Canonical Transformations

in

Quantum Field Theory

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Contents

Introduction

In this lecture notes, we discuss canonical transformations in the context of Quantum Field Theory (QFT).

The aim is not that of give a complete and exhaustive treatment of canonical transformations from a mathematical point of view. Rather, we will try to show, through some concrete examples, the physical relevance of these transformations in the framework of QFT.

This relevance is on two levels: a formal one, in which canonical transformations are an important tool for the understanding of basic aspects of QFT, such as the existence of inequivalent representations of the canonical commutation relations (see §1.2) or the way in which symmetry breaking occurs, through a (homogeneous or non-homogeneous) condensation mechanism (see Section 4), On the other hand, they are also useful in the study of specific physical problems, like the superconductivity (see §2.2) or the field mixing (see Section 5).

In the next we will restrict our attention to two specific (linear) canonical transformations: the Bogoliubov rotation and the boson translation. The reason for studying these two particular transformations is that they are of crucial importance in QFT, where they are associated to various condensation phenomena.

The plan of the lectures is the following: in Section 1 we review briefly canonical transformations in classical and Quantum Mechanics (QM) and then we discuss some general features of QFT, showing that there canonical transformations can have non-trivial meaning, whereas in QM they do not affect the physical level. In Section 2 and 3 we consider some specific problems as examples: superconductivity, QFT at finite temperature, the quantization of a simple dissipative system and the quantization of a boson field on a curved background. In all of these subjects, the ideas and the mathematical tools presented in Section 1 are applied. In Section 4 we show the connection between spontaneous symmetry breakdown and boson translation. We also show by means of an example, how macroscopic (topological) object can arise in QFT, when suitable canonical transformations are performed. Finally, Section 5 is devoted to the detailed study of the field mixing, both in the fermion and in the boson case. As an application, neutrino oscillations are discussed.

Section 1

Canonical transformations in Quantum Field Theory

1.1 Canonical transformations in Classical and Quantum Mechanics

Let us consider[1, 2] a system described by n independent coordinates $(q_1, ..., q_n)$ together with their conjugate momenta $(p_1, ..., p_n)$.

The Hamilton equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad , \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{1.1}$$

By introducing a $2n$-dimensional phase space with coordinate variables

$$(\eta_1, ..., \eta_n, \eta_{n+1}, ..., \eta_{2n}) = (q_1, ..., q_n, p_1, ..., p_n) \tag{1.2}$$

the Hamilton equations are rewritten as

$$\dot{\eta}_i = J_{ij} \frac{\partial H}{\partial \eta_j} \tag{1.3}$$

where J_{ij} is a $2n \times 2n$ matrix of the form

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{1.4}$$

and I is the $n \times n$ identity matrix.

The transformations which leave the form of Hamilton equations invariant are called *canonical transformations*. Let us consider the transformation of variables from η_i to ξ_i. We define the matrix

$$M_{ij} = \frac{\partial}{\partial \eta_j} \xi_i \tag{1.5}$$

Then we have

$$\dot{\xi}_i = M_{ij} \, J_{jk} \, M_{lk} \frac{\partial H}{\partial \xi_l} \tag{1.6}$$

Thus, the condition for the invariance of the Hamilton equations reads

$$M\,J\,M^t \,=\, J \tag{1.7}$$

The group of linear transformations satisfying the above condition is called the symplectic group.

Let us now introduce the Poisson brackets:

$$\{f\,,\,g\}_{q,p} \,=\, \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \,-\, \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}\right) \tag{1.8}$$

where f and g are function of the canonical variables. By use of the η_i variables eq.(1.2), this expression can be rewritten as

$$\{f\,,\,g\}_{q,p} \,=\, \sum_{ij} J_{ij}\,\frac{\partial f}{\partial \eta_i}\frac{\partial g}{\partial \eta_j} \tag{1.9}$$

It is thus quite clear that the Poisson bracket is invariant under canonical transformations. With this understanding we can delete the p, q subscript from the bracket. From the definition,

$$\{q_i\,,\,q_j\} \,=\, 0 \qquad , \qquad \{p_i\,,\,p_j\} \,=\, 0 \qquad , \qquad \{q_i\,,\,p_j\} \,=\, \delta_{ij} \tag{1.10}$$

We can also rewrite the Hamilton equations in terms of the Poisson brackets, as

$$\dot{q}_i \,=\, \{q_i\,,\,H\} \qquad , \qquad \dot{p}_i \,=\, \{p_i\,,\,H\} \tag{1.11}$$

The Poisson brackets provide the bridge between classical and quantum mechanics. In QM, $\hat{p}$ and $\hat{q}$ are operators and the Poisson brackets is replaced by the commutator through the replacement

$$\{f\,,\,g\} \,\rightarrow\, -\frac{i}{\hbar}[\hat{f}\,,\,\hat{g}] \tag{1.12}$$

with $[\hat{f}, \hat{g}] \equiv \hat{f}\hat{g} - \hat{g}\hat{f}$. We have

$$[\hat{q}_i\,,\,\hat{q}_j] \,=\, 0 \qquad , \qquad [\hat{p}_i\,,\,\hat{p}_j] \,=\, 0 \qquad , \qquad [\hat{q}_i\,,\,\hat{p}_j] \,=\, i\hbar\,\delta_{ij} \tag{1.13}$$

We can also rewrite the Hamilton equations in terms of the Poisson brackets, as

$$\dot{\hat{q}}_i \,=\, [\hat{q}_i\,,\,\hat{H}] \qquad , \qquad \dot{\hat{p}}_i \,=\, [\hat{p}_i\,,\,\hat{H}] \tag{1.14}$$

1.2 Inequivalent representations of the canonical commutation relations

The commutation relations defining the set of canonical variables q_i and p_i for a particular problem, are algebraic relations, essentially independent from the Hamiltonian, i.e. the dynamics. They define completely the system at a given time, in the sense that any physical quantity can be expressed in terms of them.

However, in order to determine the time evolution of the system, it is necessary to represent the canonical variables as operators in a Hilbert space. The important point is that in QM, i.e. for systems with a finite number of degrees of freedom, the choice of representation is inessential to the physics, since all the irreducible representations of the canonical commutation relations (CCR) are each other unitarily equivalent: this is the content of the Von Neumann theorem [3, 4]. Thus the choice of a particular representation in which to work, reduces to a pure matter of convenience.

The situation changes drastically when we consider systems with an infinite number of degrees of freedom. This is the case of QFT, where systems with a very large number N of constituents are considered, and the relevant quantities are those (like for example the density $n = N/V$) which remains finite in the thermodynamical limit ($N \to \infty$, $V \to \infty$).

In contrast to what happens in QM, the Von Neumann theorem does not hold in QFT, and the choice of a particular representation of the field algebra can have a physical meaning. From a mathematical point of view, this fact is due to the existence in QFT of unitarily inequivalent representations of the CCR [5, 6, 4, 7].

In the following we show how inequivalent representation can arise as a result of canonical transformations in the context of QFT: we consider explicitly two particularly important cases of linear transformations, namely the boson translation and the Bogoliubov transformation (for bosons).

- **The boson translation**

Let us consider first QM. a is an oscillator operator defined by

$$\begin{aligned} \left[a, a^\dagger\right] \;&=\; a a^\dagger - a^\dagger a \;=\; 1 \\ a|0\rangle \;&=\; 0 \end{aligned} \tag{1.15}$$

We denote by $\mathcal{H}[a]$ the Fock space built on $|0\rangle$ through repeated applications of the operator $a^\dagger$:

$$|n\rangle = (n!)^{-\frac{1}{2}}(a^\dagger)^n|0\rangle \qquad , \quad \mathcal{H}[a] = \{\sum_{n=1}^{\infty} c_n|n\rangle \,,\, \sum_{n=1} |c_n|^2 < \infty\}\,. \tag{1.16}$$

Let us now perform the following transformation on a, called Bogoliubov translation for coherent states or *boson translation*:

$$a \;\longrightarrow\; a(\theta) \;=\; a \,+\, \theta\,, \qquad \theta \in C \tag{1.17}$$

This is a canonical transformation, since it preserves the commutation relations (1.15):

$$\left[a(\theta), a^\dagger(\theta)\right] \;=\; 1 \tag{1.18}$$

We observe that $a(\theta)$ does not annihilate the vacuum $|0\rangle$

$$a(\theta)|0\rangle = \theta|0\rangle \tag{1.19}$$

We then define a new vacuum $|0(\theta)\rangle$, annihilated by $a(\theta)$, as

$$a(\theta)|0(\theta)\rangle = 0 \tag{1.20}$$

In terms of $|0(\theta)\rangle$ and $\{a(\theta)\,,\, a(\theta)^\dagger\}$ we have thus constructed a new Fock representation of the canonical commutation relations.

It is useful to find the generator of the transformation (1.17). We have[1]

$$a(\theta) \;=\; U(\theta)\, a\, U^{-1}(\theta) \;=\; a \,+\, \theta \tag{1.21}$$

$$U(\theta) = exp\,[iG(\theta)] \qquad , \qquad G(\theta) = -i(\theta^* a - \theta a^\dagger) \tag{1.22}$$

with U unitary $U^\dagger = U^{-1}$, thus the new representation is *unitarily equivalent* to the original one. The new vacuum state is given by[2]:

$$\begin{aligned} |0(\theta)\rangle &\equiv U(\theta)\,|0\rangle \\ &= exp\left[-\frac{1}{2}|\theta|^2\right]\; exp\left[-\theta a^\dagger\right]\; |0\rangle \end{aligned} \tag{1.23}$$

i.e., $|0(\theta)\rangle$ is a condensate of a-quanta; The number of a particles in $|0(\theta)\rangle$ is

$$\langle 0(\theta)|a^\dagger a|0(\theta)\rangle \;=\; |\theta|^2 \tag{1.24}$$

We now consider QFT. The system has infinitely many degrees of freedom, labelled by $\mathbf{k}$:

$$\begin{aligned} &\left[a_{\mathbf{k}}(\theta), a^\dagger_{\mathbf{q}}(\theta)\right] = \delta^3(\mathbf{k}-\mathbf{q}) \qquad , \qquad [a_{\mathbf{k}}(\theta), a_{\mathbf{q}}(\theta)] = 0 \\ &a_{\mathbf{k}}\,|0\rangle \;=\; b_{\mathbf{k}}\,|0\rangle \;=\; 0 \end{aligned} \tag{1.25}$$

We perform the boson translation for each mode separately,

$$a_{\mathbf{k}} \;\longrightarrow\; a_{\mathbf{k}}(\theta) \;=\; a_{\mathbf{k}} \,+\, \theta_{\mathbf{k}}\;,\;\; \theta_{\mathbf{k}} \in C \tag{1.26}$$

and define the new vacuum

$$a_{\mathbf{k}}(\theta)|0(\theta)\rangle = 0 \qquad \forall \mathbf{k}\;. \tag{1.27}$$

As a straightforward extension of eqs.(1.21), (1.22) we can write (since modes with different k commute among themselves):

$$a_{\mathbf{k}}(\theta) \;=\; U(\theta) a_{\mathbf{k}} U^{-1}(\theta) \;=\; a_{\mathbf{k}} \,+\, \theta_{\mathbf{k}} \tag{1.28}$$

$$U(\theta) = exp[iG(\theta)] \qquad , \qquad G(\theta) = -i \int d^3\mathbf{k}(\theta^*_{\mathbf{k}} a_{\mathbf{k}} - \theta_{\mathbf{k}} a^\dagger_{\mathbf{k}}) \tag{1.29}$$

so that we have

$$|0(\theta)\rangle \;=\; exp\left[-\frac{1}{2}\int d^3\mathbf{k}|\theta_{\mathbf{k}}|^2\right]\; exp\left[-\int d^3\mathbf{k}\theta_{\mathbf{k}} a^\dagger_{\mathbf{k}}\right]\; |0\rangle \tag{1.30}$$

[1]see Appendix

[2]see Appendix

The number of quanta with momentum $\mathbf{k}$ is

$$\langle 0(\theta)|a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}}|0(\theta)\rangle \;=\; |\theta_{\mathbf{k}}|^2 \tag{1.31}$$

Consider now the projection of $\langle 0|$ on $|0(\theta)\rangle$. We have, by using eq.(1.30)

$$\langle 0|0(\theta)\rangle \;=\; exp\left[-\frac{1}{2}\int d^3\mathbf{k}|\theta_{\mathbf{k}}|^2\right] \tag{1.32}$$

If it happens that $\int d^3\mathbf{k}|\theta_{\mathbf{k}}|^2 = \infty$, then $\langle 0|0(\theta)\rangle = 0$ and the two representations are inequivalent. A situation in which this occurs is for example when $\theta_{\mathbf{k}} = \theta\delta(\mathbf{k})$: in this case the condensation is homogeneous, i.e. the spatial distribution of the condensed bosons is uniform. Then we have

$$\int d^3\mathbf{k}|\theta_{\mathbf{k}}|^2 \;=\; \theta^2\delta(\mathbf{k})|_{\mathbf{k}=0} \tag{1.33}$$

which is infinite, in the infinite volume limit ($V \to \infty$), since the delta is $\delta(\mathbf{k}) = (2\pi)^{-3}\int d^3x\, e^{i\mathbf{kx}} = (2\pi)^{-3}V$.

Eq. (1.26) then defines a *non-unitary* canonical transformation: by acting with $U(\theta)$ on the vacuum leads out of the original Hilbert space. Thus the spaces $\mathcal{H}[a]$ and $\mathcal{H}[\alpha(\theta)]$ are orthogonal. and the representations associated to $\mathcal{H}[a]$ and $\mathcal{H}[\alpha(\theta)]$ are said to be *unitarily inequivalent.*

Note that the total number $N = \int d^3\mathbf{k} n_{\mathbf{k}}$ of $a_{\mathbf{k}}$ particles in the state $|0(\theta)\rangle$ is infinite, however the *density* remains finite

$$\frac{N}{V} \;=\; \frac{1}{V}\int d^3\mathbf{k}|\theta_{\mathbf{k}}|^2 \;=\; (2\pi)^{-3}\theta^2 \tag{1.34}$$

We can write the boson translation at the level of the field, as

$$\hat{\phi}(x) = \hat{\rho}(x) + f(x) \tag{1.35}$$

still being a canonical transformation. However (1.35) has a more general meaning of the transformation (1.26) since it includes also the cases for which $f(x)$ is not Fourier transformable and thus does not reduce to (1.26). The transformation (1.26) is called the *boson transformation.*

We will see in Section 4 how this transformation plays a central role in the discussion of symmetry breaking.

• The Bogoliubov transformation

We now consider a different example in which two different modes a and b are involved. We consider a simple bosonic system as example. The extension to the fermionic case is straightforward[5].

The canonical commutation relations for the $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ are:

$$\left[a_{\mathbf{k}}, a^{\dagger}_{\mathbf{p}}\right] = \left[b_{\mathbf{k}}, b^{\dagger}_{\mathbf{p}}\right] = \delta^3(\mathbf{k} - \mathbf{p}) \tag{1.36}$$

with all other commutators vanishing.

Denote now with $\mathcal{H}(a,b)$ the Fock space obtained by cyclic applications of $a^\dagger_{\mathbf{k}}$ and $b^\dagger_{\mathbf{k}}$ on the vacuum $|0\rangle$ defined by

$$a_{\mathbf{k}}|0\rangle = b_{\mathbf{k}}|0\rangle = 0 \tag{1.37}$$

$\mathcal{H}(a,b)$ is an irreducible representation of (1.36).

Let us consider the following (Bogoliubov) transformation:

$$\begin{aligned}\alpha_{\mathbf{k}}(\theta) &= a_{\mathbf{k}} \cosh \theta_{\mathbf{k}} - b^\dagger_{\mathbf{k}} \sinh \theta_{\mathbf{k}} \\ \beta_{\mathbf{k}}(\theta) &= b_{\mathbf{k}} \cosh \theta_{\mathbf{k}} - a^\dagger_{\mathbf{k}} \sinh \theta_{\mathbf{k}}\end{aligned} \tag{1.38}$$

The Bogoliubov transformation (1.38) is canonical, in the sense that it preserves the CCR (1.36); we have indeed

$$\left[\alpha_{\mathbf{k}}, \alpha^\dagger_{\mathbf{p}}\right] = \left[\beta_{\mathbf{k}}, \beta^\dagger_{\mathbf{p}}\right] = \delta^3(\mathbf{k}-\mathbf{p}) \tag{1.39}$$

and all the other commutators between the α's and the β's vanish.

By defining the vacuum relative to α and β as

$$\alpha_{\mathbf{k}}(\theta)\,|0(\theta)\rangle = \beta_{\mathbf{k}}(\theta)\,|0(\theta)\rangle = 0, \tag{1.40}$$

we can construct the Fock space $\mathcal{H}(\alpha,\beta)$ by cyclic applications of $\alpha^\dagger$ and $\beta^\dagger$ on $|0(\theta)\rangle$. Since the transformation (1.38) is a canonical one, also $\mathcal{H}(\alpha,\beta)$ is an irreducible representation of (1.36).

If now we assume the existence of an unitary operator $G(\theta)$[3] which generates the transformation (1.38),

$$\begin{aligned}\alpha_{\mathbf{k}}(\theta) &= U(\theta)\, a_{\mathbf{k}}\, U^{-1}(\theta) \\ \beta_{\mathbf{k}}(\theta) &= U(\theta)\, b_{\mathbf{k}}\, U^{-1}(\theta)\end{aligned} \tag{1.41}$$

where[4]

$$U(\theta) \;=\; exp[iG(\theta)] \qquad , \qquad G(\theta) = i\int d^3\mathbf{k}\,\theta_{\mathbf{k}}\left(a_{\mathbf{k}}b_{\mathbf{k}} - b^\dagger_{\mathbf{k}}a^\dagger_{\mathbf{k}}\right) \tag{1.42}$$

We have the relation[5, 8]

$$U(\theta) \;=\; exp\left[-\delta(\mathbf{0})\int d^3\mathbf{k}\, \log\cosh\,\theta_{\mathbf{k}}\right] exp\left[\int d^3\mathbf{k}\, \tanh\,\theta_{\mathbf{k}}\, a^\dagger_{\mathbf{k}}b^\dagger_{\mathbf{k}}\right] exp\left[-\int d^3\mathbf{k}\, \tanh\,\theta_{\mathbf{k}}\, b_{\mathbf{k}}a_{\mathbf{k}}\right] \tag{1.43}$$

then we have[5]

$$|0(\theta)\rangle = exp\left[-\delta(\mathbf{0})\int d^3\mathbf{k}\, \log\cosh\,\theta_{\mathbf{k}}\right] exp\left[\int d^3\mathbf{k}\, \tanh\,\theta_{\mathbf{k}}\, a^\dagger_{\mathbf{k}}b^\dagger_{\mathbf{k}}\right]|0\rangle \tag{1.44}$$

Since $\delta(\mathbf{0}) \equiv \delta(\mathbf{k})|_{\mathbf{k}=0} = \infty$, the above relation implies that $|0(\theta)\rangle$ cannot be expressed in terms of vectors of $\mathcal{H}(a,b)$, unless $\theta_{\mathbf{k}} = 0$ for any $\mathbf{k}$. This means that a generic vector of $\mathcal{H}(\alpha,\beta)$

[3] This is possible only at finite volume.

[4] see Appendix

[5] One can also consider the relation $\sum_k \to (2\pi)^{-3}V\int d^3k$ to understand naively the appearance of the $\delta(0)$ in eq.(1.43).

cannot be expressed in terms of vectors of $\mathcal{H}(a,b)$: the spaces $\mathcal{H}(\alpha,\beta)$ and $\mathcal{H}(a,b)$ are each other orthogonal.

In other words: the two irreducible representations of the CCR (1.36), $\mathcal{H}(\alpha,\beta)$ and $\mathcal{H}(a,b)$, are unitarily inequivalent each other since the transformation (1.38) cannot be generated by means of an unitary operator $G(\theta)$.

In more physical terms, one can think to the state $|0(\theta)\rangle$ as a condensed state of bosons a and b: since the vacuum should be invariant under translations, it follows that a locally observable condensation can be obtained only if an infinite number of particles are condensed in it.

1.3 Free fields and interacting fields in QFT

In this Section we consider another aspect of QFT, also connected to the existence of inequivalent representations: the difference between physical (free) and Heisenberg (interacting) fields.

First we clarify what we mean for physical fields. In a scattering process one everytime can distinguish between a first stage in which the "incoming" (or "in") particles can be identified through some measurement; a second stage, in which the particles interact; finally a third stage, where again the "outgoing" (or "out") particles can be identified. What one does everytime observe in such a process is that the sum of the energies of the incoming particles equals that of the outgoing particles.

Thus in the following we will intend for "physical" or "free" particles just these in or out particles (and the relative fields)[6]. It is worth stressing that the word "free" does not mean non-interacting, but only that the total energy of the system is given by the sum of the energies of each (observed) particle.

The Fock space of physical particles can be then constructed from the vacuum state $|0\rangle$ by the action of the creation operators corresponding to the free particles[7].

However, the space $\mathcal{H}$ so built contains also vectors with an infinite number of particles, and this implies that the basis on which it is constructed is non-numerable. It is then necessary to isolate a separable subspace $\mathcal{H}_0$ from $\mathcal{H}$ to the end of correctly represent the physical system under consideration. Without entering in the details of such a construction [5], it is here sufficient to say that $\mathcal{H}$ results to be an irreducible representations of the canonical variables obtained from the physical variables under consideration.

This fact imply the existence of infinite Fock spaces unitarily inequivalent among themselves, in correspondence of the infinite inequivalent representations of the algebra of the canonical variables (see §1.2). The choice of the representation is dictated by the physical system under consideration.

[6] In solid state physics, the physical particles are called quasiparticles.

[7] Actually, one should work with wave packets: the creation operators indeed map normalizable vectors into non-normalizable ones. However this point is inessential to the present discussion.

Let us now consider the set $\phi_i(x)$ of the physical fields under examination: they are in general column vectors and $x \equiv (t, \mathbf{x})$. These fields will satisfy some linear homogeneous equations of the kind:

$$\Lambda_i(\partial)\, \phi_i(x) = 0 \tag{1.45}$$

where the differential operators $\Lambda_i(\partial)$ are in general matrices.

Although the physical fields $\phi_i(x)$ represent particles which undergo to interaction, it is however evident that the free field equations (1.45) do not contain any information about interaction. It is then necessary to introduce other fields $\psi_i(x)$, called Heisenberg fields and the existence of which is postulated, such that they satisfy the relations for the dynamics. These relations are the Heisenberg equations and can be formally written as

$$\Lambda_i(\partial)\, \psi_i(x) = F\,[\psi_i(x)] \tag{1.46}$$

where $\Lambda_i(\partial)$ is the same differential operator of the free field equations (1.45) for the $\phi_i(x)$ and F is a functional of the $\psi_i(x)$ fields.

1.3.1 The dynamical map

The Heisenberg equations (1.46) are however only formal relations among the $\psi_i(x)$ operators, until one represents them on a given vector space.

This means that, in order to give a physical sense to the description in terms of Heisenberg fields, it is necessary to represent them in the space of the physical states and this in turn requires to represent them in terms of the physical fields $\phi_i(x)$.

The relation between Heisenberg fields and physical fields is called *dynamical map* [5], and by use of it, the Heisenberg equation (1.46) can be read as a relation between matrix elements in the Fock space of the physical particles. Such a kind of relations are also called *weak relations*, in the sense that they depend in general on the (Hilbert) space where they are represented.

Then the dynamical map is written as

$$\psi(x) \stackrel{w}{=} \mathcal{F}\,[\phi(x)] \tag{1.47}$$

where the superscript w denotes a weak equality.

A condition for the determination of the above mapping is that the interacting Hamiltonian, once rewritten in terms of the physical operators, must have the form of the free Hamiltonian (plus eventually a c-number).

Thus in general, by denoting with H the interacting Hamiltonian and with H_0 the free one, then the weak relation:

$$\langle a|H|b\rangle \;=\; \langle a|H_0|b\rangle \;+\; W_0\langle a|b\rangle\,, \tag{1.48}$$

determines the dynamical map. In eq.(1.48) W_0 is a c-number and $|a\rangle$, $|b\rangle$ are vectors in the Fock space of the physical particles.

A general form for the dynamical map is the following:

$$\begin{aligned} \psi_i(x) &= \chi_i + \sum_j Z_{ij}^{1/2}\, \phi_j(x) + \\ &+ \sum_{i,j} \int d^4y_1\, d^4y_2\, F_{ijk}(x, y_1, y_2)\; : \phi_j(y_1)\, \phi_k(y_2) : + \ldots \end{aligned} \tag{1.49}$$

where i, j, k are indices for the different physical fields, χ_i are c-number constants (different from zero only for spinless fields[8]), Z_{ij} are c-number constants called renormalization factors, the double dots denote normal ordering, ϕ denotes both the field and its hermitian conjugate, the $F_{ijk}(x, y_1, y_2)$ are c-number functions, and finally the missing terms are normal ordered products of increasing order. The functions χ_i, Z_{ij}, F_{ijk}, etc. are the coefficients of the dynamical map and can be determined in a self-consistent way.

We note that it is not necessary to have a one-to-one correspondence between the sets $\{\psi_i\}$ and $\{\phi_j\}$. Indeed, there can be physical fields which do not appear as a linear term in the dynamical map of any member of $\{\psi_i\}$. These particles are said to be *composite*, and will appear in the linear term of the dynamical map of some *products* of Heisenberg field operators. If for example, n fields ψ_i form such a product, then the composite particle is a n-body bound state.

1.3.2 The self-consistent method

We have seen how in QFT there exist two levels: on one level there are the physical fields, in terms of which the experimental observations are described; on another there are the Heisenberg fields, through which the dynamics of the physical system is described.

We have also seen that is necessary to represent the Heisenberg fields on the Fock space of physical particles, in order to attach them a physical interpretation: this is possible through the dynamical map.

For the construction of this Fock space, it is necessary to know the set of the physical field operators. However, this set is determined by the dynamics, which in turns requires the knowledge of the Fock space of physical particles!

We are then facing a problem of self-consistence[9]. The way one proceed is then the following (*self-consistent method*) [5]: on the basis of physical considerations and of intuition, one chooses a given set of physical fields (e.g. "in" fields) as candidates for the description of the physical system under consideration; then one writes the dynamical map (1.49) in terms of these fields. The problem is then to determine the coefficients of the map: to this end one considers matrix elements (on the physical Fock space) of (1.49), leaving undetermined the form of the energy spectra. The equations for the coefficients of the map are obtained from the Heisenberg equations

[8] χ_i is related to the square root of the boson condensation density.

