Path Integrals and Anomalies in Curved Space

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Abstract

In this book we study quantum mechanical path integrals in curved and flat target space (nonlinear and linear sigma models), and use the results to compute the anomalies of n-dimensional quantum field theories coupled to external gravity and gauge fields. Even though the quantum field theories need not be supersymmetric, the corresponding quantum mechanical models are often supersymmetric. Calculating anomalies using quantum mechanics is much simpler than using the full machinery of quantum field theory.

In the first part of this book we give a complete derivation of the path integrals for supersymmetric and non-supersymmetric nonlinear sigma models describing bosonic and fermionic point particles (commuting coordinates $x^i(t)$ and anticommuting variables $\psi^a(t) = e_i^a(x(t))\psi^i(t)$) in a curved target space with metric $g_{ij}(x) = e_i^a(x)e_j^b(x)\delta_{ab}$. All our calculations are performed in Euclidean space. We consider a finite time interval because this is what is needed for the applications to anomalies. As these models contain double-derivative interactions, they are divergent according to power counting, but ghost loops cancel the divergences. Only the one- and two-loop graphs are power counting divergent, hence in general the action may contain extra finite local one- and two-loop counterterms whose coefficients should be fixed. They are fixed by imposing suitable renormalization conditions. To regularize individual diagrams we use three different regularization schemes:

(i) time slicing (TS), known from the work of Dirac and Feynman

(ii) mode regularization (MR), known from instanton and soliton physics[3]

(iii) dimensional regularization on a finite time interval (DR), discussed in this book.

The renormalization conditions relate a given quantum Hamiltonian $\hat{H}$ to a corresponding quantum action S, which is the action which appears in the exponent of the path integral. The particular finite one- and two-loop counterterms in S thus obtained are different for each regularization scheme. In principle, any $\hat{H}$ with a definite ordering of the operators can be taken as the starting point, and gives a corresponding path integral, but for our physical applications we shall fix these ambiguities in $\hat{H}$ by requiring that it maintains reparametrization and local Lorentz invariance in target space (commutes with the quantum generators of these symmetries). Then there are no one-loop counterterms in the three schemes, but only two-loop counterterms. Having defined the regulated path integrals,

[3] Actually, the mode expansion was already used by Feynman and Hibbs to compute the path integral for the harmonic oscillator.

the continuum limit can be taken and reveals the correct "Feynman rules" (the rules how to evaluate the integrals over products of distributions and equal-time contractions) for each regularization scheme. All three regularization schemes give the same final answer for the transition amplitude, although the Feynman rules are different.

In the second part of this book we apply our methods to the evaluation of anomalies in n-dimensional relativistic quantum field theories with bosons and fermions in the loops (spin $0, 1/2, 1, 3/2$ and selfdual antisymmetric tensor fields) coupled to external gauge fields and/or gravity. We regulate the field-theoretical Jacobian for the symmetries whose anomalies we want to compute with a factor $\exp(-\beta\mathcal{R})$, where $\mathcal{R}$ is a covariant regulator which is fixed by the symmetries of the quantum field theory, and β tends to zero only at the end of the calculation. Next we introduce a quantum-mechanical representation of the operators which enter in the field-theoretical calculation. The regulator $\mathcal{R}$ yields a corresponding quantum mechanical Hamiltonian $\hat{H}$. We rewrite the quantum mechanical operator expression for the anomalies as a path integral on the finite time interval $-\beta \leq t \leq 0$ for a linear or nonlinear sigma model with action S. For given spacetime dimension n, in the limit $\beta \to 0$ only graphs with a finite number of loops on the worldline contribute. In this way the calculation of the anomalies is transformed from a field-theoretical problem to a problem in quantum mechanics. We give details of the derivation of the chiral and gravitational anomalies as first given by Alvarez-Gaumé and Witten, and discuss our own work on trace anomalies. For the former one only needs to evaluate one-loop graphs on the worldline, but for the trace anomalies in 2 dimensions we need two-loop graphs, and for the trace anomalies in 4 dimensions we compute three-loop graphs. We obtain complete agreement with the results for these anomalies obtained from other methods. We conclude with a detailed analysis of the gravitational anomalies in 10 dimensional supergravities, both for classical and for exceptional gauge groups.

Preface

In 1983, L. Alvarez-Gaumé and E. Witten (AGW) wrote a fundamental article in which they calculated the one-loop gravitational anomalies (anomalies in the local Lorentz symmetry of $4k+2$ dimensional Minkowskian quantum field theories coupled to external gravity) of complex chiral spin 1/2 and spin 3/2 fields and real selfdual antisymmetric tensor fields[1] [1]. They used two methods: a straightforward Feynman graph calculation in $4k+2$ dimensions with Pauli-Villars regularization, and a quantum mechanical (QM) path integral method in which corresponding nonlinear sigma models appeared. The former has been discussed in detail in an earlier book [3]. The latter method is the subject of this book. AGW applied their formulas to $N = 2B$ supergravity in 10 dimensions, which contains precisely one field of each kind, and found that the sum of the gravitational anomalies cancels. Soon afterwards, M.B. Green and J.H. Schwarz [4] calculated the gravitational anomalies in one-loop string amplitudes, and concluded that these anomalies cancel in string theory, and therefore should also cancel in $N = 1$ supergravity with suitable gauge groups for the $N = 1$ matter couplings. Using the formulas of AGW, one can indeed show that the sum of anomalies in $N = 1$ supergravity coupled to super Yang-Mills theory with gauge group $SO(32)$ or $E_8 \times E_8$, though nonvanishing, is in the technical sense exact: it can be removed by adding a local counterterm to the action. These two papers led to an explosion of interest in string theory.

We discussed these two papers in a series of internal seminars for advanced graduate students and faculty at Stony Brook (the "Friday seminars"). Whereas the basic philosophy and methods of the paper by AGW were clear, we stumbled on numerous technical problems and details. Some of these became clearer upon closer reading, some became more baffling. In a desire to clarify these issues we decided to embark on a research project: the AGW program for trace anomalies. Since gravitational and chiral anomalies only contribute at the one-worldline-loop level in the QM method, one need not be careful with definitions of the measure for the path integral, choice of regulators, regularization of divergent graphs etc. However, we soon noticed that for the trace anomalies the opposite is true: if the field theory is defined in $n = 2k$ dimensions,

[1] Just as one can shift the axial anomaly from the axial-vector current to the vector current, one can also shift the gravitational anomaly from the general coordinate symmetry to the local Lorentz symmetry [2]. Conventionally one chooses to preserve general coordinate invariance. However, AGW chose the symmetric vielbein gauge, so that the symmetry whose anomalies they computed was a linear combination of Einstein symmetry and a compensating local Lorentz symmetry.

one needs $(k+1)$-loop graphs on the worldline in the QM method. As a consequence, every detail in the calculation matters. Our program of calculating trace anomalies turned into a program of studying path integrals for nonlinear sigma models in phase space and configuration space, a notoriously difficult and controversial subject. As already pointed out by AGW, the QM nonlinear sigma models needed for spacetime fermions (or selfdual antisymmetric tensor fields in spacetime) have $N = 1$ (or $N = 2$) worldline supersymmetry, even though the original field theories were not spacetime supersymmetric. Thus we had also to wrestle with the role of susy in the careful definitions and calculations of these QM path integrals.

Although it only gradually dawned upon us, we have come to recognize the problems with these susy and nonsusy QM path integrals as problems one should expect to encounter in any quantum field theory (QFT), the only difference being that these particular field theories have a one-dimensional (finite) spacetime, as a result of which infinities in the sum of Feynman graphs for a given process cancel. However, individual Feynman graphs are power-counting divergent (because these models contain double derivative interactions just like quantum gravity). This cancellation of infinities in the sum of graphs is perhaps the psychological reason why there is almost no discussion of regularization issues in the early literature on the subject (in the 1950 and 1960's). With the advent of the renormalization of gauge theories in the 1970's also issues of regularization of nonlinear sigma models were studied. It was found that most of the regularization schemes used at that time (the time-slicing method of Dirac and Feynman, and the mode regularization method used in instanton and soliton calculations of nonabelian gauge theories) broke general coordinate invariance at intermediate stages, but that by adding noncovariant counterterms, the final physical results were still general coordinate invariant (we shall use the shorter term Einstein invariance for this symmetry in this book). The question thus arose how to determine those counterterms, and understand the relation between the counterterms of one regularization scheme and those of other schemes. Once again, the answer to this question could be found in the general literature on QFT: the imposition of suitable renormalization conditions.

As we tackled more and more difficult problems (4-loop graphs for trace anomalies in six dimensions) it became clear to us that a scheme which needed only covariant counterterms would be very welcome. Dimensional regularization (DR) is such a scheme. It had been used by Kleinert and Chervyakov [5] for the QM of a one dimensional target space on an infinite worldline time interval (with a mass term added to regulate infrared divergences). We have developed instead a version of dimensional regularization on a compact space; because the space is compact we do not need to add by hand a mass term to regulate the infrared divergences due to massless fields. The counterterms needed in such an approach are indeed covariant (both Einstein and locally Lorentz invariant).

The quantum mechanical path integral formalism can be used to compute anomalies in quantum field theories. This application forms the second part of this book. The anomalies are first written in the quantum field theory as traces of a Jacobian with a regulator, $\mathrm{Tr} J e^{-\beta\mathcal{R}}$, and then the limit $\beta \to 0$ is taken. Chiral spin 1/2 and spin 3/2 fields and selfdual antisymmetric tensor (AT) fields can produce anomalies in loop graphs with external gravitons or external gauge (Yang-Mills) fields. The treatment of the spin 3/2 and AT fields formed a major obstacle. In the article by AGW the AT fields are described by a bispinor $\psi_{\alpha\beta}$, and the vector index of the spin 3/2 field and the β index of $\psi_{\alpha\beta}$ are treated differently from the spinor index of the spin 1/2 and spin 3/2 fields and the α index of $\psi_{\alpha\beta}$. In [1] one finds the following transformation rule for the spin 3/2 field (in their notation)

$$-\delta_\eta \psi_A = \eta^i D_i \psi_A + D_a \eta_b (T^{ab})_{AB} \psi_B \tag{0.0.1}$$

where $\eta^i(x)$ yields an infinitesimal coordinate transformation $x^i \to x^i + \eta^i(x)$, and $A = 1, 2, ...n$ is the vector index of the spin 3/2 (gravitino) field, while $(T^{ab})_{AB} = -i(\delta^a_A \delta^b_B - \delta^b_A \delta^a_B)$ is the generator of the Euclidean Lorentz group $SO(n)$ in the vector representation. One would expect that this transformation rule is a linear combination of an Einstein transformation $\delta_E \psi_A = \eta^i \partial_i \psi_A$ (the index A of ψ_A is flat) and a local Lorentz rotation $\delta_{lL} \psi_{A\alpha} = \frac{1}{4}\eta^i \omega_{iAB} (\gamma^A \gamma^B)_\alpha{}^\beta \psi_{A\beta} + \eta^i \omega_{iA}{}^B \psi_{B\alpha}$. However in (0.0.1) the term $\eta^i \omega_{iA}{}^B \psi_{B\alpha}$ is lacking, and instead one finds the second term in (0.0.1) which describes a local Lorentz rotation with parameter $2(D_a \eta_b - D_b \eta_a)$ and this local Lorentz transformation only acts on the vector index of the gravitino. We shall derive (0.0.1) from first principles, and show that it is correct provided one uses a particular regulator $\mathcal{R}$.

The regulator for the spin 1/2 field λ, for the gravitino ψ_A, and for the bispinor $\psi_{\alpha\beta}$ is in all cases the square of the field operator for $\tilde{\lambda}$, $\tilde{\psi}_A$ and $\tilde{\psi}_{\alpha\beta}$, where $\tilde{\lambda}$, $\tilde{\psi}_A$ and $\tilde{\psi}_{\alpha\beta}$ are obtained from λ, ψ_A and $\psi_{\alpha\beta}$ by multiplication by $g^{1/4} = (\det e_\mu{}^m)^{1/2}$. These "twiddled fields" were used by Fujikawa, who pioneered the path integral approach to anomalies [6]. An ordinary Einstein transformation of $\tilde{\lambda}$ is given by $\delta\tilde{\lambda} = \frac{1}{2}(\xi^\mu \partial_\mu + \partial_\mu \xi^\mu)\tilde{\lambda}$, where the second derivative ∂_μ can also act on $\tilde{\lambda}$, and if one evaluates the corresponding anomaly $An_E = \mathrm{Tr}\frac{1}{2}(\xi^\mu \partial_\mu + \partial_\mu \xi^\mu) e^{-\beta\mathcal{R}}$ for β tending to zero by inserting a complete set of eigenfunctions $\tilde{\varphi}_n$ of $\mathcal{R}$

with eigenvalues λ_n, one finds

$$An_E = \lim_{\beta\to 0} \int dx\, \tilde{\varphi}_n^*(x) \frac{1}{2}(\xi^\mu \partial_\mu + \partial_\mu \xi^\mu) e^{-\beta\lambda_n} \tilde{\varphi}_n(x) \ . \tag{0.0.2}$$

Thus the Einstein anomaly vanishes (partially integrate the second ∂_μ) as long as the regulator is hermitian with respect to the inner product $(\tilde{\lambda}_1, \tilde{\lambda}_2) = \int dx\, \tilde{\lambda}_1^*(x)\tilde{\lambda}_2(x)$ (so that the $\tilde{\varphi}_n$ form a complete set), and as long as both $\tilde{\varphi}_n(x)$ and $\tilde{\varphi}_n^*(x)$ belong to the same complete set of eigenstates (as in the case of plane waves $g^{\frac{1}{4}} e^{ikx}$). One can always make a unitary transformation from the $\tilde{\varphi}_n$ to the set $g^{\frac{1}{4}} e^{ikx}$, and this allows explicit calculation of anomalies in the framework of quantum field theory. We shall use the regulator $\mathcal{R}$ discussed above, and twiddled fields, but then cast the calculation of anomalies in terms of quantum mechanics.

Twenty year have passed since AGW wrote their renowned article. We believe we have solved all major and minor problems we initially ran into. The quantum mechanical approach to quantum field theory can be applied to more problems than only anomalies. If future work on such problems will profit from the detailed account given in this book, our scientific and geographical Odyssey has come to a good ending.

Brief summary of the three regularization schemes

For experts who want a quick review of the main technical issues covered in this book, we give here a brief summary of the three regularization schemes described in the main text, namely: time slicing (TS), mode regularization (MR), and dimensional regularization (DR). After this summary we start this book with a general introduction to the subject of path integrals in curved space.

Time Slicing

We begin with bosonic systems with an arbitrary Hamiltonian $\hat{H}$ quadratic in momenta. Starting from the matrix element $\langle z|\exp(-\frac{\beta}{\hbar}\hat{H})|y\rangle$ (which we call the transition amplitude or transition element) with arbitrary but a priori fixed operator ordering in $\hat{H}$, we insert complete sets of position and momentum eigenstates, and obtain the discretized propagators and vertices in *closed* form. These results tell us how to evaluate equal-time contractions in the corresponding continuum Euclidean path integrals, as well as products of distributions which are present in Feynman graphs, such as

$$I = \int_{-1}^{0}\int_{-1}^{0} \delta(\sigma-\tau)\theta(\sigma-\tau)\theta(\tau-\sigma)\; d\sigma d\tau \; .$$

It is found that $\delta(\sigma-\tau)$ should be viewed as a Kronecker delta function, even in the continuum limit, and the step functions as functions with $\theta(0) = \frac{1}{2}$ (yielding $I = \frac{1}{4}$). Kronecker delta function here means that $\int \delta(\sigma-\tau)f(\sigma)\,d\sigma = f(\tau)$, even when $f(\sigma)$ is a product of distributions.

We show that the kernel $\langle x_{k+1}|\exp(-\frac{\epsilon}{\hbar}\hat{H})|x_k\rangle$ with $\epsilon = \beta/N$ may be approximated by $\langle x_{k+1}|(1-\frac{\epsilon}{\hbar}\hat{H})|x_k\rangle$. For linear sigma models this result is well-known and can be rigorously proven ("the Trotter formula"). For nonlinear sigma models, the Hamiltonian $\hat{H}$ is *rewritten* in Weyl ordered form (which leads to extra terms in the action for the path integral of order $\hbar$ and $\hbar^2$), and the midpoint rule follows automatically (so not because we require gauge invariance). The continuum path integrals thus obtained are phase-space path integrals. By integrating out the momenta we obtain configuration-space path integrals. We discuss the relation between both of them (Matthews' theorem), both for our quantum mechanical nonlinear sigma models and also for 4-dimensional Yang-Mills theories.

The configuration space path integrals contain new ghosts (anticommuting $b^i(\tau)$, $c^i(\tau)$ and commuting $a^i(\tau)$), obtained by exponentiating the factors $(\det g_{ij}(x(\tau)))^{1/2}$ which result when one integrates out the momenta. At the one-loop level these ghosts merely remove the overall $\delta(\sigma-\tau)$ singularity in the $\dot{x}\dot{x}$ propagator, but at higher loops they are as useful as in QCD and electroweak gauge theories. In QCD one can choose a unitary gauge without ghosts, but calculations become horrendous. Similarly one could start without ghosts and try to renormalize the theory in a consistent manner, but this is far more complicated than working with ghosts. Since the ghosts arise when we integrate out the momenta, it is natural to keep them. We stress that at any stage all expressions are finite and unambiguous once the operator $\hat{H}$ has been specified. As a result we do not have to fix normalization constants at the end by physical arguments, but "the measure" is unambiguously derived in explicit form. Several two-loop and three-loop examples are worked out, and confirm our path integral formalism in the sense that the results agree with a direct evaluation using operator methods for the canonical variables $\hat{p}$ and $\hat{x}$.

We then extend our results to fermionic systems. We define and use coherent states, define Weyl ordering and derive a fermionic midpoint rule, and obtain also the fermionic discretized propagators and vertices in closed form, with similar conclusions as for the continuum path integral for the bosonic case.

Particular attention is paid to the operator treatment of Majorana fermions. It is shown that "fermion-doubling" (by adding a full set of noninteracting Majorana fermions) and "fermion halving" (by combining pairs of Majorana fermions into Dirac fermions) yield different propagators and vertices but the same physical results such as anomalies.

Mode Regularization

As quantum mechanics can be viewed as a one-dimensional quantum field theory (QFT), we can follow the same approach in quantum mechanics as familiar from four-dimensional quantum field theories. One way to formulate quantum field theory is to expand fields into a complete set of functions, and integrate in the path integral over the coefficients of these functions. One could try to derive this approach from first principles, starting for example from canonical methods for operators, but we shall follow a different approach for mode regularization. Namely we first write down formal rules for the path integral in mode regularization without derivation, and a posteriori fix all ambiguities and free coefficients by consistency conditions.

We start from the formal sum over paths weighted by the phase factor containing the classical action (which is like the Boltzmann factor of statistical mechanics in our Euclidean treatment), and next we suitably define the space of paths. We parametrize all paths as a background trajectory, which takes into account the boundary conditions, and quantum fluctuations, which vanish at the time boundaries. Quantum fluctuations are expanded into a complete set of functions (the sines) and path integration is generated by integration over all Fourier coefficients appearing in the mode expansion of the quantum fields. General covariance demands a nontrivial measure $\mathcal{D}x = \prod_t \sqrt{\det g_{ij}(x(t))}\, d^n x(t)$. This measure is formally a scalar, but it is not translationally invariant under $x^i(t) \to x^i(t) + \epsilon^i(t)$. To derive propagators it is more convenient to exponentiate the nontrivial part of the measure by using ghost fields $\prod_t \sqrt{\det g_{ij}(x(t))} \sim \int DaDbDc\, \exp(-\int dt\, \frac{1}{2}\, g_{ij}(x)(a^i a^j + b^i c^j))$. At this stage the construction is still formal, and one regulates it by integrating over only a finite number of modes, i.e. by cutting off the Fourier sums at a large mode number M. This makes all expressions well-defined and finite. For example in a perturbative expansion all Feynman diagrams are unambiguous and give finite results. This regularization is in spirit equivalent to a standard momentum cut-off in QFT. The continuum limit is achieved by sending M to infinity. Thanks to the presence of the ghost fields (i.e. of the nontrivial measure) there is no need to cancel infinities (i.e. to perform infinite renormalization). This procedure defines a consistent way of doing path integration, but it cannot determine the overall normalization of the path integral (in QFT it is generically infinite). More generally one would like to know how MR is related to other regularization schemes. As is well-known, in QFT different regularization schemes are related to each other by local counterterms. Defining the necessary renormalization conditions introduces a specific set of counterterms of order $\hbar$ and $\hbar^2$, and fixes all of these ambiguities. We do this last step by requiring that the transition amplitude computed in the MR scheme satisfies the Schrödinger equation with an a priori fixed Hamiltonian $\hat{H}$ (the same as for time slicing). The fact that one-dimensional nonlinear sigma models are super-renormalizable guarantees that the counterterms needed to match MR with other regularization schemes (and also needed to recover general coordinate invariance, which is broken by the TS and MR regularizations) are not generated beyond two loops.

Dimensional Regularization

The dimensionally regulated path integral can be defined following steps similar to those used in the definition of the MR scheme, but the regularization of the ambiguous Feynman diagrams is achieved differently. One extends the one dimensional compact time coordinate $-\beta \leq t \leq 0$ by adding D extra non-compact flat dimensions. The propagators on the worldline are now a combined sum-integral, where the integral is a momentum integral as usual in dimensional regularization. At this stage these momentum space integrals define expressions where the variable D can be analytically continued into the complex plane. We are not able to perform explicitly these momentum integrals, but we assume that for arbitrary D all expressions are regulated and define analytic functions, possibly with poles only at integer dimensions, as in usual dimensional regularization. Feynman diagrams are written in coordinate space (t-space), with propagators which contain momentum integrals. Time derivatives d/dt become derivatives $\partial/\partial t^\mu$, but how the indices μ get contracted follows directly from writing the action in $D+1$ dimensions. We perform operations which are valid at the regulated level (like partial integration with absence of boundary terms) to cast the integrals in alternative forms. Dropping the boundary terms in partial integration is always allowed in the extra D dimension, as in ordinary dimensional regularization, but it is only allowed in the original compact time dimension when the boundary term explicitly vanishes because of the boundary conditions. Using partial integrations one rewrites the integrands such that undifferentiated $D+1$ dimensional delta functions $\delta^{D+1}(t,s)$ appear, and these allow to reduce the original integrals to integrals over fewer loops which are finite and unambiguous, and can by computed even after removing the regulator, i.e. in the limit $D=0$. This procedure makes calculations quite easy, and at the same time frees us from the task of computing the analytical continuation of the momentum integrals at arbitrary D. This way one can compute all Feynman diagrams. As in MR one determines all remaining finite ambiguities by imposing suitable renormalization conditions, namely requiring that the transition amplitude computed with dimensional regularization satisfies the Schrödinger equation with an a priori given ordering for the Hamiltonian operator $\hat{H}$, the same as used in mode regularization and time slicing. There are only covariant finite counterterms. Thus dimensional regularization preserves general coordinate invariance also at intermediate steps, and is the most convenient scheme for higher loop calculations. When extended to $N=1$ susy sigma-models, dimensional regularization also preserves worldline supersymmetry, as we show explicitly.

Contents

Part 1

Path Integrals for Quantum Mechanics in Curved Space

1

Introduction to path integrals

Path integrals play an important role in modern quantum field theory. One usually first encounters them as useful formal devices to derive Feynman rules. For gauge theories they yield straightforwardly the Ward identities. Namely, if BRST symmetry ("quantum gauge invariance") holds at the quantum level, certain relations between Green's functions can be derived from path integrals, but details of the path integral (for example, the precise form of the measure) are not needed for this purpose[1]. Once the BRST Ward identities for gauge theories have been derived, unitarity and renormalizability can be proven, and at this point one may forget about path integrals if one is only interested in perturbative aspects of quantum field theories. One can compute higher-loop Feynman graphs or make applications to phenomenology without having to deal with path integrals.

However, for nonperturbative aspects, path integrals are essential. The first place where one encounters path integrals in the study of nonperturbative aspects of quantum field theory is in the study of instantons and solitons. Here advanced methods based on path integrals have been developed. The correct measure for instantons, for example, is needed for the integration over collective coordinates. In particular, for supersymmetric nonabelian gauge theories, there are only contributions from the zero modes which depend on the measure for the zero modes, while the contributions from the nonzero modes cancel between boson and fermions.

[1] To prove that the BRST symmetry is free from anomalies, one may either use regularization-free cohomological methods, or one may perform explicit loop graph calculations using a particular regularization scheme. When there are no anomalies but the regularization scheme does not preserve the BRST symmetry, one can in general add local counterterms to the action at each loop level to restore the BRST symmetry. In these manipulations the path integral measure is usually not taken into account.

Another area where the path integral measure is important is quantum gravity. In modern studies of quantum gravity based on string theory, the measure is crucial to obtain the correct correlation functions. Finally, in lattice simulations the Euclidean version of the path integral is used to define the theory at the nonperturbative level.

In this book we study a class of simple models which lead to path integrals in which no infinite renormalization is needed, but some individual diagrams are divergent and need be regulated, and subtle issues of regularization and measures can be studied explicitly. These models are the quantum mechanical (one-dimensional) nonlinear sigma models. The one-loop and two-loop diagrams in these models are power-counting divergent, but the infinities cancel in the sum of diagrams for a given process at a given loop-level.

Quantum mechanical nonlinear sigma models are toy models for realistic path integrals in four dimensions because they describe curved target spaces and contain double-derivative interactions (quantum gravity has also double-derivative interactions). The formalism for path integrals in curved space has been discussed in great generality in several books and reviews [7, 8, 9, 10, 11, 12, 13, 14, 15]. In the first half of this book we define the path integrals for these models and discuss various subtleties. However, quantum mechanical nonlinear sigma models can also be used to compute anomalies of realistic four-dimensional and higher-dimensional quantum field theories, and this application is thoroughly discussed in the second half of this report. Quantum mechanical path integrals can also be used to compute correlation functions and effective actions, but for these applications we refer to the literature [16, 17, 18].

1.1 Quantum mechanical path integrals in curved space require regularization

The path integrals for quantum mechanical systems we shall discuss have a Hamiltonian $\hat{H}(\hat{p},\hat{x})$ which is more general than $\hat{T}(\hat{p})+\hat{V}(\hat{x})$. We shall typically be discussing models with a Euclidean Lagrangian of the form $L=\frac{1}{2}g_{ij}(x)\frac{dx^i}{dt}\frac{dx^j}{dt}+iA_i(x)\frac{dx^i}{dt}+V(x)$ where $i,j=1,..,n$. These systems are one-dimensional quantum field theories with double-derivative interactions, and hence they are not ultraviolet finite by power counting; rather, the one-loop and two-loop diagrams are divergent as we shall discuss in detail in the next section. The ultraviolet infinities cancel in the sum of diagrams, but one needs to regularize individual diagrams which are divergent. The results of individual diagrams are then regularization-scheme dependent, and also the results for the sum of diagrams are finite but scheme dependent. One must then add finite counterterms which are also scheme dependent, and which must be chosen such that certain physical requirements are satisfied (renormalization conditions). Of course, the final physical answers should be the same, no matter which scheme one uses. Since we shall be working with actions defined on the compact time-interval $[-\beta, 0]$, there are no infrared divergences. We shall also discuss nonlinear sigma models with fermionic point particles $\psi^a(t)$ with again $a = 1, .., n$. Also loops containing fermions can be divergent. For applications to chiral and gravitational anomalies the most important cases are the rigidly supersymmetric models, in particular the models with $N = 1$ and $N = 2$ supersymmetry, but non-supersymmetric models with or without fermions will also be used as they are needed for application to trace anomalies.

In the first part of this book, we will present three different regularization schemes, each with its own merit, which will produce different but equivalent ways of computing path integrals in curved space, at least perturbatively. The final answers for the transition elements and anomalies all agree.

Quantum mechanical path integrals can be used to compute anomalies of n-dimensional quantum field theories. This was first shown by Alvarez-Gaumé and Witten [1, 19, 20], who studied various chiral and gravitational anomalies (see also [21, 22]). Subsequently, Bastianelli and van Nieuwenhuizen [23, 24] extended their approach to trace anomalies. We shall in the second part of this book discuss these applications. With the formalism developed below one can now compute any anomaly, and not only chiral anomalies. In the work of Alvarez-Gaumé and Witten, the chiral anomalies themselves were directly written as a path integral in which the fermions have periodic boundary conditions. Similarly, the trace anomalies lead to path integrals with antiperiodic boundary conditions for the fermions. These are, however, only special cases, which we shall recover from our general formalism.

Because chiral anomalies have a topological character, one would expect that details of the path integral are unimportant and only one-loop graphs on the worldline contribute. In fact, in the approach of AGW this is indeed the case [2]. On the other hand, for trace anomalies, which have no topological interpretation, the details of the path integral do matter and higher loops on the worldline contribute. In fact, it was precisely because 3-loop calculations of the trace anomaly based on quantum mechanical path integrals did initially not agree with results known from other methods, that we started a detailed study of path integrals for nonlinear sigma models. These discrepancies have been resolved in the meantime, and the

[2] Their approach combines general coordinate and local Lorentz transformations, but if one directly computes the anomaly of the Lorentz operator $\gamma^{\mu\nu}\gamma_5$ one needs higher loops.

resulting formalism is presented in this book.

The reason that we do not encounter infinities in loop calculations for QM nonlinear sigma models is different from a corresponding statement for QM linear sigma models. For a linear sigma model with a kinetic term $\frac{1}{2}\dot{x}^i\dot{x}^i$, the propagator behaves as $1/k^2$ for large momenta, and vertices from $V(x)$ do not contain derivatives, hence loops $\int dk[...]$ will always be finite. For nonlinear sigma models with $L = \frac{1}{2}g_{ij}(x)\dot{x}^i\dot{x}^j$, propagators still behave like k^{-2} but vertices now behave like k^2 (as in ordinary quantum gravity) hence single loops are linearly divergent by power counting, and double loops are logarithmically divergent. It is clear by inspection of

$$\langle z|e^{-(\beta/\hbar)\hat{H}}|y\rangle \;=\int_{-\infty}^{\infty}\langle z|e^{-(\beta/\hbar)\hat{H}}|p\rangle\langle p|y\rangle\; d^n p \qquad (1.1.1)$$

that no infinities should be present: the matrix element $\langle z|\exp(-\frac{\beta}{\hbar}\hat{H})|y\rangle$ is *finite* and *unambiguous*. Indeed, we could in principle insert a complete set of momentum eigenstates and then expand the exponent and move all $\hat{p}$ operators to the right and all $\hat{x}$ operators to the left, taking commutators into account. The integral over d^np is a Gaussian and converges. To any given order in β we would then find a finite and well-defined expression[3]. Hence, also **the path integrals should be finite**.

The mechanism by which loops based on the path integrals in (1.1.8) are finite, is different in phase space and configuration space path integrals. In the phase space path integrals the momenta are independent variables and the vertices contained in $H(p, x)$ are without derivatives. (The only derivatives are due to the term $p\dot{x}$, whereas the term $\frac{1}{2}p^2$ is free from derivatives). The propagators and vertices are nonsingular functions (containing at most step functions but no delta functions) which are integrated over the finite domain $[-\beta, 0]$, hence no infinities arise. In the configuration space path integrals, on the other hand, there are divergences in individual loops, as we mentioned. The reason is that although one still integrates over the finite domain $[-\beta, 0]$, single derivatives of the propagators are discontinuous and double derivatives are divergent (they contain delta functions).

However, since the results of configuration space path integrals should be the same as the results of phase space path integrals, these infinities should not be there in the first place. The resolution of this paradox is that *configuration space path integrals contain a new kind of ghosts.* These ghosts are needed to exponentiate the factors $(\det g_{ij})^{1/2}$ which are produced when one integrates out the momenta. Historically, the cancellation of divergences at the one-loop level was first found by Lee

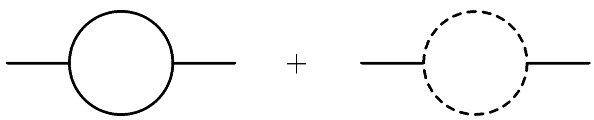
[3] This program is executed in section 2.5 to order β. For reasons explained there, we count the difference $(z - y)$ as being of order $\beta^{1/2}$.

and Yang [25] who studied nonlinear deformations of harmonic oscillators, and who wrote these determinants as new terms in the action of the form

$$\frac{1}{2} \sum_t \ln \det g_{ij}(x(t)) = \frac{1}{2}\, \delta(0) \int \mathrm{tr} \ln g_{ij}(x(t))\, dt \ . \tag{1.1.2}$$

To obtain the right hand side one may multiply the left hand side by $\frac{\Delta t}{\Delta t}$ and replace $\frac{1}{\Delta t}$ by $\delta(0)$ in the continuum limit. For higher loops, it is inconvenient to work with $\delta(0)$; rather, we shall use the new ghosts in precisely the same manner as one uses the Faddeev-Popov ghosts in gauge theories: they occur in all possible manners in Feynman diagrams and have their own Feynman rules. These ghosts for quantum mechanical path integrals were first introduced by Bastianelli [23].

In configuration space, loops with ghost particles cancel thus divergences of loops in corresponding graphs without ghost particles. Generically one has



$= \quad$ finite .

However, the fact that the infinities cancel does not mean that the remaining finite parts are unambiguous. One must regularize the divergent graphs, and different regularization schemes can lead to different finite parts, as is well-known from field theory. Since our actions are of the form $\int_{-\beta}^{0} L\, dt$, we are dealing with one-dimensional quantum field theories in a finite "spacetime", hence translational invariance is broken, and propagators depend on t and s, not only on $t-s$. In coordinate space the propagators contain singularities. For example, the propagator for a free quantum particle $q(t)$ corresponding to $L = \frac{1}{2}\dot{q}^2$ with boundary conditions $q(-\beta) = q(0) = 0$ is proportional to $\Delta(\sigma,\tau)$ where $\sigma = s/\beta$ and $\tau = t/\beta$ with $-\beta \le s,t \le 0$

$$\langle q(\sigma)q(\tau)\rangle \approx \Delta(\sigma,\tau) = \sigma(\tau+1)\theta(\sigma-\tau) + \tau(\sigma+1)\theta(\tau-\sigma) \ . \tag{1.1.3}$$

It is easy to check that $\partial_\sigma^2 \Delta(\sigma,\tau) = \delta(\sigma-\tau)$ and $\Delta(\sigma,\tau) = 0$ at $\sigma = -1, 0$ and $\tau = -1, 0$ (use $\partial_\sigma \Delta(\sigma,\tau) = \tau + \theta(\sigma-\tau)$).

It is clear that Wick contractions of $\dot{q}(\sigma)$ with $q(\tau)$ will contain a factor of $\theta(\sigma-\tau)$, and $\dot{q}(\sigma)$ with $\dot{q}(\tau)$ a factor $\delta(\sigma-\tau)$. Also the propagators for the ghosts contain factors $\delta(\sigma-\tau)$. Thus one needs a consistent, unambiguous and workable regularization scheme for products of the distributions $\delta(\sigma-\tau)$ and $\theta(\sigma-\tau)$. In mathematics the products of distributions are ill-defined [26]. Thus it comes as no surprise that in physics different regularization schemes give different answers for such integrals. For example, consider the following two familiar ways of evaluating the product of distributions: smoothing of distributions, and using Fourier transforms. Suppose one is required to evaluate

$$I = \int_{-1}^{0}\int_{-1}^{0} \delta(\sigma-\tau)\theta(\sigma-\tau)\theta(\sigma-\tau)\; d\sigma d\tau\,. \tag{1.1.4}$$

Smoothing of distribution can be achieved by approximating $\delta(\sigma-\tau)$ and $\theta(\sigma-\tau)$ by some smooth functions and requiring that at the regulated level one still has the relation $\delta(\sigma-\tau) = \frac{\partial}{\partial\sigma}\theta(\sigma-\tau)$. One obtains then $\frac{1}{3}\int_{-1}^{0}\int_{-1}^{0}\frac{\partial}{\partial\sigma}(\theta(\sigma-\tau))^3 d\sigma d\tau = \frac{1}{3}$. On the other hand, if one would interpret the delta function $\delta(\sigma-\tau)$ to mean that one should evaluate the function $\theta(\sigma-\tau)^2$ at $\sigma=\tau$ one obtains $\frac{1}{4}$. One could also decide to use the representations

$$\begin{aligned} \delta(\sigma-\tau) &= \int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}\, e^{i\lambda(\sigma-\tau)} \\ \theta(\sigma-\tau) &= \int_{-\infty}^{\infty}\frac{d\lambda}{2\pi i}\,\frac{e^{i\lambda(\sigma-\tau)}}{\lambda-i\epsilon} \qquad \text{with} \;\; \epsilon>0\;. \end{aligned} \tag{1.1.5}$$

Formally $\partial_\sigma\theta(\sigma-\tau) = \delta(\sigma-\tau) - \epsilon\,\theta(\sigma-\tau)$, and upon taking the limit ϵ tending to zero one would again expect the value $\frac{1}{3}$ for I. However, if one first integrates over σ and τ, one finds

$$I = \bigg(\int_{-\infty}^{\infty}\frac{dy}{2\pi}\frac{(2-2\cos y)}{y^2}\bigg)\bigg(\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi i}\frac{1}{\lambda-i\epsilon}\bigg)^2 = \frac{1}{4} \tag{1.1.6}$$

where we applied contour integration to $\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi i}\frac{1}{\lambda-i\epsilon} = \frac{1}{2}$. Clearly using different methods to evaluate I leads to different answers. Without further specifications integrals such as I are ambiguous and make no sense.

In the applications we are going to discuss, we sometimes choose a regularization scheme that reduces the path integral to a finite-dimensional integral. For example for time slicing one chooses a finite set of intermediate points, and for mode regularization one begins with a finite number of modes for each one-dimensional field. Another scheme we use is dimensional regularization: here one regulates the various Feynman diagrams by moving away from $d=1$ dimensions, and performing partial integrations which make the integral manifestly finite at $d=1$. Afterwards one returns to $d=1$ and computes the values of these finite integrals. One omits boundary terms in the extra dimensions; this can be justified by noting that there are factors $e^{i\mathbf{k}(\mathbf{t}-\mathbf{s})}$ in the propagators due to translation invariance in the extra D dimensions. They yield the Dirac delta functions $\delta^D(\mathbf{k_1}+\mathbf{k_2}+\cdots+\mathbf{k_n})$ upon integration over the extra space coordinates. A derivative with respect to the extra space coordinate which yields, for example, a factor $\mathbf{k_1}$ can be replaced by $-\mathbf{k_2}-\mathbf{k_3}-\cdots-\mathbf{k_n}$ due to the presence of the delta function, and this replacement is equivalent to a partial integration without boundary terms.

In time slicing we find the value $I = \frac{1}{4}$ for (1.1.4): in fact, as we shall see, the delta function is in this case a Kronecker delta which gives the product of the θ functions at the point $\sigma = \tau$. In mode regularization, one finds $I = \frac{1}{3}$ because now $\delta(\sigma-\tau)$ is indeed $\partial_\sigma \theta(\sigma-\tau)$ at the regulated level. In dimensional regularization one must first decide which derivatives are contracted with which derivatives (for example $({}_\mu\Delta_\nu)$ $({}_\mu\Delta)$ (Δ_ν)), but one does not directly encounter I in the applications[4].

As we have seen, different procedures (regularization schemes) lead to finite well-defined results for a given diagram which are in general different in different regularization schemes, but there are also ambiguities in the vertices: the finite one- and two-loop counterterms have not been fixed. The physical requirement that the theory be based on a given quantum Hamiltonian removes the ambiguities in the counterterms: for time slicing Weyl ordering of $\hat{H}$ directly produces the counterterms, while for the other schemes the requirement that the transition element satisfies the Schrödinger equation with a given Hamiltonian $\hat{H}$ fixes the counterterms. Thus in all these schemes the regularization condition is that the transition element be derived from the same particular Hamiltonian $\hat{H}$.

The first scheme, time slicing (TS), has the advantage that one can deduce it directly from the operatorial formalism of quantum mechanics. This regularization can be considered to be equivalent to lattice regularization of standard quantum field theories. It is the approach followed by Dirac and Feynman. One must specify the Hamiltonian $\hat{H}$ with an a priori fixed operator ordering; this ordering corresponds to the renormalization conditions in this approach. All further steps are finite and unambiguous. This approach breaks general coordinate invariance in target space which is then recovered by the introduction of a specific finite counterterm ΔV_{TS} in the action of the path integral. This counterterm also follows unambiguously from the initial Hamiltonian and is itself not coordinate invariant either. However, if the initial Hamiltonian is general

[4] For an example of an integral where dimensional regularization is applied, consider

$$\begin{aligned} J &= \int_{-1}^{0} d\sigma \int_{-1}^{0} d\tau \;\; ({}^\bullet\!\Delta^{\!\bullet})\,({}^\bullet\!\Delta)\,(\Delta^{\!\bullet}) \\ &= \int_{-1}^{0} d\tau \int_{-1}^{0} d\sigma \;\; [1-\delta(\sigma-\tau)]\,[\tau+\theta(\sigma-\tau)]\,[\sigma+\theta(\tau-\sigma)]\;. \qquad (1.1.7) \end{aligned}$$

One finds $J = -\frac{1}{6}$ for TS, see (2.6.35). Further $J = -\frac{1}{12}$ for MR, see (3.3.9). In dimensional regularization one rewrites the integrand as $({}_\mu\Delta_\nu)$ $({}_\mu\Delta)$ (Δ_ν) and one finds $J = -\frac{1}{24}$, see (4.1.24).

coordinate invariant (as an operator, see section 2.5) then also the final result (the transition element) will be in general coordinate invariant.

The second scheme, mode regularization (MR), will be constructed directly without referring to the operatorial formalism. It can be thought of as the equivalent of momentum cut-off in QFT, and it is close in spirit to a Wilsonian approach[5]. It is also close to the intuitive notion of path integrals, that are meant to give a global picture of the quantum phenomena (while one may view the time discretization method closer to the local picture of the differential Schrödinger equation, since one imagines the particle propagating by small time steps). Mode regularization gives in principle a non-perturbative definition of path integrals in that one does not have to expand the exponential of the interaction part of the action. However, also this regularization breaks general coordinate invariance, and one needs a different finite noncovariant counterterm ΔV_{MR} to recover it.

Finally, the third regularization scheme, dimensional regularization (DR), is the one based on the dimensional continuation of the ambiguous integrals appearing in the loop expansion. It is inherently a perturbative regularization, but it is the optimal one for perturbative computations in the following sense. It does not break general coordinate invariance at intermediate stages and the counterterm ΔV_{DR} relating it to other schemes is Einstein and local Lorentz invariant.

All these different regularization schemes will be presented in separate chapters. Since our derivation of the path integrals contains several steps which each require a detailed discussion, we have decided to put all these special discussions in separate sections after the main derivation. This has the advantage that one can read each section for its own sake. The structure of our discussions can be summarized by the flow chart in figure 1.

We shall first discuss time slicing, the lower part of the flow chart. This discussion is first given for bosonic systems with $x^i(t)$, and afterwards for systems with fermions. In the bosonic case, we first construct discretized phase-space path integrals, then derive the continuous configuration-space path integrals, and finally the continuous phase-space path integrals. We show that after Weyl ordering of the Hamiltonian operator $\hat{H}(\hat{x}, \hat{p})$ one obtains a path integral with a midpoint rule (Berezin's theorem). Then we repeat the analysis for fermions.

[5] In more complicated cases, such as path integrals in spaces with a topological vacuum (for example the kink background in Euclidean quantum mechanics), the mode regularization scheme and the momentum regularization scheme with a sharp cut-off are not equivalent (they give for example different answers for the quantum mass of the kink). However, if one replaces the sharp energy cut-off by a smooth cut-off, those schemes become equivalent [27].

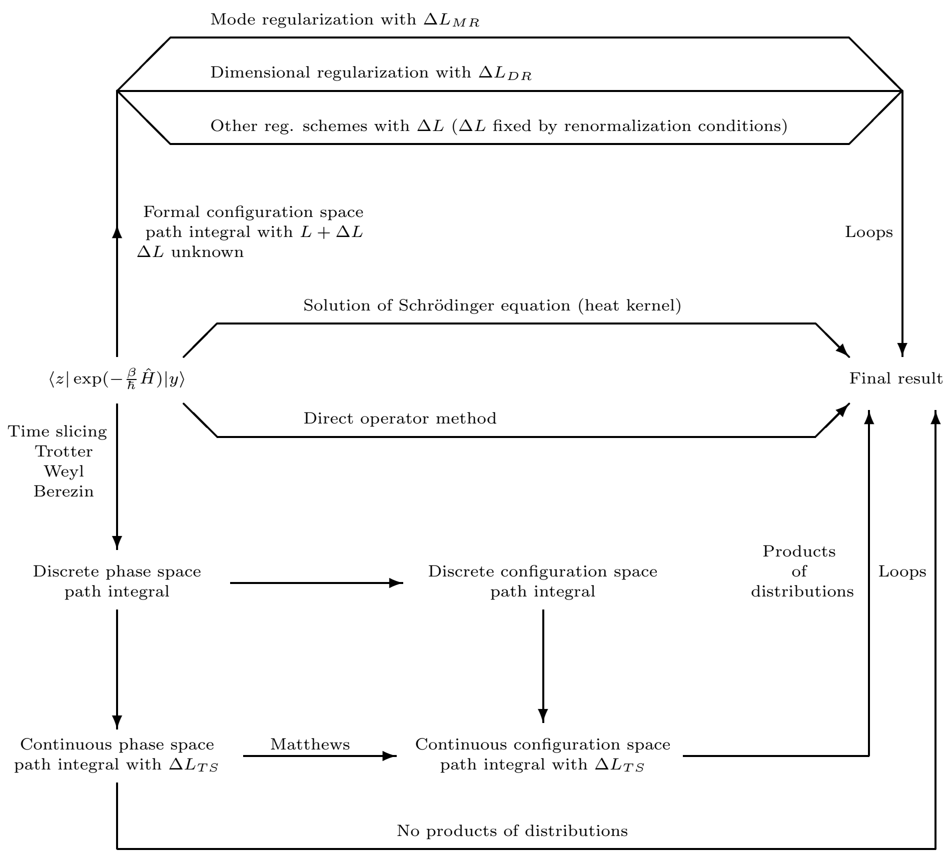


Figure 1: Flow chart.

Next we consider mode number regularization (the upper part of the flow chart). Here we define the path integrals ab initio in configuration space with the naive classical action and a counterterm ΔL which is at first left unspecified. We then proceed to fix ΔL by imposing the requirement that the Schrödinger equation be satisfied with a specific Hamiltonian $\hat{H}$. Having fixed ΔL, one can proceed to compute loops at any desired order.

Finally we present dimensional regularization along similar lines. Each section can be read independently of the previous ones.

In all three cases we define the theory by the Hamiltonian $\hat{H}$ and then construct the path integrals and Feynman rules which correspond to $\hat{H}$. The choice of $\hat{H}$ defines the physical theory. One may be prejudiced about which $\hat{H}$ makes physical sense (for example many physicists require that $\hat{H}$ preserves general coordinate invariance) but in our work one does not have to restrict oneself to these particular $\hat{H}$'s. Any $\hat{H}$, no matter how unphysical, leads to a corresponding path integral and corresponding Feynman rules. The path integral and Feynman rules depend on the regularization scheme chosen but the final result for the transitions element and correlation functions are the same in each scheme.

In the time slicing (TS) approach we shall solve some of the following basic problems: **given** a Hamiltonian operator $\hat{H}(\hat{p},\hat{x})$ with arbitrary but a-priori fixed operator ordering, find a path integral expression for the matrix element $\langle z|\exp(-\frac{\beta}{\hbar}\hat{H})|y\rangle$ [6]. (The bra $\langle z|$ and ket $|y\rangle$ are eigenstates of the position operator $\hat{x}^i$ with eigenvalues z^i and y^i, respectively. For fermions we shall use coherent states as bra and ket). One way to obtain such a path integral representation is to insert complete sets of x- and p-eigenstates (namely N sets of p-eigenstates and $N-1$ sets of x-eigenstates), in the manner first studied by Dirac [28] and Feynman [29, 30], and leads to the following result

$$\langle z|e^{-(\beta/\hbar)\hat{H}}|y\rangle \;\approx\; \int \prod_{i=1}^{N-1} dx^i \prod_{i=1}^{N} dp_i \; e^{-(1/\hbar)\int_{-\beta}^{0} L \; dt} \qquad (1.1.8)$$

where $L = -ip_i(t)\frac{dx^i}{dt} + H(p,x)$ in our Euclidean phase space approach [7]. However, several questions arise if one studies (1.1.8):

(i) Which is the precise relation between $\hat{H}(\hat{p},\hat{x})$ and $H(p,x)$? Different operator orderings of $\hat{H}$ are expected to lead to different functions $H(p,x)$. Are there special orderings of $\hat{H}$ (for example, orderings such that $\hat{H}$ is invariant under general coordinate transformations at the operator level) for which $H(p,x)$ is particularly simple?

(ii) What is the precise meaning of the measures $\Pi[dx^i][dp_i]$ in phase space and $\Pi[dx^i]$ in configuration space? Is there a normalization constant in front of the path integral? Does the measure depend on the metric? The Liouville measure $\Pi[dx^i][dp_i]$ is not a canonical invariant measure because there is one more dp than dx. Does this have implications?

(iii) Which are the boundary conditions one must impose on the paths over which one sums? One expects that all paths must satisfy the Dirichlet boundary conditions $x^i(-\beta) = y^i$ and $x^i(0) = z^i$, but are there also boundary conditions on $p_i(t)$? Is it possible to consider classical paths in phase space which satisfy boundary conditions both at $t=-\beta$ and at $t=0$?

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[6] The results are for Euclidean path integrals. All our results hold equally well in Minkowskian time, at least at the level of perturbation theory, with operators $\exp(-\frac{i}{\hbar}\hat{H}t)$ and path integrals with $\exp\frac{i}{\hbar}\int L_M dt$, where L_M is the Lagrangian in Minkowskian time, related to the Euclidean Lagrangian L by a Wick rotation.

[7] In classical mechanics $L_M = p\dot{q} - H(p,q)$ but we prefer to work in Euclidean time with actions which contain a positive definite term $+\frac{1}{2}\dot{q}^2$, and thus we define $L = -ip\dot{q} + H(p,q)$ in Euclidean time.

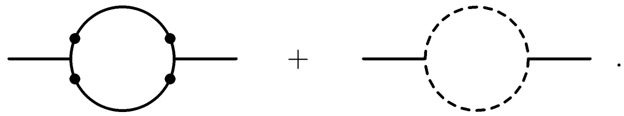
(iv) How does one compute in practice such path integrals? Performing the integrations over dx^i and dp_j for finite N and then taking the limit $N \to \infty$ is in practice hardly possible. Is there a simpler scheme by which one can compute the path integral loop-by-loop, and what are the precise Feynman rules for such an approach? Does the measure contribute to the Feynman rules?
(v) It is often advantageous to use a background formalism and to decompose bosonic fields $x(t)$ into background fields $x_{bg}(t)$ and quantum fluctuations $q(t)$. One can then require that $x_{bg}(t)$ satisfies the boundary conditions so that $q(t)$ vanishes at the endpoints. However, inspired by string theory, one can also compactify the interval $[-\beta, 0]$ to a circle, and then decompose $x(t)$ into a center of mass coordinate x_c and quantum fluctuations about it. What is the relation between both approaches?
(vi) When one is dealing with $N = 1$ supersymmetric systems, one has real (Majorana) particles $\psi^a(t)$. How does one define the Hilbert space in which $\hat{H}$ is supposed to act? Must one also impose an initial and a final condition on $\psi^a(t)$, even though the Dirac equation is only linear in (time) derivatives? We shall introduce operators $\hat{\psi}^a$ and $\hat{\psi}_a^\dagger$ and construct coherent states by contracting them with Grassmann variables $\bar{\eta}_a$ and η^a. If $\hat{\psi}_a^\dagger$ is the hermitian conjugate of $\hat{\psi}^a$, then is $\bar{\eta}_a$ the complex conjugate of η^a?
(vii) In certain applications, for example the calculation of trace anomalies, one must evaluate path integrals over fermions with antiperiodic boundary conditions. In the work of AGW the chiral anomalies came from the zero mode of the fermions. For antiperiodic boundary conditions there are no zero modes. How then should one compute trace anomalies form quantum mechanics?

These are some of the questions which come to mind if one contemplates (1.1.8) for some time. One can find in the literature discussions of some of these questions, but we have made an effort to give a consistent discussion of all of them. Answers to these questions can be found in chapter 8. New is an exact evaluation of all discretized expressions in the TS scheme as well as the derivation of the MR and DR schemes in curved space.

1.2 Power counting and divergences

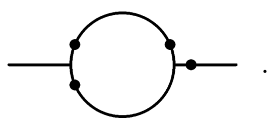
Let us now give some examples of divergent graphs. The precise form of the vertices is given later, in (2.1.82), but for the discussion in this section we only need the qualitative features of the action. The propagators we are going to use later in this book are not of the simple form $\frac{1}{k^2}$ for a scalar, rather, they have the form $\sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2} \sin(\pi n \tau) \sin(\pi n \sigma)$ due to boundary conditions. (Even the propagator for time slicing can be cast into this form by Fourier transformation). However for ultraviolet divergences, the sum of $\frac{1}{n^2}$ is equivalent to an integral over $\frac{1}{k^2}$, and in this section we analyze Feynman graphs with $\frac{1}{k^2}$ propagators. The physical justification is that ultraviolet divergences should not feel the boundaries.

Consider first the self energy. At the one loop level the self energy without external derivatives receives contributions from the following two graphs



We used the vertices from $\frac{1}{2}(g_{ij}(x) - g_{ij}(z))(\dot{q}^i\dot{q}^j + a^i a^j + b^i c^j)$. Dots indicate derivatives and dashed lines ghost particles. The two divergences are proportional to $\delta^2(\sigma - \tau)$ and cancel, but there are ambiguities in the finite part which must be fixed by suitable conditions. (In quantum field theories with divergences we call these conditions renormalization conditions). In momentum space both graphs are linearly divergent, but the linear divergence $\int dk$ cancels in the sums of the graphs, and the two remaining logarithmic divergences $\int \frac{dk k}{k^2}$ cancel by symmetric integration leaving in general a finite but ambiguous result.

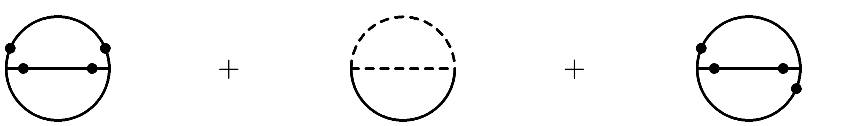
Another example is the self-energy with one external derivative



This graph is logarithmically divergent, but using symmetric integration it leaves again a finite but ambiguous part.

All three regularization schemes give the same answer for all one-loop graphs, so the one-loop counterterms are the same; in fact, there are no one-loop counterterms at all in any of the schemes if one starts with an Einstein invariant Hamiltonian [8].

At the two-loop level, there are similar cancellations and ambiguities. Consider the following vacuum graphs (vacuum graphs will play an important role in the applications to anomalies)



[8] If one would use the Einstein-noninvariant Hamiltonian $g^{1/4-\alpha}\hat{p}_i\sqrt{g}g^{ij}\hat{p}_j g^{1/4+\alpha}$, one would obtain in the TS scheme a one-loop counterterm proportional to $\hbar p_i g^{ij}\partial_j \ln g$ in phase space, or $\hbar \dot{x}^i \partial_i \ln g$ in configuration space. See appendix B.

Again the infinities in the upper loop of the first two graphs cancel, but the finite part is ambiguous. The last graph is logarithmically divergent by power counting, and also the two subdivergences are logarithmically divergent by power counting, but actual calculation shows that it is finite but ambiguous (the leading singularities are of the form $\int \frac{dkk}{k^2}$ and cancel due to symmetric integration). The sum of the first two graphs yield $(\frac{1}{4}, \frac{1}{4}, \frac{1}{8})$ in TS, MR and DR, respectively, while the last graph yields $(-\frac{1}{6}, -\frac{1}{12}, -\frac{1}{24})$. This explicitly proves that the results for power-counting logarithmically divergent graphs are ambiguous, even though the divergences cancel.

It is possible to use standard power counting methods as used in ordinary quantum field theory to determine all possibly ultraviolet divergent graphs. Let us interpret our quantum mechanical nonlinear sigma model as a particular QFT in one Euclidean time dimension. We consider a toy model of the type

$$S = \int dt \left(\frac{1}{2} g(\phi) \dot{\phi}\dot{\phi} + A(\phi)\dot{\phi} + V(\phi) \right) \tag{1.2.1}$$

where the functions $g(\phi), A(\phi)$ and $V(\phi)$ describe the various couplings. For simplicity we omit the indices i and j.

The choice $g(\phi) = 1$, $A(\phi) = 0$, $V(\phi) = \frac{1}{2}m^2\phi^2$ reproduces a free massive theory, namely an harmonic oscillator of "mass" (frequency) m. From this one deduces that the field ϕ has mass dimension $M^{-\frac{1}{2}}$. Next, let us consider general interactions and expand them in Taylor series

$$V(\phi) = \sum_{n=0}^{\infty} V_n \phi^n, \quad A(\phi) = \sum_{n=0}^{\infty} A_{n+1}\phi^n, \quad g(\phi) = \sum_{n=0}^{\infty} g_{n+2}\phi^n \ . \tag{1.2.2}$$

These expansions identify the coupling constants V_n, A_n, g_n, whose subscript indicates how many fields a given vertex contains. We easily deduce the following mass dimensions for such couplings

$$[V_n] = M^{\frac{n}{2}+1} \ ; \quad [A_n] = M^{\frac{n}{2}} \ ; \quad [g_n] = M^{\frac{n}{2}-1} \ . \tag{1.2.3}$$

The interactions correspond to the terms with $n \geq 3$, so all coupling constants have positive mass dimensions. This implies that the theory is super-renormalizable. This means that from a certain loop level on, there are no more superficial divergences by power counting. We can work this out in more detail. Given a Feynman diagram, let us indicate by L the number of loops, I the number of internal lines, V_n, A_n and g_n the numbers of the corresponding vertices present in the diagram. One can associate to the diagram a superficial degree of divergence D by

$$D = L - 2I + \sum_n (A_n + 2g_n) \tag{1.2.4}$$

reflecting the fact that each loop gives a momentum integration $\int dk$, the propagators give factors of k^{-2}, and the A_n and g_n vertices bring in at worst one and two momenta, respectively. Also, the number of loops is given by

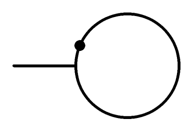
$$L = I - \sum_n (V_n + A_n + g_n) + 1 \ . \tag{1.2.5}$$

Combining these two equations we find that the degree of divergence D is given by

$$D = 2 - L - \sum_n (2V_n + A_n) \ . \tag{1.2.6}$$

Let us analyze the consequences of this formula by considering first the case with nontrivial $V(\phi)$ couplings only (linear sigma models). Then (1.2.6) shows that no divergences can ever arise. As a consequence, no ambiguities are expected either in the path integral quantization of the model. This is the class of models with $H = T(p) + V(x)$ which is extensively discussed in many textbooks [30, 11, 12, 13, 14, 15, 31, 32, 33, 34, 35].

Next, let us consider a nontrivial $A(\phi)$. From (1.2.6) we see that there is now a possible logarithmic superficial divergence in the one loop graphs with a single vertex A_n (n can be arbitrary, since the extra fields that are not needed to construct the loop can be taken as external)



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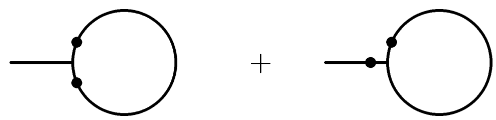
The logarithmic singularity actually cancels by symmetric integration, but the leftover finite part must be fixed unambiguously by specifying a renormalization condition. If A corresponds to an electromagnetic field, gauge invariance can be used as renormalization condition which fixes completely the ambiguity. In the continuum theory, the action $\int A_j \dot{x}^j dt$ is invariant under the gauge transformation $\delta A_j = \partial_j \lambda(x)$. Feynman [29] found that with TS one must take A_j at the midpoints $\frac{1}{2}(x_{k+1} + x_k)$ in order to obtain the Schrödinger equation with gauge invariant Hamiltonian $H = \frac{1}{2m}(\hat{p} - \frac{e}{c}\hat{A})^2 + \hat{V}$. [9] For further discussion, see for

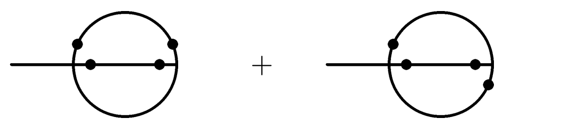
[9] To avoid confusion we mention already at this point that in our treatment of path integrals there are no ambiguities. If one takes a Hamiltonian which is gauge invariant (commutes with the generator of gauge transformations at the operator level), then the corresponding path integral uses the midpoint rule, but using another Hamiltonian, the midpoint rule does not hold.

example ref. [11], chapter 4 and 5. If the regularization scheme chosen to define the above graphs does not respect gauge invariance, one must add a local finite counterterms by hand to restore gauge invariance.

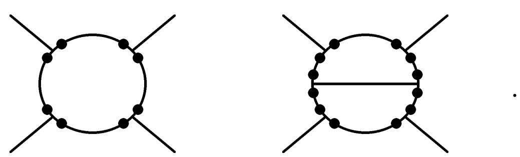
Finally, consider the most general case with $g(\phi)$. There can be linear and logarithmic divergences in one-loop graphs as in



and logarithmic divergences at two loops



Notice that (1.2.6) is independent of g_n. This implies that at the one- and two-loop level one can construct an infinity of divergent graphs form a given divergent graph by inserting g_n vertices. The following diagrams illustrate this fact



As we shall see, the nontrivial path integral measure cancels the leading divergences, but we repeat that finite ambiguities remain which must be fixed by renormalization conditions. Of course, general coordinate invariance must also be imposed, but this symmetry requirement is not enough to fix all the renormalization conditions. One can understand this from the following observation. In the canonical approach different ordering of the Hamiltonian $g_{ij}p^ip^j$ lead to ambiguities proportional to $(\partial_i g_{jk})^2$ and $\partial_i\partial_j g_{kl}$ and from them one can form the scalar curvature R. So one can always add to the Hamiltonian a term proportional to R and still maintain general coordinate invariance in target space. In fact, we should distinguish between an explicit R term in the Hamiltonian $\hat{H}$, and an explicit R term in the action which appears in the path integrals. In all three schemes we shall discuss, one always produces a term $\frac{1}{8}R$ in the action as one proceed from $\hat{H}$ to the path integral. So for a free scalar particle with $\hat{H}$ without an R term the path integral contains a term $\frac{1}{8}R$ in the potential. However, in susy theories $\hat{H}$ is obtained by evaluating the susy anticommutator $\{\hat{Q}, \hat{Q}\}$, one finds that $\hat{H}$ contains a term $-\frac{1}{8}R$, and then in the corresponding path integral one does not obtain an R term.

1.3 A brief history of path integrals

Path integrals yield a third approach to quantum physics, in addition to Heisenberg's operator approach and Schrödinger's wave function approach. They are due to Feynman [29], who developed in the 1940's an approach Dirac had briefly considered in 1932 [28]. In this section we discuss the motivations which led Dirac and Feynman to associate path integrals (with $\frac{i}{\hbar}$ times the action in the exponent) with quantum mechanics. In mathematics Wiener had already studied path integrals in the 1920's but these path integrals contained (-1) times the free action for a point particle in the exponent. Wiener's path integrals were Euclidean path integrals which are mathematically well-defined but Feynman's path integrals do not have a similarly solid mathematical foundation. Nevertheless, path integrals have been successfully used in almost all branches of physics: particle physics, atomic and nuclear physics, optics, and statistical mechanics [11].

In many applications one uses path integrals for perturbation theory, in particular for semiclassical approximations, and in these cases there are no serious mathematical problems. In other applications one uses Euclidean path integrals, and in these cases they coincide with Wiener's path integrals. However, for the nonperturbative evaluations of path integrals in Minkowski space a completely rigorous mathematical foundation is lacking. The problems increase in dimensions higher than four. Feynman was well aware of this problem, but the physical ideas which stem from path integrals are so convincing that he (and other researchers) considered this not worrisome.

Our brief history begins with Dirac who wrote in 1932 an article in a USSR physics journal [28] in which he tried to find a description of quantum mechanics which was based on the Lagrangian instead of the Hamiltonian approach. Dirac was making with Heisenberg a trip around the world, and took the trans-Siberian railway to arrive in Moscow. In those days all work in quantum mechanics (including the work on quantum field theory) started with the Schrödinger equation or operator methods in both of which the Hamiltonian played a central role. For quantum mechanics this was fine, but for relativistic field theories an approach based on the Hamiltonian had the drawback that manifest Lorentz invariance was lost (although for QED it had been shown that physical results were nevertheless relativistically invariant). Dirac considered the transition element

$$\langle x_2, t_2 | x_1, t_1 \rangle = K(x_2, t_2 | x_1, t_1) = \langle x_2 | e^{-\frac{i}{\hbar}\hat{H}(t_2 - t_1)} | x_1 \rangle \tag{1.3.1}$$

(for time independent H), and asked whether one could find an expression for this matrix element in which the action was used instead of the Hamiltonian. (The notation $\langle x_2, t_2 | x_1, t_1 \rangle$ is due to Dirac who called this element a transformation function. Feynman introduced the notation $K(x_2, t_2 | x_1, t_1)$ because he used it as the kernel in an integral equation which solved the Schrödinger equation.) Dirac knew that in classical mechanics the time evolution of a system could be written as a canonical transformation, with Hamilton's principal function $S(x_2, t_2 | x_1, t_1)$ as generating functional. This function $S(x_2, t_2 | x_1, t_1)$ is the classical action evaluated along the classical path that begins at the point x_1 at time t_1 and ends at the point x_2 at time t_2. In his 1932 article Dirac wrote that $\langle x_2, t_2 | x_1, t_1 \rangle$ *corresponds to* $\exp \frac{i}{\hbar} S(x_2, t_2 | x_1, t_1)$. He used the words "corresponds to" to express that at the quantum level there were presumably corrections so that the exact result for $\langle x_2, t_2 | x_1, t_1 \rangle$ was different from $\exp \frac{i}{\hbar} S(x_2, t_2 | x_1, t_1)$. Although Dirac wrote these ideas down in 1932, they were largely ignored until Feynman started his studies on the role of the action in quantum mechanics.

End 1930's Feynman started studying how to formulate an approach to quantum mechanics based on the action. The reason he tackled this problem was that with Wheeler he had developed a theory of quantum electrodynamics from which the electromagnetic field had been eliminated. In this way they hoped to avoid the problems of the self-acceleration and infinite self-energy of an electron which are due to the interactions of an electron with the electromagnetic field and which Lienard, Wiechert, Abraham and Lorentz had in vain tried to solve. The resulting "Wheeler-Feynman theory" arrived at a description of the interactions between two electrons in which no reference was made to any field. It is a so-called action-at-a-distance theory, in which it took a finite nonzero time to travel the distance from one electron to the others. These theories were nonlocal in space and time. (In modern terminology one might say that the fields A_μ had been integrated out from the path integral by completing squares). Fokker and Tetrode had found a classical action for such a system, given by

$$\begin{aligned} S \;=\; & -\sum_i m_{(i)} \int \Big(\frac{dx^\mu_{(i)}}{ds_{(i)}} \frac{dx^\nu_{(i)}}{ds_{(i)}} \eta_{\mu\nu} \Big)^{\frac{1}{2}} ds_{(i)} \qquad (1.3.2) \\ & - \frac{1}{2} \sum_{i \neq j} e_{(i)} e_{(j)} \int\!\!\int \delta[(x^\mu_{(i)} - x^\mu_{(j)})^2] \frac{dx^\rho_{(i)}}{ds_{(i)}} \frac{dx^\sigma_{(i)}}{ds_{(i)}} \eta_{\rho\sigma} ds_{(i)} ds_{(j)} \;. \end{aligned}$$

Here the sum over (i) denotes a sum over different electrons. So, two electrons only interact when the relativistic four-distance vanishes, and by taking $i \neq j$ in the second sum, the problem of infinite selfenergy was eliminated. Wheeler and Feynman set out to quantize this system, but Feynman noticed that a Hamiltonian treatment was hopelessly complicated[10]. Thus Feynman was looking for an approach to quantum mechanics in which he could avoid the Hamiltonian. The natural object to use was the action.

At this moment in time, an interesting discussion helped him further. A physicist from Europe, Herbert Jehle, who was visiting Princeton, mentioned to Feynman (spring 1941) that Dirac had already in 1932 studied the problem how to use the action in quantum mechanics. Together they looked up Dirac's paper, and of course Feynman was puzzled by the ambiguous phrase "corresponds to" in it. He asked Jehle whether Dirac meant that they were equal or not. Jehle did not know, and Feynman decided to take a very simple example and to check. He considered the case $t_2 - t_1 = \epsilon$ very small, and wrote the time evolution of the Schrödinger wave function $\psi(x,t)$ as follows

$$\psi(x, t+\epsilon) = \frac{1}{\mathcal{N}} \int \exp\Big(\frac{i}{\hbar} \epsilon L(x, t+\epsilon; y, t)\Big) \psi(y,t) dy \ . \tag{1.3.3}$$

With $L = \frac{1}{2} m \dot{x}^2 - V(x)$ one obtains, as we now know very well, the Schrödinger equation, provided the constant $\mathcal{N}$ is given by

$$\mathcal{N} = \Big(\frac{2\pi i \hbar \epsilon}{m}\Big)^{\frac{1}{2}} \tag{1.3.4}$$

(nowadays we call $\frac{dy}{\mathcal{N}}$ the Feynman measure). Thus, as Dirac correctly guessed, $\langle x_2, t_2 | x_1, t_1 \rangle$ was analogous to $\exp \frac{i}{\hbar} \epsilon L$ for small $\epsilon = t_2 - t_1$; however they were not equal but rather proportional.

There is an amusing continuation of this story [36]. In the fall of 1946 Dirac was giving a lecture at Princeton, and Feynman was asked to introduce Dirac and comment on his lecture afterwards. Feynman decided to simplify Dirac's rather technical lecture for the benefit of the audience, but senior physicists such as Bohr and Weisskopf did not appreciate much this watering down of the work of the great Dirac by the young and relatively unknown Feynman. Afterwards people were discussing Dirac's

[10] By expanding expressions such as $\frac{1}{\partial_x^2 + \partial_t^2 - m^2}$ in a power series in ∂_t, and using Ostrogradsky's approach to a canonical formulation of systems with higher order ∂_t derivatives, one can give a Hamiltonian treatment, but one must introduce infinitely many auxiliary fields $B, C, ...$ of the form $\partial_t A = B, \partial_t B = C, ...$ All these auxiliary fields are, of course, equivalent to the oscillators of the original electromagnetic field.

lecture and Feynman who (in his own words) felt a bit let down happened to look out of the window and saw Dirac laying on his back on a lawn and looking at the sky. So Feynman went outside and sitting down near Dirac asked him whether he could ask him a question concerning his 1932 paper. Dirac consented. Feynman said "Did you know that the two functions do not just 'correspond to' each other, but are actually proportional?" Dirac said "Oh, that's interesting". And that was the whole reaction that Feynman got from Dirac.

Feynman then asked himself how to treat the case that $t_2 - t_1$ is not small. This Dirac had already discussed in his paper: by inserting complete set of x-eigenstates one obtains

$$\begin{aligned}\langle x_{\rm f}, t_{\rm f}|x_{\rm i}, t_{\rm i}\rangle \;=\; &\int \langle x_{\rm f}, t_{\rm f}|x_{N-1}, t_{N-1}\rangle\langle x_{N-1}, t_{N-1}|x_{N-2}, t_{N-2}\rangle \ldots \\ &\ldots \langle x_1, t_1|x_{\rm i}, t_{\rm i}\rangle dx_{N-1}\ldots dx_1 \; . \end{aligned} \tag{1.3.5}$$

Taking $t_j - t_{j-1}$ small and using that for small $t_j - t_{j-1}$ one can use $\mathcal{N}^{-1} \exp \frac{i}{\hbar}(t_j - t_{j-1})L$ for the transformation function, Feynman arrived at

$$\langle x_{\rm f}, t_{\rm f}|x_{\rm i}, t_{\rm i}\rangle \;=\; \int \exp\Big[\frac{i}{\hbar}\sum_{j=0}^{N-1}(t_{j+1} - t_j)L(x_{j+1}, t_{j+1}; x_j, t_j)\Big]\frac{dx_{N-1}...dx_1}{\mathcal{N}^N} \; . \tag{1.3.6}$$

At this point Feynman recognized that one obtains the action in the exponent and that by first summing over j and then integrating over x one is summing over paths. Hence $\langle x_{\rm f}, t_{\rm f}|x_{\rm i}, t_{\rm i}\rangle$ is equal to a sum over all paths of $\exp \frac{i}{\hbar}S$ with each path beginning at $x_{\rm i}$, $t_{\rm i}$ and ending at $x_{\rm f}$, $t_{\rm f}$.

Of course one of these paths is the classical path, but by summing over all other paths (arbitrary paths not satisfying the classical equation of motion) quantum mechanical corrections are introduced. The tremendous result was that all quantum corrections were included if one summed the action over all paths. Dirac had entertained the possibility that in addition to summing over paths one would have to replace the action S by a generalization which contained terms with higher powers in $\hbar$.

Reviewing this development more than half a century later, when path integrals have largely superseded operators methods and the Schrödinger equation for relativistic field theories, one notices how close Dirac came to the solution of using the action in quantum mechanics, and how different Feynman's approach was to solving the problem. Dirac anticipated that the action had to play a role, and by inserting complete set of states he did obtain (1.3.6). However, he did not pursue the observation that the sum of terms in (1.3.6) is the action because he anticipated for large $t_2 - t_1$ a more complicated expression. Feynman, on the other hand, started by working out a few simple examples, curious to see whether Dirac was correct that the complete result would need a more complicated expression than the action, and found in this way that the truth lies in between: Dirac's transformation functions (Feynman's transition kernel K) is equal to the exponent of the action up to a constant. This constant diverges as ϵ tends to zero, but for $N \to \infty$ the result for K (and other quantities) is finite.

Feynman initially believed that in his path integral approach to quantum mechanics ordering ambiguities of the p and x operators of the operator approach would be absent (as he wrote in his PhD thesis of may 1942). However, later in his fundamental 1948 paper in Review of Modern Physics [29], he realized that the same ambiguities would be present. For our work the existence of these ambiguities is very important and we shall discuss in great detail how to remove them. Schrödinger [37] had already noticed that ordering ambiguities occur if one tries to promote a classical function $F(p,x)$ to an operator $\hat{F}(\hat{p},\hat{x})$. Furthermore, one can in principle add further terms linear and of higher order in $\hbar$ to such operators $\hat{F}$. These are further ambiguities which have to be fixed before one can make definite predictions.

Feynman evaluated the kernels $K(x_{j+1},t_{j+1}|x_j,t_j)$ for small $t_{j+1}-t_j$ by inserting complete sets of *momentum eigenstates* $|p_j\rangle$ in addition to position eigenstates $|x_j\rangle$. In this way he constructed *phase space path integrals*. We shall follow the same approach for the nonlinear sigma models we consider. It has been claimed in [11] that "... phase space path integrals have more troubles than merely missing details. On this basis they should have been left out [from the book] ....". We have come to a different conclusion: they are well-defined and can be used to *derive* the usual configuration space path integrals from the operatorial approach by adding integrations over intermediate momenta. A continuous source of confusion is the notation $dx(t)\ dp(t)$ for these phase space path integrals. Many authors, who attribute more meaning to the symbol than $dx_1 \dots dx_{N-1} dp_1 \dots dp_N$, assume that this measure is invariant under canonical transformations, and apply the powerful methods developed in classical mechanics for the Liouville measure. However, the measure $dx(t)\ dp(t)$ in path integrals is not invariant under canonical transformations of the x's and the p's because there is one more p integration then x integrations in $\prod dx_j \prod dp_j$.

Another source of confusion for phase space path integrals arise if one tries to interpret them as integrals over paths around classical solutions in phase space. Consider Feynman's expression

$$\begin{aligned} K(x_j,t_j|x_{j-1},t_{j-1}) &= \langle x_j| e^{-\frac{i}{\hbar}\hat{H}(t_j-t_{j-1})}|x_{j-1}\rangle \\ &= \int \frac{dp_j}{2\pi} \langle x_j|e^{-\frac{i}{\hbar}\hat{H}(t_j-t_{j-1})}|p_j\rangle\langle p_j|x_{j-1}\rangle\,. \quad (1.3.7) \end{aligned}$$

For $\langle x_j | e^{-\frac{i}{\hbar}\hat{H}(t_j - t_{j-1})} | x_{j-1}\rangle$ one can substitute $\exp \frac{i}{\hbar} S(x_j, t_j | x_{j-1}, t_{j-1})$ where in S one uses the classical path from x_j, t_j to x_{j-1}, t_{j-1}. In a similar way some authors have tried to give meaning to $\langle x_j | e^{-\frac{i}{\hbar}\hat{H}(t_j - t_{j-1})} | p_j\rangle$ by considering a classical path in phase space. For example in [11] the author considers two possibilities: (i) a classical path from x_{j-1}, t_{j-1} to x_j, t_j during which p is a constant and equal to p_j, or (ii) a classical path from x_{j-1} at t_{j-1} to $p = p_{j+\frac{1}{2}}$ at the midpoint $t_{j-\frac{1}{2}} \equiv t_{j-1} + \frac{1}{2}(t_j - t_{j-1})$, and then another classical path in phase space from $p = p_{j+\frac{1}{2}}$ at the midpoint $t = t_{j+\frac{1}{2}}$ to x_{j+1} at t_{j+1}. The first interpretation is inconsistent because once x_{j-1} and x_j are specified, one cannot specify also p for solutions of the Hamiltonian equations of motion. The second interpretation is not inconsistent, but impractical. On the interval $t_{j-\frac{1}{2}} \leq t \leq t_{j-1}$ one may use $p_{j-1}(x_{j-\frac{1}{2}} - x_{j-1}) - (\frac{1}{2m}p_{j-1}^2 - V(x_j))$ in the exponent for the transition element where the symbol $x_{j-\frac{1}{2}}$ denotes then the value of $x(t)$ for this classical trajectory at $t = t_{j-\frac{1}{2}}$. On the next interval $t_j \leq t \leq t_{j-\frac{1}{2}}$ one may use $p_{j-\frac{1}{2}}(x'_j - x_{j-\frac{1}{2}}) - (\frac{1}{2m}p_{j-\frac{1}{2}}^2 - V(x_{j-1}))$ where now x'_j is the value of $x(t)$ for this classical solution with $x = x_{j-\frac{1}{2}}$ at $t = t_{j-\frac{1}{2}}$ and $p = p_{j-\frac{1}{2}}$ at $t = t_{j-\frac{1}{2}}$. Thus one must solve equations of motion, and put the result into the path integral. We shall not try to interpret the transition elements in phase space in terms of classical paths, but only do what we are supposed to do: integrate over the p_j and x_j.

Yet another source of confusion has to do with path integrals over fermions for which one needs Grassmann numbers and Berezin integration [38]

$$\int d\theta = 0 \ , \qquad \int d\theta\theta = 1 \ . \tag{1.3.8}$$

Some authors claim that the notion of anticommuting classical fields makes no sense, and that only quantized fermionic fields are consistent. However, the notion of Grassmann variables is completely consistent if one uses it only at the intermediate stages to construct for example coherent states: all one does is making use of mathematical identities. We begin with fermionic harmonic oscillator operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ and construct coherent bra and kets states $|\eta\rangle$ and $\langle\bar{\eta}|$ in Hilbert space. In applications one takes traces over these coherent states using Berezin integration rules for the integrations over η and $\bar{\eta}$. One ends up with physical results which are independent from Grassmann variables, and since all intermediary steps are mathematical identities, defined by Berezin in [38, 9], there are at no point conceptual problems in the treatment of path integrals for fermions.

2
Time slicing

In this chapter we discuss quantum mechanical path integrals defined by time slicing. Our starting point is an arbitrary but fixed Hamiltonian operator $\hat{H}$. We obtain the Feynman rules for nonlinear sigma models, first for bosonic point particles $x^i(t)$ with curved indices $i = 1, .., n$ and then for fermionic point particles $\psi^a(t)$ with flat indices $a = 1, .., n$. In the bosonic case we first discuss in detail the configuration space path integrals, and then return to the corresponding phase space integrals. In the fermionic case we use coherent states to define bras and kets, and we discuss the proper treatment of Majorana fermions, both in the operatorial and in the path integral approach. Finally we compute directly the transition element $\langle z|e^{-(\beta/\hbar)\hat{H}}|y\rangle$ to order β (two-loop order) using operator methods, and compare with the results of a similar calculation based on the perturbative evaluation of the path integral with time slicing regularization. Complete agreement is found. These results were obtained in [39, 40, 41]. Additional discussions are found in [42, 43, 44, 45, 46, 47].

The quantum action, i.e. the action to be used in the path integral, is obtained from the quantum Hamiltonian by mathematical identities, and the quantum Hamiltonian is fixed by the quantum field theory whose anomalies we study in part II of the book. Hence, there is no ambiguity in the quantum action. It contains local finite counterterms of order $\hbar^2$. They were first obtained by Gervais and Jevicki [48]. Even earlier Schwinger [49] and later Christ and T.D. Lee [50] found by the same method that four-dimensional Yang–Mills theory in the Coulomb gauge has such counterterms.

2.1 Configuration space path integrals for bosons from time slicing

Consider a quantum Hamiltonian $\hat{H}(\hat{x},\hat{p})$ with a definite ordering of the operators $\hat{p}_i$ and $\hat{x}^j$. We will mostly focus on the operator

$$\hat{H}(\hat{x},\hat{p}) = \frac{1}{2} g^{-1/4} \hat{p}_i g^{ij} g^{1/2} \hat{p}_j g^{-1/4} \tag{2.1.1}$$

where $g = \det g_{ij}(x)$ and we omitted hats on $\hat{x}$ in the metric for notational simplicity. This Hamiltonian is Einstein invariant (it commutes with the generator of general coordinate transformations, see section 2.5, in particular (2.5.9)) but our methods apply also to other Hamiltonians, for example $\hat{H} = \frac{1}{2}\hat{p}_i g^{ij}\hat{p}_j$ or the nonhermitian operator $\frac{1}{2}g^{ij}\hat{p}_i\hat{p}_j$. The reason we focus on (2.1.1) is that it leads to the regulators which we use in the second part of this book to compute anomalies by quantum mechanical methods.

The essential object from which all other quantities can be calculated, is the transition element (also sometime scalled transition amplitude)

$$T(z,y;\beta) = \langle z| e^{-(\beta/\hbar)\hat{H}} |y\rangle \tag{2.1.2}$$

where $|y\rangle$ and $\langle z|$ are eigenstates of the position operator $\hat{x}^i$

$$\hat{x}^i|y\rangle = y^i|y\rangle, \quad \langle z|\hat{x}^i = \langle z| z^i \tag{2.1.3}$$

with y^i and z^i real numbers. We normalize the x and p eigenstates as follows

$$\int |x\rangle g(x)^{1/2} \langle x|\, d^n x = I \quad \rightarrow \quad \langle x|y\rangle = g(x)^{-1/2}\delta^{(n)}(x-y) \tag{2.1.4}$$

where I is the identity operator and

$$\int |p\rangle\langle p|\, d^n p = I \quad \rightarrow \quad \langle p|p'\rangle = \delta^{(n)}(p-p')\ . \tag{2.1.5}$$

The delta function $\delta^{(n)}(x-y)$ is defined by $\int \delta^{(n)}(x-y) f(y) d^n y = f(x)$. Since $\hat{x}^i$ and $\hat{p}_i$ are diagonal and real on these complete sets of orthonormal states, they are both hermitian. The Hamiltonian in (2.1.1) is then also hermitian, but we could in principle also allow nonhermitian Hamiltonians. However, we stress that $\hat{x}^i$ and $\hat{p}_i$ are always hermitian.

We have chosen the normalization in (2.1.4) in order that $T(z,y;\beta)$ will be a bi-scalar (a scalar under general coordinate transformations of z and y separately). There is no need to choose the normalization in (2.1.4) and one could also use for example $\int |x\rangle\langle x|\, d^n x = I$. However, (2.1.4) leads to simpler formulas. For example the inner product of two states $\langle\varphi|\psi\rangle$

takes the familiar form $\int \sqrt{g(x)}\varphi^*(x)\psi(x)\, d^n x$. As a consequence wave functions $\psi(x) = \langle x|\psi\rangle$ are scalar functions under a change of coordinates.

The inner product between x- and p-eigenstates yields plane waves with an extra factor $g^{-1/4}$

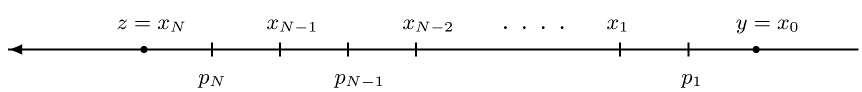
$$\langle x|p\rangle = \frac{e^{(i/\hbar)p_j x^j}}{(2\pi\hbar)^{n/2} g^{1/4}(x)} \ . \tag{2.1.6}$$

As a check note that $\int \langle p|x\rangle g(x)^{1/2} \langle x|p'\rangle\, d^n x = \delta^n(p - p')$, in agreement with the completeness relations.

We now insert N complete sets of momentum eigenstates and $N-1$ complete sets of position eigenstates into the transition element. Defining $\beta = N\epsilon$ we obtain

$$\begin{aligned}
&T(z, y; \beta) = \langle z|\left(e^{-(\epsilon/\hbar)\hat{H}}\right)^N |y\rangle \\
&= \langle z|e^{-(\epsilon/\hbar)\hat{H}}|p_N\rangle \int d^n p_N\, \langle p_N|x_{N-1}\rangle \int g(x_{N-1})^{1/2}\, d^n x_{N-1} \\
&\quad \langle x_{N-1}|e^{-(\epsilon/\hbar)\hat{H}}|p_{N-1}\rangle \int d^n p_{N-1}\, \langle p_{N-1}|x_{N-2}\rangle \int g(x_{N-2})^{1/2}\, d^n x_{N-2} \\
&\qquad \cdots\cdots \\
&\quad \langle x_1|e^{-(\epsilon/\hbar)\hat{H}}|p_1\rangle \int d^n p_1\, \langle p_1|y\rangle \ .
\end{aligned} \tag{2.1.7}$$

We have written the integration symbols between the bras and kets to which they belong in order to simplify the notation. It is natural to denote z by x_N and y by x_0. The order in which x_i and p_j appear can be indicated as follows



Although the p's occur between x's, we do not imply that a kind of mid-point rule holds. Only the ordering of the p's and x's matters.

We now *rewrite* the operators $\exp(-\frac{\epsilon}{\hbar}\hat{H}(\hat{x}, \hat{p}))$ in Weyl-ordered form. This means that after the rewriting this operator is symmetric in all $\hat{x}$ and $\hat{p}$ it contains. Weyl ordering is discussed in appendix B. As an example of such a rewriting consider the operator $\hat{x}\hat{p}$. We rewrite it as $\frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}) + \frac{1}{2}(\hat{x}\hat{p} - \hat{p}\hat{x}) = \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}) + \frac{1}{2}i\hbar$, and we denote the result by $(\hat{x}\hat{p})_W$. So in this example $\hat{x}\hat{p} = (\hat{x}\hat{p})_W + \frac{1}{2}i\hbar$. In more general cases, one has the formula $O(\hat{x}, \hat{p}) = O(\hat{x}, \hat{p})_W +$ *more*, where "*more*" may contain further operators which depends on $\hat{x}$ and $\hat{p}$, and which we again rewrite in a symmetrical way.

The reason we rewrite operators in Weyl-ordered form is that Weyl ordering leads to the midpoint rule in the following way

$$\begin{aligned}&\int \langle x_k| \left(e^{-(\epsilon/\hbar)\hat{H}}\right)_W |p_k\rangle\langle p_k|x_{k-1}\rangle \, d^n p_k \\ &\quad = \int \langle x_k|p_k\rangle \left(e^{-(\epsilon/\hbar)H(\frac{1}{2}(x_k+x_{k-1}),\, p_k)}\right)_W \langle p_k|x_{k-1}\rangle \, d^n p_k \ . \qquad (2.1.8)\end{aligned}$$

This formula is also proven in appendix B. The meaning of this equation is that one may extract the Weyl ordered operator from the matrix element and replace it by a function by replacing each $\hat{p}_i$ by $p_{k,i}$ and each $\hat{x}^i$ by $\frac{1}{2}(x_k^i + x_{k-1}^i)$. A more precise notation for the function in the second line would have been

$$\left(e^{-(\epsilon/\hbar)\hat{H}}\right)_W \left(\hat{x} \to \frac{1}{2}(x_k + x_{k-1}),\, \hat{p} \to p_k\right) \ . \qquad (2.1.9)$$

We should first Weyl order the whole operator $\exp(-\frac{\epsilon}{\hbar}\hat{H})$, and not only $\hat{H}$, and then replace $\hat{x}^i$ by $\frac{1}{2}(x_k^i + x_{k-1}^i)$ and $\hat{p}_i$ by $p_{k,i}$. For simplicity we use the notation in (2.1.8).

In general, we cannot write down a closed expression for $(\exp(-\frac{\epsilon}{\hbar}\hat{H}))_W$. However, for path integrals one may replace $(\exp(-\frac{\epsilon}{\hbar}\hat{H}))_W$ by $\exp(-\frac{\epsilon}{\hbar}\hat{H}_W)$, because the difference cancels in the path integral, as we discuss below (2.1.15). So it is sufficient to Weyl order the Hamiltonian itself. The result for the particular Hamiltonian in (2.1.1) reads as follows

$$\hat{H}(\hat{x},\hat{p}) = \left(\frac{1}{2}g^{ij}\hat{p}_i\hat{p}_j\right)_W + \frac{\hbar^2}{8}(R + g^{ij}\Gamma^k_{il}\Gamma^l_{jk}) \ . \qquad (2.1.10)$$

The definition of the scalar curvature R is given in appendix A and the Weyl ordered operator $\left(\frac{1}{2}g^{ij}\hat{p}_i\hat{p}_j\right)_W$ is according to appendix B given by

$$\left(\frac{1}{2}g^{ij}\hat{p}_i\hat{p}_j\right)_W = \frac{1}{8}g^{ij}\hat{p}_i\hat{p}_j + \frac{1}{4}\hat{p}_i g^{ij}\hat{p}_j + \frac{1}{8}\hat{p}_i\hat{p}_j g^{ij} \ . \qquad (2.1.11)$$

Note that (2.1.10) is an identity: the same operator $\hat{H}$ of (2.1.1) appears on both sides, but subsequently the right hand side is converted into a function according to (2.1.8).

