbf{p}\cdot\mathbf{k}+\frac{\mathbf{k}^2}{8m}\approx\xi\pm\frac{1}{2}\mathbf{v}\cdot\mathbf{k}+\ldots\ ,$$
$$\begin{Bmatrix} E_+ \\ E_- \end{Bmatrix} = \sqrt{\begin{Bmatrix} \xi_+^2 \\ \xi_-^2 \end{Bmatrix}+\Delta^2}\approx E\pm\frac{1}{2}\mathbf{v}\cdot\mathbf{k}\frac{\xi}{E}+\frac{1}{8}(\mathbf{v}\cdot\mathbf{k})^2\frac{\Delta^2}{E^3}+\ldots \tag{7.38}$$

with ξ, $E=\sqrt{\xi^2+\Delta^2}$ being the average values of ξ_+,ξ_-; E_+,E_-. As usual, the integral over d^3p can be split into size and directional integral according to (7.5) and we can set $\tilde{\mathbf{p}}\approx\hat{\mathbf{p}}$ and $\mathbf{v}\equiv\mathbf{p}/m\approx v_F\hat{\mathbf{p}}$.
We now rearrange the terms in the sum in such a way that we obtain combinations of single sums of the type

$$T\sum_{\omega_n}\frac{1}{i\omega_n-E_+} \tag{7.39}$$

which lead to the Fermi distribution function Eq. (he-dis)

$$T\sum_{\omega_n}\frac{1}{i\omega_n-E}=f(E)\equiv\frac{1}{e^{E/T}+1} \tag{7.40}$$

with the property

$$f(E)=1-f(-E). \tag{7.41}$$

If we drop the subscripts n and introduce the notation $\omega_\pm\equiv\omega\pm\nu/2$, the decomposition of terms in the sum of $L_{12}^{ijab}(k)$ is Eq. (he-dec)

$$\begin{aligned}
&\frac{1}{[\omega_+^2+E_+^2]\,[\omega_-^2+E_-^2]}\\
&\quad=\frac{1}{4E_+E_-}\left(\frac{1}{i\omega_++E_+}-\frac{1}{i\omega_+-E_+}\right)\left(\frac{1}{i\omega_-+E_-}-\frac{1}{i\omega_--E_-}\right)\\
&\quad=\frac{1}{4E_+E_-}\left\{-\frac{1}{E_++E_-+i\nu}\left(\frac{1}{i\omega_--E_-}-\frac{1}{i\omega_--E_-}\right)\right.\\
&\qquad+\frac{1}{E_++E_-+i\nu}\left(\frac{1}{i\omega_++E_+}-\frac{1}{i\omega_--E_-}\right)\\
&\qquad-\frac{1}{E_+-E_-+i\nu}\left(\frac{1}{i\omega_++E_+}-\frac{1}{i\omega_-+E_-}\right)\\
&\qquad\left.+\frac{1}{E_+-E_--i\nu}\left(\frac{1}{i\omega_+-E_+}-\frac{1}{i\omega_--E_-}\right)\right\}
\end{aligned} \tag{7.42}$$

We now use of (7.40) and the fact that the frequency shifts ν in $\omega_\pm$ [see (11.16)] do not appear in (12.5) since they amount to a mere translation in the infinite sum. Collecting the different terms we find Eq. (he-9.33b)

$$\begin{aligned}
&L_{12}^{ij,ab}(k)=\left[L_{21}^{ij,ab}(k)\right]^*= \\
&\quad-\int\frac{d^3p}{(2\pi)^3}\tilde{p}_i\tilde{p}_j\tilde{p}_{i'}\tilde{p}_{j'}A^0_{a'i'}A^{0*}_{b'j'}\frac{t_{a'b',ab}}{2E_-E_+}
\end{aligned} \tag{7.43}$$

$$\times \left\{ \frac{E_+ + E_-}{(E_+ + E_-)^2 + \nu^2} (1 - f(E_+) - f(E_-)) + \frac{E_+ - E_-}{(E_+ - E_-)^2 + \nu^2} (f(E_+) - f(E_-)) \right\}. \tag{7.44}$$

Eq. (he-) In the first expression we decompose

$$\frac{\omega_n^2 - \nu^2/4 + \xi_+\xi_-}{[\omega_+^2 + E_+^2]\,[\omega_-^2 + E_-^2]} = \frac{1}{2}\Bigg\{ \frac{1}{\omega_+^2 + E_+^2} + \frac{1}{\omega_-^2 + E_-^2} - (E_+^2 + E_-^2 + \nu^2 - 2\xi_+\xi_-) \frac{1}{[\omega_+^2 + E_+^2]\,[\omega_-^2 + E_-^2]} \Bigg\}. \tag{7.45}$$

When summing the first two terms we use the formula

$$T \sum_\omega \frac{1}{\omega^2 + E^2} = \frac{1}{2E}[f(-E) - f(E)] = \frac{1}{2E} \tanh \frac{E}{2T}. \tag{7.46}$$

In the last term, the right-hand factor was treated before. Replacing the factor $E_-^2 + E_+^2 + \nu^2$ once by $(E_- + E_+)^2 + \nu^2 - 2E_-E_+$ and once by $(E_- - E_+)^2 + \nu^2 + 2E_-E_+$
Eq. (he-9.32b) we obtain immediately

$$L_{11}^{ij}(k) = L_{22}^{ij}(k) = \int \frac{d^3p}{(2\pi)^3} \tilde{p}_i \tilde{p}_i \times \left\{ \frac{E_+E_- + \xi_+\xi_-}{2E_+E_-} \frac{E_+ + E_-}{(E_+ + E_-)^2 + \nu^2} (1 - f(E_+) - f(E_-)) - \frac{E_+E_- - \xi_+\xi_-}{2E_+E_-} \frac{E_+ - E_-}{(E_+ - E_-)^2 + \nu^2} (f(E_+) - f(E_-)) \right\} - \frac{\delta_{ij}}{g}. \tag{7.47}$$

For the remainder of this chapter we shall specialize on the static case and consider only the long-wavelength limit of small $\mathbf{k}$. Hence, we shall take $k_0 = 0$ and study
Eq. (he-9.35) the lowest orders in k only. At $\mathbf{k} = 0$ we find from (7.47) and (7.43(he-9.33b))

$$L_{11}^{ij}(0) = \mathcal{N}(0) \int \frac{d\hat{\mathbf{p}}}{4\pi} \hat{p}_i \hat{p}_j \int d\xi \left\{ \frac{E^2 + \xi^2}{4E^3} \left[\tan \frac{E}{2T} + 2f'(E) \right] - \frac{1}{g} \right\}. \tag{7.48}$$

Eq. (he-9.36) and

$$L_{12}^{ij,ab}(0) = -\frac{1}{2} \mathcal{N}(0) \int \frac{d\hat{\mathbf{p}}}{4\pi} \hat{p}_i \hat{p}_j \hat{p}'_i \hat{p}'_j \frac{\phi(\Delta)}{\Delta^2} A^0_{a'i'} A^{0*}_{b'j'} t_{a'b'ab} \tag{7.49}$$

Eq. (he-9.37) where we have introduced the function

$$\phi(\Delta) = \Delta^2 \left[\int_0^\infty d\xi \frac{1}{E^3} \tan \frac{E}{2T} + 2 \int_0^\infty d\xi \frac{1}{E^2} f'(E) \right]. \tag{7.50}$$

We now observe that due to the gap equation (7.18) $L_{11}^{ij}(k)$ can also be expressed
Eq. (he-9.37a) in terms this function as of

$$L_{11}^{ij}(0) = -\frac{1}{2} \mathcal{N}(0) \int \frac{d\hat{\mathbf{p}}}{4\pi} \hat{p}_i \hat{p}_j \phi(\Delta). \tag{7.51}$$

The first integral in Eq. (7.50) can be done in parts and brought to the more
Eq. (he-9.38a) convenient form

$$\phi(\Delta) = 1 - \frac{1}{2T}\int_0^\infty d\xi \frac{1}{\cosh^2(E/2T)}. \tag{7.52}$$

For $T \approx 0$, this function approaches zero exponentially. The full temperature behavior is best calculated by using the Matsubara sum of (7.36) to write Eq. (he-9.38)

$$\begin{aligned}\phi(\Delta) &= 2T\sum_{\omega_n}\int d\xi \frac{\Delta^2}{(\omega_n^2+E^2)^2} = -2\Delta^2 T\sum_{\omega_n}\frac{\partial}{\partial\omega_n^2}\int d\xi\frac{1}{\omega_n^2+\xi^2+\Delta^2} \\ &= -2\Delta^2 T\sum_{\omega_n}\frac{\partial}{\partial\omega_n^2}\frac{\pi}{\sqrt{\omega_n^2+\Delta^2}} = 2T\pi\sum_{\omega_n>0}\frac{1}{\sqrt{\omega_n^2+\Delta^2}^3}.\end{aligned} \tag{7.53}$$

Using again the variables δ and x_n from (7.27) and (7.28(he-9.26)), this becomes Eq. (he-9.39)

$$\phi(\Delta) = \frac{2}{\delta}\left\{\begin{matrix}1\\1-z^2\end{matrix}\right\}\sum_{n=0}^{\infty}\frac{1}{\sqrt{x_n^2+\left\{\begin{matrix}1\\1-z^2\end{matrix}\right\}}^3} \tag{7.54}$$

for the B and the A phase, respectively. For $T \approx T_c$, $\delta \to 0$ and Eq. (he-)

$$\phi(\Delta) \approx 2\delta^2\left\{\begin{matrix}1\\1-z^2\end{matrix}\right\}\frac{7\zeta(3)}{8}. \tag{7.55}$$

Consider the equations (7.48), (7.51(he-9.37a)) further. Let us write $L_{11}^{ij}(0)$ as follows Eq. (he-9.42)

$$L_{11}^{ij}(0) = -\frac{1}{4m^2v_F^2}\rho_{ij}. \tag{7.56}$$

where Eq. (he-9.45)

$$\rho_{ij} \equiv 3\rho\int\frac{d\hat{\mathbf{p}}}{4\pi}\hat{p}_i\hat{p}_j\phi(\Delta) \tag{7.57}$$

In the B phase, the angular integral in (7.51) can be done and we find Eq. (he-)

$$\rho_{ij}^B = \frac{2}{3}v_F^2 m^2\mathcal{N}(0)\phi(\Delta)\delta_{ij} = \rho\phi(\Delta)\delta_{ij} \equiv \rho_s^B\delta_{ij} \tag{7.58}$$

where $\phi^B(\Delta)$ is the upper of the functions (7.54) (the isotropic one). The invariant ρ_s^B will be called the *superfluid density* of the B phase. For $T \approx T_c$, (7.53) gives a behavior as Eq. (he-9.44)

$$\rho_s^B \approx 2\rho\left(1-\frac{T}{T_c}\right). \tag{7.59}$$

For $T = 0$, we see from (7.50) that $\phi = 1$ so that Eq. (he-)

$$\rho_s^B = \rho, \qquad T = 0. \tag{7.60}$$

In the A phase, the integral Eq. (he-9.45b)

$$\rho^A_{ij} \equiv 3\rho \int \frac{d\hat{\mathbf{p}}}{4\pi} \hat{p}_i \hat{p}_j \phi(\Delta) \tag{7.61}$$

can be expanded into covariants Eq. (he-9.46)

$$\rho^A_{ij} = \rho_s \left(\delta_{ij} - l_i l_j\right) + \rho^{\|}_s l_i l_j \tag{7.62}$$

Eq. (he-9.47) with the coefficients, the *superfluid densities* of the A phase

$$\frac{\rho_s}{3\rho} \equiv \int \frac{d\hat{p}}{4\pi} p_x^2 \phi(\Delta) = \frac{1}{2} \int_{-1}^{1} \frac{dz}{2} (1-z^2)\phi(\Delta) \equiv \frac{1}{2}\phi^A(\Delta)$$
$$\frac{\rho^{\|}_s}{3\rho} \equiv \int \frac{d\hat{p}}{4\pi} p_z^2 \phi(\Delta) = \frac{1}{2} \int_{-1}^{1} \frac{dz}{2} (1-z^2)\phi(\Delta). \tag{7.63}$$

The spatially averaged ϕ function in the first line will appear repeatedly in the further description of the A phase and has therefore been given an extra name

Eq. (he-9.48) $\phi^A(\Delta)$. Using the integrals

$$\int_{-1}^{1} dz \frac{1}{\sqrt{x_n^2+1-z^2}^3} = \frac{2}{x_n(x_n^2+1)} = \frac{2}{x_n^3} - \frac{2}{x_n^5} + \dots$$
$$\int_{-1}^{1} dz \frac{z^2}{\sqrt{x_n^2+1-z^2}^3} = 2\left(\frac{1}{x_n} - \arctan\frac{1}{x_n}\right) = \frac{2}{3}\frac{1}{x_n^3} - \frac{2}{5}\frac{1}{x_n^5} + \dots$$
$$\int_{-1}^{1} dz \frac{1}{\sqrt{x_n^2+1-z^2}^3} = x_n + 2\frac{x_n^2+1}{x_n} - 3(1+x_n^2)\arctan\frac{1}{x_n} = \frac{2}{5}\frac{1}{x_n^3}$$
$$-\frac{6}{35}\frac{1}{x_n^5} + \dots \quad . \tag{7.64}$$

Eq. (he-9.50) these densities are seen to have the expansions

$$\frac{\rho_s}{3\rho} = \frac{1}{2\delta} \sum_{n=0}^{\infty} \left[3x_n - \frac{2x_n}{x_n^2+1} + \left(1-3x_n^2\right)\arctan\frac{1}{x_n}\right]$$
$$= \frac{8}{15}\frac{1}{\delta}\left(\sum_{n=0}^{\infty}\frac{1}{x_n^3} - \frac{9}{7}\sum_{n=0}^{\infty}\frac{1}{x_n^5} + \dots\right) \tag{7.65}$$
$$\frac{\rho^{\|}_s}{3\rho} = \frac{1}{\delta} \sum_{n=0}^{\infty} \left[-3x_n - (3x_n^2+1)\arctan\frac{1}{x_n}\right]$$
$$= \frac{4}{15}\frac{1}{\delta}\left(\sum_{n=0}^{\infty}\frac{1}{x_n^3} - \frac{6}{7}\sum_{n=0}^{\infty}\frac{1}{x_n^5} + \dots\right). \tag{7.66}$$

Eq. (he-) For $T \approx T_c$,

$$\delta \approx \frac{10}{7\zeta(3)}\left(1 - \frac{T}{T_c}\right) \to 0, \tag{7.67}$$

Eq. (he-9.51) and the first two sums can be done with the result:

$$\rho_s \approx \frac{8}{5} c_3 \delta^2 \left(1 - \frac{9}{7}\frac{c_5}{c_3}\delta^2 + \ldots\right) \approx 2\left(1 - \frac{T}{T_c}\right) + \ldots \tag{7.68}$$

$$\rho_s{}^{\|} \approx \frac{4}{5} c_3 \delta^2 \left(1 - \frac{6}{7}\frac{c_5}{c_3}\delta^2 + \ldots\right) \approx \left(1 - \frac{T}{T_c}\right) + \ldots \tag{7.69}$$

where

$$c_k = \frac{2^k - 1}{2^k}\zeta(k) \tag{7.70}$$

are the results of the sums $\sum_{n=0}^{\infty}(2n+1)^{-k}$ ($c_3 \approx 1.0518, \quad c_5 \approx 1.0045$). The higher terms are omitted on the right-hand sides although they will be of use later).

For $T = 0$, we have again $\phi(\Delta) = 1$ and see from (7.63) (7.64(he-9.48)) that

$$\rho_s = \rho_s^{\|} = \rho, \qquad T = 0.$$

The full temperature behavior of the superfluid densities is shown in Figs. 7.2.

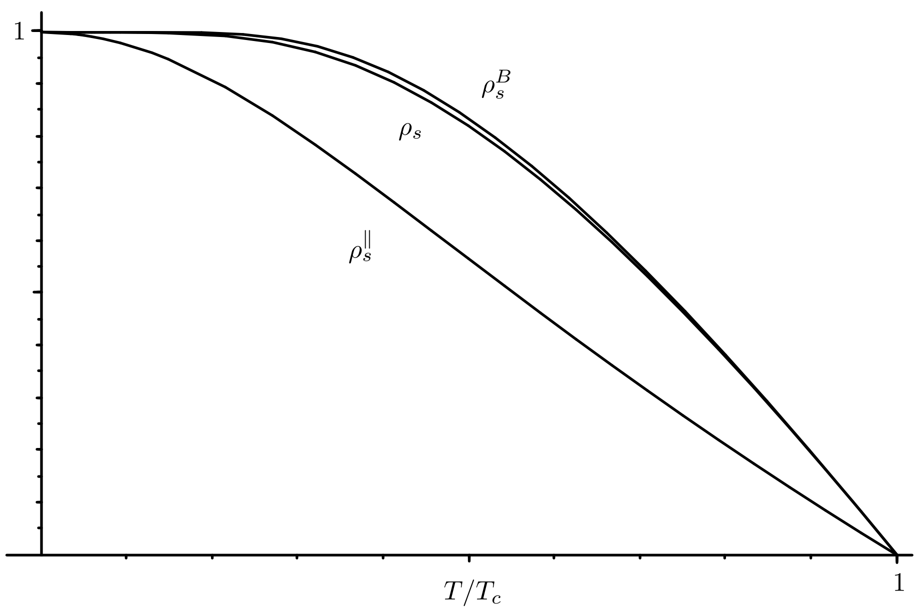


FIGURE 7.2 Temperature behavior of the superfluid densities in the A- and B-phase of superfluid ^{4}He.

Consider now the function $L_{12}^{ijab}(0)$. Here it is useful to introduce a tensor Eq. (he-9.53)

$$\rho_{ijkl} \equiv \frac{3}{2}\rho \int \frac{d\hat{\mathbf{p}}}{4\pi} \hat{p}_i \hat{p}_j \hat{p}_k \hat{p}_l \, \phi(\Delta)/\Delta^2 \tag{7.71}$$

in terms of which $L_{12}^{ijab}(0)$ can be written as Eq. (he-9.54)

$$L_{12}^{ijab}(0) = -\frac{1}{2mv_F^2}\rho_{ijkl} A^0_{a'k} A^{0*}_{b'l} \; t_{a'b'ab}. \tag{7.72}$$

In the B phase where the gap is isotropic the angular integration is trivial and we find from (7.71) Eq. (he-9.55)

$$\rho_{ijkl} = \frac{1}{10}\left(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj} + \delta_{ik}\delta_{jl}\right)\rho^B. \tag{7.73}$$

Eq. (he-9.57) In the A phase, this tensor can be expressed in terms of the three covariants

$$\begin{aligned}
\hat{A}_{ijkl} &= (\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj} + \delta_{ik}\delta_{jl}) \\
\hat{B}_{ijkl} &= \delta_{ij}l_kl_l + \delta_{ik}l_jl_l + \delta_{il}l_jl_k + \delta_{jk}l_il_l + \delta_{jl}l_il_k + \delta_{kl}l_il_j \\
\hat{C}_{ijkl} &= l_il_jl_kl_l.
\end{aligned} \tag{7.74}$$

Eq. (he-9.56) as follows

$$\rho_{ijkl} = A\hat{A}_{ijkl} + B\hat{B}_{ijkl} + C\hat{C}_{ijkl}. \tag{7.75}$$

Contracting this with $\delta_{ij}\delta_{kl}$ and $\delta_{ij}l_kl_l$ we find that the coefficients A and B are
Eq. (he-9.58) given by combinations of ρ_s and $\rho_s^{\parallel}$:

$$A = \frac{1}{8}\rho_s \tag{7.76}$$

$$A + B = \frac{1}{4}\rho_s^{\parallel}. \tag{7.77}$$

Eq. (he-9.59) The third coefficients contains another function $\gamma(\Delta)$ of the gap parameter:

$$3A + 6B + C = \frac{3}{8}\gamma \tag{7.78}$$

Eq. (he-9.60) with γ being defined by the angular integral

$$\gamma \equiv 4\rho \int \frac{d\hat{\mathbf{p}}}{4\pi}\hat{p}_z^4\phi(\Delta)\frac{\Delta_A^2}{\Delta^2}. \tag{7.79}$$

Eq. (he-9.61) Inserting (7.54) and doing the angular integrals, we find the series representation

$$\frac{\gamma(\Delta)}{\rho} = \frac{4}{\delta}\sum_{n=0}^{\infty}\left[3x_n + \frac{2}{x_n} - 3(x_n^2+1)\arctan\frac{1}{x_n}\right]. \tag{7.80}$$

By comparing this series with (7.31) and (7.65), we see that $\gamma(\Delta)$ is not a new gap
Eq. (he-9.62) function. In fact, by adding and subtracting the series for $4\log\frac{T}{T_c}$, we find

$$\frac{\gamma(\Delta)}{\rho} \equiv -4\log\frac{T}{T_c} - 2\frac{\rho_s^{\parallel}}{\rho}. \tag{7.81}$$

Eq. (he-) For $T \approx T_c$, γ starts out like

$$\gamma(\Delta) \approx 2\left(1 - \frac{T}{T_c}\right), \tag{7.82}$$

just as $\frac{\rho_s}{\rho}$. As T approaches zero, however, there is a logarithmic divergence which is due to the zeros in the gap [see (7.79)] along the $\mathbf{l}$ direction.

Let us now turn to the bending energies. For this, we expand $L_1 1(k)$ and $L_{12}(k)$ to lowest order in the momentum $\mathbf{k}$ and find

Eq. (he-9.63)

$$f = \frac{1}{4m^2}\left(\rho^{11}_{ijkl}\partial_k A^*_{ai}\partial_l A_{aj}/\Delta^2_{AB} + \mathrm{Re}\,\rho^{12}_{ijklab}\partial_k A^*_{ai}\partial_l A^*_{bj}\right). \qquad (7.83)$$

Here we have dropped in the primes on the fields since with the derivatives present the additional constant A^0_{ai} does not matter. The tensor coefficient is found by going once more through the same calculation as in Section II as

Eq. (he-9.64)

$$\begin{aligned}\rho^{11}_{ijkl} &= \frac{3\rho}{2}\int\frac{d\hat{\mathbf{p}}}{4\pi}\hat{p}_i\hat{p}_j\hat{p}_k\hat{p}_l\left[\phi(\Delta)-\frac{1}{2}\bar{\phi}(\Delta)\right]\frac{\Delta^2_{AB}}{\Delta^2} &(7.84)\\ \rho^{12}_{ijklab} &= -\frac{3\rho}{2}\int\frac{d\hat{\mathbf{p}}}{4\pi}\hat{p}_i\hat{p}_j\hat{p}_k\hat{p}_l\hat{p}_m\hat{p}_n\frac{1}{2}\bar{\phi}(\Delta)\frac{\Delta^2_{AB}}{\Delta^4}A^0_{a'm}A^0_{b'n}t_{a'b'ab} &(7.85)\end{aligned}$$

where ϕ is the same as in (7.53) while $\bar{\phi}(\Delta)$ denotes another function of the gap:

Eq. (he-9.66)

$$\bar{\phi}(\Delta) \equiv 2\pi T\Delta^4\sum_{\omega_n>0}\frac{1}{\sqrt{\omega^2+\Delta^2}^5} = \frac{2}{\delta}(1-z^2)^2\sum_{n=0}^{\infty}\frac{1}{\sqrt{x_n^2+\left\{\begin{array}{c}1-z^2\\1\end{array}\right\}}^5}. \qquad (7.86)$$

In the superconductor, this function does not appear in the hydrodynamic limit. We therefore expect a cancellation also in the B phase where the gap is isotropic. In fact, inserting

Eq. (he-9.67)

$$A_{ai} = \Delta_B\, e^{i\varphi}R_{ai}(\theta) \qquad (7.87)$$

we have

Eq. (he-9.68)

$$\begin{aligned}&\mathrm{Re}\,\rho^{12}_{ijklab}\partial_b e^{-i\varphi}R_{ai}(\theta)\partial_l e^{i\varphi}R_{bj}(\varphi)\\ &\quad= -\frac{3\rho}{2}\int\frac{d\hat{\mathbf{p}}}{4\pi}\hat{p}_i\hat{p}_j\hat{p}_k\hat{p}_l\frac{1}{2}\bar{\phi}(\Delta)\frac{\Delta^2_B}{\Delta^2}\hat{p}_m\hat{p}_n R_{a'm}R_{b'n}t_{a'b'ab}\\ &\qquad\qquad(-\partial_k\varphi\partial_l\varphi R_{ai}R_{bj}+\partial_k R_{ai}\partial_l R_{bj})\\ &\quad= -\frac{3\rho}{2}\int\frac{d\hat{p}}{4\pi}\hat{p}_k\hat{p}_l\frac{1}{2}\bar{\phi}(\Delta)\frac{\Delta^2_B}{\Delta^2}\left(-\partial_k\varphi\partial_l\varphi-\hat{p}_k\hat{p}_l\partial_k R_{ai}\partial_l R_{aj}\right). \qquad (7.88)\end{aligned}$$

But this coincides exactly with the $\bar{\phi}$ content in $\rho^{11}_{ijkl}\partial_k e^{-i\varphi}R_{ai}\partial_l e^{i\varphi}R_{aj}$.

Note that the two terms change sign for different reasons: $\partial_k\varphi_l\varphi$ because of the equality of the phases $e^{i\varphi}$, and $\partial_k R_{ai}\partial_l R_{bj}$ because of the tensor $t_{a'b'ab}$. Thus, for the B phase the result is simply

Eq. (he-9.69)

$$f = \frac{1}{4m^2}\rho_{ijkl}\partial_k A^*_{ai}\partial_l A_{aj}/\Delta^2_B \qquad (7.89)$$

with ρ_{ijkl} being the tensor discussed before in (7.71). This result is exactly the same as for a superconductor except for two additional direction vectors $\hat{p}_i\hat{p}_j$ inserted into the spatial average which are contracted with the vector indices of the fields $A^*_{ai}A_{aj}$. Inserting the decomposition (7.73) we find the energy (see Appendix 7A for details)

Eq. (he-9.70)

$$f = \frac{1}{4m^2}\frac{\rho_s^B}{2}\left[(\boldsymbol{\partial}\varphi)^2 + \frac{4}{5}(\partial_i\tilde{\theta}_j)^2 - \frac{1}{5}(\boldsymbol{\partial}\tilde{\theta})^2 - \frac{1}{5}\partial_i\tilde{\theta}_j\partial_j\tilde{\theta}_i\right]. \tag{7.90}$$

This coincides with the previous result (7.54) for $T \approx T_c$, due to (7.59), with Eq. (he-)

$$K_{23} = 2K_1. \tag{7.91}$$

The superfluid density ρ_s^B was shown before in Fig. 7.2.

In the A phase, matters are considerably more complicated. This is due to the fact that the gap size varies which prevents the $\bar{\phi}(\Delta)$ function to cancel. Consider

Eq. (he-) the field dependent parts of the ρ^{12} contribution:

$$\mathrm{Re}\, A_{a'm}{}^0 A_{b'n}{}^0 t_{a'b'ab}\partial_k A^*_{ai}\partial_l A^*_{bj}/\Delta_A^2 = \tag{7.92}$$

$$\Delta_A{}^2 \mathrm{Re}\, d_{a'}d_{b'}\phi_m\phi_n t_{a'b'ab}\left(\partial_k d_a \partial_l d_b \phi_i^\dagger\phi_j^\dagger + d_a d_b \partial_b\phi_b\phi_i^\dagger\partial_l\phi_j^\dagger\right) \tag{7.93}$$

where the mixed $\partial d\partial\phi$ derivatives vanish due to $\mathbf{d}\partial\mathbf{d} = 0$.

Contracting now the indices a', b', we see that the gradients of $\mathbf{d}$ appear with

Eq. (he-9.71) the opposite sign in the form

$$-\,\mathrm{Re}\,\Delta_A{}^2\partial_k d_a\partial_e d_b\phi_m\phi_n\phi_i^\dagger\phi_j^\dagger \tag{7.94}$$

Eq. (he-9.72) while the $\partial\phi$ derivatives keep their sign

$$\Delta_A^2 \mathrm{Re}\,\phi_m\partial_k\phi_i^*\partial_l\phi_j^\dagger. \tag{7.95}$$

Eq. (he-9.73) Using the formula (5A.5) of Appendix 5A, this expression can be cast in the form

$$\Delta_A^2\left\{(\partial_k l_m)l_i(\partial_l l_n)l_j - [(\epsilon_{mpr}l_r\partial_k l_p l_i - 2mv_{sk}\right.$$
$$\left.(\delta_{mi}l_m l_i))\left(\epsilon_{nqs}l_s\partial_q l_q l_j - 2mv_{sl}\left(\delta_{nj} - l_n l_j\right)\right)]\right\}. \tag{7.96}$$

Eq. (he-9.74) The calculation simplifies considerably by observing that an expression

$$\Delta_A^2 \mathrm{Re}\,\phi_m^*\partial_k\phi_i\phi_n\partial_e\phi_j^\dagger \tag{7.97}$$

instead of (7.95) would give exactly the same result as (7.96) except with a + sign

Eq. (he-9.75) in front of the bracket. Thus (7.95) can be written as

$$2\Delta_A^2\left(\partial_k l_m\right)l_i\left(\partial_e l_n\right)l_j\Delta_A^2 \mathrm{Re}\,\phi_m^*\partial_k\phi_i\phi_n\partial_e\phi_j^\dagger \tag{7.98}$$

Eq. (he-9.76) Now, the second piece together with (7.85) corresponds to a energy

$$-\frac{1}{4m^2}\frac{3\rho}{2}\int\frac{d\hat{\mathbf{p}}}{4\pi}\hat{p}_i\hat{p}_j\hat{p}_k\hat{p}_l$$
$$\frac{1}{2}\bar{\phi}(\Delta)\frac{\Delta_A^2}{\Delta^2}\left(\frac{\Delta_A^2}{\Delta^2}\hat{p}_m A_b^*\hat{p}_n A_{bn}\right)\partial_k A^*_{ai}\partial_l A_{aj}/\Delta_A^2 \tag{7.99}$$

which again cancels the $\bar{\phi}$ part in the ρ^{11} piece. Hence, this part of the energy e has again the form (7.89).

Eq. (he-9.77) Let us now study contribution of the first piece in (7.98) to the energy:

$$\Delta f = -\frac{1}{4m^2}\frac{3\rho}{2}\int \frac{d\hat{\mathbf{p}}}{4\pi}\hat{p}_l\hat{p}_j\hat{p}_k\hat{p}_l\hat{p}_n\bar{\phi}(\Delta)\frac{\Delta_A^4}{\Delta^4}\left(\partial_k l_m\right) l_i \left(\partial_l l_n\right) l_j \tag{7.100}$$

Eq. (he-9.78) Since $\hat{\mathbf{p}}\mathbf{l} = z$, we can introduce the tensor

$$\bar{\rho}_{ijkl} = \frac{3\rho}{2}\int \frac{d\hat{\mathbf{p}}}{4\pi}\hat{p}_i\hat{p}_j\hat{p}_k\hat{p}_l\frac{z^2}{1-z^2}\bar{\phi}(\Delta)\frac{\Delta_A^2}{\Delta^2} \tag{7.101}$$

so that the additional energy can be written as Eq. (he-9.79)

$$\Delta f = -\frac{1}{4m^2}\bar{\rho}_{ijkl}\partial_k l_i \partial_l l_j. \tag{7.102}$$

Decomposing $\bar{\rho}_{ijkl}$ in the same way as ρ_{ijkl} in (7.78), we find for the coefficients Eq. (he-9.80)

$$\begin{aligned} \bar{A} &= \frac{1}{8}\bar{\rho}_s, \\ \bar{A}+\bar{B} &= \frac{1}{4}\bar{\rho}_s^{\parallel}, \end{aligned} \tag{7.103}$$

where $\bar{\rho}_s$, $\bar{\rho}_s^{\parallel}$ are auxiliary quantities defined as Eq. (he-9.81)

$$\begin{aligned} \bar{\rho}_s &\equiv \frac{3}{4}\rho\int_{-1}^{1} dz(1-z^2)\bar{\phi}(\Delta)\frac{\Delta_A^2}{\Delta^2} \\ \bar{\rho}_s^{\parallel} &\equiv \frac{3}{2}\rho\int_{-1}^{1} dz z^4\bar{\phi}(\Delta)\frac{\Delta_A^2}{\Delta^2}. \end{aligned} \tag{7.104}$$

Note that a corresponding quantity $\bar{\gamma}$ formed with $\bar{\phi}(\Delta)$ in analogy with (7.79) need not be calculated since the covariant $\hat{C}_{ijkl}$ gives zero when the indices are contacted in (7.102).

Inserting the explicit form (7.86) for $\bar{\phi}(\Delta)$ we can partially integrate Eq. (7.104) and find Eq. (he-9.82)

$$\begin{aligned} \bar{\rho}_s &= \frac{3}{4}\rho\int_{-1}^{1} dz(1-z^2)^2 z^2\frac{2}{\delta}\sum_{n=0}^{\infty}\frac{1}{\sqrt{x_n^2+1-z^2}^5} \\ &= -\frac{3}{4}\rho\int_{-1}^{1} dz\left(\frac{1}{3}-2z^2+\frac{5}{3}z^4\right)\frac{\phi(\Delta)}{(1-z^2)} \end{aligned} \tag{7.105}$$

$$\begin{aligned} \bar{\rho}_s^{\parallel} &= \frac{3}{4}\rho\int_{-1}^{1} dz(z^4)^2(1-z^2)\frac{2}{\delta}\sum_{n=0}^{\infty}\frac{1}{\sqrt{x_n^2+1-z^2}^5} \\ &= -\frac{3}{4}\rho\int_{-1}^{1} dz\left(z^2+\frac{5}{3}z^4\right)\frac{\phi(\Delta)}{(1-z^2)}. \end{aligned} \tag{7.106}$$

The auxiliary quantities can therefore be expressed in terms of the superfluid densities as follows: Eq. (he-9.83)

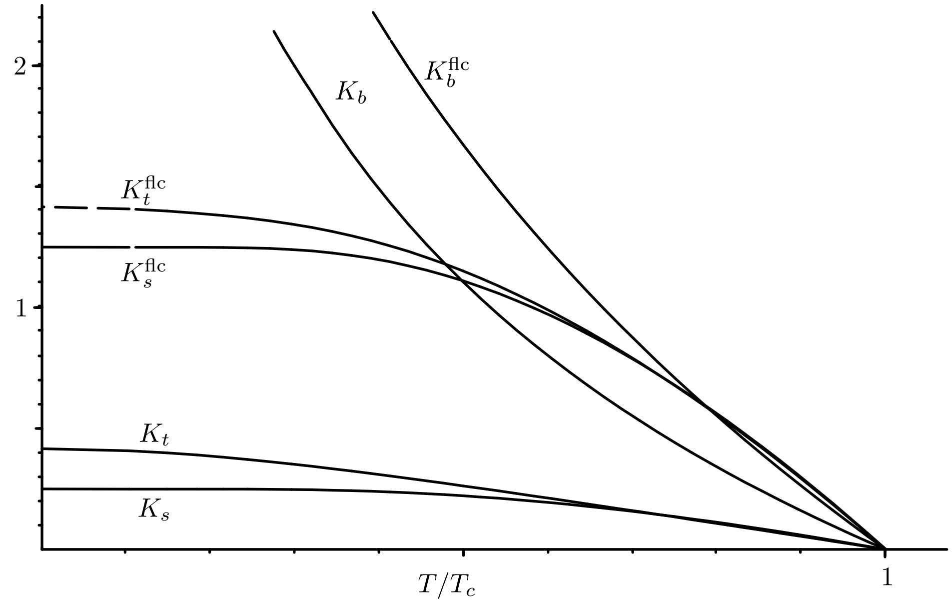


FIGURE 7.3 The superfluid stiffness functions K_t, K_b, K_s of the A-phase as a function of temperature, once without and once with Fermi liquid corrections, indicated by the superscript flc.

$$\begin{aligned} \bar\rho_s &= \frac{2}{3}\rho_s^{\|} - \frac{1}{3}\rho_s \\ \bar\rho_s^{\|} &= -\rho_s^{\|} + \frac{1}{2}\gamma. \end{aligned} \qquad (7.107)$$

If we now perform the contractions of the covariants in (7.89) and (7.102(he-9.79)), we find the energy (see Appendix 7A for details) in the form given in (5.8) but with

Eq. (he-9.84) coefficients:

$$\begin{aligned} &2mc = \frac{1}{2}\rho_s^{\|}, \quad 2c_0 m = \rho_s^{\|}, \\ &4m^2 K_1^d = \rho_s, \quad 4m^2 K_2^d = \rho_0 = \rho_s - \rho_s^{\|}, \\ &4m^2 K_s = \rho_s/4, \quad 4m^2 K_t = (\rho_s + 4\rho_s^{\|})/12, \quad 4m^2 K_b = (\rho_s^{\|} + \gamma)/2 \end{aligned} \qquad (7.108)$$

whose temperature dependence is known down to $T = 0$. The coefficients c, c_0 need no plotting since they are proportional to $\rho_s^{\|}$. The twist, bend, and splay bending

Fig. ktbs constants are displayed in Fig. 7.3. There is no need to plot K_1^d, K_2^d since K_1^d is equal to $\rho_s/4m^2$ which was plotted in Fig. (7.2). To see what K_2^d looks like we introduce, in analogy with $\rho_s^{\|}$ the longitudinal quantity

$$K_{\|}^d \equiv K_1 - K_2 \qquad (7.109)$$

which is equal to $\rho_s^{\|}/4m^2$.

If **d** is locked to **l**, the bending constants K_1^d, K_2^d change K_t, K_b, K_s into

$$\begin{aligned} K_s^l &= K_s + \rho_s = 5\rho_s/4, \\ K_t^l &= K_t + \rho_s = (13\rho_s + 4\rho_s^\parallel)/12, \\ K_b &= K_b + K_\parallel^d = (3\rho_s^\parallel + \gamma)/2. \end{aligned} \tag{7.110}$$

7.1 Fermi Liquid Corrections

In order to compare the analysis with experiment at all temperatures below T_c, the pair interaction (2.42) turns out not to be sufficient. The further T drops below T_c, the more other interactions become important. Here we shall discuss the most important of these which are due to a current-current coupling between both for the particle and the spin currents.

In Landau's theory of the normal Fermi liquid these interactions are introduced with coupling constants $F_1^s,\ F_1^a$ as follows: Eq. (he-10.1)

$$\mathcal{A}_{\text{curr-curr}} = -\frac{1}{2}\int d^4x \left[\frac{F_1^s}{2\mathcal{N}(0)}\psi^* i\tilde{\nabla}\psi\psi^*\tilde{\nabla}\psi + \frac{F_1^a}{2\mathcal{N}(0)}\psi^* i\tilde{\nabla}\sigma_a\psi\psi^* i\tilde{\nabla}\sigma^a\psi\right]. \tag{7.111}$$

Using the particle and spin currents of Eq. (4.14) and the relation $2\mathcal{N}(0)p_F^2 = 3\rho$ this can be written compactly as Eq. (he-10.2)

$$\mathcal{A}_{\text{curr-curr}} = -\frac{1}{2}\int d^4x \frac{m^2}{\rho}\left(\frac{1}{3}F_1^s j_i^2 + \frac{1}{3}F_1^a j_{ai}^2\right). \tag{7.112}$$

As in the case of the pair interaction, these quartic expressions in the fundamental fields ψ^*, ψ can be eliminated in favor of quadratic ones by introducing collective fields $\varphi_i,\ \varphi_{ai}$ and adding to the action the complete squares Eq. (he-10.3)

$$\frac{1}{2}\int d^4x \frac{m^2}{\rho}\left[\frac{1}{3}F_1^s\left(j_i + \frac{\rho}{m^2}\frac{1}{F_1^s}\varphi_i\right)^2 + \frac{1}{3}F_1^a\left(j_{ai} + \frac{\rho}{m^2}\frac{1}{F_1^a}\varphi_{ai}\right)^2\right] \tag{7.113}$$

in analogy with (3.5). Then the current-current interaction becomes Eq. (he-10.4)

$$\mathcal{A}_{\text{curr-curr}} = \int d^4x \left[j_i\varphi_i + j_{ai}\varphi_{ai} + \frac{1}{2}\frac{\rho}{m^2}\left(\frac{1}{\frac{1}{3}F_1^s}\varphi_i^2 + \frac{1}{\frac{1}{3}F_1^a}\varphi_{ai}^2\right)\right]. \tag{7.114}$$

After integrating out the Fermi fields, the trace log in (3.13) will be changed to $-i/2$ times the Trace log of the matrix Eq. (he-10.6)

$$\begin{pmatrix} i\partial_t - \xi(-i\boldsymbol{\nabla}) + \frac{i}{2m}\tilde{\nabla}_i\varphi_i + \frac{i}{2m}\tilde{\nabla}_i\sigma_a\varphi_{ai} & i\tilde{\nabla}_i\sigma_a A_{ai} \\ i\tilde{\nabla}_i\sigma_a A_{ai}^* & i\partial_t + \xi(-i\boldsymbol{\nabla}) + \frac{i}{2m}\tilde{\nabla}_i\varphi_i + \frac{i}{2m}\tilde{\nabla}_i\sigma_a\varphi_{ai} \end{pmatrix} \tag{7.115}$$

depending on φ_i, φ_{ai}.

In the hydrodynamic limit where only quadratic field dependencies are considered there is a simple method to find this dependence without going again through the loop calculations. For this we observe that a term in the action Eq. (he-10.6a)

$$\int d^4x\,(j_i\varphi_i + j_{ai}\varphi_{ai}) \tag{7.116}$$

is equivalent to adding velocity source terms to the energy density, thereby forming quantity looking like an enthalpy density, except that the roles of pressure and volume are played by momenta and velocities, i.e., Eq. (he-)

$$f \to f_{\text{ent}} \;=\; f - p_i V_i - p_{ai}.V_{ai} \tag{7.117}$$

Here $p_i \equiv mj_i$, $p_{ai} \equiv mj_{ai}$ are the momentum densities of particle and spin flow. We shall call $e \to f_{\text{ent}}$ the *flow enthalpy*. The minimum of this quantity determines the equilibrium properties of the system at externally enforced velocities V_i, V_{ai} of particles and spins:

$$V_i \equiv \varphi_i/m, \qquad V_{ai} \equiv \varphi_{ai}/m. \tag{7.118}$$

Consider, now the energy (5.8) in a planar texture which has all **l**-vectors parallel. If we want to account for the effect of the current-current interactions we must extend this expression. Recall that the earlier calculations were all done in a frame in which the normal part of the liquid was at rest. When considering nonzero velocities of the system as we now do, we have to add to the energy density the kinetic terms of the normal particle and spin flows Eq. (he-10.8)

$$\frac{\rho_n}{2}\left(v_{n_i}^{\perp 2} + v_{n_{ai}}^{\perp 2}\right) + \frac{\rho_n^{\parallel}}{2}\left(v_{n_i}^{\parallel\;2} + v_{n_{ai}}^{\parallel\;2}\right) \tag{7.119}$$

Eq. (he-10.9) where $\mathbf{v}^{\perp}$ and $\mathbf{v}^{\parallel}$ are defined by

$$\begin{aligned} \mathbf{v}^{\perp} &= \mathbf{v} - \mathbf{v}^{\parallel} \\ \mathbf{v}^{\parallel} &= \mathbf{l}\,(\mathbf{l}\cdot\mathbf{v}). \end{aligned} \tag{7.120}$$

with similar definitions for the spin velocities. The corresponding currents are

$$\mathbf{p} \;=\; m\mathbf{j} = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n, \tag{7.121}$$

$$\mathbf{p}_a \;=\; m\mathbf{j}_a = \rho_s \mathbf{v}_{sa} + \rho_n \mathbf{v}_{na}. \tag{7.122}$$

The additional terms (7.119) are necessary to guarantee the correct Galilei transformation properties of the energy density e.

We now study the equilibrium properties of the liquid. First we minimize the flow enthalpy (7.122). If topology does not enforce a nonzero superflow, both velocities $\mathbf{v}_n$ and $\mathbf{v}_s$ will be equal to a single velocity $\mathbf{v}$. Thus, in equilibrium, we may rewrite Eq. (he-10.10) the flow enthalpy density also as

$$f_{\text{ent}} = \frac{\rho}{2}\left(v_i^2 + v_{ai}^2\right) - p_i V_i - p_{ai} V_{ai}. \tag{7.123}$$

This expression is minimal at

$$v_i = V_i, \quad v_{ai} = V_{ai}, \tag{7.124}$$

Eq. (he-10.11) where it has the equilibrium

$$f_{\rm ent}\Big|_{\rm eq} = -\frac{\rho}{2}\left(V_i^2 + V_{ai}^2\right). \tag{7.125}$$

Let us compare this with the calculation of the flow enthalpy from the trace log term of the collective action. The enthalpy density is Eq. (he-10.12)

$$f_{\rm ent} = -\log\begin{pmatrix} i\partial_t - \xi(\mathbf{p}) + p_iV_i + p_i\sigma_aV_{ai} & \tilde{p}_i\sigma_aA'_{ai} \\ \tilde{p}_i\sigma_aA'^{\,*}_{ai} & i\partial_t + \xi(\mathbf{p}) - p_iV_i - p_i\sigma_aV_{ai}\end{pmatrix}. \tag{7.126}$$

The quadratic term in the fluctuating field A'_{ai} around the extremum has been calculated before and has led to the hydrodynamic limiting result Eq. (he-10.13)

$$f = \frac{\rho_s}{2}\left({v_s}_i^{\perp 2} + {v_s}_{ai}^{\perp 2}\right) + \frac{\rho_s^{\parallel}}{2}\left({v^{\parallel}_{si}}^2 + {v^{\parallel}_{sai}}^2\right). \tag{7.127}$$

In addition, there are now linear terms Eq. (he-10.14)

$$\Delta_1 f = -\rho_s\left(v_{si}V_i + v_{s\ ai}^{\perp}V_{ai}\right) - \rho_s^{\parallel}\left(v^{\parallel}_{si}V_i + v^{\parallel}_{sai}V_{ai}\right) \tag{7.128}$$

We would like to find quadratic terms in V_i, V_{ai}. They certainly have the form Eq. (he-10.15)

$$\Delta_2 f = -\frac{a}{2}\left(V_i^{\perp 2} + V_{ai}^{\perp 2}\right) - \frac{a^{\parallel}}{2}\left(V_i^{\parallel 2} + V_{ai}^{\parallel 2}\right). \tag{7.129}$$

In order to determine a and $a^{\parallel}$, we simply minimize the enthalpy in $v_{si}^{\perp,\parallel}$ and $v_{sai}^{\perp,\parallel}$, which become equal to $V_i^{\perp,\parallel}$ and $V_{ai}^{\perp,\parallel}$, respectively. At these velocities, Eq. (he-)

$$f_{\rm ent}\Big|_{\rm eq} = -\frac{\rho_s + a}{2}\left(V_i^{\perp 2} + V_{ai}^{\perp 2}\right) - \frac{\rho_s^{\parallel} + a^{\parallel}}{2}\left(V_i^{\parallel 2} + V_{ai}^{\parallel 2}\right). \tag{7.130}$$

Comparing this with (7.123) we see that Eq. (he-10.17)

$$\begin{aligned} a &= \rho_n = \rho - \rho_s, \\ a^{\parallel} &= \rho_n^{\parallel} = \rho - \rho_s^{\parallel}, \end{aligned} \tag{7.131}$$

implying that the coefficients in (7.129) are simply the normal-liquid densities. Thus, the hydrodynamic limit of the collective energy density is given by Eq. (he-10.18)

$$\begin{aligned} e = {} & \frac{\rho_s}{2}\left({v_{s\ i}^{\perp}}^2 + {v_{s\ ai}^{\perp}}^2\right) + \frac{\rho_s^{\parallel}}{2}\left({v^{\parallel}_{si}}^2 + {v^{\parallel}_{sai}}^2\right) - j_i\varphi_i - j_{ai}\varphi_{ai} \\ & -\frac{1}{2}\frac{m^2}{\rho}\left[\left(\frac{1}{\frac{1}{3}F_i^s} + \frac{\rho_n}{\rho}\right)\varphi_i^{\perp 2} + \left(\frac{1}{\frac{1}{3}F_1^s} + \frac{\rho_n^{\parallel}}{\rho}\right)\varphi_i^{\parallel 2}\right. \\ & \left. + \left(\frac{1}{\frac{1}{3}F_1^a} + \frac{\rho_n}{\rho}\right)\varphi_{ai}^{\perp 2} + \left(\frac{1}{\frac{1}{3}F_1^a} + \frac{\rho_n^{\parallel}}{\rho}\right){\varphi_{ai}^{\parallel}}^2\right]. \end{aligned} \tag{7.132}$$

We now complete the squares in the fields $\varphi_{ai}^{\perp}$ and $\varphi_{ai}^{\parallel}$, and obtain Eq. (he-10.19)

$$f = \frac{1}{2}\rho_s \left(v_{s\ i}^{\perp\ 2} + v_{s\ ai}^{\perp\ 2} \right) + \frac{\rho_s^{\parallel}}{2} \left(v_{si}^{\parallel\ 2} + v_{sai}^{\parallel\ 2} \right)$$
$$+\frac{1}{2}\frac{m^2}{s}\left(\frac{\frac{1}{3}F_a^s}{1+\frac{1}{3}F_1^s\frac{\rho_n}{\rho}} j_i^{\perp 2} + \frac{\frac{1}{3}F_a^a}{1+\frac{1}{3}F_1^a\frac{\rho_n}{\rho}} j_{ai}^{\perp 2} + \frac{\frac{1}{3}F_1^s}{1+\frac{1}{3}F_a^s\frac{\rho_n^{\parallel}}{\rho}} j_i^{\parallel 2} + \frac{\frac{1}{3}F_1^a}{1+\frac{1}{3}F_1^a\frac{\rho_n^{\parallel}}{\rho}} j_{ai}^{\parallel\ 2} \right)$$
$$-\frac{1}{2}\frac{\rho}{m^2}\left[\left(\frac{1}{\frac{1}{3}F_1^s} + \frac{\rho_n}{\rho} \right) \left(\varphi_i^{\perp} - \frac{\frac{1}{3}F_a^s}{1+\frac{1}{3}F_1^s\frac{\rho_n}{\rho}} j_i^{\perp} \right)^2 \right.$$
$$\left. + \left(\frac{1}{\frac{1}{3}F_1^a} + \frac{\rho_n}{\rho} \right) \left(\varphi_{ai}^{\perp} - \frac{\frac{1}{3}F_1^a}{1+\frac{1}{3}F_1^a\frac{\rho_n}{\rho}} j_{ai}^{\perp} \right)^2 + (\perp \to \parallel) \right]. \tag{7.133}$$

The path integrals over the fields φ_i, φ_{ai} can now be performed which makes the last terms in brackets disappear.

Finally, we allow $\mathbf{l}$ to vary in space. This will lead us to the Fermi liquid corrections to the stiffness constants $K_{s,t,b}$ [recall (5.8)]. In the presence of a nontrivial $\mathbf{l}$ Eq. (he-10.20) texture the currents acquire additional terms. The particle current j_i becomes

$$\begin{aligned} m\mathbf{j}^{\perp} &= \rho_s \mathbf{v}_s + c\,(\boldsymbol{\nabla}\times\mathbf{l})^{\perp}, \qquad 2mc = \frac{\rho_s^{\parallel}}{2}, \\ m\mathbf{j}j^{\parallel} &= \rho_s^{\parallel}\mathbf{v}_s^{\parallel} - c^{\parallel}\,(\boldsymbol{\nabla}\times\mathbf{l})^{\parallel}, \qquad 2mc^{\parallel} = \frac{\rho_s^{\parallel}}{2}, \end{aligned} \tag{7.134}$$

Eq. (he-10.21) where we have separated $\boldsymbol{\nabla}\times\mathbf{l}$ into transverse and longitudinal parts:

$$\begin{aligned} (\boldsymbol{\nabla}\times\mathbf{l})^{\perp} &= (\boldsymbol{\nabla}\times\mathbf{l}) - \mathbf{l}\,[\mathbf{l}\cdot(\boldsymbol{\nabla}\times\mathbf{l})], \\ (\boldsymbol{\nabla}\times\mathbf{l})^{\parallel} &= \mathbf{l}\,[\mathbf{l}\cdot(\boldsymbol{\nabla}\times\mathbf{l})], \end{aligned} \tag{7.135}$$

Eq. (he-10.22) respectively. The squares of the currents are

$$\begin{aligned} m^2 j_i^{\perp 2} &= \rho_s^2 \mathbf{v}^{\perp 2} + \frac{1}{4m^2}\frac{\rho_s^{\parallel 2}}{4}\left[(\boldsymbol{\nabla}\times\mathbf{l})^2 - (\mathbf{l}\cdot(\boldsymbol{\nabla}\times\mathbf{l}))^2 \right] \\ &\qquad + \frac{1}{2m}\frac{\rho_s\rho_s^{\parallel}}{2}\mathbf{v}^{\perp}(\boldsymbol{\nabla}\times\mathbf{l})^{\perp} \\ &= \rho_s^2\left[\mathbf{v}^2 - (\mathbf{l}\cdot\mathbf{v})^2 \right] + \frac{1}{4m^2}\frac{\rho_s^2}{4}\left[\mathbf{l}\times(\boldsymbol{\nabla}\times\mathbf{l}) \right]^2 \\ &\qquad + \frac{1}{2m}\frac{\rho_s\rho_s^{\parallel}}{2}\left[\mathbf{v}\cdot(\boldsymbol{\nabla}\times\mathbf{l}) - (\mathbf{v}\cdot\mathbf{l})(\boldsymbol{\nabla}\times\mathbf{l}) \right], \end{aligned} \tag{7.136}$$

$$\begin{aligned} m^2 j_i^{\parallel 2} &= \rho_s^{\parallel 2}(\mathbf{l}\cdot\mathbf{v})^2 + \frac{1}{4m^2}\frac{\rho_s^{\parallel 2}}{4}\left[\mathbf{l}\cdot(\boldsymbol{\nabla}\times\mathbf{l}) \right]^2 - \frac{1}{2m}\frac{\rho_s^{\parallel 2}}{2}(\mathbf{v}\cdot\mathbf{l})(\boldsymbol{\nabla}\times\mathbf{l}), \\ m^2 j_{ai}^{\perp 2} &= \rho_s^2 v_{ai}^{\perp 2} = \frac{\rho_s^2}{4m^2}\left\{ \epsilon_{abc} d_b \left[\nabla_i - l_i(\mathbf{l}\cdot\boldsymbol{\nabla}) \right] d_c \right\}^2 \\ &= \frac{\rho_s^2}{4m^2}\left[(\nabla_i d_a)^2 - (\mathbf{l}\cdot\boldsymbol{\nabla} d_a)^2 \right], \end{aligned} \tag{7.137}$$

$$m^2 j_{ai}^{\parallel\,2} = \rho_s^{\parallel 2} v_{ai}^{\parallel\,2} = \frac{\rho_s^{\parallel 2}}{4m^2}\left[l_i \epsilon_{abc} d_b (\mathbf{l}\cdot\boldsymbol{\nabla}) d_c\right]^2$$
$$= \frac{\rho_s^{\parallel 2}}{4m^2}\left(\mathbf{l}\cdot\boldsymbol{\nabla} d_a\right)^2 . \qquad (7.138)$$

Using these we find the energy density Eq. (he-10.23)

$$f = \frac{1}{2}\rho_s \frac{1+\frac{1}{3}F_1^s}{1+\frac{1}{3}F_1^s\frac{\rho_n}{\rho}}\left[\mathbf{v}_s^2 - (\mathbf{l}\cdot\mathbf{v}_s)^2\right] + \frac{1}{2}\rho_s^{\parallel}\frac{1+\frac{1}{3}F_1^s}{1+\frac{1}{3}F_1^s\frac{\rho_n^{\parallel}}{\rho}}(\mathbf{l}\cdot\mathbf{v}_s)^2$$
$$+\frac{1}{2}\rho_s\frac{1+\frac{1}{3}F_1^a}{1+\frac{1}{3}F_1^a\frac{\rho_n}{\rho}}\frac{1}{4m^2}\left[(\nabla_i d_a)^2 - (\mathbf{l}\cdot\boldsymbol{\nabla} d_a)^2)\right] + \frac{1}{2}\rho_s^{\parallel}\frac{1+\frac{1}{3}F_1^a}{1+\frac{1}{3}F_1^a\frac{\rho_n^{\parallel}}{\rho}}(\mathbf{l}\cdot\boldsymbol{\nabla} d_a)^2$$
$$+\frac{\rho_s^{\parallel}}{2}\frac{1+\frac{1}{3}F_a^s}{1+\frac{1}{3}F_1^s\frac{\rho_n}{\rho}}\frac{1}{2m}\left\{\mathbf{v}_s\cdot(\boldsymbol{\nabla}\times\mathbf{l}) - (\mathbf{l}\cdot\mathbf{v}_s)\left[\mathbf{l}\cdot(\boldsymbol{\nabla}\times\mathbf{l})\right]\right\}$$
$$-\frac{\rho_s^{\parallel}}{2}\frac{1}{2m}\frac{1+\frac{1}{3}F_1^s}{1+\frac{1}{3}F_1^s\frac{\rho_n^{\parallel}}{\rho}}(\mathbf{l}\cdot\mathbf{v}_s)\left[\mathbf{l}\cdot(\boldsymbol{\nabla}\times\mathbf{l})\right]$$
$$+\frac{1}{2}K_s(\boldsymbol{\nabla}\cdot\mathbf{l})^2 + \frac{1}{2}\left(K_t + \frac{1}{4m^2}\frac{\rho_s^{\parallel 2}}{4\rho^2}\frac{\frac{1}{3}F_1^3}{1+\frac{1}{3}F_1^s\frac{\rho_n^{\parallel}}{\rho}}\right)\left[\mathbf{l}\cdot(\boldsymbol{\nabla}\times\mathbf{l})\right]^2$$
$$+\frac{1}{2}\left(K_b + \frac{1}{4m^2}\frac{\rho_s^{\parallel 2}}{4\rho^2}\frac{\frac{1}{3}F_1^s}{1+\frac{1}{3}F_1^s\frac{\rho_n}{\rho}}\right)\left[\mathbf{l}\times(\boldsymbol{\nabla}\times\mathbf{l})\right]^2 . \qquad (7.139)$$

As discussed in the beginning, the mass parameter m here is the *effective mass* of the screened quasiparticles in the Fermi liquid. As a consequence, the velocity Eq. (he-10.24)

$$\mathbf{v}_s = \frac{1}{2m}\phi_i^* \overleftrightarrow{\nabla}\phi_i \qquad (7.140)$$

is not really the correct parameter of Galilean transformations. To be so, the phase change in the original fundamental fields would have to be Eq. (he-10.25)

$$\psi \to e^{im_0\mathbf{v}\cdot\mathbf{x}}\psi \qquad (7.141)$$

where m_0 is the true physical mass of the ^{3}He atoms. If we introduce the corresponding true velocity Eq. (he-10.26)

$$\mathbf{v}_{0s} = \frac{i}{2m_0}\phi_i^* \overleftrightarrow{\nabla}\phi_i \qquad (7.142)$$

with a similar expression for the spin velocity v_{0sai}, the first term in (7.139) takes the form Eq. (he-10.27)

$$\frac{1}{2}\left(\rho_s\frac{m_0}{m}\right)\frac{1}{1+\frac{1}{3}F_1^s\frac{\rho_n}{\rho}}\frac{m_0\left(1+\frac{1}{3}F_1^s\right)}{m}\left[\mathbf{v}_{0s}{}^2 - (\mathbf{l}\cdot\mathbf{v}_s)^2\right] \qquad (7.143)$$

with the other terms changing accordingly.

We now add to the energy density the kinetic energy of the normal component Eq. (he-10.28)

$$\frac{1}{2}\rho_n\left[{\mathbf{v}_{0n}}^2 - (\mathbf{l}\cdot\mathbf{v}_{0n})^2\right] + \frac{1}{2}\rho_n(\mathbf{l}\cdot\mathbf{v}_{0n})^2. \tag{7.144}$$

By Galilei invariance, the sum of the coefficients has to add up to the total density $\rho_0 = nm_0$ where n is the number of particles ($\equiv$ number of quasiparticles) per unit
Eq. (he-10.29) volume:

$$\left(\rho_s\frac{m_0}{m}\right)\frac{1}{1+\frac{1}{3}F_1^s\frac{\rho_n}{\rho}}\frac{m_0\left(1+\frac{1}{3}F_1^s\right)}{m} + \rho_n = \rho_0 = nm_0. \tag{7.145}$$

Eq. (he-10.30) At $T = 0$, the normal density vanishes, i.e., $\rho_n = 0$, and the last section gives

$$\rho_s|_{T=0} = \rho = \rho_0\frac{m}{m_0}. \tag{7.146}$$

Thus, consistency requires the following relation between the effective mass m and
Eq. (he-10.31) the atomic mass $m_c \equiv m_{^3\mathrm{He}}$:

$$m = \left(1+\frac{1}{3}F_1^s\right)m_0, \tag{7.147}$$

Eq. (he-10.32) This brings the term (7.145) to the form

$$\frac{1}{2}\rho_0\frac{\rho_s}{\rho}\frac{1}{1+\frac{1}{3}F_1^s\frac{\rho_n}{\rho}}\left[{\mathbf{v}_{0s}}^2 - (\mathbf{l}\cdot\mathbf{v}_{0s})^2\right]. \tag{7.148}$$

The prefactor can be interpreted as the *superfluid density with Fermi liquid correc-*
Eq. (he-10.33) *tions*, i.e.,

$$\rho_s^{\mathrm{FL}} \equiv \rho_0\frac{\rho_s}{\rho}\frac{1}{1+\frac{1}{3}F_1^s\frac{\rho_n}{\rho}}. \tag{7.149}$$

Eq. (he-10.34) It will be convenient to introduce the dimensionless ratio

$$\tilde{\rho}_s^{\mathrm{FL}} \equiv \frac{\rho_s^{\mathrm{FL}}}{\rho_0}. \tag{7.150}$$

At $T = 0$, this ratio goes to unity just as in the uncorrected case. For $T \approx T_c$, however, it receives a strong reduction by a factor

$$\frac{1}{1+\frac{1}{3}F_1^s} = \frac{m_0}{m},$$

Eq. (he-10.34b) i.e.,

$$\tilde{\rho}_s^{\mathrm{FL}} \equiv \frac{\rho_s}{\rho}\frac{m_0}{m}. \tag{7.151}$$

Hence near T_c the number of particles in the normal component give a true particle density if it is multiplied by the quasiparticle mass m instead of the atomic mass m_0). Specific-heat experiments[1] determine the effective mass ratios m/m_0
Eq. (he-10.35) mentioned in the beginning. They correspond to

$$\frac{1}{3}F_1^s = (2.01,\ 3.09,\ 3.93,\ 4.63,\ 5.22) \tag{7.152}$$

Eq. (he-10.36) at pressures

$$p = (0,\ 9,\ 18,\ 27,\ 34.36)\ \ \text{bar}. \tag{7.153}$$

Similarly, we may go through the Fermi liquid corrections of the spin currents which lead to Eq. (he-10.37)

$$\tilde{K}_{\rm d}^{\rm FL} \equiv \frac{K_{\rm d}^{\rm FL}}{\rho_0} = \frac{\rho_s}{\rho} \frac{1+\frac{1}{3}F_1^a}{1+\frac{1}{3}F_1^a\frac{\rho_n}{\rho}} \frac{1}{1+\frac{1}{3}F_1^s}, \tag{7.154}$$

Eq. (he-10.38)

$$\tilde{K}_{\rm d}^{\parallel{\rm FL}} \equiv \frac{K_{\rm d}^{\parallel{\rm FL}}}{\rho_0} = \frac{\rho_s^{\parallel}}{\rho} \frac{1+\frac{1}{3}F_1^a}{1+\frac{1}{3}F_1^a\frac{\rho_n^{\parallel}}{\rho}} \frac{1}{1+\frac{1}{3}F_1^s}, \tag{7.155}$$

while the coefficients c and $c^{\parallel}$ become Eq. (he-10.39)

$$\tilde{c}^{\rm FL} \equiv \frac{2m_0c^{\rm FL}}{\rho_0} = \frac{2mc}{\rho} \frac{1}{1+\frac{1}{3}F_1^s\frac{\rho_n}{\rho}}, \tag{7.156}$$

Eq. (he-10.40)

$$\tilde{c}^{\parallel{\rm FL}} \equiv \frac{2m_0c^{\parallel{\rm FL}}}{\rho_0} = \frac{2mc^{\parallel}}{\rho} \frac{1}{1+\frac{1}{3}F_1^s\frac{\rho_n^{\parallel}}{\rho}}. \tag{7.157}$$

The pure $\mathbf{l}$ parts of the bending energy lead to Eq. (he-10.41)

$$\tilde{K}_s^{\rm FL} \equiv \frac{4m_0^2 {K_s}^{\rm FL}}{\rho_0} = \frac{4m^2K_s}{\rho} \frac{1}{1+\frac{1}{3}F_1^s}, \tag{7.158}$$

$$\tilde{K}_t^{\rm FL} \equiv \frac{4m_0^2}{\rho_0}K_t^{\rm FL} = \left(\frac{4m^2K_t}{\rho} + \frac{1}{4}\frac{\rho_s^{\parallel 2}}{\rho^2} \frac{\frac{1}{3}F_1^s}{1+\frac{1}{3}F_1^s\frac{\rho_n^{\parallel}}{\rho}} \right) \frac{1}{1+\frac{1}{3}F_1^s}, \tag{7.159}$$

$$\tilde{K}_b^{\rm FL} \equiv \frac{4m_0^2}{\rho_0}K_b^{\rm FL} = \left(\frac{4m^2K_b}{\rho} + \frac{1}{4}\frac{\rho_s^{\parallel 2}}{\rho^2} \frac{\frac{1}{3}F_1^s}{1+\frac{1}{3}F_1^s\frac{\rho_n}{\rho}} \right) \frac{1}{1+\frac{1}{3}F_1^s}. \tag{7.160}$$

In the sequel, it will be convenient to define

$$\tilde{\mathbf{v}} \equiv 2m_0\mathbf{v}. \tag{7.161}$$

Then the energy density can be written in the following final form Eq. (he-10.44)

$$\begin{aligned}
\frac{4m_0^2}{\rho_0}e \equiv{}& \frac{1}{2}\tilde{\rho}_s^{\rm FL}\left[\tilde{\mathbf{v}}_s^2 - (\mathbf{l}\cdot\tilde{\mathbf{v}}_s)^2\right] + \frac{1}{2}\frac{\tilde{1}}{2}\tilde{\rho}_s^{\parallel{\rm FL}}(\mathbf{l}\cdot\tilde{\mathbf{v}}_s)^2 \\
&+\frac{1}{2}\tilde{K}_{\rm d}^{\rm FL}\left[(\nabla_i d_a)^2 - (\mathbf{l}\cdot\boldsymbol{\nabla}_a)^2\right] + \frac{1}{2}\tilde{K}_{\rm d}^{\parallel{\rm FL}}\left(\mathbf{l}\cdot\boldsymbol{\nabla}d_a\right)^2 \\
&+\tilde{c}^{\rm FL}\left\{\tilde{\mathbf{v}}\cdot(\boldsymbol{\nabla}\times\mathbf{l}) - (\tilde{\mathbf{v}}_s\cdot\mathbf{l})\left[\mathbf{l}\cdot(\boldsymbol{\nabla}\times\mathbf{l})\right]\right\} \\
&-\tilde{c}^{\parallel{\rm FL}}\left(\tilde{\mathbf{v}}_s\cdot\mathbf{l}\right)\left[\mathbf{l}\cdot(\boldsymbol{\nabla}\times\mathbf{l})\right] \\
&+\frac{1}{2}\tilde{K}_s^{\rm FL}(\boldsymbol{\nabla}\cdot\mathbf{l})^2 + \frac{1}{2}\tilde{K}_t^{\rm FL}\left[\mathbf{l}\cdot(\boldsymbol{\nabla}\times\mathbf{l})\right]^2 + \frac{1}{2}\tilde{K}_b^{\rm FL}\left[\mathbf{l}\times(\boldsymbol{\nabla}\times\mathbf{l})\right]^2.
\end{aligned} \tag{7.162}$$

FIGURE 7.4 The remaining hydrodynamic parameters of superfluid ^{3}He-A shown as a function of temperature together with their Fermi liquid corrected values.

In large containers, where **l** and **d** are locked to each other, the $\tilde{K}_{\rm d}^{\rm FL}$, $\tilde{K}_{\rm d}^{\Vert \rm FL}$ terms can be absorbed into $\tilde{K}_{s,t,b}^{\rm FL}$ which then take the *dipole-locked* values Eq. (he-10.45)

$$\tilde{K}_s^{\rm FL}|_{\rm lock} = \tilde{K}_s^{\rm FL} + \tilde{K}_{\rm d}^{\rm FL}, \tag{7.163}$$
$$\tilde{K}_t^{\rm FL}|_{\rm lock} = \tilde{K}_t^{\rm FL} + \tilde{K}_{\rm d}^{\rm FL}, \tag{7.164}$$
$$\tilde{K}_b^{\rm FL}|_{\rm lock} = \tilde{K}_t^{\rm FL} + \tilde{K}_{\rm d}^{\rm FL}. \tag{7.165}$$

Fig. XXXa The temperature dependence of all these quantities is shown in Fig. 7.4 for the experimental Fermi liquid parameters $\frac{1}{3}F_1^s = 5.22$ and $\frac{1}{3}F_1^a = -.22$.

The Fermi liquid corrections in the B phase can be applied in completely anal-
Eq. (he-10.48) ogous manner. There the energy becomes

$$\frac{4m_0^2}{\rho_0}f = \frac{1}{2}\tilde{\rho}_s^{B\,\rm FL}(\boldsymbol{\nabla}\varphi)^2 + \lambda(4+\delta)(\nabla_i\theta_j)^2 - (1+\delta)\nabla_i\theta_j\nabla_j\theta_i - (\nabla_i\theta_i)^2 \tag{7.166}$$

Eq. (he-10.49) with the dimensionless parameters

$$\tilde{\rho}_s^B = \frac{\rho_s^B}{\rho}\frac{1}{1+\frac{1}{3}F_1^s\frac{\rho_n^B}{\rho}},$$
$$\lambda = \frac{1+\frac{1}{3}F_1^a}{1+\frac{1}{3}F_1^a\frac{\rho_n^B}{\rho}+\frac{1}{3}F_1^a\frac{2}{15}\frac{\rho_s^B}{\rho}}\frac{1}{1+\frac{1}{3}F_1^s},$$
$$\delta = \frac{\frac{1}{3}F_1^a\frac{\rho_s^B}{\rho}}{1+\frac{1}{3}F_1^a\frac{\rho_n^B}{\rho}}. \tag{7.167}$$

7.2 Ground State Properties

The superfluid densities do not only characterize the hydrodynamic bending energies. They also appear in the description of the thermodynamic quantities of the

[1]See: J.W. Wheatley, Ref. Mod. Phys. *47*, 415 (1975).

ground state. For $T \approx T_c$ these can be extracted directly from the free energies (4.22). These may be used for checking the general-temperature properties to be calculated now.

7.2.1 Free Energy

Since the ground state field A^0_{ai} is constant in space and time the first two terms in Eq. (7.4) can be calculated explicitly. In energy momentum space the matrix inside the trace log is diagonal Eq. (he-9.85)

$$\begin{pmatrix} \epsilon - \xi(\mathbf{p}) & \tilde{p}_i \sigma_a A^0_{ai} \\ \tilde{p}_i \sigma_a A^{0*}_{ai} & \epsilon + \xi(\mathbf{p}) \end{pmatrix} \tag{7.168}$$

in the functional indices $\epsilon, \mathbf{p}$. In the 4×4 matrix space this can be diagonalized via a Bogoljubov transformation with the result Eq. (he-9.86)

$$\begin{pmatrix} (\epsilon - E(\mathbf{p})) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & (\epsilon + E(\mathbf{p})) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \tag{7.169}$$

where $E(\mathbf{p})$ are the quasi-particle energies (7.14). Thus the first trace log term in the expression (7.4) can be written as Eq. (he-9.87)

$$-i(t_b - t_a)V \int \frac{d\epsilon}{2\pi} \frac{d^3p}{(2\pi)^3} \log\left(\epsilon - E(\mathbf{p})\right)\left(\epsilon + E(\mathbf{p})\right). \tag{7.170}$$

The second term contributes simply Eq. (he-)

$$-\frac{1}{3g} \left\{ \begin{array}{c} 3\Delta_B^2 \\ 2\Delta_A^2 \end{array} \right\} (t_b - t_a)V. \tag{7.171}$$

After a Wick rotation this corresponds to the energy density[2] Eq. (he-9.98)

$$f = -\sum_{\omega_n} \sum_{\mathbf{p}} \log[(i\omega_n - E(\mathbf{p}))(i\omega_n + E(\mathbf{p}))] + \frac{1}{g} \left\{ \begin{array}{c} \Delta_B^2 \\ \frac{2}{3}\Delta_A^2 \end{array} \right\} + \text{const} . \tag{7.172}$$

The constant accounts for the unspecified normalization of the functional integration. It is removed by subtracting the free fermion system with $\Delta = 0,\ g = 0$ [notice that $\Delta^2 \sim e^{-1/g\mathcal{N}(0)} \to 0$ for $g \to 0$ due to (3.21), (7.22(he-9.20)), (7.24(he-9.22))]. Since the energy of the free fermion system is well-known Eq. (he-9.90)

$$f_0 = -2T \sum_{\mathbf{p}} \log\left(1 - e^{\xi(\mathbf{p})/T}\right) \tag{7.173}$$

it is sufficient to study only Eq. (he-9.91)

[2]We use again the relation $\mathcal{A} = iE/T,\ t_b - t_a = -i/T,\ \int_{-\infty}^{\infty} d\epsilon = iT \sum_{\omega_n}$

$$\Delta f = f - f_0 = \\ -T \sum_{\omega_n, \mathbf{p}} \log \frac{i\omega_n - E(p)}{i\omega_n - \xi(p)} + (E \to -E,\ \xi \to -\xi) + \frac{1}{g} \left\{ \begin{array}{c} \Delta_B^2 \\ \frac{2}{3}\Delta_A^2 \end{array} \right\}. \tag{7.174}$$

This energy difference is the condensation energy associated with the transition into the superfluid phase.

The sum over Matsubara frequency can be performed by using Cauchy's formula:

Eq. (he-9.92)

$$T \sum_{\omega_n} \log\left(1 - \frac{E}{i\omega_n}\right) = -\frac{1}{2\pi i} \int \frac{dz}{e^{z/T} + 1} \log\left(1 - \frac{E}{z}\right) \tag{7.175}$$

where the contour C encircles all poles along the imaginary axis at $z = i\omega_n$ in the positive sense but passes the logarithmic cut from $z = 0$ to E on the left if $E > 0$

Eq. (he-9.93) (see Fig. 1.1 in Part I). By deforming the contour C into C' the integral becomes

$$-\int_0^E \frac{dz}{e^{z/T} + 1} = \int_0^E dE\ n(E). \tag{7.176}$$

Eq. (he-9.94) Since

$$\frac{\partial n(E)}{\partial E} = -n(1-n)/T \tag{7.177}$$

Eq. (he-9.95) this can be calculated as

$$-\int_0^E dE n(E) = T \int_{1/2}^{n} dn' \frac{1}{1-n'} = -T \log 2(1 - n(E)). \tag{7.178}$$

Eq. (he-9.96) The expression (7.174) becomes therefore

$$\Delta f = T \sum_{\mathbf{p}} [\log(1-n)n - \log(1-n_0)n_0] + \frac{1}{g} \left\{ \begin{array}{c} \Delta_B^2 \\ \frac{2}{3}\Delta_A^2 \end{array} \right\} \tag{7.179}$$

Eq. (he-9.97) where n_0 denotes the free-fermion distribution. Alternatively, ony may write

$$\Delta f = 2T \sum_{\mathbf{p}} \{\log(1-n) - (E - \xi)\} + \frac{1}{g} \left\{ \begin{array}{c} \Delta_B^2 \\ \frac{2}{3}\Delta_A^2 \end{array} \right\} - 2T \sum_{\mathbf{p}} \log(1 - n_0) \tag{7.180}$$

The last term is recognized as minus the energy of the free system so that the first line gives the full energy of the superfluid ground state.

The explicit calculation can conveniently be done by studying Δf of (7.180) at fixed T as a function of g. At $g = 0$, $\Delta_{AB} = 0$ and $\Delta f = 0$. As g is increased to its physical value, the gap increases to Δ_{AB}. Now, since Δf is extremal in changes of Δ at fixed g and T, all g-dependence comes from the variation of the factor $1/g$,

Eq. (he-9.98) i.e.,

$$\left.\frac{\partial \Delta f}{\partial g}\right|_T = \left\{ \begin{array}{c} \Delta_B^2 \\ \frac{2}{3}\Delta_A^2 \end{array} \right\}. \tag{7.181}$$

Eq. (he-) We can therefore calculate Δf by simply performing the integral

$$\Delta f = -\int_{1/g}^{\infty} d\left(1/g'\right) \left\{ \begin{array}{c} \Delta_B^2\left(1/g'\right) \\ \frac{2}{3}\Delta_A^2\left(1/g'\right) \end{array} \right\} \tag{7.182}$$

Eq. (he-) The $1/g$-dependence of the gap is obtained directly from (7.10), (7.29(he-9.27)) as

$$\frac{1}{g\mathcal{N}(0)} - \log\left(2\frac{e^\gamma}{\pi}\frac{\omega_c}{T}\right) = \tag{7.183}$$

$$\frac{1}{\delta}\frac{3}{4}\int_{-1}^{1} dz(1-z^2)\sum_{n=0}^{\infty}\left[1\Big/\sqrt{x_n^2+\left\{\begin{array}{c}1\\1-z^2\end{array}\right\}}-1/x_n\right].$$

From this we find Eq. (he-9.101)

$$\begin{aligned} \frac{\partial}{\partial\delta^2}\left(\frac{1}{g\mathcal{N}(0)}\right)\Bigg|_T &= -\frac{1}{2\delta^2}\frac{3}{4}\int_{-1}^{1} dz(1-z^2)\phi(\delta^2,z) \\ &= -\frac{1}{2\delta^2}\phi^{B,A}(\delta^2) = -\frac{1}{2\delta^2}\left\{\begin{array}{c}\rho_s^B/\rho\\ \rho_s/\rho\end{array}\right\} \end{aligned} \tag{7.184}$$

where ρ_s^B and ρ_S^A are the superfluid densities of B and A phases, respectively. Using this we can change variables in the integration and write Eq. (he-9.102)

$$\Delta f = \mathcal{N}(0)\pi^2T^2\frac{1}{2}\int_0^{\delta^2} d\delta'^2\left\{\begin{array}{c}\phi^B(\delta'^2)\\ \frac{2}{3}\phi^A(\delta'^2)\end{array}\right\}. \tag{7.185}$$

Inserting ϕ^B from the upper part of equation (7.54) we can perform the integral with the result: Eq. (he-9.103)

$$\frac{1}{\delta^2}\int_0^{\delta^2} d\delta'^2\phi^B(\delta'^2) = \frac{4}{8}\sum_{n=0}^{\infty}\left[-\frac{1}{\sqrt{x_n^2+1}}+2\left(\sqrt{x_n^2+1}-x_n\right)\right] \tag{7.186}$$

We shall denote this angular average by $\tilde{\phi}^B$. In analogy to the relation $\phi_s^B = \rho_s^B/\rho$ [see (7.184)] we shall also write $\tilde{\phi}^B \equiv \tilde{\rho}_s^B/\rho$, and state the result (7.186) as Eq. (he-)

$$\frac{\tilde{\rho}_s^B}{\rho} \equiv \frac{4}{\delta}\sum_{n=0}^{\infty}\left[-\frac{1}{\sqrt{x_n^2+1}}+2\left(\sqrt{x_n^2+1}-x_n\right)\right]. \tag{7.187}$$

When plotted against temperature this function starts out as $(1-T/T_c)$ for $T \sim T_c$ and goes to unity for $T \to 0$.

Similarly, we may integrate the second of Eqs. (7.185), for which we use the expansion (7.65). Pleasantly, it turns out that the integral

$$\frac{1}{\delta^2}\int_0^{\delta} d\delta'^2\phi^A(\delta'^2)$$

is no new gap function but coincides with $\rho_s^{\parallel}/\rho$ of Eq. (7.66).

The condensation energy can therefore be written in the simple form Eq. (he-9.105)

$$\Delta f = -\mathcal{N}(0)\pi^2 T^2 \left\{ \begin{array}{c} \frac{1}{2}\tilde{\rho}_s^B/\rho \\ \frac{1}{3}\rho_s^{\parallel}/\rho \end{array} \right\} \delta^2. \tag{7.188}$$

For $T \to T_c$, $\tilde{\rho}_s^B$ and $\rho_s^{\parallel}$ behave both like $1 - \frac{T}{T_c}$ so that Eq. (he-9.106)

$$\Delta e \approx -\mathcal{N}\pi^2 T^2 \frac{1}{2}\left(1 - \frac{T}{T_c}\right)^2 \frac{8}{7\zeta(3)} \left\{ \begin{array}{c} 1 \\ \frac{5}{6} \end{array} \right\} \tag{7.189}$$

in agreement with our previous calculation (4.22) in the Ginzburg-Landau regime for $T \sim T_c$.

Eq. (he-) For $T \to 0$, $\tilde{\rho}^B$ and $\rho_s^{\parallel}$ both tend to ρ and

$$\delta^2 \pi^2 T^2 \to \left\{ \begin{array}{c} 3.111 \\ 4.118 \end{array} \right\} T_c^2. \tag{7.190}$$

Eq. (he-) Thus the condensation energies become at zero temperature

$$\Delta e|_{T=0} = -\left\{ \begin{array}{c} .236 \\ .209 \end{array} \right\} c_n(T_c). \tag{7.191}$$

The right-hand part of the equation has been normalized with respect to the specific

Eq. (he-9.109) heat of the liquid just above the critical temperature.

$$c_n(T_c) = -\frac{2}{3}\pi^2 \mathcal{N}(0) T_c \tag{7.192}$$

The full temperature dependence of Δe is shown in Fig 7.5.

7.2.2 Entropy

Let us now calculate the entropy. For this it is useful to note that at fixed T and $1/g$ the energy is extremal with respect to small changes in Δ. It is this condition

Eq. (he-9.110) which previously lead to the gap equation (7.9). Thus when forming

$$s = -\frac{\partial f}{\partial T} \tag{7.193}$$

we do not have to take into account the fact that Δ^2 varies with temperature.

Eq. (he-9.111) Therefore we find

$$\Delta s = -\frac{\partial \Delta f}{\partial T} = -2\sum_{\mathbf{p}} \left[\log(1 - n(\mathbf{p})) - \frac{T}{n(1-n)} \frac{\partial n}{\partial T} \right]. \tag{7.194}$$

Eq. (he-9.112) But the derivative is

$$\frac{\partial n}{\partial T} = n(1-n)\frac{E}{T^2} \tag{7.195}$$

Eq. (he-) so that the entropy becomes

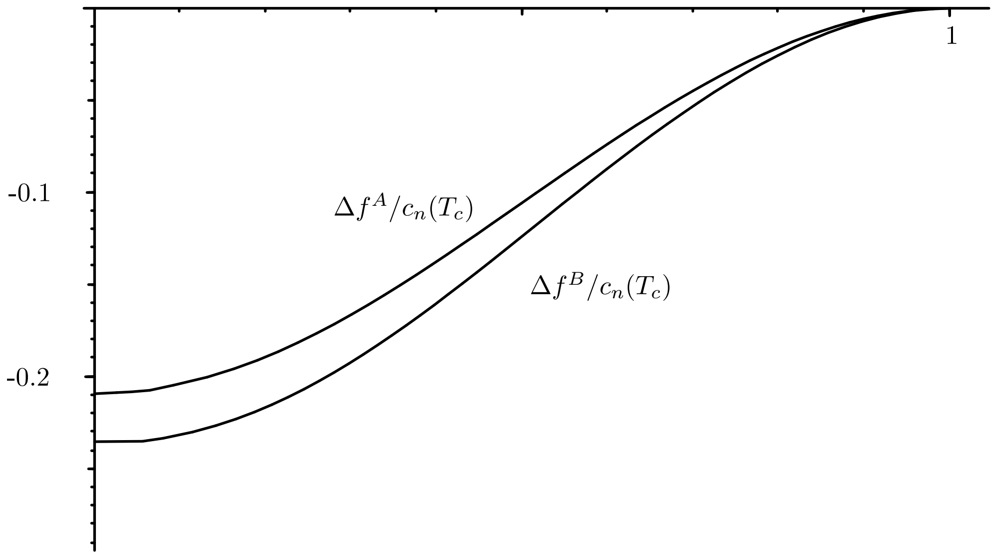


FIGURE 7.5 Condensation energies of A- and B-phases as functions of temperature.

$$\Delta s = -2\sum_{\mathbf{p}}\left[\log\left(1-n(\mathbf{p})\right) - n\frac{E(\mathbf{p})}{T}\right] \tag{7.196}$$

Eq. (he-9.114) which can be rewritten in the more familiar form

$$\Delta s = -2\sum_{\mathbf{p}}\left[(1-n)\log(1-n) + n\log n\right] \tag{7.197}$$

after having inserted the identity Eq. (he-9.115)

$$\frac{E}{T} = \log\frac{1-n}{n} \tag{7.198}$$

For the explicit calculation we differentiate (7.185) to with respect to the temperature and find Eq. (he-9.116)

$$\Delta s = -\frac{\partial \Delta f}{\partial T} = \mathcal{N}(0)\pi^2 T\int_0^{\delta^2} d\delta'^2 \left\{ \begin{array}{c} \phi^B \\ \frac{2}{3}\phi^A \end{array} \right\} + \mathcal{N}(0)\pi^2 T^2 \frac{1}{2} \left\{ \begin{array}{c} \phi^B \\ \frac{2}{3}\phi^A \end{array} \right\} \frac{\partial \delta^2}{\partial T}. \tag{7.199}$$

From Eq. (7.29) we know $\log(T/T_c)$ as a function of δ^2. Differentiation yields Eq. (he-9.117)

$$\frac{1}{T}\frac{dT}{d\delta^2} = -\frac{1}{2\delta^2}\frac{3}{4}\int_{-1}^{1} dz(1-z^2)\phi(\delta,z) = -\frac{1}{2\delta^2}\phi^{B,A} \tag{7.200}$$

so that the condensation entropy density is simply Eq. (he-9.118)

$$\Delta s^{B,A} = -\mathcal{N}(0)\pi^2 T \left\{ \begin{array}{c} 1 \\ \frac{2}{3} \end{array} \right\} \int_0^{\delta^2} d\delta'^2 \left[1 - \phi^{B,A}(\delta'^2)\right] \tag{7.201}$$

If we normalize this again with the help of $c_n(T_c)$ this can be written as Eq. (he-9.120)

$$\Delta s^{B,A}/c_n(T_c) = -\left\{ \begin{array}{c} \frac{2}{3}\,(1-\tilde{\rho}_B/\rho) \\ (1-\rho_s^{\parallel}/\rho) \end{array} \right\} \delta^2. \tag{7.202}$$

Eq. (he-9.121) For $T \to T_c$ this behaves like

$$\Delta s^{A,B}/c_n(T_c) \underset{T\approx T_c}{\approx} -\frac{3}{2}\left\{ \begin{array}{c} 1 \\ \frac{5}{6} \end{array} \right\}\left(1-\frac{T}{T_c}\right)\frac{8}{7\zeta(3)}. \tag{7.203}$$

In order to calculate the $T \to 0$ limit we consider the expansion (7.185). For $T \to 0$, $\delta \to \infty$ so that the spacings of $x_n = (2n+1)/\delta$ become infinitely narrow and the sum converges towards an integral according to the rule[3] Eq. (he-9.122)

$$\begin{aligned}\sum_{n=0}^{\infty} f(x_n) &= \frac{\delta}{2}\int dx f(x) - \frac{1}{2!\cdot 3\delta^2}\left(f'(\infty)-f'(0)\right) \\ &\quad + \left[\left(\frac{1}{2!\cdot 3}\right)^2 - \frac{1}{4!\cdot 5}\right]\frac{1}{\delta^4}\left(f'''(\infty)-f'''(0)\right)+\dots\ .\end{aligned} \tag{7.204}$$

Eq. (he-9.123) For $\tilde{\rho}^B$ this implies

$$\begin{aligned}\frac{\tilde{\rho}_B}{\rho}(\delta^2)\bigg|_{\delta^2\to 0} &= 2\int_0^{\infty} dx\left[-\frac{1}{\sqrt{x^2+1}}+2\left(\sqrt{x^2+1}-x\right)\right]-\frac{2}{3\delta^2}+\dots \\ &= 1-\frac{2}{3\delta^2}+\dots\end{aligned} \tag{7.205}$$

Eq. (he-9.124) Similarly, we can treat the series for $\rho_s^{\parallel}/\rho$ in (7.66):

$$\begin{aligned}\frac{\rho_s^{\parallel}}{\rho}(\delta^2)\bigg|_{\delta^2\to 0} &= \frac{3}{2}\int_0^{\infty} dx\left[-3x+\left(3x^2+1\right)\arctan\frac{1}{x}\right)\right]-\frac{1}{\delta^2}+\dots \\ &= 1-\frac{1}{\delta^2}+\mathcal{O}\left(\frac{1}{\delta^4}\right).\end{aligned} \tag{7.206}$$

Eq. (he-9.125) Note that for $T \to 0$, the condensation entropy densities are in *both* phases

$$\Delta s^{B,A} = -\frac{2}{3}\mathcal{N}(0)\pi^2 T. \tag{7.207}$$

Eq. (he-9.126) This is cancelled exactly with the normal entropy

$$s_n = \frac{2}{3}\mathcal{N}(0)\pi^2 T. \tag{7.208}$$

Hence the total entropy vanishes at $T = 0$, as it should. The full T behavior is plotted in Fig. 7.6. Fig. sab

It is worth pointing out that the procedure of going from sums to integrals works only if the integral over the function $f(x)$ has no singularity at $x = 0$. In the $T \to 0$

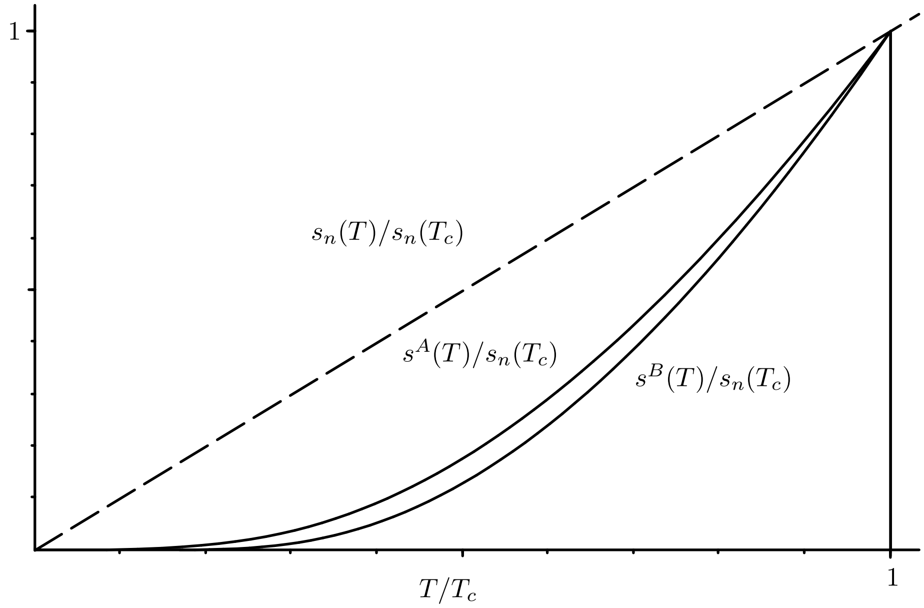


FIGURE 7.6 The temperature behavior of the condensation entropies in B- and A-phases.

limit of (7.29), for example, the following more careful limiting procedure, would be
Eq. (he-9.127) necessary

$$\begin{aligned}\sum_{n=0}^{N}\frac{1}{x_n} &= \delta\sum_{n=1}^{N}\frac{1}{2n+1} = \delta\left(\sum_{n=1}^{2(N+1)}\frac{1}{n} - \frac{1}{2}\sum_{n=1}^{N+1}\frac{1}{n}\right)\\ {}_N\underset{\text{large}}{\approx} & \ \delta\left\{\log 2(N+1+\gamma - \frac{1}{2}\left[\log(N+1)+\gamma\right]\right\}\\ &= \frac{\delta}{2}\left\{\int_{1/\delta}^{x_N}\frac{dx}{x} + \log(2e^{\gamma})\right\}. \end{aligned} \tag{7.209}$$

Thus one would obtain Eq. (he-9.127a)

$$\begin{aligned}\log\frac{T}{T_c} &\underset{T\to 0}{\longrightarrow} \frac{3}{4}\int_{-1}^{1}dz(1-z^2)\left[\int_0^\infty dx\frac{1}{\sqrt{x^2+\left\{\begin{array}{c}1\\1-z^2\end{array}\right\}}} - \int_{1/\delta}^{\infty}\frac{1}{x}\right]\\ &= \frac{3}{4}\int_{-1}^{1}dz(1-z^2)\left[-\gamma-\log\delta\left\{\frac{1}{\sqrt{1-z^2}}\right\}\right]\\ &= -\log\left(\delta e^{\gamma}\right) + \left\{\begin{array}{c}0\\ \log\left(e^{5/6}/2\right)\end{array}\right\} \end{aligned} \tag{7.210}$$

[3]Notice that this Euler-MacLaurin expansion misses exponential approaches $e^{-\delta}$

in agreement with (7.22), (7.23(he-9.21)). Mnemonically, the following rule is useful: If $\sum_{n=0}^{\infty}\frac{1}{x_n}$ appears in a sum which due to the presence of similarly divergent terms
Eq. (he-) is convergent.[4] for $n\to\infty$, it can be replaced by the integral

$$\frac{\delta}{2}\int_0^x\frac{dx'}{x'}\rightarrow\frac{\delta}{2}(\log x-\log 0). \tag{7.211}$$

Eq. (he-9.128) At the lower limit one has to substitute

$$\begin{aligned}\log 0 &\rightarrow -\log\left(2\delta e^{\gamma}\right)\\ &= -\log\frac{2\Delta e^{\gamma}/\pi}{T}\\ &= -\log\frac{2\Delta}{\Delta_{BCS}}\frac{T_c}{T}\end{aligned} \tag{7.212}$$

Eq. (he-9.128a) where Δ_{BCS} denotes the isotropic gap of the B phase at zero temperature

$$\Delta_{BCS}=\pi e^{-\gamma}T_c\sim 1.764\;T_c\;. \tag{7.213}$$

7.2.3 Specific Heat

By a further differentiation with respect to the temperature we immediately obtain
Eq. (he-9.129) the specific heat

$$\begin{aligned}\Delta c^{B,A} &= T\frac{\partial\Delta s^{B,A}}{\partial T}=\Delta s^{B,A}-\mathcal{N}(0)\pi^2T\left\{\begin{array}{c}1\\ \frac{2}{3}\end{array}\right\}\left(1-\phi^{B,A}(\delta^2)\right)T\frac{\partial\delta^2}{\partial T}\\ &= \Delta s^{B,A}+2\mathcal{N}(0)\pi^2T\frac{1-\phi^{B,A}(\delta^2)}{\phi^{B,A}(\delta^2)}\left\{\begin{array}{c}1\\ \frac{2}{3}\end{array}\right\}\delta^2.\end{aligned} \tag{7.214}$$

Eq. (he-9.130) This can be rewritten in terms of the superfluid density function as

$$\Delta c^B/c_n(T_c) = \frac{T}{T_c}\left[-\frac{3}{2}\left(1-\tilde\rho_s^B/\rho\right)+3(\rho/\rho_s^B-1)\right]\delta^2, \tag{7.215}$$

$$\Delta c^A/c_n(T_c) = \frac{T}{T_c}\left[-\left(1-\rho_s^{\parallel}/\rho\right)+2(\rho/\rho_s-1)\right]\delta^2. \tag{7.216}$$

Eq. (he-9.131) At $T=T_c$ there are finite discontinuities

$$\Delta c^B/c_n(T_c) = \frac{3}{2}\frac{8}{7\zeta(3)}=1.43\;, \tag{7.217}$$

$$\Delta c^A/c_n(T_c) = \frac{10}{7\zeta(3)}=1.19\;. \tag{7.218}$$

which can also be derived directly from Ginzburg-Landau expressions in Eqs. (3.20). For the full specific heat one has to add the normal contribution of the normal Fermi liquid to both equations (7.212), which is simply equal to T/T_c.

Eq. (he-) For $T \to 0$ we use the results (7.205), (7.206(he-9.124)) to find

$$\Delta c^{B,A}/c_n(T_c) = -T/T_c. \tag{7.219}$$

This exactly the opposite of the specific heat of the normal liquid so that the curves for the total $c^{A,B}/c_n(T_c)$ start out very flat at the origin [exponentially flat for the B phase due to the nonzero gap (i.e., a finite activation energy) and power-like for the A phase since the gap vanishes along $\mathbf{l}$]. The full temperature behavior of the specific heat is shown in Fig. 7.7

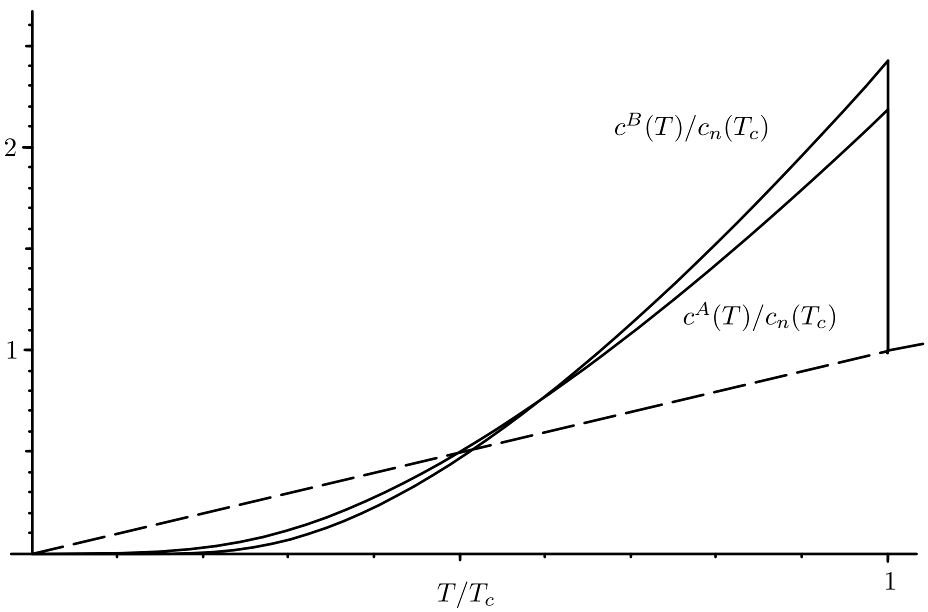


FIGURE 7.7 Specific heat of A- and B-phases as a function of temperature. The dashed line is the contribution of the normal Fermi liquid.

Certainly, all these results need strong-coupling corrections which are presently only known in the Ginzburg-Landau regime $T \to T_c$.

Appendix 7A Hydrodynamic Coefficients for $T \leq T_c$

For arbitrary temperatures $T \leq T_c$, the hydrodynamic limit is Eq. (he-B.1)

$$f = \frac{1}{4m^2}\rho_{ijkl}\partial_k A^*_{ai}\partial_l A_{aj}\frac{1}{\Delta^2_{AB}} - \left\{ \begin{array}{c} \frac{1}{4m^2}\bar{\rho}_{ijkl}\partial_k l_i \partial_l l_j \\ 0 \end{array} \right\}, \quad \left\{ \begin{array}{c} A \\ B \end{array} \right\} \text{phase} \tag{7A.1}$$

[4]If the sum diverges logarithmically, it can be made finite by subtracting an appropriate multiple of $\int_{-\omega_c}^{\omega_c} \frac{d\xi}{2\xi} \tan\frac{\xi}{2T} = \frac{2}{\delta}\sum_{n=0}^{\infty}\frac{1}{x_n} = \log\left(\frac{2\omega_c}{T}\frac{e^\gamma}{\pi}\right)$.

with A_{ai} having the forms (4.17), (4.18(he-4.11)) but being permitted to contain smooth spatial variations of the direction vectors. We now evaluate this further for the two phases:

A phase

Eq. (he-B.2) Here, we have to contract the three covariants of (7.74),

$$\begin{aligned}\hat{A}_{ijkl} &\equiv \delta_{ij}\delta_{kl}+\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl},\\ \hat{B}_{ijkl} &\equiv \delta_{ij}l_kl_l+\delta_{ik}l_jl_l+\delta_{il}l_jl_k+\delta_{kj}l_il_l+\delta_{lj}l_il_k+\delta_{kl}l_il_j,\\ \hat{C}_{ijkl} &\equiv l_il_jl_kl_l\end{aligned} \tag{7A.2}$$

Eq. (he-B.3) with

$$\partial_k\left(d_a\phi_i^*\right)\partial_l\left(d_a\phi_j\right)=\left(\partial_kd_a\partial_ld_a\right)\phi_i^*\phi_j+\partial_k\phi_i^*\partial_l\phi_j. \tag{7A.3}$$

Eq. (he-B.4) From $\hat{A}$ we find

$$\hat{A}: \qquad |\partial_i\phi_j|^2+\partial_j\phi_i{}^*\partial_i\phi_j+|\boldsymbol{\partial}\boldsymbol{\phi}|^2+2(\partial' i d_a)^2+2|\boldsymbol{\phi}\boldsymbol{\partial}d_a|^2. \tag{7A.4}$$

These gradient terms have been expanded in Appendix 5A in terms of the generic hydrodynamic gradient terms in the energy (5.8). If we use a short notation for the
Eq. (he-B.6) various invariants in that energy

$$\begin{aligned}&\hat{\rho}\equiv 4m^2\mathbf{v}_s^2\ , \quad \hat{\rho}_0\equiv -4m^2\left(\mathbf{l}\cdot\mathbf{v}_s\right)^2\ ,\\ &\hat{c}\equiv 2m\mathbf{v}_s\cdot\left(\boldsymbol{\nabla}\times\mathbf{l}\right)\ , \quad \hat{c}_0\equiv -2m\left(\mathbf{l}\cdot\mathbf{v}_s\right)\left[\mathbf{l}\cdot\left(\boldsymbol{\nabla}\times\mathbf{l}\right]\right)\ ,\\ &\hat{s}\equiv\left(\boldsymbol{\nabla}\cdot\mathbf{l}\right)^2\ , \quad \hat{t}\equiv\left[\mathbf{l}\cdot\left(\boldsymbol{\nabla}\times\mathbf{l}\right)\right]^2, \quad \hat{b}\equiv\left[\mathbf{l}\times\left(\boldsymbol{\nabla}\times\mathbf{l}\right)\right]^2,\\ &\hat{k}_1^d\equiv\left(\partial_id_a\right)^2\ , \quad \hat{k}_2^d\equiv-\left(\mathbf{l}\cdot\boldsymbol{\partial}d_a\right)^2\ .\end{aligned} \tag{7A.5}$$

it reads simply

$$\begin{aligned}f=\frac{1}{2}\Big(&\rho_s\hat{\rho}/4m^2+\varrho_0\hat{\varrho}_0/4m^2+c\hat{c}/2m+c_0\hat{c}_0/2m\\ &+K_s\hat{s}+K_t\hat{t}+K_b\hat{b}+K_1^d\hat{k}_1^d+K_2^dk_2^d\Big).\end{aligned} \tag{7A.6}$$

Eq. (he-B.5) With these invariants we can write (7A.4) as

$$\left(\hat{s}+\hat{b}+\hat{t}+2\hat{\rho}\right)+\left(\hat{b}+\hat{\rho}+\hat{\rho}_0+\hat{c}_0\right)+\left(\hat{b}+\hat{\rho}+\hat{\rho}_0+\hat{c}+\hat{c}_0\right)+4\hat{K}_1^d+2\hat{K}_2^d \tag{7A.7}$$

where parentheses indicate the different terms in (7A.4).

Eq. (he-B.7) The covariant $\hat{B}_{ijkl}$ has a very simple contribution to the **d** bending energy

$$\hat{B}: \qquad 2\left(\mathbf{l}\cdot\boldsymbol{\partial}d_a\right)^2=-2\hat{K}_2^d, \tag{7A.8}$$

as follows immediately from $\boldsymbol{\phi}\mathbf{l} = 0$. As far as the gradient terms of the $\boldsymbol{\phi}$ field are concerned we use (5A.5) to rewrite

Eq. (he-B.8)

$$\partial_k \phi_i^* \partial_l \phi_i = \left(\boldsymbol{\phi}^{(1)}\partial_k \mathbf{l}\right) l_i \left(\boldsymbol{\phi}^{(1)}\partial_l \mathbf{l}\right) l_j + (1 \leftrightarrow 2) - 4m^2 v_{sk} v_{sl}\left[\phi_i^{(1)}\phi_j^{(1)} + (1 \leftrightarrow 2)\right]$$
$$+\left\{\left[2mv_{sl}\phi_j^{(2)}(\phi^{(1)}\partial_k l) l_i + (k \leftrightarrow l,\ i \leftrightarrow j)\right] - [1 \leftrightarrow 2]\right\} \tag{7A.9}$$

and employ (5A.12) to bring the terms in curly brackets to the form

Eq. (he-B.9)

$$-2mv_{sl}\epsilon_{jmn} l_n \partial_k l_m l_i + (k \leftrightarrow l, \quad i \leftrightarrow j)\,. \tag{7A.10}$$

Contracting the pure $\mathbf{l}$ terms of (7A.9) with $\hat{B}_{ijkl}$ we find

Eq. (he-B.10)

$$\hat{B}: \quad 5[\boldsymbol{\phi}^{(1)}(\mathbf{l}\boldsymbol{\partial})\mathbf{l}]^2 + (\boldsymbol{\phi}^{(1)}\partial_k \mathbf{l})^2 + (1 \leftrightarrow 2) = 5\left(\mathbf{l}\boldsymbol{\partial} l_i\right)^2 + \left(\partial_k l_i\right)^2$$
$$= 5\hat{b} + (\hat{s} + \hat{t} + \hat{b}). \tag{7A.11}$$

The first $\mathbf{v}$ terms in (7A.9), on the other hand, contribute

Eq. (he-B.11)

$$\hat{B}: \qquad 4m^2\left(\mathbf{l}\mathbf{v}_s\right)^2 = -2\hat{\rho}_0 \tag{7A.12}$$

while the others extracted in (7A.10) add to this

Eq. (he-B.12)

$$\hat{B}: \qquad -2mv_i\epsilon_{imn} l_n (\mathbf{l}\boldsymbol{\partial}) l_m - 2m(lv_s)\epsilon_{imn} l_n \partial_i l_m$$
$$-2m(\mathbf{l}\mathbf{v}_s)\epsilon_{imn} l_n \partial_i l_m - 2mv_i\epsilon_{imn} l_n (\mathbf{l}\boldsymbol{\partial}) l_m$$
$$= -4m(\mathbf{l}\cdot\mathbf{v}_s)[\mathbf{l}\cdot(\boldsymbol{\nabla}\times\mathbf{l}) + 4m[\mathbf{v}_s\cdot(\boldsymbol{\nabla}\times\mathbf{l})] - 4m(\mathbf{l}\cdot\mathbf{v}_s)[\mathbf{l}\cdot(\boldsymbol{\nabla}\times\mathbf{l})]$$
$$= \hat{c} + 2\hat{c}_0. \tag{7A.13}$$

The contributions of the third covariant $\hat{C}_{ijkl}$, finally, are obtained by contracting four $\mathbf{l}$-vectors with (7A.8) giving

Eq. (he-B.13)

$$\hat{C}: \qquad [\boldsymbol{\phi}'(\mathbf{l}\boldsymbol{\partial})\mathbf{l}]^2 + [1 \rightarrow 2] = [(\mathbf{l}\boldsymbol{\partial})l]^2 = [\mathbf{l}\cdot(\boldsymbol{\nabla}\times\mathbf{l})]^2 = \hat{b}. \tag{7A.14}$$

Collecting all terms we obtain

Eq. (he-)

$$(A\hat{A} + B\hat{B} + C\hat{C})_{ijkl}\partial_k(d_a\phi_i^*)\partial_l(d_a\phi_j) =$$
$$4A\hat{\rho} + 2(A-B)\hat{\rho}_0 + 4A\hat{K}_1^d + 2(A-B)\hat{K}_2^d \tag{7A.15}$$
$$+(A+B)\hat{C} + 2(A+B)\hat{C}_0 + (A+B)\hat{s} + (A+B)\hat{t} + (3A+6B+C)\hat{b},$$

and the Inserting (7.76)-(7.78(he-9.59)) we obtain the energy (5.8) with the coefficients

Eq. (he-B.15)

$$\begin{aligned} 2mC &= \tfrac{1}{2}\rho_s^{\parallel}, & 2mc^{\parallel} &= 2m(c_0 - c) = \tfrac{1}{2}\rho_s^{\parallel}, \\ 4m^2 K_1^d &= \rho_s, & 4m^2 K_2^d &= \rho_0, \\ 4m^2 K_s &= 4m^2 K_t = \tfrac{1}{2}\rho_s^{\parallel}, & 4m^2 K_b &= \tfrac{3}{4}\gamma. \end{aligned}$$

We now turn to the $\bar{\rho}_{ijkl}$ term in the gradient energy (7A.1). This tensor has once more the same expansion into covariants

$$\bar{A}\hat{A}_{ijkl} + \bar{B}\hat{B}_{ijkl} + \bar{C}\hat{C}_{ijk} \tag{7A.16}$$

with the coefficients $\bar{A}$ and $\bar{B}$ given by (7.103) while $\bar{C}_{ijkl}$ does not contribute when contacting with $\partial_k l_i \partial_l l_j$ as required by (7.102). In fact, doing this contraction on (7A.16) gives Eq. (he-B.16)

$$\bar{A}(3\hat{s}+\hat{t}+\hat{b})+\bar{B}\hat{b}, \quad \bar{A}=\bar{\rho}_s/8, \quad \bar{A}+\bar{B}=\bar{\rho}_s^{\parallel}/4. \tag{7A.17}$$

This adds $-3\bar{A},-\bar{A},-(\bar{A}+\bar{B})$ to the bending constants $\frac{1}{2}4m^2K_{s,t,b}$, respectively,
Eq. (he-B.17) which therefore become

$$4m^2K_s=\rho_s/4, \quad 4m^2K_t=\left(\rho_s+4\rho_s^{\parallel}\right)/12, \quad 4m^2K_b=(\rho_s^{\parallel}+\gamma)/2 \tag{7A.18}$$

as stated in (7.108).