[9] A similar situation is that of the Lehmann-Symanzik-Zimermann formalism [9], where the "in" (resp. "out") fields are the asymptotic *weak* limit of the Heisenberg fields for $t \to -\infty$ (resp. $t \to +\infty$). In order to perform such a limit, it is necessary to know the Fock space of the "in" (resp. "out") fields.

(1.46), which hold for matrix elements of the $\psi_i(x)$ fields. Thus both the coefficients of the map and the energy spectra of the physical particles are determined.

It may however happen that the system of equations under consideration does not admit consistent solutions: this happens if the set of the physical fields introduced at the beginning is not complete[10]; it is then necessary to conveniently introduce other physical fields and to repeat the entire procedure.

It is important to note that the Heisenberg equations are not the unique condition one has to impose for the calculation of the dynamical map. Indeed, it is not necessary to postulate for the Heisenberg fields the commutation relations, rather one has to calculate them (by using the dynamical map) and to use as a condition on the coefficients of the map.

As an example of self-consistent calculation, let us consider a dynamics of nucleons[5]. Let us assume an Heisenberg equation for the nucleon field and an isodoublet of free Dirac fields, as initial set of physical field. Then, leaving the mass of the physical nucleon undetermined, we express the nucleon Heisenberg field in terms of normal ordered products of the physical nucleon field.

At this point we consider the equation for matrix elements (on the Fock space of the physical particles) of the Heisenberg equation: it is possible to show[5] that a solution does not exist, for any mass of the physical nucleon, unless another field is introduced in the set of the physical fields.

This field correspond to a composite particle, the deuteron, which will not appear in the linear part of the dynamical map for the nucleon.

1.4 Coherent and squeezed states

In §1.2 we have considered two examples of canonical transformations whose effect on the vacuum was that of producing a *condensate* of the quanta under consideration. Actually, quantities like those in eqs.(1.23), (1.30) and (1.44) represent well known objects from a mathematical point of view, since they are respectively *coherent* and *squeezed* states.

• harmonic oscillator coherent states

In the simplest case, coherent states are defined for the harmonic oscillator. In this case there are three equivalent definition for the coherent states $|\theta\rangle$ [10]:

1. as eigenstates of the harmonic-oscillator annihilation operator a:

$$a|\theta\rangle = \theta|\theta\rangle \tag{1.50}$$

with θ c-number.

[10] The existence of a complete set of physical fields implies that any other operator, including the Heisenberg fields, can be expressed in terms of them. The non-completeness of a given set of physical fields can be verified, for example, by finding a given combination of Heisenberg fields whose asymptotic (weak) limit, e.g. for $t \to -\infty$, does commute with all the "in" fields.

2. as the states obtained by the action of a *displacement* operator $U(\theta)$ on a reference state (the vacuum of harmonic oscillator):

$$\begin{aligned}&|\theta\rangle = U(\theta)|0\rangle \\ &U(\theta) = \exp\left(\theta a^\dagger - \theta^* a\right)\end{aligned} \tag{1.51}$$

3. as quantum states of minimum uncertainty:

$$\langle(\Delta p)^2\rangle\langle(\Delta q)^2\rangle = \frac{1}{4} \tag{1.52}$$

where the coordinate and momentum operators are $\hat{q} = \left(a + a^\dagger\right)/\sqrt{2}$ and $\hat{p} = -i\left(a - a^\dagger\right)/\sqrt{2}$ and

$$\begin{aligned}&\langle(\Delta f)^2\rangle \equiv \langle\theta|\left(\hat{f} - \langle\hat{f}\rangle\right)^2|\theta\rangle \\ &\langle\hat{f}\rangle \equiv \langle\theta|\hat{f}|\theta\rangle\end{aligned} \tag{1.53}$$

When $\langle\Delta q\rangle = \langle\Delta p\rangle = \frac{1}{2}$, eq.(1.52) defines coherent states, otherwise we have *squeezed states* (see below).

• **one mode squeezed states**

We now consider *one mode* squeezed states, generated by

$$\begin{aligned}a(\theta) &= U(\theta)\, a\, U^{-1}(\theta) = a \cosh\theta - a^\dagger \sinh\theta \\ a^\dagger(\theta) &= U(\theta)\, a^\dagger\, U^{-1}(\theta) = a^\dagger \cosh\theta - a \sinh\theta\end{aligned} \tag{1.54}$$

with

$$\begin{aligned}&U(\theta) = exp[iG_s(\theta)] \\ &G_s(\theta) = i(a^2 - a^{\dagger 2})\end{aligned} \tag{1.55}$$

The squeezed state (the vacuum for the $a(\theta)$ operators) is defined as

$$|0(\theta)\rangle = exp[iG_s(\theta)]0\rangle = exp\left[-\frac{1}{2}\log\cosh\theta\right] exp\left[\frac{1}{2}\tanh\theta\, a^{\dagger 2}\right]|0\rangle \tag{1.56}$$

where $a|0\rangle = 0$. We have (the $\langle\rangle$ now means expectation value on $|0(\theta)\rangle$):

$$\begin{aligned}&\langle(\Delta p)^2\rangle\langle(\Delta q)^2\rangle = \frac{1}{4} \\ &\langle(\Delta q)^2\rangle = \frac{1}{2}(\cosh\theta + \sinh\theta)^2 \\ &\langle(\Delta p)^2\rangle = \frac{1}{2}(\cosh\theta - \sinh\theta)^2\end{aligned} \tag{1.57}$$

Thus we can reduce (squeeze) the uncertainty in one component, at expense of that in the other component, which should increase.

• **two mode squeezed states**

In this case we need two sets of operators a and $\tilde{a}$, commuting among themselves. They are generated by the following Bogoliubov transformation

$$\begin{aligned} a(\theta) &= U(\theta)\, a\, U^{-1}(\theta) = a \cosh \theta - \tilde{a}^\dagger \sinh \theta \\ \tilde{a}^\dagger(\theta) &= U(\theta)\, \tilde{a}^\dagger\, U^{-1}(\theta) = \tilde{a}^\dagger \cosh \theta - a \sinh \theta \end{aligned} \tag{1.58}$$

with

$$\begin{aligned} U(\theta) = exp[iG_B(\theta)] \\ G_B(\theta) = i(a\tilde{a} - a^\dagger \tilde{a}^\dagger) \end{aligned} \tag{1.59}$$

The squeezed state (the vacuum for the $a(\theta)$ operators) is defined as

$$|0(\theta)\rangle = exp[iG_B(\theta)]0\rangle = exp\left[-\frac{1}{2}\log\cosh\theta\right] exp\left[\frac{1}{2}\tanh\theta\, a^\dagger\tilde{a}^\dagger\right]|0\rangle \tag{1.60}$$

Section 2

Examples

2.1 Superconductivity

We list here the most characteristic phenomenological features of superconductors, as they follow from experimental observations.

• A superconductor is a metal that, below a critical temperature T_c, and for not too high currents, behaves as a perfect conductor, i.e. shows zero resistivity. By Ohm's law

$$\mathbf{E} = \rho \mathbf{j} = 0 \tag{2.61}$$

the electric field vanishes inside the superconductor.

The conductivity remains infinite also when a magnetic field H is applied, provided that $H < H_c(T)$, where $H_c(T)$ is the critical magnetic field at temperature T. Experimentally, one finds the following dependence on T:

$$H_c(T) = H_c(0) \left[1 - (\frac{T}{T_c})^2\right] \tag{2.62}$$

• From Maxwell equations and eq.(2.61), we get

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E} = 0 \tag{2.63}$$

i.e., the magnetic field cannot vary with time inside the superconductor. Thus, if we start with $\mathbf{B} = 0$ and we lower the temperature to a value $T < T_c$, then we can apply an external magnetic field $H < H_c(T)$ and the magnetic field will remain zero inside the material.

However, experimentally one observes also the *Meissner effect*: by starting from $T > T_c$ with $\mathbf{B} \neq 0$, and then lowering the temperature below T_c, one observes that the magnetic field is expelled from the superconductor. Thus, for $T < T_c$, it is always $\mathbf{B} = 0$ inside the superconductor.

• The specific heat C for a superconductor decreases exponentially below T_c:

$$C \sim \exp\left[-\frac{\Delta_0}{k_B T}\right] \tag{2.64}$$

showing the presence of an energy gap $\Delta_0 \simeq 2k_BT$: photon absorption occurs only for energies $\hbar\omega > \Delta_0$.

• The condensation energy ϵ_c, defined as the difference between the ground state energy of the metal in the superconducting state and ground state energy in the normal state, is of the order of $10^{-7} - 10^{-8}$ eV per electron. This energy is very small compared with all the other energy scales of the metal, such as the energy widths or the electron interaction, or the phonon-electron interaction, which are all around few eV. Thus it is difficult to explain the origin of such a small scale, especially on the basis of perturbation theory (hierarchical problem).

• A last feature which is worth mentioning here is the *isotope effect*: it is observed that for different superconductors the critical temperature T_c is inversely proportional to the mass M of the lattice ions: $M^{\frac{1}{2}}T_c \simeq$ *const.*. Thus a stronger lattice rigidity (higher ion masses) implies a worse superconductivity (lower T_c): this fact suggests that the electron-phonon interaction is at the basis of superconductivity.

Let us now see how perfect conductivity implies the appearence of an energy gap in the quasiparticle spectrum.

An electric current inside the metal can be thought as an overall velocity $\mathbf{v}$, i.e. as a shift of momentum $\mathbf{q}$ common to all the electrons in the material. The ground state energy is then shifted by $\frac{1}{2}Mq^2$, where M is the total mass of the electron system.

If now the source of the current is switched off, the current flux will in general decrease, the energy loss manifesting into the creation of elementary excitations with energy spectrum $E(p)$. We then impose the conservation of energy and momentum as

$$\left.\begin{array}{l} \frac{1}{2}Mv^2 = \frac{1}{2}Mv'^2 + E(p) \\ M\mathbf{v} = M\mathbf{v}' + \mathbf{p} \end{array}\right\} \qquad \Rightarrow \qquad \mathbf{v}\cdot\mathbf{p} = \frac{p^2}{2M} + E(p) \tag{2.65}$$

This equation cannot be satisfied if $|\mathbf{v}|$ is smaller than

$$v_c = \min\left(\frac{p}{2M} + \frac{E(p)}{p}\right) \simeq \min\frac{E(p)}{p} \tag{2.66}$$

since M is very large. This means that for $v < v_c$ there cannot be current attenuation and the material is a perfect conductor.

If now we consider a free electron gas, whose spectrum is of the kind $E(p) = (p^2 - p_F^2)/2m$, we get $v_c = 0$. On the other hand, a relitivistic-like spectrum as

$$E(p) = \frac{1}{2m}\sqrt{(p^2 - p_F^2)^2 + 4m^2\Delta^2} \tag{2.67}$$

gives $v_c = \Delta/p_F$.

2.1.1 The BCS model

We now consider the BCS model[11, 4], which describes the most important collective effects at the basis of superconductivity. The BCS Hamiltonian is

$$\begin{aligned} H &= H_0 + \frac{g}{4V}\sum_{\mathbf{k},\mathbf{p},s,s'} U(\mathbf{k},s;\mathbf{p},s')\psi^\dagger(\mathbf{k},s)\psi^\dagger(-\mathbf{k},-s)\psi(-\mathbf{p},-s')\psi(\mathbf{p},s') \\ H_0 &= \sum_{\mathbf{k},s}\epsilon(p)\psi^\dagger(\mathbf{k},s)\psi(\mathbf{k},s) \end{aligned} \tag{2.68}$$

where V is the volume of the system. The potential $U(\mathbf{k},s;\mathbf{p},s')$ is taken to be real, even ($U(\mathbf{k},\mathbf{p}) = U(-\mathbf{k},-\mathbf{p})$) and symmetric ($U(\mathbf{k},\mathbf{p}) = U(\mathbf{p},\mathbf{k})$). It also holds $U(\mathbf{k},\mathbf{p}) = -U(-\mathbf{k},\mathbf{p})$.

The field ψ represent the electron field, and the Hamiltonian (2.68) can be thought as an effective Hamiltonian for the system of interacting electrons and phonons[4]: the interaction term in the BCS Hamiltonian takes into account the dominant effects for superconductivity, i.e. the two body correlations determined by the electron-electron elastic scattering near the Fermi surface.

In terms of electron creation and destruction operators, the BCS Hamiltionian reads

$$H = \sum_{\mathbf{k},s}(\frac{p^k}{2m}-\mu)c^\dagger_{\mathbf{k},s}c_{\mathbf{k},s} + \frac{g}{4V}\sum_{\mathbf{k},\mathbf{p},s,s'} U(\mathbf{k},s;\mathbf{p},s')c^\dagger_{\mathbf{k},s}c^\dagger_{-\mathbf{k},-s}c_{-\mathbf{p},-s}c_{\mathbf{p},s} \tag{2.69}$$

where μ is the chemical potential and the fermion operators satisfies $\{c_{\mathbf{p},s}, c^\dagger_{\mathbf{k},s'}\} = \delta_{\mathbf{pk}}\delta_{ss'}$. The equations of motion are

$$i\frac{d}{dt}c_{\mathbf{p},s}(t) = \frac{p^2}{2m}c_{\mathbf{p},s}(t) + \frac{g}{2V}\sum_{\mathbf{q},s'}U(\mathbf{p},s;\mathbf{q},s')c_{-\mathbf{q},-s'}(t)c_{\mathbf{q},s'}(t)c^\dagger_{-\mathbf{p},-s}(t) \tag{2.70}$$

At this point we observe that the operator $\Delta_V(\mathbf{p},s) \equiv \frac{1}{2V}\sum_{\mathbf{q},s'}U(\mathbf{p},s;\mathbf{q},s')c_{-\mathbf{q},-s'}(t)c_{\mathbf{q},s'}(t)$ is a c-number in the infinite volume limit[11][4] and then in this limit the dynamics gets linearized:

$$\begin{aligned} i\frac{d}{dt}c_{\mathbf{p},s}(t) &= \frac{p^2}{2m}c_{\mathbf{p},s}(t) + g\Delta(\mathbf{p},s)c^\dagger_{-\mathbf{p},-s}(t) \\ \Delta(\mathbf{p},s) &\overset{w}{=} \lim_{V\to\infty}\Delta_V(\mathbf{p},s) \end{aligned} \tag{2.71}$$

From eq.(2.71) we see that the Hamiltonian becomes quadratic

$$\begin{aligned} H_{eff} &= \sum_{\mathbf{p},s}(\frac{p^2}{2m}-\mu)c^\dagger_{\mathbf{p},s}c_{\mathbf{p},s} \\ &+ \frac{g}{2}\sum_{\mathbf{p},s}\left[\Delta(\mathbf{p},s)c^\dagger_{\mathbf{p},s}c^\dagger_{-\mathbf{p},-s} + \Delta(\mathbf{p},s)^* c_{-\mathbf{p},-s}c_{\mathbf{p},s}\right] + C \end{aligned} \tag{2.72}$$

with C a constant. The Hamiltonian (2.72) can be diagonalized by a Bogoliubov transformation. Considering the simple case in which $\Delta(\mathbf{p},s)$ is real, we have

$$\begin{aligned} c_{\mathbf{p},s} &= u(\mathbf{p},s)\,d_{\mathbf{p},s} + v(\mathbf{p},s)\,d^\dagger_{-\mathbf{p},-s} \\ c^\dagger_{\mathbf{p},s} &= u(\mathbf{p},s)\,d^\dagger_{\mathbf{p},s} + v(\mathbf{p},s)\,d_{-\mathbf{p},-s} \end{aligned} \tag{2.73}$$

[11]It is a c-number in any irreducible representation of the algebra generated by the operators $\psi(x)$, $\psi^\dagger(x)$

with $u(\mathbf{p},s)$ and $v(\mathbf{p},s)$ real and satisfying the conditions

$$\begin{aligned} u(\mathbf{p},s) &= u(-\mathbf{p},-s) \qquad , \qquad v(\mathbf{p},s) = -v(-\mathbf{p},-s) \\ u(\mathbf{p},s)^2 &+ v(\mathbf{p},s)^2 = 1 \end{aligned} \tag{2.74}$$

The second condition is the condition for the canonicity of the Bogoliubov transformation for fermions.

By requiring that H_{eff} is diagonal when expressed in terms of the quasiparticle operators $d_{\mathbf{p},s}$, we get[12]

$$u(\mathbf{p},s)^2 - v(\mathbf{p},s)^2 = \frac{p^2/2m-\mu}{\sqrt{(p^2/2m-\mu)^2+g^2\Delta(\mathbf{p},s)^2}} \tag{2.75}$$

$$2u(\mathbf{p},s)v(\mathbf{p},s) = -\frac{g\Delta(\mathbf{p},s)}{\sqrt{(p^2/2m-\mu)^2+g^2\Delta(\mathbf{p},s)^2}} \tag{2.76}$$

$$H_{eff} = \sum_{p,s} E(p,s)\, d^\dagger_{\mathbf{p},s} d_{\mathbf{p},s} + E_0 \tag{2.77}$$

$$E(p,s) = \sqrt{(p^2/2m-\mu)^2+g^2\Delta(\mathbf{p},s)^2} \tag{2.78}$$

We thus see how a energy gap has appeared in the spectrum of the quasi-particles. The ground state of the superconductor is defined as the vacuum for the quasi-particle operators $d_{\mathbf{p},s}$. We have

$$\begin{aligned} |\psi_0\rangle &= \prod_{\mathbf{p},s}\left[u(\mathbf{p},s) - v(\mathbf{p},s)\, c^\dagger_{\mathbf{p},s}c^\dagger_{-\mathbf{p},-s}\right]|0\rangle \\ d_{\mathbf{p},s}\,|\psi_0\rangle &= 0 \end{aligned} \tag{2.79}$$

where $|0\rangle$ is the vacuum for the $c_{\mathbf{p},s}$ operators. The representation $\{|\psi_0\rangle, d_{\mathbf{p},s}\}$ is a Fock representation for the quasiparticle operators $d_{\mathbf{p},s}$.

We now determine the gap function $\Delta(\mathbf{p},s)$ by using self-consistency. We have seen that $\Delta(\mathbf{p},s)$ is a c-number in any irreducible representation of the field algebra, when the limit $V\to\infty$ is performed. We can thus calculate $\Delta(\mathbf{p},s)$ on any state (for example on $|\psi_0\rangle$) and then take the limit. We have

$$\begin{aligned} \Delta(\mathbf{p},s) &= \lim_{V\to\infty}\frac{1}{2V}\sum_{\mathbf{q},s'} U(\mathbf{p},s;\mathbf{q},s')\langle\psi_0|c_{-\mathbf{q},-s'}c_{\mathbf{q},s'}|\psi_0\rangle \\ &= \frac{1}{16\pi^2}\sum_{s'}\int d^3\mathbf{q}\, U(\mathbf{p},s;\mathbf{q},s')u(\mathbf{p},s)\,v(\mathbf{p},s) \end{aligned} \tag{2.80}$$

By using eq.(2.76) we get the *gap equation*:

$$\Delta(\mathbf{p},s) = -\frac{1}{32\pi^3}\sum_{s'}\int d^3\mathbf{q}\, U(\mathbf{p},s;\mathbf{q},s')\,\frac{g\,\Delta(\mathbf{q},s)}{\sqrt{(p^2/2m-\mu)^2+g^2\Delta(\mathbf{p},s)^2}} \tag{2.81}$$

[12] Use $\Delta(\mathbf{p},s) = -\Delta(-\mathbf{p},-s)$.

This equation has a trivial solution $\Delta(\mathbf{p}, s) = 0$, corresponding to the metal in the normal state (no gap, spectrum of free Fermi gas) but also non-trivial solutions, corresponding to the superconducting phase.

We can make some assumption in order to solve eq.(2.81). Let us first assume the ground state being invariant under space inversions: this implies that $\Delta(\mathbf{p}, s)$ is a function of $|\mathbf{p}|$. If we put

$$\begin{aligned} \Delta(p) &\equiv \Delta(|\mathbf{q}|, s=+) = -\Delta(|\mathbf{q}|, s=-) \\ \bar{U}(p,q) &\equiv \frac{1}{4\pi}\int d\Omega_q U(\mathbf{p}, s=+; \mathbf{q}, s=+) \end{aligned} \tag{2.82}$$

the gap equation becomes

$$\Delta(p) = -\frac{g}{4\pi^2}\int dq\, q^2\, \bar{U}(p,q)\, \frac{\Delta(q)}{\sqrt{(q^2/2m-\mu)^2 + g^2\Delta(q)^2}} \tag{2.83}$$

In the limit of weak coupling, the integral is dominated by $q^2/2m = \mu$, i.e. $q = q_F$ and we get

$$\Delta(p) \simeq -\frac{g}{4\pi^2} q_F^2\, \bar{U}(p, q_F)\, \Delta(q) \int_0^K dq\, \frac{1}{\sqrt{(q^2/2m-\mu)^2 + g^2\Delta(q_F)^2}} \tag{2.84}$$

with K a cutoff which takes into account of the neglection of the contributions for high q. For $q = q_F$ we obtain

$$1 = -\frac{g}{4\pi^2} q_F^2\, \bar{U}(q_F, q_F) \int_0^K dq\, \frac{1}{\sqrt{(q^2/2m-\mu)^2 + g^2\Delta(q_F)^2}} \tag{2.85}$$

which has solution only for $g\bar{U}(q_F, q_F) < 0$: this means that the interaction favours the formation of electron pairs close to the Fermi surface. One thus obtains

$$\log(|g|\Delta(q_F)) = -\frac{2\pi^2}{|g| m q_F\, \bar{U}(q_F, q_F)} + C \tag{2.86}$$

whith C a constant dependent on the cutoff K, In conclusion

$$|g|\Delta \simeq C_0 \exp\left[-\frac{2\pi^2}{|g| m q_F\, \bar{U}}\right] \tag{2.87}$$

which exhibits the highly non-perturbative character of the gap and explains the origin of a hierarchically suppressed energy scale.

2.2 Thermo Field Dynamics

Thermo Field Dynamics (TFD) is an operatorial, real time formalism for field theory at finite temperature. The basic idea of TFD [12] is the transposition of the thermal averages, which are traces in statistical mechanics, to "vacuum" expectation values in a suitable Fock space, by means of the assumption:

$$\langle A \rangle = Z^{-1}(\beta) Tr \left[e^{-\beta H} A \right] = \langle 0(\beta)|A|0(\beta)\rangle \tag{2.88}$$

where $Z = Tr \left[e^{-\beta H} \right]$ is the grand-canonical partition function (H includes the chemical potential), $\beta = (k_B T)^{-1}$, and A is a generic observable.

Thus we are looking for a temperature dependent state $|0(\beta)\rangle$ (the "thermal vacuum"), which satisfies

$$\langle 0(\beta)|A|0(\beta)\rangle \; = \; Z^{-1}(\beta) \sum_n \langle n|A|n\rangle e^{-\beta E_n} \tag{2.89}$$

where

$$H|n\rangle = E_n|n\rangle \qquad , \qquad \langle n|m\rangle = \delta_{nm} \tag{2.90}$$

Now, if we expand $|0(\beta)\rangle$ in terms of $|n\rangle$ as

$$|0(\beta)\rangle \; = \; \sum_n |n\rangle \, f_n(\beta) \tag{2.91}$$

we get from eq.(2.89):

$$f_n^*(\beta) f_m(\beta) \; = \; Z^{-1}(\beta) \, e^{-\beta E_n} \, \delta_{nm} \tag{2.92}$$

This relation is not satisfied if $f_n(\beta)$ are numbers; however it resembles the orthogonality conditions for vectors. We can thus think that $|0(\beta)\rangle$ "lives" in a larger space with respect to the Hilbert space $\{ \, |n\rangle \, \}$: this space is obtained by *doubling* of the degrees of freedom, with the introduction of a fictitious dynamical system identical to the one under consideration. It is denoted by a tilde and we have:

$$\tilde{H}|\tilde{n}\rangle = E_n|\tilde{n}\rangle \qquad , \qquad \langle \tilde{n}|\tilde{m}\rangle = \delta_{nm} \tag{2.93}$$

Note that the energy is postulated to be the same of the one of the physical particles. The thermal ground state is then given by

$$|0(\beta)\rangle = Z^{-\frac{1}{2}}(\beta) \sum_n e^{-\frac{\beta}{2} E_n} |n, \tilde{n}\rangle \tag{2.94}$$

where $|n, \tilde{n}\rangle \; = \; |n\rangle \otimes |\tilde{n}\rangle$. From eq. (2.94) we note that $|0(\beta)\rangle$ contains an equal number of physical and tilde particles. In order to explicitly show some features of the thermal space $\{|0(\beta)\rangle\}$, we treat separately bosons and fermions.

2.2.1 TFD for bosons

The vacuum $|0(\beta)\rangle$ can be generated by a Bogoliubov transformation. To see this, consider two commuting sets of bosonic annihilation and creation operators:

$$\begin{aligned} \left[a_{\mathbf{k}}, a_{\mathbf{p}}^\dagger \right] &= \delta_{\mathbf{k},\mathbf{p}} \; , \quad [a_{\mathbf{k}}, a_{\mathbf{p}}] = 0 \\ \left[\tilde{a}_{\mathbf{k}}, \tilde{a}_{\mathbf{p}}^\dagger \right] &= \delta_{\mathbf{k},\mathbf{p}} \; , \quad [\tilde{a}_{\mathbf{k}}, \tilde{a}_{\mathbf{p}}] = 0 \\ [a_{\mathbf{k}}, \tilde{a}_{\mathbf{p}}] &= \left[a_{\mathbf{k}}, \tilde{a}_{\mathbf{p}}^\dagger \right] = 0 \end{aligned} \tag{2.95}$$

and the corresponding (free) Hamiltonians $H = \sum_{\mathbf{k}} \omega_k a^\dagger_{\mathbf{k}} a_{\mathbf{k}}$, $\tilde{H} = \sum_{\mathbf{k}} \omega_k \tilde{a}^\dagger_{\mathbf{k}} \tilde{a}_{\mathbf{k}}$. Let us now define the thermal operators by means of the following Bogoliubov transformation

$$
\begin{aligned}
a_{\mathbf{k}}(\theta) &= e^{-iG} a_{\mathbf{k}} e^{iG} = a_{\mathbf{k}} \cosh \theta_{\mathbf{k}} - \tilde{a}^\dagger_{\mathbf{k}} \sinh \theta_{\mathbf{k}} \\
\tilde{a}_{\mathbf{k}}(\theta) &= e^{-iG} \tilde{a}_{\mathbf{k}} e^{iG} = \tilde{a}_{\mathbf{k}} \cosh \theta_{\mathbf{k}} - a^\dagger_{\mathbf{k}} \sinh \theta_{\mathbf{k}}
\end{aligned}
\tag{2.96}
$$

where $\theta_{\mathbf{k}} = \theta_{\mathbf{k}}(\beta)$ is a function of temperature to be determined and the hermitian generator G is given by

$$
G = i \sum_{\mathbf{k}} \theta_{\mathbf{k}} \left[a^\dagger_{\mathbf{k}} \tilde{a}^\dagger_{\mathbf{k}} - \tilde{a}_{\mathbf{k}} a_{\mathbf{k}} \right] \tag{2.97}
$$

The total Hamiltonian $\hat{H}$ is defined as the difference of the physical and the tilde Hamiltonians and is invariant under the thermal transformation (2.96):

$$
\hat{H} = H - \tilde{H} \quad , \quad \left[G, \hat{H} \right] = 0\,. \tag{2.98}
$$

We notice the minus sign occurring in the total Hamiltonian $\hat{H}$. The thermal vacuum is given, in terms of the original vacuum $|0\rangle$, by

$$
|0(\theta)\rangle = e^{-iG}|0\rangle = \prod_{\mathbf{k}} \frac{1}{\cosh \theta_{\mathbf{k}}} \exp \left[\tanh \theta_{\mathbf{k}} a^\dagger_{\mathbf{k}} \tilde{a}^\dagger_{\mathbf{k}} \right] |0\rangle \tag{2.99}
$$

and is of course annihilated by the thermal operators: $a_{\mathbf{k}}(\theta)|0(\theta)\rangle = \tilde{a}_{\mathbf{k}}(\theta)|0(\theta)\rangle = 0$. Notice that the form of $|0(\theta)\rangle$ is that of a $SU(1,1)$ coherent state[8]. The vacuum $|0(\theta)\rangle$ is an eigenstate of the total Hamiltonian $\hat{H}$ with zero eigenvalue; however it is not eigenstate of the single Hamiltonians H and $\tilde{H}$.