In the discretized path integral we then encounter the functions

$$\exp\left\{-\frac{\epsilon}{\hbar}\left[\frac{1}{2}g^{ij}(\bar{x})p_{k,i}p_{k,j} + \frac{\hbar^2}{8}\left(R(\bar{x}) + g^{ij}(\bar{x})\Gamma^m_{il}(\bar{x})\Gamma^l_{jm}(\bar{x})\right)\right]\right\} \qquad (2.1.12)$$

where $\bar{x} = \frac{1}{2}(x_k + x_{k-1})$. Note that the term with the scalar curvature R and its coefficient $\frac{1}{8}$ as well as the $\Gamma\Gamma$ term are a mathematical consequence of rewriting the particular Hamiltonian in (2.1.1). If we would have started from another Hamiltonian (in particular another operator ordering) we would have found different coefficients in (2.1.12). We choose this Hamiltonian with this precise ordering since in the applications to anomalies we wish to preserve general coordinate invariance.

Up to this point we have presented the standard approach to path integrals. In the rest of this chapter we evaluate the transition element without making any of the usual approximations. We must do this because for nonlinear sigma models we need propagators of the quantum fields $q(t)$ with double time derivatives, $\langle \dot{q}(t_1)\dot{q}(t_2)\rangle$. As a result, in the evaluation of the transition element in terms of Feynman graphs we shall encounter expressions which contain products of distributions, for example $\delta(t_1 - t_2)\theta(t_1 - t_2)\theta(t_2 - t_1)$, and equal-time contractions. Such expressions are ambiguous, one can get different answers depending on how one regulates the distributions. We could at this point introduce further physical requirements which fix the ambiguities. This procedure has been used before for linear sigma models, namely if one tries to define the path integral for $L = \frac{1}{2}(\dot{x}^i)^2 + \dot{x}^i A_i(x) + V(x)$. It makes a difference whether one discretizes to $(x_k - x_{k-1})A_i(x_k)$ or $(x_k - x_{k-1})A_i(x_{k-1})$ [30, 11]. One way to fix this ambiguity is to require that the transition element be gauge-invariant, and one discovers that this is achieved by using the midpoint rule $(x_k - x_{k-1})A_i(\frac{1}{2}(x_k + x_{k-1}))$. One could also have started with a Hamiltonian operator $\hat{H}$ whose operator ordering of $\dot{x}^i A_i(x)$ is gauge-invariant, namely commutes with the operator $\hat{G}$ of gauge transformations. Using this particular $\hat{H}$, the time slicing method – with complete p and x eigenstates – produces automatically the midpoint rule, and thus gauge invariance. Similarly, by taking the particular $\hat{H}$ in (2.1.1) which is invariant under general coordinate transformations, all ambiguities are fixed, and the time slicing method – with complete p and x eigenstates – leads to well-defined and unambiguous expressions for the products of distributions. Taking at the very end the limit $\epsilon \to 0$, $N \to \infty$, $N\epsilon = \beta$, one derives the rules how to evaluate integrals over products of distributions in the continuum theory.

Since the following discussions in this chapter must be precise and therefore technical in order not to miss subtleties in the product of distributions, it may help the reader if we first give a short non-technical summary of the results to be obtained. Such a summary follows in the next three paragraphs.

We begin by integrating over the N momenta $p_1, .., p_N$. This leads to a product of N determinants $(\det g_{ij}(x_{k+1/2}))^{\frac{1}{2}}$ were $x_{k+1/2} = \frac{1}{2}(x_k + x_{k+1})$. We exponentiate these determinants using ghost fields $a_{k+1/2}, b_{k+1/2}$ and $c_{k+1/2}$. This yields discretized configuration path integrals for the transition element. We decompose x_k into background fields $x_{bg,k}$ and quantum fields q_k, and decompose the discretized action into a part $S^{(0)}$ quadratic in q_k, and an interaction part $S^{(int)}$. We require that the $x_{bg,k}$ satisfy the field equation of $S^{(0)}$ and the boundary conditions that x_{bg} be equal to z or y at the boundaries. Because the $x_{bg,k}$ satisfy the field equation of $S^{(0)}$ and not of the full action S, there will be terms in $S^{(int)}$ linear in q. We introduce external sources F and G which couple to $\frac{1}{2}(x_k + x_{k-1})$ and $(x_k - x_{k-1})$, and extract $S^{(int)}$ from the path integral, as usual in quantum field theory. Then we want to integrate over q but the action $S^{(0)}$ is not diagonal in q. In order to diagonalize $S^{(0)}$ we make an orthogonal change of integration variables with unit Jacobian. The actual integration over q can then be performed in closed form, but it requires a few relatively unknown identities for products involving sines and cosines. The final result is given in (2.1.48 – 2.1.50). By differentiation with respect to the sources F and G and similar external sources A, B, C for the ghosts, we find the discretized propagators in closed form. The results are given in (2.1.53) for $\langle \dot{q}\dot{q}\rangle$, in (2.1.69) for $\langle q\dot{q}\rangle$, in (2.1.75) for $\langle qq\rangle$, and in (2.1.78) and (2.1.79) for the ghosts. These result can also be written in the continuum limit, see (2.1.81), but if one computes diagrams in the continuum limit, one should use the rules which we **derive** from the discretized approach how to treat products of distributions.

For fermionic models with operators $\hat{\psi}^a$ and $\hat{\psi}^\dagger_a$ we begin by introducing bras $\langle\bar{\eta}|$ and kets $|\eta\rangle$ in terms of coherent states which depend on Grassmann variables $\bar{\eta}_a$ and η^a. We treat η and $\bar{\eta}$ as independent variables. This is the approach to be used for complex (Dirac) fermions, so for $N = 2$ models. We could equally well have defined $\bar{\eta}$ to be the complex conjugate of η because the only property we need is the integration over the Grassmann variables, and this integration is the same whether $\bar{\eta}$ is an independent variable or the complex conjugate of η. The transition element we wish to compute is $\langle\bar{\eta}|e^{-\frac{\beta}{\hbar}\hat{H}}|\eta\rangle$ and we insert again complete sets $|\chi_k\rangle\langle\bar{\chi}_k|$ of coherent states to arrive at a discretized path integral. After defining Weyl ordering for anticommuting operators $\hat{\psi}^a$ and $\hat{\psi}^\dagger_a$, we obtain again a midpoint rule for the variables χ^a_k. We decompose again the variables χ^a_k and $\bar{\chi}_{ka}$ into background parts ξ^a_k and $\bar{\xi}_{ka}$ and quantum parts ψ^a_k and $\bar{\psi}_{ka}$. We decompose S into a part $S^{(0)}$ that is quadratic in ψ^a_k and $\bar{\psi}_{ka}$, and the rest $S^{(int)}$. We require that ξ and $\bar{\xi}$ satisfy the field equation of $S^{(0)}$, and the boundary conditions that $\xi = \eta$ at the right and $\bar{\xi} = \bar{\eta}$ at the left. We treat also ξ and $\bar{\xi}$, ψ and $\bar{\psi}$, as independent Grassmann variables; again it makes no difference whether they are related by complex conjugation or not. We couple ψ^a_k and $\bar{\psi}_{ka}$ to external sources $\bar{K}_{ka}$ and K^a_k, complete squares and integrate over ψ^a_k and $\bar{\psi}_{ka}$. For the propagators of ψ and $\bar{\psi}$ we find the following exact discretized result

$$\langle \psi_k^a \bar{\psi}_{lb} \rangle = \begin{cases} \delta_a^b & \text{if } k \geq l \\ 0 & \text{if } k < l \end{cases} . \tag{2.1.13}$$

(Time ordering is always understood, so the case $k < l$ refers to $-\langle \bar{\psi}_{lb} \psi_k^a \rangle$). Due to the midpoint rule, we rather need the propagator for $\psi_{k+1/2}^a = \frac{1}{2}(\psi_k^a + \psi_{k-1}^a)$. It reads

$$\langle \psi_{k-1/2}^a \bar{\psi}_{lb} \rangle = \begin{cases} \delta_a^b & \text{if } k > l \\ \frac{1}{2}\delta_a^b & \text{if } k = l \\ 0 & \text{if } k < l \end{cases} . \tag{2.1.14}$$

It becomes $\theta(t - t')$ in the continuum limit, but $\theta(0)$ is now equal to $1/2$.

Finally we consider Majorana fermions $\hat{\psi}_1^a$. We add another set of **free** Majorana fermions $\hat{\psi}_2^a$, and define operator $\hat{\psi}^a = \frac{1}{\sqrt{2}}(\hat{\psi}_1^a + i\hat{\psi}_2^a)$ and $\hat{\psi}_a^\dagger = \frac{1}{\sqrt{2}}(\hat{\psi}_1^a - i\hat{\psi}_2^a)$. We can then apply the results for Dirac spinors. We consider again the transition element $\langle \bar{\eta} | e^{-\frac{\beta}{\hbar}\hat{H}} | \eta \rangle$. The Hamiltonian operator $\hat{H}$ depends on $\hat{\psi}$ and $\hat{\psi}^\dagger$, and after a Weyl ordering the midpoint rule yields the function $H(\bar{\chi}_{ka}, \chi_{k-1/2}^a)$ where $\chi_{k-1/2}^a = \frac{1}{2}(\chi_k^a + \chi_{k-1}^a)$. Because initially $\hat{H}$ depended only on ψ_1^a, the action in the final path integral contains H which depends on $(\chi_{k-1/2}^a + \bar{\chi}_k^a)/\sqrt{2}$ where $\chi_{k-1/2}^a = \eta^a + \psi_{k-1/2}^a$ and $\bar{\chi}_{ka} = \bar{\eta}_a + \bar{\psi}_{ka}$. The propagator for $\psi_{1k}^a \equiv (\psi_{k-\frac{1}{2}}^a + \bar{\psi}_{ka})/\sqrt{2}$ follows from the propagator $\langle \psi_k^a \bar{\psi}_{lb} \rangle$ for the Dirac spinors and reads

$$\langle \psi_{1k}^a \psi_{1l}^b \rangle = \frac{1}{2}\delta^{ab} \begin{cases} 1 & \text{if } k > l \\ 0 & \text{if } k = l \\ -1 & \text{if } k < l \end{cases} . \tag{2.1.15}$$

In the continuum limit this becomes $\langle \psi_1^a(t) \psi_1^b(t') \rangle = \frac{1}{2}\delta^{ab}[\theta(t-t') - \theta(t'-t)]$. With this propagator we can compute the transition element in a loop expansion, and the transition element will be used to compute anomalies. We now return to the detailed derivation of these results.

We present the proof that one may replace $(\exp(-\frac{\epsilon}{\hbar}H))_W$ by $\exp(-\frac{\epsilon}{\hbar}H_W)$ in the path integral. In general Weyl ordering and exponentiation do not commute, $(\exp(-\frac{\epsilon}{\hbar}H))_W \neq \exp(-\frac{\epsilon}{\hbar}H_W)$ and whereas H_W was easy to write down, a closed expression for $(\exp(-\frac{\epsilon}{\hbar}H))_W$ cannot be written down. One expects, however, that a suitable approximation of the kernels, containing only terms of order ϵ, suffices. It might seem that p is of order $\epsilon^{-1/2}$ due to the term $\exp(-\frac{1}{2}\epsilon p^2)$ in the action, see (2.1.12). Expansion of $\exp(-\frac{\epsilon}{\hbar}H_W)$ would contain terms of the form $\epsilon^s p^r f(x)$ for which $s \geq 2$ and such terms could still be of order ϵ if r is sufficiently large. We are now going to give an argument that p is actually of order unity, and therefore only the terms with one explicit ϵ need be retained. Hence, we may use as kernel $\exp(-\frac{\epsilon}{\hbar}H_W(\frac{1}{2}(x_k + x_{k-1}), p_k))$. In other words, the Trotter-like approximation

$$\begin{aligned}\langle x|\exp\Big(-\frac{\epsilon}{\hbar}\hat{H}\Big)|p\rangle &\simeq \langle x|\Big(1-\frac{\epsilon}{\hbar}\hat{H}\Big)|p\rangle = \Big(1-\frac{\epsilon}{\hbar}H\Big)\langle x|p\rangle \\ &\simeq \exp\Big(-\frac{\epsilon}{\hbar}H\Big)\langle x|p\rangle \end{aligned} \tag{2.1.16}$$

is still correct if used inside (2.1.8), but we repeat that H is not simply $\langle x|\hat{H}|p\rangle$ as in the usual models with $H = T(p)+V(x)$, but rather it equals H_W at the midpoints.

To prove this claim, we note that the kernels are exactly equal to

$$\begin{aligned}&\int d^n p_k\, e^{\frac{i}{\hbar}\vec{p}\cdot\vec{\Delta}x_{k-1/2}}\left(e^{-\frac{\epsilon}{\hbar}H(\bar{x}_{k-1/2},p_k)}\right)_W \\ &\Delta x_{k-1/2} \equiv x_k - x_{k-1} \\ &\bar{x}_{k-1/2} \equiv \frac{1}{2}(x_k + x_{k-1})\ .\end{aligned} \tag{2.1.17}$$

The difference between $(\exp(-\frac{\epsilon}{\hbar}H))_W$ and $\exp(-\frac{\epsilon}{\hbar}H_W)$ consists of two kinds of terms

(i) terms without a p. These are certainly of higher order in ϵ and can be omitted.

(ii) terms with at least one p.

In order to evaluate (2.1.17) one has to proceed as follows, as will be discussed in detail below. One extracts the interaction part of H from the path integral, while the terms quadratic in p and x yields the propagators. One then constructs Feynman graphs with $H^{(int)}(p,\bar{x})$ as vertices and phase-space propagators for p and $\bar{x}$. The crucial observation is now that the phase space propagators $\langle p_{k,i}p_{l,j}\rangle$ and $\langle p_{k,i}\bar{x}^j_{l-1/2}\rangle$ are both of order unity, and not of order ϵ^{-1} and $\epsilon^{-1/2}$, respectively[1]. An explicit proof is given later when we construct the discretized propagators for $\langle pp\rangle$ and $\langle px\rangle$, see eq. (2.2.6). However, already at this point one might note that the pp propagator is not only determined by the term $-\frac{\epsilon}{2}gpp$ contained in H but also by $ip\Delta x$. Completing squares, it is $p' = (p - i\Delta x/\epsilon)$ which is of order $\epsilon^{-1/2}$. In the pp propagator the singularities of the $p'p'$ and $\Delta x\Delta x$ propagators cancel each other. (The origin of the more singular nature of the $\dot{x}\dot{x}$ propagator can be understood from canonical formalism: the xx propagator contains a time-ordering step function $\theta(t-t')$, and differentiation yields a $\delta(t-t')$). As a consequence, the pp and pq

[1] We shall actually decompose $x(t) = x_{bg}(t) + q(t)$, and then we obtain propagators $p_k\bar{q}_{l-1/2}$ instead of $p_k\bar{x}_{l-1/2}$.

propagators are of order one, and this proves the Trotter formula also for nonlinear sigma models. (Already for linear sigma models with $H = T + V$ a completely rigorous proof of the Trotter formula uses Banach spaces [11], so for our nonlinear sigma models a completely rigorous proof is probably very complicated. However, we have identified the essential reason why p can be treated as being of order unity, and this is enough to justify (2.1.16)).

Using Weyl ordering and the midpoint rule to replace the operators $\exp(-\frac{\epsilon}{\hbar}\hat{H})$ by functions, we substitute the value of the various inner products. We arrive at

$$T_N(z, y; \beta) = \int \cdots \int \frac{e^{[(i/\hbar)p_N \cdot (z - x_{N-1}) - \frac{\epsilon}{\hbar} H_W(\frac{1}{2}(z + x_{N-1}), p_N)]}}{(2\pi\hbar)^n g(z)^{1/4} g(x_{N-1})^{1/4}} g(x_{N-1})^{1/2}$$

$$\frac{e^{[(i/\hbar)p_{N-1} \cdot (x_{N-1} - x_{N-2}) - \frac{\epsilon}{\hbar} H_W(\frac{1}{2}(x_{N-1} + x_{N-2}), p_{N-1})]}}{(2\pi\hbar)^n g(x_{N-1})^{1/4} g(x_{N-2})^{1/4}} g(x_{N-2})^{1/2}$$

$$\cdots\cdots$$

$$\frac{e^{[(i/\hbar)p_1 \cdot (x_1 - y) - \frac{\epsilon}{\hbar} H_W(\frac{1}{2}(x_1 + y), p_1)]}}{(2\pi\hbar)^n g(x_1)^{1/4} g(y)^{1/4}} \left(\prod_{j=1}^{n} dp_1^j \ldots dp_N^j dx_1^j \ldots dx_{N-1}^j \right) . \tag{2.1.18}$$

We note that all factors g cancel except an overall factor

$$[g(z)g(y)]^{-1/4} \ . \tag{2.1.19}$$

Furthermore we find in the exponents either coordinate differences $x_k - x_{k-1}$ or coordinate averages $\frac{1}{2}(x_k + x_{k-1})$, but the integration measure is $\prod_{j=1}^n dx_{N-1}^j \ldots dx_1^j$. In the continuum limit the exponent becomes of the form $\frac{1}{\hbar}\int (ip\dot{x} - H(x, p))dt$, but we shall not yet take the continuum limit.

We shall now go from phase space to configuration space by integrating over the momenta. We therefore assume that the terms in the Hamiltonian contain at most two momenta. Without loss of generality we put

$$H(\bar{x}, p) = \frac{1}{2} g^{ij}(\bar{x}) p_i p_j + V(\bar{x}) \ . \tag{2.1.20}$$

(We can also allow terms linear in p, but terms quartic or cubic in p we shall not consider). We find N Gaussian integrals (for $k = 1, N$)

$$\int \exp\left[-\frac{\epsilon}{2\hbar} g^{ij} \left(\frac{x_k + x_{k-1}}{2} \right) p_{k,i} p_{k,j} + \frac{i}{\hbar} p_{k,j} (x_k^j - x_{k-1}^j) \right] \prod_{j=1}^{n} dp_{k,j} \tag{2.1.21}$$

which yield N determinants upon completing squares

$$\left(\frac{2\pi\hbar}{\epsilon} \right)^{\frac{1}{2}n} [\det \ g^{ij}(\bar{x}_{k-1/2})]^{-1/2} \tag{2.1.22}$$

where we recall the notation

$$\bar{x}^i_{k-\frac{1}{2}} \equiv \frac{1}{2}(x^i_k + x^i_{k-1}) \ . \qquad (2.1.23)$$

The discretized configuration space path integral then becomes

$$T_N(z,y;\beta) = [g(z)g(y)]^{-1/4}(2\pi\hbar)^{-nN}\int d^n x_{N-1}...d^n x_1$$
$$\left[\left(\frac{2\pi\hbar}{\epsilon}\right)^{\frac{nN}{2}}\prod_{k=1}^{N}\det g_{ij}(\bar{x}_{k-1/2})^{1/2}\right]$$
$$\exp\left\{-\sum_{k=1}^{N}\frac{1}{2\hbar\epsilon}g_{ij}(\bar{x}_{k-\frac{1}{2}})\left(x^i_k - x^i_{k-1}\right)\left(x^j_k - x^j_{k-1}\right) - \frac{\epsilon}{\hbar}V(\bar{x}_{k-\frac{1}{2}})\right\} \ . \qquad (2.1.24)$$

We recall the definitions $x_N = z$ and $x_0 = y$.

Before we introduce external sources to compute discretized propagators, we remove the factors $\det g^{1/2}(\bar{x})$ from the measure by exponentiating them with new ghost fields. If there would have been factors $g^{-1/2}(\bar{x})$ instead of $g^{1/2}(\bar{x})$ we could replace them by $\int da \exp a^i g_{ij} a^j$ with commuting ghosts. However, we have $g^{1/2}$ instead of $g^{-1/2}$. By writing this as $g^{1/2} = g^{-1/2}g$, we can still exponentiate if *we introduce two anticommuting real ghosts b^i and c^i and one commuting real ghost a^i.*[2] Then the Berezin integral over b and c yields $\det g_{ij}$ and the ordinary Gaussian integral over a yields $(\det g_{ij})^{-1/2}$. Altogether one finds the following result

$$\left[\det g_{ij}(\bar{x}_{k-1/2})\right]^{1/2} = \alpha\int\left(\prod_{j=1}^{n} da^j_{k-1/2}db^j_{k-1/2}dc^j_{k-1/2}\right)$$
$$\exp\left[-\frac{\epsilon}{2\beta^2\hbar}g_{ij}(\bar{x}_{k-1/2})\left(b^i_{k-1/2}c^j_{k-1/2} + a^i_{k-1/2}a^j_{k-1/2}\right)\right] \qquad (2.1.25)$$

We define the normalization constant α such that the integral precisely yields $g^{1/2}(\bar{x}_{k-1/2})$; since we shall later do the integral over a, b, c, see (2.1.41), the normalization constant α will cancel, and for that reason we shall not bother to determine its value. The reason we have inserted the coefficients $-\epsilon(2\beta^2\hbar)^{-1}$ is that we obtain then in the continuum limit the same normalization for the ghost action as for the nonghost part, see (2.1.80). (The factor $\frac{\epsilon}{\beta}$ becomes $d\tau$ and the overall factor becomes $\frac{1}{2\beta\hbar}$).

[2] One might think that one could simply use one real anticommuting ghost α^i to obtain a result like $\int \prod_{j=1}^n d\alpha^j e^{-\alpha^k g_{kl}\alpha^l} \sim (\det g_{kl})^{+1/2}$. However, since $\alpha^k g_{kl}\alpha^l$ vanishes for symmetric g_{kl} and anticommuting α^k, one must use the slightly more complicated approach with a, b and c ghosts.

We have given the ghosts a^i, b^i and c^i which belong to $g(\bar{x}_{k-1/2})$ the subscripts $k-1/2$. This brings out clearly that the ghosts are defined by integrating out the momenta which were located between the coordinates. Similarly, we could have written the momenta as $p^j_{k-1/2}$ instead of p^j_k to indicate that they occur between x^j_k and x^j_{k-1}.

To proceed, we decompose the action for the x_k and the ghosts into a free and an interacting part

$$S = S^{(0)} + S^{(int)} \ . \tag{2.1.26}$$

The results should not depend on how one makes this split, but for practical purposes we take $S^{(0)}$ as simple as possible. Since we obtain the answer for $T(z, y; \beta)$ defined in (2.1.2) as an expansion about z (in section 2.5 we perform this calculation), we find it convenient for purposes of comparison to take the metric in $S^{(0)}$ also at the point z. We could also have taken the metric in $S^{(0)}$ at for example the midpoint $\frac{1}{2}(z+y)$, or perhaps at geodesic midpoints.

Next we decompose x^i_k into a background part $x^i_{bg,k}$ and a quantum part q^i_k

$$x^i_k = x^i_{bg,k} + q^i_k \ , \quad (k = 0, ..., N) \ . \tag{2.1.27}$$

For $k = 0$ we define $x^i_0 = y^i$ and for $k = N$ one has $x^i_N = z^i$. We shall assume that the background $x^i_{bg,k}$ satisfy the boundary conditions $x^i_{bg,0} = y^i$ and $x^i_{bg,N} = z^i$, hence q^i_k vanishes for $k = 0$ and $k = N$. Furthermore we assume that $x^i_{bg,k}$ is a solution of the $N-1$ equations of motion of $S^{(0)}$

$$g_{ij}(z)\Big(x^j_{k+1} - 2x^j_k + x^j_{k-1}\Big) = 0 \ , \quad (k = 1, ..., N-1) \ . \tag{2.1.28}$$

In the continuum limit one obtains

$$x^i_{bg}(t) = z^i + \frac{t}{\beta}(z^i - y^i) = z^i + \tau(z^i - y^i) \tag{2.1.29}$$

where $\tau = t/\beta$ and τ runs from -1 to 0. At this point the time coordinate t appears. We take it to run from $-\beta$ to 0 in order that the point z corresponds to $t = 0$, but other definitions are of course also possible.

We thus define

$$\begin{aligned} S^{(0)} = \sum_{k=1}^{N} \Big[& \frac{1}{2\epsilon} g_{ij}(z)(q^i_k - q^i_{k-1})(q^j_k - q^j_{k-1}) \\ & + \frac{\epsilon}{2\beta^2} g_{ij}(z)(b^i_{k-1/2}c^j_{k-1/2} + a^i_{k-1/2}a^j_{k-1/2})\Big] \ . \end{aligned} \tag{2.1.30}$$

By definition then, $S^{(int)} = S - S^{(0)}$. Note that $S^{(int)}$ contains terms linear in q, see (2.1.82). As already mentioned, this is due to the fact that $x_{bg}(t)$ is a solution of the field equations of $S^{(0)}$, and not of the field equation of S. Also note that $S^{(0)}$ does not depend on V so that the propagators we obtain are model independent (V independent).

We should comment on why we require $x^i(t)$ to satisfy the equations of motion of $S^{(0)}$. The reason is that the complete solution of the equations of motion of S cannot be given in closed form, so we settle for $S^{(0)}$. This has the drawback that terms linear in q will be produced if we expand $S^{(int)}$ in terms of q about x_{bg} and these give rise to tadpole diagrams. However, as we shall discuss, these tadpole diagrams are of order $\beta^{1/2}$, so to a given order in β, only a few tadpoles contribute.

The decomposition of S into $S^{(0)} + S^{(int)}$ is standard in perturbation theory, but in most cases one puts all terms proportional to q^2 into $S^{(0)}$, and not only the q^2 term with $g_{ij}(z)$. For example, in instanton physics $S^{(0)}$ contains the q^2 term in the background of the full instanton, and not for example the instanton at a particular point. Because one is dealing then with a particular background (the instanton) instead of an arbitrary metric $g_{ij}(x)$, one can determine the propagator in that background, and with this choice of $S^{(0)}$ there are no tadpoles. In our case it is impossible to determine explicitly the exact propagator in an arbitrary background, hence we settle for $S^{(0)}$ in which we use $g_{ij}(x)$ at point $x = z$.

To obtain discretized propagators, we could couple the q^i_k to external sources. However, since the discretized action only depends on $q^i_{k-1/2} = \frac{1}{2}(q^i_k + q^i_{k-1}))$ and $\frac{1}{\epsilon}(q^i_k - q^i_{k-1})$, we couple these combinations to independent real discretized external sources

$$-\frac{1}{\hbar} S_{(\text{sources, nonghost})} = \sum_{k=1}^{N} \left(F_{k-1/2,j} \frac{q^j_k - q^j_{k-1}}{\epsilon} + G_{k-1/2,j} q^j_{k-1/2} \right) . \tag{2.1.31}$$

We should now complete squares in $S^{(0)} + S_{(\text{sources, nonghost})}$ and then integrate over $dx^i_k = dq^i_k$. However, the action $S^{(0)}$ is not diagonal in q^i_k. Therefore we first make an orthogonal transformation which diagonalizes $S^{(0)}$.

We introduce modes for the quantum fluctuations by the orthogonal transformation

$$q^j_k = \sum_{m=1}^{N-1} r^j_m \sqrt{\frac{2}{N}} \sin\left(\frac{km\pi}{N}\right); \quad k = 1, .., N-1 . \tag{2.1.32}$$

The orthogonality of the real $(N-1)\times(N-1)$ matrix $O^m_k = \sqrt{\frac{2}{N}} \sin \frac{km\pi}{N}$ follows from the trigonometric formula $2\sin\alpha \sin\beta = \cos(\alpha-\beta) - \cos(\alpha+$

β). One finds

$$\sum_{m=1}^{N-1} O_j^m O_k^m = \frac{1}{N}\sum_{m=1}^{N-1}\left(\cos\frac{(j-k)m\pi}{N} - \cos\frac{(j+k)m\pi}{N}\right) \ . \qquad (2.1.33)$$

The sum over the cosines is easy. For integer p one has

$$\sum_{m=1}^{N-1}\cos\frac{pm\pi}{N} = \frac{1}{2}\sum_{m=1}^{N-1}\left(e^{\frac{ipm\pi}{N}} + e^{\frac{-ipm\pi}{N}}\right) = \frac{1}{2}\left(\sum_{m=-N+1}^{N} e^{\frac{ipm\pi}{N}} - 1 - (-)^p\right)$$

$$= \frac{1}{2}\sum_{m=1}^{2N} e^{\frac{ipm\pi}{N}} - \frac{1}{2} - \frac{1}{2}(-)^p = N\delta_{p,0} - \frac{1}{2} - \frac{1}{2}(-)^p \ . \qquad (2.1.34)$$

Using this result in (2.1.33) we find

$$\sum_{m=1}^{N-1} O_j^m O_k^m = \delta_{j,k} \qquad (2.1.35)$$

since $(-)^{j-k}$ equals of course $(-)^{j+k}$. Because O_j^m is orthogonal, we can replace $\prod_{k=1}^{N-1} dq_k^j$ by $\prod_{m=1}^{N-1} dr_m^j$.

The orthogonality of O_j^m implies that $S^{(0)}$ is diagonal in r_m^j. To demonstrate this we just evaluate $S^{(0)}$

$$\begin{aligned} S^{(0)} &= \frac{1}{2\epsilon}\sum_{k=1}^{N} g_{ij}(z)(q_k^i - q_{k-1}^i)(q_k^j - q_{k-1}^j) \\ &= \frac{1}{2\epsilon}g_{ij}(z)\sum_{k=1}^{N}\sum_{m,n=1}^{N-1}(O_k^m - O_{k-1}^m)r_m^i(O_k^n - O_{k-1}^n)r_n^j \\ &= \frac{1}{2\epsilon}g_{ij}(z)\left(\sum_{m=1}^{N-1} 2r_m^i r_m^j - \sum_{k=1}^{N-1}\sum_{m,n=1}^{N-1} O_k^m(O_{k-1}^n + O_{k+1}^n)r_m^i r_n^j\right) \ . \end{aligned} \qquad (2.1.36)$$

(We shifted $k \to k+1$ in the last term which is allowed since O_k^m vanishes for $k=0$ and $k=N$). Using $\sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$ we obtain $O_{k-1}^n + O_{k+1}^n = 2O_k^n\cos\frac{n\pi}{N}$. Using again the orthogonality of O_j^m we find

$$S^{(0)} = \frac{1}{\epsilon}\sum_{m=1}^{N-1} g_{ij}(z)r_m^i r_m^j\left(1 - \cos\frac{m\pi}{N}\right) \ . \qquad (2.1.37)$$

The path integral with $S^{(0)}$ and the external sources then becomes in the non-ghost sector

$$Z_N^{(0)}(F,G) = (\text{factor in } (2.1.24)) \int \left(\prod_{j=1}^{n} \prod_{m=1}^{N-1} dr_m^j \right) \exp E$$

$$E = -\frac{1}{\epsilon\hbar} \sum_{m=1}^{N-1} g_{ij}(z) r_m^i r_m^j \left(1 - \cos\frac{m\pi}{N}\right) \tag{2.1.38}$$

$$+ \sum_{k=1}^{N-1} \left\{ \frac{1}{\epsilon}(F_{k-1/2,j} - F_{k+1/2,j}) + \frac{1}{2}(G_{k-1/2,j} + G_{k+1/2,j}) \right\}$$

$$\times \left\{ \sum_{m=1}^{N-1} \sqrt{\frac{2}{N}} r_m^j \sin\frac{km\pi}{N} \right\} .$$

By (factor in (2.1.24)) we mean both factors in front of the exponential. Summations over $i, j = 1, n$ are always understood. For vanishing F and G one recovers the transition element $T(z, y; \beta)$.

Completing squares is now straightforward and performing the integration over dr_m^i yields

$$Z_N^{(0)}(F,G) = (\text{factor in } (2.1.24)) \left(\prod_{m=1}^{N-1} \frac{(\pi\epsilon\hbar)^{n/2}}{\sqrt{g(z)}(1 - \cos\frac{m\pi}{N})^{n/2}} \right)$$

$$\exp\left\{ \sum_{m=1}^{N-1} \frac{\epsilon\hbar}{4\,(1 - \cos\frac{m\pi}{N})} \right.$$

$$\left[\frac{2}{\epsilon}\sqrt{\frac{2}{N}} \sin\frac{m\pi}{2N} \sum_{k=0}^{N-1} \cos\left\{ \left(k + \frac{1}{2}\right)\frac{m\pi}{N} \right\} F_{k+1/2,j} \right.$$

$$\left. \left. + \sqrt{\frac{2}{N}} \cos\frac{m\pi}{2N} \sum_{k=0}^{N-1} \sin\left\{ \left(k + \frac{1}{2}\right)\frac{m\pi}{N} \right\} G_{k+1/2,j} \right]^2 \right\} . \tag{2.1.39}$$

The square denoted by $[..]^2$ is taken with $g^{ij}(z)$, so written out in full it reads $g^{ij}(z)[..]_i[..]_j$.

We similarly couple the ghosts $a_{k+1/2}^i$, $b_{k+1/2}^i$ and $c_{k+1/2}^i$ to external (commuting or anticommuting) sources as follows

$$-\frac{1}{\hbar} S_{(\text{sources, ghosts})} = \sum_{k=0}^{N-1} (A_{k+1/2,i} a_{k+1/2}^i + b_{k+1/2}^i B_{k+1/2,i} + C_{k+1/2,i} c_{k+1/2}^i) . \tag{2.1.40}$$

Completing squares and integrating over a, b, c we find, using (2.1.25),

$$Z^{(0)}(A,B,C) = g(z)^{N/2} \exp \sum_{k=0}^{N-1} \frac{\beta^2\hbar}{\epsilon} g^{ij}(z)$$

$$\left[2C_{k+1/2,i}B_{k+1/2,i} + \frac{1}{2}A_{k+1/2,i}A_{k+1/2,j}\right] . \qquad (2.1.41)$$

The factor $g(z)^{N/2}$ is due to integration over a, b, c and corresponds to the N factors $g(x_{k-1/2})^{1/2}$ which we exponentiated in (2.1.25). The integration over a, b, c cancels then the normalization constant α in (2.1.25) which we never computed for this reason.

The complete discretized transition element becomes now

$$\begin{aligned} T(z, y; \beta) = [g(z)g(y)]^{-1/4}(2\pi\hbar)^{-nN} & \left[\left(\frac{2\pi\hbar}{\epsilon}\right)^{nN/2} g(z)^{N/2}\right] \\ & \left[(\pi\epsilon\hbar)^{n(N-1)/2} g(z)^{-N/2+1/2} \prod_{m=1}^{N-1} \left(1 - \cos\frac{m\pi}{N}\right)^{-n/2}\right] \\ & \left[e^{-(1/\hbar)S^{int}} e^{-(1/\hbar)S(F,G,A,B,C)}\right]\Big|_0 \end{aligned} \qquad (2.1.42)$$

where the symbol $|_0$ indicates that all sources (F, G, A, B, C) are set to zero after differentiation. The first line is due to the factor in (2.1.24) and the various inner products and the integration over momenta and a, b and c, the second line is due to the second factor in (2.1.39) and accounts for the integration over r^j_m, while $-\frac{1}{\hbar}S[F, G, A, B, C]$ denotes the terms bilinear in external sources. In the interaction term S^{int} the quantum fields $q^j_k - q^j_{k-1}$, $q^j_{k-1/2}$, a, b and c should be replaced by derivatives with respect to the corresponding sources $F^j_{k-1/2}, G^j_{k-1/2}, A, B, C$ as usual in quantum field theory

Using the identity[3]

$$\prod_{m=1}^{N-1} 2\left(1-\cos\frac{m\pi}{N}\right) = N \tag{2.1.47}$$

we find

$$T(z,y;\beta) = \left[\frac{g(z)}{g(y)}\right]^{1/4} \frac{1}{(2\pi\hbar\beta)^{n/2}} \left(e^{-(1/\hbar)S^{int}} e^{-(1/\hbar)S[F,G,A,B,C]}\right)\Big|_0 \tag{2.1.48}$$

where we recall for completeness

$$-\frac{1}{\hbar}S[F,G,A,B,C] = \sum_{m=1}^{N-1} \frac{\epsilon\hbar}{4\left(1-\cos\frac{m\pi}{N}\right)}$$
$$\left[\frac{2}{\epsilon}\sqrt{\frac{2}{N}}\sin\frac{m\pi}{2N}\sum_{k=0}^{N-1}\cos\left\{\left(k+\frac{1}{2}\right)\frac{m\pi}{N}\right\}F_{k+1/2,j}\right.$$
$$\left.+\sqrt{\frac{2}{N}}\cos\frac{m\pi}{2N}\sum_{k=0}^{N-1}\sin\left\{\left(k+\frac{1}{2}\right)\frac{m\pi}{N}\right\}G_{k+1/2,j}\right]^2$$

---

[3] To prove this identity, consider the function

$$f(x) = \prod_{k=0}^{2N-1}\left(x-\cos\frac{k\pi}{N}\right) = (x^2-1)\left[\prod_{k=1}^{N-1}\left(x-\cos\frac{k\pi}{N}\right)\right]^2 . \tag{2.1.43}$$

The function $p(x) = -1 + (x + i\sqrt{1-x^2})^{2N}$ has zeros at the roots of unity, hence at $x = \cos\frac{k\pi}{N}$ for $k = 0, 1, \ldots, 2N-1$. In particular, its real part vanishes there. Since $Re\ p(x)$ is a polynomial in x of degree x^{2N}, we see that $f(x)$ and $Re\ p(x)$ are proportional. Since $Re\ p(x) = a_{2N}x^{2N} + \ldots$ with

$$a_{2N} = \sum_{k\ \text{even}} \binom{2N}{k} = \lim_{x\to 1}\left(\frac{1}{2}(1+x)^{2N} + \frac{1}{2}(1-x)^{2N}\right) = 2^{2N-1}, \tag{2.1.44}$$

we find $f(x) = 2^{1-2N}\ Re\ p(x)$. Furthermore, near $x = 1$ we have

$$\begin{aligned} Re\ p(x) &= x^{2N} - \binom{2N}{2}x^{2N-2}(1-x^2) + \mathcal{O}(1-x^2)^2 - 1 \\ &= (x^2-1)\left[(x^{2N}-1)/(x^2-1) + \binom{2N}{2}x^{2N-2}\right] + \mathcal{O}(1-x^2)^2 . \end{aligned} \tag{2.1.45}$$

Hence

$$2^{2N-1}\prod_{k=1}^{N-1}\left(x-\cos\frac{k\pi}{N}\right)^2 = \frac{Re\ p(x)}{x^2-1} \to \frac{x^{2N}-1}{x^2-1} + \binom{2N}{2} = N + \binom{2N}{2} = 2N^2 \tag{2.1.46}$$

as $x \to 1$. This proves the identity.

$$+\sum_{k=0}^{N-1}\frac{\beta^2\hbar}{\epsilon}g^{ij}(z)\left[2C_{k+1/2,i}B_{k+1/2,j}+\frac{1}{2}A_{k+1/2,i}A_{k+1/2,j}\right] \;. \tag{2.1.49}$$

The interactions are given by

$$\begin{aligned}
-\frac{1}{\hbar}S^{int} &= \sum_{k=1}^{N}\Big\{-\frac{1}{2\epsilon\hbar}[g_{ij}(\bar{x}_{k-1/2})-g_{ij}(z)](x_k^i-x_{k-1}^i)(x_k^j-x_{k-1}^j)\\
&-\frac{1}{2\epsilon\hbar}g_{ij}(z)[(x_k^i-x_{k-1}^i)(x_k^j-x_{k-1}^j)-(q_k^i-q_{k-1}^i)(q_k^j-q_{k-1}^j)]\\
&-\frac{\epsilon}{2\beta^2\hbar}[g_{ij}(\bar{x}_{k-1/2})-g_{ij}(z)](a_{k-1/2}^i a_{k-1/2}^j+b_{k-1/2}^i c_{k-1/2}^j)\\
&-\frac{\epsilon}{\hbar}V(\bar{x}_k)\Big\} \;.
\end{aligned}\tag{2.1.50}$$

where $V(\bar{x}_k)$ is the order $\hbar^2$ counterterm given in (2.1.12). As already discussed, the quantum fields q_k^i, $a_{k-1/2}^i$, $b_{k-1/2}^i$ and $c_{k-1/2}^i$ in S^{int} should be replaced by the corresponding differential operators with respect to the external sources; this is, of course, standard practice in quantum field theory.

The expression for $T(z,y;\beta)$ in (2.1.48) is an exact expression to order ϵ. It was derived with much labor but the final result is very simple. It contains the Feynman measure, and the factor $[g(z)/g(y)]^{\frac{1}{4}}$ is due to expanding the metric in S^{int} about z. If we had expanded about the midpoint this factor would even have been absent. Using this exact expression we can now find unambiguous Feynman rules at the discretized level.

We obtain the discretized propagators by twice differentiating the expression $\exp(-\frac{1}{\hbar}S[F,G,A,B,C])$ with respect to F,G,A,B,C and then putting external sources to zero. An easy case is the $\dot{q}\dot{q}$ propagator, by which we mean

$$\left\langle\left(\frac{q_{k+1}^i-q_k^i}{\epsilon}\right)\left(\frac{q_{k'+1}^j-q_{k'}^j}{\epsilon}\right)\right\rangle \quad ; \quad 0\le k,k'\le N \;. \tag{2.1.51}$$

According to (2.1.31) it is given by

$$\begin{aligned}
\langle\dot{q}_{k+1/2}^i\dot{q}_{k'+1/2}^j\rangle &= \frac{\partial}{\partial F_{k+1/2,i}}\frac{\partial}{\partial F_{k'+1/2,j}}\exp\Big(-\frac{1}{\hbar}S[F,G,A,B,C]\Big)\Big|_0\\
&= 2\sum_{m=1}^{N-1}\frac{\epsilon\hbar}{4(1-\cos\frac{m\pi}{N})}g^{ij}(z)\left(\frac{2}{\epsilon}\sqrt{\frac{2}{N}}\sin\frac{m\pi}{2N}\right)^2\\
&\quad\times\cos(k+1/2)\frac{m\pi}{N}\cos(k'+1/2)\frac{m\pi}{N} \;.
\end{aligned}\tag{2.1.52}$$

Twice the square of the sine cancels the factor $1-\cos$ in the denominator, and using $2\cos\alpha\cos\beta = \cos(\alpha+\beta)+\cos(\alpha-\beta)$ one finds from (2.1.34)

$$\begin{aligned}\langle \dot{q}^i_{k+1/2}\dot{q}^j_{k'+1/2}\rangle &= \frac{\hbar}{N\epsilon} g^{ij}(z)\sum_{m=1}^{N-1}\left(\cos(k+k'+1)\frac{m\pi}{N}+\cos(k-k')\frac{m\pi}{N}\right)\\ &= \frac{\hbar}{N\epsilon} g^{ij}(z)(-1+N\delta_{k,k'})\ . \end{aligned}\tag{2.1.53}$$

Since in the continuum limit $\frac{1}{\epsilon}\delta_{k,k'}$ becomes $\delta(t-t')$, we find in the continuum limit with $t = k\epsilon - \beta$

$$\langle \dot{q}^i(t)\dot{q}^j(t')\rangle = \hbar g^{ij}(z)\left(-\frac{1}{\beta}+\delta(t-t')\right)\ . \tag{2.1.54}$$

Let us compare this with the result we would have obtained naively (i.e. disregarding all subtleties involving discretizations). The naive continuum propagator is obtained from

$$\exp\left[-\frac{1}{2\hbar}\int_{-\beta}^{0} g_{ij}\dot{q}^i\dot{q}^j dt + \int_{-\beta}^{0} q^i J_i dt\right] \tag{2.1.55}$$

by the usual steps

$$\begin{aligned}\langle q^i(t)q^j(t')\rangle &= \frac{\delta}{\delta J_i(t)}\frac{\delta}{\delta J_j(t')}\exp\left[\left(-\frac{\hbar}{2}\right)g^{ij}(z)\int_{-\beta}^{0} J_i(t'')\frac{1}{\partial^2/\partial t^2}J_j(t'')dt''\right]\\ &= -\hbar g^{ij}(z)\frac{1}{\partial^2/\partial t^2}\delta(t-t')\ . \end{aligned}\tag{2.1.56}$$

Differentiating with respect to t and t' yields only the delta function: one misses the terms with $-\frac{1}{\beta}$ in (2.1.54). However, this is due to not having taken into account the boundary conditions. Imposing the boundary condition $q(0) = q(-\beta) = 0$ one must add suitable terms linear in t and t' to (2.1.56) so that the propagator vanishes at $t = 0, -\beta$ and $t' = 0, -\beta$ while still $\partial^2/\partial t^2\langle q^i(t)q^j(t')\rangle = -\hbar g^{ij}(z)\delta(t-t')$. The naive result, as one may check, is

$$\langle q^i(t)q^j(t')\rangle = -\hbar g^{ij}(z)\frac{1}{\beta}[t(t'+\beta)\theta(t-t')+t'(t+\beta)\theta(t'-t)]\ . \tag{2.1.57}$$

It follows that naively

$$\langle \dot{q}^i(t)\dot{q}^j(t')\rangle = -\hbar g^{ij}(z)\left[\frac{1}{\beta}-\delta(t-t')\right]\ . \tag{2.1.58}$$

This agrees with our discretized expression in (2.1.53), but note that **the symbol $\boldsymbol{\delta(t-t')}$ is proportional to a Kronecker delta function in**

the discretized case. Thus when one evaluates Feynman graphs, this Kronecker $\delta(t-t')$ instructs one to set everywhere in the integrand $t=t'$, and not to replace $\delta(t-t')$ by some smooth function. This will be crucial when we evaluate Feynman graphs with equal-time contractions.

Next we evaluate the $q\dot{q}$ propagator. By this we mean $\langle \frac{1}{2}(q_{k+1}+q_k)\frac{1}{\epsilon}(q_{k'+1}-q_{k'})\rangle$, of course. It is given by

$$\begin{aligned}
\langle q^i_{k+1/2}\dot{q}^j_{k'+1/2}\rangle &= \frac{\partial}{\partial G_{k+1/2,i}}\frac{\partial}{\partial F_{k'+1/2,j}}\exp\Big(-\frac{1}{\hbar}S[F,G,A,B,C]\Big)\Big|_0 \\
&= 2\sum_{m=1}^{N-1}\frac{\epsilon\hbar g^{ij}(z)}{4\left(1-\cos\frac{m\pi}{N}\right)}\frac{4}{N\epsilon}\sin\frac{m\pi}{2N}\cos\frac{m\pi}{2N} \\
&\quad\times\sin\left\{(k+\frac{1}{2})\frac{m\pi}{N}\right\}\cos\left\{(k'+1/2)\frac{m\pi}{N}\right\} \\
&= \hbar g^{ij}(z)\frac{1}{N}\sum_{m=1}^{N-1}\cos\frac{m\pi}{2N}\left(\frac{\sin\left(k+\frac{1}{2}\right)\frac{m\pi}{N}}{\sin\frac{m\pi}{2N}}\right)\cos(k'+1/2)\frac{m\pi}{N}\ .
\end{aligned} \tag{2.1.59}$$

To evaluate this series, we introduce the notation

$$\zeta=\exp\frac{i\pi}{2N} \tag{2.1.60}$$

and find then

$$\frac{1}{4N}\sum_{m=1}^{N-1}(\zeta^m+\zeta^{-m})(\zeta^{2km}+\zeta^{(2k-2)m}+\dots\zeta^{-2km})(\zeta^{(2k'+1)m}+\zeta^{-(2k'+1)m})\ . \tag{2.1.61}$$

(In the ratio $(\zeta^{(2k+1)m}-\zeta^{-(2k+1)m})/(\zeta^m-\zeta^{-m})$ only powers of ζ^2 remain). There are then four series to sum, which we write in the following four lines

$$\frac{1}{4N}\sum_{m=1}^{N-1}\left[\begin{array}{l}
\zeta^{(2k+2k'+2)m}+\zeta^{(2k+2k')m}+\ldots+\zeta^{(-2k+2k'+2)m}+ \\
\zeta^{(2k+2k')m}+\zeta^{(2k+2k'-2)m}+\ldots+\zeta^{(-2k+2k')m}+ \\
\zeta^{-(2k+2k')m}+\zeta^{-(2k-2+2k')m}+\ldots+\zeta^{-(-2k+2k')m}+ \\
\zeta^{-(2k+2k'+2)m}+\zeta^{-(2k+2k')m}+\ldots+\zeta^{-(-2k+2k'+2)m}
\end{array}\right]. \tag{2.1.62}$$

We have written the terms in the last two lines with increasing exponents, since this allows us to combine in the same column the terms in the first and fourth row, or the second and third row. Using (2.1.34)

$$\sum_{m=1}^{N-1}(\zeta^{2pm}+\zeta^{-2pm})=-1-(-)^p+2N\delta_{p,0} \tag{2.1.63}$$

we find for (2.1.62)

$$\begin{aligned}&\frac{1}{4N}\sum_{p=-k+k'+1}^{k+k'+1}(-1-(-)^p+2N\delta_{p,0})\\&+\frac{1}{4N}\sum_{p=-k+k'}^{k+k'}(-1-(-)^p+2N\delta_{p,0})\ .\end{aligned}\tag{2.1.64}$$

The terms with $(-)^p$ cancel. In the remainder we distinguish the cases $k>k'$, $k<k'$ and $k=k'$. We get then

$$\begin{aligned}&\frac{1}{4N}\Big(-(2k+1)+2N\delta_{k>k'}-(2k+1)+2N\delta_{k\geq k'}\Big)\\&=-\frac{(k+1/2)}{N}+\frac{1}{2}\delta_{k>k'}+\frac{1}{2}\delta_{k\geq k'}\end{aligned}\tag{2.1.65}$$

where $\delta_{k>k'}$ equals unity if $k>k'$ and zero otherwise. Therefore

$$\langle q^i_{k+1/2}\dot{q}^j_{k'+1/2}\rangle=\hbar g^{ij}(z)\left[-\frac{(k+1/2)}{N}+\left\{\begin{array}{cl}0 & \text{if } k<k'\\ 1/2 & \text{if } k=k'\\ 1 & \text{if } k>k'\end{array}\right\}\right].\tag{2.1.66}$$

In the continuum limit this becomes

$$\langle q^i(t)\dot{q}^j(t')\rangle=\hbar g^{ij}(z)\left[-\frac{t+\beta}{\beta}+\theta(t-t')\right]\tag{2.1.67}$$

which agrees with the naive continuum result obtained by differentiating (2.1.57)

$$\langle q^i(t)\dot{q}^j(t')\rangle=(-\hbar g^{ij}(z))\frac{1}{\beta}\left[t\theta(t-t')+(t+\beta)\theta(t'-t)\right]\ .\tag{2.1.68}$$

The discretized approach tells us that $\theta(t-t')=1/2$ at $t=t'$. However, at the point $t=t'=0$, the naive continuum propagator in (2.1.68) does not vanish and thus violates the boundary conditions. In the discretized approach one should write q_k as $q_{k-1/2}+\frac{1}{2}(q_k-q_{k-1})$ and then there are no problems at the boundary. One finds by combining (2.1.66) and (2.1.53) the following result

$$\langle q^i_{k+1}\dot{q}^j_{k'+\frac{1}{2}}\rangle=\hbar g^{ij}(z)\left[-\frac{(k+1)}{N}+\frac{1}{2}\delta_{k,k'}+\theta_{k,k'}\right]\tag{2.1.69}$$

which vanishes for $k+1=N$. The extra term $\frac{1}{2}\delta_{k,k'}$ saves the day. The reason that the continuum approach fails to give zero while the discretized approach yields the correct result is that $q^i(t)$ should be defined at midpoints, and thus never really reaches the endpoints.

Finally, we consider the qq propagators. This is the most complicated propagator. It is given by

$$
\begin{aligned}
&\langle q^i_{k+1/2} q^j_{k'+1/2}\rangle = \frac{\partial}{\partial G_{k+1/2,i}} \frac{\partial}{\partial G_{k'+1/2,j}} \exp\Big(-\frac{1}{\hbar} S[G,F,A,B,C,0]\Big)\Big|_0 \\
&= 2\sum_{m=1}^{N-1} \frac{\epsilon\hbar g^{ij}(z)}{4\,(1-\cos\frac{m\pi}{N})} \left(\sqrt{\frac{2}{N}}\cos\frac{m\pi}{2N}\right)^2 \\
&\quad\times \sin\left(k+\frac{1}{2}\right)\frac{m\pi}{N}\sin\left(k'+1/2\right)\frac{m\pi}{N} \\
&= \frac{\epsilon\hbar}{2N} g^{ij}(z) \sum_{m=1}^{N-1}\left(\cos\frac{m\pi}{2N}\right)^2 \left(\frac{\sin\left(k+\frac{1}{2}\right)\frac{m\pi}{N}}{\sin\frac{m\pi}{2N}}\right)\left(\frac{\sin\left(k'+1/2\right)\frac{m\pi}{N}}{\sin\frac{m\pi}{2N}}\right) .
\end{aligned}
\tag{2.1.70}
$$

Again we write the ratios of sines as polynomials in ζ^2, with $2k+1$ and $2k'+1$ terms, respectively. This leads to the series

$$
\begin{aligned}
&\frac{1}{4}\sum_{m=1}^{N-1}(\zeta^m+\zeta^{-m})^2(\zeta^{2km}+\zeta^{(2k-2)m}+\ldots\zeta^{-2km}) \\
&\qquad\qquad \times(\zeta^{2k'm}+\zeta^{(2k'-2)m}+\ldots\zeta^{-2km'}) \\
&=\frac{1}{4}\sum_{m=1}^{N-1}\sum_{\alpha=-k}^{k}\sum_{\beta=-k'}^{k'}(\zeta^{(2\alpha+2\beta+2)m}+2\zeta^{(2\alpha+2\beta)m}+\zeta^{(2\alpha+2\beta-2)m}) .
\end{aligned}
\tag{2.1.71}
$$

Replacing $\alpha\to-\alpha$ and $\beta\to-\beta$ in half of the terms (which yields the same result), we obtain cosines

$$
=\frac{1}{2}\sum_{m=1}^{N-1}\sum_{\alpha=-k}^{k}\sum_{\beta=-k'}^{k'}\Big[\cos(\alpha+\beta+1)\frac{m\pi}{N}+\cos(\alpha+\beta)\frac{m\pi}{N}\Big] \tag{2.1.72}
$$

and using the formula in (2.1.34) for summing cosines we obtain

$$
\begin{aligned}
&=\frac{1}{2}\sum_{\alpha=-k}^{k}\sum_{\beta=-k'}^{k'}\Big[-\frac{1}{2}-\frac{1}{2}(-)^{\alpha+\beta+1}+N\delta_{\alpha+\beta+1,0} \\
&\qquad\qquad -\frac{1}{2}-\frac{1}{2}(-)^{\alpha+\beta}+N\delta_{\alpha+\beta,0}\Big] \\
&=\frac{1}{2}\sum_{\alpha=-k}^{k}\sum_{\beta=-k'}^{k'}[-1+N(\delta_{\alpha+\beta+1,0}+\delta_{\alpha+\beta,0})] .
\end{aligned}
\tag{2.1.73}
$$

It is again easiest to consider the cases $k > k'$, $k < k'$ and $k = k'$ separately. We find

$$\langle q^i_{k+1/2} q^j_{k'+1/2} \rangle = \frac{\epsilon\hbar}{4N} g^{ij}(z) \times \left[-(2k+1)(2k'+1) + \left\{ \begin{array}{lll} 2N(2k'+1) & \text{for} & k > k' \\ N(4k+1) & \text{for} & k = k' \\ 2N(2k+1) & \text{for} & k < k' \end{array} \right\} \right] . \tag{2.1.74}$$

The last term contains a discretized theta function

$$\langle q^i_{k+1/2} q^j_{k'+1/2} \rangle = \epsilon\hbar g^{ij}(z) \Big[- \frac{(k+1/2)(k'+1/2)}{N} + \Big(k' + \frac{1}{2} \Big) \theta(k,k') + (k+1/2)\theta(k',k) - \frac{1}{4}\delta_{k,k'} \Big] . \tag{2.1.75}$$

In the continuum limit this becomes

$$\langle q^i(t) q^j(t') \rangle = -\beta\hbar g^{ij}(z) \Big[\frac{(\beta+t)}{\beta} \frac{(\beta+t')}{\beta} - \frac{(\beta+t')}{\beta}\theta(t-t') - \frac{(\beta+t)}{\beta}\theta(t'-t) \Big] \tag{2.1.76}$$

which agrees with the naive continuum propagator $-\hbar g^{ij}(z)\frac{1}{\beta}[t(t'+\beta)\theta(t-t') + t'(t+\beta)\theta(t'-t)]$ except that the value $\theta(0) = 1/2$ is now justified.

As a check we may combine the propagators for $q\dot{q}$ and qq, and check that $\langle q^i_{k+\frac{1}{2}} q^j_{k'} \rangle$ indeed vanishes for $k' = 0$ or $k' = N$. One finds by using $q_{k'+1} = q_{k'+\frac{1}{2}} + \frac{\epsilon}{2}\dot{q}_{k'+\frac{1}{2}}$ and combining (2.1.75) and (2.1.66) the following result

$$\langle q^i_{k+\frac{1}{2}} \; q^j_{k'+1} \rangle = \epsilon\hbar g^{ij}(z) \left[-\frac{(k+\frac{1}{2})(k'+1)}{N} + (k'+1)\theta(k,k') + (k+\frac{1}{2})\theta(k',k) - \frac{1}{4}\delta_{k,k'} \right] . \tag{2.1.77}$$

This expression indeed vanishes at $k'+1 = N$, while the naive continuum limit does not vanish at $t' = 0$. Similar results hold for the other endpoint. One could try to improve the naive continuum results by extending the integration region beyond $[-\beta, 0]$, and require that also for $t > 0$ and $t < -\beta$ the propagator satisfies $\partial_t^2 \langle q(t)q(t') \rangle \sim \delta(t-t')$ while still $q(0) = q(-\beta) = 0$. This is possible, but the resulting function is quite complicated. Rather, we shall derive rules for products of continuum distributions which follow directly from the corresponding discretized expressions.

Finally we determine the propagators of the ghosts. We find from (2.1.49) and (2.1.40)

$$
\begin{aligned}
\langle b^i_{k+1/2} c^j_{k'+1/2} \rangle &= -\frac{\partial}{\partial B_{k+1/2,i}} \frac{\partial}{\partial C_{k'+1/2,j}} e^{-\frac{1}{\hbar} S[F,G,A,B,C]} \Big|_0 \\
&= -\frac{2\beta^2 \hbar}{\epsilon} g^{ij}(z) \delta_{k,k'} \quad \to \quad -2\beta^2 \hbar g^{ij}(z) \delta(t-t') \qquad (2.1.78) \\
\langle a^i_{k+1/2} a^j_{k'+1/2} \rangle &= \frac{\partial}{\partial A_{k+1/2,i}} \; \frac{\partial}{\partial A_{k'+1/2,j}} e^{-\frac{1}{\hbar} S[F,G,A,B,C]} \Big|_0 \\
&= \frac{\beta^2 \hbar}{\epsilon} g^{ij}(z) \delta_{k,k'} \quad \to \quad \beta^2 \hbar g^{ij}(z) \delta(t-t') \; . \qquad (2.1.79)
\end{aligned}
$$

Again we note that the $\delta(t-t')$ in the continuum limit should be interpreted as Kronecker delta function; moreover, in the discretized approach the ghosts are only defined on midpoints.

We now summarize our results for the path integral representation of $\langle z| \exp(-\frac{\beta}{\hbar}\hat{H})|y\rangle$ for bosonic systems in configuration space. It can be written in terms of propagators and vertices from $S^{(int)}$ as follows

$$
\begin{aligned}
&\langle z| \exp\Big(-\frac{\beta}{\hbar}\hat{H}\Big)|y\rangle = \left[\frac{g(z)}{g(y)}\right]^{1/4} \frac{1}{(2\pi\beta\hbar)^{n/2}} \langle e^{-\frac{1}{\hbar}S^{(int)}} \rangle \\
&S^{int} = S - S^{(0)} \\
&-\frac{1}{\hbar} S = -\frac{1}{\beta\hbar} \int_{-1}^{0} \frac{1}{2} g_{ij}(x) \left(\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} + b^i(\tau)c^j(\tau) + a^i(\tau)a^j(\tau) \right) d\tau \\
&\qquad\qquad -\frac{\beta\hbar}{8} \int_{-1}^{0} \Big(R(x) + g^{ij}(x) \Gamma^l_{ik}(x) \Gamma^k_{jl}(x) \Big) d\tau \\
&-\frac{1}{\hbar} S^{(0)} = -\frac{1}{\beta\hbar} \int_{-1}^{0} \frac{1}{2} g_{ij}(z) \left(\frac{dq^i}{d\tau}\frac{dq^j}{d\tau} + b^i(\tau)c^j(\tau) + a^i(\tau)a^j(\tau) \right) d\tau
\end{aligned}
\qquad (2.1.80)
$$

$$
\begin{aligned}
x^i(\tau) &= x^i_{bg}(\tau) + q^i(\tau) \\
x^i_{bg}(\tau) &= z^i + \tau(z^i - y^i)
\end{aligned}
$$

$$
\begin{aligned}
\langle q^i(\sigma) q^j(\tau) \rangle &= -\beta\hbar g^{ij}(z) \Delta(\sigma,\tau) \\
\langle q^i(\sigma) \dot{q}^j(\tau) \rangle &= -\beta\hbar g^{ij}(z) (\sigma + \theta(\tau - \sigma)))
\end{aligned}
$$

$$
\begin{aligned}
\langle b^i(\sigma)c^j(\tau)\rangle &= -2\beta\hbar g^{ij}(z)\partial_\sigma^2\Delta(\sigma,\tau)\\
\langle \dot q^i(\sigma)\dot q^j(\tau)\rangle &= -\beta\hbar g^{ij}(z)(1-\delta(\tau-\sigma)))\\
\langle a^i(\sigma)a^j(\tau)\rangle &= \beta\hbar g^{ij}(z)\partial_\sigma^2\Delta(\sigma,\tau)
\end{aligned}
\qquad (2.1.81)
$$

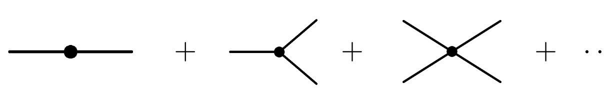
$$
\begin{aligned}
\Delta(\sigma,\tau) &= \sigma(\tau+1)\theta(\sigma-\tau)+\tau(\sigma+1)\theta(\tau-\sigma)\\
\partial_\sigma^2\Delta(\sigma,\tau) &= \delta(\sigma-\tau)\ .
\end{aligned}
$$

Since only the combination $\beta\hbar$ occurs, β counts the number of loops. To obtain a uniform overall factor $(\beta\hbar)^{-1}$ in the action we normalized the ghost actions as in (2.1.25). By expanding $\exp(-\frac{1}{\hbar}S^{int})$ and using these propagators, we can evaluate $\langle z|\exp(-\frac{\beta}{\hbar}\hat H)|y\rangle$ to any order in loops. The transition element $T(z,y;\beta)$ is due to the vacuum expectation value of $\exp(-\frac{1}{\hbar}S^{int})$: loops with internal quantum fields but no external quantum fields.

The interactions are more explicitly given by

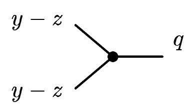
$$
\begin{aligned}
&-\frac{1}{\hbar}S^{int} = -\frac{1}{\beta\hbar}\int_{-1}^{0}\left\{\frac12 g_{ij}(x)[(z^i-y^i)(z^j-y^j)+2(z^i-y^i)\dot q^j]\right.\\
&\left.+\frac12(g_{ij}(x)-g_{ij}(z))(\dot q^i\dot q^j+b^ic^j+a^ia^j)\right\}d\tau\\
&= -\frac{1}{\beta\hbar}\int_{-1}^{0}\left[\frac12 g_{ij}(z)(z^i-y^i)(z^j-y^j)+g_{ij}(z)(z^i-y^i)\dot q^j\right.\\
&+\frac12\partial_k g_{ij}(z)(z^i-y^i)(z^j-y^j)((z-y)^k\tau+q^k)\\
&+\partial_k g_{ij}(z)(z^i-y^i)\dot q^j((z^k-y^k)\tau+q^k)\\
&+\frac14\partial_k\partial_l g_{ij}(z)((z^k-y^k)\tau+q^k)((z^l-y^l)\tau+q^l)\\
&\times[(z^i-y^i)(z^j-y^j)+2(z^i-y^i)\dot q^j]\\
&\left.+\ldots+\frac12(g_{ij}(x)-g_{ij}(z))(\dot q^i\dot q^j+b^ic^j+a^ia^j)\right]d\tau\\
&-\frac{\beta\hbar}{8}\int_{-1}^{0}\Big[R(x_{bg}+q)+g^{ij}(x_{bg}+q)\Gamma^l_{ik}(x_{bg}+q)\Gamma^k_{jl}(x_{bg}+q)\Big]d\tau\ .
\end{aligned}
\qquad (2.1.82)
$$

We now briefly discuss and check some of the terms in the action. We do this because all our later calculations will be based on this action, so we should be absolutely sure it is correct. The classical terms (the terms without q's or ghosts) yield in the path integral a factor



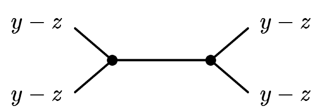
$$
= \exp\left\{ -\frac{1}{\beta\hbar} \left[\frac{1}{2} g_{ij}(z)(z^i - y^i)(z^j - y^j) \right.\right.
$$
$$
-\frac{1}{4}(z^k - y^k)\partial_k g_{ij}(z)(z^i - y^i)(z^j - y^j)
$$
$$
\left.\left. +\frac{1}{12}\partial_k\partial_l g_{ij}(z)(z^i - y^i)(z^j - y^j)(z^k - y^k)(z^l - y^l) + \ldots \right] \right\} . \tag{2.1.83}
$$

These terms are not equal to an expansion of the classical action about z because $x_{bg}(\tau)$ is only a solution of $S^{(0)}$. Rather, tree graphs with vertices which are linear in q from $S^{(int)}$ contribute as well to the order $1/\beta\hbar$ terms. Let us study this further. The term with $\dot{q}^j$ in the first line of $-\frac{1}{\hbar}S^{int}$ vanishes due to the boundary conditions, but the vertices



$$
= -\frac{1}{\beta\hbar} \int_{-1}^{0} \frac{1}{2} \partial_k g_{ij} \Big\{ (z^i - y^i)(z^j - y^j) q^k
$$
$$
-2(z^i - y^i)(z^k - y^k) q^j \Big\} d\tau
$$
$$
= \frac{1}{\beta\hbar} \Gamma_{ij;k} (z-y)^i (z-y)^j \int_{-1}^{0} q^k \; d\tau \tag{2.1.84}
$$

do contribute. (We partially integrated to obtain the second term). Two of these vertices produce a tree graph which contributes a term



$$
= -\frac{1}{\beta\hbar}\frac{1}{24} \Gamma_{ij;k}\Gamma_{i'j';k'} (z-y)^i (z-y)^{i'} (z-y)^j (z-y)^{j'} g^{kk'} \tag{2.1.85}
$$

where we used that $\int_{-1}^{0}\int_{-1}^{0} d\sigma d\tau \, \Delta(\sigma - \tau) = -\frac{1}{12}$. This indeed completes the classical action to this order in β (see (2.5.32)). Tree graphs with two q-propagators contribute at the $\beta^{3/2}$ level, and so on. Hence, the tree graph part is in good shape.

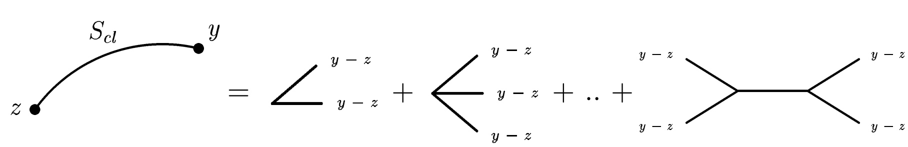


Figure 2: The expansion of the classical action evaluated for a geodesic from y to z, expanded in terms of $y - z$. The internal propagators come from the quantum fields q, and external lines denote factors $y - z$.

Next consider the one-loop part (the part independent of $\hbar$). Expanding the measure $[g(z)/g(y)]^{1/4}$ one finds a factor $1 + \frac{1}{4} g^{ij}(z-y)^k \partial_k g_{ij}(z)$ multiplying $\exp(-\frac{1}{\hbar} S_{cl}[z,y;\beta])$. This factor is canceled by the one-loop equal-time contractions from the vertices in the last line of $-\frac{1}{\hbar} S^{int}$

$$-\frac{1}{\beta\hbar}(z^k - y^k)\frac{1}{2}\partial_k g_{ij}(z) \int_{-1}^{0} \tau \, \langle \dot{q}^i \dot{q}^j + b^i c^j + a^i a^j \rangle \, d\tau \qquad (2.1.86)$$

and from the vertex in the third line

$$-\frac{1}{\beta\hbar}(\partial_k g_{ij})(z^i - y^i) \int_{-1}^{0} \langle \dot{q}^j q^k \rangle d\tau \ . \qquad (2.1.87)$$

The latter does not contribute since

$$\langle \dot{q}^j q^k \rangle \sim (\tau+1)\theta(\sigma-\tau) + \tau\theta(\tau-\sigma) = \tau + \frac{1}{2} \qquad \text{at} \quad \sigma = \tau \qquad (2.1.88)$$

which integrates to zero. In the former all $\delta(\sigma-\tau)$ cancel, as the discretized approach rigorously shows, and with

$$\langle \dot{q}^i \dot{q}^j + b^i c^j + a^i a^j \rangle_{\sigma=\tau} = -\hbar\beta g^{ij}(z) \qquad (2.1.89)$$

it yields $-1/4(z^k - y^k)\partial_k g_{ij}(z) g^{ij}(z)$, canceling the factor from the measure.

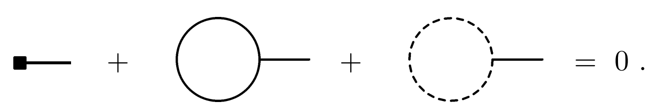


Figure 3: At the one-loop level the contributions of the measure, denoted by a black box, cancel loops with q and ghost loops. External lines denote again factors $y - z$.

There are many other one- and two-loop graphs, and the contribution from each corresponds to a particular term in the expansion of $\langle z|\exp(-\frac{\beta}{\hbar}\hat{H})|y\rangle$ about z. In particular, the two loop graph with one $\dot{q}\dot{q}$, one $\dot{q}q$ and one $q\dot{q}$ propagator agrees with the transition element only if

$$\int_{-1}^{0}\int_{-1}^{0} \delta(\sigma-\tau)\theta(\sigma-\tau)\theta(\tau-\sigma) \, d\sigma d\tau = \frac{1}{4} \ . \qquad (2.1.90)$$

This result immediately follows from the discretized approach, where $\delta(\sigma-\tau)$ is a Kronecker delta and $\theta(\sigma-\tau) = 1/2$ at $\sigma = \tau$. We shall give a complete analysis of all two-loop graphs in section 2.6.

2.2 The phase space path integral and Matthews' theorem

To obtain the phase space path integral and phase space Feynman diagrams for the transition element, we go back to the discretized expression for $T(z, y; \beta)$ in (2.1.18) with the momenta not yet integrated out, and add external sources for $q_{k-1/2} \equiv \frac{1}{2}(q_k + q_{k-1})$ and p_k because these are the variables on which H depends. Since the free equation of motion for p_k reads $g^{ij}(z)p_{k,j} = i(q_k^i - q_{k-1}^i)/\epsilon$ (see below), we denote the sources for p_k by $-iF_{k-1/2}^j$ where $F_{k-1/2}^j = g^{ij}(z)F_{k-1/2,i}$. By replacing $F_{k-1/2,j}\frac{q_k^j - q_{k-1}^j}{\epsilon}$ by $-ip_{k,j}F_{k-1/2}^j$ we can compare later the results of the phase space and the configuration space approach. Hence

$$
\begin{aligned}
Z(F, G, z, y; \beta) &= [g(z)g(y)]^{-1/4}(2\pi\hbar)^{-nN} \prod_{j=1}^{n} \left(\prod_{k=1}^{N} dp_{k,j} \prod_{l=1}^{N-1} dx_l^j \right) \\
&\exp\left[\sum_{k=1}^{N} \left\{ \frac{i}{\hbar} p_k \cdot (x_k - x_{k-1}) - \frac{\epsilon}{\hbar} H(p_k, x_{k-1/2}) \right.\right. \\
&\quad \left.\left. -iF_{k-1/2}^j p_{k,j} + G_{k-1/2,j} q_{k-1/2}^j \right\} \right] \ . \qquad (2.2.1)
\end{aligned}
$$

Next we decompose H into $H^{(0)} + H^{(int)}$ where

$$
H^{(0)} = \sum_{k=1}^{N} \frac{1}{2} g^{ij}(z) p_{k,i} p_{k,j} \ , \qquad (2.2.2)
$$

and we decompose again $x = x_{bg} + q$, but we add the term $\frac{i}{\hbar} p_{k,j}(x_{bg,k}^j - x_{bg,k-1}^j)$ to $-\frac{\epsilon}{\hbar} H^{(int)}$. The term $\frac{i}{\hbar} p_{k,j} \frac{q_k^j - q_{k-1}^j}{\epsilon}$ is part of $S^{(0)}$.

We then complete squares in the terms depending on p in the sum of $ip_k(q_k - q_{k-1}) - \epsilon H^{(0)}$ and the source terms, and perform the p-integrals. This yields

$$
\begin{aligned}
&Z(F, G, z, y; \beta) = [g(z)g(y)]^{-1/4}(2\pi\hbar)^{-nN} \\
&\left[\left(\frac{2\pi\hbar}{\epsilon} \right)^{\frac{1}{2}nN} g(z)^{\frac{1}{2}N} \right] \int \prod_{j=1}^{n} dx_{N-1}^j \ldots dx_1^j \\
&\exp\left\{ -\frac{\epsilon}{\hbar} H^{(int)} \left(p_{k,j} \to i \frac{\partial}{\partial F_{k-1/2}^j} \ , \ x_{k-1/2}^j \to x_{bg,k-\frac{1}{2}}^j + \frac{\partial}{\partial G_{k-1/2,j}} \right) \right\} \\
&\exp\left\{ \sum_{k=1}^{N} \frac{\epsilon}{2\hbar} \left[-\frac{i\hbar}{\epsilon} F_{k-1/2}^j + \frac{i}{\epsilon}(q_k^j - q_{k-1}^j) \right]^2 + G_{k-1/2,j} q_{k-1/2}^j \right\} \ . \qquad (2.2.3)
\end{aligned}
$$

The square in the last line of (2.2.3) is again taken with $g_{ij}(z)$; expanding this square we find back the terms of the configuration space path integral, multiplied by the factor

$$\exp\left\{-\frac{\hbar}{2\epsilon}\left(\sum_{k=1}^{N} g_{ij}(z)F^i_{k-1/2}F^j_{k-1/2}\right)\right\} \ . \tag{2.2.4}$$

The propagators are again obtained by differentiation with respect to the sources in (2.2.1). It follows that the discrete qq propagators are the same, while the pq propagator in the phase-space approach is equal to i times the $\dot{q}q$ propagator in the configuration space approach, in agreement with the linearized field equations $g^{ij}p_{k,j} = i(q^i_k - q^i_{k-1})/\epsilon$. However, the pp propagator is not equal to minus the $\dot{q}\dot{q}$ propagator; rather, there is an extra term proportional to $\delta_{k,k'}$ which comes from the term with F^2

$$\langle p_{k,i}p_{l,j}\rangle = -g_{ii'}(z)g_{jj'}(z)\langle \dot{q}^{i'}_{k-1/2}\dot{q}^{j'}_{l-1/2}\rangle + \frac{\hbar}{\epsilon}g_{ij}(z)\delta_{k,l} \ . \tag{2.2.5}$$

The last term cancels the singularity $\frac{\hbar}{\epsilon}g^{ij}(z)\delta_{k,l}$ in the term with $\langle \dot{q}^i_{k-1/2}\dot{q}^j_{l-1/2}\rangle$, see (2.1.53). Hence, as well-known, the phase-space propagator is nonsingular for short distances. The continuum limit reads

$$\begin{aligned}
\langle p_i(\sigma)p_j(\tau)\rangle &= \frac{1}{\beta}\hbar g_{ij}(z) \\
\langle q^i(\sigma)p_j(\tau)\rangle &= -i\hbar\delta^i_j(\sigma + \theta(\tau-\sigma)) = \langle p_j(\tau)q^i(\sigma)\rangle \\
\langle q^i(\sigma)q^j(\tau)\rangle &= -\beta\hbar g^{ij}(z)\Delta(\sigma,\tau) \ .
\end{aligned} \tag{2.2.6}$$

Using

$$\begin{aligned}
-\frac{1}{\hbar}H^{(int)} = \sum_{k=1}^{N}\Big[&-\frac{\epsilon}{2\hbar}[g^{ij}(x_{k-1/2}) - g^{ij}(z)]p_{k,i}p_{k,j} \\
&+\frac{i}{\hbar}p_{k,j}(x^j_{bg,k} - x^j_{bg,k-1}) - \epsilon\frac{\hbar}{8}(R + g^{ij}\Gamma_{ik}{}^l\Gamma_{jl}{}^k)\Big]
\end{aligned} \tag{2.2.7}$$

we can again compute the transition element loop-by-loop. In the continuum limit we find

$$\begin{aligned}
-\frac{1}{\hbar}\int_{-\beta}^{0}H^{(int)}dt &= -\frac{\beta}{\hbar}\int_{-1}^{0}\frac{1}{2}(g^{ij}(x) - g^{ij}(z))p_ip_j d\tau \\
&+\frac{i}{\hbar}\int_{-1}^{0}p_j(z^j - y^j)d\tau - \frac{\beta\hbar}{8}\int_{-1}^{0}(R + g^{ij}\Gamma_{il}{}^k\Gamma_{jk}{}^l)d\tau \ .
\end{aligned} \tag{2.2.8}$$

The tree graph with two $p(z-y)$ vertices now yields the leading term $-\frac{1}{2\beta\hbar}g_{ij}(z)(z^i - y^i)(z^j - y^j)$ in the classical action, see section 2.5, in particular (2.5.30), and the reader can check a few other graphs. Of course, the propagators as well as the vertices differ in the phase space approach (the latter contain $g^{ij}(x)$ instead of $g_{ij}(x)$ and there are no ghosts), but the result for $\langle z|\exp(-\frac{\epsilon}{\hbar}H)|y\rangle$ should be the same according to Matthews' theorem.[4] We shall later check this in a few examples.

In the phase space approach, we have trajectories $x^j(t)$ and $p_j(t)$, but we imposed only boundary conditions on $x^j(t)$, namely $x^j(0) = z^j$ and $x^j(-\beta) = y^j$. This is the correct number of boundary conditions, both in configuration and phase space, and we need only boundary conditions on x because we consider $T(z, y; \beta)$. One could also consider transition elements with a p-eigenstate at $t = 0$ and/or $t = -\beta$, and then one would need boundary conditions for the trajectories $p_j(t)$. (One could also obtain these transition elements by Fourier transform of $\langle z|\exp(-\beta\hat{H})|y\rangle$). For $T(z, y; \beta)$ no boundary condition on $p_j(t)$ are needed to make the p integrals convergent because $p_j(t)$ has no zero modes. (A zero mode is a mode which drops out of the action). There are no zero modes for $p_j(t)$ because it appears without derivatives in the action, with a leading term p^2. For $q^j(t)$, only differences $q^j_k - q^j_{k-1}$ appear in the discretized expressions of $T(z, y; \beta)$, hence we must fix the zero mode of $q^j(t)$ by suitable boundary conditions. The complete sets of p_k states were inserted between x-eigenstates, so one can view them as being defined at midpoints, not at the endpoints, and this suggests that for the computation of $T(z, y; \beta)$ one should not impose boundary conditions on the momenta as well. The most compelling reason is that there are no classical trajectories in phase space which connect two arbitrary points in phase space. (In the so-called holomorphic approach [32], one introduces variables $z \sim x + ip$ and $\bar{z} \sim x - ip$, and then one does impose separate boundary conditions at both endpoints but the classical z_{cl} and $\bar{z}_{cl}$ are then not each others complex conjugates.)

It is instructive to see how in the continuum approach the p-propagators are obtained. The kinetic terms in the phase space approach are given by

$$-\frac{1}{\hbar}S^{(0)} = \frac{i}{\hbar}\int_{-\beta}^{0} p_j \dot{q}^j \, dt - \frac{1}{\hbar}\int_{-\beta}^{0} \frac{1}{2} g^{ij}(z) p_i p_j \, dt \ . \tag{2.2.9}$$

The kinetic matrix for (p_i, q^j) is thus (replacing $g^{ij}(z)$ and $\hbar$ by unity for

[4] The original Matthews' theorem only applied to meson field theories with at most one time-derivative in the interaction [51]. It was extended to quantum mechanical models with $\dot{q}\dot{q}$ interactions and higher-time derivatives by Nambu [52]. Provided one adds the new ghosts as we have done, the equivalence between the Lagrangian and Hamiltonian approach also holds for nonlinear sigma models. For a general proof of Matthews' theorem based on path integrals see [53].

notational simplicity)

$$K = \begin{pmatrix} 1 & -i\partial_t \\ i\partial_t & 0 \end{pmatrix} , \tag{2.2.10}$$

and the Feynman (translationally invariant) propagator is its inverse

$$G = \begin{pmatrix} 0 & -\frac{i}{2}\epsilon(t-t') \\ \frac{i}{2}\epsilon(t-t') & -\frac{1}{2}(t-t')\theta(t-t') + (t \leftrightarrow t') \end{pmatrix} . \tag{2.2.11}$$

It satisfies $KG = \delta(t-t')$. To satisfy the boundary condition $q(0) = q(-\beta) = 0$, while still satisfying $KG = \delta(t-t')$, we add to G a polynomial in t, t' which is annihilated by K

$$P(t,t') = \begin{pmatrix} p_1(t') & p_2(t') \\ -itp_1(t') - iq_1(t') & -itp_2(t') - iq_2(t') \end{pmatrix} . \tag{2.2.12}$$

We then require that $(G+P)_{12}$ vanishes at $t' = 0, -\beta$, and $(G+P)_{21}$ at $t = 0, -\beta$, and $(G+P)_{22}$ at all $t = 0, -\beta$ and $t' = 0, -\beta$. (These entries correspond to $\langle pq \rangle$, $\langle qp \rangle$ and $\langle qq \rangle$, respectively). The solution is

$$P(t,t') = \begin{pmatrix} \frac{1}{\beta} & -i\left(\frac{t'}{\beta} + \frac{1}{2}\right) \\ -i\left(\frac{t}{\beta} + \frac{1}{2}\right) & -\frac{tt'}{\beta} - \frac{1}{2}(t+t') \end{pmatrix} . \tag{2.2.13}$$

Adding P to G, it is clear that the naive continuum results in (2.2.6) agree with the discretized propagators, again except when both t and t' lie on the boundaries. For these values one must again use the discretized propagators.

The Feynman propagator in (2.2.11) is given in position space, but for an analysis of the divergences in loops it is more convenient to give it in momentum space. The Fourier transform of (2.2.11) may not seem obvious, but it helps to first add a small mass term $\frac{1}{2}m^2q^2$ to $S^{(0)}$. The kinetic operator now becomes

$$K = \begin{pmatrix} 1 & -i\partial_t \\ i\partial_t & m^2 \end{pmatrix} , \tag{2.2.14}$$

and the Feynman propagator becomes

$$G(t-t'; m^2) = \begin{pmatrix} \frac{m}{2}e^{-m|t-t'|} & -\frac{i}{2}\epsilon(t-t')e^{-m|t-t'|} \\ \frac{i}{2}\epsilon(t-t')e^{-m|t-t'|} & \frac{1}{2m}e^{-m|t-t'|} \end{pmatrix} . \tag{2.2.15}$$

For small m one recovers (2.2.11), except that one finds in the qq propagator the constant $\frac{1}{2m}$. It cancels in KG, but if one were to evaluate Feynman graphs for a massive theory (with for example $V = \frac{1}{2}m^2q^2 + \lambda q^4$)

in the infinite t-interval one would need to include the contributions from this $\frac{1}{2m}$ term. The limit $m \to 0$ would then lead to infrared divergences which makes the theory ill-defined. On a finite t-interval one must specify boundary conditions, similarly to what we did in the massless theory with the matrix P in (2.2.13), and the mass singularity is removed by the boundary conditions. The Fourier transform of $G(t-t';m^2)$ is easily found

$$G(t-t';m^2) = \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} e^{ik_0(t-t')} \begin{pmatrix} \frac{m^2}{k_0^2+m^2} & \frac{-k_0}{k_0^2+m^2} \\ \frac{k_0}{k_0^2+m^2} & \frac{1}{k_0^2+m^2} \end{pmatrix} . \tag{2.2.16}$$

By decomposing m^2 into $(k_0^2+m^2) - k_0^2$ we see once again how the delta function singularity in the $\dot{q}\dot{q}$ propagator is canceled in the pp propagator: the propagator in (2.2.16) is clearly nonsingular as $t' \to t$, but the $\dot{q}\dot{q}$ propagator corresponds to the numerator k_0^2 and is singular as $t' \to t$. We can now study divergences in phase space.

All loops computed with phase space Feynman diagrams are finite because in t-space all propagators are bounded and all integration regions are finite. One can also explain by power counting methods (which are formulated in momentum space) why phase-space path integrals are convergent, while configuration space path integrals are only convergent after taking the ghosts into account. The kinetic matrix has entries unity (from the p^2 term) and k (from the $p\dot{q}$ term). Disregarding $P(t,t')$, on an infinite interval the pp propagators vanish, while the qq propagators behave like $\frac{1}{k^2}$ for large k but the pq propagators go only like $\int dk k/k^2$ and would seem to lead to a ultraviolet divergence. However, the integral over k/k^2 vanishes since it is odd in k. These results do not change if there is a mass term of the form $\frac{1}{2}m^2q^2$ present.

To illustrate the calculations in phase space by another example, consider the following Hamiltonian

$$\hat{H} = \frac{1}{2} g^{\alpha} p_i g^{1/2} g^{ij} p_j g^{-\frac{1}{2}-\alpha} . \tag{2.2.17}$$

For $\alpha = -1/4$, this is just (2.1.1), but for $\alpha \neq -1/4$ there are extra terms proportional to $\alpha + 1/4$

$$\begin{aligned} \hat{H} &= \hat{H}(\alpha = -1/4) + \hat{H}(\alpha + 1/4) ; \\ \hat{H}(\alpha + \frac{1}{4}) &= \frac{1}{2}(\alpha + \frac{1}{4}) i\hbar \left\{ p_i, g^{ij}\partial_j \ln g \right\} \\ &\quad - \frac{1}{2}(\alpha + \frac{1}{4})^2 \hbar^2 g^{ij} (\partial_i \ln g)(\partial_j \ln g) . \end{aligned} \tag{2.2.18}$$

Since the extra terms are Weyl-ordered, we can at once go to the phase-space path integral. Suppose we were to compute

$$\mathrm{Tr}\,\sigma(x)e^{-\frac{\beta}{\hbar}H} = \int dx_0 \sqrt{g(x_0)}\sigma(x_0)\langle x_0|e^{-\frac{\beta}{\hbar}H}|x_0\rangle \ . \tag{2.2.19}$$

(These kinds of expressions are found when one evaluates trace anomalies, but these interpretations do not concern us at this point). The path integral leads to

$$\langle e^{-\frac{\beta}{\hbar}\int_{-1}^{0} H^{(int)}d\tau}\ e^{-\frac{\beta}{\hbar}\int_{0}^{1} H^{(int)}(\alpha+1/4)d\tau}\rangle \tag{2.2.20}$$

with $H^{(int)}$ given by (2.2.8) and

$$\begin{aligned} -\frac{\beta}{\hbar}H^{(int)}(\alpha+\frac{1}{4}) &= -i(\alpha+\frac{1}{4})(\beta p_i)g^{ij}\partial_j \ln g \\ +\frac{1}{2}(\beta\hbar)(\alpha+\frac{1}{4})^2 & g^{ij}(\partial_i \ln g)(\partial_j \ln g) \ . \end{aligned} \tag{2.2.21}$$

Since the trace is cyclic, the result for the path integral should be α independent. To order $(\alpha+\frac{1}{4})$ there are no contributions since p_i is a quantum field whose vacuum expectation value vanishes.[5] However, to order $(\alpha+\frac{1}{4})^2$ there are two contributions: a tree graph with two $(\alpha+\frac{1}{4})$ vertices and a *pp* propagator, and further the vertex proportional to $(\alpha+\frac{1}{4})^2$. Using the *pp* propagators from (2.2.6), the sum of both contributions clearly cancels, as it should.

Upon eliminating the momenta p_j, the phase space path integral becomes a configuration space path integral, and infinities are introduced which are canceled by new ghosts, as we have discussed. The phase-space approach should yield the same finite answers as the configuration space approach. That this indeed happens is called Matthews' theorem. We illustrate it with a few examples, although a formal path integral proof can also be given, see [53].

The interaction part of the action for the phase space path integral differs from that for the configuration space path integral by the following terms

$$\begin{aligned} &-\frac{1}{\hbar}\int_{-\beta}^{0}\left\{H^{(int)}_{phase}(p,q) - H^{(int)}_{conf}(q)\right\}dt = \\ &-\frac{\beta}{\hbar}\int_{-1}^{0}\frac{1}{2}\left\{g^{ij}(x)-g^{ij}(z)\right\}p_i p_j d\tau \\ &+\frac{1}{\beta\hbar}\int_{-1}^{0}\frac{1}{2}\left\{g_{ij}(x)-g_{ij}(z)\right\}\left(\dot q^i \dot q^j + b^i c^j + a^i a^j\right)d\tau \end{aligned}$$

[5] There is also a term linear in p in $H(\alpha=-\frac{1}{4})$, namely the term $-i\int_{-1}^{0} p_j(z^j-y^j)\,d\tau$ in (2.2.7), but it does not contribute to the trace in (2.2.19) because $z=y=x_0$ in the trace.

$$
+\frac{i}{\hbar}\int_{-1}^{0} p_j(z^j-y^j)d\tau+\frac{1}{\beta\hbar}\int_{-1}^{0}\frac{1}{2}g_{ij}(x)(z^i-y^i)(z^j-y^j)d\tau
$$
$$
+\frac{1}{\beta\hbar}\int_{-1}^{0} g_{ij}(x)(z^i-y^i)\dot{q}^j d\tau \qquad \text{where} \quad x=z+\tau(z-y)+q \ . \tag{2.2.22}
$$

Inserting the complete field equation for p, namely $p_i=\frac{i}{\beta}g_{ij}(x)\dot{x}^i$, into the complete action for p, which reads $-\frac{1}{2}g^{ij}(x)p_ip_j+ip\dot{x}$, one obtains, of course, the complete action in configuration space, namely $-\frac{1}{2}g_{ij}(x)\dot{x}^i\dot{x}^j$ plus ghosts. However, if one calculates in perturbation theory, one decomposes p and q into a free part (in-and-out fields) and the rest. These free parts satisfy free field equations which differ, of course, from the full field equations and for p they read $p_i^{(0)}=\frac{i}{\beta}g_{ij}(z)\dot{q}^j$. Substituting these free field equations into the kinetic part of the phase space action (of the form $ip\dot{q}-\frac{1}{2}p^2$) one finds the free part of the configuration space action ($-\frac{1}{2}\dot{q}^2$), but **the interaction parts of the phase-space and configuration space actions differ after substituting the free p field equations.** The difference is easily calculated in our case

$$
-\frac{1}{\hbar}\Delta S=\frac{1}{\beta\hbar}\int_{-1}^{0}\frac{1}{2}\left\{g_{ij}(z)g^{jk}(x)g_{kl}(z)+g_{il}(x)-2g_{il}(z)\right\}\dot{q}^i\dot{q}^l\,d\tau
$$
$$
+\frac{1}{\beta\hbar}\int_{-1}^{0}\frac{1}{2}\left\{g_{ij}(x)-g_{ij}(z)\right\}(b^ic^j+a^ia^j)\,d\tau
$$
$$
+\frac{1}{\beta\hbar}\int_{-1}^{0}\left\{g_{ij}(x)-g_{ij}(z)\right\}\dot{q}^i(z^j-y^j)\,d\tau
$$
$$
+\frac{1}{\beta\hbar}\int_{-1}^{0}\frac{1}{2}g_{ij}(x)(z^i-y^i)(z^j-y^j)\,d\tau \ . \tag{2.2.23}
$$

Clearly, for linear sigma models with $g_{ij}=\delta_{ij}$ and no background fields ($z=y$), the actions are the same, $\Delta S=0$. The claim of Matthews' theorem is now that the effects of the extra term $\hbar g^{ij}(z)\delta(\sigma-\tau)$ in the $\dot{q}^i\dot{q}^j$ propagator cancel the effects due to ΔS. In other words the phase space approach and the configuration space approach should give the same result.