B phase

Eq. (he-B.18) Let the vacuum be given by

$$A^0_{ai}=\Delta_B R_{ai}(\theta_0)e^{-i\varphi_0}. \tag{7A.19}$$

Eq. (he-B.19) We may parametrize the oscillators around this nonzero value by letting

$$R_{ai}(\theta)=R_{aj}(\theta_0)R_{ji}(\tilde{\theta}). \tag{7A.20}$$

Eq. (he-B.20) Since the indices a of A'_{ai} are always contracted, we may also use

$$\tilde{A}_{ai}\equiv R^{-1}(\theta_0)_{aa'}A_{a'i} \tag{7A.21}$$

as an order parameter without changing the energy. With this the derivative terms
Eq. (he-B.21) of the field become simply

$$\partial_k\tilde{A}_{ai}=-iL^c_{ai}\partial_k\tilde{\theta}_c=-\epsilon_{cai}\partial_k\tilde{\theta}_c, \tag{7A.22}$$

where L^c_{ai} are the 3×3 generating matrices of the rotation group $L^c_{ai}=-i\epsilon_{cai}$.
Eq. (he-B.22) Consider now the expression (7A.1) with coefficient in the B phase being.

$$\rho_{ijkl}=\frac{3}{2}\rho_s^b\frac{1}{\Delta_B^2}\frac{1}{15}\left(\delta_{ij}\delta_{kl}+\delta_{il}+\delta_{jk}+\delta_{ik}\delta_{jl}\right). \tag{7A.23}$$

Eq. (he-B.23) The derivatives are

$$\begin{aligned}\partial_k A^*_{ai}\partial_l A_{aj} &= \partial_k\tilde{A}^*_{ai}\partial_l\tilde{A}_{aj}\\ &= \Delta_B^2\left(\partial_k\varphi\partial_l\varphi\delta_{ij}+\partial_k\tilde{R}_{ai}\partial_l\tilde{R}_{aj}\right)+\text{mixed terms}\end{aligned} \tag{7A.24}$$

Eq. (he-B.24) The mixed terms can be neglected since

$$\Delta_B^2 i\left(\partial_k\tilde{R}_{ai}\tilde{R}_{aj}\partial_l\varphi-\tilde{R}_{ai}\partial_l\tilde{R}_{aj}\partial_k\varphi\right) \tag{7A.25}$$

is antisymmetric under $(\leftrightarrow j,\ k\leftrightarrow l)$ while (7A.23) is symmetric. Contracting this
Eq. (he-B.25) with the covariant in (7A.21) gives

$$
\begin{aligned}
&\Delta_B^2 \left[(3+1+1)(\partial_i\varphi)^2 + \partial_k\tilde{\theta}_c\partial_l\tilde{\theta}_{\mathrm{d}}\epsilon_{cai}\,\epsilon_{caj}\,(\delta_{ij}\delta_{kl} + \delta_{il} + \delta_{ik}\delta_{jl})\right] \\
&= \Delta_B^2 \left\{5(\partial_i\varphi)^2 + 2(\partial_i\tilde{\theta}_j)^2 + \left[(\partial_i\tilde{\theta}_j)^2 - (\partial_i\tilde{\theta})^2\right] + \left[(\partial_i\tilde{\theta}_j)^2 - \partial_i\tilde{\theta}_j\partial_j\tilde{\theta}_i\right]\right\},
\end{aligned}
\tag{7A.26}
$$

Eq. (he-B.27) so that

$$
4m^2 f = \frac{\rho_s^B}{2}\left[(\boldsymbol{\nabla}\varphi)^2 + \frac{4}{5}(\partial_i\tilde{\theta}_j)^2 - \frac{1}{5}(\boldsymbol{\nabla}\tilde{\theta})^2 - \frac{1}{5}(\partial_i\tilde{\theta}_j\partial_j\tilde{\theta}_j)\right] \tag{7A.27}
$$

as given in (7.90).

8

Large Currents and Magnetic Fields in the Ginzburg-Landau Regime

The properties of super-flow are most easily calculated close to the critical temperature. In this regime, thermodynamic fluctuations are governed by the Ginzburg-Landau form of the energy and the depairing critical currents have been derived quite some time ago. For the sake of a better understanding of our general results to follow later we find it useful to review the well-known results.

Suppose a uniform current is set up in a container along the z direction. Since the bending energies tend to straighten out textural field lines it may be expected in equilibrium that with the current also the textures are uniform. It will be shown later in a detailed study of local stability, that this assumption is indeed justified in the B-phase. In the A-phase, on the other hand, we shall see that the textural degrees of freedom play an essential role in the flow dynamics.

We shall at first neglect this complication and proceed with a discussion of flow in uniform textures. Correspondingly the collective field will for now be assumed to
Eq. (he-15.1) have the simple form

$$\Delta_{ai}(z) = \Delta^0_{ai} e^{i\varphi(z)} \tag{8.1}$$

where Δ^0_{ai} is a constant matrix. The phase factor $e^{i\varphi(z)}$ allows for a non-vanishing
Eq. (he-15.2) matter current, which may be calculated from equation (4.3) as

$$J = i\left\{K_1|\Delta^0_{ak}|^2\delta_{ij} + K_2\left[\Delta^{0*}_{aj}\Delta^0_{ai} - (i \leftrightarrow j)\right] + K_3\left[\Delta^{0*}_{ai}\Delta_{aj} - (i \leftrightarrow j)\right]\right\}\partial_j\varphi(z). \tag{8.2}$$

Because of the smallness of strong-coupling corrections on the coefficients $K_i(\leq 3\%)$ we may assume for K the common value (3.26). The presence of a non-vanishing gradient of $\partial\varphi$ requires a new minimization of the energy. This will in general modify the normal forms (4.23) - (4A.1(he-4.19)) of the gap parameters in equilibrium.

8.1 B-Phase

For a first crude estimate of the effect of a current we shall assume only the overall size of the gap parameter (4.24) of the B phase to be changed by the current.

8.1.1 Neglecting Gap Distortion

If the current runs in z direction we find from (4.17) Eq. (he-15.3)

$$f = \frac{K}{2} 5 \left[a^2(\partial_z\varphi)^2 + (\partial_z a)^2\right] \Delta_B^{\mathrm{w}\,2} - 3\alpha\mu a^2 \Delta_B^{\mathrm{w}\,2} + 9\beta_B\beta_0\Delta_B^{\mathrm{w}\,4}. \tag{8.3}$$

For the discussions to follow it will be convenient to measure the energy densities in terms of the condensation energy of the B-phase in the weak-coupling limit. In the Ginzburg-Landau regime this is Eq. (he-15.4)

$$f_c = f_c^{B\mathrm{w}} = \frac{1}{4m^2\xi_0^2}\rho\left(1 - \frac{T}{T_c}\right)^2. \tag{8.4}$$

By using the definition (4.19) and the *temperature-dependent coherence length*

$$\xi(T) = \frac{\xi_0}{\sqrt{1 - T/T_c}}, \tag{8.5}$$

with ξ_0 from (3.18), we find the simple form Eq. (he-15.5)

$$\frac{f}{2f_c} = a^2\xi^2\left[(\partial_z\varphi)^2 + (\partial_z a)^2\right] - \alpha a^2 + \frac{1}{2}\left(\frac{6}{5}\beta_B\right)a^4. \tag{8.6}$$

If we want to study the system in the presence of a non-vanishing current it is convenient to eliminate the cyclic variable φ in favor of the "canonical momentum variable" Eq. (he-15.6)

$$j \equiv \frac{1}{2\xi}\frac{\partial}{\partial\partial_z\varphi}\frac{f}{2f_c} = a^2\xi\partial_z\varphi. \tag{8.7}$$

This has the virtue of being z-independent as follows from the equation of motion for φ.

The associated Legendre transformed energy Eq. (he-15.7)

$$g = \frac{f}{2f_c} - 2\xi j\partial_z\varphi \tag{8.8}$$

can then be used to study the remaining problem in only one variable $a(z)$ Eq. (he-15.8)

$$g = (\partial_z a)^2 - \alpha a^2 + \frac{1}{2}\left(\frac{6}{5}\beta_B\right)a^4 - \frac{j^2}{a^2}. \tag{8.9}$$

By comparing (8.6) and (8.2(he-15.2)) we see that the physical current J is determined in terms of the dimensionless quantity j up to a factor: Eq. (he-15.9)

$$\begin{aligned} J &= 2\frac{\partial f}{\partial\partial_z\varphi} = 10\Delta_B^2 K\partial_z\varphi \\ &= j\frac{\hbar}{2m\xi_0}\rho\, 2\left(1 - \frac{T}{T_c}\right)^{3/2} \equiv jJ_0\left(1 - \frac{T}{T_c}\right)^{3/2}. \end{aligned} \tag{8.10}$$

Thus the quantity j measures the current in units of Eq. (he-)

$$J_0 \left(1-\frac{T}{T_c}\right)^{3/2} = v_0 \left(1-\frac{T}{T_c}\right)^{1/2} 2\rho \left(1-\frac{T}{T_c}\right) \tag{8.11}$$

where v_0 is the following reference velocity Eq. (he-15.10)

$$v_0 \equiv \frac{1}{2m\xi_0} \tag{8.12}$$

at which the de Broglie wavelength of the quasiparticles equals the coherence length ξ_0. Consequently we shall refer to v_0 as the *coherence velocity* and to J_0 as *coherence current*. In analogy with the definition of $\xi_0(T)$ from ξ_0 in (8.5), we shall also here introduce temperature-dependent quantities which contain the Ginzburg-Landau factor $(1-T/T_c)$, the a temperature-dependent coherence velocity and current

$$v_0(T) \equiv v_0 \left(1-\frac{T}{T_c}\right)^{1/2}, \qquad J_0(T) \equiv J_0 \left(1-\frac{T}{T_c}\right)^{3/2}, \tag{8.13}$$

Eq. (he-15.10a) respectively. With the superfluid velocity

$$v_s = \frac{1}{2m}\partial_z\varphi \tag{8.14}$$

Eq. (he-15.11) we can identify the superfluid density ρ_s via the definition

$$J \equiv \rho_s v_s \tag{8.15}$$

Eq. (he-15.12) where

$$\rho_s = a^2\, 2\rho \left(1-\frac{T}{T_c}\right). \tag{8.16}$$

By writing (8.12) in the form

$$v_s = v_0(T)\xi\partial_z\varphi \tag{8.17}$$

Eq. (he-15.13) we see that the quantity

$$\kappa \equiv \xi\partial_z\varphi = j/a^2 \tag{8.18}$$

measures the superflow velocity in units of the temperature-dependent coherence
Eq. (he-15.14) velocity $v_0(T)$.

$$\kappa \equiv \frac{v_s}{v_0(T)} = \frac{v_s}{v_0}\left(1-\frac{T}{T_c}\right)^{-\frac{1}{2}}. \tag{8.19}$$

In order to be able to compare the forthcoming results with experiments we may
Eq. (he-15.15) use the parameters of Wheatley, which are reproduced in Table 1, to calculate

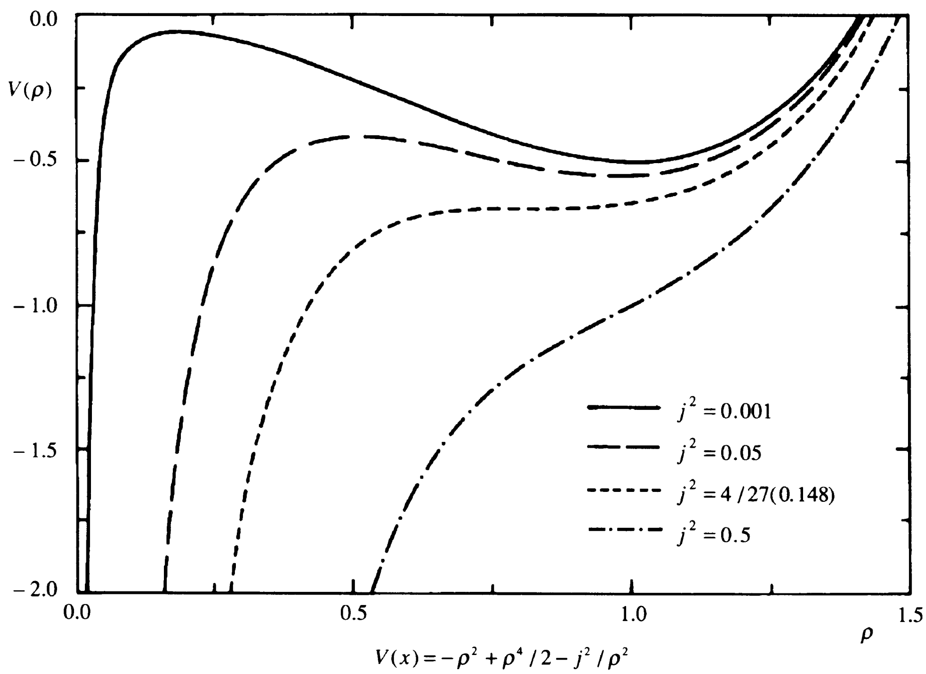


FIGURE 8.1 Shape of potential determining stability of superflow.

$$
\begin{aligned}
v_0 &= \frac{1}{2m\xi_0} = \sqrt{\frac{48}{7\zeta(3)}}\pi\frac{k_BT_c}{p_F} = 7.504\frac{k_BT_c}{p_F} \\
&\approx \left\{\begin{array}{ll} 6.25 & \text{cm/sec} \\ 15 & \text{cm/sec} \end{array}\right\} \text{ for } \left\{\begin{array}{ll} p = 0, & T_c = 1\,\text{mK} \\ p = 34.36\,\text{bar}, & T_c = 2.7\,\text{mK}. \end{array}\right\} \qquad (8.20)
\end{aligned}
$$

It is now quite simple to study the equilibrium gap configuration for a given current j. According to (8.9) the energy looks like the Lagrangian of a mass point at position a moving as a function of "time" z in a potential which is turned upside down: Eq. (he-15.16)

$$
-V(a) = -\alpha a^2 + \frac{1}{2}\left(\frac{5}{6}\beta_B\right)a^4 - \frac{j^2}{a^2}. \qquad (8.21)
$$

The shape of this potential is displayed in Fig. 8.1. For small enough current, there is a constant solution Fig. f5.1 Eq. (he-15.17)

$$
a(z) \equiv a_0 \qquad (8.22)
$$

satisfying $\partial V/\partial a = 0$. This amounts to a current Eq. (he-15.18)

$$
j^2 = a_0^4\left[\alpha - \left(\frac{6}{5}\beta_B\right)a_0^2\right]. \qquad (8.23)
$$

Obviously, this solution can exist only as long as j stays below the maximal value allowed by (8.23). By differentiation we find Eq. (he-15.19)

$$a_c^2 = \frac{2}{3}\frac{\alpha}{\frac{6}{5}\beta_B} \tag{8.24}$$

Eq. (he-15.20) with the maximal j equal to

$$j_c = \frac{2}{3}\frac{1}{\sqrt{3}}\frac{\alpha^{3/2}}{\frac{6}{5}\beta_B} \equiv a_0^2\kappa_c. \tag{8.25}$$

Eq. (he-15.21) This value determines the depairing critical current

$$J_c = J_0(T)j_c. \tag{8.26}$$

Eq. (he-15.22) At zero pressure this becomes numerically

$$J_c = 12.5\frac{\text{cm}}{\text{sec}}\rho\left(1-\frac{T}{T_c}\right)^{3/2}\frac{\alpha^{3/2}}{\frac{6}{5}\beta_B}. \tag{8.27}$$

With the values of α and β_B listed previously we find that the strong-coupling corrections provide for an increase of the critical current by a factor of about 30%. what

For completeness, let us insert (8.23) into (8.9(he-15.8)) and evaluate the total energy Eq. (he-15.23)

$$\begin{aligned} g &= -\alpha a^2 + \frac{\frac{5}{6}\beta_B}{2}a^4 - a^2\left(\alpha - \beta_B a^2\right) \\ &= -2\alpha a^2 + \frac{3}{2}\left(\frac{6}{5}\beta_B\right)a^4. \end{aligned} \tag{8.28}$$

It is cumbersome to express this analytically as a function of j since this would involve solving the cubic equation (8.9). If we, however, do not try to express it in terms of the current j but rather the parameter κ (which is the superfluid velocity in natural units) we have [see (8.7), (8.8(he-15.7)), (8.9(he-15.8))] Eq. (he-15.24)

$$\kappa^2 = \alpha - \beta_B a^2 \tag{8.29}$$

Eq. (he-15.25) and

$$g^B = -\frac{1}{2\beta_B}\left(\alpha-\kappa^2\right)\left(\alpha+3\kappa^2\right). \tag{8.30}$$

Eq. (he-15.26) Notice that the energy itself is simply

$$\frac{f^B}{2f_c} = -\frac{1}{2\beta_B}\left(\alpha-\kappa^2\right)^2. \tag{8.31}$$

8.1.2 Including a Magnetic Field

The critical currents in the B-phase depend sensitively on external magnetic fields. In order to see this consider the additional field energy Eq. (he-15.26)

$$f_{\rm mg} = g_z |H_a \Delta_{ai}|^2 \tag{8.32}$$

Eq. (he-15.27) where g_z was calculated microscopically to be

$$g_z = \frac{3}{2}\frac{\rho}{p_F^2}\frac{\xi_0^2}{v_F^2}\gamma^2\left(1+\frac{Z_0}{4}\right)^{-2} \tag{8.33}$$

Eq. (he-15.28) with γ being the magnetic dipole moment of the ^{3}He atoms

$$\gamma \approx 2.04 \times 10^4 \frac{1}{\text{gauss sec}} \tag{8.34}$$

and $Z_0 = F_0^a$ is the Fermi liquid parameter of the spin density coupling.

According to Table 2.1 its value is, at zero pressure, $-Z_0 = 2.69$. It will be useful to rewrite $f_{\rm mg}$ in a dimensionless form as Eq. (he-15.29)

$$\frac{f_{\rm mg}}{2f_c} = h^2 |\hat{H}_a \Delta_{ai}/\Delta_B|^2 \tag{8.35}$$

where $\hat{H}$ is the unit vector in the direction of the field and $h \equiv H/H_0(T)$ measures the magnetic field in terms of the following natural units Eq. (he-15.30)

$$\begin{aligned} H_0(T) &\equiv \sqrt{\frac{\frac{3}{2}\rho p_F^2}{g_z}}\sqrt{1-\frac{T}{T_c}} = \left(1+\frac{Z_0}{4}\right)\frac{v_F}{\xi_0\gamma}\sqrt{1-\frac{T}{T_c}} \\ &\equiv H_0\sqrt{1-\frac{T}{T_c}} \approx 16.4K\,\text{gauss}\sqrt{1-\frac{T}{T_c}} \end{aligned} \tag{8.36}$$

For the undistorted gap parameter (8.1), the additional magnetic energy is simply Eq. (he-15.31)

$$\frac{f_{\rm mg}}{2f_c} = h^2 a^2. \tag{8.37}$$

This enters into the expression for the equilibrium current (8.23) in the form Eq. (he-15.32)

$$j^2 = a_0^4\left[\alpha - \left(\frac{6}{5}\beta_B\right)a_0^2 - h^2\right] \tag{8.38}$$

so that the current is now maximal as Eq. (he-15.33)

$$a_0^2 = \frac{2}{3}\frac{1}{\frac{5}{6}\beta_B}\left(\alpha - h^2\right) \tag{8.39}$$

with values j_c, κ_c: Eq. (he-15.34)

$$j_c = \frac{2}{3}\frac{1}{\sqrt{3}}\frac{1}{\frac{5}{6}\beta_B}\left(\alpha - h^2\right)^{3/2}, \qquad \kappa_c = \frac{1}{\sqrt{3}}\frac{1}{\frac{5}{6}\beta_B}\sqrt{\alpha - h^2}. \tag{8.40}$$

Thus, at higher magnetic field the liquid supports less superflow. For Eq. (he-15.35)

$$ {h_c}^2 = \alpha \tag{8.41} $$

the liquid becomes normal. Notice that this result is independent of the direction of **H** with respect to the texture of the B-phase.

Let us also calculate the changes in the total energies. With (8.7) (8.18(he-15.13)) we have now Eq. (he-15.36)

$$ \kappa^2 = \alpha - \frac{6}{5}\beta_B {a_0}^2 - h^2 \tag{8.42} $$

Eq. (he-) and

$$ g^B = -\frac{1}{2\left(\frac{6}{5}\beta_B\right)}\left(\alpha - \kappa^2 - h^2\right)\left(\alpha + 3\kappa^2 + 3h^2\right) \tag{8.43} $$

Eq. (he-15.38) or

$$ \frac{f^B}{2f_c} = -\frac{1}{2\left(\frac{6}{5}\beta_B\right)}\left(\alpha - \kappa^2 - h^2\right)2 \tag{8.44} $$

8.1.3 Allowing for a Gap Distortion

Certainly, the assumption of a purely multiplicative modification of the gap was an over-simplification. For, if we look at the energy for a general gap parameter Eq. (he-15.39)

$$ \Delta^0_{ai} = \Delta^{\rm w}_B a_{ai} e^{i\varphi(z)} = \Delta^{\rm w}_B \tag{8.45} $$

complete! we find

Eq. (he-15.40)

$$ \begin{aligned} \frac{f}{2f_c} &= \frac{1}{5}\left(a^*_{ai}a_{ai} + 2a^*_{az}a_{az}\right)\left(\partial_z\varphi\right)^2 \\ &\quad -\frac{\alpha}{3}a^*a + \frac{1}{15}\{\ \beta_1 a^*. \end{aligned} \tag{8.46} $$

In the absence of field and current, the energy is invariant under the full group of independent rotations on spin and orbital indices (apart from a phase invariance $a_{ai} \to e^{i\varphi}a_{ai}$).

As a field and a current are turned on, two specific directions in these spaces are singled out. Due to the original invariance, however, the energy at the extremum cannot depend on which directions are chosen. Therefore we may pick for both **H** and **J**, the z-direction. Given the solution for the order parameter a_{ai} in this particular case, the general result may be obtained by simple performing an appropriate $\mathrm{SO}(3)_{\rm spin}\times\mathrm{SO}(3)_{\rm orbit}$ rotation into the actual directions of **H** and **J**.

We shall now determine the functional direction in which the deformation of the gap parameter has to take place. Consider at first a small current j. Then the gap parameter can be assumed to be close to the equilibrium value in the B-phase: Eq. (he-15.41)

$$ a_{ai} = a_0\delta_{ai} + a'_{ai} \equiv a_0\delta_{ai} + r_{ai} + i\, i_{ai} \tag{8.47} $$

Eq. (he-15.42) with

$$a_0 = \sqrt{\frac{\alpha}{\frac{6}{5}\beta_B}}. \tag{8.48}$$

Eq. (he-15.43) Inserting this into (8.4) we can pick up all terms up to quadratic order and find:

$$\begin{aligned} \frac{\delta^2 f}{2f_c} &= \frac{1}{5}\left\{\left(|a'_{ai}|^2 + 2|a'_{az}|^2\right) + 2a_0\left(r_{11} + r_{22} + 3r_{33}\right)\right\}\left(\partial_3\varphi\right)^2 \\ &+\frac{1}{15}\frac{\alpha}{\frac{6}{5}\beta_B}\left\{2\left(2\beta_{124} + \beta_{35}\right)R^1 + 4\beta_{12}R^3 + 2\beta_{345}R^5\right. \\ &\left.-8\beta_1 I' + 8\beta_1 I^3 - 6\beta_1 I^5 - 2\left(3\beta_1 + \beta_{35} - \beta_4\right)I^6\right\}. \end{aligned} \tag{8.49}$$

Here $R^{1,3,5,6}$ are the following quadratic forms Eq. (he-15.44)

$$\begin{aligned} R^1 &\equiv r_{11}^2 + r_{22}^2 + r_{33}^2, \\ R^3 &\equiv 2\left(r_{11}r_{22} + r_{22}r_{33} + r_{33}r_{11}\right), \\ R^{5,6} &\equiv \left(r_{12} \pm r_{21}\right)^2 + \left(r_{23} \pm r_{32}\right)^2 + \left(r_{31} \pm r_{13}\right)^2, \end{aligned} \tag{8.50}$$

with $I^{1,3,5,6}$ being the same expressions in terms of the imaginary parts i_{ai}.

Now, the linear piece involves only the real diagonal elements r_{11}, r_{22} and r_{33}. Thus only these will develop new equilibrium values. Moreover, since r_{11} and r_{22} enter symmetrically, their new values will be equal. For small currents we are just led to new gap parameter Eq. (he-15.45)

$$\Delta_{ai} = \Delta_B^{\text{w}} \begin{pmatrix} a & & \\ & a & \\ & & c \end{pmatrix} e^{i\varphi(z)}. \tag{8.51}$$

We shall now assume that this form of the distortion is present also for stronger currents up to its critical value J_c.

In order that this be really true we shall have to examine the stability of this form under small oscillations for any current. This will be done in Section 16.5, where local stability of the form (8.51) will indeed be found (up to J_c). As a side result, the analyses will provide us with the energies of all collection excitations in the presence of superflow.

In order to study the problem with distorted gap (8.51) let us, at first, neglect the strong-coupling corrections. Then the energy (8.6) takes the simple form Eq. (he-15.46)

$$\begin{aligned} \frac{f}{2f_c} &= \frac{1}{5}\xi^2\left[\left(2a^2 + 3c^2\right)\left(\partial_z\varphi\right)2 + 2a_z^2 + 3c_z^2\right] \\ &-\frac{1}{3}\left(2a^2 + c^2\right) + \frac{1}{15}\left(4a^4 + 2a^2c^2 + \frac{3}{2}c^4\right) + h^2c^2 \end{aligned} \tag{8.52}$$

where we have included the magnetic field. The current is now Eq. (he-15.47)

$$j = \frac{1}{5}\left(2a^2 + 3c^2\right)\xi\partial_z\varphi = \frac{2a^2+3c^2}{5}\kappa \tag{8.53}$$

so that the superfluid density becomes Eq. (he-15.48)

$$\rho_s^{\parallel} = \frac{1}{5}\left(2a^2 + 3c^2\right)2\rho\left(1 - \frac{T}{T_c}\right). \tag{8.54}$$

Notice that this is valid only parallel to the flow. This is why we have added a superscript $^{\parallel}$ to ρ_s. Since a and c are different, an additional small gradient of φ orthogonal to the flow would be associated with a different current Eq. (he-15.49)

$$j = \frac{1}{5}\left(4a^2 + c^2\right)\xi\partial_z\varphi \tag{8.55}$$

Eq. (he-15.50) i.e., the transverse superfluid density would rather be

$$\rho_s^{\perp} = \frac{1}{5}\left(4a^2 + c^2\right)2\rho\left(1 - \frac{T}{T_c}\right). \tag{8.56}$$

Eq. (he-15.51) The Legendre transformed energy reads

$$\begin{aligned} g = &\frac{1}{5}\xi^2\left(2a_z^2 + 3c_z^2\right) - \frac{1}{3}\left(2a^2 + c^2\right) \\ &+\frac{1}{15}\left(4a^4 + 2a^2c^2 + \frac{3}{2}c^4\right) - \frac{5j^2}{2a^2+3c^2} + h^2c^2. \end{aligned} \tag{8.57}$$

Eq. (he-) Minimizing this with respect to a and c we find two equations

$$-\frac{2}{3} + \frac{1}{15}\left[2\left(2a^2 + c^2\right) + 4a^2\right] + \frac{10j^2}{\left(2a^2+3c^2\right)^2} = 0, \tag{8.58}$$

$$-\frac{1}{3} + \frac{1}{15}\left[\left(2a^2 + c^2\right) + 2c^2\right] + 15\frac{j^2}{\left(2a^2+3c^2\right)^2} + h^2 = 0, \tag{8.59}$$

Eq. (he-15.53) which are solved by

$$a_0^2 = 1 + \frac{3}{2}h^2, \tag{8.60}$$

$$j^2 = \frac{1}{25}\left(2 + 3c_0^2 + 3h^2\right)^2\frac{1}{3}\left(1 - c_0^2 + 6j^2\right). \tag{8.61}$$

Thus, in the absence of a magnetic field, the gap parameter orthogonal to the flow is not distorted after all, Eq. (he-15.53)

$$\Delta^{\perp} \equiv \Delta_B^{\rm w}\,a_0 = \Delta_B^{\rm w}, \tag{8.62}$$

Eq. (he-15.54) whereas the gap parallel to the flow is reduced to

$$\Delta^{\parallel} \equiv \Delta_B^{\rm w}\,c_0 \tag{8.63}$$

with c_0 satisfying (8.62).

Eq. (he-15.55) The current has a maximal size for

$$c_c^2 = \frac{4}{9} - \frac{1}{13}h^2 \tag{8.64}$$

Eq. (he-15.56) where j_c, κ_c take the values

$$j_c = \frac{2}{9}\sqrt{\frac{5}{3}}\left(1-3h^2\right)^{3/2}, \quad \kappa_c = \frac{1}{3}\sqrt{\frac{5}{3}}\left(1-3h^2\right)^{1/2}. \tag{8.65}$$

The critical current is smaller than the previously calculated value by a factor of about 3/4.

The energy can be expressed most simply as a function of κ. From (8.53) and (8.62(he-15.53)) we identify Eq. (he-15.56)

$$\kappa^2 = \frac{1}{3}\left(1 - c^2 - 6h^2\right). \tag{8.66}$$

Inserting this into (8.52), which in terms of κ reads Eq. (he-15.57)

$$\begin{aligned}\frac{f}{2f_c} &= \frac{1}{5}\left(2a^2+3c^2\right)\kappa^2 \\ &\quad -\frac{1}{3}\left(2a^2+c^2\right) + \frac{1}{15}\left(4a^4+2a^2c^2+\frac{3}{2}c^4\right) + h^2c^2,\end{aligned} \tag{8.67}$$

we may evaluate only half of the quadratic terms according to the general rule that in equilibrium the quartic part is half the opposite of the quadratic one, from homogeneity. In this way we easily find Eq. (he-15.58)

$$\frac{f}{2f_c} = -\frac{1}{2}\left(1-\kappa^2-h^2\right)^2 - \frac{5}{2}\left(h^2+\frac{2}{5}\kappa^2\right)^2 \tag{8.68}$$

so that Eq. (he-15.59)

$$\begin{aligned} g &= \frac{f}{2f_c} - 2j\kappa \\ &= -\frac{1}{2}\left(1-\kappa^2-h^2\right)^2 - 2\kappa^2\left(1-3h^2-\frac{9}{5}\kappa^2\right).\end{aligned} \tag{8.69}$$

Let us now see how strong-coupling corrections modify this result. It is straightforward to calculate that then the free energy reads Eq. (he-15.60)

$$\begin{aligned} g &= \frac{1}{5}\left(2a_z^2+3c_z^2\right) \\ &\quad -\frac{1}{3}\left(2a^2+c^2\right) + \frac{1}{15}\left[\beta_{12}\left(2a^2+c^2\right)^2 + \beta_{345}\left(2a^4+c^4\right)\right] \\ &\quad -\frac{5j^2}{2a^2+3c^2} + h^2c^2. \end{aligned} \tag{8.70}$$

The local minimum is given Eq. (he-15.61)

$$-\frac{2}{3}+\frac{4}{15}\left[\beta_{12}\left(2a^2+c^2\right)+\beta_{345}a^2\right]-10\frac{j^2}{\left(2a^2+3c^2\right)^2}=0$$
$$-\frac{1}{3}+\frac{2}{15}\left[\beta_{12}\left(2a^2+c^2\right)+\beta_{345}c^2\right]-15\frac{j^2}{\left(2a^2+3c^2\right)^2}+h^2=0$$

Eq. (he-15.62) so that c and a are now related by

$$\left(4\beta_{12}+3\beta_{345}\right)a^2+\left(2\beta_{12}-\beta_{345}\right)c^2=5\left(1+\frac{3}{2}h^2\right). \tag{8.71}$$

Eq. (he-15.63) From this we find

$$\kappa^2=\frac{25j^2}{\left(2a^2+3c^2\right)^2}=\frac{5\beta_{345}}{3\left(\beta_{12}+3\beta_{345}\right)}\left(\alpha-\frac{6}{5}\beta_Bc^2-\frac{2\beta_{12}+\beta_{345}}{\beta_{345}}3h^2\right) \tag{8.72}$$

Eq. (he-15.64) so that the longitudinal gap parameter is

$$\frac{\Delta^{\parallel 2}}{\Delta_B^{\mathrm{w}\,2}}=c^2=\frac{1}{\frac{6}{5}\beta_B}\left[\alpha-\frac{9}{5}\left(\frac{4\beta_{12}}{3\beta_{345}}\right)\kappa^2-\left(1+\frac{2\beta_{12}}{\beta_{345}}\right)3h^2\right] \tag{8.73}$$

Eq. (he-15.65) similarly we find for the transversal direction

$$\frac{\Delta^{\perp 2}}{\Delta_B^{\mathrm{w}\,2}}=a^2=\frac{1}{\frac{6}{5}\beta_B}\left[\alpha+\frac{3\beta_{12}}{\beta_{345}}h^2-\left(1-\frac{2\beta_{12}}{\beta_{345}}\right)\frac{3}{5}\kappa^2\right]. \tag{8.74}$$

The current is now maximal at

$$\kappa_c^2=\frac{5}{9}\frac{5\beta_{345}}{8\beta_{12}+11\beta_{345}}\left(\alpha-\frac{4\beta_{12}+3\beta_{345}}{5\beta_{345}}3h^2\right) \tag{8.75}$$

Eq. (he-) where j_c,κ_c become

$$\begin{aligned} j_c &= \frac{1}{\frac{6}{5}\beta_B}\frac{2\sqrt{5}}{9}\sqrt{\frac{5\beta_{345}}{8\beta_{12}+11\beta_{345}}}\left(\alpha-\frac{4\beta_{12}+3\beta_{345}}{5\beta_{345}}3h^2\right)^{3/2} \\ \kappa_c &= \sqrt{\frac{5}{9}}\sqrt{\frac{5\beta_{345}}{8\beta_{12}+11\beta_{345}}}\left(\alpha-\frac{4\beta_{12}+3\beta_{345}}{5\beta_{345}}3h^2\right)^{1/2}. \end{aligned} \tag{8.76}$$

Eq. (he-15.66) The energy is found by the same trick as before

$$\frac{f}{2f_c}=-\frac{1}{2}\left\{\left(\alpha-\kappa^2-h^2\right)^2\frac{1}{\frac{6}{5}\beta_B}+\frac{5}{\beta_{345}}\left(h^2+\frac{2}{5}\kappa^2\right)^2\right\} \tag{8.77}$$

This result is rather simple since some of the strong-coupling corrections cancel in
Eq. (he-15.67) the first quadratic piece of (8.68)

$$2a^2+c^2=\frac{3}{\frac{6}{5}\beta_B}\left(\alpha-\kappa^2-h^2\right). \tag{8.78}$$

The energy g, on the other hand, looks more complicated because of the awkward
Eq. (he-15.68) form of the longitudinal superfluid density

$$\frac{\rho_s^{\parallel}}{2\rho\left(1-\frac{T}{T_c}\right)} = \frac{1}{5}\left(2a^2+3c^2\right) = \tag{8.79}$$

$$\frac{1}{\frac{6}{5}\beta_B}\left\{\alpha - \frac{4\beta_{12}+3\beta_{345}}{5\beta_{345}}3h^2 - \frac{8\beta_{12}+11\beta_{345}}{5\beta_{345}}\frac{3}{5}\kappa^2\right\} \tag{8.80}$$

entering in the additional piece Eq. (he-15.69)

$$-2j\kappa = -2\kappa^2\frac{2a^2+3c^2}{5} \tag{8.81}$$

8.2 A-phase

Before discussing the result in the B-phase further it is useful to compare them with the A-phase. Also here we shall at first assume a uniform texture Eq. (he-15.70)

$$\Delta_{ai} = \Delta_{ai}{}^0 e^{i\varphi(z)}. \tag{8.82}$$

Later we shall see, however, that this Ansatz is stable only for very small currents. Still, it is instructive to go through the same calculation as in the B-phase.

The kinetic energy has the form Eq. (he-15.71)

$$f = \frac{\kappa}{2}\xi^2\left(|\Delta_{ai}{}^0|^2 + 2|\Delta_{a3}{}^0|^2\right)\kappa^2. \tag{8.83}$$

If we suppose again that the gap parameter suffers only from a change of size we may write Eq. (he-15.72)

$$\Delta_{ai}{}^0 = \hat{\Delta}_{ai}\Delta_A = \hat{\Delta}_{ai}\Delta_B^{\mathrm{w}}\Delta_B^{\mathrm{w}}a \tag{8.84}$$

and assume for $\hat{\Delta}_{ai}$ the standard form up to spin and orbital rotation. From (8.83) we see that the bending energy is minimal if $\hat{\Delta}_{a3}$ is chosen to vanish, i.e., l points in the direction of flow. The total energy has then the form Eq. (he-15.73)

$$\begin{aligned} f &= \frac{\kappa}{2}\Delta_B^{\mathrm{w}\,2}2a^2(\partial\varphi)^2 \\ &\quad -3\alpha\mu a^2\Delta_B^{\mathrm{w}2} + 4\beta_0\beta_A a^4\Delta_B^{\mathrm{w}4}. \end{aligned} \tag{8.85}$$

The current has now the form Eq. (he-15.74)

$$J = 4m\kappa\Delta_B^{\mathrm{w}2}a^2\xi\partial_z\varphi. \tag{8.86}$$

For the sake of comparison with the previously derived results for the B-phase it is useful to measure again allenergies in units of $2f_c$ of the B-phase. Then we obtain[1] Eq. (he-15.75)

$$\frac{f}{2f_c} = \frac{2}{5}a^2\left(\partial_z\varphi\right)^2 - \frac{2}{3}\alpha a^2 + \frac{2}{9}\frac{6}{5}\beta_A a^4 \tag{8.87}$$

for which the dimensionless current is now Eq. (he-15.76)

$$j = \frac{2}{5}a^2\xi\partial_z\varphi = \frac{2}{5}a^2\kappa. \tag{8.88}$$

Eq. (he-15.77) Therefore the Legendre transformed energy

$$g = \frac{f}{2f_c} - 2j\kappa = -\frac{5j^2}{2a^2} - \frac{2}{3}\left(\alpha a^2 - \frac{2}{5}\beta_A a^4\right) \tag{8.89}$$

Eq. (he-15.78) is extremal at

$$j^2 = \frac{4}{15}a^4\left(\alpha - a^2\frac{4}{5}\beta_A\right). \tag{8.90}$$

By comparison with (8.84) we find the gap parameter as a function of the velocity
Eq. (he-15.79) κ as

$$a^2 = \frac{5}{4\beta_A}\left(\alpha - \frac{3}{5}\kappa^2\right) \tag{8.91}$$

Eq. (he-15.80) from which we may calculate

$$\Delta_A{}^2 = \Delta_B^{\mathrm{w}\,2}a^2. \tag{8.92}$$

Eq. (he-15.81) The current has a maximum at $a^2 = \frac{5}{6}\frac{\alpha}{\beta}$ with the critical values

$$\begin{aligned} j_c &= \frac{\sqrt{5}}{9}\frac{\alpha^{3/2}}{\beta_A} \\ \kappa_c &= \frac{\sqrt{5}}{3}\alpha^{1/2}. \end{aligned} \tag{8.93}$$

Eq. (he-15.82) In terms of κ the energies take the simple explicit forms:

$$\begin{aligned} \frac{f}{2f_c} &= -\frac{1}{\frac{6}{5}\beta_A}\left(\alpha - \frac{3}{5}\kappa^2\right)^2 \\ g &= \frac{f}{2f_c} - \frac{1}{\beta_A}\left(\alpha - \frac{3}{5}\kappa^2\right)\kappa. \end{aligned} \tag{8.94}$$

It is important to realize that all these results are true irrespective of the presence of a magnetic field: The **d** texture can always lower its energy by orienting itself orthogonal to **H** resulting in the absence of a magnetic energy.

8.3 Critical Current in Other Phases for $T \sim T_c$

For completeness let us analyze the energies of the Ginzburg-Landau expansion in the presence of superflow in all the above possible phases. It could happen that the presence of superflow induces a transition into a phase which at zero current is

unphysical because of its high energy. In order to eliminate this possibility we shall carry out an analysis for all known phases listed in Appendix IIIB. For each of these
Eq. (he-15.94) the order parameter may be written as

$$A_{ai} = \Delta \hat{\Delta}_{ai} \tag{8.95}$$

where $\hat{A}_{ai}$ is sometimes normalized to unity Eq. (he-15.95)

$$\mathrm{tr}\left(\hat{A}_{ai}\hat{A}_{ai}\right) = 1. \tag{8.96}$$

The energy is Eq. (he-15.96)

$$f = -\mu\Delta^2 + \beta_0\Delta^4\beta \tag{8.97}$$

where β is the combination of β_i's for the phase under consideration. This is minimal at Eq. (he-209)

$$\Delta_0^2 = \frac{\mu}{2\beta_0\beta} \tag{8.98}$$

with $f = -f_c$ and Eq. (he-15.98)

$$f_c = +\frac{\mu^2}{4\beta_0}\frac{1}{\beta} = -\rho\left(1 - \frac{T}{T_c}\right)^2 \frac{1}{2\xi_0^2 m}\frac{5}{74}\frac{1}{\beta}. \tag{8.99}$$

Let there now be an equilibrium current flowing through a uniform texture. The order parameter may be normalized as Eq. (he-15.99)

$$A^{ai} = \Delta\hat{A}_{ai}e^{i\varphi} \tag{8.100}$$

so that the bending energies are Eq. (he-15.100)

$$f = \frac{K}{2}\Delta^2\left[(\partial_i\varphi)^2 + 2(\partial_x\varphi)^2\left(A_{ax}\right)^2 + 2(\partial_y\varphi)^2(A_{ay})^2 + 2(\partial_z\varphi)^2(A_{az})^2\right] \tag{8.101}$$

In the presence of the velocity $(\partial_i\varphi)/2m$ the energy does not minimize any longer a gap value (8.98), but at a new modified order Eq. (he-15.101)

$$\Delta = \Delta_0 a \tag{8.102}$$

so that the energy can be written as Eq. (he-15.102)

$$\begin{aligned} f \;=\; & -\frac{K}{2}\Delta_0^2 a^2\Big[(\partial_i\varphi)^2 \\ & \qquad +2(\partial_x\varphi)^2 + 2(\partial_x\varphi)^2(A_{ax})^2 + 2(\partial_y\varphi)^2(A_{ay})^2 + 2(\partial_z\varphi)^2(A_{az})^2\Big] \\ & -\mu{\Delta_a}^2 a^2 + \beta_0 a^4 {\Delta_0}^4\beta. \end{aligned} \tag{8.103}$$

It is again convenient to divide out the condensation energy of the phase (12.35) in the absence of a current by substituting using Eq. (he-15.103)

$$\begin{aligned} \mu {\Delta_0}^2 &= 2f_c \\ \beta\beta_0 {\Delta_0}^4 &= f_c. \end{aligned} \tag{8.104}$$

In addition, we have from (3.26) Eq. (he-15.104)

$$\frac{K{\Delta_0}^2}{2} = \frac{3}{5}\mathcal{N}(0)\mu\frac{1}{2\beta_0\beta}{\xi_0}^2 = \frac{6}{5}f_c. \tag{8.105}$$

Eq. (he-15.105) Therefore the energy has the generic reduced form

$$\frac{f}{f_c} = \frac{6}{5}a^2{\xi_0}^2(\partial\varphi)^2 - 2a^2 + a^4 \tag{8.106}$$

Eq. (he-) with

$$\alpha = 1 + 2|\hat{A}_{ai}\hat{j}|^2 \tag{8.107}$$

Eq. (he-15.106) and $\hat{j}$ being the direction of the current. The physical current is

$$\begin{aligned} J &= f_c 4m\frac{6}{5}a^2{\xi_0}^2(\partial\varphi)\alpha \\ &= \rho\left(1-\frac{T}{T_c}\right)^{7/2}\frac{1}{\xi_0 m^2}\frac{1}{2\beta}a^2\zeta(\partial\varphi) \\ &= jJ_0. \end{aligned} \tag{8.108}$$

Here J_0 is the same quantity as introduced in (8.11) with a dimensionless reduced
Eq. (he-15.107) current

$$j = \frac{1}{2\beta}a^2\zeta(\partial\varphi). \tag{8.109}$$

Eq. (he-15.108) At a fixed j we have to minimize

$$\begin{aligned} \frac{g}{f_c} &= \frac{f - \frac{24}{5}j(\partial\varphi)}{f_c} \\ &= -\frac{24}{5}\frac{j^2}{a^2}\frac{\beta^2}{\alpha} - 2a^2 + a^4. \end{aligned} \tag{8.110}$$

Eq. (he-15.109) The equilibrium value of a lies at

$$j^2 = Ra^4\left(1-a^2\right) \tag{8.111}$$

Eq. (he-15.110) where R is the quantity

$$R = \frac{5}{R}\frac{\alpha}{\beta^2}. \tag{8.112}$$

Eq. (he-15.111) If β, α are independent of a the current is maximal for $a^2 = \frac{2}{3}$ with

[1]This result agrees, of course, with energy (8.52) of the distorted B-phase if one inserts $c = 0$ and takes $\alpha = \beta_A = 1$, i.e., the weak-coupling limit since the planar and A-phase are energetically the same.

TABLE 8.1 Parameters of the critical currents in all theoretically known phases

Phases	α	direction of current	β	β_{GL}	R_{GL}	$\frac{f_c}{f_c{}^B}$	$\frac{g_c}{f_c{}^B}$
B	$\frac{5}{3}$	x, y, z	$\beta_{12} + \frac{1}{3}\beta_{345}$	$\frac{5}{6}$	1	1	$\frac{4}{3}$
planar	1	z	$\beta_{12} + \frac{1}{2}\beta_{345}$	1	$\frac{5}{9}$	$\frac{5}{6}$	
polar	1	y, z	$\beta_{12} + \beta_{345}$	$\frac{3}{2}$	$\frac{5}{12}$	$\frac{5}{9}$	
α	$\frac{5}{3}$	x, y, z	$\beta_2 + \frac{1}{3}\beta_{345}$	$\frac{4}{3}$	$\frac{5}{27}$	$\frac{5}{18}$	
bipolar	1	z	$\beta_2 + \frac{1}{2}\beta_{345}$	$\frac{3}{2}$	$\left(\frac{5}{8}\right)^2$	$\frac{5}{9}$	
axial	1	z	β_{245}	1	$\frac{5}{12}$	$\frac{5}{6}$	
β	1	y, z	β_{234}	3	$\frac{5}{108}$	$\frac{6}{15}$	
γ	1	z	β_{124}	2	$\frac{5}{48}$	$\frac{5}{12}$	

$$j^2 = \frac{1}{3}\frac{4}{9}R. \tag{8.113}$$

Let us now calculate the parameters α, R for each of the inert phases. The results are displayed in the Table 8.1. Since α depends on the direction of the current with respect to the texture the energy has to be minimized for each of the standard forms A_{ai} of Appendix 5B. In the second column we have therefore marked the possible directions of the equilibrium current. Clearly, in the presence of strong-coupling corrections, R is modified by a factor β^2/β^2. The last column contains the condensation energy as compared to that of the B-phase. At the critical current, the energy $-g_c$ is lower than $-f_c$ by a factor $\frac{4}{3}$. Thus it might in principle happen that by increasing the current, one of the higher lying phases drops underneath a lower one. It can be checked, however, that such a crossover does not take place. For this we compare g at the critical currents Eq. (he-15.112)

$$g = -4a^2 + 3a^4 = -\frac{4}{3}\left(\frac{f_c}{f_c^B}\right) f_c^B. \tag{8.114}$$

Starting out with the B-phase, the energy drops from -1 to $-\frac{4}{3}$. In the A-phase it starts out at $-\frac{5}{6}$ and drops to $-\frac{20}{18}$. This value is underneath -1 so that there is, in principle, the changes of a crossover but we can check that the energy of the B-phase drops fast enough as to avoid a collision. Similar arguments can be applied to any other pair of phases. In order to study this behavior in detail one has to plot the energy g as a function of the current j. While g as a function of a is always on R and thus on the phase under consideration by solving the cubic equation (12.46) we find Eq. (he-15.113)

$$a^2 = \frac{1}{3} + \frac{1}{3}\cos\frac{2}{3}\varphi - \frac{1}{\sqrt{3}}\sin\frac{2}{3}\varphi \tag{8.115}$$

where Eq. (he-15.114)

$$\cos\varphi = \frac{3}{2}\sqrt{3}\frac{j}{\sqrt{R}}. \tag{8.116}$$

Eq. (he-15.115) As a check we note that for

$$\begin{aligned} j = 0, \quad \varphi = \frac{\pi}{2}, \quad a^2 = 1 \\ j = j_c, \quad \varphi = 0, \quad a^2 = \frac{2}{3} \end{aligned} \tag{8.117}$$

as it should.

Consider now the non-inert phases. Then the coefficients α and β contain one

Eq. (he-15.116) more parameters, for instance an angle θ. In addition to

$$\frac{\partial g}{\partial a} = 0 \tag{8.118}$$

Eq. (he-15.117) which leads as before to

$$\frac{j^2}{R} = a^4\left(1 - a^2\right), \tag{8.119}$$

Eq. (he-15.118) we now have to minimize g also with respect to θ

$$\frac{\partial g}{\partial \theta} = 0. \tag{8.120}$$

Eq. (he-15.119) Therefore also

$$\left(-2\frac{j^2/R}{a^2} - 2a^2 + a^4\right) f_c' + \frac{R}{R^2}\frac{j^2}{a^2} f_c = 0 \tag{8.121}$$

Eq. (he-15.120) where

$$\frac{R'}{R} = -\left(2\frac{\beta'}{\beta} - \frac{\gamma'}{\gamma}\right). \tag{8.122}$$

Eq. (he-15.121) Now, f_c depends on θ only via $\frac{1}{\beta(\theta)}$. Therefore

$$f_c' = -\frac{\beta'}{\beta} f_c \tag{8.123}$$

Eq. (he-15.122) so that (8.121) becomes

$$\frac{j^2}{R} = a^4\left(1 - \frac{a^2}{2}\right)\frac{1}{1-\tau} \tag{8.124}$$

Eq. (he-15.123) where we have abbreviated

$$\tau \equiv \frac{\beta\alpha'}{\beta'\alpha}. \tag{8.125}$$

Eq. (he-15.124) By equating (8.124) and (8.119) we find

$$a^2 = \frac{\tau}{\tau - \frac{1}{2}} \tag{8.126}$$

Eq. (he-15.125) and the relation between current and angle θ becomes

$$j^2 = \frac{1}{2}\frac{\tau^2(\theta)}{(\tau(0) - 1/2)^3} R(\theta). \tag{8.127}$$

This current is maximal if θ solves the equation Eq. (he-15.126)

$$\left(\frac{\alpha'}{\alpha} + \frac{\beta'}{\beta}\right)\left(\frac{\alpha''}{\alpha'} - \frac{\beta''}{\beta'}\right) = \frac{\alpha'}{\alpha}\left(2\frac{\alpha'}{\alpha} - \frac{5}{2}\frac{\beta'}{\beta}\right). \tag{8.128}$$

If $\alpha'' = 0$, which is often the case, this can also be written in the more convenient form Eq. (he-15.127)

$$-2\beta\left(\alpha'^2\beta' + \frac{1}{2}\alpha'\beta''\right) = \beta'\alpha\left(\beta''\alpha - \frac{5}{2}\alpha'\beta'\right). \tag{8.129}$$

As an example consider the ζ phase with Eq. (he-15.128)

$$A_{ai} = \frac{\Delta}{\sqrt{s}}\begin{pmatrix} \sin\theta\cos\phi & -i\sin\theta\sin\phi & 0 \\ i\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \\ 0 & 0 & \sqrt{2}\cos\theta \end{pmatrix}. \tag{8.130}$$

Actually, this parameterization interpolates between several phases: Eq. (he-15.129)

$$\begin{aligned}
\text{polar} &: \quad \text{all } \phi,\ \theta = 0, && A_{ai} = \Delta\begin{pmatrix} 0 & & \\ & 0 & \\ & & \sqrt{2}\end{pmatrix}, \\
\text{planar} &: \quad \text{all } \phi,\ \theta = \tfrac{\pi}{2}, && A_{ai} = \Delta\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0\end{pmatrix}, \\
B &: \quad \phi = 0,\ \sin\theta = \sqrt{2}3, && A_{ai} = \frac{\Delta}{\sqrt{3}}\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1\end{pmatrix},
\end{aligned} \tag{8.131}$$

and, certainly, the non-inert phase ζ itself. The potential energy is Eq. (he-15.130)

$$f_p = -\mu\Delta^2 + \Delta^4\beta_0\beta_\zeta \tag{8.132}$$

where Eq. (he-15.131)

$$\begin{aligned}
\beta_\zeta &= \beta_1\, 4\left(1 - 2\sin^2\phi\sin^2\theta\right)^2 + \beta_2\, 4 \\
&\quad + \beta_{35}\left[2\sin^4\theta\left(1 - \sin^2 2\phi\right) + \varphi\cos^4\theta\right] + \beta_4\left[2\sin^4\theta\left(1 + \sin^2 2\phi\right) + \varphi\cos\varphi\theta\right] \\
&= (4\beta_1 + 2\beta_{345})\,\sin^4\theta + (4\beta_1 + 4\beta_{345})\cos^4\theta + 4\beta_2 \\
&\quad + (\beta_4 - \beta_{35} - 2\beta_1)\, 2\sin^4\theta\sin^2 2\phi + 8\beta_1\sin^2\theta\cos^2\theta\cos^2\phi.
\end{aligned} \tag{8.133}$$

Minimizing this with respect to ϕ gives Eq. (he-15.133)

$$\tan^2\theta\cos^2\phi = T \equiv \frac{2\beta_1}{\beta_4-\beta_{135}-\beta_1} \tag{8.134}$$

or Eq. (he-15.134)

$$\phi = 0, \pi. \tag{8.135}$$

In the latter case, A_{ai} interpolates only between the three phases (8.131). In particular, the previously discussed distorted B-phase is contained in it.

Eq. (he-15.135) In either case, the function β becomes:

$$\beta_\zeta \;=\; \beta_4\sin^4\theta + (\beta_{1345}+\beta_1 T)\cos^4\theta + \beta_2, \tag{8.136}$$

Eq. (he-15.136)

$$\beta^{\phi=0} \;=\; \beta_{12} + \frac{1}{2}\beta_{345}\left(\sin^4\theta + 2\cos^4\theta\right). \tag{8.137}$$

Eq. (he-15.137) Consider now the bending energy. Inserting (8.131) into (3.24) gives

$$f_{\rm bend} = \frac{K}{2}\Delta^2\left[(\partial_i\varphi)^2(1+\sin^2\theta) + (\partial_z\varphi)^2(2-3\sin^2\theta)\right]. \tag{8.138}$$

The orientation of the current with respect to the texture depends on the equilibrium

Eq. (he-15.138) value of θ. If

$$\sin^2\theta \begin{array}{l} < 2/3, \\ > 2/3, \end{array} \tag{8.139}$$

the current points in x, y or in z-direction, respectively. In these two cases the

Eq. (he-15.139) bending energies are

$$f_{\rm bend} = \frac{K}{2}\Delta^2(\partial_i\varphi)^2\left\{\begin{array}{c} 1+\sin^2\theta \\ 3-2\sin^2\theta \end{array}\right\}. \tag{8.140}$$

Eq. (he-15.140) Therefore we identify

$$\alpha = \left(\begin{array}{c} 1+\sin^2\theta \\ 3-2\sin\theta \end{array}\right), \quad \sin^2\theta \begin{array}{ll} < & 2/3, \\ > & 2/3. \end{array} \tag{8.141}$$

Eq. (he-15.141) In the absence of a current the extremal value for θ is given by

$$\tan^2\theta = \frac{T\beta_1+\beta_{1345}}{\beta_4}. \tag{8.142}$$

Eq. (he-) In the Ginzburg-Landau domain

$$T = -\frac{1}{2}, \quad \tan^2\theta = \frac{3}{4}, \quad \sin^2\theta = \frac{3}{7}. \tag{8.143}$$

This value of θ lies below $\sin^2\theta = 2/3$ so that for small currents the upper of Eqs. (8.141) has to be chosen. Setting $\sin^2\theta = x$ we can now solve for the critical current.
Eq. (he-15.142) For simplicity we use only the weak-coupling values of β_i and

$$\alpha = 1 + x, \quad \alpha' = 1, \quad \alpha'' = 0$$
$$\beta = x^2 + \frac{3}{4}(1-x)^2 + 1 = \frac{7}{4}\left(x^2 - \frac{6}{7}x + 1\right)$$
$$\beta' = \frac{7}{4}\left(2x - \frac{6}{7}\right), \qquad \beta'' = \frac{7}{4}\cdot 2 \tag{8.144}$$

with these values our equation (8.129) becomes linear: Eq. (he-2.143)

$$x = \frac{1}{4}. \tag{8.145}$$

At that place $\tau = \beta\alpha'/\beta'\alpha$ is Eq. (he-15.144)

$$\tau = \frac{19}{10} \tag{8.146}$$

so that the equilibrium value of a is given by Eq. (he-15.145)

$$a^2 = \frac{\tau}{\tau - \frac{1}{2}} = \frac{19}{24}. \tag{8.147}$$

The corresponding critical current is Eq. (he-15.146)

$$j_c = \sqrt{\frac{5}{6\cdot 27}}. \tag{8.148}$$

Notice that this current is smaller than that of the B phase by factor $\sqrt{5/24} \sim 1/2$.

For consistency, we convince ourselves that at critical current the value of x is smaller than at $j = 0$ so that the direction of the current with respect to the texture and therefore the choice of the bending energy with $\alpha = 1 + x$ remains valid for all equilibrium currents.

As a cross check of this method let us confirm the critical current of the B-phase with gap distortion by using the parametrization (8.137) in the weak-coupling limit. Eq. (he-15.147)

$$\beta^{\phi=0} = \frac{1}{2}\left[1 + x^2 + 2(1-x)^2\right] = \frac{3}{2}x^2 - 2x + \frac{3}{2}. \tag{8.149}$$

Here we start out with the B-phase with Eq. (he-15.148)

$$x = \sin^2\theta = \frac{2}{3}. \tag{8.150}$$

From our previous calculation we know that $c \le a$ which says that in all currents the value of θ stays above the value implied by (8.150). Then we have to use the bending energy with Eq. (he-15.149)

$$x = 3 - 2x. \tag{8.151}$$

Inserting $\beta, \gamma, \beta', \gamma', \beta''\gamma''$ into (8.129) we find the linear equation Eq. (he-15.150)

$$x = \sin^2\theta = \frac{9}{11} \tag{8.152}$$

which is indeed larger than (8.150). The values of τ, a^2, and $R = \frac{11^3}{4\cdot(21)^2}$ found as
Eq. (he-15.151) $-\frac{42}{15}$, $\frac{28}{33}$, so that the critical current becomes

$$j^2 = R\,a^4\left(1 - a^2\right) = \frac{20}{3}\frac{1}{81} \tag{8.153}$$

as obtained before.

9

Is ^{3}He-A a Superfluid?

Equipped with the calculations of the last chapter and the topological arguments of Chapter 6 we are now ready to address ourselves to an important question: Does the superfluid ^{3}He really deserve the prefix "super" in its name (apart from the similarity in the formalism with that of the superconductor)? In order to answer this question one usually performs the Gedankenexeriment of putting the liquid in a long and wide torus, stirring it to uniform rotation along the axis, cooling it down into the A- or B-Phase, and waiting whether the liquid will slow down after a finite amount of time. How, superconductors and He-II will preserve the rotation for many years. The reason is that the order parameter describing the condensate is $\Delta_0 e^{i\varphi}$ with φ varying from zero to $2\pi N$ (where N is a very large number) when going around the torus. The liquid can slow down only by N decreasing unit by unit. But in orde to do so the order parameter has to vanish in a finite volume, for example by the formation of a narrow vortex ring at the axis which increases in radius until it hits the surface where it annihilates (see Fig. 9.1). Fig. XVII

Since such a vortex ring contains a large amount of energy (the condensation energy), the probability of this relaxation process is extremely small. Only at a very narrow place (e.g. Josephson junction) can this process be accelerated so that the relaxation takes place within minutes or seconds.

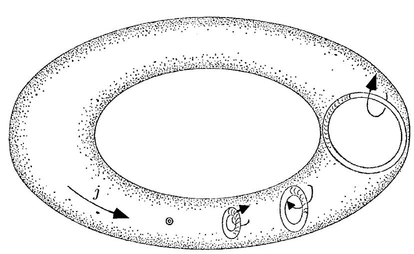


FIGURE 9.1 Superflow in a torus which can relax by vortex rings forming, increasing, and meeting their death at the surface. In a superconductor or superfluid ^{4}He, these rings have to contain a core of normal liquid and are therefore very costly in energy. This assures an extremely long lifetime of superflow. In ^{3}He-A, on the other hand, there can be coreless vortices which could accelerate the decay

The maximal size of a current which can be stable against this type of decay is reached when the kinetic energy of the superfluid reaches the order of magnitude of the condensation energy. Then the liquid may easily use up, via fluctuations, the kinetic energy to become normal so that the phase $e^{i\varphi}$ can unwind.

Obviously, the existence of a macroscopic superflow hinges on the possibility of having large flux numbers conserved topologically along the torus.

Now, in the B phase this is indeed the case. According to (6.48) and (6.49), the homotopy group describing the mapping of the axis of a torus into the parameter space of the B phase contains the group of integer numbers Z which can pile up a macroscopic superflow. In the A phase, on the other hand, one has in a large torus [see (6.45)] $\pi_1 = Z_2$. Hence, there is only one nontrivial mapping. The associated flux is of unit strength and therefore necessarily microscopic. Thus it appears as if the liquid ^{3}He is really not "super" at all in comparison with superconductors and superfluid He-II.

We shall now show that this is, fortunately, not true. There is a weaker sense, i.e., with much smaller critical currents and shorter lifetimes of stability which still can amount to hours and days, in which ^{3}He-A does *support* a stable superflow. Moreover, as the temperature drops underneath a certain value, say $T_{\rm stab}$, there are even *two separate* supercurrents, which both are topologically conserved [40]. Thus in the weaker sense ^{3}He-A really is a double superfluid. In order to understand this one has to observe that in the bulk it is not really necessary to have the overwhelming potential barrier of condensation energy in order to guarantee a stability at a macroscopic time scale. A barrier with another energy density, say $\rho_s/m^2\xi_b^2$, characterized by a length scale ξ_b much larger than the coherence length ξ_0, can easily prevent a metastable state from decaying if the volume is sufficiently large: As we argued before, such a decay can only proceed via the nucleation of a vortex tube of length L and diameter d with the energy $(\rho_s/m^2\xi_b^2)\cdot d^2L$ (for a potential barrier of the order of the dipole force $\xi_{\rm d}\sim 1000\xi_0$ and this energy corresponds to $\approx 10^{-6}$ mK per Cooper pair). The diameter d will adapt itself to the characteristic length scale of the potential barrier, i.e., $d\approx\xi_b$. Thus the energy of the vortex tube is $(\rho_s/M_2)\,L$. It is this number which enters the exponent in the Boltzmann factor
Eq. (he-11.1) dominating the decay rate

$$\frac{1}{\tau}\sim\frac{1}{\tau_0}\exp\{-(\frac{\rho_s}{m^2}\frac{1}{\xi_b^2}-f_{\rm curr})\xi_b^2L/T\} \tag{9.1}$$

where e is the energy density of the current flow and τ_0 is the characteristic time of orbital motion. This parameter varies for the decay mechanism associated with different barriers, but not by many orders of magnitude. The main effect of the smaller barrier energy lies in the significant reduction of the critical energy density which can be accumulted in the current (notice that the barrier strength parameter ξ^b cancels in the first term of the exponential). As a consequence, if we are satisfied with much smaller critical currents, potential be considered as unsurmountable. Then topological arguments can again be employed to classify stable flow configurations.

Now, the important property of ^{3}He-A is that a current, once established, does attract the l-vector into its direction via the second term in the energy (5.8) Eq. (he-11.2)

$$-\rho_0 \left(\mathbf{l}\cdot\mathbf{v}_s\right)^2. \tag{9.2}$$

It is this term which creates a potential barrier permitting a supercurrent to accumulate.

In order to simplify the discussion we shall assume the torus to be sufficiently long and wide to neglect curvature and boundaries. Thus, the fields in the energy (5.8) can be assumed to depend only on the variable z (if we assume the z axis to coincide with the axis of the torus). In order to avoid the use of constraints for respecting the curl condition (5.30) it is convenient to work directly with the parametrization of $\mathbf{l}$ and $\boldsymbol{\phi}$ in terms of Euler angles (5.3), (5.4(he-5.4)), so that $\mathbf{v}_s$ becomes, due to (5.7) Eq. (he-11.3)

$$\mathbf{v}_s = -\frac{1}{2m}\left(\boldsymbol{\nabla}\alpha + \cos\beta\,\boldsymbol{\nabla}\gamma\right). \tag{9.3}$$

We shall also express $\mathbf{d}$ in terms of directional angles as Eq. (he-11.4)

$$\mathbf{d} = \left(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta\right). \tag{9.4}$$

Since for pure z variations Eq. (he-11.5)

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{l} &= -\sin\beta\,\beta_z; \qquad \boldsymbol{\nabla}\times\mathbf{l} = -\gamma_z\mathbf{l}^{\perp} - \cos\beta\,\beta_z\frac{\mathbf{e}_z\times\mathbf{l}}{|\mathbf{e}_z\times\mathbf{l}|},\\
\mathbf{l}\cdot(\boldsymbol{\nabla}\times\mathbf{l}) &= -\sin^2\beta\,\gamma_z; \qquad \left[\mathbf{l}\times(\boldsymbol{\nabla}\times\mathbf{l})\right]^2 = \cos^2\beta\left(\beta_z^2 + \sin^2\beta\,\gamma_z^2\right),\\
\left(\nabla_i d_a\right)^2 &= \theta_z^2 + \sin^2\theta\phi_z^2; \qquad \left(\mathbf{l}\cdot\boldsymbol{\nabla}d_a\right)^2 = \cos^2\beta\left(\theta_z^2 + \sin^2\theta\,\phi_z^2\right),
\end{aligned} \tag{9.5}$$

we find the energy density Eq. (he-11.6)

$$\begin{aligned}
2f &= A(s)\alpha_z^2 + G(s)\gamma_z^2 + 2M(s)\alpha_z\gamma_z + B(s)\beta_z^2\\
&\quad + T(s)\left(\theta_z^2 + s\phi_z^2\right) + 2\frac{\rho_s^{\parallel}}{\xi_{\rm d}^{\perp 2}}\left[1 - (\mathbf{l}\cdot\mathbf{d})^2\right]
\end{aligned}$$

where the coefficients are the following functions of $s \equiv \sin^2\beta$: Eq. (he-11.7)

$$\begin{aligned}
A(s) &\equiv \rho_s^{\parallel} + \rho_0 s; \qquad \rho_s^{\parallel} \equiv \rho_s - \rho_0,\\
B(s) &\equiv K_b + (K_s - K_b)s,\\
G(s) &\equiv \rho_s^{\parallel} + \left(K_b - 2c_0 + \rho_0 - \rho_s^{\parallel}\right)s + \left(K_t - K_b + 2c_0 - \rho_0\right)s^2,\\
M(s) &\equiv \left[\rho_s^{\parallel} + \left(\rho_0 - c_0\right)s\right]\sqrt{1-s},\\
T(s) &\equiv K_1^d - K_2^d + K_2^D s.
\end{aligned} \tag{9.6}$$

Here we have dropped many factors $2m$ by going to time units t_0 in which $2m \equiv 1$, i.e.,

$$t_0 = \frac{v_F}{2p_F} = \frac{1}{2m}.$$

The energy possesses two mass currents Eq. (he-11.8)

$$J_1 \equiv -\frac{1}{2m}\frac{\partial f}{\partial \alpha_z} = -\frac{1}{2m}\left(A(s)\alpha_z + M(s)\gamma_z\right) \tag{9.7}$$

$$J_2 \equiv -\frac{1}{2m}\frac{\partial f}{\partial[(\alpha+\gamma)_z/2]} = -\frac{1}{2m}\left(G(s)\gamma_z + M(s)\alpha_z + T(\beta)s\phi_z\right) \tag{9.8}$$

which are separately conserved:

$$\partial_z J_1 = \partial_2 J_2 = 0.$$

Notice that such a conservation law is certainly not enough to stabilize a superflow since small dissipative effects neglected in (9.6) will ruin the time independence and swallow up momentum and energy. To make the following discussion as transparent as possible, let us go to units which are most natural for the problem at hand: We shall measure all lengths in units of $l_{\rm d} \equiv \xi_{\rm d}^{\perp}$, the energy in units of $f_{\rm d} \equiv \rho_s^{\parallel}/(4m^2\xi_{\rm d}^{\perp 2})$ and the current density as multipoles of $J_{\rm d} \equiv \rho_s^{\parallel}/(2m\xi_{\rm d}^{\perp})$, respectively. Physically, the $f_{\rm d}$ expression is the energy density the system would have if all $\mathbf{d}$ and $\mathbf{l}$-vectors were orthogonal, contrary to the dipole alignment force, the second is the current which flows if the Bose condensate moves with "dipole velocity" $v_{\rm d} \equiv 1/2m\xi_{\rm d}^{\perp}$ parallel to $\mathbf{l}$. Now, the energy $2f$ has again the form (9.6) except that all coefficients are divided by $\rho_s^{\parallel}$ and there is no $\rho_s^{\parallel}/\xi_{\rm d}^{\perp 2}$ in front of the dipole coupling. In the Ginzburg-Landau regime, in which the parameters of the liquid satisfy the identities

Eq. (he-11.10) (5.10), the coefficients simplify to

$$A(s) = 1+s, \quad B(s) = \frac{1}{2}(3-2s), \quad G(s) = 1-\frac{1}{2}s,$$
$$M(s) = \sqrt{1-s}, \quad T(s) = 1+s. \tag{9.9}$$

Eq. (he-11.11) Since we are interested in the system at a fixed current we study the energy

$$\begin{aligned} 2g &\equiv 2(f - j\gamma_z) \\ &= A_g j^2 + G_g\gamma_z^2 + 2Mg\gamma_z j + B\beta_z^2 + T\left(\theta_z^2 + s\phi_z^2\right) + 2\left(1 - [\mathbf{l}\cdot\mathbf{d})^2\right] \end{aligned} \tag{9.10}$$

Eq. (he-11.12) where

$$\begin{aligned} A_g &\equiv -A^{-1}, \quad M_g = M/A \\ G_g &\equiv G - M^2/A \equiv \Delta(s)/A \end{aligned} \tag{9.11}$$

Eq. (he-11.13) with

$$\begin{aligned} \Delta(s) &\equiv GA - M^2 \\ &= \frac{s}{\rho_s^{\parallel 2}}\left\{\rho_s^{\parallel}K_b + \left[\rho_s^{\parallel}(K_t - K_b) + (\rho_0 K_b - c_0^2)\right]s + \left[\rho_0(K_tK_b) + c_0^2\right]s^2\right\} \\ &\equiv s\left(\Delta_0 + \Delta_1 s + \Delta_2 s^2\right). \end{aligned} \tag{9.12}$$

Eq. (he-11.14) In the Ginzburg-Landau regime this becomes simply

$$\Delta(s)\frac{s}{2}(3-s). \tag{9.13}$$

In order to gain as much experimental flexibility as possible let us also add a magnetic field Eq. (he-)

$$e \to e + g_z\left(\mathbf{d}\cdot\mathbf{H}\right)^2. \tag{9.14}$$

It is convenient to bring this to a form in which it can be compared most easily with the dipole energy. Let H_d be the magnetic field ($H_\mathrm{d} \approx 30 0e$) at which Eq. (he-11.16)

$$g_z H d^2 = \rho_s^{\parallel}/4m^2\xi_\mathrm{d}^2. \tag{9.15}$$

If we measure H in terms of these units, say via Eq. (he-11.17)

$$h \equiv H/H_\mathrm{d} \tag{9.16}$$

we have Eq. (he-11.18)

$$f = \frac{\rho_s^{\parallel}}{4m^2\xi_\mathrm{d}^2}\left(d\cdot h\right)^2 \tag{9.17}$$

which in the energy (9.10) amounts to simply adding Eq. (he-11.19)

$$2g \to 2g - 2h^2 s. \tag{9.18}$$

In order to obtain a first estimate of the stability properties let us assume j and h to be much smaller than one (i.e., current and field energies are much smaller than the characteristic dipole values). Then the $\mathbf{d} \parallel \mathbf{l}$ alignment force causes a complete locking of these two vectors and we may set $\tau \equiv \beta, \quad \phi \equiv \gamma$. Now the energy $2g$ reads Eq. (he-11.20)

$$2g^l = A_g j^2 + G_g^l \gamma_z^2 + 2M_g \gamma_z j + B^l \beta_z^2 - 2h^2 s \tag{9.19}$$

where $G_g^l,\ B^l$ have the same form as those in (9.11) but with $K_s,\ K_t,\ K_b$ replaced by Eq. (he-11.21)

$$\begin{aligned} K_s^l &\equiv K_s + K_1^d, \\ K_t^l &\equiv K_t + K_1^d, \\ K_b^l &\equiv K_b + K_1^d - K_2^d \end{aligned} \tag{9.20}$$

as shown in Appendix 5A. In the Ginzburg-Landau regime with $\rho_s^{\parallel}$ divided out their values are Eq. (he-11.22)

$$\begin{aligned} K_s^l &= \frac{1}{2} + 2 = \frac{5}{2}, \\ K_t^l &= \frac{1}{2} + 2 = \frac{5}{2} \\ K_b^l &= \frac{3}{2} + 2 - 1 = \frac{5}{2}. \end{aligned} \tag{9.21}$$

Consider now the problem of stability of the $\mathbf{d} \parallel \mathbf{l} \parallel \mathbf{j} \parallel \mathbf{h}$ configuration with $s = 0$. Expanding the energy up to the first power in s gives[1] Eq. (he-11.23)

[1] We omit the index l and understand all K's as locked values (9.20).

$$
\begin{aligned}
2g &= j^2 + 2j\gamma_z + \frac{K_b}{g_s^{\|}}\beta_z^2 + \left(\frac{\rho_0}{\rho_s^{\|}}j^2 - 2h^2\right)s + \frac{K_b}{\rho_s^{\|}}s\gamma_z^2 - 2\frac{c_0 + \frac{1}{2}\rho_s^{\|}}{\rho_s^{\|}}s\gamma_z j \\
&= j^2 + 2j\gamma_z + \frac{K_b}{\rho_s^{\|}}\beta_z^2 + \left[\frac{\rho_0}{\rho_s^{\|}}\left(1 - \frac{\left(c_0 + \frac{\rho_s^{\|}}{2}\right)^2}{\rho_0 K_b}\right)j^2 - 2h^2\right]s \\
&\qquad + \frac{K_b}{\rho_s^{\|}}s\left(\gamma_z - \frac{c_0 + \frac{\rho_s^{\|}}{2}}{K_b}j\right)^2 .
\end{aligned}
\tag{9.22}
$$

Note that the term linear in γ_z is a pure surface term and does not influence the
Eq. (he-11.24) stability. Let us introduce the quantity

$$
K \equiv \frac{\rho_0 K_b}{\left(c_0 + \frac{\rho_s^{\|}}{2}\right)^2}. \tag{9.23}
$$

Eq. (he-11.25) Then the term proportional to β^2 is

$$
2\left[\frac{1}{2}\frac{\rho_0}{\rho_s^{\|}}\left(1 - K^{-1}\right) - \frac{h^2}{j^2}\right]\beta^2 j^2. \tag{9.24}
$$

Since the last terms in g are positive definite, the $\beta = 0$ position is stable if and
Eq. (he-11.26) only if

$$
\frac{h^2}{j^2} \leq \frac{h_c^2}{j^2} \equiv \frac{1}{2}\frac{\rho_0}{\rho_s^{\|}}(1 - K^{-1}). \tag{9.25}
$$

In the absence of a magnetic field, stability implies [37]

$$
K > 1.
$$

Eq. (he-11.27) Now, in the Ginzburg-Landau regime, this is barely satisfied:

$$
K \underset{GL}{=} \frac{10}{9}. \tag{9.26}
$$

But as the temperature decreases, ρ_0 is known to vanish. Hence one expects K to cross the line $K = 1$ eventually. If one uses the energy parameters (5.9), but with Fermi liquid corrections [38], one can argue that this will happen well within the A
Eq. (he-11.28) phase at a temperature [39]

$$
T_{\text{stab}} \equiv T(K = 1) \approx .86\, T_c. \tag{9.27}
$$

Thus we can conclude: For $T \in (T_{\text{stab}}, T_c)$, the presence of a superflow acts self-stabilizing. It creates its own potential well which prevents the free motion of $d \parallel l$ away from the direction of the current. In the parameter space SO(3) of the $d \parallel l$ phase this corresponds to a potential mountain around the equatorial region (see
Fig. XVIII Fig. 9.2.

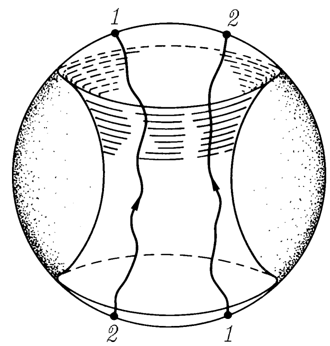


FIGURE 9.2 In the presence of a superflow in ^{3}He-A, the **l**-vector is attracted to the direction of flow. In the parameter space of ^{3}He-A this force corresponds to forbidding the equator of the sphere thereby favoring a conical section. Since diametrally opposite points are identical, the topology is infinitely connected. The figure shows an example for a closed curve with two breaks.

This mountain is sufficient to prevent the deformation of contours to the two basic ones (corresponding to integer and half-integer spin representation). For these deformations, the passage of the equator would have to be free (see Fig. 9.3). Fig. IXX

It is easy to convince oneself that the $S0_3$ sphere with forbidden equatorial regions allows for an infinity of inequivalent paths: The allowed type within the SO(3) sphere has its upper face coinciding with the lower one (except for a reflection on the axis). The parameter space becomes equivalent to a torus and $\pi_1 = Z$. Therefore *there are* again large quantum numbers which are conserved topologically in the weaker sense discussed above. There exists superflow in ^{3}He-A.

Notice that in the dipole locked regime with $\beta_0 = 0$ both currents (9.8) and (9.7(he-11.8)) coincide and are equal to Eq. (he-11.29)

$$j_1 \equiv J_1/J_\mathrm{d} = j_2 \equiv J_2/J_\mathrm{d} = -\left(\alpha_z + \gamma_z\right) \tag{9.28}$$

Topological conservation in a torus implies that $\langle \alpha_z + \gamma_z \rangle$ is pinned down at $2\pi/L$ times an integer number, say N, when going once around the axis. Hence both currents are topologically stable at a value Eq. (he-11.30)

$$j_1 = j_2 = 2\pi N/L \tag{9.29}$$

where L is the length of the torus.

What happens as the temperature drops below T_stab? Then the quadratic term becomes negative and β starts moving away from the forward direction. We shall show now that the higher orders in β stop this movement at a value $\beta_0 \neq 0$. In this case the coefficient of the last term in (9.22) becomes finite so that γ_z will be driven to an average value Eq. (he-11.31)

$$\langle \gamma_z \rangle \approx \gamma_z^0 \equiv \frac{c_0 + \frac{\rho_s^{\parallel}}{2}}{K_b} j \underset{GL}{=} \frac{3}{5} j \tag{9.30}$$

A texture with fixed angle of inclination β_0 and $\gamma_z = \gamma_z^0$ looks like a helix with constant pitch γ_z^0 (see Fig. 9.4) [41]. Fig. XX

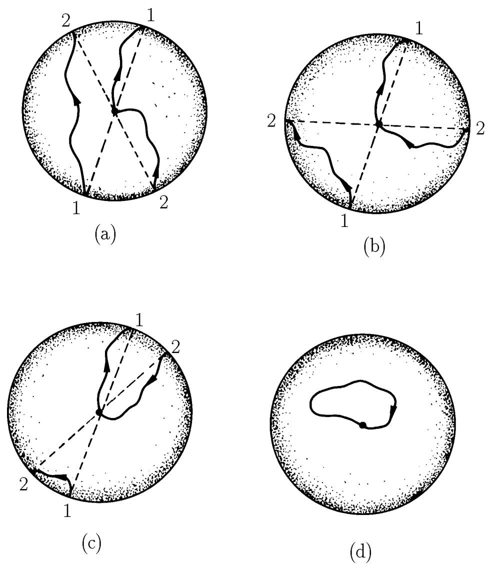


FIGURE 9.3 Doubly connected parameter space of the rotation group corresponding to integer and half-integer spin representations. Note that the continuous deformation of arbitrary contours to the two fundamental ones (either a point or a line running from a point at the surface to the diametrally opposite point) always has to pass via the equator of the sphere. An alignment force between **l** and the current which forbids the equator of the sphere therefore changes drastically the topology to being infinitely connected.

It is in this helical texture that the currents (9.7) (9.7(he-11.8)) no longer coincide and, moreover, become both conserved topologically. [40].

In order to prove the dynamic stability of the helix we first consider all stationary solutions. Since $2g$ does not depend on γ, a solution at $s \equiv s_0$ is stationary if and
Eq. (he-11.32) only if [2]

$$2g' = A'_g j^2 + 2M'_g \gamma_z j + G'_g \gamma_z^2 - 2h^2 = 0. \tag{9.31}$$

Eq. (he-11.33) This for every S_0 there are two values of γ_z at which the point s_0 is stationary:

$$\frac{\gamma_z^\pm}{j} = -\frac{M'_g}{G'_g} \pm \sqrt{\left(\frac{M'_g}{G'_g}\right)^2 - \frac{A'_g}{G'_g} + 2\frac{h^2}{j^2}\frac{1}{G'_g}}. \tag{9.32}$$

Eq. (he-11.34) Since M and G are simpler expressions than M_g and G_g, we use

[2] As we have seen in (9.22), a linear term in γ_z does not drive the system since it is a pure surface term and becomes a constant upon integration over z.

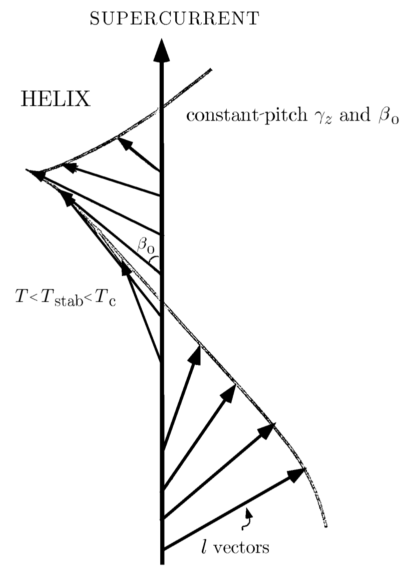


FIGURE 9.4 Helical texture in the presence of a supercurrent. The vectors show the directions of **l** which rotate around the axis of superflow when preceeding along the z-axis. The angle of inclination has a constant value β_0. The pitch of the helix is constant with a ratio $\gamma_z/j \approx (c_0 + \rho_s^{\parallel})/K_b \approx \frac{3}{5}$.

$$\begin{aligned} M_g' &= \frac{M'A - AM'}{A^2} \underset{GL}{=} -\frac{1}{2\sqrt{1-s}}, \\ G_g' &= G' - 2MM'/A + M^2A'/A^2 \\ &\underset{GL}{=} \left(5 + 6s + 9s^2 + 4s^3\right)/4(1+s)^2, \\ M_g'^2 - G_g'A_g' &= \left(M'^2 - G'A'\right)/A^2. \end{aligned} \tag{9.33}$$

to write Eq. (he-11.35)

$$\begin{aligned} \frac{\gamma_z}{j} &= \left(A^2G' - 2MM'A + M^2A'\right)^{-1} \\ &\quad \left[-(M'A - MA') \pm A\sqrt{M'^2 - G'A' + A^2G_g'^2h^2/j^2}\right] \end{aligned} \tag{9.34}$$

$$\begin{aligned} &\underset{GL}{=} \left[\sqrt{1-s}\left(5 + 6s + 9s^2 + 4s^3\right)\right]^{-1} \\ &\times \left[3 - s \pm \sqrt{(3-s)^2 - 2(1-(1+s)^2 2h^2(j^2)(5 + 6s + 9s^2 + 4s^3)(1-s)}\right]. \end{aligned} \tag{9.35}$$

This equation has two solutions if Eq. (he-11.36)

$$M'^2 - G'A' - A^2G_g'2h^2/j^2 \geq 0. \tag{9.36}$$

Consider at first the case $h = 0$. After a somewhat tedious calculation one finds Eq. (he-11.37)

$$M'^2 - G'A' = \alpha(s - s^+)(s - s^-)/4(1 - s) \tag{9.37}$$

Eq. (he-11.38)

$$s^\pm \equiv \frac{\beta}{\alpha}\left(1 \pm \sqrt{1 + 4\alpha K_b \rho_0 (1 - K^{-1})/\beta \rho_s^{\parallel 2}}\right) \tag{9.38}$$

Eq. (he-11.39) where

$$\alpha \equiv \left(\rho_0^2 2\rho_0 c_0 + g c_0^2 + 8(K_t - K_b)\rho_0\right)/\rho_s^{\parallel 2} \underset{GL}{=} 8, \tag{9.39}$$

Eq. (he-11.40)

$$\beta = \frac{2\rho_0}{(c_0 + \frac{1}{2}\rho_s^\parallel)\rho_s^{\parallel 2}}\left[3c_0 K_b (K^{-1} - 1) + (c_0 + \frac{1}{2}\rho_s^\parallel)(2K_t - \frac{1}{2}\rho_s^\parallel) - \frac{3}{2}\rho_s^\parallel K_b\right]. \tag{9.40}$$

In the absence of a magnetic field, $s^\pm$ give the boundaries of stationary solutions. Confronted in an incomplete knowledge of the parameters of the liquid we shall estimate the regions in the following fashion: Since the passage of K through unity is eventually enforced by the vanishing of ρ_0, we shall assume, for simplicity, that all coefficients have their Ginzburg-Landau values (5.10) except for ρ_0 which we assume

Eq. (he-11.41) to vary as

$$\rho_0 = \rho_s^\parallel(1 - \epsilon) = \rho_s^\parallel \frac{9}{10} K. \tag{9.41}$$

Eq. (he-11.42) Then

$$K = \frac{10}{9}(1 - \epsilon), \quad \epsilon = 1 - \frac{9}{10}K \tag{9.42}$$

Eq. (he-11.43) and

$$\alpha = 8 + \epsilon^2, \quad \beta = 3 + 6\epsilon \tag{9.43}$$

Eq. (he-11.44) so that

$$s^\pm = \left[3 + 6\epsilon \pm (1 - \epsilon)\sqrt{17 - 10\epsilon}\right]/(8 + \epsilon^2). \tag{9.44}$$

Fig. XXIa The curves $s^\pm(\epsilon)$ are shown in Fig. 9.5.

The regions above the upper and below the lower curve correspond to stationary solutions. As the lower curve drops underneath the axis ($\epsilon < 1/10$), the solution becomes meaningless. But this is precisely the region discussed before in which the $\beta = 0$ solution is stable.

In the following we shall try to keep the discussion as general as possible but find it useful to indicate a size and temperature dependence of more complicated expressions by exhibiting their generalized Ginzburg-Landau form in which only ρ_0 deviates from the values (5.10) via (9.41(he-11.41)). This limit will be indicated by a symbol $=_L$ and be referred to as L-limit. The Ginzburg-Landau case (5.10) will be exhibited with an equality sign $=_{GL}$, as before.

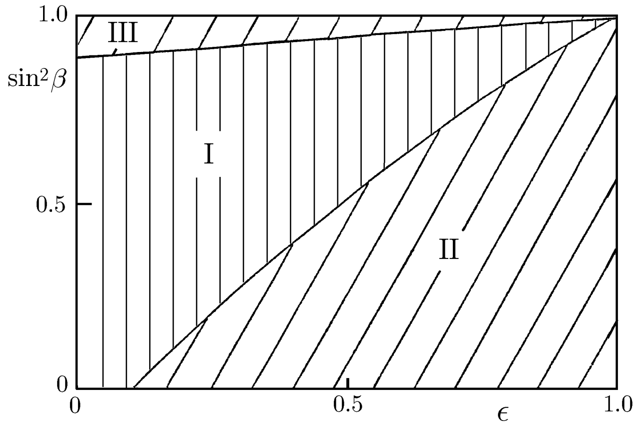


FIGURE 9.5 Three different regions in which there are equilibrium configurations of the texture at $H = 0$ (schematically).

Let us now include the magnetic field. Then the boundaries of stationary solutions are Eq. (he-11.45)

$$\begin{aligned}\alpha(s - s^{\dagger})(s - s^{-}) \\ + \frac{8h^2}{j^2}\left[\Delta_0 + 2\Delta_1 s + \left(3\Delta_2 + \frac{\rho_0}{\rho_s^{\parallel}}\Delta_1\right) s^2 + 2\frac{\rho_0}{\rho_s^{\parallel}}\Delta_2 s^3\right](1 - s) \\ \underset{GL}{=} 8s^2 - 6s - 1 + 4\frac{h^2}{j^2}\left(5 + 6s + 9s^2\right)(1 - s) \geq 0.\end{aligned} \tag{9.45}$$

This equation is no longer quadratic in s and its solution is complicated. It is gratifying to note that the physically interesting regions can easily be studied with a good approximation. First observe that at $s \geq 0$ there are stationary solutions if the magnetic field is larger than the value given by Eq. (he-11.46)

$$\alpha s^{\dagger} s^{-} + 8\frac{h_c^2}{j^2}\Delta_0 = 0. \tag{9.46}$$

This implies (see (7.16)): Eq. (he-11.46)

$$\frac{h_c^2}{j^2} = -\frac{\alpha s^{\dagger} s^{-}}{8K_b}\rho_s^{\parallel}. \tag{9.47}$$

But from (9.38) one has Eq. (he-)

$$\alpha s^{\dagger} s^{=} - 4\frac{K_b \rho_0}{\rho_s^{\parallel 2}}\left(1 - K^{-1}\right) \tag{9.48}$$

so that the value of h_c from (9.47) coincides with the critical value determined previously from the stability of the $\beta = 0$ texture (see (9.25)). Thus as h exceeds

h_c, the aligned solution destabilizes in favor of a new extremal solution with $d^{\parallel}l\!\!/ j^{\parallel}h$. The new equilibrium position can be calculated to lowest order in $\Delta h^2 \equiv h^2 h_c^2$ by expanding formula (9.45): Eq. (he-)

$$\left[8\frac{h_c^2}{j^2}(2\Delta_1 - \Delta_0) - (s^{\dagger} + s^{-})\alpha\right] s + 8\frac{\Delta h^2}{j^2} \geq 0 \tag{9.49}$$

Eq. (he-11.47) which amounts to

$$s \leq s_h^- \equiv \frac{4}{\beta - 4\frac{h_c^2}{j^2}\left(2\Delta_1 - \Delta_c\right)} \frac{\Delta j^2}{j^2}. \tag{9.50}$$

Eq. (he-11.48) Using the limiting value

$$\frac{h_c^2}{j^2} \underset{L}{=} \left(1 - 10\epsilon\right)/20 \tag{9.51}$$

Eq. (he-11.49) we can estimate the prefactor as

$$\frac{4}{\beta - 4\frac{h_c^2}{j^2}\left(2\Delta_1 - \Delta_0\right)} \underset{L}{=} \frac{10}{3 + 6\epsilon - (1 - 10\epsilon)^2/10} \tag{9.52}$$

$$= \begin{cases} 100/29 & \text{for} \quad \epsilon = 0, \quad T = T_c \\ 100/36 & \quad\quad \epsilon = 1/10, \quad T = T_{\rm stab} \end{cases} \tag{9.53}$$

which is therefore $\approx 1/3$ for all temperatures between T_c and $T_{\rm stab}$.

Within this small s region we can now solve for $\gamma_a^{\pm}$ from (9.32). Since γ_z goes with the square root of $s - s_h^-, s_h$ it is sufficient to keep, for small Δh^2, only the

Eq. (he-11.51) constants in the other terms and we find

$$\begin{aligned} \frac{\gamma_z^{\pm}}{j} &\approx \frac{c_0 + \frac{1}{2}\rho_s^{\parallel}}{K_b} \pm \frac{\rho_s^{\parallel}}{2K_b}\sqrt{\alpha(s - s_h^-)(s_h^- - s_h^{\dagger})} \\ &= \frac{c_0 + \frac{1}{2}\rho_s^{\parallel}}{K_b} \pm \frac{\rho_s^{\parallel}}{K_b}\sqrt{\frac{\beta}{2}}\left[1 + \frac{\alpha}{\beta^2}\frac{4K_b\rho_0}{\rho_s^{\parallel 2}}\left(1 - K^{-1}\right)\right]^{1/4}\sqrt{s_h^- - s}. \end{aligned} \tag{9.54}$$

Eq. (he-) If we choose, in addition, also $K \approx 1$, we have

$$\frac{\gamma_z}{j} \approx \frac{\gamma_z^0}{j} + \pm\frac{\rho_s^{\parallel}}{K_b}\sqrt{\frac{\beta}{2}}\sqrt{s_h^- - s} \tag{9.55}$$

Eq. (he-11.53) which in the L-limit reads explicitly

$$\frac{\gamma_z}{j} \underset{L}{\approx} \frac{3}{5}\left(1 \pm \sqrt{\frac{5}{4}\left[\frac{20}{9}\frac{h^2}{j^2} - (K-1)\right] - s}\right). \tag{9.56}$$

As the magnetic field increases one can solve for the external positions only numerically. The results are shown in Figs. 9.11(a)–(c) for three different values of

$\epsilon : \epsilon = 0, \quad \epsilon = .1, \epsilon.2$. Notice that the small s regions coincide if the magnetic field lines are labelled by $\Delta h^2/j^2$ rather than h^2/j^2. Let us now find out which of these positions correspond to stable extrema. The energy density can be written in the
Eq. (he-11.54) form

$$2g = \bar{B}s_z^2 + V(s, \gamma_z). \tag{9.57}$$

Eq. (he-11.55) The stationary points were determined from

$$\frac{\partial V}{\partial s}\left(s_0,\ \gamma_z^{\pm}\right) = 0. \tag{9.58}$$

If we now assume linear oscillations around this value we have Eq. (he-11.56)

$$\begin{aligned} 2\delta^2 g &= \bar{B}(\delta s_z)^2 + \frac{\partial V}{\partial \gamma_z}(s_0,\ \gamma_z^{\pm})(\delta\gamma_z) \\ &\quad + \frac{\partial^2 V}{\partial s^2}(s_0,\ \gamma_z^{\pm}) + 2\frac{\partial^2 V}{\partial s \partial \gamma_z}(s_0, \gamma_z^{\pm}) + \frac{\partial^2 V}{\partial \gamma_z}(s_0,\ \gamma_z^{\pm}). \end{aligned} \tag{9.59}$$

The second piece is a pure surface term and can be ignored. The equations of motion of (9.59) are linear. Therefore the superposition principle holds and we can test stability separately by using plane waves of an arbitrary wave vector k. With such an ansatz $2\delta^2 g$ becomes Eq. (he-11.57)

$$2\delta^2 g = \left(V'' + \bar{B}k^2\right)(\delta s)^2 + 2\dot{V}'k(\delta s)/(\delta\gamma) + \ddot{V}k^2(\delta\gamma)^2. \tag{9.60}$$

This is positive definite for all k if Eq. (he-11.58)

$$\ddot{V} \equiv \frac{\partial^2 V}{\partial \gamma_z^2} > 0 \tag{9.61}$$

and Eq. (he-11.59)

$$V''\ddot{V} - \dot{V}'^2 > 0. \tag{9.62}$$

In terms of the functions (9.6), (9.10(he-11.11)) these conditions read Eq. (he-11.60)

$$G_g > 0 \tag{9.63}$$

Eq. (he-11.61)

$$2\left(G_g''\gamma_z^{\pm 2} + 2M_g''j\gamma_z^{\pm} + A_g''j^2\right)G_g - 4\left(G_g' + M_g'j\gamma_z^{\pm}\right)^2 \geq 0. \tag{9.64}$$

Using (9.32) the second condition takes the alternative form Eq. (he-11.62)

$$D^{\pm} = \left(G_g''\gamma_z^{\pm 2} + 2M_g''j\gamma_z^{\pm} + A_g''j^2\right)\frac{G_g}{2{G_g'}^2} - \frac{(\gamma_z^+ - \gamma_z^-)^2}{4j^2} \geq 0. \tag{9.65}$$

Now it is easy to see that $G_g > 0$ for all s. Thus only (9.65) remains to be tested. Analytically, only the small s region is simple: Since Eq. (he-)

$$\frac{G_g}{2G_g'^2} \approx \frac{\rho_s^\parallel}{2K_b} s \tag{9.66}$$

we have to satisfy Eq. (he-11.63)

$$\frac{\rho_s^\parallel}{2K_b}\left\{-2\frac{\rho_0^2}{\rho_s^{\parallel 2}} + \left(c_0 - \frac{\rho_s^\parallel}{4} + 2\frac{\rho_0 c_0}{\rho_s^\parallel}\right)\frac{1}{\rho_s^\parallel}\frac{c_0+\frac{1}{2}\rho_s^\parallel}{K_b} + \left[2\left(K_t - K_b\right) + 2c_0 + \frac{\rho_s^\parallel}{2} - 2\frac{(c_0+\frac{1}{2}\rho_s^\parallel)^2}{\rho_s^\parallel}\right]\left(\frac{c_0+\frac{1}{2}\rho_s^\parallel}{K_b}\right)^2\right\} s$$
$$\geq \frac{\rho_s^{\parallel 2}}{K_b^2}\frac{\beta}{2}\sqrt{1+\frac{\alpha}{\beta^2}\frac{4K_b\rho_0}{\rho_s^{\parallel 2}}}(1-K^{-1})(s_h^- - s). \tag{9.67}$$

If $K \approx 1$, we can keep only terms linear in $K-1$, s_h^-, s. Using the generalized
Eq. (he-11.64) Ginzburg-Landau values for the parameters gives

$$\frac{1}{5}\left[\frac{1}{2}\frac{36}{25} + \frac{1}{4}\left(\frac{9}{10}\right)^2 (K-1)\right] s \geq \frac{36}{125}(s_h^- - s). \tag{9.68}$$

But on the left-hand side, $K-1$ can be neglected since it contributes higher orders
Eq. (he-11.65) in s. Thus we find that the extremal solutions, which exist for

$$s \leq s_h^-, \tag{9.69}$$

Eq. (he-11.66) are stable if

$$s > \frac{2}{3}s_h^-. \tag{9.70}$$

Eq. (he-11.67) Using (9.56), this result can also be phrased in the form

$$\left(\frac{\gamma_z - \gamma_z^0}{\gamma_z^0}\right)^2 \leq \frac{1}{3}(1-K). \tag{9.71}$$

Fig. XXIb
Fig. XXIc
In Fig. 9.6 this statement amounts to the upper third portion underneath the curve s_h^- to be stable at $s \approx 0$. In general, the stability can be decided only numerically. In Figs. 9.11(a)–(c) we have encircled the stable regions with a dashed line. Notice that for fixed h^2, the instability sets in as $\gamma_z^\dagger$, γ_z^- become too widely separated. Looking at the expression (9.65) the reason is clear: The second derivative V'' is positive but not very large. If the branches separate too much, the positivity cannot be maintained. We see that as h^2 increases, the helix is stable only up to $s \approx .3-.45$. Beyond this it collapses. For completeness, we have also indicated the stable regions on Fig. 9.7

Note that the existence of the dipole force is essential for stability. First of all, the position $s = 0$ is never stable if the vectors $\mathbf{d}$ and $\mathbf{l}$ are not coupled at all. To see this remember that the constant K of (9.23) would be, in the Ginzburg-Landau
Eq. (he-11.68) region [compare (9.20)], considerably smaller than unity:

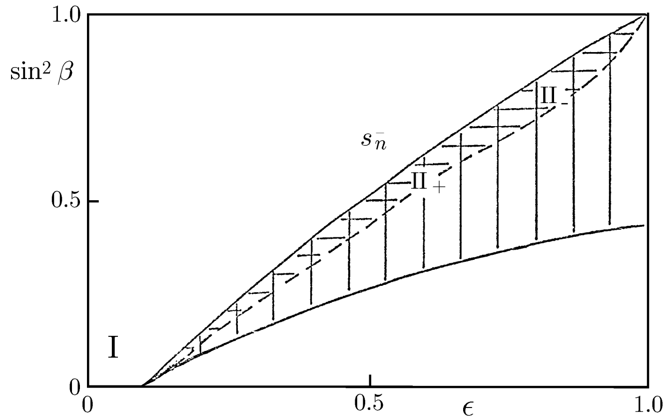


FIGURE 9.6 Regions of a stable helical texture, II- and II+. In the region of overlap there are two possible pitch values γ_z^+, γ_z^- for which a helix can be stable.

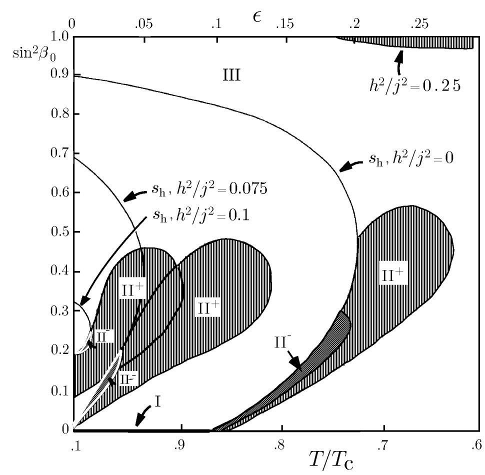


FIGURE 9.7 Regions of a stable helical texture (shaded areas). Contrary to Figs. 9.5 and 9.6, the full temperature dependence of the hydrodynamic coefficients is taken into account, including Fermi liquid corrections. The regions of a stable helical texture, II- and II+. In the region of overlap there are two possible pitch values γ_z^+, γ_z^- for which a helix can be stable.

$$K = \frac{K_b \rho_0}{(c_0 + \frac{1}{2}\rho_s^{\|})^2} = \frac{2}{3} < 1 \qquad (9.72)$$

and there is no hope that this situation reverses for smaller temperature (since $\rho_0 \to 0$ for small T). The magnetic field does not help since it couples only to d. Also the hope that a position $s \neq 0$ may be stable is futile, even though there are stationary solutions: If we calculate in the Ginzburg-Landau limit Eq. (he-11.69)

$$M'^2 - A'G' = \frac{1}{4(1-s)} - (1)\left(-\frac{1}{2}\right) \geq 0 \tag{9.73}$$

Eq. (he-11.70) this is fulfilled for all physical values $S \in (0,\ 1)$ with

$$\frac{\gamma_z^{\pm}}{j} = \frac{1}{\sqrt{1-s}} \frac{1}{3-2s-s^2}\left[3 - s \pm (1+s)\sqrt{3-s}\right]. \tag{9.74}$$

At $s = 0$, these values are 1.577, .4226. For $s \to 1$, the upper branch tends monotonously to infinity as $(1-s)^{\frac{1}{2}}$, the lower goes to zero as $(1-s)^{1/2}$. Thus $\left(\gamma_z^{\dagger} - \gamma_z^{-}\right)^2 / j^2$ increases rapidly. It is exactly for this reason why there is no hope of making $D > 0$ in (9.65). The second term is too large (remembering that the shearing apart of $\gamma_z^{\dagger}$ and γ_z^{-} was also the origin of the instability for small s in the dipole locked regime).