In order to identify the above state $|0(\theta)\rangle$ with the thermal ground state $|0(\beta)\rangle$ defined in eq.(2.94), we need an explicit relation giving θ as a function of β.

To this end let us consider the following relation

$$
\begin{aligned}
|0(\theta)\rangle &= e^{-S/2} \exp \left[\sum_{\mathbf{k}} a^\dagger_{\mathbf{k}} \tilde{a}^\dagger_{\mathbf{k}} \right] |0\rangle = e^{-\hat{S}/2} \exp \left[\sum_{\mathbf{k}} a^\dagger_{\mathbf{k}} \tilde{a}^\dagger_{\mathbf{k}} \right] |0\rangle \\
S &= - \sum_{\mathbf{k}} \left[a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \log \sinh^2 \theta_{\mathbf{k}} - a_{\mathbf{k}} a^\dagger_{\mathbf{k}} \log \cosh^2 \theta_{\mathbf{k}} \right]
\end{aligned}
\tag{2.100}
$$

where S can be interpreted as the entropy operator for the physical system (see below).

The number of physical particles in $|0(\theta)\rangle$ is given by

$$
n_{\mathbf{k}} \equiv \langle 0(\theta)| a^\dagger_{\mathbf{k}} a_{\mathbf{k}} |0(\theta)\rangle = \sinh^2 \theta_{\mathbf{k}} \tag{2.101}
$$

with a similar result for the tilde particles. By minimizing now (with respect to $\theta_{\mathbf{k}}$) the quantity

$$
\Omega = \langle 0(\theta)| \left[-\frac{1}{\beta} S + H - \mu N \right] |0(\theta)\rangle \tag{2.102}
$$

we finally get (putting $\omega_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$)

$$n_{\mathbf{k}} = \sinh^2 \theta_{\mathbf{k}} = \frac{1}{e^{\beta\omega_{\mathbf{k}}} - 1} \tag{2.103}$$

which is the correct thermal average, i.e. the Bose distribution. Thus we conclude that the thermal Fock space $\{|0(\beta)\rangle\}$ is generated from the free (doubled) Fock space $\{|0\rangle\}$ by means of the Bogoliubov transformation (2.96).

Eq.(2.103) makes possible a thermodynamical interpretation: thus Ω is interpreted as the thermodynamical potential, while S is the entropy (divided by the Boltzmann constant k_B), holding the relation [12]:

$$\begin{aligned} &\langle 0(\theta)|S|0(\theta)\rangle = -\sum_n w_n \log w_n \\ &\sum_n w_n = 1\,. \end{aligned} \tag{2.104}$$

The physical meaning of the tilde degrees of freedom is shown, for example, by the relation

$$\frac{1}{\cosh\theta_{\mathbf{k}}} a^\dagger_{\mathbf{k}}|0(\theta)\rangle = \frac{1}{\sinh\theta_{\mathbf{k}}} \tilde{a}^\dagger_{\mathbf{k}}|0(\theta)\rangle \tag{2.105}$$

which expresses the equivalence, on $|0(\theta)\rangle$, of the creation of a physical particle with the destruction of a tilde particle, which thus can be interpreted as the *holes* for the physical quanta. In other words, the thermal bath is simulated by a mirror image of the physical system, exchanging energy with it.

The tilde-symmetry, i.e. the symmetry between physical and tilde worlds, is expressed by a formal operation, postulated in TFD and called *tilde-conjugation rules*. Given A and B operators and α, β c-numbers, the tilde rules are:

$$\begin{aligned} &(AB)\tilde{} = \tilde{A}\tilde{B} \\ &(\alpha A + \beta B)\tilde{} = \alpha^* \tilde{A} + \beta^* \tilde{B} \\ &\left(A^\dagger\right)\tilde{} = \tilde{A}^\dagger \\ &\left(\tilde{A}\right)\tilde{} = A \end{aligned} \tag{2.106}$$

The vacuum state is composed of equal number of (commuting) tilde and non-tilde operators, thus being invariant under tilde conjugation: $|0(\beta)\rangle\tilde{} = |0(\beta)\rangle$.

2.2.2 Thermal propagators (bosons)

Let us now consider a (boson) real free field in thermal equilibrium. We have:

$$\begin{aligned} \phi(x) &= \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}(2\omega_k)^{\frac{1}{2}}} \left[a_{\mathbf{k}}(t)e^{i\mathbf{kx}} + a^\dagger_{\mathbf{k}}(t)e^{-i\mathbf{kx}}\right] \\ \tilde{\phi}(x) &= \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}(2\omega_k)^{\frac{1}{2}}} \left[\tilde{a}_{\mathbf{k}}(t)e^{-i\mathbf{kx}} + \tilde{a}^\dagger_{\mathbf{k}}(t)e^{i\mathbf{kx}}\right] \end{aligned} \tag{2.107}$$

where $a_{\mathbf{k}}(t) = e^{-i\omega_k t} a_{\mathbf{k}}$ and $a_{\mathbf{k}}$ are the operators of eq.(2.95). The above fields have the following commutation rules:

$$
\begin{aligned}
&[\phi(t,\mathbf{x}), \partial_t \phi(t,\mathbf{x}')] = i\delta^3(\mathbf{x}-\mathbf{x}') \\
&\left[\tilde{\phi}(t,\mathbf{x}), \partial_t \tilde{\phi}(t,\mathbf{x}')\right] = -i\delta^3(\mathbf{x}-\mathbf{x}')
\end{aligned}
\tag{2.108}
$$

and commute each other.

In TFD, and more in general in a thermal field theory (TFT) [13], the two point functions (propagators) have a matrix structure, arising from the the various possible combinations of physical and tilde fields in the vacuum expectation value. Notice that in TFD, although the physical and tilde particles are not coupled in the Hamiltonian $\hat{H}$, nevertheless they do couple in the vacuum state $|0(\theta)\rangle$. So, the finite temperature causal propagator for a free (charged) boson field $\phi(x)$ is

$$
D_0^{(ab)}(x,x') = -i\langle 0(\theta)|T\left[\phi^a(x), \phi^{b\dagger}(x')\right]|0(\theta)\rangle \tag{2.109}
$$

where the zero recalls it is a free field propagator, T denotes time ordering and the a, b indexes refers to the thermal doublet $\phi^1 = \phi$, $\phi^2 = \tilde{\phi}^\dagger$. In the present case of a real field, we will use the above definition with $\phi^\dagger = \phi$.

A remarkable feature of the above propagator is that it can be casted (in momentum representation) in the following form [5]:

$$
D_0^{ab}(k_0,\mathbf{k}) = \mathcal{B}_k^{-1}\left[\frac{1}{k_0^2 - (\omega_k - i\epsilon\tau_3)^2}\right]\mathcal{B}_k \tau_3\,, \tag{2.110}
$$

where τ_3 is the Pauli matrix $diag(1,-1)$. We note that the internal, or "core" matrix, is diagonal and coincides with the vacuum Feynman propagator. Thus the thermal propagator is obtained from the vacuum one by the action of the Bogoliubov matrix (2.96):

$$
\mathcal{B}_k = \begin{pmatrix} \cosh\theta_k & -\sinh\theta_k \\ -\sinh\theta_k & \cosh\theta_k \end{pmatrix}, \tag{2.111}
$$

The Bogoliubov matrix does affect only the imaginary part of $D_0(k)$; for example the $(1,1)$ component is $D_0^{11}(k) = \left[k_0^2 - (\omega_k - i\epsilon\tau_3)^2\right]^{-1} - 2\pi i n_k \delta(k_0^2 - \omega_k^2)$.

Another fundamental property of the matrix propagator (2.109) is that only three elements are independent, since it holds the linear relation:

$$
D^{11} + D^{22} - D^{12} - D^{21} = 0\,. \tag{2.112}
$$

This relation can be verified easily by using the inverse of (2.96) and the annihilation of the thermal operators on $|0(\beta)\rangle$. Note also that the above relation has a more general validity, being true also for a different parameterization (gauge) of the thermal Bogoliubov matrix (see next Section).

2.2.3 TFD for fermions

Consider two sets of fermionic annihilation and creation operators:

$$\begin{aligned}
&\left\{\alpha_{\mathbf{k}}, \alpha_{\mathbf{p}}^{\dagger}\right\} = \delta_{\mathbf{k},\mathbf{p}} \ , \quad \{\alpha_{\mathbf{k}}, \alpha_{\mathbf{p}}\} = 0 \\
&\left\{\tilde{\alpha}_{\mathbf{k}}, \tilde{\alpha}_{\mathbf{p}}^{\dagger}\right\} = \delta_{\mathbf{k},\mathbf{p}} \ , \quad \{\tilde{\alpha}_{\mathbf{k}}, \tilde{\alpha}_{\mathbf{p}}\} = 0 \\
&\{\alpha_{\mathbf{k}}, \tilde{\alpha}_{\mathbf{p}}\} = \left\{\alpha_{\mathbf{k}}, \tilde{\alpha}_{\mathbf{p}}^{\dagger}\right\} = 0
\end{aligned} \tag{2.113}$$

The thermal operators are defined by means of the following Bogoliubov transformation

$$\begin{aligned}
\alpha_{\mathbf{k}}(\theta) &= e^{-iG}\alpha_{\mathbf{k}}e^{iG} = \alpha_{\mathbf{k}}\cos\theta_{\mathbf{k}} - \tilde{\alpha}_{\mathbf{k}}^{\dagger}\sin\theta_{\mathbf{k}} \\
\tilde{\alpha}_{\mathbf{k}}(\theta) &= e^{-iG}\tilde{\alpha}_{\mathbf{k}}e^{iG} = \tilde{\alpha}_{\mathbf{k}}\cos\theta_{\mathbf{k}} + \alpha_{\mathbf{k}}^{\dagger}\sin\theta_{\mathbf{k}}
\end{aligned} \tag{2.114}$$

with generator G given by

$$G = i\sum_{\mathbf{k}}\theta_{\mathbf{k}}\left[a_{\mathbf{k}}^{\dagger}\tilde{a}_{\mathbf{k}}^{\dagger} - \tilde{\alpha}_{\mathbf{k}}\alpha_{\mathbf{k}}\right] \tag{2.115}$$

The thermal vacuum is a $SU(2)$ coherent state [8]:

$$\begin{aligned}
&|0(\theta)\rangle = e^{-iG}|0\rangle = \prod_{\mathbf{k}}\left[\cos\theta_{\mathbf{k}} + \sin\theta_{\mathbf{k}}\alpha_{\mathbf{k}}^{\dagger}\tilde{\alpha}_{\mathbf{k}}^{\dagger}\right]|0\rangle \\
&\alpha_{\mathbf{k}}(\theta)|0(\theta)\rangle = \tilde{\alpha}_{\mathbf{k}}(\theta)|0(\theta)\rangle = 0
\end{aligned} \tag{2.116}$$

and the entropy operator reads

$$S = -\sum_{\mathbf{k}}\left[\alpha_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}\log\sin^2\theta_{\mathbf{k}} + \alpha_{\mathbf{k}}\alpha_{\mathbf{k}}^{\dagger}\log\cos^2\theta_{\mathbf{k}}\right]$$

The number of physical particles in $|0(\theta)\rangle$ is given by

$$n_{\mathbf{k}} \equiv \langle 0(\theta)|\alpha_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}|0(\theta)\rangle = \sin^2\theta_{\mathbf{k}} \tag{2.117}$$

with a similar result for the tilde particles. By a procedure similar to that of eq.(2.102), the Fermi distribution is obtained as

$$n_{\mathbf{k}} = \sin^2\theta_{\mathbf{k}} = \frac{1}{e^{\beta\omega_{\mathbf{k}}} + 1} \tag{2.118}$$

Relations similar to (2.105) hold as

$$\frac{1}{\cos\theta_{\mathbf{k}}}\alpha_{\mathbf{k}}^{\dagger}|0(\theta)\rangle = -\frac{1}{\sin\theta_{\mathbf{k}}}\tilde{\alpha}_{\mathbf{k}}^{\dagger}|0(\theta)\rangle \tag{2.119}$$

2.2.4 Non-hermitian representation of TFD

We write the statistical average of a generic observable A by means of a statistical operator W as

$$\langle A\rangle = \frac{Tr[AW]}{Tr[W]} = \frac{Tr[W^{(1-\alpha)}AW^{\alpha}]}{Tr[W^{(1-\alpha)}W^{\alpha}]} \tag{2.120}$$

where the cyclic property of the trace and the positiveness of W have been used. The parameter α is in the range $\alpha = [0,1]$.

The operator W can be seen as a vector of a (doubled) Hilbert space, called Liouville space[13]. The above thermal average is then rewritten as expectation value in the Liouville space as

$$\langle A\rangle = \frac{(\!(W^L\|A\|W^R)\!)}{(\!(W^L\|W^R)\!)} \tag{2.121}$$

where, for a single oscillator, the left and right statistical states are coherent states given by

$$\begin{aligned} \|W^R)\!) &= \exp\left(f^{\alpha}a^{\dagger}\tilde{a}^{\dagger}\right)\|0,0)\!) \\ (\!(W^L\| &= (\!(0,0\|\exp\left(f^{(1-\alpha)}a\tilde{a}\right) . \end{aligned} \tag{2.122}$$

with $f = e^{-\beta(\omega-\mu)}$ and $\|0,0)\!)$ is the vacuum state of Liouville space, annihilated by the $a, \tilde{a}$ operators[14]. For $\alpha = 1/2$, we recover the standard TFD: in particular the states $\|W^R)\!)$ and $(\!(W^L\|$ become each other hermitian conjugates. The choice $\alpha = 1/2$ is called in TFD the *symmetric gauge*. Another useful choice is $\alpha = 1$ (*linear gauge*).

The thermal (non-hermitian) Bogoliubov transformation is now (for bosons)

$$\begin{aligned} \xi_{\mathbf{k}} &= (1-f_{\mathbf{k}})^{-\frac{1}{2}}\left[a_{\mathbf{k}} - f_{\mathbf{k}}^{\alpha}\,\tilde{a}_{\mathbf{k}}^{\dagger}\right] \\ \tilde{\xi}_{\mathbf{k}} &= (1-f_{\mathbf{k}})^{-\frac{1}{2}}\left[\tilde{a}_{\mathbf{k}} - f_{\mathbf{k}}^{\alpha}\,a_{\mathbf{k}}^{\dagger}\right] \end{aligned} \tag{2.123}$$

and

$$\begin{aligned} \xi_{\mathbf{k}}^{\sharp} &= (1-f_{\mathbf{k}})^{-\frac{1}{2}}\left[a_{\mathbf{k}}^{\dagger} - f_{\mathbf{k}}^{1-\alpha}\,\tilde{a}_{\mathbf{k}}\right] \\ \tilde{\xi}_{\mathbf{k}}^{\sharp} &= (1-f_{\mathbf{k}})^{-\frac{1}{2}}\left[\tilde{a}_{\mathbf{k}}^{\dagger} - f_{\mathbf{k}}^{1-\alpha}\,a_{\mathbf{k}}\right] \end{aligned} \tag{2.124}$$

where the ξ operators satisfy canonical commutation relations

$$\begin{aligned} &\left[\xi,\xi^{\sharp}\right] = \xi\xi^{\sharp} - \xi^{\sharp}\xi = 1\ , \quad \left[\tilde{\xi},\tilde{\xi}^{\sharp}\right] = 1 \\ &\left[\tilde{\xi},\xi^{\sharp}\right] = 0\ , \quad \left[\xi,\tilde{\xi}^{\sharp}\right] = 0 \end{aligned} \tag{2.125}$$

and the *thermal state condition*:

$$\xi\|W^R)\!) = 0\ ,\ \ \tilde{\xi}\|W^R)\!) = 0\ ,\ \ (\!(W^L\|\xi^{\sharp} = 0\ ,\ \ (\!(W^L\|\tilde{\xi}^{\sharp} = 0\,. \tag{2.126}$$

[13]For a detailed description of the properties of Liouville space see ref.[14].

[14]The time evolution in Liouville space is controlled by $\hat{H} = \omega\left(a^{\dagger}a - \tilde{a}^{\dagger}\tilde{a}\right)$.

Again, in case of thermal equilibrium and in the symmetric gauge $\alpha = 1/2$, the above Bogoliubov matrix $\mathcal{B}$ coincides with that of eq.(2.111), the ξ operators with the thermal operators $a(\theta)$ of eq.(2.96), and the $\sharp$ conjugation reduces to the usual hermitian $\dagger$ conjugation .

Section 3

Examples

3.1 Quantization of the damped harmonic oscillator

In this Section we review a recent approach [15], in which the algebraic features of the dho are emphasized and a consistent quantization scheme is obtained in the QFT framework, relying on the existence in QFT of inequivalent representations of the canonical commutation relations.

Consider the equation for a one-dimensional damped harmonic oscillator,

$$m\ddot{x} + \gamma\dot{x} + \kappa x = 0 \quad . \tag{3.1}$$

It is known since long time [16] that, in order to derive eq.(3.1) from a variational principle, the introduction of an additional variable is necessary.

It follows from this [17], that a canonical quantization scheme for the dho system requires first of all the *doubling* of the phase-space dimension (i.e. of the degrees of freedom), obtained by introducing an other variable y, mirror image of the x oscillator variable. Intuitively the y oscillator represents the (collective) degree of freedom of the bath, in which the energy dissipated by the oscillator (3.1) flows. The doubled system is of course a closed one, and it is possible to write down a Lagrangian,

$$L = m\dot{x}\dot{y} + \frac{\gamma}{2}(x\dot{y} - \dot{x}y) - \kappa xy \quad . \tag{3.2}$$

Thus eq.(3.1) is obtained by varying L with respect to y; by variation with respect to x we obtain instead

$$m\ddot{y} - \gamma\dot{y} + \kappa y = 0\,, \tag{3.3}$$

which is the time-reversed image ($\gamma \to -\gamma$) of eq.(3.1), growing as rapidly as the x decays: we thus can think to the y oscillator as the reservoir (or heat bath) associated to the x oscillator. The important point in the present approach is that one does not really needs to specify the details of the reservoir dynamics; the only requirement to be met, in order to set up the canonical formalism for the system (3.1), is that the reservoir must receive all the energy flux outgoing from the x system.

By defining the canonical momenta as $p_x \equiv \frac{\partial L}{\partial \dot{x}} = m\dot{y} - \frac{1}{2}\gamma y$; $p_y \equiv \frac{\partial L}{\partial \dot{y}} = m\dot{x} + \frac{1}{2}\gamma x$, the Hamiltonian reads

$$\begin{aligned} H &= p_x\dot{x} + p_y\dot{y} - L \\ &= \frac{1}{m}p_x p_y + \frac{1}{2m}\gamma\left(y p_y - x p_x\right) + \left(\kappa - \frac{\gamma^2}{4m}\right) xy \end{aligned} \tag{3.4}$$

At this point it is possible to quantize the system. First we introduce the canonical commutators

$$[\,x, p_x\,] = [\,y, p_y\,] = i\,\hbar \quad , \quad [\,x, y\,] = 0 = [\,p_x, p_y\,] = 0 \tag{3.5}$$

and the corresponding creation and annihilation operators

$$\begin{aligned} a &\equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}} \left(\frac{p_x}{\sqrt{m}} - i\sqrt{m}\Omega x\right) \quad ; \quad a^\dagger \equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}} \left(\frac{p_x}{\sqrt{m}} + i\sqrt{m}\Omega x\right) \quad ; \\ b &\equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}} \left(\frac{p_y}{\sqrt{m}} - i\sqrt{m}\Omega y\right) \quad ; \quad b^\dagger \equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}} \left(\frac{p_y}{\sqrt{m}} + i\sqrt{m}\Omega y\right) \quad , \end{aligned} \tag{3.6}$$

with canonical commutators

$$[\,a, a^\dagger\,] = [\,b, b^\dagger\,] = 1 \quad , \quad [\,a, b\,] = [\,a, b^\dagger\,] = 0 \quad , \tag{3.7}$$

In the above equations Ω represent the common frequency of the two oscillators, defined as

$$\Omega \equiv \left[\frac{1}{m}\left(\kappa - \frac{\gamma^2}{4m}\right)\right]^{\frac{1}{2}}$$

and it is a real quantity in the case of no overdamping $\left(\kappa > \frac{\gamma^2}{4m}\right)$.

The quantum Hamiltonian is:

$$H = \hbar\Omega\left(a^\dagger b + a b^\dagger\right) - \frac{i\hbar\gamma}{4m}\left[\left(a^2 - a^{\dagger 2}\right) - \left(b^2 - b^{\dagger 2}\right)\right] \tag{3.8}$$

By using the canonical linear transformations $A \equiv \frac{1}{\sqrt{2}}(a+b)$, $B \equiv \frac{1}{\sqrt{2}}(a-b)$, the above Hamiltonian is rewritten in a more convenient form as

$$\begin{aligned} &H = H_0 + H_I \quad , \\ &H_0 = \hbar\Omega(A^\dagger A - B^\dagger B) \quad , \quad H_I = i\hbar\Gamma(A^\dagger B^\dagger - AB) \quad , \end{aligned} \tag{3.9}$$

where $\Gamma \equiv \frac{\gamma}{2m}$ is the decay constant for the classical variable $x(t)$.

We note that the states generated by $B^\dagger$ represent the sink where the energy dissipated by the quantum damped oscillator flows

The dynamical group structure associated with the system of coupled quantum oscillators is that of $SU(1,1)$; the two mode realization of the $su(1,1)$ algebra is indeed generated by:

$$\begin{aligned} &J_+ = A^\dagger B^\dagger \quad , \quad J_- = J_+^\dagger = AB \quad , \quad J_3 = \frac{1}{2}(A^\dagger A + B^\dagger B + 1) \quad , \\ &\mathcal{C}^2 \equiv \frac{1}{4} + J_3^2 - \frac{1}{2}(J_+ J_- + J_- J_+) = \frac{1}{4}(A^\dagger A - B^\dagger B)^2 \end{aligned} \tag{3.10}$$

where $\mathcal{C}$ is the Casimir operator and the commutators are:

$$[J_+, J_-] = -2J_3 \quad , \quad [J_3, J_\pm] = \pm J_\pm \tag{3.11}$$

The above form (3.9) of the Hamiltonian is a convenient one; we have indeed:

$$H_0 = 2\hbar\Omega\mathcal{C} \quad , \quad H_I = i\hbar\Gamma(J_+ - J_-) \equiv -2\hbar\Gamma J_2 \quad , \tag{3.12}$$

and

$$[\, H_0, H_I \,] = 0 \tag{3.13}$$

which shows that H_0 is the centre of the dynamical algebra.

Let us denote by $|0\rangle$ the vacuum state for the A and B operators: $A|0\rangle = B|0\rangle = 0$. Its time evolution is controlled by H_I solely (cf. eq.(3.13)):

$$\begin{aligned} |0(t)\rangle &= \exp\left(-\frac{i}{\hbar} H\, t\right)|0\rangle = \exp\left(-\frac{i}{\hbar} H_I\, t\right)|0\rangle \\ &= \frac{1}{\cosh(\Gamma t)} \exp\left(\tanh(\Gamma t) J_+\right)|0\rangle \end{aligned} \tag{3.14}$$

For finite times t the above equation is formally correct and $|0(t)\rangle$ represents a normalized ($\langle 0(t)|0(t)\rangle = 1$) generalized $SU(1,1)$ (time dependent) coherent state [8]. However, for $t \to \infty$, the asymptotic state becomes orthogonal to the initial vacuum state:

$$\lim_{t\to\infty} \langle 0(t)|0\rangle = \lim_{t\to\infty} \exp\left(-\ln\cosh(\Gamma t)\right) \to 0 \quad . \tag{3.15}$$

This means that time evolution leads out of the original Hilbert space: the QM framework then results to be inadequate for the description of the system (3.1) since there the representations of the CCR are all unitarily equivalent to each other. It is clear that this "pathology" of the quantization of dho can find its cure only in QFT, where many unitarily inequivalent representations of CCR are allowed (see Section 1).

The obvious generalization of (3.9) to infinite degrees of freedom is

$$\begin{aligned} H_0 &= \sum_{\mathbf{k}} \hbar\Omega_{\mathbf{k}}(A_{\mathbf{k}}^\dagger A_{\mathbf{k}} - B_{\mathbf{k}}^\dagger B_{\mathbf{k}}) \quad , \\ H_I &= i\sum_{\mathbf{k}} \hbar\Gamma_k(A_{\mathbf{k}}^\dagger B_{\mathbf{k}}^\dagger - A_{\mathbf{k}} B_{\mathbf{k}}) \quad , \end{aligned} \tag{3.16}$$

and the commutation relations are now

$$[\, A_{\mathbf{k}}, A_{\mathbf{p}}^\dagger \,] = [\, B_{\mathbf{k}}, B_{\mathbf{p}}^\dagger \,] = \delta_{\mathbf{k},\mathbf{p}} \quad , \quad [\, A_{\mathbf{k}}, B_{\mathbf{p}} \,] = [\, A_{\mathbf{k}}, B_{\mathbf{p}}^\dagger \,] = 0 \quad , \tag{3.17}$$

For each $\mathbf{k}$, the group structure, denoted by $SU(1,1)_{\mathbf{k}}$, is the same of that exhibited in the case of one degree of freedom; the generators of the $su(1,1)$ algebra with different $\mathbf{k}$ do commute each other: this means that the original $SU(1,1)$ has become now a $\bigotimes_{\mathbf{k}} SU(1,1)_{\mathbf{k}}$.