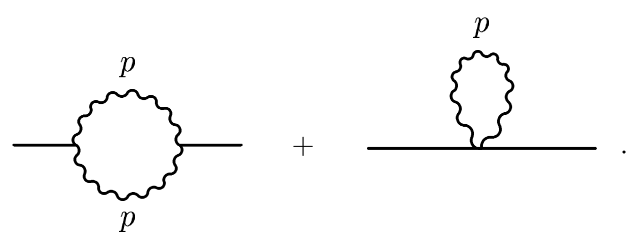
To avoid confusion, we spell out the procedure in detail. In the phase space approach one has an interaction $L^{(int)}_{phase}(p,q)$. The interactions in the configuration space approach are given by $L^{(int)}_{conf}(\dot{q},q)$. The statement that these interactions are different means that $L^{(int)}_{phase}(p,q)\neq L^{(int)}_{conf}(\dot{q},q)$ where p should be eliminated using the field equations of motion of $S^{(0)}$. This field equation reads $p_i=g_{ij}(z)\dot{q}^j$. If one changes the notation in $L^{(int)}_{phase}$ and writes $L^{(int)}_{phase}(i\dot{q},q)$, then one may use the same propagators for $\langle qq\rangle$

and $\langle q\dot{q}\rangle$ as in configuration space approach, but for $\langle\dot{q}\dot{q}\rangle$ the propagators in the phase space approach and the configuration space approach are different. The claim of Matthews' theorem is then that one may either work with $L^{(int)}_{phase}$ and the nonsingular propagator for $\langle\dot{q}\dot{q}\rangle$, or with $L^{(int)}_{conf}$ but with singular propagators for $\langle\dot{q}\dot{q}\rangle$. Green functions (with p identified with $i\dot{q}$ on external lines) should be the same.

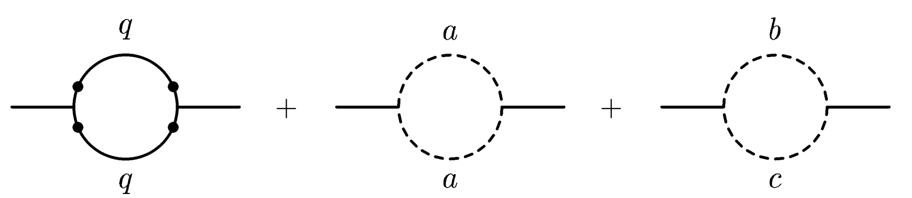
To bring out the essentials, we consider a simplified model, in which $z = y$ and $g_{ij}(z) = \delta_{ij}$ and $g_{ij}(x) - \delta_{ij} \equiv A_{ij}(q)$ and we choose an ordering of $\hat{H}$ such that there are no extra terms of order $\hbar^2$. Furthermore, the model is one-dimensional, so $i, j = 1$. This leads to $L = \frac{1}{2\beta}\dot{q}(1+A)\dot{q}$ with $A = A(q)$. We define H in the Euclidean case by $i\dot{q}p - H = -L$. This yields $H = \frac{\beta}{2}p\frac{1}{1+A}p$. Then the interaction Hamiltonian and Lagrangian read, respectively,

$$-\frac{1}{\hbar}H^{(int)} = \frac{\beta}{2\hbar}p\frac{A}{1+A}p \quad \Leftrightarrow \quad -\frac{1}{\hbar}L^{(int)} = -\frac{1}{2\beta\hbar}(\dot{q}A\dot{q} + bAc + aAa) \ . \tag{2.2.24}$$

The free field equation for p reads $\beta p = i\dot{q}$, and substitution into H^{int} produces a result which differs from $-L^{int}$ because instead of A one finds $A/(1+A)$ at vertices (and because there are no ghosts in H^{int}). However, also the $\langle\dot{q}\dot{q}\rangle$ and $\langle pp\rangle$ propagators are different. In the phase space approach the qq one-loop selfenergy receives contributions from a p-loop and a seagull graph with a pp loop



The external lines denote $A(q)$. In the configuration space approach there are $\dot{q}\dot{q}$ loops and ghost loops but no seagull graph



Matthews' theorem claims that both results are equal. Comparing both results, we see that in the configuration space approach one is left with the integrand

$$\frac{1}{4}\left\{({}^{\bullet}\Delta^{\bullet})^2 - ({}^{\bullet\bullet}\Delta)^2\right\} = \frac{1}{4}\left\{1 - 2\delta(\sigma-\tau)\right\} \ . \tag{2.2.25}$$

The factor 1 agrees with the result one obtains in the Hamiltonian approach from the p-loop, whereas the factor $\frac{1}{4}(-2)\delta(\sigma-\tau)$ agrees with result one obtains in the Hamiltonian approach from the the seagull graph. (To write the integration $\int d\tau$ of the seagull graph also as a double integral $\int d\tau \int d\sigma$ we added the factor $\delta(\tau-\sigma)$. The $\delta(\sigma-\tau)$ contracts the selfenergy graph to a seagull graph.) Hence, the extra term in the $\dot{q}\dot{q}$ propagator (the $\delta(\sigma-\tau)$) gives the same contribution as the extra vertex $(ppAA)$.

The reader may verify that also other Green's functions give the same results. For example the $p-p$ selfenergy gives the same result as minus the $\dot{q}\dot{q}$ selfenergy (the minus sign comes from the factor i in $p = i\dot{q}$) because the mixed loops (with Ap and pA propagators or with $A\dot{q}$ and $\dot{q}A$ propagators) agree, whereas in the phase-space case the loop with an AA and a pp propagator plus the seagull graph with an AA propagator gives the same result as in the configuration case the loop with an AA and a $\dot{q}\dot{q}$ propagator. There are no ghost contributions to the 1-loop pp selfenergy.

The difference between the Hamiltonian and Lagrangian approach to quantum field theories with derivative interactions historically first became a source of confusion in the 1940's when "mesotron theories" (theories with scalar fields) were studied with gradient couplings. (QED was in this respect simpler because it had no derivatives interactions, but scalar QED with $L = -|\partial_\mu\varphi - ieA_\mu\varphi|^2$ was studied and it has the same difficulties). Matthews' theorem [51] clarified the situation for these theories, and a general analysis for quantum mechanical models with double-derivatives interactions was given by Nambu [52], and later by Lee and Yang [25]. By the 1970's it had become clear that one could use the action itself to obtain the interaction vertices, and the propagators were "covariant", by which is meant that propagators of derivatives fields were equal to derivatives of propagators of the fields

$$\langle \partial_\mu A_\nu(x) \partial_\rho A_\sigma(y)\rangle = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\rho}\langle A_\nu(x) A_\sigma(y)\rangle \ . \tag{2.2.26}$$

To retain part of the canonical methods so that one could work out the radiative corrections to current algebra, one introduced the notion of a T^* product, so one wrote $\langle B(x)C(y)\rangle = \langle\Omega|T^* B(x)C(y)|\Omega\rangle$, where $|\Omega\rangle$ denotes the vacuum. This T^* operator commutes with derivatives, and is thus different from the usual time ordering symbol T which involves theta functions $\theta(x^0-y^0)$ and $\theta(y^0-x^0)$ and thus does not commute with $\frac{\partial}{\partial x^0}$

and $\frac{\partial}{\partial y^0}$. In fact, for one time derivative the results of using T^* or T are still the same because

$$\begin{aligned}
&\frac{\partial}{\partial x^0}\langle\Omega|TA_\mu(x)A_\nu(y)|\Omega\rangle \\
&\quad = \langle\Omega|T\partial_0 A_\mu(x)A_\nu(y)|\Omega\rangle + \delta(x^0-y^0)[A_\mu(x),A_\nu(y)] \\
&\quad = \langle\Omega|T\partial_0 A_\mu(x)A_\nu(y)|\Omega\rangle \\
&\quad = \langle\Omega|T^*\partial_0 A_\mu(x)A_\nu(y)|\Omega\rangle \qquad (2.2.27)
\end{aligned}$$

since $[A_\mu(x), A_\nu(y)] = 0$. But for two time derivatives the Hamiltonian (canonical) propagator with T and the Lagrangian (covariant) propagator with T^* differ. For example, for QED

$$\begin{aligned}
&\frac{\partial}{\partial x^0}\frac{\partial}{\partial y^0}\langle\Omega|TA_\mu(x)A_\nu(y)|\Omega\rangle = \frac{\partial}{\partial x^0}\langle\Omega|TA_\mu(x)\partial_0 A_\nu(y)|\Omega\rangle \\
&= \langle\Omega|T\partial_0 A_\mu(x)\partial_0 A_\nu(y)|\Omega\rangle + \delta(x^0-y^0)[A_\mu(x),\partial_0 A_\nu(y)] \\
&= \langle\Omega|T^*\partial_0 A_\mu(x)\partial_0 A_\nu(y)|\Omega\rangle + i\hbar\delta^4(x-y)\eta_{\mu\nu} \qquad (2.2.28)
\end{aligned}$$

where we used that $\partial_0 A_\nu = P(A_\nu) + \ldots$.

As we have already seen in the case of nonlinear sigma models, in the canonical approach there are extra vertices and extra terms in the propagators, but all these extra effects cancels if one computes Green's functions. In the 1960's "current algebra" was developed as a tool to deal with the strong interactions in a nonperturbative way. This was an operator formalism, which therefore used T products, but complicated noncovariant extra terms ("Schwinger terms") were found to be present in the commutation relations of (in particular the space components of) currents. To simplify the current algebras, the T^* product was introduced, and relations between the current algebra with T ordering and with T^* ordering were developed. Here, of course, the theorems by Matthews and Nambu were of some use. We shall not enter a discussion of current algebras, but instead we now study the same problems in nonabelian gauge theory. We do this for a change in Minkowski space.

Consider the nonghost sector. After adding the gauge fixing term $\mathcal{L}_{(fix)} = -\frac{1}{2}(\partial_\mu A^\mu)^2$ the Lagrange density reads

$$\begin{aligned}
\mathcal{L}(q) &= -\frac{1}{2}(\partial_\mu A^\mu)^2 - \frac{1}{4}F_{\mu\nu}^2 \\
&= \frac{1}{2}(\partial_0 A_j)^2 - \frac{1}{2}(\partial_0 A_0)^2 + \frac{1}{2}(\partial_j A_0)^2 - \frac{1}{2}(\partial_i A_j)^2 \\
&+ \partial_0 A_j A_0\wedge A_j - \partial_j A_0 A_0\wedge A_j - \frac{1}{4}(F_{ij}^2)^{int} + \frac{1}{2}(A_0\wedge A_j)^2
\end{aligned} \qquad (2.2.29)$$

where $(F_{ij}^2)^{int} = F_{ij}^2 - (\partial_i A_j - \partial_j A_i)^2$ and $A \wedge B$ denotes $f^{abc} A^b B^c$. The conjugate momenta are

$$\begin{aligned} p^j &\equiv p(A_j) = \partial_0 A_j + A_0 \wedge A_j = D_0 A_j = D_0 A^j \\ p^0 &\equiv p(A_0) = -\partial_0 A_0 = \partial_0 A^0 \ . \end{aligned} \tag{2.2.30}$$

The Hamiltonian density $\mathcal{H} = \dot{A}_\mu p^\mu - \mathcal{L}$ becomes

$$\begin{aligned} \mathcal{H}(p,q) &= \frac{1}{2}(p^j)^2 + p^j A_j \wedge A_0 - \frac{1}{2}(p^0)^2 - \frac{1}{2}(\partial_j A_0)^2 \\ &+ \partial_j A_0 A_0 \wedge A_j + \frac{1}{2}(\partial_i A_j)^2 + \frac{1}{4}(F_{ij}^2)^{int} \ . \end{aligned} \tag{2.2.31}$$

The Lagrangian density in the phase space approach is then

$$\mathcal{L}(p,q) = p^j \dot{A}_j + p^0 \dot{A}_0 - \mathcal{H}(p,q) \ . \tag{2.2.32}$$

We define $\mathcal{L}^{(0)}$ to be the terms quadratic in fields. The interactions in phase space are given by

$$\mathcal{L}^{(int)}_{phase}(p,q) = -\mathcal{H}^{(int)}(p,q) = -p^j A_j \wedge A_0 - \partial_j A_0 A_0 \wedge A_j - \frac{1}{4}(F_{ij}^2)^{int} \ . \tag{2.2.33}$$

On the other hand, in configuration space the interactions follow from (2.2.29)

$$\begin{aligned} \mathcal{L}^{(int)}_{conf}(q) &= -\frac{1}{4}(F_{\mu\nu}^2)^{int} = \frac{1}{2}(F_{0j}^2)^{int} - \frac{1}{4}(F_{ij}^2)^{int} \\ &= \Big[(\partial_j A_0 - \partial_0 A_j) A_j \wedge A_0 - \frac{1}{4}(F_{ij}^2)^{int}\Big] + \frac{1}{2}(A_0 \wedge A_j)^2 \ . \end{aligned} \tag{2.2.34}$$

The reason for grouping these terms in this way will become clear.

If one first eliminates $p^\mu = \partial S / \partial \dot{A}_\mu$ from the action $\int \mathcal{L}(p,q)$ by using the full nonlinear field equation $p^j = D_0 A^j$ and $p^0 = -\partial_0 A_0$, one recovers, of course $\mathcal{L}(q)$. One may check this by replacing p^j in $\mathcal{L}^{(int)}_{phase}(p,q)$ by $D_0 A^j$, and further by substituting $p^j = A_0 \wedge A_j$ into the terms of $\mathcal{L}(p,q)$ which are bilinear in p and q. One finds then

$$\begin{aligned} \mathcal{L}^{(int)}_{phase} - \mathcal{L}^{(int)}_{conf} &= -A_0 \wedge A_j A_j \wedge A_0 + A_0 \wedge A_j \dot{A}_j \\ &- \Big[\partial A_0 A_0 \wedge A_j + \frac{1}{2}(A_0 \wedge A_j)^2\Big] - \frac{1}{2}(A_0 \wedge A_j)^2 = 0 \ . \end{aligned} \tag{2.2.35}$$

Suppose one performs perturbation theory, using the interaction picture, both with the Hamiltonian theory in phase space and with the Lagrangian theory in configuration space. One has then the vertices from

$\mathcal{L}^{(int)}_{phase}$ and $\mathcal{L}^{(int)}_{conf}$, and the propagators as they are found from the terms bilinear in p and q, and linear in q, respectively

The propagators $\langle p^j A_\mu\rangle$ and $\langle p^0 A_\mu\rangle$ in phase space are the same as the propagators $\langle \partial_0 A^j A_\mu\rangle$ and $\langle \partial_0 A^0 A_\mu\rangle$ in configurations space. Thus one may substitute, as far as the propagators are concerned, the linear field equation of p^μ into the propagators. However, as we already discussed, the $\langle p^\mu p^\nu\rangle$ propagators are not equal to the $\langle \partial_0 A^\mu \partial_0 A^\nu\rangle$ propagators (they differ by contact terms with $\delta^4(x-y)$). To facilitate comparison, we may therefore replace everywhere (both in the phase space action and in the phase space propagators) p^μ by $\partial_0 A^\mu$, but then there are two sources of extra terms in the phase space approach:

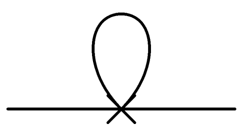
(i) the vertices $\mathcal{L}^{(int)}_{phase}(p^\mu \to \partial_0 A^\mu)$ differ from those in $\mathcal{L}^{(int)}_{conf}$ by extra terms $\mathcal{L}^{(int)\ extra}_{phase}(p,q) = -\frac{1}{2}(A_j \wedge A_0)^2$,

(ii) the propagators $\langle \partial_0 A^\mu \partial_0 A^\nu\rangle_{phase}$ in the phase space theory differ from the propagators $\langle \partial_0 A^\mu \partial_0 A^\nu\rangle_{conf}$ of the covariant Lagrangian by the extra contact terms

$$\langle \partial_0 A^\mu \partial_0 A^\nu\rangle_{phase} - \langle \partial_0 A^\mu \partial_0 A^\nu\rangle_{conf} = -i\hbar\delta^4(x-y)\eta^{\mu\nu}\ . \tag{2.2.36}$$

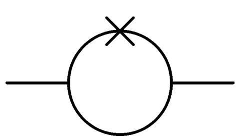
The content of Matthews' theorem is that all extra contributions cancels. We now check this in a few instructive examples.

Consider first the A_0A_0 self-energy in the phase-space approach at the one loop level. There is one extra diagram



$$\sim \frac{i}{\hbar}\int\Big(-\frac{1}{2}\Big)A_0\langle A_iA_i\rangle A_0 \tag{2.2.37}$$

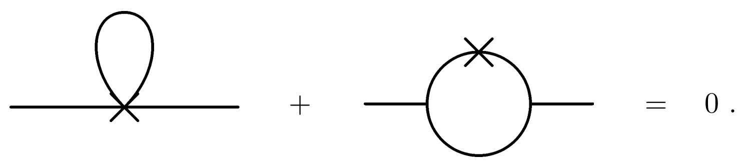
where the cross denotes that this is an extra vertex. To find the extra contributions from the propagators one first determine all interaction with one time derivative. These are given by $\mathcal{L}^{(int)} = -\partial_0 A_j A_j \wedge A_0$. These are ordinary vertices (vertices in the covariant theory), so there are no cross contributions where both extra vertices and extra propagators contribute. Two such vertices yield the following extra contribution



$$= \Big(\frac{i}{\hbar}\Big)^2\frac{1}{2!}\int\int A_0\langle A_iA_i\rangle(-i\hbar\delta^4(x-y))A_0$$

$$= \frac{i}{\hbar}\int \frac{1}{2} A_0 \langle A_i A_i \rangle A_0 \ . \tag{2.2.38}$$

The cross denotes the extra term in the propagator. Clearly the extra contributions indeed exactly cancel



The same cancellation follows for the $A_j A_j$ one-loop selfenergy.

Let us now see if the ghost sector gives extra contributions. The ghost action is

$$\mathcal{L} = -\partial^\mu b(\partial_\mu c + A_\mu \wedge c) = \dot{b}(\dot{c} + A_0 \wedge c) - \partial_j b D_j c \ . \tag{2.2.39}$$

The conjugate momenta (using left-differentiation, so $p(b) = \frac{\partial}{\partial \dot{b}} S$) are

$$p(b) = (\dot{c} + A_0 \wedge c) = D_0 c \ , \qquad p(c) = -\dot{b} \ . \tag{2.2.40}$$

The Lagrangian in phase space (for left-differentiation) is

$$\mathcal{L}_{phase} = \dot{c} p(c) + \dot{b} p(b) - (p(b) - [A_0, c]) p(c) - \partial_j b D^j c \ . \tag{2.2.41}$$

The interactions in phase space are

$$\mathcal{L}^{(int)}_{phase} = -p(c) A_0 \wedge c - \partial_j b (A_j \times c) \ . \tag{2.2.42}$$

The interactions in configuration space are

$$\mathcal{L}^{(int)}_{conf} = \partial_0 b A_0 \wedge c - \partial_j b (A_j \times c) \ . \tag{2.2.43}$$

Substituting the linearized field equation for $p(c)$ we find that there are no extra vertices

$$\mathcal{L}^{(int)}_{phase}(-p(c) = \dot{b}) - \mathcal{L}^{(int)}_{conf} = 0 \ . \tag{2.2.44}$$

There are also no contributions with an extra propagator term since these would have to come from $\langle \partial_0 b \partial_0 c \rangle$ but there are no interactions with $\partial_0 c$. Thus in the ghost sector there are no subtleties.

2.3 Path integrals for Dirac fermions

We shall now discuss the extension of the time slicing approach to fermions. We distinguish between complex (Dirac) fermions and real (Majorana) fermions. The latter have some special problems, so we begin with Dirac fermions ψ for which the conjugate momentum ($\partial\dot{\psi}\backslash\partial S = -i\psi^\dagger$ when we remove $\dot{\psi}$ from the left) is not proportional to ψ. In order to integrate in the path integral over fermions fields ψ, one must introduce Grassmann variables. It is sometimes said that path integrals with Grassmann variables are mathematically not well founded. We have two answers to such criticisms:

(i) In our approach we begin with operators (such as $\hat{\psi}$ and $\hat{\psi}^\dagger$) without any Grassmann variables. Then when we convert expressions such as $\mathrm{Tr}\, J \exp(-\beta H)$ into **discretized** path integrals we introduce Grassmann variables by means of mathematical **identities**. So the discretized path-integrals are mathematically well-defined.

(ii) The question whether the continuum limit of the path integrals exists is easier to prove for fermions than bosons because there are no convergence problems with Berezinian integration: $\int d\theta\, \theta = 1$. In particular, in our applications to anomalies we need only graphs with a given number of loops. We are then working at the level of perturbation theory, and at this level path integrals with fermions are manifestly finite and well-defined (as they are also when bosons are presents).

For fermions the problem is to evaluate again expressions such as $\mathrm{Tr}\, J \exp(-\beta H)$, where H contains now also fermions. This Hamiltonian is constructed from the Minkowskian action as $H = \dot{q}p - L$, but once it is constructed, it is a well-defined operator which acts in a well-defined Hilbert space. Hermiticity, general coordinate invariance of H and positivity of the energy define H uniquely up to corrections proportional to $\hbar^2 R$. Supersymmetry even determines the coefficient of R. In particular, there is no Wick rotation needed: all fermionic operators ψ and $\psi^\dagger$ in J and H are operators of the Minkowski theory, and $\psi^\dagger$ is equal the hermitian conjugate of ψ. When we insert complete sets of states in $\exp(-\frac{\beta}{\hbar}H)$ we will be led to a Euclidean path integral, but we arrive at this path integral by a series of identities, and not by a Wick rotation of the fermionic fields ψ and $\psi^\dagger$. The Minkowskian action for a free complex (Dirac) fermion in n dimensions is

$$S = \int \left[-(\psi^\dagger i\gamma^0) \left(\gamma^0 \frac{\partial}{\partial t} + \gamma^k \frac{\partial}{\partial x^k} \right) \psi \right] d^n x \ , \qquad (\gamma^0)^2 = -1 \ . \quad (2.3.1)$$

For one-component spinors ψ in quantum mechanics this reduces to

$$S = \int i\psi^\dagger \dot{\psi}\, dt \ . \qquad (2.3.2)$$

The conjugate momentum of ψ is $-i\psi^\dagger$ and the equal-time anticommutation relations yield $\{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = 0$ and further

$$\{-i\psi^\dagger, \psi\} = \frac{\hbar}{i} \quad \rightarrow \quad \{\psi, \psi^\dagger\} = \hbar \ . \tag{2.3.3}$$

If there are more than one pair of ψ and $\psi^\dagger$ we denote them by an index $a = 1, .., n$, as in ψ^a and $\psi_b^\dagger$. Then

$$\{\psi^a, \psi_b^\dagger\} = \hbar \, \delta_b^a \ . \tag{2.3.4}$$

In curved space this index a is a flat index, related to a curved index i by the vielbein fields $e^a{}_i(x)$ as usual in general relativity

$$\psi^a(t) = e^a{}_i(x(t)) \, \psi^i(t) \ . \tag{2.3.5}$$

It is convenient to work with flat indices for fermions and curved indices for bosons. Thus we shall be using $x^i(t)$ and $\psi^a(t)$ where in both cases i and a run form 1 to n.

Having defined the basic operators ψ^a and $\psi_b^\dagger$ we shall be using, we rewrite the trace as a path integral by inserting a complete set of states constructed from ψ^a and $\psi_b^\dagger$. In this way we shall arrive at a path integral representation of $\mathrm{Tr}\, J \exp(-\beta H)$. This path integral has the appearance of a Euclidean path integral (no $\frac{i}{\hbar}$ in front of the action). One might then, out of curiosity, wonder whether this Euclidean path integral could also have been obtained from a Minkowskian path integral by making a Wick rotation on the spinor fields ψ^a and $\psi_b^\dagger$ (and of course rotating the Minkowskian time t_M to $-it_E$ where t_E is the Euclidean time). This is a tricky question which we do not need to answer because we are using well-defined operators with given (anti) commutation relations, which were derived from the Minkowski theory but which in our applications lead to Euclidean path integrals. We repeat that we do not need to make any Wick rotation on the fermionic fields. The Wick rotation on fermionic fields has been discussed in [54].

One could try to parallel the bosonic treatment, and introduce eigenstates of $\hat{\psi}$, namely bras and kets, as well as also introduce a complete set of "momentum" eigenstates, i.e. eigenstates of $\hat{\psi}^\dagger$ which are again bras and kets. These eigenstates are coherent states as we shall see, so this approach would lead to four kinds of coherent states, with several inner products to be specified. This is one of the approaches in the literature [55], but we shall follow a simpler approach which for bosons has already been discussed in textbooks [32, 33]. Namely we shall only need one kind of coherent bras and one kind of coherent kets, and thus need only one inner product.

We begin with Dirac fermions $\hat{\psi}^a$ and $\hat{\psi}_b^\dagger$ with the usual equal-time canonical anticommutation relations

$$\{\hat{\psi}^a, \hat{\psi}_b^\dagger\} = \hbar\delta_b^a; \quad \{\hat{\psi}^a, \hat{\psi}^b\} = \{\hat{\psi}_a^\dagger, \hat{\psi}_b^\dagger\} = 0 \ . \tag{2.3.6}$$

For $N = 1$ supersymmetric system with Majorana fermions the conjugate momentum is proportional to ψ^a itself, and for certain purposes one can use Dirac brackets without having to distinguish between annihilation and creation operators. For our purposes, however, we need to be able to distinguish between $\hat{\psi}$ and $\hat{\psi}^\dagger$, and we shall later show how to do this for $N = 1$ models. (We shall either add another set of free Majorana fermions and then construct a larger Hilbert space, or combine pairs of Majorana spinors into $\hat{\psi}$ and $\hat{\psi}^\dagger$ and construct a smaller Hilbert space. Both approaches yield the same final results for physical quantities as we shall see). We repeat that in this section we restrict our attention to Dirac fermions. This is enough for $N = 2$ models.

To define fermionic coherent states without having to write factors of $\hbar^{\pm 1/2}$ all the time, it is useful to introduce rescaled variables $\hat{\psi}^a \to \hbar^{1/2}\hat{\psi}^a$ and $\hat{\psi}_a^\dagger \to \hbar^{1/2}\hat{\psi}_a^\dagger$, satisfying

$$\left\{\hat{\psi}^a, \hat{\psi}_b^\dagger\right\} = \delta_b^a \ . \tag{2.3.7}$$

Later we shall scale back to reintroduce the factors of $\hbar$. The coherent states we need are then defined by (dropping hats from now on)

$$|\eta\rangle = e^{\psi_a^\dagger \eta^a}|0\rangle \ ; \quad \psi^a|0\rangle = 0 \tag{2.3.8}$$

$$\langle\bar{\eta}| = \langle 0|e^{\bar{\eta}_a \psi^a} \ ; \quad \langle 0|\psi_a^\dagger = 0 \ . \tag{2.3.9}$$

We choose the η^a and $\bar{\eta}_a$ as **independent** complex (i.e., without reality conditions) Grassmann variables even though $\psi_b^\dagger = (\psi^b)^\dagger$. Therefore we write $\bar{\eta}_a$ instead of $\eta_a^\dagger$. (We could equally well have chosen $\bar{\eta}_a$ to be given by $(\eta^a)^\dagger$ because this does not change the result of the Grassmann integration). The state $|0\rangle$ is the Fock vacuum for the ψ's, and by definition it commutes with the Grassmann numbers: $|0\rangle\eta^a = \eta^a|0\rangle$ and $|0\rangle\bar{\eta}_a = \bar{\eta}_a|0\rangle$. The same property holds by definition for $\langle 0|$.

It is clear that these coherent states satisfy the following relations

$$\psi^a|\eta\rangle = \eta^a|\eta\rangle \ ; \ \langle\bar{\eta}|\psi_a^\dagger = \langle\bar{\eta}|\bar{\eta}_a \ . \tag{2.3.10}$$

To prove this one may expand the exponent. There are then only a finite number of terms because $\bar{\eta}_{a_1}\bar{\eta}_{a_2} = 0$ when $a_1 = a_2$.

The inner product is given by

$$\langle\bar{\eta}|\eta\rangle = e^{\bar{\eta}_a \eta^a} \ . \tag{2.3.11}$$

This relation follows from $e^A e^B = e^B e^A e^{[A,B]}$ with $A = \bar{\eta}_a \psi^a$ and $B = \psi_a^\dagger \eta^a$. Since $[A, B]$ commutes with A and B there are no further terms in the Baker-Campbell-Hausdorff formula.

Grassmann integration is defined by

$$\int d\eta^a \eta^b = \delta^{ab} \ , \quad \int d\bar{\eta}_a \bar{\eta}_b = \delta_{ab} \ , \quad \int d\eta^a = 0 \ , \quad \int d\bar{\eta}_a = 0 \ . \tag{2.3.12}$$

So, for example,

$$\int \prod_{a=1}^{n} d\bar{\eta}_a d\eta^a \Big(1 + \sum_{b=1}^{n} \eta^b \bar{\eta}_b\Big) = \delta_{n,1} \ . \tag{2.3.13}$$

Identities for one fermion are easily extended to the case of several fermions by observing that one can factorize into spaces with different a, for example

$$e^{\sum_{a=1}^{n} \psi_a^\dagger \eta^a} = \prod_{a=1}^{n} e^{\psi_a^\dagger \eta^a} = e^{\psi_1^\dagger \ \eta^1} e^{\psi_2^\dagger \ \eta^2} ... \, e^{\psi_n^\dagger \ \eta^n} \ . \tag{2.3.14}$$

The completeness relation reads

$$I = \int \Big(\prod_{a=1}^{n} d\bar{\eta}_a d\eta^a\Big) |\eta\rangle e^{-\bar{\eta}_a \eta^a} \langle\bar{\eta}| \ . \tag{2.3.15}$$

For one pair of $\bar{\eta}$ and η the completeness relation is easily checked by expanding the exponent

$$\int d\bar{\eta} d\eta (1 + \psi^\dagger \eta)|0\rangle(1 - \bar{\eta}\eta)\langle 0|(1 + \bar{\eta}\psi) = |0\rangle\langle 0| + \psi^\dagger|0\rangle\langle 0|\psi \ . \tag{2.3.16}$$

The right hand side is the identity operator in Fock space. Note the opposite sign in the exponent of the inner product and decomposition of unity. From now on we shall mean by $d\bar{\xi} d\xi$ the product

$$d\bar{\xi} d\xi \equiv \prod_{a=1}^{n} \ (d\bar{\xi}_a d\xi^a) = d\bar{\xi}_1 d\xi^1 \ldots d\bar{\xi}_n d\xi^n = d\bar{\xi}_n \ldots d\bar{\xi}_1 d\xi^1 \ldots d\xi^n \ . \tag{2.3.17}$$

We consider now the transition element between two coherent states

$$\langle\bar{\eta}| e^{-\frac{\beta}{\hbar}\hat{H}} |\eta\rangle \ . \tag{2.3.18}$$

We assume that $\hat{H}$ depends on $\hat{\psi}^a$, $\hat{\psi}_a^\dagger$ (and on $\hat{x}^i, \hat{p}_i$ which we suppress writing) with again an arbitrary but definite a priori operator ordering. In order to compute traces like $\mathrm{Tr}\, J \exp(-\frac{\beta}{\hbar} H)$, we shall use the completeness relation for coherent states to define the trace, and then all Grassmann variables are integrated over and disappear.

We begin by inserting $N-1$ complete sets of coherent states and obtain then (for clarity writing the integral signs between the coherent states to which they belong)

$$\begin{aligned}&\langle\bar\eta|e^{-\frac{\beta}{\hbar}\hat H}|\eta\rangle\\ &=\langle\bar\eta|e^{-\frac{\epsilon}{\hbar}\hat H}|\eta_{N-1}\rangle\int d\bar\eta_{N-1}d\eta_{N-1}e^{-\bar\eta_{N-1}\eta_{N-1}}\langle\bar\eta_{N-1}|e^{-\frac{\epsilon}{\hbar}\hat H}|\eta_{N-2}\rangle\\ &\quad\ldots\langle\bar\eta_1|e^{-\frac{\epsilon}{\hbar}\hat H}|\eta\rangle\ ;\quad \bar\eta\equiv\bar\eta_N\ ,\ \eta\equiv\eta_0\ ,\ \epsilon=\beta/N\ . \qquad (2.3.19)\end{aligned}$$

This is analogous to the insertion of $N-1$ sets of x eigenstates by Dirac. Next we introduce N other complete sets of coherent states which are analogous to the N complete sets of p eigenstates of Feynman. We do this to obtain a fermionic midpoint rule. We consider thus

$$\langle\bar\eta_{k+1}|e^{-\frac{\epsilon}{\hbar}\hat H}|\eta_k\rangle=\int d\bar\chi_k d\chi_k\langle\bar\eta_{k+1}|e^{-\frac{\epsilon}{\hbar}\hat H}|\chi_k\rangle e^{-\bar\chi_k\chi_k}\langle\bar\chi_k|\eta_k\rangle\ . \qquad (2.3.20)$$

Weyl ordering of fermionic operators $\hat\psi^a$ and $\hat\psi^\dagger_a$ is defined by expanding $(\bar\eta_a\hat\psi^a+\eta^a\hat\psi^\dagger_a)^N$ and retaining all terms with a given number of operators in the order they come. For one fermion one has

$$(m+n)!(\hat\psi^m\hat\psi^{\dagger n})_W=\left(\frac{\partial}{\partial\bar\eta}\right)^m\left(\frac{\partial}{\partial\eta}\right)^n(\bar\eta\hat\psi+\eta\hat\psi^\dagger)^N\ ;\quad N=m+n \qquad (2.3.21)$$

where m and n can only take the value 0 and 1. For several fermions one has for example

$$\begin{aligned}(\hat\psi^a\hat\psi^\dagger_b)_W&=\frac{1}{2}(\hat\psi^a\hat\psi^\dagger_b-\hat\psi^\dagger_b\hat\psi^a)\\ (\hat\psi^a\hat\psi^b\hat\psi^\dagger_c)_W&=\frac{1}{6}(2\hat\psi^a\hat\psi^b\hat\psi^\dagger_c-\hat\psi^a\hat\psi^\dagger_c\hat\psi^b+\hat\psi^b\hat\psi^\dagger_c\hat\psi^a+2\hat\psi^\dagger_c\hat\psi^a\hat\psi^b)\\ &=\frac{1}{2}(\hat\psi^a\hat\psi^b\hat\psi^\dagger_c+\hat\psi^\dagger_c\hat\psi^a\hat\psi^b) \qquad (2.3.22)\end{aligned}$$

where in the last line we kept $\psi^a\psi^b$ together. Weyl ordering of fermions is further explained and worked out in appendix C.

Given a Weyl ordered operator $\hat B(\hat\psi^\dagger,\hat\psi)$, the following midpoint rule holds

$$\begin{aligned}\langle\bar\eta|\hat B|\eta\rangle&=\int d\bar\chi d\chi e^{-\bar\chi\chi}\langle\bar\eta|\chi\rangle B\Big(\bar\chi,\frac{1}{2}(\chi+\eta)\Big)\langle\bar\chi|\eta\rangle\\ &=\int d\bar\chi d\chi e^{-\bar\chi\chi}\langle\bar\eta|\chi\rangle B\Big(\frac{1}{2}(\bar\eta+\bar\chi),\chi\Big)\langle\bar\chi|\eta\rangle\ . \qquad (2.3.23)\end{aligned}$$

Both formulas are true, but we shall only use the first one. The proof of this fermionic midpoint rule can either be given by following the same steps as in the bosonic case, or by starting with an operator $(\hat{\psi}^\dagger)^k$ (or $(\hat{\psi})^k$) for which (2.3.23) is obvious, and then using the property that if $\hat{A}$ is Weyl ordered then also $\frac{1}{2}(\hat{\psi}\hat{A} \pm \hat{A}\hat{\psi})$ is Weyl ordered. Repeated application of this property to $\hat{A} = (\hat{\psi}^\dagger)^k$ then proves the fermionic midpoint rule for any operator $\hat{B}$ which is a polynomial in $\hat{\psi}$ and $\hat{\psi}^\dagger$.

Next we use again the linear approximation. That is, we replace the Weyl-ordered operators $(\exp(-\frac{\epsilon}{\hbar}\hat{H}))_W$ by $\exp(-\frac{\epsilon}{\hbar}\hat{H}_W)$. In matrix elements (in particular the kernels of the path integral) these two expressions differ by terms which are of order ϵ^2 and higher, and in the path integral these extra terms do not contribute. For example, if $\hat{H} = \hat{\psi}^\dagger_a\hat{\psi}^a$ then $(\hat{H})_W = \frac{1}{2}(\hat{\psi}^\dagger_a\hat{\psi}^a - \hat{\psi}^a\hat{\psi}^\dagger_a) + n$ and the ϵ^2 terms in $(\exp(-\frac{\epsilon}{\hbar}\hat{H}))_W$ are given by $\frac{\epsilon^2}{2\hbar^2}(\hat{\psi}^\dagger_a\hat{\psi}^a\hat{\psi}^\dagger_b\hat{\psi}^b)_W$ while $\exp(-\frac{\epsilon}{\hbar}\hat{H}_W)$ yields $\frac{\epsilon^2}{2\hbar^2}(\hat{\psi}^\dagger_a\hat{\psi}^a)_W(\hat{\psi}^\dagger_b\hat{\psi}^b)_W$. The difference is $\frac{\epsilon^2 n}{8\hbar^2}$, see appendix C, equation (C.14). These terms of order ϵ^2 do not contribute for $\epsilon \to 0$. In the bosonic case one had two terms $p\Delta q$ and ϵp^2 in the exponent and whereas $p' = p - \frac{i}{\epsilon}\Delta q$ is of order $\epsilon^{-1/2}$, the phase space variable p itself is of order ϵ^0. For fermions there are not two terms in the exponent, but only one term $\bar{\chi}\Delta\chi$. Thus for fermions there are no subtleties.

After rewriting the Hamiltonian in Weyl ordered form and applying the linear approximation we arrive at

$$\langle\bar{\eta}|\exp\Big(-\frac{\beta}{\hbar}\hat{H}\Big)|\eta\rangle = \int\left(\prod_{k=1}^{N-1} d\bar{\eta}_k d\eta_k e^{-\bar{\eta}_k\eta_k}\right)\left(\prod_{k=0}^{N-1} d\bar{\chi}_k d\chi_k e^{-\bar{\chi}_k\chi_k}\right)$$
$$\prod_{k=0}^{N-1}\langle\bar{\eta}_{k+1}|\chi_k\rangle \exp\left(-\frac{\epsilon}{\hbar}H\left(\bar{\chi}_k, \frac{1}{2}(\eta_k+\chi_k)\right)\right)\langle\bar{\chi}_k|\eta_k\rangle \tag{2.3.24}$$

where $\langle\bar{\eta}_N| = \langle\bar{\eta}|$ and $|\eta_0\rangle = |\eta\rangle$. Substituting the inner products for coherent states, and using the lemma

$$\int d\bar{\eta}_k d\eta_k e^{-\bar{\eta}_k(\eta_k-\chi_{k-1})}f(\eta_k) = f(\chi_{k-1}) \tag{2.3.25}$$

we arrive at the following suggestive result

$$\langle\bar{\eta}|\exp\Big(-\frac{\beta}{\hbar}\hat{H}\Big)|\eta\rangle = \int\left(\prod_{k=0}^{N-1} d\bar{\chi}_k d\chi_k\right)$$
$$\exp\left[\bar{\eta}\chi_{N-1} - \epsilon\sum_{k=0}^{N-1}\left\{\bar{\chi}_k\left(\frac{\chi_k-\chi_{k-1}}{\epsilon}\right) + \frac{1}{\hbar}H\left(\bar{\chi}_k, \frac{\chi_k+\chi_{k-1}}{2}\right)\right\}\right] \tag{2.3.26}$$

where $\chi_{-1} \equiv \eta$.

In the continuum limit one obtains the action $S = \int \bar{\chi}\dot{\chi}\, dt_E$ for χ in the exponent of $e^{-\frac{1}{\hbar}S}$. This action could have been obtained by starting form the Minkowski action $S = \frac{i}{2}\int \bar{\chi}\dot{\chi}\, dt_M$ and then continuing $t_M \to -it_E$ to Euclidean time t_E. However, we started from the Minkowski action which we only used to derive the anticommutation relation of the ψ^a and $\psi^\dagger_a$, then we took the Hamiltonian (the usual Hamiltonian of Minkowski space) and started computing $\langle\bar{\eta}|e^{-\frac{\beta}{\hbar}\hat{H}}|\eta\rangle$ using well-defined rules. The outcome is (2.3.26), with the action $S = \int \bar{\chi}\dot{\chi}\, dt_E$ for χ. We shall refer to this as the Euclidean action for χ.

The term $\bar{\eta}\chi_{N-1}$ is the extra term which one encounters already in path integrals with bosonic coherent states [32, 33]. It comes from the inner product $\langle\bar{\eta}|\chi_{N-1}\rangle$. It is needed in the continuum theory with $\bar{\eta}\chi(0) - \int_{-\beta}^{0} \bar{\chi}(t)\dot{\chi}(t)dt$ in order that the field equation for $\chi(t)$ be given by $\frac{d}{dt}\bar{\chi}(t) = 0$ without extra boundary terms. (Note that $\chi(t=-\beta) = \eta$ and $\bar{\chi}(t=0) = \bar{\eta}$ hence $\delta\chi(t)$ vanishes at $t = -\beta$ but not at $t = 0$). However, we shall go on with the discretized approach and not yet make this (or any other further) approximations.

Since $\bar{\eta}\chi_{N-1}$ is at most linear in quantum deviations we get rid of this term by introducing again the background formalism. We decompose χ_k and $\bar{\chi}_k$ into a background part ξ_k and $\bar{\xi}_k$, and a quantum part ψ_k and $\bar{\psi}_k$

$$\chi_k^a = \xi_k^a + \psi_k^a \quad \text{with } k = -1, .., N-1\ , \quad \bar{\chi}_{ka} = \bar{\xi}_{ka} + \bar{\psi}_{ka} \quad \text{with } k = 0, .., N\ . \tag{2.3.27}$$

Again ξ_k^a and $\bar{\xi}_{ka}$ are independent complex Grassmann variables, and idem for ψ_k^a and $\bar{\psi}_{ka}$. For $k = -1$ we already defined $\chi_{-1} = \eta$. The background fermions are supposed to satisfy the boundary conditions, hence $\chi_{-1} = \xi_{-1} = \eta$ and $\psi_{-1} = 0$. Similarly $\bar{\chi}_N = \bar{\xi}_N = \bar{\eta}$ and $\bar{\psi}_N = 0$. Of course $d\bar{\chi}_{ka}d\chi_k^a = d\bar{\psi}_{ka}d\psi_k^a$, since Berezin integration is translationally invariant.

Next we split off a free part $H^{(0)}$ from H. In our applications, we always shall choose $H^{(0)} = 0$ for the fermions, so we concentrate on this case. However, nonvanishing $H^{(0)}$ can also be handled by our methods. (All our applications are to massless fermions; had there been a mass term present, we would have put it into $H^{(0)}$).

To be able to extract the interaction part of the action from the path integral, we introduce external sources $\bar{K}_{ka}$ and K_k^a which couple to ψ_k^a and $\bar{\psi}_{ka}$, and study the quadratic part of the path integral first

$$Z_N^{(0)}(K, \bar{K}) = \int \left(\prod_{k=0}^{N-1} d\bar{\psi}_{ka} d\psi_k^a\right) \exp\Bigl(-\frac{1}{\hbar}S^{(0)}\Bigr)$$

$$-\frac{1}{\hbar}S^{(0)} = -\sum_{k=0}^{N-1} \bar{\psi}_{ka}(\psi_k^a - \psi_{k-1}^a) + \sum_{k=0}^{N-1} (\bar{K}_{ka}\psi_k^a + \bar{\psi}_{ka}K_k^a)$$

$$(2.3.28)$$

where we recall that $\psi^a_{-1} = 0$. All remaining parts of $\bar{\chi}\dot{\chi}$ and $\bar{\eta}\chi_{N-1}$ in (2.3.26) combine with H into what we shall call $H^{(int)}$. Just as in the bosonic case, the kinetic terms are not diagonal. In the bosonic case we made therefore first an orthogonal transformation on the q^j_k which diagonalized the kinetic terms, but the fermionic kinetic terms are sufficiently simple that we need not first diagonalize them. By completing squares one finds

$$\begin{aligned} -\bar{\psi}_{ka}A_{kl}\psi^a_l + \bar{K}_{ka}\psi^a_k + \bar{\psi}_{ka}K^a_k \;&=\; -(\bar{\psi}_{ka} - \bar{K}_{k'a}A^{-1}_{k'k})A_{kl}(\psi^a_l - A^{-1}_{ll'}K^a_{l'}) \\ &\quad + \bar{K}_{ka}(A^{-1})_{kl}K^a_l \end{aligned} \tag{2.3.29}$$

where A is the lower triangular matrix

$$A_{kl} = \delta_{kl} - \delta_{k,l+1} \;\;;\;\; k,l = 0, N-1 \;; \qquad A = \begin{pmatrix} 1 & 0 & 0 & . & 0 \\ -1 & 1 & 0 & . & 0 \\ 0 & -1 & 1 & . & . \\ . & . & . & . & 0 \\ 0 & . & 0 & -1 & 1 \end{pmatrix} \tag{2.3.30}$$

The inverse of the matrix A is given by

$$\begin{array}{l} A^{-1}_{kl} = 0 \;\text{ if }\; k < l \\ A^{-1}_{kl} = 1 \;\text{ if }\; k \geq l \end{array} \;; \qquad A^{-1} = \begin{pmatrix} 1 & 0 & 0 & . & 0 \\ 1 & 1 & 0 & . & 0 \\ 1 & 1 & 1 & . & . \\ . & . & . & . & 0 \\ 1 & 1 & . & 1 & 1 \end{pmatrix} \tag{2.3.31}$$

(hence A^{-1} is also lower triangular). The integration over $\bar{\psi}_{ka}$ and ψ^a_k in $Z^{(0)}(K,\bar{K})$ yields unity since $\det A_{kl} = 1$. Hence

$$Z^{(0)}_N(K,\bar{K}) = \exp \sum_{k,l=0}^{N-1} \bar{K}_{ka}A^{-1}_{kl}K^a_l \;. \tag{2.3.32}$$

The propagators follow by twice differentiating $Z^{(0)}$. We find

$$\langle \psi^a_k \bar{\psi}_{lb} \rangle = \frac{\partial}{\partial K^b_l}\frac{\partial}{\partial \bar{K}_{ka}} Z^{(0)}\Big|_{K=\bar{K}=0} = A^{-1}_{kl}\delta^a_b \;. \tag{2.3.33}$$

Since H depends on $\bar{\chi}_k$ and $\frac{1}{2}(\chi_k + \chi_{k-1})$, we rather need the propagators for the $\psi_{k-1/2}$ where

$$\psi_{k-1/2} \equiv \frac{1}{2}(\psi_k + \psi_{k-1}) \;. \tag{2.3.34}$$

One clearly has

$$\langle \psi^a_{k-1/2}\bar{\psi}_{lb}\rangle = \frac{1}{2}(A^{-1}_{k,l} + A^{-1}_{k-1,l})\delta^b_a = \left\{ \begin{array}{c} 1 \text{ if } k > l \\ 1/2 \text{ if } k = l \\ 0 \text{ if } k < l \end{array} \right\} \delta^b_a \ . \tag{2.3.35}$$

The right-hand side contains the same discretized theta function as encountered in the bosonic case. The $\psi\psi$ and $\bar{\psi}\bar{\psi}$ propagators clearly vanish. In the continuum limit

$$\langle \psi^a(t)\bar{\psi}_b(t')\rangle = \theta(t-t')\delta^a_b \tag{2.3.36}$$

but when in doubt we shall go back to the discretized propagators with the discretized theta function.

The correlation function obtained from the path integral formalism are always time-ordered, an automatic consequence of the time-slicing in which the factors $\exp(-\epsilon\hat{H}/\hbar)$ move one from one time to the next. Thus the fermion propagator in (2.3.36) should be interpreted as

$$\begin{aligned} \langle \psi^a(t)\bar{\psi}_b(t')\rangle &= \delta^a_b \qquad \text{if} \ \ t > t' \\ \langle \psi^a(t)\bar{\psi}_b(t')\rangle &= \frac{1}{2}\delta^a_b \quad \ \text{if} \ \ t = t' \\ \langle \bar{\psi}_b(t')\psi^a(t)\rangle &= 0 \qquad \ \ \text{if} \ \ t' > t \end{aligned} \tag{2.3.37}$$

The equal-time propagator is now well-defined.

The fermionic path integral for the transition element in the discretized formulation reads

$$T(\bar{\eta},\eta;\beta) = \left[\exp\left(-\frac{1}{\hbar}H^{(int)}\right)\exp \bar{K}_{ka}A^{-1}_{kl}K^a_l\right]_{K=\bar{K}=0} \tag{2.3.38}$$

where $H^{(int)}$ follows from (2.3.26) and (2.3.27)

$$\begin{aligned} &-\frac{1}{\hbar}H^{int} \\ &= -\frac{\epsilon}{\hbar}\sum_{k=0}^{N-1} H\left(\bar{\psi}_{ka} \to -\frac{\partial}{\partial K^a_k}, \frac{1}{2}(\psi^a_k + \psi^a_{k-1}) \to \frac{1}{2}\left(\frac{\partial}{\partial \bar{K}_{ka}} + \frac{\partial}{\partial \bar{K}_{k-1,a}}\right)\right) \\ &\quad + \bar{\eta}_a\xi^a_{N-1} - \sum_{k=0}^{N-1}\bar{\xi}_{ka}(\xi^a_k - \xi^a_{k-1}) \ . \end{aligned} \tag{2.3.39}$$

We have suppressed the dependence of H on $\bar{\xi}_{ka}$ and $\frac{1}{2}(\xi^a_k + \xi^a_{k-1})$ for notational simplicity. All terms linear in quantum fields except those in H cancel if we require that the background fermions ξ and $\bar{\xi}$ satisfy the (discretized) equations of motion of $S^{(0)}$. In particular, the term $\bar{\eta}\psi(0)$ cancels against the term $\int \bar{\chi}\dot{\psi}$ coming form the last term in (2.3.39). The background fermions fields are then all constant because $H^{(0)}=0$. Hence $\xi_k^a=\eta^a$, for all $k=-1,0,1,..,N-1$ and $\bar{\xi}_{ka}=\bar{\eta}_a$ for all $k=0,1,..,N$. It is instructive to check that all terms linear in the quantum variables ψ_k^a and $\bar{\psi}_{k,a}$ cancel.

The path integral can now formally be written in the continuum limit as

$$\langle\bar{\eta}|e^{-\beta H}|\eta\rangle=\int d\bar{\psi}d\psi\; e^{-\int_{-\beta}^{0}\bar{\psi}\dot{\psi}\,dt-\frac{1}{\hbar}\int_{-\beta}^{0}Hdt+\bar{\eta}_a\eta^a}=\left\langle e^{-\frac{1}{\hbar}\int_{-\beta}^{0}S^{(int)}dt+\bar{\eta}_a\eta^a}\right\rangle \tag{2.3.40}$$

with propagators $\langle\psi^a(t)\bar{\psi}_b(t')\rangle=\theta(t-t')\delta_b^a$, and $S^{(int)}=H$ because we took $H^{(0)}=0$. The Hamiltonian H depends on $\bar{\eta}_a+\bar{\psi}_a(t)$ and $\eta^a+\psi^a(t)$ and $\psi(t)$ vanishes at $t=-\beta$ while $\bar{\psi}(t)$ vanishes at $t=0$. The extra term $e^{\bar{\eta}_a\eta^a}$ will play an important role in the computation of anomalies. It is equal to the inner product $\langle\bar{\eta}|\eta\rangle=e^{\bar{\eta}_a\eta^a}$.

2.4 Path integrals for Majorana fermions

In the previous section we developed a path integral formalism for Dirac spinors ψ^a and $\psi_b^\dagger$ satisfying $\{\psi^a,\psi_b^\dagger\}=\hbar\delta_a^b$. However, for many applications one needs Majorana spinors. The Dirac bracket for Majorana spinors ψ^a reads $\{\psi^a,\psi_b\}=\hbar\delta_a^b$, but one cannot directly construct a path integral formalism for Majorana spinors because we need separate operators ψ^a and $\psi_b^\dagger$ in order to construct coherent states. There are two ways to achieve this objective: either by adding an extra set of free Majorana fermions, or by combining pairs of Majorana spinors into complex spinors ψ and $\psi^\dagger$. We discuss these constructions separately. When one is dealing with one Majorana spinor (or an odd number of Majorana spinors), only the former procedure can be used.

Doubling of Majorana spinors. One way to construct separate operators ψ and $\psi^\dagger$ is to extend the set of interacting Majorana spinors ψ_1^a with $a=1,..,n$ by adding another set of free Majorana spinors ψ_2^a with again $a=1,..,n$. The Hamiltonian depends only on ψ_1^a but not on ψ_2^a. We then combine ψ_1^a and ψ_2^a into creation and annihilation operators as follows

$$\psi^a=\frac{1}{\sqrt{2}}(\psi_1^a+i\psi_2^a)\ ;\quad \psi_a^\dagger=\frac{1}{\sqrt{2}}(\psi_1^a-i\psi_2^a)\ . \tag{2.4.1}$$

For convenience we rescale the fermions such that there are no $\hbar$ in the brackets

$$\{\psi_i^a,\psi_j^b\}=\delta^{ab}\delta_{ij}\ ;\quad \{\psi^a,\psi_b^\dagger\}=\delta_a^b\ . \tag{2.4.2}$$

Further, of course, $\{\psi^a,\psi^b\}=\{\psi_a^\dagger,\psi_b^\dagger\}=0$.

Given these operators ψ and $\psi^\dagger$ we can now construct the path integral. The transition element is according to (2.3.26)

$$\langle\bar\eta|e^{-\beta H(\psi_1)}|\eta\rangle = \int \prod_{k=0}^{N-1} d\bar\chi_k d\chi_k\, e^{\bar\eta\chi_{N-1}}$$
$$\exp\Big(-\epsilon\sum_{k=0}^{N-1}\bar\chi_k(\chi_k-\chi_{k-1})/\epsilon - \frac{\epsilon}{\hbar}H_W(\bar\chi_k,(\chi_k+\chi_{k-1})/2)\Big) \tag{2.4.3}$$

where the operators $\hat\psi_1^a$ in H are first written as $(\hat\psi^a+\hat\psi_a^\dagger)/\sqrt{2}$, and again $\chi_{-1}=\eta$. After Weyl reordering with respect to ψ and $\psi^\dagger$ one then finds H_W as a function of $\bar\chi_k$ and $(\chi_k+\chi_{k-1})/2$ according to Berezin's theorem. In fact, because H originally only depended on ψ_1^a but not on ψ_2^a, it depends only on the sum of $\bar\chi_k$ and $\frac{1}{2}(\chi_k+\chi_{k-1})$. One may then introduce a background/quantum split of χ and $\bar\chi$ and H then depends only on $\psi_{1,bg}+\psi_{1,qu}$. One constructs propagators for $\psi_{1,qu}$, and the path integral becomes an integral over $\psi_{1,bg}$.

In applications one begins by considering a trace $\mathrm{Tr}\,J\exp(-\beta H)$ in a quantum field theory, and then converts this trace to a problem in quantum mechanics by representing the Dirac matrices γ^a by Majorana spinors $\sqrt{2}\,\hat\psi_1^a$ with the same anticommutation relations. One then adds free fermions $\hat\psi_2^a$ as explained above. Since J and H depend on $\hat\psi_1$, one combines $\hat\psi_1$ and $\hat\psi_2$ into $\hat\psi=(\hat\psi_1+i\hat\psi_2)/\sqrt{2}$ and $\hat\psi^\dagger=(\hat\psi_1-i\hat\psi_2)/\sqrt{2}$. The matrix elements of $\hat\psi_1$ are the same as the matrix element of γ^a. Thus the trace is still well-defined: the Hilbert space on which $\hat\psi_1$ acts is obtained by acting with $\hat\psi^\dagger$ on the vacuum annihilated by ψ. Adding a free set of fermions implies that one is considering a larger Hilbert space. One must then afterwards divide by the dimension of the subspace which is due to ψ_2. For an application where this construction is explained in great detail, see section 6.1.

Halving of Majorana spinors. The other way of constructing path integrals for Majorana spinors ψ^a is to combine pairs of Majorana spinors into complex Dirac spinors. This evidently is only possible if one has an even number of Majorana spinors. One defines then

$$\chi^A=\frac{1}{\sqrt{2}}(\psi^{2A-1}+i\psi^{2A})\ ;\quad \chi_A^\dagger=\frac{1}{\sqrt{2}}(\psi^{2A-1}-i\psi^{2A}) \tag{2.4.4}$$

where $A=1,..,n/2$ and $\{\chi^A,\chi_B^\dagger\}=\delta_B^A$. The inverse relations are given by

$$\psi^a = \frac{1}{\sqrt{2}}(\chi^{(a+1)/2}+\chi^\dagger_{(a+1)/2}) \quad \text{if } a \text{ is odd}$$

$$\psi^a = \frac{1}{\sqrt{2}}(-i\chi^{a/2} + i\chi^\dagger_{a/2}) \qquad \text{if } a \text{ is even} . \tag{2.4.5}$$

We define then again bras $|\eta\rangle$ and kets $\langle\bar{\eta}|$ by using the operators $\chi^\dagger$ and χ, respectively, to construct coherent states. The Hamiltonian depends on ψ^a, so we should first express ψ^a in terms of χ^A and $\chi^\dagger_A$, then Weyl-order this expression, and then go over to the path integral. Once again, we may then introduce a background/quantum split for the fermions and end up with a path integral for the transition element $\langle\bar{\eta}|\exp(-\beta H)|\eta\rangle$ where one integrates over $\bar{\eta}$ and η, see (6.2.47). Then one changes integration variables from the $n/2+n/2$ variables $\bar{\eta}$ and η to new variables ψ^a_1 where $a = 1, .., n$, and one ends up with a path integral over these ψ^a_1. For an application where all details are discussed see section 6.2 below (6.2.5).

We conclude this section with a discussion of how antiperiodic boundary conditions (APB) and periodic boundary conditions (PBC) arise in the continuum path integrals. We will derive these results straightforwardly from our discretized path integrals. When we compute anomalies in the second part of this report we shall explicitly perform the integrals over Grassmann variables at the discretized level, and then we shall not need to know whether in the continuum limit fermionic fields are periodic or antiperiodic. However, in the approach of Alvarez-Gaumé and Witten the continuum limit is first taken, and then one must evaluate one-loop determinants with certain boundary conditions for the quantum fields.

Let us first go back to Dirac fermions and the transition element $\langle\bar{\eta}|\exp(-\beta H)|\eta\rangle$. Recall the expression in (2.4.3). We want to take the trace of this transition element, so first we discuss how to take the trace of an operator.

The trace of an operator A is given by

$$\mathrm{Tr}\, A = \int \sqrt{g(x_0)} \prod_{i=1}^{n} dx_0^i \prod_{a=1}^{n} (d\chi^a d\bar{\chi}_a)\, e^{\bar{\chi}\chi} \langle\bar{\chi}, x_0|A|\chi, x_0\rangle \tag{2.4.6}$$

where $\prod_{a=1}^n (d\chi^a d\bar{\chi}_a)$ can be written as $d\chi^1 \dots d\chi^n d\bar{\chi}_n \dots d\bar{\chi}_1$. The states $|\chi, x_0\rangle$ and $\langle\bar{\chi}, x_0|$ contain the fermionic coherent states. The factor $\sqrt{g(x_0)}$ comes from the completeness relation $\int |x\rangle\sqrt{g(x)}\langle x|dx = I$. We take here the Grassmann variables χ and $\bar{\chi}$ as independent and not related by complex conjugation. The only property they satisfy is Berezin integration $\int d\bar{\chi}_a\, \bar{\chi}_a = 1$ and $\int d\chi^a\, \chi^a = 1$ for fixed a. Note that the order of $d\chi d\bar{\chi}$ and the sign in the exponent $\exp(\bar{\chi}\chi)$ are different in the trace formula from the completeness relation in (2.3.15). To check this trace formula, consider one pair $\chi, \bar{\chi}$. Then $\int d\chi d\bar{\chi}\, (1 + \bar{\chi}\chi)\, \langle 0|(1 + \bar{\chi}\psi)A(1 + \psi^\dagger\chi)|0\rangle$ is equal to $\langle 0|A|0\rangle + \langle 0|\psi A\psi^\dagger|0\rangle = \langle 0|A|0\rangle + \langle 1|A|1\rangle$, where $|1\rangle = \psi^\dagger|0\rangle$. We assume here that χ commutes with A, so A must have even statistics (we sometimes use the not quite correct terminology that A is commuting). This is indeed equal to TrA.[6]

We now return to the trace of the transition element

$$\mathrm{Tr}\ \mathrm{e}^{-\beta H} = \int d\eta^a d\bar{\eta}_a\, \mathrm{e}^{\bar{\eta}\eta} \langle \bar{\eta}| \exp(-\beta H)|\eta\rangle \ . \tag{2.4.7}$$

The transition element contains the expected path integral over $d\chi d\bar{\chi}$ with action $-\int_{-\beta}^{0} \bar{\chi}\dot{\chi} dt - \frac{1}{\hbar}\int_{-\beta}^{0} H dt$, plus the extra term $\bar{\eta}\chi(0)$. Performing the integration over $d\bar{\eta}$ of

$$\mathrm{e}^{\bar{\eta}\eta}\, \mathrm{e}^{\bar{\eta}\chi(0)} \tag{2.4.8}$$

leads to a factor $(\eta + \chi(0))$ which is a fermionic delta function $\delta(\eta + \chi(0))$ because $\int d\eta(\eta + \chi(0))f(\eta) = f(-\chi(0))$. Subsequent integration over η leads then to

$$\chi(0) = -\eta \qquad \text{(ABC)} \ . \tag{2.4.9}$$

Recalling that $\chi(-\beta) = \eta$, this means that the path integral is over paths $\chi(t)$ with ABC.

On the other hand, consider the trace with a matrix γ^5, $\mathrm{Tr}\, \gamma^5 \mathrm{e}^{-\beta H}$. As we show in section 6.1, the QM operator corresponding to γ^5 is $(-)^F$ where F is the fermion number operator. The path integral is as before, except that $\gamma^5|\eta\rangle = |-\eta\rangle$ [7]. The extra terms are now

$$\mathrm{e}^{-\bar{\eta}\eta}\, \mathrm{e}^{\bar{\eta}\chi(0)} \ . \tag{2.4.10}$$

Integration over $d\bar{\eta}$ now yields a factor $(-\eta + \chi(0))$. Thus one obtains the same path integral as before, but now with periodic boundary conditions

$$\chi(0) = \eta \qquad \text{(PBC)} \ . \tag{2.4.11}$$

---

[6] At the risk of confusing the non-expert reader, let us mention that one can actually distinguish between a trace and a supertrace, where by supertrace we mean the usual trace but with a minus sign for the fermionic states. The Jacobian in quantum field theory leads to a superdeterminant and a supertrace; this is not a choice but can be proven [56]. In the quantum mechanical model one can also distinguish between an ordinary trace and a supertrace. Both traces are mathematically consistent operations, and only physics can decide which one to choose. In the quantum mechanical case one needs the ordinary trace since one is taking the trace in spinor space. The $2^{n/2}$ states in spinor space split into two sets, one set with even numbers of $\psi^\dagger$ operator and the other set with an odd number. For two states $|0\rangle$ and $|1\rangle \equiv \psi^\dagger|0\rangle$, the trace of an operator A is $\langle 0|A|0\rangle + \langle 1|A|1\rangle$ while the supertrace would be $\langle 0|A|0\rangle - \langle 1|A|1\rangle$. We need a trace in the QM case because in the original formulation in terms of quantum field theory we needed a trace. The issue whether one should use a trace or a supertrace in finite temperature physics arose in the 1980's. In [57] a trace had been used, but in [58] it was argued that one needs a supertrace. It was finally settled that one needs a trace [59].

[7] The proof is as follows. Consider for simplicity 2 dimensions. Then $\gamma^5 = -i\gamma^1\gamma^2 = \psi\psi^\dagger - \psi^\dagger\psi$ with $\psi = \frac{1}{2}(\gamma^1 + i\gamma^2)$ and $\psi^\dagger = \frac{1}{2}(\gamma^1 - i\gamma^2)$. Acting with this γ^5 on $|\eta\rangle = e^{\psi^\dagger\eta}|0\rangle = (1 + \psi^\dagger\eta)|0\rangle$ yields $|-\eta\rangle$.

2.5 Direct evaluation of the transition element to order β.

In this section we shall determine the transition element $\langle z|\exp(-\frac{\beta}{\hbar}\hat{H})|y\rangle$ to order β for the Hamiltonian

$$\hat{H} = \frac{1}{2} g^{-1/4}\hat{p}_i g^{1/2} g^{ij} \hat{p}_j g^{-1/4} \ . \tag{2.5.1}$$

We shall not use path integrals but operatorial methods. We shall insert a complete set of p eigenstates, expand the exponent, move $\hat{p}$ operators to the p eigenstate and $\hat{x}$ to the x eigenstate, and in this way one gets an answer that is completely unambiguous. We shall determine the order β corrections to the flat space transition element. The operator $\hat{H}$ corresponds to the regulator of scalar field theories which preserves Einstein invariance. Because the leading term in the transition element is proportional to $\exp(-(z-y)^2/2\beta\hbar)$ we take $z-y$ to be of order $\sqrt{\beta}$. At no stage in the calculation is there any ambiguity: we move operators $\hat{p}$ next to eigenstates $|p\rangle$ or $\langle p|$ where they become c-numbers p, and operator $\hat{x}$ next to eigenstates $|x\rangle$ or $\langle x|$ where they become c-numbers x, taking carefully commutators into account. The path integral should exactly reproduce these results, and this is verified through two-loop orders in the next section.

We expand the exponent and take all terms in the expansion into account which contain none, one or two commutators. The final result will factorize into a classical part, a one-loop part which is given by the Van Vleck-Morette determinant, and a two-loop part proportional to the scalar curvature. As expected the final result preserves Einstein invariance: it is a biscalar as we shall explain.

We first prove that the operator $\hat{H}$ in (2.5.1) is Einstein invariant. We follow here DeWitt [9]. To demonstrate this, we note that infinitesimal general coordinate transformations $x^i \to x'^i \equiv x^i + \xi^i(x)$ are generated by the antihermitian operator

$$\hat{G}_E = \frac{1}{2i\hbar}(\hat{p}_k \xi^k(\hat{x}) + \xi^k(\hat{x})\hat{p}_k) \ . \tag{2.5.2}$$

Coordinates transforms then as

$$\delta\hat{x}^j = [\hat{x}^j, \hat{G}_E] = \xi^j(\hat{x}) \ . \tag{2.5.3}$$

The momenta transform as follows

$$\delta\hat{p}_j = [\hat{p}_j, \hat{G}_E] = -\frac{1}{2}\left(\hat{p}_k \frac{\partial\xi^k(\hat{x})}{\partial x^j} + \frac{\partial\xi^k(\hat{x})}{\partial x^j}\hat{p}_k\right) \tag{2.5.4}$$

which agrees with the symmetrized tensor law $p'_j = \frac{1}{2}\left\{\frac{\partial x^k}{\partial x'^j}, p_k\right\}$ for $x'^j = x^j + \xi^j$. (There is, of course, no transport term $-\xi^k\partial_k p_j$ because p_j does not depend on x^k). To simplify the notation we shall from now on omit the hats on operators. For what follows, it is useful to rewrite this result in factored form as $p'_j = \frac{\partial x^i}{\partial x'^j}(p_i + \text{more})$. To this purpose we write

$$p'_j = \frac{\partial x^i}{\partial x'^j} p_i - \frac{1}{2}\left[\frac{\partial x^i}{\partial x'^j}, p_i\right] \tag{2.5.5}$$

and rewrite the last term as follows [7, 9]

$$\begin{aligned}&\left[\frac{\partial x^i}{\partial x'^j}, p_i\right] = i\hbar \frac{\partial x'^k}{\partial x^i}\frac{\partial^2 x^i}{\partial x'^k \partial x'^j} = i\hbar \frac{\partial}{\partial x'^j} \ln\det \frac{\partial x^i}{\partial x'^k} \\ &= \frac{\partial x^i}{\partial x'^j}\left(i\hbar \frac{\partial}{\partial x^i} \ln\left|\frac{\partial x}{\partial x'}\right|\right) \ .\end{aligned}$$

Hence under a finite general coordinate transformation the momenta transform as follows

$$p'_j = \frac{\partial x^i}{\partial x'^j}\left(p_i - \frac{1}{2} i\hbar \frac{\partial}{\partial x^i}\ln\left|\frac{\partial x}{\partial x'}\right|\right) \ . \tag{2.5.6}$$

Consider now the operator $g^{1/4} p_i g^{-1/4}$ where $g = \det g_{ij}$. Using $g'(x') = \left|\frac{\partial x}{\partial x'}\right|^2 g(x)$, it is seen to transform as follows [8]

$$\begin{aligned}&(g')^{1/4} p'_i (g')^{-1/4} \\ &= g^{1/4}\left|\frac{\partial x}{\partial x'}\right|^{1/2} \frac{\partial x^j}{\partial x'^i}\left(p_j - \frac{1}{2} i\hbar \frac{\partial}{\partial x_j}\ln\left|\frac{\partial x}{\partial x'}\right|\right)\left|\frac{\partial x}{\partial x'}\right|^{-1/2} g^{-1/4} \\ &= g^{1/4}\frac{\partial x^j}{\partial x'^i} p_j g^{-1/4} = \frac{\partial x^j}{\partial x'^i}(g^{1/4} p_j g^{-1/4}) \ .\end{aligned} \tag{2.5.7}$$

Similarly

$$(g')^{-1/4} p'_i (g')^{1/4} = \left(g^{-1/4} p_j g^{1/4}\right)\frac{\partial x^j}{\partial x'^i} \ . \tag{2.5.8}$$

---

[8] The transformation rule $g'(x') = \left|\frac{\partial x}{\partial x'}\right|^2 g(x)$, if of course well-known from the tensor calculus of classical general relativity, but we should really derive this result by evaluating the commutator of $\hat{G}_E$ with $g(\hat{x})$. One can achieve this by writing $\hat{G}_E$ as the sum of the orbital part given in (2.5.2) and the following spin part $\hat{G}^{spin}(\xi) = \int d^n x\, (\xi^\lambda \partial_\lambda g_{\mu\nu} + (\partial_\mu \xi^\lambda) g_{\lambda\nu} + (\partial_\nu \xi^\lambda) g_{\mu\lambda}) \frac{\partial}{\partial g_{\mu\nu}}$. The structure of a generator as a sum of an orbital part and a spin part is well-known from the case of Lorentz symmetry. The orbital and spin generator always commute. The spin generator must therefore satisfy the same algebra as the orbital generator, $[\hat{G}_E(\xi_1), \hat{G}_E(\xi_2)] = \hat{G}_E(\partial_\nu \xi_1 \xi_2^\nu - \xi_1^\nu \partial_\nu \xi_2)$, and this fixes the spin generator [60]. In the commutator of $\hat{G}_E$ the transport term from the orbital part cancels the transport term of the spin part (so that a scalar field $\phi(\hat{x})$ actually commutes with $\hat{G}_E$). It is then straightforward to check that $\hat{H}$ commutes with $\hat{G}_E$.

Returning to the Hamiltonian, we obtain

$$\hat{H}' = \frac{1}{2}\left(g^{-1/4}p_k g^{1/4}\frac{\partial x^k}{\partial x'^i}\right)\left(\frac{\partial x'^i}{\partial x^m}\frac{\partial x'^j}{\partial x^n}g^{mn}\right)\left(\frac{\partial x^l}{\partial x'^j}g^{1/4}p_l g^{-1/4}\right) = \hat{H} \ . \tag{2.5.9}$$

Hence, we demonstrated that $\hat{H}$ is Einstein invariant, $[\hat{H}, \hat{G}_E] = 0$. In a similar manner one can demonstrate that $\hat{H}$ is Lorentz invariant when fermions are present, provided one replaces p_i by a Lorentz covariant derivatives π_i. (For $N = 2$ models the Lorentz generator is given by $\hat{J} = \frac{1}{2}\lambda_{ab}(x)\psi^a_\alpha\psi^b_\alpha = \lambda_{ab}(x)\psi^{\dagger a}\psi^b$ with $\alpha = 1, 2$ and $\delta\psi^c = [\psi^c, \hat{J}] = \lambda^c{}_b\psi^b$. It leaves the coordinates x^i inert but transforms the momenta p_j). It is defined by $\pi_i = p_i - \frac{i\hbar}{2}\omega_{iab}\psi^a_\alpha\psi^b_\alpha$ and it is Lorentz invariant if one also adds a spin term with $\partial/\partial\omega_{iab}$ to $\hat{J}$ which transforms ω_{iab} , similar to the spin term in ${\hat{G}_E}$[7]. For $N = 1$ models $\alpha = 1$, and using Dirac brackets the same results are obtained.

We turn now to the task of evaluating

$$\langle x| \exp\left(-\frac{\beta}{\hbar}\hat{H}\right)|p\rangle \tag{2.5.10}$$

with

$$\hat{H} = \frac{1}{2}g^{-1/4}\hat{p}_i g^{1/2}g^{ij}\hat{p}_j g^{-1/4} \ . \tag{2.5.11}$$

Expanding the exponent in (2.5.10), we define

$$\langle x|\hat{H}^k|p\rangle \equiv \sum_{l=0}^{2k} A^k_l(x)p^l\langle x|p\rangle \tag{2.5.12}$$

where $A^k_l(x)$ is a c-number function and p^l denotes a homogeneous polynomial of order l in the momenta.

In order to compute the transition amplitude $\langle x|\exp\left(-\frac{\beta}{\hbar}\hat{H}\right)|y\rangle$ to order β compared to the leading terms, it will turn out that we only need the terms on the right-hand side of (2.5.12) with $l = 2k$, $2k-1$, and $2k-2$. The proof will be given in (2.5.16). We find, defining $p^2 = g^{ij}(x)p_ip_j$,

$$A^k_{2k}(x)p^{2k} = \left(\frac{1}{2}p^2\right)^k \ . \tag{2.5.13}$$

Since this is the term containing the maximal number of p's, it can be easily computed because all $\hat{p}$ operators are just replaced by the corresponding c-numbers when acting on $|p\rangle$.

The next term is

$$\begin{aligned} &A^k_{2k-1}(x)p^{2k-1} \\ &= -i\hbar k\left(\frac{1}{2}p^2\right)^{k-1}\frac{1}{2}(\partial_i g^{ij})p_j - i\hbar\binom{k}{2}\left(\frac{1}{2}p^2\right)^{k-2}\frac{1}{2}g^{ij}(\partial_i g^{kl})p_jp_kp_l \ . \end{aligned} \tag{2.5.14}$$

In this expression one of the $\hat{p}$'s acts as a derivative, whereas the other $2k-1$ are replaced by the corresponding c-numbers. The first term in (2.5.14) comes about when the derivative acts within the same factor $\hat{H}$ in which it appears, and is multiplied by k since there are k factors of $\hat{H}$. The second term arises if this derivative acts on a different factor of $\hat{H}$. For this to occur there are $\binom{k}{2}$ possible combinations, and taking into account that there are two $\hat{p}$'s in $\hat{H}$ we get an extra factor 2. Notice that in both cases the terms involving a derivative acting on g cancel.

The last term we have to calculate is obtained when two of the $\hat{p}$'s act as derivatives

$$
\begin{aligned}
&A^k_{2k-2}(x)p^{2k-2} = \\
&\hbar^2 k\left(\frac{1}{2}p^2\right)^{k-1}\left[\frac{1}{32}g^{ij}(\partial_i\log g)(\partial_j\log g)+\frac{1}{8}g^{ij}(\partial_i\partial_j\log g)\right.\\
&\qquad\qquad\left.+\frac{1}{8}(\partial_i g^{ij})(\partial_j\log g)\right]\\
&-\hbar^2\binom{k}{2}\left(\frac{1}{2}p^2\right)^{k-2}\left[\frac{1}{2}g^{ij}(\partial_i\partial_k g^{kl})+\frac{1}{4}(\partial_i g^{ij})(\partial_k g^{kl})\right.\\
&\qquad\qquad\left.+\frac{1}{4}(\partial_i g^{ik})(\partial_k g^{jl})+\frac{1}{4}g^{ik}(\partial_i\partial_k g^{jl})\right]p_jp_l\\
&-\hbar^2\binom{k}{3}\left(\frac{1}{2}p^2\right)^{k-3}\left[\frac{1}{2}g^{ik}g^{jl}(\partial_i\partial_j g^{mn})+\frac{3}{4}g^{im}(\partial_i g^{kl})(\partial_j g^{jn})\right.\\
&\qquad\qquad\left.+\frac{1}{2}g^{jl}(\partial_j g^{ik})(\partial_i g^{mn})+\frac{1}{4}g^{ij}(\partial_i g^{kl})(\partial_j g^{mn})\right]p_kp_lp_mp_n\\
&-\hbar^2\binom{k}{4}\left(\frac{1}{2}p^2\right)^{k-4}\left[\frac{3}{4}g^{ij}g^{mn}(\partial_i g^{kl})(\partial_n g^{pq})\right]p_jp_kp_lp_mp_pp_q\ . \qquad (2.5.15)
\end{aligned}
$$

The first set of terms appears when both derivatives act within the same factor $\hat{H}$; again there are k terms of this kind.

The next set of terms arises when only two of the factors $\hat{H}$ play a role. There are four possibilities: (i) one $\hat{p}$ from the left factor acts on the right factor, while another $\hat{p}$ from the right factor acts within the right factor, (ii) the first $\hat{p}$ acts within the first $\hat{H}$, while the second $\hat{p}$ acts within the second $\hat{H}$, (iii) both $\hat{p}$'s come from the left $\hat{H}$, but one of them acts inside the left $\hat{H}$ while the other acts on the right $\hat{H}$, and (iv) both $\hat{p}$'s from the left $\hat{H}$ act on the right $\hat{H}$. In all cases it is easy to see that again the derivatives on g cancel.

The following set of terms in (2.5.15) comes from combinations using three factors $\hat{H}$, hence its overall factor is $\binom{k}{3}$. There are again four cases: (i) a $\hat{p}$ from the first $\hat{H}$ and $\hat{p}$ from the second $\hat{H}$ hit the third $\hat{H}$, (ii) one

$\hat{p}$ acts inside the factor $\hat{H}$ in which it appears whereas a $\hat{p}$ from another $\hat{H}$ hits the remaining $\hat{H}$ (there are 3 terms of this kind), (iii) a $\hat{p}$ from the first $\hat{H}$ hits the second $\hat{H}$, and a $\hat{p}$ from the second $\hat{H}$ hits the third $\hat{H}$, and (iv) of the two $\hat{p}$'s from the first $\hat{H}$ one acts on the second, and one on the third $\hat{H}$.

Finally, the term with $\binom{k}{4}$ in (2.5.15) involves four factors $\hat{H}$, such that one $\hat{p}$ from one $\hat{H}$ hits another $\hat{H}$, and the other $\hat{p}$ from one of the remaining factors $\hat{H}$ hits the last $\hat{H}$.

The reason further terms do not contribute can be most easily seen if we rescale $q = \sqrt{\frac{\beta}{\hbar}}p$. Then the transition amplitude becomes

$$\begin{aligned}
&\langle z|\exp\left(-\frac{\beta}{\hbar}\hat{H}\right)|y\rangle = \int d^n p \langle z|\exp\left(-\frac{\beta}{\hbar}\hat{H}\right)|p\rangle\langle p|y\rangle \\
&= g^{-1/4}(z)g^{-1/4}(y)(2\pi\hbar)^{-n}\left(\frac{\hbar}{\beta}\right)^{n/2}\int d^n q \exp\left(i\frac{q_i(z-y)^i}{\sqrt{\beta\hbar}}\right) \\
&\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\beta}{\hbar}\right)^k \sum_{l=0}^{2k} A_l^k(z)q^l \left(\frac{\beta}{\hbar}\right)^{-l/2}
\end{aligned} \tag{2.5.16}$$

where the first factor is due to the values of $\langle z|p\rangle$ and $\langle p|y\rangle$. If we consider $z-y$ of order $\sqrt{\beta}$, the q's are of order β^0. Then only the A_{2k-1}^k and A_{2k-2}^k terms contribute through order β compared to the leading term A_{2k}^k.

The sum over k in (2.5.16) can be performed for fixed l. All terms in (2.5.13)-(2.5.15) have a prefactor $(p^2)^{k-s}/(k-s)!$ with $s=0,1,2,3,4$ which leads to a factor $\exp(-p^2/2\beta k)$ after summing over k. Integration over p is then straightforward and one obtains

$$\begin{aligned}
&\langle z|\exp\left(-\frac{\beta}{\hbar}\hat{H}\right)|y\rangle = \\
&g^{-1/4}(z)g^{-1/4}(y)(4\pi^2\hbar\beta)^{-n/2}\int d^n q \exp\left(-\frac{1}{2}g^{ij}(z)q_i q_j + i\frac{q_i(z-y)^i}{\sqrt{\beta\hbar}}\right) \\
&\left[1 + \boldsymbol{i\sqrt{\beta\hbar}}\left\{\boldsymbol{\frac{1}{2}(\partial_i g^{ij})q_j} - \frac{1}{4}g^{ij}(\partial_i g^{kl})q_j q_k q_l\right\}\right. \\
&+\boldsymbol{\beta\hbar}\left\{\left[-\boldsymbol{\frac{1}{32}g^{ij}(\partial_i \log g)(\partial_j \log g)} - \boldsymbol{\frac{1}{8}g^{ij}(\partial_i\partial_j \log g)}\right.\right. \\
&\left.-\boldsymbol{\frac{1}{8}(\partial_i g^{ij})(\partial_j \log g)}\right] \\
&-\left[\frac{1}{4}g^{ij}(\partial_i\partial_k g^{kl}) + \boldsymbol{\frac{1}{8}(\partial_i g^{ij})(\partial_k g^{kl})} + \frac{1}{8}(\partial_i g^{ik})(\partial_k g^{jl})\right. \\
&\left.+\frac{1}{8}g^{ik}(\partial_i\partial_k g^{jl})\right] q_j q_l
\end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{1}{12}g^{ik}g^{jl}(\partial_i\partial_j g^{mn}) + \frac{1}{8}g^{im}(\partial_i g^{kl})(\partial_j g^{jn})\right.\\
&\left.+\frac{1}{12}g^{jl}(\partial_j g^{ik})(\partial_i g^{mn}) + \frac{1}{24}g^{ij}(\partial_i g^{kl})(\partial_j g^{mn})\right] q_k q_l q_m q_n\\
&-\left.\left.\left[\frac{1}{32}g^{ij}g^{mn}(\partial_i g^{kl})(\partial_n g^{pq})\right] q_j q_k q_l q_m q_p q_q\right\} + \mathcal{O}(\beta^{3/2})\right] \ . \qquad (2.5.17)
\end{aligned}
$$

The terms from (2.5.13) only give the leading exponential, but the terms from (2.5.14) give the two terms with $\sqrt{\beta\hbar}$, while the four sets of terms in (2.5.15) give the terms proportional to $\beta\hbar$. We have boldfaced the terms which are present if one does not take any commutators between different factors $\hat{H}$ into account. The last boldfaced term is clearly due to expanding the exponent of the first boldfaced term. The remaining terms, also of order β, are crucial to obtain the correct result for $\langle z|\exp(-\frac{\beta}{\hbar}H)|y\rangle$ to order β. To avoid confusion: if one uses Weyl ordering to evaluate the path integral rather than directly evaluating the transition element, one need not take these commutators into account, but only those which follow from Weyl ordering $\hat{H}$ itself.

We can now complete the square in the exponent and integrate out the momenta q_i, since the integral becomes just a sum of Gaussian integrals which can easily be evaluated. The problem is then to factorize the result such that it is manifestly a scalar both in z and y (a 'bi-scalar') under general coordinate transformations. We expect, of course, to find at least the classical action integrated along a geodesic. In the expansion of this functional around $x(0) = z$, one finds many of the terms in (2.5.17). However, there are terms left over. They combine into R or R_{ij}, while expansion of $g(y)$ yields terms with $\partial \log g$ or derivatives thereof. With this in mind, we write the result in a factorized form, where in one factor we put all terms which possibly can come from expanding some power of $g(y)$, while into another factor we put the expanded action and curvature terms. It is quite nontrivial, and an excellent check on the results obtained so far, that this factorization is at all possible. The resulting expression is

$$
\begin{aligned}
&\langle z|\exp\left(-\frac{\beta}{\hbar}\hat{H}\right)|y\rangle = g^{-1/4}(z)g^{-1/4}(y)(2\pi\hbar\beta)^{-n/2}\\
&\left[g^{1/2}(z) + g^{1/4}(z)(y-z)^i(\partial_i g^{1/4}(z))\right.\\
&\quad\left.+\frac{1}{2}g^{1/4}(z)(y-z)^i(y-z)^j(\partial_i\partial_j g^{1/4}(z))\right]\\
&\exp\left(-\frac{1}{2\beta\hbar}g_{ij}(z)(y-z)^i(y-z)^j\right)\\
&\times\left[1 - \frac{1}{4}\frac{1}{\beta\hbar}(\partial_k g_{ij}(z))(y-z)^i(y-z)^j(y-z)^k\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\left(\frac{1}{4}\frac{1}{\beta\hbar}(\partial_k g_{ij}(z)(y-z)^i(y-z)^j(y-z)^k\right)^2 \\
&-\frac{1}{12}\frac{1}{\beta\hbar}\left(\partial_k\partial_l g_{ij}(z)-\frac{1}{2}g_{mn}(z)\Gamma^m_{ij}(z)\Gamma^n_{kl}(z)\right) \\
&\qquad\times(y-z)^i(y-z)^j(y-z)^k(y-z)^l \\
&-\frac{1}{12}\beta\hbar R(z)-\frac{1}{12}R_{ij}(z)(y-z)^i(y-z)^j+\mathcal{O}(\beta^{3/2})\Big]
\end{aligned}
\tag{2.5.18}
$$

where the Ricci tensor is defined in Appendix A. For example, the term proportional to $(y-z)^i\partial_i g^{1/4}$ in the first pair of square brackets comes from the terms with $(\partial_j g^{ij})q_j$ and $g^{ij}(\partial_i g^{kl})q_j q_k q_k$ in (2.5.17) after integration over q.

Note that since the difference $(y-z)$ is of order $\sqrt{\beta}$, all terms are of order β or less. The terms within the first pair of square brackets are, through order β, equal to $g^{1/4}(z)g^{1/4}(y)$ and cancel the factors $g^{-1/4}(z)g^{-1/4}(y)$ in front of the whole expression. The terms with $\partial_k g_{ij}$ and its square are clearly the first two terms in an expansion of an exponent. This suggests to exponentiate all terms, yielding

$$
\begin{aligned}
&\langle z|\exp\left(-\frac{\beta}{\hbar}\hat{H}\right)|y\rangle=(2\pi\hbar\beta)^{-n/2}\exp\Big\{-\frac{\beta}{\hbar}\Big[\frac{1}{2}g_{ij}(z) \\
&+\frac{1}{4}\partial_k g_{ij}(z)(y-z)^k \\
&+\frac{1}{12}\left(\partial_k\partial_l g_{ij}(z)-\frac{1}{2}g_{mn}(z)\Gamma^m_{ij}(z)\Gamma^n_{kl}(z)\right)(y-z)^k(y-z)^l\Big] \\
&\times\frac{(y-z)^i}{\beta}\frac{(y-z)^j}{\beta} \\
&-\frac{1}{12}\beta\hbar R(z)-\frac{1}{12}R_{ij}(z)(y-z)^i(y-z)^j+\mathcal{O}(\beta^{3/2})\Big\}\ .
\end{aligned}
\tag{2.5.19}
$$

We shall now show that all terms in the exponent except the last two just correspond to an expansion around z of the classical action, which is equal to the integral along the geodesic joining z and y of the invariant line element.