Recognizing this fact we are compelled to study the effect of the dipole force with more sensitivity than implied by the assumption of dipole locking in the above discussion. Certainly, the results gained there will be valid for $h,\ j << 1$ i.e., as long as the dipole force is strong with respect to the other alignment forces. What happens if $h,\ j$ grow to comparable size? Consider again first the stability of the forward position $d^{\parallel} l^{\parallel} j^{\parallel} h$. For small, $\theta,\ \beta$ the quadratic piece in the energy can be

Eq. (he-11.71) written as

$$\begin{aligned} 2g \;\; = \;\; & \text{const} + \frac{1}{\rho_s^{\parallel}}\left[\rho_0 j^2 \beta^2 + K_b\left(\beta_z^2 + \beta^2\chi_z^2\right) - 2\left(c_0 + \frac{\rho_s^{\parallel}}{2}\right)\beta^2\gamma_z j\right. \\ & \left. + \left(K_1^d - K_2^d\right)\left(\theta_z^2 + \beta^2\phi_z^2\right) + 2\left(\theta^2 + \beta^2 - 2\theta\beta\omega(\gamma - \phi)\right) - 2h^2\beta^2\right]. \end{aligned} \tag{9.75}$$

Eq. (he-11.72) Introducing coordinates which are regular at the origin

$$\tan\gamma = \frac{u}{v}, \quad \tan\phi = \frac{u'}{v'} \tag{9.76}$$

Eq. (he-) so that

$$\gamma_z = \left(uv_z - vu_z\right)/\sqrt{u^2+v^2}; \quad \beta_z = \left(uv_z + vu_z\right)/\sqrt{u^2+v^2} \tag{9.77}$$

Eq. (he-11.73) with a similar expression for ϕ, the energy becomes

$$\begin{aligned} 2g \;\; = \;\; & \frac{1}{\rho_s^{\parallel}}\left\{\rho_0 j^2\left(u^2+v^2\right) + K_b\left(u_z^2 + v_z^2\right) - 2\left(c_0 + \frac{\rho_s^{\parallel}}{2}\right)\left(uv_z - vu_z\right) j\right. \\ & \left. + \left(K_1^d - K_2^d\right)\left(u_z'^2 + v_z'^2\right) + 2\left((u-u')^2 + (v-v')^2\right) - 2h^2\left(u'^2 + v'^2\right)\right\}. \end{aligned}$$

Since the corresponding equations of motion are linear, we can again test the stability

Eq. (he-) for all plane waves

$$u,\ u' \sim \sin kx; \qquad v,\ v' \sim \cos kx \tag{9.78}$$

Eq. (he-11.74) each of which gives

$$\begin{aligned}
2g &= \frac{1}{\rho_s^{\|}}\left(\rho_0 j^2 + K_b k^2 - 2kj\left(c_0 + \frac{\rho_s^{\|}}{2}\right) + 2\rho_s^{\|}\right)\left(u^2+v^2\right) \\
&\quad + \left[\left(K_1^d - K_2^d\right)k^2 - 2\left(h^2-1\right)\rho_s^{\|}\right]\left(u'^2+v'^2\right) \\
&\left\{\frac{1}{\rho_s^{\|}}\left(\rho_0 j^2 + K_b k^2 - 2kj\left(c_0+\frac{\rho_s^{\|}}{2}\right) - \frac{4\rho_s^{\|\,2}}{\left(K_1^d-K_2^d\right)k^2 - 2(h^2-1)\rho_s^{\|}} + 2\rho_s^{\|}\right)\right. \\
&\quad \times \left(u^2+v^2\right)\left[\left(K_1^d-K_2^d\right)k^2 - 2(h^2-1)\rho_s^{\|}\right] \\
&\quad \left.\times \left[\left(u' - \frac{2\rho_s^{\|}}{\left(K_1^d-K_2^d\right)k^2 - 2(h^2-1)\rho_s^{\|}}u\right)^2 + (u\to v)\right]\right\}.
\end{aligned} \tag{9.79}$$

Since $K_1^d - K_2^d > 0$ the second term is positive definite for all k is Eq. (he-11.75)

$$h^2 < 1. \tag{9.80}$$

Thus we remain with deciding the region in the $h,\ p$ plane for which Eq. (he-11.76)

$$\begin{aligned}
A(k) &= \rho_0 j^2 + K_b k^2 + 2\rho^{\|} - 2kj\left(c_0+\frac{\rho_s^{\|}}{2}\right) \\
&\qquad - \frac{4\rho_s^{\|\,2}}{\left(K_1^d-K_2^d\right)k^2 - 2(h^2-1)\rho_s^{\|}} \\
&= \rho_0 j^2 + K_b\left(k - \frac{c_0+\frac{\rho_s^{\|}}{2}}{K_b}j\right)^2 - K_b\left(\frac{c_0+\frac{\rho_s^{\|}}{2}}{K_b}\right)^2 \\
&\qquad + \frac{2\rho^{\|}}{s}\left(1 - \frac{2\rho_s^{\|}}{\left(K_1^d-K_2^d\right)k^2 - 2\left(h^2-1\right)\rho_s^{\|}}\right) \geq 0.
\end{aligned} \tag{9.81}$$

Notice that only the region $k \approx \frac{h^2}{K_1^d - K_2^d}$ and $k \approx \frac{c_0+\frac{\rho_s^{\|}}{2}}{K_b}j$ are dangerous. If we assume $h,\ j << 1$, also the dangerous value of k is $<< 1$ and we can expand Eq. (he-11.78)

$$\begin{aligned}
A(k) &\approx \rho_0 j^2 + K_b\left(k - \frac{c_0+\frac{\rho_s^{\|}}{2}}{K_b}j\right)^2 - K_b\left(\frac{c_0+\frac{\rho_s^{\|}}{2}}{K_b}\right)^2 \\
&\qquad + 2\rho_s^{\|}\left(-h^2 + \left(K_1^d-K_2^d\right)k^2/2\rho_s^{\|}\right) \\
&= \rho_0 j^2 + L_b^l\left(k - \frac{c_0+\frac{\rho_s^{\|}}{2}}{Kb^l}j\right)^2 - \rho_0 K^{-1} - 2\rho_s^{\|}h^2 \geq 0.
\end{aligned} \tag{9.82}$$

From this we find Eq. (he-11.79)

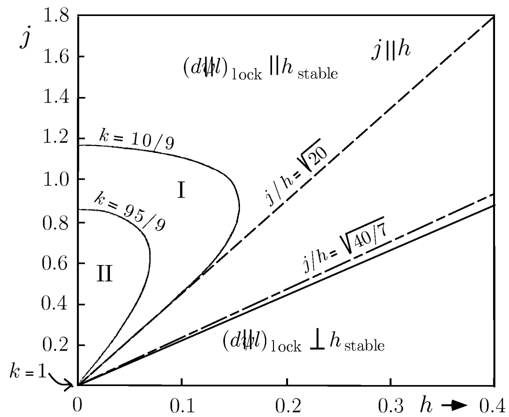


FIGURE 9.8 If the assumption of dipole locking is relaxed, the regions of stability shrink as shown in this figure. The whole region to the left of the line $j/h = \sqrt{20}$ is stable in the dipole locked limit. The finite strength of dipole locking reduces this region to I or the origin depending on whether the temperature is $T = T_c$ or $T = T_{stab}$ at which a stable helix can begin forming. For completeness, we have given also the region II for a temperature half-way between T_c and T_{stab}. Similarly, if dipole locking was perfect, the whole region below $j/h = \sqrt{\frac{40}{7}}$ would be stable with $d_{\|}l$ pointing orthogonal to the magnetic field. The finiteness of the dipole locking force reduces this region to the solid curve which becomes horizontal for large h.

$$\frac{h^2}{j} \leq -\frac{1}{2}\frac{\rho_0}{\rho_s^{\|}}\left(K^{-1}-1\right) \underset{L}{=} \frac{1}{2}\left(\frac{1}{10}-\epsilon\right) \tag{9.83}$$

in agreement with the dipole locked result (9.25), as it should. Thus the straight line (9.83) will now be tangential to the stability curve at the origin. For larger values of h, that curve bends upwards and cuts the z axis at some finite value of j. In Fig. 9.8 we have plotted the new stability curves for the generalized Ginzburg-Landau constants with $\rho_0 = \rho_s^{\|}(1-\epsilon)$ at $\epsilon = 0$, $\epsilon = .05$ and $\epsilon = .1$. Even at $h = 0$, the forward texture is stable only for $j \leq j_{\max}1.17,\ -0.83,\ 0$, respectively. The reason for the onset of stability at $h = 0$ is easy to understand: The current tries to curl up the texture in form of a helix (see the second term of (9.82)). The dipole force drags **d** behind. But the bending energies of **d** favor a uniform **d** texture. Thus, if the current is too strong, the $\mathbf{d}^{\|}\mathbf{l}$ alignment breaks. As soon as **d** and **l** are decoupled the texture destabilizes as we have discussed before in general.

Fig. XXV

The full discussion of equilibrium positions in the unlocked case is tedious. However, as j, h are small enough, say $j < \frac{3}{4}j_{\max},\ h < h_c$, the results of the dipole locked situation are perfectly applicable.

Let us now turn to the discussion of the physical content of the helix which was alluded to in the beginning of this chapter. As the helix forms at $h > h_c$, the $\beta = 0$ position turns into a potential mountain which forbids the alignment of $d^{\|}l$ with $j^{\|}h$. In the SO(3) parameter space of the dipole locked A phase, this

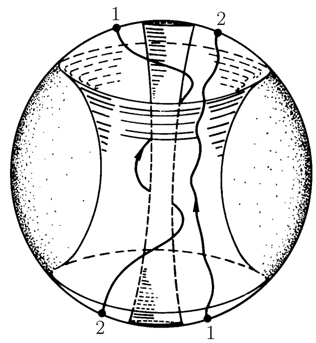


FIGURE 9.9 As a stable helix forms in the presence of a superflow in ^{3}He-A, the parameter space reduces even more. In addition to the equator being forbidden by the alignment force, a narrow cylinder along the axis is outruled as well. The topology of the remainder is doubly infinitely connected. Continuous paths can either break at the surface and continue from the diametrally opposite point or they can wind an arbitrary number of times around the central one.

amounts to removing a narrow cylindrical region running along the axis (see Fig. 9.9). Together with the potential mountain around the equator discussed before, the parameter space becomes now doubly infinitely connected: Fig. XXVI Eq. (he-11.80)

$$\pi_1 = Z + Z. \tag{9.84}$$

In addition to paths running from south to north, continuing again at the diametrally opposite point at the south, etc., also those which wind an arbitrary number of times around the narrow cylinder become topologically inequivalent. Physically, this corresponds to the fact that in a torus not only Eq. (he-11.81)

$$\langle \alpha_z + \gamma_z \rangle = 2\pi N/L \tag{9.85}$$

but also the a average pitch of the helix Eq. (he-11.82)

$$\langle \gamma_z \rangle = 2\pi M/L \tag{9.86}$$

is a topological invariant.

A consequence of this is that when increasing the magnetic field beyond h_c, or decreasing the temperature so that $h_c^2 < 0$, the value of $\langle \gamma_z \rangle \approx \left(c_0 + \frac{1}{2}\rho_s^{\parallel}\right)/K_b j = \gamma_z^0$ with which the helix begins forming (see the last term in (9.22) will be frozen. Therefore the angle of inclination β_0 will be pinned down topologically precisely at the value s_h^- (see (9.54)). In Fig. 9.10(he-XXVII) we have displayed the curves of constant $\gamma_z/\alpha_z + \gamma_z = \gamma_z^0/j$ for increasing h^2/j^2 at fixed values of ϵ until the point of collapse. These curves can be deduced from plots like those in Figs. 9.11(a)–(c). Fig. XXVII

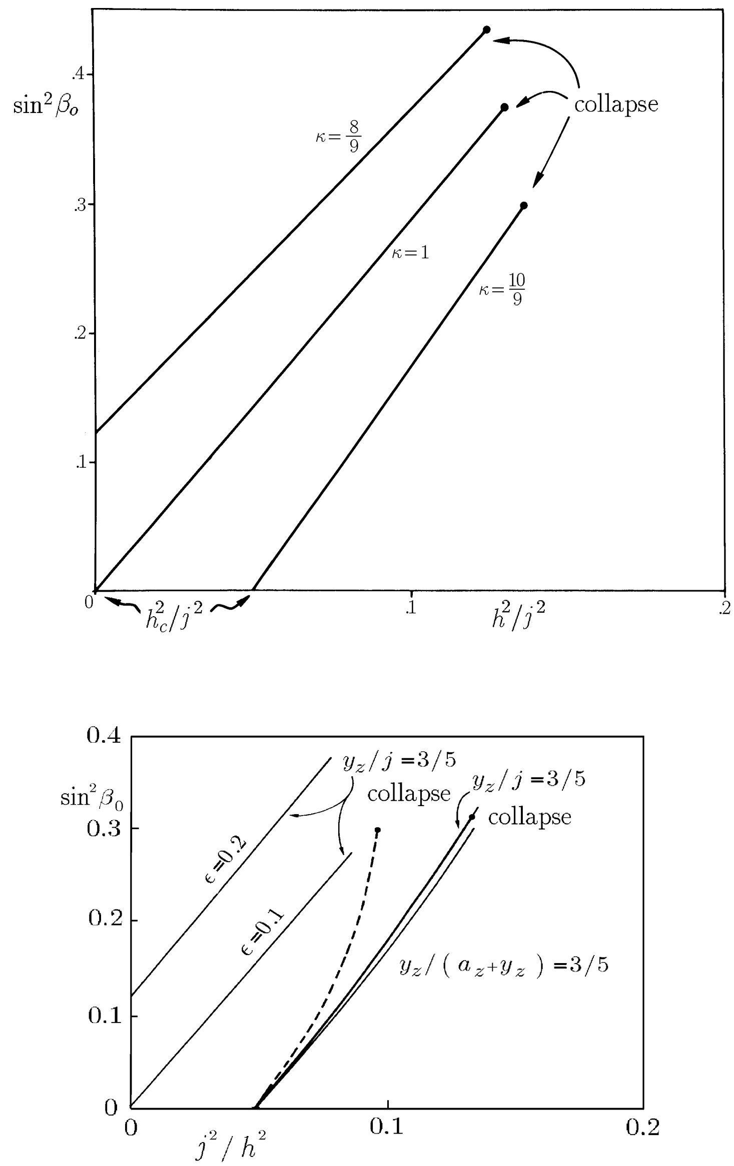


FIGURE 9.10 Angle of inclination as a function of the magnetic field at different temperatures. The values $K = \frac{10}{9}$, 1, $\frac{8}{9}$ correspond to $T = T_c$, $T = T_{stab}$, $T < T_{stab}$. As the magnetic field is increased, the helix collapses. Then the magnetic field is so strong that it tears apart the stabilizing dipole locking between 1 and d. The solid curves show the behavior if only the temperature dependence of ρ_0 is taken into account, in the dashed curves follows from the full T-dependence.

by following almost a straight line to the right starting from $\gamma_z/j = \frac{3}{5}$. The line Fig. XXII
is not exactly straight since this would show $\gamma_z/j = \frac{3}{5}$ rather than $\gamma_z/\alpha_z + \gamma_2$. The
relation is, in the Ginzburg-Landau limit, Eq. (he-11.83)

$$\frac{\gamma_z}{j} \underset{\mathrm{GL}}{=} \frac{\gamma_z/(\alpha_z+\gamma_z)}{1+s-(1+s-\sqrt{1-s})\gamma_z/(\alpha_z+\gamma_z)}$$

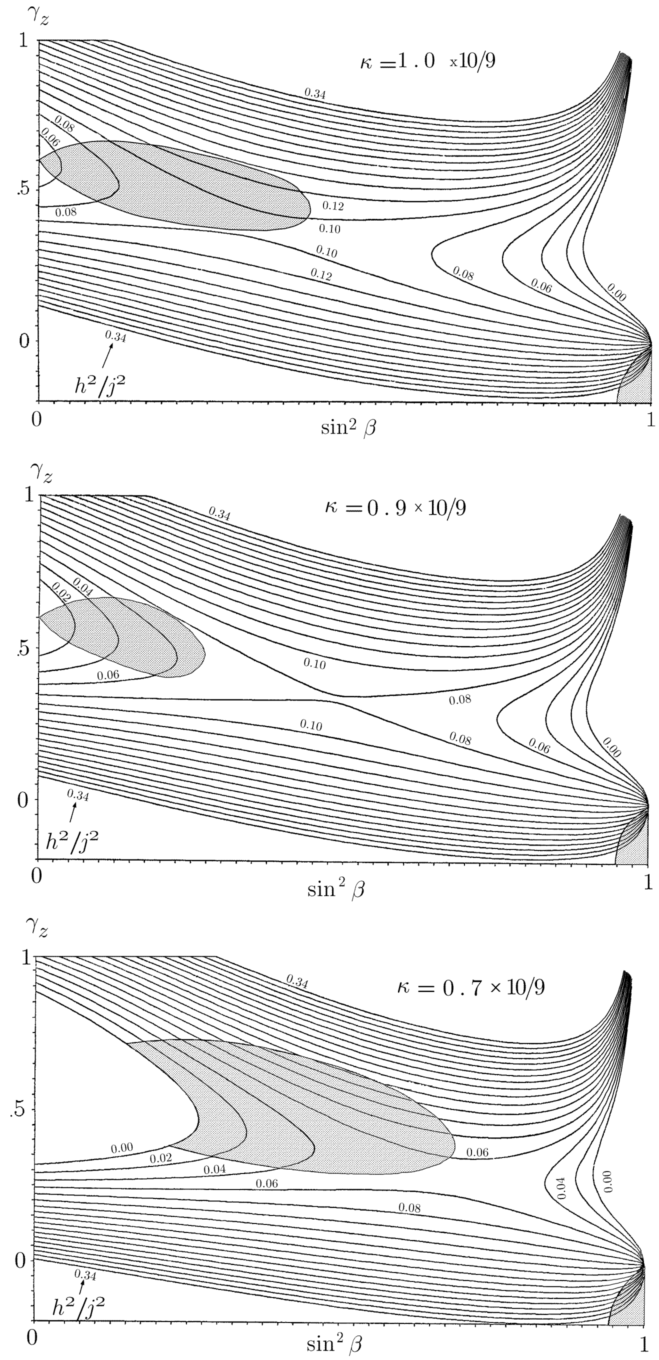


FIGURE 9.11 Pitch values for stationary helical solutions as a function of the angle of inclination β_0. The curves are lines of constant ratio between magnetic field and current. The shaded areas are regions of stability for the helical texture, the left one has l close to the directon of flow, the right one has l transverse to the flow. a) The temperature lies close to the transition point. b) at the lower temperature $T = T_{stab}$ at which the helix begins forming in a zero magnetic field. c) at a temperature below $T = T_{stab}$. The temperature dependence of the hydrodynamic coefficients is simplified assuming that only ρ_0 differs from the Ginzburg-Landau values

$$\approx \quad \frac{3}{5}\left(1-\frac{1}{10}\sin^2\beta+\dots\right) \tag{9.87}$$

so that there is very little deviation for small s.

The separate topological conservation of the two currents is intimately related with the fact that ^{3}He-A contains p-wave Cooper pairs. Remembering our discussion of Eq. (5.14) there are two current terms of different physical origin. The helix stabilizes both currents *topologically* and provides, in addition, the perfect tool for measuring their ratio. The pair current

Eq. (he-11.84)

$$J^{\rm pair} = \rho_s \mathbf{v}_s - \rho_0\,\mathbf{l}(\mathbf{l}\cdot\mathbf{v}_s) \tag{9.88}$$

Eq. (he-11.85) consists of two terms

$$\begin{aligned} J^{\rm pair} &= -\left(\rho_s-\rho_0\cos^2\beta\right)\frac{1}{2m}\left(\alpha_z+\cos\beta\gamma_z\right)\mathbf{e}_z+\rho_0\mathbf{l}^\perp\cos\beta\frac{1}{2m}\left(\alpha_z+\cos\beta\gamma_z\right)\\ &= -\frac{1}{2m}A\left(\alpha_z+\cos\beta\gamma_z\right)\mathbf{e}_z+\rho_0\mathbf{l}^\perp\cos\beta\frac{1}{2m}\left(\alpha_z+\cos\beta\gamma_z\right). \end{aligned} \tag{9.89}$$

The first flows in the z direction, the second forms stratified layers of currents whose direction changes with $\mathbf{l}^\perp$ when proceeding along the helix. Similarly, the orbital current

Eq. (he-11.86)

$$\begin{aligned} J^{orb} &= c\left(\boldsymbol{\nabla}\times\mathbf{l}\right)-c_0\mathbf{l}\left[\mathbf{l}\cdot\left(\boldsymbol{\nabla}\times\mathbf{l}\right)\right] \\ &= c_0\cos\beta\sin^2\beta\gamma_z\mathbf{e}_z-\left(c-c_0\sin^2\right)\gamma_z\mathbf{l}^\perp+c\cos\beta\beta_z\mathbf{e}_\varphi \end{aligned} \tag{9.90}$$

has three terms, the last of which which points into the azimuthal direction

$$\mathbf{e}_\varphi \equiv \left(\mathbf{e}_z\times\mathbf{l}\right)/|\mathbf{e}_z\times\mathbf{l}|$$

vanishes in equilibrium $\beta_z=0$. With $\beta=\beta_0$ and $\alpha_z,\ \gamma_z$ frozen topologically, both currents are determined. In particular, the ratio of their z components is

Eq. (he-11.87)

$$\frac{J_z^{\rm orb}}{J_z^{\rm pair}} = 2mc_0\sin^2\beta_0\cos\beta_0\frac{\gamma_z}{\left(\rho_s-\rho_0\cos^2\beta_0\right)\left(d_z+\cos\beta_0\gamma_z\right)}. \tag{9.91}$$

There is a simple way to measure $\sin^2\beta_0$. As is well-known, sound attenuation is sensitive to the angle between the l-vector and the direction of propagation of the sound [42]. In fact, if δ denotes this angle, the attenuation constant is given by

Eq. (he-11.88)

$$\begin{aligned} \alpha &\equiv \alpha_\perp\cos^4\delta+2\alpha_c\sin^2 5\cos^2\delta+\alpha_\perp\sin^4\delta, \\ &= \left(\alpha_{11}-2\alpha_c+\alpha_\perp\right)\cos^4\delta+2\left(\alpha_c\alpha_\perp\right)\cos^2\delta+\alpha_\perp. \end{aligned} \tag{9.92}$$

Eq. (he-11.89) If the helix is probed with a transverse signal, the angle δ becomes

$$\cos\delta = \sin\beta\cos\gamma. \tag{9.93}$$

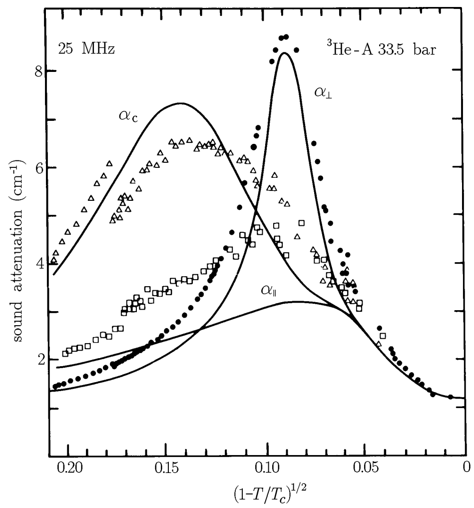


FIGURE 9.12 The sound attenuation can be parametrized in terms of three constants whose experimental measurements are shown here and compared with theoretical calculations of Ref. [42]. The most sensitive test for a helical texture can be performed in the region of largest difference between $\alpha_\perp$ and α_{11}.

Eq. (he-11.90) Therefore one has the averages

$$\langle\cos^2\delta\rangle = \sin^2\beta_0\langle\cos^2\gamma\rangle = \frac{1}{2}\sin^2\beta$$
$$\langle\cos^4\delta\rangle = \sin^4\beta_0\langle\cos^4\gamma\rangle = \frac{3}{8}\sin^4\beta \qquad (9.94)$$

Eq. (he-11.91) so that

$$\alpha = \alpha_\perp + (\alpha_c - \alpha_\perp)\sin^2\beta_0 + \frac{3}{8}\sin^4\beta. \qquad (9.95)$$

The experimental values for the coefficients are displayed in Fig. 9.12 (taken from Ref. 42). Thus, if one goes into a region of large $|\alpha_c - \alpha_\perp|$, and turns on a magnetic field, α will stay constant for $h < h_c$ (from (9.25) (9.51(he-11.48)). For $h > h_c$ it will begin to drop linearly in $\Delta h^2/j^2$ (if $\alpha_c\alpha_\perp < 0$) with a slope $\approx (\alpha_c - \alpha_\perp)\,3\Delta h^2/j^2$. It appears as if this effect has been at La Jolla. [3] Fig. XXVIII

Until now we have focussed our discussion on helical textures which may develop from a previously aligned $d^\parallel l^\parallel j^\parallel h$ configuration. A look at Figs. 9.11(a)–(c) shows that there is another domain of stability for $s \approx 1$ (open helices), as $\frac{h^2}{j^2}$ exceeds some

[3] I thank Prof. K. Maki for a discussion of this point and of the experiment performed by R. Kleinberg at La Jolla.

critical value $h_{c_2}^2/j^2$. The reason for this is obvious: If h is large enough, a potential valley is created for the **d**-vector at $\theta \approx \frac{\pi}{2}$. Dipole locking stabilizes also **l** in this position. In order to calculate the boundary in the dipole locked regime, consider
Eq. (he-11.92) the energy for $s \approx 1$:

$$
\begin{aligned}
2g &= \text{const} \quad +2\left(\frac{h^2}{j^2}-\frac{1}{2}\frac{\rho_0\rho_s^{\|}}{\rho_s^2}\right)(1-s)j^2+2\sqrt{1-s}\frac{\rho_s-c_0}{\rho_s}\gamma_z j \\
&\qquad\qquad +\frac{K_t}{\rho_s^{\|}}\gamma_z^2 \\
&= \text{const} \quad +2\left[\frac{h^2}{j^2}-\frac{\rho_0\rho_s^{\|}}{2\rho_s^2}\left(\frac{\rho_s c_0}{\rho_s}\right)^2\frac{\rho_s^{\|}}{2K_t}\right]j^2 \\
&\qquad\qquad +\frac{K_t}{\rho_s^{\|}}\left[\gamma_z+\sqrt{1s}\left(\frac{\rho_s-c_0}{\rho_s}\right)\frac{\rho_s^{\|}}{K_t}\right]^2 .
\end{aligned} \tag{9.96}
$$

Eq. (he-11.93) Thus the $\beta \approx \frac{\pi}{2}$ position is stable as long as

$$
\begin{aligned}
\frac{h^2}{j^2} \geq \frac{h_{c_2}^2}{j^2} &= \frac{\rho_s^{\|}}{2\rho_s^2}\left[\rho_0+\frac{(\rho_s-c_0)^2}{K_t}\right] \\
&=_L \frac{7}{40}\frac{(1-\epsilon)\left(1-\frac{2}{7}\epsilon\right)}{(1-\epsilon/2)^2}.
\end{aligned} \tag{9.97}
$$

This boundary is shown on Fig. 9.8

for $T \approx T_c$ (i.e., $\epsilon \approx 0$). It is important to realize, that for $h > h_{c_2}$ not only the $\beta = \frac{\pi}{2}$ position but also a whole neighborhood of it is stable. This can easily be shown: Since M_g' is diverging for $s \approx 1$ as $1/\sqrt{1-s}$, the solution of (9.34) become
Eq. (he-11.94) simply

$$
\begin{aligned}
\frac{\gamma_z}{j} &\approx \frac{1}{M_g'}\left(\frac{h^2}{j^2}-\frac{A_g'}{2}\right)\Big|_{s\approx 1} \underset{L}{=} -2\frac{2-\epsilon}{1-\epsilon}\sqrt{1-s}\left(\frac{h^2}{j^2}-\frac{1-\epsilon}{8(1-\epsilon/2)^2}\right), \\
\frac{\gamma_z^{\dagger}}{j} &\approx -2\frac{M_g'}{G_g'}\Big|_{s\approx 1} \underset{L}{=} \frac{1}{\sqrt{1-s}}\frac{1-\epsilon}{6-\frac{5}{2}\epsilon},
\end{aligned} \tag{9.98}
$$

which can be compared with Fig. 9.11(a)–(c).

Eq. (he-11.95) Now the first stability criterion (9.63) is fulfilled trivially:

$$
G_g\big|_{s=1} = \frac{K_t}{\rho_s^{\|}} > 0. \tag{9.99}
$$

The second, determinantal, criterion (9.64) on the other hand, is dominated by the
Eq. (he-11.96) singularity in M_g'' and by $\gamma_z^{\dagger}$:

$$
2M_g''\gamma_z^{\bar{+}}\frac{G_g}{2G_g'^{\,2}} \underset{\approx}{=} g_z^{+2}/4. \tag{9.100}
$$

Inserting (9.101) we see that only the negative values of γ_z on the lower branch can
Eq. (he-11.97) satisfy this under the condition:

$$\frac{h^2}{j^2} - \frac{A'_g}{2} \geq \frac{M'^2_g}{M''_g}\frac{1}{G_g}. \tag{9.101}$$

Inserting the parameters of the liquid this becomes exactly the same condition as (9.97), but now it guarantees stability of *all* positions in the neighborhood of $s \approx 1$. Notice that, contrary to $s = 0$, where $s = 0$ and $s \neq 0$ correspond to two different parameter manifolds, the point $s = 1$ is in no way special as compared to its neighborhood.

If dipole locking is relaxed, the straight boundary (9.97) in the j, h plane will curve for larger values of j (see Fig. 9.8), and approach an asymptotic line $j = j_{\max}$. In order to find $j_{\max}$, consider the terms of the energy quadratic in $\tilde{\beta} \equiv \beta - \frac{\pi}{2}$, $\tilde{\theta} \equiv \theta - \frac{\pi}{2}$, γ, ϕ: Eq. (he-11.98)

$$\begin{aligned} 2g = \text{const} + 2\left(\frac{h^2}{j^2}\tilde{\theta} - \frac{\rho_0\rho_s^{\parallel}}{2\rho_s^2}\tilde{\beta}^2\right)j^2 + K_s\tilde{\beta}_z^2 + 2\tilde{\beta}\frac{\rho_s c_0}{\rho_s}\gamma_z j \\ + \frac{K_t}{\rho_s^{\parallel}}\gamma_z^2 + \frac{K_1^d}{\rho_s^{\parallel}}\left(\theta_z^2 + \phi_z^2\right) + 2\left(\tilde{\theta} - \tilde{\beta}\right)^2 + 2\left(\gamma - \phi\right)^2 \end{aligned} \tag{9.102}$$

where now K_s, K_t are the unlocked values ($K_s =_{GL} K_t =_{GL} = \frac{1}{2}$ also $K_1^d =_{GL} 2$). For a plane wave ansatz this becomes Eq. (he-11.99)

$$2g = \text{const} \quad + \tilde{B}\tilde{\beta}^2 + \tilde{T}\tilde{\theta}^2 - 4\tilde{\theta}\tilde{\beta} + \tilde{G}\gamma^2 + 2Mjk\tilde{\beta}\gamma + \tilde{F}\phi^2 - 4\gamma\phi \tag{9.103}$$

with Eq. (he-11.100)

$$\begin{aligned} \tilde{B} &= K_s k^2 - \frac{\rho_0\rho_s^{\parallel}}{\rho_s^2}j^2 + 2, \\ \tilde{G} &= \frac{K_t}{\rho_s^{\parallel}}k^2 + 2, \\ \tilde{T} &= \frac{K_1^d}{\rho_s^{\parallel}}k^2 + 2h^2 + 2, \\ \tilde{F} &= \frac{K_1^d}{\rho_s}k^2 + 2. \end{aligned} \tag{9.104}$$

After a few quadratic completions one finds Eq. (he-11.101)

$$2g = \text{const} + \bar{B}\tilde{\beta}^2 + \tilde{T}\bar{\theta}^2 + \bar{G}\bar{\gamma}^2 + \tilde{F}\bar{\phi}^2 \tag{9.105}$$

with Eq. (he-11.102)

$$\begin{aligned} \bar{G} &\equiv \tilde{G} - 4/\tilde{F} = \frac{K_t^l}{\rho_s^{\parallel}}k^2\left(1 + \frac{K_t K_1^d}{2K_t^l\rho_s^{\parallel}}k^2\right) \Big/ \left(1 + \frac{1}{2}\frac{K_1^d}{\rho_s^{\parallel}}k^2\right) \\ \bar{B} &= \tilde{B} - \tilde{M}^2/\bar{G}j^2 - 4/\tilde{T} \\ &= \frac{K_s}{\rho_s^{\parallel}}k^2 - \frac{\rho_0\rho_s^{\parallel}}{\rho_s^2}j^2 + 2 - \left(\frac{\rho_s - c_0}{\rho_s}\right)^2\frac{k^2}{G}j^2 - \frac{2}{\frac{K_1^d}{2\rho_s^{\parallel}}k^2 + h^2 + 1} \end{aligned} \tag{9.106}$$

Eq. (he-11.104) and new angles

$$\begin{aligned}\bar{\theta} &= \tilde{\theta} - \tilde{\beta}/\tilde{T} \\ \bar{\gamma} &= \gamma - \left(Mjk/\bar{G}\right)\beta \\ \bar{\phi} &= \phi - \gamma/\tilde{F}.\end{aligned} \tag{9.107}$$

Eq. (he-11.105) Now $\tilde{F} \geq 0, \quad \tilde{T} \geq 0, \quad \bar{G} \geq 0$. Hence $2g$ is positive definite for all k if

$$\bar{B}(k) > 0. \tag{9.108}$$

For small $h^2,\ j^2$ we can expand in $h^2,\ j^2$ and k^2 and recover the dipole locked result (9.97). As h increases, the stability curve approaches the line $j = j_{\rm max}$ determined
Eq. (he-11.106) by the $h = \infty$ version of (9.108) which renders

$$j_{\rm max}^2 - \frac{2\rho_0^2/\rho_s^{\parallel}}{\rho_0 + \frac{(\rho_s - c_0)^2}{K_t^l}} \underset{GL}{=} \frac{40}{7}. \tag{9.109}$$

In fact, if the coefficients are close enough to their Ginzburg-Landau values, the
Eq. (he-11.107) value $\bar{B}(k=0)$ is the most dangerous one yielding the boundary curve:

$$j^2 \leq \frac{2\rho_s^2}{\rho_s^{\parallel}} \frac{1}{\rho_0 + \frac{(\rho_s - c_0)^2}{K_t^l}} \frac{h^2}{h^2+1} \tag{9.110}$$

which starts out as (9.97) and becomes horizontal for $h \gg 1$ (see Fig. 9.8)

A final remark concerns the possibility, that the stability discussion presented here becomes invalid due to the neglect of transverse oscillations. Certainly, these oscillations have to be included if the stability criteria developed above are to be valid. If x and y-dependence are included, the discussion of the energy becomes extremely tedious. Until now, only oscillations with very small transverse momentum have been tested. Fortunately, it turns out that at least for this limit the transverse oscillations have higher energies than the longitudinal ones so that the instabilities are always triggered along the z direction.

since the discussion of this point is very technical and does not lead to interesting physical insights, the reader is referred to Ref. [41].

9.1 Magnetic Field and Transition between A- and B-Phases

Eq. (he-15.83) At zero flow we can observe the transition between A- and B-phase at

$$\frac{\alpha}{\frac{6}{5}\beta_B}\left(\alpha - h^2\right)^2 + \frac{5}{\beta_{345}} h^4 = \frac{\alpha}{\frac{6}{5}\beta_A}. \tag{9.111}$$

Eq. (he-15.84) This is solved by

$$h_{AB}^2 = \frac{\alpha}{3}\frac{\beta_{345}}{2\beta_{12}+\beta_{345}}\left[1-f(\beta)\right] \tag{9.112}$$

Eq. (he-15.85) where

$$e(\beta) \equiv \sqrt{\frac{3\beta_B}{\beta_A\beta_{345}}}\sqrt{2\beta_{13}-\beta_{345}}. \tag{9.113}$$

In the weak-coupling limit this vanishes so that Eq. (he-15.86)

$$h_{AB}^2 = \frac{1}{6}. \tag{9.114}$$

At fixed magnetic field H this corresponds to a temperature Eq. (he-15.87)

$$1-\frac{T_{AB}}{T_c} = 6\frac{H^2}{{H_0}^2} \tag{9.115}$$

i.e., the transition temperature shifts quadratically with the magnetic field. At the polycritical point Eq. (he-15.88)

$$\beta_A = \beta_B \tag{9.116}$$

which amounts to Eq. (he-15.89)

$$3\beta_{13} = 2\beta_{345}, \quad f(\beta) = 1 \tag{9.117}$$

the transition occurs at zero magnetic field as it should. If the polycritical pressure p_{pc} is kept fixed but the temperature is slightly varied one may expand Eq. (he-15.90)

$$1-f(\beta) \sim 1-\frac{T}{T_c} \tag{9.118}$$

so that at p_{pc} the temperature T_{AB} of the transition varies linearly with H_{AB}. This is why the experimental curve of phase transition shows the most significant dependence on H in the neighborhood of the polycritical point.

The discussion must be carried out separately above T_c. For these the transition to the B phase in no magnetic field occurs not at T_c but at ${T_{AB}}^0$ which is determined by (9.115). Expanding around this temperature we find the value of $(1-f(\beta))$ at T via Eq. (he-15.91)

$$1-f(\beta) = f'|_{T_{AB}^0}\left(1-\frac{T_{AB}^0}{T}\right). \tag{9.119}$$

The magnetic field shifts the transition from T_{AB}^0 to T_{AB}^h. This is given by Eq. (he-15.92)

$$h_{AB}^2 = \frac{\alpha}{3}\frac{\beta_{345}}{2\beta_{12}+\beta_{345}}f'|_{T_{AB}^0}\left(T_{AB}^0-T_{AB}^h\right) \tag{9.120}$$

or in terms of physical magnetic fields: Eq. (he-15.93)

$$\frac{H_{AB}^2}{H_0^2} = \frac{\alpha}{3}\frac{\beta_{345}}{2\beta_{12}+\beta_{345}} f'|_{T_{AB}^0}\left(T_{AB}^0 - T_{AB}^h\right)\left(1 - \frac{T_{AB}^0}{T_c} + \frac{T_{AB}^0 - T_{AB}^h}{T_c}\right). \qquad (9.121)$$

Thus we see again that way above the polycritical pressure p_{pc} there is only a quadratic response of T_{AB}^h to the magnetic field while close to p_{pc} there is again the linear dependence discussed before.

It should be noted that in the absence of strong-coupling corrections, the order parameter of the B-phase is distorted continuously into that of the A-phase as H reaches H_{AB}. Since $a^2 = 1 + \frac{3}{2}h^2$, $c^2 = 1 - 6h^2$ become directly $a^2 = \frac{5}{4}$, $c^2 = 0$.

Thus the transition is of second order. Since it is sometimes believed that strong-coupling corrections become small for $p \to 0$ this amounts to a decreasing latent heat

Appendix 9A Generalized Ginzburg-Landau Energy

If one assumes all temperature dependence to come from $\rho_0 = \rho_s^{\parallel}(1-\epsilon) \equiv \rho_s^{\parallel}\alpha$, the
Eq. (he-D) coefficients of the energy are in the dipole locked regime:

$$\begin{aligned}
A &= 1 + \alpha s \underset{GL}{=} 1 + s \qquad (9A.1)\\
Ag &= -A^{-1}\\
Ag' &= \alpha A^{-2} \underset{GL}{=} \frac{1}{(1+s)^2}\\
Ag'' &= -2\alpha^2 A^{-3} \underset{GL}{=} -2\frac{1}{(1+s)^3}\\
Mg &= \left(1 - sA^{-1}\right)\sqrt{1-s} \underset{GL}{=} \frac{\sqrt{1-s}}{1+s}\\
Mg' &= -\left[1 + (2-3s)A^{-1} - 2(1-s)\alpha s A^{-3}\right]/2\sqrt{1-s} \underset{GL}{=} -\frac{1}{2\sqrt{1-s}}\frac{3-s}{(1+s)^2}\\
Mg'' &= -\left[1 - (4-3s)A^{-1} - 4(3s^2-5s+2)A^{-2} + 8(1-s)^2 s\alpha A^{-3}\right]/4(1-s)^{3/2}\\
&\underset{GL}{=} \frac{1}{4(1-s)^{3/2}}\frac{1}{(1+s)^4}\left(11 - 7s - 15s^2 + 3s^3\right)\\
Gg &= \frac{s}{2}\left(5 - 2s(1-s)A^{-1}\right) \underset{GL}{=} \frac{c}{2}\frac{5+3s+2s^2}{1+s}\\
Gg' &= \frac{5}{2} - s\left[2 - 3s - s(1-s)A^{-1}\right]A^{-1} \underset{GL}{=} \frac{1}{2(1+s)^2}(5 + 6s9s^2 + 4s^3)\\
Gg'' &= 2(3s-1)A^{-1} - 2(3s-2)s\alpha A^{-2} - 2s^2(1-s)\alpha A^{-3}\\
&\underset{GL}{=} \frac{2}{(1+s)^3}(s^3 + 3s^2 + 3s - 1)
\end{aligned}$$

10

Large Currents at Any Temperature $T \leq T_c$

10.1 Energy at Nonzero Velocities

For general temperatures $T \leq T_c$ we shall confine our discussion to the weak-coupling regime. Fermi liquid correction will be included at a later stage.

Adding the external source Eq. (he-11a.1)

$$\mathbf{vJ} = \mathbf{v}\psi^*(x)\frac{i}{2}\overleftrightarrow{\nabla}\psi(x) \tag{10.1}$$

to the action (3.18) gives rise, in the 2×2 matrix M of (3.24), to the additional entries Eq. (he-)

$$\begin{pmatrix} \mathbf{vp} & 0 \\ 0 & \mathbf{vp} \end{pmatrix}. \tag{10.2}$$

Therefore, the final collective action (3.27) becomes simply Eq. (he-11a.2)

$$\mathcal{A}^v = -\frac{i}{2}\mathrm{Tr}\log\begin{pmatrix} i\partial_t - \xi(\mathbf{p}) + \mathbf{vp} & A_{ai}\sigma_a\overleftrightarrow{\nabla}_i/2 \\ A^{ai*}\sigma_a\overleftrightarrow{\boldsymbol{\nabla}}_i/2 & i\partial_t + \xi(p) + \mathbf{vp} \end{pmatrix} - \frac{1}{3g}\int d^3x|A_{ai}|^2. \tag{10.3}$$

For constant field configurations, $A_{ai} \equiv A^0_{ai}$, this results in the energy density Eq. (he-11a.3)

$$\begin{aligned} g^v &\equiv -\frac{T}{V}\mathcal{A}^v \qquad (10.4)\\ &= -T\sum_{\omega_n,\mathbf{p}}\left[\log\left(i\omega_n + \mathbf{vp} - E(\mathbf{p})\right) + (E - E)\right] + \frac{1}{3g}|A^0_{ai}|^2 + \text{const} \end{aligned}$$

with Eq. (he-11a.4)

$$E(\mathbf{p}) = \sqrt{\xi^2(\mathbf{p}) + \Delta^2_\perp + (1 - r^2z^2)} \tag{10.5}$$

As in Sec. 4, it is convenient to subtract from this the free energy of the free Fermi liquid, now with the external source $\mathbf{vp}$. Then Eq. (he-11a.5)

$$g_0^v = -T \sum_{\omega_n, \mathbf{p}} \left[\log\left(i\omega_n + \mathbf{vp} - \xi(\mathbf{p})\right) + (\xi \to -\xi)\right] + \text{const} \, . \tag{10.6}$$

For $\mathbf{v} = 0$ this quantity was calculated earlier [recall Eq. (7.181)]. For $\mathbf{v} \neq 0$ we observe that we may perform a quadratic completion Eq. (he-11a.6)

$$\mp \mathbf{vp} - \xi(\mathbf{p}) = -\frac{(\mathbf{p} \pm m\mathbf{v})^2}{2m} + \mu + \frac{m}{2}\mathbf{v}^2. \tag{10.7}$$

The first term gives the same g_0 as the $\mathbf{v} = 0$ formula since the integration over $\mathbf{p}$ is merely shifted by $m\mathbf{v}$. As far as the additional kinetic energy $m\mathbf{v}^2/2$ is concerned
Eq. (he-11a.7) we may assume it to be very much smaller than $p_F^2/2m$ so that we can expand

$$\begin{aligned} g_0^v &= {g_0}^0 + T \sum_{\omega_n, \mathbf{p}} \left(\frac{e^{i\omega_n \mu}}{i\omega_n - \xi(\mathbf{p})} - \frac{e^{-i\omega_n \mu}}{i\omega_n + \xi(\mathbf{p})} \right) \frac{m}{2}\mathbf{v}^2 \\ &= {g_0}^0 - \sum_{\mathbf{p}} n(\xi) \frac{m}{2}\mathbf{v}^2 \\ &= {g_0}^0 - \frac{g}{2}\mathbf{v}^2 \end{aligned} \tag{10.8}$$

thus arriving at the usual form of a Galilean transformed energy.

10.2 The Gap Equations

We shall now specialize to considering anisotropic gaps of the same form (4.18) as
Eq. (he-11a.8) discussed previously in the Ginzburg-Landau limit, i.e.,

$$A_{ai}^0 = \Delta^0 \begin{pmatrix} a & & \\ & a & \\ & & c \end{pmatrix} = \begin{pmatrix} \Delta_\perp & & \\ & \Delta_\perp & \\ & & \Delta_\parallel \end{pmatrix} \tag{10.9}$$

Eq. (he-11a.9) so that

$$|A_{ai}^0 \hat{p}_i|^2 = \Delta^2(z) = \Delta_\perp^2 \left(1 - r^2 z^2\right) = \Delta_\perp^2 \left(1 - z^2\right) + \Delta_\parallel^2 z^2. \tag{10.10}$$

Here z is the directional cosine of the quasi particle momentum with respect to the preferred axis which lies parallel to the current, for symmetry reasons. This Ansatz permits a simultaneous discussion of B, A, planar, and polar phases. $\Delta_\perp$, $\Delta_\parallel$ are the gaps orthogonal and parallel to the flow. With the form (10.9) the last term in
Eq. (he-) the energy (10.4(he-11a.3)) becomes

$$\frac{1}{3g}|A_{ai}|^2 = \frac{1}{g}\Delta_\perp^2 \left(1 - \frac{r^2}{3} >\right) = \left(2{\Delta_\perp}^2 + {\Delta_\parallel}^2\right)/3g. \tag{10.11}$$

Eq. (he-11a.10) Minimizing g with respect to ${\Delta_\parallel}^2$ and ${\Delta_\perp}^2$ we find the two conditions:

$$\left[\frac{1}{g} - T \sum_{\omega_n, \mathbf{p}} 3z^2 \frac{1}{\left(\omega_n - ip_F z\right)^2 + E^2(\mathbf{p})}\right] \Delta_\parallel = 0 \tag{10.12}$$

$$\left[\frac{1}{g} - T \sum_{\omega_n, \mathbf{p}} \frac{3}{2}\left(1 - z^2\right) \frac{1}{\left(\omega_n - ivp_F z\right)^2 + E^2(\mathbf{p})}\right] \Delta_\perp = 0. \tag{10.13}$$

If we assume both gaps $\Delta_\parallel$ and $\Delta_\perp$ to be nonzero, there are two nontrivial gap equations which specify the equilibrium situation in the B-phase.

The other possibilities correspond to A- and planar phases ($\Delta_\parallel = 0, \quad \Delta_\perp \neq 0$) or to the planar phase ($\Delta_\parallel \neq 0, \quad \Delta_\perp = 0$), each with only one nontrivial gap equation remaining. Sometimes it will be useful to compare with the hypothetical case that the gap is free of distortion, $\Delta_\parallel = \Delta_\perp$ or $r = 0$. Then only the average ($\frac{1}{3}$ (longitudinal +2 transverse)) gap equation survives with no z weight factor in the integration and $r = 0$ inserted. Moreover, since the polar phase is physically rather uninteresting because of its weak condensation energy, we shall henceforth work with this average gap equation together with the transverse one (10.13). From the latter we shall often draw comparison with the A-phase by inserting $r = 1$. In the gap equations, the sums over Matsubara frequencies may be performed in the standard fashion Eq. (he-11a.12)

$$T\sum_{\omega_n}\frac{1}{2E}\left(\frac{1}{i\omega_n + vp_F z - E(p)} - \frac{1}{i\omega_n + vp_F z + E(p)}\right) = \frac{1}{4E}\left[\tanh\frac{E - vp_F z}{2T} + (v \to -v)\right]. \tag{10.14}$$

Decomposing the integral over momenta according to direction and size Eq. (he-11a.13)

$$\int\frac{d^3p}{(2\pi)^3} \approx \mathcal{N}(0)\int_{-1}^{1}\frac{dz}{2}\int_{-\infty}^{\infty}d\xi \tag{10.15}$$

the average and transverse gap equations, become Eq. (he-11a.14)

$$\begin{aligned}\frac{1}{g\mathcal{N}(0)} &= \int_{-1}^{1}\frac{dz}{2}\gamma,\\ \frac{1}{g\mathcal{N}(0)} &= \int_{-1}^{1}\frac{dz}{2}\frac{3}{2}\left(1 - z^2\right)\gamma,\end{aligned} \tag{10.16}$$

where γ denotes the function Eq. (he-11a.15)

$$\begin{aligned}\gamma &\equiv T\sum_{\omega_n}\int_{-\infty}^{\infty}d\xi\frac{1}{\left(\omega_n - vp_F z\right)^2 + E^2(\mathbf{p})}\\ &= \int_{-\infty}^{\infty}d\xi\frac{1}{4E}\left[\tanh\frac{E - vp_F z}{2T} + (v \to -v)\right]\end{aligned} \tag{10.17}$$

which is logarithmically divergent. It may be renormalized via the critical temperature which satisfies Eq. (he-11a.16)

$$\begin{aligned}\frac{1}{g\mathcal{N}(0)} &= \int_{-\infty}^{\infty}d\xi\frac{1}{2\xi}\tanh\frac{\xi}{2T_c} = \log\left(\frac{\omega_c}{T_c}2e^{-\gamma}/\pi\right)\\ &= \log\frac{T}{T_c} + \int_{-\infty}^{\infty}d\xi\frac{1}{2\xi}\tanh\frac{\xi}{2T}.\end{aligned} \tag{10.18}$$

Subtracting this expression on both sides the gap equations take the form Eq. (he-11a.17)

$$\log\frac{T}{T_c} = \int_{-1}^{1}\frac{dz}{2}\gamma \tag{10.19}$$
$$\log\frac{T}{T_c} = \int_{-1}^{1}\frac{dz}{2}\frac{3}{2}(1-z^2)\gamma \tag{10.20}$$

Eq. (he-11a.19) with the subtracted finite function

$$\gamma = \int_{-\infty}^{\infty}\left[\frac{1}{4E}\left(\tanh\frac{E-vp_Fz}{2T}+(v\to -v)\right)-\frac{1}{2\xi}\tanh\frac{\xi}{2T}\right] \tag{10.21}$$

For calculations it is more convenient to return to the Matsubara form (10.12), (10.13(he-11a.11)). Then the integrals over d can be performed and with the above
Eq. (he-11a.20) renormalization procedure, we find the simple expression

$$\begin{aligned}\gamma &= \pi\sum_{\omega_n}\left(\frac{1}{\sqrt{(\omega_n-ivp_Fz)^2+\Delta_\perp^2(1-r^2z^2)}}-\frac{1}{\omega_n}\right)\\ &= \frac{2}{\delta}\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{(x_n-i\nu z)^2+\Delta_\perp^2(1-r^2z^2)}}-\frac{1}{x_n}\right).\end{aligned} \tag{10.22}$$

Here the square root has to be taken with positive real part. In this and many formulas to come we have found it convenient to introduce the following dimensionless
Eq. (he-11a.21) variables

$$\begin{aligned}\delta &= \frac{\Delta_\perp}{\pi T}\\ \nu &= \frac{vp_F}{\Delta_\perp}\\ x_n &= \frac{\omega_c}{\Delta_\perp}.\end{aligned} \tag{10.23}$$

In order to check the gap equations we compare with the previously discussed Ginzburg-Landau results we take the limit $T\to T_c$. Then the variables x_n become
Eq. (he-11a.22) very large and we may approximate

$$\begin{aligned}-\log\frac{T}{T_c} &\approx 1-\frac{T}{T_c}\\ &\approx \frac{2}{\delta}\int_{-1}^{1}\frac{dz}{2}\left\{\begin{matrix}1\\ \frac{3}{2}(1-z^2)\end{matrix}\right\}\sum_{n=0}^{\infty}\left(1+\left(2\nu^2-r^2\right)z^2-2i\nu_z x_n\right)/2x_n^3\\ &= \delta^2\left[1+\left\{\begin{matrix}\frac{1}{3}\\ \frac{1}{5}\end{matrix}\right\}\left(2\nu^2-r^2\right)\right]\frac{7\zeta(3)}{8}.\end{aligned} \tag{10.24}$$

Eq. (he-11a.23) This is solved by

$$r^2 = 1-\frac{\Delta_\parallel^2}{\Delta_\perp^2} = 1-\frac{c^2}{a^2} = 2\nu^2 \tag{10.25}$$

which shows how the gap deformation increases with the current. Inserting this
Eq. (he-11a.24) back into (10.24) the transverse gap behaves like

$$\Delta_\perp{}^2 = \pi^2 T^2 \delta^2 \approx \frac{8}{7\zeta(3)} \pi^2 T_c^2 \left(1 - \frac{T}{T_c}\right) = 3.063^2 \left(1 - \frac{T}{T_c}\right) \tag{10.26}$$

i.e., it is independent of the current. Using this, the gap distortion (10.25) may be reexpressed in terms of $\kappa \equiv lfracvv_0$ [see (8.19)], and we find: Eq. (he-11a.25)

$$\begin{aligned} \nu^2 &= \frac{v^2 p_F^2}{\Delta_\perp^2} = \frac{v^2 p_F^2}{\frac{8}{7\zeta(3)} \pi^2 {T_c}^2 \left(1 - \frac{T}{T_c}\right)} \\ &= \frac{3}{2} \frac{v^2}{\left(\frac{1}{2m\xi_0}\right)^2} \frac{1}{1 - \frac{T}{T_c}} = \frac{3}{2} \frac{v^2}{v_0^2(T)} = \frac{3}{2}\kappa^2 \end{aligned} \tag{10.27}$$

which gives Eq. (he-11a.26)

$$\frac{\Delta_\parallel{}^2}{\Delta_\perp{}^2} = \frac{c^2}{a^2} = 1 - 3\kappa^2. \tag{10.28}$$

Of course, these results agree with the Ginzburg-Landau formulas (8.73) and (8.74(he-15.65)).

In the opposite limit of zero temperature the distance between neighboring values x_n toes to zero so that we may replace the sum over x_n by an integral according to the rule Eq. (he-11a.27)

$$\sum_{x_n} \sim \frac{2}{\delta} \int dx_n. \tag{10.29}$$

As in (**??**) a little care is necessary with the last sum $\sum_n 1/x_n$ as in (3.261) since each term diverges at $T = 0$. The careful replacement is Eq. (he-11a.28)

$$\frac{2}{\delta} \sum_{n=0}^{\infty} \frac{1}{x_n} \to \int_{1/\delta}^{x_N} \frac{dx}{x} + \log(2e^\gamma). \tag{10.30}$$

Therefore we obtain Eq. (he-11a.29)

$$\begin{aligned} \log \frac{T}{T_c} \underset{T\to 0}{\longrightarrow} & \, \mathrm{Re} \int_{-1}^{1} \frac{dz}{2} \left\{ \frac{1}{\frac{3}{2}(1 - z^2)} \right\} \\ & \times \left[\int_0^\infty dx \frac{1}{\sqrt{(x - i\nu z)^2 + 1 r^2 z^2}} - \int_{1/\delta}^\infty \frac{1}{x} - \log 2 - \gamma \right] \\ = -\mathrm{Re} & \int_{-1}^{1} \frac{dz}{2} \left\{ \frac{1}{\frac{3}{2}(1 - z^2)} \right\} \log\left(\sqrt{1 - (\nu^2 + r^2) z^2 - i\epsilon z} - i\nu z\right) - \log\delta - \gamma. \end{aligned} \tag{10.31}$$

Taking γ and $\log\delta$ to the other side, the $\log T$ divergence cancels and we find Eq. (he-11a.31)

$$\log \frac{\Delta_\perp(T=0)}{\Delta_{BCS}} = -\mathrm{Re} \int_{-1}^{1} \left\{ \begin{array}{c} 1 \\ \frac{3}{2}\left(1-z^2\right) \end{array} \right\} \times \log\left(\sqrt{1-\left(\nu^2+r^2\right)z^2 - i\epsilon z} - i\nu z\right) \tag{10.32}$$

where we have introduced the $T=0$ gap of BCS theory Eq. (he-11a.32)

$$\Delta_{BCS} = \pi T_c e^{-\gamma} \sim 1.7638\, T_c. \tag{10.33}$$

When calculating the logarithm, we have to be careful about using the correct square root. Taking the branch cut, as usually to the left this is specified by the $i\epsilon$ prescription.

As a cross check we see that for $\nu = 0, \quad r = 0$ (B phase) the orthogonal gap
Eq. (he-11a.33) becomes $\Delta_\perp = \Delta_{BCS}$, while for $r = 1$ (A phase) the lower equation gives

$$\log \frac{\Delta_\perp}{\Delta_{BCS}} = -\int_{-1}^{1} \frac{dz}{2}\frac{3}{2}\left(1-z^2\right)\log\sqrt{1-z^2} = \frac{5}{6} - \log 2 \tag{10.34}$$

Eq. (he-11a.34) such that

$$\Delta_\perp = \Delta_{BCS}\frac{e^{5/6}}{2} \sim 2.03 T_c \tag{10.35}$$

as it should.

While the full solution of the gap equation (10.32) at $T=0$ is quite complicated, we can see directly that for all $\nu = \frac{vp_F}{\Delta_\perp} \leq 1$, $r=0$ and $\Delta_\perp = \Delta_{BCS}$ is a solution of both equations: In fact, the real part of the logarithm vanishes identically for $r=0$.

Since $r=0$, this piece of the gap function agrees with that of the B-phase neglecting distortion altogether. In the case of the full T behavior can be found from the average between the two gap equations (10.19), (i.e., $\frac{1}{3}$ (longitudinal +2
Eq. (he-11a.35) transverse)). Then there is no z weight and we have

$$\log \frac{\Delta_\perp(T=0)}{\Delta_{BCS}} = -\mathrm{Re}\int_{-1}^{1} \frac{dz}{2} \log\left(\sqrt{1-\nu^2 z^2 i\epsilon z} - i\nu z\right). \tag{10.36}$$

Eq. (he-11a.36) There is a real part only for $\nu > 1$:

$$= \; -\int_{1/\nu}^{1} dz \log\left(\sqrt{\nu^2 z^2 - 1} + \nu z\right) \tag{10.37}$$

$$= \; -\int_{1/\nu}^{1} dz\, \mathrm{arcosh}\nu z = \left[z\, \mathrm{acosh}\nu z - 1/\nu\sqrt{\nu^2 z^2 - 1}\right]_{1/\nu}^{1} \tag{10.38}$$

Eq. (he-11a.37) so that the $T=0$ gap without distortion (superscript u) would follow as[1]

$$\log \frac{\Delta_B{}^u(T=0)}{\Delta_{BCS}} = -\theta(\nu-1)\left(\mathrm{acosh}\nu - \frac{1}{\nu}\sqrt{\nu^2-1}\right). \tag{10.39}$$

The gap remains equal to Δ_{BCS} up to $\nu = 1$. From there on it drops to zero rapidly. The place where it vanishes may be evaluated from (10.39) by inserting
Eq. (he-11a.38) $\nu = v\, p_F/\Delta_B \to \infty$ so that we find in the limit

$$-\log \Delta_{BCS} = -\log 2vp_F + 1 \tag{10.40}$$

or Eq. (he-11a.39)

$$\frac{vp_F}{\Delta_{BCS}} = \frac{e}{2} \approx 1.359 \tag{10.41}$$

which in terms of natural units reads Eq. (he-11a.40)

$$\begin{aligned}\frac{v}{v_0} = v2m\xi_0 &= v\frac{2p_F}{\pi T_c}\sqrt{\frac{7\zeta(3)}{48}} \\ &= \frac{vp_F}{\pi e^{-\gamma}T_c}2e^{-\gamma}\sqrt{\frac{7\zeta(3)}{48}} \qquad (10.42)\\ &= \frac{p_F v}{\Delta_{BCS}}.47 \approx .64. \qquad (10.43)\end{aligned}$$

For comparison we see that in the A-phase Eq. (he-11a.41)

$$\begin{aligned}\log\frac{\Delta_\perp(T=0)}{\Delta_{BCS}} &= -\mathrm{Re}\int_{-1}^{1}\frac{dz}{2}\frac{3}{2}\left(1-z^2\right)\log\left(\sqrt{1-z^2-\nu^2z^2-i\epsilon z}-i\nu z\right)\\ &= -\int_{-1}^{1}\frac{dz}{2}\frac{3}{2}\left(1-z^2\right)\log\sqrt{1-z^2}\\ &\quad -2\int_{\frac{1}{\sqrt{1+\nu^2}}}^{1}\frac{dz}{2}\frac{3}{2}\left(1-z^2\right)\log\frac{\sqrt{\left(1+\nu^2\right)z^2-1}+\nu z}{\sqrt{1-z^2}}\end{aligned}$$

so that[2] Eq. (he-11a.42)

$$\begin{aligned}\frac{\log\Delta_\perp(T=0)}{\Delta_{BCS}e^{5/6}/2} &= -\int_{1/\sqrt{1+\nu^2}}^{1}\frac{3}{2}\left(1-z^2\right)\mathrm{acosh}\frac{\nu z}{\sqrt{1-z^2}}\\ &= \frac{1}{2}\frac{\nu^2}{1+\nu^2}-\frac{1}{2}\log\left(1+\nu^2\right). \qquad (10.44)\end{aligned}$$

Here the gap decreases smoothly and hits zero at Eq. (he-11a.43)

$$\frac{vp_F}{\Delta_{BCS}} = \frac{e^{5/6}}{2}\sqrt{e} \approx 1.897 \tag{10.45}$$

or Eq. (he-11a.44)

$$\frac{v}{v_0} \approx .892. \tag{10.46}$$

The full solution of the gap equations are shown in Figs. 10.1 and **??**(he-f11.2). For

[1] Carrying the Euler-Maclaurin expansion one step further, the right-hand side in (10.39) would have an additional $-1/6\delta^2\nu\sqrt{\nu^2-1}$ which in (10.41) gives a factor $\exp\{-\frac{1}{6}\frac{\pi^2T^2}{(p_Fv)^2}\} \sim 1-\frac{2}{3}e^{2(\gamma-1)}\left(T/T_c\right)^2 \sim 1-0.29\left(T/T_c\right)^2$.

[2] To next order in $\frac{1}{\delta}$, the Euler Maclaurin expansion gives $-\frac{1}{2\delta^2}\frac{\nu^2}{(1+\nu^2)^2}$ which enters in (10.45) as a factor $e^{-\frac{\frac{1}{2}\pi^2T^2}{(p_F\nu)^2}} \approx 1-2e^{\frac{8}{3}+2\gamma}\left(\frac{T}{T_c}\right)^2 \sim 1-.44\left(\frac{T}{T_c}\right)^2$

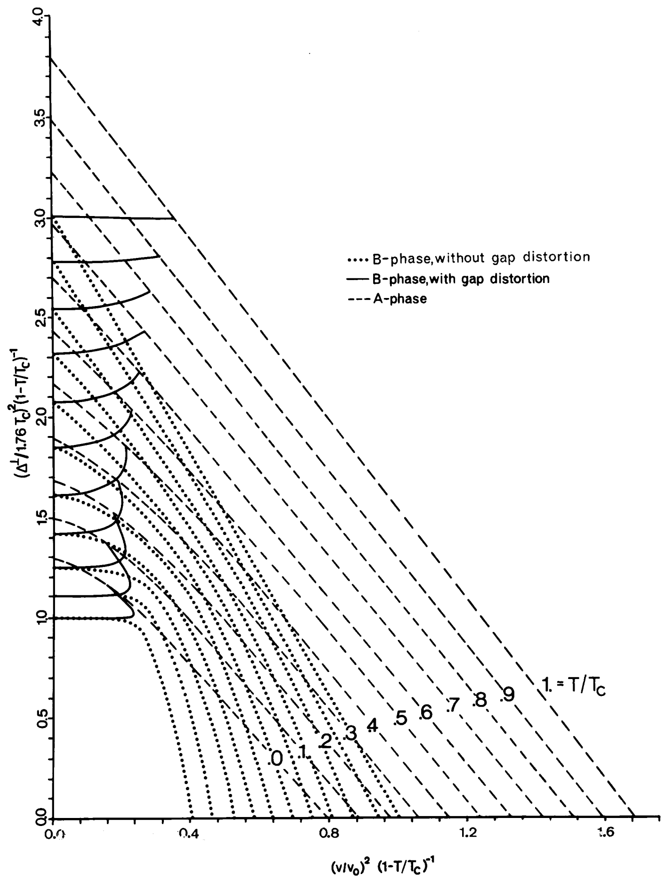


FIGURE 10.1 Velocity dependence of gap in A- and the B-phases.

comparison we also have displayed the solutions in the B-phase and the A-phase neglecting the gap distortion (i.e., using $r \equiv 0$ or $\Delta_\perp \equiv \Delta_\parallel$ and either one of the equations (10.12)).

All curves as functions of v are double valued. It will be seen later that this behavior can artifact of the neglect of Fermi liquid corrections. Once included, these will turn the lower branches anticlockwise into the region of higher velocities while distorting the upper branches at lower velocities only slightly. In the way the curves becomes single values.

For the numerical calculation we have taken formulas (10.19), (10.21) after having
Eq. (he-11a.45) performed the integrals over dz analytically: In the average gap equation we need

$$\mathrm{Re}\int_{-1}^{1}\frac{1}{\sqrt{(x-ivz)^2+1-r^2z^2}} = \mathrm{Re}\int_{-1}^{1}\frac{dz}{2}\frac{1}{\sqrt{1+x^2-2ivxz-(r^2+\nu^2)\,z^2}} \tag{10.47}$$

The square root has to be taken with positive real part i.e., with the standard choice
Eq. (he-11a.46) of the branch cut left, from zero to $i\infty$. The result is

$$\frac{\alpha_n}{\sqrt{\nu^2+r^2}} \equiv \frac{1}{\sqrt{\nu^2+r^2}} \arcsin \frac{\nu^2+r^2+i\nu x_n}{\sqrt{\nu^2+r^2+r^2+x_n^2}} \tag{10.48}$$

or Eq. (he-11a.47)

$$\alpha_n = \frac{1}{2} \arccos \left(\frac{1}{\nu^2+r^2+r^2x_n^2} \right. \tag{10.49}$$
$$\left. \times \left\{ \left(\nu^2+r^2\right) \sqrt{\left(1+x_n^2-\nu^2-r^2\right)^2+4\nu^2x_n^2} - \left(\nu^2+r^2\right)^2 - \nu^2 x_n^2 \right\} \right)$$

which lies in the interval $\left(0, \frac{\pi}{2}\right)$ so that the gap equation (10.31) becomes Eq. (he-11a.48)

$$\log \frac{T}{T_c} = \frac{2}{8} \sum_{n=0}^{\infty} \left\{ \frac{1}{\sqrt{\nu^2+r^2}} \alpha_n - \frac{1}{x_n} \right\}. \tag{10.50}$$

For $r=0$ we have Eq. (he-11a.49)

$$\alpha_n = \frac{1}{2} \cos \left(\sqrt{\left(1+x_n^2+\nu^2\right)^2-4\nu^2} - \nu^2 - x^2 \right) \tag{10.51}$$

and recover the result of the B-phase neglecting gap distortion. For the transverse gap we have to perform the integral (10.47) with an additional weight factor $\frac{3}{2}\left(1-z^2\right)$ and find Eq. (he-11a.55)

$$\log \frac{T}{T_c} = \frac{3}{2} \frac{2}{\delta} \sum_{n=0}^{\infty} \left\{ \left(1 - \frac{1}{2\left(\nu^2+r^2\right)} + \frac{\nu^2 - \frac{r^2}{2}}{\left(\nu^2+r^2\right)^2} x_n^2 \right) \frac{\alpha_n}{\sqrt{\nu^2+r^2}} \right.$$
$$\left. + \mathrm{Re} \frac{\nu^2+r^2-3i\nu x_n}{2\left(\nu^2+r^2\right)^2} \sqrt{\left(x_n-i\nu\right)^2+1-r^2} - \frac{2}{3} \frac{1}{x_n} \right\}. \tag{10.52}$$

For $r=1$ this is seen to reduce to the gap equation of the A-phase since Eq. (he-11a.56)

$$\alpha_a|_{r=1} = \arctan \frac{\sqrt{1+\nu^2}}{x_n} \tag{10.53}$$

and Eq. (he-11a.57)

$$\mathrm{Re} \frac{\nu^1+1-3i\nu x_n}{2\left(\nu^2+1\right)^2} \left(x_n-i\nu\right) = \frac{1-2\nu^2}{2\left(\nu^2+1\right)}. \tag{10.54}$$

The second term in the sum may be evaluated explicitly by defining Eq. (he-11a.58)

$$\begin{aligned} \gamma &\equiv \arctan \frac{3\nu x}{\nu^2+r^2} \in \left(0, \frac{\pi}{2}\right) \\ \beta &\equiv \arctan \frac{2\nu x}{1+x^2-\nu^2-r^2} \in (0,\pi) \end{aligned} \tag{10.55}$$

so that it becomes Eq. (he-11a.59)

$$\frac{\sqrt{\left(\nu^2+r^2\right)^2+9\nu^2x_n^2}}{2\left(\nu^2+r^2\right)^2} \left[\left(\nu^2+r^2-1-x_n^2\right)^2+4\nu^2x_n^2\right]^{\frac{1}{4}} \cos\left(\gamma+\frac{\beta}{2}\right)$$
$$= \frac{\sqrt{2\nu x_n}}{2\left(\nu^2+r^2\right)} \frac{1}{\sqrt{\sin\beta}} \frac{1}{\cos\gamma} \cos\left(\gamma+\frac{\beta}{2}\right). \tag{10.56}$$

10.3 Superfluid Densities and Currents

Eq. (he-11a.60) By constructing, the current density in the presence of the external source (10.1) is

$$-\frac{\partial g^{\nu}}{\partial v}. \tag{10.57}$$

It must, however, be noted that this is not the full current of the system. In calculating g^v we have assumed the field A^0_{ai} to be a constant in space (and time). In this way the Cooper pairs have been forced artificially to remain immobile. In full thermal equilibrium also they would want to follow the drag of the external source

Eq. (he-11a.61) and in the ground state there would be a phase modulation

$$e^{2im\mathbf{v}x}A^0_{ai} \tag{10.58}$$

rather than A^0_{ai}. Intuitively speaking, the free energy g^v allows for the movement of only the quasiparticles which are *not* bound in Cooper pairs, i.e., the normal component of the superfluid. The current associated with this flow is given a subscript

Eq. (he-11a.62) n:

$$J_n \equiv -\left.\frac{\partial g^v}{\partial \nu}\right|_{A^0_{ai}} = \text{const} . \tag{10.59}$$

Eq. (he-11a.63) The corresponding density of the normal component is defined by

$$J_n \equiv \rho_n v. \tag{10.60}$$

Eq. (he-11a.64) Since the full current would be

$$J = \varrho v \tag{10.61}$$

Eq. (he-11a.65) we may deduce that the difference

$$J_s \equiv J - J_n = (\rho - \rho_n)\, v = \rho_s v \tag{10.62}$$

may be attributed to the flow of Cooper pairs. The quantity J_s is now the supercurrent with ρ_s being the superfluid density of the liquid. According to (10.8), ρv may be obtained from $-\partial g^v_0/\partial v$ such that J_s is obtained directly from the derivative of

Eq. (he-11a.66) the condensation energy

$$J_s = \frac{\partial g^v_s}{\partial v} = \frac{\partial g^v}{\partial v} - \frac{\partial g^v_0}{\partial v} = \frac{\partial g^v}{\partial v} - (\Delta =)\,. \tag{10.63}$$

Eq. (he-11a.67) Using (10.6) we may perform the differentiation and find

$$\begin{aligned} J_s &= -T\sum_{\omega_n,\mathbf{p}} p_F z\left(\frac{1}{i\omega_n + vp_F - E} + (E \to -E)\right) - (\Delta = 0) \\ &= \frac{3}{2}\int_{-1}^{1}\frac{dz}{2}\int_{-\infty}^{\infty} d\xi\left[\left(\tanh\frac{E - vp_F z}{2T} - (v \to -v)\right) - (\Delta = 0)\right] \end{aligned} \tag{10.64}$$

where we have inserted $\mathcal{N}(0) = \frac{3}{2}\rho/oF^2$. For numerical evaluations it is more convenient to keep the Matsubara sum (10.64) and perform only the momentum integration. Then we find

Eq. (he-11a.69)

$$\begin{aligned} J_s &= \frac{3\rho}{p_F}\int_{-1}^{1}\frac{dz}{2}\int_{-\infty}^{\infty}d\xi T\sum_{\omega_n}\frac{i\omega_n + vp_F z}{\left(\omega_n - ivp_F z\right)^2 + \Delta_\perp^2\left(1-r^2z^2\right)+\xi^2} \\ &= \frac{3\rho}{p_F}\pi T\int_{-1}^{1}\frac{dz}{2}z\mathrm{Re}\sum_{\omega_n}\frac{i\omega_n + vp_F z}{\sqrt{\left(\omega_n - vp_F z\right)^2 + \Delta_\perp^2\left(1-r^2z^2\right)}} \\ &= \left[\frac{6}{\delta\nu}\int_{-1}^{1}\frac{dz}{2}z\mathrm{Re}\sum_{n=0}^{\infty}\frac{ix_n + vz}{\sqrt{\left(x_n - i\nu z\right)^2 + 1 - r^2z^2}}\right]\rho v \\ &= \rho_s v \end{aligned} \tag{10.65}$$

Thus we can identify the superfluid density parallel to the flow as Eq. (he-11a.70)

$$\begin{aligned} \frac{\rho_s^{\parallel}}{\rho} &= \frac{3}{2}\frac{1}{p_F v}\int_{-1}^{1}\frac{dz}{2}\int_{-\infty}^{\infty}d\xi\left[\left(\tanh\frac{E - vp_F z}{2T} - (v\to -v)\right) - (\Delta = 0)\right] \\ &= \frac{6}{\delta\nu}\int_{-1}^{1}\frac{dz}{2}z\mathrm{Re}\sum_{n=0}^{\infty}\frac{ix_n + \nu z}{\sqrt{\left(x_n - i\nu\right)^2 + 1 - r^2z^2}}. \end{aligned} \tag{10.66}$$

In the second expression, the integral over z yields Eq. (he-11a.71)

$$\begin{aligned} \frac{\rho_s^{\parallel}}{\rho} &= \frac{3\rho}{\delta\nu}\sum_{n=0}^{\infty}\left\{\frac{\nu}{\left(\nu^2+r^2\right)^2}\left(\nu^2 + r^2 + 3r^2x_n^2\right)\frac{\alpha_n}{\sqrt{\nu^2+r^2}}\right. \\ &\left. -\mathrm{Re}\,\frac{2}{\left(\nu^2+r^2\right)}\left(ix_n + \frac{1}{2}\nu\left(1 - 3i\frac{\nu x_n}{\nu^2+r^2}\right)\right)\sqrt{\left(x_n - i\nu\right)^2 + 1 - r^2}\right\}. \end{aligned} \tag{10.67}$$

If we neglect gap distortion, $r = 1$, we recover the result of previous calculations Eq. (he-)

$$\begin{aligned} \frac{\rho_s}{\rho} &\overset{r\equiv 0}{\equiv} \frac{3}{\delta\nu^3}\sum_{n=0}^{\infty}\left\{\alpha_n - \frac{1}{\sqrt{2}}\left[\left(v^2 - 9x_n^2\right)\left(1 - \nu^2 + x_n^2\right) - 12\nu^2x_n^2\right.\right. \\ &\left.\left. - \left(\nu^2 9x_n^2\right)\sqrt{\left(1+\nu^2+x_n^2\right)^2 - 4\nu^2}\right]\right\}. \end{aligned} \tag{10.68}$$

For $r = 1$, the last term is simplified to Eq. (he-)

$$\frac{-3x\nu}{\left(\nu^2+1\right)^2} \tag{10.69}$$

and $\rho_s^{\parallel}$ reduces to the expression for the A-phase: Eq. (he-11a.72)

$$\frac{\rho_s^{\parallel}}{\rho}\mathrm{A}\ \underline{\underline{\mathrm{phase}}} \tag{10.70}$$

$$\frac{3}{\delta}\frac{1}{\left(\nu^2+1\right)^{\frac{\Sigma}{2}}}\sum_{n=0}^{\infty}\left[\left(\nu^2 + 1 + 3x_n^2\right)\arctan\frac{\sqrt{\nu^2+1}}{x_n} - 3\sqrt{\nu^2+1}x_n\right].$$

In general, the real part has two terms. The second coincides with $-2\nu\times$ the corresponding term in the transverse gap equation (10.67). The first may be rewritten in terms of the angle β of (10.55) as Eq. (he-)

$$-\mathrm{Re}\,\frac{2ix_n}{\nu^2+r^2}\sqrt{(x_n-iv)^2+1-r^2} \qquad (10.71)$$

$$= -\mathrm{Re}\,\frac{2ix_n}{\nu^2+r^2}\left[\left(1+x_n^2-\nu^2-r^2\right)^2+4\nu^2x_n^2\right]^{\frac{1}{4}}e^{-i\beta/2}$$

$$= -\frac{2x_n}{\nu^2+r^2}\sqrt{vx_n\tan\beta/2}. \qquad (10.72)$$

?? Eq. (he-11a.72a) Let us compare the result with our Ginzburg-Landau calculation in Sec. 5.? For $T\sim T_c$, we may take the limit $x_n\to\infty$ and remain with

$$\frac{\rho_s^\|}{\rho} \approx \frac{6}{\delta}\int_{-1}^{1}\frac{dz}{2}z^2\left(1-r^2z^2\right)\sum_{n=0}^{\infty}\frac{1}{x_n^3} = 6\delta^2\left(\frac{1}{3}-\frac{r^2}{5}\right)\frac{7\zeta(3)}{8}. \qquad (10.73)$$

Eq. (he-11a.73) This coincides with our previous result if we insert (10.27)

$$\frac{\rho_s^\|}{\rho} = \frac{3}{\nu}\int_{-1}^{1}\frac{dz}{2}\int_0^\infty dx\frac{ix+\nu z}{\sqrt{(x-i\nu z)^2+1-r^2z^2}}$$

$$= \frac{3}{\nu}\int_{-1}^{1}\frac{dz}{2}z\left[\nu z-\mathrm{Re}\,i\sqrt{1-\left(\nu^2+r^2\right)z^2-i\epsilon\nu z}\right]. \qquad (10.74)$$

The square root gives a contribution only for $z^2>\frac{1}{\nu^2+r^2}$ so that $\rho_s^\|$ remains equal to ρ until $\nu^2=1-r^2$.

Since the upper branch of the gap is isotropic up to $\nu=1$, there is also an upper Eq. (he-11a.74) branch with $\rho_s^\|\equiv\rho$ up to $\nu=1$. On the lower branch one has $\nu^2>1-r^2$ and

$$\frac{\rho_s^\|}{\rho} =_{T=0} 1-\frac{3}{\nu}\mathrm{Re}\int_{1/(\nu^2+r^2)}^{1}dzz\sqrt{\left(\nu^2+r^2\right)z^2-1}$$

$$= 1-\theta\left(\nu^2+r^2-1\right)\frac{1}{\nu^2\left(\nu^2+r^2\right)}\sqrt{\nu^2+r^2-1}^3. \qquad (10.75)$$

where $\theta(z)$ is the Heaviside function. This result agrees with those of B- and A-phases for $r=0$ and 1, respectively.

10.4 Critical Currents

In the Ginzburg-Landau regime, the critical currents are known from Ch. 5. These results agree with the present calculation since $\rho_s^\|$ of (10.74) is the same as before.

In the opposite limit $T\to 0$ an exact calculation is difficult but the current can be fixed to a high accuracy by the following consideration:

Due to the distortion of the gap, the current J_s as a function of v must be below the current calculated by neglecting distortion. Now, up to $\nu=1$, both currents are

identical since the gap distortion was derived to be zero for $\nu \leq 1$. Thus $J_s(v)$ is
Eq. (he-) known to reach the value

$$J_s(v)|_{\nu=1} = \rho v_{\nu=1} = \rho \frac{\Delta_{\nu=1}}{p_F} \tag{10.76}$$

and, since also $\Delta_{\nu=1} = \Delta_{BCS}$ at $T = 0$ (see the text after (10.35)), we have the lower bound the critical current Eq. (he-11a.76)

$$J_s(v) \geq \rho \frac{\Delta_{BCS}}{p_F} \approx .47 J_0. \tag{10.77}$$

As an upper bound we may use the maximum of $J_s^{Bu}(v)$ which can easily be calculated exactly. We shall see in a moment that the critical velocity is determined by

Eq. (he-11a.77)

$$\nu_c = \frac{1}{\sqrt{1-(2^{1/3}-1)^2}} \approx 1.036. \tag{10.78}$$

Inserting this into the superfluid density (10.75) at $r = 0$ we find Eq. (he-11a.78)

$$\left.\frac{\rho_s^{Bu}}{\rho}\right|_{\nu_c} = 1 - \left(2^{1/3}-1\right)^3 \approx .982. \tag{10.79}$$

Thus leads to a critical current Eq. (he-11a.79)

$$J_c^{Bu} = \rho \left(1 - \left(2^{1/3}-1\right)^3\right) \frac{1}{\sqrt{1-(2^{1/3}-1)^2}} \frac{\Delta_{B^n}|_{\nu_c}}{p_F}. \tag{10.80}$$

But the gap at ν_c can be evaluated from (10.39) with the result Eq. (he-11a.80)

$$\begin{aligned} \log \frac{\Delta_B^{\parallel}}{\Delta_{BCS}}|_{\nu_c} &= -\log\left(\nu_c + \sqrt{\nu_c^2-1}\right) + \sqrt{1-\frac{1}{v_c^2}} \\ &= \log \nu_c + \log\left(1-\left(2^{1/3}-1\right)\right) + \left(2^{1/3}-1\right) \end{aligned} \tag{10.81}$$

so that Eq. (he-11a.81)

$$\Delta_B|_{\nu_c} \underset{T=0}{=} \Delta_{BCS} e^{2^{1/3}-1} \left[1-\left(2^{1/3}-1\right)\right] \nu_c. \tag{10.82}$$

Thus we find, altogether, a critical current Eq. (he-11a.82)

$$\begin{aligned} J_c^{Bu} &= \left[1-\left(2^{1/3}-1\right)^3\right] 2^{-1/3} e^{\left(2^{1/3}-1\right)} \frac{\Delta_{BCS}}{p_F} \rho \\ &\approx 1.0112 \frac{\Delta_{BCS}}{p_F} \rho \approx .486 J_0. \end{aligned} \tag{10.83}$$

This lies only very slightly above the lower bound. Therefore the true critical current including the effect of gap distortion is determined extremely well by Eq. (he-11a.83)

$$.470 J_0 \leq .486 J_0. \tag{10.84}$$

Notice that the critical velocity in B'' is Eq. (he-11a.84)

$$\begin{aligned} v_c &= v_c \frac{\Delta}{\Delta_{BCS}} \frac{\Delta_{BCS}}{p_F} \\ &= 2^{1/3} e^{\left(2^{1/3}-1\right)} \frac{\Delta_{BCS}}{p_F} \\ &\approx 1.029 \frac{\Delta_{BCS}}{p_F} \approx .48 v_0 \end{aligned} \tag{10.85}$$

i.e., it is reached immediately above $\nu = 1$ so that the true v_c lies between $.47 v_0$ and $.48 v_0$.

Eq. (he-11a.85) Let us now derive (10.78). Certainly, the maximum of the current has to lie at

$$\frac{d}{dv} J_s = \frac{d}{dv} \rho_s(v) + \rho_s(v) = 0. \tag{10.86}$$

In general, ρ_s is a function of ν, δ, T where ρ is itself a function of ν and T via gap
Eq. (he-11a.86) equation:

$$\log \frac{T}{T_c} = \gamma(\delta, \nu). \tag{10.87}$$

Eq. (he-11a.85) We can therefore express the derivative at fixed T as

$$\frac{\partial}{\partial v} = \frac{\partial \nu}{\partial v} \left(\frac{\partial \delta}{\partial \nu} \frac{\partial}{\partial \delta} + \frac{\partial}{\partial \nu} \right). \tag{10.88}$$

But since $\nu = \frac{v p_F}{\Delta_\perp}$ we have

$$\frac{\partial \nu}{\partial v} = \frac{p_F}{\pi T} \left(\frac{1}{\delta} - \frac{1}{\delta^2} \frac{\partial \nu}{\partial v} \frac{\partial \delta}{\partial \nu} \right) \tag{10.89}$$

Eq. (he-11a.86) or

$$\frac{\partial \nu}{\partial v} = \frac{\nu}{v} \frac{1}{1 + \frac{v}{\delta} \frac{\partial \delta}{\partial \nu}}. \tag{10.90}$$

Such that the extremal condition (10.88) may be written in terms of the natural
Eq. (he-11a.87) variables as

$$\left(\frac{\partial \delta}{\partial \nu} \frac{\partial}{\partial \delta} + \frac{\partial}{\partial \nu} \right) \rho_s + \left(\frac{1}{\delta} \frac{\partial \delta}{\partial \nu} + \frac{1}{\nu} \right) \rho_s = 0. \tag{10.91}$$

The derivative $\frac{\partial \delta}{\partial \nu}$ may be taken from (10.90) as $-\left(\partial \gamma / \partial \nu\right) / \left(\partial \gamma / \partial \delta\right)$,

We want to apply condition (10.91) to the case of zero temperature. then ρ_s becomes independent of δ (which diverges to ∞) and the first term in (10.91) is
Eq. (he-11a.88) absent. The gap equation (10.90), on the other hand, has the form

$$\log \frac{\Delta}{\Delta_{BCS}} = \gamma_0(\nu) \tag{10.92}$$

Eq. (he-11a.89) so that

$$\frac{1}{\delta}\frac{\partial \delta}{\partial \nu} = \frac{\partial \gamma_0}{\partial \nu}. \tag{10.93}$$

The critical velocity at $T = 0$ is therefore obtained from the simple relation Eq. (he-11a.90)

$$\left[\nu \frac{\partial \rho_s}{\partial \nu} + \left(v \frac{\partial \gamma_v}{\partial_\nu} + 1\right) \rho\right]_{\nu_c} = 0. \tag{10.94}$$

For the B-phase neglecting gap distortion we see from (10.75) Eq. (he-11a.91)

$$\frac{\partial \rho_s^{Bu}}{\partial \nu} = -\theta(\nu - 1)\frac{3}{\nu^4}\sqrt{\nu^2 - 1}. \tag{10.95}$$

The gap function γ_0 is taken from (10.39) so that Eq. (he-11a.92)

$$\frac{\partial \gamma_0}{\partial \nu} = -\frac{\sqrt{v^2 - 1}}{\nu^2}\theta(\nu - 1). \tag{10.96}$$

The condition (10.94) becomes Eq. (he-11a.93)

$$-3\nu \frac{\sqrt{\nu^2 - 1}}{\nu^4} + \left(1 - \frac{\sqrt{\nu^2 - 1}}{\nu}\right)\left(1 - \frac{\sqrt{\nu^2 - 1}^3}{\nu^3}\right) = 0. \tag{10.97}$$

This awkward equation is solved by setting $y \equiv \sqrt{1 - \frac{1}{\nu^2}}$ and rewriting Eq. (he-11a.94)

$$y^3 + 3y^2 + 3y - 1 = 0 \tag{10.98}$$

which has as the only real solution Eq. (he-11a.95)

$$y = 2^{1/3} - 1 \tag{10.99}$$

verifying the critical current of the previous discussion (10.78)- (10.85(he-11a.84)).

For comparison we may use (10.94) to derive also the depairing critical current for the A-phase. From (10.75) and (10.44(he-11a.42)) for $r = 1$ we see Eq. (he-11a.96)

$$\frac{\rho_s^{\parallel}}{\rho} = \frac{1}{1 + \nu^2} \tag{10.100}$$

$$\log \frac{\Delta_\perp}{\Delta_{BCS}} = \gamma_0(v) = \log \frac{e^{5/6}}{2} - \frac{1}{2}\log\left(1 + \nu^2\right) + \frac{\nu^2}{2\left(1 + \nu^2\right)} \tag{10.101}$$

which inserted into (10.94) gives Eq. (he-11a.97)

$$-\frac{2\nu^2}{\left(1 + \nu^2\right)^2} + \frac{1}{\left(1 + \nu^2\right)}\left(1 - \frac{\nu^4}{\left(1 + \nu^2\right)^2}\right) = 0. \tag{10.102}$$

This is solved by $\nu_c^2 = \frac{1}{\sqrt{2}}$. Thus the critical current is Eq. (he-11a.98)

$$\begin{aligned} J_c^A &= \frac{\rho_s^{\parallel}}{\rho} \nu_c \frac{\Delta_\perp}{\Delta_{BCS}} \frac{\Delta_{BCS}}{p_F} \rho \\ &= \left(\frac{\sqrt{2}}{\sqrt{2}+1} \right) \frac{1}{2^{1/4}} \left(\frac{e^{5/6}}{2} \sqrt{\left(\sqrt{2}-1\right)\sqrt{2}} e^{\frac{\sqrt{2}-1}{2}} \right) \frac{\Delta_{BCS}}{p_F} \rho \\ &= \sqrt{2} \left(\sqrt{2}-1\right)^{3/2} e^{\frac{\sqrt{2}-1}{2}} \frac{e^{5/6}}{2} \frac{\Delta_{BCS}}{p_F} \rho \\ &\approx .534 \frac{\Delta_{BCS}}{p_F} \rho \approx .25 J_0 \end{aligned} \tag{10.103}$$

Eq. (he-11a.99) with a critical velocity

$$\begin{aligned} v_c = \frac{J_c}{\rho_s^{\parallel}} &= \sqrt{\sqrt{2}-1} e^{\frac{\sqrt{-1}}{2}} \frac{e^{5/6}}{2} \frac{\Delta_{BCS}}{p_F} \\ &\approx .911 \frac{\Delta_{BCS}}{p_F} \approx .428 v_0 \end{aligned} \tag{10.104}$$

10.5 Ground State Energy at Large Velocities

Let us now consider the superfluid in motion. As before, we imagine bringing the liquid adiabatically from $v = 0$ to its actual velocity. This will result in an additional
Eq. (he-10.100) energy

$$g_c^{(v)} = g_c|_{v=0} + \int_0^v dv' \rho_s^{\parallel}(v') v' \tag{10.105}$$

where $g|_{v=0}$ is the previously calculated condensation energy f_c and r_c'' is the superfluid density parallel to the flow (**??**). Alternatively, we may write for the total
Eq. (he-11a.101) energy, including the moving free fermion part, $g_0 = f_0 - \frac{\rho}{2} v^2$,

$$q = f_0 + f_c - \int_0^v dv' \rho_n^{\parallel}(v') v' \tag{10.106}$$

Eq. (he-11a.102) where

$$\rho_n^{\parallel} = \rho - \rho_s^{\parallel} \tag{10.107}$$

is the density of the normal component of the liquid.

10.6 Fermi Liquid Corrections

With (**??**) the expression for the energy reaches a convenient form which permits the inclusion of the quantitatively very important Fermi liquid corrections due to the current-current coupling (11.11). By considering the form (10.91(he-11a.87)) in which the corresponding molecular fields φ_i enter into the collective action we realize that they appear on the same footing as the velocity v of the liquid (see (10.3)).

In equilibrium we expect a constant nonzero mean molecular field. For symmetry reasons, only the field parallel to the flow can contribute. Therefore we may substitute simply $\mathbf{v} \to \mathbf{v} + \varphi$ in (10.12) and add, after this, the quadratic piece of (**??**). Then the energy, corrected by the constant mean molecular field, may be written as Eq. (he-11a.103)

$$g^* = \min_{\varphi} \left[f_0 + f_c - \int_0^{\mathbf{v}+\varphi} dv' \rho_n^{\parallel}(v')v' - \tfrac{1}{2}\rho \tfrac{1}{F_1^s/3}\varphi^2 \right]. \tag{10.108}$$

Differentiating with respect to φ we see that the minimum lies at the mean field Eq. (he-11a.104)

$$\varphi = -\frac{F_1^s}{3}\frac{\rho_n(v+\varphi)}{\rho}(\mathbf{v}+\varphi) \tag{10.109}$$

Inserting this back into (10.108) the energy becomes an explicit function of the quantity Eq. (he-11a.105)

$$v^* \equiv v + \varphi \tag{10.110}$$

which may be interpreted as the local fluid velocity felt by the quasi-particles including the effects of the molecular field. In terms of v^*: Eq. (he-11a.106)

$$\begin{aligned} g^* &= f_0 + f_c - \int_0^{v^*} dv' \rho_n^{\parallel}(v')v' - \frac{1}{2}\frac{F_1^s/3}{\rho}\rho_n^{\parallel\,2}(v^*)v^{*2} \\ &= f_0 + f_c - \int_0^{v} dv' J_n(v') - \frac{1}{2}\frac{F_1^s/3}{\rho} J_n^2(v^*). \end{aligned} \tag{10.111}$$

Given an arbitrary physical velocity v, the quantity v^* may be found from (10.109), which can be rewritten in the form Eq. (he-11a.107)

$$\left(1 + \frac{F_1^s}{3}\frac{\rho_n(v^*)}{\rho}\right) v^* = v. \tag{10.112}$$

Expression (10.111) allows for a calculation of the corrected supercurrent and superfluid density. By differentiation with respect to v we find Eq. (he-11a.108)

$$\begin{aligned} J_n^*(v) &= -\frac{\partial g^*}{\partial v} = J_n(v^*)\frac{\partial v^*}{\partial v} + \frac{F_1^s/3}{\rho} J_n(v^*)\frac{\partial J_n(v^*)}{\partial v} \\ &= J_n(v^*) \left\{ \left[1 + \frac{F_1^s/3}{\rho}\frac{\partial J_n}{\partial v^*} \right] \frac{\partial v^*}{\partial v} \right\}. \end{aligned} \tag{10.113}$$

By writing (10.112) in the form Eq. (he-11a.109)

$$v^* + \frac{F_1^s/3}{\rho} J_n(v^*) = v \tag{10.114}$$

we see that the factor in curly brackets is unity. Hence the Fermi liquid corrected current equals the uncorrected one except for its being evaluated at the local velocity v^* rather than the physical v:

Eq. (he-11a.110)

$$J_n^*(v) \equiv \rho_n^*(v)v = J_n(v^*) = \rho_n(v^*)v^* \tag{10.115}$$

As in Ch. 6 we have found it convenient to introduce ρ_n^* as the corrected density of the normal component

Eq. (he-11a.111)

$$\rho_n^*(v) \equiv \rho_n(v^*)\frac{v^*}{v} = \frac{\rho_n(v^*)}{1+\frac{F_1^s}{3}\frac{\rho_n}{\rho}(v^*)} \tag{10.116}$$

which is reduced with respect to ρ_n by the ratio of v^* and v. The same reduction appears in the superfluid density. Here we have to subtract the normal current from the total one, ρv. In order to do so we have to remember that ρ contains the true mass of the ^{3}He atoms $m = m^3He$ while all quantities derived from the original action involve the effective mass $m^* = \left(1+\frac{F_1^s}{3}\right)m$.

Eq. (he-11a.112)

Therefore the supercurrent is given by

$$\begin{aligned}
J_s^*(v) &= \rho v - J_n^*(v) \\
&= \rho v - \rho_n(v^*)\,v^* \\
&= \rho v - \rho\frac{m^*}{m}\frac{\rho_n(v^*)}{\rho}\frac{v}{1+\frac{F_1^s}{3}\frac{\rho_n(v^*)}{\rho}} \qquad (10.117)\\
&= \rho v 1 - \left(1+\frac{F_1^s}{3}\right)\frac{\rho_n(v^*)}{\rho}\frac{1}{1+\frac{F_1^s}{3}\frac{\rho_n(v^*)}{\rho}}\frac{1}{1+\frac{F_1^s}{3}\frac{\rho_n(v^*)}{\rho}} \\
&= \rho v\frac{\rho_s(v^*)}{\rho}\frac{1}{1+\frac{F_1^s}{3}\frac{\rho_n(v^*)}{\rho}}. \qquad (10.118)
\end{aligned}$$

Thus the effect of Fermi-liquid corrections is to reduce the superfluid fraction ρ_s/ρ by a factor $1+\frac{F_1^s}{3}\frac{\rho_n(v^*)}{\rho}$ and to change the velocity coordinate from v to v^*.

Eq. (he-11a.113)

$$\frac{\rho_s^{\rm FL}(v)}{\rho} = \frac{\rho_s(v^*)}{\rho}\frac{1}{1+\frac{F_1^s}{3}\frac{\rho_n(v^*)}{\rho}} \tag{10.119}$$

Notice that for small velocities the integral in (10.111) may be performed so that g^* becomes simply

Eq. (he-11a.114)

$$\begin{aligned}
g^* &= f_0 + f_c - \frac{\rho}{2}v^2 + \frac{1}{2}\rho\frac{\rho_s(v^*)}{\rho}\frac{1}{1+\frac{F_1^s}{3}\frac{\rho_n(v^*)}{\rho}}v^2 \\
&= f_0 + f_c - \frac{\rho_n^*(v)}{2}v^2. \qquad (10.120)
\end{aligned}$$

As far as our Figures 1 - 5 are concerned we learn that in ∂, $\Delta^\parallel$, $\Delta^\perp$, the curves remain the same except that the v axis has to be read as v^*. The same statement holds for $\rho^\parallel$ and $\rho^\perp$ which, in addition, are reduced by the factors $\frac{1}{1+\frac{F_1^s}{3}\frac{\rho_n(v^*)}{\rho}}$, respectively.

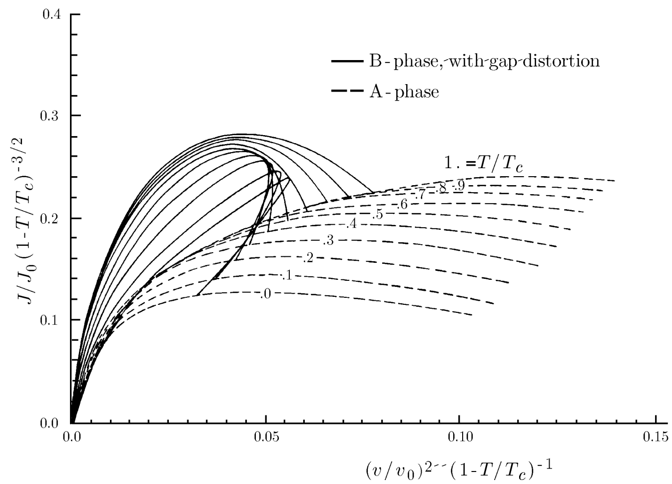


FIGURE 10.2 Current as a function of the velocity.

If experimentally it is the velocity v which is given rather than the current, the corresponding quantity v^* can easily be extracted from the $J,\ v^*$ plot (see Fig. 10.2) graphically: By rewriting Eq. (**??**) as Fig. f5

$$\frac{J_s\left(v^*\right)}{\rho} = \frac{1+\frac{F_1^s}{3}v^*}{F_1^s/3} - \frac{v}{F_1^s/3} \tag{10.121}$$

Eq. (he-11a.115)

we see that at any value of $F_1^s/3$ we may draw a straight line of slope $\frac{1+F_1^s}{3}/F_1^s/3$ and intercept $-v/F_1^s/3$. It intersects the $\frac{\partial_s}{\rho}$ curves at $v^*,\ J_s\left(v^*\right)$. The same statement holds for the reduced quantities $J/\partial_s\left(1-\frac{T}{T_c}\right)^{-3/2},\ \ \frac{v}{v_0}\left(1-\frac{T}{T_c}\right)^{-\frac{1}{2}}$ except that the right-hand side carries a factor $\frac{1}{2\left(1-\frac{T}{T_c}\right)}$.

The Fermi liquid corrections have the pleasant property of removing the double valuedness of the variables when plotted as a function of v rather than v^*. The reason is that the lower branch of $\rho_s^{\parallel}\left(v^*\right)$ corresponds, via (10.114), to a higher physical velocity v at the same v^*. This has the effect of rotating all lower branches with positive slope anticlockwise until their slopes are negative. In this all curves become single valued even at zero pressure where F_1^s takes its smallest value $\frac{F_1^s}{3} \approx 2$.

11

Collective Modes in Presence of Current at all Temperatures $T < T_c$

In Sec. 5.6 we have seen that in the neighborhood of the critical point the distorted gap parametrization of the constant order parameter (9.7) is stable under small space and time-independent fluctuations. Here we want to extend this consideration to all temperatures below T_c. For simplicity we shall only consider the weak coupling limit in which the currents were discussed in Sec. 5.6.

11.1 Quadratic Fluctuations

Eq. (he-12.1) Let us parametrize the fluctuations around the extremal field configuration by

$$A_{ai} = A^0_{ai} + A'_{ai}. \tag{11.1}$$

Inserting this into the collective action (9.3) and expanding in powers of A' up to quadratic order we find Eq. (he-12.2)

$$\begin{aligned}\delta^2\mathcal{A} &= -\frac{i}{4}\mathrm{Tr}\left[G^v\begin{pmatrix}0 & A'_{ai}\sigma_a i\tilde{\nabla}_i\\ A'^{\,*}_{ai}\sigma_a i\tilde{\nabla}_i & 0\end{pmatrix}G^v\begin{pmatrix}0 & A'_{ai}\sigma_a i\tilde{\nabla}_i\\ A'^{\,*}_{ai}\sigma_a i\tilde{\nabla}_i & 0\end{pmatrix}\right]\\ &\quad -\frac{1}{3}\int d^4x|A'_{ai}|^2.\end{aligned} \tag{11.2}$$

There are no linear terms in A' due to the gap equations (9.9), (9.10(he-11.11)). We have introduced a 4×4 matrix Eq. (he-12a.3)

$$G^v = i\begin{pmatrix} i\partial_t + \mathbf{v}\boldsymbol{\nabla} - \xi & A^0_{ai}\sigma_a\tilde{\nabla}_i\\ A^0_{ai}{}^*\sigma_a\tilde{\nabla}_i & i\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla} + \xi\end{pmatrix}^{-1} \tag{11.3}$$

which is the propagator of the pair of Fermi field $(\psi, c\psi^*)$ in the presence of the constant pair field A^0_{ai}. Its Fourier transform may be inverted explicitly as Eq. (he-13.4)

$$G^v(\epsilon, p) = \frac{i}{-(\epsilon+\mathbf{pv})^2 + E^2}\begin{pmatrix}\epsilon + \mathbf{vp} - \xi(\mathbf{p}) & -A^0_{ai}\sigma_a\tilde{p}_i\\ -A^{0*}_{ai}\sigma_a\tilde{p}_i & \epsilon + \mathbf{vp} + \xi(\mathbf{p})\end{pmatrix}. \tag{11.4}$$

We now pass from quantum mechanics to quantum statistics at constant temperature T by replacing the integral over energies $\int d\epsilon/2\pi$ by sums over Matsubara frequencies Eq. (he-13.5)

$$\int \frac{d\epsilon}{2\pi} \to iT \sum_{\omega_n} \tag{11.5}$$

replacing everywhere ϵ by $i\omega_n = i(2n+1)\pi T$. Correspondingly, we decompose the fluctuations of the pair field as usual, Eq. (he-13.6)

$$A'(x,\tau) = T \sum_{\nu_n} \int \frac{d^3k}{(2\pi)^3} e^{-i(\tau\nu_n - \mathbf{kx})} A'(\nu_n, \mathbf{k}) \tag{11.6}$$

with bosonic Matsubara frequencies Eq. (he-13.8)

$$\nu_n = 2n\pi T. \tag{11.7}$$

With the short notation

$$T \sum_{\nu_n} \int \frac{d^3k}{(2\pi)^3} f(\nu_n, \mathbf{k}) = T \sum_k f(k)$$

the quadratic piece (**??**) may be written as Eq. (he-13.9)

$$i\delta^2\mathcal{A} \approx T \sum_k \left\{ T \sum_p \frac{1}{2} \left[G\left(p - \frac{k}{2}\right) \begin{pmatrix} 0 & A'(-k)\sigma_a\hat{p}_i \\ A'^*(k)\sigma_a\hat{p}_c & 0 \end{pmatrix} \right.\right. \tag{11.8}$$
$$\left.\left. \times G\left(p + \frac{k}{2}\right) \begin{pmatrix} 0 & A'(k)\sigma_a\hat{p}_i \\ A'^*(-k)\sigma_a\hat{p}_i & 0 \end{pmatrix} \right] - \frac{1}{3g} |A'_{ai}(k)|^2 \right\}.$$

Here we have collected again frequency ν_n and momentum $\mathbf{k}$ in a single four-vector symbol k. Also, by restricting our consideration to long wavelengths with $k \ll p_F$ only, we have set $\widetilde{(p \pm k)} \approx \hat{p}_i$. After a little matrix algebra, the fluctuations can be written as Eq. (he-13.10)

$$i\delta^2\mathcal{A} \equiv -\frac{1}{2} T \sum_k \left(A'^*_{ai}(k), A'_{ai}(-k) \right) \mathcal{F}^{ij,ab}_{ab}(k) \begin{pmatrix} A'(k) \\ A'^*(-k) \end{pmatrix}_{bj} \tag{11.9}$$

with the matrix Eq. (he-)

$$\mathcal{F}^{ij,ab}(k) \equiv \begin{pmatrix} F^{ij}_{11}(k)\delta^{ab} & F^{ij,ab}_{12}(k) \\ F^{ij,ab}_{21}(k) & F^{ij}_{22}(k)\delta^{ab} \end{pmatrix} \tag{11.10}$$

whose coefficients $F_{\alpha\beta}(k)$ involve the four 2×2 submatrices of G^v Eq. (he-)

$$G^v(k) = \begin{pmatrix} G^v_{11}(k) & G^v_{12}(k) \\ G_{21}(k) & G^v_{22}(k) \end{pmatrix} \tag{11.11}$$

as follows: Eq. (he-13.12)

$$
\begin{aligned}
F_{11}^{ij}(k)\delta^{ab} &\approx \frac{1}{2}\underset{2\times 2}{\mathrm{tr}}\, T\sum_p G_{22}^v\left(p-\frac{k}{2}\right)\sigma_a\hat{p}_i G_{11}^v\left(p+\frac{k}{2}\right)\sigma_b\hat{p}_j,\\
F_{22}^{ij}(k)\delta^{ab} &\approx \frac{1}{2}\underset{2\times 2}{\mathrm{tr}}\, T\sum_p G_{11}^v\left(p-\frac{k}{2}\right)\sigma_a\hat{p}_i G_{22}^v\left(p+\frac{k}{2}\right)\sigma_b\hat{p}_j, \qquad (11.12)\\
F_{12}^{ij}(k)\delta^{ab} &\approx \frac{1}{2}\underset{2\times 2}{\mathrm{tr}}\, T\sum_p G_{12}^v\left(p-\frac{k}{2}\right)\sigma_a\hat{p}_i G_{12}^v\left(p+\frac{k}{2}\right)\sigma_b\hat{p}_j.\\
F_{21}^{ij}(k)\delta^{ab} &\approx \frac{1}{2}\underset{2\times 2}{\mathrm{tr}}\, T\sum_p G_{21}^v\left(p-\frac{k}{2}\right)\sigma_a\hat{p}_i G_{21}^v\left(p+\frac{k}{2}\right)\sigma_b\hat{p}_j.
\end{aligned}
$$

Eq. (he-13.13) Using (11.4) we find

$$
\begin{aligned}
F_{11}^{ij}(k) &= T\sum_p \frac{(i\tilde{\omega}_+ - \xi_+)\,(i\tilde{\omega}_- + \xi_-)}{(\tilde{\omega}_+^2 + E_+^2)\,(\tilde{\omega}_-^2 + E_-^2)}\hat{p}_i\hat{p}_j + \frac{1}{3g}\delta_{ij}\\
F_{22}^{ij}(k) &= F_{11}^{ij}(-k) \qquad (11.13)\\
F_{12}^{ij,ab}(k) &= T\sum_p \frac{1}{(\tilde{\omega}_+^2 + E_+^2)\,(\tilde{\omega}_-^2 + E_-^2)}\hat{p}_i\hat{p}_j\hat{p}_k\hat{p}_l t_{aba'b'}A_{a'k}^0 A_{b'l}^0\\
F_{21}^{ij,ab}(k) &= F_{12}^{ij,ak}(-k)^* \qquad (11.14)
\end{aligned}
$$

Eq. (he-13.15) either the tensor $t_{aba'b'}$ being again the trace

$$
t_{aba'b'} \equiv \frac{1}{2}\mathrm{tr}\left(\sigma^{a'}\sigma^a\sigma^{b'}\sigma^{b'}\right) = \delta_{aa'}\delta_{bb'} + \delta_{ab'}\delta_{ba'} - \delta_{ab}\delta_{a'b'} \qquad (11.15)
$$

Eq. (he-13.16) and $\xi_\pm$, $\tilde{\pi}_\pm$,$E_\pm$ abbreviating

$$
\begin{aligned}
\xi_\pm &= \xi \pm v_F\hat{\mathbf{p}}\mathbf{k},\\
\widetilde{\omega_\pm} &\equiv \omega_n \pm \nu/2 - i\mathbf{v}\hat{\mathbf{p}}p_F, \qquad (11.16)\\
E_\pm &= E\,(p\pm k/2) \equiv \sqrt{(\xi \pm v_F\hat{\mathbf{p}}\mathbf{k})^2 + |A_{ai}\hat{p}_i|^2}.
\end{aligned}
$$

If we split the energy-momentum summation into size and angular parts using the
Eq. (he-13.17) density of states $\mathcal{N}(0) = \frac{3}{2}\rho/p_F^2$. Then $F_{12}^{ij,ab}$ can be written as an angular average

$$
F_{12}^{ij,ab}(k) = \frac{\rho}{2p_F^2}3\int_{-1}^{1}\frac{d\hat{\mathbf{p}}}{4\pi}F^v(k)\hat{p}_i\hat{p}_j\hat{p}_k\hat{p}_l t_{aba'b'}A_{a'k}^0 A_{b'l}^{0*} \qquad (11.17)
$$

Eq. (he-13.18) where $F^v(k)$ is the following function

$$
F^v(k,\hat{\mathbf{p}}) = T\sum_\omega\int_{-\infty}^{\infty}d\xi\frac{1}{(\tilde{\omega}_+{}^2 + E_+^2)\,(\tilde{\omega}_-{}^2 + E_-^2)}. \qquad (11.18)
$$

It is a generalization of Yosida's function ϕ to space- and time-dependent situations.
Eq. (he-13.19) For $v = 0,\ k = 0$:

$$
F^v(k,\hat{\mathbf{p}}) \underset{v=0,k=0}{=} T\sigma_\omega\int_{-\infty}^{\infty}d\xi\frac{1}{(\omega^2+E^2)^2} = \frac{1}{2\Delta^2}\phi(\Delta^2). \qquad (11.19)
$$

11.2 Time-Dependent Fluctuations at Infinite Wavelength

Let us now specialize to fluctuations which depend only on time and not on space. Then the only preferred spatial direction is the anisotropy axis l and we may decompose the tensor F_{11}^{ij} into components parallel and orthogonal to , i.e., Eq. (he-13.20)

$$F_{11}^{ij}(\nu) = (\delta_{ij} - l_i l_j)\, F_{11}^{\perp}(\nu) - l_i l_j F^{\parallel}(\nu). \tag{11.20}$$

Alternatively we shall decompose Eq. (he-13.21)

$$F_{11}^{ij}(\nu) = \delta_{ij} F_{11}^{\perp}(\nu) - l_i l_j F_{11}^{o}(\nu) \tag{11.21}$$

with the superscript o standing for orientational part. Using the general decomposition formula for an integral of the type Eq. (he-13.22)

$$\begin{aligned} 3\int \frac{d\hat{\mathbf{p}}}{4\pi} f(\mathbf{pl})\hat{p}_i\hat{p}_j = & \\ & \left[\int_{-1}^{1} \frac{dz}{2}\frac{3}{2}\left(1-z^2\right) f(z)\right](\delta_{ij} - l_i l_j) + \left[\int_{-1}^{1} \frac{dz}{2} 3z^2 f(z)\right] l_i l_j \\ = & \left[\int_{-1}^{1} \frac{dz}{2}\frac{3}{2}(1-z^2) f(z)\right]\delta_{ij} - \left[\int_{1}^{1} \frac{dz}{2}\frac{3}{2}\left(1-3z^2\right) f(z)\right] l_i l_j \end{aligned} \tag{11.22}$$

(which may be verified immediately upon contraction with δ_{ij} and $l_i l_j$) we identify Eq. (he-13.23)

$$\begin{aligned} L_{11}^{\perp}(\nu) &= \frac{\rho}{2p_F^2}\int_{-1}^{1}\frac{dz}{2}\frac{3}{2}(1-z^2)\left[T\sum_{\omega}\int_{-\infty}^{\infty} d'\xi \frac{(i\tilde{\omega}_+ - \xi_+)(i\tilde{\omega}_- + \xi_-)}{(\tilde{\omega}_+^2 + E_+^2)(\tilde{\omega}_-^2 + E_-^2)} + \gamma\right] \\ L_{11}^{0}(\nu) &= \frac{\rho}{2p_F^2}\int_{-1}^{1}\frac{dz}{2}\frac{3}{2}(1-3z^2)\left[T\sum_{\omega}\int_{-\infty}^{\infty} d'\xi \frac{(i\tilde{\omega}_+ - \xi_+)(i\tilde{\omega}_- + \xi_-)}{(\tilde{\omega}_+^2 + E_+^2)(\tilde{\omega}_-^2 + E_-^2)} + \gamma\right] \\ & \quad -\frac{\rho}{2p_F^2}\int_{-1}^{1}\frac{dz}{2}\frac{3}{2}(1-3z^2)\gamma \end{aligned} \tag{11.23}$$

where γ is the gap function introduced earlier: Eq. (he-13.24)

$$\gamma = T\sigma_\omega \int_{-\infty}^{\infty} d\xi \frac{1}{(\tilde{\omega}^2 + E^2)} \tag{11.24}$$

Note that in $L_{11}^{o}(\nu)$, γ cancels. To keep the expressions for both coefficients as similar as possible, however, we have added and subtracted γ. The advantage of this is that the square bracket can be simplified since γ may also be summed in terms of variables $\tilde{\omega}_\pm$ and $E_\pm$ instead of $\tilde{\omega}, E$. This replacement amounts to a mere translation of the infinite sum. Taking the average of both forms we may write Eq. (he-13.25)

$$\gamma = T\sum_{\omega}\int_{-\infty}^{\infty} d\xi\ \frac{1}{2}\,\frac{\tilde{\omega}_+^2 + \tilde{\omega}_-^2 + E_+^2 + E_-^2}{(\tilde{\omega}_+^2 + E_+^2)(\tilde{\omega}_-^2 + E_-^2)}. \tag{11.25}$$

Now the numerators in (11.23) can be combined to Eq. (he-13.26)

$$F^{\perp}_{11}(\nu) = \frac{\rho}{2p_F^2}\int_{-1}^{1}\frac{dz}{2}\left(\Delta^2+\frac{\nu^2}{2}\right)F^{\nu}(\nu) \equiv \frac{\rho}{4p_F^2}\varphi^{\perp}(\nu) \tag{11.26}$$
$$F^{o}_{11}(\nu) = \frac{\rho}{2p_F^2}\int_{-1}^{1}\frac{dz}{2}\frac{3}{2}(1-3z^2)\left[\left(\Delta^2+\frac{\nu^2}{2}\right)F^{\nu}(\nu)-\gamma\right] \equiv \frac{\rho}{4p_F^2}\varphi^{o}(\nu)$$

with the same function $F^{\nu}(\nu)$ appearing as in F_{12} [see (**??**)]. On the right-hand side of (11.26(he-13.26)) we have introduced convenient dimensionless quantities $\varphi^{\perp}$, φ^{o} associated with $F^{\perp}_{11}$, F^{o}_{11}.