We have also, in parallel to (3.14),

$$|0(t)\rangle = \prod_{\mathbf{k}} \frac{1}{\cosh(\Gamma_k t)} \exp\left(\tanh(\Gamma_k t) J_+^{\mathbf{k}}\right)|0\rangle \tag{3.18}$$

It hold now all the considerations done for eq.(3.14); in particular the large t limit is the same

$$\lim_{t\to\infty} \langle 0(t)|0\rangle = \lim_{t\to\infty} \exp\left(-\sum_{\mathbf{k}} \ln\cosh(\Gamma_k t)\right) \propto \lim_{t\to\infty} \exp\left(-t\sum_{\mathbf{k}} \Gamma_k\right) = 0\,. \tag{3.19}$$

provided $\sum_{\mathbf{k}} \Gamma_k > 0$. Thus, at finite volume, the situation is the same of before; however, in the infinite volume limit, provided $\int d^3\mathbf{k}\,\Gamma_k > 0$, we have now

$$\begin{aligned} &\lim_{V\to\infty} \langle 0(t)|0\rangle = 0 \quad \forall\, t \quad , \\ &\lim_{V\to\infty} \langle 0(t)|0(t')\rangle = 0 \quad \forall\, t\,, t' \quad , \quad t\neq t' \quad . \end{aligned} \tag{3.20}$$

where we used the relation $\sum_{\mathbf{k}} \to (2\pi)^{-3} V \int d^3\mathbf{k}\,$.

These relations show the unitary inequivalence of the representations labelled by time t (see Section 1). Thus, at each time we have a copy (automorphism) of the original algebra and of the Fock space: the time evolution, induced by H_I, transforms $\{A_{\mathbf{k}}, A_{\mathbf{k}}^\dagger, B_{\mathbf{k}}, B_{\mathbf{k}}^\dagger\,;\, |0\rangle\,|\,\forall\mathbf{k}\}$ into $\{A_{\mathbf{k}}(t), A_{\mathbf{k}}^\dagger(t), B_{\mathbf{k}}(t), B_{\mathbf{k}}^\dagger(t)\,;\, |0(t)\rangle\,|\,\forall\mathbf{k}\}$.

The annihilation and creation operators at time t are defined by

$$\begin{aligned} A_{\mathbf{k}}(t) &= e^{-\frac{i}{\hbar}H_I t} A_{\mathbf{k}} e^{\frac{i}{\hbar}H_I t} = A_{\mathbf{k}}\cosh(\Gamma_k t) - B_{\mathbf{k}}^\dagger \sinh(\Gamma_k t) \quad ; \\ B_{\mathbf{k}}(t) &= e^{-\frac{i}{\hbar}H_I t} B_{\mathbf{k}} e^{\frac{i}{\hbar}H_I t} = -A_{\mathbf{k}}^\dagger \sinh(\Gamma_k t) + B_{\mathbf{k}}\cosh(\Gamma_k t) \quad , \end{aligned} \tag{3.21}$$

which is Bogoliubov transformation generated by H_I and parameterized by the time t. It is clear that, at each time t, the operators $A(t)$ and $B(t)$ are annihilators for $|0(t)\rangle$:

$$A_{\mathbf{k}}(t)|0(t)\rangle = B_{\mathbf{k}}(t)|0(t)\rangle = 0\,. \tag{3.22}$$

The situation here is very similar, with due changes, to that of Thermo Field Dynamics (TFD) (see Section 2): the vacuum (3.18) has indeed the same statistical and thermodynamical properties of the thermal vacuum of TFD as shown by the following relations, also valid in TFD:

- The particle content of the state $|0(t)\rangle$ is the same for A and B particles,

$$\langle 0(t)|A_{\mathbf{k}}^\dagger A_{\mathbf{k}}|0(t)\rangle = \langle 0(t)|B_{\mathbf{k}}^\dagger B_{\mathbf{k}}|0(t)\rangle = \sinh^2(\Gamma_k t) \quad , \tag{3.23}$$

  showing that the number $n_A - n_B$ is a constant of motion for any $\mathbf{k}$.

- The B modes can be considered as the holes (the tilde operators in TFD) for the modes A: it hold indeed the relations

$$\begin{aligned} A_{\mathbf{k}}^\dagger(t)|0(t)\rangle &= \frac{1}{\cosh(\Gamma_k t)} A_{\mathbf{k}}^\dagger|0(t)\rangle \;= \frac{1}{\sinh(\Gamma_k t)} B_{\mathbf{k}}|0(t)\rangle \quad , \\ B_{\mathbf{k}}^\dagger(t)|0(t)\rangle &= \frac{1}{\cosh(\Gamma_k t)} B_{\mathbf{k}}^\dagger|0(t)\rangle \;= \frac{1}{\sinh(\Gamma_k t)} A_{\mathbf{k}}|0(t)\rangle \quad . \end{aligned} \tag{3.24}$$

  which also show the particle content of $|0(t)\rangle$.

- It is possible to introduce formally an entropy operator S (here S generally stand for S_A or S_B)

$$S_A \equiv -\sum_{\mathbf{k}} \left[A_{\mathbf{k}}^\dagger A_{\mathbf{k}} \ln \sinh^2 (\Gamma_k t) - A_{\mathbf{k}} A_{\mathbf{k}}^\dagger \ln \cosh^2 (\Gamma_k t)\right] \quad ; \tag{3.25}$$

and similarly for S_B. It is then possible to write

$$\begin{aligned} |0(t)\rangle &= \exp\left(-\frac{1}{2}S_A\right)| I\rangle = \exp\left(-\frac{1}{2}S_B\right)| I\rangle \quad , \\ | I\rangle &\equiv \exp\left(\sum_{\mathbf{k}} A_{\mathbf{k}}^\dagger B_{\mathbf{k}}^\dagger\right)|0\rangle \end{aligned} \tag{3.26}$$

From this last equation we see how the time evolution can be expressed in terms of only one subsystem, which is then regarded as an "open" one. The formal interpretation of S as an entropy finds its justification in the following relations

$$\begin{aligned} &\langle 0(t)|S|0(t)\rangle = -\sum_{n\geq 0} W_n(t) \ln W_n(t)\,. \\ &\sum_{n\geq 0} W_n(t) = 1 \end{aligned} \tag{3.27}$$

where the $W_n(t)$ are some coefficients [5] (see also Section 2).

Differentiation of eq.(3.26) with respect to time gives

$$\frac{\partial}{\partial t}|0(t)\rangle = -\left(\frac{1}{2}\frac{\partial S}{\partial t}\right)|0(t)\rangle\,, \tag{3.28}$$

which shows that $\frac{i}{2}\hbar\frac{\partial S}{\partial t}$ is the generator of time-translations. The fact that the entropy operator S controls the time evolution is a signal of the irreversibility of such an evolution for the dissipative system under consideration. This is also clear if we consider the fact that the vacuum $|0\rangle$ is not invariant under J_2, although the Hamiltonian shows such an invariance: we thus have a spontaneous breakdown of time translational symmetry, which establish a preferential direction ("arrow") in time[15].

The increasing in entropy for the separate subsystems is expressed by the monotonical increasing of $\langle 0(t)|S|0(t)\rangle$ from $t = 0$ to $t = \infty$: however, the difference of the two entropy operators $S_A - S_B$ does commute with the Hamiltonian, thus resulting to be the conserved entropy for the complete system.

It is also shown in ref.[15] that it is possible to introduce formally a (time dependent) temperature $\beta(t)$, and a free energy $\mathcal{F}$, such that, for example for the A subsystem,

$$\mathcal{F}_A \equiv \langle 0(t)|\left(H_A - \frac{1}{\beta}S_A\right)|0(t)\rangle \tag{3.29}$$

[15]We note how the time-reversal (discrete) symmetry is also broken in the original Lagrangian (3.2).

In a adiabatic hypothesis ($\partial\beta/\partial t \simeq 0$), the minimization of the functional $\mathcal{F}_A$ with respect to $\theta_k \equiv \Gamma_k t$ gives $N_{A_{\mathbf{k}}}$, i.e. the mean number of $A_{\mathbf{k}}$ particles condensed into the vacuum:

$$\frac{\partial \mathcal{F}_A}{\partial \theta_k} = 0 \quad \Rightarrow \quad N_{A_{\mathbf{k}}}(t) = \sinh^2\left(\Gamma_k t\right) = \frac{1}{e^{\beta(t)E_k} - 1} \quad ; \tag{3.30}$$

where $E_k \equiv \hbar\Omega_k$. Finally, the energetic balance of the A-system is given as

$$dE_A - \frac{1}{\beta} d\mathcal{S}_A = 0 \tag{3.31}$$

where $\mathcal{S}_A \equiv \langle 0(t)|S_A|0(t)\rangle$ and $E_A \equiv \sum_{\mathbf{k}} E_k N_{A_{\mathbf{k}}}$. Eq.(3.31) expresses the first principle of thermodynamics for a system coupled with the environment at constant temperature and in absence of mechanical work: E_A is then interpreted as the internal energy of the A-system.

In conclusion, in the above scheme of quantization for the dho, the following fundamental line emerges: the canonical formalism for a dissipative system requires the *doubling* of the degrees of freedom in order to close the system and to deal with an isolated system. This is done by introducing a mirror (time reflected) image of the original oscillator. However, the indefinite structure of the Hamiltonian (3.9) requires, for the quantization, a larger framework than that one of QM, where only one Hilbert space is admitted.

The transition to QFT is thus necessary, and there the dissipation is seen, at a fundamental level, as transition (tunnelling) between unitarily inequivalent representations, parameterized by the time t.

The statistical nature of the dissipation arise then naturally in the above scheme, and a consistent picture of the energetic balance is given. Moreover, the strong similarity with the framework of TFD shows a deep connection with thermal systems. There is however an important difference with respect to TFD, where (see Section 2) the total Hamiltonian is given by the difference in the system and the reservoir (tilde) Hamiltonians. In contrast, the dho Hamiltonian H (3.16) contains also a mixed term, i.e. H_I, which turned out to be just the generator of the Bogoliubov transformation (3.21): it is responsible for the dissipative evolution of the system, which on the other hand is controlled by the entropy. Thus the thermodynamical interpretation of H is more that one of a *free energy* rather than of an energy.

We have also seen that the above quantization scheme naturally contains a breaking of the time reversal symmetry, resulting in a different physics for forward and backward evolution. Indeed, in the presence of the damping factor $e^{-\Gamma t}$, the forward time evolution cannot be mapped into backward time evolution by any operation (as complex conjugation in the non-dissipative case) except time-reversal $t \to -t$. For dissipative systems, however, one is not allowed to use time reversal without changing the physics: thus dissipation induces a *partition* on the time axis and positive and negative time directions must be associated with *separate* modes[16]. These two time reversed modes must be considered together, when constructing a canonical system, which requires to deal with a closed system.

[16]It is possible to show (see §3.4) that these modes are provided by hyperbolic coordinates.

In the non-dissipative case, where only oscillating factors of type $e^{\pm iEt}$ are involved, one can actually limit himself to consider, e.g., only forward time direction, the backward time direction being obtained by complex conjugation operation (or by $t \to -t$ which is now allowed since time-reversal is not broken); this is why one does not need to consider backward and forward modes as *separate* modes in the non-dissipative systems.

3.2 Quantization of boson field on a curved background

Let us now consider the problem of field quantization in a curved background. For a scalar field we have[18]

$$\left[\Box + m^2\right]\phi(x) = 0 \tag{3.32}$$

$$\Box\phi = (-g)^{-\frac{1}{2}}\partial_\mu\left[(-g)^{\frac{1}{2}}g^{\mu\nu}\partial_\nu\phi\right] \tag{3.33}$$

The scalar product needs to be generalized as follows

$$(\phi_1, \phi_2) = -i\int_\Sigma \phi_1(x)\overleftrightarrow{\partial_\mu}\,\phi_2^*(x)[-g_\Sigma(x)]^{\frac{1}{2}}d\Sigma^\mu \tag{3.34}$$

with $d\Sigma^\mu = n^\mu d\Sigma$: Σ is a spacelike hypersurface and $d\Sigma$ is the element of volume enclosed.

Let us now consider a complete set of orthonormal (in the product (3.34)) solutions of eq.(3.32), denoted by $u_i(x)$:

$$(u_i, u_j) = \delta_{ij} \qquad , \qquad (u_i^*, u_j^*) = -\delta_{ij} \qquad , \qquad (u_i, u_j^*) = 0 \tag{3.35}$$

Then the field may be expanded in this basis as

$$\phi(x) = \sum_i\left[a_i u_i(x) + a_i^\dagger u_i^*(x)\right] \tag{3.36}$$

This expansion defines a set of operators $a_i, a_i^\dagger$ with canonical commutation relations and a vacuum, denoted by $|0\rangle$.

However, since the choice of the coordinate system is arbitrary, we can choose a different basis in which decompose our field. Denoting this new set of modes by $\bar{u}_j(x)$, we can write

$$\phi(x) = \sum_j\left[\bar{a}_j\bar{u}_j(x) + \bar{a}_j^\dagger\bar{u}_j^*(x)\right] \tag{3.37}$$

where now the operators $\bar{a}_j, \bar{a}_j^\dagger$ are defined with respect a new vacuum $|\bar{0}\rangle$.

Since both sets are complete, we can relate the modes $\bar{u}_j$ to the u_i as follows

$$\bar{u}_j = \sum_i(\alpha_{ji}u_i + \beta_{ji}u_i^*) \tag{3.38}$$

$$\alpha_{ij} = (\bar{u}_i, u_j) \qquad , \qquad \beta_{ij} = -(\bar{u}_i, u_j^*) \tag{3.39}$$

These matrices are called the Bogoliubov coefficients and they have the following properties

$$\sum_k (\alpha_{ik}\alpha^*_{jk} - \beta_{ik}\beta^*_{jk}) = \delta_{ij} \tag{3.40}$$

$$\sum_k (\alpha_{ik}\beta_{jk} - \beta_{ik}\alpha_{jk}) = 0 \tag{3.41}$$

Equating the expansions (3.37) and (3.36) and using (3.38) and (3.39), we get the relation between the two sets of operators:

$$\bar{a}_j = \sum_i (\alpha^*_{ji} a_i - \beta^*_{ji} a^\dagger_i) \tag{3.42}$$

This is a canonical (Bogoliubov) transformation.

3.2.1 Rindler spacetime

Let us now consider a specific situation of particular interest, namely the one of an uniformly accelerated observer.

Consider two-dimensional Minkowski space with metric

$$ds^2 = dt^2 - dx^2 = d\bar{u}d\bar{v} \tag{3.43}$$

where $\bar{u} = t - x$ and $\bar{v} = t + x$ are the Minkowski null coordinates.

Now we perform the following coordinate transformation to the Rindler coordinates η, ξ:

$$\begin{cases} t = a^{-1}e^{a\xi}\sinh a\eta \\ x = a^{-1}e^{a\xi}\cosh a\eta \end{cases} \quad \Leftrightarrow \quad \begin{cases} \bar{u} = -a^{-1}e^{-au} \\ \bar{v} = a^{-1}e^{av} \end{cases} \quad \textit{in region R} \tag{3.44}$$

with a positive constant, $-\infty < \eta, \xi < \infty$ and $u = \eta - \xi$, $v = \eta + \xi$. The line element is now

$$ds^2 = e^{2a\xi}\left(d\eta^2 - d\xi^2\right) = e^{2a\xi}dudv \tag{3.45}$$

The coordinates (η, ξ) cover only the wedge $x > |t|$ of Minkowski space (see Fig.1). Lines of constant η are straight ($x \propto t$) while lines of constant ξ are hyperbolae

$$x^2 - t^2 = a^{-2}e^{2a\xi} = const = (proper acceleration)^{-2} \tag{3.46}$$

representing the world lines of uniformly accelerated observers. It is interesting to note that the observers' proper time is given by $\tau = e^{a\xi}\eta$.

A second Rindler wedge $x < |t|$ covering another quarter of Minkowski space is defined by

$$\begin{cases} t = -a^{-1}e^{a\xi}\sinh a\eta \\ x = -a^{-1}e^{a\xi}\cosh a\eta \end{cases} \quad \Leftrightarrow \quad \begin{cases} \bar{u} = a^{-1}e^{-au} \\ \bar{v} = -a^{-1}e^{av} \end{cases} \quad \textit{in region L} \tag{3.47}$$

Note that the regions R and L are causally disconnected: the rays $\bar{u} = 0$ and $\bar{v} = 0$ (or $u = \infty$ and $v = -\infty$) act as event horizons for the accelerated observer.

In order to consider quantization, let us look at the simple case of a massless scalar field in two dimensions. In Minkowski spacetime we have

$$\Box\,\phi(x) \;=\; (\partial_t^2 - \partial_x^2)\phi(x) = \frac{\partial^2\phi}{\partial\bar{u}\partial\bar{v}} = 0 \tag{3.48}$$

which gives the usual plane wave solutions

$$\bar{u}_k = \frac{1}{\sqrt{4\pi\omega}} e^{ikx - i\omega t} \tag{3.49}$$

$$\omega = |k| > 0 \qquad , \qquad -\infty < k < \infty \tag{3.50}$$

We denote by $|0_M\rangle$ the vacuum state associated with the expansion of the ϕ field in the basis $\bar{u}_k$.

Using the conformal invariance of Rindler space to Minkowski space (and the conformal invariance of wave equation), we can write it in Rindler coordinates as

$$e^{2a\xi}\,\Box\,\phi(x) \;=\; (\partial_\eta^2 - \partial_\xi^2)\phi(x) = \frac{\partial^2\phi}{\partial u\partial v} = 0 \tag{3.51}$$

for which we find the solutions

$$u_k = \frac{1}{\sqrt{4\pi\omega}} e^{ik\xi \pm i\omega\eta} \tag{3.52}$$

$$\omega = |k| > 0 \qquad , \qquad -\infty < k < \infty \tag{3.53}$$

where the upper signs applies in region L and the lower in region R.

We can thus define two sets of operators, each of that is complete only in one region:

$$^R u_k = \begin{cases} \frac{1}{\sqrt{4\pi\omega}} e^{ik\xi - i\omega\eta} & in\, R \\ 0 & in\, L \end{cases} \tag{3.54}$$

$$^L u_k = \begin{cases} \frac{1}{\sqrt{4\pi\omega}} e^{ik\xi + i\omega\eta} & in\, L \\ 0 & in\, R \end{cases} \tag{3.55}$$

We observe also that by making a imaginary, $^R u_k$ and $^L u_k$ can be analytically continued to regions F and P: thus they are complete (together) on all of Minkowski space and can be used for expanding the field. Thus we have:

$$\phi(x) \;=\; \sum_{k=-\infty}^{\infty} \left[a_k \bar{u}_k \,+\, a_k^\dagger \bar{u}_k^*\right] \tag{3.56}$$

$$=\; \sum_{k=-\infty}^{\infty} \left[b_{k,1}\,{}^L u_k \,+\, b_{k,1}^\dagger\,{}^L u_k^* \,+\, b_{k,2}\,{}^R u_k \,+\, b_{k,2}^\dagger\,{}^R u_k^*\right] \tag{3.57}$$

To these different expansions correspond different vacua: $|0_M\rangle$ and $|0_R\rangle$ for the modes a_k and $b_{k,1}$, $b_{k,2}$, respectively.

We now want to compare the two expansions (3.56) and (3.57). To this end, let us note that the functions ${}^R u_k$ do not go smoothly to ${}^L u_k$ passing from R to L (i.e. crossing the point $\bar{u} = \bar{v} = 0$), due to the sign change in the exponent at this point (see eqs. (3.54), (3.55)). Thus ${}^R u_k$ and ${}^L u_k$ are non-analytic at this point.

However, the following combinations

$$ {}^R u_k \,+\, e^{-\pi\omega/a}\, {}^L u^*_{-k} \tag{3.58} $$
$$ {}^R u^*_{-k} \,+\, e^{\pi\omega/a}\, {}^L u_k \tag{3.59} $$

are analytic at $\bar{u} = \bar{v} = 0$. To see this, we use eqs.(3.44) and (3.47) to rewrite ${}^R u_k$ and ${}^L u^*_{-k}$ as follows:

$$ {}^R u_k \;=\; \frac{1}{\sqrt{4\pi\omega}} exp[-i\omega u] \;=\; \frac{1}{\sqrt{4\pi\omega}} exp[\frac{i\omega}{a} ln(-a\bar{u})] \tag{3.60} $$
$$ {}^L u^*_{-k} \;=\; \frac{1}{\sqrt{4\pi\omega}} exp[-i\omega u] \;=\; \frac{1}{\sqrt{4\pi\omega}} exp[\frac{i\omega}{a} ln(a\bar{u})] \tag{3.61} $$

We thus see that the only difference is in the sign in the logarithm. Using $ln(-1) = -i\pi$, we obtain

$$ \begin{aligned} exp[-\pi\omega/a]^L u^*_{-k} \;&=\; \frac{1}{\sqrt{4\pi\omega}} exp[(-i\pi)(-i\omega/a)] exp\left[i\frac{\omega}{a} ln(a\bar{u})\right] \\ =\; \frac{1}{\sqrt{4\pi\omega}} exp\left[i\frac{\omega}{a}[ln(a\bar{u}) - ln(-1)]\right] \;&=\; \frac{1}{\sqrt{4\pi\omega}} exp\left[i\frac{\omega}{a} ln(-a\bar{u})\right] \end{aligned} \tag{3.62} $$

The expressions (3.60) and (3.62) are now the same: $exp[-\pi\omega/a]^L u^*_{-k}$ can be regarded as an analytical continuation of ${}^R u_k$ to region L. Similarly, one can see that $exp[\pi\omega/a]^L u_k$ is the analytical continuation of ${}^R u^*_{-k}$ to region L.

The combinations (3.60) and (3.61) share the positive frequency analyticity properties of the Minkowski mods $\bar{u}_k$, thus they are associated to the same vacuum state $|0_M\rangle$. Let us then write the normalized combinations

$$ \phi_\omega \;=\; N_\omega \left[e^{\pi\omega/a}\, {}^R u_k \,+\, {}^L u^*_{-k}\right] \tag{3.63} $$
$$ \phi'_\omega \;=\; N'_\omega \left[{}^R u^*_{-k} \,+\, e^{\pi\omega/a}\, {}^L u_k\right] \tag{3.64} $$

and fix the normalization factors

$$ 1 \,=\, (\phi_\omega, \phi_\omega) \,=\, N^2_\omega \left[e^{2\pi\omega/a} \,-\, 1\right] \qquad , \qquad N^2_\omega \,=\, \frac{1}{e^{2\pi\omega/a} \,-\, 1} \tag{3.65} $$

where relations (3.35) have been used. We also get $N'_\omega = N_\omega$. Then the field $\phi(x)$ can be expanded as

$$ \phi(x) = \sum_{k=-\infty}^{\infty} N_\omega \left\{ d_{k,1} \left[e^{\pi\omega/a}\, {}^R u_k \,+\, {}^L u^*_{-k}\right] \,+\, d_{k,2} \left[{}^R u^*_{-k} \,+\, e^{\pi\omega/a}\, {}^L u_k\right] \,+\, h.c. \right\} $$

where now $d_{k,1}|0_M\rangle = d_{k,2}|0_M\rangle = 0$.

We can now find the relation between the (Rindler) b modes and the (Minkowski) d_k modes, by taking the inner products $(\phi, {}^R u_k)$, $(\phi, {}^L u_k)$, first with ϕ given by (3.57) and then by (3.66). We thus get

$$b_{k,1} = (2\sinh(\pi\omega/a))^{-\frac{1}{2}} \left[e^{\pi\omega/2a} d_{k,2} + e^{-\pi\omega/2a} d^\dagger_{-k,1}\right] \tag{3.66}$$

$$b_{k,2} = (2\sinh(\pi\omega/a))^{-\frac{1}{2}} \left[e^{\pi\omega/2a} d_{k,1} + e^{-\pi\omega/2a} d^\dagger_{-k,2}\right] \tag{3.67}$$

This is a Bogoliubov transformation: we thus relate the states $|0_M\rangle$ and $|0_R\rangle$.

To understand the meaning of the above construction, consider an accelerated observer at $\xi = const.$ (for which the proper time is $\propto \eta$). In the region R, this observer will count particles (with momentum k) from the Minkowski vacuum as

$$\langle 0_M | b^\dagger_{k,2} b_{k,2} | 0_M \rangle = \frac{1}{e^{2\pi\omega/a} - 1} \tag{3.68}$$

The same result holds for an accelerated observer in region L. Since this is the Planck spectrum for a radiation at temperature $T_0 = a/2\pi k_0$.

Thus, an accelerated observer in flat space will experience the Minkowski vacuum as a thermal bath. By conformal transformation, it is possible to relate this result to the thermal bath seen by an inertial observer in curved space (Hawking effect).

Section 4

Spontaneous symmetry breaking and macroscopic objects

4.1 Spontaneous symmetry breaking

In Section 1 we have seen how QFT has a dual structure: on one side there are the fundamental entities (Heisenberg fields) in terms of which the dynamics is described, on the other the observed particles (described by free fields). This dual structure induces a sophisticated mechanism for the manifestation of the fundamental invariances of the system at phenomenological level, in the various phases in which it can be.

The qualifying aspect of a particular phase of the system is its degree of order (symmetry) with respect to the fundamental invariances of the system itself. The fact that the system can manifest, at phenomenological level, different symmetries, is a consequence of its invariance properties.

We have seen indeed that a system with infinitely many degrees of freedom admits the existence of unitarily inequivalent representations of his dynamics, i.e. of inequivalent vacua, which non-necessarily share the same invariance properties of the system.

We will see that the mechanism of boson condensation, which generate inequivalent representations, is the same at the basis of the spontaneous breakdown of symmetry (SSB) and the transition among different phases.

Let us consider a concrete example of SSB. A ferromagnet is described by a Hamiltonian which is rotationally invariant: when it develop a non-zero magnetization, then the vacuum manifests a directional order and also, since the direction of magnetization is arbitrary, vacua with different directions are degenerate in energy.

We can immediately recognize two peculiar aspects of this process: the possibility that the symmetry breaking takes place in any direction, is a sign of the original rotational invariance. The stability of this order necessitates of correlations between the parts of the system.