The classical action $S_{cl}[z,y;\beta]$ is given by

$$
S_{cl}[z,y;\beta]=\int_{-\beta}^{0}\frac{1}{2}g_{ij}[x_{cl}(t)]\frac{dx^i_{cl}(t)}{dt}\frac{dx^j_{cl}(t)}{dt}\,dt \tag{2.5.20}
$$

where x^i_{cl} satisfies the equation of motion obtained from the Euler-Lagrange variational principle

$$
D_t d_t x^i\equiv\ddot{x}^i+\Gamma^i_{jk}\dot{x}^j\dot{x}^k=0 \tag{2.5.21}
$$

together with the boundary conditions

$$x^i(-\beta) = y^i \quad , \quad x^i(0) = z^i \ . \tag{2.5.22}$$

(To avoid confusion about the notation we consider functions $x^i(t)$ and endpoints y^i and z^i). Expanding $x^i_{cl}(t)$ into a Taylor series

$$x^i_{cl}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \partial^n x^i_{cl}(0) \tag{2.5.23}$$

$$x^i_{cl}(0) = z^i \ , \ \ddot{x}^i_{cl}(0) = -\Gamma^i_{jk}(z)\dot{x}^j_{cl}(0)\dot{x}^k_{cl}(0) \ , \ \text{etc.} \tag{2.5.24}$$

we see that we can express $x^i_{cl}(t)$ into $x^i(0)$ and $\dot{x}^i(0)$. The value of $\dot{x}^i(0)$ follows from the boundary condition at $t = -\beta$. Namely, equation (2.5.23) at $t = -\beta$ yields

$$\begin{aligned}
&y^i = z^i - \beta\dot{x}^i_{cl}(0) + \frac{1}{2}\beta^2\ddot{x}^i_{cl}(0) - \frac{1}{6}\beta^3 \dddot{x}^i_{cl}(0) + \dots \\
&\dot{x}^i_{cl}(0) = (z^i - y^i)/\beta + \frac{1}{2}\beta\ddot{x}^i_{cl}(0) - \frac{1}{6}\beta^2 \frac{d}{dt}(\ddot{x}^i_{cl}(0)) + \dots \\
&= \frac{1}{\beta}(z-y)^i - \frac{1}{2}\beta\Gamma^i_{jk}(z)\dot{x}^j_{cl}(0)\dot{x}^k_{cl}(0) + \frac{1}{6}\beta^2\frac{d}{dt}(\Gamma^i_{kl}\dot{x}^k_{cl}\dot{x}^l_{cl}) + \dots \ .
\end{aligned} \tag{2.5.25}$$

Solving iteratively for $\dot{x}^i_{cl}(0)$ to order $(z-y)$ yields

$$\dot{x}^i_{cl}(0) = \frac{1}{\beta}(z-y)^i - \frac{1}{2\beta}\Gamma^i_{jk}(z-y)^j(z-y)^k \tag{2.5.26}$$

$$+\frac{1}{6\beta}(\partial_l\Gamma^i_{jk} + \Gamma^i_{sj}\Gamma^s_{kl})(z-y)^j(z-y)^k(z-y)^l + \dots \ . \tag{2.5.27}$$

From these results we can obtain an expansion of the classical action in terms of z^i and $(z-y)^i$ by Taylor expanding the Lagrangian $L(t)$

$$S_{cl}[z, y; \beta] = \int_{-\beta}^{0} \left(L(0) + t\frac{d}{dt}L(0) + \ldots \right) dt \ . \tag{2.5.28}$$

However, $L(t)$ is conserved

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j\right) &= g_{ij}\ddot{x}^i\dot{x}^j + \frac{1}{2}\partial_k g_{ij}\dot{x}^k\dot{x}^i\dot{x}^j \\
&= g_{ij}(\ddot{x}^i + \Gamma^i_{kl}\dot{x}^k\dot{x}^l)\dot{x}^j = 0 \quad \text{ for } x = x_{cl}
\end{aligned} \tag{2.5.29}$$

hence only $L(0)$ contributes in (2.5.28). We find then

$$S_{cl}[z, y; \beta] = \beta L(0) = \frac{\beta}{2}g_{ij}(z)\dot{x}^i_{cl}(0)\dot{x}^j_{cl}(0) = \frac{1}{2\beta}g_{ij}(z)$$

$$
\begin{aligned}
&\Big[(z-y)^i - \frac{1}{2}\Gamma^i_{kl}(z-y)^k(z-y)^l \\
&+\frac{1}{6}(\partial_k\Gamma^i_{lm} + \Gamma^i_{sk}\Gamma^s_{lm})(z-y)^k(z-y)^l(z-y)^m\Big] \\
&\Big[(z-y)^j - \frac{1}{2}\Gamma^j_{pq}(z-y)^p(z-y)^q \\
&+\frac{1}{6}(\partial_p\Gamma^j_{qr} + \Gamma^j_{tp}\Gamma^t_{qr})(z-y)^p(z-y)^q(z-y)^r\Big] + \ldots \ . \qquad (2.5.30)
\end{aligned}
$$

The terms quartic in $(z-y)$ have as coefficient

$$
\left[\frac{1}{8}g_{ij}\Gamma^i_{kl}\Gamma^j_{mn} + \frac{1}{6}g_{sn}(\partial_k\Gamma^s_{lm} + \Gamma^s_{tk}\Gamma^t_{lm})\right] \ . \qquad (2.5.31)
$$

The last two terms in (2.5.31) yield actually $-\frac{1}{6}$ times the first $g\Gamma\Gamma$ plus a $\partial\partial g$ term. Hence one finally arrives at

$$
\begin{aligned}
&S_{cl}[z,y;\beta] = \frac{1}{\beta}\Big[\frac{1}{2}g_{ij}(z)(z-y)^i(z-y)^j \\
&-\frac{1}{4}\partial_k g_{ij}(z)(z-y)^i(z-y)^j(z-y)^k \\
&+\left(\frac{1}{12}\partial_k\partial_l g_{mn}(z) - \frac{1}{24}g_{ij}(z)\Gamma^i_{kl}\Gamma^j_{mn}(z)\right) \\
&\times(z-y)^k(z-y)^l(z-y)^m(z-y)^n\Big] + \mathcal{O}(z-y)^5 \ . \qquad (2.5.32)
\end{aligned}
$$

These terms agree perfectly with the first four terms in (2.5.19).

The transition amplitude can then be written as

$$
\begin{aligned}
&\langle z|e^{-\frac{\beta}{\hbar}\hat{H}}|y\rangle = \frac{1}{(2\pi\hbar\beta)^{n/2}}e^{-\frac{1}{\hbar}S_{cl}[z,y;\beta]} \\
&\Big[1 - \frac{1}{24}\beta\hbar(R(z)+R(y)) \\
&-\frac{1}{24}(R_{ij}(z)+R_{ij}(y))(z-y)^i(z-y)^j + \mathcal{O}(\beta^{3/2})\Big] \ . \qquad (2.5.33)
\end{aligned}
$$

We have replaced $R(z)$ by $\frac{1}{2}(R(z)+R(y))$ which is allowed to order β, to show that the result is symmetric under exchange of z and y. These results were obtained in [41]. The transition amplitude up to order $\beta^{5/2}$ (3 loops) can be found in [61].

Of course, the composition rule for path integrals should hold

$$
\int \langle z|e^{-\frac{\beta}{\hbar}\hat{H}}|x\rangle\sqrt{g(x)}\langle x|e^{-\frac{\beta}{\hbar}\hat{H}}|y\rangle\, d^n x = \langle z|e^{-\frac{2\beta}{\hbar}\hat{H}}|y\rangle \ . \qquad (2.5.34)
$$

A quick way to check this is to use normal coordinates around x since then only the leading term in the classical action survives (for normal coordinates $\partial_i g_{jk}(x) = \partial_{(i}\partial_j g_{kl)}(x) = 0$ while $g_{ij}(z) = g_{ij}(x) - \frac{1}{3}R_{iklj}(\Gamma)(x)(z-x)^k(z-x)^l$ through order β). Taking the opposite point of view, we can impose the composition rule and find then that this fixes the coefficient of the Ricci tensor, but not that of the scalar curvature. The latter terms yield the trace anomaly in $d=2$ dimensions, and its coefficient should not be fixed by requiring the composition rule to hold because we can view $\hbar^2 R$ as a potential term in the action, and the composition rule should hold for any potential.

The terms with R_{ij} can be expressed in terms of the classical action. One should expect this: they are one loop terms and hence should be proportional to the determinant of the double derivative of the classical action (see for example the textbook by Schulman [11]). One may check that (2.5.32) yields

$$\begin{aligned} D_{ij} \equiv -\frac{\partial}{\partial z^i}\frac{\partial}{\partial y^j} S_{cl}[z,y;\beta] = \frac{1}{\beta}\Big[g_{ij}(z) - \Gamma_{jk;i}(z-y)^k \\ -\frac{3}{4}\partial_i\partial_{(k}g_{lj)}(z-y)^k(z-y)^l \\ + \left\{\partial_{(i}\partial_j g_{mn)} - \frac{1}{2}g_{st}\Gamma^s_{(ij}\Gamma^t_{\ mn)}\right\}(z-y)^m(z-y)^n\Big] \ . \end{aligned} \quad (2.5.35)$$

Hence, using the notation $D_{ij} = \frac{1}{\beta}[g_{ij} - \Delta g_{ij}]$, we have

$$\det D = \beta^{-n} g(z)\left[1 - g^{ij}\Delta g_{ij} - \frac{1}{2}g^{ij}\Delta g_{jk}g^{kl}\Delta g_{li} + \frac{1}{2}(g^{ij}\Delta g_{jk})^2 + \ldots\right] \ . \quad (2.5.36)$$

Since the first term in $-g^{ij}\Delta g_{ij}$ is equal to $-g^{ij}\Gamma_{jk;i}(z-y)^k = -\frac{1}{2}g^{ij}\partial_k g_{ij}(z-y)^k \sim g^{-1/2}(z)g^{1/2}(y)-1$, we can remove the term proportional to $(z-y)\Gamma$ from $\det D$ by replacing $g(z)$ by $g^{1/2}(z)g^{1/2}(y)$

$$\det D = \beta^{-n} g(z)^{1/2} g^{1/2}(y)[1+...] \ . \quad (2.5.37)$$

This parametrization makes sense because $g(z)^{-1/2}\left[\det \frac{\partial}{\partial z^i}\frac{\partial}{\partial y^j}S\right]g^{-1/2}(y)$ is a biscalar. If we now work out the remaining terms in (2.5.37) denoted by $\ldots$, one finds a nice surprise

$$\begin{aligned} g^{ij}\left[-\frac{3}{4}\partial_i\partial_{(k}g_{lj)} + \partial_{(i}\partial_j g_{kl)} - \frac{1}{2}g_{st}\Gamma^s_{(ij}\Gamma^t_{\ kl)} + \frac{1}{2}\Gamma_{jk,n}g^{nm}\Gamma_{ml,i}\right] \\ \times(z-y)^k(z-y)^l \ = \ -\frac{1}{6}R_{kl}(z-y)^k(z-y)^l \end{aligned} \quad (2.5.38)$$

Hence the R_{ij} terms in (2.5.33) can be written as

$$\left(1 - \frac{1}{12}R_{ij}(z-y)^i(z-y)^j\right) =$$

$$\beta^{n/2} g^{-1/4}(z) \left(\det - \frac{\partial}{\partial z^i} \frac{\partial}{\partial y^j} S_{cl} \right)^{1/2} g^{-1/4}(y) \equiv \tilde{D}^{1/2} \; . \quad (2.5.39)$$

The final result for the transition element becomes

$$\begin{aligned} \langle z| e^{-\frac{\beta}{\hbar}\hat{H}} |y\rangle &= \frac{1}{(2\pi\hbar\beta)^{n/2}} e^{-\frac{1}{\hbar} S_{cl}[z,y;\beta]} \tilde{D}^{1/2} \\ &\quad \times \left[1 - \frac{1}{24} \beta\hbar \left(R(z) + R(y) \right) + \mathcal{O}(\beta^{3/2}) \right] \; . \quad (2.5.40) \end{aligned}$$

The Einstein invariance is manifest: the transition element is a biscalar (it does not depend on the coordinates one chooses around z and y, nor on the choice of coordinates anywhere else).

The Van Vleck determinant $\tilde{D}$ is $\hbar$-independent and thus it yields the one-loop corrections. In the next section we shall directly calculate the one-loop and two-loop Feynman diagrams, and indeed obtain the Van Vleck determinant as part of the one-loop corrections. In flat space, it reduces to unity in our normalization. The factor $(2\pi\hbar\beta)^{-n/2}$ is the Feynman measure. (Since $\frac{\partial}{\partial y^j} S_{cl}[z, y; \beta] = -p_j$ where p_j is the momentum conjugate to y^j, one could interpret $\det D$ as the Jacobian for the change of variable $p(y) \to z$.)

We have thus obtained the order β corrections to the transition element $\langle z| \exp(-\frac{\beta}{\hbar}\hat{H})|y\rangle$ by direct evaluation. Another way to obtain these corrections is to use the Schrödinger equation,

$$\begin{aligned} -\hbar\frac{\partial}{\partial\beta}\langle z| \exp(-\frac{\beta}{\hbar}\hat{H})|y\rangle &= \int \langle z|\hat{H}|x\rangle \sqrt{g(x)} \langle x| \exp(-\frac{\beta}{\hbar}\hat{H})|y\rangle \, d^n x \\ = H(z)\langle z| \exp(-\frac{\beta}{\hbar}\hat{H})|y\rangle &= H(y)\langle z| \exp(-\frac{\beta}{\hbar}\hat{H})|y\rangle \; . \qquad (2.5.41) \end{aligned}$$

In the last step we used that the left-hand side of this equation is symmetric in z, y. This follows either from general arguments, or by looking at the explicit expression we obtained for $\langle z| \exp(-\frac{\beta}{\hbar}\hat{H})|y\rangle$. Since $\langle z| \exp(-\frac{\beta}{\hbar}\hat{H})|y\rangle$ is given by an expansion about z, it is evidently much easier to evaluate the action of $H(y)$ than that of $H(z)$.

The operator $H(z)$ is given by

$$H(z) = -\frac{\hbar^2}{2} g^{-1/2} \partial_i g^{1/2} g^{ij} \partial_j \; . \qquad (2.5.42)$$

Let us show in some detail how this asymmetric looking expression arises. Define $\hat{x}^i(t) = \exp\left(\frac{i}{\hbar}\hat{H}t\right) \hat{x}^i \exp\left(-\frac{i}{\hbar}\hat{H}t\right)$, and $|x, t\rangle = e^{\frac{i}{\hbar}\hat{H}t}|x\rangle$ ("moving frames") as eigenstates of $\hat{x}^i(t)$. Similarly we introduce $|p, t\rangle = e^{\frac{i}{\hbar}\hat{H}t}|p\rangle$. As before $\langle x, t|x', t\rangle = g^{-1/2}(x)\delta^{(n)}(x - x')$ and $\langle p, t|p', t\rangle = \delta^{(n)}(p - p')$.

Given a state $|\psi\rangle$ in the Hilbert space, $\psi(x,t) = \langle x,t|\psi\rangle$ is the Schrödinger wave function. In this x-representation, $\hat{p}_j(t)$ is represented when acting on $\psi(x,t)$ by

$$(p_x)_j = g^{-1/4}(x)\frac{\hbar}{i}\frac{\partial}{\partial x^j}g^{1/4}(x) \tag{2.5.43}$$

as follows from

$$\begin{aligned}
&\langle x,t|\hat{p}_j(t)|x',t\rangle = \int \langle x,t|p,t\rangle\langle p,t|\hat{p}_j(t)|x',t\rangle\, d^n p = \\
&\int p_j \langle x,t|p,t\rangle\langle p,t|x',t\rangle\, d^n p = \int p_j \frac{\exp\frac{i}{\hbar}(x-x')\cdot p}{(2\pi\hbar)^n g^{1/4}(x)g^{1/4}(x')}\, d^n p \\
&= g^{-1/4}(x)g^{-1/4}(x')\frac{\hbar}{i}\frac{\partial}{\partial x^j}\delta^{(n)}(x-x') \\
&= g^{-1/4}(x)g^{-1/4}(x')\frac{\hbar}{i}\frac{\partial}{\partial x^j}g^{1/2}(x)\langle x,t|x',t\rangle \\
&= g^{-1/4}(x)\frac{\hbar}{i}\frac{\partial}{\partial x^j}g^{1/4}(x)\langle x,t|x',t\rangle\ .
\end{aligned} \tag{2.5.44}$$

The Dirac delta function is defined by $\int \delta^{(n)}(x-x')f(x')d^n x' = f(x)$. In the last step we moved $g^{-1/4}(x')$ past $\frac{\partial}{\partial x^j}$ and then converted $g^{-1/4}(x')\delta^{(n)}(x-x')$ to $g^{-1/4}(x)\delta^{(n)}(x-x')$. A quick argument to justify (2.5.43) is to note that with the $\sqrt{g}$ in the inner product in x-space the operator $(p_x)_j$ is hermitian. Similarly one may derive

$$\langle x,t|\hat{p}_j|x',t\rangle = -g^{-1/4}(x')\frac{\hbar}{i}\frac{\partial}{\partial x'^j}g^{1/4}(x')\langle x,t|x',t\rangle\ . \tag{2.5.45}$$

It then follows that

$$\begin{aligned}
&\langle x,t|\hat{H}(t)|x',t\rangle = \langle x|\frac{1}{2}g^{-1/4}(\hat{x})\hat{p}_i g^{1/2}(\hat{x})g^{ij}(\hat{x})\hat{p}_j g^{-1/4}(\hat{x})|x'\rangle \\
&= \frac{1}{2}g^{-1/4}(x)\left(g^{-1/4}(x)\frac{\hbar}{i}\frac{\partial}{\partial x^i}g^{1/4}(x)\right)g^{1/2}(x)g^{ij}(x)\langle x|\hat{p}_j g^{-1/4}(x)|x'\rangle \\
&= H(x)\langle x|x'\rangle\ .
\end{aligned} \tag{2.5.46}$$

From (2.5.45) one finds that this expression is also equal to $H(x')\langle x|x'\rangle$, and this proves the last step of (2.5.41). One may now check that the transition element given in (2.5.33) satisfies

$$\left[H(y) + \hbar\frac{\partial}{\partial\beta}\right]\langle z|e^{-\frac{\beta}{\hbar}\hat{H}}|y\rangle = 0\ . \tag{2.5.47}$$

Already at the level of the terms of the form $(z-y)\partial g$ this is quite a good check.

2.6 Two-loop path integral evaluation of the transition element to order β.

In this section we shall explicitly verify through two loops that the path integral corresponding to the Hamiltonian $\hat{H} = \frac{1}{2} g^{-1/4} p_i g^{1/2} g^{ij} p_j g_i^{-1/4}$ for path $x^i(t)$ satisfying $x^i(-\beta) = y^i$, $x^i(0) = z^i$, reproduces to order β the results of the previous section for the matrix elements $\langle z | \exp\Big(-\frac{\beta}{\hbar} H\Big) | y \rangle$.

We recall that from (2.1.80)

$$\langle z | \exp\Big(-\frac{\beta}{\hbar} H\Big) | y \rangle \equiv Z[z, y, \beta] = \left[\frac{g(z)}{g(y)}\right]^{1/4} \frac{1}{(2\pi\beta\hbar)^{n/2}} \langle e^{-\frac{1}{\hbar} S^{int}} \rangle \tag{2.6.1}$$

where the brackets $\langle .. \rangle$ indicate that all quantum fields $q^i(\tau), b^i(\tau), c^i(\tau)$ and $a^i(\tau)$ are to be contracted using the propagators

$$\begin{aligned}
&\langle q^i(\sigma) q^j(\tau) \rangle = -\beta\hbar g^{ij}(z) \Delta(\sigma - \tau) \\
&\langle q^i(\sigma) \dot{q}^j(\tau) \rangle = -\beta\hbar g^{ij}(z)(\sigma + \theta(\tau - \sigma)) \\
&\langle \dot{q}^i(\sigma) \dot{q}^j(\tau) \rangle = -\beta\hbar g^{ij}(z)(1 - \delta(\sigma - \tau)) \\
&\langle b^i(\sigma) c^j(\tau) \rangle = -2\beta\hbar g^{ij}(z) \delta(\sigma - \tau) \\
&\langle a^i(\sigma) a^j(\tau) \rangle = \beta\hbar g^{ij}(z) \delta(\sigma - \tau) \\
&\Delta(\sigma, \tau) = \sigma(\tau + 1)\theta(\sigma - \tau) + \tau(\sigma + 1)\theta(\tau - \sigma) \ .
\end{aligned} \tag{2.6.2}$$

We also recall the definition of the interactions

$$\begin{aligned}
S^{int} &= \frac{1}{\beta} \int_{-1}^{0} \frac{1}{2} g_{ij}(x)(\dot{x}^i \dot{x}^j + b^i c^j + a^i a^j) d\tau \\
&\quad - \frac{1}{\beta} \int_{-1}^{0} \frac{1}{2} g_{ij}(z)(\dot{q}^i \dot{q}^j + b^i c^j + a^i a^j) d\tau \\
&\quad + \frac{\beta\hbar^2}{8} \int_{-1}^{0} (R + g^{ij} \Gamma^k_{il} \Gamma^l_{jk}) d\tau \ , \\
x^i(t) &= z^i + (z - y)^i \tau + q^i(\tau) \ , \quad \tau = t/\beta \ .
\end{aligned} \tag{2.6.3}$$

We shall encounter at various points ill-defined expressions to which we shall give meaning by going back to the discretized approach. Since each Feynman graph corresponds in a 1-1 fashion to terms in the answer for $\langle z | \exp(-\frac{\beta}{\hbar} \hat{H}) | y \rangle$, derived in the previous section without any ambiguities, this procedure will in a very direct way produce a list of continuum integrals for products of distributions. At the end of this section we shall check that our discretized Feynman rules produce these continuum integrals.

We shall organize the calculation as follows: first we compute all tree graphs (first from one vertex S^{int} then from two vertices S^{int}), then all one-loop graphs (first from one S^{int} then from two S^{int}), and finally all two loop graphs (first those from one S^{int}, then the one-particle reducible ones from two S^{int} and finally the one-particle irreducible ones from two S^{int}). There will be no contributions from three S^{int} vertices to order β.

Tree graphs. They consist of the classical vertices themselves and tree graphs with q-propagators. The former are given by

$$\bullet = -\frac{1}{\hbar}S^{int} = -\frac{1}{2\beta\hbar}\left(\begin{array}{c} g_{ij}(z) - \frac{1}{2}(z-y)^k\partial_k g_{ij}(z) \\ +\frac{1}{6}(z-y)^k(z-y)^l\partial_k\partial_l g_{ij}(z) \end{array}\right)(z-y)^i(z-y)^j \ . \tag{2.6.4}$$

These terms are part of the classical action $S_{cl}[z,y;\beta]$ in (2.5.32); the terms with two and three factors $(z-y)$ are already correct, while those with four $(z-y)$ and $\partial_k\partial_l g_{ij}$ are also correct, but the $\partial g\partial g(z-y)^4$ terms are lacking. They come from the tree graph with one propagator and two S^{int}. It yields

$$\begin{aligned}
&\bullet\!\!-\!\!\!-\!\!\bullet \\
&= \left(\frac{-1}{\beta\hbar}\right)^2\frac{1}{2!}\int_{-1}^{0}\int_{-1}^{0}\langle\frac{1}{2}g_{ij}(x)\dot{x}^i\dot{x}^j\frac{1}{2}g_{mn}(x)\dot{x}^m\dot{x}^n\rangle\, d\sigma d\tau \quad \text{(terms with one } \Delta) \\
&= \frac{1}{8\beta^2\hbar^2}\int_{-1}^{0}\int_{-1}^{0} d\sigma d\tau\Big[(\partial_k g_{ij})(\partial_l g_{mn})(-\hbar\beta g^{kl})\Delta(\sigma,\tau) \\
&\quad \times(z-y)^i(z-y)^j(z-y)^m(z-y)^n \\
&+4(\partial_k g_{ij}(z+(z-y)\sigma))(-\hbar\beta g^{km})\Delta^{\bullet}(\sigma,\tau)g_{nm}(z+(z-y)\tau) \\
&quad\times(z-y)^i(z-y)^j(z-y)^n \\
&+4g_{ij}(z+(z-y)\sigma)g_{mn}(z+(z-y)\tau) \\
&quad\times(-\hbar\beta g^{im}(z))\,{}^{\bullet}\Delta^{\bullet}(\sigma,\tau)(z-y)^j(z-y)^n\Big] \ .
\end{aligned} \tag{2.6.5}$$

We used the notation $\Delta^{\bullet}(\sigma,\tau) = \frac{\partial}{\partial\tau}\Delta(\sigma,\tau)$ and ${}^{\bullet}\Delta(\sigma,\tau) = \frac{\partial}{\partial\sigma}\Delta(\sigma,\tau)$ and ${}^{\bullet}\Delta^{\bullet}(\sigma,\tau) = \frac{\partial}{\partial\sigma}\frac{\partial}{\partial\tau}\Delta(\sigma,\tau)$. There are terms with three and four factors of $(z-y)$ in (2.6.5). Since the former were already accounted for, the $(z-y)^3$ terms above should vanish. This leads to a first condition on continuum integrals

$$\int_{-1}^{0}\int_{-1}^{0}\Delta^{\bullet}(\sigma,\tau)\, d\sigma d\tau + 2\int_{-1}^{0}\int_{-1}^{0}\sigma\,{}^{\bullet}\Delta^{\bullet}(\sigma,\tau)\, d\sigma d\tau = 0 \ . \tag{2.6.6}$$

Since the terms of the form $\partial_i\partial_j g_{kl}(z-y)^4$ were also already recovered in (2.6.4), also the terms from the last line in (2.6.5), with g_{ij} or g_{mn} expanded to second order, or the terms from the second line expanded to first order, should vanish. This leads to another condition

$$\int_{-1}^{0}\int_{-1}^{0}\sigma^2\,{}^{\bullet}\Delta^{\bullet}(\sigma,\tau)\, d\sigma d\tau + \int_{-1}^{0}\int_{-1}^{0}\sigma\,\Delta^{\bullet}(\sigma,\tau)\, d\sigma d\tau = 0 \ . \tag{2.6.7}$$

The $\partial g \partial g (z-y)^4$ terms from (2.6.5) are given by

$$\left(-\frac{1}{8\beta\hbar}\partial_k g_{ij}\partial_l g_{mn}\right)(z-y)^j(z-y)^n$$
$$\times\left[g^{kl}(z-y)^i(z-y)^m\int_{-1}^{0}\int_{-1}^{0}\Delta\right.$$
$$+(z-y)^i(z-y)^l g^{km}4\int_{-1}^{0}\int_{-1}^{0}\tau\,\Delta^{\bullet}$$
$$\left.+(z-y)^k(z-y)^l g^{im}4\int_{-1}^{0}\int_{-1}^{0}\sigma\tau\,{}^{\bullet}\Delta^{\bullet}\right] \tag{2.6.8}$$

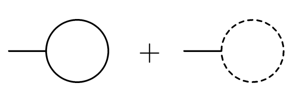
where we introduced the obvious notation $\int\int\tau\Delta^{\bullet}\equiv\int_{-1}^{0}\int_{-1}^{0}\tau\frac{\partial}{\partial\tau}\Delta(\sigma,\tau)\,d\sigma d\tau$. We get the correct terms of the form $\partial g\partial g(z-y)^4$ which occur in the classical action in (2.5.32) provided the following integrals are correct

$$\int_{-1}^{0}\int_{-1}^{0}\Delta(\sigma,\tau)\,d\sigma d\tau=-\frac{1}{12}$$
$$\int_{-1}^{0}\int_{-1}^{0}\tau\,\Delta^{\bullet}(\sigma,\tau)\,d\sigma d\tau=+\frac{1}{12}$$
$$\int_{-1}^{0}\int_{-1}^{0}\sigma\tau\,{}^{\bullet}\Delta^{\bullet}(\sigma,\tau)\,d\sigma d\tau=-\frac{1}{12}\;. \tag{2.6.9}$$

If the integrals in (2.6.7 – 2.6.9) have the values indicated, the tree graph contributions correctly reproduce the classical action to order β. We first consider all other graphs and integrals, and then we shall discuss these integrals.

We considered only the connected tree graphs because we compared the result with S_{cl} instead of $\exp(-S_{cl}/\hbar)$. The reader may have wondered why we did not consider connected tree graphs with 3 or more vertices. The reason is that these graphs do not contribute at order β. The vertices at the end of such a tree graph contain at least two factors $z-y$ (use that $\int_{-1}^{0}\dot{q}\,d\sigma=0$ since $q(\sigma)$ vanishes at the boundaries). Any vertex which is not a vertex at the ends contains at least one factor $z-y$. Thus the total number of factors $z-y$ is 5 or more, which leads to contributions of order $\beta^{3/2}$ or higher.

One-loop graphs. From the vertex S^{int} one finds, by expanding g_{ij} once, the following equal time contractions



$$=-\frac{1}{2\beta\hbar}(z-y)^k\partial_k g_{ij}(z)\int_{-1}^{0}\tau\,({}^{\bullet}\Delta^{\bullet}+\Delta^{\bullet\bullet})_{\sigma=\tau}\,d\tau(-\beta\hbar g^{ij}(z))$$

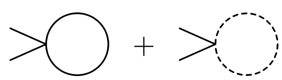
$$-\frac{1}{\beta\hbar}\partial_k g_{ij}(z)\int_{-1}^{0}(\Delta^{\bullet})_{\sigma=\tau}\, d\tau(z-y)^j(-\beta\hbar g^{ik}(z)) \ . \tag{2.6.10}$$

Since we already know from the previous section that the sum of all one-loop graphs is given by $-\frac{1}{12}R_{ij}(z)(z-y)^i(z-y)^j$, these equal-time contractions with ∂g should cancel the contribution from the measure $[g(z)/g(y)]^{1/4}$. This yields the conditions

$$\begin{aligned}\int_{-1}^{0}\tau\,({}^{\bullet}\Delta^{\bullet}+\Delta^{\bullet\bullet})_{\sigma=\tau}\, d\tau &= -\frac{1}{2}\\ \int_{-1}^{0}(\Delta^{\bullet})_{\sigma=\tau}\, d\tau &= 0 \ .\end{aligned} \tag{2.6.11}$$

These equal-time contractions are a priori ill-defined in field theory, but we deduce their value unambiguously as the limit from the discretized expressions.

By expanding g_{ij} to second order in one vertex S^{int}, one finds further equal-time one-loop contractions with $\partial\partial g$



$$\begin{aligned}&= \left(\frac{-1}{\beta\hbar}\right)\frac{1}{4}\partial_k\partial_l g_{ij}(z)\int_{-1}^{0}d\tau\,\Big[(z-y)^k\tau(z-y)^l\tau\,({}^{\bullet}\Delta^{\bullet}+\Delta^{\bullet\bullet})_{\sigma=\tau}\, g^{ij}(z)\\ &+4(z-y)^k\tau(z-y)^i(\Delta^{\bullet})_{\sigma=\tau}g^{lj}(z)\\ &+(z-y)^i(z-y)^j g^{kl}(z)\,(\Delta)_{\sigma=\tau}\ \Big]\ d\tau(-\beta\hbar) \ .\end{aligned} \tag{2.6.12}$$

There is also a term of the form $(\partial_k\partial_l g_{ij})(z-y)^k(z-y)^l g^{ij}$ coming from the measure factor $[g(z)/g(y)]^{1/4}$. Its coefficient is $-1/8$.

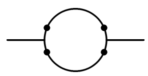
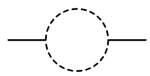
To obtain the linearized contribution

$$\begin{aligned}&-\frac{1}{12}R_{iklj}(z-y)^k(z-y)^l g^{ij}\sim\\ &\frac{1}{24}(-\partial_i\partial_j g_{kl}-\partial_k\partial_l g_{ij}+2\partial_i\partial_l g_{kj})g^{ij}(z-y)^k(z-y)^l\end{aligned} \tag{2.6.13}$$

we need the following equal-time contractions

$$\begin{aligned}\int_{-1}^{0}\tau^2\,({}^{\bullet}\Delta^{\bullet}+\Delta^{\bullet\bullet})\, d\tau &= \frac{1}{3}\\ \int_{-1}^{0}\tau\,\Delta^{\bullet}{}_{\sigma=\tau}\, d\tau &= \frac{1}{12}\\ \int_{-1}^{0}\Delta_{\sigma=\tau}\, d\tau &= -\frac{1}{6} \ .\end{aligned} \tag{2.6.14}$$

Next we consider the contributions with $\partial g \partial g$. They come from one-loop graphs with two vertices H^{int}. They can either consist of two unequal-time propagators, or one tree-graph propagator times an equal time loop. We first consider the one-loop graphs with two equal-time propagators. The contractions of two factors $(\dot{q}\dot{q} + bc + aa)$ yield (dots denote derivatives)

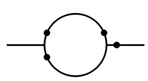
 + 

$$
\begin{aligned}
&= \left(\frac{-1}{\beta\hbar}\right)^2 \frac{1}{2!} \int_{-1}^{0} d\sigma \int_{-1}^{0} d\tau \frac{1}{2} \partial_k g_{ij} \frac{1}{2} \partial_l g_{mn} (z-y)^k \sigma (z-y)^l \tau \\
&[2 g^{im} g^{jn} \{ {}^{\bullet}\!\Delta^{\bullet}(\sigma,\tau) {}^{\bullet}\!\Delta^{\bullet}(\sigma,\tau) - {}^{\bullet\bullet}\!\Delta(\sigma,\tau) \Delta^{\bullet\bullet}(\sigma,\tau) \} (-\beta\hbar)^2] \\
&= \frac{1}{4} \partial_k g_{ij} \partial_l g_{mn} g^{im} g^{jn} (z-y)^k (z-y)^l I
\end{aligned}
\tag{2.6.15}
$$

where I is fixed by requiring that these terms complete the Ricci tensor

$$
I = \int ({}^{\bullet}\!\Delta^{\bullet}\, {}^{\bullet}\!\Delta^{\bullet} - {}^{\bullet\bullet}\!\Delta\, \Delta^{\bullet\bullet})\, \sigma\tau \, d\sigma d\tau = -\frac{5}{12} \ . \tag{2.6.16}
$$

In addition there are contributions with $q\dot{q}$ propagators (indicated by putting a dot above them) and qq propagators (without dot). Namely

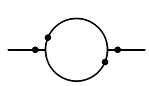


$$
\begin{aligned}
= \ & \left(\frac{-1}{\beta\hbar}\right)^2 \frac{1}{2!} \left(\frac{1}{2}\partial_k g_{ij}\right) \left(\frac{1}{2}\partial_l g_{mn}\right) g^{im} g^{jl} 8 \\
& \int_{-1}^{0} \int_{-1}^{0} \sigma\, {}^{\bullet}\!\Delta^{\bullet}(\sigma,\tau)\, {}^{\bullet}\!\Delta(\sigma,\tau) (z-y)^k (z-y)^n (-\beta\hbar)^2 d\sigma d\tau \ .
\end{aligned}
\tag{2.6.17}
$$

This should vanish, hence

$$
\int_{-1}^{0} \int_{-1}^{0} \sigma\, {}^{\bullet}\!\Delta^{\bullet}(\sigma,\tau)\, {}^{\bullet}\!\Delta(\sigma,\tau)\, d\sigma d\tau = 0 \ . \tag{2.6.18}
$$

Further,

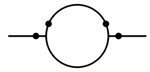


$$
\begin{aligned}
= \ & \left(\frac{-1}{\beta\hbar}\right)^2 \frac{1}{2!} \frac{1}{2} \partial_k g_{ij} \frac{1}{2} \partial_l g_{mn} g^{km} g^{il} 4 (z-y)^j (z-y)^n \\
& \int_{-1}^{0} \int_{-1}^{0} {}^{\bullet}\!\Delta(\sigma,\tau) \Delta^{\bullet}(\sigma,\tau) (-\beta\hbar)^2 d\sigma d\tau \ .
\end{aligned}
\tag{2.6.19}
$$

We need

$$
\int_{-1}^{0} \int_{-1}^{0} {}^{\bullet}\!\Delta(\sigma,\tau) \Delta^{\bullet}(\sigma,\tau)\, d\sigma d\tau = -\frac{1}{12} \ . \tag{2.6.20}
$$

Finally



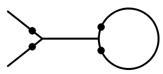
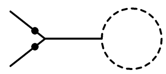
$$= \left(\frac{-1}{\beta\hbar}\right)^2 \frac{1}{2!}\frac{1}{2}\partial_k g_{ij}\frac{1}{2}\partial_l g_{mn} g^{kl} g^{im}$$

$$4\int_{-1}^{0}\int_{-1}^{0} \Delta(\sigma,\tau)\, {}^{\bullet}\!\Delta^{\bullet}(\sigma,\tau)\, d\sigma d\tau (z-y)^j (z-y)^n \tag{2.6.21}$$

We need

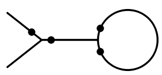
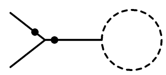
$$\int_{-1}^{0}\int_{-1}^{0} \Delta(\sigma,\tau)\, {}^{\bullet}\!\Delta^{\bullet}(\sigma,\tau)\, d\sigma d\tau = \frac{1}{12}\ . \tag{2.6.22}$$

We now record the one-particle reducible one-loop graphs with one equal-time propagator and one tree propagator. We need one vertex with one q, and the other vertex with three q's, or one q and two ghosts. In all cases we consider terms proportional $\partial_k g_{ij}\ \partial_l g_{mn}$. We then find the following results provided the integrals have the values indicated (the symbol α denotes $\partial_k g_{ij}\partial_l g_{mn}$)

 $+$ 

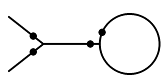
$$= -\frac{1}{48}\alpha g^{ij} g^{kl}(z-y)^m (z-y)^n I$$

$$I = \int_{-1}^{0}\int_{-1}^{1} [{}^{\bullet}\!\Delta^{\bullet}(\sigma,\sigma) + {}^{\bullet\bullet}\!\Delta(\sigma,\sigma)]\, \Delta(\sigma,\tau)\, d\sigma d\tau = -\frac{1}{12}\ , \tag{2.6.23}$$

 $+$ 

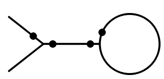
$$= \frac{1}{24}\alpha g^{ij} g^{km}(z-y)^l (z-y)^n I$$

$$I = \int_{-1}^{0}\int_{-1}^{0} [{}^{\bullet}\!\Delta^{\bullet}(\sigma,\sigma) + {}^{\bullet\bullet}\!\Delta(\sigma,\sigma)]\, \Delta^{\bullet}(\sigma,\tau)\, \tau\, d\sigma d\tau = \frac{1}{12}\ , \tag{2.6.24}$$



$$= \frac{1}{24}\alpha g^{ik} g^{jl}(z-y)^m (z-y)^n I$$

$$I = \int_{-1}^{0}\int_{-1}^{0} \Delta^{\bullet}(\sigma,\sigma)\, {}^{\bullet}\!\Delta(\sigma,\tau)\, d\sigma d\tau = \frac{1}{12}\ , \tag{2.6.25}$$



$$= -\frac{1}{12}\alpha g^{ik} g^{jm}(z-y)^l (z-y)^n I$$

$$I = \int_{-1}^{0}\int_{-1}^{0} \Delta^{\bullet}(\sigma,\sigma)\, {}^{\bullet}\!\Delta^{\bullet}(\sigma,\tau)\, \tau\, d\sigma d\tau = -\frac{1}{12}\ . \tag{2.6.26}$$

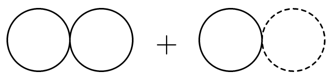
If all the integrals in (2.6.10 – 2.6.26) have the values indicated, the one-loop contributions correctly reproduce the Van Vleck-Morette determinant in (2.5.37).

Two loop contributions. The two loop graphs should reproduce the terms of order $\beta\hbar$ in the transition element. These were found to be given by

$$-\frac{\beta\hbar}{12}R(z) \tag{2.6.27}$$

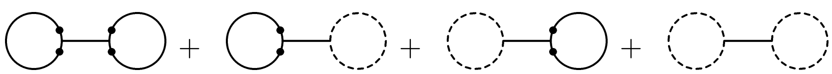
We quote again the various graphs and below them the values which the corresponding integrals should have.

First there is the figure 8 graph due to one vertex

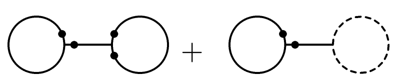


$$\begin{aligned}
&= -\frac{1}{\beta\hbar}\int \frac{1}{4}\partial_k\partial_l g_{ij}\langle q^k q^l(\dot q^i \dot q^j + b^i c^j + a^i a^j)\rangle d\tau(-\beta\hbar)^2 \\
&= \frac{1}{24}\beta\hbar\partial_k\partial_l g_{ij}(z)(g^{kl}g^{ij}I_1 - g^{ik}g^{jl}I_2) \\
I_1 &= \int_{-1}^{0}\Delta(\tau,\tau)\,[{}^{\bullet}\Delta^{\bullet}(\tau,\tau) + {}^{\bullet\bullet}\Delta(\tau,\tau)]\,d\tau = -\frac{1}{6} \\
I_2 &= \int_{-1}^{0}\Delta^{\bullet}(\tau,\tau)\,\Delta^{\bullet}(\tau,\tau)\,d\tau = \frac{1}{12}\ .
\end{aligned} \tag{2.6.28}$$

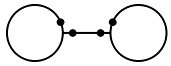
Next there are the products of two equal-time loops connected by an unequal-time propagator



$$\begin{aligned}
&= \beta\hbar\frac{1}{96}\alpha g^{kl}g^{ij}g^{mn}\,(\partial_k g_{im})\,(\partial_l g_{jn})\,I \\
I &= \int_{-1}^{0}\int_{-1}^{0}[{}^{\bullet}\Delta^{\bullet}(\sigma,\sigma) + \Delta^{\bullet\bullet}(\sigma,\sigma)]\,\Delta(\sigma,\tau)\,[{}^{\bullet\bullet}\Delta(\tau,\tau) + \Delta^{\bullet\bullet}(\tau,\tau)]\,d\sigma d\tau \\
&= -\frac{1}{12}\ ,
\end{aligned} \tag{2.6.29}$$



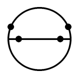
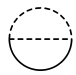
$$\begin{aligned}
&= -\beta\hbar\frac{1}{24}\alpha g^{ij}g^{lm}g^{kn}\,(\partial_m g_{ij})\,(\partial_n g_{kl})\,I \\
I &= \int_{-1}^{0}\int_{-1}^{0}\{{}^{\bullet}\Delta^{\bullet}(\sigma,\sigma) + {}^{\bullet\bullet}\Delta(\sigma,\sigma)\}\Delta^{\bullet}(\sigma,\tau)\Delta^{\bullet}(\tau,\tau)d\sigma d\tau \\
&= \frac{1}{12}\ ,
\end{aligned} \tag{2.6.30}$$



$$= \beta\hbar\frac{1}{24}\alpha g^{ik}g^{jm}g^{ln}\left(\partial_m g_{ij}\right)\left(\partial_n g_{kl}\right) I$$

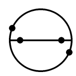
$$I = \int_{-1}^{0}\int_{-1}^{0} {}^{\bullet}\!\Delta(\sigma,\sigma)\,{}^{\bullet}\!\Delta^{\bullet}(\sigma,\tau)\,\Delta^{\bullet}(\tau,\tau)\,d\sigma d\tau = -\frac{1}{12}\ . \qquad (2.6.31)$$

Finally there are the two-loop graphs with the form of a setting sun. They come from all possible contractions of $q(\dot{q}\dot{q}+bc+aa)$ times $q(\dot{q}\dot{q}+bc+aa)$

 + 

$$= -\beta\hbar\frac{1}{16}\alpha g^{kl}g^{im}g^{jn}\left(\partial_k g_{ij}\right)\left(\partial_l g_{mn}\right) I$$

$$\begin{aligned} I &= \int_{-1}^{0}\int_{-1}^{0} \Delta(\sigma,\tau)\left[{}^{\bullet}\!\Delta^{\bullet}(\sigma,\tau)\,{}^{\bullet}\!\Delta^{\bullet}(\sigma,\tau) - {}^{\bullet\bullet}\!\Delta(\sigma,\tau)\,\Delta^{\bullet\bullet}(\sigma,\tau)\right] d\sigma d\tau \\ &= \frac{1}{4}\ , \end{aligned} \qquad (2.6.32)$$



$$= \beta\hbar\frac{1}{12}\alpha g^{km}g^{in}g^{jl}\left(\partial_k g_{ij}\right)\left(\partial_l g_{mn}\right) I$$

$$I = \int_{-1}^{0}\int_{-1}^{0} {}^{\bullet}\!\Delta^{\bullet}(\sigma,\tau)\,\Delta^{\bullet}(\sigma,\tau)\,{}^{\bullet}\!\Delta(\sigma,\tau)\,d\sigma d\tau = -\frac{1}{6}\ . \qquad (2.6.33)$$

The sum of all these two-loop contributions plus the contribution from the counterterm should be equal to $-\beta\hbar\frac{1}{12}R$. Different regularization schemes lead to different counterterms, and thereby the one- and two-loop graphs give different results if one uses different regularization methods. In time slicing the counterterm is $-\frac{\beta}{\hbar}\frac{\hbar^2}{8}\left(R + \Gamma^k_{il}\Gamma^l_{jk}g^{ij}\right)$. Hence, the sum of all two-loop graphs evaluated above should be equal to $\frac{\beta\hbar}{24}R + \frac{\beta\hbar}{8}(\Gamma\Gamma)$. Expanding the $\Gamma\Gamma$ term one finds two structures of the form $\partial g\partial g$, while R contains three more structures with $\partial g\partial g$, and two structures of the form $\partial\partial g$. All these terms match:

$$\begin{aligned} 2-\text{loop} &= \frac{1}{24}(\Box g - \partial^\alpha g_\alpha) + \frac{1}{96}\Big(\partial_\alpha g\partial^\alpha g - 4\partial_\alpha g g^\alpha + 4g_\alpha g^\alpha \\ &\quad -6(\partial_\alpha g_{\beta\gamma})^2 + 8\partial_\alpha g_{\beta\gamma}\partial_\beta g_{\alpha\gamma}\Big) \\ \left(\frac{1}{8}-\frac{1}{12}\right)R &= \frac{1}{24}(\Box g - \partial^\alpha g_\alpha + \frac{1}{2}\partial_\alpha g_{\beta\gamma}\partial_\beta g_{\alpha\gamma} - \frac{3}{4}(\partial_\alpha g_{\beta\gamma})^2 \\ &\quad +\frac{1}{4}\partial_\alpha g\partial^\alpha g - \partial_\alpha g g^\alpha + g_\alpha g^\alpha) \\ \frac{1}{8}\Gamma\Gamma &= \frac{1}{8}\left(\frac{1}{2}\partial_\alpha g_{\beta\gamma}\partial_\beta g_{\alpha\gamma} - \frac{1}{4}(\partial_\alpha g_{\beta\gamma})^2\right)\ . \end{aligned} \qquad (2.6.34)$$

Contraction are performed with the metric g^{ij}, for example $(\partial_i g_{jk})^2 = g^{ii'}g^{jj'}g^{kk'}(\partial_i g_{jk})(\partial_{i'}g_{j'k'})$.

We now discuss the integrals we encountered. We first make a list.

Trees

$$\int\!\!\int \Delta^{\bullet}(\sigma,\tau) + 2\int\!\!\int \sigma\, {}^{\bullet}\Delta^{\bullet}(\sigma,\tau) = 0$$
$$\int\!\!\int \sigma^2\, {}^{\bullet}\Delta^{\bullet}(\sigma,\tau) + \int\!\!\int \sigma\,\Delta^{\bullet}(\sigma,\tau) = 0$$
$$\int\!\!\int \Delta(\sigma,\tau) = -\frac{1}{12}\ , \quad \int\!\!\int \tau\,\Delta^{\bullet}(\sigma,\tau) = \frac{1}{12}\ , \quad \int\!\!\int \sigma\, {}^{\bullet}\Delta^{\bullet}(\sigma,\tau)\tau = -\frac{1}{12}$$

One-loop

$$\int\!\!\int ({}^{\bullet}\Delta^{\bullet}\ {}^{\bullet}\Delta^{\bullet} - {}^{\bullet\bullet}\Delta\Delta^{\bullet\bullet})\ \sigma\tau = -\frac{5}{12}\ , \quad \int\!\!\int \sigma\ {}^{\bullet}\Delta^{\bullet}\ {}^{\bullet}\Delta = 0$$
$$\int\!\!\int\ {}^{\bullet}\Delta\Delta^{\bullet} = -\frac{1}{12}\ , \quad \int\!\!\int \Delta\, {}^{\bullet}\Delta^{\bullet} = \frac{1}{12}$$
$$\int \tau({}^{\bullet}\Delta^{\bullet}(\tau,\tau) + \Delta^{\bullet\bullet}(\tau,\tau)) = -\frac{1}{2}\ , \quad \int \tau^2({}^{\bullet}\Delta^{\bullet}(\tau,\tau) + \Delta^{\bullet\bullet}(\tau,\tau)) = \frac{1}{3}$$
$$\int \Delta(\tau,\tau) = -\frac{1}{6}\ , \quad \int \Delta^{\bullet}(\tau,\tau) = 0\ , \quad \int \Delta^{\bullet}(\tau,\tau)\tau = \frac{1}{12}$$

Two-loop

$$\int\!\!\int ({}^{\bullet}\Delta^{\bullet}(\sigma,\sigma) + {}^{\bullet\bullet}\Delta(\sigma,\sigma))\Delta(\sigma,\tau) = -\frac{1}{12}$$
$$\int\!\!\int ({}^{\bullet}\Delta^{\bullet}(\sigma,\sigma) + {}^{\bullet\bullet}\Delta(\sigma,\sigma)\Delta^{\bullet}(\sigma,\tau)\tau = \frac{1}{12}$$
$$\int\!\!\int (\Delta^{\bullet}(\sigma,\sigma)\, {}^{\bullet}\Delta(\sigma,\tau)) = \frac{1}{12}$$
$$\int\!\!\int \Delta^{\bullet}(\sigma,\sigma)\, {}^{\bullet}\Delta^{\bullet}(\sigma,\tau)\tau = -\frac{1}{12}$$
$$\int \Delta(\tau,\tau)({}^{\bullet}\Delta^{\bullet}(\tau,\tau) + {}^{\bullet\bullet}\Delta(\tau,\tau)) = -\frac{1}{6}\ , \quad \int \Delta^{\bullet}(\tau,\tau)\Delta^{\bullet}(\tau,\tau) = \frac{1}{12}$$
$$\int\!\!\int ({}^{\bullet}\Delta^{\bullet}(\sigma,\sigma) + {}^{\bullet\bullet}\Delta(\sigma,\sigma))\Delta(\sigma,\tau)({}^{\bullet}\Delta^{\bullet}(\tau,\tau) + {}^{\bullet\bullet}\Delta(\tau,\tau)) = -\frac{1}{12}$$
$$\int\!\!\int ({}^{\bullet}\Delta^{\bullet}(\sigma,\sigma) + {}^{\bullet\bullet}\Delta(\sigma,\sigma))\Delta^{\bullet}(\sigma,\tau)\Delta^{\bullet}(\tau,\tau) = \frac{1}{12}$$
$$\int\!\!\int {}^{\bullet}\Delta(\sigma,\sigma)\, {}^{\bullet}\Delta^{\bullet}(\sigma,\tau)\Delta^{\bullet}(\tau,\tau) = -\frac{1}{12}$$
$$\int\!\!\int \Delta(\sigma,\tau)({}^{\bullet}\Delta^{\bullet}(\sigma,\tau)\, {}^{\bullet}\Delta^{\bullet}(\sigma,\tau) - {}^{\bullet\bullet}\Delta(\sigma,\tau)\Delta^{\bullet\bullet}(\sigma,\tau)) = \frac{1}{4} \qquad \text{(with TS)}$$
$$\int\!\!\int {}^{\bullet}\Delta^{\bullet}(\sigma,\tau)\Delta^{\bullet}(\sigma,\tau)\, {}^{\bullet}\Delta(\sigma,\tau) = -\frac{1}{6} \qquad \text{(with TS)}\ . \tag{2.6.35}$$

Only if all these integrals have the values indicated is there complete agreement between the Feynman diagram result and the operator approach result.

Using the naive continuum limits

$$
\begin{aligned}
&\Delta(\sigma,\tau) = \sigma(\tau+1)\theta(\sigma-\tau) + \tau(\sigma+1)\theta(\tau-\sigma) \\
&\Delta^{\bullet}(\sigma,\tau) = \sigma + \theta(\tau-\sigma)\ , \quad \Delta^{\bullet}(\tau,\tau) = \tau + 1/2 \\
&{}^{\bullet}\Delta^{\bullet}(\sigma,\tau) = 1 - \delta(\sigma-\tau)\ , \quad {}^{\bullet\bullet}\Delta(\sigma,\tau) = \delta(\sigma-\tau) \\
&{}^{\bullet}\Delta^{\bullet}(\sigma,\sigma) = 1 - \delta(\sigma-\sigma)\ , \quad {}^{\bullet\bullet}\Delta(\sigma,\sigma) = \delta(\sigma-\sigma)
\end{aligned}
\tag{2.6.36}
$$

we find complete agreement for the transition element provided we interpret $\delta(\sigma-\tau)$ as a Kronecker delta. The expressions $\delta(\sigma-\sigma)$ cancel always since they only appear in the combination ${}^{\bullet}\Delta^{\bullet}(\sigma,\sigma) - {}^{\bullet\bullet}\Delta(\sigma,\sigma)$. In the next chapters we discuss two other regularization schemes; these also lead to complete agreement.

3
Mode regularization

In this section we discuss path integrals defined by mode regularization (MR). Ideally, one would like to derive mode regularization from first principles, namely starting from the transition amplitude defined as the matrix element of the evolution operator $\langle x_{\rm f}^k|\exp(-\beta\hat{H})|x_{\rm i}^k\rangle$, as done in the time slicing regularization of the previous section (we set $\hbar = 1$ in this chapter). However, a derivation along those lines seems quite laborious and will not be attempted. We find it easier to take a more pragmatic approach, and present a quick construction of the mode-regulated path integral. This can be done by recalling general properties of quantum field theories (QFT) in d dimensions as a guideline, and specializing those properties to the simpler context of one dimension.

General theorems for quantum field theories with local Lagrangians guarantee the possibility of constructing a consistent perturbative expansion by renormalizing the infinities away. Renormalization is usually achieved by adding local counterterms with infinite coefficients to the original Lagrangian. At the same time *finite local counterterms* relate different regularization schemes to each other. More precisely, the finite counterterms, left undetermined after the removal of divergences, are fixed by imposing a sufficient number of renormalization conditions. All regularization schemes should then produce the same physical results. The renormalization program through counterterms is performed iteratively, loop by loop.

Let us consider the simple case of a scalar QFT, which is enough for our purposes. One can classify the interactions as non-renormalizable, renormalizable, and super-renormalizable according to the mass dimension of the corresponding coupling constant being negative, zero, and positive, respectively. Coupling constants with negative mass dimensions render the theory non-renormalizable, since one is forced to introduce an infinite number of counterterms (of a structure not contained in the original

Lagrangian) to cancel the divergences of the Feynman graphs. These theories are generically considered to be effective field theories, like the Fermi theory of the weak interactions, gravity, and supergravity. Renormalizable interactions allow instead for infinities to be removed with the use of a finite number of counterterms, though at each loop the coefficients of the counterterms receive additional infinite contributions, as in QED and in the Standard Model. Finally, super-renormalizable interactions generate a perturbative expansion which can be made finite at any loop by counterterms that can appear only up to a finite loop order, like perturbative $\lambda\phi^3$ theory in four dimensions.

Our one-dimensional nonlinear sigma model is super-renormalizable (recall the explicit power counting exercise presented in the introduction). Thus the QFT theorems guarantee that one needs to consider counterterms only up to a finite loop order. In addition, we will see that there is no need to cancel infinities thanks to the inclusion of the extra vertices coming form the measure. Therefore only *finite* counterterms can appear. We will show that they appear only up to 2 loops. As described above they are needed to satisfy the renormalization conditions. The precise renormalization conditions that we impose are contained in the following requirement: the transition amplitude computed with the regulated path integral must satisfy the Schrödinger equation with a given Hamiltonian operator. Without loss of generality, we choose this operator to be the one containing the covariant Laplacian without any additional coupling to the scalar curvature (if desired, extra couplings can always be introduced by including them into the potentials V and A_i). This renormalization condition fixes completely the counterterm V_{MR} as well as the overall normalization of the path integral. In this way, also the MR scheme is fully specified and can be used in applications.

Mode regularization for one dimensional nonlinear sigma models was introduced in [23, 24, 60]. The latter reference contain the correct counterterm V_{MR}. This regularization was used in [61] to compute the transition amplitude at three loops. Related references are [62, 63]. A previous use of mode regularization for quantizing nonlinear sigma models was attempted in [64].

3.1 Mode regularization in configuration space

We start from a general classical action in Euclidean time for the fields x^i with $i = 1, .., n$

$$S = \int_{t_\mathrm{i}}^{t_\mathrm{f}} dt \left[\frac{1}{2} g_{ij}(x) \frac{dx^i}{dt}\frac{dx^j}{dt} + iA_i(x)\frac{dx^i}{dt} + V(x)\right] \tag{3.1.1}$$

and try to define directly the transition amplitude as a path integral

$$\langle x^k_{\rm f}, t_{\rm f} | x^k_{\rm i}, t_{\rm i} \rangle = \int_{\rm BC} \mathcal{D}x \ {\rm e}^{-S} \tag{3.1.2}$$

$$\mathcal{D}x = \prod_{t_{\rm i}<t<t_{\rm f}} \sqrt{\det g_{ij}(x(t))} \ d^n x(t) \tag{3.1.3}$$

where BC indicates the boundary conditions at initial and final time $x^k(t_{\rm i}) = x^k_{\rm i}$ and $x^k(t_{\rm f}) = x^k_{\rm f}$. The measure $\mathcal{D}x$ is formally a scalar since it is the product of scalar measures. The action is also a scalar and the transition element should therefore be a scalar when properly defined.

Usually one considers in QFT the path integral representation for the transition amplitude from the in-vacuum to the out-vacuum, which corresponds to an infinite propagation time. In quantum mechanics one can afford to be more general, and ask for the transition amplitude between an arbitrary initial state $|\Psi_{\rm i}\rangle$ at time $t_{\rm i}$ and an arbitrary final state $|\Psi_{\rm f}\rangle$ at time $t_{\rm f}$. For simplicity we consider initial and final states as eigenstates of the position operator, since a general transition amplitude is then given by

$$\langle \Psi_{\rm f}, t_{\rm f} | \Psi_{\rm i}, t_{\rm i} \rangle = \int d^n x_{\rm f} \sqrt{g(x_{\rm f})} \int d^n x_{\rm i} \sqrt{g(x_{\rm i})} \ \Psi^*_{\rm f}(x_{\rm f}) \ \langle x_{\rm f}, t_{\rm f} | x_{\rm i}, t_{\rm i} \rangle \ \Psi_{\rm i}(x_{\rm i}) \ . \tag{3.1.4}$$

Note again that the transition amplitude on the right hand side, given in (3.1.2), is formally a scalar since the measure factors for integrating over the initial and final points are not included into (3.1.3). Therefore they appear in (3.1.4).

The nontrivial measure in (3.1.3) is not translationally invariant under $x^i(t) \to x^i(t) + \epsilon^i(t)$. This makes it difficult to generate the perturbative expansion: one cannot complete squares and shift integration variables to derive the propagators as usual. A standard trick to obtain a translationally invariant measure is to introduce ghost fields and exponentiate the nontrivial factor appearing in (3.1.3)

$$\prod_{t_{\rm i}<t<t_{\rm f}} \sqrt{\det g_{ij}(x(t))} = \int DaDbDc \ {\rm e}^{-S_{gh}} \tag{3.1.5}$$

$$S_{gh} = \int_{t_{\rm i}}^{t_{\rm f}} dt \ \frac{1}{2} g_{ij}(x)(a^i a^j + b^i c^j) \tag{3.1.6}$$

where the translationally invariant measures for the ghosts are given by

$$Da = \prod_{t_{\rm i}<t<t_{\rm f}} d^n a(t) \ , \quad Db = \prod_{t_{\rm i}<t<t_{\rm f}} d^n b(t) \ , \quad Dc = \prod_{t_{\rm i}<t<t_{\rm f}} d^n c(t) \ . \tag{3.1.7}$$

The ghosts a^i are commuting while the ghosts b^i and c^i are anticommuting, so they reproduce the same measure factor that is also obtained by integrating out the momenta in phase space.

Up to this point the whole construction is completely formal and we should try to give it a concrete meaning. Thus, we must introduce a regularization scheme to define the path integral and evaluate it unambiguously. The regularization will bring along a corresponding counterterm ΔV which will be used to satisfy the renormalization conditions mentioned before. In particular, the counterterm will restore the symmetries which may be accidentally broken by the regularization (one may recall that no anomalies are expected in quantum mechanics). The regularization that we choose to present in this chapter is equivalent to a cut-off in the loop momenta. Since the momenta on a compact space are discrete this scheme is called mode regularization.

To get started it is convenient to shift and rescale the time parameter in order to extract the total propagation time β out of the action $S = \frac{1}{\beta} S'$. We do this by defining $t = t_{\rm f} + \beta\tau$ with $\beta = t_{\rm f} - t_{\rm i}$, so that $-1 \leq \tau \leq 0$. The full rescaled action reads

$$S' = \int_{-1}^{0} d\tau \left[\frac{1}{2} g_{ij}(x)\dot{x}^i \dot{x}^j + i\beta A_i(x)\dot{x}^i + \beta^2 \Big(V(x) + V_{MR}(x) \Big) \right] \quad (3.1.8)$$

where $\dot{x}^i = \frac{dx^i}{d\tau}$. We have denoted by V_{MR} the counterterm required by mode regularization. Note that $\exp(-\frac{1}{\beta} S')$ is now the weight factor for the sum over paths, so that the total propagation time β plays here a role similar to the Planck constant $\hbar$ (which we have set to one) and can be used to count the number of loops. In the loop expansion generated by β the potentials V and V_{MR} start contributing only at two loops, while A_i starts at one loop[1]. From now on we drop the prime on S.

For an arbitrary metric $g_{ij}(x)$ one is only able to calculate the path integral in a perturbative expansion in β and in the coordinate displacements $\xi^i \equiv x^i_{\rm i} - x^i_{\rm f}$. Thus we start by parametrizing

$$x^i(\tau) = x^i_{bg}(\tau) + q^i(\tau) \quad (3.1.9)$$

where $x^i_{bg}(\tau)$ is a background trajectory and $q^i(\tau)$ the quantum fluctuations. After choosing a coordinate system in which one carries out the computations, the background trajectory is most conveniently taken to satisfy the free equations of motion in the chosen reference frame. It is a

[1] Reintroducing $\hbar$ one can see that the classical potentials V and A_i are of order $\hbar^0$, while the counterterm V_{MR} will turn out to contribute only at the two-loop level (order $\hbar^2$). Thus if one uses $\hbar$ to count loops, V appears at the tree level, but if one uses β then V starts contributing at two loops. In Feynman graphs one might represent V by a cross to indicate that it is a term of order β^2.

function linear in τ connecting the initial point $x_{\rm i}^i$ to the final point $x_{\rm f}^i$, enforcing the correct boundary conditions

$$x_{bg}^i(\tau) = x_{\rm f}^i - \xi^i \tau \ , \qquad \text{with} \qquad \xi^i \equiv x_{\rm i}^i - x_{\rm f}^i \tag{3.1.10}$$

where $x_{\rm f} = z$ and $x_{\rm i} = y$ was the notation used in the previous chapter. Note that by free equations of motion we mean those arising from (3.1.8) by neglecting the potentials $V + V_{MR}$ (which are explicitly of order β^2) and A_i (which is explicitly of order β), and by keeping the constant leading term in the expansion of the metric $g_{ij}(x)$ around the final point $x_{\rm f}^i$ (thus making the space effectively flat). Of course one could have taken any other point to expand about. Also, one could use the exact solution of the classical equations of motion as the background trajectory, but this cannot change the result of the computation. It would just correspond to a different parametrization of the space of paths.

The quantum fields $q^i(\tau)$ in (3.1.9) should vanish at the time boundaries since the boundary conditions are already included in $x_{bg}^i(\tau)$. Therefore they can be expanded in a sine series. For the ghosts we use the same Fourier expansion. This cannot be justified with the same rigor as for the fields x, but we can give the following arguments. First of all, in (3.1.3) there are no factors $(\det g_{ij})^{1/2}$ at the end points; they do not appear because we want to introduce them explicitly later in (3.1.4) in order that the transition amplitude be a biscalar. Since the factors $(\det g_{ij})^{1/2}$ were exponentiated with ghosts, we do not need ghosts at the end points, and the way to achieve this is to impose as boundary conditions that they vanish at the end points. Another argument is that with these boundary condition, the expansion into modes defines a well-defined functional space, at least as well defined as for the x. Of course, any choice of functional space is a priori equally acceptable, since the role of the ghost is to remove ambiguities in the $\dot{x}\dot{x}$ propagators: one might even prefer to use cosines instead of sines in the expansion of the ghosts, but one would get the same answers. To conclude: we expand all ghosts into a series with sines[2].

Hence

$$\phi^i(\tau) = \sum_{m=1}^{\infty} \phi_m^i \sin(\pi m \tau) \tag{3.1.11}$$

where ϕ^i stands for all the quantum fields q^i, a^i, b^i, c^i. The functional space of paths is now concretely defined by the space of all Fourier co-

[2] A suggestive way to interpret this is by considering a background/quantum split, where the background carries the boundary conditions implied by the classical equation of motion, while the quantum part is required to vanish at the time boundaries so not to modify the boundary conditions of the background. In our case the classical solutions of the ghost field equations are $a^i = b^i = c^i = 0$.

efficients $\phi^i_m = (q^i_m, a^i_m, b^i_m, c^i_m)$. Similarly, the path integral measure is properly defined in terms of integration over the Fourier coefficients ϕ^i_m as follows

$$\mathcal{D}x = \prod_{t_\mathrm{i}<t<t_\mathrm{f}} \sqrt{\det g_{ij}(x(t))}\, Dx = Dq \int DaDbDc \; \mathrm{e}^{-\frac{1}{\beta}S_{gh}}$$
$$DqDaDbDc = \lim_{M\to\infty} A \prod_{m=1}^{M}\prod_{i=1}^{n} m\, dq^i_m da^i_m db^i_m dc^i_m \;, \qquad (3.1.12)$$

where A is a constant, and the ghosts have been rescaled $(a,b,c) \to \frac{1}{\beta}(a,b,c)$ to normalize the ghost action as $S_{gh} = \int_{-1}^0 d\tau\; \frac{1}{2} g_{ij}(x)(a^i a^j + b^i c^j)$. Note that we have used $Dx \equiv \prod_\tau d^n x(\tau) = \prod_\tau d^n q(\tau) \equiv Dq$ which is formally justified by the translational invariance of these free measures. In any case the second line in (3.1.12) defines precisely what we mean by path integration. Note also that with this definition the path integral for a free particle in Cartesian coordinates reduces to

$$\int \mathcal{D}x \;\exp\Big(-\frac{1}{\beta}S_{free}\Big) = A\; \exp\Big(-\frac{1}{2\beta}\delta_{ij}\xi^i\xi^j\Big) \qquad (3.1.13)$$

where

$$S_{free} = \int_{-1}^0 d\tau\; \frac{1}{2}\delta_{ij}(\dot{x}^i\dot{x}^j + a^i a^j + b^i c^j)\;. \qquad (3.1.14)$$

(We used that the set $\sqrt{2}\sin(\pi m\tau)$ is orthonormal and that Grassmann integration yields $\int db\, dc\, b\, c = -1$). It is well-known that $A = (2\pi\beta)^{-\frac{n}{2}}$, however this value can also be deduced from the consistency requirement of satisfying the "renormalization conditions", as will be shown later on. Note that any other constant metric in (3.1.13) and (3.1.14), as for example the choice $g_{ij}(x_\mathrm{f})$ we are going to use, does not change the normalization of the measure in (3.1.12): the Jacobian for the change of variables of the commuting fields q^i, a^i is exactly canceled by the corresponding Jacobian for the anticommuting fields b^i, c^i (i.e., this linear change of variables has unit super-Jacobian).

The way to implement mode regularization is now quite clear and already suggested by (3.1.12): limiting the integration for each field up to a finite mode number M gives a natural regularization of the path integral. One computes all quantities of interest at finite M. This necessarily gives a finite and unambiguous result. Then one sends $M \to \infty$ to reach the continuum limit. This regularization is enough to resolve all ambiguities in the product of distributions, as we shall see.

We now start to describe in detail the perturbative expansion and give the formulas for the propagators in mode regularization. The perturbative expansion is generated by splitting the action into a quadratic part S_2, which defines the propagators, and an interacting part S_{int}, which gives the vertices[3]. We do this splitting by expanding the action about the final point $x^i_{\rm f}$. Recalling that

$$\begin{aligned} x^i(\tau) &= x^i_{\rm f} - \xi^i\tau + q^i(\tau)\ , \quad (\xi^i \equiv x^i_{\rm i} - x^i_{\rm f}) \\ \dot{x}^i(\tau) &= \dot{q}^i(\tau) - \xi^i \end{aligned} \tag{3.1.15}$$

we obtain

$$S = S_2 + S_{int} \tag{3.1.16}$$

where

$$S_2 = \int_{-1}^0 d\tau\, \frac{1}{2}\, g_{ij}(x_{\rm f})\,(\xi^i\xi^j + \dot{q}^i\dot{q}^j + a^ia^j + b^ic^j) \tag{3.1.17}$$

$$\begin{aligned} S_{int} &= \int_{-1}^0 d\tau \left[\frac{1}{2}\,(g_{ij}(x) - g_{ij}(x_{\rm f}))\,(\dot{x}^i\dot{x}^j + a^ia^j + b^ic^j) \right. \\ &\quad \left. + i\,\beta A_i(x)\dot{x}^i + \beta^2(V(x) + V_{MR}(x))\right] . \end{aligned} \tag{3.1.18}$$

Note that also a term linear in $\dot{q}^i$ appears in S_2, but due to the boundary conditions on q^i its integral vanishes, and thus has been dropped. Inserting the mode expansions (3.1.11) into S_2 one obtains

$$S_2 = \frac{1}{2} g_{ij}\,(x_{\rm f})\xi^i\xi^j + \frac{1}{4} g_{ij}(x_{\rm f}) \sum_{m=1}^{M} \Big(\pi^2 m^2 q^i_m q^j_m + a^i_m a^j_m + b^i_m c^j_m\Big) \tag{3.1.19}$$

The propagators are easily obtained by using this S_2 in the path integral, adding sources, and completing squares as usual. As an example, let's see in detail the derivation for the mode regulated propagator $\langle q^i(\tau)q^j(\sigma)\rangle$. Using the mode expansion (3.1.11) we obtain

$$\begin{aligned} \langle q^i(\tau)q^j(\sigma)\rangle &= \Big\langle \sum_{m=1}^{M} q^i_m \sin(\pi m\tau) \sum_{n=1}^{M} q^j_n \sin(\pi n\sigma)\Big\rangle \\ &= \sum_{m=1}^{M}\sum_{n=1}^{M} \langle q^i_m q^j_n\rangle \sin(\pi m\tau)\sin(\pi n\sigma)\ . \end{aligned} \tag{3.1.20}$$

Adding sources for the q^i_m modes, completing squares and shifting integration variables produces the correlator

$$\langle q^i_m q^j_n\rangle = \beta g^{ij}(x_{\rm f})\delta_{mn}\frac{2}{\pi^2 m^2} \tag{3.1.21}$$

[3] In the previous chapter the quadratic part was called the free part and denoted by $S^{(0)}$.

which is just the inverse of the quadratic form Q appearing in the exponent $(\exp[-\frac{1}{2}\,\phi\, Q\,\phi])$. Using (3.1.21) into (3.1.20) one gets

$$\langle q^i(\tau) q^j(\sigma)\rangle = \beta g^{ij}(x_{\rm f}) \sum_{m=1}^{M} \frac{2}{\pi^2 m^2} \sin(\pi m \tau) \sin(\pi m \sigma) \ . \tag{3.1.22}$$

To check the normalization, note that acting with the field operator $-\frac{1}{\beta} g_{ki}(x_{\rm f}) \frac{\partial^2}{\partial \tau^2}$ produces the Dirac delta function in the space of functions that vanish at the boundaries, $\delta(\tau,\sigma) = \sum 2\, \sin(\pi m\tau)\sin(\pi m\sigma)$. Similarly one obtains the ghost propagators. Thus we get the following list of propagators

$$\begin{aligned} \langle q^i(\tau) q^j(\sigma)\rangle &= -\beta\ g^{ij}(x_{\rm f})\ \Delta(\tau,\sigma) \\ \langle a^i(\tau) a^j(\sigma)\rangle &= \beta\ g^{ij}(x_{\rm f})\ \Delta_{gh}(\tau,\sigma) \\ \langle b^i(\tau) c^j(\sigma)\rangle &= -2\,\beta\ g^{ij}(x_{\rm f})\ \Delta_{gh}(\tau,\sigma) \end{aligned} \tag{3.1.23}$$

where Δ and Δ_{gh} are regulated by the mode cut-off

$$\Delta(\tau,\sigma) = \sum_{m=1}^{M} \left[-\frac{2}{\pi^2 m^2} \sin(\pi m\tau)\sin(\pi m\sigma)\right] \tag{3.1.24}$$

$$\Delta_{gh}(\tau,\sigma) = \sum_{m=1}^{M} 2\, \sin(\pi m\tau)\sin(\pi m\sigma) \ . \tag{3.1.25}$$

Note that at the regulated level (M big, but fixed) one has the relation $\Delta_{gh}(\tau,\sigma) = {}^{\bullet\bullet}\Delta(\tau,\sigma) = \Delta^{\bullet\bullet}(\tau,\sigma)$, where as usual left and right dots indicate derivatives with respect to left and right variables. These functions have the following limiting value for $M \to \infty$

$$\Delta(\tau,\sigma) \ \to\ \tau(\sigma+1)\theta(\tau-\sigma) + \sigma(\tau+1)\theta(\sigma-\tau) \tag{3.1.26}$$

$$\Delta_{gh}(\tau,\sigma) \ \to\ \delta(\tau-\sigma) \ . \tag{3.1.27}$$

Conversely, the Fourier transform of these relations yields back (3.1.24) and (3.1.25).

More generally, in loop computations one needs also the propagators for $\langle \dot q^i(\tau) q^j(\sigma)\rangle$, $\langle q^i(\tau) \dot q^j(\sigma)\rangle$ and $\langle \dot q^i(\tau) \dot q^j(\sigma)\rangle$, so that it is useful to explicitly record the corresponding formulas

$${}^{\bullet}\Delta(\tau,\sigma) = \sum_{m=1}^{M} \left[-\frac{2}{\pi m} \cos(\pi m\tau)\sin(\pi m\sigma)\right] \tag{3.1.28}$$

$$\Delta^{\bullet}(\tau,\sigma) = \sum_{m=1}^{M} \left[-\frac{2}{\pi m} \sin(\pi m\tau)\cos(\pi m\sigma)\right] \tag{3.1.29}$$

$${}^{\bullet}\Delta^{\bullet}(\tau,\sigma) = \sum_{m=1}^{M} \left[-2\, \cos(\pi m\tau)\cos(\pi m\sigma)\right] \tag{3.1.30}$$

whose limiting values for $M \to \infty$ can be computed as

$$
\begin{aligned}
{}^\bullet\!\Delta(\tau,\sigma) &\to \sigma + \theta(\tau-\sigma) && (3.1.31)\\
\Delta^{\!\bullet}(\tau,\sigma) &\to \tau + \theta(\sigma-\tau) && (3.1.32)\\
{}^\bullet\!\Delta^{\!\bullet}(\tau,\sigma) &\to 1 - \delta(\tau-\sigma)\ . && (3.1.33)
\end{aligned}
$$

In addition, at coinciding times $\sigma = \tau$ one has

$$
\begin{aligned}
\Delta(\tau,\tau) &\to \tau(\tau+1) && (3.1.34)\\
{}^\bullet\!\Delta(\tau,\tau) &\to \tau + \frac{1}{2} && (3.1.35)\\
\Delta^{\!\bullet}(\tau,\tau) &\to \tau + \frac{1}{2}\ . && (3.1.36)
\end{aligned}
$$

These limiting values, and in fact all formal expressions in the limit $M \to \infty$, are the same as in time-slicing. However at finite M these regularized propagators have different properties from the propagators which are regularized by time-slicing. Consider as an example the expression

$$
I = \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\ {}^\bullet\!\Delta(\tau,\sigma)\, {}^\bullet\!\Delta^{\!\bullet}(\tau,\sigma)\, \Delta^{\!\bullet}(\tau,\sigma)\ .
$$

With time-slicing the result is $I(TS) = -1/6$. However, with mode number regularization one obtains a different answer $I(MR) = -1/12$. To derive this result we use that at the regulated level boundary terms in partial integration are well-defined

$$
\int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\ {}^\bullet\!\Delta(\tau,\sigma)\, {}^\bullet\!\Delta^{\!\bullet}(\tau,\sigma)\, \Delta^{\!\bullet}(\tau,\sigma) = \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\ \frac{1}{2}\partial_\sigma({}^\bullet\!\Delta(\tau,\sigma))^2\, \Delta^{\!\bullet}(\tau,\sigma)\ .
$$

We can partially integrate with ∂_σ without encountering boundary terms because ${}^\bullet\!\Delta(\tau,\sigma)$ vanish at the boundary points $\sigma = 0$ and $\sigma = -1$. We obtain then $-\frac{1}{2}({}^\bullet\!\Delta(\tau,\sigma))^2\Delta^{\!\bullet\bullet}(\tau,\sigma)$ in the integrand. Next we may replace $\Delta^{\!\bullet\bullet}(\tau,\sigma)$ by ${}^{\bullet\bullet}\!\Delta(\tau,\sigma)$ because this relation is clearly satisfied at the regulated level. Finally we combine $-\frac{1}{2}({}^\bullet\!\Delta(\tau,\sigma))^2\,{}^{\bullet\bullet}\!\Delta(\tau,\sigma) = -\frac{1}{6}\partial_\tau({}^\bullet\!\Delta(\tau,\sigma))^3$. The integration over τ can be performed, and at this point one may take the continuum limit because the integrand is finite and well-behaved. This yields

$$
I(MR) = -\frac{1}{6}\int_{-1}^{0} d\sigma\ ({}^\bullet\!\Delta)^3\Big|_{\tau=-1}^{\tau=0} = -\frac{1}{6}\int_{-1}^{0} d\sigma\,((\sigma+1)^3 - \sigma^3) = -\frac{1}{12}\ .
$$

3.2 The two loop amplitude and the counterterm V_{MR}

We now compute the transition amplitude at the two-loop level, using mode regularization. We count the coordinate displacement $\xi^i = x_{\rm i}^i - x_{\rm f}^i$

as order $\sqrt{\beta}$. More precisely, we evaluate all graphs which contribute to order β; these are not only the two-loop graphs but also one-loop graphs with vertices of order β and tree graphs with vertices of order β^2. We take ξ^i of order $\sqrt{\beta}$ because at the end we will use the resulting transition amplitude to evolve wave functions, and a Gaussian integral over the displacements ξ^i will make them effectively of order $\sqrt{\beta}$. Taking this into account we can Taylor expand the interaction potentials in S_{int} given in (3.1.18) around the final point $x^i_{\rm f}$. We classify the vertices as

$$S_{int} = S_3 + S_4 + \dots \tag{3.2.1}$$

with

$$\begin{aligned} S_3 &= \int_{-1}^{0} d\tau \, \Big[\frac{1}{2} \partial_k g_{ij} (q^k - \xi^k \tau)(\xi^i \xi^j - 2\, \xi^i \dot{q}^j + \dot{q}^i \dot{q}^j + a^i a^j + b^i c^j) \Big] - i\beta A_i \xi^i \\ & \hspace{30em} (3.2.2) \\ S_4 &= \int_{-1}^{0} d\tau \, \Big[\frac{1}{4} \partial_k \partial_l g_{ij} (q^k q^l + \xi^k \xi^l \tau^2 - 2\, q^k \xi^l \tau) \\ &\qquad \times (\xi^i \xi^j - 2\, \xi^i \dot{q}^j + \dot{q}^i \dot{q}^j + a^i a^j + b^i c^j) \\ &\qquad + i\beta \partial_j A_i (q^j - \xi^j \tau)(\dot{q}^i - \xi^i) \Big] + \beta^2 (V + V_{MR}) \; . \qquad (3.2.3) \end{aligned}$$

In this expansion all geometrical quantities, like g_{ij} and $\partial_k g_{ij}$, as well as A_i, V, V_{MR}, and derivatives thereof, are constants since they are evaluated at the final point $x^i_{\rm f}$, but for notational simplicity we do not exhibit explicitly this dependence, as no confusion can arise. Each term $\frac{1}{\beta} S_n$ contributes effectively as $\beta^{\frac{n}{2}-1}$. For example, S_3 is of order $\beta^{\frac{3}{2}}$ because ξ is of order $\beta^{\frac{1}{2}}$ and each q is also of order $\beta^{\frac{1}{2}}$ because the q propagator is of order β. Similarly for the ghost fields and their propagator. Note also that a term originating from the expansion of the velocity $\dot{x}^i = \dot{q}^i - \xi^i$ in the A_i term of (3.2.2) integrates to zero and has been canceled. To obtain all corrections to the amplitude of a free particle to order β^2 we need at most the vertex S_4. (Two loops come from terms with β^{L-1} with $L = 2$).

Thus, the perturbative quantum expansion reads:

$$\begin{aligned} \langle x^k_{\rm f}, t_{\rm f} | x^k_{\rm i}, t_{\rm i} \rangle &= \int_{\rm BC} \mathcal{D}x \ \exp\Big[-\frac{1}{\beta} S \Big] = A \ {\rm e}^{-\frac{1}{2\beta} g_{ij} \xi^i \xi^j} \langle {\rm e}^{-\frac{1}{\beta} S_{int}} \rangle \\ &= A \ {\rm e}^{-\frac{1}{2\beta} g_{ij} \xi^i \xi^j} \Big(\Big\langle 1 - \frac{1}{\beta} S_3 - \frac{1}{\beta} S_4 + \frac{1}{2\beta^2} S_3^2 \Big\rangle + O(\beta^{\frac{3}{2}}) \Big) \\ &= A \ {\rm e}^{-\frac{1}{2\beta} g_{ij} \xi^i \xi^j} \exp\Big(- \frac{1}{\beta} \langle S_3 \rangle - \frac{1}{\beta} \langle S_4 \rangle + \frac{1}{2\beta^2} \langle S_3^2 \rangle_c + O(\beta^{\frac{3}{2}}) \Big) \end{aligned} \tag{3.2.4}$$

where the brackets $\langle\cdots\rangle$ denote the averaging with the free action S_2, and amount to using the propagators given in (3.1.23). In fact, we have extracted the coefficient A together with the exponential of the quadratic action S_2 evaluated on the background trajectory so that the normalization of the remaining path integral is such that $\langle 1\rangle = 1$. In the last line only connected graphs appear in the exponent; this is indicated by the subscript c where it is needed.

Using standard Wick contractions one gets

$$\begin{aligned}
-\frac{1}{\beta}\langle S_3\rangle &= -\frac{1}{\beta}\frac{1}{2}\partial_k g_{ij}\Big[\beta\xi^k g^{ij}\,\mathbf{I_1} + 2\beta\xi^i g^{jk}\mathbf{I_2} + \frac{1}{2}\xi^i\xi^j\xi^k\Big] + iA_i\xi^i \\
&= -\frac{1}{4\beta}\partial_k g_{ij}\xi^i\xi^j\xi^k + iA_i\xi^i\ . \qquad (3.2.5)
\end{aligned}$$

On the right hand side there are terms without quantum fields and terms due to the contraction of two quantum fields. The latter contributions are denoted by $\mathbf{I_1}$ and $\mathbf{I_2}$, and correspond to the Feynman diagrams in (3.2.8) and (3.2.9). For example $\mathbf{I_1}$ is due to $\int_{-1}^0 d\tau\,\tau\,\langle\dot q\dot q + aa + bc\rangle$. Similarly

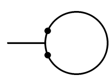
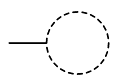
$$\begin{aligned}
-\frac{1}{\beta}\langle S_4\rangle &= -\frac{1}{\beta}\frac{1}{4}\partial_k\partial_l g_{ij}\Big[\beta^2(g^{ij}g^{kl}\,\mathbf{I_3} + 2g^{ik}g^{jl}\mathbf{I_4}) \\
&\quad -\beta(g^{ij}\xi^k\xi^l\,\mathbf{I_5} + g^{kl}\xi^i\xi^j\mathbf{I_6} + 4g^{jk}\xi^i\xi^l\,\mathbf{I_7}) + \frac{1}{3}\xi^i\xi^j\xi^k\xi^l\Big] \\
&\quad -i\partial_j A_i\Big(-\beta g^{ij}\,\mathbf{I_8} - \frac{1}{2}\xi^i\xi^j\Big) - \beta(V+V_{MR}) \\
&= \partial_k\partial_l g_{ij}\Big[\frac{\beta}{24}(g^{ij}g^{kl} - g^{ik}g^{jl}) + \frac{1}{24}(2g^{jk}\xi^i\xi^l - g^{ij}\xi^k\xi^l - g^{kl}\xi^i\xi^j) \\
&\quad -\frac{1}{12\beta}\xi^i\xi^j\xi^k\xi^l\Big] + \frac{i}{2}\partial_j A_i\xi^i\xi^j - \beta(V+V_{MR})\ . \qquad (3.2.6)
\end{aligned}$$

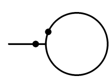
Now in the expression for the connected graphs with two S_3 vertices one does not need terms corresponding to ξ^6 because they could only come from squaring the classical contributions in S_3, and would correspond to disconnected graphs. Thus we find

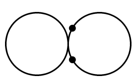
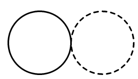
$$\begin{aligned}
\frac{1}{2\beta^2}\langle S_3^2\rangle_c &= \frac{1}{2\beta^2}\frac{1}{4}\partial_k g_{ij}\partial_l g_{mn}\Big[-\beta^3\Big(2g^{kl}g^{im}g^{jn}\,\mathbf{I_9} + 4g^{km}g^{il}g^{jn}\,\mathbf{I_{10}} \\
&\quad +g^{kl}g^{ij}g^{mn}\,\mathbf{I_{11}} + 4g^{ki}g^{jl}g^{mn}\,\mathbf{I_{12}} + 4g^{ki}g^{lm}g^{jn}\,\mathbf{I_{13}}\Big) \\
&\quad +\beta^2\Big(4\xi^i\xi^m(g^{kl}g^{jn}\,\mathbf{I_{14}} + g^{kn}g^{jl}\,\mathbf{I_{15}}) + 2\xi^k\xi^l g^{im}g^{jn}\,\mathbf{I_{16}} \\
&\quad +8\xi^k\xi^m g^{il}g^{jn}\,\mathbf{I_{17}} + 2\xi^i\xi^j(g^{kl}g^{mn}\,\mathbf{I_{18}} + 2g^{km}g^{ln}\,\mathbf{I_{19}}) \\
&\quad +4\xi^k\xi^i(g^{jl}g^{mn}\,\mathbf{I_{20}} + 2g^{jm}g^{ln}\,\mathbf{I_{21}})\Big) - \beta\Big(\xi^i\xi^j\xi^m\xi^n g^{kl}\,\mathbf{I_{22}}
\end{aligned}$$

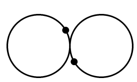
$$
\begin{aligned}
&\quad +4\xi^k\xi^i\xi^m\xi^n g^{jl}\,\mathbf{I_{23}} + 4\xi^k\xi^i\xi^l\xi^m g^{jn}\,\mathbf{I_{24}}\Big)\Big] \\
&= \partial_k g_{ij}\partial_l g_{mn}\Big[-\frac{\beta}{96}\Big(6g^{kl}g^{im}g^{jn} - 4g^{km}g^{il}g^{jn} - g^{kl}g^{ij}g^{mn} \\
&\quad +4g^{ki}g^{jl}g^{mn} - 4g^{ki}g^{lm}g^{jn}\Big) + \frac{1}{48}\Big(2\xi^i\xi^m(g^{kl}g^{jn} - g^{kn}g^{jl}) \\
&\quad +\xi^k\xi^l g^{im}g^{jn} - \xi^i\xi^j(g^{kl}g^{mn} - 2g^{km}g^{ln}) \\
&\quad +2\xi^k\xi^i(g^{jl}g^{mn} - 2g^{jm}g^{ln})\Big) \\
&\quad +\frac{1}{96\beta}\Big(\xi^i\xi^j\xi^m\xi^n g^{kl} - 4\xi^k\xi^i\xi^m\xi^n g^{jl} + 4\xi^k\xi^i\xi^l\xi^m g^{jn}\Big)\Big]\,.
\end{aligned}
\tag{3.2.7}
$$

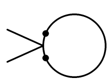
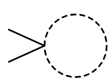
These results inserted into (3.2.4) give the transition amplitude at the two-loop approximation. The integrals needed for computing the various Feynman diagrams are evaluated using mode regularization, namely first they are computed at finite M (and so without ambiguities) and then the $M \to \infty$ limit is taken. We first list them here, and then explain in the next section how the computations in mode regularization are most easily performed.

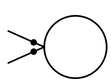
$\mathbf{I_1} =$  $+$  $\displaystyle = \int_{-1}^{0} d\tau\,\tau\,({}^{\bullet}\!\Delta^{\bullet} + {}^{\bullet\bullet}\!\Delta)|_\tau = 0 \qquad (3.2.8)$

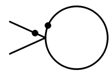
$\mathbf{I_2} =$  $\displaystyle = \int_{-1}^{0} d\tau\,{}^{\bullet}\!\Delta|_\tau = 0 \qquad (3.2.9)$

$\mathbf{I_3} =$  $+$  $\displaystyle = \int_{-1}^{0} d\tau\,\Delta|_\tau\,({}^{\bullet}\!\Delta^{\bullet} + {}^{\bullet\bullet}\!\Delta)|_\tau = -\frac{1}{6} \qquad (3.2.10)$

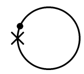
$\mathbf{I_4} =$  $\displaystyle = \int_{-1}^{0} d\tau\,{}^{\bullet}\!\Delta^2|_\tau = \frac{1}{12} \qquad (3.2.11)$

$\mathbf{I_5} =$  $+$  $\displaystyle = \int_{-1}^{0} d\tau\,\tau^2\,({}^{\bullet}\!\Delta^{\bullet} + {}^{\bullet\bullet}\!\Delta)|_\tau = -\frac{1}{6} \qquad (3.2.12)$

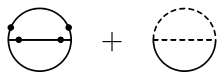
$\mathbf{I_6} =$  $\displaystyle = \int_{-1}^{0} d\tau\,\Delta|_\tau = -\frac{1}{6} \qquad (3.2.13)$

$\mathbf{I_7} =$ 

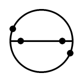
$$= \int_{-1}^{0} d\tau\, \tau\, {}^{\bullet}\!\Delta|_{\tau} = \frac{1}{12} \tag{3.2.14}$$

$\mathbf{I_8} =$ 

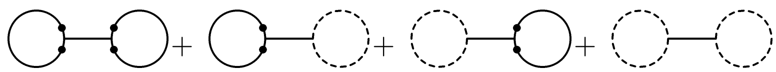
$$= \int_{-1}^{0} d\tau\, {}^{\bullet}\!\Delta|_{\tau} = 0 \tag{3.2.15}$$

$\mathbf{I_9} =$ 

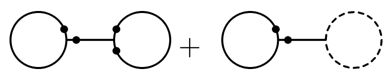
$$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\, \Delta\, ({}^{\bullet}\!\Delta^{\bullet 2} - {}^{\bullet\bullet}\!\Delta^{2}) = \frac{1}{4} \tag{3.2.16}$$

$\mathbf{I_{10}} =$ 

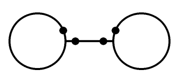
$$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\, {}^{\bullet}\!\Delta\, {}^{\bullet}\!\Delta^{\bullet}\, \Delta^{\bullet} = -\frac{1}{12} \tag{3.2.17}$$

$\mathbf{I_{11}} =$ 

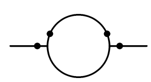
$$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\, ({}^{\bullet}\!\Delta^{\bullet} + {}^{\bullet\bullet}\!\Delta)|_{\tau}\, \Delta\, ({}^{\bullet}\!\Delta^{\bullet} + {}^{\bullet\bullet}\!\Delta)|_{\sigma} = -\frac{1}{12} \tag{3.2.18}$$

$\mathbf{I_{12}} =$ 

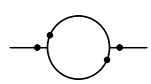
$$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\, {}^{\bullet}\!\Delta|_{\tau}\, {}^{\bullet}\!\Delta\, ({}^{\bullet}\!\Delta^{\bullet} + {}^{\bullet\bullet}\!\Delta)|_{\sigma} = \frac{1}{12} \tag{3.2.19}$$

$\mathbf{I_{13}} =$ 

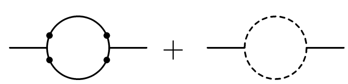
$$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\, {}^{\bullet}\!\Delta|_{\tau}\, {}^{\bullet}\!\Delta^{\bullet}\, \Delta^{\bullet}|_{\sigma} = -\frac{1}{12} \tag{3.2.20}$$

$\mathbf{I_{14}} =$ 

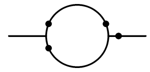
$$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\, {}^{\bullet}\!\Delta^{\bullet}\, \Delta = \frac{1}{12} \tag{3.2.21}$$

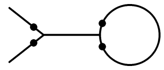
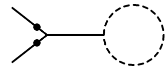
$\mathbf{I_{15}} =$ 

$$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\, {}^{\bullet}\!\Delta\, \Delta^{\bullet} = -\frac{1}{12} \tag{3.2.22}$$

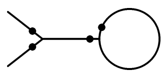
$\mathbf{I_{16}} =$ 

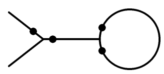
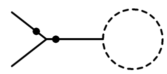
$$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\, \tau\, ({}^{\bullet}\!\Delta^{\bullet 2} - {}^{\bullet\bullet}\!\Delta^{2})\, \sigma = \frac{1}{12} \tag{3.2.23}$$

$$\mathbf{I_{17}} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\, \tau\, {}^{\bullet}\!\Delta^{\bullet}\, {}^{\bullet}\!\Delta = 0 \qquad (3.2.24)$$

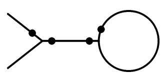
$$\mathbf{I_{18}} =$$  $$+$$ 

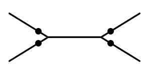
$$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\, \Delta\, ({}^{\bullet}\!\Delta^{\bullet} + {}^{\bullet\bullet}\!\Delta)|_{\sigma} = -\frac{1}{12} \qquad (3.2.25)$$

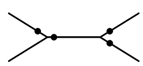
$$\mathbf{I_{19}} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\, \Delta^{\bullet}\, \Delta^{\bullet}|_{\sigma} = \frac{1}{12} \qquad (3.2.26)$$

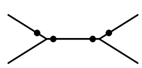
$$\mathbf{I_{20}} =$$  $$+$$ 

$$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\, \tau\, {}^{\bullet}\!\Delta\, ({}^{\bullet}\!\Delta^{\bullet} + {}^{\bullet\bullet}\!\Delta)|_{\sigma} = \frac{1}{12} \qquad (3.2.27)$$

$$\mathbf{I_{21}} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\, \tau\, {}^{\bullet}\!\Delta^{\bullet}\, \Delta^{\bullet}|_{\sigma} = -\frac{1}{12} \qquad (3.2.28)$$

$$\mathbf{I_{22}} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\, \Delta = -\frac{1}{12} \qquad (3.2.29)$$

$$\mathbf{I_{23}} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\, \tau\, {}^{\bullet}\!\Delta = \frac{1}{12} \qquad (3.2.30)$$

$$\mathbf{I_{24}} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\, \tau\, {}^{\bullet}\!\Delta^{\bullet}\, \sigma = -\frac{1}{12}\ . \qquad (3.2.31)$$

These are the tree, one- and two-loop graphs which contribute to the transition amplitude to order β or less. Dots denote derivatives, and the cross on $\mathbf{I_8}$ denotes $\partial_j A_i$. Dotted lines denote ghosts, solid internal lines denote q-propagators, and external lines denote factors of ξ. Note how ghost graphs combine with divergent graphs without ghosts to yield finite results. This aspect will be discussed at length in the next section.