It is a pleasant feature of the B phase that the γ term in (11.26) does not contribute due to the simultaneous validity of the longitudinal and the transversal gap equation (9.9), (9.10(he-11.11)). Thus the B phase acts as if there were no gap distortion at all. This is not so in the A-phase where only the transversal gap equation is available and γ does contribute!

Let us now perform a tensorial decomposition of F_{12}. Generalizing (11.22) we

Eq. (he-13.27) may now decompose

$$3\int\frac{d\hat{\mathbf{p}}}{4\pi}F(\mathbf{pl})\,\hat{p}_i\hat{p}_j\hat{p}_k\hat{p}_l = A\left(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)$$
$$+B\left(\delta_{ij}l_kl_l+\delta_{ik}l_jl_j+\delta_{il}l_jl_k+\delta_{ik}l_il_l+\delta_{jl}l_il_k+\delta_{kl}l_il_j\right)+Cl_il_jl_kl_l \tag{11.27}$$

Eq. (he-13.28) where $A,\ B,\ C$ are the following angular projections of F

$$\begin{aligned} A &= \frac{3}{8}\int_{-1}^{1}\frac{dz}{2}(1-z^2)^2F(z),\\ B &= -\frac{3}{8}\int_{-1}^{1}\frac{dz}{2}(1-z^2)(1-5z^2)F(z),\\ C &= \frac{3}{8}\int_{-1}^{1}\frac{dz}{2}(3-30z^2+35z^4)F(z),\\ &= -7B-\frac{3}{8}\int_{-1}^{1}\frac{dz}{2}(1-3z^2)F(z). \end{aligned} \tag{11.28}$$

For the purpose of obtaining the final results in the simplest possible form it is

Eq. (he-13.29) convenient to use the alternative dimensionless projections

$$\begin{aligned} \sigma_1(\nu) &\equiv 2(A+B)\Delta_\perp^2 = \int_{-1}^{1}\frac{dz}{2}3z^2(1-z^2)F^{\nu}(\nu)\Delta_\perp^2\\ \sigma_2(\nu) &\equiv 4A\Delta_\perp^2 = \int_{-1}^{1}\frac{dz}{2}\frac{3}{2}(1-z^2)^2F^{v}(\nu)\Delta_\perp^2\\ \sigma_3(\nu) &\equiv 2(3A+6B+C)\Delta_\perp^2 = 6\int_{-1}^{1}\frac{dz}{2}z^4F^{v}(\nu)\Delta_\perp^2. \end{aligned} \tag{11.29}$$

Notice that due to (7.21) the functions A and B at $\nu=0$ contain the information

explain c? on the orthogonal superfluid density, since

Eq. (he-13.30)

$$\begin{aligned} \rho_s^{\perp} &= \left[8A+2c^2(A+B)\right]_{\nu=0}\Delta_\perp^2\\ &= 2\sigma_2(0)+c^2\sigma_1(0). \end{aligned} \tag{11.30}$$

Eq. (he-13.31) We now evaluate the full tensor $F_{12}^{ijab}(\nu)$ in terms of $\sigma_{1,2,3}$. Contracting (11.27) with $A^0_{a'k}A^0_{b'l}t_{aba'b'}$ and A^0_{ai} from (9.7) we find

$$\begin{aligned}
&A\{2(\delta_{ai}\delta_{bj}+\delta_{aj}\delta_{bi}-3\delta_{ab}\delta_{ij})\\
&\quad+(1-c)\left[2(2l_al_b-\delta_{ab})+2(\delta_{ai}l_bl_j+\delta_{bi}l_al_j+(i\leftrightarrow j)-4\delta_{ab}l_il_j\right]\\
&\qquad+(1-c)^2\left[-(\delta_{ab}-2l_al_b)(\delta_{ij}+2l_il_j)\right]\Big\}\\
&+B\{[(2l_al_b-\delta_{ab})\delta_{ij}+2(\delta_{ai}l_bl_j+\delta_{bi}l_al_j+(i\leftrightarrow j))-5\delta_{ab}l_il_j]\\
&+(1-c)\left[2(2l_al_b-\delta_{ab})(3l_il_j+\delta_{ij})+2(\delta_{ai}l_bl_j+\delta_{bi}l_al_j+(i\leftrightarrow j))-4\delta_{ab}l_il_j\right]\\
&\qquad+(1-c)^2\left[(2l_al_b-\delta_{ab})\delta_{ij}+5(2l_al_b-\delta_{ab})l_il_j\right]\Big\}\\
&+C\left\{c^2(2l_al_b-\delta_{ab})l_il_j\right\}.
\end{aligned}\tag{11.31}$$

Collecting terms of equal tensorial properties this becomes Eq. (he-13.32)

$$\begin{aligned}
&2A[\delta_{ai}\delta_{bj}+(i\leftrightarrow j)]\\
&+\left[-2A-c^2(A+B)\right]\delta_{ab}\delta_{ij}+\left[-2A+2c^2(A+B)\right]l_al_b\delta_{ij}\\
&+\left[2A+c^2(A+B)-c^2(3A+6B+C)\right]\delta_{ab}l_il_j\\
&+[-2A+2(A+B)]\left[\delta_{ai}l_bl_j+\delta_{ai}l_al_j+(i\leftrightarrow j)\right]\\
&+\left[4(1-c)^2(A+B)-6(1-c^2)B+2c^2C\right]l_al_bl_il_j.
\end{aligned}\tag{11.32}$$

Contracting F_{11}, F_{12} with the fluctuating fields $A'_{ai}=\Delta_\perp d_{ai}$ and noticing that the contributions from $F_{22},\ \ F_{21}$ are complex conjugate to each other gives Eq. (he-13.33)

$$\begin{aligned}
i\delta^2\mathcal{A}=&-\frac{\Delta_\perp^2}{4p_F^2}\rho VT\sum_{\nu_n}\{\varphi^\perp(\nu_n)|d(\nu_n)|^2-\varphi^0(\nu_n)|d^Tl|^2\\
&+\frac{\sigma_2}{2}(d_{aa}d_{bb}+d_{ai}d_{ia}+c.c.)-\frac{1}{2}(c^2\sigma_i+\sigma_2)(d_{ai}d_{ai}+c.c.)\\
&+\frac{1}{2}(2c^2\sigma_1-\sigma_2)\left[(l^Td)_i(l^Td)_i+c.c.\right]+\frac{1}{2}(\sigma_1+\sigma_2-c^2\sigma_3)\left[(l^Td)_al^Td_b+c.c.\right]\\
&+2(2\sigma_1-\sigma_2)\left[(d^Tl)_a(l^Td)_a+c.c.\right]\\
&+\left[-(1+4c+c^2)\sigma_1+\frac{3}{2}\sigma_2+c^2\sigma_3\right]\left[(l^Tdl)(l^Tdl)+c.c.\right]\}.
\end{aligned}$$

As a cross check of our calculation we may verify that this expansion reduces in the static case $\nu_n=0$ and the Ginzburg-Landau limit $T\sim T_c$ to the previous expressions (5.6) (5.7(he-5.7)): We equate $d(0)=\frac{1}{T}d$ and $i\delta^2\mathcal{A}=-\delta^2 fV/T$ and rewrite Eq. (he-13.34)

$$\frac{\Delta_\perp^2\rho}{4p_F^2}\approx\frac{1}{6}\frac{p_F^2}{m^2\xi^2}\frac{\rho}{4p_F^2}\approx 2f_c\frac{1}{6}\frac{1}{1-\frac{T}{T_c}}.\tag{11.33}$$

Then we observe that for $T\sim T_c$ Eq. (he-13.35)

$$F(0)\Delta_\perp^2\approx\frac{1}{2}\phi(\Delta^2)\approx\left(1-\frac{T}{T_c}\right)\tag{11.34}$$

so that $\sigma_{1,2,3}$ have the extremely simple Ginzburg-Landau limits Eq. (he-13.36)

$$\sigma_i \approx i \cdot \frac{2}{5}\left(1 - \frac{T}{T_c}\right). \tag{11.35}$$

Inserting this together with (11.33) into (11.33(he-13.33)) we indeed recover the quadratic fluctuations of (6.44) and (6.44).

11.3 Normal Modes

It is pleasant to re alize that also the new formula (11.33), which is valid for all $T \leq T_c$ and $\nu_n \neq 0$ can be diagonalized on the same subspaces of real and imaginary
Eq. (he-13.37) parts of $d_{ai}(\nu_n) \equiv r_{ai}(\nu_n) + i i_{ai}(\nu_n)$

$$\begin{array}{lll} r_{11}, \quad r_{22}, \quad r_{33}; & r_{12}r_{21}; & r_{\frac{1}{2}3}r_{3\frac{1}{2}} \\ i_{11}, \quad i_{22}, \quad i_{33}; & i_{12}i_{21}; & i_{\frac{1}{2}3}i_{3\frac{1}{2}} \end{array} \tag{11.36}$$

Eq. (he-13.38) on which the curly brackets in the sum (11.33) take the form

$$\begin{aligned} R &= \begin{pmatrix} \lambda^\perp + c^2\sigma_1 - \sigma_2 & -\sigma_2 & -2c\sigma_1 \\ -\sigma_2 & \lambda^\perp + c^2\sigma_1 - \sigma_2 & -2c\sigma_1 \\ -2c\sigma_1 & -2c\sigma_1 & \lambda^\perp - \lambda^0 + 2\sigma_1 - c^2\sigma_2 \end{pmatrix}, \\ R^{12} &= \begin{pmatrix} \lambda^\perp + c^2\sigma_1 - \sigma_2 & \sigma_2 \\ \sigma_2 & \lambda^\perp + c^2\sigma_1 - \sigma_2 \end{pmatrix}, \\ R^{\,13}_{\,23} &= \begin{pmatrix} \lambda^\perp - \lambda^0 - c^2\sigma_3 & 2c\sigma_1 \\ 2c\sigma_1 & \lambda^\perp + c^2\sigma_1 - 2\sigma_2 \end{pmatrix}, \end{aligned} \tag{11.37}$$

with similar matrices for the imaginary parts, except that the σ- terms appear with reversed sign. These matrices serve two purposes. On the one hand, we can now verify the stability under static fluctuations for all temperatures $T \neq T_c$ by finding the eigenvalues at $\nu_n = 0$. On the other hand, the matrices contain informations on the energy of collective excitations at infinite wavelengths: By continuing analytically from the discrete values ν_n to physical frequencies

$$\nu_n \to -i(\omega + i\epsilon) \tag{11.38}$$

these energies are given by the frequency ω at which the matrices become singular. The corresponding eigenvectors are the normal modes of the order parameter fluctuations $d_{ai}(\omega)$.

To embark in this calculation it is useful to express the functions $\varphi^\perp$, φ^0 of
Eq. (he-13.39) (11.26) in terms of the functions σ_i as follows [see (11.26)]

$$\varphi^\perp(\nu_n) = \int_{-1}^{1} \frac{dz}{2}\frac{3}{2}(1-z^2)(1-r^2z^2)2F\Delta_\perp^2 + \frac{\nu_n^2}{2\Delta_\perp^2}\int \frac{dz}{2}\frac{3}{2}\left(1-z^2\right)2F\Delta_\perp^2. \tag{11.39}$$

Eq. (he-13.40) Using (11.29) this becomes

$$\varphi^{\perp}(\nu_n) = c^2\sigma_1 + 2\sigma_2 + 2\frac{\nu_n^2}{4\Delta_\perp^2}\left(\sigma_1 + 2\sigma_2\right). \tag{11.40}$$

Similarly, we find Eq. (he-13.41)

$$\begin{aligned} \varphi''(\nu_n) &= 2\sigma_1 + \sigma_3 + 2\frac{\nu_n^2}{4\Delta_\perp^2}(2\sigma_1 + \sigma_3) \\ &\equiv \varphi(\nu_n)^{\perp} - \varphi^0(\nu_n) \end{aligned} \tag{11.41}$$

Inserting these relations into (11.37) we find Eq. (he-13.42)

$$\begin{aligned} R &= \begin{pmatrix} 3\sigma_2 - 2w^2(\sigma_1+2\sigma_2) & \sigma_2 & 2c\sigma_1 \\ \sigma_2 & 3\sigma2 - 2w^2(\sigma_1+2\sigma_2) & 2c\sigma_1 \\ 2c\sigma_1 & 2c\sigma_1 & 2c^2\sigma_3 - 2w^2(2\sigma_1+\sigma_3) \end{pmatrix}, \\ R^{12} &= \begin{pmatrix} \sigma_2 - 2w^2(\sigma_1+2\sigma_2) & \sigma_2 \\ \sigma_2 & \sigma_2 - 2w^2(\sigma_1+2\sigma_2) \end{pmatrix}, \\ R^{\,13}_{\,23} &= \begin{pmatrix} 2\sigma_1 - 2w^2(2\sigma_1+\sigma_3) & 2c\sigma_1 \\ 2c\sigma_1 & 2c^2\sigma_1 - 2w^2(\sigma_1+2\sigma_2) \end{pmatrix}. \end{aligned} \tag{11.42}$$

For the imaginary parts of d_{ai} the fluctuation matrices are Eq. (he-13.43)

$$\begin{aligned} I &= \begin{pmatrix} \sigma_2+2c^2\sigma_1-2w^2(\sigma_1+2\sigma_2) & -\sigma_2 & -2c\sigma_1 \\ -\sigma_2 & \sigma_2+2c^2\sigma_1-2w^2(\sigma_1+2\sigma_2) & -2c\sigma_1 \\ -2c\sigma & -2c\sigma_1 & 4\sigma_1-2w^2(2\sigma_1+\sigma_3) \end{pmatrix} \\ I^{12} &= \begin{pmatrix} 3\sigma_2+2c^2\sigma_1-2w^2(\sigma_1+2\sigma_2) & -\sigma_2 \\ -\sigma_2 & 3\sigma_2+2c^2\sigma_1-2w^2(\sigma_1+2\sigma_2) \end{pmatrix} \\ I^{\,13}_{\,23} &= \begin{pmatrix} 2(\sigma_1+c^2\sigma_3)-2w^2(2\sigma_1+\sigma_3) & -2c\sigma_1 \\ -2c\sigma_1 & 4\sigma_2-2w^2(\sigma_1+2\sigma_3) \end{pmatrix}. \end{aligned} \tag{11.43}$$

Here we have introduced the dimensionless frequency variable Eq. (he-13.44)

$$w^2 \equiv -\frac{\nu_n^2}{4\Delta_\perp^2} = \frac{(\omega + i\epsilon)^2}{4\Delta_\perp^2}, \tag{11.44}$$

for brevity. It is now straightforward to determine the places of vanishing determinants: For R^{12}, I^{12} we immediately find Eq. (he-13.45)

$$\begin{aligned} w^2_{\substack{1\\2}} &= \left\{ \begin{matrix} 0 \\ \frac{\sigma_2}{\sigma_1+2\sigma_2} \end{matrix} \right\} \begin{matrix} (1,-1) \\ (1,1) \end{matrix} \\ w^2_{\substack{1\\2}} &= \left\{ \begin{matrix} \frac{c^2\sigma_1+2\sigma_2}{\sigma_1+2\sigma_2} \\ \frac{c^2\sigma_1+\sigma_2}{\sigma_1+2\sigma_2} \end{matrix} \right\} \begin{matrix} (1,-1) \\ (1,1) \end{matrix} \end{aligned} \tag{11.45}$$

respectively. Behind each eigenvalue we have written down the corresponding eigenvector. For $R^{\,13}_{\,23}$, the formula is more complicated: Eq. (he-13.46)

$$w^2_{\frac{1}{2}} = \left\{ \begin{array}{c} 0 \\ \frac{\sigma_1}{(\sigma_1+2\sigma_2)(2\sigma_1+\sigma_3)}\left[(2c^2+1)\sigma_1+2\sigma_2+c^2\sigma_3\right] \end{array} \right\} \begin{array}{l} (1,-1/c) \\ (\sigma_1+2\sigma_2, \quad c(2\sigma_1+\sigma_3)). \end{array}$$

Eq. (he-13.47) For I^{13}_{23} the roots can no longer be taken explicitly. Here we find

$$w^2_{\frac{1}{2}} = \frac{\sigma_2}{\sigma_1+2\sigma_2} + \frac{\sigma_1+c^2\sigma_3}{2(2\sigma_1+\sigma_3)} \pm \frac{1}{2(\sigma_1+2\sigma_2)(2\sigma_1+\sigma_3)} \times\sqrt{[(2\sigma_1+\sigma_3)2\sigma_2-(\sigma_1+c^2\sigma_3)(\sigma_1+2\sigma_2)]+4c^2\sigma_1^2(2\sigma_1+\sigma_3)(\sigma_1+2\sigma_2)}. \tag{11.46}$$

In the 3×3 subspaces the eigenvalues look simple only for the imaginary components
Eq. (he-13.48) of d_{ai}. First we observe that $(1,1,c)$ is an eigenvector of I with

$$w_1^2 = 0 \qquad (1,1,c). \tag{11.47}$$

For $c = 1$, this is the pure phase oscillation of zero sound. By adding the second
Eq. (he-13.49) and the last column times c to the first, the determinant of I can be written as

$$|I| = -2w^2(\sigma_1+2\sigma_2) \times \left| \begin{array}{lcc} 1 & 0 & -2c\sigma_1 \\ 1 & \sigma_2+2c^2\sigma_1-2w^2(\sigma_1+2\sigma_2) & -2c\sigma_1 \\ c & -2c\sigma_1 & 4\sigma_1-2w^2(2\sigma_1+\sigma_3) \end{array} \right|. \tag{11.48}$$

Eq. (he-13.49a) The remaining determinant has the form

$$4(\sigma_1+2\sigma_2)(2\sigma_1+\sigma_3)w^4-4w^2\left[(c^2\sigma_1+\sigma_2)(2\sigma_1+\sigma_3)+(2+c^2)\sigma_1(\sigma_1+12\sigma_2)\right] +4(c^2\sigma_1+\sigma_2)(2+c^2)\sigma_1=0 \tag{11.49}$$

Eq. (he-13.50) so that the remaining two solutions are

$$w_2^2 = \frac{(2+c^2)\sigma_1}{2\sigma_1+\sigma_3}, \qquad w_3^2 = \frac{c^2\sigma_1+\sigma_2}{\sigma_1+2\sigma_2}. \tag{11.50}$$

For the real 3×3 matrix R, finally, we can find a trivial eigenvector $(1,-1,0)$ with
Eq. (he-13.51) eigenvalue

$$w_1^2 = \frac{\sigma_2}{\sigma_1+2\sigma_2}. \tag{11.51}$$

It is degenerate with the second of the eigenvalues of R^{12}.

Eq. (he-13.52) By subtracting in the determinant of R the second from the first row we obtain

$$|R| = \left[2\sigma_2-2W^2(\sigma_1+2\sigma_2)\right] \times \left| \begin{array}{ccc} 1 & -1 & 0 \\ \sigma_2 & 2\sigma_2-2w^2(\sigma_1+2\sigma_2) & 2c\sigma_1 \\ 2c\sigma_1 & 2c\sigma_1 & 2c^2\sigma_3-2w^2(\sigma_1+\sigma_3) \end{array} \right|,$$

Eq. (he-13.53) so that the remaining two roots are found from

$$w^4(\sigma_1+2\sigma_2)(2\sigma_1+\sigma_3) - w^2\left[(\sigma_1+2\sigma_2)c^2\sigma_3 + (2\sigma_1+\sigma_3)2\sigma_2\right]$$
$$+2c^2(\sigma_2\sigma_3-\sigma_1^2) = 0 \qquad (11.52)$$

which is solved by Eq. (he-13.54)

$$w^2_{\substack{2\\3}} = \frac{1}{2}\frac{c^2\sigma_3}{2\sigma_1+\sigma_3} + \frac{\sigma_2}{\sigma_1+2\sigma_2} \pm \frac{1}{2(\sigma_1+2\sigma_2)(\sigma_1+\sigma_3)} \qquad (11.53)$$
$$\times\sqrt{[(\sigma_1+2\sigma_2)c^2\sigma_3-(2\sigma_1+\sigma_3)2\sigma_2]^2+8c^2\sigma_1^2(\sigma_1+2\sigma_2)(2\sigma_1+\sigma_3)}.$$

All these equations are transcendental since the right-hand sides depend again on w^2. They can, however, be solved quite simply in an iterative fashion.

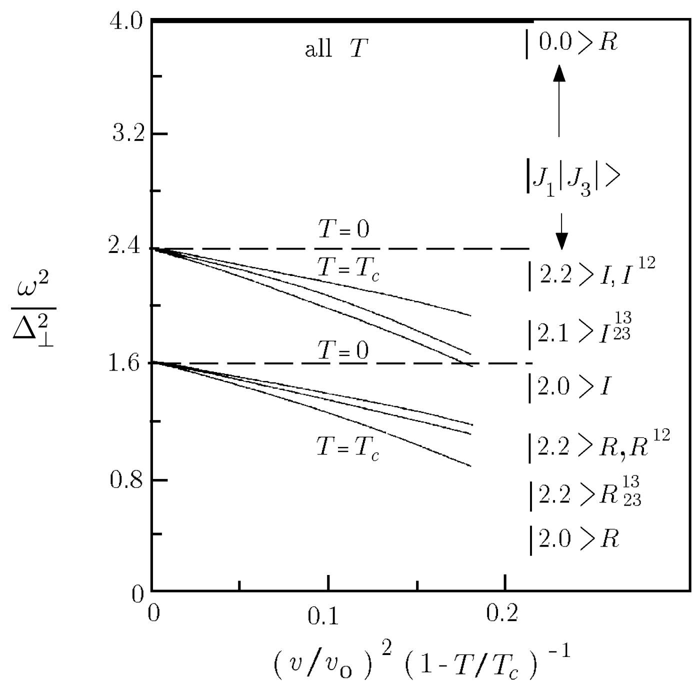


FIGURE 11.1 Collective frequencies of B-phase in the presence of superflow of velocity v at zero and slightly below the critical temperaure T_c (Ginzburg-Landau regime). Near T_c, there is a considerable splitting between the levels of different $|J_3|$. The quantum numbers of angular momentum are displayed at the right end of each curve. The gap distortion $r^2 \equiv 1-\Delta_\parallel^2/\Delta_\perp^2$ is related to the superfluid velocity v by $r^2 = 3(v/v_0)^2(1-T/T_c)^{-1}$ $r^2 = 3(v/v_0)^2(1-T/T_c)^{-1}$.

11.4 Simple Limiting Results at Zero Gap Deformation

Before attempting a numerical solution of these equations we may extract several results right away: For small current the asymmetry parameter r vanishes at all temperatures. As a consequence, $\sigma_{1,2,3}$ becomes independent of z and we find immediately from integrating (11.29) that the functions $\sigma_1 : \sigma_2 : \sigma_3$ have a fixed ration $1:2:3$. As a result we find the well-known collective frequencies of the B phase, at all temperatures:

Eq. (he-13.55)

$$
\begin{array}{lllll}
R^{12}: & \left\{\begin{array}{c} 0 \\ \frac{2}{5} \end{array}\right\}, & \omega^2 = \left\{\begin{array}{c} 0 \\ \frac{8}{5}\Delta_\perp^2 \end{array}\right\} & & \begin{array}{l} (1,-1) \\ (1,1) \end{array} \\
R^{13}_{23}: & \left\{\begin{array}{c} 0 \\ \frac{2}{5} \end{array}\right\}, & \omega^2 = \left\{\begin{array}{c} 0 \\ \frac{8}{5}\Delta_\perp^2 \end{array}\right\} & & \begin{array}{l} (1,-1) \\ (1,1) \end{array} \\
R: & \left\{\begin{array}{c} \frac{2}{5} \\ 1 \\ \frac{2}{5} \end{array}\right\}, & \omega^2 = \left\{\begin{array}{c} \frac{8}{5}\Delta_\perp^2 \\ 4\Delta_\perp^2 \\ \frac{8}{5}\Delta_\perp^2 \end{array}\right\} & & \begin{array}{l} (1,-1,0) \\ (1,1,1) \\ (1,1,-2) \end{array} \\
I^{12}: & \left\{\begin{array}{c} 1 \\ \frac{3}{5} \end{array}\right\}, & \omega^2 = \left\{\begin{array}{c} 4\Delta_\perp^2 \\ \frac{12}{5}\Delta_\perp^2 \end{array}\right\} & & \begin{array}{l} (1,-1) \\ (1,1) \end{array} \\
I^{13}_{23}: & \left\{\begin{array}{c} 1 \\ \frac{3}{5} \end{array}\right\}, & \omega^2 = \left\{\begin{array}{c} 4\Delta_\perp^2 \\ \frac{12}{5}\Delta_\perp^2 \end{array}\right\} & & \begin{array}{l} (1,-1) \\ (1,1) \end{array} \\
I: & \left\{\begin{array}{c} \frac{3}{5} \\ \frac{3}{5} \\ 0 \end{array}\right\}, & \omega^2 = \left\{\begin{array}{c} \frac{12}{5}\Delta_\perp^2 \\ \frac{12}{5}\Delta_\perp^2 \\ 0 \end{array}\right\} & & \begin{array}{l} (1,-1,0) \\ (1,1,-2) \\ (1,1,1) \end{array}
\end{array}
$$

where the eigenvectors have again been marked in each case. Moreover, since at $T=0$ there is no gap deformation for $\nu \leq 1$ these results remain true for all velocities up to $\leq v_n$. It is useful to classify this symmetric situation in terms of angular momentum. The real and imaginary 3×3 matrices contain a $J=0,\ J=1$, and $J=2$ tensor with the correspondence

Eq. (he-13.57)

$$
\begin{array}{rcll}
\frac{1}{\sqrt{3}}(1,1,1) & = & |00\rangle & \\
\frac{1}{\sqrt{2}}(1,-1,0) & = & \frac{1}{\sqrt{2}}\left(|2,2\rangle + |2,-2\rangle\right)) & R, I \\
\frac{1}{\sqrt{6}}(1,1,-2) & = & |2,0\rangle & \\
\frac{1}{\sqrt{2}}(1,-1) & = & |1,0\rangle & \\
\frac{1}{\sqrt{2}}(1,1) & = & \frac{1}{\sqrt{2}}\left(|2,2\rangle - |2,-2\rangle\right) & R^{12}, I^{12} \\
\frac{1}{\sqrt{2}}(1,-1) & = & \frac{1}{\sqrt{2}}\left(|1,1\rangle + |1,-1\rangle\right) & R^{13}, I^{13} \\
\frac{1}{\sqrt{2}}(1,1) & = & \frac{1}{\sqrt{2}}\left(|2,1\rangle + |2,-1\rangle\right) & \\
\frac{1}{\sqrt{2}}(1,-1) & = & \frac{1}{\sqrt{2}}\left(|1,1\rangle - |1,-1\rangle\right) & R^{23}, I^{23} \\
\frac{1}{\sqrt{2}}(1,1) & = & \frac{1}{\sqrt{2}}\left(|2,1\rangle - |2,-1\rangle\right) &
\end{array}
$$

explaining the degeneracies among the 5 real $J=2$ modes the three real $\omega^2=0$ Goldstone modes of $J=1$, the 5 imaginary $J=2$ modes $\frac{12}{5}\Delta_\perp^2$ and the 3 imaginary $J=1$ modes $4\Delta_\perp^2$.

Now, if a current is turned on the levels of different $|J_3|$ within each multiplets split up. Using the explicit forms of analytically continued σ_i functions, to be discussed in the next section, we find for T close to T_c the level structure displayed on Fig. 11.1.

11.5 Static Stability

In order to verify static stability we have to take the matrices R, I before analytic continuation at zero Matsubara frequency $\nu_n = 0$ and calculate their eigenvalues. These are found as Eq. (he-13.58)

$$R \;:\; 2\sigma_2, 2\sigma_2 + c^2\sigma_3 \pm \sqrt{(2\sigma_2 - c^2 r_3)^2 + 8c^2\sigma_1^2},$$

$$R^{12} \;:\; \left\{ \begin{array}{c} 0 \\ 2\sigma_2 \end{array} \right\} \qquad \begin{array}{l} (1,-1) \\ (1,1) \end{array}$$

$$R^{13}_{23} \;:\; \left\{ \begin{array}{c} 0 \\ (1+c^2)\sigma_1 \end{array} \right\} \qquad \begin{array}{l} (1,-1/c) \\ (1,c) \end{array} \tag{11.54}$$

$$I \;:\; 2(c^2\sigma_1 + \sigma_2), 2(c^2+2)\sigma_1, 0$$

$$I^{12} \;:\; \left\{ \begin{array}{c} 2c^2\sigma_1 + 4\sigma_2 \\ 2c^2\sigma_1 + 2\sigma_2 \end{array} \right\} \qquad \begin{array}{l} (1,-1) \\ (1,1) \end{array}$$

$$I^{13}_{23} \;:\; \sigma_1 + 2\sigma_2 + c^2\sigma_3 \pm \sqrt{(\sigma_1 - 2\sigma_2 + c^2)^2 + 4c^2\sigma_1^2}. \tag{11.55}$$

The eigenvectors are marked if they are simple. Using the result of Sec. ?? we can verify that all nonzero values remain positive for all subcritical velocities thus guaranteeing static stability.

12
Fluctuation Coefficients

We have seen in the last section that all properties of quadratic fluctuations at finite wavelengths are expressible in terms of the functions $\sigma_{1,2,3}(\omega^2)$ which in turn are angular projections of (see (11.13) (11.29(he-13.29))

Eq. (he-14.1)

$$F^v(\nu_n, \mathbf{k}) = T \sum_\omega \int_{-\infty}^{\infty} d\xi \frac{1}{(\tilde{\omega}_+^2 + E_+^2)(\tilde{\omega}_-^2 + E_-^2)} \tag{12.1}$$

at $\mathbf{k} = 0$. For the particular case of static fluctuations $F^\nu(0,0)$ reduces directly to the standard Yosida function

Eq. (he-14.2)

$$F^v(0,0) = \frac{1}{2\Delta_\perp^2}\phi^v(0,0). \tag{12.2}$$

It can then easily be checked that in this case the projection $\sigma_i(0)$ are positive thus guaranteeing the stability of static fluctuation frequencies (11.54): First close to T_c, all nonzero eigenvalues are positive since σ_i have the simple form (11.35). Moreover, as the temperature reaches zero, the gap becomes uniform and

Eq. (he-14.3)

$$F^v\Delta_\perp^2 \to \frac{1}{2} \tag{12.3}$$

for subcritical velocities so that $\sigma_i(0)$ are positive members with the same rations $1:2:3$. Inserting this together with $c = 1$ into (11.54) all eigenvalues become again positive. By monotony of the gap distortion at fixed velocity $\frac{v^2}{\nu_0^2}\left(1 - \frac{T}{T_c}\right)^{-1}$ (see Figs.) as a function of temperature we conclude for stability for all temperatures $T \le T_c$ and all subcritical velocities.

For dynamic fluctuations, let us continue F^v analytically in the frequency $-\nu_n$. For this we decompose

$$\frac{1}{\tilde{\omega}_-^2 + E_-^2}\frac{1}{\tilde{\omega}_+^2 + E_+^2} \tag{12.4}$$

as in (7.42) and use the summation formula (7.40(he-dis)) which now gives

Eq. (he-14.4)

$$\frac{1}{2E}T\sum_\omega \frac{1}{i\tilde{\omega}_\pm \pm E} = \frac{1}{2E}\left[1 \pm \frac{1}{2}\left(\tanh\frac{E + vp_Fz}{2T} + \tanh\frac{E - vp_Fz}{2T}\right)\right]. \tag{12.5}$$

Again we have made use of the fact that the frequency shift ν_n in $\omega_\pm$ (see (11.16) does not appear in (12.5) since it amounts to a mere translation in the infinite sum. Collecting the different terms we find

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Eq. (he-14.5)

$$\begin{aligned}
&\frac{1}{4E_+E_-}\Bigg\{\frac{E_+{+}E_-}{(E_+{+}E_-)^2{+}\nu_n^2}\Big[\frac{1}{2}\Big(\tanh\frac{E_+ + vp_F z}{2T}{+}(v\to -v)\Big){+}(E_+\leftrightarrow E_-)\Big]\\
&\qquad -\frac{E_+{-}E_-}{(E_+{-}E_-)^2{+}\nu_n^2}\Big[\frac{1}{2}\Big(\tanh\frac{E_+ + vp_F z}{2T}{+}(v\to -v)\Big){-}(E_+\leftrightarrow E_-)\Big]\Bigg\}\\
&= \left\{\frac{1}{4E_-}\frac{E_-^2 - E_+^2 + \nu_n^2}{(E_+^2 + E_-^2 + \nu_n^2)^2 - 4E_+^2E_+^2}\left[\tanh\frac{E_+ + vp_F z}{2T} + (v\to -v)\right]\right\}\\
&\qquad\qquad + \{E_+ \leftrightarrow E_-\}.
\end{aligned} \tag{12.6}$$

We now perform a shift in the integration variable so that

Eq. (he-14.6)

$$\begin{aligned}
E_-^2 &= \xi^2 + \Delta^2\\
E_+^2 &= (\xi + v_F\hat{\mathbf{p}}\mathbf{k})^2 + \Delta^2.
\end{aligned} \tag{12.7}$$

Then F takes the form

Eq. (he-14.7)

$$\begin{aligned}
&F^v(\nu_n, \mathbf{k}, \hat{\mathbf{p}}) =\\
&\quad\int_{-\infty}^{\infty} d\xi \frac{\nu_n^2 + (2\xi + v_F\hat{\mathbf{p}}\mathbf{k})\, v_F\hat{\mathbf{p}}\mathbf{k}}{(2\xi + v_F\hat{\mathbf{p}}\mathbf{k})^2(\nu_n^2 + v_F^2(\hat{\mathbf{p}}\mathbf{k})^2) + \nu_n^2(\nu_n^2 + v_F^2(\hat{\mathbf{p}}k)^2 + 4\Delta^2)}\\
&\qquad\times\frac{1}{2}\left(\tanh\frac{E + vp_F z}{2T} + (v\to -v)\right).
\end{aligned} \tag{12.8}$$

As a cross check we verify that for $\nu_n = 0$ this reduces to

Eq. (he-14.8)

$$F^v(\nu_n = 0, \mathbf{k}, \hat{\mathbf{p}}) {=} {-}\int_{-\infty}^{\infty} d\xi \frac{1}{4\xi^2 - v_F^2(\hat{\mathbf{p}}\mathbf{k})^2}\frac{1}{2}\left(\tanh\frac{E + vp_F z}{2T} {+} (z{\to}{-}z)\right). \tag{12.9}$$

At $\mathbf{k} = 0$ this can be partially integrated to

Eq. (he-14.9)

$$\begin{aligned}
&F^v(0, 0, \hat{\mathbf{p}}) = \\
&\frac{1}{8}\int_{-\infty}^{\infty} d\xi \frac{1}{E^3}\left[\left(\tanh\frac{E + vp_F z}{2T} - \frac{1}{2TE^2} - \frac{1}{\cosh^2\frac{E+vp_F z}{2T}}\right) + (v\to -v)\right].
\end{aligned} \tag{12.10}$$

Upon using the expansion (12.5) and its derivative

Eq. (he-14.10)

$$\begin{aligned}
&\frac{1}{8T}\left[\frac{1}{\cosh^2\frac{E+vp_F z}{2T}} + (v\to -v)\right]\\
&\qquad = \frac{d}{dE}T\sum_\omega \frac{E}{\tilde{\omega}^2 + E^2} = T\sum_\omega\left[\frac{1}{\tilde{\omega}^2 + E^2} - \frac{2E^2}{(\tilde{\omega}^2 + E^2)^2}\right]
\end{aligned} \tag{12.11}$$

we recover the Yosida function in the presence of superflow (**??**) governing the superfluid densities:

Eq. (he-14.11)

$$F^v(0,0,\hat{\mathbf{p}}) = \int_{-\infty}^{\infty} d\xi \frac{1}{E^2} T \sum_{\omega} \frac{E^2}{(\tilde{\omega}^2+E^2)^2}$$
$$= \phi^v(\Delta^2)/2\Delta_\perp^2. \tag{12.12}$$

Eq. (he-14.12) The expression (12.8) can readily be continued analytically to physical frequencies ω by merely replacing

$$\nu_n^2 \to -(\omega+i\epsilon)^2. \tag{12.13}$$

Eq. (he-14.13) Let us now turn to the calculation of the functions. For this we consider the continued expression at infinite wavelength

$$F^v(\omega,0,\hat{p}) = \int_{-\infty}^{\infty} \frac{1}{E(4E^2-\omega^2)} \frac{1}{2} \left(\tanh \frac{E+vp_F z}{2T} \leftarrow (v \to -v) \right). \tag{12.14}$$

Eq. (he-14.14) The temperature region close to T_c is explored most easily by inserting the expansion (12.5). Then the integral over ξ can be done and we find

$$F^v(\omega,0,\hat{p}) = \frac{1}{4}\int_{-\infty}^{\infty} d\xi T \sum_{\omega} \frac{1}{(\omega_n - ivp_F z)^2+\omega^2/4}$$
$$\times \left(\frac{1}{\xi^2+\Delta^2-\omega^2/4} - \frac{1}{(\omega_n - ivp_F z)^2+\xi^2+\Delta^2} \right)$$
$$= \frac{\pi}{4}\frac{1}{\sqrt{\Delta^2-\omega^2/4}}\frac{1}{\omega}\left(\tanh\frac{\omega/2+v}{2T} + (v\to -v)\right) \tag{12.15}$$
$$-\frac{\pi T}{4}\sum_{\omega_m}\frac{1}{(\omega_m - ivp_F z)^2} + \frac{\omega^2}{4}\frac{1}{\sqrt{(\omega_n - ivp_F z)^2+\Delta^2}}.$$

Eq. (he-14.15) Using the previously introduced dimensionless variables (11.44) this may be rewritten as

$$F^v(\omega,0,\hat{p})\Delta_\perp^2 =$$
$$\frac{\pi}{8}\frac{1}{\sqrt{1-r^2z^2-w^2}}\left\{ \tanh\left[\frac{\pi}{2}(w-\nu z)\delta\right] + (\nu \to -\nu)\right\}$$
$$-\frac{1}{2\delta}\mathrm{Re}\sum_{n=0}^{\infty}\frac{1}{(x_n - i\nu z)^2+w^2}\frac{1}{\sqrt{(x_n - ivz)^2+1-r^2z^2}} \tag{12.16}$$

where the square root has to be taken with positive real part.

Eq. (he-14.16) In the limit $T \to T_c$, $\delta \to 0$ and the sum is suppressed by one power of δ as compared with the first term so that we may use the simple expression

$$F^v(\omega,0,\hat{p})\Delta_\perp^2 \underset{T\to T_c}{=} \frac{\pi^2\delta}{8}\frac{1}{\sqrt{1-r^2z^2-w^2}}. \tag{12.17}$$

Eq. (he-14.17) For $T \to 0$, the integral is found easily from (12.14) if the velocity v is by $v < \Delta_{BCS}/p_F \approx v_c$. Then $\tanh\frac{E\pm p_F z}{2T} = 1$ and we have

$$F^v(\omega,0,\hat{p}) = \frac{1}{4}\int_{-\infty}^{\infty} d\xi \frac{1}{\sqrt{\xi^2+\Delta^2}}\frac{1}{\xi^2+\Delta^2(1-w^2)}. \tag{12.18}$$

Eq. (he-14.18) It is useful to remove the square root by an auxiliary integration, writing

$$F^v(\omega,0,\hat{p}) = \frac{1}{4\pi}\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} \frac{1}{\xi^2+\mu^2+\Delta^2}\frac{1}{\xi^2+\Delta^2(1-w^2)}. \tag{12.19}$$

Using Feynman's formula Eq. (he-14.19)

$$\frac{1}{AB} = 2\int_0^1 dss\frac{1}{[sA+(1-s^2)B]^2} \tag{12.20}$$

this becomes Eq. (he-14.20)

$$F^v(\omega,0,\hat{p}) = \frac{1}{\pi}\int_0^1 ds\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\xi d(\mu s)\frac{1}{(\xi^2+s^2\mu^2+\Delta^2-(1-s^2)w^2)^2}. \tag{12.21}$$

Due to rotational invariance in the $(\xi, s\mu)$-plane this can be evaluated in polar coordinates to give Eq. (he-14.21)

$$2\int_0^1 ds\int_0^{\infty} dr\frac{r}{(r^2+\Delta^2-(1-s^2)w^2)^2} = \int_0^{\infty} ds\frac{1}{sA^2-(1-s^2)w^2}. \tag{12.22}$$

Thus we arrive at the simple integral representation Eq. (he-14.22)

$$F^v(\omega,0,\hat{\mathbf{p}})\Delta_\perp^2 = \frac{1}{2}\int_0^{\infty} ds\frac{1}{s^2w^2+1-r^2z^2-w^2} \tag{12.23}$$

which can be integrated to Eq. (he-14.23)

$$F^v(\omega_1,0,\hat{\mathbf{p}})\Delta_\perp^2 = \frac{1}{2}\frac{1}{\sqrt{1-r^2z^2-w^2}}\frac{1}{w}\arcsin\frac{\omega}{\sqrt{1-r^2z^2}}. \tag{12.24}$$

We can now proceed to calculate the $\sigma_{1,2,3}$ functions. Consider first the limit $T \to T_c$. Straight-forward integration yields, with the overall factor Eq. (he-14.24)

$$\alpha \equiv \pi^2\delta/4 = \pi\Delta_\perp/4T \tag{12.25}$$

the expressions Eq. (he-14.25)

$$\begin{aligned}\sigma_1(w^2) \underset{T\to T_c}{=} & \alpha\frac{3}{2}\int_{-1}^{1}\frac{dz}{2}z^2(1-z^2)\frac{1}{\sqrt{1-w^2-r^2z^2}} \\ &= \frac{3}{4r^5}\Big\{\Big[-\frac{3}{4}(1-w^2)^2+r^2(1-w^2)\Big]\,l \\ &\qquad + \Big[\frac{3}{4}(1-w^2)-\frac{r^2}{2}\Big]\,r\sqrt{1-w^2r^2}\Big\},\end{aligned} \tag{12.26}$$

$$\sigma_2(w^2) \underset{T\to T_c}{=} \alpha\frac{3}{4}\int_{-1}^{1}\frac{dz}{2}(1-z^2)^2\frac{1}{\sqrt{1-w^2-r^2z^2}} \tag{12.27}$$

$$= \frac{3}{4r^5}\frac{3}{8}\left\{\left[(1-w^2)^2 - \frac{8}{3}r^2(1-w^2-r^2)\right] l \right.$$
$$\left. + \left[-(1-w^2) + 2r^2\right] r\sqrt{1-w^2-r^2}\right\},$$

$$\sigma_3(w^2) \underset{T\to T_c}{=} 3\int_{-1}^{1}\frac{dz}{2}z^4\frac{1}{\sqrt{1-w^2-r^2z^2}} \tag{12.28}$$
$$= \frac{3}{4r^5}\left\{\left[-\frac{3}{4}(1-w^2)^2 + r^2(1-w^2)\right] l \right.$$
$$\left. + \left[\frac{3}{4}(1-w^2) + \frac{r^2}{2}r^2\right] r\sqrt{1-w^2-r^2}\right\}.$$

Eq. (he-14.26) Here l is the fundamental integral

$$l(w^2) \equiv r\int_{-1}^{1}\frac{dz}{2}\frac{1}{\sqrt{1-w^2-r^2z^2}} = \arcsin\frac{r}{\sqrt{1-w^2}}. \tag{12.29}$$

This formula may be used as long as $w^2 < 1-r^2$. For w^2 between $1-r^2$ and l there
Eq. (he-14.27) is an imaginary part whose sign is controlled by the $i\epsilon$ prescription in ω:

$$l(w^2) = \frac{\pi}{2} + \frac{i}{r}\log\frac{r+\sqrt{w^2-(1-r^2)}}{\sqrt{1-w^2}}. \tag{12.30}$$

It may in principle give rise to a width of the collective excitation due to pair breaking along directions where the gap is not maximal.

12.1 Stability of Super-Flow in the B-phase under Small Fluctuations for $T \sim T_c$

Let us finally investigate the important question whether the Ansatz (10.47) for the distorted order parameter is a local minimum of the free energy for all currents up to J_c. Previously, we had shown this form to develop for infinitesimal currents. We shall now study, for all currents up to the critical value J_c, the small fluctuations in the 18 parameter field space A_{ai}.

With the time driving term of the collective action being of the simple pure damping form, it will be sufficient to consider only static fluctuations. While it is a disadvantage of the Ginzburg-Landau regime that there are no properly oscillating modes which could easily be detected experimentally. On the other hand, there is the advantage of a simple parametrization of strong-coupling corrections. In the later chapter 12 shall study the physically more yielding dynamic fluctuation problem for all temperatures $T < T_c$. But then we shall be forced to neglect strong-coupling effects.

Eq. (he-15.94) Let us parametrize the static fluctuations in the form

$$A_{ai} = \Delta_B^{w.c.}\left[a\left(\delta_{ai} + rl_a l_i\right) + d_{ai}\right] \tag{12.31}$$

Eq. (he-15.95) where

$$r \equiv \frac{c}{a} - 1 \tag{12.32}$$

and c and a are the equilibrium values (10.61), (10.62) of the gap parameters in the presence of a current (we shall leave out the magnetic field, for simplicity). Inserting (12.31) into the energy we obtain the potential terms for $r = 0$ Eq. (he-15.96)

$$\begin{aligned}\delta^2 e/2f_c|_{pot} \;&=\; -\frac{\alpha}{3}|d_{ai}|^2 + \frac{a^2}{15}\Big[(3\beta_1 + \beta_{35})\left(d_{ai}^2 + d_{ai}^{*\,2}\right) \\ &+ (6\beta_B - 6\beta_1 + 2\beta_4)\,|d|^2 + (4\beta_1 + 2\beta_2)\,|t|^2 + \beta_2\left(t^2 + t^{*2}\right) \\ &+ 2\beta_{35} d_{ij} d_{ji}^* + \beta_4 \left(d_{ij} d_{ji} + h.c.\right)\Big]\,. \end{aligned} \tag{12.33}$$

Here t denotes the trace of d_{ai}. The linear terms have been left out since they are all of the form $t + t^*$ and cancel at the extremum. Moreover, with the equilibrium value of a^2 being $\alpha/\left(\frac{6}{5}\beta_B\right)$, the first term simply cancels the $6\beta_B$ term inside the bracket.

Neglecting strong-coupling corrections, the expression simplifies to Eq. (he-15.97)

$$\begin{aligned}&\delta^2 f/2f_c|_{pot} = \\ &\frac{a^2}{15}\left\{5|d|^2 + (d_{ij}d_{ji} + h.c.) + \left(t^2 + t^{*2}\right) - \frac{3}{2}\left(d_{ai}^2 + h.c.\right)\right\}.\end{aligned} \tag{12.34}$$

The piece containing the gap distortion gives an additional Eq. (he-15.98)

$$\begin{aligned}&\frac{a^2}{15}\Big\{\beta_1\left[4r^2|d_{33}|^2 + 4r\left(td_{33}^* + h.c.\right)\right] \\ &+\beta_2\left[2r^2|d_{33}|^2 + 2r\left(td_{33}^* + h.c.\right) + 2r\,(2+r)\,|d|^2\right] \\ &+\beta_3\left[2r^2|d_{33}|^2 + 2r\left(d_{i3}d_{3i}^* + c.c.\right) + 2r(2+r)|d_{a3}|^2\right] \\ &\beta_4\left[2r^2|d_{3i}|^2 + 4r|d_{3i}|^2 + 2r(2+r)|d_{a3}|^2\right] \\ &\beta_5\left[2r^2|d_{33}|^2 + 2r\left(d_{3i}d_{i3}^* + c.c.\right) + 2r(2+r)|d_{3i}|^2\right] \\ &\beta_1 r(2+r)\left(d^2 + h.c.\right) \\ &\beta_2\left[r^2\left(d_{33}^2 + h.c.\right) + 2r\left(ta_{33} + h.c.\right)\right] \\ &\beta_3\left[r^2\left(d_{3i}^2 + c.c.\right) + 2r\left(d_{3i}^2 + h.c.\right)\right] \\ &\beta_4\left[r^2\left(d_{33}^2 + h.c.\right) + 2r\left(d_{i3}d_{3i} + h.c.\right)\right] \\ &\beta_5 r(2+r)\left(d_{a3}^2 + c.c.\right)\Big\}\end{aligned} \tag{12.35}$$

Without strong-coupling effects this simplifies considerably leaving only Eq. (he-15.99)

$$\begin{aligned}&\frac{1}{15}\Big\{2\left(c^2-1\right)\left(|d|^2 + 2|d_{a3}|^2\right) \\ &-\frac{1}{2}\left(c^2-1\right)\left(d^2 + h.c.\right) + 2\,(c-1)^2\left(d_{33}^2 + h.c.\right) \\ &+2\,(c-1)\left(td_{33} + h.c.\right) + \left(c^2-1\right)\left(d_{3i}^2 + h.c.\right) \\ &-\left(c^2-1\right)\left(d_{a3}^2 + h.c.\right)\Big\}\end{aligned} \tag{12.36}$$

Eq. (he-15.100) where we have made use of $a = 1$ so that

$$r = c - 1. \tag{12.37}$$

Eq. (he-15.101) The result can be written in matrix form

$$15\frac{\delta^2 f}{2f_c} = r_{ai}R_{ai,a'i'}r_{ai} + i_{ai}I_{ai,a'i'}i_{a'i'} \tag{12.38}$$

Eq. (he-15.102) where we have separated d into real and imaginary parts

$$d_{ai} = r_{ai} + ii_{ai} \tag{12.39}$$

The matrix R may be decomposed as $R \times R^{12} \times R^{13} \times R_{23}$ where R is a 3×3 submatrix acting only in the space $\begin{pmatrix} r_{11} \\ r_{23} \\ r_{33} \end{pmatrix}$ while $R^{12},\ R^{13}\ , R^{23}$ are 2×2 blocks in

Eq. (he-15.103) the subspaces

$$\begin{pmatrix} r_{12} \\ r_{21} \end{pmatrix} \begin{pmatrix} r_{13} \\ r_{31} \end{pmatrix} \begin{pmatrix} r_{23} \\ r_{32} \end{pmatrix}. \tag{12.40}$$

An analogous decomposition holds for I. Collecting the different contribution we

Eq. (he-15.104) find

$$\begin{aligned}
R &= \begin{pmatrix} 5+c^2 & 2 & 2c \\ 2 & 5+c^2 & 2c \\ 2c & 2c & 9c^2-3 \end{pmatrix} \\
R^{12} &= \begin{pmatrix} c^2+1 & 2 \\ 2 & c^2+1 \end{pmatrix} \\
R^{\substack{13\\23}} &= \begin{pmatrix} 3c^2-1 & 2c \\ 2c & 3c^2-1 \end{pmatrix} \\
I &= \begin{pmatrix} 1+3c^2 & -2 & -2c \\ -2 & 1+3c^2 & -2c \\ -2c & -2c & 1+3c^2 \end{pmatrix} \\
I^{12} &= \begin{pmatrix} 3c^2+5 & -2 \\ -2 & 3c^2+5 \end{pmatrix} \\
I^{\substack{13\\23}} &= \begin{pmatrix} -1+9c^2 & -2c \\ -2c & 7+c^2 \end{pmatrix}
\end{aligned} \tag{12.41}$$

In the absence of a current, we have $c = 1$ and can recover immediately the eigen-

Eq. (he-15.105) values:

$$\begin{aligned}
&R : (10,\ 4,\ 4) \\
&R^{12,13,23} : (0, 4) \\
&I : (0,\ 6,\ 6) \\
&I^{12,\ 13,\ 23} : (6, 10)\,.
\end{aligned} \tag{12.42}$$

We observe the occurrence of 4 Nambu-Goldstone modes corresponding to overall phase oscillations (sound) and three vibrations of the order parameter θ, one for the length and two for the direction.

These correspond to the residual part of the original $\mathrm{SO}(3)_{spin}\times$ $\mathrm{SO}(3)_{orbit} \times U(1)_{phase}$ symmetry left unbroken by the isotropic parameter A^0_{ai}, of the B-phase.

The strong-coupling corrections change the eigenvalues only slightly. Since the Nambu-Goldstone are a consequence of the symmetry of the action and A^0_{ai}, their eigenvalues remain exactly zero. Collecting the different terms in (12.35) we find the corrected matrices Eq. (he-15.106)

$$
\begin{aligned}
R &= 4\begin{pmatrix} \beta_{12345} & \beta_{12} & \beta_{12} \\ \beta_{12} & \beta_{12345} & \beta_{12} \\ \beta_{12} & \beta_{12} & \beta_{12345} \end{pmatrix} \frac{\alpha}{\frac{6}{5}\beta_B} \\
R^{12,13,23} &= 2\beta_{345}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{\alpha}{\frac{6}{5}\beta_B} \\
I &= -4\beta_1 \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \frac{\alpha}{\frac{6}{5}\beta_B} \\
I^{12,13,23} &= 2\begin{pmatrix} -6\beta_1 - \beta_{35} + \beta_4 & \beta_{35} - \beta_4 \\ \beta_{35} - \beta_4 & -6\beta_1 - \beta_{35} + \beta_4 \end{pmatrix} \frac{\alpha}{\frac{6}{5}\beta_B}
\end{aligned}
\tag{12.43}
$$

with eigenvalues Eq. (he-15.107)

$$
\begin{aligned}
&R : (12\beta_B, \quad 4\beta_{345}, \quad 4\beta_{345}) \frac{\alpha}{\frac{6}{5}\beta_B} \\
&R^{12,13,23} : \beta_{345}(0,4)\frac{\alpha}{\frac{6}{5}\beta_B} \\
&I : -2\beta_1(0,6,6)\frac{\alpha}{\frac{6}{5}\beta_B} \\
&I^{12,13,23} : (-12\beta_1, \quad -12\beta_1 + 4\left(\beta_4 - \beta_{35}\right)) \frac{\alpha}{\frac{6}{5}\beta_B}.
\end{aligned}
\tag{12.44}
$$

Remember that $\frac{\alpha}{\frac{6}{5}\beta_B}\Delta_B^{\mathrm{w}}$ represents the corrected gap value of the B-phase.

Notice that if β_{345} were zero there would be two more zero-frequency modes in R. This fact is associated with the accidental degeneracy of polar and planar phase at $\beta_{345} = 0$: the two modes correspond to linear interpolations between these two phases.

Let us now turn on the current. Then we have to add the fluctuations from the term Eq. (he-15.108)

$$
-\frac{5j^2}{|d|^2 + |d_{a3}|^2}
\tag{12.45}
$$

which in equilibrium contribute inside the curly brackets of (12.35) Eq. (he-15.109)

$$\frac{3\kappa^2}{2a^2+3c^2}\left[\left(|d_{a1}|^2+|d_{a2}|^2+3|d_{a3}|^2\right)\left(2a^2+3c^2\right)\right.$$
$$-4a^2\left(r_{11}^2+r_{22}^2\right)-36c^2{r_{33}}^2$$
$$\left.-8r_{11}r_{22}a^2-24ac\left(r_{11}+r_{22}\right)r_{33}\right]. \tag{12.46}$$

Eq. (he-15.110) Without strong-coupling corrections $a=1$ this adds directly

$$\left(1-c^2\right)\left\{\frac{1}{2+3c^2}\begin{pmatrix} 3c^2-2 & -4 & -12c \\ -4 & 3c^2-2 & -12c \\ -12c & -12c & 6-27c^2 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & 3 \end{pmatrix}\right.$$
$$\left.\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\right\} \tag{12.47}$$

Eq. (he-15.111) into $R,\ I,\ R^{12},\ I^{12},\ R^{13}_{23},\ I^{13}_{23}$ so that we obtain the new matrices:

$$R=\begin{pmatrix} 22c^2+8 & 2\left(c^2+4\right) & 2c\left(9c^2-4\right) \\ 2\left(c^2+4\right) & 22c^2+8 & 2c\left(9c^2--4\right) \\ 2c\left(9c^2-4\right) & 2c\left(9c^2-4\right) & 6c^2\left(9c^2-4\right) \end{pmatrix}$$
$$R^{12}=\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$
$$R^{12}_{23}=\begin{pmatrix} 2 & 2c \\ 2c & 2c^2 \end{pmatrix}$$
$$I=\begin{pmatrix} 2\left(1+c^2\right) & -2 & -2c \\ -2 & 2\left(1+c^2\right) & -2c \\ -2c & -2c & 4 \end{pmatrix}$$
$$I^{12}=\begin{pmatrix} 2c^2+6 & -2 \\ -2 & 2c^2+6 \end{pmatrix}$$
$$I^{13}_{2o3}=\begin{pmatrix} 6c^2+2 & -2c \\ -2c & 8 \end{pmatrix}. \tag{12.48}$$

Eq. (he-15.112) The eigenvalues are now

$$R:\left(\frac{1}{5}\left[\left(27c^4+8\right)\pm\frac{1}{3}\sqrt{\left(9c^2\right)^4-8\left(9c^2\right)^3-48\left(9c^2\right)^2+512\left(9c^2\right)+576}\right],4c^2\right)$$
$$R^{12}:(0,4)$$
$$R^{13}_{23}:\left(0,2\left(1+c^2\right)\right)$$
$$I:\left(0,\frac{2}{5}\left(2+c^2\right),\frac{2}{5}\left(2+c^2\right)\right)$$
$$I:\left(2c^2+4,2c^2+8\right)$$
$$I^{13}_{23}:\left(3c^2+5\pm\sqrt{c^4-\frac{14}{9}c^2+1}\right). \tag{12.49}$$

for increasing current, $c^2 = 1 - 3\kappa^2$ decreases and with it also the eigenfrequencies. At the critical current $\kappa_c^2 = 5/27$ the value of c^2 drops to 4/9 and the eigenvalues becomes

Eq. (he-15.113)

$$
\begin{aligned}
&R : \left(0, \frac{16}{3}, \frac{16}{9}\right) \\
&R^{12} : (0, 4) \\
&R^{\substack{13 \\ 23}} : \left(0, \frac{26}{9}\right) \\
\\
&I : \left(0, \frac{44}{45}, \frac{44}{45}\right) \\
&I^{12} : \left(\frac{44}{9}, \frac{62}{9}\right) \\
&I^{\substack{13 \\ 23}} : (4.2, 7.04)\,.
\end{aligned}
\tag{12.50}
$$

The zero eigenvalue in R signalizes the instability for decay into the planar (or A) phase.

Conclusion

We have only presented an introduction into the wide field of ^{3}He physics which has been developed in recent years. The methods used in describing the physical properties of the superfluid run hand in hand with those which are popular nowadays in particle physics and field theory. For a particle physicist it can be rewarding to study some of the phenomena and their explanations since it may provide him with a more transparent understanding of σ-type of models. Also, the visualization of functional field spaces in the laboratory may lend a more realistic appeal to topological considerations which have becomes a current tool in the analysis of solutions of gauge field equations.

Finally, there may even be direct applications of superfluid ^{3}He in particle physics [11]. Due to the fact that the condensate is characterized by two vectors $\mathbf{L}$ and $\mathbf{S}$, there is a vector $\mathbf{L} \times \mathbf{S}$ which T invariant but parity violating. How, if there are neutral currents of this symmetry type in weak interactions they will, in general, build up a small electric dipole moment in the Cooper pairs. This has to be aligned necessarily with $\mathbf{L} \times \mathbf{S}$. In the condensed phase of the superfluid, this very small dipole moment can pile up coherently and might result in an observable microscopid dipole moment. This could lead to a more sensitive test than those available right now. Unfortunately, the uncertainty in the Cooper pair wave function is, at present, an obstacle to a reliable estimate of the effect. Also, the detection of the resulting macroscopic dipole moment may be hampered by competing orientational effects.

Notes and References

Notes and References

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Preface

Since its discovery in 1972 [1], the superfluid phases of ^{3}He have attracted increasing experimental and theoretical attention. On the one hand, there is the practical challenge of achieving and maintaining ultra-low temperatures. The observed phenomena show a macroscopic system in an anisotropic quantum state with rather interesting collection excitations. Many surprising properties have been found and probably wait for their discovery in the future. On the other hand, there is a beautiful field of application for many theoretical methods developed in recent years in different branches of physics. At the microscopic level there is the problem of passing from the fundamental action involving ^{3}He atoms to an alternative, equivalent, form in which the collective excitations can be studied most directly. A very convenient method is provided by Feynman's [2] path integral description of quantum phenomena. In this formulation no operators appear and the problem can be solved exactly by mere changes of integration variables. The necessary technical framework was invented a long time ago in field theory [3]. It was introduced into many-body theory only a few years later [4] and has recently found new and interesting applications. The reasons for this revival are the following:
In high-energy physics the proliferation of particles which a decade ago were called elementary has led to postulating a new set of more fundamental objects, called quarks. Within an underlying quark world the strongly interacting particles, the mesons and baryons, may be considered as collective excitations. Very similar collective excitations are found in in various macroscopic quantum systems (for example supercvonductors, superfluid ^{4}He and ^{4}He). The main difference with respect to the quark system is that in these systems the collective excitation spectrum is much simpler due to the rather short range of the fundamental interaction. In contrast to this the physics of the quark system is governed by infinite range forces as manifested by the phenomenon of quark confinement, and this gives rise to a very complex spectrum of collective excitations. For the quark physicist, the obove many-body systems are an important and elementary testing ground for some aspects of his theoretical tools. The important common feature of the two so different domains of physics is that the ground state of the systems is of lower symmetry than the fundamental action defining the theory. Associated with it there are massless particles called Nambu-Goldstone bosons. In superfluid ^{4}He they are known as zero sound, in hadron physics they are the almost massless pions, the slight mass being due to a small symmetry breaking term in the action) [5]. A first crude application of the path integral methods to quark theories has rendered correctly many interesting properties of low lying mesons [6]. In nuclear physics, the interplay be-

tween fundamental constituents (the nucleons) and their collective excitations (pair and multipole vibrations) has been the subject of many investigations [7]. Also, there changes of field variables from fundamental to collective via path integrals have solved a number of problems [8]. These methods could provide for the most economic and theoretically satisfactory approach to fission problems [9]. Also in abstract field theory the connection between different field formulations for one and the same sytem has recently becomes an important object of study. Most results have been found only in one space and one time dimensions. Some of them may be relevant to the understanding of physical systems in higher dimensions but with restricted geometries, for example layers of superfluid ^{4}He or ^3He. The first work in this direction was the original exact solution of the Thirring model in which the four-fermion theory $\underline{1} + \underline{1}$ dimensions was reduced to a free massless boson theory. Later, quantum electrodynamic in the same space was shown to be equivalent to a free boson [11] with a mass. Finally, it was discovered [12] that introducing a fermion mass into the Thirring model is exactly equivalent to adding a interaction term proportional to the cosine of the field to the free bose theory (whose field equation leads to the Sine-Gordon theory). This work stimulated a great number of investigations whose most spectacular success was finally the discovery of the exact analytic S-Matrix as well as the determination of all physical states in the Hilbert space for many two-dimensional theories [13]. Also these transformations from Fermi to Bose fields are examples for the transition from fundamental to collective field variables any may be performed most simply via path integrals. Once for correct collective field theory is obtained, there is the interesting problem of extracting information from it, reexplaining known and predicting new experimental phenomena. In the hydrodynamic limit of the theory, the superfluid behaves like a combination of a perfect fluid and a liquid crystal. In the absence of flow, there are various kinds of novel extended field configurations, (called textures, just as in liquid crystals [14], which can be created in the laboratory and which show characteristic responses to a variety of external signals [mainly nuclear magnetic resonance (NMR)]. The investigation of such textures has many parallels with recent studies of extended objects in nonlinear field theories, in particular the non-ablian gauge theories of the Yang-Mills type [15] which are now believed to constitute the correct fundamental theory of strong and weak interactions between quarks (and therefore hadrons). In both fields, topological arguments are helpful in achieving some understanding of the great variety of observable phenomena. Field configurations which classically may be stable for topological reasons can decay via thermodynamic or quantum mechanical fluctuations. Also here is a wide field of applications of path integral methods (completely independent of the previously discussed change of integration variables), whose development goes hand in hand with that in several other branches of theoretical physics. The decay of superflow in superconductors [16], first-order phase transitions like the boiling of a superheated liquid [17], decay of "false vacua" in field theory [18], and the communications of an infinity of equivalent vacua in Yang-Mills theory [19] are prime examples. Superfluid ^{3}He with its several different phase transitions may offer the possibility of studying experimentally the duality

FIGURE 12.1 The remaining hydrodynamic parameters of superfluid ^{3}He-A are shown as a function of temperature together with their Fermi liquid corrected values.

properties between ordered and a disordered phases which have recently come under intensive study in theoretical models [20] as well as in gauge theories [21]. Here the theoretical problem is to gain information on one side of the phase boundary (say on the disordered phase) by studying the extended objects known in the other side (say in the ordered phase). In gauge theory this is of prime importance: The theory is valid only in the phase in which quarks are almost free but physics takes place in the strong-coupling regime where quarks can never become free. It appears that the knowledge of vortex-like solutions to the gauge field equations may help in deriving some of the strong-interaction properties of hadrons [22]. The equivalent problem in ^{3}He would be to try and predict the properties of the B phase by knowing the properties of the textures in the A phase. This is not an impossible task since there are always vortex-like objects whose cores contain the phase in question. In fact, in some systems like ^{4}He the phase transition seems to be related to the accumulation of such vortons, the other phase being reached when the "cores" of the vortex lines have spread over the whole liquid. Related to this is the solid-liquid phase transition of melting. Here lines of crystal defects proliferate and liberate the atoms from their crystal sites [23]. Certainly, we are as yet quite far from a complete understanding of all these phenomena. A detailed study of the superfluid ^{3}He, however, may generate much insight in these phenomena which could be harder to obtain otherwise.

Fig. XXX

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Part IV

Hadronization of Quark Theories

In this Part we shall study a simple model of quantum field theory which shows how quark theories can be converted into bilocal field theories via functional techniques. The new basic field quanta of the converted theory are approximate the quark-antiquark meson bound states. They are obtained by solving the Bethe-Salpeter bound-state equation in the ladder approximation. They will be called *bare mesons*. Mesonic Feynman graphs are developed which strongly resemble dual diagrams. In the limit of heay gluon masses, the bilocal fields become local and describe π, ρ, A_1, and σ-mesons in a chirally invariant Lagrange density, which has been known for a long time from the so-called SU(3)-symmetric linear σ-model.

Many interesting relations are found between meson and quark properties such as $m_\rho^2 \approx 6M^2$, where M is the non-strange quark mass *after* spontaneous breakdown of chiral symmetry. There is a simple formula linking these quark masses with the small *bare masses* appearing as parameters in the Lagrange density. The quark masses also determine the vacuum expectations of scalar densities.

1

Introduction

In attempting to understand the physics of strongly interacting particles, the *hadrons*, two fundamentally different theoretical approaches have been developed. One of them, the dual approach, is based on complete democracy among all strongly interacting particles. Within this approach, an elaborate set of rules asures the construction of certain lowest order vertex functions for any number of mesons [1]. The other approach, assumes the existence of a local field equation involving fundamental quarks bound together by vector gluons [2]. Here strong interaction effects on electromagnetic and weak currents of hadrons can be analyzed in a straight-forward fashion without detailed dynamical computations [3]. Either approach has its weakness where the other is powerful. Dual models have, until now, given no access to currents while quark theories have left the problem of mesonic vertex function intractable. Not even an approximate bound state ucalculation is available (except in $1+1$ dimensions [4] or by substituting the field couplings by simple ad-hoc forces [5].

At present there is hope that the problems connected with quark models are of a purely technical nature. A Lagrangian field theory of Yang-Mills type seems to have a good chance of defining a true fundamental theory of elementary particles. Dual models, on the other hand, seem to be of a more phenomenological character. Once the fundamental vertices are determined, it is difficult to find next corrections and to extend the prescriptions to what might be called a complete theory. If this could be done it would certainly have to be phrased in terms of local infinite-component or multi-local fields [6].

It would be very pleasing if both models were, in fact, essentially equivalent both being different languages for one and the same underlying dynamics. In this case one could use one or the other depending on whether one wants to answer short- or long-distance questions concerning quarks.

In order to learn how a translation between the different languages might operate we shall consider, in these lectures, the simplified field theory in which quarks are colorless, have N flavours, and are held together by vector gluons of arbitrary mass μ. This theory incorporates several realistic features of strong interactions, for example current algebra and PCAC. Moreover, the case $N = 1$ and $\mu = 0$ includes ordianry quantum electrodynamics (Q.E.D.). This will provide a good deal of intuition as well as the possibility of a detailed test of our results.

We shall demonstrate how functional methods can be employed to transform the local quark gluon theory into a new completely equivalent field theory involving only bilocal fields. The new free field quanta coincide with quark-antiquark bound states when calculated by ladder exchanges only. They may be considered as *bare mesons*. Accordingly, the transition from the local quark- to the bilocal meson-theory will be named *mesonization*. In the special case of QED, bare mesons are positronium atoms in ladder approximation.

The functional technique will ensure that bare mesons have exactly the correct interactions among each other in order that mesonization preserves the equivalence to the original quark gluon theory. It is simple to establish the connection between classes of Feynman graphs involving quarks and gluons with single graphs involving mesons. The topology of meson graphs is the same as that of dual diagrams. It is interesting to observe the appearence of a current-meson field identity for photons just as employed in phenomenological discussions of vector meson dominance. Moreover, since the theory is bilocal, this identity can be extended to bilocal currents which are measured in deeply inelastic electromagnetic and weak interactions.

The limit of a very heavy gluon mass can be mesonized most simply. Here the bilocal fields become local and describe only a few mesons with the quantum numbers of $\sigma-$, $\pi-$, $\rho-$, A_1- mesons. The Lagrange density coincides with that of the standard chirally invariant σ-model which is known to account quite well for the low-energy aspects of meson physics. Here mesonization renders additional connection between quark and meson properties. It also makes transparent the connection between the very small bare quark masses (which describe the explicit breakdown of chiral symmetry) and the mechanical quark masses (which include the dynamic effects due to spontaneous symmetry violations).

2

Abelian Quark Gluon Theory

Consider now a system of N quarks $\psi(x)$ held together by one gluon field $G^\nu(x)$ of mass μ via a Lagrangian

$$\begin{aligned}\mathcal{L}(x) &= \bar\psi(x)\,(i\partial\!\!\!/ - \mathcal{M})\,\psi(x) + g\bar\psi(x)\gamma_\nu\psi(x)G^\nu(x)\\ &\quad -\frac{1}{4}F_{\mu\nu}^2(x) + \frac{\mu^2}{2}G_\nu^2.\end{aligned}\tag{2.1}$$

Here $F_{\mu\nu}$ is the usual curl $\partial_\mu G_\nu - \partial_\nu G_\mu$. In the special case in which $N=1,\ \ \mu=0$, and $g^2=4\pi\alpha$, this Lagrangian describes quantum electrodynamics. In other cases it may be considered as a model field theory which carries many interesting properties of strong interactions, for example approximate $SU(3)$ symmetry, chiral $SU(3)\times SU(3)$ current algebra, PCAC, and scaling up to small corrections. Certainly, this model will never be able to confine quarks, give symmetric baryon wave functions, and explain infinitely rising meson trajectories. For this it would have to contain an additional, exactly conserved, color symmetry with $G^\nu(x)$ being its non-abelian gauge mesons. Before attempting to deal with this far more complicated situation we shall develop [10] our tools for the less realistic but much simpler model (2.1) without color.

The generating functional of all time ordered Green's function is

$$Z\,[\eta,\ \bar\eta, j^\nu] = \text{const }\times\int\mathcal{D}\psi\mathcal{D}\bar\psi\mathcal{D}Ge^{i\int dx\left(\mathcal{L}+\bar\psi\eta+\bar\eta\psi+gj^\nu G_\nu\right)}\tag{2.2}$$

The exponent is quadratic in $G^\nu(x)$, such that the functional integration over the gluon field can be performed [7] [8] using formulas (1.31) and (1.32) whose relativistic generalization was given in Section 2.5. The result is

$$Z\,[\eta,\bar\eta,j^\nu] = \text{const }\times\int\mathcal{D}\psi\mathcal{D}\bar\psi e^{i\mathcal{A}[\psi,\psi,\eta,\eta,j^\nu]}\tag{2.3}$$

with an action

$$\begin{aligned}\mathcal{A}\left[\psi,\bar\psi,\eta,\bar\eta j^\nu\right] &= \int dxdy\Big\{\left(\mathcal{L}(x)+\bar\psi(x)\eta(x)+\bar\eta(x)\psi(x)\right)\delta(x-y)\\ &\quad -\frac{i}{2}g^2D(x-y)\left(\bar\psi(x)\gamma^\nu\psi(x)+j^\nu(x)\right)\left(\bar\psi(y)\gamma_\nu\psi(y)+j_\nu(y)\right)\Big\}.\end{aligned}\tag{2.4}$$

By employing the Fierz identity:

$$\begin{aligned}\gamma^{\nu}_{\alpha\beta} \otimes \gamma_{\nu_{\gamma\delta}} &= 1_{\alpha\delta} \otimes 1_{\alpha\beta} + (i\gamma_5)_{\alpha\delta} \otimes (i\gamma_5)_{\gamma\beta} \\ &\quad -\frac{1}{2}\gamma^{\nu}_{\alpha\beta} \otimes \gamma_{\nu_{\gamma\beta}} - \frac{1}{2}(\gamma^{\nu}\gamma_5)_{\alpha\delta} \otimes (\gamma_{\nu}\gamma_5)_{\gamma\beta} \qquad (2.5)\end{aligned}$$

the quartic quark interaction term can be written in a different fashion

$$\frac{i}{2}g^2 D(x-y)\Big\{\bar{\psi}(x)\psi(y)\bar{\psi}(y)\psi(x) + \bar{\psi}(x)i\gamma_5\psi(y)\bar{\psi}(y)i\gamma_5\psi(x) \\ -\frac{1}{2}\bar{\psi}(x)\gamma^{\nu}\psi(y)\bar{\psi}(y)\gamma_{\nu}\psi(x) - \frac{1}{2}\bar{\psi}(x)\gamma^{\nu}\gamma_5\psi(y)\bar{\psi}(y)\gamma_{\nu}\gamma_5\psi(x)\Big\}. \qquad (2.6)$$

This will be written short as

$$\frac{i}{2}g^2 D(x-y)\bar{\psi}_{\alpha}(x)\psi_{\delta}(y)\xi_{\alpha\delta,\gamma\beta}\bar{\psi}_{\gamma}(y)\psi_{\beta}(x), \qquad (2.7)$$

where the matrix $\xi_{\alpha\delta,\gamma\beta}$ denotes the right-hand side of Eq. (2.5).

This is the point where our elimination of quark fields in favor of new bilocal fields starts.

Let $S(x,y)$, $P(x,y)$, $V^{\nu}(x,y)$, $A^{\nu}(x,y)$ be a set of hermitian auxiliary fields, i.e.

$$S(x,y) = S(y,x), \quad P(x,y) = P^{*}(y,x), \quad \text{etc.} \qquad (2.8)$$

With these field, we can construct the following functional identities [9]

$$\begin{aligned}&\int \mathcal{D}S(x,y)e^{-\frac{i}{2}|S(x,y)+ig^2D(x-y)\bar{\psi}(y)\psi(x)|^2/ig^2D(x-y)} = \text{const.}\ , \\ &\int \mathcal{D}P(x,y)e^{-\frac{i}{2}|P(x,y)+ig^2D(x-y)\bar{\psi}(y)i\gamma_5\psi(x)|^2/ig^2D(x-y)} = \text{const.}\ , \\ &\int \mathcal{D}V(x,y)e^{i|V^{\nu}(x,y)-\frac{i}{2}g^2D(x-y)\bar{\psi}(y)g^{\gamma}\psi(x)|^2/ig^2D(x-y)} = \text{const.}\ , \\ &\int \mathcal{D}A(x,y)e^{iA^{\nu}|S(x,y)+\frac{i}{2}g^2D(x-y)\bar{\psi}(y)\gamma^{\nu}\gamma_5\psi(x)|^2/ig^2D(x-y)} = \text{const.}\ , \qquad (2.9)\end{aligned}$$

which are independent of the fields $\psi(x)$. If we now multiply $Z[\eta,\ \bar{\eta}, j^{\nu}]$ in (2.3) by these constants, make use of (2.6), all quartic quark terms are seen to cancel. The generating functional becomes

$$Z[\eta,\ \bar{\eta},\ j^{\nu}] = \text{const} \times \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}S\mathcal{D}P\mathcal{D}V\mathcal{D}Ae^{i\tilde{\mathcal{A}}} \qquad (2.10)$$

where the new action $\tilde{\mathcal{A}}$ is obtained as an integral

$$\tilde{\mathcal{A}}\left[\psi,\bar{\psi},S,P,V,A,\eta,\bar{\eta},j^{\nu}\right] = \int dxdy\mathcal{L}(x,y) \qquad (2.11)$$

over a bilocal Lagrange density:

$$\begin{aligned}\mathcal{L}(x,y) \equiv & \left\{\bar{\psi}(x)\left(i\partial\!\!\!/ - \mathcal{M}\right)\psi(x) + \bar{\psi}(x)\eta(x) + \bar{\eta}(x)\psi(x)\right\}\delta^{(4)}(x-y) \\ & -\bar{\psi}(x)m(x,y)\psi(y) - \frac{i}{2}g^2 D(x-y)j^\nu(x)j_\nu(y) \\ & -\left\{\frac{1}{2}|S|^2 + \frac{1}{2}|P|^2 - |V|^2 - |A|^2\right\}\frac{1}{ig^2 D(x-y)}. \end{aligned} \tag{2.12}$$

Here $m(x,y)$ has been introduced as an abbreviation for the combined field

$$\begin{aligned} m(x,y) \equiv & S(x,y) + P(x,y)i\gamma_5 \\ & + \left(V^\nu(x,y) + \delta^{(4)}(x-y)\int dz i g^2 D(x-z) j^\nu(z)\right)\gamma_\nu + A^\nu(x,y)\gamma_\nu\gamma_5. \end{aligned} \tag{2.13}$$

Due to (2.8), the matrix $m(x,y)$ is self-adjoint in the sense

$$\left(\overline{m(x,y)}\right)_{\alpha\beta} \equiv \gamma_{0_{\alpha\alpha'}}\left(m^*(x,y)^T\right)_{\alpha'\beta'}\gamma_{0_{\beta'\beta}} = m_{\alpha\beta}(y,x). \tag{2.14}$$

At this place it is worth remarking that the Lagrangian (2.12) shows its equivalence to the previous form (2.4) also quite directly. Extremizing the action we obtain the Euler-Lagrange equations for the fields S, P, V, A are seen to be dependent fields coinciding with the corresponding bilocal quark expressions

$$\begin{aligned} S(x,y) &= -ig^2 D(x-y)\bar{\psi}(y)\psi(x) \\ P(x,y) &= -ig^2 D(x-y)\bar{\psi}(y)i\gamma_5\psi(x) \end{aligned} \tag{2.15}$$

$$V^\nu(x,y) = \frac{i}{2}g^2 D(x-y)\bar{\psi}(y)\gamma^\nu\psi(x) \tag{2.16}$$

$$A^\nu(x,y) = \frac{i}{2}g^2 D(x-y)\bar{\psi}(y)\gamma^\nu\gamma_5\psi(x), \tag{2.17}$$

which show the relation with with the corresponding bilocal quark expressions if fluctuationns are neglected.

Inserting these relations back into (2.12) reproduces (2.4). In the action (2.11), quark fields enter only in quadratic form such that they can be integrated according to formula (1.32). In the relativistic fermion version discussed in Section 2.5, the matrix A is given by [compare (2.127)]

$$A(x,y) = \left(-\partial\!\!\!/ - \mathcal{M}\right)\delta^{(4)}(x-y) - m(x,y). \tag{2.18}$$

Hence $A^{-1}(x,y) \equiv -iG(x,y)$ is the Green function associated with the equation

$$\int dy\left[\left(i\partial\!\!\!/ - \mathcal{M}\right)\delta^{(4)}(x-y) - m(x,y)\right]G(y,z) = i\delta^{(4)}(x-z). \tag{2.19}$$

With this notation, the quark integration brings the functional (2.10) to the form

$$Z\left[\eta,\bar{\eta},j^\nu\right] = \text{const} \times \int \mathcal{D}m(x,y)e^{i\mathcal{A}[\eta,\bar{\eta},j^\nu]} \tag{2.20}$$

with

$$\mathcal{A}\left[m,\eta,\bar{\eta},j^{\nu}\right] = \int dxdy \Big\{ -i\mathrm{Tr}\left(\ln iG^{-1}\right)(x,y)\delta(x-y) \tag{2.21}$$
$$-\frac{1}{2}\mathrm{Tr}\left(m(x,y)\xi^{-1}m(y,x)\right)\frac{1}{ig^2 D(x-y)} + i\bar{\eta}(x)G(x,y)\eta(y) - ig^2\frac{[\delta^{(4)}(x-y)]^2}{D(0)}$$
$$-\frac{2}{D(0)}V^{\nu}(x,x)D(x-y)j_{\nu}(y) + \int dzdz' D(z-x)D(y-z')j^{\nu}(z)j_{\nu}(z')\Big\}.$$

Here we have introduced, for brevity, the notation

$$\mathcal{D}m(x,y) \equiv \mathcal{D}S\mathcal{D}P\mathcal{D}V\mathcal{D}V. \tag{2.22}$$

Note that the effect of the matrix ξ^{-1} defined in Eq. (2.6) is simply to devide the projections into S,P,V,A by $4,-4,-2,2$, respectively.[1] The trace refers only to Dirac indices. The new functional (2.20) is identical to the original one in Eq. (2.2). As a consequence, a quantum theory based on the action (2.21) must be completely equivalent to the original quantized quark gluon theory.

A word is in order concering the internal symmetry $SU(N)$ among the N quarks $(i=1,\ldots N)$ under consideration. Since the gluon is an $SU(N)$ singlet, the interaction in Eq. (2.1) is $g\sum_{i=1}^{N}\bar{\psi}_i\gamma^{\nu}\psi_i G_{\nu}(x)$. In the Fierz transformed version (2.6) the indices i and j appear separated

$$\frac{i}{2}g^2 D(x-y)\bar{\psi}^j(x)\psi_i(y)\xi\bar{\psi}^i(y)\psi_j(x).$$

Hence, in the presence of N quarks, the fields $m(x,y)$ have to be thought of a matrices in $SU(N)$ space $m(x,y)_i{}^j$. This carries over to the action with the traces including Dirac as well as $SU(N)$ indices.

Let us now develop a quantum theory for the new action. In general, the field $m(x,y)$ may oscillate around some constant non-zero vacuum expectation value $m_0\delta^{(4)}(x-y)$. It is convenient to subtract such a value from $m(x,y)$ and introduce the field

$$m'(x,y) \equiv m(x,y) - m_0\delta^{(4)}(x,y). \tag{2.23}$$

With this and the definition

$$M \equiv \mathcal{M} + m_0, \tag{2.24}$$

Eq. (2.19) can be rewritten as

$$\int dy\left[\left(i\partial\!\!\!/ - M\right)\delta^{(4)}(x-y) - m'(x,y)\right]G(y,z) = i\delta^{(4)}(x-z). \tag{2.25}$$

[1] Since $\frac{1}{4}1\otimes 1, -\frac{1}{4}(i\gamma_5)\otimes(i\gamma_5), \frac{1}{4}\gamma^{\nu}\otimes\gamma_{\nu}, -\frac{1}{4}\gamma^{\nu}\gamma_5\otimes\gamma_{\nu}\gamma_5$ are the corresponding projection operators.

Now let us assume that the oscillations $m'(x,y)$ are sufficiently small as to permit a perturbation expansion for $G(x,y)$:

$$G(x,y) = \\ G_M(x,y) - i\left(G_M m' G_m\right)(x,y) - \left(G_M m' G_M m' G_M\right)(x,y) + \dots \quad (2.26)$$

where $G_M(x,y)$ are the usual propagators of a free fermion of mass M.

$$G_M(x,y) \equiv G_M(x-y) \equiv \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{i}{\not{p} - M}.$$

Using this expansion, the action (2.21) takes the form[2]

$$\mathcal{A}\left[m', \eta, \bar\eta, j^\nu\right] = \mathcal{A}_1\left[m'\right] + \mathcal{A}_2\left[m'\right] + \mathcal{A}_{\rm int}\left[m'\right] + \mathcal{A}_{\rm ext}\left[m', \eta, \bar\eta, j^\nu\right], \quad (2.27)$$

$$\mathcal{A}_1\left[m'\right] \equiv \int dxdy \mathrm{tr}\left[G_M(x-y)m'(x,y) - \xi^{-1}\xi^{-1}m'(x,y)m_0\delta(y,x)/ig^2D(x-y)\right] (2.28)$$

and $\mathcal{A}_2$ being quadratic in m'

$$\mathcal{A}_2\left[m'\right] \equiv \int dxdytr\left[\frac{i}{2}G_M m G_M m'(x,y) - \frac{1}{2}\xi^{-1}m'(x,y)m'(y,x)/ig^2D(x,y)\right]. (2.29)$$

The term $\mathcal{A}_{int}[m']$ collects all remaining powers in m'

$$\mathcal{A}_{int}[m'] \equiv \int dx \mathrm{Tr}\left[-\sum_{n=3}^{\infty} \frac{(-i)^{n+1}}{n}\left(G_M m'\right)^n(x,x)\right]. \quad (2.30)$$

The last piece $\mathcal{A}_{ext}$, finally, contains all interactions with the external sources

$$\mathcal{A}_{ext}\left[m', \eta, \eta, j^\nu\right] = \\ \int dxdy\left\{i\bar\eta(x)G(x,y)\eta(y) - \frac{i}{2}g^2D(x-y)j^\nu(x)j_\nu(y) - \frac{2}{D(0)}\right. \quad (2.31) \\ \left. V^\nu(x,x)D(x-y)j_\nu(y) - ig^2\frac{\delta^{(4)}(0)\delta^{(4)}(x-y)}{D(0)}\int dzdz'D(z-x)D(y-z')j^\nu(z)j_\nu(z')\right\}.$$

For the quantization we shall adopt an interaction picture. As usual, the quadratic part of the action, $\mathcal{A}_2[m']$, serves for the construction of free-particle Hilbert space. According to the least action principle, the free equation of motion are obtained from $\delta\mathcal{A}_2[m']/\delta m'(x,y) = 0$ rendering

$$m'(x,y) = g^2\xi D(x-y)\left(G_M m' G_M\right)(x,y). \quad (2.32)$$

Going to momentum space

$$m'(p_{2,1}) \equiv \int dx_2 dx_1 e^{i(x_2p_2 - x_1p_1)} m'(x_2, x_1)$$

[2]A trivial additive constant has been dropped.

and introducing relative and total momenta

$$P \equiv (p_2 + p_1)/2, \quad q \equiv (p_2 - p_1)/2$$

together with the notation

$$m'(P|q) \equiv m'(p_2,\ p_1)$$

the field equation becomes

$$m'(P|q) = \xi g^2 \int \frac{d^4P'}{(2\pi)^4} D(P'-P)\, G_M\left(P' + \frac{q}{2}\right) m'(P'|q)\, G_M\left(P' - \frac{q}{2}\right). \quad (2.33)$$

In this form we easily recognize the Bethe Salpeter equation [11] in ladder approximation for the vertex functions of quark-antiquark bound states

$$\Gamma^H(P|q) \equiv N_H\left(P + \frac{q}{2}\right) \int dz e^{iPz} \langle 0|T\psi\left(\frac{z}{2}\right)\bar{\psi}\left(-\frac{z}{2}\right)|0\rangle G_M\left(P' - \frac{q}{2}\right) \quad (2.34)$$

where N_H is some normalization factor. As a consequence our free field $m'(x,y)$ can be expanded in a complete set of ladder bound state solutions. These are the bare quanta spanning the Hilbert space of the interaction picture. Because of their bound quark-antiquark nature, they are called bare mesons. In the special case of QED, the "quarks" are electrons and the bare mesons are positronium atoms.

For mathematical reasons it is convenient to solve (2.33) for *fixed* $q^2 \in (0,\ 4M^2)$ and all possible coupling constants g^2, to be called $g_H^2(q^2)$, i. e.

$$\Gamma^H(P|q) = \xi g_H^2\left(q^2\right) \int \frac{d^4P'}{(2\pi)^4} D(P-P')\, G_M\left(P' + \frac{q}{2}\right) \Gamma^H\left(P' - \frac{q}{2}\right). \quad (2.35)$$

A useful normalization condition is

$$-i\int \frac{d^4P}{(2\pi)^4} \mathrm{Tr}\left[G_M\left(P + \frac{q}{2}\right)\Gamma^H(P|q)\bar{\Gamma}^{H'}(P|q)\right] = \epsilon^H \delta^{HH'}. \quad (2.36)$$

Here we have allowed for a sign factor $\epsilon^H(q)$ which cannot be absorbed in the normalization N_H of (2.34). It may take the values $+1$, -1 or zero. Then the expansion of the free field $m'(x,y)$ in terms of meson creation and annihilation operators $a_H^+(\mathbf{q}),\ a_H(\mathbf{q})$ can be written as

$$\begin{aligned} m'_{\alpha\beta}(x,y) \quad = \quad & \int \frac{d4^4q}{(2\pi)^4} \sum_H \delta^{(+)}\left(g_H^2\left(q^2\right) - g^2\right) \int \frac{d^4P}{(2\pi)^4} \\ & \left\{e^{-i(q(x+y)/2+P(x-y))}\Gamma^H(P|q) n_H a_H(\mathbf{q})\right. \\ & \left. e^{-i(q(x+y)/2-P(x-y))}\bar{\Gamma}^H(P|-q) n_H^* a_H^+(\mathbf{q})\right\} \end{aligned} \quad (2.37)$$

where n_H are appropriate factors giving $a_H(\mathbf{q})$ the standard normalization

$$\left[a_H(\mathbf{q}),\quad a_{H'}^+(\mathbf{q}')\right] = (2\pi)^3\delta^{(3)}(\mathbf{q}-\mathbf{q}')\, 2\omega^H(\mathbf{q})\epsilon^H(q). \quad (2.38)$$

Now the sign factor $\epsilon^H(q)$ appears at the norm of the mesonic state $a_H^+|0\rangle \equiv |H\rangle$. In general there will be many states with unphysical norms since the "bare mesons" are produced by ladder diagrams only and may not be directly related to physical particles. This situation presents no fundamental difficulty. There are many interactions among bare mesons which are capable of excluding unphysical states from the S-matrix. In fact, the equivalence of the mesonized theory to the healthy original quark gluon version is a guarantee for physical results (on shell).

The propagator of the free field $m'(x,y)$ can be found most directly by adding an external disturbance to the free action

$$\mathcal{A}_2[m'] \to \mathcal{A}_2[m'] - \int dxdy \mathrm{Tr}\left[m'(x,y)J(y,x)\right]. \tag{2.39}$$

This current enters the equation of motion as

$$\begin{aligned} m'(x,y) &= \xi g^2 D(x-y)\left(G_M m' G_M\right)(x,y) \\ &\quad -\xi i g^2 D(x-y) J(x,y). \end{aligned} \tag{2.40}$$

The propagator $\mathbf{G}_{\alpha\beta,\ \alpha'\beta'}(xy,\ \ x'y') \equiv \dot{m}'_{\alpha\beta}(x,y)\dot{m}'{}_{\alpha\beta}(x',y')$ is then defined as the solution of (2.40) for the δ-function disturbance

$$J_{\alpha\beta}(x,y) = i\delta(x-x')\delta(y-y')\delta_{\alpha\alpha'}\delta_{\beta\beta'}. \tag{2.41}$$

It satisfies the inhomogeneous Bethe-Salpeter equation

$$\begin{aligned} &\mathbf{G}_{\alpha,\ \beta,\ \alpha'\beta'}(x,y;\ \ x'y') = \\ &\quad \xi_{\alpha\beta,\ \alpha'\beta'} D(x-y) \int d\bar{x} d\bar{y} G_M(x-\bar{x})_{\alpha\bar{\alpha}} \\ &\quad \mathbf{G}_{\bar{\alpha}\bar{\beta},\ \alpha'\beta'}(\bar{x},\bar{y};\ x'y')\, G_M(\bar{y}-y)_{\bar{\beta}\beta} + \xi_{\alpha\beta,\ \beta'\alpha'} g^2 D(x-y)\delta'(y-y'). \end{aligned} \tag{2.42}$$

This is immediately recognized as the equation for the two-quark transition matrix in ladder approximation (see Eq. (3A.22) in App. A.

An explicit representation of the Green's function in terms of the solutions $\Gamma^H(P|q)$ of the homogeneous equation (2.35) can now be given.

If $\mathbf{G}_{\alpha\beta,\ \alpha'\beta'}(P,P'|q)$ denotes the Fourier transform

$$\begin{aligned} &(2\pi)^4 \delta^{(4)}(q-q')\, \mathbf{G}_{\alpha\beta,\alpha'\beta'}(P,P'|q) \equiv \\ &\quad \int dxdydx'dy' e^{i[P(x-y)+q(x+y)/2--P'(x'-y')-q')x_+y')/2]} \mathbf{G}_{\alpha\beta,\alpha'\beta'}(x,y;\ x'y') \end{aligned} \tag{2.43}$$

it can be written as the sum over all meson solutions:

$$\mathbf{G}_{\alpha\beta,\alpha'\beta'}(P,P'|q) = -ig \sum_H \epsilon_H(q) \frac{\Gamma^H_{\alpha\beta}(P|q)\bar{\Gamma}^H_{\beta'\alpha'}(P'|-q)}{g_H^2(q^2)-g^2} \tag{2.44}$$

where the sum comprises possible integrals over a continuous set of solutions. If quarks and gluons were scalars, the sum would be discrete for $q^2 \in (0, 4M^2)$ since the kernel of the integral equation (2.35) would be of the Fredholm type. A more detailed discussion is given in Appendix A. Here we only note that a power series expansion of the denominator

$$\mathbf{G}_{\alpha\beta,\alpha'\beta'}(P,P'|q) = \\ -i\sum_{n=1}^{\infty}\sum_{H}\left(\frac{g^2}{g_H^2(q^2)}\right)^n \epsilon_H(q)\Gamma^H_{\alpha\beta}(P|q)\bar{\Gamma}^H_{\beta'\alpha'}(P'|-q) \tag{2.45}$$

renders explicit the exchange of one, two, three, etc. gluons. Hence one additional gluon can be inserted (or removed) by multiplying (or dividing) (2.44) by a factor $g^2/g_H^2(q^2)$. This fact will be of use later on.

Seen microscopically in terms of quarks and gluons, the free meson propagator (2.44) is given by the sum of ladders (see Fig. 2.1) Graphically, it will be represented

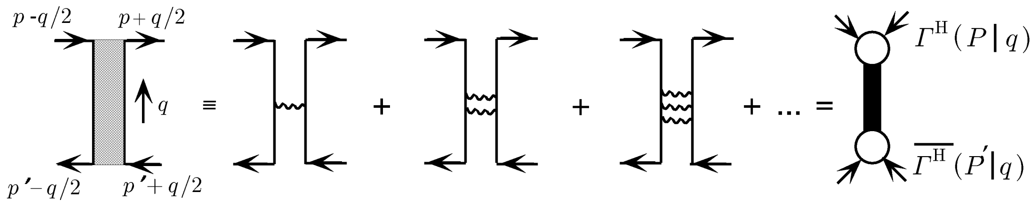

FIGURE 2.1

by a wide band. In the last term of Fig. 2.1 we have also given a visualisation of the expansion (2.44). Here, the fat line denotes the propagator

$$\Delta_H(q) = -i\epsilon_H(q)\frac{g^2}{g_H^2(q^2)-g^2} \tag{2.46}$$

while upper and lower bubbles stand for the Bethe Salpeter vertices $\Gamma^H(P|q)$ and $\Gamma^H(P'|-q)$, respectively. This picture suggests another way of representing the new bilocal theory in terms of an *infinite component* meson field depending only on the average position $X=(x+y)/2$. For this we simply expand the *interacting* field $m'(P|q)$ in terms of the complete set of free vertex function

$$m'(P|q) = \sum_H \Gamma^H(P|q)m_H(q). \tag{2.47}$$

Inserting this expansion into (2.27), the free action becomes directly

$$\mathcal{A}_2[m'] = \frac{1}{2}\int dX m_H(X)\left(1-g_H^2\left(q^2\right)/g^2\right)m_H(X) \tag{2.48}$$

implying the free propagator (2.46) for the field $m_H(X)$. With this understanding of the free part of the action we are now prepared to interprete the remaining pieces.

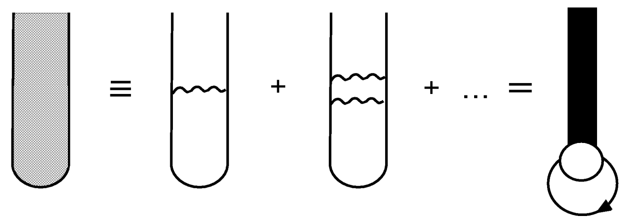

FIGURE 2.2

Consider first the linear part $\mathcal{A}_1[m']$. The first term in it can graphically be presented as shown in Fig. 2.2. When attached to other mesons it produces a tadpole correction. When interpreted within the underlying quark gluon picture, such a correction sums up all rainbow contributions to the quark propagator (see Fig. 2.3). Also the second term in $\mathcal{A}_1[m']$ has a straight-forward interpretation. First

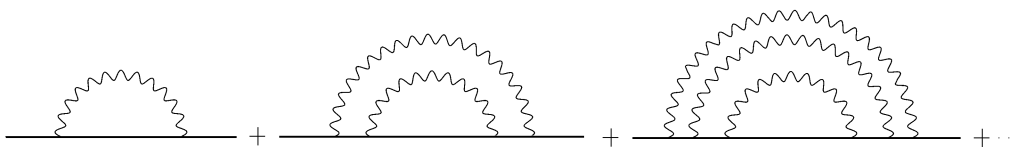

FIGURE 2.3

of all, the division by $\xi i g^2 D(x-y)$ has the effect of removing one rung from the ladder sum (such that the ladder starts with no rung, one rung, etc.) and creating two open quark legs. This can be seen directly from (2.35) and (2.44): Suppose a meson line ends at the interaction $-\int dxdy \text{Tr}\left[m'(x,y)\xi^{-1}m_0\right]/ig^2D(x-y)\delta(x-y)$. then the factor $\left[\xi i g^2 D(x-y)\right]^{-1}$ applied to $\mathbf{G}(P,P'|q)$ gives (leaving out irrelevant indices)

$$\left[\xi g^2 D\right]^{-1}\mathbf{G} = -ig^2\sum_H \epsilon_H \frac{\left[\xi g^2 D\right]^{-1}\Gamma^H\bar{\Gamma}^H}{g_H^2(q^2)-g^2}. \tag{2.49}$$

Using (2.35) this yields

$$= -ig^2\sum_H \epsilon_H \frac{g_H^2\left(q^2\right)}{g^2}\frac{\left(G_M\Gamma^H G_M\right)\bar{\Gamma}^H}{g_H^2\left(q^2\right)-g^2}. \tag{2.50}$$

As discussed before, the factor $g_H^2\left(q^2\right)/g^2$ amounts to the removal of one rung. Multiplication by $-m_0$ and integration over $\int dP|(2\pi)^4$ yields the total contribution of this meson graph

$$im_0\sum_H \frac{g_H^2\left(q^2\right)}{g_H^2\left(q^2\right)-g^2}$$
$$\int\frac{d^4P}{(2\pi)^4}\text{Tr}\left[G_M\left(P+\frac{q}{2}\right)\Gamma^H(P|q)G_M\left(P-\frac{q}{2}\right)\right]\epsilon_H(q)\bar{\Gamma}^H(P'|-q). \tag{2.51}$$

As far as quarks and gluons are concerned, this amounts to the insertion of a mass term m_0 on top of a ladder graph with one rung removed (this being indicated by a slash in Fig. 2.4. The quark gluon picture leads us to expect that m_0 must be a

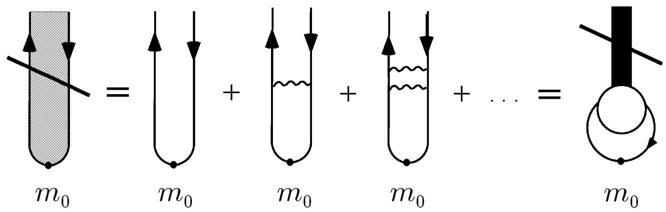

FIGURE 2.4

cutoff dependent quantity cancelling the logarithmic divergence in every upper loop of the ladder sum of Fig. 2.2. Numerically, m_0 is most easily calculated by cancelling the infinite contributed by $\mathcal{A}_1[m']$ to the equation of motion (2.33). If we include $\mathcal{A}_1[m']$, this equation reads

$$
\begin{aligned}
&m_0(2\pi)^4\delta^{(4)}(q) + m'(P|q) = \\
&\quad\left[\xi i g^2 \int \frac{d^4P'}{(2\pi)^4} D(P-P')G(P')\right](2\pi)^4\delta^{(4)}(q) \\
&\quad + \xi g^2 \int \frac{d^4P'}{(2\pi)^4} D(P-P')G_M\left(P'+\frac{q}{2}\right)m'(P'|q)G_M\left(P'-\frac{q}{2}\right). \qquad (2.52)
\end{aligned}
$$

The first term on the right-hand side is exactly the usual self energy $\Sigma(P)(2\pi)^4\delta^{(4)}(q)$ in second order

$$
\Sigma_{\alpha\beta}(P) \equiv -\xi_{\alpha\beta,\gamma\delta}\, i \int \frac{d^4P'}{(2\pi)^4}\frac{1}{(P-P')^2-\mu^2}\frac{1}{\not{P}-M}. \qquad (2.53)
$$

Normalizing $\Sigma(P)$ on mass shell one find the usual expression

$$
\Sigma(P) = \Sigma_0 + \Sigma_1 \cdot (\not{P} - M) + \Sigma_R(P) \qquad (2.54)
$$

where Σ_R is the regularized self-energy. The cutoff dependent term

$$
\Sigma_0 = \frac{3}{4\pi}\frac{g^2}{4\pi}M\left(\log\Lambda^2/M^2 + \frac{1}{2}\right) \qquad (2.55)
$$

must be balanced by choosing $m_0 = -\Sigma_0$ on the left-hand side of (2.52). Also, the second term Σ_1 is cutoff dependent:

$$
\Sigma_1 = \frac{1}{4\pi}\frac{g^2}{4\pi}\left(\log\Lambda^2/M^2 + \frac{9}{2} + 2\log\mu^2/M^2\right) \qquad (2.56)
$$

and a renormalization is necessary to cancel this infinity. Most economic is the introduction of an appropriate wave function counter term $(Z_2-1)\,\bar\psi(i\partial\!\!\!/-M)$ in the original Lagrangian (2.1). Such a term would enter Eq. (2.19) as

$$\int dy\,\{(i\partial-\mathcal{M}) \tag{2.57}$$
$$\delta^{(4)}(x-y)+(Z_2-1)\,(i\partial\!\!\!/-M)\,\delta^{(4)}(x-y)-m(x,y)\Big\}\,G(y,x)=i\delta^{(4)}(x-y).$$

Instead of (2.23), $m(x,y)$ should now be assumed to oscillate around $\left[m_0+\left(Z_2^{-1}-1\right)(i\partial\!\!\!/-M)\right]\delta(x-y)$. By defining a new $m'(x,y)$ via

$$m'(x,y)\equiv m(x,y)-\left[m_0+\left(Z_2^{-1}-1\right)(i\partial\!\!\!/-M)\right]\delta^{(4)}(x-y) \tag{2.58}$$

the full action (2.27) is obtained exactly as before except for the linear part $\mathcal{A}$ in which the new wave function renormalization term enters together with m_0:

$$\mathcal{A}_1[m']=\int dxdytr\,\{G_M(x-y)m'(x,y)$$
$$-\xi^{-1}m'(x,y)\left[m_0+\left(Z_2^{-1}-1\right)(i\partial\!\!\!/-M)\right]\delta(x-y)/ig^2D(x-y)\Big\}$$

By choosing

$$Z_2^{-1}-1=-\Sigma_1 \tag{2.59}$$

the cutoff dependent term Σ_1 is exactly compensated in the equation of motion (2.52). After this renormalization procedure, only the finite term $\Sigma_R(P)$ is left. the regularized action is

$$\mathcal{A}_1[m']_R=\int dxdy\Sigma_R(x-y)m'(x,y)/ig^2D(xy). \tag{2.60}$$

Using the expansion (2.47), this can be rewritten as

$$\mathcal{A}_1[m']_R=\Sigma_H dX f_H(-\Box)m_H(X) \tag{2.61}$$

with

$$f_H\left(q^2\right)=i\int\frac{d^4P}{(2\pi)^4}\mathrm{Tr}\left[\sigma_R(P\!\!\!/)G_M\left(P+\frac{q}{2}\right)\Gamma^H(P|q)G_M\left(P-\frac{q}{2}\right)\right]\frac{g_H^2\left(q^2\right)}{g^2}. \tag{2.62}$$

By momentum conservation, the tadpole momentum always vanishes such that only $f_H(0)$ is needed eventually.