The first point indicates that the degenerate vacua are each other connected (at least formally) by the invariance transformations for the dynamics. The second point means that the magnetization, i.e. the macroscopic order, is stable independently from local deviations: thus the deviation of a spin should be reacted by the other spins in a way that globally it is compensated. The way in which such a correction acts is through a wave which propagates on the entire domain. In QFT this domain is generally infinite, and this means that the mode associated to the correlation is gapless.

It is important to understand in which cases it does exist a unitary representation of an invariance transformation for the dynamics. This analysis will let us understand better the difference between QM and systems with an infinite number of degrees of freedom.

We can define an invariance transformation for the dynamics as an *automorphism* of the algebra $\mathcal{A}$ of the canonical variables (Heisenberg algebra). By definition, the automorphism does preserve the algebraic relations ($A'B' = (AB)'$, $A^{\dagger'} = A^{\dagger}$ *etc.*) and in particular the commutation relations, then it is a canonical transformation: the equations of motion, which in the Heisenberg representation are algebraic relation among the elements of $\mathcal{A}$, are invariant under its action.

A transformation $T : \mathcal{H} \to \mathcal{H}$ in QM is an *exact symmetry* if $\forall \alpha, \beta \in \mathcal{H}$

$$|\langle \alpha, \beta \rangle|^2 \;=\; |\langle \alpha', \beta' \rangle|^2 \tag{4.1}$$

i.e. if it leaves invariant the transition probabilities. It is an important result due to Wigner[19], that any transformation of exact symmetry can be described in terms of an operator $U : \mathcal{H} \to \mathcal{H}$ which is unitary or anti-unitary. This implies that the transformation T induces a corresponding transformation $T^{\mathcal{A}}$ in the algebra $\mathcal{A}$ of canonical variables

$$T^{\mathcal{A}} :\; A \in \mathcal{A} \to A' \;=\; UAU^{-1} \,\in \mathcal{A} \tag{4.2}$$

which, due to the unitarity of U, is an automorphism. Thus, in QM any exact symmetry is an automorphism of the Heisenberg algebra. The point is now if a generic automorphism of $\mathcal{A}$ can be written in the form (4.2). In QM, as a direct consequence of the Von Neumann theorem, this is always possible[4]: any automorphism of $\mathcal{A}$ is an exact symmetry, i.e. it can be expressed by means of a unitary operator U. The words *exact symmetry* and *invariance* are synonyms in QM.

The situation is quite different for infinite systems, for which the existence of inequivalent representations is admitted. The existence of a unitary realization of the automorphism as in (4.2) is not anymore guaranteed by the Von Neumann theorem, and when it does not exist, we have *spontaneous breakdown of the symmetry.*

4.1.1 Spontaneous breakdown of continuous symmetries

The dynamics of a system of Heisenberg fields $\psi(x)$ is determined by the Lagrangian density $\mathcal{L}(\psi(x), \partial\psi(x))$, where ψ stands for the set of the fields ψ_i $i = 1...n$. If $G(\theta^a)$ is a continuous group in the parameters θ^a, of (purely functional) transformations of the fields, then the variation of $\mathcal{L}(\psi(x), \partial\psi(x))$ under these transformations is given by

$$\partial^{\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\psi)}\frac{\delta\psi}{\delta\theta^a}\right]\delta\theta^a \;=\; \delta\mathcal{L}(x) \tag{4.3}$$

and the Noether current is

$$j^a_\mu(x) \;=\; \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\frac{\delta \psi}{\delta \theta^a} \tag{4.4}$$

The observable charges are given by

$$Q^a(t) \;=\; \int d^3x \, j^a_0(x) \tag{4.5}$$

and we have

$$\frac{d}{dt}Q^a(t) \;=\; \int d^3x \, \delta\mathcal{L}(x) \tag{4.6}$$

Thus the charges Q^a are conserved when $\mathcal{L}$ is invariant under the transformations of G. In this case we have $[H, Q^a] = 0$, where H is the Hamiltonian.

If G is a compact Lie group, the charges Q^a form a basis of the associated Lie algebra

$$[Q^a, Q^b] = C^{abc}Q^c \tag{4.7}$$

where C^{abc} are the structure constants.

Now, it is possible to see[4] that necessary and sufficient condition for the transformations of G to be exact symmetries, is $Q^a|0\rangle = 0$. In such a case, it exist an unitary representation of G obtained by exponentiation of the charges Q^a:

$$U(\theta) = exp[i\theta^a Q^a] \tag{4.8}$$

If for some a, $Q|0\rangle \neq 0$ in a representation of $\mathcal{A}$, then $|0\rangle$ is not anymore invariant under transformations of G and the symmetry is spontaneously broken. If the vacuum is invariant under spatial translations we have that the action of Q^a on it is not confined to the Hilbert space of the representation under consideration. We have indeed[20]:

$$|Q|0\rangle|^2 = \langle 0|Q^2|0\rangle = \int d^3x \int d^3y \langle 0|j_0(x)j_0(y)|0\rangle \tag{4.9}$$

and, using the translational invariance of the vacuum state, i.e. $e^{i\mathbf{P}\cdot\mathbf{x}}|0\rangle = |0\rangle$, we get (a part from a phase factor)

$$\begin{aligned}\langle 0|j_0(x)j_0(y)|0\rangle \;&=\; \langle 0|e^{i\mathbf{P}\cdot\mathbf{x}}j_0(0)e^{-i\mathbf{P}\cdot\mathbf{x}}e^{i\mathbf{P}\cdot\mathbf{y}}j_0(0)e^{-i\mathbf{P}\cdot\mathbf{y}}|0\rangle \\ &=\; \langle 0|j_0(0)e^{i\mathbf{P}\cdot(\mathbf{x}-\mathbf{y})}j_0(0)|0\rangle = F(\mathbf{x}-\mathbf{y})\end{aligned} \tag{4.10}$$

and $Q|0\rangle$ is not normalizable

$$|Q|0\rangle|^2 = \int d^3x F(\mathbf{x}) \int d^3y = \infty \tag{4.11}$$

On the other hand, such a result is not surprising: in §1.2 we have seen that the action of the Bogoliubov generator $G_c(\theta)$ on the vacuum has the same effect, and this is to be interpreted as a change of representation of the canonical variables.

Thus, when the invariance is spontaneously broken, we see how the transformations of G connect inequivalent and degenerate vacua (representations).

4.2 SSB and symmetry rearrangement

We have seen that necessary and sufficient condition for having breakdown of the invariance under a continuous group G in a specific representation built on the vacuum $|0\rangle$ is the following

$$[H, Q^a] = 0 \qquad , \qquad Q^a|0\rangle \neq 0 \tag{4.12}$$

As a consequence, the action of G leads out of the Fock space $\{|0\rangle\}$ and the vacua $|0(\theta)\rangle = exp[i\theta^a Q^a]|0\rangle$, on which are built the inequivalent representations connected by transformations of G, are degenerate.

When SSB is present, in the spectrum of the physical particles appear *gapless* excitations (Goldstone bosons) which generate the *long range correlations* necessary for the stability of the ordered state. It is possible to see that the transition between the inequivalent vacua is controlled by transformations of the Goldstone fields.

This is the *rearrangement of symmetry.* In more general terms, this is defined as follows: consider a continuous group G of transformations on Heisenberg fields, under which the Heisenberg equations are invariant,

$$\psi(x) \to \psi'(x) \;=\; G[\psi(x)] \tag{4.13}$$

Let $\psi(x) = \psi[x; \phi]$ be the dynamical map, through which the Heisenberg fields are expressed in terms of the physical fields ϕ. If then exists a continuous group g of transformations on the physical fields

$$\phi(x) \to \phi'(x) \;=\; g[\phi(x)] \tag{4.14}$$

inducing the transformations G on the Heisenberg fields in the following way,

$$G[\psi(x)] \;=\; \psi[\,x; g[\phi(x)]\,] \tag{4.15}$$

and if G and g are different groups, then we will say that the invariance G has been rearranged in terms of the group g.

This means that, in the presence of SSB, the fundamental invariance of the theory is not lost, but is realized (rearranged) in a different symmetry at the level of the observable fields. The link between the two different levels of symmetry is provided by the dynamical map.

4.2.1 The rearrangement of symmetry in a phase invariant model

Consider the simple model of a boson complex scalar field exhibiting spontaneous symmetry breaking. We have

$$\mathcal{L} = -\partial^\mu \psi^\dagger \partial_\mu \psi \,+\, \mu^2 \psi^\dagger \psi \,+\, \lambda(\psi^\dagger \psi)^2 \tag{4.16}$$

The Hamiltonian is given by

$$H = \int d^3x \left[\dot{\psi}^\dagger \dot{\psi} \,-\, \nabla\psi^\dagger \nabla\psi \,-\, \mu^2 \psi^\dagger \psi \,-\, \lambda(\psi^\dagger \psi)^2\right] \tag{4.17}$$

This Lagrangian is invariant under global phase transformations:

$$\psi(x) \to \psi'(x) = e^{i\theta}\,\psi(x)$$
$$\delta\psi = i\psi \quad , \quad \delta\psi^\dagger = -i\psi^\dagger \tag{4.18}$$

where the second line gives the variation for infinitesimal θ. The Noether current associated with the above invariance is

$$j_\mu = -i\,\psi^\dagger\,\overleftrightarrow{\partial}_\mu\,\psi \tag{4.19}$$

The Noether charge Q is also the generator of the above transformation

$$Q \,=\, -i\int d^3x\,\left[\psi^\dagger\dot{\psi} \,-\, \dot{\psi}^\dagger\psi\right] \tag{4.20}$$

When the vacuum is not invariant under the action of Q, we have spontaneous symmetry breaking:

$$Q\,|0\rangle \,\neq\, 0 \tag{4.21}$$

Associated with this breaking of symmetry, it appears the (gapless) Goldstone boson χ (we have only one Goldstone since we have only one generator Q)

$$\partial^2\,\chi(x) \,=\, 0 \tag{4.22}$$

It is possible to show[5] by use of path integral, that the general form of the dynamical map for a theory with phase invariance is

$$\psi(x) \,=\, e^{\frac{i}{c}\chi(x)}F[\rho,\partial\chi] \tag{4.23}$$

with F being a functional of the quasiparticle field and of the derivatives (only) of the Goldstone boson. The field $\chi(x)$ appears in the phase. This is the *rearrangement of symmetry*: the phase transformation of the Heisenberg field $\psi(x)$ is controlled, at the phenomenological level, only by the boson translation of the Goldstone field $\chi(x)$:

$$\chi(x) \,\to\, \chi(x) \,+\, c\theta \tag{4.24}$$

We now define Φ as the following quantity

$$\Phi(x) \,=\, \frac{1}{2}(\psi(x) \,+\, \psi^\dagger(x)) \tag{4.25}$$

and χ_H as

$$\chi_H \equiv \delta\Phi \,=\, \frac{i}{2}(\psi(x) \,-\, \psi^\dagger(x)) \tag{4.26}$$

The field χ_H is the interpolating Heisenberg field for the Goldstone boson χ; we also have $\psi = \Phi - i\chi_H$. The order parameter is given by

$$v \,=\, \frac{1}{2}\langle 0|(\psi(x) \,+\, \psi^\dagger(x))|0\rangle \tag{4.27}$$

Thus we have the first terms for the dynamical maps of Φ and χ_H:

$$\begin{aligned} \Phi &= v + Z_1 \rho + \cdots \\ \chi_H &= Z_2 \chi + \cdots \end{aligned} \tag{4.28}$$

and consequently for ψ

$$\psi = (v + Z_1 \rho) - i Z_2 \chi + \cdots \tag{4.29}$$

Here Z_1 and Z_2 are renormalization constants.

It is interesting to see which kind of informations can be extracted from the dynamical map (4.29) in the linear approximation, i.e. when we disregard the dots.

We require that the Hamiltonian (4.17) is diagonalized at operator level (since we consider a linear approximation) when the expansion (4.29) is used.

We then get the usual relation

$$v^2 = -\frac{\mu^2}{2\lambda} \tag{4.30}$$

indicating that λ should be negative (attractive interaction).

The Hamiltonian becomes

$$H = \int d^3x \left[\dot{\rho}^2 - (\nabla\rho)^2 + 2\mu^2 Z_1^2 \rho^2 + \dot{\chi}^2 - (\nabla\chi)^2\right] \tag{4.31}$$

The dynamical map of the generator Q has the form

$$Q = \int d^3x \, c \, \dot{\chi}(x) \tag{4.32}$$

since $[\chi(x), \dot{\chi}(y)]_{t=t'} = i\delta^3(x-y)$.

Thus, considering $\partial_\mu j^\mu = 0$, and $\partial^2\chi = 0$, we write

$$j_\mu = c\,\partial_\mu \chi(x) + ... \tag{4.33}$$

We have seen that this operator induces the boson transformation (4.24) on the Goldstone field χ (rearranged as phase transformation at the level of the Heisenberg field: $\psi \to e^{i\theta}\psi$).

4.3 The boson transformation and the description of macroscopic objects

Consider an interacting boson field $\psi(x)$ satisfying the following Heisenberg equation:

$$\Lambda(\partial)\,\psi(x) = J[\psi](x) \tag{4.34}$$

where $x \equiv (t, \mathbf{x})$ and J is a certain functional of the ψ fields, describing their interactions.

The equations for the physical field $\varphi(x)$ is:

$$\Lambda(\partial)\,\varphi(x) \;=\; 0 \tag{4.35}$$

Now, equation (4.34) can be recasted in the following integral form (Yang-Feldman equation):

$$\psi(x) \;=\; \varphi(x) \,+\, \Lambda^{-1}(\partial)\, J[\psi](x) \tag{4.36}$$

The inverse of the free field operator is the Green function for the $\varphi(x)$ field. Eq.(4.36) can be solved by iteration, thus giving an expression for the Heisenberg field $\psi(x)$ in terms of powers of the $\varphi(x)$ field; this is the *dynamical map*:

$$\psi(x) \;=\; F\,[x;\varphi] \tag{4.37}$$

This is expression is however *not* unique. Consider indeed a c-number function $f(x)$, satisfying the same equations of the free field:

$$\Lambda(\partial)\, f(x) \;=\; 0 \tag{4.38}$$

Then the Heisenberg equation (4.34) is satisfied also by

$$\psi(x) \;=\; \varphi(x) \,+\, f(x) + \Lambda^{-1}(\partial)\, J[\psi](x) \tag{4.39}$$

whose dynamical map is written as

$$\psi_f(x) \;=\; F\,[x;\varphi + f] \tag{4.40}$$

It is possible to show[12, 5] that this new map is obtained from that in eq.(4.37), by the *boson field transformation*

$$\varphi(x) \;\to\; \varphi(x) \,+\, f(x)\,. \tag{4.41}$$

Eqs.(4.40), (4.41) express the *boson transformation theorem*[12, 5]. It is now clear that a given dynamics, eq.(4.34), contains an internal freedom, represented by the possible choices of the function $f(x)$ among the solutions of the free field equations (4.38).

Among these solutions, we can distinguish between Fourier transformable ones and Fourier non-transformable ones. In the first case, the boson transformation (4.41) essentially give rise to an homogeneous condensation.

More interesting is the case associated with solutions which are non Fourier transformable, since in this case the translation (4.41) generate a local (inhomogeneous) condensation, which can be associated to a macroscopic object. The absence of Fourier transform for the $f(x)$ is associated to the singularities carried by this function: they can be either *divergent singularities* or *topological singularities.* The first are associated to a divergence of $f(x)$ for $|\mathbf{x}| = \infty$, at least in some direction. The second means that $f(x)$ is not single-valued, i.e. is path dependent. In this case, the macroscopic object described by the order parameter, will carry a topological singularity.

Also notice that the free field equations (4.38) are linear: thus every linear combination of the solutions is also a valid choice for the boson transformations; these choices are however physically inequivalent, since they describe, as we will see, different "sectors" for the macroscopic objects.

This is clear when we consider the structure of the Fock space: as we have seen in §1.2, the boson field translation changes the structure of the original Hilbert space. It is possible to show[12, 5] that in general, the structure of the Fock space in presence of extended object is given by $|0\rangle_f \equiv |0\rangle \otimes |0\rangle_q$ where $|0\rangle$ is the vacuum for the physical particles and $|0\rangle_q$ that relative to the quantum-mechanical operators associated with the macroscopic objects.

The vacuum expectation value of eq.(4.39) gives the classical behaviour of the macroscopic object:

$$\begin{aligned} \phi_f(x) &\equiv \langle 0|\psi_f(x)|0\rangle \\ &= f(x) + \langle 0| \left[\Lambda^{-1}(\partial)\, J[\psi_f](x)\right] |0\rangle \end{aligned} \tag{4.42}$$

Notice that the order parameter $\phi(x)$ contains quantum effects, due to the contractions of the physical fields occurring in the second term of eq.(4.42). One may disregard these contributions in a first approximation: this is obtained at the tree level, i.e. when a product of normal ordered products of the $\varphi(x)$ fields is taken equal to a normal product. Thus at the tree level, the only contributions from the second hand side of eq.(4.42) are from the $f(x)$, since it holds $\langle 0|J[\psi_f]|0\rangle = J[\phi_f]$. In this limit, $\phi(x)$ given by eq.(4.42) is the solution of the (non-linear) classical equation:

$$\Lambda(\partial)\,\phi_f(x) \;=\; J[\phi_f](x) \tag{4.43}$$

Beyond the tree level, in general the form of this equation changes: quantum corrections can affect the macroscopic level in a relevant way.

When we consider $\phi(x)$ at tree level, the boson method essentially provides a linearization of eq.(4.43), through eq.(4.42), which becomes a series in $f(x)$ only, since all the terms containing $\varphi(x)$ are normal ordered, thus having zero expectation value on $|0\rangle$.

The boson transformation however does not give only the equations for the order parameter, i.e. eq.(4.43). The dynamical map (4.39) describes indeed, at higher orders, the dynamics of one or more quantum physical particles in the potential generated by the macroscopic object $\phi_f(x)$.

To see this, we rewrite the dynamical map in presence of the macroscopic object, eq.(4.40), by expanding around the classical field $\phi(x)$ (including in general quantum corrections):

$$\begin{aligned} \psi_f(x) &= \phi_f(x) + \int d^4y\,\varphi(y)\,\frac{\delta}{\delta f(y)}\phi_f(x) + \frac{1}{2} : \int d^4y_1\, d^4y_2\,\varphi(y_1)\,\varphi(y_2)\,\frac{\delta}{\delta f(y_1)}\frac{\delta}{\delta f(y_2)}\phi_f(x) : + \dots \\ &\equiv \phi_f(x) + \psi_f^{(1)}(x) + \frac{1}{2} : \psi_f^{(2)}(x) : + \dots \end{aligned} \tag{4.44}$$

By acting with $\int d^4y\,\varphi(y)\frac{\delta}{\delta f(y)}$ on eq.(4.43), we get[17]

$$\Lambda(\partial)\,\psi_f^{(1)}(x) - \int d^4y\, V_f(x,y)\,\psi_f^{(1)}(y) \;=\; 0$$

[17] We use $\int d^4y\,\varphi(y)\frac{\delta}{\delta f(y)}G(x) = \int d^4z\,\frac{\delta G(x)}{\delta\phi(z)}\int d^4y\,\varphi(y)\frac{\delta\phi_f(z)}{\delta f(y)} = \int d^4z\,\frac{\delta G(x)}{\delta\phi(z)}\,\psi_f^{(1)}(z)$

$$with \quad V_f(x,y) \,=\, \frac{\delta}{\delta\phi_f(y)}\,\langle 0|J[\psi_f]|0\rangle \tag{4.45}$$

$V_f(x,y)$ is the (self-consistent) potential: it is induced by the macroscopic object which is self-consistently created in the quantum field system.

Eq.(4.45) describe a particle under the influence of the potential $V_f(x,y)$. It is possible to derive a complete hierarchy of such equations, describing at higher order, scattering of two or more particles in the potential of the macroscopic object. For example, the equation of next order with respect to eq.(4.45) is:

$$\Lambda(\partial)\,\psi^{(2)}(x) \,-\, \int d^4y\, V_f(x,y)\,\psi^{(2)}(y) \,=\, \int d^4y_1\, d^4y_2\, V_f(x,y_1,y_2)\,\psi^{(1)}(y_1)\,\psi^{(1)}(y_2)$$

$$with \quad V_f(x,y_1,y_2) \,=\, \frac{\delta}{\delta\phi(y_1)}\,\frac{\delta}{\delta\phi(y_2)}\,\langle 0|J[\psi]|0\rangle \tag{4.46}$$

4.3.1 Solitons in $1+1$-dimensional $\lambda\phi^4$ model

In order to show how the above theory works, we consider an exactly solvable mode exhibiting non-trivial topological solutions, namely the $1+1$-dimensional $\lambda\phi^4$ model for a scalar boson field $\psi(x)$:

$$\mathcal{L} = \partial^i\psi\partial_i\psi \,-\, \frac{1}{2}\mu^2\psi^2 \,-\, \frac{1}{4}\lambda\,\psi^4 \tag{4.47}$$

The equation of motion reads

$$(\partial^2 + \mu^2)\psi(x) \,=\, -\lambda\,\psi^3(x) \tag{4.48}$$

Here ψ is an Heisenberg field. If spontaneous symmetry breaking occurs, we have a non-zero expectation value for the Heisenberg field, denoted as

$$v \,\equiv\, \langle 0|\psi(x)|0\rangle \tag{4.49}$$

We then define the Heisenberg operator $\rho(x)$ as

$$\psi(x) \,=\, v \,+\, \rho(x) \tag{4.50}$$

Then the equation (4.47) is rewritten as

$$(-\partial^2 - m^2)\rho(x) \,=\, \frac{3}{2}mg\rho^2(x) \,+\, \frac{1}{2}g^2\rho^3(x) \tag{4.51}$$

$$\lambda v^2 = -\mu^2 \qquad , \qquad g = \sqrt{2\lambda} \qquad , \qquad m^2 = 2\lambda v^2 \tag{4.52}$$

We now denote the asymptotic in-field by $\rho^{in}(x)$: by definition it satisfies the free-field equation associated to (4.51),

$$\left[\partial^2 \,+\, m^2\right]\rho^{in}(x) \,=\, 0 \tag{4.53}$$

Now we consider the *dynamical map* for the Heisenberg field $\rho(x)$: this is an expansion in terms of the field operators of the asymptotic fields, in our case $\rho^{in}(x)$. We remember that such an expansion is valid in a weak sense, i.e. when the expectation value on states of the Fock space of asymptotic particles are taken.

By using the tree approximation - i.e. we write all products of normal products as normal products without contractions -, we get

$$\rho(x) = \rho^{in}(x) - \frac{3}{2}mgi \int d^2y \Delta(x-y) : [\rho^{in}(y)]^2 :$$
$$-\frac{1}{2}g^2 i \int d^2y \Delta(x-y) : [\rho^{in}(y)]^3 : -\frac{9}{2}m^2 g^2 \int d^2y d^2z \Delta(x-y)\Delta(y-z) : \rho^{in}(z)[\rho^{in}(y)]^2 : \quad (4.54)$$

where $\Delta(x)$ is the propagator for the free field ρ^{in}, i.e.: $\partial^2\Delta(x) = i\delta^2(x)$. We now put in eq.(4.51) the expansion

$$\rho(x) = \sum_{n=1}^{\infty} \rho^{(n)}(x) \quad (4.55)$$

where n denotes the order of the normal products and $\rho^{(1)} = \rho^{in}$. We then get (in the tree approximation) the recurrence relation[18]:

$$\begin{aligned} \rho^{(n)}(x) &= -\frac{3}{2}mgi \int d^2y \Delta(x-y) : \sum_{i+j=n} \rho^{(i)}(y)\rho^{(j)}(y) : \\ &- \frac{1}{2}g^2 i \int d^2y \Delta(x-y) : \sum_{i+j+k=n} \rho^{(i)}(y)\rho^{(j)}(y)\rho^{(k)}(y) : \end{aligned} \quad (4.56)$$

At this point we perform the boson transformation:

$$\rho^{in}(x) \to \rho^{in}(x) + f(x) \quad (4.57)$$
$$\left[\partial^2 + m^2\right] f(x) = 0 \quad (4.58)$$

Denoting by ρ_f the boson-transformed (Heisenberg) field ρ, we have

$$\langle 0|\psi_f(x)|0\rangle = \phi_f(x) = v + \langle 0|\rho_f(x)|0\rangle \quad (4.59)$$

We consider the static case, then eq.(4.58) becomes

$$\left[\frac{d^2}{dx^2} - m^2\right] f(x) = 0 \quad (4.60)$$

and, by taking the vacuum expectation value of the dynamical map eq.(4.54), we get

$$\begin{aligned} \phi_f(x) &= v + f(x) + \frac{3}{2}mg \int dy K(x-y) f^2(y) \\ &+ \frac{1}{2}g^2 \int dy K(x-y) f^3(y) + \frac{9}{2}m^2 g^2 \int dy K(x-y) f(y) \int dz K(y-z) f^2(z) + \ldots \end{aligned} \quad (4.61)$$

[18] We use the formula (valid for absolutely converging series):

$$(\sum_n a_n)^k = \sum_n \sum_{\substack{i_1,\ldots i_k \\ i_1+..+i_k=n}} a_{i_1} \cdots a_{i_k}$$

where the Green's function $K(x)$ is defined by

$$\left(\frac{d^2}{dx^2} - m^2\right) K(x-y) = \delta(x-y) \tag{4.62}$$

The recurrence relation becomes

$$\begin{aligned} \rho_f^{(n)}(x) &= \frac{3}{2}mg \int dy K(x-y) : \sum_{i+j=n} \rho_f^{(i)}(y)\rho_f^{(j)}(y) : \\ &+ \frac{1}{2}g^2 \int dy K(x-y) \int dy K(x-y) : \sum_{i+j+k=n} \rho_f^{(i)}(y)\rho_f^{(j)}(y)\rho_f^{(k)}(y) : \end{aligned} \tag{4.63}$$

As a solution of eq.(4.60) we choose $f(x)$ which diverges at $x = -\infty$ and is regular at $x = \infty$. We have

$$f(x) = A\, e^{-mx} \tag{4.64}$$

$$K(x) = -\theta(-x)\frac{1}{m}\sinh(mx) \tag{4.65}$$

Noticing that $\rho_f^{(1)}(x) = f(x)$, we get from the vacuum expectation value of (4.63)[19]:

$$\rho_f^{(n)}(x) = C_n (e^{-mx})^n \tag{4.66}$$

$$C_n = \frac{1}{m^2(n^2-1)}\left[\frac{3}{2}mg \sum_{i+j=n} C_i C_j + \frac{1}{2}g^2 \sum_{i+j+k=n} C_i C_j C_k\right] \quad , \ (n \geq 2) \tag{4.67}$$

$$C_1 = A$$

Solving eq.(4.67), and using $v = m/g$, we have

$$C_n = 2v\left(\frac{A}{2v}\right)^n \tag{4.68}$$

which gives

$$\begin{aligned} \phi_f(x) &= v + 2v\sum_{n=1}^{\infty}\left(\frac{A}{2v}e^{-mx}\right)^n \\ &= v\frac{1 + (A/2v)e^{-mx}}{1 - (A/2v)e^{-mx}} \end{aligned} \tag{4.69}$$

By choosing $A = -2ve^{ma}$, we obtain

$$\phi_f(x) = v \tanh\left[\frac{m}{2}(x-a)\right] \tag{4.70}$$

[19] We use

$$-\int_x^{\infty} dy\, \frac{1}{m}\sinh(my)\, e^{-mny} = \frac{e^{-mnx}}{m^2(n^2-1)}$$

which is the well known static solution of the non-linear equation

$$(\partial^2 + \mu^2)\phi_f(x) = -\lambda\,\phi_f^3(x) \tag{4.71}$$

If on the other hand we choose $A = 2ve^{ma}$, we get another solution of eq.(4.71):

$$\phi_f(x) = v\coth\left[\frac{m}{2}(x-a)\right] \tag{4.72}$$

Multiple-soliton solutions of eq.(4.71) can be obtained by f which are superposition of f corresponding to single-soliton solutions[20].