Now we come to the task of imposing the "renormalization conditions" which fix the overall normalization of the path integrals as well as the counterterm V_{MR}. We **require** that the transition amplitude (3.2.4) should yield the correct time evolution of an arbitrary wave function $\Psi(x,t)$

$$\Psi(x_{\rm f},t_{\rm f}) = \int d^n x_{\rm i} \sqrt{g(x_{\rm i})} \langle x_{\rm f}^i, t_{\rm f} | x_{\rm i}^i, t_{\rm i}\rangle \Psi(x_{\rm i},t_{\rm i}) \tag{3.2.32}$$

and verify whether $\Psi(x_{\rm f},t_{\rm f})$ solves the Schrödinger equation with a given Hamiltonian. We consider the Hamiltonian in the coordinate representation with the covariant Laplacian $\nabla_A^2 = g^{ij}(\nabla_i + A_i)(\partial_j + A_j)$ and without any coupling to the scalar curvature

$$H = -\frac{1}{2}\nabla_A^2 + V \ . \tag{3.2.33}$$

This Hamiltonian can arise as a possible quantization of the classical model in (3.1.1) and thus is a consistent requirement. It is the x-space representation of the abstract operator in (2.1.1) with A_i and V terms added.

Since the transition amplitude is given in terms of an expansion around the final point $(x_{\rm f},t_{\rm f})$, we Taylor expand the wave function $\Psi(x_{\rm i},t_{\rm i})$ and the measure $\sqrt{g(x_{\rm i})}$ in eq. (3.2.32) about that point, perform the integration over $d^n x_{\rm i}$, and match the various terms. Thus we insert

$$\begin{aligned} \Psi(x_{\rm i},t_{\rm i}) &= \Psi(x_{\rm f},t_{\rm f}) - \beta\partial_t\Psi(x_{\rm f},t_{\rm f}) + \xi^i\partial_i\Psi(x_{\rm f},t_{\rm f}) \\ &\quad + \frac{1}{2}\xi^i\xi^j\partial_i\partial_j\Psi(x_{\rm f},t_{\rm f}) + O(\beta^{\frac{3}{2}}) \\ \sqrt{g(x_{\rm i})} &= \sqrt{g(x_{\rm f})}\Big(1 + \xi^i \Gamma_{ik}{}^k + \frac{1}{2}\xi^i\xi^j(\partial_i\Gamma_{jk}{}^k + \Gamma_{ik}{}^k\Gamma_{jl}{}^l) + O(\xi^3)\Big)\Big|_{x_{\rm f}} \end{aligned} \tag{3.2.34}$$

as well as (3.2.4) into (3.2.32). In the last expansion we have used that $\frac{1}{\sqrt{g}}\partial_i\sqrt{g} = \frac{1}{2}g^{mn}\partial_i g_{mn} = \Gamma_{ik}{}^k$. All quantities are now evaluated at the point $(x_{\rm f},t_{\rm f})$. For notational simplicity we do not indicate this dependence from now on, as no confusion can arise. The integrals over $d^n x_{\rm i} = d^n\xi$ give Gaussian averages since the transition amplitude (3.2.4) contains the exponential factor $\mathrm{e}^{-\frac{1}{2\beta}g_{ij}\xi^i\xi^j}$. These averages are easily carried out using "Wick contractions" with the basic "propagator" $\langle\xi^i\xi^j\rangle = \beta g^{ij}$. This also explains why we counted $\xi^i \sim \sqrt{\beta}$ in the expansion of the wave functions in (3.2.34).

From the various terms in the expansion of (3.2.32) we find the following. The leading term (order β^0) fixes A

$$\Psi = A(2\pi\beta)^{\frac{n}{2}}\Psi \qquad \to \qquad A = (2\pi\beta)^{-\frac{n}{2}} \ . \tag{3.2.35}$$

This yields the Feynman measure, as expected.

The terms of order β involve the counterterm V_{MR}. We fix it by requiring that (3.2.32) yields the prescribed Schrödinger equation for Ψ. At order β one finds

$$\beta\Big[-\partial_t\Psi + \frac{1}{2}\nabla_A^2\Psi - \Big(V + V_{MR} - \frac{1}{8}R + \frac{1}{24}g^{ij}g^{mn}g_{kl}\Gamma_{im}{}^{k}\Gamma_{jn}{}^{l}\Big)\Psi\Big] = 0\ . \tag{3.2.36}$$

For example, $V + V_{MR}$ comes from (3.2.6), and the A^2 term in $\frac{1}{2}\nabla_A^2\Psi$ comes from expanding $\exp\Big(-\frac{1}{\beta}\langle S_3\rangle\Big)$ in (3.2.4). Thus fixing

$$V_{MR} = \frac{1}{8}R - \frac{1}{24}g^{ij}g^{mn}g_{kl}\Gamma_{im}{}^{k}\Gamma_{jn}{}^{l} \tag{3.2.37}$$

gives the correct Schrödinger equation with the Hamiltonian in (3.2.33).

Higher order terms in β yield equations which must be automatically satisfied, since we have completely fixed all the "free" parameters entering mode regularization. This can be explicitly checked. A closely related check is that applying MR to evaluate trace anomalies in 4 dimensions (a three-loop calculation) produces the correct results.

We see here a difference with the TS method: in TS we first determined the counterterm from Weyl ordering, and then we did loop calculations. In MR we needed first to do loop calculations to order β to fix the counterterm, but then one can go ahead and do further loop calculations.

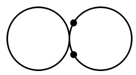
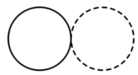
To summarize, we have described the mode regularization scheme for computing the path integral and have derived the corresponding counterterm V_{MR}. With precisely this counterterm the path integrals will produce a solution of the Schrödinger equation with Hamiltonian $H = -\frac{1}{2}\nabla_A^2 + V$. Any Hamiltonian for the Schrödinger equation can always be cast in the form (3.2.33) with suitable A_i and V. In particular, the mode regulated path integral with V_{MR} gives a general coordinate invariant results for the transition element. We stress that given an arbitrary but fixed Hamiltonian $\hat{H}$ we obtain always the same V_{MR} in the action for the path integral, but of course the action which corresponds to $\hat{H}$ will look different for different $\hat{H}$'s. The total action for the path integral is the sum of:
(i) the sigma model action in (3.1.1),
(ii) the counterterm V_{MR} (*(i)* + *(ii)* produce now the covariant Hamiltonian in (3.2.33)),
(iii) the extra terms present when the Hamiltonian is noncovariant (or contains an additional coupling to the scalar curvature). These are given by the extra terms in V and A_i by which the noncovariant Hamiltonian differs from the covariant Hamiltonian.

Thus, the mode regularization method can handle any Hamiltonian which is at most quadratic in the momenta.

3.3 Calculation of Feynman graphs in mode regularization

In this section we analyze in detail mode regularization, and explain how to efficiently evaluate Feynman diagrams.

First of all, all possible divergences are canceled by the ghost contributions. This is seen in diagrams like those in eqs. (3.2.8) or (3.2.10). Let's consider for example the case of I_3 in (3.2.10)

$I_3 =$  $+$  $$= \int_{-1}^{0} d\tau\, \Delta|_\tau \, ({}^{\bullet}\Delta^{\bullet} + {}^{\bullet\bullet}\Delta)|_\tau \qquad (3.3.1)$$

where we must insert for the Δ's on the right-hand side the discretized propagator as given in (3.1.24). At finite M each of the two diagrams produce a finite result since each one corresponds to a finite sum of finite integrals. However, only the sum of the two diagrams has a finite limit for $M \to \infty$. To compute this final value one can evaluate both terms at finite M, combine them, and then take the limit $M \to \infty$. This way of proceeding, though correct, is extremely laborious.

An easier way to proceed is to use partial integration to cast the integral into a form which can be computed directly and without ambiguities by taking the $M \to \infty$ limit inside the integral. Along the way one may use simple identities valid at the regulated level, like the following one

$$\begin{aligned}
({}^{\bullet}\Delta^{\bullet} + {}^{\bullet\bullet}\Delta)|_\tau &= \sum_{m=1}^{M} \Big[-2\,\cos^2(\pi m\tau) + 2\,\sin^2(\pi m\tau)\Big] \\
&= \partial_\tau \sum_{m=1}^{M} \Big[-\frac{2}{\pi m}\,\sin(\pi m\tau)\cos(\pi m\tau)\Big] \\
&= \partial_\tau({}^{\bullet}\Delta|_\tau)\ . \qquad (3.3.2)
\end{aligned}$$

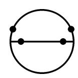
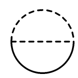
Thus we compute

$$\begin{aligned}
I_3 &= \int_{-1}^{0} d\tau\, \Delta|_\tau \, ({}^{\bullet}\Delta^{\bullet} + {}^{\bullet\bullet}\Delta)|_\tau = \int_{-1}^{0} d\tau\, \Delta|_\tau \, \partial_\tau({}^{\bullet}\Delta|_\tau) \\
&= -\int_{-1}^{0} d\tau\, \partial_\tau(\Delta|_\tau)\, {}^{\bullet}\Delta|_\tau\ . \qquad (3.3.3)
\end{aligned}$$

In the partial integration no boundary terms are picked up since both $\Delta|_\tau$ and ${}^{\bullet}\Delta|_\tau$ vanish at $\tau = -1, 0$. In fact, notice that at the regulated level ${}^{\bullet}\Delta|_\tau$ always vanishes at those boundaries, even though its limit for $M \to \infty$ is discontinuous at those points, see eq. (3.1.35) (continued along the whole line $-\infty < \tau < \infty$, the function ${}^{\bullet}\Delta|_\tau$ limits to the periodic triangular "sawtooth"). Finally, the last integral in (3.3.3) can be computed directly in the continuum limit, since only step functions and no delta functions arise in single derivatives acting on Δ. Thus for $M \to \infty$ one can use the limits (3.1.34)–(3.1.36) directly inside the integral to obtain

$$I_3 = -\int_{-1}^{0} d\tau \, [\partial_\tau(\tau^2+\tau)] \Big(\tau + \frac{1}{2}\Big) = -2\int_{-1}^{0} d\tau \, \Big(\tau + \frac{1}{2}\Big)^2 = -\frac{1}{6} \ . \quad (3.3.4)$$

Let us next discuss the computations of the diagrams in (3.2.16) and (3.2.17), whose values differ in all three different regularization schemes discussed in this book. First we look at

$I_9 =$  $+$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \, \Delta \, ({}^{\bullet}\Delta^{\bullet 2} - {}^{\bullet\bullet}\Delta^2) \ . \quad (3.3.5)$$

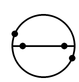
The minus sign is due to the closed ghost loop. Using partial integration we compute

$$\begin{aligned} I_9 \ &= \iint \Delta \, ({}^{\bullet}\Delta^{\bullet}\, {}^{\bullet}\Delta^{\bullet} - {}^{\bullet\bullet}\Delta \, {}^{\bullet\bullet}\Delta) \\ &= \iint (-{}^{\bullet}\Delta \, \Delta^{\bullet}\, {}^{\bullet}\Delta^{\bullet} - \Delta \, \Delta^{\bullet}\, {}^{\bullet\bullet}\Delta^{\bullet} + {}^{\bullet}\Delta \, {}^{\bullet}\Delta \, {}^{\bullet\bullet}\Delta + \Delta \, {}^{\bullet}\Delta \, {}^{\bullet\bullet\bullet}\Delta) \ . \quad (3.3.6) \end{aligned}$$

There are no boundary contributions because Δ vanishes at the boundaries. Now we notice that the second and forth term cancel because at the regulated level we can exchange two left derivatives with two right ones (i.e. ${}^{\bullet\bullet}\Delta = \Delta^{\bullet\bullet}$, see eq. (3.1.24)). Once again we see how the ghosts cancel a potential divergence. The first term in (3.3.6) equals $-I_{10}$ while the remaining third term gives

$$\iint {}^{\bullet}\Delta \, {}^{\bullet}\Delta \, {}^{\bullet\bullet}\Delta = \iint {}^{\bullet}\Delta \, {}^{\bullet}\Delta \, \Delta^{\bullet\bullet} = -2 \iint {}^{\bullet}\Delta^{\bullet}\, {}^{\bullet}\Delta \, \Delta^{\bullet} = -2I_{10} \ . \quad (3.3.7)$$

The boundary terms cancel because ${}^{\bullet}\Delta(\tau,\sigma)$ vanish at $\sigma = 0, -1$. Thus $I_9 = -3I_{10}$. So let us look at

$I_{10} =$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \, {}^{\bullet}\Delta \, {}^{\bullet}\Delta^{\bullet} \, \Delta^{\bullet} \ . \quad (3.3.8)$$

By using partial integration we obtain

$$\begin{aligned} I_{10} \ &= \iint {}^{\bullet}\Delta \, {}^{\bullet}\Delta^{\bullet} \, \Delta^{\bullet} = \frac{1}{2}\iint \Delta^{\bullet} \, ({}^{\bullet}\Delta^2)^{\bullet} = -\frac{1}{2}\iint \Delta^{\bullet\bullet} \, ({}^{\bullet}\Delta^2) \\ &= -\frac{1}{2}\iint {}^{\bullet\bullet}\Delta \, {}^{\bullet}\Delta^2 = -\frac{1}{6}\int_{-1}^{0}\int_{-1}^{0} d\tau \, d\sigma \, {}^{\bullet}({}^{\bullet}\Delta^3) = -\frac{1}{6}\int_{-1}^{0} d\sigma \, {}^{\bullet}\Delta^3 \Big|_{\tau=-1}^{\tau=0} \\ &= -\frac{1}{6}\int_{-1}^{0} d\sigma \, ((\sigma+1)^3 - \sigma^3) = -\frac{1}{12} \ . \quad (3.3.9) \end{aligned}$$

Again we first used ${}^{\bullet\bullet}\!\Delta = \Delta^{\bullet\bullet}$ and then used that ${}^\bullet\!\Delta(0,\sigma) = \sigma + 1$ and ${}^\bullet\!\Delta(-1,\sigma) = \sigma$, see (3.1.31). In this last step one should be careful in checking that the discretized functions really limit to the above values (up to sets of points of zero measure). Indeed one can verify that

$$\begin{aligned} {}^\bullet\!\Delta(0,\sigma) &= \sum_{m=1}^{M} \Big[-\frac{2}{\pi m} \sin(\pi m \sigma) \Big] \quad \rightarrow \quad \sigma + 1 && (3.3.10) \\ {}^\bullet\!\Delta(-1,\sigma) &= \sum_{m=1}^{M} \Big[-\frac{2}{\pi m} (-1)^m \sin(\pi m \sigma) \Big] \quad \rightarrow \quad \sigma \ . && (3.3.11) \end{aligned}$$

Thus we obtained $I_{10} = -1/12$ and $I_9 = 1/4$.

To summarize, in computing mode-regulated integrals it is convenient to use partial integration together with the following identities valid at finite M

$$\begin{aligned} &{}^{\bullet\bullet}\!\Delta(\tau,\sigma) = \Delta^{\bullet\bullet}(\tau,\sigma) && (3.3.12) \\ &{}^\bullet\!\Delta^\bullet(\tau,\tau) + {}^{\bullet\bullet}\!\Delta(\tau,\tau) = \partial_\tau({}^\bullet\!\Delta(\tau,\tau)) && (3.3.13) \\ &\partial_\tau(\Delta(\tau,\tau)) = 2\Delta^\bullet(\tau,\tau) && (3.3.14) \\ &\Delta^\bullet(\tau,\tau) = 0 \ \text{ at } \tau = -1, 0 && (3.3.15) \end{aligned}$$

and the following limits for $M \to \infty$

$$\begin{aligned} \Delta(\tau,\sigma) &\rightarrow \tau(\sigma+1)\theta(\tau-\sigma) + \sigma(\tau+1)\theta(\sigma-\tau) && (3.3.16) \\ {}^\bullet\!\Delta(\tau,\sigma) &\rightarrow \sigma + \theta(\tau-\sigma) && (3.3.17) \\ \Delta^\bullet(\tau,\sigma) &\rightarrow \tau + \theta(\sigma-\tau) && (3.3.18) \\ \Delta(\tau,\tau) &\rightarrow \tau^2 + \tau && (3.3.19) \\ \Delta^\bullet(\tau,\tau) = {}^\bullet\!\Delta(\tau,\tau) &\rightarrow \tau + \frac{1}{2} \ . && (3.3.20) \end{aligned}$$

4
Dimensional regularization

In this chapter we discuss path integrals defined by dimensional regularization (DR). In contrast to the previous TS and MR schemes, this type of regularization seems to have no meaning outside perturbation theory. However it leads to the simplest set up for perturbative calculations. In fact the associated counterterm V_{DR} turns out to be covariant, and the additional vertices obtained by expanding V_{DR}, needed at higher loops, can be obtained with relative easiness (using for example Riemann normal coordinates).

Dimensional regularization is based on the analytic continuation in the number of dimensions of the momentum integrals corresponding to Feynman graphs ($1 \to D+1$ with arbitrary complex D, in our case). At complex D we assume that the regularization of UV divergences is achieved by the analytic continuation as usual. The limit $D \to 0$ is taken at the end. Again one does not expect divergences to arise in quantum mechanics when the regulator is removed ($D \to 0$), and thus no infinite counterterms are necessary to renormalize the theory: potential divergences are canceled by the ghosts.

To derive the dimensional regularization scheme, one can employ a set up quite similar to the one described in the previous chapter for mode regularization. The only difference will be the prescriptions to regulate ambiguous diagrams.

One novelty of the dimensional regularization described in this chapter is that it addresses UV regularization on a compact space, namely on a one-dimensional segment corresponding to the finite time $\beta = t_{\mathrm{f}} - t_{\mathrm{i}}$. On such a space there cannot be infrared divergences, so that occasional mixing between IR and UV divergences (which sometimes occurs in infinite space) does not arise. On a compact space the momenta are discrete, and the Feynman graphs contain discrete sums $\sum_{k_n}[..]$ rather then continuous integrals $\int dk[..]$. The latter are easily extended to arbitrary D and computed, but the former are more difficult to treat. In general we have not been able to compute explicitly the combined sum and integrals in complex $D + 1$ dimensions, and test if poles arise only at some integer value of D. However, assuming that to be the case, we will show how one can compute the regulated graphs directly at $D \to 0$ and with relative easiness.

Dimensional regularization for bosonic nonlinear sigma models with finite propagation time and with the correct counterterm V_{DR} which we present in the following section was developed in [65]. It was extended to fermions and to supersymmetric nonlinear sigma models in [18]. In the infinite propagation time limit, dimensional regularization was previously employed in [5] and the corresponding covariant counterterm was identified in [66]. An extended use of DR for computing trace anomalies in 6 dimensions is described in [67, 68]. Moreover DR has been employed in [17, 18] to describe quantum field theories in a gravitational background within the worldline formalism. Additional discussions have been presented in [69].

For pedagogical purposes it is useful to first read the chapter on mode regularization, but the expert reader interested in learning directly the DR scheme can start here.

4.1 Dimensional regularization in configuration space

We start from the classical action in Euclidean time for the fields x^i with $i = 1, .., n$

$$S = \int_{t_\mathrm{i}}^{t_\mathrm{f}} dt \left[\frac{1}{2} g_{ij}(x) \frac{dx^i}{dt} \frac{dx^j}{dt} + i A_i(x) \frac{dx^i}{dt} + V(x) \right] \tag{4.1.1}$$

and aim to quantize the theory by defining directly the transition amplitude as a path integral

$$\langle x_\mathrm{f}^k, t_\mathrm{f} | x_\mathrm{i}^k, t_\mathrm{i} \rangle = \int_{\mathrm{BC}} \mathcal{D}x \ \mathrm{e}^{-S} \tag{4.1.2}$$

$$\mathcal{D}x = \prod_{t_\mathrm{i} < t < t_\mathrm{f}} \sqrt{\det g_{ij}(x(t))} \, d^n x(t) \tag{4.1.3}$$

where BC indicates the boundary conditions at initial and final time, $x^k(t_\mathrm{i}) = x_\mathrm{i}^k$ and $x^k(t_\mathrm{f}) = x_\mathrm{f}^k$. Since this quantum theory is super-renormalizable, we first proceed formally and derive the Feynman graphs, then we introduce the dimensional regularization procedure to give a meaning to the ambiguous integrals and compute them, and finally we calculate the transition amplitude at two loops. Imposing the same "renormalization conditions" as used in the TS and MR schemes will determine the counterterm V_{DR} and the overall normalization of the path integral.

The measure $\mathcal{D}x$ in (4.1.3) is formally a scalar under general coordinate transformations, but the factor $\prod_t \sqrt{\det g_{ij}(x(t))}$ is field dependent and makes the measure unsuitable to generate the perturbative expansion. Thus it is useful to introduce ghost fields a^i, b^i, c^i (with a^i commuting and b^i, c^i anticommuting) to exponentiate this field dependent factor. No boundary conditions should be imposed on the path integral for the ghosts as they are auxiliary algebraic fields (by this we mean that no initial or final point value for these fields can be specified in the transition amplitude (4.1.2)).

It is also convenient to shift and rescale the time parameter t, so that the total propagation time β can be extracted from the action $S \to \frac{1}{\beta}S$. Defining $t = t_{\rm f} + \beta\tau$ with $\beta = t_{\rm f} - t_{\rm i}$, so that $-1 \leq \tau \leq 0$, and rescaling suitably the ghost fields one obtains the following complete action

$$\begin{aligned} S \;=\; \int_{-1}^{0} d\tau \Big[& \frac{1}{2} g_{ij}(x)(\dot{x}^i \dot{x}^j + a^i a^j + b^i c^j) \\ & + i\beta A_i(x)\dot{x}^i + \beta^2 \Big(V(x) + V_{DR}(x) \Big) \Big] \end{aligned} \qquad (4.1.4)$$

where $\dot{x}^i = \frac{dx^i}{d\tau}$. We have denoted by V_{DR} the counterterm which is needed for dimensional regularization. Note that since $\exp(-\frac{1}{\beta}S)$ is the weight factor for the sum over paths, the time β plays a role analogous to the Planck constant $\hbar$ (which is set to one in this chapter) and can be used to count the number of loops.

For an arbitrary metric $g_{ij}(x)$ one can calculate the path integral in a perturbative expansion in β and the coordinate displacements $\xi^i \equiv x_{\rm i}^i - x_{\rm f}^i$. We perform a background/quantum split and parametrize

$$x^i(\tau) = x^i_{bg}(\tau) + q^i(\tau) \qquad (4.1.5)$$

where $x^i_{bg}(\tau)$ is a background trajectory and $q^i(\tau)$ the quantum fluctuations. After choosing the coordinate system to be employed for carrying out the computations, the background trajectory is taken to satisfy the free equations of motion $g_{ij}(x_{\rm f})\partial_\tau^2 x^j(\tau) = 0$. It incorporates the correct boundary conditions

$$x^i_{bg}(\tau) = x^i_{\rm f} - \xi^i \tau \;, \qquad \text{with} \qquad \xi^i \equiv x^i_{\rm i} - x^i_{\rm f} \;. \qquad (4.1.6)$$

Note that by free equations of motion we mean the ones arising from (4.1.4) by neglecting the potentials $V + V_{DR}$ (which are explicitly of order β^2) and A_i (which is explicitly of order β), and by keeping the constant leading term in the expansion of the metric $g_{ij}(x)$ around the final point $x^i_{\rm f}$ (thus making the space effectively flat). Of course one could as well take any other point to linearize the metric. For the ghost fields one can as well perform a background/quantum split. However the background ghost fields vanish as a consequence of their algebraic equation of motion.

The quantum fields are all taken to vanish at the time boundaries since the boundary conditions are already included in the background configurations. Therefore they can be expanded in a Fourier sine series

$$\phi^i(\tau) = \sum_{m=1}^{\infty} \phi^i_m \sin(\pi m \tau) \tag{4.1.7}$$

where ϕ^i stands for all the quantum fields q^i, a^i, b^i, c^i. The functional space of paths is now defined as the space of all Fourier coefficients $\phi^i_m = (q^i_m, a^i_m, b^i_m, c^i_m)$. Similarly, the path integral measure is defined in terms of integration over the Fourier coefficients ϕ^i_m. Thus we obtain the following path integral

$$\langle x^k_{\rm f}, t_{\rm f} | x^k_{\rm i}, t_{\rm i} \rangle = \int_{\rm BC} DqDaDbDc \; {\rm e}^{-\frac{1}{\beta}S} \tag{4.1.8}$$

$$\begin{aligned} S = \int_{-1}^{0} d\tau \biggl[\frac{1}{2} g_{ij}(x)(\dot{x}^i \dot{x}^j + a^i a^j + b^i c^j) \\ + i\beta A_i(x)\dot{x}^i + \beta^2 (V(x) + V_{DR}(x)) \biggr] \end{aligned} \tag{4.1.9}$$

$$x^i(\tau) = x^i_{bg}(\tau) + q^i(\tau) \tag{4.1.10}$$

$$DqDaDbDc = A \prod_{m=1}^{\infty} \prod_{i=1}^{n} m \, dq^i_m da^i_m db^i_m dc^i_m, \tag{4.1.11}$$

where A is a constant which will fixed later on (we will find again the Feynman measure $A = (2\pi\beta)^{-\frac{n}{2}}$).

The perturbative expansion is generated by splitting the action into a quadratic part S_2 which defines the propagators, and an interacting part S_{int} which gives the vertices. If the theory is free and S_{int} vanishes, there would not be any real reason to introduce a regularization. However, when the theory is interacting with a nontrivial field dependent metric, one must regulate the ambiguous Feynman graphs. In dimensional regularization these graphs are extended to $D+1$ dimensions. To recognize how to uniquely do this extension in each Feynman graph, we introduce D extra infinite regulating dimensions $\mathbf{t} = (t^1, \ldots, t^D)$ and extend directly the action. After having obtained from this action the corresponding Feynman diagrams, and in principle computed them at arbitrary D, one takes the limit $D \to 0$. Introducing $t^\mu \equiv (\tau, \mathbf{t})$ with $\mu = 0, 1, \ldots, D$ and $d^{D+1}t = d\tau d^D\mathbf{t}$, the action in $D+1$ dimensions reads

$$S \;=\; \int_\Omega d^{D+1}t \biggl[\frac{1}{2}\, g_{ij} \left(\partial_\mu x^i \partial_\mu x^j + a^i a^j + b^i c^j \right)$$

$$+i\beta A_i \partial_0 x^i + \beta^2 (V + V_{DR})\Big] \qquad (4.1.12)$$

where $\Omega = I \times R^D$ is the region of integration containing the finite interval $I = [-1, 0]$. Note that the contraction of the indices μ in the term quadratic in derivatives tells us how momenta get contracted in higher dimensions. Note also that the coupling to the abelian gauge field A_i is not modified in higher dimensions ($\partial_0 x^i \equiv \partial_\tau x^i = \dot{x}^i$). In addition, in $D+1$ dimensions the background solution (4.1.6) is left unchanged, so that the split $S = S_2 + S_{int}$ is given by

$$S_2 = \frac{1}{2} g_{ij}\,(x_{\rm f})\xi^i \xi^j + \int_\Omega d^{D+1}t\, \frac{1}{2}\, g_{ij}(x_{\rm f})\,(\partial_\mu q^i \partial_\mu q^j + a^i a^j + b^i c^j) \qquad (4.1.13)$$

$$\begin{aligned} S_{int} = & \int_\Omega d^{D+1}t \Big[\frac{1}{2}\,(g_{ij}(x) - g_{ij}(x_{\rm f}))\,(\partial_\mu x^i \partial_\mu x^j + a^i a^j + b^i c^j) \\ & + i\beta A_i(x) \partial_0 x^i + \beta^2 (V(x) + V_{MR}(x)) \Big] \,. \end{aligned} \qquad (4.1.14)$$

A term linear in $\partial_0 q^i$ appearing in S_2 integrates to zero and thus has been dropped.

The regulated propagators are given by

$$\langle x^i(t) x^j(s) \rangle = -\beta\, g^{ij}(x_f)\, \Delta(t,s) \qquad (4.1.15)$$

$$\langle a^i(t) a^j(s) \rangle = \beta\, g^{ij}(x_f)\, \Delta_{gh}(t,s) \qquad (4.1.16)$$

$$\langle b^i(t) c^j(s) \rangle = -2\beta\, g^{ij}(x_f)\, \Delta_{gh}(t,s) \qquad (4.1.17)$$

where

$$\Delta(t,s) = \int \frac{d^D\mathbf{k}}{(2\pi)^D} \sum_{m=1}^{\infty} \frac{-2}{(\pi m)^2 + \mathbf{k}^2} \sin(\pi m \tau) \sin(\pi m \sigma)\, \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})} \qquad (4.1.18)$$

$$\begin{aligned} \Delta_{gh}(t,s) &= \int \frac{d^D\mathbf{k}}{(2\pi)^D} \sum_{m=1}^{\infty} 2 \sin(\pi m \tau) \sin(\pi m \sigma)\, \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})} \\ &= \delta(\tau,\sigma)\, \delta^D(\mathbf{t}-\mathbf{s}) = \delta^{D+1}(t,s) \,. \end{aligned} \qquad (4.1.19)$$

Here

$$\delta(\tau,\sigma) = \sum_{m=1}^{\infty} 2 \sin(\pi m \tau) \sin(\pi m \sigma) \qquad (4.1.20)$$

is the Dirac delta on the space of functions vanishing at $\tau, \sigma = -1, 0$. Note that the function $\Delta(t,s)$ satisfies the relation (Green's equation)

$$\partial_\mu^2 \Delta(t,s) = \Delta_{gh}(t,s) = \delta^{D+1}(s,t) \,. \qquad (4.1.21)$$

Formally, the $D \to 0$ limits of these propagators are the usual ones

$$
\begin{aligned}
\Delta(\tau,\sigma) &= \tau(\sigma+1)\theta(\tau-\sigma) + \sigma(\tau+1)\theta(\sigma-\tau) && (4.1.22)\\
\Delta_{gh}(\tau,\sigma) &= {}^{\bullet}\!\Delta^{\!\bullet}(\tau,\sigma) = \delta(\tau,\sigma) && (4.1.23)
\end{aligned}
$$

where dots on the left/right side denote derivatives with respect to the first/second variable, respectively. However, such limits can be used only after one has defined the integrands in an unambiguous form by making use of the manipulations allowed by the regularization scheme. It is difficult to compute the integrals in Feynman graphs for arbitrary D. However, this is not strictly necessary. We can use various manipulations which are identities at the regulated level (and thus can be safely performed) to cast the integrals in alternative forms. This way one tries to reach a form which can be unambiguously computed by removing the regulator $D \to 0$. This is the same strategy we used in MR to compute quickly the various regulated integrals.

In particular, in DR one can often use partial integration without the need of including boundary terms: this is always allowed in the extra D dimension because of momentum conservation, while it can be achieved along the finite time interval direction whenever there is an explicit function vanishing at the boundary $\tau = -1, 0$ (for example the propagator of the coordinates $\Delta(t,s)$). Along the way one may find terms of the form $\partial_\mu^2 \Delta(t,s)$ which according to eq. (4.1.21) give Dirac delta functions. The latter can be safely used only at the regulated level, i.e. in $D+1$ dimensions. By performing such partial integrations one tries to arrive at a form of the integrals which are unambiguous even in the limit $D \to 0$. At this point they can be safely and easily calculated in this limit.

We will give more details on how to compute integrals in DR in section 4.3. For the moment an explicit example will suffice to describe how the above rules are concretely used:

$$
\begin{aligned}
I_{10} &= \int_{-1}^{0}\! d\tau \int_{-1}^{0}\! d\sigma \;\; ({}^{\bullet}\!\Delta)\; (\Delta^{\!\bullet})\; ({}^{\bullet}\!\Delta^{\!\bullet}) \to \int d^{D+1}t \int d^{D+1}s \; ({}_{\mu}\Delta)\; (\Delta_{\nu})\; ({}_{\mu}\Delta_{\nu})\\
&= \int d^{D+1}t \int d^{D+1}s \; ({}_{\mu}\Delta)\; {}_{\mu}\!\left(\frac{1}{2}(\Delta_{\nu})^2\right)\\
&= -\frac{1}{2} \int d^{D+1}t \int d^{D+1}s \; ({}_{\mu\mu}\Delta)\; (\Delta_{\nu})^2\\
&= -\frac{1}{2} \int d^{D+1}t \int d^{D+1}s \; \delta^{D+1}(t,s)\; (\Delta_{\nu})^2 = -\frac{1}{2} \int d^{D+1}t \; (\Delta_{\nu})^2|_t\\
&\to -\frac{1}{2} \int_{-1}^{0}\! d\tau \; (\Delta^{\!\bullet})^2|_\tau = -\frac{1}{24}
\end{aligned}
\qquad (4.1.24)
$$

where the symbol $|_\tau$ means that one should set $\sigma = \tau$. We have extended the notation to ${}_{\mu}\Delta$ and Δ_{μ} to indicate derivatives with respect to the first or second variable. Thus $I_{10}(DR) = -\frac{1}{24}$. In MR this integral was equal to $I_{10}(MR) = -\frac{1}{12}$. The difference between both schemes occurred when we obtained $({}_{\mu\mu}\Delta)\ (\Delta_\nu)^2$. In both schemes we can still replace ${}_{\mu\mu}\Delta$ by $\Delta_{\mu\mu}$, but in DR we then set $\Delta_{\mu\mu}$ equal to the delta function (a distribution), while in MR $\Delta_{\mu\mu} = {}^{\bullet}\!\Delta^{\bullet}$ is still regulated and yields ${}^{\bullet}\!\Delta^{\bullet}({}^{\bullet}\!\Delta)^2 = \frac{1}{3}\partial_\tau({}^{\bullet}\!\Delta)^3$.

Thus, we see that the rules of computing in DR are quite similar to those used in MR, except for the different options allowed for partial integrations. In DR the rule for contracting which index with which index follows directly from the extended action in (4.1.12). Thus only certain partial integrations lead to the Green equation $\partial_\mu^2\Delta(t,s) = \delta^{D+1}(s,t)$ in $D+1$ dimensions. At the same time the complex number D is the regulator, so that the discrete mode sums in (4.1.7) and (4.1.11) are really summed up to infinity. Instead in MR one regulates by cutting off all mode sums at a large mode number M and then performs partial integrations: now all derivatives are of the same nature and different options of partial integrations arise. This explains the origin of the differences between these two regularizations.

4.2 Two loop transition amplitude and the counterterm V_{DR}

We now compute the transition amplitude in dimensional regularization at the two-loop level (to order β), treating the coordinate displacement ξ^i as being of order $\beta^{\frac{1}{2}}$. The perturbative expansions is precisely of the same form as given in section 3.2 to which we refer. The only difference is in the calculation of the integrals $I_1, .., I_{24}$, which must be evaluated with the rules of dimensional regularization just described.

Proceeding to this task, one notices that all these integrals computed in DR acquire the same value as in MR, except I_9 and I_{10}. We already described in (4.1.24) how to compute $I_{10}(DR) = -\frac{1}{24}$. As for I_9 we obtain $I_9(DR) = -3I_{10}(DR) = \frac{1}{8}$. We will discuss these integrals in the next section.

Thus, it is straightforward to compute the difference between the DR and MR transition amplitude without counterterms

$$\begin{aligned}
&\Delta\langle x_{\rm f}^k, t_{\rm f}|x_{\rm i}^k, t_{\rm i}\rangle \equiv \langle x_{\rm f}^k, t_{\rm f}|x_{\rm i}^k, t_{\rm i}\rangle(DR) - \langle x_{\rm f}^k, t_{\rm f}|x_{\rm i}^k, t_{\rm i}\rangle(MR) \\
&= A\ \mathrm{e}^{-\frac{1}{2\beta}g_{ij}\xi^i\xi^j}\,\frac{1}{2\beta^2}\,\frac{1}{4}\partial_k g_{ij}\partial_l g_{mn}(-\beta^3)\Big[2g^{kl}g^{im}g^{jn}\,[\mathbf{I_9}(DR) - \mathbf{I_9}(MR)] \\
&+4g^{km}g^{il}g^{jn}\,[\mathbf{I_{10}}(DR) - \mathbf{I_{10}}(MR)]\Big]\ . \qquad (4.2.1)
\end{aligned}$$

After integration over ξ^i one finds

$$\int d^n\xi\,\Delta\langle x_{\rm f}^k, t_{\rm f}|x_{\rm i}^k, t_{\rm i}\rangle =$$

$$
\begin{aligned}
&= \beta\Big(\frac{1}{32} g^{kl} g^{im} g^{jn} \partial_k g_{ij} \partial_l g_{mn} - \frac{1}{48} g^{km} g^{il} g^{jn} \partial_k g_{ij} \partial_l g_{mn}\Big) \\
&= \frac{\beta}{24} g^{ij} g^{mn} g_{kl} \Gamma_{im}{}^{k} \Gamma_{jn}{}^{l} \ . \qquad (4.2.2)
\end{aligned}
$$

Recalling eqs. (3.2.36) and (3.2.37), this implies that the complete counterterm necessary to satisfy the renormalization conditions in dimensional regularization is covariant and equals

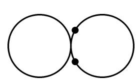
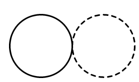
$$
V_{DR} = \frac{1}{8} R \qquad (4.2.3)
$$

while the value of the constant A is again fixed to be $A = (2\pi\beta)^{-\frac{n}{2}}$. This result shows that general covariance as well as gauge invariance are automatically preserved by dimensional regularization on the finite interval.

4.3 Calculation of Feynman graphs in dimensional regularization

In this section we analyze in some detail the principles to be followed in the application of dimensional regularization and explain through examples how to evaluate efficiently all Feynman diagrams.

First of all, all possible divergences are canceled by ghost contributions. This is seen in diagrams like those in eqs. (3.2.8) or (3.2.10). Let's consider for example the case of I_3 in (3.2.10) which is regulated in DR as follows

$I_3(DR) =$  $+$  $= \int d^{D+1}t \, \Delta|_t \, ({}_{\mu}\Delta_{\mu} + {}_{\mu\mu}\Delta)|_t \ .$ (4.3.1)

Note that we use here $\Delta_{gh} = {}_{\mu\mu}\Delta$ for the ghost propagator, i.e. the Green equation (4.1.21). Because there is a Δ with two derivatives, we shall use various allowed manipulations to arrive at an expression where all Δ carry at most one derivative, and then we can take the limit $D \to 0$ without encountering ambiguities or divergences. Recall that we denote by a subscript 0 the derivative along the original compact time direction. By inspecting formulas (4.1.18) and (4.1.19), one gets the following identity for $\Delta(t,s)$

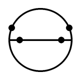
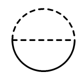
$$
({}_{\mu}\Delta_{\mu} + {}_{\mu\mu}\Delta)|_t = {}_{0}({}_{0}\Delta|_t) \ . \qquad (4.3.2)
$$

Inserting this identity into (4.3.1), one can partially integrate the ∂_0 without picking up boundary terms and obtains

$$
I_3(DR) \ = \ -\int d^{D+1}t \, \partial_0(\Delta|_t) \, {}_{0}\Delta|_t \quad \to \quad -\int_{-1}^{0} d\tau \, \partial_\tau(\Delta|_\tau) \, {}^{\bullet}\!\Delta|_\tau
$$

$$= -\int_{-1}^{0} d\tau\, \partial_\tau(\tau^2+\tau)\left(\tau+\frac{1}{2}\right) = -\frac{1}{6}\ . \tag{4.3.3}$$

Let's now discuss the integral

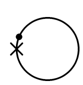
$I_9 =$  $+$  $= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\, \Delta\,({}^{\bullet}\Delta^{\bullet 2} - {}^{\bullet\bullet}\Delta^2)\ . \quad (4.3.4)$

In dimensional regularization

$$\begin{aligned}
I_9(\mathrm{DR}) &= \int_{-1}^{0}\!d\tau\int_{-1}^{0}\!d\sigma \ \ \Delta\ ({}^{\bullet}\Delta^{\bullet 2} - \Delta_{gh}^2)\ \to \\
\to &\quad \int d^{D+1}t\int d^{D+1}s\ \Delta({}_\mu\Delta_\nu\ {}_\mu\Delta_\nu - {}_{\mu\mu}\Delta\ {}_{\nu\nu}\Delta) = \\
&= \int\!\!\int\Big(-({}_\mu\Delta)\ (\Delta_\nu)\ ({}_\mu\Delta_\nu) - \Delta\ (\Delta_\nu)\ ({}_{\mu\mu}\Delta_\nu) \\
&\qquad +({}_\mu\Delta)\ ({}_\mu\Delta)\ ({}_{\nu\nu}\Delta) + \Delta\ ({}_\mu\Delta)\ ({}_{\mu\nu\nu}\Delta)\ \Big) \\
&= \int\!\!\int\Big(-({}_\mu\Delta)\ (\Delta_\nu)\ ({}_\mu\Delta_\nu) + ({}_\mu\Delta)\ ({}_\mu\Delta)\ ({}_{\nu\nu}\Delta)\Big) \\
&= -I_{10}(DR) + \int d^{D+1}t\int d^{D+1}s\ ({}_\mu\Delta)^2\delta^{D+1}(t,s) \\
&= -I_{10}(DR) + \int d^{D+1}t\ ({}_\mu\Delta)^2|_t \\
&= -3I_{10}(DR)\ .
\end{aligned} \tag{4.3.5}$$

We used the identity ${}_{\mu\mu}\Delta_\nu = \Delta_{\nu\mu\mu}$, obvious from (4.1.18), and recognized that the second and fourth term in the second line cancel. Finally we used the last-but-one line of (4.1.24) which tells that $I_{10}(DR) = -\frac{1}{2}\int d^{D+1}t\ ({}_\mu\Delta)^2|_t$. Thus, $I_9(DR) = -3I_{10}(DR)$ which is the same relation as in MR. The value of $I_{10}(DR)$ was already obtained in (4.1.24) and differs from $I_{10}(MR)$.

Finally, we study the integral I_8 which is related to gauge invariance

$\mathbf{I_8} =$  $= \int_{-1}^{0} d\tau\ {}^{\bullet}\Delta|_\tau\ . \quad (4.3.6)$

where the cross indicates the location of the vertex and denotes the factor $\partial_j A_i$ in (3.2.3) (there are no ξ^i factors, so no external lines according to our graphical notation). Computationally this diagram is rather simple. By power counting it is logarithmically divergent, and one could get any value for this diagram by using different prescriptions. Symmetric integration gives zero, but asymmetric integration schemes may easily produce a nonvanishing answer. All three schemes, DR, MR and TS, gives the same answer which preserves gauge invariance. In DR one can write it as

$$I_8(DR) = \int d^{D+1}t \; ({}_0\Delta)|_t \quad \to \quad \int_{-1}^{0} d\tau \, {}^{\bullet}\!\Delta|_\tau = 0 \tag{4.3.7}$$

which is directly computed in the $D = 0$ limit.

4.4 Path integrals for fermions

In this section we describe the dimensional regularization of fermionic path integrals. We shall discuss explicitly path integrals for Majorana fermions on a circle with periodic (PBC) or antiperiodic boundary conditions (ABC), as these are the only boundary conditions that will be directly needed in the applications to anomalies. First we consider the path integral with ABC and describe how to extend dimensional regularization to fermions. The requirement that a two-loop computation with DR reproduces known results (namely those obtained by time slicing) fixes once for all the two loop counterterm due to fermions. As we shall see this counterterm vanishes in DR. Since counterterms are due to ultraviolet effects, the infrared vacuum structure and the related boundary conditions on the fields should not matter in their evaluation. Therefore the same counterterm should apply to the fermionic path integral with PBC as well. No higher-loop contributions to the counterterm are expected as the model is super-renormalizable, just like the purely bosonic case. We end the section by presenting the essential formulas for the fermionic path integral with PBC.

Let us consider the $N = 1$ supersymmetric model written in terms of fermions with flat target-space indices

$$\begin{aligned} S \;=\; \int_{-1}^{0} d\tau \Big[& \frac{1}{2} g_{ij}(x)(\dot{x}^i \dot{x}^j + a^i a^j + b^i c^j) + \frac{1}{2}\psi_a(\dot{\psi}^a + \dot{x}^i \omega_i{}^a{}_b(x)\psi^b) \\ & + \beta^2 \Big(V(x) + V_{CT}(x) + V'_{CT}(x) \Big) \Big] \end{aligned} \tag{4.4.1}$$

where $V'_{CT}(x)$ denotes the additional counterterm which may arise from the fermions ψ^a in the chosen regularization scheme. This $N = 1$ model is described in detail in appendix D. It is classically supersymmetric if all the potential terms which are multiplied by β^2 are set to zero (note also that the ghosts are set to zero by their algebraic equations of motion and can be eliminated). Supersymmetry may be broken by boundary conditions, e.g. periodic for the bosons and antiperiodic for the fermions. To start with we assume antiperiodic boundary conditions (ABC) for the Majorana fermions $\psi^a(0) = -\psi^a(-1)$. Majorana fermions realize the

Dirac gamma matrices in a path integral context, and ABC compute the trace over the Dirac matrices[1].

Now we may explicitly compute by time slicing the transition amplitude for going from the background point x at time $t = 0$ back to the same point x at a later time $t = \beta$ using ABC for the Majorana fermions. In the two loop approximation this computation gives

$$Z \equiv \mathrm{tr}\,\langle x|e^{-\beta\hat{H}}|x\rangle = \frac{2^{\frac{n}{2}}}{(2\pi\beta)^{\frac{n}{2}}}\Big(1 - \frac{\beta}{24}R + O(\beta^2)\Big) \tag{4.4.2}$$

where the trace on the left-hand side is only over the Dirac matrices and where

$$\hat{H} = \hat{Q}^2 = -\frac{1}{2}\not{D}\not{D} = -\frac{1}{2}\Big(D^i D_i + \frac{1}{4}R\Big) \tag{4.4.3}$$

is the supersymmetric Hamiltonian of the $N = 1$ model. This Hamiltonian is the square of the supercharge $\hat{Q}$, realized by the Dirac operator

$$\hat{Q} = \frac{i}{\sqrt{2}}\not{D} = \frac{i}{\sqrt{2}}\gamma^a e_a{}^i D_i\ , \quad D_i = \partial_i + \frac{1}{4}\omega_{iab}\gamma^a\gamma^b \tag{4.4.4}$$

with ω_{iab} the spin connection (see appendix A). Note that there is an explicit coupling to the scalar curvature arising in (4.4.3). Thus one needs to use $V = -\frac{1}{8}R$ in the action together with the time slicing counterterms $V_{TS} = \frac{1}{8}(R + g^{ij}\Gamma^k_{il}\Gamma^l_{jk})$ and $V'_{TS} = \frac{1}{16}g^{ij}\omega_i{}^{ab}\omega_{jab}$ (see eq. ...). For convenience we will later rederive this value of V'_{TS}.

Now we want to reproduce eq. (4.4.2) with a path integral over Majorana fermions in dimensional regularization. This will unambiguously fix the additional counterterm $V'_{DR}(x)$ due to the fermions. Note that in dimensional regularization the potential $V = -\frac{1}{8}R$ cancels exactly the counterterm $V_{DR} = \frac{1}{8}R$ due to the bosons.

We focus directly on the regularization of the Feynman graphs arising in perturbation theory. To recognize how to dimensionally continue the various Feynman graphs we extend the action to D dimensions as follows

$$\begin{aligned} S \;=\; & \int_\Omega d^{D+1}t\Big[\frac{1}{2}\,g_{ij}(\partial_\mu x^i\partial_\mu x^j + a^i a^j + b^i c^j) \\ & +\frac{1}{2}\bar{\psi}_a\gamma^\mu(\partial_\mu\psi^a + \partial_\mu x^i\omega_i{}^a{}_b\psi^b) + \beta^2 V'_{DR}\Big] \end{aligned} \tag{4.4.5}$$

[1] One may avoid the path integral over Majorana fermions by explicitly using a matrix valued action: one drops the kinetic term for fermions and replaces the potential term $\dot{x}^i\omega_{iab}\psi^a\psi^b$ by the matrix $\frac{1}{2}\dot{x}^i\omega_{iab}\gamma^a\gamma^b$. The path integral then requires an explicit time ordering prescription to evaluate the exponential of the matrix valued action and maintain gauge invariance ($T\mathrm{e}^{-\frac{1}{\beta}S}$) [24].

where $\Omega = I \times R^D$ is the region of integration containing the finite interval $I = [-1, 0]$ and γ^μ are the gamma matrices in $D+1$ dimensions ($\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$). As before $t^\mu = (\tau, \mathbf{t})$ with $\mu = 0, 1, \ldots, D$. Here we assume that we can continue to those Euclidean integer dimensions where Majorana fermions can be defined. The Majorana conjugate is defined by $\bar\psi_a = \psi_a^T C_\pm$ with a suitable charge conjugation matrix $C_\pm$ such that $\bar\psi^a \gamma^\mu \psi^b = -\bar\psi^b \gamma^\mu \psi^a$. This can be achieved for example in 2 dimensions[2]. This requirement guarantees that the coupling $\omega_{iab}\psi^a\psi^b = -\omega_{iab}\psi^b\psi^a$ in (4.4.1) is nonvanishing when extended to $D+1$. The actual details how to represent $C_\pm$ and the gamma matrices in $D+1$ dimensions are not important. These gamma matrices only serve as a book-keeping device to keep track how derivatives are going to be contracted in higher dimensions. Apart from the above requirements, no additional Dirac algebra for γ^μ in $D+1$ dimensions is needed.

The bosonic and ghost propagators are as in the previous sections. The fermionic fields with ABC on the worldline, $\psi^a(0) = -\psi^a(-1)$, can be expanded in half-integer modes

$$\psi^a(\tau) = \sum_{r \in Z + \frac{1}{2}} \psi_r^a \, \mathrm{e}^{2i\pi r\tau} \tag{4.4.6}$$

and have the following unregulated propagator

$$\begin{aligned} \langle \psi^a(\tau)\psi^a(\sigma)\rangle &= \beta\delta^{ab}\Delta_{AF}(\tau-\sigma) \\ \Delta_{AF}(\tau-\sigma) &= \sum_{r\in Z+\frac{1}{2}} \frac{1}{2\pi r i} \mathrm{e}^{2i\pi r(\tau-\sigma)} \ . \end{aligned} \tag{4.4.7}$$

Note that the Fourier sum defining Δ_{AF} is conditionally convergent for $\tau \neq \sigma$ and yields

$$\Delta_{AF}(\tau-\sigma) = \frac{1}{2}\epsilon(\tau-\sigma) \tag{4.4.8}$$

where $\epsilon(x) = \theta(x) - \theta(-x)$ is the sign function (with the value $\epsilon(0) = 0$ obtained by symmetrically summing the Fourier series). The function Δ_{AF} satisfies

$$\partial_\tau \Delta_{AF}(\tau-\sigma) = \delta_{AF}(\tau-\sigma) \tag{4.4.9}$$

where $\delta_{AF}(\tau-\sigma)$ is the Dirac's delta function on functions with antiperiodic boundary conditions

$$\delta_{AF}(\tau-\sigma) = \sum_{r\in Z+\frac{1}{2}} \mathrm{e}^{2i\pi r(\tau-\sigma)} \ . \tag{4.4.10}$$

[2] In Euclidean 2d one can choose $\gamma^1 = \sigma^3$, $\gamma^2 = \sigma^1$ and $C_+ = 1$. Recall that $C_\pm$ is defined by $C_\pm \gamma^\mu C_\pm^{-1} = \pm\gamma^{\mu\,T}$.

The dimensionally regulated propagator obtained by adding the extra coordinates reads

$$\langle \psi^a(t)\bar{\psi}^b(s)\rangle = \beta\, \delta^{ab}\Delta_{AF}(t,s) \tag{4.4.11}$$

where the function

$$\Delta_{AF}(t,s) = -i\int \frac{d^D\mathbf{k}}{(2\pi)^D}\sum_{r\in Z+\frac{1}{2}} \frac{2\pi r\gamma^0 + \vec{\gamma}\cdot\mathbf{k}}{(2\pi r)^2+\mathbf{k}^2}\, \mathrm{e}^{2i\pi r(\tau-\sigma)}\mathrm{e}^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})} \tag{4.4.12}$$

satisfies

$$\gamma^\mu \frac{\partial}{\partial t^\mu}\Delta_{AF}(t,s) = -\frac{\partial}{\partial s^\nu}\Delta_{AF}(t,s)\gamma^\nu = \delta_{AF}(\tau-\sigma)\delta^D(\mathbf{t}-\mathbf{s})\ . \tag{4.4.13}$$

These are the essential relations needed to extend DR to fermions. They keep track of which derivative is contracted to which vertex to produce the $D+1$ delta function. The delta function is only to be used in $D+1$ dimensions, as we assume that the regularization due to the extra dimensions is taking place[3]. By using partial integration one casts the various loop integrals in a form which can be computed by sending first $D \to 0$. Then one can use $\gamma^0 = 1$ and no extra factors arise from the Dirac algebra in $D+1$ dimensions.

We are now ready to perform the two-loop calculation in the $N=1$ nonlinear sigma model using DR. The bosonic vertices together with the ghosts, V and V_{DR} give the same contribution calculated in section 3.2. The overall normalization of the fermionic path integral gives the extra factor $2^{\frac{n}{2}}$ which equals the number of components of a Dirac fermion in n (even) dimensions. This already produces the full expected result in (4.4.2).

Thus the sum of the additional fermion graphs arising from the cubic vertex contained in $\Delta S = \int_{-1}^0 d\tau\, \frac{1}{2}\, \dot{x}^i\omega_{iab}\psi^a\psi^b$ and the extra counterterm V'_{DR} must vanish at two loops. The cubic vertex arise by evaluating the spin connection at the background point x and reads $\Delta S_3 = \frac{1}{2}\omega_{iab}\int_{-1}^0 d\tau\, \dot{q}^i\psi^a\psi^b$. Using Wick contractions we identify the following contribution to $\langle e^{-\frac{1}{\beta}S^{int}}\rangle$

$$\frac{1}{2\beta^2}\langle(\Delta S_3)^2\rangle = \frac{1}{2\beta^2}(-2)\Big(\frac{1}{2}\omega_{iab}\Big)^2(-\beta^3)$$

[3] Again, we are not able to show this in full generality, and at this stage this rule is taken as an assumption which has turned out to be consistent in all the examples we have been dealing with. One way to prove it explicitly would be to compute all integrals arising in perturbation theory at arbitrary D and then check the location of the poles.

$$\times \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \, {}^\bullet\!\Delta^\bullet(\tau,\sigma)[\Delta_{AF}(\tau,\sigma)]^2 \ . \quad (4.4.14)$$

Using DR this graph is regulated by

$$\begin{aligned}
&\int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \, {}^\bullet\!\Delta^\bullet(\tau,\sigma)[\Delta_{AF}(\tau,\sigma)]^2 \rightarrow \\
&\qquad \rightarrow \quad -\int\!\!\int {}_\mu\Delta_\nu(t,s) \ \mathrm{tr}\,[\gamma^\mu \Delta_{AF}(t,s)\gamma^\nu \Delta_{AF}(s,t)] \quad (4.4.15)
\end{aligned}$$

(note the minus sign obtained in exchanging t and s in the last propagator; it is the usual minus sign arising for fermionic loops). We can partially integrate ∂_μ without picking boundary terms and obtain

$$\begin{aligned}
&2\int\!\!\int \Delta_\nu(t,s) \ \mathrm{tr}\,[(\gamma^\mu \partial_\mu \Delta_{AF}(t,s))\gamma^\nu \Delta_{AF}(s,t)] \\
&= 2\int\!\!\int \Delta_\nu(t,s) \ \mathrm{tr}\,[\delta^{D+1}(t,s)\gamma^\nu \Delta_{AF}(s,t)] \\
&= 2\int \Delta_\nu(t,t) \ \mathrm{tr}\,[\gamma^\nu \Delta_{AF}(t,t)] \\
&\rightarrow \quad 2\int_{-1}^{0} d\tau \, \Delta^\bullet(\tau,\tau)\Delta_{AF}(0)] = 0 \qquad (4.4.16)
\end{aligned}$$

because $\Delta_{AF}(0) = \frac{1}{2}\epsilon(0) = 0$ (and $\gamma^0 = 1$ at $D = 0$). As this example shows, the Dirac gamma matrices in $D+1$ are just a book-keeping device to keep track where one can use the Green equation (4.4.13).

Thus no contribution arises from the fermions at this order, and this implies that the extra counterterm must vanish

$$V'_{DR} = 0 \ . \qquad (4.4.17)$$

This is what one expects to preserve supersymmetry: the counterterm V_{DR} is exactly canceled by the tree level potential $V = -\frac{1}{8}R$ needed to have the correct coupling to the scalar curvature in the Hamiltonian (4.4.3) while no extra contribution to the counterterm arises from fermions. Thus dimensional regularization **without any counterterm** and **without extra order $\boldsymbol{\beta}^2$ tree level potential** preserves the supersymmetry of the classical $N = 1$ action

$$S = \int_{-1}^{0} d\tau \left[\frac{1}{2} g_{ij}(x)\dot{x}^i\dot{x}^j + \frac{1}{2}\psi_a(\dot{\psi}^a + \dot{x}^i \omega_i{}^a{}_b(x)\psi^b)\right] \ . \qquad (4.4.18)$$

In fact the amount of curvature coupling in the Hamiltonian H brought in by DR is of the exact amount to render it supersymmetric at the quantum level.

To compare with TS, we can compute the graph (4.4.14) using the TS rules. Now we must use that ${}^{\bullet}\!\Delta^{\bullet}(\tau,\sigma) = 1 - \delta(\tau,\sigma)$. The Dirac delta functions is ineffective as $\epsilon(0) = 0$, but the rest gives

$$
\begin{aligned}
\frac{1}{2\beta^2}\langle(\Delta S_3)^2\rangle(TS) &= \frac{1}{2\beta^2}(-2)\Big(\frac{1}{2}\omega_{iab}\Big)^2(-\beta^3)\int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\,\frac{1}{4} \\
&= \frac{\beta}{16}(\omega_{iab})^2
\end{aligned}
\qquad (4.4.19)
$$

This is canceled by using an extra counterterm $V'_{TS} = \frac{\beta^2}{16}(\omega_{iab})^2$ which at this order contributes with a term $-\frac{1}{\beta}V'_{TS}$ (evaluated at the background point x).

Let us conclude this section by considering briefly the case of Majorana fermions with PBC. Now the mode expansion of $\psi^a(\tau)$ requires integer modes

$$\psi^a(\tau) = \sum_{n\in Z}\psi^a_n\,\mathrm{e}^{2i\pi n\tau}\ . \qquad (4.4.20)$$

The zero modes ψ^a_0 of the kinetic operator (∂_τ) are treated separately, and the unregulated propagator in the sector of periodic functions orthogonal to the zero mode reads

$$\langle\psi^a(\tau)\psi^a(\sigma)\rangle = \beta\delta^{ab}\Delta_{PF}(\tau-\sigma) \qquad (4.4.21)$$

$$\Delta_{PF}(\tau-\sigma) = \sum_{n\neq 0}\frac{1}{2\pi n i}\mathrm{e}^{2i\pi n(\tau-\sigma)} \qquad (4.4.22)$$

where the function Δ_{PF} satisfies

$$\partial_\tau\Delta_{PF}(\tau-\sigma) = \delta_{PF}(\tau-\sigma) - 1 \qquad (4.4.23)$$

with $\delta_{PF}(\tau-\sigma)$ the Dirac's delta on periodic functions. Its continuum limit can be obtained by summing up the series and reads (for $(\tau-\sigma)\in[-1,1]$)

$$\Delta_{PF}(\tau-\sigma) = \frac{1}{2}\epsilon(\tau-\sigma) - (\tau-\sigma)\ . \qquad (4.4.24)$$

Curved indices

It is interesting to consider the case of fermions with curved target-space indices. This is equivalent to the case of fermions with flat target-space indices: it is just a change of integration variables in the path integral. However it is an useful exercise to work out since some formulas will become simpler. The classical $N = 1$ supersymmetric sigma model is written as

$$S = \int_{-1}^{0} d\tau \frac{1}{2} g_{ij}(x) \Big[\dot{x}^i \dot{x}^j + \psi^i (\dot{\psi}^j + \dot{x}^l \Gamma_{lk}{}^j(x) \psi^k) \Big] \ . \tag{4.4.25}$$

The fermionic term could also be written more compactly in terms of the covariant derivative $\frac{D}{d\tau}\psi^j = \dot{\psi}^j + \dot{x}^l \Gamma_{lk}{}^j(x)\psi^k$. Note that the action is written in terms of the metric and Christoffel connection and there is no need of introducing the vielbein and spin connection. Writing out the Christoffel connection in terms of the metric shows also that the coupling to the metric g_{ij} is linear (see appendix D, eq. (D.7)).

The bosonic part of the path integral has been already described and goes on unchanged. For the fermionic part we can now derive the correct path integral measure by taking into account the jacobian from the change of variable from the free measure with flat indices

$$\begin{aligned} D\psi^a &= D(e^a{}_i(x)\psi^i) = \mathrm{Det}^{-1}(e^a{}_i(x)) D\psi^i \\ &= \left(\prod_{-1 \le \tau < 0} \frac{1}{\sqrt{\det g_{ij}(x(\tau))}} \right) D\psi^i \ . \end{aligned} \tag{4.4.26}$$

Note the inverse determinant which is due to the Grassmann nature of the integration variables. The extra factor appearing in the measure can be exponentiated using bosonic ghosts $\alpha^i(\tau)$ with the same boundary condition of the fermions (ABC or PBC) and it leads to the extra term in the ghost action

$$S_{gh}^{extra} = \int_{-1}^{0} d\tau \, \frac{1}{2} g_{ij}(x) \alpha^i \alpha^j \ . \tag{4.4.27}$$

One can check that the counterterms of dimensional regularization are left unchanged. The full quantum action for the $N = 1$ supersymmetric sigma model reads

$$S = \int_{-1}^{0} d\tau \frac{1}{2} g_{ij}(x) \Big[\dot{x}^i \dot{x}^j + a^i a^j + b^i c^j + \psi^i (\dot{\psi}^j + \dot{x}^l \Gamma_{lk}{}^j(x)\psi^k) + \alpha^i \alpha^j \Big] \tag{4.4.28}$$

and appears in the path integral as

$$Z = \int DxDaDbDcD\psi D\alpha \; e^{-\frac{1}{\beta}S} \ . \tag{4.4.29}$$

Supersymmetry is not broken by boundary conditions if one uses periodic boundary conditions for both bosons and fermions. Then the effect of the ghosts cancels by themselves (they have the same boundary conditions) and can be eliminated altogether

$$\left(\prod_{-1 \le \tau < 0} \sqrt{\det g_{ij}(x(\tau))} \right) \left(\prod_{-1 \le \tau < 0} \frac{1}{\sqrt{\det g_{ij}(x(\tau))}} \right) = 1 \ . \tag{4.4.30}$$

One can now recognize that the potential divergence arising in the bosonic $\dot{x}\dot{x}$ contractions are canceled by the fermionic $\psi\dot{\psi}$ contractions. The remaining UV ambiguities are treated by dimensional regularization as usual. This scheme seems to be the best one to test, for example, that the Witten index (i.e. the gravitational contribution to the abelian chiral anomaly for a spin 1/2 field) does not get higher order corrections in worldline loops, and is thus β independent.

If one used ABC, the ghosts have different boundary conditions, their cancellation is not complete and they should be kept.

Part 2
Applications to Anomalies

5
Introduction to anomalies

We now start the second part of this book, namely the computation of anomalies in higher dimensional quantum field theories by using quantum mechanical (QM) path integrals. As we shall see, the ordinary Dirac action for a chiral fermion in n dimensions has anomalies which can be computed by using an $N = 1$ supersymmetric (susy) nonlinear sigma model in one (timelike) dimension. Although this relation between a nonsusy quantum field theory (QFT) and a susy QM system may seem surprising at first sight, it becomes plausible if one notices that the Dirac operator $\gamma^\mu D_\mu$ contains Dirac matrices γ^m (where $\gamma^\mu = \gamma^m e_m{}^\mu$ with $e_m{}^\mu$ the inverse vielbein field) satisfying the same Clifford algebra $\{\gamma^l, \gamma^m\} = 2\delta^{lm}$ with $l, m = 1, .., n$ flat indices as the equal-time anticommutation rules of a fermionic quantum mechanical point particle $\psi^a(t)$ with $a = 1, .., n$, namely $\{\psi^a(t), \psi^b(t)\} = \hbar\delta^{ab}$. [1] This suggests a representation of operators which appear in the QFT (γ^m) in terms of QM operators ($\psi^a(t)$), namely

$$\gamma^m \;\leftrightarrow\; \sqrt{\frac{2}{\hbar}}\psi^a(t) \;. \tag{5.0.1}$$

It is also natural to represent the coordinates x^μ in the QFT by a corresponding point particle $x^i(t)$ in QM. Hence one is led to suspect that the expression for the anomaly in terms of the operators $\frac{\partial}{\partial x^\mu}$, γ^m etc. of the quantum field theory can be rewritten as an expression in terms of the operators of a corresponding QM model with bosonic $x^i(t)$ and fermionic $\psi^a(t)$. These QM models are often supersymmetric. Of course, it had been known long before the 1980's that many calculations in field theory

[1] To distinguish objects in quantum field theory from objects in quantum mechanics, we use vectors indices μ (curved) and m (flat) in field theory, and vector indices i (curved) and a (flat) for the point particle in quantum mechanics. We are always in Euclidean space with metric $\delta_{mn} = (1, \ldots, 1)$ in tangent space, unless stated otherwise.

can be simplified by just using first quantization (with point particles) instead of second quantization [70, 71], and thus one might expect that also the calculation of anomalies may drastically simplify if one uses quantum mechanics. This is indeed the case.

The anomalies we shall compute are chiral anomalies for n-dimensional chiral fermions and selfdual antisymmetric tensor fields (AT) coupled to external gravitational and gauge fields, and trace anomalies for various fields coupled to gravity in 2 and 4 dimensions. These anomalies are anomalies in the local Lorentz, chiral and scale symmetry of the fields. (As we shall discuss, we only use regularization schemes that maintain Einstein (general coordinate) invariance so there are no separate Einstein anomalies). Before analyzing the formalism of quantum mechanics to calculate anomalies, it is useful to demonstrate that such anomalies really exist. We therefore start this chapter in section 5.1 with an explicit computation of anomalies in the simplest case: two dimensional field theories (one space and one time dimension). In this case we use a regularization scheme that is special for 2 dimensions: analytic regularization. To use this scheme we must use Minkowski space, so in this subsection we are in Minkowski space. First we calculate the chiral anomaly for a complex one-component fermion coupled to an external Maxwell or Yang-Mills field. Then we compute the gravitational anomaly for a real one-component chiral fermion coupled to an external gravitational field. Finally we compute the trace anomaly for a real nonchiral (two-component) fermion coupled to external gravity. These calculations will confirm the existence of chiral, gravitational and trace anomalies in 2 dimensions, and in the rest of the book we calculate similar anomalies in higher dimensions, using quantum mechanics.

In section 5.2 we discuss general aspects of the approach of calculating anomalies in field theories by using quantum mechanics. Field theories coupled to external gravitational fields will lead to quantum mechanical nonlinear sigma models, while field theories coupled to gauge fields will lead to linear sigma models. As we already observed, Dirac matrices correspond to fermionic point particle operators $\psi^a(t)$. To describe the anomalies in terms of quantum mechanical operators, we shall use Fujikawa's approach [6]. In this approach the anomaly for a QFT is given by the trace of the regulated Jacobian associated with a given symmetry, and this Jacobian is an expression which depends on $\frac{\partial}{\partial x^\mu}$, external fields $A_\mu^\alpha(x)$ and $e_\mu^m(x)$, Lie-algebra matrices T_α, and Dirac matrices γ^m. First we shall construct an explicit expression for the anomaly in terms of quantum mechanical operators, and then we switch to quantum mechanical path integrals. At that point we take over all results of the first part of the book on the construction of properties of path integrals for nonlinear sigma models.

In section 5.3 we give a brief history of anomalies.

5.1 The simplest case: anomalies in 2 dimensions

In this section we present an explicit calculation of chiral, gravitational and trace anomalies in a toy model: the theory of massless fermions coupled to external gauge and gravitational fields in 2 dimensions. In this model the full effective action can be computed explicitly and its response to gauge and gravitational transformations can be easily studied.

a. The chiral anomaly

We start discussing the classical Lagrangian of Dirac and Weyl fermions coupled to a gauge field. Then we proceed to analyze three typical cases. In *(i)* we present the calculation of the chiral anomaly due to a Weyl fermion. This is an example of a gauge anomaly, i.e. an anomaly in a current which is coupled to a gauge field. The corresponding effective action is not gauge invariant, and there is no local counterterm that can be added to the effective action to restore gauge invariance. The anomaly is thus a genuine anomaly. It satisfies certain consistency conditions, and thus it is called a "consistent anomaly". One particular consequence of the consistency conditions is that the anomaly cannot be gauge invariant. Nevertheless, it can be related to a "covariant anomaly", which is gauge invariant but cannot be interpreted as the gauge variation of an effective action. *(ii)* Then we add the contribution of another Weyl fermion, but with opposite chirality. The two Weyl fermions with opposite chiralities make up a Dirac fermion. The vector current of the Dirac fermion is coupled to a $U(1)$ gauge field A_μ. The action has a local vector symmetry $U_V(1)$ and a rigid chiral symmetry $U_A(1)$. [2] The total anomaly cancels in the $U_V(1)$ symmetry and the full effective action is $U_V(1)$ gauge invariant. *(iii)* However, the $U_A(1)$ symmetry is anomalous and we compute the corresponding anomaly. It is invariant under transformations of the gauge group $U_V(1)$ and it is an example of a rigid anomaly since the corresponding current is not coupled to gauge fields (although we could couple it to an external axial vector field, see previous footnote.) It is again a genuine anomaly and it satisfies again consistency conditions.

Let us start describing the classical Lagrangian of a massless Dirac field

[2] Actually, in 2 dimensions, the gauge field A_μ can be decomposed into light-cone components A_+ and A_- which couple to left-moving and right-moving massless fermions. Then this model has even a local $U_A(1)$ symmetry, but in our discussion we consider the $U_A(1)$ symmetry only as a rigid symmetry.

λ coupled to a $U_V(1)$ gauge field A_μ

$$\mathcal{L} = -\bar{\lambda}\,\gamma^\mu(\partial_\mu - iA_\mu)\lambda\ , \qquad \bar{\lambda} \equiv \lambda^\dagger i\gamma^0 \tag{5.1.1}$$

where the Dirac matrices satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, with $(\gamma^0)^2 = -1$. The classical symmetries are the $U_V(1)$ gauge transformations with infinitesimal local parameter $\alpha(x)$

$$\begin{aligned} \delta\lambda(x) &= i\alpha(x)\lambda(x) \\ \delta\bar{\lambda}(x) &= -i\alpha(x)\bar{\lambda}(x) \\ \delta A_\mu(x) &= \partial_\mu\alpha(x) \end{aligned} \tag{5.1.2}$$

and the axial $U_A(1)$ transformations with infinitesimal constant parameter β

$$\begin{aligned} \delta\lambda(x) &= i\beta\gamma_5\lambda(x) \\ \delta\bar{\lambda}(x) &= i\beta\bar{\lambda}(x)\gamma_5 \\ \delta A_\mu(x) &= 0\ . \end{aligned} \tag{5.1.3}$$

The chiral matrix γ_5 is chosen to satisfy $\gamma_5^2 = 1$ and $\{\gamma_5, \gamma^\mu\} = 0$. This model exists in any even spacetime dimension n, since then one can construct a matrix γ_5 with the required properties. In this section we restrict our attention to $n = 2$ by taking $\eta^{11} = -\eta^{00} = 1$ and $\gamma_5 \equiv \gamma^1\gamma^0$. We will also use the antisymmetric tensor $\epsilon_{\mu\nu}$, which we normalize to $\epsilon^{01} = -\epsilon_{01} = 1$. With this normalization one can verify that the following identity is satisfied by the gamma matrices: $\gamma^\mu\gamma_5 = \epsilon^{\mu\nu}\gamma_\nu$. We choose the following representation of the gamma matrices

$$\gamma^0 = -i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \gamma^1 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.1.4}$$

and represent the Dirac spinor as

$$\lambda = 2^{-\frac{1}{4}} \begin{pmatrix} \lambda_L \\ \lambda_R \end{pmatrix} \tag{5.1.5}$$

where the Weyl components λ_L and λ_R are eigenstates of γ_5 with eigenvalues $+1$ and -1, respectively. It is useful to eliminate completely the gamma matrices from the Lagrangian (5.1.1). Using light-cone coordinates $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$ one obtains

$$\mathcal{L} = i\lambda_L^\dagger(\partial_+ - iA_+)\lambda_L + i\lambda_R^\dagger(\partial_- - iA_-)\lambda_R\ . \tag{5.1.6}$$

It is evident that one can consider a model with a single Weyl fermion, for example the left moving fermion λ_L, by setting the other chirality to zero. In this case the two classical symmetries discussed above are not independent.

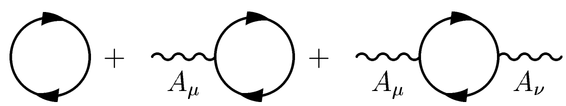
i) Now let us consider the path integral quantization of the left moving fermion λ_L

$$S[\lambda_L, \lambda_L^\dagger, A] = \int d^2x \; i\lambda_L^\dagger(\partial_+ - iA_+)\lambda_L$$
$$\int \mathcal{D}\lambda_L \mathcal{D}\lambda_L^\dagger \; e^{iS[\lambda_L,\lambda_L^\dagger,A]} = e^{\,iW_L[A]} \; . \tag{5.1.7}$$

The one-loop effective action $W_L[A]$ is a functional of the gauge field A_μ, and is formally given by the logarithm of a functional determinant

$$W_L[A] = -i \log \mathrm{Det}\,(i\partial_+ + A_+) \; . \tag{5.1.8}$$

For our purposes it is simpler to view the effective action perturbatively, namely as the sum of all one loop graphs with external gauge fields

$W_L[A] =$ 

$+ \cdots$ (5.1.9)

The first graph is a constant. It can be removed by a suitable normalization of the path integral. The second graph vanishes by symmetric integration. Thus let us take a closer look at the third graph. Expanding (5.1.7) to second order in A_μ we obtain

$$iW_L^{(2)}[A] = \frac{1}{2}\langle (iS_{int})^2 \rangle \tag{5.1.10}$$
$$= -\frac{1}{2}\int d^2x d^2y \; A_+(x)\Big\langle \lambda_L^\dagger(x)\lambda_L(x)\, \lambda_L^\dagger(y)\lambda_L(y)\Big\rangle A_+(y) \tag{5.1.11}$$

where we have split $S = S_0 + S_{int}$, with $S_0 = \int d^2x \; i\lambda_L^\dagger \partial_+ \lambda_L$ the free action which yields the propagator, and $S_{int} = \int d^2x \; A_+ \lambda_L^\dagger \lambda_L$ which describes the interaction with the gauge field A_μ.

The propagator is readily obtained

$$\langle \lambda_L(x)\lambda_L^\dagger(y)\rangle = \frac{1}{\partial_+}\delta^2(x-y) = \int \frac{d^2p}{(2\pi)^2}\, e^{ip\cdot(x-y)}\, \frac{2ip_-}{p^2 - i\epsilon} \tag{5.1.12}$$

where $p \cdot x = p_+ x^+ + p_- x^-$, $p^2 \equiv p_\mu p^\mu = -2p_+p_-$ with $p_\pm = \frac{1}{\sqrt{2}}(p_0 \pm p_1)$ and $-i\epsilon$ is the Feynman prescription that enforces the correct boundary conditions. It is easier to Fourier transform to momentum space by setting

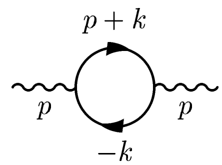
$A_+(x) = \int \frac{d^2p}{(2\pi)^2}\, e^{-ip\cdot x} A_+(p)$. We obtain [3]

$$W_L^{(2)}[A] = \frac{i}{2}\int \frac{d^2p}{(2\pi)^2} A_+(p)U(p)A_+(-p) \tag{5.1.13}$$

with

$$U(p) \;\equiv\; \int d^2x\, e^{-ip\cdot x}\, \langle \lambda_L^\dagger(x)\lambda_L(x)\, \lambda_L^\dagger(0)\lambda_L(0)\rangle$$

$$=$$



$$= \int \frac{d^2k}{(2\pi)^2}\, \frac{2(p_- + k_-)}{(p+k)^2 - i\epsilon}\, \frac{2k_-}{k^2 - i\epsilon}$$

$$= \int \frac{dk_- dk_+}{(2\pi)^2}\, \frac{1}{p_+ + k_+ + \frac{i\epsilon}{2(p_- + k_-)}}\, \frac{1}{k_+ + \frac{i\epsilon}{2k_-}}\ . \tag{5.1.14}$$

We now perform analytic regularization [72]. This scheme is suitable for a chiral theory. One can first perform the integral over k_+ by using a contour in the complex k_+-plane. To get a nonvanishing result, the two poles at

$$k_+ = -\frac{i\epsilon}{2k_-}\ , \quad k_+ = -p_+ - \frac{i\epsilon}{2(p_- + k_-)} \tag{5.1.15}$$

must be on opposite sides of the real k_+-axis, otherwise one could close the contour on the side without poles, obtaining a vanishing result. Let us first assume $p_- > 0$. Then the two poles are on opposite sides if $k_- < 0$ and $k_- + p_- > 0$, and

$$U(p) = \frac{1}{(2\pi)^2}\int_{-p_-}^{0} dk_-\ 2\pi i\, \frac{1}{p_+} = \frac{i}{2\pi}\frac{p_-}{p_+}\ . \tag{5.1.16}$$

Similarly, when $p_- < 0$, the two poles are on opposite sides if $k_- > 0$ and $k_- + p_- < 0$, and the same final result is obtained

$$U(p) = \frac{1}{(2\pi)^2}\int_{0}^{-p_-} dk_-\ (-2\pi i)\, \frac{1}{p_+} = \frac{i}{2\pi}\frac{p_-}{p_+}\ . \tag{5.1.17}$$

The effective action to this order is thus

$$W_L^{(2)}[A] = -\frac{1}{4\pi}\int \frac{d^2p}{(2\pi)^2} A_+(p)\frac{p_-}{p_+}A_+(-p)\ . \tag{5.1.18}$$

[3] The notation $A_+(x)$ for the function and $A_+(p)$ for its Fourier transform should not cause confusion, as we indicate the arguments explicitly.

One may check that for abelian gauge fields higher order contributions to the effective action vanish, so this result is exact. For nonabelian gauge fields one finds nonlocal terms with 3,4,5 ... external A_μ's, but only the term with two A_μ fields contributes to the anomaly [73].

Let us now analyze the gauge invariance. Under a gauge transformation

$$\delta A_\mu(x) = \partial_\mu \alpha(x) \quad \rightarrow \quad \delta A_\mu(p) = -ip_\mu \alpha(p) \tag{5.1.19}$$

the effective action is not gauge invariant

$$\begin{aligned} \delta W_L[A] &= \frac{i}{2\pi} \int \frac{d^2p}{(2\pi)^2}\, \alpha(p)\, p_- A_+(-p) \\ &= \frac{1}{2\pi} \int d^2x\, \alpha(x)\, \partial_- A_+(x) \; . \end{aligned} \tag{5.1.20}$$

Thus, it seems that the gauge symmetry has an anomaly. However, before deciding that this is a true anomaly, one must make sure that there does not exists a local counterterm whose variation cancels the anomaly. Since the anomaly is Lorentz invariant, we consider the most general Lorentz-invariant local counterterm with the correct dimension

$$\begin{aligned} W_{loc}[A] &= \beta \int d^2x\, A_\mu(x) A^\mu(x) = -2\beta \int d^2x\, A_-(x) A_+(x) \\ &= -2\beta \int \frac{d^2p}{(2\pi)^2}\, A_-(p) A_+(-p) \end{aligned} \tag{5.1.21}$$

where β is an arbitrary parameter. Its gauge variation is easily computed

$$\delta W_{loc}[A] = 2i\beta \int \frac{d^2p}{(2\pi)^2}\, \alpha(p) \Big(p_- A_+(-p) + p_+ A_-(-p) \Big) \; . \tag{5.1.22}$$

Clearly, no value of β can make the effective action $W_L[A] + W_{loc}[A]$ gauge invariant. Thus the final conclusion is that there is an anomaly in the gauge symmetry.

Let us pause for a moment and make various comments.

• To compute the effective action we have used analytic regularization which regulates the logarithmic divergent graph in (5.1.14) and automatically removes the divergence. This is one possible renormalization condition. For other renormalization conditions we may need to add the local counterterm in (5.1.21) with a particular value of β.

• One-loop effective actions which are computed with different regularization schemes can only differ by local counterterms. The addition of the most general local counterterm allows one to scan all possible regularizations at once. If the anomaly does not vanish for any possible counterterm, it means that the anomaly is not an artefact of the chosen regularization scheme. It is a genuine effect appearing in the quantum theory.

• One can write the gauge variation of the effective action also as follows

$$\delta W[A] = \int d^2x\, \delta A_\mu(x) \frac{\delta W[A]}{\delta A_\mu(x)} = -\int d^2x\, \alpha(x) \partial_\mu J^\mu(x) \qquad (5.1.23)$$

where $J^\mu(x) \equiv \frac{\delta W[A]}{\delta A_\mu(x)}$ is sometimes called the induced current. (It corresponds to the expectation value of the current coupled to the gauge field, $j^\mu = i\bar{\lambda}\gamma^\mu\lambda$, namely $J^\mu = \langle j^\mu \rangle$). The gauge anomaly then reads

$$\partial_\mu J^\mu(x) = -\frac{1}{2\pi} \partial_- A_+(x)\ . \qquad (5.1.24)$$

One may note that this expression is not gauge invariant.

• The consistency conditions are integrability conditions which follow from applying the commutator algebra of the symmetries to the effective action. The algebra of the gauge symmetry in eq. (5.1.2) is abelian and reads

$$[\delta(\alpha_1), \delta(\alpha_2)] = 0 \qquad (5.1.25)$$

while the anomaly can be denoted by

$$\delta(\alpha) W[A] \equiv \mathcal{A}[\alpha, A]\ . \qquad (5.1.26)$$

Combining these two equations gives the consistency condition for the anomaly of a chiral fermion

$$\delta(\alpha_1)\mathcal{A}[\alpha_2, A] = \delta(\alpha_2)\mathcal{A}[\alpha_1, A]\ . \qquad (5.1.27)$$

The anomaly is given by

$$\mathcal{A}[\alpha, A] = \frac{1}{2\pi} \int d^2x\, \alpha(x) \partial_- A_+(x) \qquad (5.1.28)$$

and clearly satisfies (5.1.27).

• A consequence of the consistency conditions is that the consistent gauge anomaly cannot be gauge invariant. This is immediately obvious by inspection of (5.1.24). More generally one can prove this property as follows. Let us introduce the shift transformation

$$\delta_s A_\mu = s_\mu \qquad (5.1.29)$$

as a trick to study the gauge current J^μ, since then $\delta_s W[A] = \int d^2x\, s_\mu J^\mu$ (recall eq. (5.1.23)). It is clear that

$$[\delta_s, \delta(\alpha)] A_\mu = 0 \qquad (5.1.30)$$

if one defines $\delta(\alpha)s_\mu = 0$. However, evaluating this commutator on the effective action produces

$$[\delta_s, \delta(\alpha)]W[A] = \delta_s\mathcal{A}[\alpha, A] - \int d^2x\,(\delta(\alpha)J^\mu)s_\mu = 0 \tag{5.1.31}$$

which shows that the current J^μ must transform nontrivially under a gauge transformation and thus cannot be gauge invariant (unless the anomaly $\mathcal{A}$ vanishes or is A_μ independent).

• One can introduce another anomaly, called the covariant anomaly, which is not obtained by varying the effective action, but which is gauge covariant (or rather gauge invariant in our case). It is by definition the divergence of a "covariant current" obtained by adding a suitable local (in general noncovariant) term $\tilde{J}^\mu$ to the consistent current J^μ. For $J_- = \frac{1}{2\pi}\frac{\partial_-}{\partial_+}A_+$ it is clear that $\tilde{J}_- = -\frac{1}{2\pi}A_-$ yields a gauge-invariant current

$$J_- + \tilde{J}_- = \frac{1}{2\pi}\frac{1}{\partial_+}(\partial_- A_+ - \partial_+ A_-) \tag{5.1.32}$$

whose anomaly is covariant

$$\partial_+(J_- + \tilde{J}_-) = \frac{1}{2\pi}(\partial_- A_+ - \partial_+ A_-)\ . \tag{5.1.33}$$

We stress that this "covariant anomaly" cannot be obtained as the gauge variation of the effective action, as it does not satisfy the consistency conditions.

ii) Let us now add the contribution of a right handed fermion to the previous model. The total action is the sum of the chiral actions and the path integral factorizes. So we only need to consider the extra terms coming from

$$\begin{aligned} S[\lambda_R, \lambda_R^\dagger, A] &= \int d^2x\ i\lambda_R^\dagger(\partial_- - iA_-)\lambda_R \\ \int \mathcal{D}\lambda_R\mathcal{D}\lambda_R^\dagger\ e^{iS[\lambda_R,\lambda_R^\dagger,A]} &= e^{\,iW_R[A]}\ . \end{aligned} \tag{5.1.34}$$

The calculation is quite similar to the one described above and produces the following contribution to the effective action

$$W_R[A] = -\frac{1}{4\pi}\int \frac{d^2p}{(2\pi)^2}A_-(p)\frac{p_+}{p_-}A_-(-p)\ . \tag{5.1.35}$$

The sum $W_L[A] + W_R[A]$ is still not gauge invariant, but adding the local counterterm W_{loc} in (5.1.21) with $\beta = -\frac{1}{4\pi}$ makes the final effective action gauge invariant

$$\begin{aligned} W[A] &= W_L[A] + W_R[A] + W_{loc}[A] \\ &= -\frac{1}{4\pi}\int \frac{d^2p}{(2\pi)^2}\Big(p_+A_-(p) - p_-A_+(p)\Big)\frac{1}{p_+p_-} \\ &\qquad \times\Big(p_+A_-(-p) - p_-A_+(-p)\Big) \\ &= \frac{1}{4\pi}\int d^2x F_{+-}\frac{1}{\partial_+\partial_-}F_{+-}\ . \end{aligned} \tag{5.1.36}$$

Equivalently, the induced gauge current is conserved: $\partial_\mu J^\mu(x) = 0$. This is an example of anomaly cancellation. One may notice that our simple model turned out to have just vectorial couplings. A manifestly gauge invariant regularization can be used in similar models, and it is enough to guarantee absence of anomalies. For example, in the case above a Pauli-Villars regularization maintains gauge invariance and would have produced automatically the correct local counterterm in the effective action. In models with chiral coupling, where a manifest gauge invariant regularization is lacking, cancellation of anomalies must be checked by hand, as in the Standard Model of particle physics.

iii) Finally, let us discuss the anomaly in a global current. The model above with a Dirac fermion enjoys at the classical level the axial symmetry given in eq. (5.1.3). It is quite simple to see that this symmetry is anomalous. The Noether current associated to this symmetry is $j_5^\mu = i\bar\lambda\gamma^\mu\gamma_5\lambda$. Thanks to the reducibility of the Lorentz group in 2 dimensions, we can actually use a trick to reinterpret A_μ also as the gauge field (or source) coupled to j_5^μ. In fact, we can substitute $A_\mu = B^\nu\epsilon_{\nu\mu}$ (so $A_+ = B_+$ but $A_- = -B_-$) and use $\epsilon_{\nu\mu}\gamma^\mu = \gamma_\nu\gamma_5$

$$iA_\mu\bar\lambda\gamma^\mu\lambda = iB^\nu\epsilon_{\nu\mu}\bar\lambda\gamma^\mu\lambda = iB_\mu\bar\lambda\gamma^\mu\gamma_5\lambda\ . \tag{5.1.37}$$

The transformation

$$\delta(\beta)B_\mu(x) = \partial_\mu\beta(x) \tag{5.1.38}$$

gauges the symmetry in (5.1.3) and can be used to test the conservation of the axial current $j_5^\mu = i\bar\lambda\gamma^\mu\gamma_5\lambda$. On one hand we can compute the variation of the effective action $W[A]$ as

$$\begin{aligned} \delta(\beta)W[A] &= \int d^2x\,\Big(\delta(\beta)B_\mu(x)\Big)\frac{\delta W[A(B)]}{\delta B_\mu(x)} \\ &= -\int d^2x\,\beta(x)\partial_\mu\langle j_5^\mu(x)\rangle\ . \end{aligned} \tag{5.1.39}$$

On the other hand the explicit form of $W[A]$ computed in (5.1.36) yields, using $\delta(\beta)A_+ = \partial_+\beta$ but $\delta(\beta)A_- = -\partial_-\beta$,

$$\delta(\beta)W[A] = -\frac{1}{\pi}\int d^2x\,\beta(x)F_{+-}(x)\ . \tag{5.1.40}$$

Thus

$$\partial_\mu \langle j_5^\mu \rangle = \frac{1}{\pi} F_{+-}(x) = -\frac{1}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu}(x) \ . \tag{5.1.41}$$

This is the anomaly in the global axial $U_A(1)$ current which is manifestly invariant under the $U_V(1)$ gauge group, as expected since the $U_V(1)$ gauge symmetry is not anomalous. Since the only local counterterm which depends on A_+ and A_- and is Lorentz invariant is $\delta\mathcal{L} = \beta A_+ A_-$, and since its coefficient was already fixed by requiring cancellation of the vector anomaly, one cannot remove the chiral (or better the axial vector) anomaly by a local counterterm: the axial anomaly is a genuine anomaly. The 4-dimensional analogue of this anomaly is the original ABJ anomaly which is gauge invariant and related to pion decay.

b. The gravitational anomaly

We follow [1] and construct the gravitational anomaly for a real chiral spin 1/2 field coupled to gravity in 1+1 dimensions.