Let us now proceed to the discussion of the interaction part $\mathcal{A}_{int}[m']$ of Eq. (2.30). Take as an example the term of the third order in m'. If a meson line ends at every m', it can be represented graphically as shown in Fig. 2.5. Employing the expansion

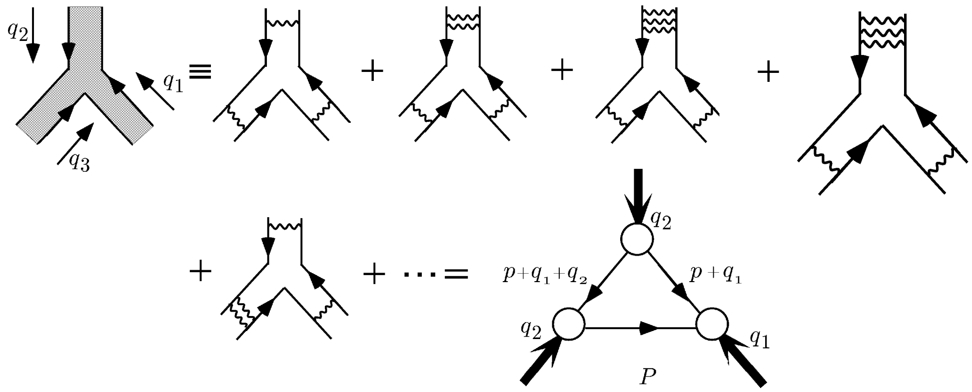

FIGURE 2.5

(2.47), this interaction term can be rewritten as

$$\mathcal{A}_{int}^{3hadr}[m'] = -\frac{1}{3} \sum_{H_1 H_2 H_3} \int \frac{d^4 q_3}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{d^4 q_1}{(2\pi)^4} (2\pi)^4 \delta\left(q_1 + q_2 + q_3\right)$$
$$\times \int \frac{d^4 P}{(2\pi)^4} tr \left[\Gamma^H \left(P - \frac{q_3}{2} | q_3 \right) G_M \left(P + q_1 q_2 \right) \Gamma^{H_2} \left(P + q_1 + \frac{q-2}{2} | q_2 \right) \right.$$
$$\left. G_M \left(P + q_1 \right) \Gamma^{H_1} \left(P + \frac{q_1}{2} | q_1 \right) G_M(P) \right] m_{H_3}(q_3) m_{H_2}(q_2) m_{H_1}(q_1)$$
$$\frac{1}{3} \sum_{H_1 H_2 H_3} \int dX \upsilon_{H_3 H_2 H_1} \left(i\partial_X^{H_3},\ i\partial_X^{H_2},\ i\partial_X^{H_1} \right) m_{H_3}(X) m_{H_2}(X) m_{H_1}(X) \quad (2.63)$$

with a vertex function $\upsilon_{H_3 H_2 H_1} \left(i\partial_X^{H_3},\ i\partial_X^{H_2},\ i\partial_X^{H_1} \right)$ whose derivatives $\partial_X^{H_i}$ are to be applied only to the argument of the corresponding field $m_{H_i}(X)$. A corresponding formula holds for every power of m'.

Notice that the flow of the quark lines in every interaction is anticlockwise. When drawing up mesonic Feynman graphs it may sometimes be more convenient to draw a clockwise flow. A simple identity helps to write down directly the corresponding Feynman rules. Consider a graph for a three meson interaction and cross the upper band downwards Usee Fig. 2.6). The interaction appears now with the mesonic bands in anticyclic order, and the fermion lines in the meson vertex flowing clockwise. This is topologically compensated by twisting every band once. Mathematically, this deformation displays the following identity of the vertex functions

$$\upsilon_{H_3 H_2 H_1}\left(q_3, q_2, q_1\right) = \eta_{H_3} \eta_{H_2} \eta_{H_1} \upsilon_{H_1 H_2 H_3}\left(q_1 q_2 q_3\right) \quad (2.64)$$

where the phase η_H denotes the charge parity of the meson H. This phase may be absorbed in the propagator characterizing the twisted band.

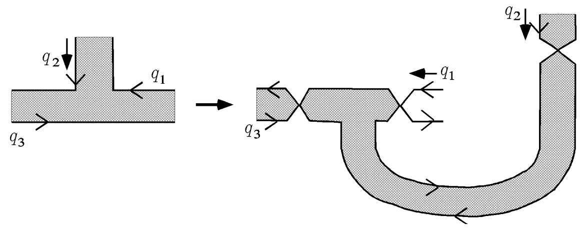

FIGURE 2.6

The proof of this identity (2.64) is quite simple. Let C be the charge conjugation matrix. Then the vertices satisfy:

$$C\Gamma^H(P|q)\,C^{-1} = \eta_H \Gamma^H(-P|q)^T . \tag{2.65}$$

Inserting now CC^{-1} between all factors in (2.63) and observing $c\gamma^\mu c^{-1} = -\gamma^{\mu T}$ one has

$$\begin{aligned}
&v_{H_3H_2H_1}(q_3,\ q_2,\ q_1) = \\
&\quad -\eta_{H_3}\eta_{H_2}\eta_{H_1}\int \frac{dP}{(2\pi)^4} tr\left\{\Gamma^{H_3}\left(-P+\frac{q_3}{2}|q_3\right)^T\left(\frac{i}{-\not{P}-\not{q}_1/\not{q}_2-M}\right)^T\right.\\
&\quad \left.\Gamma^{H_2}\left(-P-q_1-\frac{q_2}{2}|q_2\right)^T\left(\frac{i}{-\not{P}-\not{q}_1-M}\right)^T\Gamma^{H_1}\left(-P-\frac{q_1}{2}|q_1\right)\left(\frac{i}{-\not{P}-M}\right)^T\right\}.
\end{aligned}$$

Taking the transpose inside the trace and changing the dummy variable P to $-P$, the vertices appear in anticyclic order and the right-hand side coincides indeed with $\eta_{H_3}\eta_{H_2}\eta_{H_1}v_{H_1H_2H_3}(q_1,\ q_2,\ q_3)$. Twisted propagaotrs are physically very important. They describe the strong rearrangement collisions of quarks and certain classes of cross-over gluon lines. Fig. 2.7 shows some twisted graphs together with their quark gluon contents. In meson scattering rearragement collisions (Fig. 2.7(a)) have rougly the same coupling strength as direct (untwisted) exchanges. In QED, on the other hand, they provide for the main molecular binding forces.

The exchange of two twisted meson lines (Fig. 2.7b) seems to be an important part of diffraction scattering (Pomeron).

Two more examples are shown in Fig. 2.8. Notice that in the pseudoscalar channel these graphs incorporate the effect of the Adler triangle anomaly.

In this connection it is worth pointing out that all fundamental meson vertices are planar graphs as far as the quark lines are concerned. Non-planar graphs are generated by building up loops involving twisted propagators. With propagator bands, their twisted modifications and planar fundamental couplings meson graphs are seen to possess exactly the same topology as the graphs used in dual models [12] except for the stringent dynamcial property of duality itself: In the present mesonized theory one still must sum s and t channel exchanges and they are by no means the same. Only after introduction of color and the ensuing linearly rising mass spectra one can hope to account also for this particular aspect of strong interactions.

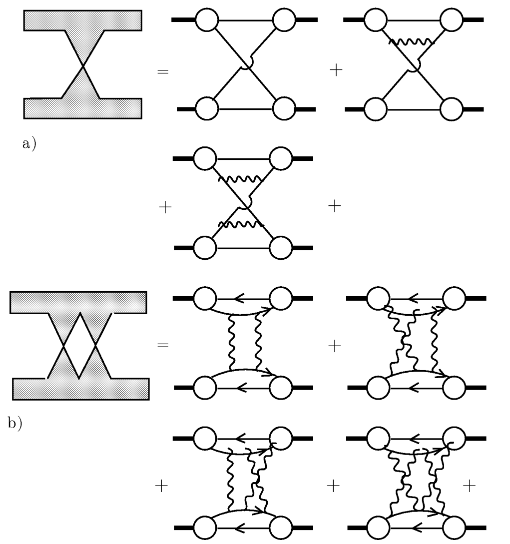

FIGURE 2.7

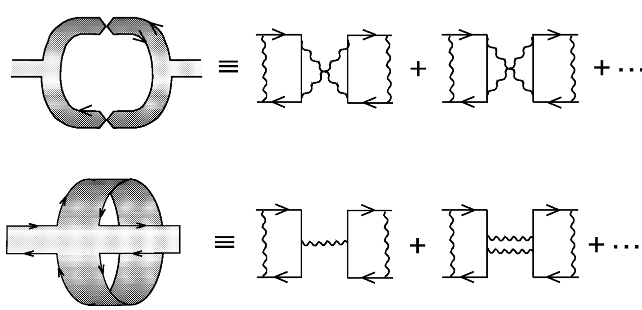

FIGURE 2.8

The similarity in topology should be exploited for a model study of an important phenomenon of strong interactions. the Okubo, Zweig, and Iizuka rule. Obviously all meson couplings derived by mesonization exactly respect this rule. All violations have to come from graphs of the so called cylinder type [13] (for example Fig. 2.7b). If it is true that the topological expansion [12] is the correct basis for explaining this rule[3], it may also provide the appropriate systematics for organizing the mesonized perturbation expansion.

Let us finally discuss the external sources. From $\mathcal{A}_{ext}$ in (2.41) we see that external fermion lines are connected via the full propagator G which after expansion in powers of m' amounts to radiation of any number of mesons (see Fig. 2.9) These

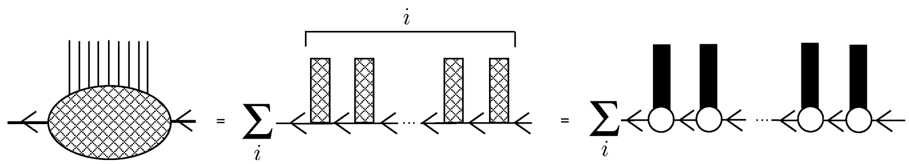

FIGURE 2.9

mesons then interact among each other as quantum fields. Diagrammatically, every bubble carries again a factor $\Gamma^H(P|q)$.

It has to be watched out that mesons are always emitted to the right of each line. For example, the lowest order quark-quark scattering amplitude should initially be drawn as shown in Fig. 2.10 in order to avoid phase errors due to twisted bands. Then

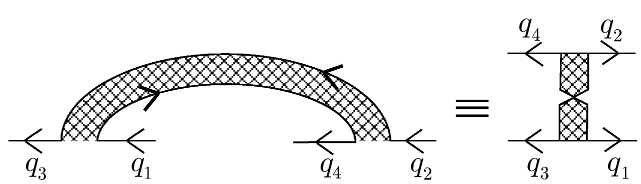

FIGURE 2.10

the graphical rules yield directly the expression (2.44) as they should. Afterwards, arbitrary deformations can be performed if all twisted factors η_H are respected.

External gluons interact with mesons according to the third term in Eq. (2.31)

$$-\frac{2}{g^2D(0)}\int dxdyV^\nu(x,x)g^2D(x-y)j_\nu(y). \tag{2.66}$$

Hence every external gluon enters the mesonic world only via an intermediate vector particle and there is a current field identity as has been postulated in phenomenological treatments of vector mesons (VMD). Here one finds a non-trivial coupling

[3]See the forth of Ref. 14).

between the gluon and the vector mesons: As discussed before, the division by $g^2 D$ amounts to a removal of one rung from the ladder of the incoming meson propagator and takes care of the direct coupling of the gluon to the quarks *without* the ladder corrections. This effect was shown to be accounted for a factor $g_H^2(q^2)/g^2$ in the propagator sum (2.44). Thus the direct coupling of the vector meson field $m_H(x)$ to an external gluon field $G_\nu^{ext}(x)$ can be written as:

$$g \sum_H \int \frac{d^4 q}{(2\pi)^4} \int \frac{d^4 P}{(2\pi)^4} \mathrm{Tr} \left\{ \gamma^\nu G_M \left(P + \frac{q}{2} \right) \Gamma^H(P|q) G_M \left(P - \frac{q}{2} \right) \right\} \frac{g_H^2(q^2)}{g^2} m_H(q) G_\nu^{ext}(-q) \tag{2.67}$$

In a mesonic graph, the removal of one rung will be indicated by a slash. As an example, the lowest order contribution to the quark gluon form factor is illustrated in Fig. 2.11. The slash guarantees the presence of the direct coupling. The free

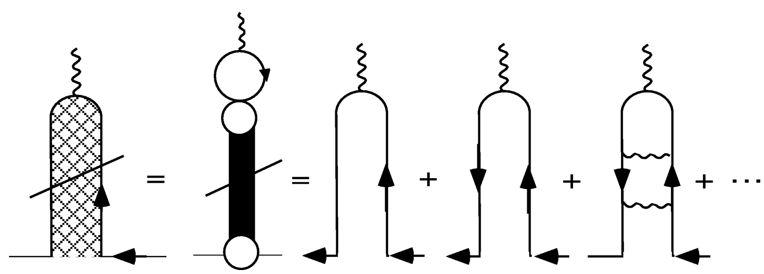

FIGURE 2.11

propagator of external gluon is given by the second term of Eq. (2.31). The lowest radiative corrections consist in an intermediate slashed vector mesons (see Fig. 2.12). Here the slash is important to ensure the presence of one single quark loop.

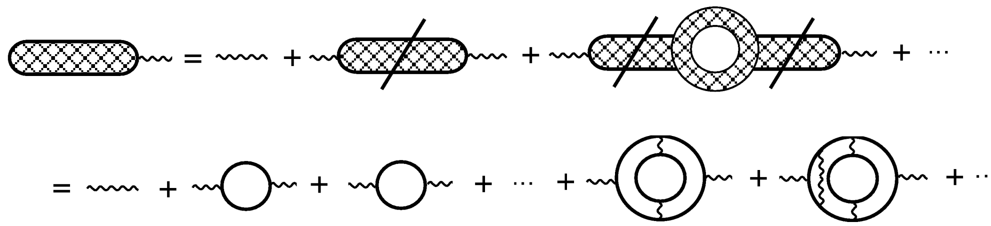

FIGURE 2.12

The divergent last term in the external action (2.31) has no physical significance since it contributes only to the external gluon mass and can be cancelled by an appropriate counter term.

A final remark concerns the bilocal currents as measured in deep inelastic electron and neutrino scattering. These are vector currents of the type

$$j^{\nu}(x,y) \equiv \bar{\psi}(x)\gamma^{\nu}\psi(y). \tag{2.68}$$

It is obvious, that also for bilocal currents there is a current-field identity with the bilocal field $V^{\mu}(x,y)$. In fact, if one would have added an external source term $C_{\nu}(x,y)$ in the quark action:

$$\Delta\mathcal{A}_{ext} \equiv \int dxdy\bar{\psi}(x)\gamma^{\nu}\psi(y)C_{\nu}(x,y) \tag{2.69}$$

this would appear in the mesonized version in the form

$$\Delta\mathcal{A}_{ext} = \int dxdy\frac{1}{ig^2D(x-y)}V^{\nu}(x,y)C_{\nu}(x,y) \tag{2.70}$$

which proves our statement. Again, a rung has to be removed in order to allow for the pure quark contribution (see Fig. 2.13)

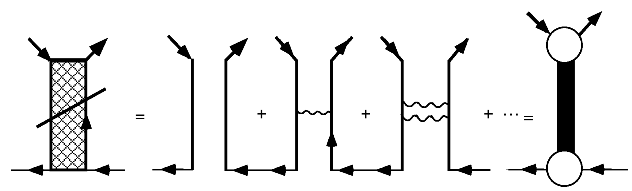

FIGURE 2.13

Bilocal currents carry direct information on the properties of Regge trajectories [15]. Therefore the present bilocal field theory seems to be the appropriate tool for the construction of a complete field theory of Reggeons [16], which is again equivalent to the original quark gluon theory. Technically, such a construction would proceed via analytic continuation of the propagators (2.34) in the angular momentum (and the principal quantum number) of the mesons H. The result would be a "reggeonized" quark gluon theory. The corresponding Feynman graphs would guarantee unitary in all channels. Present attempts at such a theory enforces at channel unitarity only [17]. Also, they are asymptotically valid by consturction and apparently have a chance of approximating nature only at energies unaccessible in the near future [4]

[4]See, for example, D. Amati and R. Jengo, Physics Letter B 54 81974).

3

The Limit of Heavy Gluons

As an illustration of the mesonization procedure we now discuss in detail the limit of very heavy gluons [18], [34]. Apart from its simplicity, this limit is quite attractive on physical grounds since it may yield a reasonable approximation to low energy meson interactions. This is suggested by the following arguments:

Suppose hadrodynamics follows a colored quark gluon theory. In this theory the color degree of freedom is very important for generating a potential between quarks rising at long distances which can explain the observed great number of high mass resonances. However, as far as low-energy interactions among the lowest lying mesons are concerned, color seems to be a rather superfluous luxury:

First, many fundamental aspects of strong interaction dynamics such as chiral $SU(3) \times SU(3)$ current algebra PCAC (together with the low-energy theorems derived from both) and the approximate light cone algebra are independent of color.

Second, there is no statistics argument concerning the symmetry of the meson wave function as there is for baryons [19].

Third, high-lying resonances are known to contribute very little in most dispersion relations of low-energy amplitudes. For example, the low-energy value of the isospin odd $\pi\pi$ scattering amplitude is given by a dispersion integral over the mesons ρ and σ with $\approx 90\%$ accuracy [20]. Similarly, $\pi\rho$ scattering is saturated by the intermediate mesons π and A_1. By looking at all scattering combinations one can easily convince oneself that the resonances $\pi,\ \pi,\ \sigma,\ A_1$ form an approximately closed “subworld” of mesons as far as dispersion relations are concerned. As a consequence, it would not at all be astonishing if the neglect of color in a quark gluon theory would not change the dynamics when restricting the attention to this mesonic “subworld” [1] The point is now that in the limit of a large gluon mass $\mu \to \infty$, exactly this restricted set of mesons appears as particles in the mesonized quark gluon theory (2.1) *without* color. Thus it might be considered as some approximation to the low-energy aspects of the colored version. Indeed, we shall see that the mesonized theory coincides exactly with the well-known chirally invariant

[1] There is one estimate concerning the electromagnetic decay of π^0. $\gamma\gamma$ which is based on short distance arguments and therefore depends on color [21]. However, the same decay can be estimated also via intermediate distance arguments, namely by using the coupling $\rho\omega\pi$ and vector meson dominance such that color does not come in.

σ model. This model has proven in the past to be an appropriate tool for the rough description of low-energy meson physics [23]. Our derivation of the σ model via mesonization will render several new relations between meson and quark properties [18]. We shall at first confine ourselves to $SU(2)$ quarks only, such that symmetry breaking may be neglected. The extension to broken $SU(3)$ will pe performed afterwards.

In order to start with the derivation observe that in the limit $\mu \to \infty$, the gluon propagator approaches a δ-function:

$$iD(x-y) \to \frac{1}{\mu^2}\delta(x-y). \tag{3.1}$$

The equation of motion (2.32) forces $m'(x,y)$ to become a local field $m'(x)$:

$$m'(x,y) \to m'(x)\delta(x-y) \tag{3.2}$$

which satisfies the free field equation

$$m'(x) = -i\frac{g^2}{\mu^2}\xi \int dy G_M(x-y)m'(y)G_M(y-x). \tag{3.3}$$

In the local limit, the action without external sources takes the form

$$\mathcal{A}[m'] = \int dxtt \left\{ G_M(x,x)m'(x) - \frac{1}{2}\left(G_M m' G_M m'\right)(x,x) \right.$$
$$\left. + \sum_n \frac{(-i)^{n-1}}{n}\left(G_M m'\right)^n(x,x) - \frac{\mu^2}{g^2}\frac{1}{2\xi}m'(x)^2 - \frac{1}{\xi}m'(x)m_0 \right\} \tag{3.4}$$

where $\left(G_M m' G_M m'\right)(x,x)$ stands short for $\int dy G_M(x-y)m'(y)G_M(x-y)$ etc. As before in the general discussion, the constant m_0 is determined by the vanishing of the tadpole parts in (3.4) which amounts to balancing the constant contributions in the wave equation. Due to the singularity of $G_M(x-y)$ for $x \to y$ this condition has a meaning only if a cutoff is introduced such that $G_M(0)$ is finite:

$$\begin{aligned}
\left[G_M(0)\right]_{\alpha\beta} &= \int \frac{d^4P}{(2\pi)^4}\left[\frac{i}{\not{P}-M}\right]_{\alpha\beta} = M\int_0^\Lambda \frac{d^4P_E}{(2\pi)^4}\left(P_E^2+M^2\right)^{-1}\delta_{\alpha\beta} \\
&= M\frac{\pi^2}{(2\pi)^4}\left(\Lambda^2 - M^2\log\Lambda^2/M^2\right)\delta_{\alpha\beta} \equiv MQ\delta_{\alpha\beta}.
\end{aligned} \tag{3.5}$$

Here the dP^0 integration has been Wick-rotated by 90^0 such that the momentum $P^\mu = (P^0, \mathbf{P})$ becomes $(iP^4, \mathbf{P})$ with $P^4 \in (-\int, \infty)$ along the integration path. The new real momentum $(P^4,\ \mathbf{P})$ has been denoted by P_E^μ and its euclidean scalar product by $P_E^\mu = P^{42} + \mathbf{P}^2 = -P^2$. The tadpoles can now be cancelled by setting m_0 equal

$$m_0 = 4\frac{g^2}{\mu^2}QM. \tag{3.6}$$

Remembering the relation to the bare quark mass $m_0 = M - \mathcal{M}$, this determines the connection between the "true" quark mass M and the bare mass $\mathcal{M}$ contained in the Lagrangian:

$$M = \mathcal{M} + 4\frac{g^2}{\mu^2}QM. \tag{3.7}$$

Equation (3.8) is often called "gap equation" because of its analoguous appearance in the theory of superconductivity [24].

Consider now the free part $\mathcal{A}_2[m']$ of the action. Performing again a decomposition of type (2.13) but with the local field $m'(x)$, it can be written in the form

$$\begin{aligned} \mathcal{A}_2[m'] \quad &= \quad \int dx \mathrm{tr}_{SU(2)} \left\{ \frac{1}{2} m'_i(x) J_{ij}(i\partial) m_j(x) \right. \\ & \qquad \left. \frac{\mu^2}{2g^2} \left(S^2_{(x)} + P^2_{(x)} - 2V^2(x) 2A^2(x) \right) \right\} \end{aligned} \tag{3.8}$$

where $m'_i(x)(i = 1, 2, 3, 4)$ stands short for the fields [2] $S(x),\ P(x),\ V(x),\ A(x)$ and the trace runs only over internal $SU(2)$ indices. The coefficients $J_{ij}(q)$ are given by the integrals

$$J_{ij}(q) \equiv -4 \int \frac{d^4 P_e}{(2\pi)^4} \frac{1}{\left(P + \frac{q}{2}\right)^2_E + M^2} \frac{1}{\left(P - \frac{q}{2}\right)^2_E + M^2} t_{ij}(P|q) \tag{3.9}$$

where $t_{ij}(P|q)$ denotes the Dirac traces

$$t_{ij}(P|q) \equiv \frac{1}{4} tr_{Dirac} \left\{ \Gamma_i \left(\not{P} + \frac{\not{q}}{2} + M \right) \Gamma_j \left(\not{P} - \frac{\not{q}}{2} + M \right) \right\} \tag{3.10}$$

with $\Gamma_i\ (i = 1, 2, 3, 4)$ abbreviating the standard Dirac covariants $1, i\gamma_5,\ \gamma^\nu,\ \gamma^\nu\gamma_5$ The traces are displayed in Appendix A Eq. (3A.39). Some of them grow quadratically in P. The corresponding integrals $J_{ij}(q)$ are quadratically divergent for large cutoffs. The others diverge lograthmically. If one introduces the basic integral

$$L \equiv \int \frac{d^4 P_E}{(2\pi)^4} \frac{1}{(P_E^2 + M^2)^2} = \frac{\pi^2}{(2\pi)^4} \left(\log \frac{\Lambda^2}{M^2} - 1 \right) \tag{3.11}$$

the divergent parts of j_{ij} are (see App. A)

$$\begin{aligned} J_{ss}(q) \quad &= \quad Q + L\left(\frac{q^2}{2} - 2M^2\right) \\ J_{pp}(q) \quad &= \quad Q + L\frac{q^2}{2}; \quad J_{PA^\nu} = -iLMq^\nu \\ J_{V^\mu V^\nu}(q) \quad &= \quad -\frac{1}{3}\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) L \\ & \qquad J_{A^\mu A^\nu}(q) = -\frac{1}{3}\left[\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) - 6M^2 g^{\mu\nu}\right] L; \quad J_{A^\mu P} = iLMq^\mu \end{aligned} \tag{3.12}$$

[2]The Lorentz indices of V^ν and A^ν fields are suppressed.

with al other integrals vanishing.

If we neglect the finite contributions as compared with these divergent ones, the action $\mathcal{A}_2[m']$ is seen to correspond to the local Lagrangian[3]

$$\begin{aligned}\mathcal{L}(x) \;=\; & \mathrm{tr}_{SU(2)}\left\{\frac{1}{2}S'(x)\left[4Q-2\left(\Box+4M^2\right)L-\frac{\mu^2}{g^2}\right]S'(x)\right.\\ & +\frac{1}{2}P(x)\left[4Q-2\Box L-\frac{\mu^2}{g^2}\right]P(x)\\ & \frac{1}{2}V_\mu(x)\left[\frac{4}{3}\left(\Box g^{\mu\nu}-J^\mu\partial^\nu\right)L+\frac{2\mu^2}{g^2}\right]V_\nu(x)\\ & +\frac{1}{2}A_\mu(x)\left[\frac{4}{3}\left(\Box g^{\mu\nu}-\partial^\mu\partial^\nu\right)L+8M^2g^{\mu\nu}L+\frac{2\mu^2}{g^2}\right]A_\nu(x)\\ & \left.+2ML\left(\partial_\mu P(x)A^\mu(x)+A^\mu(x)\partial_\mu P(x)\right)\right\}. \end{aligned} \tag{3.13}$$

If we respect the gap equation (3.7) in this Lagrangian, the quadratically divergent terms Q can be eliminated. The mixed terms can be removed by introducing a new field $\widetilde{A^\mu}(x)$ via

$$A^\mu(x)=\widetilde{A^\mu}(x)+\lambda\partial^\mu P \tag{3.14}$$

and fixing λ as

$$\lambda=-3M/m_A^2 \tag{3.15}$$

where m_A^2 stands short for

$$m_A^2=m_V^2+6M^2 \tag{3.16}$$

with

$$m_V^2=3\mu^2/(2g^2L). \tag{3.17}$$

This substitution produces additional kinetic terms for the pseudoscalar fields which now appears with a factor

$$\begin{aligned}&-\mathrm{tr}_{SU(2)}\left(P(x)\Box P(x)\right)\left(1+\frac{2}{3}m_A^2\lambda^2+4M_\lambda\right)L\\ &\quad=\mathrm{tr}_{SU(2)}\left(\partial_\mu P(x)\partial^\mu P(x)\right)Z_P^{-1}L.\end{aligned} \tag{3.18}$$

Using (3.15), this renormalization factor becomes

$$Z_P^{-1}=1-6M^2/m_A^2. \tag{3.19}$$

[3]Since $m(x)$ and $m'(x)$ differ only by a Dirac scalar constant $m_0 1_{\alpha,\beta}$ there is no difference between prmed and unprimed fields except for $S'(x)=S(x)-m_0$

After this diagonalization, the Lagrangian reads

$$\begin{aligned}\mathcal{L}(x) \;=\;& \mathrm{tr}_{SU(2)}\Big\{\partial_\mu S'\partial^\mu S' - \Big(4M^2 + \frac{1}{3}m_V^2\mathcal{M}/M\Big)S'^2 \\ & -\partial_\mu P\partial^\mu P Z_P^{-1} - \frac{1}{3}m_V^2\mathcal{M}/MP^2 \\ & -\frac{1}{3}F_V^{\mu\nu}F_{\mu\nu}^V + \frac{2}{3}m_V^2V_\mu^2 \\ & -\frac{1}{3}F_{\tilde{A}}^{\mu\nu}F_{\mu\nu}^{\tilde{A}} + \frac{2}{3}m_A^2\tilde{A}_\mu^2\Big\} \times L \end{aligned} \tag{3.20}$$

where ${F^{\mu\nu}}_{V,\tilde{A}}$ are the usual field tensors of vector and axial vector fields. The particle content of this free Lagrangian is now obvious. There are vector mesons of mass m_V^2, axial-vector mesons of mass m_A^2 and scalar and pseudoscalar mesons of mass

$$m_S^2 \;=\; 4M^2 + \frac{1}{3}m_V^2\mathcal{M}/M \tag{3.21}$$

$$m_P^2 \;=\; \frac{1}{3}m_V^2\mathcal{M}/MZ_P. \tag{3.22}$$

With (3.17), the constant (3.19) can also be written as

$$Z_P^{-1} = m_V^2/m_A^2. \tag{3.23}$$

As we have argued before, there is a good chance that the fields $P,\ V,\ S, A$ describe approximately the lowest lying mesons $\pi,\ \rho\ ,\sigma,\ A$. Let us test this hypothesis as far as the masses are concerned. Since experimentally $m_{A_1}^2 \approx 2m_\rho^2$ the factor Z_π becomes ≈ 2. Furthermore, Eq. (3.16) determines the quark mass as:

$$6M^2 = m_{A_1}^2 - m_\rho^2; \quad M \approx 310\text{MeV} \tag{3.24}$$

in good agreement with other estimates [25]. The small pion mass yields via (3.22)

$$\mathcal{M} \approx 15\text{MeV} \tag{3.25}$$

Thus the bare quark mass has to be extremely small. Also this result has been obtained by many authors [26]. It is common to all models in which the smallness of the pion mass is related to the approximate conservation of the axial current (PCAC).

The scalar meson finally is pedicted from (3.21) to have a mass

$$m_T \approx 2M \sim 620\text{MeV}. \tag{3.26}$$

This agrees well with the oberved broad resonance in $\pi\pi$ scattering. [27] [20]

One disagreement with experiment appears in connection with the $SU(2)$ singlet pseudoscalar mass (the η meson). According to (3.22) it should be degenerate with the pion. The resolution of this problem will bei discussed later when the theory has been extended to $SU(3)$.

After these first encouraging results we shall rename the fields $P\ldots,\ V,\ S,\ A$ by the corresponding particle symbols

$$\sqrt{L}P \equiv \pi, -\sqrt{L}S' \equiv \sigma', -\sqrt{\frac{2}{3}}LV^\mu \equiv \rho^\mu, -\sqrt{\frac{2}{3}}LA^\mu \equiv A_1^\mu \tag{3.27}$$

where a normalization factor has been introduced in order to bring the kinetic terms in the Lagrangian to a conventional form.

A comment is in order concerning the appearance of a quadratic divergence in equations (3.7), (3.12). Such a strong divergence indicates, that the limiting procedure $\mu \to \infty$ of Eq. (3.1) has been performed too carelessly. In fact, if one inserts (3.1) into the action (2.4), the theory becomes of the $\left(\bar{\psi}\psi\right)^2$ type and thus non-renormalizable. In order to keep the renormalizability while dealing with a large gluon mass $\mu^2 \gg \Lambda^2$. Then the quadratic divergence becomes actually of the logarithmic type (compare (2.53)):

$$\begin{aligned} Q &= \int \frac{d^4P_E}{(2\pi)^4}\frac{1}{P_E^2+M^2}\frac{1}{P_E^2+\mu^2} \\ &= \frac{\pi^2}{\mu^2-M^2}\left[\mu^2\log\left(\frac{\Lambda^2}{\mu^2}+1\right)-M^2\log\left(\frac{\Lambda^2}{M^2}+1\right)\right] \end{aligned} \tag{3.28}$$

(which in the careless limit $\mu^2 \to \infty$ reduces again to (3.5)). The logarithmic divergence (3.11) on the other hand becomes in this more careful treatment independent of the cutoff which is replaced by the large gluon mass

$$L = \int \frac{d^4P_E}{(2\pi)^4}\frac{1}{\left(P_E^2+M^2\right)^2}\frac{1}{(P_E^2+\mu^2)} \tag{3.29}$$

$$\frac{\pi^2}{(2\pi)^4}\frac{\mu^2}{\left(\mu^2-M^2\right)^2}\left[\mu^2\log\frac{\mu^2}{M}+M^2-\mu^2\right] \approx \frac{\pi^2}{(2\pi)^4}\log\frac{\mu^2}{M^2} \tag{3.30}$$

Hence all our results refer to a renormalizable theory if one reads both Q and L as logarithmic expression once in the cutoff and once in the gluon mass, respectively.

Let us now proceed to study the interaction terms. The n'th order contribution to the action is given by

$$\mathcal{A}_n[m'] = \frac{(-i)^{n-1}}{n}\int dx \mathrm{Tr}\left(G_M m'\right)^n \tag{3.31}$$

In momentum space this can be written as the one loop integral

$$\begin{aligned} \mathcal{A}_n[m'] &= 4\frac{(-1)^{n-1}}{n}\int\frac{dq_n}{(2\pi)^4}\cdots\frac{dq_1}{(2\pi)^4}(2\pi)^4\delta\left(q_n+\ldots+q_1\right) \\ &\int\frac{d^4P_E}{(2\pi)^4}\frac{1}{\left(P+q_n+dots+q_1\right)_E^2+M^2}\cdots\frac{1}{\left(P+q_1\right)_E^2+M^2}t_{i_n\ldots i_1}\left(P|q_{n-1},\ldots,q_1\right) \\ &\mathrm{tr}_{SU(2)}\left[m'_{i_n}(q_n)\cdot\ldots\cdot m'_{i_1}(q_1)\right] \end{aligned} \tag{3.32}$$

where $t_{i_n \ldots i_1}(P|q_{n-1}, \ldots, q_1)$ is the generalization of the tensor (3.10)

$$t_{i_n \ldots i_1}(P|q_{n-1}, \ldots, q_1) \equiv \tag{3.33}$$
$$\frac{1}{4}\mathrm{Tr}\left[\Gamma_{i_n}\left(\not{P} + \not{q}_{n-1} + \ldots + \not{q}_1 + M\right)\Gamma_{i_{n-1}} \ldots \Gamma_{i_2}\left(\not{P} + \not{q}_1 + M\right)\Gamma_{i_1}(\not{P} + M)\right]$$

The result is hard to evaluate in general (except in a $1+1$ dimensional space). With the approximation of a large cutoff one may however, neglect again all contributions which do not diverge. This considerable simplifies the results. Since $t_{in \ldots i-1}(\Pi|q_{n-1}, \ldots, q_1)$ are polynomials in P of order n, the integral is seen to converge for $n > 4$. For $n = 4$ there is a logarithmic divergence with only the leading momentum behaviour of $t_{i_n} \ldots i_1$ contributing. For $n \leq 3$ also lower powers in momentum P of $t_{i_n} \ldots i_1(P|q_{n-1}, \ldots, q_1)$ diverge logarithmically. A simple but somewhat tedious calculation of all the integrals (see App. B) yields the remaining terms in the Lagrangian. They can be written down in a most symmetric fashion by employing the unshifted fields [4] $S(x) \equiv M * S'(x)$ rather than S', or in renormalized form

$$\sigma(x) = -\sqrt{L}M + \sigma'(x). \tag{3.34}$$

Then the Lagrangian reads

$$\begin{aligned}\mathcal{L}(x) \;=\; & Tr_{SU(2)}\left\{\left[(D_\mu(D_\mu\sigma)^2 + (D_\mu\pi)^2\right] + M_0^2\left(\sigma^2 + \pi^2\right)\right.\\ & -\frac{2}{3}\gamma^2\left[\sigma^4 + \pi^4 - 2\sigma\pi\sigma\pi\right] - \frac{1}{2}F_{\mu\nu}^{V^2} - \frac{1}{2}F_{\mu\nu}^{A^2}\\ & \left.+m_V^2\left(V_\mu^2 + A_\mu^2\right) - \frac{2}{3}m_V^2\sqrt{L}\mathcal{M}\right\}.\end{aligned} \tag{3.35}$$

Here $D_\mu\sigma$ and $D_\mu\pi$ are the usual covariant derivatives:

$$\begin{aligned}D_\mu\sigma &= \partial_\mu\sigma - \gamma[V_\mu\sigma] - \gamma\{A_\mu\pi\}\\ D_\mu\pi &= \partial_\mu\pi - i\gamma[V_\mu\pi] + \gamma\{A_\mu\sigma\}\end{aligned} \tag{3.36}$$

and $F_{\mu\nu}^V, \;\; F_{\mu\nu}^A$ are the covariant curls

$$\begin{aligned}F_{\mu\nu}^V &= \partial_\mu V_\nu - \partial_\nu V_\mu - i\gamma[V_\mu,\; V_\nu] - i\gamma[A_\mu,\; A_\nu]\\ F_{\mu\nu}^A &= \partial_\mu A_\nu - \partial_\nu A_\mu - i\gamma[V_\mu,\; A_\nu] - i\gamma[A_\mu, V_\nu]\end{aligned} \tag{3.37}$$

The constant γ denotes

$$\gamma = \sqrt{\frac{3}{2L}}. \tag{3.38}$$

[4]Notice that with this notation $m(x) = m_0 + m'(x) = (M - \mathcal{M}) + m'(x) = -\mathcal{M} + S + P_i\gamma_5 + V^\mu\gamma_\mu + A^\mu\gamma_\mu\gamma_s$

It describes the direct coupling of the vector mesons to the currents, i. e. it coincides with the coupling conventionally denoted by γ_ρ. Here γ has its origin in the renormalization of the fields. The mass term stands short for

$$2M_0^2 \equiv 2M^2 - \frac{1}{3}m_V^2\mathcal{M}/M. \tag{3.39}$$

Actually, the so defined mass quantity has an intrinsic significance. This can be seen by deriving the Lagrangian in a different fashion from the beginning. Consider the tadpole terms of the action

$$\mathcal{A}_1\left[m_1\right] = \int dxtr\left\{G_M(x,x)m'(x) - \frac{1}{\xi}m'(x)m_0\right\}. \tag{3.40}$$

In the former treatment we have eliminated m_0 completely by giving the quarks a mass M satsifying the gap equation

$$M - \mathcal{M} \equiv m_0 = 4\frac{g^2}{\mu^2}QM. \tag{3.41}$$

Instead, we could have introduced an auxiliary mass M_0 satisfying the equation

$$M_0 = 4\frac{g^2}{\mu^2}Q_0M_0 \tag{3.42}$$

where Q_0 is the same function of M_0 as Q is of M. The connection between this M_0 and the other masses is obtained by inserting $M = M_0\delta M$ into Q:

$$Q = Q_0 - 2M_0\delta M\left(1 + \delta M/2M_0\right)L \tag{3.43}$$

which holds exactly in δM with only small corrections for large cutoffs (notice that at this accuracy $L_0 = L$). Inserting this into (3.41) we find

$$\mathcal{M} = 4\frac{g^2}{\mu^2}L2M_0M\left(1 + \delta M/2M_0\right)\delta M \tag{3.44}$$

and using m_V^2 from

$$\mathcal{M} = \frac{12}{m_V^2}M_0M\left(1 + \delta M/2M_0\right)\delta M. \tag{3.45}$$

If now $m(x)$ is split in a different fashion

$$m(x) = \tilde{m_0} + m''(x) \tag{3.46}$$

with a new $\tilde{m_0} = M_0 - \mathcal{M}$ then the propagator $G(x,y)$ would have an expansion

$$G(x,y) = G_{M_0}(x-y) - i\left(G_{M_0}m''G_{M_0}\right)(x,y) + . \tag{3.47}$$

For this reason, the derivation of all Lagrangian terms yields exactly the same results as before only with m'', M_0, L_0, and Q_0 occurring rather than m', M, L and Q, respectively. There are only two differences: First, due to the gap equation (3.42), the scalar and pseudoscalar mass terms become $4M_0^2$ and O rather than (3.21), (3.22) second, the tadpole terms in this derivation do *not* cancel completely. Instead one finds from (3.40)

$$\begin{aligned}\mathcal{A}_1\,[m'] &= \int dxtr\left\{\left(4Q_0M_0 - \frac{M_0-\mathcal{M}}{g^2/\mu^2}\right)m''(x)\right\} \\ &= \int dx\frac{\mu^2}{g^2}tr\,\{\mathcal{M}m''(x)\} = \frac{2}{3}m_x^2\int dxtr\,\{\mathcal{M}m''(x)\}\,. \qquad (3.48)\end{aligned}$$

These tad pole terms provide exactly the necessary additional shifts in the fields which are needed in order to bring the scalar and pseudoscalar masses from $4M_0^2$ and O to their correct values m_σ^2 and m_π^2. The symmetric form (3.35) of the Lagrangian is again reached by introducing the original unprimed fields

$$S(x) \equiv M0 + S''(x), \quad \sigma = -\sqrt{L}M_0 + \sigma''. \qquad (3.49)$$

Then the mass term appears as an $SU(3)\times SU(3)$ invariant

$$2M_0^2\left(\sigma^2+\pi^2\right).$$

[5] Notice now this coincides exactly with the former calculation which rendered (see 4.38)

$$\left(2M^2 - \frac{1}{3}m_V^2\mathcal{M}/M\right)\left(\sigma^2+\pi^2\right).$$

Inserting here $M = M_0 + \delta M$ and (3.45) gives

$$\begin{aligned}2M^2 - \frac{1}{3}m_V^2\mathcal{M}/M &= 2M_0^2 + 4M_0\delta M + 2(\delta M)^2 - 4M_0\left(1+\frac{\delta M}{M_0}\right)\delta M \\ &= 2M_0^2. \qquad (3.50)\end{aligned}$$

Hence the $SU(3)$ symmetric mass M_0 defined by the gap equations (3.42) coincides with the mass M_0 introduced as an abbreviation to the mass combination (3.39).

[5]With this substitution, the unprimed field S really coincides with the formerly introduced field S since now

$$\begin{aligned}m(x) &= (M_0-\mathcal{M}) + S''(x) + P(x)i\gamma_5 + \ldots \\ &= -\mathcal{M} + S(x) + P(x)i\gamma_5 + dots\end{aligned}$$

while before

$$\begin{aligned}m(x) &= (M-\mathcal{M}) + S'(x) + P(x)i\gamma_5 + \ldots \\ &= -\mathcal{M} + S(x) + P(x)i\gamma_5 + \ldots\end{aligned}$$

The Lagrangian (3.35) is recognized as the standard chirally invariant σ model. Its symmetry transformations are for isospin

$$\delta\sigma = i\left[\alpha,\sigma\right] \quad ; \; \delta\pi = i\left[\alpha,\pi\right]$$
$$\delta V^\mu = i\left[\alpha, V^\mu\right] + \frac{1}{\gamma}\partial^\mu\alpha \quad ; \; \delta A^\mu = i\left[\alpha, A^\mu\right]. \tag{3.51}$$

For axial transformations the fields change according to

$$\bar\delta\sigma = -\left\{\bar\alpha,\pi\right\} \quad ; \; \bar\delta\pi = \left\{\bar\alpha,\sigma\right\}$$
$$\bar\delta V^\mu = i\left[\bar\alpha, A^\mu\right] \quad ; \; \bar\delta A^\mu = i\left[\bar\alpha, V^\mu\right] + \frac{1}{\gamma}\partial^\mu\bar\alpha. \tag{3.52}$$

The only term in the Lagrangian which is not invariant is the last linear term. In fact from

$$\bar\delta\mathcal{L} = i\frac{2}{3}m_V^2\mathcal{M}\sqrt{L}\left\{\bar\alpha\pi\right\} \equiv i\left\{\bar\alpha,\ \partial A\right\} \tag{3.53}$$

one finds

$$\partial A(x) = f_\pi m_\pi^2 Z_\pi^{-1/2}\pi(x). \tag{3.54}$$

Introducing the conventional pion decay constant via

$$\partial A(x) \equiv f_\pi m_\pi^2 Z_\pi^{-1/2}\pi(x) \tag{3.55}$$

one can read off

$$f\pi m_\pi^2 = Z_\pi^{1/2}\frac{2}{3}m_V^2\sqrt{L}\mathcal{M}. \tag{3.56}$$

Inserting m_π^2 from (3.22) this gives

$$f_\pi = Z_\pi^{-1/2}2M\sqrt{L}. \tag{3.57}$$

By squaring this and using $\gamma = \sqrt{\frac{3}{2L}}$, one obtains

$$f\pi^2 = \frac{6M^2}{Z_\pi}\frac{1}{\gamma^2} = \frac{m\rho^2}{\gamma^2}\frac{m_A^2 - m\rho^2}{m_A^2} \tag{3.58}$$

which for $m_A^2 \approx 2m_\rho^2$ renders the well known KSFR relation. The model has the usual predictions

$$g_{\rho\pi\pi} = \gamma_\rho\left(1 - \frac{m_A^2 - m\rho^2}{2m_A^2}\right) \approx \frac{3}{4}\gamma_\rho \tag{3.59}$$

and

$$\begin{aligned} g_{A,\rho\pi} &= \frac{1}{2m_\rho}\gamma^2 Z_\pi f_\pi \approx \frac{m_\rho}{2f_\pi} \approx 4; h_{A,\rho\pi} = 0 \\ g_{A,\sigma\pi} &= \gamma Z_\pi^{1/2} \approx \frac{m_\rho}{f_\pi} \approx 8 \qquad (3.60) \\ g_{\sigma\pi\pi} &= \frac{2}{m_\sigma}\frac{1}{3}\gamma^2 f_\pi Z_\pi^{3/2} \approx \frac{m_\sigma}{f_\pi}\sqrt{2} \approx 9. \qquad (3.61) \end{aligned}$$

When compared with experiment, the only real defect consists in the d-wave $A, \rho\pi$ coupling, $h_A, \rho\pi$, being absent, additional chirally invariant terms are needed in the Lagrangian, for example, the so-called δ-term:

$$\delta \mathrm{Tr}\left[\left(F_{\mu\nu}^A + F_{\mu\nu}^V\right) D^\mu (\sigma + i\pi) D^\nu (\sigma - i\pi) - (V \to -V, \pi \to -\pi)\right]. \qquad (3.62)$$

Such terms appear in our derivation if the approximation of large μ^2 is improved by terms which do not grow logarithmically in μ.

Let us now determine the couplings of $\pi,\ \sigma,\ \rho, A_1, A_1$ to external quark fields. The external propagation proceeds via

$$i\bar\eta G\eta = i\bar\eta G_{M_0}\eta + \bar\eta G_{M_0} m'' G_{M_0} m'' G_{M_0}\mu - i\dots \qquad (3.63)$$

If one defines the couplings by

$$\begin{aligned} &\mathcal{L}\alpha g_{\pi QQ}\bar\Psi i\gamma_5\tau_a\Psi\pi^a + g_{\sigma QQ}\bar\psi\tau_a\psi\sigma^a \\ &+ g_{VQQ}\bar\Psi\gamma^\mu\frac{\tau_a}{2}\Psi V_\mu^a + g_{AQQ}\bar\Psi\gamma^\mu\gamma_5\frac{\tau_a}{2}\psi A_\mu^a \qquad (3.64) \end{aligned}$$

and can read off

$$g_{\pi QQ} = \frac{1}{2\sqrt{L}}Z_\pi^{1/2} = \frac{M}{f_\pi} \quad ; \quad g_{\sigma QQ} = \frac{1}{2\sqrt{L}} = \frac{M}{f_\pi Z_\pi^{1/2}}$$
$$g_{\rho QQ} = g_{A_1 QQ} = \gamma. \qquad (3.65)$$

We see the vector coupling to quarks agree with vector-meson dominance. Due to PCAC also the Goldberger Treiman relation is respected

$$g_{\pi QQ} = g_A\frac{M}{f_\pi} \qquad (3.66)$$

since the axial charge of the quark is $g_A = 1$. Since the quark mass is $M \sim m_{N/3}$, the pionic coupling to quarks is considerably smaller than to nucleons. Numerically

$$\frac{g_{\pi\psi\psi}^2}{4\pi} \approx \frac{1}{17}\frac{g_{\pi NN}^2}{4\pi} \approx .86. \qquad (3.67)$$

The σ meson couples even weaker

$$g_{\sigma QQ}^2 \approx .43.$$

Vector- and axial-vector mesons, on the other hand, couple as strongly as to nucleons which is an expression of universality:

$$\frac{g^2_{\rho QQ}}{4\pi} = \frac{g^2_{AQQ}}{4\pi} = \frac{\gamma^2}{4\pi} \approx 2.6 \left(\approx \frac{g^2_{\rho NN}}{4\pi} \right) \tag{3.68}$$

We are now ready to extend our consideration to $SU(3)$ (and higher groups). In this case the explicit symmetry breaking in the Lagrangian is too large to be neglected. Thus the bare masses $\mathcal{M}$ of the quarks have to be consdiered as a matrix

$$\mathcal{M} \approx \begin{pmatrix} \mathcal{M}^n & & \\ & \mathcal{M}^d & \\ & & \mathcal{M}^s \end{pmatrix} \tag{3.69}$$

The derivation of the Lagrangian presented above (via the gap equation (3.42) has shown complete $SU(3)$ symmetry of M_0^2. Hence when extending from $SU(2)$ to $SU(3)$, no change occurs except in the last symmetry breaking term of (3.35). As a consequence, the mass expressions for m_p^2 and m_s^2 remain as they are only that the renormalization constants Z_p become more complicated $SU(3)$ dependent quantities due to the involved mixing of pseudoscalar and axialvector mesons. For a complete discussion of this $SU(3) \times SU(3)$ invariant chiral Lagrangian the reader is referred to the review articles [23]. Here we only give a few results:
A best fit to π K meson masses requries[6]

$$\mathcal{M} \approx \begin{pmatrix} 15 & & \\ & 15 & \\ & & 435 \end{pmatrix} \text{MeV.} \tag{3.70}$$

Thus the explicit symmetry breakdown of $SU(3)$ caused by the bare masses is quite large. The standard parameter [28] C characterizes this:

$$C \equiv \frac{\mathcal{M}^8}{\mathcal{M}^0} = \frac{\frac{1}{\sqrt{3}}\left(\mathcal{M}^n + \mathcal{M}^d - 2\mathcal{M}^s\right)}{\sqrt{\frac{2}{3}}\left(\mathcal{M}^n + \mathcal{M}^d + \mathcal{M}^s\right)} \approx -1.28(\approx -\sqrt{2}). \tag{3.71}$$

Inserting into (3.45) we find the shifts in the quark masses caused by dynamics

$$\delta M = \begin{pmatrix} 7 & & \\ & 7 & \\ & & 127 \end{pmatrix} \text{MeV} \tag{3.72}$$

and hence for the "physical" quark masses

$$M \approx \begin{pmatrix} 3/2 & & \\ & 3/2 & \\ & & 432 \end{pmatrix} \text{MeV.} \tag{3.73}$$

[6]For other determinations of see Ref. 26.

Thus contrary to the large explicit $SU(3)$ violation the bare masses $\mathcal{M}$, the physical quark masses M show only the moderate violation

$$C' \equiv \frac{M^8}{M^0} = \frac{\frac{1}{\sqrt{3}}\left(M^n + M^d - 2M^s\right)}{\sqrt{\frac{2}{3}}\left(M^n + M^d + M^s\right)} \approx -16 \tag{3.74}$$

Since the quark masses M are produced almost completely by dynamical effects we expect some symmetry breakdown to appear also in the vacuum. A measure of this is provided by the expectation values of the scalar quark densities

$$\langle 0|\tilde{U}^a|0\rangle \equiv \langle 0|\bar{\psi}(x)\frac{\lambda^a}{2}\psi(x)|0\rangle. \tag{3.75}$$

In the mesonized theory, the scalar densities are identical with the scalar fields, up to a factor:

$$S^i_j(x) = -\frac{g^2}{\mu^2}\bar{\psi}^i(x)\psi_j(x) \tag{3.76}$$

as can be seen most easily by considering the equations of constraint (2.15) following from the Lagrangian (2.12) in the large $-\mu$ limit. Hence

$$\begin{aligned}\langle 0|\tilde{u}^a|0\rangle &= -\frac{\mu^2}{g^2}\sum_{i,j}\lambda^{aj}_i\langle 0|S^i_j|0\rangle \\ &= -\frac{\mu^2}{2g^2}\mathrm{tr}(\mathcal{M}\lambda^a).\end{aligned} \tag{3.77}$$

Inserting (3.17) and (3.57) the factor becomes simply

$$\frac{1}{2}\frac{\mu^2}{g^2} = \frac{1}{3}\frac{1}{\frac{2}{3}\frac{g^2}{\mu^2}L}L = \frac{1}{3}m_V^2 Z_\pi \frac{f_\pi^2}{4M^2} \approx f_\pi^2$$

such that

$$\begin{aligned}&\langle 0|\tilde{u}^0|0\rangle \approx f_\pi^2 M^0 = -f_\pi^2\sqrt{\frac{2}{3}}\left(M^n + M^d + M^s\right) \approx -8\times 10^{-3}\mathrm{GeV}^3 \\ &\langle 0|\tilde{u}^8|0\rangle \approx -f_\pi^2 M^8 = c'\langle 0|\tilde{u}^0|0\rangle.\end{aligned} \tag{3.78}$$

This shows that the $SU(3)$ violation in the vacuum equals that in the quark masses $\approx -16\%$. [7] Notice that the three results (3.22), (3.45) and (3.76) are in complete agreement with what one obtains by very general consideratons using only chiral symmetry and PCAC (see App. C).

The extension of the Lagrangian to $SU(3)$ produces additional defects which are well known from general discussions of chiral $SU(3)\times SU(3)$ symmetry [23]. For

[7] In Ref. cite30, $SU(3)$ breaking in the vacuum was neglected. For a more general discussion and earlier references see Ref. [29].

example the vector mesons ω_1, φ are not mixed (almost) ideally as they should but φ remains close to an $SU(3)$ singlet. In general discussions, additional terms have been added to chiral Lagrangian in order to account for this. There are the so called "current mixing terms":

$$\mathrm{Tr}\left\{\left(F^V_{\mu\nu}+F^A_{\mu\nu}\right)^2(\sigma+i\pi)(\sigma-o\pi)+(A\to -A,\pi\to -\pi)\right\} \tag{3.79}$$

as well as "mass mixing terms"

$$\mathrm{Tr}\left\{(V^\mu+A^\mu)^2(\sigma+i\pi)(\sigma-i\pi)+(V^\mu-A^\mu)^2(\sigma-i\pi)(\sigma+i\pi)\right\}. \tag{3.80}$$

In our derivation these arises as a next correction to the $\mu^2\to\infty$ limit. Another problem is the degeneracy of the ideally mixed isosinglet pseudoscalar meson $\mu'_{\rm ideal}$ with the pion. In order to account for the fact that the $\eta'\,(\Xi_0)$ meson is almost a pure $SU(3)$ singlet and much heavier than the other pseudoscalar mesons one needs some chirally symmetric term

$$\det(\sigma+i\pi)+\det(\sigma-i\pi). \tag{3.81}$$

Such a term breaks PCAC for the ninth axial current. It is well known [30] that the quark gluon triangle anomaly operates in the singlet channel and might be capable of producing such a PCAC violation. In fact, if this was not true, quantum electrodynamics would possess an exactly massless Goldstone boson [31] with η quantum numbers. Also the term (3.81) will appear when μ is not any more very large.

It is obvious that corrections to the $\mu^2\to\infty$ approximation will become even more important if one tries to extend the consideration to $SU(4)$ since then vector and pseudoscalar masses are quite heavy. In addition, the narrow width of the $SU(4)$ vector meson ψ/J seems to indicate that short-distance parts of the gluon propagator are being probed. Thus the colorless quark gluon theory itself cannot be considered any more a realistic approximation to the colored theory.

At this place we should remark that present explanations of electromagnetic mass differences require also an breakdown of $SU(2)$ symmetry in $\mathcal{M}$ [32]. This is conventionally parametrized by

$$d\equiv\frac{\mathcal{M}^3}{\mathcal{M}^0}=\frac{\mathcal{M}^n-\mathcal{M}^d}{\sqrt{\frac{2}{3}}\,(\mathcal{M}^n+\mathcal{M}^d+\mathcal{M}^s)}. \tag{3.82}$$

From meson masses (as well as from the electromagnetic $\eta\to 3\pi$ decay) one finds [33]

$$d\approx -3 \tag{3.83}$$

This amounts to the bare quark masses

$$\mathcal{M}\approx\begin{pmatrix}10 & & \\ & 20 & \\ & & 435\end{pmatrix}\mathrm{MeV} \tag{3.84}$$

giving the "true" masses

$$M \approx \begin{pmatrix} 310 & & \\ & 315 & \\ & & 432 \end{pmatrix} \text{MeV}. \tag{3.85}$$

Thus the $SU(2)$ breaking of the vacuum is very small

$$d' \equiv \frac{\langle 0|\tilde{u}^3|0\rangle}{\langle 0|\tilde{u}^0|0\rangle} = \frac{M^3}{M^0} \approx -.6 \tag{3.86}$$

With all parameters fixed numerically we should finally check whether the approximation of a large gluon mass is self consistent. From (3.57) we have

$$L \approx \frac{1}{2}\frac{f\pi^2}{M^2} \approx .046. \tag{3.87}$$

Inserting this into (3.29) we calculate

$$\log \frac{\mu^2}{M^2} \sim (4\pi)^2 \times 0.046 \approx 7.3 \tag{3.88}$$

and hence

$$\mu^2 \approx 1500 M^2 \gg M^2 \tag{3.89}$$

or

$$\mu \approx 12\text{BeV}. \tag{3.90}$$

It is gratifying to note that this value is much larger than the mass of the vector mesons. In this way it is assured that higher powers of $q^2/\left(P_E^2 + M^2\right)$ which were neglected in the derivation of the Lagrangian remain really small as compared to unity for *all* mesons of the theory (see App. B).

We should point out that the quark gluon theory in the limit $\mu^2 \to \infty$ coincides with the well-known Nambu-Jona Lasinio [24] model which has proven in the past to be a convenient tool of studying the spontaneous breakdown of chiral symmetry and the dynamical generation of PCAC. Those authors have demonstrated the close analogy of the dynamic structure of this model with that of superconductivity. As we have mentioned before the equation (3.41) removing the tadpoles in the action is analogous to the gap eqution for superconductors.

A similar analogy to superconductors exists also for the hdronized theory. The classical version of it corresponds exactly to the classical Ginzburg-Landau equation for type II superconductors in which the gap is allowed to be space time dependent. In fact, the classical mesonized theory can be derived alternatively by assuming such a dependence in the gap equation [34], [18]. The advantage of our functional derivation is that the mesonized theory is not merely some classical approximation but becomes upon quantization completely equivalent to the original quark gluon theory.

A final comment concerns the Okubo-Zweig-Iizuka rule. As argued in the general section, the meson Lagrangian exactly respects this rule. This can be checked directly for all interaction terms in (3.35). Violations of this rule are all coming from meson loops to straight-forward estimates for the size of such violations.

Outlook

We have shown that in the abence of color, quark gluon theories can successfully be mesonized. The resulting quantum field theory incorporates correctly amyn features of strong interactions. Its basic fields are bilocal and the Feynman rules are topologically similar to dual diagrams. Our considerations haven take place at a rather formal level. Certainly, there are many problems which have been left open. For example, there is need for an understanding of the non-trivial gauge properties of the bilocal theory. Also, a consistent renormalization procedure will have to be developed in future investigations.

The inclusion of color is the challenging problem left open by this investigation. If color quark gluon theory is really equivalent to some kind of dual model the corresponding mesonization program should not produce bilocal but multilocal fields which are characterized by the position of a whole string rather that just its end points. A field theory should be constructed for gauge invariant objects like

$$\bar{\psi}(x) \exp\left(ig \int_x^y G^\nu(z) dz_\nu\right) \psi(y)$$

which depend on the whole path from x to y.

The difficulty in a direct generalization of the previous procedure is the self-interactionj of the gluons. Only after the infrared behaviour of gluon propagators will be known, bare mesons can be constructed inside the corresponding potential well and the "mesonization" methods can serve for the determination of the complete residual interactions.

It is hoped that mesonic Feynman rules in the presence of color will follow a pattern similar to that found here for the non-abelian theory.

Let us finally mention that an interesting field of applications of our methods lies in solid-state physics. Semi-conductors in which conduction and valence bean dhave only small separations may show a phase transition to what is called excitonic insulator. The critical phenomena taking place inside such an exciton system will find their most appropriate description by studying the scaling properties of the bilocal field theory.

Appendix 3A Remarks on the Bethe-Salpeter Equation

Consider the four Fermion Green's funcion

$$G^{(4)}_{\alpha\beta,\alpha'\beta'}(x, y; x'y') \equiv \langle 0|T\left(\psi_\alpha(x)\bar{\psi}_\beta(y)\bar{\psi}_{\alpha'}(x')\psi_{\beta'}(y')\right)|0\rangle \tag{3A.1}$$

which becomes in the interaction picture

$$G^{(4)}_{\alpha\beta,\alpha'\beta'}(x,y;x',y') \equiv \left[\langle 0|Te^{ig\int d^4z\bar\psi(z)\gamma^\mu\psi(z)G_\mu(z)}|0\rangle\right]^{-1}$$
$$\times\langle 0|T\left(e^{ig\int d^4z\bar\psi(z)\gamma^{\mu A}\psi(z)G_\mu(z)}\psi_\alpha(x)\bar\psi_\beta(y)\psi_{\alpha'}(x')\psi_{\beta'}(y')\right)|0\rangle \qquad (3A.2)$$

Expanding the exponential and keeping only the ladder exchanges corresponding to the Feynman graph in Fig. 3.1), we obtain

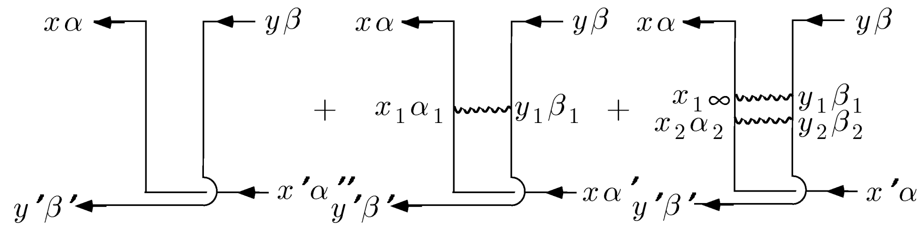

FIGURE 3.1

$$G^{(4)}_{\alpha\beta,\alpha'\beta'}(xy,x'y') = G_{\alpha\alpha'}(x-x')\,G_{\beta'\beta}(y'-y) \qquad (3A.3)$$
$$+g^2\int dx_1dy_1G_{\alpha\alpha_1}(x-x_1)\,\gamma^\mu_{\alpha_1\alpha_1'}G_{\alpha_1'\alpha'}(x_1-x')\,G_{\beta'\beta_1'}(y'-y_1)\,\gamma_{\mu_{\beta_1'\beta_1}}G_{\beta_1\beta}(y_1-y)\,D(x-y)$$
$$+g^4\int dx_1dy_1dx_2dy_2G_{\alpha\alpha_1}(x-x_1)\,\gamma^{\mu_1}_{\alpha_1\alpha_1'}G_{\alpha_1'\alpha_2}(x_1-x_2)\,\gamma^{\mu_2}_{\alpha_2\alpha_2'}G_{\alpha_2'\alpha_2}(x_2-x')$$
$$D(x_1-y_1)\,D(x_2-y_2)\,G_{\beta'\beta_2'}(y'-y_2)\,\gamma_{\mu_{2_{\beta_2'\beta_2}}}G_{\beta_2\beta_1'}(y_2-y_1)\,\gamma_{\mu_{1_{\beta_1'\beta_1}}}G_{\beta'\beta}(y_1-y)$$
$$+\ldots \qquad (3A.4)$$

The series can be summed to the integral equation

$$G^{(4)}_{\alpha\beta,\alpha'\beta'}(xy,x'y') = \qquad (3A.5)$$
$$G_{\alpha\alpha'}(x-x')G_{\beta'\beta}(y'-y) + g^2\int dx_1dy_1G_{\alpha\alpha_1}(x-x_1)\,\gamma^\mu_{\alpha_1\alpha_1'}$$
$$G^{(4)}_{\alpha_1'\beta_1',\alpha'\beta'}(x,y_1,x'y')\,\gamma\mu_{\beta_1'\beta_1}G_{\beta_1\beta}(y_1-y)\,D(x_1y_1)$$

With the abbreviation

$$\xi_{\alpha_1\beta_1,\beta_1'\alpha_1'} = \gamma^\mu_{\alpha_1\alpha_1'}\gamma\mu_{\beta_1'\beta_1} = 1_{\alpha_1\beta_1}1_{\beta_1'\alpha_1'} + (i\gamma_5)_{\alpha_1\beta_1}(i\gamma_5)_{\beta_1'\alpha_1'}$$
$$-\frac{1}{2}\gamma^\mu_{\alpha_1\beta_1}\gamma_{\mu\beta_1'\alpha_1'} - \frac{1}{2}(\gamma^\mu\gamma_5)_{\alpha_1\beta_1}(\gamma_\mu\gamma_5)_{\beta_1'\alpha_1'}. \qquad (3A.6)$$

This can be written as

$$G^{(4)}_{\alpha\beta,\alpha'\beta'}(xy_1x'y') =$$
$$G_{\alpha\alpha'}(x-x')\,G_{\beta'\beta}(y'-y) + g^2\int dx_1dy_1G_{\alpha\alpha_1}(x-x_1)$$
$$\xi_{\alpha_1\beta_1,\beta_1'\alpha_1}D(x_1-y_1)\,G^{(4)}_{\alpha_1'\beta_1',\alpha'\beta'}\Gamma_{\beta_1\beta}(y_1-y) \qquad (3A.7)$$

or, symbolically:

$$G^{(4)} = GG^T + GG^T \xi g^2 D G^{(4)}. \tag{3A.8}$$

The transition matrix T is defined by removing the external particle poles in the connected part of $G^{(4)}$

$$G^{(4)}_{\alpha\beta,\alpha'\beta'}\left(xy, x'y'\right) \equiv \tag{3A.9}$$

$$G_{\alpha\alpha'}\left(x-x'\right) G_{\beta\beta'}\left(y'-y\right) + \int dx_1 dy_1 dx_2 dy_2 G_{\alpha\alpha_1}\left(x-x_1\right)$$

$$G_{\beta'\beta_2}\left(y'-y_2\right) T_{\alpha_1\beta_1,\alpha_2\beta_2}\left(x_1y_1, x_2y_2\right) G_{\alpha_1\alpha}\left(x_2-x'\right) G_{\beta_1\beta}\left(y_1-y\right) \tag{3A.10}$$

which may be abbreviated by

$$G^{(4)} = GG^T + GG^T T GG^T. \tag{3A.11}$$

From (3A.7) and (3A.9) the transition matrix satisfies the integral equation

$$T_{\alpha_1\beta_1,\alpha_2\beta_2}\left(x_1y_1, x_2y_2\right) =$$

$$\xi_{\alpha_1\beta_1,\beta_2\alpha_2} g^2 D\left(x_1-y_1\right) \delta\left(x_1-y_1\right)$$

$$\int dx_1' dy_1' G_{\alpha_1\alpha_1'}\left(x_1-x_1'\right) T_{\alpha_1'\beta_1,\alpha_2\beta_2}\left(x_1'y_1'x_2y_2\right) G_{\beta'\beta_1}\left(y_1'-y_2\right)$$

which is seen to coincide with the equation (2.42) for the propagator of the bilocal field. In a short notation, this equation can be written as

$$T = \xi g^2 D + \xi g^2 D G G^T T. \tag{3A.12}$$

The perturbation expansion

$$T = \xi g^2 D + \xi g^2 D G G^T \xi g^2 D + \dots \tag{3A.13}$$

reveals the one, two etc. photon exchanges of the ladder diagrams. In momentum space the four particle Green's function is defined by

$$(2\pi)^4 \delta^4\left(q'-q\right) G^{(4)}_{\alpha\beta,\alpha'\beta'}\left(P, P'|q\right) i\left[\left(P+\frac{g}{2}\right)x + \left(P'-\frac{g'}{2}\right)y'\right.$$

$$\left. - \left(P-\frac{q}{2}\right)y - \left(P'+\frac{q'}{2}\right)x\right] G^{(4)}_{\alpha\beta\alpha'\beta'}\left(xy, x'y'\right) \equiv \int dx dy dx' dy' e \tag{3A.14}$$

where the momenta are indicated in (3.2). The corresponding scattering matrix $T\left(P, P'|q\right)$ satisfies the integral equation The corresponding scattering matrix $T\left(P_1 P'|q\right)$ satisfies the integral equation

$$T\left(P, P'|q\right) = \tag{3A.15}$$

$$\xi g^2 D(P-P') + \xi g^2 \int \frac{dP''}{(2\pi)^4} D(P-P'') G\left(P''+\frac{g}{2}\right) T\left(P'', P'|q\right) G\left(P''-\frac{q}{2}\right).$$

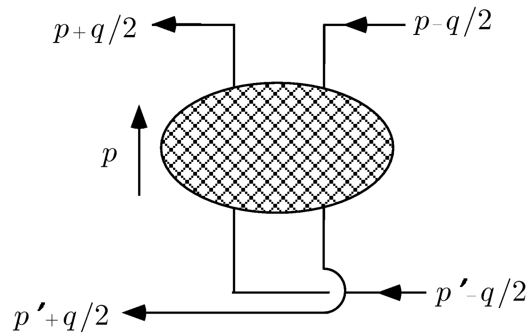

FIGURE 3.2

The ladder exchange is in general expected to produce quark anti-quark bound states. Suppose $|H(\mathbf{q})\rangle$ is one of them. Inserting it into (3A.1) as an intermediate state gives for $x_0, y_0 > x_0', y_0'$ a contribution

$$G^{(4)}_{\alpha\beta,\alpha'\beta'}(x, y; x'y') = \tag{3A.16}$$

$$\int \frac{d^3q}{2E_q(2\pi)^3}\theta\left(R_0 - R_0' - \frac{1}{2}(|z_0| + |z_d'|)\right)$$

$$\times\langle 0|T\left(\psi_\alpha(x)\bar{\psi}_\beta(y)\right)|H(\mathbf{q})\rangle\langle H(\mathbf{q})|T\left(\bar{\psi}_{\alpha'}(x')\psi_{\beta'}(y')\right)|0\rangle \tag{3A.17}$$

where $R \equiv (+y)/2$ and $z = x - y$. The θ function is non-zero if

$$\min(x_0, y_0) > \max(x_0' y_0').$$

Using the integral representation

$$\theta(x_0) = \frac{i}{2\pi}\int da e^{-ax_0}\frac{1}{a + i\epsilon}$$

we have

$$\theta\left(R_0 - R_0' - \frac{1}{2}(|z_0| + |z_0'|)\right) = \tag{3A.18}$$

$$\frac{i}{2\pi}\int dq_0 e^{-i(q_0 - E_q)\left(R_0 - R_0'\right)}$$

$$e^{i(q_0 - E_q)\frac{1}{2}\left(z_0| + k_0'|\right)}\frac{1}{q_0 - E_q + i\epsilon}.$$

Introducing Bethe-Salpeter wave functions

$$\begin{aligned}\phi_{\alpha\beta}(x, y|q) &\equiv \langle 0|T\left(\psi_\alpha(x)\bar{\psi}_\beta(y)\right)|H(\mathbf{q})\rangle \\ \bar{\phi}_{\beta'\alpha'}(y'x'|-q) &\equiv \langle H(\mathbf{q})|T\left(\psi_{\beta'}(y')\bar{\psi}_{\alpha'}(x')\right)|0\rangle\end{aligned} \tag{3A.19}$$

and their momentum space versions

$$\psi_{\alpha\beta}(x,y|q) = e^{-i(E_1 R_0 - \mathbf{qR})} \int \frac{d^4P}{(2\pi)^4} e^{-iP(x-y)} \phi_{\alpha\beta}(P)|q) \quad (3A.20)$$

$$\bar{\psi}_{\alpha'\beta'}(x',y'|-q) = e^{i\left(E_1 R_0' - \mathbf{qR}'\right)} \int \frac{d^4P}{(2\pi)^4} e^{-iP'(x'-y')} \phi_{\alpha'\beta'}(P')|-q) \quad (3A.21)$$

the four-particle Green's function in momentum space is seen to exhibit a pole at $q_0 \approx E_q$

$$G^{(4)}_{\alpha\beta\alpha'\beta'}(P,P'|q) \approx \frac{-i}{2E_q(q_0 - E_q + i\epsilon)} \psi_{\alpha\beta}(P|q)\, \bar{\phi}_{\beta'\alpha'}(P|-q)\,. \quad (3A.22)$$

The opposite time ordering $x_0, y_0 < x_0', y_0'$ contributes a pole at $q_0 = -E_q$. Both poles can be collected by writing in (3A.22) the factor

$$-\frac{i}{q^2 - M_H^2 + i\epsilon}.$$

This factorization is consistent with the integral equation only for a specific normalization of the Bethe-Salpeter wave functions. In orde to see thsi write (3A.8) in the form

$$\begin{aligned} G^{(4)} &= GG^T + GG^T \xi g^2 D G^{(4)} \\ &= \left(1 - GG^T \xi g^2 D\right)^{-1} GG^T \\ &= GG^T \left(1 - \xi g^2 D GG^T\right)^{-1}. \end{aligned} \quad (3A.23)$$

Suppose now that a solution is found for different values of the coupling constant g^2. Then the variation of $G^{(4)}$ for small changes of g^2 is

$$\begin{aligned} \frac{\partial G^{(4)}}{\partial g^2} &= \left(1 - GG^T \xi g^2 D\right)^{-1} GG^T \xi D \\ &\quad \times GG^T \left(1 - \xi g^2 D\right)^{-1} \\ &= G^{(4)} \xi D G^{(4)} \end{aligned} \quad (3A.24)$$

If one goes in the vicinity of the pole $q^2 \approx M_H^2(g^2)$ this becomes

$$-\frac{\partial}{\partial g^2} \frac{i}{s - M_H^2(g^2)} \phi_H^g(P|q) \bar{\phi}_H^g(P|-q) = \quad (3A.25)$$

$$\frac{i}{S - M_H^2(g^2)} \phi_H^g(P|q) \int \frac{d\bar{P} d\bar{P}'}{(2\pi)^8} \bar{\phi}_H^g\left(\bar{P}/-q\right) \xi D\left(\bar{P} - \bar{P}'\right) \phi_H^g\left(\bar{P}'|q\right)$$

$$\bar{\phi}_H^g\left(\bar{P}'|-q\right) \frac{i}{S - M_H^2(g^2)}. \quad (3A.26)$$

This can be true at the double pole t $q^2 = M_H^2(g^2)$ only if

$$\frac{\partial M_H^2(g^2)}{\partial g^2} = -i\int \frac{dPdP'}{(2\pi)^4(2\pi)^4}\phi_H(P|q)\xi D(P-P')\bar{\phi}_H(P'|-q). \tag{3A.27}$$

If we go over to the Bethe Salpeter vertex function (2.34)

$$\begin{aligned} \Gamma^H(P|q) &= N_H G_M^{-1}\left(P+\frac{q}{2}\right)\phi^H(P|q)G_M^{-1}\left(P-\frac{q}{2}\right) \\ \Gamma^H(P|-q) &= N_H^* G_M^{-1}\left(P-\frac{q}{2}\right)\bar{\phi}^H(P|-q)G_M^{-1}\left(P+\frac{q}{2}\right) \end{aligned} \tag{3A.28}$$

this amounts to

$$g_V^2\left(q^2\right)\frac{\partial M_H^2(g^2)}{\partial g^2} = \tag{3A.29}$$

$$i|N_H|^2\int\frac{dPdP'}{(2\pi)^4(2\pi)^4}\mathrm{Tr}\left\{G_M\left(P+\frac{q}{2}\right)\Gamma^H(P|q)\right.$$
$$\left.G_M\left(P-\frac{q}{2}\right)g_H^2\left(q^2\right)D(P-P')G_M\left(P'-\frac{q}{2}\right)\bar{\Gamma}^H(P'|-q)\,G_M\left(P'+\frac{q}{2}\right)\right\} \tag{3A.30}$$

Using the integral equation (2.35) this reduces to the normalization

$$\frac{g_H^2(q^2)}{\frac{\partial g_H^2(q^2)}{\partial q^2}} = -i|N_H|^2\int\frac{d^4P}{(2\pi)^4} \tag{3A.31}$$

$$\times\mathrm{Tr}\left[G_M\left(P+\frac{q}{2}\right)\Gamma^H(P|q)G_M\left(P-\frac{q}{2}\right)\bar{\Gamma}^H(P|-q)\right]. \tag{3A.32}$$

This determines $|N_H|^2$ as

$$|N_H|^2 = \frac{g_H^2(q^2)}{\frac{\partial g_H^2(q^2)}{\partial q^2}}. \tag{3A.33}$$

Notice that this normalization is defined for all q^2 with some $N_H(q^2)$. For real $\Gamma(P|q)$ one may choose $N_H(q^2)$ real such that

$$\bar{\Gamma}(P|-q) = \Gamma(P|-q)$$

(Both satisfy the same integral equation). The orthogonality of $\Gamma^H(P|q)$ and $\bar{\Gamma}^{H'}(P|-q)$ for different mesons is proved as usual by considering (2.35) once for $(\xi g^2 D)^{-1}\Gamma^H$ and once for $(\xi g^2 D)^{-1}\bar{\Gamma}^{H'}$, multiplying the first by $\bar{\Gamma}^{H'}$ and the second by Γ^H, taking the trace and subtracting the results from each other (assuming no degeneracy of $g_H(q^2)$ and $g_{H'}(q^2)$. The normalization (3A.33) is seen to be consistent with the expansion of the T matrix given in (2.44)

$$T_{\alpha\beta,\alpha'\beta'}(P,P'|q) = -ig^2\sum_H\frac{\Gamma_{\alpha\beta}^H(P|q)\bar{\Gamma}_{\beta'\alpha'}^H(P'|-q)}{g_H^2(q^2)-g^2}. \tag{3A.34}$$

If q^2 runs into a ple M_H^2 this expression is singular as

$$T_{\alpha\beta,\alpha'\beta'}\left(P,P'|q\right) \approx -ig^2 \frac{1}{\left(q^2-M_H^2\right)\frac{\partial g_H^2(q^2)}{\partial q^2}} \Gamma^H_{\alpha\beta}(P|q)\bar{\Gamma}^H_{\beta'\alpha'}\left(P'|-q\right). \tag{3A.35}$$

According to (3A.11) this produces a singularity in $G^{(4)}$ (in short notation)

$$\begin{aligned} G^{(4)} &\approx -i\frac{g^2}{\frac{\partial g_H^2(q^2)}{\partial q^2}} \frac{1}{q^2-M_H^2}\left(G\Gamma^H G\right)\left(G\bar{\Gamma}^H G\right) \\ &= -i\frac{g^2}{\frac{\partial g_H^2(q^2)}{\partial q^2}} \frac{1}{q^2-M_H^2}|N_J|^2\phi^H\bar{\phi}^H \end{aligned} \tag{3A.36}$$

which coincides with (3A.22) by virtue of (3A.33).

For completeness we now give the Bethe-Salpeter equation (2.32) the form projected into the different covariants:

$$\begin{aligned} &m'(P|q) = \\ &\quad S(P|q)+P(P|q)i\gamma_5+V^\mu(P|q)\gamma_\mu+A^\mu(P|q)\gamma_\mu\gamma_5. \end{aligned} \tag{3A.37}$$

If $m_i(P|q)(i=1,2,3,4)$ abbreviates S,P,V,A, one has

$$\begin{aligned} m_i(P|q) &= -4\xi_i g^2\sum_{j=1}^{4}\frac{d^4P'}{(2\pi)^4}\frac{1}{(P-P')_E^2+\mu^2}\frac{1}{\left(P'+\frac{q}{2}\right)_E^2+M^2} \\ &\quad \frac{1}{\left(P'-\frac{q}{2}\right)_E^2+M^2}t_{ij}\left(P'|q\right)m_j\left(P'|q\right) \end{aligned} \tag{3A.38}$$

with $t_{ij}(P|q)$ being the traces defined in (3.10) and $\xi_i = (4,\ +4,\ -2,\ -2)$. Explicitly, one finds

$$\begin{aligned} t_{ss}(P|q) &= P^2-\frac{q^2}{4}+M^4, \quad t_{SV}(P|q)=t_{VS}=2MP^\mu \\ t_{PP}(P|q) &= P^2-\frac{q^2}{4}-M^2, \quad t_{PA^\mu}(P|q)=-t_{A^\mu P}=iMq^\mu \\ t_{V^\mu V^\nu}(P|q) &= -\left(P^2-\frac{q^2}{4}-M^2\right)g^{\mu\nu}+2P^\mu P^\nu-\frac{1}{2}q^\mu q^\nu \qquad (3A.39) \\ t_{A^\mu A^\nu}(P|q) &= -\left(P^2-\frac{q^2}{4}-M^2\right)g^{\mu\nu}+2P^\mu P^\nu-\frac{1}{2}q^\mu q^\nu-2M^2g^{\mu\nu} \qquad (3A.40) \end{aligned}$$

with all other traces vanishing. Notice that in the Bethe-Salpeter equation for m' there is no tensor contribution due to the absence of such a term in the Fierz transform of $\gamma^\mu\otimes\gamma_\mu$. The integrals in (3A.37) go directly over into (3.9) for large gluon mass μ.

Appendix 3B Vertices for Heavy Gluons

$\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ for large μ. As discussed in the text, all higher vertices remain finite when the cutoff and μ to to infinity in the order $\Lambda^2 \gg \mu^2 \gg M^2$ and will consequently be neglected.

Consider first $\mathcal{A}_2$ as described in (3.10) and (3A.39). The intgrals $J_{ij}(q)$ are evaluated by expanding

$$\left[\left(P+\frac{q}{2}\right)^2_E + M^2\right]^{-1}\left[\left(P-\frac{q}{2}\right)^2_E + M^2\right]^{-1}\left[\left(P-\frac{q}{2}\right)^2_E + M^2\right]^{-1} = \tag{3B.1}$$
$$\left[P_E^2+M^2\right]^{-2}\left\{1+\frac{q^2/2}{P_E^2+M^2}+\frac{(P_q)^2}{(P_E^2+M^2)^2}+\mathcal{O}\left(\frac{M^4}{(P_E^2+M^2)^2}\frac{q^4}{(P_E^2+M^2)^2)}\right)\right\}.$$

Since $t_{ij}(P|q)$ grow at most like P_E^2 [see (3A.39)] the terms $\mathcal{O}(M^4/E^4,\ q^4/P_E^4)$ contribute finite amounts upon integration and will be neglected. At this place we have assumed q^2 to remain of the same order of M^2. Actually, this is not true for vector- and axialvector meson fields[8] but since numerically $m_\rho^2,\ \ m_{A_1}^2 < \frac{1}{100}\mu^2$, the neglected terms are indeed very small.

The following integrals are needed in addition to (3.5) (3.11) (neglecting finite amounts)

$$\begin{aligned}
\int \frac{d^4P_E}{(2\pi)^4}\frac{1}{(P_E^2+M^2)^2}P^2 &= -(Q-M^2L)\\
\int \frac{d^4P_E}{(2\pi)^4}\frac{1}{(P_E^2+M^2)^2}P_\mu P_\nu &= -(Q-M^2L)\frac{g_{\mu\nu}}{4}\\
\int \frac{d^4P_E}{(2\pi)^4}\frac{1}{(P_E^2+M^2)}P_\mu P_\nu &= -L\frac{g_{\mu\nu}}{4}\\
\int \frac{d^4P_E}{(2\pi)^4}\frac{1}{(P_E^2+M^2)^4}P_\mu P_\nu P_\lambda P_\kappa &= \frac{L}{24}\left(g_{\mu\nu}g_{\lambda\kappa}+g_{\mu\lambda}g_{\nu\kappa}+g_{\mu\kappa}g_{\nu\lambda}\right).
\end{aligned}\tag{3B.2}$$

The results are displayed in Eq. (3.12).

There is one subtlety connected with gauge invariance when evaluating the integrals $J_W(q)$ and $J_{AA}(q)$. In fact, the first of these integrals coincides with the standard photon solf-energy graph in quantum electro-dynamics. There the cutoff procedure is known to produce a non-gauge invariant result. The cutoff calculation yields:

$$\begin{aligned}
J_{V^\mu V^\nu}(q) &= -\frac{1}{3}\left(q^2 g_{\mu\nu}-q_\mu q_\nu\right)L-\frac{1}{2}(Q+M^2L)g_{\mu\nu}\\
J_{A^\mu A^\nu} &= -\frac{1}{3}\left(\left(q^2-6M^2\right)g_{\mu\nu}-q_\mu q_\nu\right)L-\frac{1}{2}(Q+M^2L)g_{\mu\nu}.
\end{aligned}\tag{3B.3}$$

[8] $m_\rho^2 = 6M^2;\ \ m_A^2 \approx 12M^2;\ \ \mu^2 \approx 144\text{GeV}^2$

There are many equivalent ways to enforce gauge invariance. The simplest one proceeds via dimensional regularization. If one evaluates the integrals Q and L in $D = 4 - \epsilon$ dimensions with a small $\epsilon > 0$, then

$$\begin{aligned} L &= \int \frac{d^D p}{(2\pi)^4} \frac{1}{P_E^2 + M^2)^2} = \frac{\pi^{D/2}}{(2\pi)^4} \frac{1}{(M^2)^{2-(D/2)}} \frac{\Gamma\left(2 - \frac{D}{2}\right)}{\Gamma(2)} \\ &= \frac{\pi^2}{(2\pi)^4} \frac{2}{\epsilon} + \mathcal{O}(\epsilon) \\ Q &= \int \frac{d^D P_E}{(2\pi)^4} \frac{1}{P_E^2 + M^2} = \frac{\pi^{D/2}}{(2\pi)^4} \frac{1}{(M^2)^{1-(D/2)}} \frac{\Gamma\left(1 - \frac{D}{2}\right)}{\Gamma(2)} \\ &= \frac{\pi^2}{(2\pi)^4} M^2 \left(-\frac{2}{\epsilon}\right) + \mathcal{O}(\epsilon). \end{aligned} \tag{3B.4}$$

Hence at the pole $\epsilon = 0, Q$ and L become related such that $Q + M^2 L = 0$, cancelling the last terms in (3B.3). Notice that when dealing with the renormalizable theory with large gluon mass $\mu^2 \ll \Lambda^2$, this cancellation is still present while the other Q integrals in $J_{ij}(q)$ become unrelated with the L integrals, the first being essentially $\mu^2 \log \Lambda^2/\mu^2$, the other $\log \mu^2/M^2$.

Consider now the interacation terms $\mathcal{A}_3$. Here the traces grow at most as P_E^2. Thus as far as the divergent contributions are concerned, the denominators in the intetrals (3.27) can be approximated as

$$\begin{aligned} &\frac{1}{(P + q_2 + q_1)_E^2 + M^2} \frac{1}{(P + q_1)_E^2 + M^2} \frac{1}{P_E^2 + M^2} = \\ &\frac{1}{(P_E^2 + M^2)^3} \left\{ 1 + \frac{P(2q_1 + q_2)}{P_E^2 + M^2} + O\left(\frac{M^2}{P_E^2 + M^2}, \quad \frac{q_1^2}{P_E^2 + M^2}\right) \right\}. \end{aligned} \tag{3B.5}$$

Since this expression decreases at least as $1/p_E^6$ the traces have to be known only with respect to their leading P_E^2 and P_E^2 behaviours. These are

$$\begin{aligned} t_{SSS}(P|q_2 q_1) &\simeq 3P^2 M, t_{SPP} \\ t_{SPA^\mu}(P|q_2 q_1) &\simeq iP^2 P^\mu + 2iP^\mu [P(q_1 + q_2)] - iP^2 q_2^\mu \\ t_{SA^\mu P}(P|q_2 q_1) &\simeq -iP^2 P^\mu - iP^2 (2q_1 + q_2)^\mu \\ t_{SSV^\mu}(P|q_2 q_1) &\simeq P^2 P^\mu - P^2 q_2^\mu + 2P^\mu [P(q_1 + q_2)] = t_{PPV^\mu}(P|q_2 q_1) \\ t_{SV^\mu S}(P|q_2 q_1) &\simeq P^2 P^\mu + P^2 (2q_1 + q_2)^\mu = t_{PV^\mu P}(P|q_2 q_1) \\ t_{SV^\mu V^\nu}(P|q_2 q_1) &\simeq 4MP^\mu P^\nu - MP^2 g^{\mu\nu} \\ t_{SA^\mu A^\nu}(P|q_2 q_1) &\simeq 4MP^\mu P^\nu - 3MP^2 g^{\mu\nu} \\ t_{V^\mu V^\lambda V^\kappa}(P|q_1 q_2) &\simeq 4P^\mu P^\lambda P^\kappa - P^2 \left(P^\mu g^{\lambda\kappa} + P^\lambda g^{\mu\kappa} + P^\kappa g^{\mu\lambda}\right) \\ &\quad + 2P^\mu P^\lambda q_1^\kappa + 2P^\lambda P^\kappa (q_1 + q_2)^\mu + 2P^\mu P^\kappa (2q_1 + q_2)^\lambda \\ &\quad - 2P^\kappa P(q_1 + q_2) g^{\mu\lambda} - 2P^\mu P q_1 g^{\lambda\kappa} - P^2 \left(-g^{\mu\lambda} q_2 + g^{\lambda\kappa} q_2 + g^{\mu\kappa}(2q_1 + q_2)\right) \end{aligned}$$

(3B.6)

(3B.7)

Using (3B.2) one obtains exactly the third order terms in the Lagrangian Eq. (3.29) (if this is written in the σ'-form).

The fourth order couplings in $\mathcal{A}_4$ are the simplest to evaluate. Here only the leading P^4 behaviour of $t_{ij}\,(P|q_3q_2q_1)$ contributes proportional to L and the propagator can directly be used in the form $[P_E^2+M^2]^{-4}$.

$$\begin{aligned} t_{SSSS}\,(P|q_3q_2q_1) &\simeq t_{SSPP}\simeq t_{PPPP}\simeq P^4 \\ t_{SSV^\mu V^\nu}\,(P|q_3q_2q_1) &\simeq t_{PPV^\mu V^\nu}\simeq t_{SSA^\mu A^\nu} \\ &\simeq t_{PPA^\mu A^\nu}\simeq -it_{SPA^\mu V^\nu}\simeq -it_{PSA^\mu V^\nu} \\ &\simeq 2P^2P^\mu P^\nu - P^4 g^{\mu\nu}. \end{aligned} \tag{3B.8}$$

Appendix 3C Some Algebra

Here we want to compare some of our results with traditional derivations [28], [29] obtained by purely algebraic considerations together with PCAC.

The vector and axial-vector currents

$$V_\mu^a(x)=\bar\psi(x)\gamma_\mu^a\frac{\lambda^a}{2}\psi(x);\quad A_\mu^a(x)=\bar\psi(x)\gamma_\mu\gamma_5\frac{\lambda^a}{2}4(x) \tag{3C.1}$$

generate chiral $SU(3)\times SU(3)$ under which the quark gluon Lagrangian transforms as

$$\mathcal{L}=\mathcal{L}_{\text{chiralinvariant}}-u^0-cu^8-du^3 \tag{3C.2}$$

where

$$\begin{aligned} u^0+cu^8+du^3 &= \bar\psi\mathcal{M}\psi\equiv\sum_a\mathcal{M}^a\bar\psi\frac{\lambda^a}{2}\psi \\ &= \sqrt{\frac{2}{2}}\left(\mathcal{M}^u+\mathcal{M}^d+\mathcal{M}^s\right)\bar\psi\frac{\lambda^0}{2}\psi \\ &\quad +\frac{1}{\sqrt3}\left(\mathcal{M}^u\ \mathcal{M}^d-2\mathcal{M}^s\right)\bar\psi\frac{\lambda^8}{2}\psi \\ &\quad +\left(\mathcal{M}^u-\mathcal{M}^d\right)\bar\psi\frac{\lambda^3}{2}\psi. \end{aligned}$$

Hence

$$\begin{aligned} u^a &\equiv \mathcal{M}^0\bar\psi\frac{\lambda^a}{2}\psi(=\mathcal{M}^0\tilde u^a \\ c &\equiv \frac{\mathcal{M}^8}{\mathcal{M}^0},\quad d=\frac{\mathcal{M}^3}{\mathcal{M}^0}. \end{aligned} \tag{3C.3}$$

Defining also the pseudoscalar densities

$$\upsilon^a\equiv\mathcal{M}^0\bar\psi i\gamma_5\frac{\lambda^a}{2}\psi \tag{3C.4}$$

then u^a and υ^a form the $(3\bar3\pm(\bar33)$ representation of $SU(3)\times SU(3)$:

$$[Q_5^a,u^a]=id^{abc}\upsilon^c,\left[Q_5^a,\upsilon^b\right]=-id^{abc}u^c. \tag{3C.5}$$

From the equation of motion one finds the conservation law [9]

$$\begin{aligned}
\partial^\mu V^a_\mu(x) &= \partial^\mu \bar\psi \gamma_\mu \frac{\lambda^a}{2}\psi = \bar\psi\left[\frac{\lambda^a}{2}\mathcal{M}\right]\psi \\
&= if^{abc}\mathcal{M}^b\bar\psi\frac{\lambda^c}{2}\psi \quad a=0,1\ldots 8 \\
\partial^\mu A^a_\mu(x) &= \partial^\mu \bar\psi\gamma_\mu\gamma_5\frac{\lambda^a}{2}\psi = -\bar\psi\left[\frac{\lambda^a}{2}\mathcal{M}\right]\gamma_5\psi \\
&= id^{abc}\mathcal{M}^b\bar\psi i\gamma_5\frac{\lambda^c}{2}\psi \quad a=1\ldots 8 \qquad (3C.6)
\end{aligned}$$

Let us neglect $SU(2)$ breaking in $\mathcal{M}$. By taking (3C.6) between vacuum and pseudoscalar meson states one finds

$$\begin{aligned}
f_\pi m_\pi^2 &= \frac{1}{\sqrt3}\left(\sqrt2\mathcal{M}^0+\mathcal{M}^8\right)\frac{m_Y^2}{3}\sqrt{L}Z_\pi^{1/2} \\
f_K m_K^2 &= \frac{1}{\sqrt3}\left(\sqrt2\mathcal{M}^0+\mathcal{M}^8\right)\frac{m_Y^2}{3}\sqrt{L}Z_K^{1/2} \qquad (3C.7)
\end{aligned}$$

etc. for the other members of the multiplet, where one has used [see the pseudoscalar version of (3.76)]:

$$\begin{aligned}
&\langle 0|\partial^\mu A^a_\mu|\pi\rangle \equiv f_\pi m_\pi^2; \\
&\langle 0|\bar\psi_{i\gamma_5}\frac{\lambda^\pi}{2}\psi|\pi\rangle = \qquad (3C.8)\\
&\frac{\mu^2}{2g^2}\frac{1}{2g^2}\frac{1}{\sqrt{L}}\langle 0|\pi|\pi\rangle = \frac{\mu^2}{2g^2}\frac{Z_\pi^{1/2}}{\sqrt{L}} = \frac{m_V^2}{3}\sqrt{L}Z_\pi^{1/2}
\end{aligned}$$

etc. By writing $\mathcal{M}$ as

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}^u & & \\ & \mathcal{M}^d & \\ & & \mathcal{M}^s\end{pmatrix} = \frac{1}{2\sqrt3}\begin{pmatrix}\sqrt2 m?^0+\mathcal{M}^8 & & \\ & \sqrt2\mathcal{M}^0+\mathcal{M}^8 & \\ & & \sqrt2\mathcal{M}^0-2\mathcal{M}^8\end{pmatrix} \qquad (3C.9)$$

Eq. (3C.7) takes the form

$$\begin{aligned}
f_\pi^2 m_\pi^2 &= \left(\mathcal{M}^n+\mathcal{M}^d\right)/2Z_\pi^{1/2}\frac{2}{3}m_V^2\sqrt{L} \\
f_K^2 m_K^2 &= \left(\mathcal{M}^n+\mathcal{M}^s\right)/2Z_\pi^{1/2}\frac{2}{3}m_V^2\sqrt{L} \qquad (3C.10)
\end{aligned}$$

which agrees with (3.56).

By evaluating (3C.5) between vacuum states and saturating the commutator with pseudoscalar intermediate state one finds

$$\begin{aligned}
f_\pi\mathcal{M}_0\frac{\mu^2}{2g^2}\frac{Z_\pi^{1/2}}{\sqrt{L}} &= \frac{1}{\sqrt3}\left(\sqrt2\langle 0|u^0|0\rangle+\langle 0|u^8|0\rangle\right) \\
f_K\mathcal{M}_0\frac{\mu^2}{2g^2}\frac{Z_K^{1/2}}{\sqrt{L}} &= \frac{1}{\sqrt3}\left(\sqrt2\langle 0|u^0|0\rangle-\frac12\langle 0|u^8|0\rangle\right) \qquad (3C.11)
\end{aligned}$$

[9] +)

and similar for the other partners of the multiplet. Inserting the result of Eq. (3.76)

$$\langle 0|u^a|0\rangle = \mathcal{M}^0\langle 0|\tilde{u}^a|0\rangle = \frac{1}{2}\frac{\mu^2}{g^2}\mathcal{M}^0 M^a \tag{3C.12}$$

and writing M in the same way as $\mathcal{M}$ in (3C.9) brings (3C.11) to the form

$$\begin{aligned} f_\pi Z_\pi^{1/2} &= \sqrt{L}\left(M_u + M^d\right) \\ f_K Z_K^{1/2} &= \sqrt{L}\left(M^u + M^s\right) \end{aligned} \tag{3C.13}$$

which agrees exactly with (3.56) (written there in $SU(3)$ matrix form). Considerations of this type have led to the determination [28], [29], [33]

$$\begin{aligned} &C \approx -1.28 \\ &\text{or} \\ &\frac{\left(\mathcal{M}^u + \mathcal{M}^d\right)/2}{\mathcal{M}^s} \approx \frac{1}{29}. \end{aligned} \tag{3C.14}$$

Including also $SU(2)$ violation in such a consideration gives [33]

$$\begin{aligned} &d \approx -.03 \\ &\text{or} \\ &\frac{\mathcal{M}_u - \mathcal{M}_d}{\mathcal{M}_u + \mathcal{M}_d} \approx -\frac{1}{4}. \end{aligned} \tag{3C.15}$$

There are numerous extensions to $SU(4)$ [35] but they have to be viewed with great caution since it is hard to see how the large pseudoscalar and vector masses occuring there can dominate the idvergence of the axial current and the vector current, respectively.

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Part V

Exactly Solvable Field Theoretic Models via Collective Quantum Fields

1

Low-Dimensional Models

1.1 The Pet Model in One Time Dimension

Consider the extremely simple case of a fundamental theory

$$H = (a^\dagger a)^2/2 \tag{1.1}$$

where $a^\dagger$ denotes the creation operator of either a boson or a fermion. In the first case the spectrum is

$$E_n = \frac{n^2}{2} \tag{1.2}$$

for the states

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle, \qquad n = 0, 1, 2, \dots\ , \tag{1.3}$$

in the second

$$E_0 \;=\; 0 \quad \text{for} \quad |0\rangle \tag{1.4}$$

$$E_1 \;=\; \frac{1}{2} \quad \text{for} \quad |1\rangle = a^\dagger|0\rangle. \tag{1.5}$$

The Lagrangian corresponding to H is

$$\mathcal{L}(t) = a^\dagger(t)i\partial_t a(t) - \left[a^\dagger(t)a(t)\right]^2/2 \tag{1.6}$$

and the generating functional of all Green functions becomes

$$\begin{aligned} Z[\eta^\dagger,\eta] &= \langle 0T\exp\left[i\int dt(\eta^\dagger a + a^\dagger\eta)\right]0\rangle \\ &= N\int \mathcal{D}a^\dagger Da\exp\left[i\int dt\left(\mathcal{L} + \eta^\dagger a + a^\dagger\eta\right)\right]. \end{aligned} \tag{1.7}$$

A collective field may be introduced via the formula

$$\exp\left\{-i\int dt[a^\dagger a(t)]^2/2\right\} = \int \mathcal{D}\rho(t)\exp\left\{\frac{i}{2}\int dt\left[\rho^2(t) - 2\rho(t)a^\dagger a(t)\right]\right\} \tag{1.8}$$

or by adding to (1.7) in the exponent

$$i/2 \int dt \left[\rho(t) - a^\dagger(t)a(t)\right]^2$$

and integrating functionally over the ρ-field.

Thus the generating functional Z can be rewritten as

$$Z[\eta^\dagger, \eta = N \int \mathcal{D}a^\dagger \mathcal{D}a \mathcal{D}\rho \times \exp\left\{dt\left[a^\dagger(t)i\partial_t a(t) - \rho(t)a^\dagger(t)a(t) + \frac{\rho^2(t)}{2} + \eta^\dagger(t)a(t) + a^\dagger(t)\eta(t)\right]\right\}. \quad (1.9)$$

The collective field describes the particle density: Functional derivation of the action in (1.9) displays the dependence

$$\rho(t) = a^\dagger(t)a(t). \quad (1.10)$$

Integrating out the $a^\dagger, a$ fields gives

$$Z[\eta^\dagger, \eta] \;=\; N \int \mathcal{D}\rho \exp\left\{i\mathcal{A}[\rho] - \int dtdt'\eta^\dagger(t)G_\rho(t,t')\eta(t')\right\} \quad (1.11)$$

with the collective filed action

$$\mathcal{A}[\rho] = \pm i \mathrm{Tr} \log\left(iG_\rho^{-1}\right) + \int dt \frac{\rho^2(t)}{2}, \quad (1.12)$$

where G_ρ denotes the propagator of the fundamental particles in a classical $\rho(t)$ field satisfying

$$\left[i\partial_t - \rho(t)\right] G_\rho(t,t') = i\delta(t-t'). \quad (1.13)$$

The Green function can be solved by introducing an auxiliary field

$$\varphi(t) = \int^t \rho(t')dt' \quad (1.14)$$

in terms of which

$$G_\rho(t,t') = e^{-i\varphi(t)}e^{i\varphi(t')}G_0(t-t') \quad (1.15)$$

with G_0 being the free-field propagator of the fundamental particles. At this point one has to specify the boundary condition on $G_0(t-t')$. They have to be adapted to the physical properties of the system. The generating functional is supposed to describe the amplitude for vacuum to vacuum transitions in the presence of the source fields $\eta^\dagger, \eta$. The propagation of the free particles must take place in the same vacuum. If $a_0^\dagger, a_0$ describes a free particle, their time ordered product in the free vacuum is

$$G_0(t-t') = \langle 0|T\left(a_0(t)a_0^\dagger(t')\right)|0\rangle = \Theta(t-t') \quad (1.16)$$

Using (1.15) we find

$$G_\rho(t,t') = e^{-\tau\varphi(t)} e^{i\varphi(t')}\Theta(t-t'). \tag{1.17}$$

Equipped with this knowledge we can readily calculate the Tr log term in (1.12). The functional derivative is certainly

$$\frac{\delta}{\delta\rho(t)}\left\{\pm i\mathrm{Tr}\log(iG_\rho^{-1})\right\} = \mp G_\rho(t,t')|_{t'=t+\epsilon} = 0 \tag{1.18}$$

where the $t' \to t$ limit is specified in such a way that the field $\rho(t)$ couples, in Eq. (1.9), to

$$a^\dagger(t)a(t) = \pm T\left(a(t)a^\dagger(t')\right)|_{t'=t+\epsilon} \hat{=} \pm G^\rho(t,t')|_{t'=t+\epsilon}. \tag{1.19}$$

Hence, the Θ function in (1.17) makes the functional derivative vanish and the tr log becomes an irrelevant constant. The generating functional is the simply

$$\begin{aligned} Z[\eta^\dagger,\eta] = N \int \mathcal{D}\varphi(t) \\ \exp\left\{\frac{i}{2}\int dt\dot\varphi(t)^2 - \int dtdt'\eta^\dagger(t)\eta(t')e^{-i\varphi(t)}e^{-i\varphi(t')}\Theta(t-t')\right\} \end{aligned} \tag{1.20}$$

where

$$\mathcal{D}\rho = \mathcal{D}\varphi \det\left(\dot\delta(t-t')\right) = \mathrm{const}\cdot\mathcal{D}\varphi. \tag{1.21}$$

has been used. Observe that it is $\varphi(t)$ which becomes a convenient dynamical plasmon variable, not $\rho(t)$ itself.

The original theory has been transformed into a new one involving plasmons of zero mass. At this point we take advantage of equivalence between functional and quantized operator formulation by considering the plasmon action in the exponent of (1.21) directly as a quantum field theory. The first term may be associated with a Lagrangian

$$\mathcal{L}_0(t) = \frac{1}{2}\dot\varphi(t)^2 \tag{1.22}$$

describing free plasmons.

The Hilbert space of the corresponding Hamiltonian $H = p^2/2$ consists of plane waves which are eigenstates of the functional momentum operator $p = -i\partial/\partial\varphi$:

$$\{\varphi|p\} = \frac{1}{\sqrt{2\pi}}e^{ip\varphi} \tag{1.23}$$

normalized according to

$$\int_{-\infty}^{\infty} d\varphi\,\{p|\varphi\}\,\{\varphi|p'\} = \delta(p-p'). \tag{1.24}$$

In the operator version (**??**) then, the generating functional reads

$$Z[\eta^\dagger,\eta] = \frac{1}{\{0|0\}} \left\{0|T\exp\left[-\int dtdt'\eta^\dagger(t)\eta(t')e^{-i\varphi(t)}e^{i\varphi(t')}\Theta(t-t')\right]|0\right\} \tag{1.25}$$

where $\varphi(t)$ are free field operators. Notice that it is the zero-functional momentum state between which Z is taken. Due to the norm (1.24) there is an infinite normalization factor which has formally been taken out.