4.3.2 Vortices in superfluids

The transformation (4.24) can be read as the boson transformation when θ is interpreted as the cylindrical angle[21],

$$\chi(x) \to \chi(x) + c\,n\,\theta(x) \tag{4.73}$$

For a static f indeed, $\partial^2\chi = 0$ becomes the Laplace eq. for θ, and also the order parameter is single valued, as should be, since the phase of $\phi = {}_f\langle 0|\psi|0\rangle_f$ is $e^{in\theta}$, which is single valued.

After the boson transformation, the current becomes

$$j_i = \frac{1}{c}\nabla_i\chi + n\nabla_i\theta \tag{4.74}$$

where the dots are ignored. This generates the macroscopic current

$$J_i(x) = {}_f\langle 0|j_i|0\rangle_f = n\nabla_i\theta = \frac{n}{r}\mathbf{e}_i \qquad , \quad i = 1,2 \tag{4.75}$$

This is the superfluid vortex. We have $J_3 = 0$.

The quantization of the flux is expressed as

$$\int_{\partial S} d\mathbf{s}\cdot\mathbf{J} = 2\pi n \tag{4.76}$$

or

$$Q_T = \int_S dS\mathbf{e}_3\cdot(\nabla\times\mathbf{J}) = 2\pi n \tag{4.77}$$

From eq.(4.75) we see that the superfluid current does diverge at $r = 0$; this however does not happen in practice since the current should be well defined everywhere, thus it is zero at the vortex center.

[20] This is possible due to the linearity of the equation for f.

[21] Then θ satisfies $\nabla^2\theta = 0$.

Section 5

Mixing transformations in Quantum Field Theory

5.1 Fermion mixing

In this Section we will consider the mixing transformations for fermion fields in QFT [22]. Since we have in mind neutrinos, for which we will consider flavour oscillations, we will specialize in the following discussion to neutrino Dirac fields. However, the scheme has general validity for any Dirac fields.

Let us consider mixing of two flavour fields [23] (for extension to three flavours see ref.[22]) which we will denote by $\nu_e(x)$, $\nu_\mu(x)$. The mixing relations, originally proposed by Pontecorvo, are

$$\begin{aligned} \nu_e(x) &= \nu_1(x)\ \cos\theta + \nu_2(x)\ \sin\theta \\ \nu_\mu(x) &= -\nu_1(x)\ \sin\theta + \nu_2(x)\ \cos\theta\ , \end{aligned} \tag{5.1}$$

where $\nu_e(x)$ and $\nu_\mu(x)$ are the (Dirac) neutrino fields with definite flavours. $\nu_1(x)$ and $\nu_2(x)$ are the (free) neutrino fields with definite masses m_1 and m_2, respectively. The fields $\nu_1(x)$ and $\nu_2(x)$ are written as

$$\nu_i(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} e^{i\mathbf{k}\cdot\mathbf{x}} \left[u^r_{\mathbf{k},i}(t)\alpha^r_{\mathbf{k},i} + v^r_{-\mathbf{k},i}(t)\beta^{r\dagger}_{-\mathbf{k},i} \right], \quad i = 1, 2\ . \tag{5.2}$$

where $u^r_{\mathbf{k},i}(t) = e^{-i\omega_{k,i}t}u^r_{\mathbf{k},i}$ and $v^r_{\mathbf{k},i}(t) = e^{i\omega_{k,i}t}v^r_{\mathbf{k},i}$, with $\omega_{k,i} = \sqrt{\mathbf{k}^2 + m_i^2}$. The $\alpha^r_{\mathbf{k},i}$ and the $\beta^r_{\mathbf{k},i}$, $i = 1, 2$, $r = 1, 2$ are the annihilation operators for the vacuum state $|0\rangle_{1,2} \equiv |0\rangle_1 \otimes |0\rangle_2$: $\alpha^r_{\mathbf{k},i}|0\rangle_{12} = \beta^r_{\mathbf{k},i}|0\rangle_{12} = 0$. The anticommutation relations are:

$$\{\nu_i^\alpha(x), \nu_j^{\beta\dagger}(y)\}_{t=t'} = \delta^3(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}\delta_{ij}\ , \qquad \alpha, \beta = 1, .., 4\ , \tag{5.3}$$

and

$$\{\alpha^r_{\mathbf{k},i}, \alpha^{s\dagger}_{\mathbf{q},j}\} = \delta_{\mathbf{kq}}\delta_{rs}\delta_{ij}; \qquad \{\beta^r_{\mathbf{k},i}, \beta^{s\dagger}_{\mathbf{q},j}\} = \delta_{\mathbf{kq}}\delta_{rs}\delta_{ij}, \quad i, j = 1, 2\ . \tag{5.4}$$

All other anticommutators are zero. The orthonormality and completeness relations are:

$$\sum_\alpha u^{r\alpha*}_{\mathbf{k},i} u^{s\alpha}_{\mathbf{k},i} = \sum_\alpha v^{r\alpha*}_{\mathbf{k},i} v^{s\alpha}_{\mathbf{k},i} = \delta_{rs} \,, \qquad \sum_\alpha u^{r\alpha*}_{\mathbf{k},i} v^{s\alpha}_{-\mathbf{k},i} = \sum_\alpha v^{r\alpha*}_{-\mathbf{k},i} u^{s\alpha}_{\mathbf{k},i} = 0 \,,$$
$$\sum_r (u^{r\alpha*}_{\mathbf{k},i} u^{r\beta}_{\mathbf{k},i} + v^{r\alpha*}_{-\mathbf{k},i} v^{r\beta}_{-\mathbf{k},i}) = \delta_{\alpha\beta} \,. \tag{5.5}$$

Eqs.(5.1) (or the ones obtained by inverting them) relate the respective Hamiltonians $H_{1,2}$ (we consider only the mass terms) and $H_{e,\mu}$ [23]:

$$H_{1,2} = m_1 \, \bar{\nu}_1 \nu_1 + m_2 \, \bar{\nu}_2 \nu_2 \tag{5.6}$$

$$H_{e,\mu} = m_{ee} \, \bar{\nu}_e \nu_e + m_{\mu\mu} \, \bar{\nu}_\mu \nu_\mu + m_{e\mu} \, (\bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e) \tag{5.7}$$

where $m_{ee} = m_1 \cos^2\theta + m_2 \sin^2\theta$, $m_{\mu\mu} = m_1 \sin^2\theta + m_2 \cos^2\theta$ and $m_{e\mu} = (m_2 - m_1)\sin\theta\cos\theta$.

We have seen in the previous Chapter, that in QFT the basic dynamics, i.e. the Lagrangian and the resulting field equations, is given in terms of Heisenberg (or interacting) fields. The physical observables are expressed in terms of asymptotic in- (or out-) fields, also called physical or free fields. In the LSZ formalism of QFT [9, 7], the free fields, say for definitiveness the in-fields, are obtained by the weak limit of the Heisenberg fields for time $t \to -\infty$. The meaning of the weak limit is that the realization of the basic dynamics in terms of the in-fields is not unique so that the limit for $t \to -\infty$ (or $t \to +\infty$ for the out-fields) is representation dependent.

Typical examples are the ones of spontaneously broken symmetry theories, where the same set of Heisenberg field equations describes the normal (symmetric) phase as well as the symmetry broken phase. We also observed that, since observables are described in terms of asymptotic fields, unitarily inequivalent representations describe different, i.e. physically inequivalent, phases. It is therefore of crucial importance, in order to get physically meaningful results, to investigate with much care the mapping among Heisenberg or interacting fields and free fields, i.e. the dynamical map (see §1.3). Only in a very rude and naive approximation we may assume that interacting fields and free fields share the same vacuum state and the same Fock space representation.

With this warnings, mixing relations such as the relations (5.1) deserve a careful analysis, since they actually represent a dynamical mapping. It is now our purpose to investigate the structure of the Fock spaces $\mathcal{H}_{1,2}$ and $\mathcal{H}_{e,\mu}$ relative to $\nu_1(x)$, $\nu_2(x)$ and $\nu_e(x)$, $\nu_\mu(x)$, respectively. In particular we want to study the relation among these spaces in the infinite volume limit. We expect that $\mathcal{H}_{1,2}$ and $\mathcal{H}_{e,\mu}$ become orthogonal in such a limit, since they represent the Hilbert spaces for free and interacting fields, respectively [7]. In the following, as usual, we will perform all computations at finite volume V and only at the end we will put $V \to \infty$.

Our first step is the study of the generator of eqs.(5.1) and of the underlying group theoretical structure.

Eqs.(5.1) can be put in the form[22] [22]:

$$\nu_e^\alpha(x) = G^{-1}(\theta, t) \, \nu_1^\alpha(x) \, G(\theta, t)$$

[22] We use here the notation $G(\theta)$ for the generator, instead of $U(\theta)$.

$$\nu_\mu^\alpha(x) = G^{-1}(\theta,t)\; \nu_2^\alpha(x)\; G(\theta,t)\;, \qquad (5.8)$$

where $G(\theta,t)$ is given by

$$G(\theta,t) = exp\left[\theta \int d^3\mathbf{x} \left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right]\;, \qquad (5.9)$$

and is (at finite volume) an unitary operator: $G^{-1}(\theta,t) = G(-\theta,t) = G^\dagger(\theta,t)$, preserving the canonical anticommutation relations (5.3).

Eq.(5.9) follows from $\frac{d^2}{d\theta^2}\nu_e^\alpha = -\nu_e^\alpha\;,\quad \frac{d^2}{d\theta^2}\nu_\mu^\alpha = -\nu_\mu^\alpha$ with the initial conditions $\nu_e^\alpha|_{\theta=0} = \nu_1^\alpha$, $\frac{d}{d\theta}\nu_e^\alpha|_{\theta=0} = \nu_2^\alpha$ and $\nu_\mu^\alpha|_{\theta=0} = \nu_2^\alpha$, $\frac{d}{d\theta}\nu_\mu^\alpha|_{\theta=0} = -\nu_1^\alpha$.

We note the time dependence of the generator G: it represent an important feature and will be carefully considered in §5.3; in the following however it will be often omitted since it is inessential to the present discussion.

We also observe that $G(\theta)$ is an element of $SU(2)$. Indeed, it can be write as

$$G(\theta) = exp[\theta(S_+ - S_-)]\;. \qquad (5.10)$$

with

$$S_+ \equiv \int d^3\mathbf{x}\; \nu_1^\dagger(x)\nu_2(x)\quad,\quad S_- \equiv \int d^3\mathbf{x}\; \nu_2^\dagger(x)\nu_1(x)\; = (S_+)^\dagger\;, \qquad (5.11)$$

$$S_3 \equiv \frac{1}{2}\int d^3\mathbf{x}\left(\nu_1^\dagger(x)\nu_1(x) - \nu_2^\dagger(x)\nu_2(x)\right)\;, \qquad (5.12)$$

together with the Casimir, which is just the total charge,

$$S_0 \equiv \frac{1}{2}\int d^3\mathbf{x}\left(\nu_1^\dagger(x)\nu_1(x) + \nu_2^\dagger(x)\nu_2(x)\right)\;, \qquad (5.13)$$

closing the $su(2)$ algebra:

$$[S_+, S_-] = 2S_3\quad,\quad [S_3, S_\pm] = \pm S_\pm\quad,\quad [S_0, S_3] = [S_0, S_\pm] = 0\;. \qquad (5.14)$$

It is interesting to look at the momentum expansion of the above generators:

$$S_+(t) \equiv \sum_{\mathbf{k}} S_+^{\mathbf{k}}(t) = \sum_{\mathbf{k}}\sum_{r,s}(u^{r\dagger}_{\mathbf{k},1}(t)u^s_{\mathbf{k},2}(t)\;\alpha^{r\dagger}_{\mathbf{k},1}\alpha^s_{\mathbf{k},2} + \qquad (5.15)$$
$$+v^{r\dagger}_{-\mathbf{k},1}(t)u^s_{\mathbf{k},2}(t)\;\beta^r_{-\mathbf{k},1}\alpha^s_{\mathbf{k},2} + u^{r\dagger}_{\mathbf{k},1}(t)v^s_{-\mathbf{k},2}(t)\;\alpha^{r\dagger}_{\mathbf{k},1}\beta^{s\dagger}_{-\mathbf{k},2} + v^{r\dagger}_{-\mathbf{k},1}(t)v^s_{-\mathbf{k},2}(t)\;\beta^r_{-\mathbf{k},1}\beta^{s\dagger}_{-\mathbf{k},2})\;,$$

$$S_-(t) \equiv \sum_{\mathbf{k}} S_-^{\mathbf{k}}(t) = \sum_{\mathbf{k}}\sum_{r,s}(u^{r\dagger}_{\mathbf{k},2}(t)u^s_{\mathbf{k},1}(t)\;\alpha^{r\dagger}_{\mathbf{k},2}\alpha^s_{\mathbf{k},1} + \qquad (5.16)$$
$$+v^{r\dagger}_{-\mathbf{k},2}(t)u^s_{\mathbf{k},1}(t)\;\beta^r_{-\mathbf{k},2}\alpha^s_{\mathbf{k},1} + u^{r\dagger}_{\mathbf{k},2}(t)v^s_{-\mathbf{k},1}(t)\;\alpha^{r\dagger}_{\mathbf{k},2}\beta^{s\dagger}_{-\mathbf{k},1} + v^{r\dagger}_{-\mathbf{k},2}(t)v^s_{-\mathbf{k},1}(t)\;\beta^r_{-\mathbf{k},2}\beta^{s\dagger}_{-\mathbf{k},1})\;,$$

$$S_3 \equiv \sum_{\mathbf{k}} S_3^{\mathbf{k}} = \frac{1}{2}\sum_{\mathbf{k},r}\left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^r_{\mathbf{k},1} - \beta^{r\dagger}_{-\mathbf{k},1}\beta^r_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\alpha^r_{\mathbf{k},2} + \beta^{r\dagger}_{-\mathbf{k},2}\beta^r_{-\mathbf{k},2}\right)\;, \qquad (5.17)$$

$$S_0 \equiv \sum_{\mathbf{k}} S_0^{\mathbf{k}} = \frac{1}{2} \sum_{\mathbf{k},r} \left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},1} - \beta^{r\dagger}_{-\mathbf{k},1}\beta^{r}_{-\mathbf{k},1} + \alpha^{r\dagger}_{\mathbf{k},2}\alpha^{r}_{\mathbf{k},2} - \beta^{r\dagger}_{-\mathbf{k},2}\beta^{r}_{-\mathbf{k},2} \right) . \tag{5.18}$$

We observe that the operatorial structure of eqs.(5.15) and (5.16) is the one of the rotation generator and of the Bogoliubov generator. These structures will be exploited in the following. Using these expansions it is easy to show that the $su(2)$ algebra does hold for each momentum component:

$$\begin{aligned} &[S^{\mathbf{k}}_{+}, S^{\mathbf{k}}_{-}] = 2S^{\mathbf{k}}_3 \quad , \quad [S^{\mathbf{k}}_3, S^{\mathbf{k}}_{\pm}] = \pm S^{\mathbf{k}}_{\pm} \quad , \quad [S^{\mathbf{k}}_0, S^{\mathbf{k}}_3] = [S^{\mathbf{k}}_0, S^{\mathbf{k}}_{\pm}] = 0 \, , \\ &[S^{\mathbf{k}}_{\pm}, S^{\mathbf{p}}_{\pm}] = [S^{\mathbf{k}}_3, S^{\mathbf{p}}_{\pm}] = [S^{\mathbf{k}}_3, S^{\mathbf{p}}_3] = 0 \quad , \quad \mathbf{k} \neq \mathbf{p} \, . \end{aligned} \tag{5.19}$$

This means that the original $su(2)$ algebra given in eqs.(5.14) splits into k disjoint $su_{\mathbf{k}}(2)$ algebras, given by eqs.(5.19), i.e. we have the group structure $\bigotimes_{\mathbf{k}} SU_{\mathbf{k}}(2)$.

To establish the relation between $\mathcal{H}_{1,2}$ and $\mathcal{H}_{e,\mu}$ we consider the generic matrix element ${}_{1,2}\langle a|\nu_1^\alpha(x)|b\rangle_{1,2}$ (a similar argument holds for $\nu_2^\alpha(x)$), where $|a\rangle_{1,2}$ is the generic element of $\mathcal{H}_{1,2}$. By using eq. (5.8), we obtain:

$${}_{1,2}\langle a|G(\theta)\; \nu_e^\alpha(x)\; G^{-1}(\theta)|b\rangle_{1,2} \;=\; {}_{1,2}\langle a|\nu_1^\alpha(x)|b\rangle_{1,2} \, . \tag{5.20}$$

Since the operator field ν_e is defined on the Hilbert space $\mathcal{H}_{e,\mu}$, eq.(5.20) shows that $G^{-1}(\theta)|a\rangle_{1,2}$ is a vector of $\mathcal{H}_{e,\mu}$, so $G^{-1}(\theta)$ maps $\mathcal{H}_{1,2}$ to $\mathcal{H}_{e,\mu}$: $G^{-1}(\theta) : \mathcal{H}_{1,2} \mapsto \mathcal{H}_{e,\mu}$. In particular for the vacuum $|0\rangle_{1,2}$ we have (at finite volume V):

$$|0(t)\rangle_{e,\mu} = G^{-1}(\theta, t)\; |0\rangle_{1,2} \, . \tag{5.21}$$

$|0(t)\rangle_{e,\mu}$ is the vacuum for $\mathcal{H}_{e,\mu}$, which we will refer to as the flavour vacuum. Due to the linearity of $G(\theta, t)$, we can define the flavour annihilators, relative to the fields $\nu_e(x)$ and $\nu_\mu(x)$ as[23]

$$\begin{aligned} \alpha^r_{\mathbf{k},e}(t) &\equiv G^{-1}(\theta,t)\; \alpha^r_{\mathbf{k},1}\; G(\theta,t) \, , \\ \alpha^r_{\mathbf{k},\mu}(t) &\equiv G^{-1}(\theta,t)\; \alpha^r_{\mathbf{k},2}\; G(\theta,t) \, , \\ \beta^r_{\mathbf{k},e}(t) &\equiv G^{-1}(\theta,t)\; \beta^r_{\mathbf{k},1}\; G(\theta,t) \, , \\ \beta^r_{\mathbf{k},\mu}(t) &\equiv G^{-1}(\theta,t)\; \beta^r_{\mathbf{k},2}\; G(\theta,t) \, . \end{aligned} \tag{5.22}$$

The flavour fields are then rewritten into the form:

$$\begin{aligned} \nu_e(\mathbf{x},t) &= \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} e^{i\mathbf{k}\cdot\mathbf{x}} \left[u^r_{\mathbf{k},1}(t)\alpha^r_{\mathbf{k},e}(t) \;+ v^r_{-\mathbf{k},1}(t)\beta^{r\dagger}_{-\mathbf{k},e}(t) \right] \\ \nu_\mu(\mathbf{x},t) &= \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} e^{i\mathbf{k}\cdot\mathbf{x}} \left[u^r_{\mathbf{k},2}(t)\alpha^r_{\mathbf{k},\mu}(t) \;+ v^r_{-\mathbf{k},2}(t)\beta^{r\dagger}_{-\mathbf{k},\mu}(t) \right] \end{aligned} \tag{5.23}$$

Note that the above expansion for a field is a general one for Heisenberg fields [5]. We observe that $G^{-1}(\theta) = exp[\theta(S_- - S_+)]$ is just the generator for generalized coherent states of $SU(2)$ [8]:

[23] The annihilation of the flavour vacuum at each time is expressed as: $\alpha^r_{\mathbf{k},e}(t)|0(t)\rangle_{e,\mu} = G^{-1}(\theta,t)\, \alpha^r_{\mathbf{k},1}|0\rangle_{1,2} = 0$.

the flavour vacuum is therefore an $SU(2)$ (time dependent) coherent state. Let us now obtain the explicit expression for $|0\rangle_{e,\mu}$ and investigate the infinite volume limit of eq.(5.21).

Using the Gaussian decomposition, $G^{-1}(\theta)$ can be written as [8]

$$exp[\theta(S_- - S_+)] = exp(-tan\theta\ S_+)\ exp(-2ln\ cos\theta\ S_3)\ exp(tan\theta\ S_-) \tag{5.24}$$

where $0 \leq \theta < \frac{\pi}{2}$. Eq.(5.21) then becomes

$$|0\rangle_{e,\mu} = \prod_{\mathbf{k}} |0\rangle^{\mathbf{k}}_{e,\mu} = \prod_{\mathbf{k}} exp(-tan\theta\ S^{\mathbf{k}}_+) exp(-2ln\ cos\theta\ S^{\mathbf{k}}_3)\ exp(tan\theta\ S^{\mathbf{k}}_-)|0\rangle_{1,2}\ . \tag{5.25}$$

The final expression for $|0\rangle_{e,\mu}$ in terms of $S^{\mathbf{k}}_\pm$ and $S^{\mathbf{k}}_3$ is [22]:

$$\begin{aligned}|0\rangle_{e,\mu} = \prod_{\mathbf{k}} \Big[& 1 + \sin\theta\cos\theta \left(S^{\mathbf{k}}_- - S^{\mathbf{k}}_+\right) + \frac{1}{2}\sin^2\theta\cos^2\theta \left((S^{\mathbf{k}}_-)^2 + (S^{\mathbf{k}}_+)^2\right) + \\ & - \sin^2\theta S^{\mathbf{k}}_+ S^{\mathbf{k}}_- + \frac{1}{2}\sin^3\theta\cos\theta \left(S^{\mathbf{k}}_-(S^{\mathbf{k}}_+)^2 - S^{\mathbf{k}}_+(S^{\mathbf{k}}_-)^2\right) + \frac{1}{4}\sin^4\theta (S^{\mathbf{k}}_+)^2(S^{\mathbf{k}}_-)^2 \Big] |0\rangle_{1,2}\ .\end{aligned} \tag{5.26}$$

The state $|0\rangle_{e,\mu}$ is normalized to 1 (see eq.(5.21)).

Let us now compute ${}_{1,2}\langle 0|0\rangle_{e,\mu}$. We obtain

$$\begin{aligned}{}_{1,2}\langle 0|0\rangle_{e,\mu} &= \prod_{\mathbf{k}} \left(1 - \sin^2\theta\ {}_{1,2}\langle 0|S^{\mathbf{k}}_+ S^{\mathbf{k}}_-|0\rangle_{1,2} + \frac{1}{4}\sin^4\theta\ {}_{1,2}\langle 0|(S^{\mathbf{k}}_+)^2(S^{\mathbf{k}}_-)^2|0\rangle_{1,2}\right) \\ &= \prod_{\mathbf{k}} \left(1 - \sin^2\theta\ |V_{\mathbf{k}}|^2\right)^2 \equiv \prod_{\mathbf{k}} \Gamma(k) = e^{\sum_{\mathbf{k}} ln\ \Gamma(k)}.\end{aligned} \tag{5.27}$$

where the function $V_{\mathbf{k}}$ is defined in eq.(5.33) and plotted in Fig.1 . Note that $|V_k|^2$ depends on k only through its modulus, it is always in the interval $[0, 1[$ and $|V_{\mathbf{k}}|^2 \to 0$ when $k \to \infty$.

By using the customary continuous limit relation $\sum_{\mathbf{k}} \to \frac{V}{(2\pi)^3} \int d^3k$, in the infinite volume limit we obtain

$$\lim_{V\to\infty}\ {}_{1,2}\langle 0|0\rangle_{e,\mu} = \lim_{V\to\infty}\ e^{\frac{V}{(2\pi)^3}\int d^3k\ ln\ \Gamma(k)} = 0 \tag{5.28}$$

since $\Gamma(k) < 1$ for any value of k and of the parameters m_1 and m_2.

Notice that (5.28) shows that the orthogonality between $|0\rangle_{e,\mu}$ and $|0\rangle_{1,2}$ is due to the infrared contributions which are taken in care by the infinite volume limit and therefore high momentum contributions do not influence the result (for this reason here we do not need to consider the regularization problem of the UV divergence of the integral of ln $\Gamma(k)$). Of course, this orthogonality disappears when $\theta = 0$ and/or when $m_1 = m_2$ (because in this case $V_{\mathbf{k}} = 0$ for any k and no mixing occurs in Pontecorvo theory).

Eq.(5.28) expresses the unitary inequivalence (see §1.2) in the infinite volume limit of the flavour and the mass representations and shows the non-trivial nature of the mixing transformations (5.1). In other words, the mixing transformations induce a physically non-trivial structure in the flavour vacuum which indeed turns out to be an $SU(2)$ generalized coherent state. In §5.3 we will see how such a vacuum structure may lead to phenomenological consequences in the neutrino oscillations, which possibly may be experimentally tested. From eq.(5.28) we also see that eq.(5.21) is a purely formal expression which only holds at finite volume.