The classical Lagrangian reads

$$\mathcal{L} = -\frac{1}{2} e \bar{\lambda}\, \gamma^m e_m{}^\mu \partial_\mu \lambda \ , \qquad \bar{\lambda} \equiv \lambda^T i \gamma^0 \tag{5.1.42}$$

and is invariant under general coordinate transformations given by

$$\begin{aligned} \delta\lambda &= \xi^\mu \partial_\mu \lambda \\ \delta e_\mu{}^m &= \xi^\nu \partial_\nu e_\mu{}^m + \partial_\mu \xi^\nu e_\nu{}^m \\ \delta e &= \partial_\mu(\xi^\mu e) \ . \end{aligned} \tag{5.1.43}$$

It is also invariant under local Lorentz transformations given by

$$\begin{aligned} \delta\lambda &= \frac{1}{4} \lambda^{mn} \gamma_{mn} \lambda \ , \qquad\qquad \gamma_{mn} \equiv \frac{1}{2}[\gamma_m, \gamma_n] \\ \delta e_\mu{}^m &= \lambda^m{}_n e_\mu{}^n \\ \delta e &= 0 \ . \end{aligned} \tag{5.1.44}$$

To prove the last statement, note that the Lorentz variation

$$\delta\mathcal{L} = -\frac{1}{2} e \bar{\lambda}\, \gamma^m e_m{}^\mu \frac{1}{4} (\partial_\mu \lambda^{pq}) \gamma_{pq} \lambda \tag{5.1.45}$$

vanishes since a Majorana spinor satisfies the identity $\bar{\lambda}\gamma_m\lambda = 0$ while $\gamma^m \gamma_{pq} = \delta_p^m \gamma_q - \delta_q^m \gamma_p$. In higher dimensions, or for a complex (=Dirac) spinor in 2 dimensions, one needs a term with the spin connection

$$\mathcal{L} = -e \bar{\lambda}\, \gamma^m e_m{}^\mu \left(\partial_\mu + \frac{1}{4} \omega_\mu{}^{pq} \gamma_{pq} \right) \lambda \ , \qquad \bar{\lambda} \equiv \lambda^\dagger i \gamma^0 \tag{5.1.46}$$

but in 2 dimensions this term vanishes for real λ, as we explained before.

Consider now a chiral fermion satisfying $(1-\gamma_5)\lambda = 0$. Since $\gamma_5 = \gamma^1\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, this field has only an upper component which we denote by λ_-, so $\lambda = \begin{pmatrix} \lambda_- \\ 0 \end{pmatrix}$. The action for $\lambda_- = \lambda_L$ reduces to

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{2} e\lambda^T\, i\gamma^0 (\gamma^m e_m{}^\mu)\partial_\mu \lambda \\
&= \frac{i}{2} e\lambda^T (e_0{}^\mu + \gamma_5 e_1{}^\mu)\partial_\mu\lambda \\
&= \frac{i}{2} e\lambda_L^T (e_0{}^\mu + e_1{}^\mu)\partial_\mu\lambda_L \\
&= \frac{i}{\sqrt{2}} e\lambda_- (e_+{}^\mu \partial_\mu)\lambda_- \ , \qquad\qquad e_\pm{}^\mu = \frac{1}{\sqrt{2}}(e_0{}^\mu \ \pm e_1{}^\mu) \ .
\end{aligned}
\tag{5.1.47}
$$

There follow now a few typical two-dimensional manipulations which allow us to write the action such that it only depends on one component of the gravitational field [74]. First we write $e_+{}^\mu\partial_\mu$ as $e_+{}^{\tilde{+}}\partial_+ + e_+{}^{\tilde{-}}\partial_-$ where $\partial_\pm = \frac{1}{\sqrt{2}}(\partial_0 \pm \partial_1)$ and hence $e_m{}^{\tilde{\pm}} = \frac{1}{\sqrt{2}}(e_m{}^0 \pm e_m{}^1)$. Then we extract the field $e_+{}^{\tilde{+}}$ and redefine λ_- such that it absorbs $e_+{}^{\tilde{+}}$

$$
\mathcal{L} \ = \ \frac{i}{\sqrt{2}}(\sqrt{e\, e_+{}^{\tilde{+}}}\lambda_-)(\partial_+ + h_{++}\partial_-)(\lambda_- \sqrt{e\, e_+{}^{\tilde{+}}}) \tag{5.1.48}
$$

where

$$
h_{++} = \frac{e_+{}^{\tilde{-}}}{e_+{}^{\tilde{+}}} \ . \tag{5.1.49}
$$

Clearly, h_{++} is Lorentz invariant, and also

$$
\tilde{\lambda}_- = \sqrt{e\, e_+{}^{\tilde{+}}}\lambda_- \tag{5.1.50}
$$

is Lorentz invariant because vielbeins rotate twice as fast as spinors under Lorentz rotations. [4]

One can find a simpler expressions for the spinor $\tilde{\lambda}_-$ by using the explicit form of the inverse vielbein

$$
e\, e_m{}^\mu = \begin{pmatrix} e_{\tilde{-}}{}^- & -e_{\tilde{+}}{}^- \\ -e_{\tilde{-}}{}^+ & e_{\tilde{+}}{}^+ \end{pmatrix} \ , \qquad e_\mu{}^m = \begin{pmatrix} e_{\tilde{+}}{}^+ & e_{\tilde{+}}{}^- \\ e_{\tilde{-}}{}^+ & e_{\tilde{-}}{}^- \end{pmatrix} \ . \tag{5.1.51}
$$

[4] The Lorentz invariance is manifest from the matching of $+$ and $-$ indices. Some people write $+\!\!+$ and $=$ for the indices of vector fields, and other $\sqrt{+}$ and $\sqrt{-}$ for the indices of spinor fields. We use only indices $+$ and $-$ but the reader need to remember that vectors and spinors transforms differently.

Then

$$\tilde{\lambda}_- = \sqrt{e_{\tilde{-}}{}^-}\lambda_- \ , \qquad\qquad h_{++} = -\frac{e_{\tilde{+}}{}^-}{e_{\tilde{-}}{}^-} \tag{5.1.52}$$

and the Lagrangian reads

$$\mathcal{L} \ = \ \frac{i}{\sqrt{2}}\tilde{\lambda}_-\partial_+\tilde{\lambda}_- + \frac{1}{\sqrt{2}}h_{++}T_{--} \ , \qquad T_{--} = i\tilde{\lambda}_-\partial_-\tilde{\lambda}_- \ . \tag{5.1.53}$$

The $\tilde{\lambda}_-$ are left moving fields which transforms as follows under general coordinate transformations

$$\begin{aligned} \delta\tilde{\lambda}_- \ &= \ \xi^\mu\partial_\mu\tilde{\lambda}_- + \frac{1}{2}\frac{1}{\sqrt{e_{\tilde{-}}{}^-}}(\partial_-\xi^\alpha e_\alpha{}^-)\lambda_- \\ &= \ \xi^\mu\partial_\mu\tilde{\lambda}_- + \frac{1}{2}(\partial_-\xi^- - \partial_-\xi^+ h_{++})\tilde{\lambda}_- \ . \end{aligned} \tag{5.1.54}$$

In particular, under transformations with ξ^-, $\tilde{\lambda}_-$ transforms as a "half-vector" (due to the factor $\frac{1}{2}$), and this forms the starting point for conformal field theory where $\tilde{\lambda}_-$ has conformal spin $\frac{1}{2}$.

The field h_{++} transforms as follows under general coordinate transformations

$$\begin{aligned} \delta h_{++} \ &= \ \xi^\alpha\partial_\alpha h_{++} - \frac{1}{e_{\tilde{-}}{}^-}(\partial_+\xi^\alpha e_\alpha{}^-) + \frac{e_{\tilde{+}}{}^-}{(e_{\tilde{-}}{}^-)^2}(\partial_-\xi^\alpha e_\alpha{}^-) \\ &= \ \xi^\alpha\partial_\alpha h_{++} + (\partial_+\xi^+)h_{++} - \partial_+\xi^- + (\partial_-\xi^+)h_{++}^2 - (\partial_-\xi^-)h_{++} \ . \end{aligned} \tag{5.1.55}$$

So, to lowest order in h_{++}, we have $\delta h_{++} = -\partial_+\xi^- = \partial_+\xi_+$ (since $\eta^{+-} = -1$).

A similar treatment of right-moving fermions satisfying $(1+\gamma_5)\lambda = 0$ shows that they couple only to

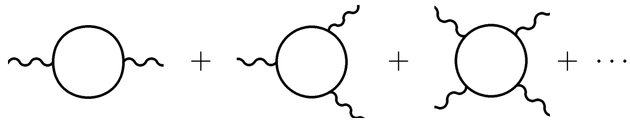
$$h_{--} = -\frac{e_{\tilde{-}}{}^+}{e_{\tilde{+}}{}^+} \ . \tag{5.1.56}$$

As the third field which parametrizes the space of symmetric vielbeins we take

$$\begin{aligned} h_{+-} \ &= \ h_{-+} = e_{\tilde{+}}{}^m e_{\tilde{-}}{}^n\eta_{mn} = -e_{\tilde{+}}{}^+ e_{\tilde{-}}{}^- - e_{\tilde{+}}{}^- e_{\tilde{-}}{}^+ \\ \delta h_{+-} \ &= \ \partial_+\xi_- + \partial_-\xi_+ + \dots \ . \end{aligned} \tag{5.1.57}$$

All 3 fields h_{++}, h_{--} and h_{+-} are Lorentz invariant.

Let us now compute the effective action for the theory with $\tilde{\lambda}_-$. To one-loop order it is given by the sums of graphs

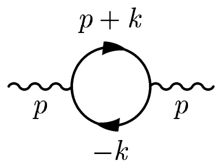


$$+ \cdots \qquad (5.1.58)$$

The anomaly resides only in the first graph, so we only evaluate this graph. Afterwards we will comment on the graphs with 3 and more h-fields. The $\tilde{\lambda}$ propagator is of course unchanged

$$\frac{1}{\partial_+}\delta^2(x-y) = \frac{\partial_-}{\partial_+\partial_-}\delta^2(x-y) = \int \frac{d^2k}{(2\pi)^2}\frac{2ik_-}{k^2 - i\epsilon}e^{ik(x-y)} \ . \qquad (5.1.59)$$

Hence



$$\sim \iint d^2x d^2y \, h_{++}(x)\langle T_{--}(x)T_{--}(y)\rangle h_{++}(y)$$

$$\sim \iint d^2xd^2y \, h_{++}(x)h_{++}(y)\langle\tilde{\lambda}_-(x)\partial_-\tilde{\lambda}_-(x)\, \lambda_-(y)\partial_-\tilde{\lambda}_-(y)\rangle$$

$$\sim \int d^2p \, h_{++}(p)h_{++}(-p)\int \frac{d^2k}{(2\pi)^2}(2k_- + p_-)^2\frac{k_- + p_-}{(k+p)^2 - i\epsilon}\frac{k_-}{k^2 - i\epsilon}$$

$$\sim \int d^2p \, h_{++}(p)h_{++}(-p)$$

$$\times \int \frac{dk_+dk_-}{(2\pi)^2}(2k_- + p_-)^2\frac{1}{k_+ + p_+ + \frac{i\epsilon}{2(k_- + p_-)}}\frac{1}{k_+ + \frac{i\epsilon}{2k_-}} \ . \qquad (5.1.60)$$

We use again contour integration for the integral over k_+, which is non-vanishing (for $p_+ > 0$) when $-p_- < k_- < 0$. We are then left with an integral of the form

$$\int_{-p_-}^{0} dk_- \frac{(2k_- + p_-)^2}{p_+} \sim \frac{p_-^3}{p_+} \ . \qquad (5.1.61)$$

Hence the effective action is proportional to

$$S_{eff} \sim \int d^2p \, h_{++}(p)h_{++}(-p)\frac{p_-^3}{p_+} \ . \qquad (5.1.62)$$

We are now in a position to check whether the effective action is still gauge (Einstein) invariant. Using the linearized transformation rules

$$\delta h_{++}(p) = \partial_+\xi_+ + \cdots = -ip_+\xi_+(p) + \cdots \qquad (5.1.63)$$

we find

$$\delta S_{eff} \sim \int d^2p\, \xi_+(p) p_-^3 h_{++}(-p) \ . \tag{5.1.64}$$

The result is rigidly Lorentz invariant (+ and − indices match; rigid Lorentz transformations act on curved indices).

So the effective action is not gauge invariant, but one should still check that its variation cannot be canceled by the variation of a suitable local counterterm in the action. The most general counterterm whose variation has the same number of fields as S_{eff} and is Lorentz invariant reads

$$\begin{aligned}\Delta S \ = \ \int d^2p\, [& A h_{++}(p) p_- p_- h_{+-}(-p) \\ & + B h_{++}(p) p_+ p_- h_{--}(-p) \\ & + C h_{+-}(p) p_+ p_- h_{+-}(-p) \\ & + D h_{+-}(p) p_+ p_+ h_{--}(-p)] \ . \end{aligned} \tag{5.1.65}$$

One may check that for no value of the constants A, B, C, D the variation of ΔS can cancel δS_{eff}. Hence there exists a genuine gravitational anomaly in 1+1 dimensions for chiral spinors.

One expects that for nonchiral spinors, there is no genuine gravitational anomaly. Consider the sum of the actions for λ_L and λ_R. The effective action action is now a sum of an effective action for $\tilde{\lambda}_-$ depending on h_{++}, and another effective action for $\tilde{\lambda}_+$ depending on h_{--}. The variation of this sum of effective actions is proportional to

$$\begin{aligned}\delta S_{eff} \ &= \ \delta(S^L_{eff} + S^R_{eff}) \\ &\sim \ \int d^2p\, [\xi_+(p) p_-^3 h_{++}(-p) + \xi_-(p) p_+^3 h_{--}(-p)] \end{aligned} \tag{5.1.66}$$

Now however there is a local counterterm whose variation cancels δS_{eff}, namely

$$\begin{aligned}\Delta S \ \sim \ \int d^2p\, [& -4 h_{++}(p) p_- p_- h_{+-}(-p) \\ & + 2 h_{++}(p) p_+ p_- h_{--}(-p) \\ & + 4 h_{+-}(p) p_+ p_- h_{+-}(-p) \\ & - 4 h_{+-}(p) p_+ p_+ h_{--}(-p)] \ . \end{aligned} \tag{5.1.67}$$

In fact, using appendix A, one may show that the effective action $S_{eff} + \Delta S$ is given by

$$S_{eff} + \Delta S \sim \int d^2p\, \frac{R(p)R(-p)}{p_+ p_-} \sim \int d^2x\, e R(x) \frac{1}{\Box} R(x) \tag{5.1.68}$$

where $R(p) \sim p_+^2 h_{--}(p) + p_-^2 h_{++}(p) - 2p_+p_-h_{+-}(p)$ is the linearized form of the scalar curvature.

c. The trace anomaly

Finally we demonstrate also the presence of a trace anomaly in 2 dimensions. We consider a real nonchiral fermion. The classical action is independent of h_{+-} so that the classical stress tensor is traceless: $T_{+-} = 0$. (We obtain T_{+-} by varying h_{+-} in the classical action). At the quantum level, S_{eff} is still independent of h_{-+}, so T_{+-} still vanishes at the one-loop level. However, if we make the effective action Einstein invariant by adding the local counterterm ΔS, the effective action starts depending on h_{-+}, and thus there is a trace anomaly. From the expression for ΔS, and $\delta R \sim 2p_+p_-\delta h_{+-} + \cdots$ we see that the trace anomaly is given by

$$T_\mu{}^\mu \sim R \qquad \text{in } 1+1 \text{ dimensions.} \tag{5.1.69}$$

We have seen in concrete two dimensional models the various aspects of anomalies: anomalies in gauge and gravitational currents, anomalies in rigid currents, and cancellation of anomalies. In higher dimensions the full effective action is not explicitly calculable in closed form, but one can still study the anomalous behavior of the various Feynman graphs. In the remaining part of the book we will use the general method based on (susy) quantum mechanics to compute the anomalies.

5.2 How to calculate anomalies using quantum mechanics

Anomalies arise when a classical action has a symmetry but the corresponding effective action is no longer invariant under this symmetry. The anomaly is then by definition the variation of the effective action under the symmetry. At the one-loop level the anomaly is a local polynomial in the fields and derivatives of the fields with finite coefficients. In the path integral for quantum field theories the anomaly appears if one makes an infinitesimal change of integration variables which amounts to a symmetry transformation. The action in the path integral is invariant under this change of variables, but if there is an anomaly the Jacobian $1 + \mathrm{Tr}\, J$ for an infinitesimal change of integration variables is not unity. This is a true anomaly only if it cannot be removed by adding local counterterms to the action without spoiling other symmetries. The infinitesimal part of the Jacobian, $\mathrm{Tr}\, J$, is the trace of an operator summed over all points in spacetime (i.e. the trace of an infinite dimensional matrix) which must be properly defined by regularization. Thus the expression for the anomaly in terms of the operators of the QFT is

$$An = \lim_{\beta \to 0} \mathrm{Tr}\, J \mathrm{e}^{-\beta \mathcal{R}} \tag{5.2.1}$$

where $\mathcal{R}$ is the regulator (also an operator) [6]. The trace is over a complete set of states, for which we can take for example the set of plane waves, or the set of eigenfunctions of the regulator $\mathcal{R}$. (In the latter case one would prefer to use a positive-definite self-adjoint regulator $\mathcal{R}$ because then the eigenfunctions form a complete orthonormal set with positive eigenvalues).

Consider first the case that the integration variable in the path integral which we use to describe the spinor field in the quantum field theory is an Einstein scalar $\lambda(x)$ (and of course a Lorentz spinor). Then a natural choice as regulator $\mathcal{R}$ for the Jacobian of the n-dimensional Dirac action is the square of the Dirac operator, $\mathcal{R} \sim \not{D}\not{D}$. Using standard manipulations with Dirac matrices, one can simplify $\not{D}\not{D}$ to $\not{D}\not{D} = D^\mu D_\mu + \frac{1}{4}R$, where the second D_μ on the right hand side is given by $D_\mu = \partial_\mu + \frac{1}{4}\omega_\mu{}^{mn}\gamma_m\gamma_n$ while the first D^μ on the right hand side contains an extra term with a Christoffel connection which acts on the index μ of the second D_μ. One can remove this Christoffel term by rewriting $D^\mu D_\mu$ as $\frac{1}{\sqrt{g}}D_\mu\sqrt{g}g^{\mu\nu}D_\nu$ where both D_μ and D_ν now only contain the spin connection[5]

$$\mathcal{R} \sim \frac{1}{\sqrt{g}}D_\mu\sqrt{g}g^{\mu\nu}D_\nu + \frac{1}{4}R\ , \quad D_\mu = \partial_\mu + \frac{1}{4}\omega_{\mu mn}\gamma^m\gamma^n \tag{5.2.2}$$

Replacing γ^m by $\sqrt{\frac{2}{\hbar}}\,\psi^a$ and making also the usual identification[6]

$$\frac{\hbar}{i}\frac{\partial}{\partial x^j} \ \longleftrightarrow\ g^{1/4}\,p_j\,g^{-1/4} \tag{5.2.3}$$

we interpret the regulator $\mathcal{R}$ as ($\frac{1}{\hbar}$ times) the Hamiltonian of a system with bosonic point particles $x^i(t)$ and fermionic point particles $\psi^a(t)$. We multiply $\mathcal{R}$ in (5.2.2) by $\left(\frac{\hbar}{i}\right)^2$ to replace ∂_μ by p_i, and by a factor 1/2

[5] The expansion proceeds as follows: $\not{D}\not{D} = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}D_\mu D_\nu + \frac{1}{2}[\gamma^\mu,\gamma^\nu]D_\mu D_\nu = D^\mu D_\mu + \frac{1}{4}[\gamma^\mu,\gamma^\nu][D_\mu,D_\nu]$ where all derivatives D_μ are fully gravitationally covariant, so for example $[D_\mu,\gamma^\nu]=0$ and $D_\mu e_\nu{}^m = 0$. The second term yields a curvature $\frac{1}{8}\gamma^\mu\gamma^\nu R_{\mu\nu}{}^{mn}(\omega)\gamma_{mn}$. Since we always take for $\omega_\mu{}^{mn}$ the usual spin connection $\omega_\mu{}^{mn}(e)$ which corresponds to the Christoffel connection via the vielbein postulate, see appendix A, then $R_{\mu\nu}{}^{mn}(\omega)$ satisfies the cyclic identity $R_{\mu[\nu mn]}=0$ and the Ricci tensor is symmetric. It follows that $\gamma^\nu\gamma_{mn}R_{\mu\nu}{}^{mn} = 2e^\nu_m\gamma_n R_{\mu\nu}{}^{mn}$ since $\gamma^\nu{}_{mn}R_{\mu\nu}{}^{mn}=0$ due to the cyclic identity. Further, $\gamma^\mu(e^\nu_m\gamma_n R_{\mu\nu}{}^{mn}(\omega)) = e^\nu_m e^\mu_n R_{\mu\nu}{}^{mn}(\omega) = R$ because the Ricci tensor is symmetric. The final result is the term $\frac{1}{4}R$ in $\not{D}\not{D}$. Moreover in general relativity one proves that the covariant divergence of a contravariant vector density $\sqrt{g}v^\mu$ is equal to the ordinary derivative, i.e. $D_\mu(\sqrt{g}v^\mu) = \partial_\mu(\sqrt{g}v^\mu)$. This yields (5.2.2).

[6] We derived this relation in (2.5.43); we recall that it follows from hermiticity of p_j and the hermiticity of $g^{-1/4}\frac{\hbar}{i}\frac{\partial}{\partial x^j}g^{1/4}$ with the inner product $\langle\psi|\varphi\rangle = \int\sqrt{g}\,\psi^*(x)\varphi(x)\,d^nx$.

to obtain the conventional normalization $H = \frac{1}{2}p^2 + \cdots$. This leads to the following Hamiltonian for the bosonic point particle $x^i(t)$ and the fermionic point particle $\psi^a(t)$

$$\hat{H} = \frac{1}{2}g^{-1/4}\Big(p_i - \frac{i}{2}\omega_i{}^{ab}\psi_a\psi_b\Big)g^{1/2}g^{ij}\Big(p_j - \frac{i}{2}\omega_j{}^{cd}\psi_c\psi_d\Big)g^{-1/4} - \frac{\hbar^2}{8}R \ . \tag{5.2.4}$$

Note that x, p and ψ are all operators in this expression. To avoid confusion we mention that we shall later Weyl-order this Hamiltonian, which will produce another term with R.

We now ask the crucial question: which quantum-mechanical nonlinear sigma model leads to this Hamiltonian? The answer is the $N = 1$ supersymmetric nonlinear sigma model which in Minkowskian time is given by

$$L = \frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j + \frac{i}{2}\psi_a(\dot{\psi}^a + \dot{x}^j\omega_j{}^{ab}\psi_b) \ . \tag{5.2.5}$$

In this expression x and ψ are of course classical functions of t, not operators. We discuss this model in appendix D, where it is shown that the classical action $S = \int L\,dt$ is supersymmetric. For this model $p_j(x) = \frac{\partial}{\partial\dot{x}^j}L = g_{jk}\dot{x}^k + \frac{i}{2}\omega_j{}^{ab}\psi_a\psi_b$ and the conjugate momentum of the fermion is given by $\pi_a(\psi) = \frac{\partial}{\partial\dot{\psi}^a}L = -\frac{i}{2}\psi_a$. The classical Hamiltonian is given by

$$H_{cl} = \dot{x}^j p_j + \dot{\psi}^a\pi_a - L \ . \tag{5.2.6}$$

The terms with $\dot{\psi}$ cancel in this expression and elimination of $\dot{x}$ yields the following result

$$H_{cl} = \frac{1}{2}\Big(p_i - \frac{i}{2}\omega_i{}^{ab}\psi_a\psi_b\Big)g^{ij}\Big(p_j - \frac{i}{2}\omega_j{}^{cd}\psi_c\psi_d\Big) \ . \tag{5.2.7}$$

The term $-\frac{\hbar^2}{8}R$ is absent in this expression because it is a quantum effect, as it is clear from the $\hbar^2$ in (5.2.4).

To write this classical Hamiltonian at the quantum level as an operator which is general coordinate and local Lorentz invariant (meaning it should commute with the operators which generate infinitesimal general coordinate and local Lorentz transformations), one must add the factors with $g^{-1/4}$ and $g^{1/2}$ as in (5.2.4). We discussed this in the beginning of section 2.5. The scalar curvature is Einstein and locally Lorentz invariant by itself, so its coefficient is not fixed by requiring Einstein and local Lorentz invariance only. However the term $-\frac{\hbar^2}{8}R$ in (5.2.4) with this precise coefficient is fixed by susy (on the worldline). The argument goes as follows. The quantum susy generator[7] is given by

$$\hat{Q} = \psi^c e_c^i \Big(g^{1/4} p_i g^{-1/4} - \frac{i}{2} \omega_i{}^{ab} \psi_a \psi_b \Big)$$
$$= \Big(g^{-1/4} p_i g^{1/4} - \frac{i}{2} \omega_i{}^{ab} \psi_a \psi_b \Big) \psi^c e_c^i \; . \tag{5.2.8}$$

It commutes with the generator

$$G_E = \frac{1}{2i\hbar} \Big(p_k \xi^k(x) + \xi^k(x) p_k \Big) + \text{terms acting on } e_a^i \text{and } \omega_{iab} \tag{5.2.9}$$

of general coordinate transformations in (2.5.2), (2.5.7) [9]. This fixes the factors $g^{1/4}$ and $g^{-1/4}$ in Q. It also commutes with the generator of local Lorentz rotations, $J = \{\frac{1}{2}\lambda_{ab}(x)\psi^a\psi^b +$ terms acting on e_a^i and $\omega_{iab}\}$ because $\psi^c e_c^i$ and $(g^{1/4} p_i g^{-1/4} - \frac{i}{2}\omega_i{}^{ab}\psi_a\psi_b)$ are separately locally Lorentz invariant, see below eq. (2.5.9). Defining $\hat{H} = \frac{1}{2}\{\hat{Q}, \hat{Q}\}$ one finds (5.2.4) including the term $-\frac{\hbar^2}{8}R$. Thus whereas Einstein and local Lorentz symmetry do not fix the coefficient of the R term in the quantum Hamiltonian, rigid susy of the QM model does.

Once the action for the quantum field theory in n dimensions is given, the consistent regulator for the Jacobian can be constructed. By a consistent regulator we mean a regulator which produces consistent anomalies, namely anomalies which satisfy the consistency conditions which follow from the fact that the anomalies are the gauge variation of the one-loop effective action Γ. If there are no anomalies, the effective action (due to fermion loops with external Yang-Mills or gravitational fields) is gauge invariant, but if there are anomalies, the gauge variation with parameter $\lambda^a(x)$ of the effective action Γ leads to the consistent anomaly: $\delta(gauge, \lambda^a(x))\ \Gamma = \lambda^a(x) An_a(x)$. No ambiguities about the coefficient of the term with the scalar curvature R in the regulator exist: it follows straightforwardly from working out $\not{D}\not{D}$ as we showed. However, the action of the quantum field theory itself may contain nonminimal terms with scalar or other curvatures. In that case the regulator will inherit the same terms. In [75] consistent regulators for quantum field theories are constructed using Pauli-Villars regularization of the action.

However, it is not necessary to use consistent regulators for the purpose of calculating anomalies; one may also use for example covariant regulators. When anomalies cancel with one regulator, they also cancel for

[7] In general a generator of a symmetry is the space integral of the time component of the corresponding Noether current. For quantum mechanics there are no space coordinates, so the charge is the current. The susy Noether current is most easily obtained by making a susy variation with a local (time-dependent) susy parameter $\epsilon(t)$, and collecting all terms proportional to $\dot{\epsilon}$. This yields the expression for Q in the text. The order of the factors with $g^{\pm 1/4}$ in $\hat{Q}$ is determined by the transformation rule $(g^{1/4} p_i g^{-1/4})' = \frac{\partial x^j}{\partial x'^i}(g^{1/4} p_j g^{-1/4})$.

another regulator (possibly after adding local counterterms to the action) and working with covariant regulators has the advantage that calculations are simpler. For this reason we shall use covariant regulators to evaluate the anomalies in the local Lorentz symmetry and gauge symmetry of loops with chiral fermions and selfdual antisymmetric tensors.

The anomaly in the field theory is proportional to $\mathrm{Tr}\, J\mathrm{e}^{-\beta\mathcal{R}}$ where $1+J$ is the infinitesimal Jacobian for the symmetry whose anomaly we want to compute. For example, for the rigid chiral symmetry $\delta\psi \sim \gamma^5\psi$ of massless Dirac actions the Jacobian is proportional to γ^5 and we shall construct a quantum mechanical representation for γ^5. One is then led to the expression

$$An = \lim_{\beta\to 0} \mathrm{Tr}\, \gamma^5 \mathrm{e}^{-\frac{\beta}{\hbar}H} \tag{5.2.10}$$

as an operator expression in the quantum mechanical model. This expression can now be rewritten as a quantum mechanical path integral by inserting complete sets of states as explained in the first part of this book. In this way we see how the problem of evaluating a functional trace in n dimensions gets mapped into a problem in susy quantum mechanics. We shall systematically calculate the following anomalies

1. The usual abelian γ^5 anomaly for complex (Dirac) spin 1/2 fields coupled to external gravity (the gravitational contribution to the chiral anomaly). The transformation rules for this rigid symmetry multiply fermions by $i\alpha\gamma^5$. The Feynman graphs which yield this anomaly are fermion loops with external gravitons at all vertices, except one vertex where the axial vector current is present. Taking the divergence of this axial vector current (contracting the vertex with the momentum which flows in or out at this vertex) produces the anomaly. This anomaly will be shown to be present only in $4k$ dimensions. The calculation of this anomaly is very simple, but for didactical reasons we shall spell out each step in detail. As a curious technical point we already mention that only loops with scalar point particles x^i contribute, but no loops with fermions or ghosts.

2. Next we compute the same γ^5 anomaly for spin 1/2 fields, but now coupled to external Yang-Mills fields instead of external gravitons fields. We call this the abelian chiral anomaly, to distinguish it from the gauge anomaly for chiral fermions coupled to Yang–Mills fields which corresponds to $\mathrm{Tr}\ \gamma^5\, T_\alpha$ in Fujikawa's approach. The latter is called the nonabelian chiral anomaly. In that case the gauge fields are also transformed under symmetry transformations. To deal with the internal symmetry generators T_α in the quantum

mechanical model we introduce new ghosts c^* and c. The corresponding nonlinear sigma model is discussed in appendix E. A few technical problems are encountered and solved: one must take traces only over one-particle states, and to achieve this we construct a suitable projection operator [40]. We can then give a full path integral treatment of these anomalies. (In the work by Alvarez-Gaumé and Witten [1], and also earlier work by us [24], the internal sector was still treated with operatorial methods).

3. Then we consider "Einstein-Lorentz anomalies" for chiral spin 1/2 fields. These are also called gravitational anomalies and are anomalies in either Einstein or local Lorentz symmetry. The Feynman graphs which yield the effective action are polygons with fermions in the loop and gravitons sticking out. Just as with vector and axial vector symmetry in gauge theories, one can push the anomaly from the Einstein to the Lorentz sector or back [2]. We shall find it advantageous to consider a suitable linear combination of these two local symmetries to compute its anomaly. The Jacobian becomes then covariant: it can be written in terms of covariant derivatives $J = \frac{1}{2}(D_\mu \xi^\mu + \xi^\mu D_\mu)$ in which, as we shall see, the Christoffel connections cancel. It will turn out that gravitational anomalies only exist in $4k+2$ dimensions.

4. Next we consider mixed gravitational and nonabelian chiral Yang–Mills anomalies, corresponding to loops with chiral spin 1/2 fields coupled to external gravitational and gauge fields. The Jacobians are proportional to $\frac{1}{2}(D_\mu \xi^\mu + \xi^\mu D_\mu)$ and $i\eta^\alpha T_\alpha \gamma^5$, respectively. As a particular case they contain the purely gravitational anomaly as well as the purely nonabelian chiral anomaly. These anomalies correspond to a breakdown of the reparametrization and gauge invariances of the effective action for chiral fermions coupled to gravity and nonabelian gauge fields. These anomalies are fatal: in four dimensions they imply a breakdown of renormalizability and unitarity of the QFT, and one should try to find a collection of fields for which the anomalies cancel each other. In higher dimensions both gauge and gravitational quantum field theories are not renormalizable, but it is believed that anomalies should still cancel in order that the theory still makes sense.

5. After these studies of anomalies for spin 1/2 fields we turn to spin 3/2 fields. For readers unfamiliar with supergravity we give a short self-contained discussion of the quantization and the ghost structure of supergravity. We then compute the gravitational contribution to the abelian γ^5 anomaly for spin 3/2 in $4k$ dimensions. The corre-

sponding Feynman graphs consist of a spin 3/2 loop with gravitons sticking out, and at one vertex the axial current is present. From a technical point of view this calculation is amusing because it combines the results in 1 and 2.

6. Next we calculate the gravitational anomaly for spin 3/2 fields, corresponding to loops of chiral 3/2 fields coupled to external gravity. Now we are dealing with spin 3/2 loops with gravitons at all vertices. So this section is the spin 3/2 counterpart of the discussion in section 3. Again there is an anomaly only in $4k+2$ dimensions. Spin 3/2 fields do not couple in supergravity to Yang–Mills fields, hence there is no discussion of mixed anomalies for loops with spin 3/2 fields.

7. Finally we discuss gravitational anomalies due to loops with selfdual antisymmetric tensor fields coupled to external gravity. Again couplings to an arbitrary Yang–Mills group does not exist. The problem that there is no covariantly gauge fixed action for selfdual antisymmetric tensor fields was circumvented by Alvarez-Gaumé and Witten [1] by using ordinary (unconstrained, namely non-selfdual) fields in loops, and coupling only one vertex to the selfdual part of the stress tensor. It seems not well-known that there exist **local** actions for selfdual antisymmetric tensor fields in even dimensions. These actions can be found in [76, 77, 78], and one can use them to calculate the selfdual tensor anomalies in $4k+2$ dimensions in exactly the same way as the other anomalies[8]. One obtains the same result as AGW [78].

After these chiral anomalies we turn to trace anomalies. Here the situation is much more delicate: one needs to evaluate higher-loop graphs and the calculations depend very much on the precise definition of the measure, Hamiltonian, and Feynman rules. Let us once again state that by Feynman rules we mean not only certain formal expressions for the propagators and vertices, but also the precise rules how to compute integrals over products of these. The precise rules were in great details derived in part I, and we shall now reap the fruits of that labor. We consider

1. trace anomalies for scalar and spin 1/2 fields in 2 dimensions,

2. trace anomalies for spin 0, spin 1/2, and spin 1 fields in 4 dimensions. For spin 1 fields we need to include the contributions to the spin 1 trace anomaly which come from the Faddeev-Popov ghosts.

[8] Covariant actions for selfdual antisymmetric fields can be formulated at the classical level [79] but their covariant quantization remains problematic. When quantized noncovariantly they reduce to the actions in [76, 77, 78] and thus lead to the same anomaly computation [80].

Before turning to the calculation of these anomalies, we want to test the QM approach in a case where we know beforehand that there should be no anomalies. This case is Einstein symmetry. Consider the Einstein transformation of a scalar field φ, given by $\delta_E\varphi = \xi^\mu(x)\partial_\mu\varphi(x)$. It simplifies the analysis if one takes instead the variable $\tilde{\varphi} = g^{1/4}\varphi$ as fundamental variable. The inner product in the space of variables $\tilde{\varphi}$ is $(\tilde{\varphi}_1, \tilde{\varphi}_2) = \int \tilde{\varphi}_1^*(x)\tilde{\varphi}_2(x)\, dx$ without extra factors of $\sqrt{g}$ (the usual factor of $\sqrt{g}$ has been absorbed into the definition of $\tilde{\varphi}$). Then $\delta_E\tilde{\varphi} = \xi^\mu\partial_\mu\tilde{\varphi} + \frac{1}{2}(\partial_\mu\xi^\mu)\tilde{\varphi}$ because $\tilde{\varphi}$ is a scalar half-density, and we can write this as $\delta_E\tilde{\varphi} = \frac{1}{2}(\xi^\mu\partial_\mu + \partial_\mu\xi^\mu)\tilde{\varphi}$. The Jacobian is now

$$J = \frac{1}{2}(\xi^\mu\partial_\mu + \partial_\mu\xi^\mu) \tag{5.2.11}$$

where the derivative ∂_μ can act past ξ^μ. The anomaly is then

$$An = \lim_{\beta\to 0} \mathrm{Tr}\, J\mathrm{e}^{-\beta\mathcal{R}}\ . \tag{5.2.12}$$

We now show that this symmetry has no anomaly. As regulator we consider an arbitrary operator $\mathcal{R}$ with complete set of eigenfunctions $\tilde{\varphi}_N$ with eigenvalues λ_N^2. One finds then

$$\begin{aligned} An(E) &= \frac{1}{2}\mathrm{Tr}(\xi^\mu\partial_\mu + \partial_\mu\xi^\mu)e^{-\beta\mathcal{R}} \\ &= \frac{1}{2}\int\sum_N \tilde{\varphi}_N^*(\xi^\mu\partial_\mu + \partial_\mu\xi^\mu)e^{-\beta\lambda_N^2}\tilde{\varphi}_N\, d^nx \\ &= \frac{1}{2}\int \partial_\mu\Big(\sum_N \tilde{\varphi}_N^*\xi^\mu\tilde{\varphi}_N e^{-\beta\lambda_N^2}\Big)d^nx \end{aligned} \tag{5.2.13}$$

as long as the complete set contains both $\tilde{\varphi}_N$ and $\tilde{\varphi}_N^*$ (plane waves are an example). In general the λ_N^2 increase fast enough with increasing N so that the sum over N converges, and assuming that $\xi^\mu(x)$ vanishes for large x one finds that Einstein symmetry indeed has no anomaly.

In practice one can calculate anomalies by using a complete set of plane waves, as shown by Fujikawa for chiral anomalies [6]. For Einstein symmetries a two parameter class of regulators has been considered in [81]

$$\mathcal{R} = -g^{-\alpha}\partial_\mu g^{\mu\nu}g^\beta\partial_\nu g^{-\alpha}\ . \tag{5.2.14}$$

This operator is hermitian in the space of fields $\tilde{\varphi}$ with inner product $\langle\tilde{\varphi}|\tilde{\psi}\rangle = \int dx\, \tilde{\varphi}^*(x)\tilde{\psi}(x)$. Hence it can be diagonalized and $e^{-\beta\mathcal{R}}$ becomes then $e^{-\beta\lambda_N^2}$ when acting on $\tilde{\psi}_N$. The explicit calculation of the Einstein anomaly with this regulator using a complete set of plane waves is tedious but straightforward (one has to use the Baker-Campbell-Hausdorff theorem), and the result is that the Einstein anomaly given by (5.2.12) indeed vanishes for arbitrary α and β.

So let us now repeat this calculation using the QM model, and check that the Einstein anomaly still vanishes. This provides a test of the method. We start from the field $\tilde{\varphi}$ and represent[9] ∂_μ by $\frac{i}{\hbar}p_i$ (we recall our notation that $\mu, \nu \cdots$ denote indices in the QFT, and $i, j \cdots$ corresponding indices in the QM model). The operator $\frac{1}{2}(\xi^\mu \partial_\mu + \partial_\mu \xi^\mu)$ turns into

$$J = \frac{i}{2\hbar}(\xi^i(x)p_i + p_i \xi^i(x)) \tag{5.2.15}$$

which is Weyl ordered. We can then rewrite the trace as a path integral, as explained in part I of this book. We recall that we replace x by $x_0 + q$ where the quantum fluctuations q vanish at the endpoints $t = -\beta$ and $t = 0$. It further simplifies the analysis if we write J as $\exp(J)$ and later take the term linear in ξ. It is then convenient to integrate out the momenta and obtain a path integral in configuration space[10]. The Einstein anomaly (if nonvanishing) is then obtained by expanding

$$An = \lim_{\beta \to 0} \int dx_0 \sqrt{g(x_0)} \frac{1}{(2\pi\beta\hbar)^{n/2}} \left\langle e^{-\frac{1}{\hbar}S^{int} - \frac{1}{\beta}\int_{-\beta}^{0} \frac{1}{\hbar}\xi^i(x) g_{ij}(x) \frac{dx^j}{dt} dt} \right\rangle \tag{5.2.16}$$

and keeping the terms linear in ξ^i. The factor $1/\beta$ in front of the second term in the exponent is important, so let us explain in detail its origin. We write J as N times $\frac{\epsilon}{\beta}J$ where $\beta = N\epsilon$. Then we exponentiate. The sum over the N terms with ϵ turns into an integral $\frac{1}{\beta}\int_{-\beta}^{0} dt$, and this yields the result.

The interactions were discussed in part I of this book and read

$$\begin{aligned} -\frac{1}{\hbar}S^{int} &= -\frac{1}{\beta\hbar}\int_{-1}^{0} \frac{1}{2}[g_{ij}(x_0+q) - g_{ij}(x_0)](\dot{q}^i \dot{q}^j + b^i c^j + a^i a^j) d\tau \\ &\quad - \frac{\beta\hbar}{8}\int_{-1}^{0} (R + g^{ij}\Gamma^l_{ik}\Gamma^k_{jl}) d\tau \ . \end{aligned} \tag{5.2.17}$$

(Because in the trace the initial and final point coincide, the classical trajectory $x_{cl}(t)$ is simply x_0.) Expanding $g_{ij}(x_0+q) - g_{ij}(x_0)$ we find in normal coordinates (in which $\partial_k g_{ij}(x_0) = 0$) terms of the form $\frac{1}{\beta\hbar} q^k q^l R_{iklj}(\dot{q}^i \dot{q}^j + b^i c^j + a^i a^j)$ and higher order terms. The term with ξ

[9] With the Fujikawa variables $\tilde{\varphi}$ the scalar product is given by $\langle \tilde{\varphi} | \tilde{\psi} \rangle = \int d^n x \, \tilde{\varphi}^*(x) \tilde{\psi}(x)$ and the hermitian operator p_i is simply represented by $\frac{\hbar}{i}\partial_i$.

[10] Details are as follows. The phase space action contains $\frac{1}{\hbar}\int_{-\beta}^{0}(ip_i \dot{q}^i - \frac{1}{2}g^{ij}p_i p_j)\, dt + \frac{i}{\beta\hbar}\int_{-\beta}^{0} p_i \xi^i \, dt$. Completing squares and integrating over p_i yields (5.2.16).

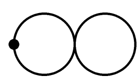
can be rewritten as

$$
\begin{aligned}
-V_\xi &= -\frac{1}{\beta\hbar}\int_{-1}^{0} \xi^i(x_0+q)g_{ij}(x_0+q)\dot q^j\, d\tau \\
&= -\frac{1}{\beta\hbar}\int_{-1}^{0} q^k\dot q^j[\partial_k\xi^i(x_0)g_{ij}(x_0)+\xi^i(x_0)\partial_k g_{ij}(x_0)]\, d\tau+\cdots \\
&= -\frac{1}{\beta\hbar}\Big(\int_{-1}^{0} q^k\dot q^j\, d\tau\Big)g_{ij}(x_0)D_k\xi^i(x_0)+\cdots \\
&= -\frac{1}{\beta\hbar}\Big(\int_{-1}^{0} q^k\dot q^j\, d\tau\Big)D_k\xi_j(x_0)+\cdots
\end{aligned}
\tag{5.2.18}
$$

where the terms denoted by $\cdots$ contain more q fields. We rescaled $t=\beta\tau$, but this did not change the prefactor $\frac{1}{\beta}$ because $\dot q^j(t)\, dt=\dot q^j(\tau)\, d\tau$. Because q vanishes at the endpoints ($\tau=0$ and $\tau=-1$) the integral $\int_{-1}^{0} q^k\dot q^j\, d\tau$ is antisymmetric in k and j.

The Einstein anomaly is given by the β independent terms, hence the factor $(\beta\hbar)^{-n/2}$ in the Feynman measure should be compensated by factors $\beta\hbar$ produced by loops. Since we expect no anomaly in the Einstein transformations, the terms in the final expression of order $(\beta\hbar)^0$ should vanish. We need Feynman graphs with precisely one vertex V_ξ and any number of other vertices. All vertices are proportional to $\frac{1}{\beta\hbar}$ (or $\beta\hbar$, see the last term in S^{int}) and the q propagators to $\beta\hbar$.

In $n=2$ dimensions there is a factor $(\beta\hbar)^{-1}$ in the measure, hence the sum of all Feynman graphs with one factor $\beta\hbar$ should vanish[11]. There is one graph which could possibly contribute



where the dot denotes the vertex V_ξ. The vertex in the middle contains R_{iklj}. It is of order $\beta\hbar$, and if it would be nonvanishing, there would be an Einstein anomaly. However, the result must be of the form of $D_m\xi_n$ times a curvature, and this product always vanishes since the curvature can have at most two indices, hence it is either Ricci curvature R_{mn} or $g_{mn}R$, and in both cases the contraction with the antisymmetric $D_m\xi_n$ vanishes.

We could go on to check that also in $n=4$ dimensions the QM approach yields vanishing Einstein anomaly. Now the sums of all graphs

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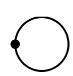
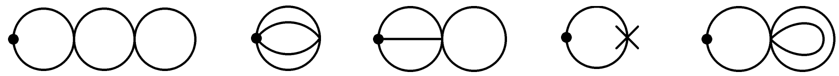
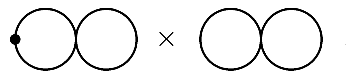
[11] There is even a graph of zeroth order in $\beta\hbar$, namely  . If nonvanishing this would yield a divergent contribution proportional to $\frac{1}{\beta}$ to the anomaly. Fortunately its contribution vanishes due to $\int {}^{\bullet}\!\Delta(\tau,\tau)\, d\tau=0$, see section 2.5.

proportional to $(\beta\hbar)^2$ should vanish. The graphs to be analyzed are the irreducible graphs



and the product of graphs



The cross in the fourth graph indicates the counterterm, and all vertices on the far left come from V_ξ. We shall encounter similar graphs in the chiral and trace anomalies, and since we are mostly interested in nonvanishing anomalies we leave the analysis that there is no Einstein anomaly in $d = 4$ as an exercise.

We close this section by briefly reviewing the algorithm of refs. [75] for determining those regulators $\mathcal{R}$ which produce consistent anomalies. The regulators we shall use in the next sections are covariant regulators, not consistent regulator, so for this reason the discussion of how to construct a consistent regulator is not needed: we include it for the sake of interest. (To be precise, we will actually employ consistent regulators in the computation of the anomalies in the nongauged $U(1)$ axial symmetry and in the the trace anomalies). The basic idea is to regulate the quantum theory by the Pauli-Villars (PV) method. In a path integral context one introduces PV fields which are designed to keep the measure of the path integral invariant. Then one computes the anomalies which are now only due to the noninvariance of the PV mass term. From this computation one reads off the regulators and quantum integration variables to be used in Fujikawa's scheme in order to reproduce the same anomalies. Since the PV method yields consistent anomalies, being a Feynman graph computation, one obtains "consistent" regulators. The method goes as follows. Let us denote by ϕ the original fields and by ψ the PV fields. The regulated action has the generic form

$$\mathcal{L} = \frac{1}{2}\phi^T T\mathcal{O}\phi + \frac{1}{2}\psi^T T\mathcal{O}\psi + \frac{1}{2}M\psi^T T\psi \tag{5.2.19}$$

where M is the regulating mass of the Pauli-Villars fields. For reasons to become clear we denote the kinetic term by $T\mathcal{O}$ instead of $\mathcal{O}$. The mass term should only be quadratic in the quantum fields, but T may depend on background fields (on the metric, for example). The invariance of the original action under $\delta\phi = K\phi$ is extended to an invariance of the massless part of the PV action by $\delta\psi = K\psi$, so that only the PV mass term breaks the symmetry (if one can find a symmetrical mass term, then the symmetry will be anomaly free. We assume for simplicity that the transformation rules are linear in ϕ). We refer to $T\mathcal{O}$ as the kinetic matrix and to T as the mass matrix, and they both may depend on background fields which may get transformed under the symmetry variation. The anomalous response of the path integral Z to a symmetry variation is now due to the mass term only, since the measure of the PV fields is defined in such a way that at the one loop level its Jacobian cancels the Jacobian from the fields ϕ. One obtains[12]

$$\begin{aligned}\delta Z \sim \mathrm{Tr}\Big[\tfrac{1}{2}\Big(TK + K^T T + \delta T\Big) M \Big(TM + T\mathcal{O}\Big)^{-1}\Big] &= \\ = \mathrm{Tr}\Big[\Big(K + \tfrac{1}{2}T^{-1}\delta T\Big)\Big(1 + \tfrac{\mathcal{O}}{M}\Big)^{-1}\Big] . \qquad (5.2.20)\end{aligned}$$

We replaced $K^T T$ by TK, since T and $T\mathcal{O}$ are symmetric, and we used the ψ propagator from (5.2.19). In the limit M to infinity the function $(1 + \frac{\mathcal{O}}{M})^{-1}$ of the regulator $\mathcal{O}$ leads to same anomaly as $e^{\frac{\mathcal{O}}{M}}$, hence one identifies the regulator $\mathcal{R}$ as well as the infinitesimal Jacobian J

$$\mathcal{R} = \mathcal{O} \ , \quad J = K + \frac{1}{2}T^{-1}\delta T \ . \qquad (5.2.21)$$

The variables $\tilde{\varphi}$ (and $\tilde{\lambda}$ for fermions) have a mass term $m\tilde{\varphi}\tilde{\varphi}$ for which the mass matrix T is constant, so that the infinitesimal Jacobian simplifies to the naive one, namely $J = K$.

For many cases the regulator $\mathcal{O}$ is enough, while in other cases (typically when $\mathcal{O}$ is a first order differential operator, as for fermions) one has to improve it. One way to do this is achieved by inserting the identity $1 = (1 - \frac{\mathcal{O}}{M})^{-1}(1 - \frac{\mathcal{O}}{M})$ into (5.2.20), and using the invariance of the massless part of the action (5.2.19) which implies

$$T\mathcal{O} + \frac{1}{2}\delta T\mathcal{O} + \frac{1}{2}T\delta\mathcal{O} = 0 \ . \qquad (5.2.22)$$

The product $(K + \frac{1}{2}T^{-1}\delta T)(1 - \frac{\mathcal{O}}{M})$ can be simplified, and one obtains

$$\begin{aligned}&\delta Z \sim \mathrm{Tr}\Big[\Big(K + \tfrac{1}{2}T^{-1}\delta T + \tfrac{1}{2}\delta\mathcal{O}M^{-1}\Big)\Big(1 - \tfrac{\mathcal{O}^2}{M^2}\Big)^{-1}\Big] \\ &\to \quad \mathcal{R} = -\mathcal{O}^2 \ , \quad J = K + \tfrac{1}{2}T^{-1}\delta T + \tfrac{1}{2}\delta\mathcal{O}M^{-1} \ . \qquad (5.2.23)\end{aligned}$$

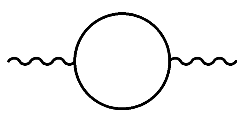
[12] First vary the mass term, then integrate out all ψ fields, and write the resulting determinants as a product of the determinant of $(T\mathcal{O}+TM)$ and a determinant with δT and K, finally re-exponentiate the first determinant to yield back the original action for ψ.

The regulator is now $\mathcal{O}^2$ but the Jacobian has acquired an extra term $\frac{1}{2}\delta\mathcal{O}M^{-1}$. For many applications this last term can be omitted when M tends to infinity.

For the variables $\tilde{\varphi}$ and the spinors $\tilde{\lambda}$, the operator $T\mathcal{O}$ is given by $g^{-\frac{1}{4}}\partial_\mu\sqrt{g}g^{\mu\nu}\partial_\mu g^{-\frac{1}{4}}$ and $g^{\frac{1}{4}}\not{D}g^{-\frac{1}{4}}$, respectively, and $T=1$ in both cases. Then one obtains the regulator for $\tilde{\varphi}$ we have used. For $\tilde{\lambda}$ the regulator becomes instead $-g^{\frac{1}{4}}\not{D}\not{D}g^{-\frac{1}{4}}$, and this regulator will be used in the next section. Furthermore in the basis with $\tilde{\varphi}$ one has $\frac{\hbar}{i}\partial_\mu = p_\mu$ without extra factors $g^{\frac{1}{4}}$.

5.3 A brief history of anomalies

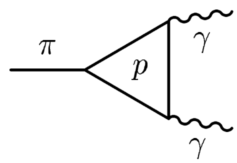
When physicists tried to compute radiative corrections to processes in QED in the 1930's, they of course stumbled on divergences and other inconsistencies. Even the simplest loop diagrams presented enormous difficulties, and some physicists (Heisenberg and Pauli at one time or another, and also Dirac and Oppenheimer) blamed QED itself for these difficulties. In the 1940's the problems became more focused. A diagram which exhibited very clearly some difficulties was the photon selfenergy diagram due to an electron loop (we use of course modern terminology)



$$\partial_\mu\langle 0 \mid T j^\mu_{em}(x) j^\nu_{em}(y) \mid 0\rangle = 0\ ? \tag{5.3.1}$$

Gauge invariance required that this diagram be transversal, and on-shell it should vanish because the photon should remain massless, but Tomonaga and collaborators found it to be infinite, as well as not gauge invariant [82]. They studied the e^2 corrections to the Klein-Nishina formula for Compton scattering and reported that "there is an infinity containing [the] electromagnetic potential bilinearly ... in the ... the vacuum polarization effect. [It] cannot be subtracted by amalgamation [removal by renormalization] as in the case of mass-type and charge-type infinities". This divergence could be identified as a photon mass, but unlike the mass divergence of the electron which could be "amalgamated" into an already existing electron mass, the photon mass divergence could not be dealt with in the same way because there is no photon mass in Maxwell's equations [83]. Oppenheimer commented in a note attached to this article: "As ... Schwinger and others have shown, the very greatest care must be taken in evaluating such selfenergies lest, instead of the zero value they should have, they give non-gauge covariant, noncovariant, in general infinite results ... . I would conclude ... [that] ... the difficulties ... result from ... an inadequate identification, of light quantum self-energies." [83]

Motivated by this problem, two of Tomonaga's collaborators, Fukuda and Miyamoto [84], examined the next simplest diagram, namely the triangle diagram.



It was supposed to describe the decay $\pi \to p\bar{p} \to \gamma + \gamma$. They considered the cases that the neutral meson (π^0, Yukawa's U particle) was a scalar, pseudoscalar or pseudovector, with couplings $fU\bar{\psi}\psi$, $fU\bar{\psi}\gamma_5\psi$ and $(f/2m)\bar{\psi}\gamma_5\gamma^\mu\psi U_\mu$, respectively, where m is the proton mass. They found two problems
1) the results were not gauge covariant since bare A_μ appeared in the result
2) the results for the decay into two photons of a pseudovector U_μ and a pseudoscalar U particle were not the same if they set $U_\mu = \partial_\mu U$, even though the interactions were the same after partial integration and using the Dirac equation of motion.
They concluded: "Evidently these inconsistent results arise from the mathematical difficulty of obtaining [a] definite expression using the singular function of Jordan and Pauli. At present we know [of] no appropriate prescription which makes one free from ambiguities of this kind". The singular function in question was $D(x) = \int \frac{d^3k}{(2\pi)^3} \frac{\sin(kx-\omega t)}{2\omega}$ which appears in the equal-time canonical commutation relations.

Steinberger [85], then a theorist at Princeton, heard from Yukawa (who was visiting Princeton) about the work of Fukuda and Miyamoto (see footnote 11 of his article) and he applied the brand new Pauli-Villars regularization scheme [86] to the triangle graph and an array of other problems. Tomonaga was of course also quite interested in these consistency problems, and with coworkers he also applied the Pauli-Villars regularization scheme to the calculation of the triangle graph [87]. The conclusion of these studies was a partial success: the scheme did maintain gauge invariance and Lorentz invariance, and it led to a finite result for the triangle graph, but the actual value for this finite result seemed to depend on how the calculations were performed, and the relation between pseudovector and pseudoscalar couplings was still not satisfied [85, 87]. In modern terms: there was a chiral anomaly! However, this was not yet fully understood at that time. Rather, it seemed to lead to the perplexing conclusion that the lifetime of the neutral pion was ambiguous: "We see that there remains still some ambiguity how to use the regulator, and this ambiguity would be solved only by some experiment which could detect the γ- decay of [the] neutretto" [87]. (Neutretto was another name for π^0).

Schwinger made in 1951 a fresh attack on the problem of gauge invariance of the photon selfenergy and the triangle diagrams. He introduced a regularization scheme (point splitting) which preserves gauge invariance at all intermediate stages. As he wrote in "On gauge invariance and vacuum polarization" [71]: "This paper is based on the elementary remark that the extraction of gauge invariant results from a formally gauge invariant theory is ensured if one employs methods of solution that involve only gauge covariant quantities". He then proceeded to solve the equations of motion of an electron in an electromagnetic field

$$\frac{dx^\mu}{ds} = 2\pi_\mu \; ; \qquad \frac{d\pi_\mu}{ds} = e(F_{\mu\nu}\pi^\nu + \pi^\nu F_{\mu\nu}) + \frac{1}{2}e\sigma^{\lambda\nu}\frac{\partial F_{\lambda\nu}}{\partial x^\mu} \tag{5.3.2}$$

where $\pi_\mu = p_\mu - eA_\mu$, and s is the proper time. He found that the photon selfenergy did vanish on-shell, so gauge invariance was preserved. However, he also concluded that the pseudovector coupling gave the same result for the triangle graph describing π^0 decay as the pseudoscalar coupling, namely

$$\mathcal{L}_{eff} = \frac{\alpha}{\pi}\frac{f}{m}\pi^0 \vec{E}\cdot\vec{H} \; . \tag{5.3.3}$$

Although this was the result which seemed to solve the earlier problems, we now know that the pseudovector and pseudoscalar couplings should not be the same: there is an axial anomaly! It has been argued that he moved the anomaly from the right-hand side of the anomaly equation to the left-hand side [88] [13].

In the 1950's and 1960's field theory fell from favor, and alternative physical theories took the limelight: Regge theory, the S-matrix program of Chew, and current algebra. Although the first two alternatives were

[13] In section 5 of ref. [71] he used "point splitting", a regularization scheme in x-space that is completely equivalent to the Pauli-Villars scheme in momentum space. According to this scheme the axial current is written as

$$\bar{\psi}\Big(x + \frac{1}{2}\epsilon\Big)\gamma_5\gamma_\mu\Big(\exp ie\int_{x-\frac{1}{2}\epsilon}^{x+\frac{1}{2}\epsilon} A_\mu dx^\mu\Big)\psi\Big(x - \frac{1}{2}\epsilon\Big) \tag{5.3.4}$$

and the exponential factor (later called a Wilson line) is added to keep electromagnetic gauge invariance. Schwinger defined $\partial_\mu[\mathrm{tr}\,\gamma_5\gamma^\mu G(x,x)]$ by $\lim_{x',\,x''\to x}[(\partial'_\mu - ieA_\mu(x')) + (\partial''_\mu + ieA_\mu(x''))]\mathrm{tr}\,\gamma_5\gamma^\mu G(x',x'')$ because "[this] structure is dictated by the requirement that only gauge covariant quantities be employed". However if one adds a Wilson line, one should use ordinary instead of covariant derivatives. If one would have required in the Pauli-Villars scheme that the $U(1)$ vector gauge invariance is maintained, the ambiguities in this scheme would also have been fixed, and one would have obtained the same result as point splitting.

meant to replace field theory, it was natural to try to build field theoretical models which gave a representation of current algebra and in which the consistency of current algebra could be tested. In fact, many of the physicists who worked on current algebra in those days later helped to create modern quantum gauge field theory.

One such attempt was a beautiful little article in 1960 by Gell-Mann and Levy on the linear sigma model [89], in which PCAC (the partially conserved axial-vector current relation) was satisfied: $\partial_\mu j_5^\mu = f_\pi m_\pi^2 \pi(x)$ where f_π is the π-decay constant (93 MeV). The model contained, in addition to the nucleons, the three pions $\pi^\pm, \pi^0$ and a scalar meson σ, with an $SO(4)$ symmetry which was spontaneously broken, giving the nucleons a mass. If a term linear in σ was added to the action, this explicit symmetry breaking also gave the pions a mass. This model became obligatory reading for graduate students at Utrecht University (where one of us obtained his PhD). In Stony Brook B. Lee started studying the renormalization program of spontaneously broken field theories and wrote an influential small book [90] on the renormalization of this model.[14] G. 't Hooft heard B. Lee at the Cargèse summer school lecture on this topic, and upon returning to Utrecht, he decided to start applying these ideas to gauge theories, with well-known consequences.

In 1969 two important articles were submitted for publication within two weeks from each other, one by Bell and Jackiw [92], and the other by Adler [93]. Bell and Jackiw noted that the amplitude for $\pi^0 \to \gamma\gamma$ could be parametrized as follows

$$T^{\mu\nu}(p,q) = \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta T(k^2) \tag{5.3.5}$$

where p and q were the on-shell photon momenta, and $k = p + q$ was the pion momentum. They used the linear sigma model and considered both the case with k^2 off-shell as well as the case with $k^2 + m^2 = 0$ for an on-shell pion. Their amplitude satisfied gauge invariance ($p_\mu T^{\mu\nu}(p,q) = q_\nu T^{\mu\nu}(p,q) = 0$) as well as Bose symmetry ($T^{\mu\nu}(p,q) = T^{\nu\mu}(q,p)$). They noted that Steinberger had calculated $T(k^2)$ using the same graphs that occur in the linear sigma model and had found a nonzero result ($T(0) = g4\pi^2/m$). On the other hand Veltman and Sutherland [94] had found that $T(0) = 0$ if one used an off-mass-shell pion field that was equal to the divergence of the axial current (PCAC). The puzzle that $T(0)$ should on the one hand be nonvanishing and on the other hand be vanishing was the problem Bell and Jackiw decided to tackle. They noted that the problem

[14] Because there were no direct axial-vector couplings in this model, no problems with the chiral anomaly were encountered. (However, the chiral symmetry between pions and σ meson allowed one to define an axial vector current, and its renormalization was also studied [91]).

was "in the same tradition as that of the photon mass, noncanonical terms in commutators – Schwinger terms – and violations of the Jacobi identity." They claimed that this "demonstrates in a very simple example [the linear σ model] the unreliability of the formal manipulations common to current-algebra calculations", but then they went on to "develop a variation which respects PCAC, as well as Lorentz and gauge invariance, and find that indeed the explicit perturbation calculation also then yields $T(0) = 0$".[15] In their appendix they noted the hallmark of an anomaly: "Since the integral is linearly divergent a shift of variable picks up a surface term". (The procedure which yielded $T(0) = 0$ amounted to adding a nonlocal counter term to the action [95], but this violates renormalizability).

Adler just studied the AVV triangle graph in spinor QED, and took the results as they came: " ... we demonstrate the uniqueness of the triangle diagrams [by imposing vector gauge invariance] ... and discuss a possible connection between our results and the $\pi^0 \to 2\gamma$ and $\eta \to 2\gamma$ decays ... [The] partial conservation of the axial-vector current ... must be modified in a well-defined manner, which completely alters the PCAC predictions for the π^0 and the η two-photon decays". Here is the axial anomaly in all its glory: it could not be clearer. He used an explicit expression for the triangle graph which Rosenberg had obtained already in 1963 [96]. Rosenberg considered electromagnetic properties of neutrinos in the V-A theory, and expanded the amplitude for the triangle graph coupled to two photons and a neutrino current in form factors, some of which were divergent and others which were finite. Then he imposed vector gauge invariance, and this expressed the divergent form factors in terms of convergent ones. However, Rosenberg did not study whether the (naive) axial vector Ward identity failed in the case of the triangle graphs; that was done by Bell and Jackiw, and Adler.

With the demonstration of 't Hooft in 1971 that nonabelian pure gauge theories are renormalizable, it was realized that anomalies would spoil renormalizability and unitarity [97]. Thus one had to make sure that anomalies (more precisely anomalies in the gauge transformations of chiral spin 1/2 fields, the quarks and leptons) would cancel. In the Standard Model the gauge group $SU(3)$ has no anomalies because it does not couple to chiral quarks, while $SU(2)$ has no anomalies because all of its representations are pseudoreal. Only the $U(1)$ hypercharge gauge symmetry is potentially anomalous, but its anomalies cancel because the sum of electric charges of all quarks and leptons in a given family cancels[16]. Thus

[15] This variation was the old Pauli-Villars regularization scheme, applied to the Steinberger calculation, but with *mass-dependent* coupling constants for the extra regulator-fermions.

[16] Triangle graphs with one $U(1)$ gauge field and two $SU(2)$ gauge fields are proportional to the sum of the hypercharges of the left-handed doublets. This sum clearly vanishes:

the threat of anomalies in the Standard Model was averted.

Having settled the issue of the chiral anomalies in nongravitational theories, it was realized first by Kimura, and later by Delbourgo and Salam, and then by Eguchi and Freund (who corrected a factor two in the paper by Delbourgo and Salam) that one could also encounter anomalies if one couples fermions to external gravity instead of external electromagnetism [98]. These authors considered triangle graphs in four dimensions with nonchiral (Dirac) fermions in the loop, with one vertex given by the axial current $\bar{\psi}\gamma_5\gamma_\mu\psi$ and the other two vertices given by $h^{\mu\nu}T_{\mu\nu}$ where $T_{\mu\nu}$ is the stress tensor for fermions. They found indeed anomalies of the form $\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu}{}^{mn}R_{\rho\sigma mn}$.

This in turn lead to a related problem: if one couples chiral fermions to external gravity, are there anomalies in the conservation of the stress tensor which are the counterpart of the anomalies in the gauge invariance of chiral gauge theories? It was soon realized that the nonconservation of the stress tensor is closely related to the presence of local Lorentz anomalies and the symmetry of the stress tensor; in fact we discuss the precise relation in section 6.3. It was then found that gravitational contributions to the chiral anomaly do cancel in the Standard Model[17], while local Lorentz anomalies can only occur in $4k+2$ dimensions, and thus yield no potential problems for the Standard Model. Also in the minimally supersymmetric Standard Model all non gravitational and gravitational contribution to the chiral anomalies cancel, because the two Higgsinos have opposite electric charge. However, one can also write down models in which the nongravitational anomalies cancel, but the gravitational anomalies do not cancel. Thus (external) gravity fits remarkably well with the Standard Model and its minimal supersymmetric extension. All these anomalies were treated in a uniform way, and for all dimensions at once, in the fundamental paper by Alvarez-Gaumé and Witten [1], on which part of this book is based.

In addition to anomalies in chiral models there are also trace anomalies which occur when (rigid or local) scale invariance of the classical action is broken at the quantum level. For rigid scale transformations this was first shown by Coleman and Jackiw in 1971 [99], while the breakdown of local

$\frac{1}{6}\times 3\times 2+(-\frac{1}{2})\times 2 = 0$. Furthermore, triangle graphs with three $U(1)$ gauge fields are proportional to the sum of the cubes of the hypercharges of all left-handed fermions (rewriting right handed fermions as charge conjugates of left handed fermions), which also vanishes: $(\frac{1}{6})^3\times 6+(-\frac{2}{3})^3\times 3+(\frac{1}{3})^3\times 3+(-\frac{1}{2})^3\times 2+(1)^3 = 0$.

[17] These triangle graphs with one $U(1)$ gauge field and two gravitons are proportional to the sum of the hypercharges of all left-handed fermions (rewriting right handed fermions as charge conjugates of left handed fermions), which is also the sum of their electric charges because the hypercharge is the average electric charge for each multiplet. Again this sum vanishes.

(Weyl) scale invariance for massless vectors and spinors in 4 dimensions coupled to gravity was first observed by Capper and Duff [100]. In the latter case the most general form of the trace anomaly was found to be given by [101]

$$\begin{aligned} T_\mu{}^\mu &= aR \qquad\qquad (d=2) \\ T_\mu{}^\mu &= aR^2 + bR^2_{\mu\nu} + cR^2_{\mu\nu\rho\sigma} + d\Box R + e(F^a_\mu)^2 \qquad (d=4)\ . \end{aligned} \tag{5.3.6}$$

The term $\Box R$ could be removed by a local counterterm $\Delta\mathcal{L} \sim R^2$, but the other terms were genuine anomalies. The coefficients in the $d=4$ trace anomaly are not all independent, but rather, as required by the consistency conditions, they combine as follows [102]

$$T_\mu{}^\mu \;=\; \alpha\Big(C^2_{\mu\nu\rho\sigma} + \frac{2}{3}\Box R\Big) + \beta\Big(\epsilon\epsilon RR\Big) + \gamma(F^a_{\mu\nu})^2 \tag{5.3.7}$$

where $C^2_{\mu\nu\rho\sigma} = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2$ is the square of the Weyl tensor, and $(\epsilon\epsilon RR) = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2$ yields the Euler invariant.

The constants a in $d=2$ and α, β, γ in $d=4$ also parametrize the one loop divergences due to matter loops with external gravity [103].

For scalars an improvement term $\sim R\varphi^2$ can be added to the action which then becomes classically Weyl invariant, but a genuine trace anomaly develops at the quantum level. In a theory with N_S real scalars, N_F spin 1/2 Dirac fermions, and N_V real vectors fields the $d=4$ trace anomaly is given by

$$\begin{aligned} \alpha &= \frac{1}{120(4\pi)^2}\Big(N_S + 6N_V + 12N_F\Big) \\ \beta &= -\frac{1}{360(4\pi)^2}\Big(N_S + 11N_V + 62N_F\Big)\ . \end{aligned} \tag{5.3.8}$$

It follows from unitarity that all coefficients in α must be positive, so that trace anomalies cannot cancel in rigidly susy $d=4$ models. In models where the scale invariance is already explicitly broken at the classical level, one can nevertheless define a trace anomaly by

$$An(\text{Weyl}) = g^{\mu\nu}\langle T_{\mu\nu}\rangle_{reg} - \langle g^{\mu\nu}T_{\mu\nu}\rangle_{reg}\ . \tag{5.3.9}$$

For example, using dimensional regularization, one uses $\gamma^\mu\gamma_\mu = g^{\mu\nu}g_{\mu\nu} = n$ in the first term, but $\gamma^\mu\gamma_\mu = g^{\mu\nu}g_{\mu\nu} = 4$ in the second term. Moreover it was found in [104] that one can write a scalar in $d=4$ either as φ or a rank two antisymmetric gauge field $\varphi_{\mu\nu}$, but the trace anomalies are different. A rank 3 antisymmetric gauge field is dual to nothing, but it nevertheless yields a nonvanishing trace anomaly. If one reduces by trivial Kaluza-Klein reduction $d=10$ type IIB supergravity one finds in $d = 4$ not the usual 70 scalars, but rather 63 scalars, 7 fields $\varphi_{\mu\nu}$ and one field $\varphi_{\mu\nu\rho}$. One can use index theorems to compute the axial and conformal anomalies for arbitrary spin in gravity and supergravity [105]. By considering background fields with $R_{\mu\nu} = 0$ the trace anomaly becomes proportional to $(\alpha + \beta)R^2_{\mu\nu\rho\sigma}$. The combined trace anomaly for these spin $2, \frac{3}{2}, 1, \frac{1}{2}, 0$ fields then cancels in $N = 8$ $d = 4$ supergravity.

6

Chiral anomalies from susy quantum mechanics

6.1 The abelian chiral anomaly for spin 1/2 fields coupled to gravity in $4k$ dimensions

As a first application of the formalism we have developed, we shall compute the anomaly in the chiral symmetry $\delta\lambda = i\alpha\gamma_5\lambda$ for a massless Dirac fermion λ in n dimensions coupled to external gravity (n is even) [98]. The real parameter α is an infinitesimal constant and γ_5 is proportional to the product $\gamma^1 \ldots \gamma^n$ and hermitian (hence $(\gamma_5)^2 = 1$). This anomaly is sometimes called the gravitational chiral anomaly, although a more precise name would be the gravitational contribution to the abelian chiral anomaly. If there is an anomaly, the axial vector current is no longer conserved at the quantum level.

The Lagrangian of the field theory in n Minkowski dimensions is given by

$$\mathcal{L} = -e\,\bar{\lambda}\, e_m^{\mu} \gamma^m D_\mu \lambda\ , \qquad \bar{\lambda} = \lambda^\dagger i\gamma^0\ , \qquad (\gamma^0)^2 = -1 \tag{6.1.1}$$

where $e = (\det e_\mu^{m})$, e_m^{μ} is the inverse of the vielbein field e_μ^{m}, and $D_\mu\lambda = \partial_\mu\lambda + \frac{1}{4}\omega_{\mu mn}(e)\gamma^m\gamma^n$ with $\omega_{\mu mn}(e)$ the spin connection of appendix A. In Minkowski space the chiral transformation law $\delta\lambda = i\alpha\gamma_5\lambda$ implies that $\delta\bar{\lambda} = i\alpha\bar{\lambda}\gamma_5$ because $\bar{\lambda} = \lambda^\dagger i\gamma^0$, but in Euclidean space λ and $\bar{\lambda}$ are independent complex spinors and then $\delta\bar{\lambda} = i\alpha\bar{\lambda}\gamma_5$ follows from requiring chiral invariance of the action[1].

If in the path integral

$$Z[e_\mu^{m}] = \int \mathcal{D}\lambda\mathcal{D}\bar{\lambda}\, e^{-\int d^n x\, \mathcal{L}} \tag{6.1.2}$$

[1] In the Euclidean case, α can even be complex, but since only α and not α^* appears in the transformation laws of λ and $\bar{\lambda}$, the fact that α may be complex does not enlarge the symmetry group.

one makes a chiral change of integration variables $\lambda' = (1 + i\alpha\gamma_5)\lambda$ and $\bar{\lambda}' = \bar{\lambda}(1 + i\alpha\gamma_5)$ with local $\alpha(x)$, one obtains the Jacobian we shall compute, and a term $\int(\partial_\mu\alpha)j_5^\mu dx$ in the action, where the Noether current j_5^μ is the axial-vector current. Since the path integral does of course not change under a change of integration variables, the Jacobian yields the expectation value of the divergence of the axial-vector current. The corresponding Feynman graphs are single loops with λ in the loops, and gravitons sticking out from all vertices, except at one vertex where one has $(\partial_\mu\alpha)j_5^\mu$. One could have gauged the axial $U(1)$ symmetry by introducing a gauge field with coupling $-A_\mu j_5^\mu$ at this vertex. Then the chiral anomaly causes a breakdown of the gauge invariance of the effective action under $\delta A_\mu = \partial_\mu\alpha$.

The infinitesimal chiral transformations of λ and $\bar{\lambda}$ are equal: they are given for both λ and $\bar{\lambda}$ by the matrix

$$J = i\alpha\gamma_5 \ . \tag{6.1.3}$$

Its trace yields the Jacobian, but one should regulate this trace. Of course, one can compute Feynman diagrams with the spin 1/2 field in the loop, with at one vertex the axial-vector current, while at the other vertices external gravitons couple to the fermion. There are vertices with one, two, or more gravitons. Clearly, a background field formalism is called for, which takes the sum of all vertices into account at once. Such a background field formalism has been developed by Fujikawa [6], who showed that in path integrals the anomaly is given by the regulated Jacobian

$$An = \lim_{\beta\to 0} \mathrm{Tr}\, J\, e^{-\beta\mathcal{R}} \ . \tag{6.1.4}$$

As regulator we use a covariant regulator which is obtained as follows. As spin 1/2 fields we take $\tilde{\lambda} = g^{\frac{1}{4}}\lambda$ and $\tilde{\bar{\lambda}} = g^{\frac{1}{4}}\bar{\lambda}$ (recall that if one takes these fields as integration variables in the path integral, the Einstein anomalies are immediately seen to be absent for any selfadjoint regulator). The field operator for $\tilde{\lambda}$ and $\tilde{\bar{\lambda}}$ is $g^{\frac{1}{4}}\not{D}g^{-\frac{1}{4}}$. The covariant regulator is proportional to the square of this Dirac operator. Hence

$$\mathcal{R} = -\frac{1}{2}g^{\frac{1}{4}}\not{D}^2 g^{-\frac{1}{4}} \tag{6.1.5}$$

and this regulator is the same for $\tilde{\lambda}$ and $\tilde{\bar{\lambda}}$ because the Dirac operator stands between $\tilde{\lambda}$ and $\tilde{\bar{\lambda}}$. (The factor 1/2 is conventional and could have been absorbed into β). One could now directly calculate the trace in (6.1.4) using this regulator. However, for higher dimensions n, the calculations become progressively more complicated due to the algebra of the many Dirac matrices, and a simpler method than the Fujikawa method is needed. This is the method of supersymmetric quantum mechanics (susy QM). As we discussed in chapter 5, one uses a representation of the operators x^μ, ∂_μ and γ^m (which are the only ingredients entering in J and $\mathcal{R}$) in terms of quantum mechanical operators $\hat{x}^i, \hat{p}_i$ and $\hat{\psi}^a$, with the same (anti)-commutation relations in a Hilbert space with the same dimension and with the same hermiticity properties. The regulator becomes in the QM model the Hamiltonian of a simple susy QM model, the so-called $N = 1$ model. As we shall see, the presence of fermions in the QM model will remove all factors of β from the measure, and as a consequence, in the limit β tending to zero, only one-loop graphs need be computed. For this particular anomaly, loops with QM ghosts or QM fermions do not even contribute, but this is in general not the case.

Underlying this approach is the fact that all different representations of the canonical (anti) commutation relations which preserve the hermiticity of the operators are unitarily equivalent[2]. Since the anomaly we are going to calculate is proportional to a trace in a Hilbert space, and traces are invariant under similarity transformations, the anomaly does not depend on the representation chosen. The representation in terms of QM leads to particularly simple calculations, and this is the reason why we transform the quantum field theory problem into a problem in quantum mechanics.

We choose to work in Euclidean space because in this case the Gaussian integrals we need to evaluate are well-defined. One could have started in Minkowski space, but then one would need at some point to make a Wick rotation to evaluate these Gaussian integrals, so it is easier to start from the beginning in Euclidean space.

The matrix γ_5 denotes the product of all Dirac matrices, and in order that $(\gamma_5)^2 = +1$ we normalize it as follows

$$\gamma_5 = (-i)^{n/2} \gamma^1 \dots \gamma^n \tag{6.1.6}$$

where $\{\gamma^m, \gamma^k\} = 2\delta^{mk}$ and all γ's are hermitian (including γ_5). For $n = 2$, with $\gamma^1 = \sigma^1$ and $\gamma^2 = \sigma^2$, γ_5 equals the Pauli matrix σ^3, while in $n = 4$ we have $\gamma_5 = -\gamma^1\gamma^2\gamma^3\gamma^4$. The anomaly can be written as $\mathrm{Tr}\,\gamma_5 \exp(-\beta\mathcal{R})$, where the regulator $\mathcal{R}$ which preserves Einstein (general coordinate) and local Lorentz invariance is given by

$$\mathcal{R} \;=\; -\frac{1}{2} g^{\frac{1}{4}} \not{D} \not{D} g^{-\frac{1}{4}}$$

[2] For the quantum mechanical variables p and q this is a theorem due to von Neumann [106]. For the fermionic extension with $\psi^a \sim \gamma^m$ one can use finite group theory to prove that there is only one faithful irreducible representation of the Clifford algebra in even dimensions [56, 107], hence the dimension of the fermionic part of the Hilbert space is fixed and is equal to $2^{n/2}$.

$$
\begin{aligned}
&= -\frac{1}{2}\Big(g^{\frac{1}{4}} g^{\mu\nu} D_\mu^{(\omega,\Gamma)} D_\nu g^{-\frac{1}{4}} + \gamma^m \gamma^n \frac{1}{4} R_{mnpq}(\omega)\gamma^p\gamma^q\Big) \\
&= -\frac{1}{2} g^{-\frac{1}{4}} D_\mu \sqrt{g} g^{\mu\nu} D_\nu\, g^{-\frac{1}{4}} - \frac{1}{8} R \qquad (6.1.7) \\
&(\text{with } \not{D} = \gamma^\mu D_\mu\ ,\ \ \gamma^\mu = e^\mu_m \gamma^m\ ,\ \ D_\mu = \partial_\mu + \frac{1}{4}\omega_{\mu mn}(e)\gamma^m\gamma^n)\ .
\end{aligned}
$$

In the second line we have written $D_\mu^{(\omega,\Gamma)}$ for the first derivative because it contains in addition to the spin connection a Christoffel symbol that acts in the index ν of the derivative D_ν. To obtain the third line, we used that in general relativity the covariant derivative of a contravariant vector density equals the ordinary derivative. Hence, no Christoffel connections are present in D_μ, but D_μ contains of course terms with the spin connection.

We represent $\mathcal{R}$ in terms of quantum mechanical operators $\hat{x}^i, \hat{p}_i$ and $\hat{\psi}^a$ and denote the result by $\hat{H}$. Of course, $[\hat{p}_i, \hat{x}^j] = \frac{\hbar}{i}\delta_i^j$. However this does not fix the x-representation of $\hat{p}_j$ completely, namely $(p_x)_j = g^\alpha \frac{\hbar}{i}\frac{\partial}{\partial x^j} g^{-\alpha}$ is still possible for arbitrary α. Hermiticity of $(p_x)_j$ fixes the factors of g^α. The relation is then $\frac{\hbar}{i}\partial_\mu = p_\mu$ without extra factors of $g^{\frac{1}{4}}$ because we use $\tilde{\lambda}$ as basic fields, and $\frac{\hbar}{i}\frac{\partial}{\partial x^\mu}$ is hermitian if $\int \bar{\tilde{\lambda}}_1 \tilde{\lambda}_2\, d^n x$ is the inner product. We find then the Hamiltonian discussed in appendix B, eq. (B.25).

The Dirac matrices $\gamma^m (m = 1,..,n)$ can be viewed as operators in a $2^{n/2}$ dimensional linear vector space, with anticommutation relations $\{\gamma^m, \gamma^k\} = 2\delta^{mk}$. Of course, we must take n to be an even number if we want to define a matrix γ_5. In the QM model, we introduce corresponding operators $\psi_1^a (a = 1,..,n)$ satisfying $\{\psi_1^a, \psi_1^b\} = \delta^{ab}$. The reason for the subscript 1 will become clear shortly. Hence $\gamma^m \leftrightarrow \sqrt{2}\psi_1^a$. Flat vector indices $m, n...$ in quantum field theory correspond to indices $a, b...$ in the QM model. In a given dimension of spacetime there may or may not exist a Majorana representation of the Dirac matrices, but we always use a hermitian representation of the Dirac matrices in Euclidean space, and hence the ψ_1^a are hermitian.

In our formalism, we need operators $\psi_a^\dagger$ and ψ^b satisfying $\{\psi_a^\dagger, \psi^b\} = \delta_a{}^b$ (and $\{\psi^a, \psi^b\} = \{\psi_a^\dagger, \psi_b^\dagger\} = 0$). We therefore introduce *new* operators $\psi_2^a (a = 1,..,n)$ which are *free* (i.e., the Hamiltonian $\hat{H}$ (to be constructed) will be independent of ψ_2^a) and satisfy $\{\psi_2^a, \psi_2^b\} = \delta^{ab}$ and $\{\psi_1^a, \psi_2^b\} = 0$. We then define

$$
\psi^a \equiv (\psi_1^a + i\psi_2^a)/\sqrt{2}\ ; \qquad \psi_a^\dagger = (\psi_1^a - i\psi_2^a))/\sqrt{2}\ . \qquad (6.1.8)
$$

In particular, $\gamma^m \leftrightarrow (\psi^a + \psi_a^\dagger)$. The operators ψ^a and $\psi_b^\dagger$ then indeed have the desired anticommutation relations. (At this point, ψ_1^a and ψ_2^a have been introduced without any considerations involving canonical quantization. Hence, in $\{\psi^a, \psi_b^\dagger\} = \delta^a{}_b$ there are no factors $\hbar$. This simplifies the notation. One could rescale the ψ_a and $\psi_a^\dagger$ with factors $\sqrt{\hbar}$ to revert to the usual normalization of fermion fields).

The space in which ψ_2^a acts has also dimension $2^{n/2}$, just like the space for ψ_1^a. So we take as Hilbert space the direct product of the spaces for ψ_1^a an ψ_2^a. **In the Hamiltonian $\boldsymbol{\psi_2^a}$ is absent** but when we convert the operator expression to a path integral, we will find terms $-\int_{-1}^0 \bar{\psi}_a \dot{\psi}^a d\tau$ in the action, so that effectively terms with ψ_2^a are present in the action which appears in the path integrals. The linear vector space obtained by acting with $\psi_a^\dagger$ on the ψ-vacuum, has dimension 2^n. (The ψ-vacuum is defined by $\psi^a|0\rangle = 0$). In traces over the direct product of both spaces we therefore divide by hand by $2^{n/2}$, since the original problem only involved the space of ψ_1^a.

There is a more minimal but also more cumbersome way of deriving the results without extra ψ_2^a. We can begin with operators $\hat{\psi}_1^a$ satisfying $\{\hat{\psi}_1^a, \hat{\psi}_1^b\} = \delta^{ab}$, but then we can combine pairs of them into ψ and $\bar{\psi}^\dagger$. For example $(\psi_1^1 + i\hat{\psi}_1^2)/\sqrt{2} = \psi^{\mathrm{I}}$ and $(\psi_1^1 - i\hat{\psi}_1^2)/\sqrt{2} = \psi_{\mathrm{I}}^\dagger$. The Hilbert space has now dimension $2^{n/2}$. The final Feynman rules for the approach with extra ψ_2^a and the approach in which one combines Majorana spinors differ, but physical results (such as the transition element) are the same. We shall discuss and use both approaches. The approach in which one combines spinors is purely deductive and uses only the original Hilbert space, but the approach with ψ_2^a is algebraically somewhat simpler.

The γ_5 anomaly for the field λ in the QFT can now be written in the QM model as the trace in (6.1.4) with $J = \gamma_5$ in (6.1.6)

$$\begin{aligned} An &= \lim_{\beta\to 0} \frac{1}{2^{n/2}} (-i)^{n/2}\, \mathrm{Tr} \prod_{a=1}^{n} (\hat{\psi}^a + \hat{\psi}_a^\dagger)\, e^{-\frac{\beta}{\hbar}\hat{H}} \\ \hat{H} &= \frac{1}{2} g^{-1/4} \pi_i \sqrt{g} g^{ij} \pi_j g^{-1/4} - \frac{\hbar^2}{8} R \\ \pi_i &= \hat{p}_i - \frac{i\hbar}{2} \omega_{iab}(e) \hat{\psi}_1^a \hat{\psi}_1^b \end{aligned} \tag{6.1.9}$$

where ψ_1^a and ψ_1^b are to be written in terms of ψ^a and $\psi_a^\dagger$ using (6.1.8). We have redefined β such that it has the dimensions of a time. We shall first compute (6.1.9) which is proportional to the anomaly for the field λ in (6.1.1), but at the end we must add to this result the contribution from $\bar{\lambda}$. Since these two contributions are equal (because the Jacobians and the regulators are equal), we shall just multiply the final result by a factor $-2i\alpha$ to obtain the correct normalization. (The minus sign is due to the fact that the traces over fermions acquire a minus sign. This in turn is due to the fact that the Jacobian for bosonic and fermionic fields in quantum field theory is a super-determinant, see the appendix of [56]).

Using the trace formula in (2.4.6)

$$\mathrm{Tr}\, A = \int \sqrt{g(x_0)} \prod_{i=1}^{n} dx_0^i \prod_{a=1}^{n} (d\chi^a d\bar{\chi}_a)\, e^{\bar{\chi}\chi} \langle \bar{\chi}, x_0 | A | \chi, x_0 \rangle \qquad (6.1.10)$$

and the completeness relation in (2.3.15)

$$I = \int \Big(\prod_{a=1}^{n} d\bar{\eta}_a d\eta^a \Big) |\eta\rangle e^{-\bar{\eta}_a \eta^a} \langle \bar{\eta}| \qquad (6.1.11)$$

we obtain for the chiral anomaly (omitting the overall factor $-2i\alpha$ for the time being, and not yet taking the limit of vanishing β)

$$\begin{aligned} An \;=\; & \frac{(-i)^{n/2}}{2^{n/2}} \int \left(\prod_{i=1}^{n} dx_0^i \right) \sqrt{g(x_0)} \prod_{a=1}^{n} (d\bar{\eta}_a d\eta^a d\chi^a d\bar{\chi}_a) \\ & e^{\bar{\chi}\chi} \langle \bar{\chi}| \prod_{a=1}^{n} (\hat{\psi}^a + \hat{\psi}_a^\dagger) |\eta\rangle\, e^{-\bar{\eta}\eta} \langle \bar{\eta}, x_0 | e^{-\frac{\beta}{\hbar}\hat{H}} | \chi, x_0 \rangle \;. \end{aligned} \qquad (6.1.12)$$

Since $\prod(\hat{\psi}^a + \hat{\psi}_a^\dagger)$ is already Weyl ordered (each factor is separately Weyl ordered, and different factors anticommute), we can at once evaluate the first matrix element. For the matrix element $\langle \bar{\eta}, x_0 | e^{-\frac{\beta}{\hbar}\hat{H}} | \chi, x_0 \rangle$ we substitute the result derived in chapter 2. We recall that this involves a Dirac action with fields ψ and $\bar{\psi}$, together with an extra term $\bar{\eta}\chi$, see (2.3.40). We found that

$$\langle \bar{\eta}, x_0 | e^{-\frac{\beta}{\hbar}\hat{H}} | \chi, x_0 \rangle = \frac{1}{(2\pi\beta\hbar)^{n/2}} \Big\langle e^{-\frac{1}{\hbar}\int_{-\beta}^{0} S^{(int)} dt + \bar{\eta}_a \chi^a} \Big\rangle \qquad (6.1.13)$$

where in general $S^{(int)}$ depends on both $\bar{\eta}_a + \bar{\psi}_a(t)$ and $\chi^a + \psi^a(t)$, with propagator $\langle \psi^a(t) \bar{\psi}_b(t') \rangle = \theta(t - t') \delta^a{}_b$.