We can now trace the generation of all Green functions of fundamental particles by forming functional derivatives with respect to $\eta^\dagger,\eta$. First

$$\begin{aligned}\langle 0|Ta(t)a^\dagger(t')|0\rangle &= -\frac{\delta^{(2)}Z}{\delta\eta^\dagger(t)\delta\eta(t')}|_{\eta^\dagger,\eta=0}\\ &= \frac{1}{\{0|0\}}\left\{0|e^{-i\varphi(t)}e^{i\varphi(t')}|0\right\}\Theta(t-t').\end{aligned} \tag{1.26}$$

Inserting the time translation operator

$$e^{iHt} = e^{i\frac{p^2}{2}t} \tag{1.27}$$

the matrix element (1.26) becomes

$$\begin{aligned}&\frac{1}{\{0|0\}}\left\{0|e^{-ip^2}2e^{-i\varphi(0)}e^{-i\frac{p^2}{2}(t-t')}e^{i\varphi(0)}e^{-i\frac{p^2}{2}t'}\right\}\\ &\quad= \frac{1}{\{0|0\}}\left\{0|e^{-i\varphi(0)}e^{-i\frac{p^2}{2}(t-t')}e^{i\varphi(0)}|0\right\}.\end{aligned} \tag{1.28}$$

But the state $e^{i\varphi(0)}|0\,\}$ is an eigenstate of p with momentum $p=1$ such that (1.28) equals

$$\frac{1}{\{0|0\}}\{1|1\}\,e^{-i(t-t')/2} = e^{-i(t-t')/2} \tag{1.29}$$

and the Green function (1.26) becomes

$$\langle 0Ta(t)a^\dagger(t')|0\rangle = e^{-i(t-t')/2}\Theta(t-t'). \tag{1.30}$$

This coincides exactly with the result of a calculation within the fundamental fields $a^\dagger(t),a(t)$:

$$\begin{aligned}\langle 0Ta(t)a^\dagger(t')|0\rangle &= \Theta(t-t')\langle 0|e^{i(a^\dagger a)^2t/2}a(0)e^{-\frac{i}{2}(a^\dagger a)^2(t-t')}a^\dagger(0)e^{-i(a^\dagger a)^2t'/2}|0\rangle\\ &= \Theta(t-t')e^{-i(t-t')/2}.\end{aligned} \tag{1.31}$$

Observe that nowhere in the calculation has Fermi or Bose statistics been used. This becomes relevant for higher Green functions. Expanding the exponent (1.26) to n'th order gives

$$Z^{[n]}\left[\eta^\dagger,\eta\right]=\frac{1}{\{0|0\}}\frac{(-)^n}{n!}\int dt_1dt_1'\cdots dt_ndt_n'\eta^\dagger(t_1)\eta(t_1')\cdots\eta^\dagger(t_n)\eta(t_n')$$
$$\times\left\{0|Te^{-i\varphi(t_1)}e^{i\varphi(t_1')}\cdots e^{-i\varphi(t_n)}e^{i\varphi(t_n')}|0\right\}\Theta(t-1-t_1')\cdots\Theta(t_n-t_n').(1.32)$$

The Green function

$$\langle 0Ta(t_1)\cdot a(t_n)a^\dagger(t_n')\cdots a^\dagger(t_1')|0\rangle \tag{1.33}$$

is obtained by forming the derivative $(-i)^{2n}\delta^{(2n)}\mathcal{Z}[\eta^\dagger\eta]/\delta\eta^\dagger(t_1)\cdots\delta\eta^\dagger(t_n)\delta\eta(t_n')\cdots\delta\eta(t)$. There are $(n!)^2$ contributions due to the product rule of differentiation, $n!$ of them being identical thereby canceling the factor $1/n!$ in (1.32). The other correspond, from the point of view of combinatorics, to all wick contractions of (1.33), each contraction being associated with a factor $e^{-i\varphi(t)}e^{i\varphi(t')}$. In addition, the Grassmann nature of source fields η causes a minus sign to appear if the contractions deviating by an odd permutation from the natural order $11', 22', 33', \ldots$ For example

$$\begin{aligned}&\langle 0Ta(t_1)a(t_2')a^\dagger(t_2')a^\dagger(t_1')|0\rangle\\&=\langle 0T\dot a(t_1)a(\ddot t_2)\ddot a^\dagger(t_2')\dot a^\dagger(t_1')|0\rangle\pm\langle 0|T\dot a(t_1)\ddot a(t_2)\ddot a^\dagger(t_2')\dot a^\dagger(t_1')|0\rangle\\&=\frac{1}{\{0|0\}}\left\{0|Te^{-i\varphi(t_1)}e^{-i\varphi(t_2)}e^{i\varphi(t_2')}e^{i\varphi(t_1')}|0\right\}\\&=\left[\Theta(t_1-t_1')\Theta(t_2-t_2')\pm\Theta(t_1-t_2')\Theta(t_2-t_1')\right]\end{aligned}\tag{1.34}$$

where the upper sign holds for bosons, the lower for fermions. The lower sign enforces the Pauli exclusion principle: If $t_1>t_2>t_2'>t_1'$ the two contributions cancel reflecting the fact that no two fermions $a^\dagger(t_2')a^\dagger(t_1')$ can be created successively on the particle vacuum. For bosons one may insert again the time translation oparator (1.27) and complete sets of states $\int dp|p\,\}\{\,p|=1$ with the result:

$$\begin{aligned}&\frac{1}{\{0|0\}}\int dpdp'\left\{0|e^{-i\varphi(0)}e^{-i\frac{p^2}{2}(t_1-t_2)}e^{-i\varphi(0)}e^{-i\frac{p^2}{2}(t_2-t_2')}e^{i\varphi(0)}e^{-i\frac{p^2}{2}(t_2'-t_1')}e^{i\varphi(0)}|0\right\}\\&\qquad=e^{-i(t_1-t_2)/2}e^{-i2(t_2-t_2')}e^{-i(t_2'-t_1')/2}.\end{aligned}\tag{1.35}$$

where $\left\{0|e^{-i\varphi(0)}|p\right\}=\delta(1-p)$, $\left\{p|e^{-i\varphi(0)}|p'\right\}=\delta(p+1-1)$ has been used. This again agrees with an operator calculation like (1.31).

We now understand how the collective quantum field theory works in this model. Its Hilbert space is very large consisting of states of *all* functional momenta $|p\rangle$. When it comes to calculating the Green functions of the fundamental fields, however, only a small portion of this Hilbert space is used. A fermion can make plasmon transitions back and forth between ground state $|0\,\}$ and the momentum one state $|1\,\}$ only, due to the anticommutativity of the fermion source fields $\eta^\dagger,\eta$. Bosons, on the other hand, can connect all states of integer momentum $|n\,\}$. No other states are ever reached. The collective basis is overcomplete as far as the description of the underlying system is concerned. Strong selection rules, $p \to p \pm 1$, together with the source statistics make sure that only a small subspace becomes involved in the dynamics of the fundamental system. That such a projection is compatible with unitarity is ensured by the conservation law $a^\dagger a = \text{const}$. In higher dimensions, there have to be infinitely many conservation laws (one for every space point).

Actually, in the boson case, the overcompleteness can be removed by defining the collective Lagrangian in (1.21) on a cyclic variable, i.e., one takes (1.22) on $\varphi \in [0, 2\pi)$ and extends it periodically. The path integral (1.21) is the integrated accordingly. In this case the Hilbert space would be grated containing only integer momenta $p = 0, \pm 1, \pm 2, \ldots$ coinciding with the multi-boson states.

the following observations may be helpful in understanding the structure of the collective theory: It may sometimes be convenient to build all Green functions not on the vacuum state $|0\rangle$ but on some other reference state $|R\rangle$ for which we may choose the excited state $|n\rangle$. In the operator language this amounts to a generating functional

$$ {}^nZ[\eta^\dagger, \eta] = \langle n|T \exp\left\{ i \int dt \left[\eta^\dagger(t) a(t) + a^\dagger(t)\eta(t) \right] \right\} |n\rangle. \tag{1.36} $$

This would reflect itself in the boundary condition of G_0 for bosons

$$ \begin{aligned} {}^nG_0(t-t') &= \langle n|T\left(a_0(t) a_0^\dagger(t') \right)|n\rangle \\ &= (n+1)\Theta(t-t') + n\Theta(t'-t). \end{aligned} \tag{1.37} $$

For fermions, only $n = 1$ would be an alternative with

$$ {}^1G_0(t-t') = \langle 1|T\left(a_0(t) a_0^\dagger(t') \right)|1\rangle = -\Theta(t'-t). \tag{1.38} $$

As a consequence of (1.37) or (1.38), formula (1.18) would become

$$ \frac{\delta}{\delta\rho(t)} \left\{ \pm i \operatorname{Tr}\log\left(iG_\rho^{-1} \right) \right\} = -\left\{ \begin{array}{c} n \\ 1 \end{array} \right\} \tag{1.39} $$

Integrating this functionally gives

$$ \pm i \operatorname{Tr}\log\left(iG_\rho^{-1} \right) = -\left\{ \begin{array}{c} n \\ 1 \end{array} \right\} \int_{-\infty}^{\infty} \rho(t) dt \tag{1.40} $$

so that the functional form of (1.36) reads, according to (1.12):

$$ \begin{aligned} \left\{ \begin{array}{c} n \\ 1 \end{array} \right\} Z[\eta^\dagger, \eta] &= \int \mathcal{D}\varphi \exp\left[i \int dt \left(\frac{\dot\varphi^2}{2} - \left\{ \begin{array}{c} n \\ 1 \end{array} \right\} \dot\varphi \right) dt \right] \\ &\times \exp\left[- \int dt dt' \eta^\dagger(t)\eta(t') e^{-i\varphi(t)} e^{i\varphi/t')} \left[\left\{ \begin{array}{c} n+1 \\ 0 \end{array} \right\} \Theta(t-t') + \left\{ \begin{array}{c} n \\ -1 \end{array} \right\} \Theta(t'-t) \right] \right] \end{aligned} \tag{1.41} $$

Now the collective Lagrangian is

$$\mathcal{L}(t) = \frac{\cdot\varphi^2}{2} - \left\{ \begin{array}{c} n \\ 1 \end{array} \right\} \dot{\varphi}$$
$$= \frac{1}{2}\left(\dot{\varphi} - \left\{ \begin{array}{c} n \\ 1 \end{array} \right\}\right)^2 - \frac{1}{2}\left\{ \begin{array}{c} n^2 \\ 1 \end{array} \right\} \tag{1.42}$$

with the functional canonical momentum

$$p = \dot{\varphi} - \left\{ \begin{array}{c} n \\ 1 \end{array} \right\}$$

the Hamiltonian takes the form

$$H = \left(\dot{\varphi} - \left\{ \begin{array}{c} n \\ 1 \end{array} \right\}\right)\dot{\varphi} - \mathcal{L}$$
$$= \frac{\dot{\varphi}^2}{2} = \frac{\left(p + \left\{ \begin{array}{c} n \\ 1 \end{array} \right\}\right)^2}{2}. \tag{1.43}$$

Thus the spectrum is the same as before but the momenta are shifted by n (or 1) units accounting for the fundamental particles contained in the reference state $|R\rangle$ of (1.36). In the collective quantum field theory, this reference state corresponds now to functional momentum zero:

$$\left\{ \begin{array}{c} n \\ 1 \end{array} \right\} Z[\eta^\dagger, \eta] = \frac{1}{\{0|0\}}\{0|T \exp\left[-\int dt dt' \eta^\dagger(t)\eta(t') e^{-i\varphi(t)} e^{i\varphi(t')}\right.$$
$$\times \left.\left[\left\{ \begin{array}{c} n \\ -1 \end{array} \right\}\Theta(t-t') + \left\{ \begin{array}{c} n \\ -1 \end{array} \right\}\Theta(t'-t)\right]\right]|0\}. \tag{1.44}$$

In fact, the one-particle Green function becomes

$$\left\{ \begin{array}{c} n \\ 1 \end{array} \right\} G(t,t') = -\frac{\delta^{(2)}}{\delta\eta^\dagger(t)\delta\eta(t')}\left\{ \begin{array}{c} n \\ 1 \end{array} \right\} Z[\eta^\dagger, \eta]$$
$$= \frac{1}{\{0|0\}}\{0|T e^{-i\varphi(t)} e^{i\varphi(t')}|0\}$$
$$\times \left[\left\{ \begin{array}{c} n+1 \\ 0 \end{array} \right\}\Theta(t-t') + \left\{ \begin{array}{c} n \\ -1 \end{array} \right\}\Theta(t'-t)\right]. \tag{1.45}$$

Inserting the times translation operator corresponding to (1.43) this yields for $t > t'$

$$\left\{ \begin{array}{c} n \\ 1 \end{array} \right\} G(t,t') = \exp\left[-i\left\{ \begin{array}{c} n+1/2 \\ 3/2 \end{array} \right\}(t-t')\right]\left\{ \begin{array}{c} n+1 \\ 0 \end{array} \right\}$$
$$= \left\{ \begin{array}{c} (n+1)\exp[-i(n+1/2)(t-t') \\ 0 \end{array} \right\} \tag{1.46}$$

and for $t < t'$

$$\begin{Bmatrix} n \\ 1 \end{Bmatrix} G(t,t') = \exp\left[-i\begin{Bmatrix} n-1/2 \\ 1/2 \end{Bmatrix}(t-t')\right]\begin{Bmatrix} n \\ -1 \end{Bmatrix}$$
$$= \begin{Bmatrix} n\exp[-i(n-1/2)(t-t') \\ -\,e^{-i(t-t')/2} \end{Bmatrix} \tag{1.47}$$

In agreement with a direct operator calculation.

The appearance of the additional derivative term $\dot\varphi$ in the Lagrangian (1.42) can be understood in an alternative fashion. The reference state $|n\rangle$ of ${}^nZ$ in (1.36) can be generated in the original generating functional by applying successively derivatives $-\delta^{(2)}/\delta\eta^\dagger(t)\delta\eta(t')$, letting $t' \to -\infty,\ t \to \infty$ and absorbing an infinite phase $\exp[-i\Delta E \times (2\infty)]$, into the normalization constant where ΔE is the energy difference between $|n\rangle$ and $|0\rangle$:

$$ {}^nZ[\eta^\dagger,\eta]|_{\eta^\dagger=\eta=0} \propto \frac{\delta^{(n)}}{(\delta\eta^\dagger(+\infty))^n}\frac{\delta^{(n)}}{(\delta\eta(-\infty))^n}{}^0Z[eta^\dagger,\eta]|_{\eta^\dagger=\eta=0}. \tag{1.48}$$

Each such pair of derivatives brings down a Green function

$$e^{-i\varphi(t)}e^{i\varphi(t')}\Theta(t-t') = \exp\left[-i\int_{t'}^{t}\dot\varphi(tt')\right]\Theta(t-t'). \tag{1.49}$$

As $t' \to -\infty,\ t \to \infty$ this becomes for n factors

$$\exp\left[-in\int_{-\infty}^{\infty}\cdot\varphi(t)dt\right] \tag{1.50}$$

in agreement with the derivative term in (1.41).

While the functional Schrödinger picture is useful in understanding what happens in the Hilbert space of the collective field theory, it is quite awkward to apply to more than one dimension, in particular to the relativistic situation where the time does not play a special role. A more direct and easily generalizable method for the evaluation of fermion propagators in the collective theory consists in the following procedure: One brings the products of exponentials in (1.32) to normal order by using Wick's, contraction formula in the functional form (**??**). Let the "charges" of the incoming and outgoing fermions be $q_i = +1$ and $q_i) - 1$, respectively.

Then the matrix element to be calculated in (1.32) are

$$\{0|T\exp\left[i\sum_i q_i\varphi(t_i)\right]|0\} = \{0|T\exp\left[dt\varphi(t)\partial_i q_i(t-t_i)\right]|0\} \tag{1.51}$$

where we have numbered the times as $t_1, t_2, t_3, t_4, \ldots$ rather than $t_1, t'_1, t_2, t'_2, \ldots$ etc. Now from (**??**) one has

$$\{0|Te^{i\sum q_i\varphi(t_i)}|0\} = \exp\left[-\frac{1}{2}\int dtdt'\sum_i q_i\delta(t-t_i)\dot\varphi(t)\dot\varphi(t')(t')\sum_j q_j\delta(t-t_j)\right]$$

$$\begin{aligned}&\times\{0|T:\exp\left[i\int dt\varphi(t)\sum_i q_i\delta(t'-t_i)\right]:|0\}\\ =\ &\exp\left[-\frac{1}{2}\sum_{ij} q_iq_j\dot\varphi(t_i)\dot\varphi(t_j)\right].\end{aligned} \tag{1.52}$$

where common dots denote again a propagator of a φ-field. It is well defined after introducing a small regulator mass κ:

$$\begin{aligned}\dot\varphi(t)\dot\varphi(t') &= \int\frac{dE}{2\pi}\frac{i}{E^2-\kappa^2+i\epsilon}e^{-iE(t-t')}\\ &= \frac{1}{2\kappa}e^{-\kappa|t-t'|}=\frac{1}{2\kappa}-\frac{i}{2}|t-t'|+\mathcal{O}(\kappa).\end{aligned} \tag{1.53}$$

As $\kappa\to 0$ this expression vanishes unless the sum of all charges is zero: $\sum_i q_i = 0$. Thus one finds the general result for (1.32):

$$\{0|T\exp\left[i\sum_{q_i}\varphi(t_i)\right]|0\} = \delta_{\Sigma_{q_i},0}\exp\left[\frac{i}{2}\sum_{i>j}q_iq_j|t_i-t_j|\right] \tag{1.54}$$

In particular, the two point function (1.26) agrees with the Schrödinger calculation (1.30).

1.2 The Generalized BCS Model in a Degenerate Shell

A less trivial but completely transparent example is provided by the BCS degeneratshelf model used in nuclear physics to describe the energy levels of some nuclei in which pairing forces are dominant (for example Sn and Pb isotopes [31]). For the understanding of the structure of collective theory it will be useful to consider at first both bosons and fermions as well as a more general interaction and impose the restriction to fermions and to the particular BCS pairing force at a later stage. This more general Hamiltonian reads

$$\begin{aligned}H &= H_0+H_{\rm int}=\epsilon\Sigma_{i=1}^{\Omega}({a_i}^+a_i+{b_i}^+b_i)-\frac{V}{2}\{\Sigma_{i,j}{a_i}^+{b_i}^+b_ja_j\}\\ &\pm\frac{V}{4}g\left[\Sigma_i({a_i}^+a_i+{b_i}^+b_i)\pm\Omega\right]\end{aligned} \tag{1.55}$$

where $g=0$ reduces to the actual BCS model in the case of fermions. The model can be completely solved by introducing quasi-spin operators

$$\begin{aligned}&L^+=\Sigma_{i=1}^{\Omega}{a_i}^+{b_i}^+L^-=\Sigma_{i=1}^{\Omega}b_ia_i=(L^+)^+ \\ &L_3=\frac{1}{2}\{\Sigma_i({a_i}^+a_i+{b_i}^+b_i)\pm\Omega\}=\frac{1}{2}\Sigma_i{a_i}^+a_i\pm b_i{b_i}^+=\frac{1}{2}\{N\pm\Omega\}\end{aligned} \tag{1.56}$$

where N counts the total number of particles. These operators generate the group $SU(1,1)$ or $SU(2)$ for bosons or fermions, respectively:

$$\begin{aligned}[L_3,L^\pm] &= \pm L^\pm\\ [L^+,L^-] &= \mp 2L_3\end{aligned} \tag{1.57}$$

using

$$L^+L^- = L^2 \mp L_3 \pm {L_3}^2 \tag{1.58}$$

we can write

$$\begin{aligned} H &= 2\varepsilon L_3 \mp \epsilon\Omega - V(L^2 \pm {L_3}^2 \mp g{L_3}^2) \\ &= 2\varepsilon L_3 - V(L^2 \pm (1-g){L_3}^2) \mp \varepsilon\Omega. \end{aligned} \tag{1.59}$$

Notice that the interaction term is $SU(1,1)$ or $SU(2)$ symmetric for $g = 1$. The irreducible representation of the algebra (1.57) consist of states

$$|n[\Omega,\nu]\rangle = N_n(L^+)^n|0[\Omega,\nu]\rangle \tag{1.60}$$

where the seniority label ν denotes the presence of ν unpaired particles ${a_i}^+$ or ${b_j}^+$, i.e. those which are orthogonal to the configurations $(L^+)^n|0\rangle$. For $\nu = 0$ the spectrum of L_3 in an irreducible representation is

$$\pm\frac{\Omega}{2}, \pm\frac{\Omega}{2} + 1, \quad \pm\frac{\Omega}{2}, +2, \dots. \tag{1.61}$$

This continues ad infinitum for bosons due to the non-compact topology of $SU(1,1)$ while it terminates for fermions at $\Omega/2$ corresponding to a finite spin $\Omega/2$. The invariant Casimir operator

$$L^2 \equiv {L_1}^2 + {L_2}^2 \mp {L_3}^2 \tag{1.62}$$

characterizing the representation has the eigenvalue $\Omega/2(1 \mp \Omega/2$ showing in the fermion case again the quasi-spin $\Omega/2$. If ν unpaired particles are added to a vacuum, the eigenvalues start at $\pm(\Omega+\nu)/2$. Thus the quasi-spin is reduced to $(\Omega - \nu/2$. If $\nu = \Omega$ unpaired fermions are present, the state is quasi-spin symmetric, for example:

$$|0[,\Omega,\Omega]\rangle = {b_1}^+{b_2}^+ \cdot \ldots \cdot {b_\Omega}^+|0\rangle. \tag{1.63}$$

Due to the many choices of unpaired particles with the same total number the levels show considerable degeneracies and one actually needs another label for their distinction. This has been dropped for brevity.

On the states $|n[\Omega\nu]\rangle$ the energies are from (1.59) and using $N = 2n + \nu$:

$$E = e(N \pm \Omega) - V\left[\frac{\Omega \pm \nu}{2}\left(1 \mp \frac{\Omega \pm \nu}{2}\right) \pm \frac{(1-g)}{4}(N \pm \Omega)^2\right] \mp \varepsilon\Omega. \tag{1.64}$$

A typical level scheme for fermions of $\Omega = 8$ with $\varepsilon = 0$ is displayed on Fig. XVII. If the single particle energy ε is non-vanishing, the scheme is distorted via a linear dependence on L_2 lifting the right- and depressing the left-hand side. For an attractive potential and given total particle number N, the state with $\nu = 0$ is the ground state with the higher seniorities having higher energies:

$$E_{N\Omega\nu} - E_{N\Omega 0} = V\left(\Omega \mp 1 \pm \frac{\nu}{2}\right)\nu. \tag{1.65}$$

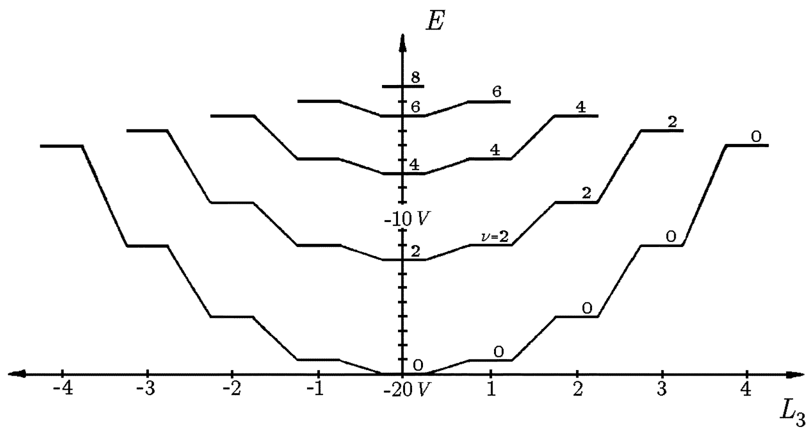

FIGURE 1.1 .
The figure shows the level scheme of the BCS model in a single degenerate shell of multiplicity $\Omega = 8$. The abscissa denotes the third component of quasi-spin. The index ν at each level stands for the number of unpaired particles ("seniority").

The Lagrangian of the model is from (1.55)

$$\begin{aligned}\mathcal{L}(t) &= \Sigma_i(a_i{}^+(t)(i\partial_t - \varepsilon)a_i(t) + b_i{}^+(t)(i\partial_t - \varepsilon)b_i(t)) \\ &\quad + \frac{V}{2}\left\{\Sigma_{i,j}a_i{}^+b_i{}^+b_ja_i\right\} \mp \frac{V}{4}g\left\{\Sigma_i(a_i{}^+a_i \pm b_ib_i{}^+)\right\}^3 \qquad (1.66)\end{aligned}$$

and the generating functional:

$$\begin{aligned}Z[\eta^+, \eta, \lambda] &= \int \prod_i Da_i{}^+Da_iDb_i{}^+Db_i \\ &\quad \times \exp\left[i\int dt\left\{\mathcal{L} + \Sigma_i\eta_i{}^+a_i + a_i{}^+\eta_i + \lambda_i{}^+b_i + b_i{}^+\lambda_i\right\}\right]. \qquad (1.67)\end{aligned}$$

The quartic terms in the exponential can be removed by introducing a complex field $S = S_1 + iS_2$ and a real field S_3', adding

$$- V\left\{\left|S(t) - \Sigma_ia_i{}^+ + b_i{}^+\right|^2 \mp g\left[S_3'(t) - \frac{1}{2}\Sigma_i(a_i{}^+a_i \pm b_ib_i{}^+\right]^2\right\} \qquad (1.68)$$

and integrating Z functionally over $DS = DS_1DS_2DS_3$. The addition of (1.68) changes $\mathcal{L}$ to:

$$\begin{aligned}\mathcal{L}(t) &= \Sigma_i\{a_i{}^+(i\partial_t - \varepsilon \mp gVS_3{}')a_i \mp b_i(i\partial_t + \varepsilon \pm gVS_3{}')b_i{}^+\} \\ &\quad + VS^+\Sigma_ia_i{}^+b_i{}^+\Sigma_ib_ia_iVS - V(|S|^2 \mp gS_3{}'^2) \pm \varepsilon\Omega. \qquad (1.69)\end{aligned}$$

By using the more convenient two-spinor notation for fundamental fields and sources

$$f_i \equiv \begin{pmatrix} a_i \\ b_i{}^+ \end{pmatrix}; \quad f_i{}^+ \equiv (a_i{}^+, b_i)$$

$$j_i \equiv \begin{pmatrix} n_i \\ \lambda_i{}^+ \end{pmatrix}; \quad j_i{}^+ \equiv (\eta_i{}^+, \lambda_i) \tag{1.70}$$

the generating functional can be rewritten as

$$Z[j^+j] = \int \prod_i Df_i^+ DS \exp\left[i\int dt\left\{\mathcal{L} + \Sigma_i(j_i^+ f_i + f_i^+ j_i)\right\}\right] \tag{1.71}$$

with

$$\begin{aligned}\mathcal{L} &= \Sigma_{i=1}^{\Omega} f_i{}^+(t)\begin{pmatrix} i\partial_t - \varepsilon \mp gVS_3{}' & VS^+ \\ VS & \mp(i\partial_t + \varepsilon \pm gVS_3{}') \end{pmatrix} f_i(t) \\ &\quad -V(|S|^2 \mp gS_3{}'^2) \pm \varepsilon\Omega. \end{aligned} \tag{1.72}$$

Now the fundamental fields $f_i{}^+, f_i$ can be integrated out yielding the collective action [32]

$$\mathcal{A}[S] = \pm itr\log(iG_x{}^{-1}) - V(S_1{}^2 + S_2{}^2 \mp g - S_3{}'^2) \pm \varepsilon\Omega \tag{1.73}$$

where G_s is the matrix collecting the Green's functions of the particles in the external field S

$$G_s(t,t')_{ij} = \begin{pmatrix} \overbrace{a_i(t)a_j{}^+}(t_i{}') & \overbrace{a_i(t)b_j{}^+}(t_i{}') \\ \overbrace{b_i{}^+(t)a_j{}^+}(t') & \overbrace{b_i{}^+(t)b_j{}^+}(t') \end{pmatrix}. \tag{1.74}$$

Its equation of motion, multiplied by $\left\{\begin{smallmatrix}\sigma^3 \\ 1\end{smallmatrix}\right\}$

$$\begin{pmatrix} i\partial_t - \varepsilon \mp gVS_3{}' & VS^+ \\ \mp VS & i\partial_t + \epsilon \pm gVS_3{}' \end{pmatrix} G_S(t,t') = i\left\{\begin{matrix}\sigma^3 \\ 1\end{matrix}\right)\delta(t-t') \tag{1.75}$$

may be solved by an Ansatz

$$G_s(t,t') = U^+(t)G_0(t,t')U(t') \tag{1.76}$$

where G_0 is a solution of (1.75) for $S=0,\ S_3{}'=0, \varepsilon=0$. Before we proceed it is useful to absorb ε and g into $S_3{}'$ by defining the more symmetric variable

$$\mp S_3 = \mp gS_3{}' - \frac{\varepsilon}{V}. \tag{1.77}$$

Then equ. (1.75) reads

$$\left(i\partial_t + V\left\{\begin{matrix}-iS_2 \\ S_1\end{matrix}\right\}\sigma^1 + V\left\{\begin{matrix}iS_1 \\ -S_2\end{matrix}\right\}\sigma^2 \mp VS_3\sigma^3\right)U^+(t)G_0U(t') = i\left\{\begin{matrix}\sigma^3 \\ 1\end{matrix}\right\}\delta(t-t'). \tag{1.78}$$

It is solved if U satisfies $U^+\left\{\begin{smallmatrix}\sigma^3 \\ 1\end{smallmatrix}\right\} = \left\{\begin{smallmatrix}\sigma^3 \\ 1\end{smallmatrix}\right\}$ and the differential equation

$$iU^+(t)U^+(t)^{-1} = -V\left(\left\{\begin{matrix}-iS_2 \\ S_1\end{matrix}\right\}\sigma^1 + \left\{\begin{matrix}iS_1 \\ S_2\end{matrix}\right\}\sigma^2 \mp VS_3\sigma^3\right). \tag{1.79}$$

The condition $U^{+}\left\{ {\sigma^3 \atop 1} \right\} U = \left\{ {\sigma^3 \atop 1} \right\}$ can be met by parametrizing U in terms of Euler angles

$$U(t) = e^{i\alpha\frac{\sigma_3}{2}} e^{\left\{ {-\tilde{\beta} \atop i\beta} \right\} \frac{\sigma^2}{2}} e^{i\gamma\frac{\sigma_3}{2}}. \tag{1.80}$$

As should be expected from the above discussion of the operators L_i, the matrices U form a subgroup of the Lorentz group $SL(2, C)$. In the Bose case this subgroup is $SU(1,1)$ in the Fermi case $SU(2)$. The equ. (1.78) implies the differential equations for the Euler angles

$$\begin{aligned} \tilde{\omega}_1 &\equiv \dot{\tilde{\beta}} \sin\gamma + \dot{\alpha} \sinh \tilde{\beta} \cos\gamma = 2VS_1 \\ \tilde{\omega}_2 &\equiv \dot{\tilde{\beta}} \cos\gamma - \dot{\alpha} \sinh \tilde{\beta} \sin\gamma = 2VS_2 \\ \tilde{\omega}_3 &\equiv \dot{\alpha} \cosh \tilde{\beta} + \dot{\gamma} = 2VS_3 \end{aligned} \tag{1.81}$$

and

$$\begin{aligned} \omega_1 &\equiv -\dot{\beta} \sin\gamma + \dot{\alpha} \sin\beta \cos\gamma = -2VS_2 \\ \omega_2 &\equiv \dot{\beta} \cos\gamma + \dot{\alpha} \sin\beta \sin\gamma = -2VS_2 \\ \omega_3 &\equiv \dot{\alpha} \cos\beta + \dot{\gamma} = -2VS_3. \end{aligned} \tag{1.82}$$

The left-hand sides of (1.83) are recognized as the standard Euler equations for the angular velocities ω_i in a body-fixed reference frame.

The upper equations follow from the lower by replacing $\beta \to - - i\tilde{\beta}, S_1 \to -iS_2, S_2 \to iS_1, S_3 \to -S_3$. Since this transition can be done at any later stage it is convenient to avoid the clumsy distinction of different cases and focus attention to the Fermi case only.

In the Fermi case, the matrix $U(1)$ us unitary and coincides with the well-known representation matrices $D^{1/2}_{m'm}(\alpha\beta\gamma)$ of the rotation group.[1]. They can be solved formally as

$$U(t) = T \exp\left[-i \int_{\infty}^{t} 2VS\sigma dt'\right] \tag{1.83}$$

Given this $U(t)$ we can now proceed to evaluate the $tr \log$ term in (1.73). By differentiation with respect to S we find:

$$\frac{\delta}{\delta S_k(t)} [-itr \log(iG_s^{-1})] = V\Sigma_i tr(\sigma^k G_s^{ii}(t, t')|_{t'=t+\varepsilon}. \tag{1.84}$$

The right-hand side can be calculated in terms of Euler angles by inserting (1.80). In addition one has to choose the reference state for $Z[\eta^+, \eta]$ by specifying the boundary condition on G_0. Since G_0 represents the same matrix of Green's functions as (1.74), except with free oscillators a_0^+, ${b_0}^+$ of zero energy, this is easily done. Let us choose

[1] For the conventions see: A.R. Edmonds, *Angular Momentum in Quantum Mechanics*, Princeton University Press

as reference state $|R\langle$ one of the quasi-spin symmetric states of seniority $\nu = \Omega$, say (1.63). Then G_0 has to have the form

$$G_0{}^{ij}(t,t') = \begin{pmatrix} \Theta(t-t') & 0 \\ 0 & \Theta(t-t') \end{pmatrix}^{\delta^{ij}} . \tag{1.85}$$

As a consequence $G_0^{ij}(t,t')|_{t'=t+\varepsilon} = 0$ such that also (1.84) vanishes and $-itr\log(iG_s{}^{-1}$ becomes an irrelevant constant.

Hence the generating functional in the quasi-spin symmetric reference state (1.63) is

$${}^{R}Z[j^+j] = \int DS \exp\left[\int dtVS(t)^2 - \int dtdt'\Theta(t-t')\Sigma_i j_i{}^+(t)U^+(t)U(t')j_i(t')\right] . \tag{1.86}$$

As in the case of the trivial model it is now convenient to change variables and integrate directly over the Euler angles $\alpha\beta\gamma$ rather than $S_1S_2S_3$. Using the derivatives

$$\begin{aligned} -\frac{1}{2V}\frac{\delta S_i(t)}{\delta q_j(t')} &\equiv A(t)_{ij}\delta(t-t') + B(t)_{ij}\dot{\delta}(t-t') \\ &= \begin{pmatrix} 0 & \dot\alpha\cos\beta\cos\gamma & -\dot\beta\cos\gamma - \dot\alpha\sin\beta\sin\gamma \\ 0 & \dot\alpha\cos\beta\sin\gamma & -\dot\beta\sin\gamma + \dot\alpha\sin\beta\cos\gamma \\ 0 & -\dot\alpha\sin\beta & 0 \end{pmatrix}_{ij} & (1.87) \\ &\quad \times\delta(t-t') + \begin{pmatrix} \sin\beta\cos\beta\gamma & -\sin\gamma & 0 \\ \sin\beta\sin\gamma & \cos\gamma & 0 \\ \cos\beta & 0 & 1 \end{pmatrix}_{ij} \delta(t-t') & (1.88) \end{aligned}$$

one calculates the functional determinant as the determinant of the second matrix B. This can be seen most easily by multiplication with the constant (functional) matrix $\int dt'\Theta(t'-t'')$ which diagonalizes the $\dot\delta(t-t')$ and brings the δ term completely to the right of the (functional diagonal: $\delta\Theta' = \Theta$. The determinant of such a matrix equals the determinant of the diagonal part only. Thus, up to an irrelevant factor, one has

$$DS = \text{const} D\alpha D\beta D\gamma \sin\beta \tag{1.89}$$

corresponding to the standard measure of the rotation group. Inserting now (1.83) into (1.86) we find

$$\begin{aligned} Z[j^+,j] &= \int D\alpha D\cos\beta D\gamma \exp\left[i\int dt\left\{-\frac{1}{4V}\left(\omega_1{}^2 + \omega_2{}^2 + \frac{1}{g}(\omega_3 - 2\varepsilon)^2\right) - \varepsilon\Omega\right]\right. \\ &\quad \times\exp\left[o\int dtdt'\Theta(t-t')\Sigma_i j_i{}^+(t)U^+(t)U(t')j_i(t')\right] . \end{aligned} \tag{1.90}$$

The collective Langrangian becomes:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4V}\left(\omega_1^2 + \omega_2{}^2 + \frac{1}{g}(\omega_3 - 2\varepsilon)^2\right) - \varepsilon\Omega & (1.91) \\ &= -\frac{1}{4V}\left\{(\dot\beta^2 + \dot\alpha^2\sin^2\beta) + \frac{1}{g}(\dot\gamma + \dot\alpha\cos\beta)^2\right\} + \frac{\varepsilon}{Vg}(\dot\gamma + \dot\alpha\cos\beta) - \frac{\varepsilon^2}{Vg} - \varepsilon\Omega. \end{aligned}$$

This has the standard form

$$\mathcal{L} = \frac{1}{2}\dot{q}^i g_{ij}(q)\dot{q}^j + a_i(q)\dot{q}^i - v(q) \tag{1.92}$$

with the metric

$$g_{ij}(q) = -\frac{1}{2V}\begin{bmatrix} \sin^2\beta + \frac{1}{g}\cos^2\beta & 0 & \frac{1}{g}\cos\beta \\ 0 & 1 & 0 \\ \frac{1}{g}\cos\beta & 0 & \frac{1}{g} \end{bmatrix} \tag{1.93}$$

$$g^{ij}(q) \equiv (g^{-1}(q))^{ij} = -2V\frac{g}{\sin^2\beta}\begin{bmatrix} \frac{1}{g} & 0 & -\frac{1}{g}\cos\beta \\ 0 & 1 & 0 \\ -\frac{1}{g}\cos\beta & 0 & \sin^2\beta + \frac{1}{g}\cos^2\beta \end{bmatrix} \tag{1.94}$$

of determinant

$$g \equiv \det(g_{ij}) = -\frac{1}{8V^3}\frac{1}{g}\sin^2\beta$$

in the space labelled again by $q^i \equiv (\alpha, \beta, \gamma)$.
The Hamiltonian in such a curved space is given by [33]

$$H = H_1 + H_2 + H_3 + v(q) + \frac{1}{2}a^i a_i(q) \tag{1.95}$$

with

$$\begin{aligned} H_1 &= -\frac{1}{2}g^{-1/2}\frac{\partial}{\partial q^i}\left(g^{1/2}g^{ij}\frac{\partial}{\partial q^j}\right) \\ H_2 &= \frac{i}{2}g^{-1/2}\left[\frac{\partial}{\partial q^i}g^{1/2}g^{ij}a_j(q)\right] \end{aligned} \tag{1.96}$$

$$H_3 = ia_i(q)g^{ij}g^{ij}\frac{\partial}{\partial q^i}. \tag{1.97}$$

Here we find H_1 as the standard asymmetric-top Hamiltonian,

$$H_1 = V\left(\frac{\partial^2}{\partial\beta^2} + \cot\beta\frac{\partial}{\partial\beta} + (g + \cot\beta)\frac{\partial^2}{\partial\gamma^2} + \frac{1}{\sin^2\beta}\frac{\partial^2}{\partial\alpha^2} - 2\frac{\cos\beta}{\sin^2\beta}\frac{\partial^2}{\partial\alpha\partial\gamma}\right). \tag{1.98}$$

Since

$$a_i = \frac{\epsilon}{Vg}(\cos\beta, 0, 1) \tag{1.99}$$

the second part, H_2, vanishes and the third part becomes

$$H_3 = -2\varepsilon i\partial_\gamma. \tag{1.100}$$

The resulting Hamiltonian is exactly the Schrödinger version of the quasi-spin form (1.59) with

$$\begin{aligned} L^{\pm} &= e^{\pm i_\gamma}\left[\pm\partial_\beta + \cot\beta i\partial_a - i\tfrac{i}{\sin\beta}\partial_\gamma\right] \\ L_3 &= -i\partial_\gamma. \end{aligned} \tag{1.101}$$

The eigenfunctions of H coincide with the rotation matrices

$$D^j_{m'm}(\alpha,\beta,\gamma) = e^{i\alpha m'+\gamma m)} d^j_{m'm}(\beta). \tag{1.102}$$

The energy eigenvalues of H_1 are well-known

$$E^1_{jm} = -V[j(j+1) - m^2(1-g)] \tag{1.103}$$

such that the full energies are

$$E_{jm} = 2\varepsilon m - V[j(j+1) - (1-g)m^2] + \varepsilon\Omega. \tag{1.104}$$

This coincides with the fermion part of the spectrum (1.64) if m, j are set equal to

$$m = (N-\Omega)/2, \quad j = \frac{\Omega - \nu}{2} \tag{1.105}$$

as is necessary due to (1.57), (1.62).

For $g = 1, \varepsilon = 0$ the spectrum is degenerate as the Lagrangian (1.92) is rotationally invariant. It may be worth mentioning that in this case the Lagrangian can also be written as a standard σ-model in the time dimension. In order to see this use $i\dot{U}^+U = -iU^+\dot{U} = \omega_i\sigma_i/2$ to bring (1.83) to the form

$$\mathcal{L} = -\frac{1}{4V}({\omega_i}^2 + {\omega_2}^2 + {\omega_3}^2) = -\frac{1}{2V} tr(\dot{U}^+UU^+\dot{U}).$$

If one now defines σ and π fields as

$$U = \sigma + i\pi \cdot \sigma,$$

where $\sigma^2 + \pi^2 = 1$ due to unitary of U, the Lagrangian takes the familiar expression:

$$\mathcal{L} = -\frac{1}{V}(\dot{\sigma}^2 + \pi^2). \tag{1.106}$$

It is instructive to exhibit the original quasi-spin operators and their algebra within the collective Lagrangian. For this we add a coupling to external currents

$$\Delta H = -2V \int L_i(t) l_i(t) dt. \tag{1.107}$$

to the Hamiltonian (1.55) where L_i are the operators (1.57). In the Lagrangian (1.69) this amounts to

$$\Delta\mathcal{L}(t) = 2V L_i(t) l_i(t) dt, \tag{1.108}$$

which modifies (1.72) by adding the matrix

$$V f^+(t) \begin{pmatrix} l_3 & l^+ \\ l & l_3 \end{pmatrix} f(t). \tag{1.109}$$

This has the effect of replacing

$$S_i \to \tilde{S}_i \equiv S_i + l_i$$

in the tr log term of (1.73).

Performing a shift in the integration $DS \to D(S+l)$ we can also write

$$\mathcal{A}[S,l] = +itrlog(iG_s^{-1}) - V\left((S_1 - l_1)^2 + (S_2 - l_2)^2 - \frac{1}{g}\left(S_3 + \frac{\varepsilon}{V} - l_3\right)^2\right). \tag{1.110}$$

The Green's function involving angular momentum operators can now be generated by differentiating

$$Z[0,0,l_i] = \int DS \exp\{i\mathcal{A}[S,l]\}$$

with respect to δl_i:

$$L_i \hat{=} -\frac{i}{2V}\frac{\delta}{\delta l_i}. \tag{1.111}$$

In the reference state $|R\rangle$ where the tr log term vanishes, $-i/2V\eta/\delta l_{\frac{1}{2}}, -\frac{i}{2V}\sigma/\sigma l_3$ generate from (1.108) the fields $S_{\frac{1}{2}} - l_{\frac{1}{2}}, (S_3 + \varepsilon/V - l_3)/g$ in the functional integral.

In the fermion case, this implies for $l = 0$, using equs. (1.80)

$$\begin{aligned} L^{\pm} &= -\frac{1}{2V}(\omega_1 \pm i\omega_2) = -\frac{1}{2V}(\pm i\dot{\beta} + \dot{\alpha}\sin\beta)e^{\pm i\gamma} \\ L_3 &= -\frac{1}{2Vg}(\omega_3 - 2\varepsilon) = -\frac{1}{2Vg}(\dot{\alpha}\cos\beta + \dot{\gamma} - 2\varepsilon) \end{aligned} \tag{1.112}$$

which are exactly the angular momenta of the Lagrangian (1.92) with moments of intertia

$$I_{1\top 2} = -\frac{1}{2V}, I_3 = -\frac{1}{2Vg}. \tag{1.113}$$

Inserting the canonical momenta of (1.92)

$$\begin{aligned} P_\alpha &= -\frac{1}{2V}\left(\dot{\alpha}\sin^2\beta + \frac{1}{g}(\dot{\gamma} + \dot{\alpha}\cos\beta - 2\varepsilon)\cos\beta\right) \\ &= -\frac{1}{2V}\dot{\alpha}\sin^2\beta + \cos\beta p_\gamma = -i\partial_\alpha \\ P_\beta &= -\frac{1}{2V}\dot{\beta} = -i\sin^{-1/2}\beta\partial_\beta\sin^{1/2}\beta = -i\partial_\beta - \frac{i}{2}\cot\beta \\ P_\gamma &= -\frac{1}{2V_g}(\dot{\gamma} + \dot{\alpha}\cos\beta - 2\varepsilon) = -i\partial_\gamma \end{aligned} \tag{1.114}$$

we recover the differential operators (1.101).

The quasi-spinalgebra can now be verified by applying the derivatives:

$$-\frac{1}{4V^2}\left(\frac{\delta}{\delta l_j(t+\varepsilon)}\frac{\delta}{\delta l_i(t)} - \frac{\delta}{\delta l_i(t)} - \frac{\delta}{\delta l_i(t+\varepsilon)}\frac{\delta}{\delta l_j(t)}\right)Z|_{t=0} = \frac{1}{2V}\varepsilon_{ijk}\frac{\delta}{\delta l_k}Z|_{l=0}. \tag{1.115}$$

What would have happened in this model if we had not chosen the symmetric reference state $|R\langle$ to specify the boundary condition on G_0? Consider for example the vacuum state $|0\langle$. Then the Green's function becomes for $S = 0$:

$$G_0^{ij}(t,t') = \begin{pmatrix} \Theta(t-t') & 0 \\ 0 & -\Theta(t'-t) \end{pmatrix} \delta^{ij} \Bigg) \delta^{ij} \tag{1.116}$$

rather than (1.85). In this case *there is* a contribution of $-i\text{trlog}(iG_s{}^{-1})$ since from (1.84) and (1.76):

$$\frac{\delta}{\delta S_i}[-itrlog(iG_s{}^{-1})] = -V\Omega tr\left(\sigma^i U^+(l)\frac{-1+\sigma^3}{2}U(t')\right)|_{t'=t}. \tag{1.117}$$

Now (1.80) implies

$$U^+(t)\sigma^3 U(t) = \cos\beta\sigma_3 + \sin\beta(\cos\gamma\sigma_1 + \sin\gamma\sigma_2)$$

yielding for the right hand side of (1.117) the expression

$$-V\Omega\left\{\begin{array}{c} n_1 \\ n_2 \\ n_3 \end{array}\right\} \equiv -V\Omega\left\{\begin{array}{c} \sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{array}\right\}. \tag{1.118}$$

Observe that due to the differential equations (1.83) the unit vector n_i can be found to satisfy the equation of motion

$$\dot{\mathbf{n}} = 2V\mathbf{n}\times\mathbf{S}. \tag{1.119}$$

We can now proceed and find $-i\text{trlog}iG_s{}^{-1}$ by functionally integrating (1.117). We shall do so in terms of the Euler variables $\alpha\beta\gamma$. Using (1.117), (1.118), (1.88), and the chain rule of differentiation

$$\begin{aligned} \frac{\delta}{\delta q_j(t')}[-i\text{trlog}G_s{}^{-1}] &= \sum_i \int dt \frac{\delta S_i(t)}{\delta q_j(t')}\frac{\delta}{\delta S_i(t)}[-i\text{trlog}iG_s{}^{-1}] \\ &= -V\Omega\sum_i\int dt n_i(t)\frac{\delta S_i(t)}{\delta q_j(t')} \end{aligned} \tag{1.120}$$

$$\begin{aligned} \frac{\delta}{\delta q_i(t)}[-i\text{trlog}iG_s{}^{-1}] &= \frac{\Omega}{2}\sum_i\int dt\,(n_i(t)A_{ij}(t)\delta(t-t') + n_i(t)B_{ij}(t)\delta(t-t')) \\ &= \frac{\Omega}{2}[(0,0,-\dot\beta\sin\beta(t'))_j + \int dt(1,0,\cos\beta(t))_j\delta(t-t')]. \end{aligned} \tag{1.121}$$

Partial integration renders for the second part in brackets

$$(1,0,\cos\beta(t))\delta(t-t')|_{t=-\infty}^{t=\infty} + (0,0,\dot\beta\sin\beta(t')). \tag{1.122}$$

With the boundary condition $\cos\beta(\pm\infty) = 1$ one has therefore

$$\frac{\delta}{\delta(\alpha,\beta,\gamma)(t)}[-i\mathrm{trlog}iG_s{}^{-1}] = \frac{\Omega}{2}(1,0,1)[\delta(\infty - t) - \delta(-\infty - t)]. \qquad (1.123)$$

This pure boundary contribution can immediately be functionally integrated with the result:

$$-i\mathrm{trlog}iG_s^{-1} = \frac{\Omega}{2}\int_{\infty}^{\infty}(\dot{\alpha}(t) + \dot{\gamma}(t)dt. \qquad (1.124)$$

Hence the exponent of the generating functional $Z[j^+j]$ on the reference state $|0\rangle$ becomes

$$i\int dt\left\{-\frac{1}{4V}\left({\omega_1}^2 + {\omega_2}^2 + \frac{1}{g}(\omega_3 - 2\varepsilon)^2\right) + \frac{\Omega}{2}(\dot{\alpha}+\dot{\gamma}) - \varepsilon\Omega\right\}$$
$$-\int dtdt'\sum_i {j_i}^+(t)\left\{U^+(t)\frac{1+\sigma^3}{2}U(t')\Theta(t-t')U^+(t)\frac{1-\sigma^3}{2}U(t')\Theta(t'-t)\right\}j_i(t') \qquad (1.125)$$

other than (1.90). As in the case of the Pet model in the last section, the Hamiltonian s changed quite trivially. The canonical momenta P_α, P_γ become

$$\begin{aligned} P_\alpha &= -\frac{1}{2V}\left[\dot{\alpha}\sin^2\beta + \frac{\cos\beta}{g}(\dot{\gamma} + \dot{\alpha}\cos\beta - 2\varepsilon)\right] + \frac{\Omega}{2} \\ &= -\frac{1}{2V}\dot{\alpha}\sin^2\beta + \cos\beta p_\gamma - \frac{\Omega}{2}(\cos\beta - 1) = -i\partial_\alpha \qquad (1.126) \\ P_\gamma &= -\frac{1}{2Vg}(\dot{\gamma} + \dot{\alpha}\cos\beta - 2\varepsilon) + \frac{\Omega}{2} = -i\partial_\gamma. \end{aligned}$$

The additional term can be removed by multiplying all eigenfunctions belonging to (1.127) by a phase $\exp[-i\Omega/2(\alpha+\gamma)]$ thereby reducing them to the previous case. In the present context it is really superfluous to discuss such trivial surface terms. We are doing this only because these terms do become important at that moment at which the transition to the true BCS model is made by letting $g \to 0$.

This will be discussed in the next section.

1.3 The Hilbert Space of Generalized BCS Model

Let us now study in which fashion the Hilbert of all rotational wave functions imbeds the fermion theory. For this consider the generation of Green's functios by functional derivation of ${}^RZ[j^+, j]$, with the reference state $|R\rangle$ being the quasi-spin symmetric one (1.61), for simplicity.

The resulting one-particle Green's function will have to coincide with

$$G^{ij}_{mm'}(t,t') = \langle 0|b_\Omega \cdot \ldots \cdot b_1 \begin{pmatrix} Ta_i(t)a_j^+(t') & Ta_i(t)b_j(t') \\ Tb_i^+(t)a_j^+(t') & Tb_i^+(t)b_j(t') \end{pmatrix}_{mm'} b_1 + \cdot \ldots \cdot {b_\Omega}^+|0\rangle. \qquad (1.127)$$

If we differentiate (1.90) accordingly, we find

$$G^{ij}_{mm'}(t,t') = \int D\alpha D\cos\beta D\gamma \delta^{ij}(U^+(t)U(t')_{mm'}\Theta(t-t')\exp[i\int dt\mathcal{L}(t)]. \quad (1.128)$$

This can be calculated most easily by going to the Schrödinger picture

$$G^{ij}_{mm'}(t,t') = \sum_k \{R|D^{1/2}_{km}(\alpha\beta\gamma(t))D^{1/2}_{km'}(\alpha\beta\gamma(t'))|R\}\delta^{ij}\Theta(t-t'). \quad (1.129)$$

Since the reference state is symmetric, it must be associated with the wave function $\{\alpha\beta\gamma(t)|R\} = D^0_{00}(\alpha\beta\gamma(t)) \equiv 1/\sqrt{8\pi^2}$

$$E_R \equiv E_{0;0} = \varepsilon\Omega. \quad (1.130)$$

Inserting the time translation operator[2]

$$D(\alpha\beta\gamma(t)) = e^{iHt}D(\alpha\beta\gamma(0))e^{iHt} \quad (1.131)$$

with H in the differential form (1.95) one finds a phase

$$e^{i\Delta E(t-t')}, \quad (1.132)$$

where ΔE is the energy difference between the state $|jm\rangle = |1/21/2\rangle$ and the reference state $|R\rangle = |0,0\rangle$

$$\Delta E = E_{1/21/2} - E_{0,0} = \varepsilon - V\left(\frac{1}{2}+\frac{g}{4}\right) \quad (1.133)$$

and the integral

$$\sum_k \int d\alpha d\cos\beta d\gamma \{R|\alpha\beta\gamma\}D^{1/2^*}_{km}(\alpha\beta\gamma)D^{1/2}_{km'}(\alpha\beta\gamma)\{\alpha\beta\gamma|R\} = \delta_{mm'}. \quad (1.134)$$

This coincides exactly with the result one would obtain from (1.127) by using the original operator (1.55) and observing the energy spectrum (1.64).
Notice that the orthogonality relation together with the Grassmann algebra ensure the validity of the anticommutation rules among the operators. For higher Green's functions the functional derivatives amount again to the contractions as in (1.34), except that now the contractions are associated with

$$\begin{aligned} \overset{1}{f}_{mi(t)}\overset{1}{f}_{m'}(t') &= D^{1/2}_{mm'}(U^+(t)U(t')\Theta(t-t')\delta^{ij} \\ &= \sum_k D^{1/2^*}_{km}(U^+(t))D^{1/2}_{km'}(U(t))\Theta(t-t')\delta^{ij} \end{aligned} \quad (1.135)$$

where $f_{1/2i}, f_{-1/2i}$ stands for $(a_i, {b_i}^+)$.
We can now proceed and construct the full Hilbert space by piling up operators a_i^+ or

[2]The Schrödinger angles $\alpha\beta\gamma$ coincide with the time dependent angles $\alpha(t),\beta(t),\gamma(t)$ at $t=0$.

b_j on the reference state $|R\rangle = {b_1}^+ \dots {b_\Omega}^+|0\rangle$. First we shall go to true vacuum state of $a^+, b^+ : |0\rangle$, i.e. we shall calculate ${}^0Z[j^+, j]$ in this state. For this we obviously have to bring down successively ${b_1}^+(\infty)\cdot\ldots\cdot b_\Omega(-\infty)b_\Omega(-\infty)\cdot\ldots\cdot b_1(-\infty)$ by forming the functional derivatives:

$$Z^0[0,0] \propto \frac{\delta^{2\Omega}}{\delta j_{-1/2,1}(\infty)\cdot\ldots\cdot\delta j^+_{-1/2,1}(-\infty)} {}^R Z[j^+, j]|_{j=0} \tag{1.136}$$

in the functional (1.90). Of the resulting $n!$ contractions, only one combination survives, since all indices i, j are different and the Kronecker δ^{ij} permits only one set of contractions. The result is

$${}^0Z[0,0] = N\int D\alpha D\cos\beta D\gamma \exp[i\int dt\mathcal{L}(t)][D^{1/2}_{-1/2-1/2}U^+(\infty)U(-\infty))]^\Omega. \tag{1.137}$$

But from the coupling rules of angular momenta and the group property one has:

$$\begin{aligned}[D^{1/2}_{-1/2-1/2}(U^+(\infty)U(-\infty))]^\Omega &= D^{\Omega/2}_{-\Omega/2-\Omega/2}(U^+(\infty)U(-\infty)) \\ &= \sum_k D^{\Omega/2^*}_{k-\Omega/2}((\infty))D_{k-\Omega/2}(U(-\infty)).\end{aligned} \tag{1.138}$$

Going to the Schrödinger picture and inserting the time translation operator (1.131) one finds an infinite phase $\exp[i(E_R - E_0)2\infty]$ which can be absorbed in the normalization factor N. Here $E_0 = E_{\Omega/2,-\Omega/2}$ is the energy of the ground state $|0\rangle$ which has $|jm\rangle = |\Omega/2 - \Omega/2\rangle$. The eigenfunction $D(\alpha,\beta,\gamma)$ now appear both at $t = 0$ and the functional (1.137) becomes in the Schrödinger picture

$${}^0Z[0,0] = \sum_{k=-\Omega/2}^{\Omega/2}\int d\alpha d\beta d\gamma \sin\gamma\{0k|\alpha\beta\gamma\}\{\alpha\beta\gamma|0k\} \tag{1.139}$$

with the vacuum wave functions

$$\{\alpha\beta\gamma|0,k\} = D^{\Omega/2}_{k,-\Omega/2}(\alpha\beta\gamma) = e^{i(k\alpha-\Omega\gamma/2)}d^{\Omega/2}_{k,-\Omega/2}(\beta). \tag{1.140}$$

It is easy to verify, how an additional unpaired particle a^+, added to the vacuum, decreases $\Omega/2 \to (\Omega-1)/2$ and raises the third component of quasi-spin by 1/2 unit. Differentiating (1.88) by $-\delta^2/\delta j_{1/21}(\infty)\delta j^+_{1/21}(-\infty)$ in addition to (1.136) one finds a different set of contractions. Picturing them within the original fermion language, there are

$$\begin{aligned}&\langle R|T({b_1}^+(+\infty)\cdot\ldots\cdot {b_\Omega}^+(+\infty)a_1(+\infty){a_1}^+(-\infty)b_\Omega(-\infty)\cdot\ldots\cdot b_1(-\infty))|R\rangle \\ &= \langle R|T(\overset{1}{b_1}{}^+(\infty)\cdot\ldots\cdot \overset{2}{b_\Omega}{}^+(\infty)\,\overset{3}{a_1}(\infty)\,\overset{3}{a_1}{}^+(-\infty)\,\overset{2}{b_\Omega}(-\infty)\cdot\ldots\cdot \overset{1}{b_1}(-\infty))|R\rangle \\ &+\langle R|T(\overset{1}{b_1}{}^+(\infty)\cdot\ldots\cdot \overset{2}{b_\Omega}{}^+(\infty)\,\overset{3}{a_1}(\infty)\,\overset{1}{a_1}{}^+(-\infty)\,\overset{2}{b_\Omega}(-\infty)\cdot\ldots\cdot \overset{3}{b_1}(-\infty))|R\rangle.\end{aligned} \tag{1.141}$$

Employing the explicit formulas

$$D^{\Omega/2}_{-\Omega/2-\Omega/2}(\alpha\beta\gamma) = e^{-\Omega(\alpha+\gamma)/2}\left(\cos\frac{\beta}{2}\right)^{\Omega}$$
$$D^{1/2}_{1/21/2}(\alpha\beta\gamma) = e^{(\alpha+\gamma)/2}\cos\frac{\beta}{2} \qquad (1.142)$$
$$D^{1/2}_{-1/21/2}(\alpha\beta\gamma)D^{1/2}_{1/2-1/2}(\alpha\beta\gamma) = -\sin^2\frac{\beta}{2}$$

the r.h.s. of (**??**) becomes

$$= e^{-\Omega(\alpha+\gamma)/2}\left(\cos\frac{\beta}{2}\right)^{\Omega} e^{(\alpha+\gamma)/2}\cos\frac{\beta}{2}$$
$$+ e^{-(\Omega-1)(\alpha+\gamma)/2}\left(\cos\frac{\beta}{2}\right)^{\Omega-1}\sin^2\frac{\beta}{2}$$
$$= e^{-(\Omega-1)(\alpha+\gamma)/2}\left(\cos\frac{\beta}{2}\right)^{\Omega-1} = D^{(\Omega-1)/2}_{-(\Omega-1)/2,-(\Omega-1)/2}(\alpha\beta\gamma) \qquad (1.143)$$

and therefore, in analogy to (1.137), (1.139)

$$a_1^{+|0\rangle}Z[j^+,j]|_{j=0} = N\int D\alpha D\cos\beta D\gamma D^{(\Omega-1)/2}_{-(\Omega-1)/2,-(\Omega-1)/2}(\alpha\beta\gamma)\exp(i\int\mathcal{L}dt)$$
$$= \sum_{k=-(\Omega-1)/2}^{(\Omega-1)/2}\int dt\alpha d\cos\beta d\gamma\{a_1k|\alpha\beta\gamma\}\{\alpha\beta\gamma|a_1^+k\} \qquad (1.144)$$

with the Schrödinger wave functions

$$\{\alpha\beta\gamma|{a_1}^+k\} \equiv D^{(\Omega-1)/2}_{k,-(\Omega-1)/2}(\alpha\beta\gamma). \qquad (1.145)$$

In a similar fashion we may work our way through the whole Hilbert space!

2

Massive Thirring Model in 1+1 Dimensions

et us also study an example of a quantum filed theory in two spacetime dimensions, the massive Thirring model.

In $1+1$ dimensional field theories, such as the Thirring model, the technique presented here leads to an exact translation from the Fermi fields ψ to collective Bose fields $\varphi(x), \lambda(x)$. Here, one has to add the complete square Eq. (3.23)

$$\Delta\mathcal{A} = \frac{g}{2}\int d^2x \left\{\bar{\psi}(x)\gamma^\mu\psi(x) - A^\mu\right\}^2 \tag{2.1}$$

which eliminates the $-\frac{g}{2}\left(\bar{\psi}\gamma^\mu\psi\right)^2$ interaction. Integrating out the fermions renders the collective action Eq. (3.24)

$$\mathcal{A}_{\text{coll}}[A] = -i\text{Tr}\log\left(i\partial\!\!\!/ - gA\!\!\!/\right) + \frac{g}{2}\int d^4x A_\mu^2(x). \tag{2.2}$$

Now one can make use of the fact that in two dimensions A^μ has only two components and can be written as Eq. (3.25)

$$A^\mu(x) = \frac{1}{\sqrt{g}}\left(\partial^\mu\varphi(x) - \epsilon^{\mu\nu}\partial_\nu\lambda(x)\right) \tag{2.3}$$

so that Eq. ()

$$\frac{g}{2}A_\mu^2(x) = \frac{1}{2}\left(\partial_\mu\varphi(x)\right)^2 - \frac{1}{2}\left(\partial_\mu\lambda(x)\right)^2. \tag{2.4}$$

The trace log term can be expanded with only one loop contributing giving Eq. ()

$$\frac{g^2}{2\pi}\left[\left(g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\partial^2}\right)A_\nu(x)\right]^2. \tag{2.5}$$

Hence the collective action is Eq. (3.26)

$$\mathcal{A}_{\text{coll}}[\varphi,\lambda] = \int d^4x\left[\frac{1}{2}\left(\partial\varphi(x)\right)^2 - \frac{1}{2}\left(1+\frac{g}{\pi}\right)\left(\partial\lambda\right)^2\right]. \tag{2.6}$$

Since this transformation from the ψ to the φ, λ-field description is exact, one can calculate also the Green functions of the original fermion fields ψ. For this, an external current interaction Eq. ()

$$\int d^4x \left(\bar{\psi}(x)\eta(x) + \text{c.c.}\right) \tag{2.7}$$

has to be added in the exponent of the generating functional (2.10). Before integrating out the fermions, a quadratic completion is necessary giving an additional

Eq. (3.27) external current piece in (2.2):

$$\mathcal{A}_{\text{ext curr}} = i \int d^4x d^4y \bar{\eta}(x) \left(\frac{i}{i\not\partial - g\not A}\right)(x,y)\eta(y). \tag{2.8}$$

Eq. (3.28) With the decomposition (2.3) the Green function can be solved by

$$\begin{aligned} \frac{i}{i\not\partial - g\not A}(x,y) &= \frac{i}{i\not\partial - \sqrt{g}\not\partial(\varphi + \gamma_5\lambda)}(x,y) \\ &= e^{-i\sqrt{g}\varphi(x)+\gamma_5\lambda(x)} \frac{i}{i\not\partial}(x,y) e^{i\sqrt{g}(\varphi(y)+\gamma_5\lambda(y)} \\ &= e^{-i\sqrt{g}(\varphi(x)+\gamma_5\lambda(x))} \frac{1}{4\pi i} \frac{\not x - \not y}{(x-y)^2 + i\epsilon} e^{i\sqrt{g}(\varphi)+\gamma_5\lambda(y))}. \end{aligned} \tag{2.9}$$

Forming functional derivatives

$$\frac{\delta}{\delta\bar{\eta}(x)}\frac{\delta}{\delta\eta(y)}$$

Eq. (3.29) one finds the Green function of the original fermions in the form

$$\frac{1}{4\pi i}\frac{\not x}{x^2 + i\epsilon}\langle 0| : e^{-i\sqrt{g}(\varphi(x)+\gamma_5\lambda(x))} :: e^{i\sqrt{g}(\varphi(0)+\gamma_5\lambda(0))} : |0\rangle. \tag{2.10}$$

Eq. (3.30) With the standard rule for calculating the exponential of free fields

$$\frac{1}{4\pi i}\frac{\not x}{x^2 + i\epsilon}\langle 0| : e^{-i\alpha\varphi(x)} :: e^{i\alpha\varphi(y)} : |0\rangle = e^{\alpha^2\langle\varphi(x)\varphi(y)\rangle} \tag{2.11}$$

Eq. (3.31a) (this follows from Wick's theorem) and using the expectations [1]

$$\langle 0|\varphi(x)\varphi(0)|0\rangle = -\frac{1}{4\pi}\log(\mu^2 x^2) \tag{2.12}$$

$$\langle 0|\lambda(x)\lambda(0)|0\rangle = \frac{1}{1+g/\pi}\frac{1}{4\pi}\log(\mu^2 x^2) \tag{2.13}$$

Eq. (3.32) the vacuum expectation value in (2.10) becomes

$$\left(\frac{x^2}{\mu^2}\right)^{\left(-g+\frac{g}{1+g/\pi}\right)\frac{1}{4\pi}} = \left(\frac{x^2}{\mu^2}\right)^{-\frac{g^2}{1+g/\pi}\frac{1}{4\pi}} \tag{2.14}$$

with the exponent displaying the dynamically generated anomalous dimension of the field ψ, which vanishes in the free-fields case.

Notice that a mass term in the original action $m\bar{\psi}\psi$ would amount to

$$m\frac{\delta}{\delta\mu(x)}\frac{\delta}{\delta\bar{\mu}(x')}|_{x=x'}.$$

Eq. (3.33) Due to the result (2.9) this is equivalent to setting

$$m\bar{\psi}\psi = m\left(\psi_1^*\psi_2 + \psi_2^*\psi_1\right) \to m\left(e^{i2\sqrt{g}\lambda(x)}\psi_{01}^*\psi_{02} + \text{c.c.}\right) \tag{2.15}$$

where ψ_0 are free fields.

It is a pleasant accident of the two-dimensional world that all matrix elements of products of many $\psi_{01}^*\psi_{02},\ \psi_{02}^*\psi_{01}$ can also be using exponentials of the Bose fields φ [due to (2.11), compare with (2.9(3.28)) for $g = 0$], namely: Eq. (3.34)

$$\psi_{01}^*\psi_{02} \simeq e^{i\sqrt{4\pi}\varphi}. \tag{2.16}$$

Moreover, the matrix elements of Eq. (3.34a)

$$e^{i(2\sqrt{g}\lambda + \sqrt{4\pi}\varphi)} \tag{2.17}$$

are, again due to (2.11), (2.12(3.31a)), (2.13(3.31b)) the same as those of Eq. ()

$$e^{-i\sqrt{\frac{4\pi}{1+g/\pi}}\varphi}. \tag{2.18}$$

Because of this accidental situation the mass term can be "fitted" by the operator Eq. (3.34b)

$$m\bar{\psi}\psi \simeq 2m\cos\left(\sqrt{\frac{4\pi}{1+(g/\pi)}}\varphi\right) \tag{2.19}$$

rendering the well-known sine-Gordon bosonic description of the massive Thirring model.

Similarly, the Schwinger model can be converted into a single free "plasmon" Bose field of mass

$$m_{\text{plasmon}} = \frac{e^2}{\pi}$$

by integrating out the Fermi fields in the generating functional [22].

Also in $1+0$ dimensions the method has been applied to field theories of nuclear excitations and to models [23].

[1]Here μ is an infrared renormalization mass. See J.A. Swieca, Fortschr. Phys. *25*, 303 (1977).

3

$O(N)$-Symmetric Four-Fermi Interaction in $2+\epsilon$ Dimensions

Another class of field theories which become soluble by introducing collective quantum fields is obtained by introducing a large number N of identical fields, assuming for them an $O(N)$-symmetric four-field, and taking the limit $N \to \infty$. For Dirac fermions, the model Lagrangian density reads at finite N:

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m_0\right)\psi_a + \frac{g_0}{2N}\left(\bar{\psi}_a\psi_a\right)^2 \tag{3.1}$$

where the index a runs from 1 to N. This model is called the *Gross-Neveu model* since these authors gave a first thorough study of it [?], although it had been discussed earlier by Vaks and Larkin and by Anselm [?]. For later studies see [?].

At the mean-field level, the effective action becomes

$$\Gamma\left[\psi\bar{\psi}\right])\mathcal{A}\left[\psi,\bar{\psi}\right] = \int dx\left\{\bar{\Psi}\left(i\partial\!\!\!/ - m_0\right)\Psi + \frac{g_0}{2N}\left(\bar{\Psi}_a\Psi_a\right)^2\right\} \tag{3.2}$$

and yields an equation of motion

$$\left(i\partial\!\!\!/ - m_0 + \frac{g_0}{N}\bar{\Psi}_a\Psi_a\right)\bar{\Psi}_b(x) = 0. \tag{3.3}$$

This equation can only have a trivial solution $\Psi_a = 0$ since a non-vanishing fermion field expectation in the ground state

$$\bar{\Psi}_a = \langle 0|\psi_a|0\rangle \tag{3.4}$$

would imply that the state $|0\rangle$ contains a coherent mixture of bosonic and fermionic excitations. Such a state does not exist in nature, a fact which is considered to be the consequence of a so-called a *superselection rule*. This is in contrast to a Bose filed theory where $\bar{\Psi}_a$ can be nonzero.

The following discussion will show that for large N, the model exhibits a spontaneous symmetry breakdown very similar to the BCS model. The symmetry to be broken will not be the $O(N)$-symmetry of the interaction but a discrete one. For the sake of generality we shall carry on the discussion in any dimension D. The generating functional is written in terms of fermionic anticommuting sources as

$$Z[\eta, \bar{\eta}] = e^{iW[\eta_1 \bar{\eta}]} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{i\mathcal{A}\left[\psi, \bar{\psi}\right] + i\left(\bar{\psi}\eta + \text{c.c.}\right)}$$

We now introduce a collective field $\sigma \sim g\bar{\psi}\psi$ and rewrite this as

$$Z[\eta, \bar{\eta}] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\sigma e^{i\int d^D x \left\{\left[\bar{\psi}_a (i\partial\!\!\!/ - m_0 - \sigma)\psi_a + \bar{\eta}\psi + \text{c.c.}\right] - i\frac{N}{2g_0}\sigma^2\right\}}. \tag{3.5}$$

Integrating out the fields ψ according to the rules in Part I we express the partition function in terms of the collective density field σ,

$$Z[\eta, \bar{\eta}] = \int \mathcal{D}\sigma e^{i\mathcal{A}_{coll}[\sigma] - \bar{\eta} G_\sigma \eta}, \tag{3.6}$$

with the collective action

$$\mathcal{A}_{\text{coll}}[\sigma] = N\left[-\frac{N}{2g_0}\sigma^2 - itr\log\left(i\partial\!\!\!/ - m_0 - \sigma\right]\right). \tag{3.7}$$

In the limit $N \to \infty$, the fluctuations are squeezed into the extremum of the exponent. This leads to the effective action

$$\frac{1}{N}\Gamma\left[\Sigma, \psi, \bar{\psi}\right] = -\frac{1}{2g_0}\Sigma^2 - i\text{Tr}\log\left(i\partial\!\!\!/ - m_0 - \Sigma\right) + \frac{1}{N}\Psi_a\left(i\partial\!\!\!/ - m_0 - \Sigma\right)\Psi_a \tag{3.8}$$

Extremization of Γ yields the equations of motion:

$$\left(i\partial\!\!\!/ - m_0 - \Sigma\right), \Psi_a = 0 \tag{3.9}$$

$$\Sigma(x) = g_0 \text{Tr}\left(\frac{i}{i\partial\!\!\!/ - m_0 - \Sigma}\right)(x, x) - \frac{g_0}{N}\bar{\Psi}_a(x)\Psi_a(x) \tag{3.10}$$

where the trace runs over the Dirac indices. From what we said before, the field expectation Ψ must vanish such that we remain only with a single equation called gap equation because of its first analogous appearance in the theory of superconductivity:

$$\Sigma(x) = g_0 \text{Tr}\left(\frac{i}{i\partial\!\!\!/ - m_0 - \Sigma}\right)(x, x). \tag{3.11}$$

Thus, as far as the extremum is concerned, we may study only the purely collective part of the exact action

$$\frac{1}{N}\Gamma[\Sigma] = -\frac{1}{2g_0}\Sigma^2 - i\text{Tr}\log\left(i - \partial\!\!\!/ - m_0 - \Sigma\right). \tag{3.12}$$

Let us seek for an extremal constant solution Σ_0 Then gap equation reduces to

$$\begin{aligned}\Sigma^0 &= ig_0 \operatorname{tr} \int \frac{d^Dp}{(2\pi)^D} \frac{\not p + m_0 + \Sigma^0}{p^2 - (m_0+\Sigma_0)^2} \\ &= \operatorname{tr}(1)\; g_0 \int \frac{d^Dp_E}{(2\pi)^D} \frac{1}{p_E^2 + (m_0+\Sigma_0)^2} \left(m_0 + \Sigma^0\right),\end{aligned} \tag{3.13}$$

or

$$1 = \operatorname{tr}(1)\; g_0 \int \frac{d^Dp_E}{(2\pi)^D} \frac{1}{p_E^2 + (m_0+\Sigma_0)^2} \left(\frac{m_0}{\Sigma_0^D} + 1\right) \tag{3.14}$$

The Dirac matrices have dropped out except for the unit matrix such that we can work in any desired number dimensions. We only need to know the dimension of the Dirac matrices which is $2^{D/2}$ for even D. In this form we may continue Eq. (3.4) to any non-integer value of D.

For a constant Σ, the effective action gives rise to an effective potential

$$\frac{1}{N} v(\Sigma) = -\frac{1}{N}\Gamma[\Sigma] = \frac{1}{2g_0}\Sigma^2 - \operatorname{tr}(1)\frac{1}{2} \int \frac{d^Dp_E}{(2\pi)^D} \log\left[p_E^2 + (m_0+\Sigma)^2\right]. \tag{3.15}$$

The last term is obtained from the Tr log in (3.8) by the following calculation

$$\begin{aligned}\int \frac{d^Dp}{(2\pi)^D} \log(\not p - m_0 - \Sigma) &= \frac{1}{2} \int \frac{d^Dp}{(2\pi)^D} \left[\log(\not p - m_0 - \Sigma) + \log(-\not p - m_0 - \Sigma)\right] \\ &= \frac{1}{2} \int \frac{d^Dp}{(2\pi)^D} \log\left[-p^2 + (m_0+\Sigma)^2\right] \\ &= \frac{i}{2} \int \frac{d^Dp_E}{(2\pi)^D} \log\left[p_E^2 + (m_0+\Sigma)^2\right].\end{aligned}$$

The integral is performed with the help of formula (**??**) yielding

$$\frac{1}{N} v(\Sigma) = \frac{1}{2g_0}\Sigma^2 - 2^{\frac{D-2}{2}} \mu^{D-2} \frac{1}{2} \bar{S}_D \Gamma(D/2)\Gamma(1-D/2) \frac{2}{D} \left(\frac{m_0+\Sigma}{\mu}\right)^D \mu^2. \tag{3.16}$$

The arbitrary mass scale μ will be important for a study of the theory in the limit $m_0 = 0$.

We now focus attention upon the dimensional neighbourhood of $D = 2$ spacetime dimensions, settind

$$D = 2 + \epsilon, \quad \text{with} \quad \epsilon > 0. \tag{3.17}$$

Then

$$\frac{1}{N} v(\Sigma) = \frac{\mu^\epsilon}{\Sigma} \left[\frac{\Sigma^2}{g_0 \mu^\epsilon} - b_\epsilon \left(\frac{m_0+\Sigma}{\mu}\right)^{2+\epsilon} \mu^2\right], \tag{3.18}$$

where the constant b_ϵ stands for

$$b_\epsilon = \frac{2}{D} 2^{\epsilon/2} S_D \Gamma(D/2)\Gamma(1-D/2) = \frac{2}{D}\frac{1}{(2\pi)^{D/2}}\Gamma(1-D/2). \tag{3.19}$$

For small ϵ, b_ϵ behaves as [recall Eq. (**??**)]

$$b_\epsilon \sim -\frac{1}{\pi\varepsilon}\left[1 - \frac{\epsilon}{2}\log\left(2\pi e^{-\gamma}\right)\right] + \mathcal{O}(\epsilon). \tag{3.20}$$

Therefore, the bare parameters must somehow diverge in order to obtain a finite theory in two dimensions.

For simplicity, let us focus attention to the massless case, $m = 0$. Then a renormalized coupling constant can be defined via

$$\frac{1}{g_0 \mu^\epsilon} - b_\epsilon = \frac{1}{g}. \tag{3.21}$$

The limit $\epsilon \to 0$ can now be taken at a finite g and we obtain the renormalized potential

$$\frac{1}{N} v(\Sigma) \to \frac{1}{2}\left[\frac{\Sigma^2}{g} + \frac{\Sigma^2}{\pi}\log(\Sigma/\mu)\right]. \tag{3.22}$$

For arbitrary $4 > \epsilon > 0$ we may write a renormalized v as

$$\frac{1}{N} v(\Sigma) = \frac{\mu^\epsilon}{2}\left\{\frac{\Sigma^2}{g} + b_\epsilon \Sigma^2 \left[g - \left(\frac{\Sigma}{\mu}\right)^\epsilon\right]\right\} \tag{3.23}$$

which reduces to (3.22) for $\epsilon \to 0$.

Let us now study the possibility of a nontrivial solution $\Sigma = \Sigma_0$ for the gap equation. There is no need to perform the integral in (3.12), since the gap equations determines the minimum of $v(\Sigma)$. Thus we may simply differentiate (3.18) for $m = 0$, and obtain

$$1 = g_0 \mu^\epsilon b_\epsilon \frac{D}{2}\left(\frac{\Sigma_0}{\mu}\right)^\epsilon. \tag{3.24}$$

In the renormalized version (3.23) this reads

$$1 = g b_\epsilon \left[\left(1 + \frac{\epsilon}{2}\right)\left(\frac{\Sigma_0}{\mu}\right)^\epsilon - 1\right], \tag{3.25}$$

and becomes in the limit $\epsilon \to 0$:

$$1 = -\frac{1}{\pi} g \left(\frac{1}{2} + \log\frac{\Sigma_0}{\mu}\right). \tag{3.26}$$

The vacuum expectation of the collective field is therefore given by

$$\Sigma_0 = \mu e^{-\left(\frac{1}{2} + \frac{\pi}{g}\right)}. \tag{3.27}$$

The question arises whether this non-trivial solution corresponds to the true ground state of the problem. For this, we differentiate v once more and find

$$\frac{1}{N}v''(\Sigma) = \frac{1}{g_0} - b_\epsilon \frac{D}{2}(D-1)\,\Sigma^\epsilon. \tag{3.28}$$

Inserting (3.27) this yields the condensation energy

$$\frac{1}{N}v''(\Sigma) = -\frac{1}{g_0}(D-2) = \epsilon b_\epsilon \frac{D}{2}(D-2)\Sigma_0^\epsilon, \tag{3.29}$$

which is positive for $D > 2$ if

$$g_0 < 0. \tag{3.30}$$

What does this condition mean for the renormalized coupling g? Using (3.21) we see that $g_0 < 0$ and $g_0 > 0$ amount to $g > g^*$ and $g < g^*$, respectively, with

$$g^* \equiv \epsilon. \tag{3.31}$$

Thus there exists a critical value of the renormalized coupling $g^* = -b_\epsilon^{-1}$, above which the model has a phase with a non-vanishing field expectation Σ^0 if $g > g^*$. If the renormalized coupling lies below g^*, the bare coupling constant is positive and only the trivial solution $\Sigma_0 = 0$ has a positive $v''(\Sigma_0)$ indication.

What are the physical properties of the two solutions? Looking back at the effective action (3.8) we see that Σ_0 increased the fermion mass term to

$$M = m_0 + \Sigma_0. \tag{3.32}$$

The effect of Σ_0 is most drastic, of the bare mass $m_0 = 0$ vanishes. Then, for $g < g^*$, the massless input fermions remain massless. For $g > g^*$, on the other hand, the fermions aquire a mass $M = \Sigma_0$ via fluctuations. One speaks of a *spontaneous generation of a fermion mass*. The result may also be phrased differently: In the $g < g^*$-phase, the fermions have long-range correlations, in the $g > g$-phase, the spontaneously generated mass limits their correlations to a range $1/M$.

The spontaneous mass generation is closely related to the fact that the model displays, for zero initial mass, the phenomenon of spontaneous symmetry breakdown (recall Chapters **??**, **??**). Indeed, for $m_0 = 0$, the Lagrangian (3.1) possesses an additional symmetry called γ_5-*invariance*. In two dimensions we may choose the following γ-matrices:

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^1 \\ \gamma^1 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma^2 \end{aligned} \tag{3.33}$$

which satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} = 2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.34}$$

The hermitian γ^5-matrix is defined in analogy with the four dimensional case as

$$\gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.35}$$

The γ_5-transformation T may now be introduced as

$$\psi \underset{T}{\rightarrow} \gamma_5\psi \tag{3.36}$$

which obviously satisfies $T^2 = 1$. Under T:

$$\bar{\psi} \underset{T}{\rightarrow} \psi^\dagger\gamma_5\gamma_0 \rightarrow -\bar{\psi}\gamma_5. \tag{3.37}$$

Hence:

$$\begin{aligned} \bar{\psi}\psi &\rightarrow -\bar{\psi}\psi \\ \bar{\psi}\gamma^\mu\psi &\rightarrow -\bar{\psi}\gamma^\mu\psi \end{aligned} \tag{3.38}$$

If the bare mass m_0 in (3.1) is zero, the Lagrangian is invariant under T.

Also the action in the exponents of (3.5) and (3.6) are invariant, if we assign to $\sigma \sim g\bar{\psi}\psi$, in accordance with (3.35), (3.36) the transformation

$$\sigma \underset{T}{\rightarrow} -\sigma. \tag{3.39}$$

Thus the $m_0 = 0$ collective action (3.7) is symmetric in σ. It is precisely this γ_5 symmetry which is broken by the non-vanishing expectation value $\langle\sigma\rangle = \epsilon^0$. We may compare this result with our Bose discussion in the last sections. There we found that the continuous $O(N)$ symmetry could not be broken in 2 (and 1) dimensions due to the fluctuations. Here we see that contrary to this a descrete symmetry can be broken in two dimensions. This is a general feature. The impossibilities of the spontaneous breakdown of a continous symmetry is related to the fact that if it were to take place there would have to be massless Nambu-Goldstone bosons in two dimensions. But it can be shown that such excitations cannot exist in such a reduced space-time. For a discrete symmetry there need not be any such bosons and this is what makes the spontaneous breakdown possible.

Such a behavior is found for many quasi-two dimensional systems, even though the physics in these systems is taking place at $\epsilon = 0$, and the mechanism causing the transition is quite different from the one described here. The prime example is ^{4}He which can easily be prepared as a surface layer of a few atoms thickness. It shows superflow properties below some temperature T^* and behaves normally for $T > T^*$. The transition is caused by a separation of bound pairs of vortices. The transition is very similar to the phase transition of a two-dimensional electron gas seen as follows: By rescaling the field as

$$\psi \to \frac{1}{\sqrt{g}}\psi \tag{3.40}$$

we can bring the path integral to the normalized form (leaving out the sources)

$$Z[0] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{i\frac{1}{g}\int d^Dx\left[\bar{\psi}(i\partial\!\!\!/-m_0)\psi+\frac{1}{2N}\left(\bar{\psi}\psi\right)^2\right]}.$$

The above calculation of this functional proceeded in a Wick-rotated form, such that the result is equal to the euclidean path integral

$$Z[0] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{\frac{1}{g}\int d^Dx_E\left[\bar{\psi}\left(i\partial\!\!\!/_E+m_0\right)\psi-\frac{1}{2N}\left(\bar{\psi}\psi\right)^2\right]}, \tag{3.41}$$

in which the time has been continued to imaginary values

$$t = -i\tau, \quad \tau \text{ real.} \tag{3.42}$$

The continued variable τ may be considered as a second spatial coordinate. Thus the exponent takes the typical form of a purely thermal fluctuation problem, which is usually

$$Z[0] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-\frac{1}{T}\int d^Dx\mathcal{H}} \tag{3.43}$$

where T is the temperature and $\mathcal{H}$ is the Hamiltonian density. The analogy is now obvious: The coupling constant g in (3.41) plays the same role as the temperature in (3.43). The euclidean version of the field-theoretic Lagrange density in one space and one time dimension becomes the Hamiltonian density in two space dimensions.

It is worth pointing out an important structural property of the final result: The characteristic parameter of the two-dimensional system is the fermion mass which is determined from the coupling strength and the arbitrary mass parameter via (3.27), (**??**) as

$$M = \mu e^{-\left(\frac{1}{2}+\frac{\pi}{g}\right)}. \tag{3.44}$$

The original theory with $m_0 = 0$ had only a single free parameter, namely the coupling strength g_0. For the purpose of renormalizing the massless theory we introduced the auxiliary mass parameter μ. The renormalized coupling g depends on the choice of μ, and (3.21) should more explicitly be written as

$$\frac{1}{g_0\mu^\epsilon} - b_\epsilon = \frac{1}{g(\mu)}. \tag{3.45}$$

In this way, a system with a single parameter g_0 has been recharacterized by two parameters μ and $g(\mu)$.

This increase of parameters is certainly an artifact. There must be a relation between μ and $g(\mu)$ such that different pairs $(\mu, q(\mu))$ correspond to the same set of Green's functions, i. e., to the same physical theory. Indeed, such a relation follows from (3.21). At a fixed g_0, we can plot curves in the (μ, g) plane which correspond to one and the same theory. In the limit $\epsilon \to 0$ this relation between μ and g has a subtlety. It goes over into the mass relation (3.44).

The fermion mass is a physically observable finite quantity. There are infinitely many pairs of parameter μ and coupling $g(\mu)$ which lead to the same fermion mass, and this mass is the most economic single parameter by which all properties of the theory can be expressed. Let us illustrate this by calculating two physical quantities:

1) We reexpress the potential (3.22) in terms of M. For this we write the renormalized gap equation in the form (with $E^0 = M$)

$$\left(\frac{1}{g} + \frac{1}{2\pi}\right) + \frac{1}{\pi}\log\frac{M}{\mu} = 0. \tag{3.46}$$

Multiplying by $\Sigma^2/2$, and subtract the result from the potential (3.22), then we find

$$\frac{1}{N}v(\Sigma) = \frac{\Sigma^2}{2\pi}\left[\log\left(\frac{\Sigma}{M}\right) - \frac{1}{2}\right]. \tag{3.47}$$

This has indeed the desired property that neither μ nor $g(\mu)$ appear but only the single parameter M. The same property can, of course, be verified for $D > 2$ dimensions. Here we may combine (3.25) with (3.23) and find

$$\begin{aligned} \frac{1}{N}v(\Sigma) &= \frac{\mu^\epsilon}{2}\left[\frac{\Sigma^2}{g_0\mu^\epsilon} - b_\epsilon\left(\frac{\Sigma}{\mu}\right)^{2+\epsilon}\mu^2\right] \\ &= \frac{M^\epsilon}{2}b_\epsilon\Sigma^2\left[1 + \frac{\epsilon}{2} - \left(\frac{\Sigma}{M}\right)^\epsilon\right]. \end{aligned} \tag{3.48}$$

For $\epsilon \to 0$, this reduces to (3.47)

2) We calculate the scattering amplitude for fermions. As we know from the discussion in the Bose case, this is given entirely by the exchange of Σ-propagators. These can be extracted from the effective action (3.8) always at $m_0 = 0$ by forming the second functional derivative at Σ_0 [compare the discussion leading to (**??**)]. The quadratic piece in $\Sigma' \equiv \Sigma - \Sigma_0 = \Sigma - M$ is

$$\delta^2\Gamma = -\frac{N}{2}\left[\frac{\Sigma'^2}{g_0} + i\ \mathrm{Tr}\left(\frac{i}{i\partial\!\!\!/ - M}\Sigma'\frac{i}{i\partial\!\!\!/ - M}\Sigma'\right)\right]. \tag{3.49}$$

From this we extract the propagator

$$\dot{\sigma}'\dot{\sigma}' = -\frac{1}{N}\frac{i}{\frac{1}{g_0} + \Pi(q)}, \tag{3.50}$$

where the self-energy of the σ' field is

$$\begin{aligned}\Pi(q) &= i\ \mathrm{tr}\int\frac{d^Dk}{(2\pi)^D}\frac{i}{\not k - M + i\eta}\frac{i}{(\not k - \not q) - M + i\eta}\\ &= -i\ \mathrm{tr}\int\frac{d^Dk}{(2\pi)^D}\frac{[(\not k + M)(k-q+M)]}{(k^2-M^2)\,[(k-q)^2-M^2]}\\ &= -2^{D/2}\int\frac{d^Dk_E}{(2\pi)^D}\frac{k(k-q)_E - M^2}{(k_E^2+M^2)\,[(k-q)_E^2+M^2]}.\end{aligned} \tag{3.51}$$

In the last line we have gone to the euclidean form. The denominator can be treated with the help of the Feynman formula (**??**), and we may write [compare (**??**)]

$$\Pi(q) = -2^{D/2}\int\frac{d^Dk_E}{(2\pi)^D}\int_0^1 dx\frac{k(k-q)_E - M^2}{\left(k_E^2 - 2k_Eq_Ex + q_E^2x + M^2\right)^2}. \tag{3.52}$$

The integrand can be regrouped to

$$\frac{(k-qx)_E^2 + (k-qx)_Eq_E(2x-1) - q_E^2x(1-x) - M^2}{\left[(k-qx)_E^2 + q_E^2x(1-x) + M^2\right]^2}. \tag{3.53}$$

Upon integration, the second term in the numerator will vanish since it is odd in the shifted variable $k - qx$. Therefore, we remain with the integral

$$\begin{aligned}\Pi(q) = -2^{D/2}&\int_0^1 dx\int\frac{d^Dk_E}{(2\pi)^D}\\ &\times\left\{\frac{1}{k_E^2+q_E^2x(1-x)+M^2} - 2\frac{M^2+q_E^2x(1-x)}{\left[k_E^2+q_E^2x(1-x)+M^2\right]^2}\right\}.\end{aligned} \tag{3.54}$$

This can be integrated with the rules (**??**), leading to

$$\begin{aligned}\Pi(q) &= -2^{D/2}\bar S_D\left[\frac{1}{2}\frac{\Gamma(D/2)\Gamma(1-D/2)}{\Gamma(1)} - 2\frac{1}{2}\frac{\Gamma(D/2)\Gamma(2-D/2)}{\Gamma(2)}\right]\\ &\quad\times\int_0^1 dx\left[q_E^2x(1-x)+M^2\right]^{D/2-1} \qquad (3.55)\\ &= -2^{D/2-1}\bar S_D(D-1)\Gamma(D/2)\Gamma\left(1-D/2\right)M^\epsilon\int_0^1 dx\left[\frac{q_E^2}{M^2}x(1-x)+1\right]^{D/2-1}\\ &= -\frac{D(D-1)}{2}b_\epsilon M^\epsilon\int_0^1 dx\left[\frac{q_E^2}{M^2}x(1-x)+1\right]^{-\epsilon/2}\end{aligned} \tag{3.56}$$

where we have introduced the parameter b_ϵ of (3.19). We may now calculate the propagator in terms of the renormalized coupling g

$$\begin{aligned}\dot\sigma'\dot\sigma' &= -\frac{i}{N}\mu^{-\epsilon}\frac{1}{\frac{1}{g_0\mu^\epsilon} - \frac{D(D-1)}{2}b_\epsilon\left(\frac{M}{\mu}\right)^\epsilon\int_0^1 dx\left[\frac{g_\epsilon^2}{M^2}x(1-x)+1\right]^{\epsilon/2}}\\ &= -\frac{i}{N}\mu^{-\epsilon}\frac{1}{\frac{1}{g} + b_\epsilon\left\{1 - \frac{D(D-1)}{2}\left(\frac{M}{\mu}\right)^\epsilon\int_0^1 dx\left[\frac{q_E^2}{M^2}x(1-x)+1\right]^{\epsilon/2}\right\}}.\end{aligned} \tag{3.57}$$

The expression in brackets behaves for small ϵ as

$$
\begin{aligned}
1-\left(1+\frac{\epsilon}{2}\right)(1+\epsilon)\left\{1+\frac{\epsilon}{2}\int_0^1 dx\log\left[\frac{q_E^2}{M^2}x(1-x)+1\right]+\epsilon\log\frac{M}{\mu}\right\}\\
=-\frac{\epsilon}{2}\left\{\left[\int_0^1 dx\log\left(\frac{q_E^2}{M^2}x(1-x)+1\right)\right]+3+2\log\frac{M}{\mu}\right\},
\end{aligned}
\tag{3.58}
$$

so that the denominator of (3.50) becomes, in $D=2$ dimensions,

$$
\frac{1}{g}+\frac{1}{\mu^\epsilon}=\frac{1}{g}+\frac{1}{2\pi}\left\{\int_0^1 dx\log\left[\frac{q_E^2}{M^2}x(1-x)+1\right]+3+2\log\frac{M}{\mu}\right\}.
\tag{3.59}
$$

Let us now check that this result can be expressed completely in terms of M. For this purpose we subtract again the gap equation (3.26) and find

$$
\frac{1}{2\pi}\int_0^1 dx\left\{\log\left[\frac{q_E^2}{M^2}x(1-x)+1\right]+2\right\}
\tag{3.60}
$$

and the σ' propagator becomes indeed:

$$
\dot\sigma'\dot\sigma' \;=\; G_{\sigma'}=-\frac{i}{N}\frac{1}{\dfrac{1}{g_0}+\Pi(q)}
\tag{3.61}
$$

$$
=\;-\frac{i}{N}\frac{2\pi}{\displaystyle\int_0^1 dx\log\left[\frac{q_E^2}{M^2}x(1-x)+1\right]+2}
\tag{3.62}
$$

The integral in the denominator has been performed before [see (**??**)] with the result

$$
J(z)=\int_0^1 dx\log\left[zx(1-x)+1\right]=-2+2\Theta\coth\Theta
\tag{3.63}
$$

where

$$
\Theta=\operatorname{atanh}\sqrt{\frac{q_E^2}{q_E^2+4M^2}},\quad \sinh\Theta=\sqrt{\frac{q_E^2}{4M^2}}.
\tag{3.64}
$$

The section $J(z)$ is monotonously increasing in q^2z_E, with the minimum lying at the origin. There

$$
G'_\sigma|_{q_E^2=0}=-i\frac{\pi}{N},
\tag{3.65}
$$

and this decreases smoothly in size for growing euclidean momentum.

We may now ask whether there exists a scalar ground state in the fermion antifermion scattering amplitude, which is usually called σ particle in the analogy with a resonance of $\pi^\dagger\pi^-$ in the proton proton scattering amplitude which is seen at roughly 700 MeV (and which is the origin of using the name σ for the collective field

$\sigma \sim \bar{\psi}\psi$. This particle would have to manifest itself in a pole in the $\sigma'\sigma'$ at timelike q^2 i. e. of some negative value of $q_E^2 = -s = -M_\sigma^2$. Indeed, the denominator (3.63) is seen to vanish for

$$\delta = 4M^2 \tag{3.66}$$

as can be seen by continuing (3.140) to $0 < S < 4M^2$ using

$$\Theta \coth \Theta = \bar{\Theta} \coth \bar{\Theta} \tag{3.67}$$

In conclusion we have seen that the number of parameters characterizing the theory has remained the same: The bare coupling g_0 which becomes undefined for $\epsilon \to 0$ has turned into the finite fermion mass M which is independent on the particular renormalization procedure. It should be noted that in this way a dimensionless quantity g_0 has been replaced by another quantity with the dimension of a mass. This process is often referred to as dimensional transmutation. It was first observed in the microscopic theory of superconductivity. There are many superconductors with different coupling strenghts g and mass parameters μ (which are a characteristic for the phonon spectrum; see Chapter **??**, but there is only one quantity specific for the superconductivity properties which is the critical temperature T_c. Theories with the same T_c are identical superconductors independent on what g or μ they were derived from. There is one important difference, however, between the different cases: In the fundamental Lagrangian (3.1), μ and $g(\mu)$ were completely irrelevant parameters which were not detectable separately by any physical experiment. Only their combination M is in a superconductor, on the other hand, both quantities are properties of the microscopic substructure and can both be measured. This points at an important physical aspect of the renormalization procedure: Every theory which need coupling constant renormalization has a redundancy in its parameterization via mass parameter μ and renormalized coupling constant renormalization has a redundancy in its parameterization via mass parameter μ and renormalized coupling strength g. This redundancy cannot be resolved at the level of the theory itself. But there may be a more microscopic theory in which both parameters μ and $g(\mu)$ aquire separate physical significance. Until now, theoretical physics has gone precisely this way. Every theory which was considered to be microscopic turned out later to be a phenomenological description of even more microscopic substructures. We may write

$$\Theta = \arctan \sqrt{\frac{s}{s-4M^2}}, \quad \sin \Theta = \sqrt{\frac{s}{4M^2}}. \tag{3.68}$$

Close to $4M^2$ the propagator behaves as

$$G_\sigma = -\frac{2i}{N} \sqrt{\frac{s}{s-4M^2}}. \tag{3.69}$$

Thus we see that there is no proper particle pole at $s = 4M^2$, but only a branch cut which runs from $s = 4M^2$ to infinity which is present in every scattering amplitude and commonly referred to as the elastic cut. Only for finite N will there be a proper bound-state pole before the cut starts.

Let us finally take a short look and see what physics is described for $g < g^*$ or $g^0 > 0$. Here the fermions remain massless and there are long-range correlations. The potential may still be parametrized in the form (3.18), (3.23) with $m_0 = 0$:

$$\begin{aligned} \frac{1}{N} v(\epsilon) &= \frac{\mu^\epsilon}{2}\left[\frac{\epsilon^2}{g_0\mu^\epsilon} - b_\epsilon \left(\frac{\sigma}{\mu}\right)^{2+\epsilon} \mu^2\right] \\ &= \frac{\mu^2}{2}\left\{\frac{\epsilon^2}{g} - b_\epsilon \left[\left(\frac{\epsilon}{\mu}\right)^{2+\epsilon} \mu^2 - \epsilon^2\right]\right\} \end{aligned} \tag{3.70}$$

There remains the arbitrary mass parameter μ with the renormalized coupling g depending on the choice of μ. There is now no fermion mass in terms of which the result can be expressed in a renormalization independent fashion. Nevertheless, it is still possible to substitute the pair of parameters $\mu, g(\mu)$, by a single one whose dimension is mass. For this we may simply turn the sign of Eq. (3.24) and define M by

$$1 \equiv -g_0\mu^\epsilon b_\epsilon \frac{D}{2}\left(\frac{M}{\mu}\right)^\epsilon \tag{3.71}$$

Now v can be rewritten as

$$\frac{1}{N} V = -\frac{M^\epsilon}{2} b_\epsilon \Sigma^2 \left[\frac{D}{2} + \left(\frac{\epsilon}{M}\right)^\epsilon\right] \tag{3.72}$$

and M is a measure for the deviation of the potential from its quadratic shape. The minimum lies at the origin corresponding to the massless fermions.

Notice that this potential exists only for ϵ truly larger than zero. For $\epsilon \to 0$ no finite limit remains which is due to the fact that $1/g_0\mu^\epsilon - b_a$ can only be compensated to become a finite quantity $1/g$ in the limit $\epsilon \to 0$ if g_0 is negative.

If we calculate (3.59) for vanishing fermion mass we obtain

$$\begin{aligned} \dot\sigma'\dot\sigma' &= -\frac{i}{N}\mu^{-\epsilon}\left\{\frac{1}{g_0\mu^\epsilon} - \frac{D(D-1)}{2} b_\epsilon \left(\frac{q_E^2}{\mu^2}\right)^{\epsilon/2} \int_0^1 dx\, [x(1-x)]^{\epsilon/2}\right\}^{-1} \\ &= -\frac{i}{N}\mu^{-\epsilon}\left\{\frac{1}{g_0\mu^\epsilon} - \frac{D(D-1)}{2} b_\epsilon \frac{\Gamma(1+\epsilon/2)^2}{\Gamma(2+\epsilon)} \left(\frac{q_E^2}{\mu^2}\right)^{\epsilon/2}\right\}^{-1} \end{aligned} \tag{3.73}$$

This may be expressed in terms of the auxiliary mass parameter (3.71) as

$$\dot\sigma'\dot\sigma' = \frac{i}{N}\left(\frac{D}{2} b_\epsilon M^\epsilon\right)^{-1} \left[1 + \frac{\Gamma^2(1+\epsilon/2)}{\Gamma(1+\epsilon)} \left(\frac{q_E^2}{M^2}\right)^{\epsilon/2}\right]^{-1}. \tag{3.74}$$

It should be pointed out that if we had calculated the $\dot\sigma'\dot\sigma'$ propagator by expanding around the wrong ground state solution, say $\Sigma'' = 0$ for $g_0 < 0,\ g > g^*$, the resulting propagator would show this mistake. We can see this directly from (3.73) which is singular at euclidean momentum by having an unphysical tachyon

$$g_E^2 = \left[\frac{1}{g_0 b_\epsilon} \frac{\Gamma(2+\epsilon)}{\Gamma^2(1+\epsilon/2)} \frac{2}{D(D-1)} \right]^{2/\epsilon}. \tag{3.75}$$

This may also be expressed in terms of renormalized quantities as

$$\frac{g_E^2}{\mu^2} = \left[\left(1 + \frac{1}{g b_\epsilon}\right) \frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon/2)} \frac{2}{D(D-1)} \right]^{2/\epsilon}. \tag{3.76}$$

When going to Minkowski space this amounts to a particle pole at

$$q^2 = -q_E^2.$$

This is similar to the situation in ϕ^4 theory in four dimensions. Also there we found such a particle with an imaginary mass which travels faster than the speed of light and is therefore unphysical (tachyon). There it appeared for very large g^2, here for very small g^2. Since a tachyon can have states with arbitrary negative energy, there must be another ground state for the theory which lies lower than the $\phi^0 = 0$ field configuration.

It can be argued that for finite N, positive couplings g_0 correspond to another interesting physical phase for which the collective field $\sigma \sim \frac{g_0}{N}\bar{\psi}_a\psi_a$ is no longer appropriate. Instead, a field of the type $g_0\psi_a\psi_a$ does allow for an economic description of this situation. This will becomes clearer after the next section.

There is one more observation we should make in the massive phase. We may express the potential (3.23) also using g^* and find the form

$$\frac{1}{N} v(\Sigma) = \frac{\mu^\epsilon}{2} \frac{1}{g} \left[\left(1 - \frac{g}{g^*}\right) \Sigma^2 + \frac{g}{g^*} \left(\frac{\Sigma}{\mu}\right)^\epsilon \right] \epsilon^2. \tag{3.77}$$

This form exhibits very nicely the unstable origin for $g > g^*$ and the stabilization due to the term $\Sigma^{2+\epsilon}$. The potential looks very similar to the previously discussed ϕ^4 theory. In fact, for $\epsilon \to 2$ $(D \to 4)$ it takes exactly this form. The minimum lies at $\Sigma = \Sigma_0 = M$, where M is the fermion mass

$$\frac{M}{\mu} = \left[\frac{2}{D} \frac{g^*}{g} \left(\frac{g}{g^*} - 1 \right) \right]^{1/\epsilon} \tag{3.78}$$

in terms of which the potential may be written in a natural parametrization

$$\frac{1}{N} v(\Sigma) = \frac{\mu^\epsilon}{2} M^2 \frac{1}{g} \left(\frac{g}{g^*} - 1 \right) \left[-\left(\frac{\Sigma}{M}\right)^2 + \frac{2}{D} \left(\frac{\Sigma}{M}\right)^D \right]. \tag{3.79}$$

We argued before that g, g^* play the role of temperature T and critical temperature T_c in surface layers. There the mass goes with

$$\frac{M}{\mu} \sim \left(\frac{T}{T_c} - 1 \right)^{1/\epsilon} \hat{=} \frac{M}{\mu} \sim \left(\frac{g}{g^*} - 1 \right)^{1/\epsilon}. \tag{3.80}$$

It vanishes at the critical point in which case v takes on a pure power behavior

$$\frac{1}{N}v(\Sigma) \underset{\Sigma\to 0}{\sim} \mu^D \left(\frac{\Sigma}{\mu}\right)^D \tag{3.81}$$

This power can also be seen at arbitrary T if Σ is increased to be much larger than the mass scale M [ultraviolet (UV) limit of the theory].

Note that in the opposite limit small Σ [infrared (IR) limit], the power behavior is

$$\frac{1}{N}v(\Sigma) \underset{\Sigma\to 0}{\sim} -\Sigma^2 \tag{3.82}$$

which corresponds to the $g \to 0$ (free field) limit of the theory. One says, the theory behaves IR free. Such UV and IR power behaviors are typical at a critical point. They have been the subject of extended experimental and theoretical investigation over the past decade. It will be worthwile to dedicate the next chapter to the corresponding physical phenomena.

Before we come to that, let us shortly indicate what happens to this model if there is a fermion mass from the beginning, a case which we discarded for the sake of simplicity. We may assume m_0 to be a positive, since otherwise its sign can be changed by a simple γ_5 transformation under which $m_0\bar{\psi}\psi \to -m_0\bar{\psi}\psi$. Looking at (3.18) we see that the gap equation becomes

$$\frac{\Sigma^0}{g_0\mu^\epsilon} = \frac{D}{2}b_\epsilon \left(\frac{m_0+\Sigma^0}{\mu}\right)^{1+\epsilon} \mu \tag{3.83}$$

which has a solution $\Sigma^0 > 0$ for $g_0 < 0$ and $-m_0 < \Sigma^0 < 0$ for $g_0 > 0$. In other words, for repulsive interaction the mass becomes larger and for attractive interaction smaller. The second derivative is at Σ^0

$$\begin{aligned} \frac{1}{N}v'' &= \mu^\epsilon \left[\frac{1}{g_0\mu^\epsilon} - b_\epsilon \frac{D}{2}(D-1)\left(\frac{m_0+\Sigma^0}{\mu}\right)^{D-2}\right] \\ &= \mu^\epsilon \frac{1}{g_0\mu^\epsilon}\left[1-(D-1)\frac{\Sigma^0}{m_0+\Sigma^0}\right] \\ &= -\frac{1}{g_0}\frac{(D-2)(\Sigma^0+m_0)-(D-1)_{m_0}}{m_0+\sigma^0} \end{aligned} \tag{3.84}$$

which shows the stability regions. The renormalization is affected by introducing $\Delta \equiv m_0 + \Sigma$ (whose equilibrium value $\Delta^0 = m_0 + \Sigma$ is the total fermion mass) such that

$$\frac{1}{N}v(\epsilon) = \frac{\mu^\epsilon}{2}\left[\frac{\Delta^2 - 2\Delta m_0 + m_0^2}{g_0\mu^\epsilon} - b_\epsilon \left(\frac{\Delta}{\mu}\right)^{2+\epsilon} \mu^2\right]. \tag{3.85}$$

We may renormalize the coupling again via (3.21). The term $2\Delta m_0/g_0\mu^\epsilon$ is made finite by defining the renormalized mass as

$$\frac{m_0}{g_0\mu^\epsilon} = \frac{m}{g}, \tag{3.86}$$

i.e.,

$$\frac{m_0}{m} = \frac{g_0\mu^\epsilon}{g} = 1 - g_0\mu_\epsilon b_\epsilon = (1+gb_\epsilon)^{-1}. \tag{3.87}$$

The term $m_0^2/g_0\mu^\epsilon$ is not finite for $\epsilon \to 0$ but this does not matter since it differs from m^2/g by a trivial additive constant in $v(\Sigma)$. Thus we find the renormalized potential

$$\frac{1}{N}v(\epsilon) = \frac{\mu^2}{2}\left\{\frac{(\Delta-m)^2}{g} - b_\epsilon\left[\left(\frac{\Delta}{\mu}\right)^{2+\epsilon}\mu^2 - \Delta^2\right]\right\}. \tag{3.88}$$

The renormalized gap equation becomes

$$\Lambda = gb_\epsilon\left[\frac{D}{2}\left(\frac{M}{\mu}\right)^\epsilon - 1\right]\frac{M}{M-m} \tag{3.89}$$

where we have set Δ^0 equal to the final fermion mass M. Using this, the potential becomes

$$\begin{aligned}\frac{1}{N}v(\Sigma) &= \frac{\mu^2}{2}b_\epsilon\left\{(\Delta-m)^2\left(\frac{D}{2}\left(\frac{M}{\mu}\right)^\epsilon - 1\right)\frac{M}{M-m}\left[\left(\frac{\Delta}{\mu}\right)^{2+\epsilon}\mu^2 - \Delta^2\right]\right\}\\ &= \frac{M^\epsilon b_\epsilon}{\epsilon}\left[(\Delta-m)2\frac{D}{2}\frac{M}{M-m} - \left(\frac{\Delta}{M}\right)^{2+\epsilon}M^2\right]\\ &\quad -\frac{\mu^\epsilon b_\epsilon}{2}\left[(\Delta-m)^2\frac{M}{M-m} - \Delta^2\right]\end{aligned} \tag{3.90}$$

It is now parametrized in terms of two finite mass parameters, initial and final fermion mass m and M respectively.