Let us now return to the dynamical map, eqs.(5.22): it can be calculated explicitly, thus giving the flavour annihilation operators

$$\begin{aligned}
\alpha^r_{\mathbf{k},e}(t) &= \cos\theta \; \alpha^r_{\mathbf{k},1} \; + \; \sin\theta \; \sum_s \left[u^{r\dagger}_{\mathbf{k},1}(t) u^s_{\mathbf{k},2}(t) \; \alpha^s_{\mathbf{k},2} \; + \; u^{r\dagger}_{\mathbf{k},1}(t) v^s_{-\mathbf{k},2}(t) \; \beta^{s\dagger}_{-\mathbf{k},2} \right] \\
\alpha^r_{\mathbf{k},\mu}(t) &= \cos\theta \; \alpha^r_{\mathbf{k},2} \; - \; \sin\theta \; \sum_s \left[u^{r\dagger}_{\mathbf{k},2}(t) u^s_{\mathbf{k},1}(t) \; \alpha^s_{\mathbf{k},1} \; + \; u^{r\dagger}_{\mathbf{k},2}(t) v^s_{-\mathbf{k},1}(t) \; \beta^{s\dagger}_{-\mathbf{k},1} \right] \\
\beta^r_{-\mathbf{k},e}(t) &= \cos\theta \; \beta^r_{-\mathbf{k},1} \; + \; \sin\theta \; \sum_s \left[v^{s\dagger}_{-\mathbf{k},2}(t) v^r_{-\mathbf{k},1}(t) \; \beta^s_{-\mathbf{k},2} \; + \; u^{s\dagger}_{\mathbf{k},2}(t) v^r_{-\mathbf{k},1}(t) \; \alpha^{s\dagger}_{\mathbf{k},2} \right] \\
\beta^r_{-\mathbf{k},\mu}(t) &= \cos\theta \; \beta^r_{-\mathbf{k},2} \; - \; \sin\theta \; \sum_s \left[v^{s\dagger}_{-\mathbf{k},1}(t) v^r_{-\mathbf{k},2}(t) \; \beta^s_{-\mathbf{k},1} \; + \; u^{s\dagger}_{\mathbf{k},1}(t) v^r_{-\mathbf{k},2}(t) \; \alpha^{s\dagger}_{\mathbf{k},1} \right]
\end{aligned} \tag{5.29}$$

Without loss of generality, we can choose the reference frame such that $k = (0, 0, |\mathbf{k}|)$. In this case the spins decouple and we have the simpler expressions:

$$\begin{aligned}
\alpha^r_{\mathbf{k},e}(t) &= \cos\theta \; \alpha^r_{\mathbf{k},1} \; + \; \sin\theta \; \left(U^*_{\mathbf{k}}(t) \; \alpha^r_{\mathbf{k},2} \; + \; \epsilon^r \; V_{\mathbf{k}}(t) \; \beta^{r\dagger}_{-\mathbf{k},2} \right) \\
\alpha^r_{\mathbf{k},\mu}(t) &= \cos\theta \; \alpha^r_{\mathbf{k},2} \; - \; \sin\theta \; \left(U_{\mathbf{k}}(t) \; \alpha^r_{\mathbf{k},1} \; - \; \epsilon^r \; V_{\mathbf{k}}(t) \; \beta^{r\dagger}_{-\mathbf{k},1} \right) \\
\beta^r_{-\mathbf{k},e}(t) &= \cos\theta \; \beta^r_{-\mathbf{k},1} \; + \; \sin\theta \; \left(U^*_{\mathbf{k}}(t) \; \beta^r_{-\mathbf{k},2} \; - \; \epsilon^r \; V_{\mathbf{k}}(t) \; \alpha^{r\dagger}_{\mathbf{k},2} \right) \\
\beta^r_{-\mathbf{k},\mu}(t) &= \cos\theta \; \beta^r_{-\mathbf{k},2} \; - \; \sin\theta \; \left(U_{\mathbf{k}}(t) \; \beta^r_{-\mathbf{k},1} \; + \; \epsilon^r \; V_{\mathbf{k}}(t) \; \alpha^{r\dagger}_{\mathbf{k},1} \right)
\end{aligned} \tag{5.30}$$

where $\epsilon^r = (-1)^r$ and

$$\begin{aligned}
U_{\mathbf{k}}(t) &\equiv u^{r\dagger}_{\mathbf{k},2}(t) u^r_{\mathbf{k},1}(t) = v^{r\dagger}_{-\mathbf{k},1}(t) v^r_{-\mathbf{k},2}(t) \\
V_{\mathbf{k}}(t) &\equiv \epsilon^r \; u^{r\dagger}_{\mathbf{k},1}(t) v^r_{-\mathbf{k},2}(t) = -\epsilon^r \; u^{r\dagger}_{\mathbf{k},2}(t) v^r_{-\mathbf{k},1}(t) \, .
\end{aligned} \tag{5.31}$$

We have:

$$V_{\mathbf{k}} = |V_{\mathbf{k}}| \; e^{i(\omega_{k,2}+\omega_{k,1})t} \quad , \quad U_{\mathbf{k}} = |U_{\mathbf{k}}| \; e^{i(\omega_{k,2}-\omega_{k,1})t} \tag{5.32}$$

$$\begin{aligned}
|U_{\mathbf{k}}| &= \left(\frac{\omega_{k,1}+m_1}{2\omega_{k,1}} \right)^{\frac{1}{2}} \left(\frac{\omega_{k,2}+m_2}{2\omega_{k,2}} \right)^{\frac{1}{2}} \left(1 + \frac{|\mathbf{k}|^2}{(\omega_{k,1}+m_1)(\omega_{k,2}+m_2)} \right) \\
|V_{\mathbf{k}}| &= \left(\frac{\omega_{k,1}+m_1}{2\omega_{k,1}} \right)^{\frac{1}{2}} \left(\frac{\omega_{k,2}+m_2}{2\omega_{k,2}} \right)^{\frac{1}{2}} \left(\frac{\mathbf{k}}{(\omega_{k,2}+m_2)} - \frac{\mathbf{k}}{(\omega_{k,1}+m_1)} \right)
\end{aligned} \tag{5.33}$$

$$|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1 \tag{5.34}$$

We thus see that, at the level of annihilation operators, the structure of the mixing transformation is that of a Bogoliubov transformation nested into a rotation. The two transformations however cannot be disentangled, thus the mixing transformations (5.30) are essentially different from the Bogoliubov transformations previously encountered.

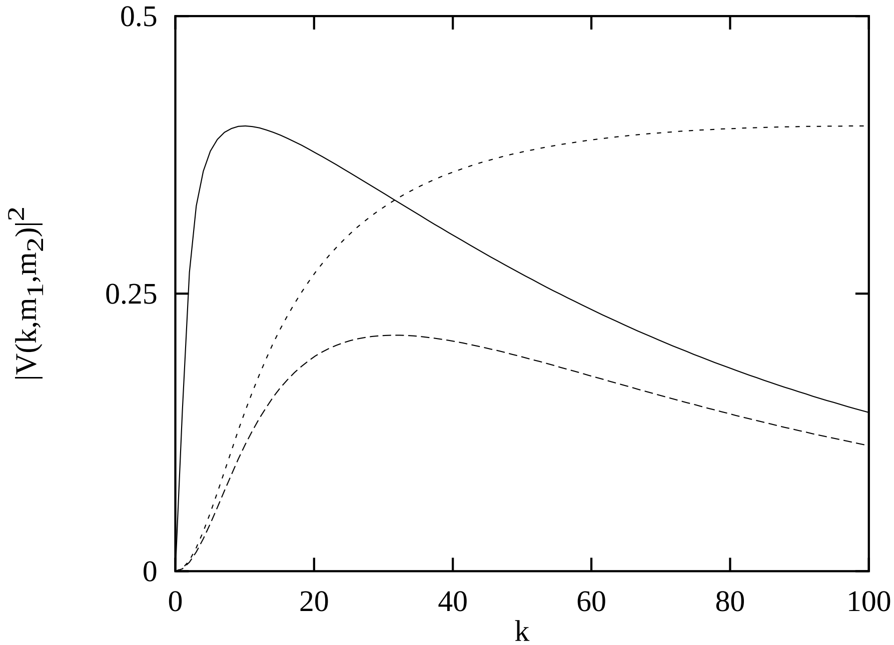

Figure 5.1: The fermion condensation density $|V_k|^2$ in function of k and for sample values of the parameters m_1 and m_2.
Solid line: $m_1 = 1$, $m_2 = 100$
Long-dashed line: $m_1 = 10$, $m_2 = 100$
Short-dashed line: $m_1 = 10$, $m_2 = 1000$

It is possible to exhibit the full explicit expression of $|0\rangle^{\mathbf{k}}_{e,\mu}$ (at time $t = 0$) in the reference frame for which $\mathbf{k} = (0, 0, |\mathbf{k}|)$:

$$|0\rangle^{\mathbf{k}}_{e,\mu} = \prod_r \Big[(1 - \sin^2\theta\, |V_{\mathbf{k}}|^2) - \epsilon^r \sin\theta\, \cos\theta\, |V_{\mathbf{k}}|\, (\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}) + \tag{5.35}$$
$$+ \epsilon^r \sin^2\theta\, |V_{\mathbf{k}}|\, |U_{\mathbf{k}}| \left(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}\right) + \sin^2\theta\, |V_{\mathbf{k}}|^2\, \alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big]\, |0\rangle_{1,2}$$

We see that the expression of the flavour vacuum $|0\rangle_{e,\mu}$ appears to be rather complicated, since it involves four different particle-antiparticle "couples", two of which with different masses: this in particular makes complicated the time dependence of such a state due to the presence of different phase factors associated to the various terms. It is interesting to compare $|0\rangle_{e,\mu}$ with the BCS superconducting ground state[11], which involves only one kind of couple and is generated by a simple Bogoliubov transformation (cf. eq.(2.79).

The condensation density is given by

$$_{e,\mu}\langle 0|\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},1}|0\rangle_{e,\mu} = \sin^2\theta\, |V_{\mathbf{k}}|^2 \tag{5.36}$$

with a similar result for $\alpha^r_{\mathbf{k},2}$, $\beta^r_{\mathbf{k},1}$ and $\beta^r_{\mathbf{k},2}$.

5.2 Boson mixing

Let us now discuss the case of boson mixing. Consider two charged boson fields $\phi_i(x)$, $i = 1, 2$ and their conjugate momenta $\pi_i(x) = \partial_0 \phi_i^\dagger(x)$, satisfying the usual commutation relations with non-zero commutators given by:

$$
\begin{aligned}
\left[\phi_i(x), \pi_i(y)\right]_{t=t'} &= \left[\phi_i^\dagger(x), \pi_i^\dagger(y)\right]_{t=t'} = i\delta^3(\mathbf{x}-\mathbf{y}) \\
\left[a_{\mathbf{k},i}, a_{\mathbf{p},i}^\dagger\right] &= \left[b_{\mathbf{k},i}, b_{\mathbf{p},i}^\dagger\right] = \delta^3(\mathbf{k}-\mathbf{p})
\end{aligned}
\tag{5.37}
$$

$$
\phi_i(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega_{k,i}}} \left(a_{\mathbf{k},i}\, e^{-ik.x} + b_{\mathbf{k},i}^\dagger\, e^{ik.x}\right) \tag{5.38}
$$

$$
\pi_i(x) = i \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{\omega_{k,i}}{2}} \left(a_{\mathbf{k},i}^\dagger\, e^{ik.x} - b_{\mathbf{k},i}\, e^{-ik.x}\right) \tag{5.39}
$$

with $k.x = \omega t - \mathbf{k}\cdot\mathbf{x}$. Now we define mixing relations as:

$$
\begin{aligned}
\phi_A(x) &= \phi_1(x)\ \cos\theta + \phi_2(x)\ \sin\theta \\
\phi_B(x) &= -\phi_1(x)\ \sin\theta + \phi_2(x)\ \cos\theta
\end{aligned}
\tag{5.40}
$$

We generically denote the mixed fields with A and B. As for fermions we put eqs.(5.40) into the form:

$$
\begin{aligned}
\phi_A(x) &= G^{-1}(\theta,t)\ \phi_1(x)\ G(\theta,t) \\
\phi_B(x) &= G^{-1}(\theta,t)\ \phi_2(x)\ G(\theta,t)
\end{aligned}
\tag{5.41}
$$

and similar ones for π_A, ϕ_B, where $G(\theta,t)$ is given by

$$
G(\theta,t) = exp\left[-i\,\theta \int d^3\mathbf{x} \left(\pi_1(x)\phi_2(x) - \pi_2^\dagger(x)\phi_1^\dagger(x) - \pi_2(x)\phi_1(x) + \pi_1^\dagger(x)\phi_2^\dagger(x)\right)\right] \tag{5.42}
$$

and is (at finite volume) an unitary operator: $G^{-1}(\theta,t) = G(-\theta,t) = G^\dagger(\theta,t)$. Exactly like in the fermion case, $G(\theta,t)$ can be written as

$$
G(\theta,t) = exp[\theta(S_+ - S_-)]\ . \tag{5.43}
$$

where now

$$
S_+ = S_-^\dagger \equiv -i \int d^3\mathbf{x}\ (\pi_1(x)\phi_2(x) - \pi_2^\dagger(x)\phi_1^\dagger(x)) \tag{5.44}
$$

which together with

$$
S_3 \equiv \frac{-i}{2} \int d^3\mathbf{x} \left(\pi_1(x)\phi_1(x) - \pi_2(x)\phi_2(x) + \pi_2^\dagger(x)\phi_2^\dagger(x) - \pi_1^\dagger(x)\phi_1^\dagger(x)\right) \tag{5.45}
$$

$$S_0 = \frac{Q}{2} \equiv \frac{-i}{2} \int d^3\mathbf{x} \left(\pi_1(x)\phi_1(x) - \pi_1^\dagger(x)\phi_1^\dagger(x) + \pi_2(x)\phi_2(x) - \pi_2^\dagger(x)\phi_2^\dagger(x) \right) \tag{5.46}$$

close the $su(2)$ algebra associated to the rotation (5.40): $[S_+, S_-] = 2S_3$, $[S_3, S_\pm] = \pm S_\pm$, $[S_0, S_3] = [S_0, S_\pm] = 0$.

The expansions of S_+ and S_- are:

$$S_+(t) = \int d^3\mathbf{k} \left(U_\mathbf{k}(t)\, a^\dagger_{\mathbf{k},1} a_{\mathbf{k},2} - V^*_\mathbf{k}(t)\, b_{-\mathbf{k},1} a_{\mathbf{k},2} + V_\mathbf{k}(t)\, a^\dagger_{\mathbf{k},1} b^\dagger_{-\mathbf{k},2} - U^*_\mathbf{k}(t)\, b_{-\mathbf{k},1} b^\dagger_{-\mathbf{k},2} \right) \tag{5.47}$$

$$S_-(t) = \int d^3\mathbf{k} \left(U^*_\mathbf{k}(t)\, a^\dagger_{\mathbf{k},2} a_{\mathbf{k},1} - V_\mathbf{k}(t)\, a^\dagger_{\mathbf{k},2} b^\dagger_{-\mathbf{k},1} + V^*_\mathbf{k}(t)\, b_{-\mathbf{k},2} a_{\mathbf{k},1} - U_\mathbf{k}(t)\, b_{-\mathbf{k},2} b^\dagger_{-\mathbf{k},1} \right) \tag{5.48}$$

with $U_\mathbf{k} \equiv |U_\mathbf{k}|\; e^{i(\omega_{k,1}-\omega_{k,2})t}$, $V_\mathbf{k} \equiv |V_\mathbf{k}|\; e^{i(\omega_{k,1}+\omega_{k,2})t}$ and

$$|U_\mathbf{k}| \equiv \frac{1}{2}\left(\sqrt{\frac{\omega_{k,1}}{\omega_{k,2}}} + \sqrt{\frac{\omega_{k,2}}{\omega_{k,1}}} \right) \quad , \qquad |V_\mathbf{k}| \equiv \frac{1}{2}\left(\sqrt{\frac{\omega_{k,1}}{\omega_{k,2}}} - \sqrt{\frac{\omega_{k,2}}{\omega_{k,1}}} \right) \;,$$
$$|U_\mathbf{k}|^2 - |V_\mathbf{k}|^2 = 1 \tag{5.49}$$

Then one can put $|U_\mathbf{k}| \equiv \cosh\sigma_\mathbf{k}$, $|V_\mathbf{k}| \equiv \sinh\sigma_\mathbf{k}$ with $\sigma_\mathbf{k} = \frac{1}{2}\ln\left(\frac{\omega_1}{\omega_2}\right)$.

Like in the fermion case, the generator of boson mixing transformations does not leave invariant the vacuum of the fields $\phi_{1,2}(x)$, say $|0\rangle_{1,2}$, since it induces an $SU(2)$ (bosonic)coherent state structure resulting in a new state $|0\rangle_{A,B}$:

$$|0(t)\rangle_{A,B} = G^{-1}(\theta, t)\; |0\rangle_{1,2} \tag{5.50}$$

Clearly, the annihilator operators for the vacuum $|0(t)\rangle_{A,B}$ are given by $a_{\mathbf{k},A} \equiv G^{-1}(\theta,t)\; a_{\mathbf{k},1}\; G(\theta,t)$, etc.. We have for example

$$a_{\mathbf{k},A}(t) = \cos\theta\; a_{\mathbf{k},1} \;+\; \sin\theta\; \left(U_\mathbf{k}(t)\; a_{\mathbf{k},2} \;+\; V_\mathbf{k}(t)\; b^\dagger_{-\mathbf{k},2} \right) \;. \tag{5.51}$$

Similar expressions can be obtained for $a_{\mathbf{k},B}$, $b_{\mathbf{k},A}$ and $b_{\mathbf{k},B}$. From eq.(5.51) and similar, we see how the only difference with respect to fermion mixing, is in the (internal) Bogoliubov transformation, which now, as due for bosons, has coefficients which satisfy hyperbolic relations (cf.eq.(5.49)).

The condensation density of the vacuum is given by

$${}_{1,2}\langle 0| a^\dagger_{\mathbf{k},A} a_{\mathbf{k},A} |0\rangle_{1,2} = \sin^2\theta\; |V_\mathbf{k}|^2 = \sin^2\theta\; \sinh^2\left[\frac{1}{2}\ln\left(\frac{\omega_{k,1}}{\omega_{k,2}} \right) \right] \tag{5.52}$$

which appears to be very different from the corresponding quantity in the fermion case. We observe (see Fig.2) that still the main contribution to the condensate comes from the infrared region, although now it is maximal at zero and, most important, not limited to be less than one.

It is also interesting to see how the above scheme works in the case of neutral boson fields $\phi_i(x)$, $i = 1,2$ and their conjugate momenta $\pi_i(x) = \partial_0\phi_i(x)$, with the following non-zero commutators:

$$[\phi_i(x), \pi_i(y)]_{t=t'} = i\delta^3(\mathbf{x}-\mathbf{y}) \;, \qquad \left[a_{\mathbf{k},i}, a^\dagger_{\mathbf{p},i}\right] = \delta^3(\mathbf{k}-\mathbf{p}) \tag{5.53}$$

$$\phi_i(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega_{k,i}}} \left(a_{\mathbf{k},i}\, e^{-ik.x} + a^\dagger_{\mathbf{k},i}\, e^{ik.x} \right) \tag{5.54}$$

$$\pi_i(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{\omega_{k,i}}{2}}\; i \left(-a_{\mathbf{k},i}\, e^{-ik.x} + a^\dagger_{\mathbf{k},i}\, e^{ik.x} \right) \tag{5.55}$$

The mixing generator is still given by $G(\theta, t) = exp[\theta(S_+ - S_-)]$ and the $su(2)$ operators are now realized as

$$\begin{aligned}
S_+ &\equiv -i \int d^3\mathbf{x}\; \pi_1(x)\phi_2(x) \;\; , \quad S_- \equiv -i \int d^3\mathbf{x}\; \pi_2(x)\phi_1(x) \\
S_3 &\equiv \frac{-i}{2} \int d^3\mathbf{x}\, (\pi_1(x)\phi_1(x) - \pi_2(x)\phi_2(x)) \;, \\
S_0 &\equiv \frac{-i}{2} \int d^3\mathbf{x}\, (\pi_1(x)\phi_1(x) + \pi_2(x)\phi_2(x))
\end{aligned} \tag{5.56}$$

The realization in terms of the creation and annihilation operator is slightly different with respect to the charged field case, being now

$$(S_+ - S_-) = \int d^3k \left(U_k\; a^\dagger_{\mathbf{k},1} a_{\mathbf{k},2} - V^*_{\mathbf{k}}\; a_{-\mathbf{k},1} a_{\mathbf{k},2} + V_{\mathbf{k}}\; a^\dagger_{\mathbf{k},2} a^\dagger_{-\mathbf{k},1} - U^*_{\mathbf{k}}\; a^\dagger_{\mathbf{k},2} a_{\mathbf{k},1} \right) \tag{5.57}$$

where the hyperbolic coefficients $U_{\mathbf{k}}$ and $V_{\mathbf{k}}$ are the same of the ones defined in eq.(5.49).

Also the annihilator for the mixed field has a similar form to that of eq.(5.51),

$$a_{\mathbf{k},A}(t) = \cos\theta\; a_{\mathbf{k},1} \;+\; \sin\theta\; \left(U_{\mathbf{k}}(t)\; a_{\mathbf{k},2} \;+\; V_{\mathbf{k}}(t)\; a^\dagger_{-\mathbf{k},2} \right) \;. \tag{5.58}$$

The condensation density is exactly the same as in eq.(5.52).

5.3 Green's functions and neutrino oscillations

• The usual picture for neutrino oscillations (Pontecorvo)

In the original Pontecorvo and collaborators treatment[23], the mixing relations (5.1) are assumed to hold also at the level of states - i.e. the vacuum is taken to be the same for flavour and mass eigenstates - :

$$\begin{aligned}
|\nu_e\rangle &= \cos\theta\, |\nu_1\rangle \,+\, \sin\theta\, |\nu_2\rangle \\
|\nu_\mu\rangle &= -\sin\theta\, |\nu_1\rangle \,+\, \cos\theta\, |\nu_2\rangle \;,
\end{aligned} \tag{5.59}$$

where the states $|\nu_i\rangle$, $i = 1, 2$ are eigenstates of the Hamiltonian: $H|\nu_i\rangle = \omega_i|\nu_i\rangle$. Then the time evolution gives

$$\begin{aligned}
|\nu_e(t)\rangle &= e^{-iHt}|\nu_e\rangle \,=\, e^{-i\omega_1 t}\, \cos\theta\, |\nu_1\rangle \,+\, e^{-i\omega_2 t}\, \sin\theta\, |\nu_2\rangle \\
|\nu_\mu(t)\rangle &= e^{-iHt}|\nu_\mu\rangle \,=\, -e^{-i\omega_1 t}\, \sin\theta\, |\nu_1\rangle \,+\, e^{-i\omega_2 t}\, \cos\theta\, |\nu_2\rangle \;,
\end{aligned} \tag{5.60}$$

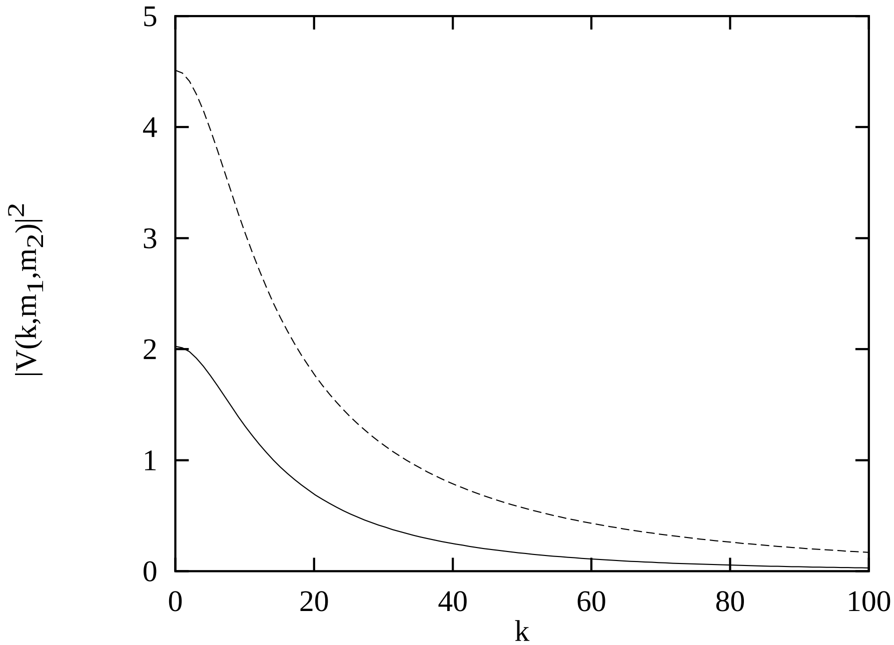

Figure 5.2: The boson condensation density $|V_{\mathbf{k}}|^2$ in function of k and for sample values of the parameters m_1 and m_2.
Solid line: $m_1 = 10$, $m_2 = 100$
Dashed line: $m_1 = 10$, $m_2 = 200$

We thus have at time t, the flavour oscillations

$$\begin{aligned} P_{\nu_e\to\nu_e}(t) &= |\langle\nu_e|\nu_e(t)\rangle|^2 \\ &= 1 - \sin^2 2\theta\ \sin^2\left(\frac{\Delta\omega}{2}t\right)\ . \end{aligned} \tag{5.61}$$

The number of electron neutrinos therefore oscillates in time with a frequency given by the difference $\Delta\omega = \omega_2 - \omega_1$. This oscillation is a flavour oscillation since we have at the same time:

$$\begin{aligned} P_{\nu_e\to\nu_\mu}(t) &= |\langle\nu_\mu|\nu_e(t)\rangle|^2 \\ &= \sin^2 2\theta\ \sin^2\left(\frac{\Delta\omega}{2}t\right) \end{aligned} \tag{5.62}$$

The conservation of probability reads as

$$P_{\nu_e\to\nu_e}(t)\ +\ P_{\nu_e\to\nu_\mu}(t)\ =\ 1 \tag{5.63}$$

• **Neutrino oscillations in QFT**

We now consider neutrino oscillations in the framework of QFT. We discuss this by using Green's functions. In order to discuss flavour oscillations is sufficient to consider the following Lagrangian

$$\mathcal{L} = \bar\nu_e\,(i\ \partial\!\!\!/ - m_e)\,\nu_e + \bar\nu_\mu\,(i\ \partial\!\!\!/ - m_\mu)\,\nu_\mu -\ m_{e\mu}\ (\bar\nu_e\nu_\mu + \bar\nu_\mu\nu_e)\ . \tag{5.64}$$

Generalization to a higher number of flavours is straightforward. This Lagrangian can be fully diagonalized by substituting for the fields the mixing relations

$$\begin{aligned} \nu_e(x) &= \nu_1(x)\,\cos\theta + \nu_2(x)\,\sin\theta \\ \nu_\mu(x) &= -\nu_1(x)\,\sin\theta + \nu_2(x)\,\cos\theta\,, \end{aligned} \tag{5.65}$$

where θ is the mixing angle and $m_e = m_1 \cos^2\theta + m_2 \sin^2\theta$, $m_\mu = m_1 \sin^2\theta + m_2 \cos^2\theta$, $m_{e\mu} = (m_2 - m_1) \sin\theta \cos\theta$. ν_1 and ν_2 therefore are non-interacting, free fields, anticommuting with each other at any space-time point. Their expansions are given in eqs.(5.2)

The fields ν_e and ν_μ are thus completely determined through eq.(5.65). We have seen that it is possible to expand the flavour fields ν_e and ν_μ in the same basis as ν_1 and ν_2,

$$\nu_e^\alpha(x) = G_\theta^{-1}(t)\,\nu_1^\alpha(x)\,G_\theta(t) = V^{-\frac{1}{2}} \sum_{\mathbf{k},r} \left[u^r_{\mathbf{k},1} e^{-i\omega_{k,1}t} \alpha^r_{\mathbf{k},e}(t) + v^r_{-\mathbf{k},1} e^{i\omega_{k,1}t} \beta^{r\dagger}_{-\mathbf{k},e}(t)\right] e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{5.66}$$

$$\nu_\mu^\alpha(x) = G_\theta^{-1}(t)\,\nu_2^\alpha(x)\,G_\theta(t) = V^{-\frac{1}{2}} \sum_{\mathbf{k},r} \left[u^r_{\mathbf{k},2} e^{-i\omega_{k,2}t} \alpha^r_{\mathbf{k},\mu}(t) + v^r_{-\mathbf{k},2} e^{i\omega_{k,2}t} \beta^{r\dagger}_{-\mathbf{k},\mu}(t)\right] e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{5.67}$$

by means of the generator (5.9). The flavour annihilation and creation operators are given in eq.(5.30).