In chapter 2 we discussed all aspects of the path integral at the discretized level, but we shall now use a continuum notation, and only go back to discretized expressions when this is necessary to resolve ambiguities. This simplifies the notation, but it should be stressed that all our continuum expressions stand for more complicated but well-defined discretized expressions.

Since in our case the Hamiltonian depends only on ψ_1^a, the expectation value of $\exp\Big(-\frac{1}{\hbar} S^{int}\Big)$ will only depend on x_0 and $(\chi^a + \bar{\eta}_a)/\sqrt{2}$, but not on $\chi^a - \bar{\eta}_a$ (except for the boundary term $\exp(\bar{\eta}\chi)$). Hence, three of the four Grassmann integrations over η, $\bar{\eta}$, χ and $\bar{\chi}$ will be very simple. Finally, the measure factor $[g(z)/g(y)]^{1/4}$ in the transition element in (2.1.80) becomes unity since $z = y = x_0$. After integrating out the p's, we obtain

$$\begin{aligned} An &= \frac{1}{2^{n/2}} \int dx_0 \sqrt{g(x_0)}\, d\bar{\eta} d\eta d\chi d\bar{\chi}\, e^{\bar{\chi}\chi}\, e^{-\bar{\eta}\eta}\, e^{\bar{\eta}\chi} \\ & \left[(-i)^{n/2} e^{\bar{\chi}\eta} \prod_{a=1}^{n} (\eta^a + \bar{\chi}_a)\right] \frac{1}{(2\pi\beta\hbar)^{n/2}} \langle e^{-\frac{1}{\hbar} S^{int}(x_0, \frac{\bar{\eta}+\chi}{\sqrt{2}})} \rangle \ . \end{aligned} \tag{6.1.14}$$

The factor in square brackets comes form the matrix element of γ_5 and the rest of the second line comes from the transition element $\langle \bar{\eta}, x_0 | e^{-\frac{\beta}{\hbar}\hat{H}} | \chi, x_0 \rangle$.

We take a closer look at the action. Rewriting the quantum Hamiltonian in (6.1.9) in Weyl-ordered form yields

$$H_W = \left(\frac{1}{2} g^{ij} \pi_i \pi_j\right)_S + \frac{\hbar^2}{8} g^{ij} \left(\Gamma^l_{ik} \Gamma^k_{jl} + \frac{1}{2} \omega_i{}^{ab} \omega_{jab}\right) \tag{6.1.15}$$

because the scalar curvature R from Weyl ordering the bosonic sector, see (B.25), cancels the scalar curvature R in (6.1.9). In appendix C we show that Weyl ordering of the fermions in the $N = 2$ Hamiltonian gives a contribution $\frac{1}{8}\omega\omega$. The result in (6.1.15) refers to an $N = 1$ model, see appendix D, and for that reason a factor 1/2 appears in front of the $\omega\omega$ term.

Having integrated over p_i, one finds in the path integral the following action

$$\begin{aligned} -\frac{1}{\hbar} S &= -\frac{1}{\beta\hbar} \int_{-1}^{0} \frac{1}{2} g_{ij}(x_0 + q)(\dot{q}^i \dot{q}^j + b^i c^j + a^i a^j)\, d\tau + \bar{\eta}\psi(0) \\ & - \int_{-1}^{0} \bar{\psi}_a \dot{\psi}^a\, d\tau - \int_{-1}^{0} \frac{1}{2} \psi_1^a \dot{q}^i \omega_{iab}(x_0 + q) \psi_1^b d\tau \\ & - \frac{\beta\hbar}{8} \int_{-1}^{0} (\Gamma\Gamma + \frac{1}{2}\omega\omega)\, d\tau \end{aligned} \tag{6.1.16}$$

where $\psi_1 = (\psi + \bar{\psi})/\sqrt{2}$. This is the $N = 1$ model in appendix D, eq. (D.7), but in Euclidean space, and with the order $\hbar^2$ counterterms and the extra term $\bar{\eta}\psi(0)$ whose presence we derived in (2.3.26) and whose role is to cancel boundary terms in the ψ field equation. Substituting $\bar{\psi}_a = \bar{\eta}_a + \bar{\psi}_{qu,a}$ and $\psi^a = \chi^a + \psi^a_{qu}$ with constant background fermions χ^a and $\bar{\eta}_a$, one finds

$$\begin{aligned} -\frac{1}{\hbar} S &= -\frac{1}{\beta\hbar} \int_{-1}^{0} \frac{1}{2} g_{ij}(x_0 + q)(\dot{q}^i \dot{q}^j + b^i c^j + a^i a^j)\, d\tau \\ & + \bar{\eta}\chi - \int_{-1}^{0} \bar{\psi}_{qu,a} \dot{\psi}^a_{qu} d\tau \end{aligned}$$

$$-\frac{1}{2}\int_{-1}^{0}\dot{q}^i\omega_{iab}(x_0+q)(\psi^a_{1,bg}+\psi^a_{1,qu})(\psi^b_{1,bg}+\psi^b_{1,qu})\,d\tau$$
$$-\frac{\beta\hbar}{8}\int_{-1}^{0}(\Gamma\Gamma+\frac{1}{2}\omega\omega)\,d\tau \tag{6.1.17}$$

where $\psi^a_{1,bg}=(\bar{\eta}_a+\chi^a)/\sqrt{2}$ and $\psi^a_{1,qu}=(\psi^a_{qu})+\bar{\psi}^a_{qu})/\sqrt{2}$.[3] Note that the term $\bar{\eta}\psi_{qu}(0)$ (with undetermined $\psi_{qu}(0)$) has canceled. The terms $-\frac{1}{\beta\hbar}\int_{-1}^{0}\frac{1}{2}g_{ij}(x_0)(\dot{q}^i\dot{q}^j+b^ic^j+a^ia^j)\,d\tau-\int_{-1}^{0}\bar{\psi}_{qu,a}\dot{\psi}^a_{qu}\,d\tau$ yield the propagators, and the rest yields the vertices. These results were derived in chapter 2, and the reader may look there for more details on the derivation.

After doing the loop integrations, the result for $\langle\exp(-\frac{1}{\hbar}S^{int})\rangle$ will only depend on $\psi^a_{1,bg}=(\chi^a+\bar{\eta}_a)/\sqrt{2}$. Hence we can first do the $\bar{\chi}$ and η integrals, while the $\chi-\bar{\eta}$ integral will effectively remove ψ^a_2 from the trace. We shall then be left with an integral over $\psi^a_{1,bg}=(\chi^a+\bar{\eta}_a)/\sqrt{2}$. There follows now an orgy of Grassmann integrations. Readers who are only interested in the final result may jump to (6.1.24).

In the $\bar{\chi},\eta$ sector we find the following integral (use $\int d\bar{\eta}d\eta d\chi d\bar{\chi}=\int d\bar{\eta}d\chi d\bar{\chi}d\eta$)

$$\int d\bar{\chi}d\eta\, e^{-\bar{\eta}\eta}\, e^{\bar{\chi}\chi}\, e^{\bar{\chi}\eta}\prod_{a=1}^{n}(\eta^a+\bar{\chi}_a)\ . \tag{6.1.18}$$

The last factor is a fermionic delta function $\delta(\eta+\bar{\chi})$, hence $\exp(\bar{\chi}\eta)$ can be replaced by unity. For the same reason we can make the following rewriting in the exponent

$$-\bar{\eta}\eta+\bar{\chi}\chi=-\frac{1}{2}(\eta-\bar{\chi})(\chi-\bar{\eta})\ . \tag{6.1.19}$$

Using

$$\begin{aligned} d\bar{\chi}d\eta &= d\bar{\chi}^n\ldots d\bar{\chi}^1 d\eta^1\ldots d\eta^n \qquad (6.1.20)\\ &= 2^n d(\bar{\chi^n}+\eta^n)\ldots d(\bar{\chi^1}+\eta^1)d(\eta^1-\bar{\chi^1})\ldots d(\eta^n-\bar{\chi^n}) \end{aligned}$$

(*not* with a factor 2^{-n} because we need the super Jacobian) and pulling $\prod(\eta+\bar{\chi})$ to the left past $d(\eta-\bar{\chi})$, we obtain a factor $(-)^n$ times $\int d(\bar{\chi}+\eta)\prod(\eta+\bar{\chi})=1$. Then we get

$$2^n\int d(\eta-\bar{\chi})(-)^n e^{-\frac{1}{2}(\eta-\bar{\chi})(\chi-\bar{\eta})}=\prod_{a=1}^{n}(\chi^a-\bar{\eta}_a)\ . \tag{6.1.21}$$

[3] More precisely, in the discretized approach the interactions depend on $\psi^a_{k-1/2}=(\psi^a_k+\psi^a_{k-1})/2$ and $\bar{\psi}_{k,a}$, and then this leads to $\psi^a_{1,k}=(\psi^a_{k-1/2}+\bar{\psi}_{k,a})/\sqrt{2}$ and $\psi^a_{1,qu}=(\psi^a_{qu,k-1/2}+\bar{\psi}_{k,a})/\sqrt{2}$. From these discretized results we derived the propagators in chapter 2.

Hence we end up with another fermionic delta function, which again will make the corresponding Grassmann integral trivial.

At this point we have obtained

$$
\begin{aligned}
An \;=\; & \frac{(-i)^{n/2}}{(4\pi\beta\hbar)^{n/2}} \int dx_0 \sqrt{g(x_0)}\, d\bar{\eta} d\chi \prod (\chi - \bar{\eta})\, e^{\bar{\eta}\chi} \\
& \Big\langle \exp\Big[-\frac{1}{\beta\hbar} \int_{-1}^{0} \frac{1}{2} (g_{ij}(x_0+q) - g_{ij}(x_0))(\dot{q}^i \dot{q}^j + b^i c^j + a^i a^j)\, d\tau \\
& -\frac{1}{2} \int_{-1}^{0} \dot{q}^i \omega_{iab}(x_0+q)(\psi^a_{1,bg} + \psi^a_{1,qu})(\psi^b_{1,bg} + \psi^b_{1,qu})\, d\tau \\
& -\frac{\beta\hbar}{8} \int_{-1}^{0} (\Gamma\Gamma + \frac{1}{2}\omega\omega)\, d\tau \Big] \Big\rangle \;. \qquad (6.1.22)
\end{aligned}
$$

We combined the bosonic measure $(2\pi\beta\hbar)^{-n/2}$ with the factor $2^{-n/2}$ which accounted for the dimension of the space in which ψ^a_2 acts. **The "extra term" $e^{\bar{\eta}\chi}$ in the action is annihilated by the fermionic delta function** $\prod(\chi^a - \bar{\eta}_a)$, and we proceed to do the $(\chi - \bar{\eta})$ integrals.

We perform once more the transition from the variables $\bar{\eta}$ and χ to $\chi + \bar{\eta}$ and $\bar{\eta} - \chi$. For any function F of $\frac{\chi+\bar{\eta}}{\sqrt{2}}$ one has

$$
\begin{aligned}
& \int d\bar{\eta} d\chi \prod (\chi - \bar{\eta}) F\Big(\frac{\chi + \bar{\eta}}{\sqrt{2}}\Big) \\
& = \int 2^n d(\bar{\eta} + \chi) d(\chi - \bar{\eta}) \prod (\chi - \bar{\eta}) F\Big(\frac{\chi + \bar{\eta}}{\sqrt{2}}\Big) \\
& = \int 2^n d(\chi + \bar{\eta}) F\Big(\frac{\chi + \bar{\eta}}{\sqrt{2}}\Big) = \int 2^n d(\sqrt{2}\psi^a_{1,bg}) F(\psi^a_{1,bg}) \\
& = \int 2^{n/2} \prod_{a=1}^{n} d\psi^a_{1,bg} F(\psi^a_{1,bg}) \;. \qquad (6.1.23)
\end{aligned}
$$

We used that $d(\bar{\eta}+\chi)d(\chi-\bar{\eta}) \equiv d(\bar{\eta}+\chi)^n \ldots d(\bar{\eta}+\chi)^1 d(\chi-\bar{\eta})^1 \ldots d(\chi-\bar{\eta})^n$ equals $d(\chi+\bar{\eta})d(-\bar{\eta}+\chi) \equiv d(\chi+\bar{\eta})^1 \ldots d(\chi+\bar{\eta})^n d(-\bar{\eta}+\chi)^n \ldots d(-\bar{\eta}+\chi)^1$ to do the integral over $\prod(\chi - \bar{\eta}) = (\chi - \bar{\eta})^1 \ldots (\chi - \bar{\eta})^n$.

Next we rescale $\psi_{1,qu}$ and $\psi_{1,bg}$ by a factor $(\sqrt{\beta\hbar})^{-1}$. **The rescaling of $\psi^a_{1,bg}$ removes the $\beta\hbar$ dependence of the measure** (use $d\psi = d(\psi'/\sqrt{\beta\hbar}) = \sqrt{\beta\hbar} d\psi'$) and will have enormous consequences. The rescaling of ψ_1 adds a factor $1/\beta\hbar$ to the vertices with fermions and a factor $\beta\hbar$ to the propagators of the fermions. Dropping the primes on ψ' we arrive at

$$
An \;=\; \frac{(-i)^{n/2}}{(2\pi)^{n/2}} \int \prod_{i=1}^{n} dx_0^i \sqrt{g(x_0)} \prod_{a=1}^{n} d\psi^a_{1,bg} \Big\langle \exp\Big(-\frac{1}{\hbar} S^{int}\Big) \Big\rangle
$$

$$
\begin{aligned}
-\frac{1}{\hbar}S^{int} \;=\; & -\frac{1}{\beta\hbar}\int_{-1}^{0}\frac{1}{2}(g_{ij}(x_0+q)-g_{ij}(x_0))(\dot{q}^i\dot{q}^j+b^ic^j+a^ia^j)\,d\tau \\
& -\frac{1}{\beta\hbar}\int_{-1}^{0}\frac{1}{2}\dot{q}^i\omega_{iab}(x_0+q)(\psi^a_{1,bg}+\psi^a_{1,qu})(\psi^b_{1,bg}+\psi^b_{1,qu})d\tau \\
& -\frac{\beta\hbar}{8}\int_{-1}^{0}(\Gamma\Gamma+\frac{1}{2}\omega\omega)\,d\tau\ . \qquad (6.1.24)
\end{aligned}
$$

The expectation value $\langle ... \rangle$ indicates that all quantum fields ($q^i, \psi^a_{1,qu}$ and a, b, c ghosts) must be contracted using the propagators of chapter 2. However, in the end we must take the limit $\beta \to 0$, and since all propagators are proportional to $\beta\hbar$ and all vertices to $\frac{1}{\beta\hbar}$ (or even $\beta\hbar$ for the $\Gamma\Gamma + \frac{1}{2}\omega\omega$ term), we conclude:

1) only one-loop graphs survive the $\beta\hbar \to 0$ limit. (At higher loops there are more propagators than vertices).

2) the a, b, c ghosts do not contribute at the one-loop level because their vertices involve at least 3 quantum fields.

3) the $\Gamma\Gamma + \frac{1}{2}\omega\omega$ term can be discarded as it is of higher order in β.

4) there are no terms linear in quantum fields, so no tadpoles, because the integral of $\dot{q}^i\omega_{iab}(x_0)\psi^a_{1,bg}\psi^b_{1,bg}$ vanishes due to the boundary conditions on q^i.

5) we can for convenience choose a frame with $\omega_{iab}(x_0) = 0$. Then $\omega_{iab}(x_0+q)$ is at least linear in quantum fields. Expanding $\omega_{iab}(x_0+q)$ to first order, one can set $\psi^a_{1,qu} = 0$ in the one-but-last line of (6.1.24).

6) the only remaining vertex is

$$
\begin{aligned}
-\frac{1}{\hbar}S^{int} \;&=\; -\frac{1}{\beta\hbar}\int_{-1}^{0}\frac{1}{2}\dot{q}^iq^j\partial_j\omega_{iab}(x_0)\psi^a_{1,bg}\psi^b_{1,bg}d\tau \\
&=\; -\frac{1}{\beta\hbar}\frac{1}{4}\int_{-1}^{0}q^i\dot{q}^jR_{ijab}(\omega(x_0))\psi^a_{1,bg}\psi^b_{1,bg}d\tau \qquad (6.1.25)
\end{aligned}
$$

where $R_{ijab}(\omega)(x_0) = \partial_i\omega_{jab}(e(x_0)) + \omega_{iac}\omega_j{}^c{}_b - (i \leftrightarrow j)$ and we used that $\int q^i\dot{q}^j d\tau$ is antisymmetric in i and j. (Since q^i vanishes at the endpoints we are allowed to partially integrate).

Hence, we need only compute closed q-loops, with q-propagators and $R_{ij}(x_0) \equiv R_{ijab}(\omega(x_0))\psi^a_{1,bg}\psi^b_{1,bg}$ sticking out of each vertex. Then

$$
An = \frac{(-i)^{n/2}}{(2\pi)^{n/2}}\int dx_0\sqrt{g(x_0)}\,d\psi_{1,bg}\,\langle e^{-\frac{1}{\beta\hbar}\frac{1}{4}R_{ij}(x_0)\int_{-1}^{0}q^i\dot{q}^jd\tau}\rangle\ . \qquad (6.1.26)
$$

This formula contains the chiral anomaly for any dimension, and the explicit evaluation of the expression $\langle ... \rangle$ is far simpler than the corresponding Feynman graph calculation. Expanding the exponent, one obtains disconnected graphs: sums of *products* of closed q loops are found, which yield terms like $(\mathrm{tr}R^2)^2$ and $\mathrm{tr}R^4$, for example. If one writes the result for $\langle ... \rangle$ as $\exp[-\frac{1}{\hbar}W(\text{loops})]$, then $W(\text{loops})$ contains only single closed loops (because $\exp[-\frac{1}{\hbar}W(\text{loops})]$ is the generating functional for connected graphs). To obtain the final formula for the anomaly one must expand the exponent, and one finds then back the products of closed loops.

The evaluation of the sum of connected closed loops in $W(\text{loops})$ yields a sum of graphs, each with k vertices and k propagators. The propagators read

$$\langle q^i(\sigma) q^j(\tau) \rangle = -\beta\hbar g^{ij}(x_0)\Delta(\sigma,\tau) \tag{6.1.27}$$

where $\Delta(\sigma,\tau)$ was defined (1.1.3). The $g^{ij}(x_0)$ contract the first two indices of the curvatures to a trace over k curvature tensors. Hence

$$\begin{aligned} -\frac{1}{\hbar}W(\text{loops}) \; &= \; \sum_{k=2}^{\infty} \frac{1}{k!}\, \frac{1}{4^k} (\mathrm{tr}R^k)(k-1)!\, 2^{k-1} \int_{-1}^{0} d\tau_1 \cdots \int_{-1}^{0} d\tau_k \\ &\quad {}^\bullet\!\Delta(\tau_1,\tau_2)\, {}^\bullet\!\Delta(\tau_2,\tau_3) \ldots {}^\bullet\!\Delta(\tau_k,\tau_1) \; . \end{aligned} \tag{6.1.28}$$

The factor $(k-1)!$ states that one can contract the k vertices in $(k-1)!$ ways, while the symmetry of each vertex in both q's yields a factor 2^{k-1} (partial integration is allowed since $q^i(\sigma) = 0$ at the end points). There remains an overall factor $1/2$. Because $\mathrm{tr}\, R = 0$ we started the summation at $k = 2$.

The integrals

$$\begin{aligned} I_k \; &= \; \int_{-1}^{0} d\tau_1 \cdots \int_{-1}^{0} d\tau_k \,(\tau_2 + \theta(\tau_1 - \tau_2)) \\ &\quad (\tau_3 + \theta(\tau_2 - \tau_3)) \ldots (\tau_1 + \theta(\tau_k - \tau_1)) \end{aligned} \tag{6.1.29}$$

are most easily evaluated by first computing the generating function $\sum_{k=1}^{\infty} \frac{y^k}{k} I_k$. In fact, the expression for the anomaly has precisely this structure, with $y = \frac{R}{2}$. The first few I_k are easily evaluated. One finds $I_1 = 0$, and

$$\begin{aligned} I_2 \; &= \; \int_{-1}^{0} d\tau_1 \int_{-1}^{0} d\tau_2 \,(\tau_2 + \theta(\tau_1 - \tau_2))(\tau_1 + \theta(\tau_2 - \tau_1)) \\ &= \; \frac{1}{4} + 2 \int_{-1}^{0} d\tau_1 \int_{\tau_1}^{0} \tau_2 \, d\tau_2 = -\frac{1}{12} \end{aligned} \tag{6.1.30}$$

etc. Using induction (see appendix A.4 of [40]), the general result is found

$$\sum_{k=2}^{\infty} \frac{y^k}{k} I_k = \ln \frac{y/2}{\sinh y/2} = -\frac{1}{3!}\Big(\frac{y}{2}\Big)^2 + \dots \ . \tag{6.1.31}$$

Using this result, we find for the gravitational contribution to the *chiral anomaly of a Dirac fermion* in n dimensions due to the transformation law $\delta\lambda = i\alpha\gamma^5\lambda$ and $\delta\bar{\lambda} = i\alpha\bar{\lambda}\gamma^5$

$$\begin{aligned} An &= (-2i\alpha)\frac{(-i)^{n/2}}{(2\pi)^{n/2}} \int dx_0^i \sqrt{g(x_0)}\, d\psi_{1,bg}^a \ \exp\left[\frac{1}{2}\mathrm{tr}\ln\left(\frac{R/4}{\sinh R/4}\right)\right] \\ R &= R_{ij} = R_{ijab}\psi_{1,bg}^a\psi_{1,bg}^b \ . \end{aligned} \tag{6.1.32}$$

The factor $-2i\alpha$ is of course due to the Jacobian in (6.1.3), and the factor $1/2$ in the exponent is the overall factor $1/2$ mentioned below (6.1.28). Furthermore, the factor $(-i)^{n/2}$ is due to the definition of γ_5, and the factor $(2\pi)^{-n/2}$ is due to the Feynman measure. Since only the term with precisely n factors $\psi_{1,bg}$ can contribute to the Grassmann integral (so only the terms proportional to $R^{n/2}$), we can absorb the overall normalization factor into the trace

$$\begin{aligned} An &= (-2i\alpha)\left(\int \prod_{i=1}^{n} dx_0^i \sqrt{g(x_0)}\right) \\ &\quad \times \left(\int \prod_{a=1}^{n} d\psi_{1,bg}^a\right) \exp\frac{1}{2}\mathrm{tr}\ln\left[\frac{-iR/8\pi}{\sinh(-iR/8\pi)}\right] \end{aligned} \tag{6.1.33}$$

The matrix $-iR_{jkab}\psi_{1,bg}^a\psi_{1,bg}^b$ is real, so the anomaly is purely imaginary. In the path integral in Euclidean space, we find $\langle(\partial_\mu\alpha)j_5^\mu\rangle - \langle 2i\alpha\gamma_5\rangle = 0$, but j_5^μ has no definite reality properties in Euclidean space. In Minkowski space one has i times the action in the path integral: now $i\langle(\partial_\mu\alpha)j_5^\mu\rangle$ is antihermitian. The anomaly we have computed is imaginary, both in Euclidean and in Minkowski space. (Making a Wick rotation the factor i from the ϵ symbol cancels the factor i from $d^n x$.) Only traces with an even number of Riemann tensors are present (because $x^{-1}\sinh x$ is even in x. As a consequence, no factors of i survive in the expansion of the exponent.) Since each R_{ij} contains two ψ's, this means that **there is only a gravitational contribution to the chiral anomaly in $\mathbf{n} = \mathbf{4k}$ dimensions.** In particular there is a gravitational contribution to the chiral anomaly in $d = 4$ but not in $d = 2$ or $d = 10$. For $n = 4$, the terms with four ψ_1^a yields $\epsilon^{abcd}R_{ijab}R_{ijcd}$ times a factor $\frac{1}{3!}\ \frac{1}{(8\pi)^2}\ \frac{1}{2} = \frac{1}{192\,(2\pi)^2}$, times $(-2i\alpha)$, which is the correct result for Dirac spinors [98].

This result contains the complete dependence on all gravitational fields (not only the leading term) since we used the spin connections $\omega_{iab}(e)$

as external fields. Although we put ω_{iab} at the point x_0 equal to zero, the complete γ_5 anomaly does not contain further terms with bare ω's. This follows either from a direct calculation, or from the fact that chiral transformations and local Lorentz transformations commute. Since we used a regulator which preserves Einstein and local Lorentz symmetry, the anomaly must be locally Lorentz invariant:

$$\delta_{lL}\, An(\text{chiral}) = 0 \ . \tag{6.1.34}$$

The same holds for the Einstein symmetry. Hence, (6.1.33) is the complete answer.

Note also that one obtains in the exponent a sum of terms, so that the anomaly corresponds to sums of products of traces over Riemann tensors. For example, in $d = 8$ one finds two terms with four curvatures, proportional to $\text{tr}R^4$ and $(\text{tr}R^2)^2$. This is different from the Yang-Mills contribution to the abelian chiral anomaly which always has the form $\text{Tr}(F^{n/2})$. [4] In Feynman diagram language this means that disconnected graphs contribute to the gravitational chiral anomaly. The group theoretical reason is that the Lorentz generators do not commute with the Dirac matrices, whereas the Yang-Mills generators of course commute with the Dirac matrices.

One might worry that our procedure of introducing free ψ_2^a at the beginning violates local Lorentz invariance. Since we started with a regulator which is locally Lorentz invariant, and all other steps were mathematical identities, local Lorentz invariance cannot be lost. As a check one might repeat the calculations with $\omega_{iab}(x_0)$ not vanishing. One should find that terms with bare ω's cancel. In fact, if one defines that the ψ_2^a are inert under local Lorentz transformations, the action preserves local Lorentz symmetry because the ψ_2^a do not couple, and then local Lorentz invariance should remain preserved at all stages.

In the next anomaly we use an alternative approach in which one does not add free ψ_2^a ("doubling") but combines pairs of spinors into ψ and $\bar{\psi}$ ("halving"). One could repeat the calculations of this section using halving instead of doubling; the answer should be the same.

A comment on supersymmetry. Starting from the non-susy Dirac action in n dimensional space, we found an action for the QM path integral which turns out to be the $N = 1$ susy QM model for spinors ψ_1^a plus

[4] Jumping ahead, we shall see that for spin 1/2 the Yang-Mills contributions to the abelian chiral anomaly are due to the factor $\text{Tr}\, e^{\frac{1}{2}F}$, whereas the gravitational contributions are due to the factor $\exp[\frac{1}{2}\text{tr}\ln(\frac{R/4}{\sinh R/4})]$. Because in the latter case the trace occurs in the exponent, one obtains an expression of the form $\exp[\,a\,\text{tr}R^2+b\,\text{tr}R^4+\cdots]$. Expanding one obtains products of traces. From $\text{Tr}\, e^{\frac{1}{2}F}$ one obtains of course only a single trace.

terms of order $\hbar^2$ due to Weyl ordering. This classical $N = 1$ action (with Euclidean time) is given by

$$L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + \frac{1}{2} \psi_1^a (\dot{\psi}_1^a + \dot{x}^i \omega_{iab}(e) \psi_1^b) \tag{6.1.35}$$

and is invariant under

$$\delta x^i = -i\epsilon\, \psi_1^a e_a{}^i \ , \quad \delta \psi_1^a = i e_i{}^a \dot{x}^i \epsilon - \delta x^j \omega_j{}^a{}_b \psi_1^b \ . \tag{6.1.36}$$

(In one-dimensional worldspace there are Euclidean Majorana spinors). In flat space $L = \frac{1}{2}(\dot{x}^i)^2 + \frac{1}{2}\psi_1^i \dot{\psi}_1^i$ is clearly invariant under $\delta x^i = -i\epsilon \psi_1^i$ and $\delta \psi_1^i = i\dot{x}^i \epsilon$ (because $\delta L = -\dot{x}^i(i\epsilon \dot{\psi}_1^i) + (i\dot{x}^i \epsilon)\dot{\psi}_1^i = 0$), while in curved space one just covariantizes these rules ($\delta \psi_1^a + \delta x^j \omega_j{}^a{}_b \psi_1^b$ is covariant under local Lorentz transformations, see (D.54)). We refer to appendix D for more details. The $\Gamma\Gamma + \omega\omega/2$ terms we found in the regulator were quantum effects due to regularization by time slicing. They did not contribute to the gravitational γ_5 anomaly, but they do contribute to the trace anomaly. Using dimensional regularization these noncovariant terms are absent. Another way of obtaining the regulator $\mathcal{R}$ would have been to first construct the supersymmetry generator Q at the quantum level, whose operator ordering is fixed by requiring that it be Einstein and local Lorentz invariant. We discussed this in chapter 5. Then $\hat{H} = \hat{Q}\hat{Q}$ is clearly a supersymmetric, Einstein and locally Lorentz invariant regulator. This reproduces the regulator in (6.1.9); in particular the coefficient of the curvature in that expression is fixed by supersymmetry. We did not impose supersymmetry from the beginning, but rather we chose a regulator which was the square of the field operator of the fermions. So we took the action of the quantum field theory as our starting point, and this fixed the R term in the quantum action.

Let us now discuss the relation between our calculations and those of [1]. One can either evaluate the Feynman graphs with l external sources R_{ij} and then summing over l, or directly evaluate the propagator in a gravitational background and using that the sum of one-loop graphs corresponds to the determinant of the field operator. The former approach is the most natural at this point, since we have already determined the propagators and interaction vertices. However, it needs some special tricks [40]. The latter approach is used in [1] and we sketch in this footnote the connection. If one goes back to the discretized path integral with $z = y = x_0$ and adds a final integration with dx_0 to the $N-1$ integrations $dx_1 ... dx_{N-1}$, one obtains for $N \to \infty$ a continuous path integral over $Dx(\tau)$ with periodic boundary conditions (PBC). The part quadratic in quantum fields

q yields then the one-loop determinant

$$\left\langle \exp\Big(-\frac{1}{\beta\hbar}\frac{1}{4}R_{ij}(x_0)\int_{-1}^{0} q^i \dot{q}^j\, d\tau\Big)\right\rangle = \left[\frac{\det(-g_{ij}(x_0)\partial_\tau^2 + \frac{1}{2}R_{ij}(x_0)\partial_\tau)}{\det(-g_{ij}(x_0)\partial_\tau^2)}\right]^{-1/2} . \tag{6.1.37}$$

Transforming $g_{ij}(x_0)$ to δ_{ij} and diagonalizing the hermitian $n \times n$ matrix $\frac{1}{2}R_{ijab}\psi_1^a\psi_1^b$ with eigenvalues $(y_1, .., y_n)$, one obtains for the ratio of the two determinants

$$\prod_{k=1}^{n}\prod_{n=-\infty}^{\infty}{}'\Big(1+\frac{iy_k}{2\pi n}\Big) = \prod_{k=1}^{n}\prod_{n=1}^{\infty}\Big(1+\frac{y_k^2}{4\pi^2 n^2}\Big) = \prod_{k=1}^{n}\frac{\sinh(y_k/2)}{y_k/2} . \tag{6.1.38}$$

The prime indicates that one should omit the zero mode with $n = 0$ (one can regularize the infrared divergence corresponding to $n = 0$ by giving the quantum field a small mass. Then for $n = 0$ there is a nonvanishing contribution to both determinants which cancels in the ratio). This can be rewritten as $\exp \operatorname{tr} \ln \left(\frac{\sinh \frac{R}{4}}{\frac{R}{4}}\right)$. Bringing this to power $-\ 1/2$ yields indeed the expected result.

6.2 The abelian chiral anomaly for spin 1/2 fields coupled to Yang-Mills fields in $2k$ dimensions

The next anomaly we consider is the abelian chiral (γ_5) anomaly in loops with Dirac fermions, coupled to external Yang-Mills fields instead of external gravitational fields. This is the same calculation as in the previous section, but with gravity replaced by Yang-Mills fields. We can again make a local chiral transformation of the integration variables of the path integral, and find then that the anomaly is equal to the divergence of the abelian axial-vector current $\bar{\lambda}\gamma_5\gamma^\mu\lambda$. There is now no metric $g_{ij}(x)$, so in the QM approach we are now dealing with *linear* sigma models, and no a, b, c ghosts will be present. The regulator for the quantum field theory will contain new objects, namely the matrices for the generators in the representation of the gauge group for the fermions. These matrices will be denoted by $(T_\alpha)^M{}_N$, and in the QM model new internal ghosts must be introduced. These ghosts will be denoted by $\hat{c}^M$ and $\hat{c}^*{}_M$ and satisfy the anticommutation relations $\{\hat{c}^M, \hat{c}^*{}_N\} = \delta^M{}_N$. We shall omit the hats most of the time.

As we shall explain, the ghosts c and c^* act only in the one-particle subspace of the whole Fock space obtained by acting with one c^* on the c-vacuum, and the matrices T_α are represented in this subspace by $c^*{}_M(T_\alpha)^M{}_N c^N$. (It is also possible to represent T_α with ghosts satisfying commutation relations, as $d^*{}_M(T_\alpha)^M{}_N d^N$ where $[d^M, d^*{}_N] = \delta^M{}_N$, but this has the disadvantage that the Fock space becomes infinite dimensional. Incidentally, note that one can not represent the Dirac matrices $(\gamma^m)^\alpha{}_\beta$ with either commuting or anticommuting ghosts as $d^\dagger\gamma^m d$ or $c^\dagger\gamma^m c$, since they satisfy *anti*commutation relations). In their pioneering article, Alvarez-Gaumé and Witten [1] used an operator (Hamiltonian) approach for the internal ghost sector, but we shall treat the ghosts on equal footing with the nonghost sector, namely by a path integral approach.

The action of the quantum field theory we consider is given by

$$\mathcal{L} = -\bar\lambda\gamma^\mu D_\mu\lambda \ ; \quad D_\mu\lambda^M = \partial_\mu\lambda^M + gA_\mu{}^\alpha(T_\alpha)^M{}_N\lambda^N \tag{6.2.1}$$

with $[T_\alpha, T_\beta] = f_{\alpha\beta}{}^\gamma T_\gamma$ and $[D_\mu, D_\nu] = gF_{\mu\nu}{}^\alpha T_\alpha$. The regulator in the quantum field theory, as obtained in the previous section, or from the construction of [75], is $\mathcal{R} \sim \not{D}\not{D} = D_\mu D^\mu + \frac{1}{2}\gamma^\mu\gamma^\nu(gF_{\mu\nu})$. The massless Dirac action has the rigid chiral symmetry

$$\delta\lambda = i\alpha\gamma_5\lambda \ , \quad \delta\bar\lambda = i\alpha\bar\lambda\gamma_5 \tag{6.2.2}$$

and in quantum field theory the anomaly is given by

$$An = -2i\alpha \lim_{\beta\to 0} \mathrm{Tr}\,\gamma_5\, e^{\frac{\beta}{\hbar}\mathcal{R}} \ . \tag{6.2.3}$$

We shall omit for the time being the overall prefactor $-2i\alpha$. The operators γ_5 and $\mathcal{R}$ depend on $x^\mu, \partial_\mu, \gamma^\mu$ (in flat space we do not distinguish between flat and curved indices) and $(T_\alpha)^M{}_N$, and the trace is over the internal indices M, N as well as over the spinor indices, and, of course, over all points in spacetime.

To construct the corresponding QM model, we represent again x^μ by $\hat x^i$, the hermitian operator $\frac{\hbar}{i}\partial_\mu$ by $\hat p_i$, and γ^m by the hermitian operators $\sqrt{2}\hat\psi_1^a$, with $[\hat x^i, \hat p_j] = i\hbar\delta^i{}_j$ and $\{\hat\psi_1^a, \hat\psi_1^b\} = \delta^{ab}$. This yields again the regulator in (6.1.7), but now in flat space and coupled to Yang-Mills fields. The matrices $(T_\alpha)^M{}_N$ are represented by operators

$$\hat T_\alpha = \hat c^*{}_M(T_\alpha)^M{}_N\hat c^N \tag{6.2.4}$$

which act in the sector of the one-particle states $|N\rangle$ (omitting hats)

$$|N\rangle = c^*{}_N|0\rangle \ , \quad c^M|0\rangle = 0 \ . \tag{6.2.5}$$

There are, of course, also two-particle etc. states $c^*{}_M c^*{}_N|0\rangle$ etc., but if we start with a one-particle state, the operators $\hat T_\alpha$ will never bring us outside this subspace. The action of $\hat T_\alpha$ on $|N\rangle$ is just like the matrix $(T_\alpha)^M{}_N$ acts on vectors in the carrier space of the representation R, and products of the operators $\hat T^\alpha$ lead to matrix multiplication.

To construct operators $\psi_a^\dagger$ and ψ^a from the ψ_1^a, we could again add a set of free fermions ψ_2^a and proceed as in the previous section. The reader may follow this approach as an exercise; the answer for the anomaly will be the same. Here we follow an alternative approach: we combine pairs of hermitian fermions into Dirac spinors. Namely, we define

$$
\begin{aligned}
\chi^A &= \frac{1}{\sqrt{2}}(\psi_1^{2A-1} + i\psi_1^{2A}) \; ; \quad \chi_A^\dagger = \frac{1}{\sqrt{2}}(\psi_1^{2A-1} - i\psi_1^{2A}) \\
\psi_1^a &= \frac{1}{\sqrt{2}}(\chi^{(a+1)/2} + \chi^\dagger_{(a+1)/2}) \qquad \text{for } a \text{ odd} \\
\psi_1^a &= \frac{-i}{\sqrt{2}}(\chi^{a/2} - \chi^\dagger_{a/2}) \qquad \text{for } a \text{ even} \; . \qquad (6.2.6)
\end{aligned}
$$

In order to define a matrix γ_5, we need an even number of dimensions $n = 2k$. The indices a run from 1 to n as always but $A = 1, \ldots n/2$, and

$$
\{\chi^A, \chi_B^\dagger\} = \delta^A{}_B \; ; \quad \{\chi^A, \chi^B\} = \{\chi_A^\dagger, \chi_B^\dagger\} = 0 \; . \qquad (6.2.7)
$$

Weyl ordering will be defined with respect to the operators χ and $\chi^\dagger$.

We first must write γ_5 as an operator constructed from the χ and $\chi^\dagger$. Recalling the definition $\gamma_5 = (-i)^{n/2}\gamma^1\gamma^2 \cdots \gamma^n$ and $\gamma^m \sim \sqrt{2}\psi^a$ we get

$$
\begin{aligned}
\gamma_5 &= (-i)^{n/2}(\chi^1 + \chi_1^\dagger)(-i)(\chi^1 - \chi_1^\dagger) \cdots \\
&\quad \cdots (\chi^{n/2} + \chi^\dagger_{n/2})(-i)(\chi^{n/2} - \chi^\dagger_{n/2}) \\
&= (-1)^{n/2} \prod_{A=1}^{n/2} (\chi^A + \chi_A^\dagger)(\chi^A - \chi_A^\dagger) \\
&= \prod_{A=1}^{n/2} (\chi^A \chi_A^\dagger - \chi_A^\dagger \chi^A) = \prod_{A=1}^{n/2} (1 - 2\chi_A^\dagger \chi^A) \qquad (6.2.8)
\end{aligned}
$$

where we used $\chi^A \chi_A^\dagger = 1 - \chi_A^\dagger \chi^A$. Since $\chi_A^\dagger \chi^A$ for fixed A is a projection operator, we can also write this as

$$
\gamma_5 = \prod_{A=1}^{n/2} e^{-i\pi \chi_A^\dagger \chi^A} = e^{-i\pi \sum_{A=1}^{n/2} \chi_A^\dagger \chi^A} = (-)^F \qquad (6.2.9)
$$

where $F = \sum_{A=1}^{n/2} \chi_A^\dagger \chi^A$ is the fermion number operator. Indeed, the operator $\exp(-i\pi \chi_A^\dagger \chi^A)$ for fixed A is equal to

$$
1 + \left(-i\pi + \frac{(-i\pi)^2}{2!} \cdots \right) \chi_A^\dagger \chi^A = 1 + (e^{-i\pi} - 1)\chi_A^\dagger \chi^A = 1 - 2\chi_A^\dagger \chi^A \; . \quad (6.2.10)
$$

Hence the Jacobian can be written in two ways

$$J = \prod_{A=1}^{n/2} (1 - 2\chi_A^\dagger \chi^A) = (-)^F \tag{6.2.11}$$

In ref. [1] $(-)^F$ is used; we shall also use this expression[5].

Next we consider the regulator $\mathcal{R}$. The Hamiltonian of the QM model is obtained from the Euclidean regulator $\mathcal{R} \sim D_\mu D^\mu + \frac{1}{2}\gamma^\mu\gamma^\nu(gF_{\mu\nu})$ of the quantum field theory by replacing all operators of the quantum field theory by corresponding operators in the QM model, and reads, after multiplication by $\frac{1}{2}(\hbar/i)^2$,

$$\begin{aligned} \hat{H} &= \frac{1}{2}\hat{\pi}_i\hat{\pi}_j\delta^{ij} - \frac{\hbar}{2}^2 \hat{\psi}_1^a\hat{\psi}_1^b gF_{ab}{}^\alpha(A(\hat{x}))\hat{c}^* T_\alpha \hat{c} \\ D_\mu &= \partial_\mu + gA_\mu{}^\alpha T_\alpha \quad \rightarrow \quad \hat{\pi}_i = \hat{p}_i - i\hbar g A_\mu{}^\alpha(\hat{x})\hat{c}^* T_\alpha \hat{c}\ . \end{aligned} \tag{6.2.12}$$

The operator $\hat{H}$ is an operator constructed from the action of the quantum field theory in Euclidean space. The operators $\hat{\psi}_1^a$ are hermitian, and if one defines that $\hat{c}_M^*$ is the hermitian conjugate of $\hat{c}^M$, then $\hat{H}$ is formally hermitian. Since we are in flat (Euclidean) space, the indices of π_i and π_j are contracted with δ^{ij}, so we are dealing with *linear* sigma models, and there is no difference between curved indices i, j and flat indices a, b. The $\hat{\psi}_1^a$ in (6.2.12) are understood to be expressed in terms of $\hat{\chi}$ and $\hat{\chi}^\dagger$ by (6.2.6). Note that there are no counterterms generated if we rewrite the fermions $\hat{\psi}_1^a$ in $\hat{H}$ into Weyl ordered form because F_{ab} is antisymmetric whereas Weyl ordering produces only an extra term with δ^{ab}. The ghosts $\hat{c}$ and $\hat{c}^*$ should not be rewritten in Weyl ordered form for reasons soon to be explained. We drop again hats.

The anomaly is now given by

$$An = \lim_{\beta\to 0} \mathrm{Tr}'\, e^{-i\pi\chi^\dagger\chi}\, e^{-\frac{\beta}{\hbar}H} \tag{6.2.13}$$

where the prime indicates that we are evaluating the trace only over the one-particle ghost sector (the states $|N\rangle = c^*{}_N|0\rangle$). To write the trace as an unconstrained trace, we introduce the one-particle ghost projection operator P_{gh}. We claim that [40]

$$P_{gh} = :xe^{-x}:\ , \quad x \equiv c^*{}_M c^M \tag{6.2.14}$$

where : : indicates normal ordering with respect to $c^*{}_M$ and c^N. Indeed, on the vacuum $P_{gh}|0\rangle = 0$, while on the one-particle states

$$P_{gh}|N\rangle = :x - x^2 + \ldots :|N\rangle = :x:|N\rangle = |N\rangle \tag{6.2.15}$$

[5] In string theory, the operator $(-)^F$ is part of the so-called GSO projection operator.

Of course $: x^2 :$ vanishes on $|N\rangle$ because it contains two annihilation operators which stand to the right of the two creation operators. On two-particle states

$$|M,N\rangle \equiv c^*{}_M c^*{}_N |0\rangle \tag{6.2.16}$$

P_{gh} vanishes

$$\begin{aligned} P_{gh}|M,N\rangle &= : x - x^2 + \frac{1}{2}x^3 + \ldots : |M,N\rangle \\ &= : x - x^2 : |M,N\rangle \\ &= (2- : x^2 :)|M,N\rangle = 0 \end{aligned} \tag{6.2.17}$$

since $: x^2 : |M,N\rangle = c^*{}_P c^*{}_Q c^Q c^P c^*{}_M c^*{}_N |0\rangle = 2|M,N\rangle$. The reader may check that P_{gh} also vanishes on 3-particle states. (Only $: x - x^2 + \frac{1}{2}x^3 :$ contributes, and yields $3 - 3.2 + \frac{1}{2}3! = 0$). For a proof that P_{gh} vanishes on all ghost states, see section 12 of [40].

The anomaly can thus be written as the following unconstrained trace

$$An = \lim_{\beta\to 0} \mathrm{Tr}\, e^{-i\pi\chi^\dagger_A\chi^A} : c^*{}_M c^M e^{-c^*{}_N c^N} : e^{-\frac{\beta}{\hbar}H} \ . \tag{6.2.18}$$

To write out the trace on a basis of fermionic coherent states we introduce decompositions of unity in the internal (Yang-Mills ghost) space and in the fermionic $(\chi, \chi^\dagger)$ space

$$\begin{aligned} I_{gh} &= \int d\bar{\eta}_{gh} d\eta_{gh}\, |\eta_{gh}\rangle e^{-\bar{\eta}_{gh}\eta_{gh}} \langle\bar{\eta}_{gh}| \\ I_f &= \int d\bar{\eta}_f d\eta_f\, |\eta_f\rangle e^{-\bar{\eta}_f\eta_f} \langle\bar{\eta}_f| \end{aligned} \tag{6.2.19}$$

as we discussed in section (2.4). The coherent states denoted by a subscript gh (for ghosts) are constructed from the operators c^M and $c^*{}_N$, and the coherent states with a subscript f (for fermion) are constructed from the operators χ^A and $\chi^\dagger_A$. The trace over internal ghost states and fermionic states of a bosonic operator $\hat{A}$ is given by, respectively,

$$\begin{aligned} \mathrm{tr}_{gh}\, \hat{A} &= \int d\chi_{gh} d\bar{\chi}_{gh}\, e^{\bar{\chi}_{gh}\chi_{gh}}\, \langle\bar{\chi}_{gh}|\hat{A}|\chi_{gh}\rangle \\ \mathrm{tr}_f\, \hat{A} &= \int d\chi_f d\bar{\chi}_f\, e^{\bar{\chi}_f\chi_f}\, \langle\bar{\chi}_f|\hat{A}|\chi_f\rangle \end{aligned} \tag{6.2.20}$$

as we discussed in the previous section. Whenever we write a multiple integral such as $d\eta$, it is ordered as $d\eta^1..d\eta^n$, while the integrals over barred fermions such as $d\bar{\eta}$ are ordered in the opposite order $d\bar{\eta}^n..d\bar{\eta}^1$. Recall that χ_f consists of χ^A_f with $A = 1, \ldots, n/2$, while χ_{gh} contains χ^M_{gh} with $M = 1, \ldots \dim R$, where $\dim R$ is the dimension of the Yang-Mills representation of the fermions in the original QFT. Hence (omitting the symbol $\lim_{\beta\to 0}$ for the time being)

$$An = \mathrm{tr}_{x_0}\mathrm{tr}_f\,\mathrm{tr}_{gh}\,\langle x_0, \bar\chi_{gh}, \bar\chi_f| e^{-i\pi\chi_f^\dagger\chi_f} P_{gh} I_{gh} I_f e^{-\frac{\beta}{\hbar}H} |\chi_f, \chi_{gh}, x_0\rangle \tag{6.2.21}$$

where $\mathrm{tr}\, x_0 = \int \prod_{i=1}^n dx_0^i$. This trace factorizes into a ghost trace and a fermionic trace.

The ghost part yields

$$\int d\chi_{gh} d\bar\chi_{gh}\, e^{\bar\chi_{gh}\chi_{gh}}\, d\bar\eta_{gh} d\eta_{gh}\, e^{-\bar\eta_{gh}\eta_{gh}}\, \langle\bar\chi_{gh}|P_{gh}|\eta_{gh}\rangle\langle\bar\eta_{gh}|e^{-\frac{\beta}{\hbar}H}|\chi_{gh}\rangle\ . \tag{6.2.22}$$

Since $P_{gh} = :xe^{-x}:$ projects the coherent state $|\eta_{gh}\rangle$ onto its one-particle part, $P_{gh}|\eta_{gh}\rangle = c_M^* \eta_{gh}^M|0\rangle$, the first matrix element is easily computed and yields

$$\langle\bar\chi_{gh}|P_{gh}|\eta_{gh}\rangle = \bar\chi_{gh}\eta_{gh}\ . \tag{6.2.23}$$

Note that in this case we did not use Weyl ordering to evaluate this matrix element. We begin with the integral over $\bar\chi_{gh}$

$$\begin{aligned}&\int d\bar\chi_{gh}\, e^{\bar\chi_{gh}\chi_{gh}}\bar\chi_{gh}\eta_{gh} = \\ &= \int d\bar\chi_{gh,\dim R}\dots d\bar\chi_{gh,1}\Big(\sum_M \bar\chi_{gh,M}\eta_{gh}^M\Big)e^{\bar\chi_{gh}\chi_{gh}}\ .\end{aligned} \tag{6.2.24}$$

For given M, each $\bar\chi_{gh,N}$ integral yields χ_{gh}^N, except the integral which contains $\bar\chi_{gh,M}\eta_{gh}^M$ in the integrand, which yields η_{gh}^M. Hence the $\bar\chi_{gh}$ integrals yield

$$\sum_{M=1}^{\dim R} \chi_{gh}^{\dim R}\dots\chi_{gh}^{M+1}\,\eta_{gh}^M\,\chi_{gh}^{M-1}\dots{\chi_{gh}}^1\ . \tag{6.2.25}$$

Next we perform the η_{gh} integrations

$$\int d\eta_{gh}^1\dots\int d\eta_{gh}^{\dim R} e^{-\bar\eta_{gh}\eta_{gh}}\sum_M\Big(\prod_{N>M}\chi_{gh}^N\Big)\eta_{gh}^M\Big(\prod_{N<M}\chi_{gh}^N\Big)\ . \tag{6.2.26}$$

Again each $d\eta_{gh}$ integral yields a factor $\bar\eta_{gh}$, except in one case where the η_{gh}^M in the integrand yields unity. The result will be called $P_{\bar\eta,\chi}^{gh}$ and reads

$$\begin{aligned}P_{\bar\eta,\chi}^{gh} \;=\; &\sum_{M=1}^{\dim R}\Big(\bar\eta_{gh,1}\chi_{gh}^1\Big)\dots\Big(\bar\eta_{gh,M-1}\,\chi_{gh}^{M-1}\Big)\\ &\qquad\Big(\bar\eta_{gh,M+1}\chi_{gh}^{M+1}\Big)\dots\Big(\bar\eta_{gh,\dim R}\,\chi_{gh}^{\dim R}\Big)\ .\end{aligned} \tag{6.2.27}$$

Clearly this operator deletes in an arbitrary function of $\bar{\eta}_{gh}$ and χ_{gh} all terms with two or more $\bar{\eta}_{gh}$'s and χ_{gh}'s. It is thus a kind of projection operator onto terms which are linear in (or independent of) $\bar{\eta}$ and χ. We denote it by $P^{gh}_{\bar{\eta},\chi}$. We interrupt the discussion of the ghost sector at this point and first perform the trace in the fermionic sector.

In the fermionic sector we must first evaluate the matrix element of γ_5. We find

$$\begin{aligned}\langle\bar{\chi}_f|e^{-i\pi\chi^\dagger_A\chi^A}|\eta_f\rangle &= \langle\bar{\chi}_f|\prod_{A=1}^{n/2}(1-2\chi^\dagger_A\chi^A)|\eta_f\rangle \\ &= e^{\bar{\chi}_f\eta_f}\prod_{A=1}^{n/2}(1-2\bar{\chi}_{f,A}\eta^A_f) = e^{-\bar{\chi}_f\eta_f}\ . \end{aligned} \tag{6.2.28}$$

To obtain this result, we used the definition of coherent states and the identity

$$1-2\bar{\chi}_{f,A}\eta^A_f = e^{-2\bar{\chi}_{f,A}\eta^A_f}\ . \tag{6.2.29}$$

The Grassmann integral over $\bar{\chi}_f$ and η_f can now be done, yielding

$$\begin{aligned}&\int d\chi_f d\bar{\chi}_f\, e^{\bar{\chi}_f\chi_f}\, d\bar{\eta}_f d\eta_f\, e^{-\bar{\eta}_f\eta_f}\, e^{-\bar{\chi}_f\eta_f} \\ &= \int d\chi_f d\bar{\eta}_f\left(\int d\eta_f d\bar{\chi}_f\, e^{\bar{\chi}_f(\chi_f-\eta_f)}\, e^{-\bar{\eta}_f\eta_f}\right) \\ &= \int d\chi_f d\bar{\eta}_f\left(\int d\eta_f\ (\chi^{n/2}_f-\eta^{n/2}_f)\ldots(\chi^1_f-\eta^1_f)e^{-\bar{\eta}_f\eta_f}\right) \\ &= \int d\bar{\eta}_f d\chi_f\, e^{-\bar{\eta}_f\chi_f}\ . \end{aligned} \tag{6.2.30}$$

In the last line we used that $\prod(\chi_f-\eta_f)$ is a fermionic delta function, and replaced $\exp(-\bar{\eta}_f\eta_f)$ by $\exp(-\bar{\eta}_f\chi_f)$. We also canceled a sign factor $(-)^{n/2}$ by interchanging $d\chi_f$ and $d\bar{\eta}_f$. This yields

$$\begin{aligned}An\ =\ &\int\left(\prod_i dx^i_0\right)\left(d\chi_{gh}d\bar{\eta}_{gh}P^{gh}_{\bar{\eta},\chi}\right)\left(d\bar{\eta}_f d\chi_f e^{-\bar{\eta}_f\chi_f}\right) \\ &\langle x_0,\bar{\eta}_{gh},\bar{\eta}_f|e^{-\frac{\beta}{\hbar}H}|\chi_f,\chi_{gh},x_0\rangle\ . \end{aligned} \tag{6.2.31}$$

The regulated trace of γ^5 contains the transition element

$$\langle x_0,\bar{\eta}_{gh},\bar{\eta}_f|e^{-\frac{\beta}{\hbar}H}|\chi_f,\chi_{gh},x_0\rangle = \frac{1}{(2\pi\beta\hbar)^{n/2}}\langle e^{-\frac{1}{\hbar}S^{int}}\rangle \tag{6.2.32}$$

where in the exponent the extra terms $\bar{\chi}_{gh}\psi_{gh}(0)$ and $\bar{\chi}_f\psi_f(0)$ are present. The action S is obtained by inserting complete sets of states. For the fermions we use coherent states depending on $\psi^A_{f,k}$, $\bar\psi_{f,k,A}$ and $\psi^M_{gh,k}$, $\bar\psi_{gh,k,M}$. Then the operators $\hat\psi_1$ and $\hat c^*$ and $\hat c$ are replaced by $\psi_1 \equiv (\psi_f+\bar\psi_f)/\sqrt{2}$ and ψ_{gh} and $\bar\psi_{gh}$. The momenta are integrated out from

$$\int_{-1}^{0}\left[\frac{i}{\hbar}p_j\dot x^j-\frac{\beta}{\hbar}\left\{\frac{1}{2}(p_j-i\hbar gA_j{}^\alpha\bar\psi_{gh}T_\alpha\psi_{gh})^2\right.\right.$$
$$\left.\left.-\frac{\hbar^2}{2}\psi_1^a\psi_1^b gF_{ab}{}^\alpha\bar\psi_{gh}T_\alpha\psi_{gh}\right\}\right]d\tau\ . \qquad (6.2.33)$$

Then the terms quadratic in A^α_j cancel and only the familiar $\dot x^j A^\alpha_j$ interaction is left. In the path integral the integration variables are $\psi^A_f, \bar\psi_{f,A}$, $\psi^M_{gh}, \bar\psi_{gh,M}$ and q^i. We make a decomposition into background fields and quantum fields as follows

$$\begin{aligned}\psi^A_f &= \chi^A_f+\psi^A\ , \qquad \bar\psi_{f,A}=\bar\eta_{f,A}+\bar\psi_A\\ \psi^M_{gh} &= \chi^M_{gh}+c^M\ , \qquad \bar\psi_{gh,M}=\bar\eta_{gh,M}+\bar c_M\end{aligned} \qquad (6.2.34)$$

and find then along the same lines as in (6.1.16)[6]

$$\begin{aligned}-\frac{1}{\hbar}S \ = \ & -\frac{1}{\beta\hbar}\int_{-1}^{0}\frac{1}{2}\dot q^i\dot q^i\,d\tau+(\bar\eta_{gh}\chi_{gh}+\bar\eta_f\chi_f)\\ & -\int_{-1}^{0}\left(\bar\psi_A\dot\psi^A+\bar c_M\dot c^M\right)d\tau\\ & -\int_{-1}^{0}\dot q^j gA_j{}^\alpha(x_0+q)(T_\alpha)\,d\tau\\ & +\beta\hbar\int_{-1}^{0}\frac{1}{2}\psi_1^a\psi_1^b gF_{ab}{}^\alpha(x_0+q)(T_\alpha)\,d\tau\end{aligned} \qquad (6.2.35)$$

where

$$(T_\alpha)\equiv\bar\psi_{gh,M}(T_\alpha)^M{}_N\psi^N_{gh}=(\bar\eta_{gh,M}+\bar c_M)(T_\alpha)^M{}_N(\chi^N_{gh}+c^N) \qquad (6.2.36)$$

and ψ^a_1 is expressed in terms of $\psi^A_f, \bar\psi_{fA}$ as in (6.2.6), which themselves are further decomposed as in (6.2.34). Thus $q^i, \psi^A, \bar\psi_A, c^M$ and $\bar c_M$ are the quantum variables, and x_0, χ^A_f, $\bar\eta_{f,A}$, χ^M_{gh}, $\bar\eta_{gh,M}$ the background variables.

The coupling $-\dot q A^\alpha(T_\alpha)$ came from integrating out the momenta, and the $\bar c c$ terms in (T_α) combine with the kinetic term $\bar c_M\dot c^M$ to a covariant derivative

$$D_\tau c^M=\dot c^M+\dot q^j gA_j{}^\alpha(x_0+q)(T_\alpha)^M{}_N c^N \qquad (6.2.37)$$

---

[6] Recall that except in the vertices, all terms linear in quantum fields cancel. In particular $\exp[\bar\eta\psi_f(0)-\int\bar\psi_f\dot\psi_f\,d\tau]$ becomes equal to $\exp[\bar\eta\chi-\int\bar\psi\dot\psi\,d\tau]$.

but note that there are also background fields in the interaction term.

The anomaly now reduces to

$$
\begin{aligned}
An \;=\; & \frac{1}{(2\pi\beta\hbar)^{n/2}} \int dx_0 d\chi_{gh} d\bar{\eta}_{gh} P^{gh}_{\bar{\eta},\chi}\, e^{\bar{\eta}_{gh}\chi_{gh}}\, d\bar{\eta}_f d\chi_f \\
& \Big\langle \exp\Big(-\int_{-1}^{0} \dot{q}^j g A_j{}^{\alpha}(x_0+q)(\bar{\eta}_{gh}+\bar{c})T_{\alpha}(\chi_{gh}+c)\, d\tau\Big) \\
& \exp\Big(\beta\hbar \int_{-1}^{0} \frac{1}{2}\psi_1^a\psi_1^b g F_{ab}{}^{\alpha}(\bar{\eta}_{gh}+\bar{c})T_{\alpha}(\chi_{gh}+c)\, d\tau\Big)\Big\rangle \qquad (6.2.38)
\end{aligned}
$$

The term $\exp(\bar{\eta}_{gh}\,\chi_{gh})$ in the ghost sector of this expression is the "extra term" in the action which remains after substitution of (6.2.34), but the corresponding term $\exp(\bar{\eta}_f\chi_f)$ in the fermionic sector has canceled with the factor $\exp(-\bar{\eta}_f\chi_f)$ in (6.2.31). *We rescale the fermionic variables $\bar{\eta}_{f,A}$, χ_f^A, ψ^A and $\bar{\psi}_A$, but not the ghost variables*, by a factor $(\beta\hbar)^{-1/2}$ (so one sets $\chi_f^A = \chi'^A_f/\sqrt{\hbar\beta}$ and then drops the prime). Then the measure becomes $\beta\hbar$ independent

$$
\begin{aligned}
An \;=\; & \frac{1}{(2\pi)^{n/2}} \int dx_0 d\bar{\eta}_f d\chi_f \int d\chi_{gh} d\bar{\eta}_{gh} P^{gh}_{\bar{\eta},\chi} e^{\bar{\eta}_{gh}\chi_{gh}} \\
& \Big\langle \exp\Big(-\int_{-1}^{0} \dot{q}^j g A_j{}^{\alpha}(x_0+q)(\bar{\eta}_{gh}+\bar{c})T_{\alpha}(\chi_{gh}+c)\, d\tau\Big) \\
& \exp\Big(\frac{1}{2}\int_{-1}^{0} \psi_1^a\psi_1^b g F_{ab}{}^{\alpha}(x_0+q)(\bar{\eta}_{gh}+\bar{c})T_{\alpha}(\chi_{gh}+c)\, d\tau\Big)\Big\rangle\ . \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (6.2.39)
\end{aligned}
$$

After this rescaling the fermion and boson propagators $\langle\psi^A\bar{\psi}_B\rangle$ and $\langle q^i q^j\rangle$ are proportional to $\beta\hbar$, but the ghost propagators $\langle c^M\bar{c}_N\rangle$ are $\beta\hbar$ independent[7]. All vertices are $\hbar\beta$ independent. It follows that *in the limit of vanishing β, only graphs with ghost propagators or without any propagators contribute.* Hence we may set the quantum fields q, ψ^A and $\bar{\psi}_A$ equal to zero, and replace ψ_1^a and ψ_1^b by their background values, which we denote by ψ_{bg}^a and ψ_{bg}^b. Then only the vertex with $\frac{1}{2}\psi^a_{bg}\psi^b_{bg} g F^{\alpha}_{ab}(x_0)(\bar{\eta}_{gh}+\bar{c})T_{\alpha}(\chi_{gh}+c)$ will contribute, but the vertex with with $\dot{q}^i g A_i^{\alpha}(x_0)(\bar{\eta}_{gh}+\bar{c})T_{\alpha}(\chi_{gh}+c)$ does not contribute since we set $q=0$.

The propagator due to $\int_{-1}^{0}\bar{c}_M\dot{c}^M d\tau$ is the same as for $\int_{-1}^{0}\bar{\psi}_{qu,A}\dot{\psi}^A_{qu}d\tau$

[7] More precisely, having introduced external sources K and $\bar{K}$ which couple to the fermionic quantum integration variables $\bar{\psi}$ and ψ, completed squares, and integrated out the fermionic quantum variables, S^{int} depends on $\chi + \frac{\delta}{\delta K}$ and $\eta - \frac{\delta}{\delta \bar{K}}$. By rescaling $\frac{\delta}{\delta K}$ and $\frac{\delta}{\delta \bar{K}}$ the same way as $\bar{\eta}$ and χ, we must also rescale K and $\bar{K}$ in the source term $\bar{K}AK$. This produces the $\beta\hbar$ dependence of the $\psi\bar{\psi}$ propagator, and the $\beta\hbar$ independence of the vertices.

given by (2.3.36), namely

$$\langle c^M(\sigma)\bar{c}_N(\tau)\rangle = \delta^M{}_N \theta(\sigma-\tau) \; . \tag{6.2.40}$$

It follows that closed ghost loops do not contribute (a closed loop always moves somewhere backwards in time.) *Only tree graphs with ghosts can contribute*, with at one end a field $\bar{\eta}_{gh}$ and at the other end a field χ_{gh}. In fact, only terms with precisely one $\bar{\eta}_{gh}$ and one χ_{gh} contribute due to the operator $P^{gh}_{\bar{\eta},\chi}$.

The ghost tree graphs are obtained by expanding

$$\exp(\frac{1}{2}\psi^a_{bg}\psi^b_{bg} g F_{ab}{}^{\alpha}(x_0)\int_{-1}^{0}(\bar{\eta}_{gh}+\bar{c})T_\alpha(\chi_{gh}+c)d\tau \tag{6.2.41}$$

and contracting the vertices with ghost propagators. If one has k vertices one obtains

$$\frac{1}{k!}k!\left(\frac{1}{2}\right)^k \int_{-1}^{0}\ldots\int_{-1}^{0} d\sigma_1\ldots d\sigma_k\, \bar{\eta}_{gh} F \underbrace{c\bar{c}} F \ldots \underbrace{c\bar{c}} F \chi_{gh} \tag{6.2.42}$$

where $F \equiv \psi^a_{bg}\psi^b_{bg} g F_{ab}{}^{\alpha}(x_0) T_\alpha$ is a matrix in the internal symmetry space, and where the factor $k!$ in the numerator is due to the fact that one can order the k vertices into a tree in precisely $k!$ ways. The integral over the σ_k yields

$$\int_{-1}^{0} d\sigma_1 \ldots \int_{-1}^{0} d\sigma_k\, \theta(\sigma_1-\sigma_2)\ldots\theta(\sigma_{k-1}-\sigma_k) = \frac{1}{k!} \; . \tag{6.2.43}$$

Hence, the ghost trees yield

$$\bar{\eta}_{gh}\exp\Big(\frac{1}{2}F\Big)\chi_{gh} \; . \tag{6.2.44}$$

The term without any F vertices is provided by expanding the factor $\exp(\bar{\eta}_{gh}\chi_{gh})$ in (6.2.39). Due to the $P^{gh}_{\bar{\eta},\chi}$, only terms with precisely one $\bar{\eta}_{gh,M}$ and χ^M_{gh} with the same M will contribute, and they yield a trace in the space of the representation of the fermions

$$\int d\chi_{gh} d\bar{\eta}_{gh}\, (\sum_M \prod_{N\neq M} \bar{\eta}_{gh,N}\chi^N_{gh})\, \bar{\eta}_{gh}\, \mathrm{e}^{\frac{1}{2}F}\chi_{gh} = \mathrm{Tr}\, \mathrm{e}^{\frac{1}{2}F} \; . \tag{6.2.45}$$

All ghost variables are now gone.

The anomaly becomes

$$An = \frac{1}{(2\pi)^{n/2}}\int dx_0 d\bar{\eta}_f d\chi_f \,\mathrm{Tr}\exp\left(\frac{1}{2}\psi^a_{bg}\psi^b_{bg} g F_{ab}{}^{\alpha}(x_0)T_\alpha\right) \tag{6.2.46}$$

where ψ^a_{bg} is expressed in term of the constant background fields χ^A_f and $\bar\eta_{f,A}$ (introduced in (6.2.34)) as in (6.2.6). Finally we transform integration variables from $d\bar\eta_{f,A}d\chi^A_f$ to $d\psi^a_{bg}$. We find[8]

$$d\bar\eta_{f,n/2}\dots d\bar\eta_{f,1}d\chi^1_f\dots\chi^{n/2}_f = (-i)^{n/2}d\psi^1_{bg}\dots d\psi^n_{bg}\ . \tag{6.2.47}$$

The final result for the abelian chiral anomaly for complex spin 1/2 fields coupled to external Yang-Mills gauge fields in $n=2k$ dimensions is given by

$$\begin{aligned} An &= (-2i\alpha)\frac{(-i)^{n/2}}{(2\pi)^{n/2}}\int dx_0 d\psi^1_{bg}\dots d\psi^n_{bg}\,\mathrm{Tr}\exp\left(\frac{1}{2}\psi^a_{bg}\psi^b_{bg}gF_{ab}\right) \\ &= \frac{(-2i\alpha)}{(2\pi)^{n/2}}\left(\frac{ig}{2}\right)^{n/2}\epsilon^{a_1\dots a_n}\int \mathrm{Tr}(F_{a_1a_2}\dots F_{a_{n-1}a_n})dx_0 \end{aligned} \tag{6.2.48}$$

where we reinstated the factor $(-2i\alpha)$ mentioned before (6.2.3). A factor $(-)^{n/2}$ has been canceled by another factor $(-)^{n/2}$ arising from the formula

$$\int d\psi^1_{bg}\dots d\psi^n_{bg}\,\psi^{a_1}_{bg}\dots\psi^{a_n}_{bg} = (-)^{n/2}\epsilon^{a_1\cdots a_n}\ . \tag{6.2.49}$$

Thus we obtain the familiar result that the divergence of the axial-vector current is proportional to $\epsilon F..F$, but the great advantage of this expression is that it yields the result for all n in a simple compact formula. The anomaly is proportional to the totally symmetrized trace of the generators of the gauge group in the representation of the spin 1/2 fields, a d-symbol, and if for a particular group the d-symbol vanishes, there is no corresponding anomaly. Again the anomaly is imaginary. For example, in 2 dimensions, there is only a $U(1)$ anomaly because $\mathrm{tr}\,T_\alpha=0$ for the representations of simple Lie algebras, and the factor $T_\alpha=i$ of the $U(1)$ group cancels the factor $(-i)^{n/2}$, leaving only the i in $-2i\alpha$.

On the other hand, when the symmetrized trace over a product of generators in a particular representation of the gauge group is nonvanishing, there is an abelian chiral Yang-Mills anomaly in that representation . For certain groups and representations the trace over an odd number of generators vanishes (for example for real representation of $SO(N)$) and in these cases there are only anomalies possible in $4k$ dimensions.

We obtained the Hamiltonian H in (6.2.12) from the regulator $\mathcal{R}=\not{D}\not{D}$ of the QFT by replacing the operators of the QFT by corresponding

[8] As a check note that for $n=2$ one finds

$$d\psi^1_{bg}d\psi^2_{bg} = d((\chi+\bar\eta)/\sqrt{2})d((-i)(\chi-\bar\eta)/\sqrt{2}) = 2id(\chi+\bar\eta)d(\chi-\bar\eta)\ .$$

Furthermore $d(\chi+\bar\eta)d(\chi-\bar\eta)=\frac{1}{2}d\bar\eta d\chi$ (and not $2d\bar\eta d\chi$).

operators in the QM model. When we followed these steps for the case that $D_i = \partial_i + \frac{1}{4}\omega_i{}^{mn}\gamma_m\gamma_n$ we found the N=1 susy model. This suggest that also for $D_i = \partial_i + gA_i^\alpha T_\alpha$ there is a corresponding susy model. There is indeed such a model and this model is discussed in appendix E. The interaction term in (6.2.35) with the Yang–Mills curvature is needed to supersymmetrize the interaction with $\dot{x}A$. It is interesting to note that once again an ordinary non-supersymmetric quantum gauge field theory has produced a supersymmetric QM model that yields its anomalies.

6.3 Lorentz anomalies for chiral spin 1/2 fields coupled to gravity in $4k+2$ dimensions

Another important anomaly concerns the violation of the conservation of the stress tensor at the quantum level. Actually, there are two local symmetries which can be violated at the quantum level: Einstein (general coordinate) invariance and local Lorentz symmetry. The anomalies in Einstein and local Lorentz symmetry can be moved from one to the other, just like anomalies in the vector or axial-vector gauge invariance [2]. In principle, one should consider at the quantum level the Noether current for the rigid BRST symmetry which gets contributions from all local symmetries in the classical action: Einstein symmetry, local Lorentz symmetry (and local supersymmetry if spin 3/2 fields are present). We consider, however, external gravitational fields, and then the quantum actions still have classical Einstein and local Lorentz symmetries. Even when we consider spin 3/2 fields and add a gauge fixing term for the local supersymmetry (which is needed to be able to construct propagators for the spin 3/2 fields), we still preserve Einstein and local Lorentz gauge symmetry in the quantum action if the gauge fixing term for local supersymmetry preserves these spacetime symmetries.

The gravitational anomalies we shall obtain are covariant anomalies: they depend only on Riemann curvatures and do not contain terms with bare ω_{iab}. We achieve this by using regulators which are both Einstein and locally Lorentz invariant, and which are vector-like (treat left-handed and right-handed spinors the same way), but in the Jacobian an extra factor γ_5 appears to take into account that we compute the anomalies for chiral complex Dirac fermions. (Chiral fermions satisfy $\tilde{\lambda} = \frac{1}{2}(1+\gamma_5)\tilde{\lambda}$, but the term with $\frac{1}{2}\tilde{\lambda}$ does not contribute, leaving only the term with $\frac{1}{2}\gamma_5\tilde{\lambda}$). As a result, these anomalies will not satisfy the consistency conditions which are present if the anomaly is the response of the effective action under a gauge transformation. One could construct a consistent regulator to be sure that the anomalies satisfy the consistency conditions. However, proceeding this way is extremely tedious, because the regulator is not manifestly Lorentz invariant. It is much simpler to use instead regulators which are also Lorentz invariant, since then the anomaly will be also Lorentz covariant. This can be done, but these regulators are not "consistent" and the anomaly will not satisfy the consistency conditions. This is not a problem because there is a well-defined procedure to obtain the consistent anomaly from the covariant one. As to the Jacobian, we shall consider a particular combination of general coordinate and local Lorentz transformations which leads to covariant transformation laws, and then we shall prove that if one uses also our covariant regulator, one obtains in this way (twice) the local Lorentz anomaly.

We shall now first define the covariant transformation law. Then we shall determine the regulator by requiring that a certain identity involving the Jacobian and the regulator holds. These are issues which are not explicitly discussed in [1] and which have confused us for a very long time, but through the work of Endo [132, 109] we finally clarified these issues.

We begin with the concept of a covariant Einstein transformation, denoted by δ_{cov}. This is a combination of an ordinary Einstein (= general coordinate) transformation with the usual parameter ξ^μ and a local Lorentz transformation with composite parameter $\lambda_{mn} = \xi^\mu \omega_{\mu mn}$

$$\delta_{cov}(\xi) = \delta_E(\xi) + \delta_{lL}(\xi^\mu \omega_{\mu mn}) \tag{6.3.1}$$

where the ordinary Einstein and local Lorentz transformation on the vielbein are given as usual by

$$\begin{aligned} \delta_E(\xi) e_\mu{}^m &= \xi^\nu \partial_\nu e_\mu{}^m + (\partial_\mu \xi^\nu) e_\nu{}^m \\ \delta_{lL}(\lambda_{mn}) e_\mu{}^m &= \lambda^m{}_n e_\mu{}^n \ . \end{aligned} \tag{6.3.2}$$

The covariant Einstein transformation on the vielbein is then

$$\begin{aligned} \delta_{cov}(\xi) e_\mu{}^m &= \xi^\nu \partial_\nu e_\mu{}^m + (\partial_\mu \xi^\nu) e_\nu{}^m + \xi^\nu \omega_\nu{}^m{}_n e_\mu{}^n \\ &= \xi^\nu D_\nu(\omega) e_\mu{}^m + (\partial_\mu \xi^\nu) e_\nu{}^m \\ &= \xi^\nu (D_\nu(\omega) e_\mu{}^m - D_\mu(\omega) e_\nu{}^m) + \xi^\nu D_\mu(\omega) e_\nu{}^m + (\partial_\mu \xi^\nu) e_\nu{}^m \\ &= D_\mu(\omega)(\xi^\nu e_\nu{}^m) = D_\mu(\omega) \xi^m \ . \end{aligned} \tag{6.3.3}$$

We used the vielbein postulate $D_\nu(\omega) e_\mu{}^m - D_\mu(\omega) e_\nu{}^m = 0$. The notation $D_\mu(\omega)$ indicates that this derivative contains spin connections but no Christoffel symbols, and $\xi^m = \xi^\nu e_\nu{}^m$.

Another symmetry which plays a role in the computation of the anomalies is a symmetrized version of δ_{cov}. This symmetrized covariant Einstein transformation of the vielbein field is defined by

$$\delta_{sym}(\xi) e_\mu{}^m = \frac{1}{2}\Big(D_\mu(\omega) \xi^m + D^m(\Gamma) \xi_\mu \Big) \ . \tag{6.3.4}$$

It is a combination of a covariant Einstein transformation and a local Lorentz transformation with parameter $\lambda_{mn} = \frac{1}{2}(D_m\xi_n - D_n\xi_m) \equiv D_{[m}\xi_{n]}$. Namely,

$$\begin{aligned}
\delta_{sym}(\xi)e_\mu{}^m &= D_\mu\xi^m - \frac{1}{2}(D_\mu\xi^m - D^m\xi_\mu) \\
&= \delta_{cov}(\xi)e_\mu{}^m + \frac{1}{2}(D^m\xi_n e_\mu^n - e_\mu^n D_n\xi^m) \\
&= \delta_{cov}(\xi)e_\mu{}^m + \delta_{lL}(D_{[m}\xi_{n]})e_\mu{}^m \ . \qquad (6.3.5)
\end{aligned}$$

Its physical meaning is clear: if one begins with symmetric vielbein fields, then the particular combination of Einstein and local Lorentz transformations which constitutes δ_{sym} preserves the symmetry of the vielbein fields.

The anomaly due to δ_{cov} is the response of the effective action Γ under a covariant Einstein transformation. Using the chain rule we find

$$\begin{aligned}
An_{cov}(\xi) &= \delta_{cov}(\xi)\Gamma[e_\mu{}^m] = \int dx\,(\delta_{cov}(\xi)e_\mu{}^m(x))\frac{\delta\Gamma}{\delta e_\mu{}^m(x))} \\
&= \int dx\, e\, D_\mu\xi^m\, T_m{}^\mu = -\int dx\, e\,\xi^m(D_\mu T_m{}^\mu) \qquad (6.3.6)
\end{aligned}$$

where we defined the stress tensor $T_m{}^\mu$ by

$$T_m{}^\mu = \frac{1}{e}\frac{\delta\Gamma}{\delta e_\mu{}^m(x)} \ . \qquad (6.3.7)$$

For $\mathcal{L} = -\frac{1}{2}\sqrt{g}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$ this definition yields the usual normalization $T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi + \cdots$. Hence the covariant divergence of the stress tensor $D_\mu T_m{}^\mu$ is the covariant Einstein anomaly.

The local Lorentz anomaly is the response of the effective action under a local Lorentz transformation

$$\begin{aligned}
An_{lL}(\lambda_{mn}) &= \delta_{lL}(\lambda_{mn})\Gamma[e_\mu{}^m] = \int dx\,\lambda^m{}_n e_\mu{}^n(x)\frac{\delta\Gamma}{\delta e_\mu{}^m(x)} \\
&= \int dx\, e\,\lambda^m{}_n e_\mu{}^n T_m{}^\mu = \int dx\, e\,\lambda_{mn}T^{mn} \\
&= \int dx\, e\,\lambda_{mn}T_A^{mn} \qquad (6.3.8)
\end{aligned}$$

where $T_A^{mn} = \frac{1}{2}(T^{mn} - T^{nm})$ indicates the antisymmetric part of T^{mn}. Hence the antisymmetric part of the stress tensor of the effective action is the local Lorentz anomaly.

We can also define an anomaly due to a symmetric Einstein transformation

$$An_{sym}(\xi) = \delta_{sym}(\xi)\Gamma[e_\mu{}^m] = \int dx\,(\delta_{sym}(\xi)e_\mu{}^m(x))\frac{\delta\Gamma}{\delta e_\mu{}^m(x))}$$

$$= \int dx\, e\, \frac{1}{2}\Big(D_\mu(\omega)\xi^m + D^m(\Gamma)\xi_\mu\Big)T_m{}^\mu$$
$$= -\int dx\, e\, \xi_\nu(D_\mu T^{\mu\nu}) \qquad (6.3.9)$$

where $T_S^{\mu\nu} = \frac{1}{2}(T^{\mu\nu} + T^{\nu\mu})$.

By applying (6.3.5) to the effective action we obtain the following relation between the anomalies

$$An_{cov}(\xi) = An_{sym}(\xi) - An_{lL}(D_{[m}\xi_{n]}) \ . \qquad (6.3.10)$$

We can restate the results we have obtained for anomalies as two theorems.
Theorem I: the stress tensor is covariantly conserved if and only if the effective action is invariant under a covariant Einstein transformation (see (6.3.6).
Theorem II: the stress tensor is symmetric if and only if the effective action is locally Lorentz invariant (see (6.3.8).
When there are matter fields φ on which Γ also depends, so $\Gamma = \Gamma[e_\mu^m, \varphi]$, these two theorems remain true "on-shell", namely when the matter field equations $\frac{\partial\Gamma}{\partial\varphi} = 0$ are satisfied. Again, covariant Einstein and local Lorentz anomalies break these symmetries. The anomaly (nonconservation) of $T_{(S)}^{\mu\nu} = \frac{1}{2}(T^{\mu\nu} + T^{\nu\mu})$ is sometimes called the "general coordinate anomaly" [108], but note that is really a combination of a covariant Einstein anomaly (which itself is a combination of an ordinary Einstein anomaly and a local Lorentz anomaly) and the local Lorentz anomaly itself.

In the path integral formalism, the transformation of the fields in the measure yields the Jacobian, which in turn yields the anomaly. So we now study how spin 1/2 field transforms under these symmetries. For a spinor half-density $\tilde{\lambda} = g^{\frac{1}{4}}\lambda$, the covariant translation is given by

$$\delta_{cov}\tilde{\lambda} = \xi^\mu\partial_\mu\tilde{\lambda} + \frac{1}{2}(\partial_\mu\xi^\mu.)\tilde{\lambda} + \frac{1}{4}(\xi^\mu\omega_\mu{}^{mn})\gamma_{mn}\tilde{\lambda} \ . \qquad (6.3.11)$$

The dot in $(\partial_\mu\xi^\mu.)$ indicates that the derivative ∂_μ does not act to the right of the dot. For later purposes we write the expression for δ_{cov} in terms of the following derivative

$$\tilde{D}_\mu(\omega) \equiv g^{\frac{1}{4}} D_\mu(\omega) g^{-\frac{1}{4}} \ . \qquad (6.3.12)$$

It is straightforward to verify that

$$\delta_{cov}\tilde{\lambda} \ = \ \frac{1}{2}(\xi^\mu D_\mu(\omega) + D_\mu(\omega)\xi^\mu)\tilde{\lambda}$$

$$
\begin{aligned}
&= \frac{1}{2}(\xi^\mu D_\mu(\omega,\Gamma) + D_\mu(\omega,\Gamma,\Gamma)\xi^\mu)\tilde{\lambda} \\
&= \frac{1}{2}(\xi^\mu \tilde{D}_\mu(\omega) + \tilde{D}_\mu(\omega,\Gamma)\xi^\mu)\tilde{\lambda}\ . \qquad (6.3.13)
\end{aligned}
$$

In the second line the notation $D_\mu(\omega,\Gamma)$ indicates that this derivative contains one Christoffel symbol (for the density character of $\tilde{\lambda}$), and $D_\mu(\omega,\Gamma,\Gamma)$ contains two Christoffel symbols (one for the density character of $\tilde{\lambda}$ and another for the index μ on ξ^μ). These Γ terms cancel as one easily verifies using $\frac{1}{2}\partial_\mu \ln g = \Gamma^\nu_{\mu\nu}$ and $D_\mu(\omega,\Gamma)\tilde{\lambda} = D_\mu(\omega)\tilde{\lambda} - \frac{1}{2}\Gamma^\nu_{\mu\nu}\tilde{\lambda}$. In the third line the Γ in $\tilde{D}_\mu(\omega,\Gamma)$ acts on ξ^μ and the tilde on $\tilde{D}_\mu$ takes care of the density character of $\tilde{\lambda}$. This proves that the third line is equal to the second line.

We shall use covariant Einstein transformations to compute anomalies for the following reason. The first line in (6.3.13) contains the same operator $D_\mu(\omega)$ in the first term and in the second term. The Jacobian is thus Weyl-ordered, and it may be replaced by a function in the path integral formalism according to Berezin's theorem. However, we use the third line in (6.3.13) to derive a property of the regulator we are going to use.

Consider the regulator

$$
\mathcal{R} = \tilde{\not{D}}\tilde{\not{D}}/M^2\ . \qquad (6.3.14)
$$

The operator $\tilde{\not{D}}$ is the field operator for $\tilde{\lambda}$. This regulator satisfies a crucial identity

Identity : $$\mathrm{Tr}\,\gamma_5\Big(\not{\xi}\tilde{\not{D}}(\omega) + \tilde{\not{D}}(\omega)\not{\xi}\Big)\, e^{\mathcal{R}} = 0\ . \qquad (6.3.15)$$

We stress that all operators $\tilde{D}_\mu(\omega)$ in this expression are the same, and given by

$$
\tilde{\not{D}} = g^{\frac{1}{4}}\gamma^\mu(\partial_\mu + \frac{1}{4}\omega_\mu{}^{mn}\gamma_{mn})g^{-\frac{1}{4}}\ . \qquad (6.3.16)
$$

The proof of (6.3.15) is trivial: use in the second term $\gamma_5\tilde{\not{D}}(\omega) = -\tilde{\not{D}}(\omega)\gamma_5$, then use cyclicity to move $\tilde{\not{D}}(\omega)$ to the right, and finally pull $\tilde{\not{D}}(\omega)$ past $e^{\mathcal{R}}$ (which is possible because $\mathcal{R}$ only depends on $\tilde{\not{D}}(\omega)$). One obtains then minus the first term.

Next we expand the identity in (6.3.15) as follows

$$
\begin{aligned}
0 &= \mathrm{Tr}\,\gamma_5\Big(\not{\xi}\tilde{\not{D}}(\omega) + \tilde{\not{D}}(\omega)\not{\xi}\Big)\, e^{\mathcal{R}} \\
&= \mathrm{Tr}\,\gamma_5\gamma^\mu\gamma^\nu\Big(\xi_\mu\tilde{D}_\nu(\omega) + \tilde{D}_\mu(\omega,\Gamma)\xi_\nu)\Big)\, e^{\mathcal{R}} \\
&= \mathrm{Tr}\,\gamma_5\Big[\Big(\xi^\mu\tilde{D}_\mu(\omega) + \tilde{D}_\mu(\omega,\Gamma)\xi^\mu\Big) + \gamma^{\mu\nu}\Big(\xi_\mu\tilde{D}_\nu(\omega) - \tilde{D}_\nu(\omega,\Gamma)\xi_\mu\Big)\Big]\, e^{\mathcal{R}} \\
&= \mathrm{Tr}\,\gamma_5\Big[\Big(\xi^\mu\tilde{D}_\mu(\omega) + \tilde{D}_\mu(\omega,\Gamma)\xi^\mu\Big) + \gamma^{\mu\nu}\Big(D_\mu(\Gamma)\xi_\nu.\Big)\Big]\, e^{\mathcal{R}}\ . \qquad (6.3.17)
\end{aligned}
$$

Again the dot in $(D_\mu(\Gamma)\xi_\nu.)$ indicates that $(D_\mu(\Gamma)\xi_\nu)$ is a function and that $D_\mu(\Gamma)$ does not act beyond ξ_ν. The Γ in $\tilde{D}_\mu(\Gamma)$ acts on the index of ξ_ν. The first term is just twice the Jacobian for $\delta_{cov}\tilde{\lambda}$ (see the third line in (6.3.13)), while the second term is four times the Jacobian for a local Lorentz transformation with parameter $D_{[m}\xi_{n]}$. Hence we have found another relation between the gravitational anomalies of spin 1/2 fields

$$An^E_{cov}(\xi) + 2An^{lL}(D_{[m}\xi_{n]}) = 0 \; . \tag{6.3.18}$$

However, **this relation only holds if one uses $\tilde{\not{D}}\tilde{\not{D}}$ as regulator for the spin 1/2 field**.

Finally, we return to our original question: what anomaly are we going to calculate, and what regulator must we use to compute the covariant anomaly the easiest way? The answer follows from (6.3.18). The regulator is $\mathcal{R} = \tilde{\not{D}}\tilde{\not{D}}/M^2$, and we compute the covariant Einstein anomaly. For this particular regulator the covariant Einstein anomaly is equal to -2 times the local Lorentz anomaly.

We shall now construct the quantum mechanical model for this anomaly. On the basis with inner product $\langle\tilde{\lambda}_1, \tilde{\lambda}_2\rangle = \int \tilde{\tilde{\lambda}}_1\tilde{\lambda}_2\, d^n x$ the operator $\frac{\hbar}{i}\frac{\partial}{\partial x^\mu}$ is hermitian. Hence in the corresponding QM model $\frac{\hbar}{i}\frac{\partial}{\partial x^\mu}$ is replaced by p_i. The Jacobian $J = \frac{1}{2}(\xi^\mu D_\mu(\omega) + D_\mu(\omega)\xi^\mu)$ from the first line of (6.3.13) corresponds then to $\frac{i}{2\hbar}(\xi^i(x)\pi_i + \pi_i\xi^i(x))$. The regulator $\mathcal{R} = \tilde{\not{D}}\tilde{\not{D}} = g^{-1/4}D_\mu(\omega)\sqrt{g}g^{\mu\nu}D_\nu(\omega)g^{-1/4} + \frac{1}{4}R$ becomes then the Hamiltonian derived before.

$$\begin{aligned}
An(\text{grav}) \;&\equiv\; An_{cov} = -2\,\text{Tr}\,\frac{1}{2}(D_\mu\xi^\mu + \xi^\mu D_\mu)\left(\frac{1+\gamma_5}{2}\right)e^{-\frac{\beta}{\hbar}\tilde{\mathcal{R}}} \\
&=\; -\frac{2i}{\hbar}\,\text{Tr}\,\frac{1}{2}(\pi_i\xi^i + \xi^i\pi_i)\left(\frac{1+\gamma_5}{2}\right)e^{-\frac{\beta}{\hbar}H} \\
\pi_i \;&=\; p_i - \frac{i\hbar}{2}\omega_{iab}\psi_1^a\psi_1^b \\
H \;&=\; \frac{1}{2}g^{-1/4}\pi_i\sqrt{g}g^{ij}\pi_j g^{-1/4} - \frac{\hbar^2}{8}R \; .
\end{aligned} \tag{6.3.19}$$

The factor -2 in front takes into account that $\tilde{\lambda}$ and $\tilde{\tilde{\lambda}}$ are independent fields in Euclidean space. The operator $\tilde{\mathcal{R}}$ should act in the space of chiral spinors, and in the space of nonchiral spinors one needs the projection operator $\frac{1}{2}(1+\gamma^5)$ to project on the chiral subspace. The term with $-\frac{\hbar^2}{8}R$ is due to expanding $-\not{D}\not{D}/2$. Note that the operator $\tilde{\not{D}}$ does not map the space of chiral spinors into itself, rather it maps chiral spinor into antichiral spinors and vice-versa. However $\mathcal{R}$ maps chiral spinors into chiral spinors and antichiral spinors into antichiral spinors. Note that $\pi_i\xi^i + \xi^i\pi_i$ is already Weyl ordered. Thus, in the path integral we will obtain simply the function $\pi_i\xi^i + \xi^i\pi_i = 2\pi_i\xi^i$. This is one reason why we chose $\tilde{\lambda}$ as basic field variable. Another reason is that the Jacobian for Einstein transformations becomes a total derivative on a basis with $\tilde{\lambda}$, see chapter 5. Only the term with γ_5 will contribute (for nonchiral spinors there is no gravitational anomaly). So we must evaluate

$$An\ (\text{grav}) = -\frac{i}{2\hbar}\mathrm{Tr}\,\gamma_5(\pi_i\xi^i + \xi^i\pi_i)e^{-\frac{\beta}{\hbar}H}\ . \tag{6.3.20}$$

We would like to bring the operator $\pi_i\xi^i + \xi^i\pi_i$ into the exponent, as a term which is added to H. At the end we then expand in terms of ξ^i and take the term linear in ξ^i. To achieve this we decompose the operator $\frac{-i}{2\hbar}(\pi_i\xi^i + \xi^i\pi_i)$ into N times $\mathcal{O} = \frac{-i\epsilon}{2\hbar\beta}(\pi_i\xi^i + \xi^i\pi_i)$ where $N\epsilon = \beta$, and making use of the cyclicity of the trace, and the fact that γ_5 and H commute, we write the trace as

$$\begin{aligned}&\mathrm{Tr}\,\gamma_5\Big(\mathcal{O}e^{-\frac{N\epsilon}{\hbar}H} + e^{-\frac{\epsilon}{\hbar}H}\mathcal{O}e^{-(N-1)\frac{\epsilon}{\hbar}H}\\ &\qquad + \ldots + e^{-\frac{(N-1)\epsilon}{\hbar}H}\mathcal{O}e^{-\frac{\epsilon}{\hbar}H}\Big); \qquad \epsilon = \frac{\beta}{N}\ .\end{aligned} \tag{6.3.21}$$

Instead of $\mathcal{O}$ we write $\exp\mathcal{O}$ and obtain then a path integral with modified Hamiltonian $H + \frac{i}{2\beta}(\pi_i\xi^i + \xi^i\pi_i)$. (Strictly speaking one should use the Baker-Campbell-Hausdorff theorem to combine $\mathrm{e}^{-\frac{\epsilon}{\hbar}H}\mathrm{e}^{\mathcal{O}}$, but the terms involving commutators are of higher order in ϵ or ξ and can be neglected). Inserting complete sets of x, p eigenstates and coherent states for the fermions ψ_1^a, ψ_2^a (with a free set ψ_2 added as explained in section 6.1), one obtains the phase space path integral.