The Pairing Model and Dynamically Generated Goldstone Bosons

The model discussed in the last section is somewhat uninteresting, since the symmetry which is broken is discrete. It is instructive to consider a slightly modified situation in which there is a spontaneous breakdown of a continuous symmetry. From the Nambu-Goldstone theorem we then expect the occurence of a massless particle. Again we consider N fields ψ_a in $D = 2+\epsilon$ dimensions, but now we take the Lagrangian to be

$$\mathcal{L} = \bar\psi_a\left(i\partial\!\!\!/ - m_0\right)\psi_a + \frac{g_0}{2N}\left(\bar\psi_a C\bar\psi_a^T\right)\left(\psi_b^T C\psi_b\right). \tag{3.91}$$

Here C is the 4×4 matrix of charge conjugation which is defined by

$$C\gamma^\mu C^{-1} = -\gamma^{\mu T}. \tag{3.92}$$

In two dimensions, where the γ-matrices have the explicit form (3.69), we may use $C = \gamma^1$:

$$C = \gamma^1 = -i\sigma^2. \tag{3.93}$$

It is the same matrix which was introduced in the four-dimensional discussion in Eq. (**??**) as the 2×2-submatrix c of the 4×4 charge conjugation matrix C.

Note that

$$\left(\bar\psi_a C \bar\psi_a^T\right)^\dagger = \psi_a^T C \psi_a \tag{3.94}$$

such that $g_0 < 0$ amounts to an attractive potential. Now we introduce a collective field by adding to $\mathcal{L}$ the term

$$\frac{N}{2g_0}\left(\Delta - \frac{g_0}{N}\psi_b^T C\psi_b\right)^2,$$

leading to the partition function

$$\begin{aligned} Z[\eta,\bar\eta] = &\int \mathcal{D}\psi\mathcal{D}\bar\psi\mathcal{D}\Delta \\ &\times \exp\left\{i\int d^Dx\left[\bar\psi_a(i\not\partial - m_0)\psi_a + \frac{1}{2}\left(\Delta^\dagger\psi_a^T C\psi_a^\dagger + \text{c.c.}\right) + \bar\psi\eta + \bar\eta\psi - \frac{N}{2g_0}|\Delta|^2\right]\right\}. \end{aligned} \tag{3.95}$$

In order to integrate out the Fermi fields we rewrite the free part of Lagrangian in the matrix form

$$\frac{1}{2}\left(\psi^T C, \bar\psi\right)\begin{pmatrix} 0 & i\not\partial - m_0 \\ i\not\partial - m_0 & 0 \end{pmatrix}\begin{pmatrix} \psi \\ C\bar\psi^T \end{pmatrix} \tag{3.96}$$

which is the same as $\bar\psi\,(i\not\partial - m_0)\,\psi$ since

$$\begin{aligned} \psi^T C C\bar\psi^T &= -\psi^T\bar\psi^T = \bar\psi\psi \\ \psi^T C \overset{\leftrightarrow}{\not\partial} C\bar\psi^T &= +\psi^T \overset{\leftrightarrow T}{\not\partial}\ \bar\psi^T\bar\psi^T = \bar\psi\not\partial\psi \end{aligned} \tag{3.97}$$

But then the interaction with Δ can be combined with (3.81) in the form

$$\frac{1}{2}\phi_i^T G_\Delta^{-1}\phi \tag{3.98}$$

where

$$\phi = \begin{pmatrix} \psi \\ C\bar\psi^T \end{pmatrix}, \quad \phi^T = \left(\psi^T, \bar\psi C^{-1}\right) \tag{3.99}$$

denotes the doubled fermion field and

$$iG_\Delta^{-1} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} \Delta & i\partial\!\!\!/ - m_0 \\ i\partial\!\!\!/ - m_0 & \Delta^\dagger \end{pmatrix} = -\left(iG_\Delta^{-1}\right)^T \tag{3.100}$$

is the inverse propagator in the presence of the external field Δ. Observe that ϕ is a quasi-real field since ϕ^* is similar to ϕ via

$$\begin{aligned} \phi^* &= \begin{pmatrix} \psi^* \\ C\bar\psi^{T^*} \end{pmatrix} \\ &= \begin{pmatrix} 0 & C\gamma^0 \\ C\gamma^0 & 0 \end{pmatrix} \begin{pmatrix} \psi \\ C\bar\psi^T \end{pmatrix} \\ &= \begin{pmatrix} 0 & C\gamma^0 \\ C\gamma^0 & 0 \end{pmatrix} \phi \end{aligned} \tag{3.101}$$

For a quasi-real field, G_Δ^{-1} must be an antisymmetric matrix in the combined spinor *plus* functional space, which it is indeed:

$$\begin{aligned} \begin{pmatrix} C\Delta & C(\partial\!\!\!/ - m_0) \\ C(\partial\!\!\!/ - m_0) & C\Delta^\dagger \end{pmatrix}^T &= \begin{pmatrix} C^T\Delta & (\partial\!\!\!/^T - m_0)C^T \\ (\partial\!\!\!/^T - m_0)C^T & C^T\Delta^\dagger \end{pmatrix} \\ &= -\begin{pmatrix} C\Delta & C(\partial\!\!\!/ - m_0) \\ C(\partial\!\!\!/ - m_0) & C\Delta^\dagger \end{pmatrix} \end{aligned} \tag{3.102}$$

since $C\partial\!\!\!/^T C^{-1} = \partial\!\!\!/$. The additional negative sign with respect to $C\gamma^{\mu^T} C^{-1} = -\gamma^\mu$ in spinor space comes from the fact that by a partial integration the derivative ∂ is essentially an antisymmetric functional matrix $\frac{1}{2}\left(\vec\partial - \overleftarrow{\partial}\right)$. In momentum space, $iG^{-1}(p',p) = \delta^{(2)}(p'+p)iG^{-1}(p)$ is antidiagonal and an antisymmetric functional matrix. This is necessary to have a nonzero kinetic part in the Lagrangian, which reads in terms of the field $\phi(p)$:

$$\int d^Dp' d^Dp\, \phi(p')iG^{-1}(p',p)\phi(p) = \int d^Dp\, \phi(-p)iG^{-1}(p)\phi(p), \tag{3.103}$$

showing that the functional matrix between the fields must be antisymmetric: $G^T(p,p') = -G(p,p')$ which it does.

We can now perform the functional integral over the fermion fields according to the rule (**??**), leading to

$$Z[j] = \int \mathcal{D}\Delta\mathcal{D}\Delta^\dagger e^{Ni\mathcal{A}[\Delta] + \frac{1}{2}j_a^T G_\Delta j_a} \tag{3.104}$$

where $\mathcal{A}[\Delta]$ is the collective action

$$\mathcal{A}[\Delta] = -\frac{1}{2}|\Delta|^2 - \frac{i}{2}\mathrm{Tr}\log iG_\Delta^{-1} \tag{3.105}$$

and j_a is the doubled version of the external source in analogy to (3.85)

$$j = \begin{pmatrix} \bar{\eta}^T \\ C^{-1}\eta \end{pmatrix} \tag{3.106}$$

This is chosen so that

$$\bar{\psi}\eta + \bar{\eta}\psi = \frac{1}{2}\left(j^T\phi - \phi^T j\right) \tag{3.107}$$

and a quadratic completion can be performed according to

$$\frac{1}{2}\phi^T iG_\Delta^{-1}\phi + \frac{1}{2}(j^T\phi - \phi^T j) = \frac{1}{2}(\phi^T + ij^T G_\Delta^T)iG_\Delta^{-1}(\phi + iG_\Delta j) - \frac{i}{2}j^T G_\Delta j \tag{3.108}$$

Note the sign change in front of $\frac{1}{2}j^T G_\Delta j$ in Eq. (3.104) with respect to the Bose case, in accordance with the negative relative sign of the source term $\frac{1}{2}(j^T\phi - \phi^T j)$. In the limit $N \to \infty$ we obtain from (3.105) the effective action

$$\frac{1}{N}\Gamma[\Delta, \Psi] = \frac{1}{2g_0}|\Delta|^2 - \frac{i}{2}\mathrm{Tr}\log iG_\Delta^{-1} + \frac{1}{N}\bar{\Psi}_a iG_\Delta^{-1}\Psi_a \tag{3.109}$$

in the same way as in the last chapter for the simpler model with a real σ-field.

The ground state has $\Psi = 0$ such that Δ satisfies the gap equation

$$\frac{1}{g_0} = \frac{1}{2}\mathrm{Tr}G_{\Delta^0} \tag{3.110}$$

where we may assume Δ^0 to be real. For simplicity we shall from now on consider only the case of zoo initial mass $m_0 = 0$.

Then the Green's function is inverted as follows

$$G_{\Delta_0}(x,y) = \int \frac{d^D p}{(2\pi)^D} e^{-ip(x-y)} \frac{i}{p^2 - \Delta_0} \begin{pmatrix} \Delta_0 & \not{p} \\ \not{p} & -\Delta_0 \end{pmatrix} \begin{pmatrix} C^{-1} & 0 \\ 0 & C^{-1} \end{pmatrix} \tag{3.111}$$

as we can verify by multiplying with (3.86). Thus the gap equation (3.110) is simply

$$\frac{1}{g_0} = 2^{D/2} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 + M^2} \tag{3.112}$$

where we have introduced the notation

$$M \equiv \Delta_0 \tag{3.113}$$

to indicate the significance of Δ_0 as a spontaneously generated fermion mass. Also, we have taken the trace in Dirac space to be $2^{D/2}$ in D dimensions.

The integral can be performed just as before and we find

$$\frac{1}{g_0\mu^\epsilon} = -b_\epsilon \frac{D}{\epsilon}\left(\frac{M}{\mu}\right)^\epsilon. \tag{3.114}$$

The effective potential is now for $\Psi = 0$

$$\frac{1}{N}v(\Delta) = \frac{1}{2g_0}|\Delta|^2 + \frac{i}{2}\int \frac{d^Dp}{(2\pi)^D}\log\begin{pmatrix} -\Delta^\dagger & \not p \\ \not p & -\Delta \end{pmatrix}. \tag{3.115}$$

The eigenvalues of the matrix are easily seen to be $\pm\sqrt{p^2 - |\Delta|^2}$, each of them occurring twice, such that we obtain after a Wick rotation

$$\frac{1}{N}v(\Delta) = \frac{1}{g_0}|\Delta|^2 - \frac{1}{2}2^{D/2}\int \frac{d^Dp}{(2\pi)^p}\log\left(p^2 + |\Delta|^2\right)$$

Performing the integral gives

$$\begin{aligned}\frac{1}{N}v(\Delta) &= \frac{1}{2g_0}|\Delta|^2 - 2^{\frac{D}{2}-1}\bar{S}_D\frac{1}{2}\Gamma(D/2)\Gamma\left(1-\frac{D}{2}\right)\frac{2}{D}|\Delta|^2 \\ &= \frac{\mu^\epsilon}{2}\left[\frac{|\Delta|^2}{g_0\mu^\epsilon} + b_\epsilon\left(\frac{|\Delta|}{\mu}\right)^{2+\epsilon}\mu^\epsilon\right]\end{aligned}$$

from which the gap equation (3.114) can again be recovered by differentiation. Stability is insured for $g_0 < 0$, i.e., for attractive interactions. For $\Delta^0 = 0$, only the trivial solution $\Delta^0 = 0$ is stable.

For $\Delta^0 \neq 0$, we may use (3.114) and express the potential in terms of M rather than the bare coupling constant g_0:

$$\frac{1}{N}v(\Delta) = \frac{M^\epsilon}{2}b_\epsilon\left[\frac{D}{2}|\Delta|^2 - \left(\frac{|\Delta|}{M}\right)^D\right]. \tag{3.116}$$

As before, $v(\Delta)$ can be rewritten in terms of the renormalized coupling

$$\frac{1}{g_0\mu^\epsilon} = \frac{1}{g} - b_\epsilon, \tag{3.117}$$

in the alternative form

$$\frac{1}{N}v(\Delta) = \frac{\mu^\epsilon}{2}\left\{\frac{|\Delta|^2}{g} - b_\epsilon\left[1 - \left(\frac{|\Delta|}{M}\right)^{D-2}\right]|\Delta|^2\right\}. \tag{3.118}$$

From either expression, we may extract the following limit $\epsilon \to 0$ as:

$$\begin{aligned}\frac{1}{N}v(\Delta) &= \frac{1}{2\pi}\left(\log\frac{|\Delta|}{M} - \frac{3}{2}\right)|\Delta|^2 \\ &= \frac{1}{2}\left[\frac{|\Delta|^2}{g} + \frac{1}{\pi}|\Delta|^2\log\frac{|\Delta|}{\mu}\right]\end{aligned} \tag{3.119}$$

in analogy to (3.22) and (3.47) .

Let us now study the propagator of the complete Δ-field. For small deviations $\Delta' \equiv \Delta - \Delta_0$ away from the ground state value we find from (3.109) the quadratic term

$$\frac{1}{N}\delta^2\Gamma = -\frac{1}{2}\left\{\frac{|\Delta|^2}{g_0} + \frac{i}{2}\mathrm{Tr}\left[\left(\begin{array}{cc} \Delta'^\dagger & \\ & \Delta' \end{array}\right) G_M \left(\begin{array}{cc} \Delta'^\dagger & \\ & \Delta' \end{array}\right) G_M\right]\right\}. \tag{3.120}$$

The trace term may be written more explicitly as

$$\frac{i}{2}\left[M^2(\Delta'^2+\Delta'^{*2})2^{D/2}\int\frac{d^Dk}{(2\pi)^D}\frac{i}{k^2-M^2}\frac{i}{(k-q)^2-M^2}\right.$$
$$\left. +2|\Delta'|^2\int\frac{d^Dk}{(2\pi)^D}\frac{i}{k^2-M^2}\frac{i}{(k-q)^2-M^2}\mathrm{tr}[\not k(\not k-\not q)]\right]. \tag{3.121}$$

In a Wick-rotated form, this becomes

$$\frac{1}{2}\left\{M^2\left(\Delta'^2+\Delta'^{\dagger 2}\right)\tilde{\Pi}(q_E^2/M^2)+2|\Delta'|^2\left[\Pi(q_E^2/M^2)-M^2\tilde{\Pi}(q_E^2/M^2)\right]\right\}, \tag{3.122}$$

where $\Pi(q_E^2/M^2)$ is the previous self-energy (3.51). The slightly simpler quantity $\tilde{\Pi}\,(q_E^2/M^2)$ stands for

$$\tilde{\Pi}\left(q_E^2/M^2\right) \;=\; i\,2^{D/2}\int\frac{d^Dk}{(2\pi)^D}\frac{i}{k^2-M^2}\frac{i}{(k-q)^2-M^2},$$

and is calculated as follows:

$$\begin{aligned}\tilde{\Pi}\left(q_E^2/M^2\right) &= 2^{D/2}\int\frac{d^Dk_E}{(2\pi)^D}\int_0^1 dx\frac{1}{\left[k_E^2+q_E^2x(1-x)+M^2\right]^2}\\ &= 2^{D/2}\bar{S}\frac{1}{2}\Gamma(D/2)\Gamma(2-D/2)\int_0^1 dx\left[q_E^2x(1-x)+M^2\right]^{\frac{D}{2}-2}\\ &= -\frac{D}{2}b_\epsilon(1-D/2)\int_0^1 dx\left[q_E^2x(1-x)+M^2\right]^{D/2-2}\end{aligned}$$

As a result, the action for the quadratic deviations from Δ_0 can be written as

$$\frac{1}{N}\delta^2\Gamma = -\frac{1}{2}\left[\left(\frac{1}{g_0}+A\right)|\Delta'|^2+\frac{1}{2}B\left(\Delta'^2+\Delta'^{*2}\right)\right], \tag{3.123}$$

with the coefficients

$$\begin{aligned} A &= \Pi(q_E^2/M^2)-\tilde{\Pi}(q_E^2/M^2)\\ &= -\frac{D}{2}b_\epsilon M\left[(D-1)J_1^\epsilon\left(q_E^2/M^2\right)+(1-D/2)J_2^\epsilon(q_E^2/M^2)\right], \end{aligned} \tag{3.124}$$

$$\begin{aligned} B &= \tilde{\Pi}(q_E^2/M^2)\\ &= \frac{D}{2}b_\epsilon(1-D/2)M^\epsilon J_2^\epsilon\left(q_E^2/M^2\right)\end{aligned} \tag{3.125}$$

and the integrals

$$J_1^\epsilon(z) = \int_0^1 dx\,[zx(1-x)+1]^{D/2-1}\,,\; J_2^\epsilon(z) = \int_0^1 dx\,[zx(1-x)+1]^{D/2-2}\,. \qquad (3.126)$$

Thus the propagators of real and imaginary parts of the field Δ' are

$$\dot{\Delta}'_{\rm re}\;\dot{\Delta}'_{\rm re} \;=\; -\frac{i}{N}\frac{1}{\frac{1}{g_0}+A+B} \qquad (3.127)$$

$$\dot{\Delta}'_{\rm im}\;\dot{\Delta}'_{\rm im} \;=\; -\frac{i}{N}\frac{1}{\frac{1}{g_0}+A-B} \qquad (3.128)$$

and for the complex fields Δ', $\Delta'^\dagger$:

$$\dot{\Delta}'^\dagger\;\dot{\Delta}'^\dagger \;=\; -2\frac{i}{N}\frac{1}{\left(\frac{1}{g_0}+A\right)^2-B^2}(-B), \qquad (3.129)$$

$$\dot{\Delta}'\;\dot{\Delta}'^\dagger \;=\; -2\frac{i}{N}\frac{1}{\left(\frac{1}{g_0}+A\right)^2-B^2}\left(\frac{1}{g_0}+A\right). \qquad (3.130)$$

The expressions (3.127)–(3.130) can be made finite by using the gap equation (3.114). The term involving $1/g_0$,

$$\frac{1}{g_0}+A = \frac{D}{2}b_\epsilon M^\epsilon\left\{\left[1-(D-1)J_1^\epsilon\left(q_E^2/M^2\right)\right]-(1-D/2)\,J_2^\epsilon\left(q_E^2/M^2\right)\right\}, \qquad (3.131)$$

depends then only on the parameter M, and remains finite for $\epsilon\to 0$, where it becomes

$$\frac{1}{g_0}+A \to \frac{1}{2\pi}\left\{\left[J_1^0\left(q_E^2/M^2\right)+2\right]-J_2^0\left(q_E^2/M^2\right)\right\}. \qquad (3.132)$$

The other matrix element B needs no renormalization, and has the $\epsilon\to 0$ -limit

$$B \underset{\epsilon\to 0}{\longrightarrow} \frac{1}{2\pi}J_2^0\left(q_E^2/M^2\right). \qquad (3.133)$$

We now observe that there is a zero mass excitation in the imaginary part of the Δ'-field, the component of Δ which points orthogonal to the real ground state value $\Delta^0 = M$. To show this we consider the denominator of the propagator (3.128):

$$\frac{1}{g_0}+A-\;B = -\frac{D}{2}b_\epsilon M^\epsilon\left\{\left[1-(D-1)J_1^\epsilon\left(q_E^2/M^2\right)\right]-(2-D)J_2^\epsilon\left(q_E^2/M^2\right)\right\}. \qquad (3.134)$$

By expanding it in powers of $z = q_E^2/M^2 \approx 0$

$$J_1^\epsilon \;\sim\; 1+\frac{D-2}{12}z+\mathcal{O}(z^2),$$

$$J_2^\epsilon \;\sim\; 1+\frac{D-4}{12}z+\mathcal{O}(z^2), \qquad (3.135)$$

we find

$$\frac{1}{g_0}+A-B = \frac{D}{2}b_\epsilon\left\{\left[1-(D-1)\left(1+\frac{D-2}{12}z\right)\right]-(2-D)\left(1+\frac{D-4}{12}z\right)\right\}+\mathcal{O}(z^2)$$
$$= -\frac{D}{2}b_\epsilon\frac{D-2}{4}z+\mathcal{O}(z^2), \tag{3.136}$$

such that the propagator of (3.128) becomes, expressed with proper Minkowski square momentum $q^2 = -q_E^2$,

$$\dot{\Delta}'_{\rm im}\ \dot{\Delta}'_{\rm im} = -\frac{1}{N}\frac{2}{D(D-2)b_\epsilon}4M^2\frac{i}{q^2} + \text{regular part at } q^2 = 0.$$

Since $b_\varepsilon < 0$ for $D > \varepsilon$ the residue is positive

$$\text{Res} = \frac{1}{N}\frac{2}{D(D-2)b_\epsilon}4M^2 \to \frac{4\pi}{N}M^2, \tag{3.137}$$

so that the propagator exhibits a proper particle pole at $q^2 = 0$. The positive sign is necessary for a positive norm of the corresponding particle state in the Hilbert space.

In the limit $\epsilon \to 0$, the explicit form of (3.136) may be obtained as follows:

$$\frac{1}{g_0} + A - B \underset{\epsilon\to 0}{\to} \frac{1}{2\pi}[J(z) + 2 - 2J_2^0(z)]. \tag{3.138}$$

The integral $J(z)$ is known from Eq. (3.63) as

$$J(z) = -2 + 2\Theta\coth\Theta, \tag{3.139}$$

where

$$\Theta = \text{atanh}\frac{\sqrt{z}}{\sqrt{z+4}}, \quad \tanh\Theta = \sqrt{\frac{z}{z+4}} \tag{3.140}$$

The integral $J_2^0(z)$ can be calculated as

$$J_2^0(z) = \frac{1}{z}\frac{2\sqrt{z}}{\sqrt{z+4}}\text{atanh}\frac{\sqrt{z}}{\sqrt{z+4}} = \frac{2}{z+4}(J(z)+2), \tag{3.141}$$

so that with $q^2 = -q_E^2$

$$\dot{\Delta}'_{\rm im}\ \dot{\Delta}'_{\rm im} = \frac{i}{N}2\pi\frac{4M^2-q^2}{q^2}\frac{1}{J(-q^2/M^2)+2}$$
$$= \frac{i}{N}2\pi\frac{4M^2-q^2}{q^2}\frac{1}{2\theta\coth\theta}. \tag{3.142}$$

The real part has for $\epsilon \to 0$ the propagator

$$\dot{\Delta}'_{\rm re}\ \dot{\Delta}'_{\rm re} = -\frac{i}{N}2\pi\frac{1}{J(-q^2/M^2)+2}. \tag{3.143}$$

When finding the pole at $q^2 = 0$, we have said that this particle was a Nambu-Goldstone boson. In order to justify this association, we have to exhibit the continuous symmetry which has been broken spontaneously by the ground state solution. Looking back at the original Lagrangian (3.91), we see that it is invariant under global gauge transformations

$$\begin{aligned} \psi &\rightarrow e^{i\alpha}\psi, \qquad \alpha = \text{const.} \\ \bar{\psi} &\rightarrow e^{-i\alpha}\psi. \end{aligned} \tag{3.144}$$

Similarly, the collective action (3.104) remains invariant if the collective field transforms as

$$\Delta \rightarrow e^{2i\alpha}\Delta \tag{3.145}$$

This invariance has been used before when we chose a real ground state expectation Δ_0. Any other phase would have given the same physical result. Of course, once this phase is chosen, the invariance (3.144) is gone. Thus the zero-mass particle is indeed a Nambu-Goldstone particle. It corresponds to excitation whose long-wavelength limit reduces to a pure global gauge transformation.

Strictly speaking, this zero mass boson can only exist in dimensions $D > 2$, as follows from a very general theorem due to Coleman.[1] Indeed, we have seen before in the Bose case that fluctuations prevent the spontaneous breakdown of a continuous symmetry, which might be resent at the mean-field level. Thus we may conclude that if fluctuations were included in the collective field, the theory would also exhibit this general feature in two space-time dimensions. In the limit $N \rightarrow \infty$ there are no fluctuations in Δ. Thus Coleman's theorem should be satisfied after including all $1/N$ corrections. Things are more subtle, however. In two dimensios, there exists a critical coupling strength where a quasi-ordered state exists. This will be discussed in Subsection (**??**). The model above was first used in four dimensions by Nambu and Jona-Lasinio to study the spontanoeus breakdown of chiral symmetry.[2]

These authors were the first to point out the existence of a Nambu-Goldstone boson for spontaneously broken continous symmetry. Then inspiration came from the theory of superconductivity which had just been invented by Bardeen, Cooper, and Schrieffer.[3] Actually, in that model one has to turn off magnetism to see the Nambu-Goldstone bosons. The vector potential of the magnetic field removes this mode, thereby acquiring a short range. This is the famous Meissner-Higgs effect, discussed in Section **??**.

The physical interpretation of the field Δ is the following: Due to the attraction for $g_0 < 0$, the fermions form bound state pairs, called, *Cooper pairs*, which are bosons and can form a condensate, just as before the bosonic ϕ fields in ϕ^4 theory. In fact, the effective potential for the Δ-field looks qualitatively very similar to that of the bosonic potential $v(\Delta)$ of the O(N)-symmetric theory for negative m^2.

[1] S. Coleman, Comm. Math. Phys. *31*, 259 (1973)
[2] Y. Nambu and G. Jona-Lasinio, Phys. Rev. *122*, 345 (1961)
[3] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. *108*, 1175 (1957)

The origin is unstable and there is a new minimum at $\Delta^0 \neq 0$ with an arbitrary phase [see (3.98) with $m_0 = 0$]. Just as in the previous model with an interaction $(g_0/2N)\left(\bar{\psi}_a\psi_a\right)^2$, the opposite sign $g_0 > 0$ does not lead to a spontaneous symmetry breakdown, and massless fermions remain massless.

Finally we must justify why we have called the vacuum expectation $\Delta^0 = M$ a fermion mass. Looking back at the collective effective action (??) we see that the Δ^0 fields appear in the form

$$\begin{aligned}&\bar{\Psi}i\partial\!\!\!/\Psi - \frac{1}{2}M\left(\Psi^T C\Psi + \bar{\Psi}C\bar{\Psi}^T\right)\\ &= \frac{1}{2}\left(\Psi^T C, \bar{\Psi}\right)\begin{pmatrix} M & i\partial\!\!\!/ \\ i\partial\!\!\!/ & M\end{pmatrix}\begin{pmatrix}\Psi \\ C\bar{\Psi}^T\end{pmatrix}.\end{aligned} \tag{3.146}$$

There is a simple transformation which brings this to the canonical Dirac form. Let us introduce the γ^5 matrix in 2 dimensions as

$$\gamma^5 \equiv \gamma^0\gamma^1. \tag{3.147}$$

Then

$$\begin{aligned}\Psi' &= \frac{1-\gamma_5}{2}\Psi + \frac{1+\gamma_5}{2}C\bar{\Psi}^T\\ \bar{\Psi} &= \bar{\Psi}\frac{1+\gamma_5}{2} + \Psi^T C\frac{1-\gamma_5}{2}\end{aligned} \tag{3.148}$$

has the property that

$$\begin{aligned}\bar{\Psi}'\Psi &= \frac{1}{2}\left(\Psi^T C\Psi\bar{\Psi} + \bar{\Psi}C\bar{\Psi}^T\right)\\ &\quad -\frac{1}{2}\left(\Psi^T C\gamma_5\Psi - \bar{\Psi}\gamma^5 C\bar{\Psi}^T\right)\end{aligned} \tag{3.149}$$

since

$$\left(\frac{1+\pm\gamma_5}{2}\right)^2 = \frac{1\pm\gamma_5}{2}.$$

But the γ_5 contribution vanishes due to

$$C\gamma_5^T C^{-1} = C\left(\gamma^0\gamma'\right)^T C^{-1} \tag{3.150}$$

$$= -\gamma^0\gamma' = -\gamma_5. \tag{3.151}$$

In particular representation (??), γ_5 becomes σ_3, and the ransformation reads explicitly

$$\Psi' = \begin{pmatrix} -\Psi_1^\dagger \\ \Psi_2\end{pmatrix}, \qquad \bar{\Psi}' = \left(\Psi_2^\dagger, -\Psi_1\right) \tag{3.152}$$

so that have

$$\begin{aligned}
\bar\Psi'\Psi' &= -\Psi_2^\dagger\Psi_1^\dagger - \Psi_1\Psi_2 \\
&= \frac{1}{2}\left(\Psi^T C\Psi + \text{c.c.}\right) \\
&= \frac{1}{2}(\Psi_2,\Psi_1)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} + \text{c.c.} \\
&= -\Psi_1\Psi_2 + \text{c.c.} \qquad (3.153) \\
\bar\Psi\gamma^{\substack{0\\1}} \overset{\leftrightarrow}{\partial}_{\substack{0\\1}} \Psi &= \Psi^\dagger \begin{Bmatrix} 1 \\ \sigma_3 \end{Bmatrix} \partial_{\substack{0\\1}} \Psi \\
&= \Psi_1 \overset{\leftrightarrow}{\partial}_{\substack{0\\1}} \Psi_1^\dagger \pm \Psi_2 \overset{\leftrightarrow}{\partial}_{\substack{0\\1}} \Psi_2^\dagger \\
&= \Psi_1^\dagger \overset{\leftrightarrow}{\partial}_{\substack{0\\1}} \Psi_1 \pm \Psi_2^\dagger \overset{\leftrightarrow}{\partial}_{\substack{0\\1}} \psi_2 \qquad (3.154)
\end{aligned}$$

which may be compared directly with (3.146) using $\gamma^0 = \sigma', \gamma' = -i\sigma^2, e = \gamma'$. In this context it should be mentioned that the whole model could have been written in terms of ϕ' fields from the outset. If we supplement the identity (3.149) by

$$\bar\Psi'\gamma_5\Psi' = -\frac{1}{2}\left(\Psi^T C\Psi - \bar\Psi C\bar\Psi^T\right) + \frac{1}{2}\left(\Psi^T C\gamma_5\Psi + \bar\Psi\gamma_5 C\Psi^T\right), \qquad (3.155)$$

where the second parenthesis again vanishes, we see that the exponent in (**??**) can be written for zero sources and mass as

$$\begin{aligned}
&\bar\Psi'_a i\partial\!\!\!/\Psi_a - \frac{1}{2}\Delta_{\text{re}}\left(\Psi_a^T C\Psi_a + \bar\Psi_a C\bar\Psi_a^T\right) - \frac{1}{2}\text{Im}\,\Delta\left(\Psi_a^T C\Psi_a - \bar\Psi_a C\bar\Psi_a^T\right) \\
&\qquad = \bar\Psi'_a\left(i\partial\!\!\!/ - \sigma - i\pi\gamma_5\right)\Psi_a - \frac{N}{2g_0}(\sigma^2+\pi^2) \qquad (3.156)
\end{aligned}$$

where we have identified

$$\begin{aligned}
\sigma &\equiv \Delta_{\text{re}} \\
\pi &= -\text{Im}\,\Delta. \qquad (3.157)
\end{aligned}$$

Notice that the invariance under global gauge transformation (3.144) becomes in terms of Ψ' fields an invariance under

$$\begin{aligned}
\Psi' &\equiv \begin{pmatrix} -\Psi_1^\dagger \\ \Psi_2 \end{pmatrix} \to \begin{pmatrix} -e^{i\alpha}\Psi_1 \\ e^{-i\alpha}\Psi_2 \end{pmatrix} \\
&= \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}\Psi' = e^{i\alpha\gamma_5}\Psi'. \qquad (3.158)
\end{aligned}$$

Such transformations involving γ_5 are referred to as *chiral* and play an important role in particle physics. Under the chiral transformation, $\bar\Psi'\Psi'$ and $\bar\Psi' i\gamma_5\Psi$ behave like a vector in a plane

$$\begin{aligned}
\bar\Psi'\Psi' &\to \bar\Psi' e^{2i\alpha}\gamma' \qquad (3.159) \\
&= \cos 2\alpha\,\bar\Psi'\Psi' + \sin 2\alpha\,\bar\Psi' i\gamma_5\Psi' \\
\bar\Psi' i\gamma_5\Psi' &\to \cos 2\alpha\,\bar\Psi' i\gamma_5\Psi' - \sin 2\alpha\,\bar\Psi'\Psi'. \qquad (3.160)
\end{aligned}$$

Correspondingly, the transformation (3.145) becomes via (3.155)

$$\begin{pmatrix} \sigma \\ \pi \end{pmatrix} \rightarrow \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{pmatrix} \begin{pmatrix} \sigma \\ \pi \end{pmatrix}, \tag{3.161}$$

so that the form (3.172) is indeed chirally invariant.

The ground state breaks chiral invariance since σ acquires an expectation value $\sigma_0 = M$. The Nambu-Goldstone boson generated by this phase transition is the massless field π. It is for this reason that the chiral invariance is believed to be an important principle of strong interactions among elementary particles. There is a particle in nature, the pion, whose mass is a particle in nature, the pion, whose mass is 135 MeV and lies much lower than any other strongly interacting particle. One therefore interpretes the pion as an almost Nambu-Goldstone particle of the underlying Lagrangian. It was really in this context that Nambu initiated the study of chiral symmetry in particle physics.

Finally, let us remark that the inclusion of an initial fermion mass $m_0 \neq 0$ is possible but will not be cone since it merely makes the discussion more involved while adding little to the understanding of the model.

3.0.1 Relation between Pairing and Gross-Neveu Model

Both models discussed in the last chapter showed spontaneous mass generation in the limit of $B \rightarrow \infty$ only for one sign of the bare coupling constant, the Gross-Neveu model only for an interaction

$$\frac{g_0}{N} \left(\bar{\psi}_a \psi_a \right)^2, g_0 < 0. \tag{3.162}$$

the pairing model only for

$$\frac{g_0}{N} \psi_a^T C \psi_a \bar{\psi}_b c \bar{\psi}_b^T, \; g_0 < 0. \tag{3.163}$$

For particles, the latter case means attraction the first repulsion. In either case, the opposite sing of g_0 leaves the fermions massless for $N \rightarrow \infty$. Let us now point out that these two exactly soluble models seem to be the idealized descriptions of two phases of the $N = 1$ model

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g_0}{2} \left(\bar{\psi} \psi \right)^2 \tag{3.164}$$

as long as $D > 2$.[4] In this model, we still can introduce the collective field σ according to the rules of Chapter 6. But for $N = 1$ the field σ will fluctuate. Suppose now that fluctuations do not completely destroy the fact that for $g_0 < 0$ there is solution in which a mass is generated spontaneously, i.e., that the $N \rightarrow \infty$ limit does give a qualitatively correct description of the system for $g_0 < 0$.

[4] I. Ojima and R. Fukuda, Progr. Theor. Phys. *57*, 1720 (1977)

Then we might tend to believe that also the $N = 1$ version of the pairing model

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g_0}{2} \psi^T C \psi \bar{\psi} C \bar{\psi}^T \tag{3.165}$$

should have a solution for $g_0 < 0$ which resembles that of the $N \to \infty$ limit, i.e., in which there are massive fermions but Cooper pair bound states which carry massless Nambu-Goldstone bosons. But the we may conclude that these solutions are just two different phases of one and the same theory. For this we just note that the interactions in the Lagrangians (3.164) and (3.165) go over into each other by a simple change of sign of the coupling apart from a factors 2. This follows directly by rewriting the interaction in terms of spin up and spin down components of the ψ field $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$:

$$\begin{aligned} (\bar{\psi}\psi)^2 &= \left(\psi_2^\dagger \psi_1^\dagger + \psi_1^\dagger \psi_2\right)^2 \\ &= 2\psi_2^\dagger \psi_1 \psi_1^\dagger \psi_2 \end{aligned} \tag{3.166}$$

Since $\psi_1^2 = \psi_2^2 = 0$, because of anticommutativity while

$$\begin{aligned} |\psi^\dagger C \psi|^2 &= |-\psi_1\psi_2 + \psi_2\psi_1|^2 \\ &= 4(\psi_2\psi_1)^\dagger \psi_2\psi_1 \\ &= -4\psi_2^\dagger \psi_1 \psi_1^\dagger \psi_2. \end{aligned} \tag{3.167}$$

Hence

$$\left(\bar{\psi}\psi\right)^2 = -\frac{1}{2} |\psi^T C \psi|^2 \tag{3.168}$$

such that our conclusion follows.

The $N = 1$ model seems to have two diferent phases: One for $g_0 < 0$ or $g > g^*$ in which case there are massive fermions and a spontaneously broken γ_5 invariance; the system remains symmetric under global gauge transformations. In the second case $g_0 > 0, g < g^*$, there are again massive fermions but, in addition, there are massless Nambu-Goldstone modes due to a spontaneously broken gauge transformation. Physically, the two phases will be distinguished by the long-range correlations which are present in the second phase.

For $N \to \infty$ either one of the phases is the exact and only consistent solution depending on how one distributes the indices over the four Fermi fields.

We pointed out before the analogy between the coupling constant in ths model and the temperature in systems with two space dimensions. Accepting this, the behavior of this model looks very similar to that found experimentally in thin films of ^{4}He. There is a phase transition of a certain temperature T_c. Above T_c there are only short-range correlations (the system is normal). Below there are long-range correlations (the system is super-fluid) due to the goldstone excitations of the condensate. We have said before that in exactly two dimesnions there can be no

Nambu-Goldstone bosons. In a thin film, however, this theorem is circumvented by a finite thickness of the film. This acts just as if the dimension was $D = 2+\epsilon$ with some small number ϵ. Due to this fact the second phase $g_0 > 0$, for which the renormalized $g(T)$ must lie only in a narrow range $(0, g^*)$, [or $(0, T_\epsilon)$] which is of order ϵ does really arise. Moreover, one may expect the effective value of g^*, to grow with the thickness d of the film such that also T_c grows with d. This fact is indeed observed experimentally.

In exactly two dimensions there is a subtlety due to the possibility of macroscopic quantum fluctuations in which quantum vortices and antivortices can form just as the vortex lines in superconductors. They attract each other by a logarithmic potential just like an electron gas by Coulomb interaction. At a temperature T_c, the long-range correlations due to this Coulomb interaction breaks down by screening caused by the dissociation of vortex pairs into free vortices. Thus there is a phase transition even though there can be no condensate. [5]

3.0.2 Comparison with O(N)-Symmetric ϕ^4-Model

After having observed the possibility of spontaneously generating a mass in a massles theory via fluctuations we may look once more back at the scalar ϕ^4 version of the O(N) model in $D = 4 - \epsilon$ dimensions. In the massless case the potential is

$$v(\Phi, \Delta) = \frac{1}{2}\Delta\Phi_a^2 - \frac{N}{4g_0}\Delta^2 + \frac{1}{2}N\bar{S}_D\frac{1}{2}\Gamma(D/2)\Gamma(1-D/2)\frac{2}{D}\Delta^{D/2} \tag{3.169}$$

which may be written as

$$\frac{1}{N}v(\Phi, \Delta) = \frac{1}{2N}\Delta\Phi_a^2 - \frac{1}{4g_0}\Delta^2 + \frac{b_\epsilon}{4}\Delta^{D/2} \tag{3.170}$$

with

$$\begin{aligned} b_\epsilon &= \frac{4}{D}\bar{S}_D\frac{1}{2}\Gamma(D/2)\Gamma(1-D/2) \\ &= \frac{4}{D}\bar{S}_D\frac{1}{\epsilon} \approx \frac{1}{8\pi^2}\frac{1}{\epsilon} \end{aligned} \tag{3.171}$$

The gap equation reads

$$\frac{1}{N}\Phi_a^2 - \frac{1}{g_0}\Delta - b_\epsilon\frac{D}{4}\Delta^{\frac{D}{2}-1} = 0. \tag{3.172}$$

A renormalized coupling constant may be introduced as

$$\frac{1}{g_0\mu^{-\epsilon}} = b_\epsilon + \frac{1}{g} \tag{3.173}$$

[5] J. M. Kosterlitz and D. J. Thouless, J. Phys. C *6*, 1181 (1973)
V. Ambegaokar and S. Teitel, Phys. Rev. B *19*, 1667 (1979).

and v becomes

$$\frac{1}{N}v(\Phi,\Delta) = \frac{1}{2N}\Delta\Phi_a^2 - \frac{\mu^{-\epsilon}}{4}\left\{\frac{\Delta^2}{g} + b_\epsilon\left[\left(\frac{\Delta}{\mu^2}\right)^{-\epsilon/2} - 1\right]\Delta^2\right\}. \tag{3.174}$$

In the renormalized form, $g_0 >$ or $g_0 < 0$ implies $g < g^*$ or $g > g^*$ with $g^* = b_\epsilon^{-1} \approx 8\pi^2\epsilon$. We now see that there is a O(N) symmetric phase with $\Phi_a = \vartheta\Delta^0 \neq 0$ for $g_0 < 0$ where the bosons acquire a mass. For $g_0 < 0$ there is only the solution $\Phi_a = 0$, $\Delta^0 = 0$ which is again a symmetric phase, but contrary to the previous one this is massless. As far as v'' is concerned, both phases are stable. Consider, however, the excitations: In the massless phase with $g < g^*$ we may calculate the propagator $\dot\Sigma'\dot\Sigma'$ from the quadratic variation

$$\delta^2\Gamma[\Phi,\Sigma] = \frac{1}{2}\Sigma'\,\Gamma^{(2)}\,\Sigma' \tag{3.175}$$

where in euclidean space

$$\Gamma^{(2)} = -\frac{N}{2}\left[\frac{1}{g_0} + I(q)\right] \tag{3.176}$$

with [see (**??**), (**??**), and (**??**)]

$$I(q) = \int\frac{d^Dk}{(2\pi)^D}\frac{1}{k_E^2(k+q)_E^2} = \left(1-\frac{D}{2}\right)\frac{D}{k}b_\epsilon\mu^{-\epsilon}\left(\frac{q_E^2}{\mu^2}\right)^{-\epsilon/2} \frac{\Gamma(1-\epsilon/2)^2}{\Gamma(2-\epsilon)} = c_\epsilon\mu^{-\epsilon}\left(\frac{q_E^2}{\mu^2}\right)^{\epsilon/2} \tag{3.177}$$

so that

$$\dot\Sigma'\,\dot\Sigma' = -\frac{2i}{N}\mu^\epsilon\frac{1}{1/g_0\mu^{-\epsilon} + c_\epsilon\left(g_\epsilon^2/\mu^2\right)^{-\epsilon/2}} \tag{3.178}$$

with $c_\epsilon = -\left(1-\frac{D}{2}\right)\frac{D}{4}b_\epsilon \sim \frac{1}{8\pi^2\epsilon}$. We now observe that for $g_0 > 0$, i.e. $g < g^*$, this is a physically acceptable quantity. There are no tachyons. For $g_0 < 0$, however, or $g > g^*$, there is a tachyon pole at

$$\frac{q_E^2}{\mu^2} = \left(\frac{1}{g_0\mu^{-\epsilon}c_\epsilon}\right)^{-2/\epsilon} \tag{3.179}$$

as an indication that we have expanded around the wrong vacuum $\Delta^0 = 0$. We must insert the spontaneously generated mass and find that this solution must be rejected since it has a tachyon pole. The same observation was maded before at

$D = 4$ in which $g^* = O$ where any $g > 0$ is larger than g^*. Alternatively, we can see that in the $D \sim 4$ -equation

$$\frac{1}{g_0 \mu^{-\epsilon}} = \frac{1}{g} - \bar{S}_4 \frac{1}{\epsilon} \tag{3.180}$$

a finite renormalized coupling g can only be achieved in the limit $\epsilon \to 0$ for negative $g_0 < 0$ in which case the ϕ^4 potential turns the wrong way around. In this case the only consistent solution for $\epsilon \to 0$ is the free one with $g = 0$. We now realize the difference with the $N \to \infty$ Fermi case. There were two possible consistent phases, one in which a mass was spontaneously generated which was the phase which was the one with $g > g^*, g_0 < 0$ and another one for $g < g^*$ where the fermions remain massless. Here only the phase which remains massless with $\delta < g^*, g_0 > 0$ is acceptable.

The four dimensional theory has no consistent $m = 0$ ground state, except for $g = 0$. The three dimensional does, however, have one for any $g < g^* \sim 8\pi^2\epsilon$. In the Gross-Neveu model we pointed out the existence of certain power laws in the massive phase. One concerned the physical mass as a function of $g - g^*$ or $T - T^c$ [see (3.79)], the other was a power law for $v(\Sigma) \sim \Sigma^0$ at the critical point $g = g^*$, i.e., at $T = T_c$. For $T \neq T_c$, this power was still found in the UV limit $\Sigma \gg M$, while for $\Sigma \ll M$ we found the IR free point law $v \sim \Sigma^2$.

We now shall see that the O(N)-symmetric ϕ^4 theory shows quite a similar power behavior which, however, is opposite as far as IR and UV limits are concerend. For this the mass parameter m_0^2 has to be concerned as being proportional to $(T/T_c - 1)$, and the critical theory is that with $m_0 = 0$. In order to see this consider the potential (3.169)

$$\begin{aligned} \frac{1}{N} v(\Phi, \Delta) &= \frac{1}{2N} \Delta \Phi_a^2 + \frac{m_0^2}{2g_0} \Delta - \frac{1}{4g_0} \Delta^2 \\ &\quad - \frac{b_\epsilon}{4} \Delta^{D/2} - \frac{m_0^4}{4g_0}, \end{aligned} \tag{3.181}$$

where Δ is the function of Φ^2 for which $v_\Delta = 0$:

$$\frac{1}{N} \Phi_a^2 = \frac{1}{g_0} \Delta - \frac{m_0^2}{g_0} + \frac{b_\epsilon}{2} \frac{D}{2} \Delta^{\frac{D}{2} - \epsilon}. \tag{3.182}$$

Suppose we are in the normal phase $\Phi_a = 0, \Delta^0 \neq 0, m_0^2 \to 0$. Then from (3.163) we see Δ to behave as a function of $m_0^2 \to 0$ as

$$\Delta \sim (m_0^2)^{1/(1-\epsilon/2)}. \tag{3.183}$$

Inserting this into (3.181), we see that the minimal value of v has the power behavior

$$v_{\min} \sim (m_0^2)^{1+1/(1-\epsilon/2)}. \tag{3.184}$$

At the critical point $m_0 = 0$ we have

$$\frac{1}{N}\Phi_a^2 = \frac{1}{g_0}\Delta + b_\epsilon \frac{D}{4}\Delta^{D/2-1}. \tag{3.185}$$

Contrary to the Gross-Neveu model there is no pure power behavior. Only if also $g_0 = 0,\ g = 0$ (free theory) or $g_0 = \infty\ \ (g = g^*)$ a pure power remains:

$$\begin{array}{llll} \Phi_a^2 \sim \Delta, & g_0 = 0 & g = 0 & (\text{or}\ \ \Delta \to \infty), \\ \Phi_a^2 \sim \Delta^{D/2-1}, & g_0 = \infty & g = g^* & (\text{or}\ \ \Delta \to 0). \end{array} \tag{3.186}$$

The same behavior is found at any g for $\Delta \to \infty$ (ultraviolet limit) or $\Delta \to \infty$ (infrared limit), respectively. In the renormalized form of (3.186)

$$\frac{1}{N}\Phi_a^2 = \mu^{-\epsilon}\left\{\frac{1}{g} - b_\epsilon\left[1 - \frac{D}{4}\left(\frac{\Delta}{\mu}\right)^{\epsilon/2}\right]\right\}\Delta, \tag{3.187}$$

it is the arbitrary scale parameter μ which separates these two limits. For small Δ the potential itself behaves as

$$v(\Phi) \sim (\Phi^2)^{1+1/(1-\epsilon/2)}, \tag{3.188}$$

as determined by the first and last term in (3.169). The small-Δ behavior can be collected in the single formula valid for m_0^2

$$v(\Phi) \sim \left[m_0^2\left(\frac{\Phi^2}{m_0^2} + \frac{N}{g_0}\right)\right]^{1+1/(1-\epsilon/2)}, \tag{3.189}$$

which allows from writing (3.182) as

$$\left(\frac{1}{N}\Phi_a^2 + \frac{m_0^2}{g}\right) \sim b_\epsilon \frac{D}{4}\Delta^{D/2-1}, \tag{3.190}$$

valid for small and vanishing Δ and reinserting, this into (3.181). If Φ is interpreted as magnetization M and $m_0^2 \approx (T/T_c - 1)$ as the deviation of the temperature from the critical value, this corresponds to a general power law.

$$v(M) \sim M^{\delta+1} f\left(\frac{T/T^c - 1}{M^{2/\beta}}\right) \tag{3.191}$$

which was first observed experimentally by Widom in magnetic systems. In our case

$$\beta = \frac{1}{2}, \qquad \frac{\delta+1}{2} = \frac{2-\epsilon/2}{1-\epsilon/2}. \tag{3.192}$$

For large Δ (UV) there is again free field power behavior

$$\frac{1}{N}\Phi_a^2 + \frac{m_0^2}{g_0} \sim \Delta \tag{3.193}$$

$$v \sim \Phi^2 \left(\frac{\Phi_a^2}{N} + \frac{m_0^2}{g} \right). \tag{3.194}$$

Noce that there is great similarity with the Gross-Neveu model as far as power behaviors are concerned. But there is a difference as to the scales. In the GN model, there is only one scale, the fermion mass M or the distance, of g from g^* (i.e., T from T_c). The mass M separates UV and IR limits. Once at T_c where $M = 0$ there are pure powers. Away from T_c there are different powers in the UV and IR limit with $M = 0$ agreeing with the UV limit. In the ϕ^4 theory, the massless theory at the point $m_0^2 = 0$ still has freedom in the coupling. It can be anywhere between zero and g^*. At both ends there are pure powers. In between there are powers in the IR and UV limit. If m_0 is taken away from zero, there are more power laws in $m_0^2 \sim (T/T^c - 1)$ for fields and potential.

3.1 Finite-Temperature Properties

It is useful to study also the behavior of the Gross-Neveu model at a finite temperature. The thermal properties of this model will closely resemble those of a superconductor. For this we we confine the imaginary-time variable τ to the interval $\tau \in (0, \hbar\beta)$ with $\beta = 1/k_B T$, and take the fields to be antiperiodic under $\tau \to \tau + \hbar\beta$. Equivalently, we may think of this model as a nonlinear σ-model on an infinitely long spatial strip with antiperiodic boundary conditions, whose width along the τ-axis is β. In the limit $N \to \infty$, we can study the effects of temperature exactly. For simplicity, we consider the model only for a vanishing initial bare mass m_0, The corresponding effective potential of the Σ-field in Eq. (3.15),

$$\frac{1}{N} v(\Sigma) = -\frac{1}{N} \Gamma[\Sigma] = \frac{1}{2g_0} \Sigma^2 - \text{tr}(1) \frac{1}{2} \int \frac{d^D p_E}{(2\pi)^D} \log \left[p_E^2 + \Sigma^2 \right] \tag{3.195}$$

is generalized to finite temperature T by exchanging the momentum integral by a sum over Matsubara frequencies $\omega_m = 2\pi m T/\hbar, \quad m = 0, \ \pm 1 \ , \pm 2, \ \ldots$:

$$\int \frac{d^D p_E}{(2\pi)^D} \to \frac{d^{D-1} p}{(2\pi)^{D-1}} \frac{1}{\hbar\beta} \sum_{\omega_m = -\infty}^{\infty}, \tag{3.196}$$

thus becoming (in natural units with $k_B = 1$ and $\hbar = 1$)

$$\frac{1}{N} v(\Sigma) = \frac{1}{2g_0} \Sigma^2(x) - 2^D \frac{1}{2} \int \frac{d^{D-1} p}{(2\pi)^{D-1}} T \sum_{m=-\infty}^{\infty} \log(\omega_m^2 + p_E^2 + \Sigma^2). \tag{3.197}$$

The gap equation is obtained by minimizing this action and becomes [compare (3.14)]

$$\frac{1}{g_0} \ = \ 2^{D/2} \int \frac{d^{D-1} p_E}{(2\pi)^{D-1}} T \sum_{m=-\infty}^{\infty} \frac{1}{\omega_m^2 + p_E^2 + \Sigma^2}, \tag{3.198}$$

the solution being called Σ_0.

Using the well-known summation formula

$$T\sum_{m=-\infty}^{\infty}\frac{1}{\omega_m^2+\Omega^2}=\frac{1}{2\Omega}\tanh\left(\frac{\Omega}{2T}\right), \tag{3.199}$$

the gap equation becomes

$$\frac{1}{g_0} = 2^{D/2}\int\frac{d^{D-1}p_E}{(2\pi)^{D-1}}\frac{1}{2\Omega}\tanh\left(\frac{\Omega}{2T}\right). \tag{3.200}$$

We shall renormalize this by adding and subtracting, on the right-hand side, the zero-temperature limit:

$$\frac{1}{g_0} = 2^{D/2}\int\frac{d^Dp_E}{(2\pi)^D}\frac{1}{p_E^2+\Sigma^2}+2^{D/2}\int\frac{d^Dp_E}{(2\pi)^D}\frac{1}{2\Omega}\left[\tanh\left(\frac{\Omega}{2T}\right)-1\right]. \tag{3.201}$$

Near two dimensions, the first integral can be written as

$$2^{D/2-1}S_D\Gamma(D/2)\Gamma(1-D/2)(\Sigma^2)^{\frac{D}{2}-1}=b_\epsilon\frac{D}{2}\Sigma^\epsilon\approx\epsilon\mu^\epsilon-\frac{1}{\pi}\left[\log\frac{\Sigma}{\mu}+\frac{1}{2}\right]+\mathcal{O}(\epsilon) \tag{3.202}$$

[recall Eqs. (3.16), (3.19) according to which $b_\epsilon\approx(2/D)(2\pi)^{-D/2}$, and using $\Gamma(1-D/2)\sim 1/\pi\epsilon$]. The renormalized gap equation reads therefore at $\epsilon=0$:

$$\frac{1}{g_R(\mu^2)}=\frac{1}{\pi}\log\frac{\Sigma}{\mu}+\frac{1}{\pi}\int_0^\infty\frac{dp}{\Omega}\left[\tanh\left(\frac{\Omega}{2T}\right)-1\right] \tag{3.203}$$

It is convenient to express the zero-temperature part of this equation without the arbitrary scale parameter μ, using the renormalization invariant mass

$$M\equiv\Sigma_0=\mu\exp\left[-\frac{\pi}{g_R(\mu)}-\frac{1}{2}\right]$$

which solves the zero-temperature gap equation. Then we arrive at the finite equation

$$\frac{1}{\pi}\log\left(\frac{\Sigma}{\Sigma_0}\right)=\frac{1}{\pi}S_1\left(\frac{\Sigma}{2\pi T}\right) \tag{3.204}$$

with the function [bosonic version of this will appear later in Eq. (**??**)]

$$\begin{aligned} S_1\left(\frac{\Sigma}{2\pi T}\right) &= \int_0^\infty\frac{dp}{\Omega}\left[\tanh\left(\frac{\Omega}{2T}\right)-1\right] \\ &= -2\int_0^\infty dp\,\frac{1}{\Omega}\left[e^{\Omega/T}+1\right]^{-1}. \end{aligned} \tag{3.205}$$

This accounts for all finite-temperature effects. This function depends only on the dimensionless ratio Σ/T. We have divided this by one more factor 2π for later convenience. The ratio $\Sigma/2\pi T$ will in the following be denoted by Σ_T, i.e.,

$$\Sigma_T=\frac{\Sigma}{2\pi T}. \tag{3.206}$$

The solution $\Sigma(T) \equiv M(T)$ of (3.204) is now the temperature-dependent fermion mass. For $T = 0$, the function $S_1(\Sigma_T)$ vanishes and $\Sigma(0) = \Sigma_0 = M$. As the temperature rises, the fermion mass $M(T)$ decrease, until it vanishes at a certain critical temperature T_c. The value of T_c is found by assuming $\Sigma(T)$ to be small, and approximating the right-hand side of (3.205) by

$$2\int_0^\infty \frac{dp}{2\pi}\left[\frac{1}{p}\tanh\left(\frac{p}{2T}\right) - \frac{1}{\sqrt{p^2+\Sigma^2}}\right]. \tag{3.207}$$

Integrating the first term by parts gives

$$\frac{1}{\pi}\left\{\log\left(\frac{p}{2T}\right)\tanh\left(\frac{p}{2T}\right)\Big|_0^\infty - \int_0^\infty dx \log x \cosh^{-2} x\right\}. \tag{3.208}$$

The integral is convergent and gives $-\log(4e^\gamma/\pi)$ where $\gamma = 0.577\ldots$ is Euler's number. This follows at once from the formula

$$\int_0^\infty dx\, x^{\mu-1}\cosh^{-2} ax = \frac{4}{(2a)^\mu}\left(1-2^{2-\mu}\right)\Gamma(\mu)\zeta(\mu-1) \tag{3.209}$$

in the limit $\mu \sim 1$, using $\zeta(0) = -1/2$ and $\zeta'(0) = -\frac{1}{2}\log(2\pi)\Gamma'(1) = -\gamma$. The second term in (3.207) can be integrated directly with the result $\operatorname{asinh}(p/\Sigma) = \log\left[p/\Sigma + \sqrt{p^2/\Sigma^2+1}\right]$. Hence we find for (3.207)

$$\frac{1}{\pi}\left\{\left[\log\left(\frac{p}{2T}\right)\tanh\left(\frac{p}{2T}\right) - \log\left(\frac{p}{\Sigma}+\sqrt{\frac{p^2}{\Sigma^2}+1}\right)\right]_0^\infty + \log\left(\frac{4e^\gamma}{\pi}\right)\right\} = \frac{1}{\pi}\log\left(\Sigma\frac{2e^\gamma}{\pi T}\right), \tag{3.210}$$

and (3.205) determines the critical temperature by the equation

$$\log\left(\frac{\Sigma}{\Sigma_0}\right) = \log\left(\Sigma\frac{2e^\gamma}{\pi T}\right), \tag{3.211}$$

or

$$T_c = 2\Sigma_0\frac{e^\gamma}{\pi} = 2M\frac{e^\gamma}{\pi}. \tag{3.212}$$

At this temperature, the fermion mass $M(T)$ vanishes.

In order to study the full behavior of $M(T)$ as a function of T, the right-hand side of the gap equation (3.204) has to be evaluated numerically. For this purpose, we shall derive a more useful form of the gap equation. Let us go back once more to the original form (3.198) and rewrite it for $\epsilon \sim 0$ as

$$\begin{aligned}\frac{1}{g_0} - 2^{D/2}\int\frac{d^Dp}{(2\pi)^D}\frac{1}{p^2+\Sigma^2} &= \frac{1}{g} - b_\epsilon\mu^\epsilon + \frac{1}{2\pi}\left[\log\left(\frac{\Sigma^2}{\mu^2}\right)+\frac{1}{2}\right]\\ &= \frac{1}{\pi}S_1(\Sigma_T) = T\sum_{m=0}^\infty\frac{1}{\sqrt{\omega_m^2+\Sigma^2}} - \int_0^\infty\frac{d\omega_m}{2\pi}\frac{1}{\sqrt{\omega_m^2+\Sigma^2}}.\end{aligned} \tag{3.213}$$

This can again be written as in (3.204),

$$\log\left(\frac{\Sigma}{\Sigma_0}\right) = S_1(\Sigma_T), \tag{3.214}$$

where S_1 is found in the form

$$S_1(\Sigma_T) = \left(\sum_{m=0}^{\infty} - \int_{-1/2}^{\infty} dm\right) \frac{1}{\sqrt{\left(m+\frac{1}{2}\right)^2 + \Sigma_T^2}}, \tag{3.215}$$

where we have used again the dimensionless ratio $\Sigma_T = \Sigma/2\pi T$. It is useful to reorganize the sum as follows:

$$\begin{aligned} S_1(\Sigma_T) &= \sum_{m=0}^{\infty} \left(\frac{1}{\sqrt{\left(m+\frac{1}{2}\right)^2 + \Sigma_T^2}} - \frac{1}{m+\frac{1}{2}}\right) \\ &+ \sum_{m=0}^{\infty} \frac{1}{m+\frac{1}{2}} - \int_{-1/2}^{\infty} dm \frac{1}{\sqrt{\left(m+\frac{1}{2}\right)^2 + \Sigma_T^2}}. \end{aligned} \tag{3.216}$$

The integral up to some large $m = M$ gives $\operatorname{arcsin}(M/\Sigma_T) \to \log(2M/\Sigma_T)$. The sum over $1/\left(m+\frac{1}{2}\right)$ is $\psi(M+1/2)$ and has the limit $\gamma + \log 2 + \log M$. Thus we obtain for $S_1(\Sigma_T)$ the convergent sum

$$\begin{aligned} S_1(\Sigma_T) &= \sum_{m=0}^{\infty} \left[\frac{1}{\sqrt{\left(m+\frac{1}{2}\right)^2 - \Sigma_T^2}} - \frac{1}{m+\frac{1}{2}}\right] + \log(2e^{\gamma}\Sigma_T) \\ &\equiv \tilde{S}^1(\Sigma_T) + \log(2e^{\gamma}\Sigma_T). \end{aligned} \tag{3.217}$$

The logarithm of Σ_T cancels a similar term on the left-hand side of the gap equation (3.204), and using the connection (3.212) between Σ_0 and the critical temperature, and substituting M for Σ_0, we obtain the gap equation in a form most suitable for a numerical evaluation:

$$\log\left(\frac{T}{T_c}\right) = \tilde{S}_1(\Sigma_T). \tag{3.218}$$

This can be used to calculate T/T_c and $M(T) = \Sigma = 2\pi T \Sigma_T$ as a function of Σ_T. The resulting function $M(T)$ is plotted in Fig. 3.1. It is quite easy to calculate the

FIGURE 3.1 Solution of the temperature dependent gap equation, showing the decrease of the fermion mass $M(T) = \Sigma(T)$ with increasing temperature T/T_c.

way in which $M(T)$ vanishes as T approaches the critical temperature. We simply expand

$$\begin{aligned} \tilde{S}_1(\Sigma_T) &= \sum_{k=1}^{\infty} \binom{-1/2}{k} \Sigma_T^{2k} \sum_{m=0}^{\infty} \frac{1}{\left(m+\frac{1}{2}\right)^{2k+1}} \\ &= -\frac{\Sigma_T^2}{2} \sum_{m=0}^{\infty} \frac{1}{\left(m+\frac{1}{2}\right)^3} + \frac{3}{8}\Sigma_T^4 \sum_{m=0}^{\infty} \frac{1}{\left(m+\frac{1}{2}\right)^5} - \dots . \end{aligned} \tag{3.219}$$

Expanding near T_c the logarithm as $\log(T/T_c) \sim T/T_c - 1$, we find in a first approximation

$$M(T) = \Sigma \approx \pi T_c \sqrt{\frac{8}{7\zeta(3)}} \sqrt{1 - \frac{T}{T_c}}. \tag{3.220}$$

In the opposite limit of low temperatures, the series (3.219) converges very slowly. It is, however, easy to find out how Σ behaves near $T = 0$ by expanding in (3.205)

$$\tanh\left(\frac{\Omega}{2T}\right) - 1 = 2\sum_{\tilde{m}=1}^{\infty} (-)^{\tilde{m}} e^{-\tilde{m}\Omega/T}. \tag{3.221}$$

Using the integral

$$\int_{-\infty}^{\infty} dp\, e^{\tilde{m}\sqrt{p^2+\Sigma^2}/T} = 2K_0\left(\tilde{m}\Sigma/T\right), \tag{3.222}$$

where $K_0(z)$ is the associated Bessel function, we find the alternative expression

$$\log\left(\frac{\Sigma}{\Sigma_0}\right) = S_1(\Sigma_T) = \frac{1}{\Sigma_T} 2 \sum_{\tilde{m}=1}^{0} (-)^{\tilde{m}} K_0\left(2\pi\tilde{m}\Sigma_T\right). \tag{3.223}$$

For small T, $K_0\left(\tilde{m}\Sigma/T\right)$ has the asymptotic behaviour

$$K_0\left(\frac{\tilde{m}\Sigma}{T}\right) \sim \sqrt{\frac{\pi}{2\tilde{m}\Sigma/T}} e^{-\tilde{\Sigma}/T}, \tag{3.224}$$

so that we can expand

$$\Sigma = \Sigma_0 - \sqrt{\frac{2\pi}{\Sigma_0/T}} e^{-\Sigma/T} + \mathcal{O}\left(e^{-2\Sigma/T}\right) \tag{3.225}$$

and see that Σ approaches its $T = 0$ -value Σ_0 exponentially fast from below (see Fig. 3.1).

It is instructive to go through the same discussion once more in D dimensions. For this it is convenient to rewrite the gap equation (3.213) in accordance with the general procedure of dimensional regularization in Section **??** as

$$\frac{1}{g_0} - 2^{D/2} \int_0^{\infty} \int \frac{d^D p}{(2\pi)^D} e^{-\tau(p^2+\Sigma^2)} = \frac{\Sigma^{D-2}}{\pi} 2^{1-D/2} \pi^{(1-D)/2} \Gamma(3/2 - D/2) S_1(\Sigma_T). \tag{3.226}$$

The left-hand side corresponds to the zero-temperature gap equation and is integrated directly to

$$\frac{1}{g_0} - b_\Sigma \frac{D}{2}\Sigma^\epsilon = \frac{\mu^\epsilon}{g(\mu)} - b_\epsilon\left(\frac{D}{2}\Sigma^\epsilon - \mu^\epsilon\right), \tag{3.227}$$

while

$$\begin{aligned} S_1(\Sigma_T) &= \pi\Sigma^{2-D}2^{D-1}\pi^{(D-1)/2}\left[\Gamma\left(\frac{3}{2}-\frac{D}{2}\right)\right]^{-1} \\ &\times\int_0^\infty d\tau\int\frac{d^{D-1}p}{(2\pi)^{D-1}}\left(T\sum_{m=-\infty}^{\infty}-\int_{-\infty}^{\infty}d\omega_r\right)\exp\left[-\tau\left(\omega_m^2+p^2+\Sigma^2\right)\right]. \end{aligned} \tag{3.228}$$

The p-integrals can now be done with the result

$$\frac{1}{g_0} - 2^{D/2}\frac{1}{2^D}\pi^{D/2}\int_0^\infty\frac{d\tau}{\tau^{D/2}}e^{-\tau\Sigma^2} = \Sigma^{D-2}2^{1-D/2}\tau^{\frac{1}{2}-\frac{D}{2}+}\Gamma\left(\frac{3}{2}-\frac{D}{2}\right)S_1(\Sigma_T), \tag{3.229}$$

where

$$S_1(\Sigma_T) = 2\pi\Sigma^{2-D}\left[\Gamma\left(\frac{3}{2}-\frac{D}{2}\right)\right]^{-1}\int_0^\infty\frac{d\tau}{\tau^{(D-2)/2}}\left(T\sum_{m=0}^{\infty}-\int_0^\infty d\omega_m\right)e^{-\tau(\omega_m^2+\epsilon^2)}. \tag{3.230}$$

By performing the τ-integral in $S_1(\Sigma_T)$, we find

$$\begin{aligned} S_1(\Sigma_T) &= 2\pi\Sigma^{2-D}\left(T\sum_{m=0}^{\infty}-\int_0^\infty d\omega_m\right)(\omega_m^2+\Sigma^2)^{(D-3)/2} \\ &= \Sigma_T^{2-D}\left(\sum_{m=0}^{\infty}-\int_{-1/2}^{\infty}dm\right)\left[\left(m+\frac{1}{2}\right)^2+\Sigma_T^2\right]^{(D-3)/2}. \end{aligned} \tag{3.231}$$

Let us expand the sum over m formally. Its its contributions to $S_1(\Sigma_T)$ is

$$S_1(\Sigma_T)|_{\rm sum\ part} = \Sigma_T^{2-D}\left[1+\sum_{k=D}\binom{(D-3)/2}{k}\Sigma_T^{2k}\zeta\left(2k+3-D,1/2\right)\right]. \tag{3.232}$$

The integral over ω_m in (3.231) adds to this the $T=0$ -limit

$$-2^{D/2-1}\pi^{(D-1)/2}\left[\Gamma\left(\frac{3}{2}-\frac{D}{2}\right)\right]^{-1}b_\epsilon\frac{D}{2}\pi = -\left(\frac{\sqrt{\pi}}{2}\right)\left[\Gamma\left(\frac{3}{2}-\frac{D}{2}\right)\right]^{-1}\Gamma\left(1-\frac{D}{2}\right). \tag{3.233}$$

For D near an even dimension $\bar{D}$, say $D=\bar{D}+\epsilon$, the kth term with $k=\bar{D}/2-1$ has an $1/\epsilon$-singularity form $\zeta\left(1+\bar{D}-D,1/2\right)\sim(1/\epsilon)\left[1+\epsilon\psi\left(1/2\right)\right]$. This is cancelled by a singularity of opposite sign in the b_ϵ-term. Observing that

$$
\begin{aligned}
&\Sigma_T^{2-D}\left(\begin{array}{c}(D-3)/2\\ k\end{array}\right)_{k=\frac{\bar{D}}{2}}\Sigma_T^{\bar{D}-2}\zeta\left(1-\epsilon,\frac{1}{2}\right)\\
&=\frac{\Gamma((D-2)/2)}{\Gamma(\bar{D}/2)\Gamma(1/2+\epsilon/2)}\Sigma^{\epsilon}\zeta\left(1-\epsilon,\frac{1}{2}\right) \qquad (3.234)\\
&=\Big[(3/2-D/2)\,\Gamma(\bar{D}/2)\Gamma(1/2)\Big]^{-1}\\
&\frac{\pi}{\cos\pi\left(\bar{D}/2-1\right)}-\frac{-1}{\epsilon}\left\{1+\frac{\epsilon}{2}\left(-2\log\Sigma_T+\psi(1/2)\right)\right\}\\
&\frac{(-1)^{\tau/2}}{\Gamma(\bar{D}/2)}\frac{\sqrt{\pi}}{\epsilon}\left\{1+\frac{\epsilon}{2}\left(-2\log\Sigma_T+\psi(1/2)-\psi\left(3/2-\bar{D}/2\right)\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\frac{1}{2}\right)\sqrt{\pi}\left[\Gamma\left(\frac{3}{2}-\frac{D}{2}\right)\right]^{-1}\Gamma\left(1-\frac{D}{2}\right)\\
&=-\left(\frac{1}{2}\right)\sqrt{\pi}\left[\frac{\Gamma}{3/2-\bar{D}/2}\right]^{-1}(-)^{\bar{D}/2}\frac{\Gamma(1+\epsilon/2)\Gamma(1-\epsilon/2)}{\Gamma(D/2)}\frac{2}{\epsilon}\\
&=-\frac{(-)^{\bar{D}/2}}{\Gamma(\bar{D}/2)\Gamma(3/2-\bar{D}/2)}\frac{\sqrt{\pi}}{\epsilon}\left\{1-\frac{\epsilon}{2}\left(\psi(\bar{D}/2)+\psi\left(\frac{3}{2}-\frac{\bar{D}}{2}\right)\right)\right\}, \qquad (3.235)
\end{aligned}
$$

and adding the two terms gives

$$
\begin{aligned}
&-\frac{(-)^{\bar{D}/2}}{\Gamma(\bar{D}/2)\Gamma(3/2-\bar{D}/2)}\frac{\sqrt{\pi}}{2}\left\{2\log\Sigma_T-\psi\left(\frac{1}{2}\right)-\psi\left(\frac{\bar{D}}{2}\right)\right\}\\
&=-\frac{(-)^{\bar{D}/2}}{\Gamma(\bar{D}/2)\Gamma(3/2-D/2)}\frac{\sqrt{\pi}}{2}\left\{\log(2e^{\gamma}\Sigma_T)-\left(\psi\left(\frac{\bar{D}}{2}\right)+\gamma\right)/2\right\}, \qquad (3.236)
\end{aligned}
$$

where for $\bar{D}=2$, $\psi(\bar{D}/2)+\gamma=0$ and for $\bar{D}>2$, $\psi(\bar{D}/2)+\gamma=1+\frac{1}{\bar{D}/2}+\ldots\frac{1}{\bar{D}/2-1}$. Altogether we have for even D

$$
\begin{aligned}
S_1(\Sigma_T) \;=\; &-\frac{(-)^{\bar{D}/2}}{\Gamma(\bar{D}/2)\Gamma(3/2-\bar{D}/2)}\frac{\sqrt{\pi}}{2}\left\{\log(2e^{\gamma}\Sigma_T)-\left(\psi(D/2)+\gamma\right)/2\right\}\\
&+\sum_{k\neq\bar{D}(2-1}^{\infty}\left(\begin{array}{c}(D-3)/2\\ k\end{array}\right)\Sigma_T^{2k+2-D}\zeta\left(2k+3-D,\frac{1}{2}\right). \qquad (3.237)
\end{aligned}
$$

By taking the negative powers of Σ_T^2 out of the sum we can split

$$
\begin{aligned}
S_1(\Sigma_T) \;=\; &-\frac{(-)^{D/2}}{\Gamma(D/2)\Gamma(9/2-D/2)}\frac{\sqrt{\pi}}{2}\left\{\log\left(2e^{\gamma}\Sigma_T\right)-\left(\psi(D/2)+\gamma\right)/2\right\}\\
&+\sum_{k}^{D/2-2}\left(\begin{array}{c}(D-3)/2\\ k\end{array}\right)\Sigma_T^{2k+2-D}\zeta\left(2k+3-D,\frac{1}{2}\right)+\tilde{S}_1(\Sigma_T) \qquad (3.238)
\end{aligned}
$$

where the sum $\tilde{S}_1$ can be obtained from (**??**) via $D/2$ subtractions (**??**)

$$\tilde{S}_1(\Sigma_T) = \Sigma_T^{2-D} \sum_{m=0}^{D} \left\{ \left[\left(m+\frac{1}{2}\right)^2 + \Sigma_T^2 \right]^{(D-3)/2} - \left(m+\frac{1}{2}\right)^{D-3} - (D-3)/2\Sigma_T^2 \left(m+\frac{1}{2}\right)^{D-5} + \dots \right\}. \quad (3.239)$$

This sum is convergent and is a power series in Σ_T^2 with $\tilde{S}_1(0) = 0$. It is the generalization of (**??**) to any even D. Let us now generalize the other expression for $S_1(\Sigma_T)$ (**??**) and (**??**) to arbitrary D. Since $S_1(\Sigma_T)$ gives by finite temperature correction to the gap equation with the normalization

$$\begin{aligned} \Sigma^{-1}\Delta_T \mathrm{tr}(i\partial\!\!\!/ - \Sigma)(0) &= 2^{D/2} \int \frac{d^{D-1}}{(2\pi)^{D-1}} \frac{1}{p^2+\Sigma^2} \\ &= \frac{\Sigma^{D-2}}{\pi} 2^{1-D/2} \pi^{(1-D)/2} \Gamma\left(\frac{3}{2}-\frac{D}{2}\right) S_1(\Sigma_T). \end{aligned} \quad (3.240)$$

By comparison with (**??**), we can identify

$$\begin{aligned} S_1(\Sigma_T) &= 2^{D-2}\pi^{(D+1)/2}\Sigma^{2-D} \left[\Gamma\left(\frac{3}{2}-\frac{D}{2}\right)\right]^{-1} \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \frac{1}{\Omega} \left[\tanh\left(\frac{\Omega}{2T}\right) - 1\right] \\ &+ \pi \left[\Gamma\left(\frac{3}{2}-\frac{D}{2}\right)\Gamma\left(D-\frac{1}{2}\right)\right]^{-1} \Sigma^{2-D} \int_0^\infty dp\, p^{D-2} \frac{1}{\Omega} \left[\tanh\left(\frac{\Omega}{2T}\right) - 1\right], \end{aligned} \quad (3.241)$$

where $(1/\Omega)\left[\tanh\left(\frac{\Omega}{\pi}\right) - 1\right]$ is conveniently rewritten as $-2\left[1+e^{\Omega/T}\right]^{-1}$. Expanding this in powers of $e^{-\Omega/T}$ we obtain

$$S_1(\Sigma_T) = 2\pi \left[\Gamma\left(\frac{3}{2}-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-\frac{1}{2}\right)\right]^{-1} \sum_{\tilde{m}=1}^{\infty} (-)^{\tilde{m}} \int_0^\infty dp\, p^{D-2} \Omega^{-1} e^{-\tilde{m}\Omega/T}. \quad (3.242)$$

Using the integral representation for the Bessel function $K_\nu(z)$

$$K_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{\Gamma(1/2)}{\Gamma(\nu+(1/2))} \int_0^\infty ds s^{2\nu} (s^2+1)^{-1/2} e^{-z\sqrt{s^2+1}}, \quad (3.243)$$

we find finally the expansion

$$S_1(\Sigma_T) = \pi \left[\Gamma\left(\frac{3}{2}-\frac{D}{2}\right)\Gamma\left(\frac{1}{2}\right)\right]^{-1} 2^{D/2} \sum_{\tilde{m}=1}^{\infty} (-)^{\tilde{m}} \left(2\pi\tilde{m}\Sigma_T\right)^{1-D/2} K_{D/2-1}(2\pi\tilde{m}\Sigma_T), \quad (3.244)$$

which for $D = 2$ reduces properly to (3.222). Let us now calculate the finite temperature correction to the free energy for all dimensions D. By integrating the identity (**??**) and (**??**) in $\Sigma^2/2$ we see that

$$\left(\frac{1}{N}\right) v(\Sigma_T) = \frac{\Sigma^2}{2g} - \mathrm{Tr}\log(i\partial\!\!\!/ - \epsilon) - \Delta_T \log(i\partial\!\!\!/ - \epsilon) \tag{3.245}$$

with

$$\begin{aligned}\Delta_T \mathrm{Tr}\log(i\partial\!\!\!/ - \Sigma) &= \int \frac{d^{D-1}}{(2\pi)^{D-1}} \frac{1}{2} \left(T \sum_{m=-\infty}^{\infty} - \int_{-\infty}^{\infty} d\omega_m \right) \log\left(\omega_m^2 - \Omega^2\right) \\ &= 2^{D/k} T \int \frac{d^{D-1}}{(2\pi)^{D-1}} \log\left(1 + e^{-\Omega/T}\right).\end{aligned} \tag{3.246}$$

We shall introduce again a function $S_0(\Sigma_T)$ so that

$$\Delta_T \mathrm{Tr}\log(i\partial\!\!\!/ - \Sigma) = -\frac{\Sigma^D}{\pi} 2^{-D/2} \pi^{1/2-D/2} \Gamma\left(\frac{1}{2} - \frac{D}{2}\right) S_0(\Sigma_T), \tag{3.247}$$

and find for $S_0(\Sigma_T)$ the following expressions

$$S_0(\Sigma_T) = 4\pi\Sigma^{-D} \left[\Gamma\left(\frac{1}{2} - \frac{D}{2}\right)\Gamma\left(\frac{D}{2} - \frac{1}{2}\right)\right]^{-1} + T\int_0^\infty dp\, p^{D-2} \log\left(1 + e^{-\Omega/T}\right), \tag{3.248}$$

or alternativeley:

$$S^0(\Sigma_T) = \Sigma_T^{-D} \left(\Sigma_{m=0}^{\infty} - \int_{-1/2}^{0} dm \right) \left[\left(m + \frac{1}{2}\right)^2 + \Sigma_T^2 \right]^{(D-1)/2} \tag{3.249}$$

Since $S_0(\Sigma_T)$ is related to $S_1(\Sigma_T)$ by

$$S_0(\Sigma_T) = \frac{D-1}{2} \Sigma_T^{-D} \int d\Sigma_T^2 \left(\Sigma_T^{D-1} S_1(\Sigma_T)\right) \tag{3.250}$$

we can integrate (**??**) and see that the latter expression is separated into a convergent sum

$$\begin{aligned}\tilde{S}_0(\Sigma_T) &= \Sigma_T^{-D} \sum_{m=0}^{\infty} \Bigg\{ \left[\left(m + \frac{1}{2}\right)^2 + \Sigma_T^2 \right]^{(D-1)/2} \\ &\quad - \left(m - \frac{1}{2}\right)^{D-1} - \left[\frac{D-1}{2}\right] \Sigma_T^2 \left(m + \frac{1}{2}\right)^{D-3} + \dots \Bigg\},\end{aligned} \tag{3.251}$$

with $D/2 + 1$ subtractions, plus a logarithm as well as negative powers of Σ_T:

$$\frac{(-)^{D/2}}{\Gamma(D/2)\Gamma(1/2 - D/2)} \frac{\sqrt{\pi}}{2} \left\{ \log(2^\gamma \Sigma_T) - \left(\psi\left(\frac{D}{2} + \frac{1}{2}\right) + \gamma)/2\right) \right\}. \tag{3.252}$$

Thus we find

$$S_0(\Sigma_T) = 2\pi 2^{D/2}\left[\frac{1}{2}-\frac{D}{2}\Gamma(\tfrac{1}{2})\right]^{-1}\sum_{\tilde m=1}^{\infty}(-)^{\tilde m}\left(2\pi\tilde m\Sigma_T\right)^{-D/2}K_{D/2}(2\pi\tilde m\Sigma_T) \quad (3.253)$$

This has to satisfy $d\Sigma_T^D S_0/d\Sigma_T = -(1-D)\Sigma_T^{D-1}S_1$, and a comparison with (**??**) shows that it does, since

$$\left[z^{D/2}K_{D/2}(z)\right]' = -z^{D/2}K_{D/2-1}(z). \quad (3.254)$$

In the high-temperature limit, we can use the small-z approximation

$$K_{D/2}(z)\sim\frac{1}{2}\Gamma\left(\frac{D}{2}\right)\left(\frac{z}{2}\right)^{-D/2} \quad (3.255)$$

and find

$$S_0(\Sigma_T)\to\pi^{1-D}\Sigma_T^{-D}\left[\frac{\Gamma(D/2)}{\Gamma(1/2-D/2)}\Gamma\left(\frac{1}{2}\right)\right]. \quad (3.256)$$

The sum can be expressed as

$$\left(1-2^{1-D}\right)\Sigma_{\tilde m=1}^{\infty}(\tilde m)^{-D} = \left(1-2^{1-D}\right)\zeta(D). \quad (3.257)$$

For $D=2$ this gives

$$\begin{aligned} S_0(\Sigma_T) &\to -\frac{T^2}{\Sigma^2}\zeta(2) \\ \Delta_T(\Sigma_T) &\to \frac{1}{\pi}T^2\zeta(2) = \frac{-T^2\pi}{6} \end{aligned} \quad (3.258)$$

which is the well-known free energy of a hot (or massless) Fermi Gas in two dimensions. It could have been obtained directly by dimensional regularization of

$$\begin{aligned} \Delta_T &= -\int\frac{dp}{2}T\Sigma_m\log(\omega_m^2+p^2) \\ &= -T\Sigma_m\sqrt{\omega_m^2} = -4\pi T^2\zeta\left(-1,\frac{1}{2}\right) = T^2\frac{\pi}{6}. \end{aligned} \quad (3.259)$$

Since $\zeta\left(-1,\frac{1}{2}\right) = (2^{-1}-1)\zeta(-1) = -\frac{1}{24}$. In four dimensions, the result is

$$\begin{aligned} S_0(\Sigma_T) &= \frac{1}{\pi^3}\Sigma_T^{-4}\left[\Gamma\left(-\frac{3}{2}\right)\sqrt{\pi}\right]^{-1}\sum_{\tilde m=1}^{\infty}(-)^{\tilde m}\tilde m^{-4} \\ &= \Sigma^{-4}2^4\pi T^4\left[\Gamma\left(-\frac{3}{2}\right)\sqrt{\pi}\right]^{-1}\frac{7}{8}\zeta(4) \\ \Delta_T(\Sigma_T) &\to -4T^4\frac{7}{8}\frac{\pi^2}{g^D} \end{aligned} \quad (3.260)$$

which follows also from

$$\begin{aligned}\Delta_T v &= -2\int \frac{d^3p}{(2\pi)^3} T \sum_m \log\left(\omega_m^2 + p^2\right) \\ &= 2\int_0^\infty \frac{d\tau}{\tau} \int \frac{d^3p}{(2\pi)^3} T \sum_m e^{-\tau(\omega_m^2+p^2)} \\ &= (4\pi^{3/2})^{-1}\Gamma\left(-\tfrac{3}{2}\right) T \sum_m = T^4 4\pi^{3/2}\Gamma\left(-\tfrac{3}{2}\right)\zeta\left(-3, \tfrac{1}{2}\right).\end{aligned} \tag{3.261}$$

It is the fermion equivalent to the *Stephan-Boltzmann law* for the free energy (k = internal energy density)

$$f = -\frac{1}{3} = -4\left(\frac{2}{3}\frac{\pi}{60}\right)\frac{7}{8}T^4. \tag{3.262}$$

In proper units

$$f = -\frac{1}{3} = -4\left(\frac{2}{3}\frac{\sigma}{c}\right)\frac{7}{8}T^4 \tag{3.263}$$

where

$$\sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2} \approx 5.67\times 10^{-5}\frac{\mathrm{g}}{\sec^\circ \mathrm{K}^4} \tag{3.264}$$

is the Stephan Boltzmann constant. The factor 4 accounts for the two polarization degree of freedom of each particles and antiparticles and the factor 7/8 for the fermion nature recall that the Black-body radiation of photons has a free energy

$$f = -\frac{1}{3}u = -2\left(\frac{2}{3}\frac{T}{c}\right)T^4 = -2\frac{\pi^2}{90}T^4 \tag{3.265}$$

where the factor 2 accounts for the two polarization degrees of photons. In contrast to the fermion case, there are no extra antiparticles.

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correspond to an imaginary mass, i. e. to a particle which travels with speed faster than light (tachyon). In order to see this one only has to note that the denominator is positive at $q^2 =$ remember $-2m^2 > 0$ and goes to $-\infty$ for $q^2 > +\infty$. This tachyon pole is very much related to the complexity of the effective potential for high values of the field: A tachyon has the property that its energy can be made arbitarily negative. Therefore any ground state calculated from the effective potential can at most be metastable. The high value of q^2 at hwich the tachyon pole appears reflects itself in the high field value for which the effective potential becomes complex. Singularities of this type where first discussed by Landau in quantum electrodynamics QED and are named after him.[6]

Notes and References

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Part V

Liquid Crystals

1

Field Theory of Liquid Crystals

2

Introduction

liquid crystal is a system of rod-like molecules which bahave under translations in the same way as the molecules in an ordinary liquid while their molecular orientations can undergo phase transitions into states of long-range order, a typical property of crystals. In this part of the book we shall focus our attention on molecules whose shape strongly deviates from spherical symmetry but which mechanically have no dipole properties, i.e. a reversal of the direction of the principal axis remains is energetically negligible. Examples for such systems are given by p, p'-azoxyanisole shortly called PAA the formula or p-methoxybenzylidene-p-n-butylaniline, usually

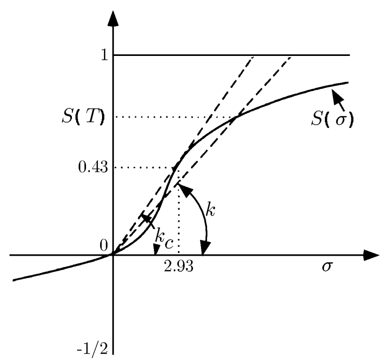

FIGURE 2.1 Molecualar structure of PAA

abbreviated as $MBBA$

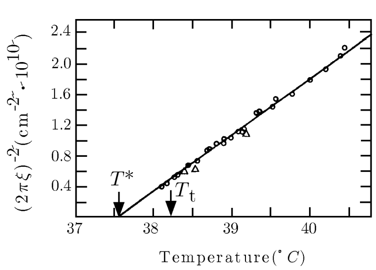

FIGURE 2.2 Molecualar structure of $MBBA$

For more elongated molecules it may happen that the atomic array exhibits a slight screw-like structure. This is the case in many derivatives of steroids whose prime example is cholesterol. Such molecules violate mirror reflection symmetry.

A rather satisfactory description of the long-range correlations in all such system can be given by means of a collective field theory. It is constructed by using the lowest non-vanishing multipole moment of the molecules as a local field characterizing the orientation of the molecules and expanding the free energy in a power series in this field and its derivatives. The thermodynamic properties are then obtained by calculating the partition function for all fluctuating field configurations.

$$Z = \sum_{\substack{field \\ configurations}} e^{-Energy/kT}. \tag{2.1}$$

If the system is not extremely close to a critical point, where fluctuations become important, the partition function can be approximated by the field configuration which extremizes the energy (saddle point method). This is equivalent to considering the collective field as a mean field variable of the Landau type [2].

If a satisfactory microscopic description of the system is known, the collective field theory can be *derived* from the microscopic one via simple path integral techniques. This has been done successfully for other systems such as superconductors [3] and superfluid ^3He [2] [3].

If a satisfactory microscopic description of the system is known, the collective field theory can be *derived* from the microscopic one via simple path integral techniques. This has been done successfully for other systems such as superconductors [2] and superfluid $^3He^{2,3}$. In liquid crystals, however, where the status of microscopic theory is not as satisfactory it is convenient to build up a collective field theory purely on phenomenological grounds.

For this it is best to proceed backwards by starting with a Landau mean field description in terms of a non-fluctuating order parameter [4], [5]. Theoretical statements concerning the long-range properties of the system can be compared with experiment and will be reliable as long as fluctuations are strongly suppressed by the Boltzmann factor in **??**. If critical points are approached, however, where fluctuations do become important these may simply be included by assuming the order parameter itself to be the fluctuating collective field.

2.1 Mean Field Theory of Nematic Order

2.2 Uniform Order

The lowest non-vanishing multipole moment of the elongatd molecules is of the quadrupole type. Thus a traceless symmetric tensor $Q_{\alpha\beta}$ is the appropriate order parameter for a Landau expansion [3] [5]. To lowest approximation, any other physical property which is described by the same type of tensor must be a multiple of this order parameter $Q_{\alpha\beta}$. Examples are the deviations of the dielectric tensor $\epsilon_{\alpha\beta}$ or the magnetic permeability $\mu_{\alpha\beta}$ from the isotropic value

$$\begin{aligned} \delta\epsilon &= \epsilon_{\alpha\beta} - \epsilon_0\delta_{\alpha\beta} \\ \delta_\mu &= \mu_{\alpha\beta} - \mu_0\delta_{\alpha\beta} \end{aligned} \tag{2.2}$$

For if Q vanishes there can be no orientational preference and $\delta^{\alpha\beta}\epsilon = 0, \quad \delta\mu = 0$. For small Q one can expand

$$\begin{aligned} \delta\epsilon_{\alpha\beta} &= M^{\epsilon}_{\alpha\beta\gamma\delta}Q_{\gamma\delta} + \dots \\ \delta\epsilon_{\alpha\beta} &= M^{\epsilon}_{\alpha\beta\gamma\delta}Q_{\gamma\delta} + \dots \end{aligned} \tag{2.3}$$

where from symmetry arguments $M_{\alpha\beta\gamma\delta}$ can only have the general form

$$M_{\alpha\beta\gamma\delta} = a\delta_{\alpha\beta}\delta_{\gamma\delta} + \frac{b}{2}\left(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\right). \tag{2.4}$$

But applied to a symmetric traceless tensor $Q_{\gamma\delta}$, the a term vanishes while the b term simply gives $bQ\alpha\beta$. Hence, the deviations of electric and magnetic permeability are proportional to $Q_{\alpha\beta}$. This makes all properties of the order parameter observable via some couplings

$$\delta H_{\rm int} = \int \frac{1}{2}\left(\xi_E Q_{\alpha\beta}E_\alpha E_\beta + \xi_M Q_{\alpha\beta}H_\alpha H_\beta\right) d^3x. \tag{2.5}$$

It will be convenient to choose the normalization of $Q_{\alpha\beta}$ such that

$$Q_{\alpha\beta} \equiv \delta\epsilon_{\alpha\beta} \quad \text{, i.e.} \quad \xi_E \equiv 1. \tag{2.6}$$

Locally, the symmetric order parameter may be diagonalized by a rotation and has the form

$$Q_{\alpha\beta} = \begin{pmatrix} -Q_1 & & \\ & -Q_2 & \\ & & (Q_1+Q_2) \end{pmatrix}. \tag{2.7}$$

If $Q_1 \neq Q_2$ the order is called biaxial, if $Q_1 = Q_2$ it is called unaxial. Suppose Q_1 and Q_2 are of similar magnitude and both are positive or negative. In the first case the dielectric tensor has two small and one larger component. This corresponds to an ellipsoid of rod-like shape. For the opposite situation, $Q_1 \approx Q_2 < 0$, the order corresponds to a disc. For the molecular systems discussed before we expect the rod-like option to have the lower energy. This will, in fact, emerge one very general grounds, except for small regions of temperature and pressure (close to the critical point in the phase diagram).

Let us now expand the free energy in powers of $Q_{\alpha\beta}$. On invariance grounds we can have terms

$$I_2 = {\rm tr}Q^2 \tag{2.8}$$

$$I_3 = {\rm tr}Q^3 \tag{2.9}$$

$$
\begin{aligned}
I_4 &= \mathrm{tr} Q^4 \quad , \quad I_2{}^2 && (2.10)\\
I_5 &= \mathrm{tr} Q^5 \quad , \quad I_2 I_3 && (2.11)\\
I_6 &= \mathrm{tr} Q^6 \quad , \quad I_3{}^2, I_2{}^3 && (2.12)\\
&\vdots \qquad . && (2.13)
\end{aligned}
$$

But for traceless symmetric tensors there is only one independent invariant of forth as well as of fifth order since

$$I_4 = \frac{1}{2} I_2{}^2 \ , \ I_5 = \frac{5}{6} I_2 I_3. \tag{2.14}$$

At sixth order there are two invariants, which may be taken as $I_3{}^2$ and $I_2{}^3$. Then, for space and time independent order parameter, the free energy may be expanded as [7]

$$f = \frac{1}{2}\left(a_2 I_2 + a_3 I_3 + \frac{a_4}{2} I_2{}^2 + a_5 I_2 I_3 + \frac{a_6}{2} I_3^2 + \frac{a_6'}{3} I_2{}^3 \right) + O(Q^7) \tag{2.15}$$

Phase transitions take place roughly at room temperature. They are caused by the fact that the coefficient of the quadratic invariant vanishes at some temperature T_* and can be expanded in a small neighbourhood as

$$a_2 \approx a_2^0 \left(\frac{T}{T_*} - 1 \right). \tag{2.16}$$

Such a behaviour may be drived from any simple microscopic model [6] (see App. A). The values of T_* and a_2^0 usually depend on pressure.

Also from model calculations one finds that the other coefficients are all of the same order as a_2^0. The only exception is a_3 which is found to be small on experimental grounds

$$\frac{a_3^2}{a_2^0 a_4} \ll 1. \tag{2.17}$$

Moreover, by increasing the pressure to several hundred atmospheres, the unequality can be improved by a factor 4 or more and there is hope that the point $a_3 = 0$ can be reached at some pressure $P*$ [8]. In the following we shall assume the existence of a point (P^*, T_*) in the PT diagram where both a_2 and a_3 vanish. The neighbourhood of this point will be particularly accessible to theoretical investigations. Within the PT diagram, the lines of constant a_2 and a_3 can be used to define a local coordinate frame whose axes cross at $(P_1^*,\ T_*)$ at a non-zero angle (see Fig. 1). In models, the coefficient a_3 is negative at low pressure such that the a_3 axis points roughly into the direction of increasing P. With this mapping in mind we may picture all results directly in the $(a_3,\ a_2)$ plane with the a_3 axis pointing to the right, and only a slight d stortion has to be imagined in order to transfer the phase diagrams to the PT plane.

Before starting it is useful to realize that the expansion 2.15, while being a complicated sixth order polynomial in the eigenvalues $Q_1,\ Q_2$ of the diagonalized order parameter, is a simple third order polynomial in I_2, I_3. It is therefore convenient to treat it directly as such. One only has to keep in mind the allowed range of I_2, I_3: First of all, I_2 is positive definite. Second, I_3 is bounded by

$$I_3{}^2 \leq \frac{1}{6} I_2{}^3. \tag{2.18}$$

The boundary is precisely reached for the uniaxial phase[1], one with $I_3 = I_2^{3/2}/\sqrt{6}$ of positive, rod-like order, the other with $I_3 = -I_2^{3/2}/\sqrt{6}$ of negative, disc-like order. Only between these boundaries are I_2, I_3 independent corresponding to a biaxial phase. The domain is shown in Fig. 2.

With this simplifying view of the expansion 2.15 let us, for a moment, consider the expansion only up to the forth power and look for the minimum in I_2 and I_3. Since $\partial f/\partial I_3 = a_3$ there is no extremum in the allowed domain of Fig. 2 except for $a_3 = 0$. There the transition is of second order: For $a_2 > 0,\ T > T_*$ one hase only $I_2 = 0$ and hence $Q = 0$ which is the isotropic phase. For $a_2 < 0,\ \ T < T_*$ one finds $I_2 = -a_2/a_4$ and the system is ordered. Since I_3 is not specified, the order can be anywhere on the biaxial line in Fig. 2 between the rod-like and disc-like end points. The energy is

$$f = -\frac{a_2{}^2}{4a_4} = -\frac{a_2^{0\,2}}{4a_4}\left(\frac{T}{T_*} - 1\right)^2. \tag{2.19}$$

The specific heat has the usual jump

$$\Delta c = -T\frac{\partial^2 f}{\partial T^2} = -\frac{1}{T_*}\frac{1}{2}\frac{a_2^{0\,2}}{a_4} \tag{2.20}$$

when passing from $T > T_*$ to $T < T_*$.

The situation is quite different in the presence of the cubic term $a_3 \neq 0$. Since there cannot be any minimum for independent I_2 and I_3, it must necessarily lie at the uniaxial boundaries (there must exist a minimum since F is continuous in Q_1, Q_2 and eventually $F \to \infty$ for $Q_1,\ Q_2 \to \infty$). Let us insert the particular uniaxial parametrization

$$\alpha\beta = \varphi\epsilon^{(0)}(\mathbf{n}) \equiv \varphi\sqrt{\frac{3}{2}}\left(n_\alpha n_\beta - \frac{1}{3}\delta_{\alpha\beta}\right) \tag{2.21}$$

where $\mathbf{n}$ is an arbitrary unit vector ($\varphi > 0$ rod-like, $\varphi < 0$ disc-like). Then we find, using $\mathrm{tr}(\epsilon^{(0)2}) = 1,\ \ \mathrm{tr}(\epsilon^{(0)3}) = 1/\sqrt{6}$,

$$f = \frac{1}{2}a_2\varphi^2 + \frac{1}{2\sqrt{6}}a_3\varphi^3 + \frac{a_4}{4}\varphi^4. \tag{2.22}$$

[1]Because of

$$\mathrm{tr}\begin{pmatrix} -Q_1 & & \\ & -Q_1 & \\ & & 2Q_1 \end{pmatrix} = \pm\sqrt{6}\left[\mathrm{tr}\begin{pmatrix} -Q_1 & & \\ & -Q_1 & \\ & & 2Q_1 \end{pmatrix}^2\right]^{3/2} \quad \text{for } \ Q_1 0.$$

This energy is minimal at $\varphi = 0$ with $f = 0$ and at

$$\varphi_{\gtrless} = -\frac{3a_3}{4\sqrt{6}a_4}\left(1 \pm \sqrt{1 - \frac{96a_2a_4}{9{a_3}^2}}\right) \tag{2.23}$$

which are the solutions of

$$f' = \left(a_2 + \frac{3}{2\sqrt{6}}a_3\varphi + a_4\varphi^2\right)\varphi = 0. \tag{2.24}$$

Combining (2.24) with (2.22) we see that the energy at $\varphi_{\gtrless}$ is

$$f = -\frac{1}{4}\varphi^3\left(\frac{a_3}{\sqrt{6}} + a_4\varphi\right). \tag{2.25}$$

The energy vanishes at a point $\varphi \neq 0$, if $\varphi_{\gtrless}$ satisfies

$$\varphi = -\frac{a_3}{\sqrt{6}a_4} \tag{2.26}$$

From (2.23) we see that this happens at a temperature T_c at which

$$a_2 = a_2^0\left(\frac{T_c}{T_*} - 1\right) = \frac{{a_3}^2}{12a_4}. \tag{2.27}$$

At this point the potential has the usual symmetric double-well form entered around $\frac{\varphi}{2}$ (see Fig. 3). Now the transition is of first order: As T passes the temperature T_c which lies above T_*, the order jumps discontinuously from the old minimum at $\varphi = 0$ to the new one at $\varphi = \varphi_>$ (see Fig. 4) entropy changes by

$$\begin{aligned}\Delta s &= -T\left(\left.\frac{\partial f}{\partial T}\right|_{T=T_c+\epsilon} - \left.\frac{\partial f}{\partial T}\right|_{T=T_c-\epsilon}\right) && (2.28)\\ &= \frac{1}{2}\frac{T}{T_*}a_2^0\varphi_>^2 && (2.29)\\ &= -\frac{1}{T_*}\frac{a_2^0a_2}{a_4} && (2.30)\end{aligned}$$

giving a latent heat

$$\Delta q = \frac{T}{T_*}\frac{{a_2^0}^2}{a_4}\left(\frac{T_c}{T_*} - 1\right) \tag{2.31}$$

From 2.23 it is obvious that for $a_3 < 0$ the order is positive uniaxial, for $a_3 A 0$ negative. Thus up to quartic order we arrive a the phase diagram shown in Fig. 4.

Energetically, the higher powers of the free energy are negligible as long as $\varphi_>$ is sufficiently small. From 2.26 and 2.27 we see that at the transition

$$\varphi_> = \sqrt{\frac{2a_2^0}{a_4}}\sqrt{\frac{T_c}{T_*} - 1}. \tag{2.32}$$

Since a_2^0 and a_4 are roughly of equal size, the corrections in the energy are of order $O\left(\sqrt{T_c/T_* - 1}\right)$. Experimentally we shall se that the temperature precocity of the first order transition $T_c/T_* - 1$ is extremely small, usually in the range $\frac{1}{400}$. Thus higher terms in the expansion can change the enerty by a fex percent only.

Also the latent heat 2.28 is supprssed by the factor $\frac{T_c}{T_*} - 1$ with repsect to the natural scale $\frac{a_2^{0\,2}}{a_4}$.

First order transitions with these properties are usually referred to as being weakly of first order.

while for $a_2 > 0$ and close to the critical point $(P^*, \ T_*)$ the higher orders are rather insignificant, they do become relevant for $a_2 < 0$, in particular in some neighbourhood of the $a_3 = 0$ line where the different phases are unspecified. In order to get a qualitative picture let us neglect the a_6' term which could give only slight quantitative changes but which would make the following discussion much more clumsy. Varying f independently with respect to I_2 and I_3 we find the extremality conditions

$$
\begin{aligned}
a_2 + a_4 I_2 + a_5 I_3 &= 0 \\
a_3 + a_5 I_2 + a_6 I_3 &= 0
\end{aligned}
\tag{2.33}
$$

For

$$
a_4 a_6 - {a_5}^2 > 0 \tag{2.34}
$$

or

$$
a_4 a_6 - {a_5}^2 < 0 \tag{2.35}
$$

this can be solved by

$$
\begin{pmatrix} I_2 \\ I_3 \end{pmatrix} = \frac{1}{a_4 a_6 - {a_5}^2} \begin{pmatrix} -a_6 & a_5 \\ a_5 & -a_4 \end{pmatrix} \begin{pmatrix} a_2 \\ a_3 \end{pmatrix}. \tag{2.36}
$$

We shall exclude the accidental equality sign since $a_4, \ a_5, \ a_6$ are rather invariable material constants. The extremum is a minimum only under the condition 2.34. We then have to see whether I_2 and I_3 remain inside the allowed domain ${I_2}^3 \geq 6{I_3}^2$. For this we simply map the position of the boundaries into $(a_2, \ \ a_3)$ plane. On the rod-like and disc-like boundaries,

$$
\begin{aligned}
a_2 &= -a_4 I_2 \mp a_5 \frac{1}{\sqrt{6}} I_2^{3/2} \\
a_3 &= -a_5 I_2 \mp a_6 \frac{1}{\sqrt{6}} {I_2}^{3/2}.
\end{aligned}
\tag{2.37}
$$

We may form two combinations

$$
\begin{aligned}
a_2 a_5 - a_3 a_4 &= \pm (a_4 a_6 - {a_5}^2) \frac{1}{\sqrt{6}} {I_2}^{3/2} \\
a_2 a_6 - a_3 a_5 &= -(a_4 a_6 - {a_5}^2) I_2.
\end{aligned}
\tag{2.38}
$$

Inserting **??** into 2.38 gives

$$a_2 a_5 - a_3 a - 4 = \pm \left(a_4 a_6 - {a_5}^2 \right) \frac{1}{\sqrt{6}} \left[\frac{a_2 a_6 - a_3 a - 5}{-(a_4 a_6 - {a_5}^2)} \right]^{3/2} . \tag{2.39}$$

If the right-hand side were absent, this would give a straight line

$$a_2 = \frac{a_4}{a_5} a_3 \tag{2.40}$$

in the a_2, $_3$ diagram. But we can easily see that the right-hand side gives only a correction of order ${a_3}^{3/2}$ to this result: Inserting the lowest approximation 2.40, the right-hand side becomes

$$\pm \left(a_4 a_6 - {a_5}^2 \right) \frac{1}{\sqrt{6}} \left(-\frac{a_2}{a_4} \right)^{3/2} \tag{2.41}$$

such that, up to order ${a_2}^{3/2}$

$$a_3 a_4 = a_2 a_5 \mp \left(a_4 a_6 - {a_5}^2 \right) \frac{1}{\sqrt{6}} \left(-\frac{a_2}{a_4} \right)^{3/2} + \ldots \quad . \tag{2.42}$$

The two boundary curves are displayed on Fig. 5. Between these brances the order is biaxial with a well determined ratio $\frac{Q_1}{Q_2}$. One may envisage the effect of the higher powers in the free energy expansion as having slightly rotated the vertical degenerate line in Fig. 4 and opened it up into the two branches of Fig. 5 generating a hole domain for the biaxial phase. Since the order parameter moves continuously towards the uniaxial boundary the transition uniaxial to biaxial is of second order.

If the determinant $a_4 a - 6 - {a_5}^2$ becomes smaller, the biaxial region shrinks. For negative sign, it disappears and the two uniaxial regions overlap. Since only one of them can have the lower energy there must be a line at which the transition takes place. This is found most easily by considering the uniaxial energy in the parametrization 2.21 where it reads

$$\begin{aligned} 2f &= a_2 \varphi^2 + \frac{a_3}{\sqrt{6}} \varphi^3 + \frac{a_4}{2} \varphi^4 \\ & \frac{a_5}{\sqrt{6}} \varphi^5 + \frac{a_6}{12} \varphi^6 \end{aligned} \tag{2.43}$$

where $\frac{a_6}{12}$ can be thought of containing also $\frac{a_6'}{3}$ of the last term.

The minimum lies at

$$\left(a_2 + \frac{3}{2\sqrt{6}} a_3 \varphi + a_4 \varphi^2 + \frac{5}{2\sqrt{6}} \varphi^3 + a_6 \varphi^4 \right) \varphi = 0 \tag{2.44}$$

up to a_4 this was solved by

$$\varphi_{\gtrless} = -\frac{3}{4\sqrt{6}} \frac{a_3}{a_4} \left(\pm \sqrt{1 - \frac{96 a_2 a_4}{9 {a_3}^2}} \right) \tag{2.45}$$

and only $\varphi_>$ gave the minimum the other the maximum. As a_5, a_6 are turned on, the maximum may become a minimum in order to see where this happens let us assume a_3 to be very small as compared with $a_2 a_4$. Then φ is given for $a_3 >< 0$ by

$$\varphi = \pm\sqrt{\frac{-a_2}{a_4}}\left(1 + \pm\frac{3}{4\sqrt{6}}\sqrt{\frac{{a_3}^2}{-a_2 a_4}} + \dots\right). \tag{2.46}$$

Inserting this back into the energy we find that the two energies becomes equal at

$$a_2 = a_3\frac{a_4}{a_5} + \frac{1}{\sqrt{6}}{a_3}^{3/2}{a_5}^{-1/2} + O(a_3^2). \tag{2.47}$$

For small a_5 this reduces back to the line $a_3 = 0$. While for $a_5 = 0$ the transition was of second order it is now of first order with a latent heat

$$\Delta q = -\frac{a_3}{2a_4}T\frac{\partial a_2}{\partial T} - T\frac{\partial a_3}{\partial T}\frac{1}{2a_5}(a_2 a_5 - a_3 a_4). \tag{2.48}$$

Let us finally calculate the correction to the isotropic-uniaxial curve 2.27 which was shown in Fig. 4. For small a_3 we find

$$a_2 = \frac{1}{12}\frac{{a_3}^2}{a_4} + \frac{1}{36}\frac{{a_3}^3 a_5}{{a_4}^3} + O({a_3}^4) \tag{2.49}$$

which may be used to calculate a small correction to the latent heat 2.31.