The bilinear mixed term of eq.(5.64) generates four non-zero two point causal Green's functions for the mixed fields ν_e, ν_μ. They are compactly written in the matrix form

$$\begin{pmatrix} S^{\alpha\beta}_{ee}(x,y) & S^{\alpha\beta}_{\mu e}(x,y) \\ S^{\alpha\beta}_{e\mu}(x,y) & S^{\alpha\beta}_{\mu\mu}(x,y) \end{pmatrix} \equiv {}_{1,2}\langle 0| \begin{pmatrix} T\left[\nu_e^\alpha(x)\bar{\nu}_e^\beta(y)\right] & T\left[\nu_\mu^\alpha(x)\bar{\nu}_e^\beta(y)\right] \\ T\left[\nu_e^\alpha(x)\bar{\nu}_\mu^\beta(y)\right] & T\left[\nu_\mu^\alpha(x)\bar{\nu}_\mu^\beta(y)\right] \end{pmatrix} |0\rangle_{1,2}\,, \tag{5.68}$$

where T denotes time ordering. Use of (5.65) gives S_{ee} in momentum representation as

$$S_{ee}(k_0,\mathbf{k}) = \cos^2\theta\, \frac{\not{k} + m_1}{k^2 - m_1^2 + i\delta} \;+\; \sin^2\theta\, \frac{\not{k} + m_2}{k^2 - m_2^2 + i\delta}\,, \tag{5.69}$$

which is just the weighted sum of the two propagators for the free fields ν_1 and ν_2. It coincides with the Feynman propagator obtained by resumming (to all orders) the perturbative series

$$S_{ee} = S_e \left(1 \,+\, m_{e\mu}^2\, S_\mu S_e \,+\, m_{e\mu}^4\, S_\mu S_e S_\mu S_e \,+\, ...\right) = S_e \left(1 - m_{e\mu}^2\, S_\mu S_e\right)^{-1}\,, \tag{5.70}$$

where the "bare" propagators are defined as $S_{e/\mu} = (\not{k} - m_{e/\mu} + i\delta)^{-1}$. In a similar way, one computes $S_{e\mu}$ and $S_{\mu e}$.

The transition amplitude for an electronic neutrino created by $\alpha^{r\dagger}_{\mathbf{k},e}$ at time $t = 0$ into the same particle at time t is given by

$$\mathcal{P}^r_{ee}(\mathbf{k},t) = i u^{r\dagger}_{\mathbf{k},1} e^{i\omega_{k,1}t}\, S^>_{ee}(\mathbf{k},t)\, \gamma^0 u^r_{\mathbf{k},1}\,. \tag{5.71}$$

Here, $S^>_{ee}(\mathbf{k},t)$ denotes the unordered Green's function (or Wightman function) in mixed $(\mathbf{k},t)$ representation. The upper script $>$ (or $<$) is related with the corresponding θ function. The explicit expression for $S^>_{ee}(\mathbf{k},t)$ is

$$S^{>\,\alpha\beta}_{ee}(\mathbf{k},t) = -i \sum_r \left(\cos^2\theta\, e^{-i\omega_{k,1}t}\, u^{r,\alpha}_{\mathbf{k},1}\, \bar{u}^{r,\beta}_{\mathbf{k},1} \,+ \sin^2\theta\, e^{-i\omega_{k,2}t}\, u^{r,\alpha}_{\mathbf{k},2}\, \bar{u}^{r,\beta}_{\mathbf{k},2}\right)\,. \tag{5.72}$$

The probability amplitude (5.71) is independent of the spin orientation and given by

$$\mathcal{P}_{ee}(\mathbf{k},t) = \cos^2\theta \; + \sin^2\theta \, |U_{\mathbf{k}}|^2 \, e^{-i(\omega_{k,2}-\omega_{k,1})t} \; . \tag{5.73}$$

For different masses and $\mathbf{k} \neq 0$, $|U_{\mathbf{k}}|$ is always < 1 (see eq.(5.33 and fig.(5.1)). Notice that $|U_{\mathbf{k}}|^2 \rightarrow 1$ in the relativistic limit $\mathbf{k} \gg \sqrt{m_1 m_2}$: only in this limit the squared modulus of $\mathcal{P}_{ee}(\mathbf{k},t)$ reproduces the Pontecorvo oscillation formula.

Of course, it should be $\lim_{t\to 0^+} \mathcal{P}_{ee}(t) = 1$. However, one obtains instead

$$\mathcal{P}_{ee}(\mathbf{k},0^+) = \cos^2\theta + \sin^2\theta \, |U_{\mathbf{k}}|^2 < 1 \; . \tag{5.74}$$

This means that the *choice of the state* $|0\rangle_{1,2}$ in (5.68) and in the computation of the Wightman function is *not* the correct one. We thus realize the necessity to work in the correct representation for the flavour fields, i.e. we have to calculate the Green's functions on the *flavour vacuum* $|0\rangle_{e,\mu}$.

We now show that the correct definition of the Green's function matrix for the fields ν_e, ν_μ is the one which involves the non-perturbative vacuum $|0\rangle_{e,\mu}$, i.e.

$$\left(\begin{array}{cc} G^{\alpha\beta}_{ee}(x,y) & G^{\alpha\beta}_{\mu e}(x,y) \\ G^{\alpha\beta}_{e\mu}(x,y) & G^{\alpha\beta}_{\mu\mu}(x,y) \end{array} \right) \equiv \, {}_{e,\mu}\langle 0(y_0)| \left(\begin{array}{cc} T\left[\nu^\alpha_e(x)\bar{\nu}^\beta_e(y)\right] & T\left[\nu^\alpha_\mu(x)\bar{\nu}^\beta_e(y)\right] \\ T\left[\nu^\alpha_e(x)\bar{\nu}^\beta_\mu(y)\right] & T\left[\nu^\alpha_\mu(x)\bar{\nu}^\beta_\mu(y)\right] \end{array} \right) |0(y_0)\rangle_{e,\mu} \, . \tag{5.75}$$

Notice that here the time argument y_0 (or, equally well, x_0) of the flavour ground state, is chosen to be equal on both sides of the expectation value. We observe that transition matrix elements of the type ${}_{e,\mu}\langle 0|\alpha_e \exp\left[-iHt\right] \alpha_e^\dagger|0\rangle_{e,\mu}$, where H is the Hamiltonian, do not represent a physical transition amplitudes: they actually vanish (in the infinite volume limit) due to the unitary inequivalence of flavour vacua at different times (see below). Therefore the comparison of states at different times necessitates a *parallel transport* of these states to a common point of reference. The definition (5.75) includes this concept of parallel transport, which is a sort of "gauge fixing": a rich geometric structure underlying the mixing transformations (5.65) is thus uncovered. This geometric features include also Berry phase and a gauge structure associated to the mixing transformations.

In the case of $\nu_e \rightarrow \nu_e$ propagation, we now have (for $\mathbf{k} = (0,0,|\mathbf{k}|)$):

$$\begin{aligned} G_{ee}(k_0,\mathbf{k}) \;\; &= \;\; S_{ee}(k_0,\mathbf{k}) \; + \; 2\pi \, i \, \sin^2\theta \Big[|V_{\mathbf{k}}|^2 \; (\not{k} + m_2) \; \delta(k^2 - m_2^2) \\ & \qquad - |U_{\mathbf{k}}||V_{\mathbf{k}}| \, \sum_r \Big(\epsilon^r u^r_{\mathbf{k},2} \, \bar{v}^r_{-\mathbf{k},2} \, \delta(k_0 - \omega_2) \; + \; \epsilon^r v^r_{-\mathbf{k},2} \, \bar{u}^r_{\mathbf{k},2} \, \delta(k_0 + \omega_2) \Big) \Big] \, , \end{aligned} \tag{5.76}$$

where we used $\epsilon^r = (-1)^r$. Comparison with eq.(5.69) shows that the difference between the full and the perturbative propagators is in the imaginary part.

The Wightman functions for an electron neutrino are $iG^{>\alpha\beta}_{ee}(t,\mathbf{x};0,\mathbf{y}) = {}_{e,\mu}\langle 0|\nu^\alpha_e(t,\mathbf{x}) \; \bar{\nu}^\beta_e(0,\mathbf{y})|0\rangle_{e,\mu}$, and $iG^{>\alpha\beta}_{\mu e}(t,\mathbf{x};0,\mathbf{y}) = {}_{e,\mu}\langle 0|\nu^\alpha_\mu(t,\mathbf{x}) \; \bar{\nu}^\beta_e(0,\mathbf{y})|0\rangle_{e,\mu}$. These are conveniently expressed in terms of anticommutators at different times as

$$iG^{>\alpha\beta}_{ee}(\mathbf{k},t) \;\; = \;\; \sum_r \Big[u^{r,\alpha}_{\mathbf{k},1} \, \bar{u}^{r,\beta}_{\mathbf{k},1} \Big\{ \alpha^r_{\mathbf{k},e}(t), \alpha^{r\dagger}_{\mathbf{k},e} \Big\} \, e^{-i\omega_{k,1}t} + \; v^{r,\alpha}_{-\mathbf{k},1} \, \bar{u}^{r,\beta}_{\mathbf{k},1} \Big\{ \beta^{r\dagger}_{-\mathbf{k},e}(t), \alpha^{r\dagger}_{\mathbf{k},e} \Big\} \, e^{i\omega_{k,1}t} \Big] \, , \tag{5.77}$$

$$iG^{>\alpha\beta}_{\mu e}(\mathbf{k},t) \;\; = \;\; \sum_r \Big[u^{r,\alpha}_{\mathbf{k},2} \, \bar{u}^{r,\beta}_{\mathbf{k},1} \; \Big\{ \alpha^r_{\mathbf{k},\mu}(t), \alpha^{r\dagger}_{\mathbf{k},e} \Big\} \, e^{-i\omega_{k,2}t} + \; v^{r,\alpha}_{-\mathbf{k},2} \, \bar{u}^{r,\beta}_{\mathbf{k},1} \; \Big\{ \beta^{r\dagger}_{-\mathbf{k},\mu}(t), \alpha^{r\dagger}_{\mathbf{k},e} \Big\} \, e^{i\omega_{k,2}t} \Big] \tag{5.78}$$

Here and in the following $\alpha^{r\dagger}_{\mathbf{k},e}$ stands for $\alpha^{r\dagger}_{\mathbf{k},e}(0)$. These relations show that the definition of the transition amplitudes singles out one anticommutator by time :

$$\begin{aligned}\mathcal{P}^r_{ee}(\mathbf{k},t) &\equiv i\,u^{r\dagger}_{\mathbf{k},1}e^{i\omega_{k,1}t}\,G^{>}_{ee}(\mathbf{k},t)\,\gamma^0 u^r_{\mathbf{k},1} = \left\{\alpha^r_{\mathbf{k},e}(t),\alpha^{r\dagger}_{\mathbf{k},e}\right\}\\ &= \cos^2\theta + \sin^2\theta\,\left[|U_{\mathbf{k}}|^2 e^{-i(\omega_{k,2}-\omega_{k,1})t} + |V_{\mathbf{k}}|^2 e^{i(\omega_{k,2}+\omega_{k,1})t}\right],\end{aligned} \tag{5.79}$$

$$\begin{aligned}\mathcal{P}^r_{\bar{e}e}(\mathbf{k},t) &\equiv i\,v^{r\dagger}_{-\mathbf{k},1}e^{-i\omega_{k,1}t}\,G^{>}_{ee}(\mathbf{k},t)\,\gamma^0 u^r_{\mathbf{k},1} = \left\{\beta^{r\dagger}_{-\mathbf{k},e}(t),\alpha^{r\dagger}_{\mathbf{k},e}\right\}\\ &= \epsilon^r\,|U_{\mathbf{k}}||V_{\mathbf{k}}|\,\sin^2\theta\,\left[e^{i(\omega_{k,2}-\omega_{k,1})t} - e^{-i(\omega_{k,2}+\omega_{k,1})t}\right],\end{aligned} \tag{5.80}$$

$$\begin{aligned}\mathcal{P}^r_{\mu e}(\mathbf{k},t) &\equiv i\,u^{r\dagger}_{\mathbf{k},2}e^{i\omega_{k,2}t}\,G^{>}_{\mu e}(\mathbf{k},t)\,\gamma^0 u^r_{\mathbf{k},1} = \left\{\alpha^r_{\mathbf{k},\mu}(t),\alpha^{r\dagger}_{\mathbf{k},e}\right\}\\ &= |U_{\mathbf{k}}|\,\cos\theta\,\sin\theta\left[1 - e^{i(\omega_{k,2}-\omega_{k,1})t}\right],\end{aligned} \tag{5.81}$$

$$\begin{aligned}\mathcal{P}^r_{\bar{\mu} e}(\mathbf{k},t) &\equiv i\,v^{r\dagger}_{-\mathbf{k},2}e^{-i\omega_{k,2}t}\,G^{>}_{\mu e}(\mathbf{k},t)\,\gamma^0 u^r_{\mathbf{k},1} = \left\{\beta^{r\dagger}_{-\mathbf{k},\mu}(t),\alpha^{r\dagger}_{\mathbf{k},e}\right\}\\ &= \epsilon^r\,|V_{\mathbf{k}}|\,\cos\theta\,\sin\theta\left[1 - e^{-i(\omega_{k,2}+\omega_{k,1})t}\right].\end{aligned} \tag{5.82}$$

All other anticommutators with $\alpha^\dagger_e$ vanish. Notice that in the perturbative case, there were only two non-zero amplitudes, i.e. $\mathcal{P}_{ee}$ and $\mathcal{P}_{\mu e}$. In conclusion, we thus find that the probability amplitude is now correctly normalized: $lim_{t\to 0^+}\mathcal{P}_{ee}(\mathbf{k},t) = 1$, and $\mathcal{P}_{ee}$, $\mathcal{P}_{\mu e}$, $\mathcal{P}_{\bar{\mu}e}$ go to zero in the same limit $t \to 0^+$. Moreover,

$$|\mathcal{P}^r_{ee}(\mathbf{k},t)|^2 + |\mathcal{P}^r_{\bar{e}e}(\mathbf{k},t)|^2 + \left|\mathcal{P}^r_{\mu e}(\mathbf{k},t)\right|^2 + \left|\mathcal{P}^r_{\bar{\mu}e}(\mathbf{k},t)\right|^2 = 1\,, \tag{5.83}$$

as the conservation of the total probability requires. We also note that the above transition probabilities are independent of the spin orientation.

For notational simplicity, we now drop the momentum and spin indices. The momentum is taken to be aligned along the quantization axis, $\mathbf{k} = (0,0,|\mathbf{k}|)$. It is also understood that antiparticles carry opposite momentum to that of the particles. At time $t = 0$ the vacuum state is $|0\rangle_{e,\mu}$ and the one electronic neutrino state is

$$|\nu_e\rangle \equiv \alpha^\dagger_e|0\rangle_{e,\mu} = \left[\cos\theta\,\alpha^\dagger_1 + |U|\,\sin\theta\,\alpha^\dagger_2 - \epsilon\,|V|\,\sin\theta\,\alpha^\dagger_1\alpha^\dagger_2\beta^\dagger_1\right]|0\rangle_{1,2}\,. \tag{5.84}$$

In this state a multiparticle component is present, disappearing in the relativistic limit $k \gg \sqrt{m_1 m_2}$: in this limit the (quantum-mechanical) Pontecorvo state is recovered. The time evoluted of $|\nu_e\rangle$ is given by $|\nu_e(t)\rangle = e^{-iHt}|\nu_e\rangle$.

Notice that the flavour vacuum $|0\rangle_{e,\mu}$ is not eigenstate of the free Hamiltonian H. It "rotates" under the action of the time evolution generator: one indeed finds $\lim_{V\to\infty}\ {}_{e,\mu}\langle 0\mid 0(t)\rangle_{e,\mu} = 0$. Thus at different times we have unitarily inequivalent flavour vacua (in the limit $V\to\infty$): this expresses the different particle content of these (coherent) states and it is direct consequence of the fact that flavour states are not mass eigenstates. The flavour content of the time evoluted electronic neutrino state is found to be

$$|\nu_e(t)\rangle = \Big[\eta_1(t)\ \alpha_e^\dagger\ +\ \eta_2(t)\ \alpha_\mu^\dagger\ +\ \eta_3(t)\ \alpha_e^\dagger\alpha_\mu^\dagger\beta_e^\dagger\ +\ \eta_4(t)\ \alpha_e^\dagger\alpha_\mu^\dagger\beta_\mu^\dagger\Big]\,|0\rangle_{e,\mu}\,, \tag{5.85}$$

with $\sum_{i=1}^{4}|\eta_i(t)|^2\ =\ 1$. Let $|0(t)\rangle_{e,\mu}\ \equiv\ e^{-iHt}\ \ |0\rangle_{e\mu}$ and define the charge operators as $Q_{e/\mu}\ \equiv\ \alpha_{e/\mu}^\dagger\alpha_{e/\mu} - \beta_{e/\mu}^\dagger\beta_{e/\mu}$. We have

$$\begin{aligned}
{}_{e,\mu}\langle 0(t)|Q_e|0(t)\rangle_{e,\mu}\ &=\ {}_{e,\mu}\langle 0(t)|Q_\mu|0(t)\rangle_{e,\mu}\ =\ 0\,, && (5.86)\\
\langle\nu_e(t)|Q_e|\nu_e(t)\rangle\ &=\ \left|\left\{\alpha_e(t),\alpha_e^\dagger\right\}\right|^2\ +\ \left|\left\{\beta_e^\dagger(t),\alpha_e^\dagger\right\}\right|^2\,, && (5.87)\\
\langle\nu_e(t)|Q_\mu|\nu_e(t)\rangle\ &=\ \left|\left\{\alpha_\mu(t),\alpha_e^\dagger\right\}\right|^2\ +\ \left|\left\{\beta_\mu^\dagger(t),\alpha_e^\dagger\right\}\right|^2\,. && (5.88)
\end{aligned}$$

Charge conservation is ensured at any time: $\langle\nu_e(t)|\,(Q_e\ +\ Q_\mu)\,|\nu_e(t)\rangle\ =\ 1$. The oscillation formula for the flavour charges then readily follows

$$\begin{aligned}
P_{\nu_e\to\nu_e}(\mathbf{k},t)\ &=\ \left|\left\{\alpha^r_{\mathbf{k},e}(t),\alpha^{r\dagger}_{\mathbf{k},e}\right\}\right|^2\ +\ \left|\left\{\beta^{r\dagger}_{-\mathbf{k},e}(t),\alpha^{r\dagger}_{\mathbf{k},e}\right\}\right|^2 && (5.89)\\
&=\ 1-\sin^2(2\theta)\left[|U_\mathbf{k}|^2\ \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}t\right)+|V_\mathbf{k}|^2\ \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}t\right)\right]\,,
\end{aligned}$$

$$\begin{aligned}
P_{\nu_e\to\nu_\mu}(\mathbf{k},t)\ &=\ \left|\left\{\alpha^r_{\mathbf{k},\mu}(t),\alpha^{r\dagger}_{\mathbf{k},e}\right\}\right|^2\ +\ \left|\left\{\beta^{r\dagger}_{-\mathbf{k},\mu}(t),\alpha^{r\dagger}_{\mathbf{k},e}\right\}\right|^2 && (5.90)\\
&=\ \sin^2(2\theta)\left[|U_\mathbf{k}|^2\ \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}t\right)+|V_\mathbf{k}|^2\ \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}t\right)\right]\,.
\end{aligned}$$

This result is exact. The additional contribution to the usual oscillation formula, does oscillate with a frequency which is the sum of the frequencies of the mass components. For particular values of the masses and/or of the momentum, eqs.(5.89), (5.90) show that resonance is possible in vacuum. In the relativistic limit $k\gg\sqrt{m_1m_2}$ the traditional oscillation formula is recovered.

Appendix

Derivation of some relations

Derivation of eqs.(1.21), (1.22)

From eqs.(1.21) we have, for θ real,

$$\frac{d}{d\theta}a(\theta) \,=\, i\,U(\theta)\,[\frac{dG}{d\theta},a]\,U^{-1}(\theta) \,=\, 1 \tag{A.1}$$

which has (1.17) as a solution, when the initial condition $a(\theta)|_{\theta=0} = a$ is taken.

For θ complex, we can redefine the a operator by including in it the phase of θ, then the proof is the same as above.

Derivation of eq.(1.23)

Eq.(1.23) follows from the following relation:

$$U(\theta) \,=\, \exp\left(-\frac{1}{2}|\theta|^2\right)\, exp\left(-\theta a^\dagger\right)\, exp\left(\theta^* a\right) \tag{A.2}$$

This can be proved for θ real, by using $e^A B e^{-A} = e^{[A,\,\cdot\,]}B$ and the definition of $U(\theta)$ eq.(1.22):

$$\frac{d}{d\theta}\left[U(\theta)e^{-\theta a}\right] \,=\, -\,U(\theta)\,e^{-\theta a}\left(e^{\theta a}a^\dagger e^{-\theta a}\right) \,=\, -\,U(\theta)\,e^{-\theta a}\frac{d}{d\theta}\left(\theta a^\dagger + \frac{1}{2}\theta^2\right) \tag{A.3}$$

Derivation of eq.(1.42)

Consider $\alpha(\theta)$ as given by eq.(1.41). We have

$$\frac{\partial}{\partial\theta_{\mathbf{k}}}\alpha_{\mathbf{k}}(\theta) \,=\, iU\,\left[\frac{dG}{d\theta_{\mathbf{k}}},a_{\mathbf{k}}\right]\,U^{-1} \,=\, iU\,ib^\dagger_{\mathbf{k}}\,U^{-1} \,=\, -\beta^\dagger_{\mathbf{k}}(\theta) \tag{A.4}$$

$$\frac{\partial^2}{\partial\theta_{\mathbf{k}}^2}\alpha_{\mathbf{k}}(\theta) \,=\, -i\,U\,\left[\frac{dG}{d\theta_{\mathbf{k}}},b^\dagger_{\mathbf{k}}\right]\,U^{-1} \,=\, \alpha_{\mathbf{k}}(\theta) \tag{A.5}$$

By use of the initial conditions $\alpha_{\mathbf{k}}(\theta)|_{\theta_{\mathbf{k}}=0} = a_{\mathbf{k}}$ and $\frac{\partial}{\partial\theta_{\mathbf{k}}}\alpha_{\mathbf{k}}(\theta)|_{\theta_{\mathbf{k}}=0} = -b^\dagger_{\mathbf{k}}$, eq.(A.5) has (1.38) as solution.

Derivation of eq.(4.54)

Eq.(4.54) follows from eq.(4.48) after defining the inverse of the differential operator as

$$(-\partial_x^2 - m^2)^{-1} F[\rho(x)] \equiv (-i) \int d^2y \Delta(x-y) \ : F[\rho(y)] : \tag{A.6}$$

i.e., as the convolution integral with the propagator when acting on a certain functional of the fields.

Then eq.(4.48) gives, at different levels of iteration,

$$\begin{aligned} \rho^0(x) &= \rho^{in}(x) \qquad\qquad\qquad (A.7)\\ \rho^I(x) &= \rho^0(x) + \frac{3}{2} mg(-i) \int d^2y \Delta(x-y) : [\rho^0(y)]^2 : + \frac{1}{2} g^2(-i) \int d^2y \Delta(x-y) : [\rho^0(y)]^3 : \\ \rho^{II}(x) &= \rho^0(x) + \frac{3}{2} mg(-i) \int d^2y \Delta(x-y) : [\rho^I(y)]^2 : + \frac{1}{2} g^2(-i) \int d^2y \Delta(x-y) : [\rho^I(y)]^3 : \end{aligned}$$

Then $\rho^{II}(x)$ contains a g^2 term coming from the term $: [\rho^I(y)]^2 ::$

$$\frac{3}{2} mg(-i) \int d^2y \Delta(x-y) : \left[\rho^0(y) + \frac{3}{2} mg(-i) \int d^2z \Delta(y-z) : [\rho^0(z)]^2 : +.. \right]^2 : \tag{A.8}$$

from which we get the third term in the dynamical map eq.(4.54).

Spinor wave functions

$$u^r_{k,i} = A_i \begin{pmatrix} \xi^r \\ \\ \frac{\bar{\sigma} \cdot \bar{k}}{\omega_{k,i} + m_i} \xi^r \end{pmatrix} \quad , \quad v^r_{k,i} = A_i \begin{pmatrix} \frac{\bar{\sigma} \cdot \bar{k}}{\omega_{k,i} + m_i} \xi^r \\ \\ \xi^r \end{pmatrix}$$

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \quad A_i \equiv \left(\frac{\omega_{k,i} + m_i}{2\omega_{k,i}} \right)^{\frac{1}{2}} \quad , \quad i = 1, 2 \ , \quad r = 1, 2 \ .$$

$$v^{1\dagger}_{-k,1} u^1_{k,2} = -v^{2\dagger}_{-k,1} u^2_{k,2} = A_1 A_2 \left(\frac{-k_3}{\omega_{k,1} + m_1} + \frac{k_3}{\omega_{k,2} + m_2} \right)$$

$$v^{1\dagger}_{-k,1} u^2_{k,2} = \left(v^{2\dagger}_{-k,1} u^1_{k,2} \right)^* = A_1 A_2 \left(\frac{-k_1 + ik_2}{\omega_{k,1} + m_1} + \frac{k_1 - ik_2}{\omega_{k,2} + m_2} \right)$$

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