All ordered phases described here are referred to as nematic.

described here are referred to as nematic.

2.3 Bending Energy

The order parameters discussed in the last section were independent of space and time. In the laboratory, such configurations are difficult to realize. External boundaries usually do not permit a uniform order but enforce spatial variations. The system tries, however, to keep the variations as smooth as possible. It exerts resistance to local deformations. In order to parametrize the restoring forces one expands the free energy in powers of the derivatives of the collective field $Q_{\alpha\beta}$. If the fields bend sufficiently smooth, the expansion may be terminated after the lowest derivative.

Due to rotational invariance, there can only be the following *bending energies*

$$f_{\text{bend}} = \frac{b}{2}\partial_\gamma Q_{\alpha\beta}\partial_\gamma Q_{\alpha\beta} + \frac{c_1}{2}\partial_\alpha Q_{\alpha\gamma}\partial_\beta Q_{\beta\gamma} + \frac{c_2}{2}\partial_\alpha Q_{\beta\gamma}\partial_\beta Q_{\alpha\gamma}. \tag{2.50}$$

As far as the total energy $F = \int d^3x f$ is concerned, the latter two terms may be collected into one, say the first among them by substituting $c_1 \to c_1 + c_2 \equiv c$.

In the ordered phase which is usually of the rod-like type we may use the parametrization 2.21 and split the gradient of $Q_{\alpha\beta}$ into variation of the size φ and the direction $\mathbf{n}$.

In the bulk liquid the size of the order parameter φ is caught in the potential minimum at $\varphi_>$ (see Fig. 3) and only the direction $\mathbf{n}$ will vary from point to point. Then we can find from 2.50 the purely directional bending energy.

$$\begin{aligned} f_{\text{bend,dir}} &= \frac{3}{4}\varphi^2 \left[a\partial_\gamma(n_\alpha n_\beta)\partial_\gamma(n_\alpha n_\beta)\right. \\ &\quad \left. + c_1\partial_\alpha(n_\alpha n_\gamma)\partial_\beta(n_\beta n_\gamma) + c_2\partial_\alpha(n_\beta n_\gamma)\partial(n_\alpha n_\gamma)\right] \end{aligned} \tag{2.51}$$

Since ${n_\alpha}^2 = 1$ one may use $n_\alpha \partial_\gamma n_\alpha = 0$ and collect

$$\begin{aligned} f_{\text{bend,dir}} &= \frac{3}{4}\varphi^2 \left\{b2n^2_{\alpha,\beta} + c_1\left[(\partial\mathbf{n})^2 + (\mathbf{n}\partial n)^2\right]\right. \\ &\quad \left. + c_2\left[(\partial_\alpha n_\beta)(\partial_\beta n_\alpha) + (\mathbf{n}\partial n)^2\right]\right\}. \end{aligned} \tag{2.52}$$

We now rewrite

$$n^2_{\alpha,\beta} = (\nabla\cdot n)^2 + (n\cdot(\nabla\times n))^2 + (n\times(\nabla\times n))^2 \tag{2.53}$$

and

$$n_{\alpha,\gamma}n_{\gamma,\alpha} = (\nabla)^2 + \partial_\alpha(n_\beta\partial_\beta n_\alpha) - \partial_\beta(n_\beta\partial_\alpha n_\alpha) \tag{2.54}$$

such that

$$\begin{aligned} f_{bend,dir} &= \frac{3}{2}\varphi^2\left\{\left(b+\frac{c}{2}\right)(\nabla\cdot\mathbf{n})^2 + b\,(\mathbf{n}\cdot(\nabla\times\mathbf{n}))^2\right. \\ &\quad \left. + \left(b+\frac{c}{2}\right)(\mathbf{n}\times(\nabla\times\mathbf{n}))^2\right\} \\ &\quad + c_2\left[\partial_\alpha(n_\beta\partial_\beta n_\alpha) - \partial_\beta(n_\beta\partial_\alpha n_\alpha)\right]. \end{aligned} \tag{2.55}$$

The latter is again a pure surface term.

The coefficients

$$\begin{aligned} K_1 &\equiv K_s = 3\left(b+\frac{c}{2}\right)\varphi^2 \\ K_2 &\equiv K_t = 3b\varphi^2 \\ K_3 &\equiv K_b = 3\left(b+\frac{c}{2}\right)\varphi^2 \end{aligned} \tag{2.56}$$

are known as Frank constants of textural bending. The subscripts $s,\ t,\ b$ stand for splay, twist, and bend and indicate that each term dominates a certain class of distortions of the directional field. They are shown in Fig. 7a-c). The experimental values of $K_{1,2,3}$ lies in the order of 5 to 10×10^{-7} dynes, for example [1]

$$\begin{aligned} &\text{MBBA},\ T\approx 22^0\ C \qquad K_{1,2,3} = (5.3\pm .5\ ,\ 2.2\pm .7\ ,\ 7.45\pm 1.1)\times 10^{-7}\text{dynes} \\ &\text{PAA},\ T\approx 125^0\ C \qquad K_{1,2,3} = (4.5\ ,\ 2.9\ ,\ 9.5)\times 10^{-7}\text{dynes}. \end{aligned} \tag{2.57}$$

For topological reasons the field configurations may have singularities often called defects. In their neighbourhood, also the size φ has spatial variations. The same thing happens near boundaries or at the interface between two phases. The derivative terms for these variations are found from 2.50 by calculating

$$
\begin{aligned}
\partial_\gamma Q_{\alpha\beta}\partial_\gamma Q_{\alpha\beta} &= (\partial\varphi)^2 + (\partial n)^2 \text{terms} \\
\partial_\alpha Q_{\alpha\gamma}\partial_\beta Q_{\beta\gamma} &= \partial_\alpha\varphi\partial_\beta\varphi\frac{1}{2}\left(n_\alpha n_\beta + \frac{1}{3}\delta_{\alpha\beta}\right) \\
& 3\left[(\partial_\alpha n_\alpha)n_\gamma + n_\alpha(\partial_\alpha n_\gamma)\right]\left(n_\beta n_\gamma - \frac{1}{3}\delta_{\beta\gamma}\right)\partial_\beta\varphi + (\partial_n)^2 \text{terms} \\
& \frac{1}{2}\left(\mathbf{n}\partial\varphi\right)^2 + \frac{1}{6}(\partial\varphi)^2 + 2(\mathbf{n}\partial\varphi)\partial\mathbf{n} - (\mathbf{n}\partial n_\gamma)\partial_\gamma{}^\varphi \qquad (2.58) \\
\partial_\alpha Q_{\beta\gamma}\partial_\beta Q_{\alpha\gamma} &= \partial_\alpha\varphi\partial_\beta\varphi\frac{1}{2}(n_\alpha n_\beta + \frac{1}{3}\delta_{\alpha\beta}) \\
& +3\left[(\partial_\alpha n_\beta)n_\gamma + n_\beta(\partial_\alpha n_\gamma)\right]\left(n_\alpha n_\gamma - \frac{1}{3}\delta_{\alpha\gamma}\right)\partial_\beta)\partial_\beta\varphi + (\partial_n)^2 \text{terms} \\
&= \frac{1}{2}(\mathbf{n}\partial\varphi)^2 + \frac{1}{6}(Q\varphi)^2 + 2(\mathbf{n}\partial n_\gamma)\partial_\gamma\varphi - (\mathbf{n}\partial\varphi)\partial\mathbf{n}
\end{aligned}
$$

such that the remaining bending energies are

$$
\begin{aligned}
f_{\text{bend}} &= f_{\text{bend,dir}} + \frac{1}{2}\left(b + \frac{c}{6}\right)(\partial\varphi)^2 + \frac{c}{2}\left(\mathbf{n}\nabla\varphi\right)^2 \\
& + \frac{2c_1 - c_2}{2}\varphi\left(\mathbf{n}\nabla\varphi\right)\left(\nabla\mathbf{n}\right) + \frac{2c_2 - c_1}{2}\varphi\left(\mathbf{n}\nabla n_\alpha\right)\nabla_\alpha\varphi \qquad (2.59)
\end{aligned}
$$

2.4 Light Scattering

In bending energies determine the length scale at which local field fluctuations take place. These in turn are directly observable in light scattering experiments.

Consider at first the region $T > T_c$. There the order parameter vanishes such that the field $Q_{\alpha\alpha\beta}$ fluctuates around zero. If the temperature is sufficiently far above T_c (precisely how far we shall see soon), the quadratic term in the energy strongly confines such fluctuations and we can study their properties by considering only the quadratic term in the free energy

$$
f_2 = \frac{a_2}{2}Q^2 + \frac{b}{2}\left(\partial_\gamma Q_{\alpha\beta}\right)2 + \frac{c}{2}\partial_\alpha Q_{\alpha\gamma}\partial_\beta Q_{\beta\gamma} + \text{surface terms.} \qquad (2.60)
$$

Obviously, b/a_2 and c/a_2 have the dimension of a length square and it is useful to define the squares of the so-called coherence lengths

$$
\begin{aligned}
{\xi_1}^2(T) &\equiv \frac{b}{a_2}\frac{b}{a_2^0}\frac{1}{\dfrac{T}{T_*} - 1} \equiv \xi_1^{02}\frac{1}{\dfrac{T}{T_*} - 1} \\
{\xi_2}^2(T) &= \frac{c}{a_2} = \frac{c}{a_2^0}\frac{1}{\dfrac{T}{T_*} - 1} \equiv \xi_2^{02}\frac{1}{\dfrac{T}{T_*} - 1}
\end{aligned} \qquad (2.61)
$$

which increase as the temperature approaches T_* from above. These length scales will turn out to control the range of local fluctuations. Let us expand Q in a fourier series

$$Q_{\alpha\beta}(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{q}} e^{iqx} Q_{\alpha\beta}(a) \tag{2.62}$$

where $Q_{\alpha\beta}{}^*(q) = Q_{\alpha\beta}(-q)$. Then the total energy becomes

$$F_2 = \frac{1}{2} \sum_{\mathbf{q}} Q_{\alpha\beta}(-\mathbf{q}) \left[(a + b\mathbf{q}^2) g_{\alpha\alpha'} + c q_\alpha q_{\alpha'}\right] Q_{\alpha'\beta}(\mathbf{q}). \tag{2.63}$$

The spin orbit coupling term c can be diagonalized most easily on states of fixed helicity. The spin matrix for the tensor field is

$$(S^\gamma Q)_{\alpha\beta} = -i \left(\epsilon_{\gamma\alpha\alpha'} Q_{\alpha'\beta} + (\alpha \leftrightarrow \beta)\right). \tag{2.64}$$

The helicity is defined as the projection of $\mathbf{s}$ along $\mathbf{q}$

$$H = S^\gamma \hat{q}_\gamma. \tag{2.65}$$

We now realize that

$$\begin{aligned} &(HQ(-\mathbf{q}))_{\alpha\beta} (HQ(\mathbf{q}))_{\alpha\beta} = \\ &\quad (\hat{q}_\gamma \epsilon_{\gamma\alpha\alpha'} Q_{\alpha'\beta}(-q) + (\alpha, \beta)) (\hat{q}_\delta \epsilon_{\delta\alpha\alpha''} Q_{\alpha''\beta}(q) + (\alpha, \beta)) \\ &\quad = 4Q_{\alpha\beta}(-\mathbf{q}) Q_{\alpha\beta}(\mathbf{q}) - 6Q_{\alpha\beta}(-\mathbf{q}) \hat{q}_\alpha \hat{q}_{\alpha'} Q_{\alpha'\beta}(\mathbf{q}) \end{aligned} \tag{2.66}$$

such that (2.62) can be rewritten as

$$F^2 = \frac{1}{2} \sum_{\mathbf{q}} \left\{ \left[a + \left(b + \frac{2}{3} c\right) q^2\right] |Q_{\alpha\beta}(\tau)|^2 - \frac{c}{6} |HQ(q)|^2 \right\}. \tag{2.67}$$

This is obviously diagonal on eigenstates of helicity. These are easily constructed. First those of unit agular momentum: For this one simply takes the spherical combinations of unit vectors

$$\begin{aligned} \epsilon^{(1)} &= \frac{1}{\sqrt{2}} (\hat{x} + i\hat{y}), \\ \epsilon^{(-1)} &= -\frac{1}{\sqrt{2}} (\hat{x} + i\hat{y}), \\ \epsilon^{(0)} &= \hat{z}, \end{aligned} \tag{2.68}$$

which are eigenstates of S^3 and $(\mathbf{S})^2$:

$$S^3 \epsilon^{\pm 1} = \pm 1 \epsilon^{\pm 1}, \quad S^3 \epsilon^0 = 0, \tag{2.69}$$

$$\mathbf{S}^2 \epsilon^{\pm 1} = 2\epsilon^{\pm 1}, \quad \mathbf{S}^2 \epsilon^0 = 0, \tag{2.70}$$

and rotates them into the direction of $\mathbf{q}$ by

$$R(\hat{\mathbf{q}}) = e^{-i\varphi L_3} e^{-i\theta L_2} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \tag{2.71}$$

where φ and θ are the polar angles of $\hat{q}$. Then $\hat{x}, \hat{y}$ turn into a local triped $l^{(1)}, \;\; l^{(2)}, \hat{\mathbf{q}}$ of unit vectors such that $\epsilon^{\begin{pmatrix} +1 \\ 0 \\ -1 \end{pmatrix}}(q) \equiv R(\hat{q})\epsilon^{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}$ diagonalize H:

$$\begin{aligned} -i\hat{q}_\gamma \epsilon_{\gamma\alpha\beta}\left(l_\beta^{(1)} \pm i l_\beta^{(2)}\right) &= i\hat{\mathbf{q}}x\left(l^{(1)} \pm i l^{(2)}\right) \\ &= \pm i\left(\mathbf{l}^{)1)} \pm i\mathbf{l}^{(2)}\right) \\ -i\hat{q}_\gamma \epsilon_{\gamma\alpha\beta}\hat{\mathbf{q}}_\beta &= 0 \end{aligned} \tag{2.72}$$

with eigenvalues $\pm 1, 0$. Now we couple two of these symmetrically and obtain the angular momentum two helicity tensors.

$$\begin{aligned} &\epsilon_{\alpha\beta}^{(2)}(\hat{\mathbf{q}}) = \epsilon_{\alpha\beta}^{(-2)*}(\hat{\mathbf{a}}) = l_\alpha l_\beta = \epsilon^{(2)}\begin{pmatrix} 1 \\ \mathbf{q} \end{pmatrix} \\ &\epsilon_{\alpha\beta}^{(1)}(\mathbf{q}) = -\epsilon_{\alpha\beta}^{(-1)*}(\hat{\mathbf{q}}) = \frac{1}{\sqrt{2}}\left(l_\alpha \hat{q}_\beta + l_\beta \hat{q}_\alpha\right) = -\epsilon^{(1)}\left(-\hat{\mathbf{q}}\right) \\ &\epsilon_{\alpha\beta}^{(0)}(\mathbf{q}) = \sqrt{\frac{3}{2}}\left(\hat{q}_\alpha \hat{q}_\beta - \frac{1}{3}\delta_{\alpha\beta}\right) \equiv \epsilon^{(0)}\left(-\hat{\mathbf{q}}\right) \end{aligned} \tag{2.73}$$

where we have defined $l \equiv \frac{1}{\sqrt{2}}\left(l^{(1)} + i l^{(2)}\right)$. Since $l^2 = 0, \;\; ll^* = 1$ we varify directly the orthogonality

$$\operatorname{tr}\left(\epsilon^{(m)}(\hat{\mathbf{q}})\epsilon^{(m')*}(\hat{q})\right) = \delta_{mm'}. \tag{2.74}$$

The completeness relation is found to be

$$\sum_m \epsilon_{\alpha\beta}^{(m)}(\hat{\mathbf{q}})\epsilon_{\gamma\delta}^{(m)}(\hat{\mathbf{q}}) = I_{\alpha,\beta,\gamma,\delta} \tag{2.75}$$

where

$$I_{\alpha,\beta,\gamma,\delta} = \frac{1}{2}\left(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\right) - \frac{1}{3}\delta_{\alpha\beta}\delta_{\gamma\delta} \tag{2.76}$$

is the projection into the space of symmetric traceless tensors of spin 2, as it should.

The energy can now be diagonalized by expanding $Q_{\alpha\beta}(x)$ in terms of these $\epsilon_{\alpha\beta}^{(m)}(q)$ eigenmodes as

$$Q_{\alpha\beta}(x) = \sum_{\mathbf{q}, m=-1,\dots,2}\left(e^{i\mathbf{q}x}\epsilon_{\alpha\beta}^{m}\left(\hat{\mathbf{q}}\right)\varphi^{(m)}(\mathbf{q}) + c.c.\right) \tag{2.77}$$

where the energy takes the form

$$F = \int d^3x f = \sum_{\mathbf{q},m} \tau^{(m)}(q)|\varphi^{(m)}(q)|^2 \tag{2.78}$$

with

$$\begin{aligned}\tau^{(m)}(q) &= a_2 + \left[b + \left(\frac{3}{2} - \frac{m^2}{6}\right)c\right]q^2 \\ &= a_2\left\{1 + \left({\xi_1}^2 + \left(\frac{2}{3} - \frac{m^2}{6}\right){\xi_2}^2\right)q^2\right\}.\end{aligned} \tag{2.79}$$

We can now calculate the correlation functions of the field. If we rewrite

$$Z = \sum_{Q} e^{-\frac{1}{kT}F} \tag{2.80}$$

in the diagonalized form we have

$$Z = \sum_{\varphi^{(m)}} \exp\left\{-\frac{1}{kT}\sum_{\mathbf{q},m}\tau^{(m)}(q)|\varphi^{(m)}(\mathbf{q})|^2\right\}. \tag{2.81}$$

Therefore the correlation functions are[2]

$$\langle\varphi^{(m)}(\mathbf{q})\varphi^{(m)*}(q')\rangle = \delta_{q,q'}\frac{kT/2}{\tau^{(m)}(q)}. \tag{2.82}$$

Translating this back to $Q_{\alpha\beta}$ we may write

$$\langle Q_{\alpha\beta}(q)Q_{\gamma\delta}(q')^*\rangle = \delta_{\mathbf{q},\ \mathbf{q}'}\sum_{m}\frac{kT}{\tau^{(m)}(q)}\epsilon^{(m)}_{\alpha\beta}(\hat{\mathbf{q}}')\,\epsilon^{(m)*}_{\gamma\delta}(\hat{\mathbf{q}}) \tag{2.83}$$

or in x space

$$\langle Q_{\alpha\beta}(x)Q_{\gamma\delta}(x')\rangle = kT\sum_{q} e^{i\mathbf{q}(\mathbf{x}-\mathbf{x})}\frac{1}{\epsilon^{(m)}(q)}\epsilon^{(m)}_{\alpha\beta}(\hat{\mathbf{q}})\epsilon^{(m)*}_{\gamma\delta}(\hat{\mathbf{q}})\,. \tag{2.84}$$

These correlations are observable in inelastic scattering of visible light. With the identification $Q_{\alpha\beta} = \delta\epsilon_{\alpha\beta}$ there is an electric coupling

$$H_{\text{int}} = \frac{1}{2}\int d^3x E_\alpha Q_{\alpha\beta} E_\beta. \tag{2.85}$$

Let E_{in} be the field incoming light of momentum k_{in} and frequency ω. For a given fixed dielectric configuration $\epsilon(\mathbf{x})$, the polarization of the medium is given by

$$P(\mathbf{x})e^{-i\omega t} = \frac{1}{4\pi}(\epsilon(\mathbf{x}) - 1)E_{\text{in}}e^{-i(\omega t - \mathbf{k}_{\text{in}}\mathbf{x})}. \tag{2.86}$$

[2]The factor $\frac{1}{2}$ is due to the dependence of φ and φ^*, $\varphi^{(m)}(-q) = \varphi^{(m)*}(q)$.

Since $\mathbf{p}$ may be considered as a density of radiating dipoles, these will emit light in a spherical wave which at a large radius R away from the sample is

$$E_{\text{out}}(\mathbf{x}') = \frac{\omega^2}{c^2}\frac{1}{R}e^{ikR}P_{\perp}(\mathbf{x}) \tag{2.87}$$

where $k = \frac{\omega}{c}$, $R = |\mathbf{x}' - \mathbf{x}|$ and $P_\perp$ is the component transverse to the direction of the outgoing wave. Expanding R around $x = 0$, $kR \approx kR_0 - \mathbf{k}_{\text{out}}\mathbf{x}$ and integrating over the sample, the scattering amplitude A for incoming and outgoing polarization directions ϵ_{in}, ϵ_{out} is given by

$$\epsilon_{\text{out}}\mathbf{E}_{\text{out}}(x') = \frac{E_{\text{in}}}{R_0}e^{ikR_0}A \tag{2.88}$$

with

$$A = \frac{\omega^2}{4\pi c^2}\epsilon^*_{\text{out}}\left[\int d^3x e^{-i\mathbf{qx}}\left(\epsilon_{\alpha\beta}(\mathbf{x}) - 1\right)\right]\epsilon_{\text{in}\beta} \tag{2.89}$$

where $q = k_{\text{out}} - k_{\text{in}}$ is the momentum transfer. The square of A gives directly the differential cross section per unit solid angle

$$\frac{d\tau}{d\Omega} = |A|^2. \tag{2.90}$$

Eliminating the direct beam from the spatially constant part of ϵ we may write

$$\frac{d\tau}{d\Omega} = \frac{\Omega^4}{(4\pi c^2)^2}\epsilon^*_{\text{out}}\delta\epsilon(q)\epsilon_{\text{in}}\epsilon^*_{\text{in}}\delta\epsilon^*(q)\epsilon_{\text{out}}. \tag{2.91}$$

In the present case, the dielectric tensor has thermodynamic fluctuations and we have tr eplace $\delta\epsilon(q)\delta\epsilon^*(q)$ by the correlation function (2.85). This gives

$$\frac{d\tau}{d\Omega} = \frac{\omega^4}{(4\pi c^2)^2}\frac{kT}{2}\sum_m \frac{1}{\epsilon^{(m)}(q)}|\epsilon_{\text{out}}\epsilon^{(m)}(q)\epsilon_{\text{in}}|^2. \tag{2.92}$$

Let the incoming beam run in the z direction with the outgoing being rotated by an angle θ towards the y axis (see Fig. 8). Then

$$k_{\text{in}} = k(0,0,1), \quad k_{\text{out}} = k, \quad (0, \sin\theta, \cos\theta). \tag{2.93}$$

The momentum transfer is

$$\mathbf{q} = q\left(0, \cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right) \tag{2.94}$$

with

$$q^2 = 2\mathbf{k}^2(1 - \cos\theta). \tag{2.95}$$

For an incoming polarization vertical to the scattering plane, i.e. along the x axis which have

$$\epsilon_{\text{in}} = \epsilon_V = (1,0,0). \tag{2.96}$$

Let the final polarization be inclined by an angle φ against the vertical direction, i.e.

$$E_{\mathrm{out}} = (\cos\varphi, -\sin\varphi\cos\theta, \sin\varphi\sin\theta). \tag{2.97}$$

The tensors $\epsilon^{(m)}(\hat{\mathbf{q}})$ are all given in terms of $\hat{\mathbf{q}}$ and l which may be taken as

$$l = \frac{1}{\sqrt{2}}\left(1, -i\sin\frac{\theta}{2}, -i\cos\frac{\theta}{2}\right) = (l^*)^*\,. \tag{2.98}$$

In this way we find

$$\begin{aligned}
\epsilon^*_{\mathrm{out}}\epsilon^{(+pm2)}(\hat{\mathbf{q}})\epsilon_V &= \frac{1}{2}\left(\cos\varphi \mp \varphi\sin\varphi\sin\frac{\theta}{2}\right)\\
\epsilon^*_{\mathrm{out}}\epsilon^{(\pm1)}(\hat{\mathbf{q}})\epsilon_V &= -\frac{1}{2}\sin\varphi\cos\frac{\theta}{2}\\
\epsilon^*_{\mathrm{out}}\epsilon^{(0)}(\hat{\mathbf{q}})\epsilon_V &= -\frac{1}{\sqrt{6}}\cos\varphi.
\end{aligned} \tag{2.99}$$

If the initial polarization had been horizontal

$$\epsilon_{\mathrm{in}} = \epsilon_H = (0,1,0) \tag{2.100}$$

these scalar products would read

$$\begin{aligned}
\epsilon^*_{\mathrm{out}}\epsilon^{(\pm2)}(\hat{\mathbf{q}})\epsilon_H &= \frac{1}{2}\sin\frac{\theta}{2}\left(\mp i\cos\varphi - \sin\varphi\sin\frac{\theta}{2}\right)\\
\epsilon^*_{\mathrm{out}}\epsilon^{(\pm1)}(\hat{\mathbf{q}})\epsilon_H &= \frac{1}{2}\cos\frac{\theta}{2}\cos\varphi\\
\epsilon^*_{\mathrm{out}}\epsilon^{(0)}(\hat{\mathbf{q}})\epsilon_H &= -\frac{1}{\sqrt{6}}\sin\varphi\left(1+\cos^2\frac{\theta}{2}\right).
\end{aligned} \tag{2.101}$$

Inserting this into (2.92) we find the cross section for vertical incidence:

$$\begin{aligned}
\frac{d\sigma_V}{d\Omega} &= \frac{\omega^4}{(4\pi c^2)^2}\frac{kT}{2}\left[\frac{1}{6T^{(0)}(q)}\cos^2\varphi + \frac{1}{4}\left(\frac{1}{\tau^{(1)}(q)} + \frac{1}{\tau^{(-1)}(a)}\right)\cos^2\frac{\theta}{2}\sin^2\varphi\right.\\
&\left. +\frac{1}{4}\left(\frac{1}{\tau^{(2)}(q)} + \frac{1}{\tau^{(-2)}(q)}\right)\left(-\sin^2\frac{\theta}{2}\sin^2\varphi\right)\right]
\end{aligned} \tag{2.102}$$

and

$$\begin{aligned}
\frac{d\sigma_H}{d\Omega} &= \frac{\omega^4}{(4\pi c^2)^2}\frac{kT}{2}\left[\frac{1}{6T^{(0)}(q)}\sin^2\varphi\left(1+\cos^2\frac{\theta}{2}\right)\right.\\
&+ \frac{1}{4}\left(\frac{1}{\tau^{(1)}(q)} + \frac{1}{\tau^{(-1)}(a)}\right)\cos^2\frac{\theta}{2}\cos^2\varphi\\
&\left.+ \frac{1}{4}\left(\frac{1}{\tau^{(2)}(q)} + \frac{1}{\tau^{(-2)}(q)}\right)\sin^2\left(1-\cos^2\frac{\theta}{2}\sin^2\varphi\right)\right].
\end{aligned} \tag{2.103}$$

The experimental results show very little q dependence such that one may conclude that for visible light,

$$\xi_1 q \ll 1 \quad , \quad \xi_2 q \ll 1 \tag{2.104}$$

i.e. the wave length is much larger than both coherence lengths. If we therefore neglect $\xi_1, \ \ \xi_2$ for a moment, we see

$$\tau^{(0)} \approx \tau^{(\pm 1)} \approx \tau^{(\pm 2)} \tag{2.105}$$

such that the intensity of the scattered light goes like

$$I_V \sim \frac{1}{a_2}\left[\frac{1}{6}\cos^2\varphi + \frac{1}{2}\cos^2\frac{\theta}{2}\sin^2\varphi + \frac{1}{2}\left(1 - \sin^2\frac{\theta}{2}\sin^2\varphi\right)\right]. \tag{2.106}$$

For final polarizations vertical or horizontal to the scattering plane at a scattering angle $= 90^0$ this gives

$$\begin{aligned} I_{VV} &\sim \frac{2}{3a_2} \\ I_{HV} &\sim \frac{1}{2a_2} \end{aligned} \tag{2.107}$$

such that

$$\frac{I_{VV}}{I_{HV}} \sim \frac{4}{3}. \tag{2.108}$$

This ratio is approximately observed experimentally for T sufficiently above T_* [9]. As T approaches T_*, the coherence length grows larger and athe q dependence has a chance of becoming observable. Expanding $\tau^{(m)-1}(a)$ to lowest order in $(\xi q)^2$ we find

$$\tau^{(m)}(q)^{-1} = a_2^{-1}\left[1 - \left({\xi_1}^2 + \left(\frac{2}{3} - \frac{m^2}{6}\right){\xi_2}^2\right) + \dots\right] \tag{2.109}$$

such that the intensities I_{VV} , I_{HV} behave as

$$\begin{aligned} I_{VV} &\sim \frac{1}{6}\left[1 - \left({\xi_1}^2 + \frac{2}{3}{\xi_2}^2\right)q^2\right] + \frac{1}{2}\left[1 - {\xi_1}^2 q^2\right] + \dots \\ I_{HV} &\sim \frac{1}{4}\left[1 - \left({\xi_1}^2 + \frac{2}{3}{\xi_2}^2\right)q^2\right] + \frac{1}{4}\left[1 - {\xi_1}^2 q^2\right] + \dots \end{aligned} \tag{2.110}$$

with their ratio being

$$ {I_{VV}}^{-1}/I_{HV}^{-1} \sim \frac{3}{4}\left(1 - \frac{1}{12}{\xi_2}^2 q^2 + \dots\right). \tag{2.111}$$

This result is in agreement with experiment [9], with ${\xi_2}^2 > 0$ (in principle, c could have been negative).

As the temperature drops towards T_*, the intensity of scattered light increases like $a_2^{-1} \sim \left(\frac{T}{T_*} - 1\right)^{-1}$, (see (2.110) as a manifestation of increasing fluctuations.

for a comparison with the data it is most convenient to plot the inverse intensity against temperature which must behave for large enough T (a few ^0C above T_*) as

$$a_2^0\left(\frac{T}{T_*}-1\right)+\xi^2 q^2 \tag{2.112}$$

i.e. as a straight line, where ξ^2 is a combination of ${\xi_1}^2$ and ${\xi_2}^2$ depending on the polarizations (see Figs. 9, 10). Comparing such lines at different q values it is possible to deduce the size of the coherence lengths, for example in MBBA (see Figs. 10, 11)

$$\xi(T)\approx 5.5x\frac{1}{\sqrt{\frac{T}{T_*}-1}}\text{Å}. \tag{2.113}$$

As the temperature hits T_c which usually lies one half to one ^0C above T_*, the data points jump down to very small values, i.e. the intensity suddenly grows large (see Fig. 9).

It is easy to explain this phenomenon. At T_c, the size of the order paraemter jumps from $\varphi = 0$ to $\varphi = \varphi_> \neq 0$. But tue to rotational invariance of the energy, there is an infinite number of points in the $Q_{\alpha\beta}$ parameter space with the same energy, namely all those which differ only by a rotation of the direction vector $\mathbf{n}$. Due to this continuous degeneracy, there will be very strong directional fluctuations. It is these which scatter light at a much larger rate than before. Let us calculate the cross section: From (2.21) we have

$$\frac{d\sigma}{d\Omega}=\frac{\omega^4}{(4\pi^2c^2)^2}kT\frac{3}{4}\varphi^2|\epsilon^*_{\mathrm{out}\,\beta}\delta(n_\alpha n_\beta)\epsilon_{\mathrm{in}\,\alpha}|^2. \tag{2.114}$$

Thus we must calculate the correlation function

$$\langle\delta(n_\alpha n_\beta)\delta(n_\gamma n_\delta)\rangle=\langle\delta n_\alpha\delta n_\gamma\rangle n_\beta n_\delta+\langle\delta n_\alpha\delta n_\delta\rangle n_\beta n_\gamma+\langle\delta n_\beta\delta n_\gamma\rangle n_\alpha n_\delta+\langle\delta n_\beta\delta n_\delta\rangle n_\alpha n_\gamma. \tag{2.115}$$

Thus we need the correlation function $\langle\delta n_\alpha\delta n_\beta\rangle$.

In order to find this let us consider the bending energy (2.55) for the Fourier transformed field

$$\delta n(x)=\frac{1}{\sqrt{V}}\sum_{\mathbf{q}}e^{i\mathbf{qx}}\delta n(\mathbf{q}). \tag{2.116}$$

Then

$$F=\frac{1}{2}\sum_{\mathbf{q}}\left[K_1 q_\alpha q_\beta+K_2(nt\times q)_\alpha(n\times q)_\beta+K_3(\mathbf{nq})^2\delta_{\alpha\beta}\right]\delta n_\alpha(-\mathbf{q})\delta n_\alpha(\mathbf{q}). \tag{2.117}$$

In order to simplify the discussion suppose the system has an average orientation $n\|z$. Then

$$F=\frac{1}{2}\sum_{\mathbf{q}}\left[K_1 q_\alpha q_\beta+K_2 q_{\perp\alpha}q_{\perp\beta}++K_3{q_z}^2\delta_{\alpha\beta}\right]\delta n_\alpha(-\mathbf{q})\delta n_\alpha(\mathbf{q}) \tag{2.118}$$

where $q_\perp \equiv (-q_2, \quad q_1, \quad 0)$. The fluctuations can have only x and y components.[3] We can diagonalize this expression by introducing two orthogonal unit vectors $e_1(q) = (\hat{q}_1, \quad \hat{q}_2, \quad 0)$ and $e_2(\mathbf{q}) = \hat{q}_\perp$. If we decompose

$$\delta n(q) = e_1(\mathbf{q})\delta n_1(\mathbf{q}) + e_2(\mathbf{q})\delta n_2(\mathbf{q}) \tag{2.119}$$

we find the diagonal form

$$F = \frac{1}{2}\sum_{a=1,2}\left(K_a {q_\perp}^2 + K_3 {q_z}^2\right)|\delta n_a(q)|^2. \tag{2.120}$$

Thus the fluctuations of δn_1 and δn_2 diverge for $q \to 0$. The liquid crystal becomes opaque.

In this fashion, the bending constants $K_1, \ K_2, \ K_3$ can be measured with values for which examples were quoted before.

2.5 Interfacial Tension between Nematic and Isotropic Phase

At the different lines of first order phase transition, the order parameter moves from one value to another. Due to the derivative terms in the free energy, this change cannot take place abruptly but must be distributed over a length scale of the order of ξ in order to save gradient energies. It is a simple application of mean field theory to calculate the energy stored in the interface.

Experimentally this quantity can be measured in the form of a surface tension. This may be deduced to light scattering experiments [12] or, more directly, by looking at the curvature radius of a droplet of one phase embedded inside the other [13]. In this way, the surface tension was found for MBBA to be

$$\sigma \approx 2.3 \times 10^{-2} \quad erg/cm^2 \quad [12] \tag{2.121}$$

$$\sigma \approx 1.6 \times 10^{-2}\mathrm{erg/cm}^2. \quad [13] \tag{2.122}$$

An earlier measurement in another compound (PAP) gave a value two order of magnitude smaller than that and seems to be too small to be correct.

For the calculation it is convenient to go to natural dimensionless quantities and introduce a renormalized field

$$\varphi_{\alpha\beta} = -\frac{a_4}{a_3}\sqrt{\frac{8}{3}}Q_{\alpha\beta} \tag{2.123}$$

a temperature parameter

$$\tau = \frac{4a_4 b}{3{a_3}^2}\frac{a_2^0}{b}\left(\frac{T}{T_*} - 1\right) \tag{2.124}$$

[3]Since $\frac{1}{2}\delta\left(n_\alpha n_\alpha\right)^2 = \delta n_\alpha n_\alpha = 0$

a length scale

$$\xi_c \equiv \sqrt{\frac{g}{2}\frac{4a_4 b}{3{a_3}^2}} \tag{2.125}$$

and a energy density

$$\tilde{f} = \frac{2^g}{g}\frac{{a_4}^3}{{a_3}^4} f \equiv \frac{f}{\kappa}. \tag{2.126}$$

This has the simple expansion

$$\begin{aligned} \tilde{f} &= \tau {\varphi_{\alpha\beta}}^2 + \frac{2}{g}{\xi_c}^2 \left[(\partial_\gamma \varphi_{\alpha\beta})^2 + \frac{{\xi_2}^2}{{\xi_1}^2} \partial_\alpha \varphi_{\alpha\gamma} \partial_\beta \varphi_{\beta\gamma} \right] \\ &\quad - \frac{\sqrt{6}}{3} \varphi_{\alpha\beta}\varphi_{\beta\gamma}\varphi_{\gamma\alpha} + \frac{1}{8}\left({\varphi_{\alpha\beta}}^2\right)^2 + \dots\, . \end{aligned} \tag{2.127}$$

The nematic phase with the order parameter

$$\varphi_{\alpha\beta} = \varphi \epsilon^{(0)}_{\alpha\beta}(\mathbf{n}) \tag{2.128}$$

has now a potential energy

$$\tilde{f} = \tau\varphi^2 - \frac{1}{3}\varphi^3 + \frac{1}{8}\varphi^4 \tag{2.129}$$

with a first order transition from $\varphi = 0$ to $\varphi_>$

$$\varphi_> = \frac{1/3}{2(1/8)} = \frac{4}{3} \tag{2.130}$$

at

$$\tau_c = \frac{(1/3)^2}{4(1/8)} = \frac{2}{g}. \tag{2.131}$$

This corresponds to $T = T_c$ via (2.124). We can therefore rewrite

$$\frac{2}{g} = \frac{4a_4 b a_2^0}{3a_3^2 b}\left(\frac{T_c}{T_*} - 1\right) = \frac{\frac{2}{g}{\xi_c}^2}{\xi_1^2(T_c)} \tag{2.132}$$

which reveals the length scale ξ_c as the coherence length ξ_1 at the transition temperature.

Consider now a planar interface in the xy plane with the nematic and normal phase for $z \gg 0$ and $z \ll 0$, respectively. If we assume all gradients to point along the z axis, the bending energy is

$$\tilde{f}_{bend} = \frac{2}{g}{\xi_c}^2 \left[(\partial_z \varphi_{\alpha\beta})^2 + \frac{{\xi_2}^2}{{\xi_1}^2} \partial_z \varphi_{z\gamma} \partial_z \varphi_{z\gamma} \right]. \tag{2.133}$$

With the order parameter (2.128) and $\varphi \neq 0$, this is minimized by letting $\mathbf{n}$ point orthogonal to the z axis. Then (2.133) becomes

$$\frac{2}{g}{\xi_c}^2 \left(1 + \frac{1}{6}\frac{{\xi_2}^2}{{\xi_1}^2}\right)(\partial_z \varphi)^2 . \tag{2.134}$$

Therefore the total energy across the interface reads at $T = T_c$

$$\tilde{f} = \frac{2}{g}\xi_c^{t2}\left(\partial_z\varphi\right)^2 + \tau_c\varphi^2 - \frac{1}{3}\varphi^3 + \frac{1}{8}\varphi^4 \tag{2.135}$$

where we have introduced the transverse coherence length at $T = T_c$

$$\xi^z{}_c = \xi_1{}^2 + \frac{1}{6}\xi_2{}^2|_{T=T_c}. \tag{2.136}$$

If we adapt $\sqrt{\frac{2}{g}}\xi_c{}^z$ as our length scale we may rewrite

$$\begin{aligned} \tilde{f} &= (\partial_z\varphi)^2 + V(\varphi) \\ &= (\partial_z\varphi)^2 + V_0\varphi^2(\varphi - \varphi_0)^2 \end{aligned} \tag{2.137}$$

where $V_0 = \frac{1}{8}$, $\varphi_0 = \frac{3}{4}$. The potential term (see Fig. 12) is of the standard symmetric double well form with minima at $\varphi = 0$ and $\varphi = \frac{3}{4}$. Inside the interface, the order parameter has to move from one value to the other while keeping the total energy minimal, i.e. it has to satisfy the Euler Lagrange differential equation

$$\partial_z{}^2\varphi = V'(\varphi). \tag{2.138}$$

This precisely the same as the equation of motion of a point particle at position φ as a function of "time" z but in the reversed potential. The solution corresponds to a mass point rolling "down" the hill from $\varphi = 0$ through the "valley" at $\varphi = \varphi_0/2$ up to the other hill at $\varphi = \varphi_0$. the total "engerg" of this motion is conserved, i.e.

$$(\partial_z\varphi)^2 - V(\varphi) \quad \text{const.} \tag{2.139}$$

Far away from the interface, $\varphi = 0$ or φ_0 and $V = 0$ such that const $= 0$. Thus we may integrate

$$z = \int_0^\varphi \frac{d\varphi'}{\sqrt{V(\varphi')}} \tag{2.140}$$

which is solved by

$$\varphi = \frac{1}{2}\varphi_0\left\{1 + th\frac{z\sqrt{V_0}\varphi_c}{2}\right\} \tag{2.141}$$

i.e. here

$$\varphi(z) = \frac{2}{3}\left\{\frac{1 + thz}{3\sqrt{2}}\right\}. \tag{2.142}$$

The total free energy for this parameter is found simply as

$$\begin{aligned} \tilde{f} &= \int_{-\infty}^{\infty} dz\left[(\partial z\varphi)^2 + V(\varphi)\right](??)(??) \\ &= 2\int_0^{\infty\varphi_0} dzV(\varphi) = 2\int_0^{\varphi_0} d\varphi\sqrt{V(\varphi)} \\ &= 2\sqrt{V_0}\int_0^{\varphi_0} d\varphi\ \varphi(\varphi - \varphi_0) = \frac{\sqrt{V_0}}{3}\varphi_0{}^3 = \frac{16}{81}\sqrt{2}. \end{aligned} \tag{2.143}$$

This is the surface tension which back in physical units reads

$$\sigma = \frac{16}{81}\sqrt{2}\sqrt{\frac{2}{9}}\xi^z{}_c\kappa. \tag{2.144}$$

The value fo κ involves a_3 and a_4 and which individually are somewhat hard to determine. But here is a simple experimental quantity which contains κ rather directly: The latent heat of the trasition. In MBBA, for example, one finds [15] [16]:

$$\Delta q = 0.3kJ/\text{mol} \approx 1.2J/g = 1.2 \cdot 10^7 \frac{erg}{g}. \tag{2.145}$$

Within the present natural units, the latent heat is found from (**??**) as

$$\begin{aligned} \Delta q &= \kappa \frac{\partial \tau}{\partial T} T \left(\left.\frac{\partial \tilde{f}}{\partial \tau}\right|_{\varphi=\varphi_c} - \left.\frac{\partial \tilde{f}}{\partial \tau}\right|_{\varphi=0} \right) \\ &= \kappa \frac{2}{g} \left(\frac{\xi_c}{\xi^0} \right)^2 \frac{T_c}{T_*} \varphi_c^2 \approx \frac{32}{81} \frac{1}{\frac{T_c}{T_*} - 1}. \end{aligned} \tag{2.146}$$

Comparing this with (2.144) we find the simple relation

$$\sigma = \Delta q \frac{1}{3} \xi_c^t \left(\frac{T_c}{T_*} - 1 \right). \tag{2.147}$$

For MBBA we may insert on the right-hand side Δq from (2.145) and

$$\xi_c{}^t \approx 150\text{Å} \quad , \quad \frac{T_c}{T_*} - 1 \approx \frac{1}{400} \tag{2.148}$$

and find $\sigma \approx 1.5 \times 10^{-2}$ erg/cm^2 in reasonable agreement with the experimental values (2.121), (2.122).

2.6 Cholesteric Liquid Crystals

The collective field theory developed up to this point is able to describe an ensemble of rod-like, disc-like or biaxial order. In the introduction it was mentioned that in cholesterol and similar compounds the molecular array exhibits a slight screw like distortion. This violates mirror reflection (parity) symmetry. In order to describe such systems we have to add a parity violating piece to the energy. To lowest order there exists the following quadratic term with this property:

$$f_{p\nu} = -2d\epsilon_{\alpha\beta\gamma} Q_{\alpha\beta} \partial_\gamma Q_{\beta\gamma}. \tag{2.149}$$

This may be written alterntively in terms of the spin matrix (2.63) as

$$f_{p.\nu.} = -dQ_{\alpha\beta} \left(\mathbf{S} i \partial Q \right)_{\alpha\beta}. \tag{2.150}$$

For the Fourier transformed field, this becomes ($q \equiv |\mathbf{q}|$)

$$f_{p.\nu.} = -d \sum_q Q_{\alpha\beta}(-\mathbf{q}) q \left(HQ(\mathbf{q}) \right)_{\alpha\beta}. \tag{2.151}$$

2.7 Small Fluctuations above T_c

If the temperature lies far enough above T_c (say a few ${}^0C$) the fluctuations are dominated by the quadratic part of the free energy. Obviously, the normal modes are still given by the different helicity tensors $\epsilon^{(2)}(\hat{\mathbf{q}})$ but now the energies behave as

$$\begin{aligned}
\tau^{(0)}(q) &= a_2\left(1+\frac{2}{3}{\xi_1}^2q^2\right) \\
\tau^{(\pm 1)}(q) &= a_2\left[1+\left({\xi_1}^2+\frac{1}{2}{\xi_2}^2\right)\left(q^2+\frac{d}{b+\frac{c}{2}}q\right)\right] \qquad (2.152) \\
\tau^{(\pm 2)}(q) &= a_2\left[1+{\xi_1}^2\left(q^2\mp 2\frac{d}{b}q\right)\right]. \qquad (2.153)
\end{aligned}$$

These can be rewritten as

$$\begin{aligned}
\tau^{(\pm 1)} &= a_2\left[1-\frac{d^2/{a_2}^2}{4(\xi_1^2+\frac{1}{2}{\xi_2}^2)}\right]+a\left({\xi_1}^2+\frac{1}{2}{\xi_2}^2\right)\left(q\mp\frac{d}{2b+c}\right)^2 \qquad (2.154) \\
\tau^{(\pm 2)} &= a_2\left[1-\frac{d^2/{a_2}^2}{{\xi_1}^2}\right]+a_2{\xi_1}^2\left(q\mp\frac{d}{b}\right)^2. \qquad (2.155)
\end{aligned}$$

The quantity $\frac{d}{b}$ has the dimension length -1 and may be used to define a new characteristic parameter ξ_h as

$$\frac{d}{b}\equiv{\xi_n}^{-1}. \qquad (2.156)$$

Then $\tau^{(\pm 1)},\tau^{(\pm 2)}$) take the form

$$\begin{aligned}
\tau^{(\pm 1)}(q) &= a_2\left[\left(1-\frac{1}{4}\frac{{\xi_1}^2/{\xi_n}^2}{1+{\xi_2}^2/2{\xi_1}^2}\right)+{\xi_1}^2\left(1+{\xi_2}^2/2{\xi_1}^2\right)\left(q\mp q^{(1)}\right)^2\right] \\
\tau^{(\pm 2)}(q) &= a_2\left[\left(1-{\xi_1}^2/{\xi_n}^2\right)+{\xi_1}^2\left(q\mp q^{(2)}\right)^2\right]. \qquad (2.157)
\end{aligned}$$

While $\tau^{(0)}(q)$ is unaffected by the parity violating d term, the helicity one and two fluctuations now are strongest for the non-vanishing momenta (see Fig. 13)

$$\begin{aligned}
q^{(1)} &\equiv \frac{1}{2\xi_n}\frac{1}{1+{\xi_2}^2/2{\xi_1}^2} \\
q^{(2)} &= \frac{1}{\xi_n}. \qquad (2.158)
\end{aligned}$$

This fact will be seen to give rise to a number of distinctive physical properties of cholesteric systems.

2.8 Some Experimental Facts

As far as Raleigh scattering far above T_c is concerned, the momentum transfers are so small that the result

$$\frac{I_{VH}{}^{-1}}{I_{VV}{}^{-1}} \approx \frac{4}{3} \tag{2.159}$$

is still expected to be true. Experimentally a slight deviation (1.448±.94) is observed which has not yet been explained (see Fig. 14).

The most striking difference with respect to the nematic case, however, consists in the following. The data points of I^{-1} do no longer end at a precocious phase transition at $T_c > T_*$. Instead, they turn off the straight line and can now be followed down to below T_* (see Fig. 15) by half a degree where they suddenly jump down to small values as the ordered phase is reached. These values are, however, much ($\approx$ 10 times) larger than those in the nematic ordered phase, i.e. the scattered ligth intensity is much smaller. This indicates a lower level of degeneracy of orientational degrees of freedom as compared to the nematic phase. There is another characteristic feature which was observed by Reinitzer [18] in his first investigations of such systems. The liquid appears in a bright blue color. For this reason, this temperature regime is referred to as the blue phase.

When pressed into a thin layer betwee two glass plates, the liquid forms a great number of domains, called plaquelets, some of them blue [19].

As the temperature is lowered by one more degree, the colors suddenly disappear and the intensity of scattered light jumps up once more. Now the liquid shows the same degree of opaqueness as nematic ordered phases. This temperature regime is called the cholesteric phase.

If the liquid is subjected to more detailed optical investigations it reveals several important phenomena.

1. The refractive indices for ordinary and extraordinary light rays are equal in the blue phase but differ by about one percent in the cholesteric phase [19] (see Fig. 16).

2. The cholesteric phase shows a single strong Bragg reflex of circularly polarized light at normal incidence at barely UV wave lengths.

   Thus, the liquid is capable of importing a certain momentum [4]

$$q = 2k_0 = \frac{4\pi}{\lambda_R} \tag{2.160}$$

   upon the incoming light of momentum k_0 and wave length λ_R. The quantity $P = 4\pi/q$ is referred to as optical pitch.

[4]If the light is observed outside the medium, λ_R has to be replaced by λ_R/n.

3. for oplique incidence there are also reflexes of higher order $2q$, $3q$, at Bragg angles θ: [5]
$$\lambda_R = \frac{P}{m} \sin\theta \tag{2.161}$$
($\theta = 90^0$, normal incidence). But now the polarizations are elliptical.

4. Also the blue phase gives Bragg reflexes but with a larger pitch P_{blue} which is about two times larger than that in the cholesteric phase (this is why the color is blue rather than UV). Moreover, the plaquelets described above reflect light at wave lengths which are integer fractions of the above pitch P_{blue} and of $P_{blue} \cdot \sqrt{2}$. As a matter of fact, the directions of reflexes can be fitted by the same Bragg condition as those in a bcc lattice
$$\left(\frac{\sin\theta}{\lambda_R/P_b}\right)^2 = \frac{{m_1}^2 + m^{22} + {m_3}^2}{2} \tag{2.162}$$
where the Miller indices can take integer values with even. The presence of lattice planes $(1,1,0)$, $(2,0,0,)$, $(2,0,0)$ has apparently been established [20].

5. There is one more important observation [20]. The wave length of reflected light remains constant for about half a degree Celsius. Then it has jump to a higher value and increases even more for another half degree before it falls back to a low value as the cholesteric phase is reached. The jump is present only for samples of shorter pitch (see Figs. 11, 12). We shall now try to understand these properties theoretically.

2.9 Mean Field Description of Cholesteric Phase

In the presence of the parity violating term (2.150), the ground state is much harder to determine than in nematics even at the mean field level. The reason is that a constant field configuration can no longer give the lowest energy. For the following discussion let us truncate the free energy after the quartic term, for simplicity. In the natural units introduced before we may write the free energy density as
$$\begin{aligned}\tilde{f} &= (\tau + 2\alpha){\varphi_{\alpha\beta}}^2 - \frac{\sqrt{6}}{3}\varphi_{\alpha\beta}\varphi_{\beta\gamma}\varphi_{\gamma\alpha} + \frac{1}{8}\left(\varphi_{\alpha\beta}^2\right)^2 \\ &\quad +2\alpha\xi_n^2\left[(\partial_\gamma\varphi_{\alpha\beta})^2 + \frac{{\xi_2}^2}{{\xi_1}^2}\partial_\alpha\varphi_{\alpha\gamma}\partial_\beta\varphi_{\beta\gamma}\right] \\ &\quad -4\alpha\xi_n\epsilon_{\alpha\beta\gamma}\varphi_{\alpha\beta}\partial_\gamma\varphi_{\beta\delta}.\end{aligned} \tag{2.163}$$

Here we have introduced the additional dimensionless parameter
$$2\alpha \equiv \frac{4a_4 b}{3{a_3}^2}\frac{d^2}{b^2} = \frac{2}{g}{\xi_c}^2/{\xi_n}^2 \tag{2.164}$$

[5] If the light is observed outside the medium, λ_R has to be replaced by λ_R/n.

and shifted τ to $\tau + 2\alpha$, i.e. in the cholesteric case we define

$$\tau = 2\alpha \equiv \frac{4a_4 b}{3{a_3}^2}\frac{a_2^0}{b}\left(\frac{T}{T_*} - 1\right) \equiv \frac{2}{g}{\xi_c}^2/{\xi_1}^2(T). \tag{2.165}$$

The parameter α measures the cholesteric lenght scale χ_4 with respect to the coherence lenght ξ_c and will be called cholestericity. Obviously, the limit $\alpha \to 0, d = 0$ brings us back to the nematic case [see (**??**)] in which case (2.165) coincides with the previous definition (2.124) and $2\alpha{\xi_n}^2$ has to be replaced by $\frac{2}{g}{\xi_c}^2$.

We have seen in the last chapter that at the level of small fluctuations the last term in (2.163) gives a preference to the helicity two ($q \approx q^{(2)}$) mode (see Fig. 13) we may therefore expect a lower energy for an ansatz:

$$\varphi_{\alpha\beta} = \frac{1}{\sqrt{2V}}\left(\epsilon^{(2)}(\hat{\mathbf{q}})e^{i\mathbf{qx}}\varphi^{(2)} + C.C.\right). \tag{2.166}$$

Inserting this into (2.163) we find

$$\tilde{f} = \tau{\phi^{(2)}}^2 + \frac{1}{8}{\varphi^{(2)}}^4 + 2\alpha\left(\frac{q}{q^{(2)}} - 1\right)^2{\varphi^{(2)}}^2. \tag{2.167}$$

There is no cubic term since the product of three $\epsilon^{(2)}, \;\; \epsilon^{(2)*}$ tensors always vanishes. For details of the calculation see the Appendix B. The energy is minimized by setting $q = q^{(2)}$ where it becomes

$$\tilde{f} = \tau\varphi^{(2)} + \frac{1}{8}{\varphi^{(2)}}^4 \tag{2.168}$$

This is to be compared with the helicity zero expression:

$$\begin{aligned}\tilde{f} \;\;&=\;\; (\tau + 2\alpha){\varphi^{(0)}}^2 - \frac{1}{3}{\varphi^{(0)}}^3 + \frac{1}{8}{\varphi^{(0)}}^4 \\ &+2d{\xi_n}^2\left(1 + \frac{2}{3}\frac{{\xi_1}^2}{{\xi_2}^2}\right)q^2{\varphi^{(0)\prime}}^2\end{aligned} \tag{2.169}$$

which is minimal at $q = 0$.

We now realize that for large enough α the energy (2.168) is always lower than (2.169). For if $2\alpha > \frac{2}{g}$, the energy (2.169) vanishes for $\tau > \frac{2}{g} - 2\alpha$ while (2.167) has a second order phase transition at $\tau = 0$ and starts being negative for $T < 0$. But this is by far not the lowest possible energy. In order to see this let us combine both helicities linearly and take

$$\varphi_{\alpha\beta} = \frac{1}{\sqrt{V}}\left[\epsilon^{(0)}(n)\varphi^{(0)} + \frac{1}{\sqrt{2}}\left(\epsilon^{(2)}(\hat{\mathbf{q}})e^{i\mathbf{q}x}\varphi^{(2)} + c.c.\right)\right]. \tag{2.170}$$

Now the energy has the form (see App. B)

$$\begin{aligned}\tilde{f} \;\;&=\;\; (\tau + 2\alpha)\,{\varphi^{(0)}}^2 + \tau|\varphi^{(2)}|^2 - \frac{{\varphi^{(0)}}^3}{3} - \varphi^{(0)}|\varphi^{(2)}|^2\left(3|l\cdot n|^2 - 1\right) \\ &+\frac{1}{8}\left[\left({\varphi^{(0)}}^2 + |\varphi^{(2)}|^2\right) + 6\varphi^{(0)}|\varphi^{(2)}|^2\right].\end{aligned} \tag{2.171}$$

The two modes are coupled at the cubic level. This gives rise to a linear asymmetry for the $\varphi^{(0)}$ amplitude such that it is pulled out of the equilibrium position to a new minimum thereby reducing the remaining quartic potential for $\varphi^{(2)}$. This effect is strongest if the cubic term is maximal and the quartic term minimal which happens for

$$\mathbf{nl} = 0. \tag{2.172}$$

Thereby we are confronted with

$$\tilde{f} = (\tau + 2\alpha)x^2 + \tau y^2 - \frac{x^3}{3} + xy^2 + \frac{1}{8}\left(x^2 + y^2\right)^2. \tag{2.173}$$

Here we have changed variables from $\varphi^{(0)}$, $\varphi^{(2)}$ to x and y, for convenience. We now minimize $\tilde{f}$ with respect to x and y and find

$$(\tau + 2\alpha)\,x - \frac{1}{2}x^2 + \frac{1}{2}y^2 + x(x^2 + y^2) = 0 \tag{2.174}$$

$$\tau y + xy + y(x^2 + y^2) = 0. \tag{2.175}$$

From these two equations we find

$$y^2 = 3x^2 - 4\alpha x \tag{2.176}$$

which inserted back into (2.172) gives

$$x^2 + (1 - \alpha)x + \tau = 0 \tag{2.177}$$

i.e.

$$x_{1,2} = -\left[\frac{1-\alpha}{2} \pm \sqrt{\frac{(1-\alpha)^2}{4} - \tau}\right]. \tag{2.178}$$

At the extrema, the energy is

$$\tilde{f}_{ext} = 2\left(x^2 + \frac{x^3}{3} - \alpha\tau x\right) \tag{2.179}$$

$$= 2\left[-\tau^2 + \frac{1}{3}(1-\alpha)\tau - \frac{4}{3}\left(\frac{(1-\alpha)^2}{4} - \tau\right) \cdot \left(\frac{1-\alpha}{2} \pm \sqrt{\frac{(1-\alpha)^2}{4} - \tau}\right)\right] \tag{2.180}$$

and we see that the + sign corrresponds to the lower value.

The phase transition takes place at $\tau_c = \tau_c(\alpha)$ where $tildef$ vanishes. Instead of solving $f = 0$ from (**??**) it is more convenient to take $f = 0$ (2.179) and combine it with (2.177) to get two linear equations

$$x = -\frac{\tau + \alpha - \alpha^2}{\alpha + \frac{1}{3}} \tag{2.181}$$

and

$$x = \frac{\left(\alpha + \frac{1}{3}\right)\tau}{\tau + (\alpha - 1)/3}. \tag{2.182}$$

Once by eliminating the lowest and once the highest power in x. Comparing both we obtain

$$g\tau^2 + 2(g\alpha - 1)\tau - 3\alpha(1-\alpha)^2 = 0 \tag{2.183}$$

as the line in the $\alpha, \quad \tau$ plane where $\tilde{f}$ vanishes.

For $\alpha <$, this happens first at a value $\tau_c > 0$ which for $\alpha = 0$ is the nematic value $\frac{2}{9}$, and decreases down to zero at $\alpha = 1$. Above $\alpha = 1$, the curve (2.181) does not correspond t a minimum. In that region the phase transition takes place at $\tau = 0$ and is of second order as can be seen directly from (**??**). The energy for small $\tau \preceq 0$ becomes for $\alpha > 1$:

$$\tilde{f} = -2\tau^2\left(1 + \frac{4}{3(\alpha-1)}\right) + O(\tau^3). \tag{2.184}$$

The full behaviour of $\tilde{f}$ as a function of temperature τ and cholestericity α is shown in form of a contour plot in Fig. 17.

The order parameters x and y are displayed in Fig. 18 and 19. The lines of constant x are straight: $\epsilon = x\alpha - (x^2 + x)$. The ordered phase for $\alpha > 0$ is referred to as cholesteric phase.

Notice that for $\alpha \to \infty$, the helicity zero component becomes more and more suppressed and only $\varphi^{(2)} = y$ survives.

What happens if also the helicity one component is admitted. In order to study this let us assume all fields to vary only along the z axis. For symmetry reasons, we may take $l = \frac{1}{\sqrt{2}}\left(\hat{x} + i\hat{y}\right)$. Then $\mathbf{n} = \hat{\mathbf{z}}$ from (2.172). If we now expand

$$\varphi_{\alpha\beta}(z) = \frac{1}{\sqrt{V}}\left[\epsilon^{(0)}(\hat{z})\varphi^{(0)}(z) + \frac{1}{\sqrt{z}}\left(\epsilon^{(1)}(\hat{z}) + \epsilon^{(2)}(\hat{z})\varphi^{(2)}(z) + c.c.\right)\right] \tag{2.185}$$

with real $\epsilon^{(0)}$ and complex $\varphi^{(1)}, \quad \varphi^{(2)}$ fields, we obtain from (2.163)

$$\begin{aligned}\tilde{f} \;=\; & (\tau + 2\alpha)\left({\varphi^{(0)}}^2 + |\varphi^{(1)}|^2 + |\varphi^{(2)}|^2\right) \\ & -\frac{1}{3}{\varphi^{(0)}}^3 - \frac{1}{2}\varphi^{(0)}\left(|\varphi^{(1)}|^2 - 2|\varphi^{(2)}|^2\right) - \frac{\sqrt{3}}{4}\left(\varphi^{(2)*}{\varphi^{(1)}}^2 + c.c.\right) \\ & +\frac{1}{8}\left({\varphi^{(0)}}^2 + |\varphi^{(1)}|^2 + |\varphi^{(2)}|^2\right)^2 \\ & +2\alpha\xi n^2\left(r_0\left(\partial_z\varphi^{(0)}\right)^2 + r_1\left(\partial_z\varphi^{(1)}|^2 + |\partial_z\varphi^{(2)}|^2\right)\right. \\ & \left.-2\alpha\xi_n(\varphi^{(1)*}\frac{\partial}{2}\varphi^{(1)}\varphi^{(1)} + 2\varphi^{(2)*}\frac{\partial}{2}\varphi^{(2)}\right).\end{aligned} \tag{2.186}$$

Here we have introduced the convenient abbreviations

$$r_0 \equiv 1 + \frac{2}{3}\frac{c_1 + c_2}{b} = \frac{4r_1 - 1}{}$$

$$r_1 \equiv 1 + \frac{c_1 c_2}{2b} = 1 + \frac{c}{2b} \tag{2.187}$$

which are accessible in the ordered phase via the ratio of Frank constants

$$r_1 \equiv \frac{K_1 + K_3}{2K_2} = \frac{K_3 + K - b}{2K_t}. \tag{2.188}$$

In momentum space, the quadratic terms can be rewritten after a quadratic completion as $(q \equiv q_z)$

$$\begin{aligned} \tilde{f} &= \sum_q \left(\tau + 2\alpha 2\alpha r_0 q^2\right) |\varphi_{(0)}(q)|^2 \\ &\quad + \left(\tau + 2\alpha\left(1 - \frac{1}{4r_1}\right) + 2\alpha r_1 \left(q\xi_n \mp \frac{1}{2r_1}\right)^2\right) |\varphi^{(n)}(q)|^2 \\ &\quad + \left(\tau + 2\alpha\left(q\xi_n \mp 1\right)^2\right) |\varphi^{(\pm 2)}(q)|^2. \end{aligned} \tag{2.189}$$

For very large α, this certainly is minimal at the former solution with $q = \frac{1}{\xi_n}$ and no $\varphi^{(1)}$ component can be present (just as the $\varphi^{(0)}$ for $\alpha \to \infty$). But experimentally, α is quite moderate: In a typical cholesteric system one has $\xi_1^0 \approx$ Å and $\xi_n \approx 2000/4\pi$Å such that $\alpha \approx .21$. Therefore, $\varphi^{(1)}$ could be present. From the energy we see that the amplitude $\varphi^{(1)}$ enters only in second and for the order. Thus there can be a second order phase transition with $\varphi^{(1)} \neq 0$ developing from the previous solution with $\varphi^{(0)}, \;\; \varphi^{(2)} \neq 0$ along a line in the $\alpha, \; \tau$ diagram where the coefficient of the quadratic term becomes negative:

$$\begin{aligned} D &\equiv \tau + 2\alpha\left(1 - \frac{1}{4r_1}\right) + 2\alpha r_1 \left(1 - \frac{1}{2r_1}\right)^2 \\ &\quad -\frac{1}{2}x - \frac{\sqrt{3}}{2}y + \frac{1}{4}\left(x^2 + y^2\right) \preceq 0. \end{aligned} \tag{2.190}$$

Inserting the solutions (2.176) and (2.178) we find that this cannot happen. At $\alpha = 0$. $x = -\frac{1}{2} + \sqrt{\frac{1}{2} - \tau}, \;\; y = -\sqrt{3}x$ and $x^2 + x + \tau = 0$ such that $D = 0$. But for all allowed $\alpha > 0, \tau$ in the cholesteric phase we can verify that $\tau + \alpha - \frac{x}{2} - \frac{\sqrt{3}}{2}y + \frac{1}{4}\left(x^2 + y^2\right)$ starts out with $0(\alpha^2)$ and is always > 0. But this ensures alsoe $D > 0$ since the first line in (2.190) is $\tau + \alpha\left(1 + \frac{r_1}{2}\right)$ and $r_1 > 0$.

Let us take a look at the cholesteric order parameter with $\varphi^{(0)}, \;\; \varphi^{(2)}$. It may be written in matrix form as

$$\begin{aligned} \varphi_{\alpha\beta} &= \varphi^{(0)} \frac{1}{\sqrt{6}} \begin{pmatrix} -1 & & \\ & -1 & \\ & & 2 \end{pmatrix} + \frac{1}{\sqrt{2}} \varphi^{(2)} \left(\frac{1}{2} \begin{pmatrix} 1 & i & 0 \\ \epsilon & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} e^{iqz} + c.s. \right) \\ &= \begin{pmatrix} -\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{2}} \cos qz, & -\frac{\varphi^{(2)}}{\sqrt{2}} \sin qz & 0 \\ -\frac{\varphi^{(2)}}{\sqrt{2}} \sin qz & -\frac{1}{\sqrt{6}}\varphi^{(0)} - \frac{1}{\sqrt{2}}\varphi^{(2)} \cos qz & 0 \\ 0 & 0 & \frac{2}{\sqrt{6}}\varphi^{(0)} \end{pmatrix}. \end{aligned} \tag{2.191}$$

This has to be added to $\epsilon_0\delta_{\alpha\beta}$ in order to obtain the dielectric tensor which is usually parametrized as

$$\epsilon = \begin{pmatrix} \bar{\epsilon} + \delta\cos 2kz & -\delta\sin 2kz & 0 \\ -\delta\sin 2kz & \bar{\epsilon} - \delta\cos 2kz & 0 \\ 0 & 0 & \epsilon_3. \end{pmatrix} \tag{2.192}$$

Notice that the mixing between $\varphi^{(0)}$ and $\varphi^{(2)}$ induces, in general, biaxiality. The local eigenvalues are now all three different $\bar{\epsilon}+\delta_1, \bar{\epsilon}-b1, \epsilon_3$.

In order to interprete the order parameter (2.194) physically it is useful to realize the following: Suppose the helicity zero rod-like form $\epsilon^{(0)}(n) = \sqrt{\frac{3}{2}}\left(n_\alpha n_\beta - \frac{1}{3}\delta_{\alpha\beta}\right)$ is taken with the direction

$$n(z) = (\cos kz, \quad -\sin kz) \tag{2.193}$$

rotating in the z plane while proceeding along the z direction. Then $\epsilon_{(0)}(n(z))$ becomes

$$\begin{aligned} \epsilon^{(0)}\left(n(z)\right) &= \sqrt{\frac{3}{2}}\begin{pmatrix} \cos^2 kz - \frac{1}{3} & -\sin kz\cos kz & 0 \\ \sin kz\cos kz & \sin^2 kz - \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \\ &= \sqrt{\frac{3}{2}}\begin{pmatrix} \frac{1}{6} + \frac{1}{2}\cos 2kz & -sin2kz & 0 \\ \sin 2kz & \frac{1}{6} - \frac{1}{2}\cos 2kz & 0 \\ 0 & 0 & -\frac{2}{6} \end{pmatrix} \\ &= \frac{1}{2}\left(-\epsilon^{(0)}(z) + \sqrt{3}\frac{1}{\sqrt{2}}\left(\epsilon^{(2)}(z)e^{iqz} + c.c.\right)\right) \end{aligned} \tag{2.194}$$

Thus it is precisely of the form (2.170) with the particular ratio

$$\frac{\varphi^{(2)}}{\varphi^{(1)}} = \frac{y}{x} = \sqrt{3}. \tag{2.195}$$

In this case we may interprete the solution (2.188) as a purely transverse helical configuration of rod-like molecules. In the following section we shall see how these parameters can be measured in optical experiments. It will turn out that the experimental biaxiality remains small: the eigenvalue $\bar{\epsilon}-\delta$ is usually equal to ϵ_3 (example: $\bar{\epsilon} = 2.745, \quad \delta = .315, \quad \epsilon_3 = 2.430$ [21]).

Thus experimentally $\varphi^{(\epsilon)} \approx \sqrt{3}\varphi^{(0)}$. Looking back at (2.176) we notice that for $\alpha = 0$ this is automatically true (as it should since $\alpha = 0$ corresponds to the absence of the parity violating term). the uniaxiality remains approximately true for the typical experimental value .21.

$$\frac{4}{\sqrt{3}|x|} = 1 - \frac{2}{3}\alpha/|x| + O(\alpha^2). \tag{2.196}$$

Thus we find for the ratio of the dielectric eigenvalues in (**??**).

$$\frac{\bar{\epsilon}-\delta}{\epsilon_3}1 + \frac{3}{8}\left(\frac{1}{\sqrt{\frac{1}{4}-\tau}} - \frac{2}{3}\right). \tag{2.197}$$

2.10 Maier Saupe Model and Generalizations

The simplest microscopic model for the description of phase transitions in liquid crystals was constructed by Maier and Saupe. It is based on the standard molecular field approximation invented a long time ago by P. Weiss to explain ferromagnetism. By construction, their model was confined to nematic systems. The molecules are assumed to be non-polar, rod-like objects. If the direction of the body axis is denoted by the unit vector **n**, the instantaneous orientation may be characterized by the traceless tensor

$$Q^{mol}_{\alpha\beta} = \epsilon^{(0)}_{\alpha\beta}(\mathbf{n} = \sqrt{\frac{3}{2}}\left(n_\alpha n_\beta - \frac{1}{3}\delta_{\alpha\beta}\right). \tag{2.198}$$

In the normal phase, this field fluctuates around zero. Below the phase transition, however, there is a non-vanishing average order

$$Q_{\alpha\beta} = \langle Q^{mol}_{\alpha\beta}\rangle = S\epsilon^{(0)}(\mathbf{m}) = S\sqrt{\frac{3}{2}}\left(m_\alpha m_\beta - \frac{1}{3}\delta_{\alpha\beta}\right). \tag{2.199}$$

This is due to the intermolecular forces which tend to align the instantenous value (2.198) with the average value (2.199). The interaction may be approximated by an orientational energy

$$H_{or} = -A_0 Q^{mol}_{\alpha\beta} Q_{\alpha\beta} \tag{2.200}$$

with some coupling strength A_0. Inserting this into Boltzmann's distribution law, one finds the self-consistency relation

$$\begin{aligned} Q_{\alpha\beta} &= \langle Q_{\alpha\beta}\rangle = Z^{-1}\int \frac{d\hat{n}}{4\pi} Q^{mol}_{\alpha\beta} e^{-H_{or}/kT} \\ Z &= \int \frac{d\hat{n}}{4\pi} e^{-H_{or}/kT} \end{aligned} \tag{2.201}$$

or, putting m in z direction, $m_z \equiv z$

$$\begin{aligned} S &= Z^{-1}\frac{3}{2}\int_0^1 dz\left(z^2 - \frac{1}{3}\right) e^{\frac{3}{2}A^0 S\left(z^2 - \frac{1}{3}\right)/kT} \\ z &= \int_0^1 dz e^{\frac{3}{2}A_0 S\left(z^2-\frac{1}{3}\right)/kT}. \end{aligned} \tag{2.202}$$

With $\sigma = S/\kappa$ and $\kappa = \frac{kT}{\frac{3}{2}A^0}$, this takes the form

$$\kappa\sigma = -\frac{1}{2} + \frac{3}{2}\frac{1}{3(r)}\frac{d}{d\sigma}J(\sigma) \tag{2.203}$$

where

$$J(\sigma) = \int_0^1 dz e^{\sigma z^2} \tag{2.204}$$

is related to Dawson's integral

$$D(x) = \int_0^x e^{y^2} dy = x + \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} + \frac{x^7}{7 \cdot 3!} + \dots \tag{2.205}$$

as

$$J(\sigma) = \frac{1}{-\sqrt{\sigma}} D(\sqrt{\sigma}). \tag{2.206}$$

By a partial integration one sees that

$$J(\sigma) = e^{\sigma} - 2\sigma \frac{d}{d\sigma} J(\sigma) \tag{2.207}$$

such that equ. (2.203) can be written as

$$\begin{aligned} S = \kappa\sigma \quad &= \quad -\frac{1}{2} + \frac{2}{3}\frac{1}{2}\left(\frac{e^{\sigma}}{\sqrt{\sigma}D(\sqrt{\sigma})} - \frac{1}{\sigma}\right) \\ & \quad -\frac{1}{2} + \frac{3}{2}\left(\frac{1}{3} + \frac{4}{45}\sigma - \frac{2^3}{3^3 \cdot 5 \cdot 7}\sigma^2 + \frac{2^4}{3^4 \cdot 5^2 \cdot 7}\sigma^3 + \dots\right) \\ & \quad S(\sigma) \end{aligned} \tag{2.208}$$

$$f = \frac{1}{2} A_0 S^2 - kT \log Z = \frac{kT}{3}\left(\kappa\sigma^2 - 3 \log Z\right) \tag{2.209}$$

where Z is the partition function of (2.202)

$$Z = \int_0^1 dz e^{\sigma\left(z^2 - \frac{1}{3}\right)} = e^{-\frac{1}{3}\sigma} \frac{1}{\sqrt{\sigma}} D(\sqrt{\sigma}). \tag{2.210}$$

The solutions are found grahically. Equ. (2.208) is solved by the intersection of the straight lines $\kappa\sigma$ with the curves $S(\sigma)$, when plotting simultaneously $f(\sigma)$ we see the $\kappa >\ll \kappa_c = .147$, the only solution is $S = \sigma = 0$ (normal phase). At $\kappa = \kappa_0$, the order parameter jumps, in a first order phase transition, to

$$S_c = .43, \quad \sigma_c = S_c/\kappa_c = 2.93 \tag{2.211}$$

and for $\kappa \to 0$ (i.e. $T \to 0$), φ approaches unity corresponding to a perfect order (see Fig. A1). The free energy density corresponding to equ. (2.202) is

$$f \quad = \quad \frac{1}{2} A_0 s^2 - kT \log Z = \frac{kT}{3}\left(\kappa\sigma^2 - 3 \log Z\right) \tag{2.212}$$

such that

$$3 \log Z) - \sigma + 3 \log J(\sigma). \tag{2.213}$$

For small values of T we may use (**??**) and expand the free energy in powers of σ:

$$\frac{3f}{kT} = \left(\kappa - \frac{2}{15}\right)\sigma^2 - \frac{8}{3^3 \cdot 5 \cdot 7}\sigma^3 + \frac{2^2}{3^3 \cdot 5^2 \cdot 7}\sigma^4 + \dots \ . \tag{2.214}$$

Conventionally one denotes the temperature at which the quadratic term changes sign by T_*, i.e.

$$\kappa - \frac{2}{15} \equiv \frac{2}{3}\frac{kT_*}{A_0}\left(\frac{T}{T_*} - 1\right) \equiv \kappa^*\left(\frac{T}{T_*} - 1\right) \tag{2.215}$$

such that

$$\kappa^* = \frac{2kT_*}{3A_0} \equiv \frac{2}{15}. \tag{2.216}$$

With this notation, the expression (2.21) amounts to the Landau De Gennes free energy expansion for the nematic liquid crystal. If terms beyond the forth powers are neglected, the first order nature of the transition H is seen to arise from the cubic term at a transition temperature T_t where

$$\frac{(\text{cubic term})^2}{4 \cdot \text{quadratic} \cdot \text{quartic term}} = \frac{\text{cubic term}}{2 \cdot \text{quartic term}}$$

i.e.

$$\frac{T_c}{T_*} - 1 = \frac{(8/3^3 \cdot 5 \cdot 7)^2}{4(2/15)(4/3^3 \cdot 5^2 \cdot 7)} = \frac{10}{63}, \quad \sigma_c = 5 \tag{2.217}$$

or $\frac{T_c}{T_*} = \frac{73}{63} \approx 1.159$ $\left(\text{with } S_c = \kappa_c\sigma_c = \kappa^*\frac{T_c}{T_*}\sigma_c \approx \frac{2}{3}\frac{73}{63}\right)$ which lies somewhat higher than the exact expression $\frac{T_c}{T_*} = \kappa_c\frac{15}{2} \approx 1.1$. Experimentally, T_c lies much closer to T $\left(\frac{T_c}{T_*} \approx 1.0025\right)$ which shows that the cubic coefficient ot the theory is too large with respect to quartic and quadratic ones, a well known weakness of the model.

The whole framework may be generalized to cholesteric liquid crystals. For this purpose we find it useful to rewrite the free energy density (2.18) in another form using the following auxiliary field quantity:

$$\sqrt{\frac{15}{8\pi}}n_\alpha n_\beta Q_{\alpha\beta} \equiv Q(\mathbf{n}). \tag{2.218}$$

Now

$$\frac{f}{kT} = \frac{4\pi}{3\kappa}\int\frac{d\hat{n}}{4\pi}Q^2 - \log Z \tag{2.219}$$

with a partition function

$$Z = \int\frac{d\hat{n}}{4\pi}e^{\sqrt{\frac{4\pi}{5}}\frac{2}{3}Q(n)/\kappa}. \tag{2.220}$$

Expanding in powers of   we see

$$\begin{aligned} \frac{f}{kT} &= \frac{1}{3\kappa}\left(\kappa - \frac{2}{15}\right)4\pi\int\frac{d\tilde{n}}{4\pi}Q^2 \\ &\quad - \left(\frac{2}{3\kappa}\right)^3\sqrt{\frac{4\pi}{5}}^3\int\frac{d\hat{n}}{4\pi}Q^3 \\ &\quad + \left(\frac{2}{3\kappa}\right)^4\sqrt{\frac{4\pi}{5}}^4\left[\frac{1}{24}\int\frac{d\hat{n}}{4\pi}Q^4 - \frac{1}{8}\left(\int\frac{d\hat{n}}{4\pi}Q^2\right)^2\right] + \dots . \end{aligned} \tag{2.221}$$

If we perform the angular averages

$$\begin{aligned}
\int \frac{d\hat{n}}{4\pi} Q^2 &= \frac{15}{8\pi}\frac{2}{15} Q_{\alpha\beta}Q_{\alpha\beta} \\
\int \frac{d\hat{n}}{4\pi} Q^3 &= \left(\frac{15}{8\pi}\right)^{3/2} \frac{8}{105} Q_{\alpha\beta}Q_{\beta\gamma}Q_{\gamma\alpha} \\
\int \frac{d\hat{n}}{4\pi} Q^4 &= \left(\frac{15}{8\pi}\right)^{2} \frac{36}{945} \left(Q_{\alpha\beta}{}^2\right)^2
\end{aligned} \tag{2.222}$$

this becomes

$$\begin{aligned}
\frac{f}{kT} &= \frac{1}{3\kappa^2}\left(\kappa - \frac{2}{15}\right)\mathrm{tr}Q^2 - \frac{\sqrt{6}}{3\cdot 5\cdot 7}\left(\frac{2}{3\kappa}\right)^3 \mathrm{tr}Q^3 \\
&\quad + \frac{1}{700}\left(\frac{2}{3\kappa}\right)^4 (\mathrm{tr}Q^2)^2 + \dots \quad .
\end{aligned} \tag{2.223}$$

Inserting here $Q_{\alpha\beta} = \kappa\sigma\epsilon^{(0)}(m)$, $\mathrm{tr}\epsilon^{(0)2} = 1$, $\mathrm{tr}\epsilon^{(0)^3} = \frac{1}{\sqrt{6}}$, we directly recover (2.21).

This is the appropriate starting point for our generalization to cholesteric liquid crystals. We proceed in two steps. First we allow for spatially varying field configurations. if these are very smooth, it is sufficient to take only lowest order derivatives into account and use the bending energies ($b,\ c > 0$)

$$\begin{aligned}
\frac{f_{bend}}{kT} &= \frac{b}{2}\left(\partial_\gamma Q_{\alpha\beta}\right)^2 + \frac{c}{2}\partial_\alpha Q_{\alpha\gamma}\partial_\beta Q_{\beta\gamma} \\
& \int d\hat{n}\left[\frac{c+b}{2}(\partial Q)^2 - \frac{b}{4}(\mathbf{S}\partial Q)^2\right]
\end{aligned} \tag{2.224}$$

where

$$\mathbf{S} = -i\mathbf{n}\times\partial_n \tag{2.225}$$

is the generator of rotations on the directional vector $\mathbf{n}$. Second we introduce a small parity violating derivative term:

$$\frac{f'p.v.}{kT} = -d\epsilon_{\alpha\beta\gamma}Q_{\alpha\beta}\partial_\gamma Q_{\beta f} = -i\int dnQS\partial Q. \tag{2.226}$$

Now the energy $f_{tot} = f + f_{der} + f_{p.v.}$ is sufficiently general to describe the cholesteric phase transition.

Let us construct the cholesteric ground state. For this we consider small fluctuations and expand Q into normal modes

$$Q_{\alpha\beta}(x) = \sum_{\mathbf{q}}\left(\sum_m \epsilon^{(m)}_{\alpha\beta}(\hat{q})e^{iqx}S^{(m)}(\mathbf{q}) + c.c.\right). \tag{2.227}$$

For the traceless symmetric field $Q_{\alpha\beta}$ there are five helicity tensors: $\epsilon^{(\pm 2)}, \epsilon^{(\pm 1)}, \epsilon^{(0)}$ as defind before,

$$\epsilon^{(2)}_p\alpha\beta(\hat{q})l_\alpha l_\beta = \epsilon^{(-2)*}(\hat{\mathbf{q}})$$

$$\epsilon_{\alpha\beta}^{(1)}(\hat{\mathbf{q}}) = \frac{1}{\sqrt{2}}\left(l_\alpha q_\beta + l_\beta q_\alpha\right) = -\epsilon^{(-1)*}(\hat{\mathbf{q}}) \tag{2.228}$$
$$\epsilon_{\alpha\beta}^{(0)}(\hat{\mathbf{q}}) = \sqrt{\frac{3}{2}}\left(\hat{q}_\alpha q_\beta - \frac{1}{3}\alpha_{\alpha\beta}\right) = \epsilon^{(0)*}(\hat{\mathbf{q}})$$

and the quadratic part of the free energy becomes

$$\begin{aligned}
\frac{f_2}{kT} &= \sum_q \left\{ \frac{1}{2}\left[a + \left(b + \frac{2}{3}c\right)q^2\right] |S^{(0)}(\mathbf{q}|^2 \right. \\
&+\frac{1}{2}\left[\left(a - \frac{d^2}{4\left(b+\frac{c}{2}\right)}\right) + \left(b + \frac{c}{2}\right)\left(q \mp \frac{d}{2\left(b+\frac{c}{2}\right)}\right)^2\right] |S^{(\pm 1)}(\mathbf{q})|^2 \\
&\left. + \frac{1}{2}\left[\left(a - \frac{d^2}{b}\right) + b\left(q \mp \frac{d}{b}\right)^2\right] |S^{(\pm 2)}(\mathbf{q}|^2 \right\}
\end{aligned} \tag{2.229}$$

where

$$A = \frac{2}{3\kappa^2}\left(\kappa - \frac{2}{15}\right) = \frac{4}{45}\left(\frac{T}{T_*} - 1\right)\frac{1}{\kappa^2}. \tag{2.230}$$

From light scattering experiments one find $S^{(2)}$ and $S^{(0)}$ to be the modes of largest fluctuations such that we may conclude $<> 0$. The $S^{(2)}$ mode has a non-vanishing momentum

$$q_n = \frac{d}{b} = \frac{1}{\xi_n} \tag{2.231}$$

which gives rise to normal reflection of circularly polarized light of wavelength $\lambda_R = 4\pi\xi_n$.

The cholesteric ground state may now be found from a superposition of the dominant $m = 2$ and $m = 0$ modes

$$Q_{\alpha\beta} = S^{(0)}\epsilon_{\alpha\beta}^{(0)} + S^{(2)}\left(\epsilon_{\alpha\beta}^{(2)} e^{iq_n x} + c.c.\right). \tag{2.232}$$

Therefore

$$\sqrt{\frac{4\pi}{5}}\frac{2}{3}Q(n) = \left(z^2 - \frac{1}{2}\right)S^{(0)} + \sqrt{\frac{2}{3}}\left(1 - z^2\right)S^{(2)}\cos\left(q_n x + \delta\right) \tag{2.233}$$

where δ is an arbitrary phase. Averaged over a period this results in a free energy $\left(\delta^{(m)} \equiv \kappa\sigma^{(m)}\right)$

$$\begin{aligned}
\frac{f}{kT} &= \frac{1}{3}\kappa\sigma^{(0)2} + \frac{2}{3}\kappa\left(1 - \frac{d^2}{\kappa b}\right)\sigma^{(2)} \\
&\quad -\frac{1}{2\pi}\int_0^{2\pi} d\delta \log \int_0^1 dz e^{\left(z^2 - \frac{1}{3}\right)\sigma^{(0)} + \sqrt{\frac{2}{3}}\left(1-z^2\right)\sigma^{(2)}\cos\delta} \\
&= \frac{\kappa}{3}{\sigma^{(0)}}^2 + \frac{1}{3}\sigma^{(0)} \\
&\quad +\frac{2}{3}\kappa\left(1 - \frac{d^2}{\kappa b}\right){\sigma^{(2)}}^2 - \frac{1}{2\pi}\int_0^{2\pi} d\delta \log J\left(\sigma^{(0)},\ \sigma^{(2)},\ \delta\right)
\end{aligned} \tag{2.234}$$

where

$$J = \int_0^1 dz e^{z^2\left(\sigma^{(0)} - \sqrt{\frac{2}{3}}\sigma^{(2)}\cos\delta\right)} \tag{2.235}$$

is the generalization of the previous integral (2.204). In equilibrium, we now have the equations

$$\begin{aligned} \kappa\sigma^{(0)} &= -\frac{1}{2} + \frac{2}{3}\frac{1}{2\pi}\int_0^{2\pi} d\delta \frac{1}{J}\frac{\partial}{\partial\sigma^{(0)}}J & (2.236) \\ 2\kappa\left(1 - \frac{d^2}{\kappa b}\right)\sigma^{(2)} &= \frac{3}{2}\frac{1}{2\pi}\int_0^{2\pi} d\delta \frac{1}{J}\frac{\partial}{\partial\sigma^{(2)}}J. & (2.237) \end{aligned}$$

The first equation can again be expressed in the same fashion as before in (2.208 except that σ has to be replaced by $\sigma^{(0)} - \sqrt{\frac{2}{3}}\sigma^{(2)}\cos\delta$ and the average has to be taken

$$\kappa\sigma^{(0)} = \frac{1}{2\pi}\int_0^{2\pi} d\delta S\left(\sigma^{(0)} - \sqrt{\frac{2}{3}}\sigma^{(2)}\cos\delta\right) \tag{2.238}$$

while (2.237) has an additional weight factor $(\cos\delta)\cdot\sqrt{\frac{2}{3}}$:

$$2\kappa\left(1 - \frac{d^2}{\kappa b}\right)\sigma^{(2)} = \sqrt{\frac{2}{3}}\frac{1}{2\pi}\int_0^{2\pi} d\delta\cos\delta S\left(\sigma^{(0)} - \sqrt{\frac{2}{3}}\sigma^{(2)}\right). \tag{2.239}$$

In order to establish contact with previous calculations of the cholesteric free energy it is useful to go to the natural variables

$$\begin{aligned} \kappa - \frac{2}{15} &= \frac{2}{21}(\tau + 2\alpha) \\ 2\alpha &= \frac{21}{2}\frac{3}{2}\frac{d^2}{b} & (2.240) \end{aligned}$$

$$\frac{f}{kT} = \frac{25}{56}\tilde{f} \tag{2.241}$$

$$\sigma^{(0)} = \frac{15}{4}x \quad , \quad \sigma^{(2)} = \frac{15}{4}\frac{y}{\sqrt{2}}. \tag{2.242}$$

Then $\tilde{f}$ has the standardized expansion

$$\tilde{f} = (\tau + 2\alpha)x^2 + \tau y^2 - \frac{1}{3}x^3 + xy^2 + \frac{1}{8}\left(x^2 + y^2\right)^2 + \ldots \tag{2.243}$$

in agreement with (**??**). We may now solve the equations

$$\begin{aligned} \left[(\tau + 2\alpha) + \frac{7}{5}\right]x &= \frac{21}{5}\frac{1}{2\pi}\int^{2\pi} 0 d\delta S\left(\frac{15}{4}\left(x - \frac{1}{\sqrt{3}}y\cos\delta\right)\right) \\ 2\left[\tau + \frac{7}{5}\right]y &= \sqrt{3}\frac{21}{5}\frac{1}{2\pi}\int_0^{2\pi} d\delta\cos\delta S\left(\frac{15}{4}\left(x - \frac{1}{\sqrt{3}}y\cos\delta\right)\right) & (2.244) \end{aligned}$$

by iteration. The results for $x,\ y,\ \tilde{f}$ are shown in Fig. A2-4 in a contour plot where they are compared with the values obtained previously from the Landau De Gennes expansion (**??**) up to the quartic terms.

The main defect of the model is the large size of the cubic term with the actual transition being much weaker of first order than that in the model. Now, can this aspect be improved?

In the discussion of the free energy we have seen that for small cubic term a_3 there is a region of biaxial order.

Thus the largeness of a_3 in the model seems to be connected with the assumption of uniaxial molecules at the microscopic level. Let us see whether this is, in fact, true. Consider again the namatic free energy(2.28)

$$\begin{aligned} \frac{f}{kT} &= \frac{1}{3\kappa}\left(\kappa-\frac{2}{15}\right)4\pi\int\frac{dn}{4\pi}Q^2(n)-\left(\frac{2}{3\kappa}\right)^2\frac{1}{6}\sqrt{\frac{4\pi}{5}}^3\int\frac{dn}{4\pi}Q^3(n) \\ &+\left(\frac{2}{3\kappa}\right)^4\left(\frac{4\pi}{5}\right)^2\left[\frac{1}{24}\int\frac{dn}{4\pi}Q^4(n)-\frac{1}{8}\left(\int\frac{dn}{4\omega}Q^2(n)\right)^2\right]. \end{aligned} \tag{2.245}$$

The integral over **n** correspond to averaging over all microscopic orientation of the rod-like uniaxial molecules such that the orientational energy is proportional to

$$Q(n)=\sqrt{\frac{15}{8\pi}}\sqrt{\frac{2}{3}}Q^{mol}_{\alpha\beta}Q_{\alpha\beta}=\sqrt{\frac{15}{8\pi}}\left(n_\alpha n_\beta-\frac{1}{3}\delta_{\alpha\beta}\right)Q_{\alpha\beta}. \tag{2.246}$$

Suppose now the microscopic order parameter is biaxial

$$Q^{mol}_{\alpha\beta}=\sqrt{\frac{3}{2}}\left[\left(n_\alpha n_\beta-\frac{1}{3}\delta_{\alpha\beta}\right)+\epsilon\left(m_\alpha m_\beta-\frac{1}{3}\delta_{\alpha\beta}\right)\right] \tag{2.247}$$

where m is another unit vector orthogonal to n. If $n,\ m$ point in z and x direction,

$$\begin{aligned} \sqrt{\frac{2}{3}}Q^{mol}_{\alpha\beta} &= \begin{pmatrix} -\frac{1}{3} & & \\ & -\frac{1}{3} & \\ & & \frac{2}{3}\end{pmatrix}+\epsilon\begin{pmatrix} -\frac{2}{3} & & \\ & -\frac{1}{3} & \\ & & -\frac{1}{3}\end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 2\epsilon-1 & & \\ & -(1+\epsilon) & \\ & & -(2-\epsilon)\end{pmatrix}. \end{aligned} \tag{2.248}$$

By an appropriate choice of ϵ we can now simulate any desired ratio for the three principal axes for the molecules. The spatial averages are a little more involved. Let us parametrize n and m in terms of angles as

$$\begin{aligned} \mathbf{n} &= (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta) \\ \mathbf{m} &= (\cos\theta\cos\varphi\cos\gamma-\sin\varphi\sin\gamma, \\ &\quad \cos\theta\sin\varphi\cos\gamma+\cos\varphi\sin\gamma,\ -\sin\theta\cos\gamma)\,. \end{aligned} \tag{2.249}$$

Then the directional average must be performed as an integral

$$\begin{aligned}\langle\rangle_{dir} &= \langle\rangle_z\langle\rangle_\gamma \\ &\int \frac{dn}{4\pi}\int_0^{2\pi}\frac{d\gamma}{2\pi} = \int_1^1 \frac{dz}{2}\int_0^{2\pi}\frac{2\varphi}{2\pi}\int_0^{2\pi}\frac{d\gamma}{2\pi}.\end{aligned} \tag{2.250}$$

Since the resulting invariants to be formed from $Q_{\alpha\beta}$ are unique up to quartic power we may use the specific form $Q_{\alpha\beta})\begin{pmatrix} -Q & & \\ & -Q & \\ & & 2Q\end{pmatrix}$ and substitute, at the end, $Q^2 = \frac{1}{6}\mathrm{tr}Q^2,\quad Q^3 = \frac{1}{6}\mathrm{tr}Q^3,\quad Q^4 = \frac{1}{36}\left(\mathrm{tr}Q^2\right)^2$. Then

$$\begin{aligned}\sqrt{\frac{2}{3}}Q^{mol}_{\alpha\beta}Q_{\alpha\beta} &= Q\left[\left(-n_x{}^2 - n_\gamma{}^2 + 2n_z\right) + \epsilon\left(-mx^2 - m_y{}^2 + 2m_z{}^2\right)\right] \\ &= Q\left(3n_z{}^2 - 1 + \epsilon\left(3m_z{}^2 - 1\right)\right) \\ &= Q\left(3z^2 - 1 + \epsilon\left(3\cos^2\alpha\left(1 - z^2\right) - 1\right)\right) \\ &= Q\left(3z^2 a - b\right)\end{aligned} \tag{2.251}$$

where we have set

$$a = 1 - \epsilon\cos^2\gamma \quad , \quad b = 1 - \epsilon\left(3\cos^2\gamma - 1\right). \tag{2.252}$$

the averages in z are easily performed and give

$$\begin{aligned}\langle Q^2(n)\rangle &= \frac{15}{8\pi}\quad \frac{trQ^2}{6}\langle\frac{9}{5}a^2 - 2ab + b^2\rangle_\gamma \\ \langle Q^3(n)\rangle &= \left(\frac{15}{8\pi}\right)^{3/2}\quad \frac{trQ^3}{6}\langle\frac{27}{7}a^3 - \frac{27}{5}a^2b + 3ab^2 - b^3\rangle_\gamma \\ \langle Q^4(n)\rangle &= \left(\frac{15}{8\pi}\right)^2\quad \frac{(\mathrm{tr}Q^2)^2}{36}\langle\frac{81}{9}a^4 - \frac{108}{7}a^3b + \frac{54}{5}a^2b^2 - \frac{12}{3}ab^3 + b^4\rangle_\gamma\end{aligned} \tag{2.253}$$

For $\epsilon = 0, a = b = 1$ and we obtain back the previous results (2.222). Now there is an additional average over γ. For this we remember $\langle\cos^{2n}\gamma\rangle_\gamma = \frac{(2n-1)!!}{2n!!}$, i.e. $\frac{1}{2},\ \frac{3}{8},\ \frac{5}{16},\ \frac{35}{(8\cdot 16)}$ for $n_1 = 1, 2, 3, 4$. Then we write $a = 1 - \epsilon d$, $b = 1 - \epsilon\beta$ with $\alpha = \cos^2\gamma,\quad \beta = 3\cos^2\gamma - 1$ and calculate

$$\begin{aligned}\langle\alpha\rangle &= \langle\beta\rangle = \frac{1}{2}, \langle\alpha^2\rangle = \frac{3}{8}, \langle\alpha\beta\rangle = \frac{5}{8}, \langle\beta^2\rangle = \frac{11}{8} \\ \langle\alpha^3\rangle &= \frac{5}{16}, \langle\alpha^2\beta\rangle = \frac{9}{16}, \langle\alpha\beta^2\rangle = \frac{17}{16}, \langle\beta^3\rangle = \frac{29}{16} \\ \langle\alpha^4\rangle &= \frac{35}{128}, \langle\alpha^3\beta\rangle = \frac{65}{128}, \langle\alpha^2\beta^2\rangle = \frac{123}{128}, \langle\alpha\beta^3\rangle = \frac{233}{128}, \langle\beta^4\rangle = \frac{467}{128}.\end{aligned} \tag{2.254}$$

Then we find

$$\langle a^2\rangle \quad = \quad 1 - \epsilon + \frac{3}{8}\epsilon^2\langle ab\rangle = 1 - \epsilon + \frac{5}{8}\epsilon^2\langle b^2\rangle = 1 - \epsilon + \frac{11}{8}\epsilon^2$$

$$
\begin{aligned}
\langle a^3 \rangle &= 1 - \frac{3}{2}\epsilon + \frac{9}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 \\
\langle a^2 b \rangle &= 1 - \frac{3}{2}\epsilon + \frac{13}{8}\epsilon^2 - \frac{9}{16}\epsilon^3 \\
\langle a b^2 \rangle &= 1 - \frac{3}{2}\epsilon + \frac{21}{8}\epsilon^2 - \frac{17}{16}\epsilon^3 \\
\langle b^3 \rangle &= 1 - \frac{3}{2}\epsilon + \frac{33}{8}\epsilon^2 - \frac{29}{16}\epsilon^3 \\
\langle a^4 \rangle &= 1 - 2\epsilon + \frac{18}{8}\epsilon^2 - \frac{5}{4}\epsilon^3 + \frac{35}{128}\epsilon^4 \\
\langle a^3 b \rangle &= 1 - 2\epsilon + 3\epsilon^2 - 2\epsilon^3 + \frac{65}{128}\epsilon^4 \\
\langle a^2 b^2 \rangle &= 1 - 2\epsilon + \frac{34}{8}\epsilon^2 - \frac{52}{16}\epsilon^3 + \frac{123}{8 \cdot 16}\epsilon^4 \\
\langle a b^3 \rangle &= 1 - 2\epsilon + 6\epsilon^2 - \frac{80}{16}\epsilon^3 + \frac{233}{8 \cdot 16}\epsilon^4 \\
\langle b^4 \rangle &= 1 - 2\epsilon + \frac{66}{8}\epsilon^2 - \frac{29}{4}\epsilon^3 + \frac{467}{8 \cdot 16}\epsilon^4 .
\end{aligned}
\tag{2.255}
$$

Combining these we obtain the following correction factors to the $\epsilon = 0$ terms of (2.253)

$$
\begin{aligned}
&\left(1 - \epsilon + \epsilon^2\right) \\
&\left(1 - \frac{3}{2}\epsilon - \frac{3}{2} + \epsilon^3\right) \\
\left(1 - 2\epsilon + 3\epsilon^2 - 2\epsilon^3 + \epsilon^4\right) &= \left(1 - \epsilon + \epsilon^2\right)^2 .
\end{aligned}
\tag{2.256}
$$

Going back to (2.221) we see that the first coeffecients two multiply directly the $\epsilon = 0$ coefficients of (2.223) while the quartic term receives a combined correction factor

$$
a_4 \rightarrow a_4 \left(\frac{7}{2}\left(1 - \epsilon + \epsilon^2\right)^2 - \frac{5}{2}\left(1 - \epsilon + \epsilon^2\right)^2 \right) = a_4 \left(1 - \epsilon + \epsilon^2\right)^2 . \tag{2.257}
$$

Since the cubic factor may be written as $\left(\epsilon - \frac{1}{2}\right)(\epsilon + 1)(\epsilon - 2)$ we see that we can indeed make it arbitrary small, for example by choosing $\epsilon \approx \frac{1}{2}$. Notice that the values $\epsilon = \frac{1}{2}, \quad \epsilon = -1, \quad \epsilon = 2$ correspond to

$$
\sqrt{\frac{3}{2}} Q^{mol} = \frac{1}{2} \begin{pmatrix} 0 & & \\ & -1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & 0 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \tag{2.258}
$$

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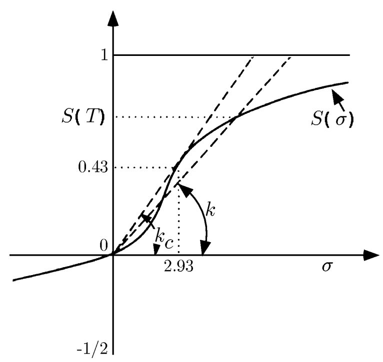


FIGURE 2.3

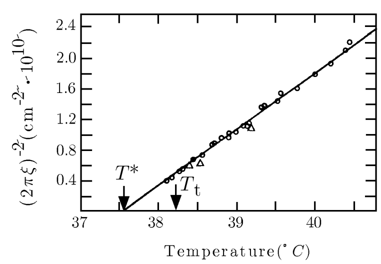
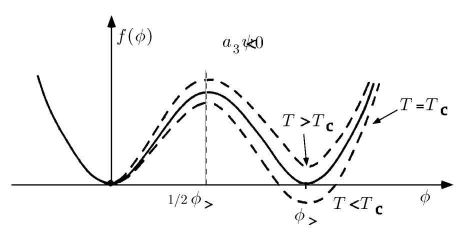


FIGURE 2.4



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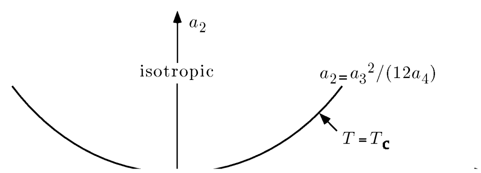

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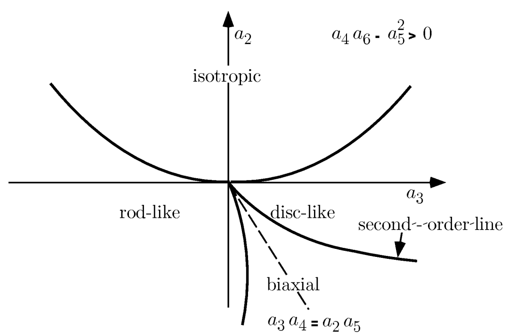

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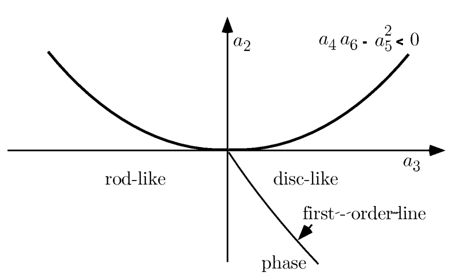

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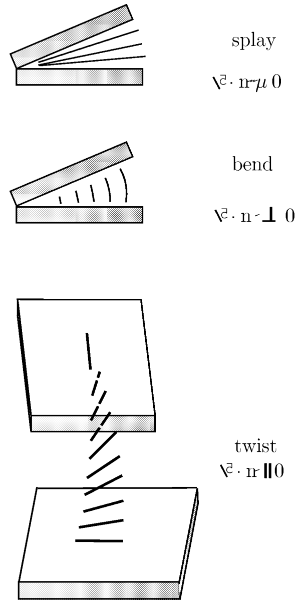

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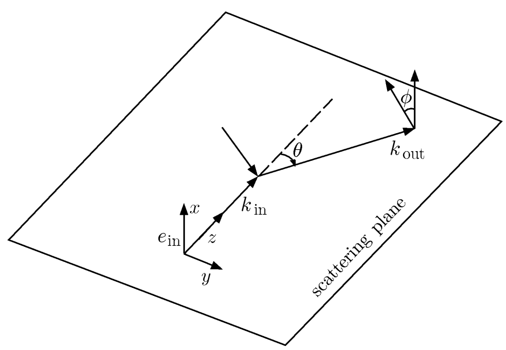

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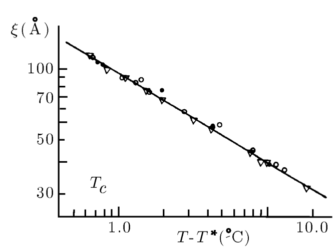

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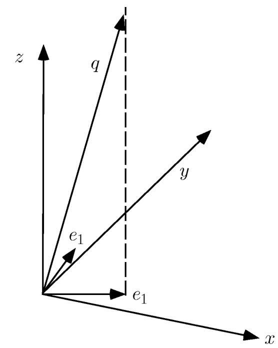

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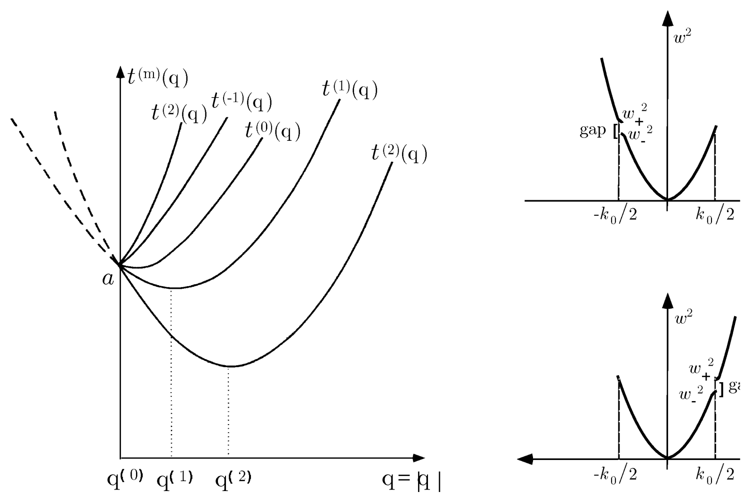


FIGURE 2.13

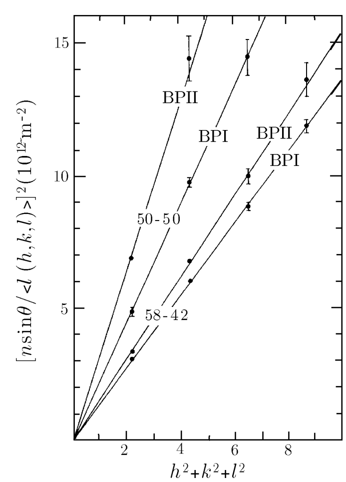
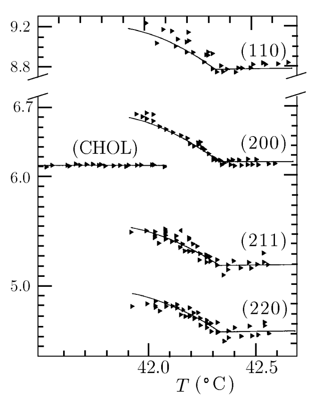


FIGURE 2.14



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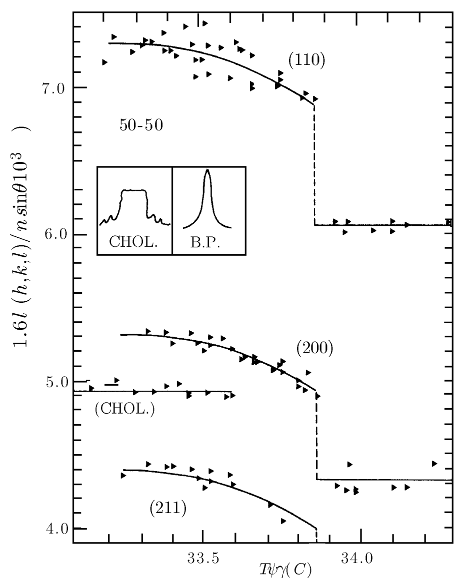

FIGURE 2.16