The next step is to integrate out the momenta from

$$-\frac{\epsilon}{2\hbar}g^{ij}(\bar{x}_k)\pi_{k,i}\pi_{k,j} + \frac{i}{\hbar}p_{k,i}(x_k^i - x_{k-1}^i) - \frac{i\epsilon}{\beta\hbar}\pi_{k,i}\xi^i(\bar{x}_k) \tag{6.3.22}$$

where $\bar{x}_k = (x_k + x_{k-1})/2$. This yields, as before, the interaction term in the covariant derivative in $-\frac{1}{2}\int_{-1}^0 \psi_1^a \frac{D}{D\tau}\psi_1^a d\tau$, while the gravitational Jacobian $\xi^i\pi_i$ is replaced by $\frac{1}{\beta\hbar}\int_{-1}^0 \xi^i g_{ij}(x)\dot{x}^j\, d\tau$. Similarly to (6.1.24) we arrive at

$$\begin{aligned}An(\text{grav}) = &\frac{(-i)^{n/2}}{(2\pi\beta\hbar)^{n/2}}\int\prod_{i=1}^n dx_0^i\sqrt{g(x_0)}\prod_{a=1}^n d\psi_{1,bg}^a\\ &\Big\langle \exp\Big(- \frac{1}{\beta\hbar}\frac{1}{2}\int_{-1}^0 (g_{ij}(x_0+q) - g_{ij}(x_0))(\dot{q}^i\dot{q}^j + b^ic^j + a^ia^j)d\tau\Big)\\ &\exp\Big(- \int_{-1}^0 \frac{1}{2}\dot{q}^i\omega_{iab}(x_0+q)(\psi_{1,bg}^a + \psi_{qu}^a)(\psi_{1,bg}^b + \psi_{qu}^b)d\tau\Big)\end{aligned}$$

$$\exp\left(-\frac{\beta\hbar}{8}\int_{-1}^{0}(\Gamma\Gamma+\frac{1}{2}\omega\omega)d\tau\right)$$
$$\exp\left(\frac{1}{\beta\hbar}\int_{-1}^{0}\dot{q}^i\xi^j(x_0+q)g_{ij}(x_0+q)d\tau\right)\Bigg\rangle \ . \tag{6.3.23}$$

The last exponent does not contain a term linear in q since $\int \dot{q}d\tau = 0$. Expanding ξ^j and g_{ij} to first order in q, the last term can be written as $\frac{1}{\beta\hbar}\int_{-1}^{0}\dot{q}^i q^k D_k\xi_i(x_0)d\tau$, where $\int_{-1}^{0}\dot{q}^i q^k d\tau$ is antisymmetric in i and k. Rescaling the ψ_1^a as before, the factors $\frac{1}{\hbar\beta}$ in the measure cancel, while the vertex with $\dot{q}\omega\psi_1\psi_1$ acquires a factor $\frac{1}{\hbar\beta}$. If we also choose a local Lorentz frame in which $\omega_{iab}(x_0)=0$, there is only the vertex with $q^i\dot{q}^j R_{ijab}\psi_1^a\psi_1^b$ which we encountered before, and a new vertex of the form $(D_i\xi_j)q^i\dot{q}^j$. They combine into the vertex

$$-\frac{1}{\hbar}S^{(int)} = -\frac{1}{\beta\hbar}\left[\frac{1}{4}R_{ijab}(x_0)\psi_1^a\psi_1^b - D_i\xi_j\right]\int_{-1}^{0}q^i\dot{q}^j d\tau \ . \tag{6.3.24}$$

Hence, the gravitational anomaly is obtained from the abelian chiral anomaly in (6.1.33) by adding $-D_{[i}\xi_{j]}$ to $\frac{1}{4}R_{ijab}\psi_1^a\psi_1^b$, where $D_{[i}\xi_{j]} = \frac{1}{2}(D_i\xi_j - D_j\xi_i)$.

We conclude that the gravitational anomaly of a complex chiral spinor in n dimensions is given by

$$\begin{aligned} &An(\text{grav, spin } 1/2) = \\ &= i\int dx_0^i\sqrt{g(x_0)}d\psi^a_{1,bg}\exp\frac{1}{2}\mathrm{tr}\ln\left(\frac{-i\tilde{R}/8\pi}{\sinh(-i\tilde{R}/8\pi)}\right)\Bigg|_{\text{term linear in }\xi} \\ &\tilde{R}\equiv R_{ijab}\psi_1^a\psi_1^b - 4D_i\xi_j \ . \end{aligned} \tag{6.3.25}$$

This anomaly is (-2) times the local Lorentz anomaly as we showed in (6.3.18). We absorbed the factors $(-i)^{n/2}$ into the exponent because then $-i\tilde{R}$ is hermitian. One factor i is left in front because one must expand the exponent to first order in ξ, hence, once again, the anomaly is purely imaginary. Expanding the term in the exponent, and retaining the terms linear in ξ, one is now left with an odd number of R terms. Hence **there are only gravitational anomalies in $\mathbf{n = 4k + 2}$ dimensions** (there were only gravitational chiral anomalies in $n = 4k$ dimensions). So, when one discusses gravitational anomalies for chiral spinors in string models in 10 dimensions, one means local Lorentz anomalies (anomalies in the conservation of the stress tensor if it has been made symmetric by a suitable Lorentz transformation) but not gravitational contributions to the rigid abelian chiral anomaly.

6.4 Mixed Lorentz and non-abelian gauge anomalies for chiral spin 1/2 fields coupled to gravity and Yang-Mills fields in $2k$ dimensions

We consider a complex chiral (Weyl) spin 1/2 field in $n = 2k$ dimensions, coupled to both external gravitational fields and to external Yang-Mills fields with gauge group G with antihermitian generators T_α. Mixed anomalies can occur, namely anomalies in the Einstein-Lorentz symmetry and in the Yang-Mills symmetry of one loop graphs coupled both to gravitons and to gauge bosons. We determine them in this section. We shall again switch to nonchiral complex Dirac fermions on the basis $\tilde{\lambda} = g^{\frac{1}{4}}\lambda$, and use a projection operator $\frac{1}{2}(1+\gamma^5)$ in the Jacobian. As before only the term with $\frac{1}{2}\gamma^5$ contributes.

The first time anomalies can appear is in $(k+1)$-polygon graphs; for example in triangle graphs in 4 dimensions. There can then be graphs with $(k+1)$ external gravitons, or with k gravitons and one external gauge field, or with $(k-1)$ gravitons and two gauge fields, up to $(k+1)$ external gauge fields. Given a graph with r external gauge fields, the trace over the generators T_α of the gauge fields yields a factor

$$\mathrm{Tr}_S(T_{\alpha_1}T_{\alpha_2}\dots T_{\alpha_r}) \tag{6.4.1}$$

where the subscript S indicates that one should totally symmetrize with respect to the indices $\alpha_1, \alpha_2 \dots \alpha_r$. If and only if this trace does not vanish there can be an anomaly in this graph. However, even if the symmetrized trace is nonvanishing, there need not be an anomaly. For example, for $r = 0$, we already saw that purely gravitational anomalies can only occur in $n = 4k+2$ dimensions. Graphs with only external gauge fields (the case $r = k+1$) are anomalous whenever the symmetrized trace with $r = n/2+1$ matrices is nonvanishing. In 4 dimensions this occurs whenever there is a cubic Casimir operator (the $d_{\alpha\beta\gamma}$ symbol) in the gauge group, and in n dimensions whenever there is a rank $(k+1)$ Casimir operator. The general case is most easily explained if one has the explicit result in hand, so we first derive the general formulas for mixed gravitational and gauge anomalies.

The Dirac operator in the n-dimensional field theory is given by

$$\not{D} = e_m^\mu \gamma^m D_\mu \ , \quad D_\mu = \partial_\mu + \frac{1}{4}\omega_{\mu mn}(e)\gamma^m\gamma^n + gA_\mu^\alpha T_\alpha \ . \tag{6.4.2}$$

The corresponding nonlinear sigma model is a combination of the $N = 1$ nonlinear sigma model of section 6.1 and appendix D, and the linear sigma model with extra ghosts of appendix E. It reads in Minkowski time

$$L_M \ = \ \frac{1}{2}g_{ij}(x)\dot{x}^i\dot{x}^j + \frac{i}{2}\psi_a\Big(\dot{\psi}^a + \dot{x}^k\omega_k{}^a{}_b\psi^b\Big)$$

$$
\begin{aligned}
&+ ic^*_A[\dot c^A + \dot x^k A^\alpha_k(x)(T_\alpha)^A{}_B c^B] \\
&+ \frac{1}{2}\psi^i\psi^j F_{ij}{}^\alpha c^*_A (T_\alpha)^A{}_B c^B \ .
\end{aligned}
\tag{6.4.3}
$$

The sum of the first three terms and the sum of the last three terms are separately supersymmetric. In Euclidean space we obtain

$$
\begin{aligned}
L_E \ = \ & \frac{1}{2}g_{ij}(x)\dot x^i \dot x^j + \frac{1}{2}\psi_a\Big(\dot\psi^a + \dot x^k \omega_k{}^a{}_b\psi^b\Big) \\
& + c^*_A[\dot c^A + \dot x^k A^\alpha_k(x)(T_\alpha)^A{}_B c^B] \\
& - \frac{1}{2}\psi^i\psi^j F_{ij}{}^\alpha c^*_A (T_\alpha)^A{}_B c^B \ .
\end{aligned}
\tag{6.4.4}
$$

The regulator for the n-dimensional spinor $\tilde\lambda = g^{1/4}\lambda$ of the quantum field theory is given by

$$
\begin{aligned}
\mathcal{R} \ &= \ -g^{1/4}\not{D}\not{D} g^{-1/4} \\
&= \ -g^{-1/4}D_\mu g^{1/2} g^{\mu\nu} D_\nu g^{-1/4} - \frac{1}{4}R - \frac{1}{2}\gamma^\mu\gamma^\nu F^\alpha_{\mu\nu}T_\alpha
\end{aligned}
\tag{6.4.5}
$$

where D_μ and D_ν contain spin and gauge connections but no Christoffel symbol. We derived this regulator in section 6.1, see (6.1.7).

The Jacobian for Einstein–Lorentz transformations and for separate Yang–Mills transformations of $\tilde\lambda$ and its conjugate field is given by combining (6.3.13) and a gauge transformation with parameter η

$$
J = \Big(\frac{1}{2}(\xi^\mu D_\mu + D_\mu \xi^\mu) - \eta^\alpha T_\alpha\Big)\,\frac{1+\gamma_5}{2} \ .
\tag{6.4.6}
$$

where we repeat that the derivative D_μ contains a spin connection and a gauge connection. The mixed anomalies in the QFT are then

$$
An(\text{mixed}) = (-2)\lim_{\beta\to 0} \text{Tr}\ J e^{-\frac{\beta}{\hbar}\mathcal{R}} \ .
\tag{6.4.7}
$$

In the corresponding QM approach the anomalies are given by

$$
An(\text{mixed}) = \text{Tr}\ \Big[\Big(\gamma_5 \frac{-i}{2\hbar}(\pi_i\xi^i + \xi^i\pi_i) + \gamma_5\eta^\alpha c^* T_\alpha c\Big) e^{-\frac{\beta}{\hbar}H}\Big]
\tag{6.4.8}
$$

where π_i was defined in (6.3.19).

We exponentiate the Jacobian as in section 6.3. After integrating out the momenta, we find then in the action the nonlinear sigma model of appendix E, together with the term $\frac{1}{\beta\hbar}\int_{-1}^0 \dot q^i \xi^j(x_0+q) g_{ij}(x_0+q)d\tau$ which we already found in section 6.3 and the term

$$
\int_{-1}^0 \eta^\alpha(x_0+q)(\bar\eta_{gh} + \bar c_{gh})T_\alpha(\chi_{gh} + c_{gh})d\tau
\tag{6.4.9}
$$

which is new. Inspection of (6.2.39) shows then that only the combination

$$\frac{1}{2}\tilde{F} = \Big(\frac{1}{2}\psi^a_{1,bg}\psi^b_{1,bg}gF^\alpha_{ab} + \eta^\alpha\Big)(\bar{\eta}_{gh} + \bar{c}_{gh})T_\alpha(\chi_{gh} + c_{gh}) \tag{6.4.10}$$

appears. From (6.3.24) we also find that only the combination

$$\tilde{R}_{ij} = R_{ijab}\psi^a_{1,bg}\psi^b_{1,bg} - 2(D_i\xi_j - D_j\xi_i) \tag{6.4.11}$$

occurs. The mixed anomaly is then given by the same formulas as derived in sections 6.2 and 6.3 but with $\tilde{F}$ and $\tilde{R}$ instead of F and R, where now

$$\frac{1}{2}\tilde{F} = \frac{1}{2}\psi^a_{1,bg}\psi^b_{1,bg}gF^\alpha_{ab}T_\alpha + \eta^\alpha T_\alpha \tag{6.4.12}$$

according to (6.2.48) and $\tilde{R}$ is given in (6.4.11)

$$\begin{aligned} An(\text{mixed, spin } 1/2) \;=\; & \frac{(-i)^{n/2}}{(2\pi)^{n/2}}\int dx^i_0\sqrt{g(x_0)}d\psi^a_{1,bg} \\ & \left(\text{Tr } \mathrm{e}^{\frac{1}{2}\tilde{F}}\right)\exp\frac{1}{2}\text{tr } \ln\left(\frac{\tilde{R}/4}{\sinh\tilde{R}/4}\right) \;. \end{aligned} \tag{6.4.13}$$

The trace tr is over matrices $\tilde{R_{ij}}$ but the trace Tr is in the space of the representation of the fermion (the space with matrices $(T_\alpha)_M{}^N$). The anomalies are obtained by expanding both factors and extracting the terms linear in ξ^m or η^α. A given anomaly depends in general both on F and R.

At this point we can make a consistency check between the abelian chiral anomaly of section 6.2 and the gauge anomaly of this section. Consider a local chiral $U(1)$ gauge transformation (for example the $U(1)$ of the Standard Model). We can view it as an abelian chiral transformation (with $T_\alpha = -i$) and evaluate $\text{Tr}\,\mathrm{e}^{\frac{1}{2}F}$. We can also treat it as a gauge anomaly in which case we introduce ghosts to construct $c^*T_\alpha c$ with T_α a constant, and then must evaluate $\text{Tr}\,\mathrm{e}^{\frac{1}{2}\tilde{F}}$ by taking the term linear in η. The answer should be the same[9], and it is the same because Tr $\frac{(F/2)^n}{n!}$ equals the term linear in η in Tr $\frac{(\tilde{F}/2)^{n+1}}{(n+1)!}$.

Consider now a one-loop graph with a complex chiral spin 1/2 field in the loop and with p external gravitons and q external gauge fields. The integration over $\psi^a_{i,bg}$ requires $p + q = \frac{1}{2}n + 1$, so the first time anomalies are possible in n dimensions is in polygon graphs with $\frac{1}{2}n + 1$

[9] The abelian chiral anomaly we computed in section 6.2 referred to a nonchiral fermion with Jacobian $-2i\alpha\gamma_5$. In this section we have used a chiral fermion by inserting the projection operator $\frac{1}{2}(1 + \gamma_5)$ in the trace. Hence for comparison one should take $\frac{1}{2}$ times the abelian chiral anomaly of a nonchiral fermion.

sides. The gauge variation of a graviton yields a factor $D_{[m}\xi_{n]}$, while the gauge variation of a Yang–Mills field yields a factor $\eta^\alpha T_\alpha$ in (6.4.13). Since the second factor in (6.4.13) is even in $\tilde{R}$, **there are only anomalies (both gravitational or gauge anomalies) if there are an even number of external gravitons (p even)**.

In 10 dimensions, the first time anomalies appear is in hexagon graphs. There is a purely gravitational anomaly with 6 external gravitons, because 10 is in the set $4k+2$. There is a mixed anomaly with 4 external gauge fields and 2 gravitons if the gauge group is such that $\mathrm{Tr}F^4$ is nonvanishing in the representation of the fermions. There is always a mixed anomaly with 2 external gauge fields and 4 gravitons, because every gauge group has a quadratic Casimir operator, but a purely gauge anomaly only exists if $\mathrm{Tr}F^6$ is nonvanishing.

In 4 dimensions the first time anomalies appear is in triangle graphs. There is no purely gravitational anomaly, only a mixed anomaly if $G = U(1)$ (the abelian chiral gravitational anomaly of section 6.1), and only a pure gauge anomaly if G has a nonvanishing $d_{\alpha\beta\gamma}$ symbol in the representation of the fermions.

6.5 The abelian chiral anomaly for spin 3/2 fields coupled to gravity in $4k$ dimensions.

In this section we extend the calculation of the gravitational corrections to the abelian chiral anomaly ("the γ_5 anomaly") from the case of the spin 1/2 field to the case of spin 3/2. This introduces supergravity, because the only consistent interactions for spin 3/2 with other fields are the interactions of supergravity models. Earlier models, with spin 3/2 only coupled to spin 1, turned out to be inconsistent or trivial, while the couplings of spin 3/2 fields to spin 2 are consistent if they are given by $N = 1$ supergravity. One can also couple spin 3/2 fields to spin 1 and spin 0 fields, but only if at the same time one couples the spin 3/2 to gravity and these interactions are given by a supergravity model with $N > 1$ (N is the number of real spin 3/2 fields). In this section we consider only loops with spin 3/2 fields in the loop and external gravity fields. The spin 2 and 3/2 interactions may be part of a more complicated supergravity model, but that does not make a difference for the computation of the one-loop anomalies. Only the minimal gravitational couplings of gravity to spin 3/2 contribute, so torsion due to gravitinos may be ignored. We assume that the reader has no knowledge of supergravity, and start from the beginning.

In $3+1$ dimensional Minkowski spacetime, the real spin 3/2 field ${\psi_\mu}^\alpha$(with $\mu = 0,..,3$ the vector index and $\alpha = 1,..,4$ the corresponding 4-component spinor index) is the gauge field for local supersymmetry

(=supergravity). The classical action for the spin 3/2 field in $3+1$ dimensional Minkowski spacetime coupled to external gravitational fields reads

$$\mathcal{L}_{3/2} = -\frac{1}{2}e\bar{\psi}_\mu \gamma^{\mu\rho\sigma} D_\rho \psi_\sigma \tag{6.5.1}$$

$$\begin{aligned} e &= \det e_\mu{}^m \ ; \quad \gamma^\mu = e_m{}^\mu \gamma^m \\ D_\rho \psi_\sigma &= \partial_\rho \psi_\sigma + \frac{1}{4}\omega_{\rho mn}(e)\gamma^m \gamma^n \psi_\sigma \\ \bar{\psi} &= \psi^T C \ ; \quad C\gamma^\mu C^{-1} = -(\gamma^\mu)^T \end{aligned}$$

where $\gamma^{\mu\rho\sigma}$ equals the totally antisymmetrized product of γ^μ, γ^ρ and γ^σ (so $\gamma^{\mu\rho\sigma} = \frac{1}{6}(\gamma^\mu\gamma^\rho\gamma^\sigma$+5 other terms)), and γ^m are constant 4×4 matrices. The form of the action for the spin 3/2 gauge field is fixed by requiring invariance of the free spin 3/2 action in flat spacetime under $\delta\psi_\mu = \partial_\mu \epsilon$; in fact, it already follows from requiring that the residue of the free field propagator be without ghosts [56]. Thus, also for local supersymmetry, gauge invariance follows from unitarity. (When one does not treat gravity as external, one needs also, of course, the Einstein action, and all the other paraphernalia of supergravity. In particular, the spin connection $\omega_{\rho mn}$ contains then torsion terms bilinear in gravitinos. For the calculation of anomalies with spin 3/2 loops, we can restrict ourselves to external gravitational fields and use the spin connection $\omega_{\rho mn}(e)$ of appendix A which only depends on the external vielbein field $e_\mu{}^m$). In principle one should add a Christoffel connection in the definition of $D_\rho\psi_\sigma$ but it cancels in $\mathcal{L}_{3/2}$ because $D_\rho\psi_\sigma$ appears in the action only as antisymmetric in ρ and σ.

This classical spin 3/2 action is gauge invariant by itself provided the background fields are Ricci-flat (Einstein spaces with $R_{\mu\nu} = 0$): it is locally supersymmetric. Under $\delta\psi_\sigma = D_\sigma\epsilon, \delta e_\mu{}^m = 0$ one obtains, using that the commutator of two covariant derivatives is a curvature,

$$\begin{aligned} \delta\mathcal{L}_{3/2} &= -\frac{1}{2}e\bar{\psi}_\mu\gamma^{\mu\rho\sigma}\frac{1}{2}(\frac{1}{4}R_{\rho\sigma mn}\gamma^m\gamma^n\epsilon) \\ &\quad -\frac{1}{2}e(D_\mu\bar{\epsilon})\gamma^{\mu\rho\sigma}D_\rho\psi_\sigma \ . \end{aligned} \tag{6.5.2}$$

After partial integration (using that D_μ commutes with e and γ^μ because we have omitted the torsion terms in the spin connection) one finds

$$\begin{aligned} \delta\mathcal{L}_{3/2} &= -\frac{1}{16}e\bar{\psi}_\mu\gamma^{\mu\rho\sigma}\gamma^m\gamma^n\epsilon R_{\rho\sigma mn} + \frac{1}{16}e\bar{\epsilon}\gamma^{\mu\rho\sigma}R_{\mu\rho mn}\gamma^m\gamma^n\psi_\sigma \\ &= -\frac{1}{16}eR_{\rho\sigma mn}\bar{\psi}_\mu\{\gamma^{\mu\rho\sigma}\gamma^m\gamma^n + \gamma^m\gamma^n\gamma^{\mu\rho\sigma}\}\epsilon \ . \end{aligned} \tag{6.5.3}$$

(We used $\bar{\epsilon}\gamma^{\mu\rho\sigma}\gamma^m\gamma^n\psi_\sigma = -\bar{\psi}_\sigma\gamma^n\gamma^m\gamma^{\sigma\rho\mu}\epsilon$; this relation follows from $\bar{\epsilon} = \epsilon^T C$ and the property $C\gamma^{m\,T} = -\gamma^m C$). In the anticommutator $\{\gamma^{\mu\rho\sigma}, \gamma^{mn}\}$ there are only terms with a totally antisymmetric product of five Dirac matrices or terms with one Dirac matrix. The terms with a totally antisymmetric product of three Dirac matrices cancel in the anticommutator. (They survive in the commutator). Those with five Dirac matrices do not contribute in 3+1 dimensions because a tensor with 5 indices which is totally antisymmetric vanishes in less then 5 dimensions, and also because of the cyclic identity of the Riemann tensor. In higher dimensions, these terms still vanish due to the cyclic identity of the Riemann tensor. The variations with one Dirac matrix can only contract with a Ricci tensor $R_{\mu\nu}$. In fact, this could have been anticipated because there are not enough free indices to contract with a full Riemann tensor. One finds $\delta\mathcal{L}_{3/2} = \frac{1}{2}e(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)(\bar{\psi}^\mu\gamma^\nu\epsilon)$. When $R_{\mu\nu} = 0$, one calls the gravitational field Ricci flat. Hence: for Ricci flat backgrounds the gravitino action is gauge invariant (locally supersymmetric).

If one treats the gravitational field dynamically, there is no longer a restriction to Ricci-flatness, provided one also transforms the gravitational field (the vielbein field ${e_\mu}^m$) under local supersymmetry. It transforms as $\delta {e_\mu}^m = \bar{\epsilon}\gamma^m\psi_\mu$, and also this variation is multiplied by the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$; in fact, the sum of all local supersymmetry variations cancel. From now on we continue with arbitrary gravitational fields; for spin 3/2 loops coupled to gravity the spacetime symmetries (general coordinate and local Lorentz transformations) of the fields (vielbein ${e_\mu}^m$ and gravitino ψ_μ) are as usual, and the classical action is also invariant under local supersymmetry if one transforms ${e_\mu}^m$ as we discussed. The previous derivation with external gravitational fields was given for readers who are not familiar with supergravity. Readers who want an introduction to supergravity are referred to [56].

Because there is a local supersymmetry one must add a gauge fixing term and a ghost action. Without gauge fixing term the kinetic operator cannot be inverted, and no graphs with spin 3/2 in the loop can be constructed. With gauge-fixing term the local supersymmetry gets broken, but we shall choose the local supersymmetry gauge-fixing term such that the classical spacetime symmetries remain unbroken. As gauge fixing term the most convenient choice is $\gamma^\mu\psi_\mu = 0$ leading to

$$\mathcal{L}(\text{fix}) = \frac{1}{4}e\bar{\psi}_\mu\gamma^\mu \not{D}\gamma^\nu\psi_\nu \tag{6.5.4}$$

which preserves general coordinate invariance and local Lorentz symmetry but breaks local supersymmetry. The corresponding Faddeev-Popov ghost action follows then from applying $\delta\psi_\mu = D_\mu\epsilon$ to $\gamma^\mu\psi_\mu$ and contracting with the antighost of supersymmetry

$$\mathcal{L}(FP\,ghost) = -e\bar{b}_\alpha(\not{D}c)^\alpha \tag{6.5.5}$$

where b_α and c^α are real commuting ghosts (the spinor index α will be dropped below). (One should also vary the vielbein in $\gamma^\mu = e^\mu_m\gamma^m$, but this yields a term $(\bar{b}\gamma^m\psi_\mu)(\bar{\psi}_m\gamma^\mu c)$ in the ghost action which does not contribute at the one-loop level. It is used, however, to fix the chiral symmetry transformation rules, see below).

To obtain (6.5.4) in the exponent of the path integral, one starts from the gauge fixing term $\delta[\gamma\cdot\psi - F]$ with F an independent anticommuting Majorana spinor, and then one inserts unity into the path integral as follows: $I = \int[dF]exp(\bar{F}\not{D}F)(\det\not{D})^{-\frac{1}{2}}$. Integration over F yields $(\det\not{D})^{1/2}$ which cancels the factor $(\det\not{D})^{-1/2}$. The normalization factor $(\det\not{D})^{-1/2}$ can be exponentiated to give another ghost, the so-called Nielsen-Kallosh ghost. It is really a commuting complex ghost (a pair B, C of real ghosts[10]) and an anticommuting real ghost A, as we now explain, just like the a, b, c ghosts of the QM model. Because the Dirac action for one real commuting ghost vanishes, one writes $(\det\not{D})^{-1/2}$ as $(\det\not{D})^{1/2}/(\det\not{D})$ and then exponentiation of $(\det\not{D})^{1/2}$ gives a real anticommuting A ghost while $\det\not{D}^{-1}$ yields commuting B,C ghosts. The total Nielsen-Kallosh ghost action reads

$$\mathcal{L}(NK\,ghosts) = -\frac{e}{2}\bar{A}\not{D}A - \frac{e}{2}\bar{B}\not{D}C\ . \tag{6.5.6}$$

The classical action together with its gauge fixing term can be written in a very simple form

$$\mathcal{L}(3/2) = \frac{e}{4}\bar{\psi}_\mu\gamma^\sigma\not{D}\gamma^\mu\psi_\sigma\ . \tag{6.5.7}$$

This result follows from the identity $-\frac{1}{2}\gamma^{\mu\rho\sigma} + \frac{1}{4}\gamma^\mu\gamma^\rho\gamma^\sigma = \frac{1}{4}\gamma^\sigma\gamma^\rho\gamma^\mu$. This action is clearly invariant under rigid chiral transformations

$$\delta\psi_\mu = i\alpha\gamma_5\psi_\mu, \quad \delta\bar{\psi}_\mu = i\alpha\bar{\psi}_\mu\gamma_5 \tag{6.5.8}$$

with real constant α, because the field operator contains an odd number of Dirac matrices. Each ghost action in (6.5.5) and (6.5.6) is by itself chirally invariant, for arbitrary chiral weights. One may fix these weights by considering the full quantum supergravity action (with Einstein action and various couplings between ghosts and gravitinos). Requiring that the whole ghost sector (b, c, A, B, C) be also invariant under γ_5 transformations fixes then the relative chiral weights of the various ghost fields,

[10] A pair of real ghosts can, of course, be replaced with one complex (Dirac) ghost.

and the net result is that, as far as γ_5 transformations are concerned, **the ghost sector acts as if it contained only one anticommuting complex chiral spin 1/2 field with opposite chiral weight as the gravitino**[11]. So the final result for the anomaly will be the result of $\mathrm{Tr}\,\gamma_5$ for ψ_μ minus the result of $\mathrm{Tr}\,\gamma_5$ for one chiral Dirac fermion (a chiral Dirac fermion is equivalent to a nonchiral real (Majorana) fermion; one can rewrite one in terms of the other in four dimensions).

We now consider *the gravitational contribution to the abelian chiral (γ_5) anomaly.* Since the gravitino in $N = 1$ supergravity has no Yang-Mills index, it can only couple to gravity. Hence we can only consider the chiral anomaly, due to a spin 3/2 loop with (infinitely many) external gravitons at the vertices, and the abelian chiral current as one of its vertices. In the path integral approach, the anomaly in the conservation of this current is, up to an overall constant $-2i\alpha$,

$$An = \mathrm{Tr}\,\gamma_5\, e^{\beta R}\ . \tag{6.5.9}$$

In addition there is a trace for the ghost which we add later. We must now discuss the regulator for the spin 3/2 field.

A mass term for the gravitino which preserves both Einstein invariance and local Lorentz symmetry is given by $\mathcal{L}(m) = -\frac{e}{2} m \bar\psi_m \psi^m$ where $\psi_m = e_m{}^\mu \psi_\mu$ has flat indices. It follows that $\tilde\psi_m = e^{1/2}\psi_m$ is the field whose mass term is proportional to the unit matrix. One could then use its field operator to yield the regulator. Namely, if $R_m{}^n$ is the full kinetic operator for $\tilde\psi_m$, then its square is the regulator which preserves general coordinate and local Lorentz symmetry

$$\begin{aligned}
\mathcal{R} &= R_m{}^p R_p{}^n \\
R_m{}^n &= g^{1/4}\gamma^n\gamma^\rho\gamma_m D_\rho g^{-1/4} \\
D_\mu &= \delta^n_m \delta^\beta_\alpha \partial_\mu + \frac{1}{4}\omega_{\mu pq}(\gamma^p\gamma^q)^\alpha{}_\beta \delta^n_m + \omega_{\mu m}{}^n \delta^\beta_\alpha\ .
\end{aligned} \tag{6.5.10}$$

[11] The details are as follows. Variation of the vielbein in the gauge fixing term $\gamma^\mu\psi_\mu$, and contracting the result with the antighost to obtain the ghost action as usual, produces a term $(\bar b\gamma^m\psi_\mu)(\bar\psi_m\gamma^\mu c)$ in the ghost action. Chiral invariance requires then that the Faddeev–Popov ghosts b and c have the same chiral weight as the gravitino. The operator $(\not{D})^{-\frac{1}{2}}$ is exponentiated by means of Nielsen–Kallosh ghosts; since it acts in the space which contains $\gamma\cdot\psi$, the spinors in this space have opposite chirality of the gravitino. We find then for the total effective chiral weight in ghost space (defining the chiral weight as the weight one chiral complex anticommuting spinor should have in order to reproduce the same anomaly): $-1(b) - 1(c) - 1(A) + 1(B) + 1(C) = -1$. Physically the role of the ghosts is as follows: the Faddeev–Popov ghosts b and c remove as usual the unphysical longitudinal and time components of the vector-spinor field ψ^α_μ. This corresponds to the gauge symmetry $\delta\psi^\alpha_\mu = \partial_\mu\epsilon^\alpha + \ldots$. On-shell a physical massless spin 3/2 particle should have two polarizations with helicity $\pm 3/2$. This is indeed achieved because on-shell $\gamma^\mu\psi_\mu = 0$. The Nielsen–Kallosh ghosts remove the $\gamma\cdot\psi$ part from the gravitino.

Actually the mass term which does not lead to tachyons is not $\bar{\psi}_m \psi^m$, but rather $\bar{\psi}_m \gamma^{mn} \psi_n$. The corresponding regulator would then be the square of $(T^{-1})_m{}^s R_s{}^n$ where T^{-1} is the inverse of γ^{mn}. These operators are all difficult to work with.

A much simpler regulator which yields the same chiral anomalies is the Dirac operator (see Alvarez-Gaumé and Witten [1])

$$
\begin{aligned}
\mathcal{R}_{3/2}(D) &\sim g^{1/4} \not{D} \not{D} g^{-1/4} \\
&= \left(g^{-1/4} D_\mu \sqrt{g} g^{\mu\nu} D_\nu g^{-1/4} \right)^\alpha{}_\beta \, \delta_m{}^n \\
&\quad + \frac{1}{2} \Big(\frac{1}{4} \gamma^\mu \gamma^\nu R_{\mu\nu pq}(\omega) \gamma^p \gamma^q \Big)^\alpha{}_\beta \, \delta_m{}^n \\
&\quad + \frac{1}{2} (\gamma^\mu \gamma^\nu)^\alpha{}_\beta R_{\mu\nu m}{}^n(\omega) \ .
\end{aligned}
\tag{6.5.11}
$$

The term with four Dirac matrices can be simplified to $\frac{1}{4}R$.[12] It acts still in the combined vector-spinor space, and D_μ contains again a spinor and a vector connection, as in (6.5.10). Hence, the anomaly becomes (up to a factor $-2i\alpha$)

$$
An = \mathrm{Tr}\, \gamma_5 \, e^{\mathcal{R}_{3/2}(D)} - \mathrm{tr}\, \gamma_5 \, e^{\mathcal{R}_{1/2}(D)}
\tag{6.5.12}
$$

where the first trace Tr is over vector and spinor indices of the gravitino, while the second trace tr is only over spinor indices of the ghost. We have added the subscripts 3/2 and 1/2 to $\mathcal{R}(D)$ to stress that $\mathcal{R}_{3/2}(D)$ has an extra term $\omega_\mu{}^{mn}$ with respect to $\mathcal{R}_{1/2}(D)$. The trace with $\mathcal{R}_{3/2}(D)$ has a minus sign because it is due to an anticommuting ghost field, as we explained above.

Alvarez-Gaumé and Witten give a general proof that one may use $\not{D}\not{D}$ for the γ_5 anomaly of spin 3/2 fields. We present here a direct proof, see also [109]. We do this for general dimensions. The spin 3/2 action in n dimensions reads

$$
\mathcal{L}_0 = -\frac{e}{2} \bar{\psi}_\mu \gamma^{[\mu} \gamma^\nu \gamma^{\rho]} \, D_\nu \psi_\rho \ .
\tag{6.5.13}
$$

Again, this results follows from the fact that in flat space this is the only action that is invariant under $\delta\psi_\mu = \partial_\mu \epsilon$. After adding a gauge fixing term

$$
\mathcal{L}_{\text{fix}} = \frac{n-2}{8} e \bar{\psi}_\mu \gamma^\mu \gamma^\nu \gamma^\rho D_\nu \psi_\rho
\tag{6.5.14}
$$

---

[12] Write $\gamma^\nu \gamma^p \gamma^q$ as $\gamma^{\nu pq} + e^{\nu p} \gamma^q + \eta^{pq} \gamma^\nu - e^{\nu q} \gamma^p$. Then $\gamma^{\nu pq}$ does not contribute due to the cyclic identity. There remain only Ricci tensors $R_{\mu\nu}$, and since these are symmetric (recall that we dropped the torsion terms), the remaining two Dirac matrices $\gamma^\mu \gamma^\nu$ can be replaced by $g^{\mu\nu}$.

and choosing a new basis for the spin 3/2 fields

$$\chi_\mu = \psi_\mu - \frac{1}{2}\gamma_\mu\gamma\cdot\psi \qquad \rightarrow \qquad \psi_\mu = \chi_\mu - \frac{1}{n-2}\gamma_\mu\gamma\cdot\chi \tag{6.5.15}$$

the action becomes a sum of Dirac actions

$$\mathcal{L}_0 + \mathcal{L}_{\text{fix}} = -\frac{1}{2}e\bar{\chi}_\mu \not{D}\chi^\mu = -\frac{1}{2}e\bar{\chi}_m \not{D}\chi^m \ . \tag{6.5.16}$$

The field χ_m transforms of course in the same way as ψ_m under spacetime transformations and γ_5 transformations, and the regulator for the spin 3/2 field χ_m is thus $g^{1/4}\not{D}\not{D}g^{-1/4}$, for the same reasons as for the spin 1/2 field.[13]

We are now ready to compute the gravitational contribution to the γ^5 anomaly. We do this in n dimensions for one complex nonchiral gravitino. This is the procedure we have also followed for the spin 1/2 case. (In 3+1 dimensions, a chiral gravitino is complex and can be rewritten as a real nonchiral gravitino, but in other dimensions this is not always true). We repeat the steps taken in the case of the γ_5 anomalies for spin 1/2. We continue to write the term with the spin 1/2 Lorentz generator in terms of ψ_1^a, but the term with the spin 1 generator we treat like the internal generator of a Yang-Mills group[14]. Hence, the internal matrix $A_i{}^\alpha T_\alpha$ with T_α now the Lorentz generators, is represented by $c^*{}_a\omega_i{}^a{}_b c^b$ (just like $A_i{}^\alpha T_\alpha$ was represented by $A_i{}^\alpha c^* T_\alpha c$). The spin 1/2 term $\frac{1}{4}\omega_{\mu mn}\gamma^m\gamma^n$ becomes $\frac{1}{2}\omega_{iab}\psi_1^a\psi_1^b$. Note that although in (6.5.10) the spin connection terms which act on the flat spinor and flat vector index of the gravitinos appear on equal footing, we treat them differently in the QM model. Hence the QM treatment for spin 3/2 combines the gravitational and Yang-Mills treatment for spin 1/2 .

The ghosts require again a one-particle projection operator, and this yields upon combining (6.1.24) and (6.2.39)

$$\begin{aligned} An \ &= \ \frac{(-i)^{n/2}}{(2\pi)^{n/2}}\int dx_0\sqrt{g(x_0)}d\psi_{1,bg}\int d\chi d\bar{\eta}\, P^{gh}_{\bar{\eta},\chi}\, e^{\bar{\eta}\chi} \\ &\quad \Big\langle \exp\Big(-\frac{1}{\beta\hbar}\int_{-1}^{0}\frac{1}{2}(g_{ij}(x_0+q)-g_{ij}(x_0))(\dot{q}^i\dot{q}^j + b^i c^j + a^i a^j)\, d\tau\Big) \end{aligned}$$

[13] One can write down an Einstein and locally Lorentz invariant mass term for χ_m, namely $e\bar{\chi}_m\chi^m$. This is not the mass term which is without ghosts and tachyons, but it serves our purposes to construct a regulator [75].

[14] The deeper reason these separate treatments of the vector part and the spinor part of the generators make sense is that the trace of a direct product is the product of the traces. In group theory one uses this simple fact to compute traces over products of generators in a given representation R (such as the $d^{(R)}_{abc}$ symbols) which are built from direct products of the fundamental representation F (yielding a relation between $d^{(R)}_{abc}$ and $d^{(F)}_{abc}$).

$$\exp\Big(-\frac{1}{\beta\hbar}\frac{1}{2}\int_{-1}^{0}\dot{q}^i\omega_{iab}(x_0+q)\psi_1^a\psi_1^b\,d\tau\Big)$$
$$\exp\Big(-\int_{-1}^{0}\dot{q}^i\omega_i{}^a{}_b(x_0+q)c^*{}_a c^b\,d\tau\Big)$$
$$\exp\Big(\frac{1}{2}\int_{-1}^{0}c_a^* R^a{}_{bcd}(\omega(x_0+q))c^b\psi_1^c\psi_1^d\,d\tau\Big)$$
$$\exp\Big(-\frac{\beta\hbar}{8}\int_{-1}^{0}(\Gamma\Gamma+\frac{1}{2}\omega\omega)\,d\tau\Big)\Big\rangle \qquad (6.5.17)$$

where as in (6.1.24) ψ_1 stands for $\psi_{1,bg}+\psi_{1,qu}$, and as in (6.2.39) c (c^*) stands for $\chi+c_{qu}$ ($\bar{\eta}+c^*_{qu}$). Recall that we only rescaled the ψ_1^a but not the ghosts; this removed the $\beta\hbar$ from the measure. The term $\frac{1}{4}\hbar\gamma^\mu\gamma^\nu R_{\mu\nu m}{}^n$ in the regulator $-\frac{1}{\hbar}(-\frac{\hbar^2}{2}\mathcal{R})$ becomes a term $\beta\hbar\int_{-1}^{0}\frac{1}{2}c^*Rc\psi_1\psi_1$ in the action and becomes $\hbar\beta$ independent after the rescaling of ψ_1^a, see the one-but-last line. The term with $\Gamma\Gamma+\frac{1}{2}\omega\omega$ does not contribute, since it is proportional to $\hbar\beta$, and also the terms with $\dot{q}^i\dot{q}^j+b^ic^j+a^ia^j$ and $\dot{q}^i\omega_i{}^a{}_b c_a^* c^b$ do not contribute for the same reason as before. The vertex with $\frac{1}{\hbar\beta}\dot{q}^i\omega_{iab}\psi_1^a\psi_1^b$ yields the vertex $\int d\tau\,\dot{q}^i q^j R_{ijab}\psi_1^a\psi_1^b$ upon expanding ω_{iab}, but the vertex with $\int d\tau\,\dot{q}^i\omega_{iab}c^{*a}c^b$ does not contribute because the propagator for $\dot{q}^i$ brings in a factor β whereas this vertex is β-independent.

In fact, **only closed q-loops with $R\psi_1\psi_1$ at the vertices, or ghost trees with $R\psi_1\psi_1$ at the vertices contribute**. As we have seen, the latter give a factor $\mathrm{tr}\exp(\frac{1}{2}R_{..ab}\psi^a_{1,bg}\psi^b_{1,bg})$, and the former give the factor with $\exp\frac{1}{2}\mathrm{tr}\ln[\frac{R}{4}/\sinh(\frac{R}{4})]$. The final result is

$$\begin{aligned} An(\gamma_5,\,\mathrm{spin}\,3/2) \;&=\; (-2i\alpha)\frac{(-i)^{\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}}\int\left(\prod_{i=1}^{n}dx_0^i\sqrt{g(x_0)}\right)\left(\prod_{a=1}^{n}d\psi^a_{1,bg}\right) \\ &\qquad \left[\left(\mathrm{tr}\,e^{\frac{1}{2}R}\right)-1\right]\exp\frac{1}{2}\mathrm{tr}\ln\left(\frac{R/4}{\sinh(R/4)}\right) \\ R \;&\equiv\; R_m{}^n{}_{ab}\psi^a_{1,bg}\psi^b_{1,bg} \end{aligned} \qquad (6.5.18)$$

where the factor $-2i\alpha$ mentioned above (6.5.9) has been reinserted. The first trace corresponds to the result in (6.2.48) and contains the contributions due to the vector index m of the gravitino $\psi_m{}^\alpha$; the exponent with the second trace corresponds to the result in (6.1.32) and takes care of the contributions due to the spinor index α of the gravitino $\psi_m{}^\alpha$, and the factor -1 accounts for the contributions of the supersymmetry ghosts. In the Yang-Mills case the trace could be over any representation of the gauge group, and we denoted this trace by Tr. Because both traces in (6.5.18) are over the indices m and n of R_{mn}, we denote both by the same symbol tr.

As an application we compute the γ_5 anomaly for spin 3/2 loops in 4 dimensions. The answer is known to be -21 times the same anomaly for a spin 1/2 loop [110]. Expanding the factor $\left(\operatorname{tr} e^{\frac{1}{2}R}\right) - 1$ gives a contribution $\frac{1}{8}\mathrm{tr}R^2$. The second exponent gives the contribution of a spin 1/2 loop, which is $-\frac{1}{2}\mathrm{tr}\left(\frac{R}{4}\right)^2\frac{1}{3!} = -\frac{1}{192}\mathrm{tr}R^2$. This second contribution is multiplied by $(\mathrm{tr}I) - 1 = 3$ (after gauge fixing the field ψ_m represents 4 spin 1/2 spinors, but the ghosts remove one spin 1/2 spinor). Then one finds for $n = 4$

$$\begin{aligned} An(\gamma_5, \text{ spin } 3/2 \text{ in } n=4) &= \frac{2i\alpha}{4\pi^2}\int\Big[\frac{1}{8}\mathrm{tr}R^2 - 3\frac{1}{192}\mathrm{tr}R^2\Big] \\ &= \frac{2i\alpha}{4\pi^2}\int\Big[\frac{21}{192}\mathrm{tr}R^2\Big] = -21 An(\gamma_5, \text{ spin } 1/2 \text{ in } n=4)\ . \quad (6.5.19) \end{aligned}$$

where $\mathrm{tr}R^2$ is equal to $\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu}{}^{mn}R_{\rho\sigma mn}$. This is indeed the correct result.[15]

6.6 Lorentz anomalies for chiral spin 3/2 fields coupled to gravity in $4k+2$ dimensions

We now discuss the anomaly in the combined Einstein and local Lorentz symmetries when a chiral spin 3/2 field in a loop couples to an external gravitational field. We take the spin 3/2 field to be complex. In certain dimensions, chiral spinors can be real (Majorana-Weyl spinors) and for these cases one must divide the result for a complex chiral gravitino by a factor 2.

It is preferable to have a covariant expression for the spin 3/2 transformation rule under spacetime transformations, and a covariant expression for the corresponding Jacobian, because then the answer for the anomaly will be a relatively simple expression involving only curvatures. This can be achieved by taking certain linear combinations of Einstein transformations and local Lorentz transformations. For spin 1/2 fields, we already explained this before. The case of spin 3/2 is more complicated. To wet the appetite of the reader for this problem, we first quote the transformation rules[16] used by Alvarez-Gaumé and Witten [1]

$$\delta_{AW}\psi_m \;=\; \frac{1}{2}(\xi^\mu D_\mu + D_\mu\xi^\mu)\psi_m + [(D_m\xi^n) - (D^n\xi_m)]\psi_n$$

[15] In [1] one finds the result $\prod_{i=1}^{n/2}\frac{x_i/2}{\sinh(x_i/2)}$ instead of $\exp[\frac{1}{2}\mathrm{tr}\ln(\frac{R/4}{\sinh(R/4)})]$ where R denotes the matrix $R_m{}^n = R_m{}^n{}_{ab}\psi^a_{1,bg}\psi^b_{1,bg}$ which is made block diagonal with blocks of the type $\left(\begin{smallmatrix}0 & x_i\\ -x_i & 0\end{smallmatrix}\right)$ along the diagonal. These formulas agree and the factor 1/2 in the exponent is correct, as one may check. For example, $\sum_i(\frac{x_i}{2})^2 = \frac{1}{2}\mathrm{tr}(\frac{R}{4})^2$.

[16] It took us many years to find a clear rigorous derivation of these rules. We thank R. Endo whose help was essential.

$$
\begin{aligned}
D_m\xi^n &= e_m{}^\mu(\partial_\mu\xi^n + \omega_\mu{}^n{}_p\xi^p) \ , \qquad \xi^n = e_\mu{}^n\xi^\mu \\
D_\mu\psi^a_m &= \partial_\mu\psi^a_m + \omega_{\mu m}{}^n\psi^a_n + \frac{1}{4}\omega_m{}^{pq}(\gamma_p\gamma_q)^a{}_b\psi^b_m \ .
\end{aligned} \tag{6.6.1}
$$

The spin 3/2 field ψ_m, the so-called gravitino, is a Lorentz vector-spinor. This explains the term $\omega_{\mu m}{}^n\psi_n$ in the last line. The spin 3/2 transformation rule contains thus both a covariant translation and **an extra covariant local Lorentz transformation acting only on the vector index of the gravitino**. Its meaning is at first sight rather mysterious.

We now proceed to derive this transformation rule. Several steps in the derivation are identical to the steps taken in the spin 1/2 case, but there are new aspects due to the spin one index of the gravitino, and in order not to have to refer back all the time to the spin 1/2 case, we give a complete derivation of the spin 3/2 case from scratch.

An Einstein transformation of a spin 3/2 gravitino with a flat index, $\psi_m = e_m{}^\mu\psi_\mu$, is given by

$$
\delta_E(\xi)\psi_m = \xi^\mu\partial_\mu\psi_m \ . \tag{6.6.2}
$$

We aim at covariant transformation rules and covariant regulators for reasons explained before. Hence we prefer to consider the following transformation rule

$$
\delta_{cov}(\xi)\psi_m = \xi^\mu D_\mu\psi_m = \xi^\mu\partial_\mu\psi_m + \frac{1}{4}\xi^\mu\omega_{\mu rs}\gamma^r\gamma^s\psi_m + \xi^\mu\omega_{\mu m}{}^n\psi_n \ . \tag{6.6.3}
$$

This is a linear combination of Einstein transformations $\xi^\mu\partial_\mu\psi_m$ and local Lorentz transformation with parameter $\xi^\mu\omega_{\mu mn}$, namely $\frac{1}{4}\xi^\mu\omega_{\mu rs}\gamma^r\gamma^s\psi_m + \xi^\mu\omega_{\mu m}{}^n\psi_n$. We called this contribution a covariant Einstein transformation in section 6.3. So $D_\mu\psi_m$ is completely covariant and includes the spin connection for both (vector and spinor) indices of ψ_m. When we use path integral techniques to convert the trace $\mathrm{Tr}Je^{\mathcal{R}}$ into a path integral, the corresponding operator becomes a corresponding function provided it is Weyl ordered. This is Berezin's theorem which we discussed in part I of this book. Thus, rather then $\delta\psi_m = \xi^\mu D_\mu\psi_m$ we would like to use the transformation rule $\delta\psi_m = \frac{1}{2}(\xi^\mu D_\mu(\omega) + D_\mu(\omega)\xi^\mu)\psi_m$. Note that the same operator $D_\mu(\omega)$ should appear in the term $\xi^\mu D_\mu(\omega)$ as in the term $D_\mu(\omega)\xi^\mu$. This is not a covariant expression, of course, because for it to be covariant would also need a Christoffel symbol $\Gamma^\nu_{\nu\mu}$ to take care of the index of ξ^μ. **However, if we take $\tilde{\psi}_m = g^{1/4}\psi_m$ as basic variable, the transformation rule $\delta\tilde{\psi}_m = \frac{1}{2}(\xi^\mu D_\mu(\omega) + D_\mu(\omega)\xi^\mu)\tilde{\psi}_m$ is covariant**. Let us prove this.

Consider a covariant Einstein transformation, a combination of an Einstein transformation with parameter ξ^μ and a local Lorentz transformation with composite parameter $\lambda^{mn} = \xi^\mu {\omega_\mu}^{mn}$. Then the field $\tilde{\psi}_m$ transforms as follows

$$\delta_{cov}(\xi)\tilde{\psi}_m = \xi^\mu \partial_\mu \tilde{\psi}_m + \frac{1}{2}(\partial_\mu \xi^\mu)\tilde{\psi}_m + \frac{1}{4}\xi^\mu \omega_{\mu rs}\gamma^r\gamma^s\tilde{\psi}_m + \xi^\mu {\omega_{\mu m}}^n \tilde{\psi}_n \ . \tag{6.6.4}$$

The term $\frac{1}{2}(\partial_\mu \xi^\mu)\tilde{\psi}_m$ is needed in the transformation rule of a half-density according to the rules of tensor calculus in general relativity. We can rewrite this results as follows

$$\delta_{cov}(\xi)\tilde{\psi}_m = \frac{1}{2}(\xi^\mu D_\mu(\omega) + D_\mu(\omega)\xi^\mu)\tilde{\psi}_m \tag{6.6.5}$$

where both the derivatives can act on $\tilde{\psi}_m$. This is indeed a covariant transformation law in Weyl ordered form and we used it in the spin 1/2 case, but it does not lead to the easiest way to compute the anomaly for spin 3/2. The easiest way to derive the anomaly is to use the law used in [1], and we now proceed to derive it.

The crux to the derivation of the transformation law in (6.6.1) is an identity satisfied by the regulator $\mathcal{R}$ where

$$\mathcal{R} = -\frac{1}{2}\tilde{\not{D}}\tilde{\not{D}} \ , \qquad \tilde{\not{D}} = g^{1/4}\gamma^\mu D_\mu(\omega)g^{-1/4} \tag{6.6.6}$$

and $D_\mu(\omega)$ is the derivative which also appears in $\delta_{cov}\tilde{\psi}_m$. The operator $\tilde{\not{D}}$ is the field operator for $\tilde{\psi}_m$ in the Dirac action $\mathcal{L} = \bar{\psi}^m \sqrt{g}\not{D}\psi_m = \bar{\tilde{\psi}}^m \tilde{\not{D}}\tilde{\psi}_m$. One obtains the second form of $\mathcal{L}$ by changing variables from ψ_m to $\tilde{\psi}_m = g^{1/4}\psi_m$. We can write $\delta_{cov}\tilde{\psi}_m$ in terms of derivatives $\tilde{D}_\mu(\omega)$ as follows

$$\delta_{cov}(\xi)\tilde{\psi}_m = \frac{1}{2}(\xi^\mu \tilde{D}_\mu(\omega) + \tilde{D}_\mu(\omega,\Gamma)\xi^\mu)\tilde{\psi}_m \tag{6.6.7}$$

where the Γ in $\tilde{D}_\mu(\omega,\Gamma)$ acts on the index ξ^μ as usual. It is easy to show that this expression is the same as (6.6.5), because the two terms $\frac{1}{2}\xi^\mu g^{1/4}\partial_\mu g^{-1/4}$ cancel the term $\frac{1}{2}{\Gamma_{\mu\nu}}^\mu \xi^\nu$. We showed this already for the spin 1/2 case in (6.3.13) We are now ready to derive the identity we need. It reads

Lemma : $$An_{cov}(\xi^\mu) = -2{An_{lL}}^{(1/2)}(D_{[m}\xi_{n]}) \ . \tag{6.6.8}$$

In other words, the anomaly $An = \mathrm{Tr}Je^{\mathcal{R}}$ with $\mathcal{R}$ given by (6.6.6) and J the Jacobian for δ_{cov} is the same as minus twice the anomaly with the same $\mathcal{R}$ but J due to a local Lorentz transformation which **only acts on the spin 1/2 index** of the gravitino.

The proof of this lemma is the same as in the spin 1/2 case, see (6.3.15), but since there are now also terms acting on the vector index of the gravitino, we shall present the complete proof for spin 3/2 case. Consider the expression

$$\mathrm{Tr}\,\gamma_5\Big(\not{\xi}\tilde{\not{D}}^{(2\omega)} + \tilde{\not{D}}^{(2\omega)}\not{\xi}\Big)\, e^{\tilde{\not{D}}^{(2\omega)}\tilde{\not{D}}^{(2\omega)}} \tag{6.6.9}$$

where we repeat that $\tilde{\not{D}}^{(2\omega)} = \gamma^\mu \tilde{D}^{(2\omega)}_\mu$ has no Γ-term, but two ω-terms, one acting on the spin 1/2 index and the other acting on the spin 1 index. We pull the second $\tilde{\not{D}}^{(2\omega)}$ to the left past the matrix γ_5 (this yields a minus sign), and then, using cyclicity of the trace, and commuting $\tilde{\not{D}}^{(2\omega)}$ past $e^{\tilde{\not{D}}^{(2\omega)}\tilde{\not{D}}^{(2\omega)}}$ one obtains zero

$$\begin{aligned}&\mathrm{Tr}\,\gamma_5\Big(\not{\xi}\not{D}^{(2\omega)} + \not{D}^{(2\omega)}\not{\xi}\Big)\, e^{\tilde{\not{D}}^{(2\omega)}\tilde{\not{D}}^{(2\omega)}}\\ &\quad = \mathrm{Tr}\,\gamma_5\Big(\not{\xi}\not{D}^{(2\omega)} - \not{\xi}\not{D}^{(2\omega)}\Big)\, e^{\tilde{\not{D}}^{(2\omega)}\tilde{\not{D}}^{(2\omega)}} = 0\ .\end{aligned} \tag{6.6.10}$$

Next we rewrite (6.6.9). We pull the two Dirac matrices to the left, and taking symmetric and antisymmetric parts, we obtain

$$\begin{aligned}0 &= \Big(\not{\xi}\tilde{\not{D}}^{(2\omega)} + \tilde{\not{D}}^{(2\omega)}\not{\xi}\Big) = \gamma^\mu\gamma^\nu\Big(\xi_\mu \tilde{D}^{(2\omega)}_\nu + \tilde{D}^{(2\omega,\Gamma)}_\mu \xi_\nu\Big)\\ &= \Big(\xi^\mu \tilde{D}^{(2\omega)}_\mu + \tilde{D}^{(2\omega,\Gamma)}_\mu \xi^\mu\Big) + \gamma^{\mu\nu}\Big(\xi_\mu \tilde{D}^{(2\omega)}_\nu - \tilde{D}^{(2\omega,\Gamma)}_\nu \xi_\mu\Big)\ .\end{aligned} \tag{6.6.11}$$

The Γ-term in the last $\tilde{D}^{(2\omega,\Gamma)}_\nu$ derivative acts on the vector index of ξ_μ. Note now that in

$$\xi_\mu \tilde{D}^{(2\omega)}_\nu - \tilde{D}^{(2\omega,\Gamma)}_\nu \xi_\mu = -\Big(D^{(\Gamma)}_\nu \xi_\mu\Big) \tag{6.6.12}$$

the derivative no longer acts past ξ_μ. The Γ-term cancels after contracting with $\gamma^{\mu\nu}$. Thus, inside the regulated trace one has the identity

$$\Big(\xi^\mu \tilde{D}^{(2\omega)}_\mu + \tilde{D}^{(2\omega,\Gamma)}_\mu \xi^\mu\Big) - \gamma^{\mu\nu}(\partial_\nu \xi_\mu) = 0\ . \tag{6.6.13}$$

The last term can be written with flat indices as follows

$$\gamma^{\mu\nu}(\partial_\nu \xi_\mu) = \gamma^{mn}(D^{(\omega)}_n \xi_m) \tag{6.6.14}$$

where $D^{(\omega)}_n \xi_m = e_n{}^\mu(\partial_\mu \xi_m + \omega_{\mu m}{}^n \xi_n)$. In the first term we can replace $\tilde{D}_\mu$ by D_μ and drop the Γ-term in $\tilde{D}^{(2\omega,\Gamma)}_\nu$ because these three Γ-term cancel each other. So finally

$$\Big(\xi^\mu D^{(2\omega)}_\mu + D^{(2\omega)}_\mu \xi^\mu\Big) - \gamma^{mn}(D^{(\omega)}_n \xi_m) = 0\ . \tag{6.6.15}$$

Recalling the definition of δ_{cov} in (6.6.5), we have found

$$\mathrm{Tr}\Big[\delta_{cov}(\xi^\mu) + 2\delta_{lL}^{(1/2)}\Big(\frac{D_m^{(\omega)}\xi_n - D_n^{(\omega)}\xi_m}{2}\Big)\Big]e^{\mathcal{R}} = 0 \ . \qquad (6.6.16)$$

This concludes the proof of the lemma in (6.6.8).

To calculate the gravitational anomalies, Alvarez-Gaumé and Witten did not use δ_{cov} to obtain the Jacobian, but rather $2\delta_{sym}$, where δ_{sym} is the same combinations of symmetries as in the spin 1/2 case

$$2\delta_{sym}(\xi) = 2\delta_{cov}(\xi) + 2\delta_{lL}(\frac{D_m\xi_n - D_n\xi_m}{2}) \ . \qquad (6.6.17)$$

For the spin 3/2 field a local Lorentz transformation contains a part which acts on the spin 1/2 index and also a part which acts on the vector index

$$2\delta_{sym}\tilde{\psi}_m = 2\delta_{cov}\tilde{\psi}_m + 2\delta_{lL}^{1/2}\tilde{\psi}_m + 2\delta_{lL}^{1}\tilde{\psi}_m \ . \qquad (6.6.18)$$

Using (6.6.16) this can also be written as

$$\delta_{AW} = 2\delta_{sym} = \delta_{cov}(\xi^\mu) + 2\delta_{lL}^{(1)}(D_{[m}\xi_{n]}) \qquad (\text{spin } 3/2) \ . \qquad (6.6.19)$$

This is precisely the mysterious transformation law in (6.6.1)!

We now turn to the calculation of the gravitational anomaly for complex chiral spin 3/2 fields in n dimensions using

$$\delta_{AW}\ \tilde{\psi}_m = \frac{1}{2}(\xi^\mu D_\mu + D_\mu\xi^\mu)\tilde{\psi}_m + [(D_m\xi^n) - (D^n\xi_m)]\tilde{\psi}_n \ . \qquad (6.6.20)$$

There is, of course, one question left. Are the transformations laws in the spin 1/2 and spin 3/2 case the same combinations of Einstein and local Lorentz transformations? Obviously they should be the same if one wants to study cancellation of anomalies in theories with different spin contents. The spin 1/2 transformation law we used to compute the gravitational anomaly was

$$\delta_{cov}(\xi)\tilde{\lambda} = \frac{1}{2}(\xi^\mu D_\mu + D_\mu\xi^\mu)\tilde{\lambda} \ . \qquad (6.6.21)$$

We proved the identity

$$\delta_{cov}(\xi)\tilde{\lambda} + 2\delta_{lL}(D_{[m}\xi_{n]})\tilde{\lambda} = 0 \ . \qquad (6.6.22)$$

We also encountered another combination of symmetries

$$\delta_{sym}(\xi)\tilde{\lambda} = \delta_{cov}(\xi)\tilde{\lambda} + 2\delta_{lL}(D_{[m}\xi_{n]})\tilde{\lambda} \ . \qquad (6.6.23)$$

Hence, for spin 1/2, $\delta_{cov}\tilde{\lambda}$ is twice $\delta_{sym}\tilde{\lambda}$

$$\delta_{cov}\tilde{(\xi)}\lambda = 2\delta_{sym}(\xi)\tilde{\lambda} \ . \tag{6.6.24}$$

For spin 3/2 fields, we have just derived that the AWG law is twice δ_{sym}

$$\delta_{AW}\tilde{\psi}_m = 2\delta_{sym}(\xi)\tilde{\psi}_m \ . \tag{6.6.25}$$

Hence, if one uses $\delta_{cov}\tilde{\lambda}$ to compute anomalies in the spin 1/2 case, we should use $\delta_{AW}\tilde{\psi}_m$ to compute the same anomalies in the spin 3/2 case.

The calculation is similar to the calculation for the spin 1/2 case in section 6.3, except that we treat the last term in (6.6.20) as a Yang–Mills symmetry, so with extra ghosts according to the methods of section 6.2. As in (6.4.6) the covariant derivative D_μ contains both a spin connection acting on the spinor index and a spin connection acting on the vector index of the gravitino; the whole D_μ becomes the covariant conjugate momentum π_i, see (6.4.8). After integrating out the momenta one obtains a term $\dot{q}^i\omega_{iab}\psi_1^a\psi_1^b$ which yields $\dot{q}^i q^j R_{ijab}\psi_1^a\psi_1^b$, as in (6.5.17).

The contribution from $\delta_{cov}(\xi)$ combines with a term $R_{ijkl}\psi^i\psi^j\dot{q}^k q^l$ in the action into the combination

$$\Big(\frac{1}{4}R_{ijkl}\psi^i\psi^j - D_{[k}\xi_{l]}\Big)\dot{q}^k q^l \ . \tag{6.6.26}$$

We encountered this combination in the spin 1/2 case. The contribution from $2\delta^{(1)}_{lL}(D_{[m}\xi_{n]})$ combines with a term $\psi^i\psi^j R_{ijkl}c^{*k}c^l$ into

$$\Big(\frac{1}{4}R_{ijkl}\psi^i\psi^j - D_{[k}\xi_{l]}\Big)c^{*k}c^l \ . \tag{6.6.27}$$

So, thanks to the extra spin 1 Lorentz transformation in (6.6.1), the final answer only depends on the combination $\Big(\frac{1}{4}R_{ijkl}\psi^i\psi^j - D_{[k}\xi_{l]}\Big)$, both in the spin 1/2 sector and in the spin 1 sector. This is closely related to the descent equations from two dimensions higher [111, 2].

We can then directly write down the result for the gravitational anomaly for a complex chiral gravitino in n dimensions

$$An(\text{grav, spin } 3/2) = \frac{(-i)^{\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}}\int\left(d^n x_0^i\sqrt{g(x_0)}\right)\left(\prod_{a=1}^{n} d\psi^a_{1,bg}\right)$$
$$\Big[\Big(\mathrm{tr}\, e^{\frac{1}{2}\tilde{R}}\Big) - 1\Big]\exp\frac{1}{2}\mathrm{tr}\ln\left(\frac{\tilde{R}/4}{\sinh(\tilde{R}/4)}\right) \tag{6.6.28}$$

where everywhere $\tilde{R}$ stands for $R_{abcd}\psi^a_{1,bg}\psi^b_{1,bg} - 2(D_a\xi_b - D_b\xi_a)$. Thus the relative normalizations of the two terms in (6.6.20) is just so that the

Einstein-Lorentz anomaly in $4k+2$ dimensions is obtained from the chiral anomaly (more precisely for the gravitational contribution to the chiral $U(1)$ anomaly in $4k+4$ dimensions) by the *uniform* shift $R \to R - \frac{1}{4}D\xi$.[17]

The last factor comes from the spin 1/2 sector, see (6.3.25), and takes into account the transformation law $\delta\tilde{\psi}_m = \frac{1}{2}(\xi^n D_n + D_n \xi^n)\tilde{\psi}_m$. The first factor takes into account the vector index of $\tilde{\psi}_m$, see (6.2.48), and gets its contributions from $\delta\tilde{\psi}_m = (D_m\xi^n - D^n\xi_m)\tilde{\psi}_n$. The Yang–Mills curvature F of (6.2.46) is replaced by R. Finally the term -1 is due to the ghost sector; as we have discussed, we need to subtract one chiral complex ghost. If one is dealing with real chiral spin 3/2 fields, one needs to divide the result by 2.

6.7 Lorentz anomalies for selfdual antisymmetric tensor fields coupled to gravity in $4k+2$ dimensions

In addition to chiral fermions, also selfdual (or antiselfdual) antisymmetric tensor gauge fields in $4k+2$ dimensions can produce gravitational (Lorentz) anomalies. From string theory one already knows an example: a chiral boson in 2 dimensions is a selfdual antisymmetric tensor ($\partial_\mu\varphi = \epsilon_{\mu\nu}\partial^\nu\varphi$ implies $(\partial_0 + \partial_1)\varphi = 0$), and in string theory such a field has a gravitational anomaly. To discuss the higher dimensional case, we first need some formalism for antisymmetric tensor (AT) gauge fields.

The field strength and the Lagrangian for an arbitrary antisymmetric tensor gauge field with p indices (a p-form) in Minkowski space are defined by

$$\begin{aligned} F_{\mu_1\ldots\mu_{p+1}} &= \partial_{\mu_1} A_{\mu_2\ldots\mu_{p+1}} \pm p \text{ cyclic permutations} \\ \mathcal{L} &= -\frac{e}{2\cdot(p+1)!} F_{\mu_1\ldots\mu_{p+1}} F_{\nu_1\ldots\nu_{p+1}}\, g^{\mu_1\nu_1}\ldots g^{\mu_{p+1}\nu_{p+1}} \end{aligned} \tag{6.7.1}$$

with $e = \det e_\mu^m$. For a scalar and a vector field these definitions yield the Klein-Gordon and Maxwell actions, respectively. The stress tensor follows from the coupling to gravity

$$\mathcal{L} = \frac{1}{2} h^{\mu\nu} T_{\mu\nu}(F) + O(h^2) \tag{6.7.2}$$

[17] The D_μ's in $\frac{1}{2}(\xi^\mu D_\mu + D_\mu \xi^\mu)$ lead again to a term with $\dot{q}^i \xi^j (x_0+q) g_{ij}(x_0+q)$ in the action which can be written in the form $\dot{q}^i q^k D_k \xi_i$, see the last term in (6.3.23). Together with the term with $R_{ijab}\psi_1^a\psi_1^b$ due to expanding the $\dot{q}^i\omega_{iab}$ term in the action, these two contributions yield the term with $\exp \frac{1}{2} \mathrm{tr} \ln \left(\frac{\tilde{R}/4}{\sinh(\tilde{R}/4)}\right)$. On the other hand the term $c_a^*(R^a{}_{bcd}\psi_1^c\psi_1^d)c^b$ coming from the commutator $\gamma^\mu\gamma^\nu[D_\mu, D_\nu]$ in the regulator and the last term in (6.6.20) together produce $\mathrm{tr}\, e^{\frac{1}{2}\tilde{R}}$ as in (6.5.17) and (6.5.18).

and reads in flat space

$$T_{\mu\nu}(F) = \frac{1}{p!} F_{\mu\mu_1\ldots\mu_p} F_\nu{}^{\mu_1\ldots\mu_p} - \frac{1}{2\cdot(p+1)!}\eta_{\mu\nu}(F_{\mu_1\ldots\mu_{p+1}})^2 \tag{6.7.3}$$

where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. With these normalizations the kinetic term has the standard form $\mathcal{L} = \frac{1}{2\cdot p!}(\partial_t A_{\mu_1\ldots\mu_p})^2 + \ldots$ and the stress tensor for a field strength with $2k+1$ indices in $4k+2$ dimensions is traceless. Generalizing the Lorentz gauge for a vector field, we add a gravitationally covariant gauge fixing term for the abelian gauge symmetry

$$\mathcal{L}_{fix} = -\frac{e}{2\cdot(p-1)!}(g^{\mu\mu_1} D_\mu A_{\mu_1\ldots\mu_p})^2 \tag{6.7.4}$$

and find then a diagonal kinetic term

$$\mathcal{L} + \mathcal{L}_{fix} = -\frac{e}{2\cdot p!}(D_\mu A_{\mu_1\ldots\mu_p})^2 \ . \tag{6.7.5}$$

We shall use tensors with flat indices, and in that case the covariant derivatives will contain spin connections instead of Christoffel connections. Faddeev-Popov ghosts will also be needed in general, but they will not contribute to the chiral anomalies (they do contribute to the trace anomalies).

Consider a one-loop graph with an antisymmetric tensor field in the loop coupled to external gravity. Let the field strength be selfdual; we shall denote such fields by selfdual AT. In Minkowski spacetime this is only possible in $4k+2$ dimensions

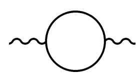
$$F_{\mu_1\ldots\mu_{2k+1}} = \frac{e}{(2k+1)!}\epsilon_{\mu_1\ldots\mu_{2k+1}\nu_1\ldots\nu_{2k+1}} F^{\nu_{2k+1}\ldots\nu_1} \ . \tag{6.7.6}$$

(On the other hand, in Euclidean space this is only possible in $4k$ dimensions, instantons being an example with $k = 1$ [18]). We shall consider real selfdual AT in Minkowski space; if $A_{\mu_1\ldots\mu_{2k}}$ is complex, the anomaly is twice as large. In general no covariant action is known for a selfdual antisymmetric tensor field with which one can easily compute[19] although actions which are not (manifestly) covariant exist [76, 77, 78] whose field equations are the duality conditions (which, together with the Bianchi identity $\partial_{[\mu} F_{\mu_1\ldots\mu_{2k+1}]} = 0$ imply the field equation $\nabla^{\mu_1} F_{\mu_1\ldots\mu_{2k+1}} = 0$). They yield the correct gravitational anomalies [78], but because they are not covariant they look unusual.

[18] In Euclidean space $\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\epsilon^{\alpha\beta\sigma\tau} = +(\delta_\mu^\sigma\delta_\nu^\tau + \cdots)$, but also in Minkowski space $\frac{1}{3!}\epsilon_{\mu\nu\rho\alpha\beta\gamma}\epsilon^{\alpha\beta\gamma\sigma\tau\kappa} = +(\delta_\mu^\sigma\delta_\nu^\tau\delta_\rho^\kappa + \cdots)$ because interchanging $\alpha\beta\gamma$ and $\sigma\tau\kappa$ yields a minus sign that compensates the minus sign in $\epsilon_{012345} = -\epsilon^{012345}$.

[19] A classical covariant action with scalar fields in the denominator does exist [79], but it is not clear how to covariantly gauge fix it.

The first question that one would like to be answered is: are there really anomalies in the conservation of the stress tensor for selfdual antisymmetric tensor fields? We already mentioned an example: chiral bosons in two dimensions. In 1+1 dimensions a selfdual scalar satisfies the equation $\partial_\mu \varphi = e\epsilon_{\mu\nu}\partial^\nu \varphi$, which in flat space light cone coordinates becomes $\partial_- \varphi = (\partial_t - \partial_x)\varphi = 0$. This defines what is known as a chiral boson. In conformal field theory a real chiral boson can be fermionized to a complex chiral fermion, and since chiral fermions do have gravitational (Lorentz) anomalies in $4k+2$ dimensions, as we saw in section 6.3, we have obtained an example of a selfdual antisymmetric tensor with a gravitational anomaly. One can also directly compute the diagram which contains the anomaly, similar to the calculation of the anomaly of a spin 1/2 field in 2 dimensions which we performed in section 5.2. The corresponding diagram for a scalar in the loop reads

$V(p) =$  $= \int d^2x \, \langle T_{++}(x) T_{++}(y) \rangle e^{ip(x-y)}$. (6.7.7)

If $\partial_-\varphi = 0$, on-shell the coupling to gravity reduces to $\frac{1}{2} h^{++} T_{++}$ where $T_{++} = \partial_+\varphi\partial_+\varphi$, and we can compute $V(p)$ using either x-space methods of conformal field theory or momentum space methods [1].

A non-manifestly covariant action describing a chiral boson in two dimensions has been introduced by Floreanini and Jackiw in [76] and coupled to gravity in [77, 135]. It can be used to prove explicitly the existence of a gravitational anomaly for this bosonic system, as we shall briefly review now. The action describing a chiral boson coupled to gravity is

$$\mathcal{L} = \dot{\varphi}\varphi' - F\varphi'\varphi' \tag{6.7.8}$$

where dot and prime indicate derivatives with respect to time $x^0 = \tau$ and space $x^1 = \sigma$, and $F = \frac{e_0{}^+}{e_1{}^+} = -\frac{E_-{}^1}{E_-{}^0}$. It is convenient for the moment to denote by $E_a{}^\mu$ the inverse of the vielbein $e_\mu{}^a$. The equation of motion reads

$$\frac{\partial}{\partial \sigma}\Big(\dot{\varphi} - F\varphi'\Big) = 0 \tag{6.7.9}$$

and with suitable spacelike boundary conditions it gives the correct chiral equation in curved space

$$\dot{\varphi} - F\varphi' = 0 \qquad \rightarrow \qquad E_-{}^\mu \partial_\mu \varphi = 0 \ . \tag{6.7.10}$$

The Lagrangian is not manifestly covariant. Nevertheless it is invariant under the following general coordinate transformations

$$\begin{aligned} \delta\varphi &= (\xi^1 + F\xi^0)\varphi' \\ \delta e_\mu{}^a &= \xi^\nu \partial_\nu e_\mu{}^a + (\partial_\mu \xi^\nu) e_\nu{}^a \end{aligned} \tag{6.7.11}$$

from which it follows

$$\delta F = \xi^\nu \partial_\nu F + \partial_0 \xi^1 + F(\partial_0 \xi^0 - \partial_1 \xi^1) - F^2 \partial_1 \xi^0 \ . \tag{6.7.12}$$

Note that on-shell the transformation rule of the chiral boson φ coincides with the usual transformation rule of a scalar field.

To compute the gravitational anomaly it is convenient to express F in terms of the variable $h_{--} = \frac{e_1{}^+ - e_0{}^+}{e_1{}^+ + e_0{}^+} = -\frac{e_{\tilde{-}}{}^-}{e_{\tilde{+}}{}^-}$, already used in section 5.1, see eq. (5.1.56). One then finds $F = \frac{1-h_{--}}{1+h_{--}}$. It is now easy to extract from the Lagrangian the linearized coupling to h_{--}

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int} = (\dot{\varphi}\varphi' - \varphi'\varphi') + 2h_{--}\varphi'\varphi' + \cdots \ . \tag{6.7.13}$$

The free propagator is given by

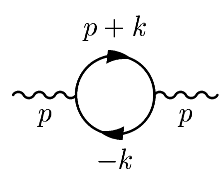
$$\begin{aligned} \langle \varphi(x)\varphi(y)\rangle &= -\frac{i}{2}\Big(\partial_1(\partial_1 - \partial_0)\Big)^{-1} \delta^2(x-y) \\ &= \int \frac{d^2p}{(2\pi)^2}\, e^{ip\cdot(x-y)}\, \frac{i}{\sqrt{2}} \frac{p_+}{p_1} \frac{1}{p^2 - i\epsilon} \end{aligned} \tag{6.7.14}$$

where $p\cdot x = p_+ x^+ + p_- x^-$, $p^2 \equiv p_\mu p^\mu = -2p_+p_-$, $p_\pm = \frac{1}{\sqrt{2}}(p_0 \pm p_1)$ and $-i\epsilon$ is the Feynman prescription giving the correct causal boundary conditions. The leading term of the effective action is then

$$\begin{aligned} W^{(2)}[h] &= \frac{i}{2}\langle S_{int}^2 \rangle \\ &= \frac{i}{2}\int\!\!\int d^2x d^2y\ 2h_{--}(x)\Big\langle \varphi'(x)\varphi'(x)\,\varphi'(y)\varphi'(y)\Big\rangle 2h_{--}(y) \\ &= \frac{i}{2}\int \frac{d^2p}{(2\pi)^2}\ h_{--}(p)U(p)h_{--}(-p) \end{aligned} \tag{6.7.15}$$

where

$$U(p) \equiv 4\int d^2x\, e^{-ip\cdot x}\,\Big\langle \varphi'(x)\varphi'(x)\,\varphi'(0)\varphi'(0)\Big\rangle =$$



$$= -4\int \frac{d^2k}{(2\pi)^2}\, \frac{(p_1+k_1)(p_+ + k_+)}{(p+k)^2 - i\epsilon}\, \frac{k_1 k_+}{k^2 - i\epsilon} \tag{6.7.16}$$

Analytic regularization can be employed as in section 5.2 to obtain

$$U(p) = \frac{i}{24\pi}\Big(\frac{p_+^3}{p_-} - 3p_+^2\Big) \ . \tag{6.7.17}$$

Up to the local term $-3p_+^2$, which can be canceled by a counterterm, the resulting effective action

$$W^{(2)}_{eff} = -\frac{1}{48\pi}\int \frac{d^2p}{(2\pi)^2} h_{--}(p)\Big(\frac{p_+^3}{p_-} - 3p_+^2\Big)h_{--}(-p) \qquad (6.7.18)$$

produces the expected gravitational anomaly, as in (5.1.62) for the chiral fermion (there we looked at the opposite chirality). These calculations confirm that there is a genuine gravitational anomaly for this selfdual antisymmetric tensor field. This construction can be extended to $4k+2$ dimensions to calculate the correct gravitational anomalies for selfdual AT fields using the Feynman rules obtained from an action [78].

Let us now sketch how we are going to compute the gravitational anomaly of the real selfdual AT in $n = 4k+2$ dimensions, using quantum mechanics and following [1]. First we add a whole array of other real AT which are not selfdual and which therefore have no anomalies: $F = 0, F_\mu = \partial_\mu A, F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \ldots, F_{\mu_1\ldots\mu_n} = 0$. The reason we add these AT is based on a simple but useful fact: one can use bispinors[20] $\psi_{\alpha\beta}$ to describe their field strengths all at once

$$\begin{aligned}
\psi_{\alpha\beta} &= \frac{1}{2^{n/4}}\sum_{l=0}^{n}\frac{1}{l!}(\gamma^{m_l\ldots m_1})_{\alpha\beta}\,F_{m_1\ldots m_l}\ , \quad n = 4k+2\\
F_{m_1\ldots m_l} &= \frac{1}{2^{n/4}}\,\psi_{\alpha\beta}(\gamma_{m_1\ldots m_l})^{\beta\alpha}\ . \qquad (6.7.19)
\end{aligned}$$

For example in 2 dimensions we have $\psi_{\alpha\beta} = \frac{1}{\sqrt{2}}\gamma^\mu_{\alpha\beta}\partial_\mu\varphi$. Chirality of $\psi_{\alpha\beta}$, defined by $(\gamma_5)_\alpha{}^{\alpha'}\psi_{\alpha'\beta} = \psi_{\alpha\beta}$, implies selfduality of the AT. Since the AT only couple to gravity by means of their field strength $F_{m_1\ldots m_l}$ we can build Feynman graphs if we know the vertices for the interaction of $F_{m_1\ldots m_l}$ with gravity, and the propagators of $F_{m_1\ldots m_l}$. Knowing the vertices and propagators of $F_{m_1\ldots m_l}$ we can construct those for $\psi_{\alpha\beta}$. The calculation of Feynman graphs along these lines was performed in [1], and discussed in a textbook [3]. Here we are interested in the QM approach to these problems. We shall write down a covariant transformation law for $\tilde\psi_{\alpha\beta} \equiv g^{\frac14}\psi_{\alpha\beta}$, compute the corresponding Jacobian, and use a regulator $\exp(-\beta\tilde{\mathcal{R}})$ with $\mathcal{R} = \tilde{\not{D}}\tilde{\not{D}}$ in the trace, just as for the chiral spin 1/2 and 3/2 fields. Here $\tilde{\not{D}}$ is a Dirac operator for the bispinor which will be

[20] Bispinors are also called Dirac-Kähler fermions. They were introduced by Kähler in 1960 [112] and discussed by Banks et al. [113]. Actually, already in 1929 C. Lanczos, nowdays best known for his work in classical mechanics, had studied a modification of the Dirac equation of 1928 with quaternions [114]. He found that this modified theory described an antisymmetric tensor. Quaternions can be represented by the four Pauli matrices $(I, \vec\sigma)$ as the bispinor $\sigma^\mu{}_{\alpha\dot\alpha}$.

described below (see eq. (6.7.36)). For the Jacobian the two indices α and β of $\psi_{\alpha\beta}$ are treated differently, just as the spinor and vector indices of the gravitino. We call the corresponding spaces the α space and the β space. We find again a covariant translation in both the α space and β space, while in the β space we find an additional Lorentz transformation which acts on the spinor index. As in the spin 3/2 case, one needs this extra term in the tranformation law in order that the action of the nonlinear sigma model only contains the combination $\tilde{R}_{ij} = R_{ij} - 2(D_i\xi_j - D_j\xi_i)$. Hence, the only difference from the spin 3/2 case is that the extra Lorentz transformation acts on a spinor index instead of a vector index. We use ghosts c^* and c in β space, and fermions ψ_1^a in α space, again as in the case of spin 3/2. [21] Finally we reduce the trace to Feynman graphs in quantum mechanics and find again the anomaly as a product of a factor for the α trace and a factor for the β trace.

We now give the details. First we discuss the properties of the gamma matrices γ_m in $4k+2$ dimensions. We consider $n = 4k+2$ dimensional Euclidean spaces because the regulator regulates in all directions in momentum space only if we use Euclidean space. In the space with 2,10,18,... dimensions a symmetric Majorana representation exists [107]: all Dirac matrices γ_m can be chosen as real and symmetric $2^{n/2} \times 2^{n/2}$ matrices satisfying $\{\gamma_m, \gamma_n\} = \delta_{mn}$ with $\delta_{mn} = (+1, ..., +1)$.[22] In all even dimensions there is a charge conjugation matrix C_+ satisfying $C_+\gamma_m C_+^{-1} = +\gamma_m^T$, which is related to the usual charge conjugation matrix C_- satisfying $C_-\gamma_m C_-^{-1} = -\gamma_m^T$ by $C_+ = C_-\gamma_5$ as one easily checks. This matrix C_+ is the unit matrix in our case, and we will use it to raise and lower spinor indices. Hence we need not be careful whether the spinor indices are up or down. Furthermore, it is now clear that $\psi_{\alpha\beta}$ in (6.7.19) is real for real $F_{m_1...m_l}$ (actually, as explained below, we use in Euclidean space complex field strengths). The chirality matrix denoted by γ_5 is given

[21] In [1] the extra spinor index β is treated on equal footing with the spinor index α, by introducing a second set of Grassmann variables ψ_2^a in addition to the Grassmann variables ψ_1^a in α space, rather then treating β space as an internal space and using ghosts c_α^* and c^β. In the susy model underlying the approach of [1], the ψ_1^a and ψ_2^a appear symmetrically, and this yields an $N=2$ model. It can be obtained by dimensional reduction from the $N=(1,1)$ model in $1+1$ dimensions, see appendix D.

[22] From supergravity or string theory one knows that there exists a Majorana representation in ten-dimensional Minkowski space with a real matrix γ_5 [133]. Interchanging the matrix γ_0 and γ_5, one obtains a Majorana representation in Euclidean space. The chirality matrix in Euclidean space is equal to $i\gamma_0$, hence purely imaginary and antisymmetric. The same results hold in two dimensions with σ_1 and σ_3. In six Euclidean dimensions, an example of a purely imaginary antisymmetric representation is given by the set of matrices $\gamma_k \otimes \sigma_2$, $i\gamma_1\gamma_2\gamma_3 \otimes \sigma_1$, $i\gamma_1\gamma_2\gamma_3 \otimes \sigma_3$, and $\gamma_5 \otimes I$, where the γ_μ form a Majorana representation in four Minkowski dimensions.

by $\gamma_5 = (-i)^{n/2}\gamma_1 ... \gamma_n$ (where $\gamma_n = -i\gamma_0$ such that $\gamma_n^2 = 1$) so that γ_5 is purely imaginary and antisymmetric with square unity. In the opposite case, Euclidean spaces with 6,14,22,... dimensions, one can choose an antisymmetric purely imaginary representation of the Dirac matrices. Then C_- equals the unit matrix. We shall continue below with the case of 2,10,18,... dimensions, but there is a parallel treatment for the cases $n = 6, 14, 22, ...$

We define

$$\gamma_{m_1...m_p} = \frac{1}{p!}\Big(\gamma_{m_1}...\gamma_{m_p} \pm (p!-1) \text{ permutations}\Big) \ . \tag{6.7.20}$$

So $\gamma_{m_1...m_p}$ is the totally antisymmetric part of the product of p Dirac matrices with strength one. Two formulas one needs are

$$\begin{aligned} (\gamma_{m_1...m_p})^T &= \gamma_{m_p...m_1} \\ \mathrm{Tr}\,\gamma_{m_1...m_p}\gamma_{n_p...n_1} &= 2^{n/2}(\delta_{m_1n_1}...\delta_{m_pn_p} \pm (p!-1)\text{perms of}(n_1...n_p)) \ . \end{aligned} \tag{6.7.21}$$

For example for $n = 2$ and $p = 2$ one has $(\gamma_{12})^T = \gamma_{21}$ and

$$\mathrm{Tr}\,\gamma_{mn}\gamma_{rs} = 2(\delta_{ms}\delta_{nr} - \delta_{mr}\delta_{ns}) \ . \tag{6.7.22}$$

The propagator of an arbitrary AT in Euclidean space is

$$\begin{aligned} \langle A_{\mu_1...\mu_p}(x)A_{\nu_1...\nu_p}(y)\rangle &= \int \frac{d^nk}{(2\pi)^n}e^{ik(x-y)}\frac{1}{k^2} \\ &\times(\delta_{\mu_1\nu_1}...\delta_{\mu_p\nu_p} \pm \ (p!-1) \text{ perms of}(\nu_1...\nu_p)) \ . \end{aligned} \tag{6.7.23}$$

(and $\langle\varphi(x)\varphi(y)\rangle = \int \frac{d^nk}{(2\pi)^n}e^{ik(x-y)}\frac{1}{k^2}$ for a scalar). Our metric is such that $k^2 = k_1^2 + \cdots + k_n^2$ in n-dimensional Euclidean space. For our purposes we need the propagator of two field strengths in momentum space

$$\begin{aligned} \langle F_{\mu_1...\mu_{p+1}}(k)F_{\nu_1...\nu_{p+1}}(-k)\rangle = \frac{1}{k^2}(k_{\mu_1}k_{\nu_1}\delta_{\mu_2\nu_2}...\delta_{\mu_{p+1}\nu_{p+1}} \\ \pm \text{ all permutations of } \mu_i \text{ and all cyclic permutations of } \nu_i) \ . \end{aligned} \tag{6.7.24}$$

This propagator has $(p+1)(p+1)!$ terms.

The propagator of the tensors $F_{\mu_1...\mu_{p+1}}$ determines the propagator of the bispinor. We claim that the latter is given by

$$\langle\psi_{\alpha\beta}(k)\psi_{\gamma\delta}(-k)\rangle = \frac{1}{2k^2}\Big[(\gamma_5 \not{k})_{\alpha\gamma}(\gamma_5 \not{k})_{\beta\delta} + k^2\delta_{\alpha\gamma}\delta_{\beta\delta}\Big] \ . \tag{6.7.25}$$

To prove this formula we insert the definition of the bispinors in (6.7.19) into the left-hand side

$$\frac{1}{2^{n/2}}\sum_{l=0}^{n}\langle F^{\mu_1...\mu_l}(k)F_{\nu_1...\nu_l}(-k)\rangle(\gamma_{\mu_l...\mu_1})_{\alpha\beta}(\gamma^{\nu_l...\nu_1})_{\gamma\delta}\,\frac{1}{l!}\frac{1}{l!}$$

$$= \frac{1}{2^{n/2}} \sum_{l=0}^{n} \frac{1}{k^2} (k^{\mu_1} k_{\nu_1} \delta^{\mu_2}_{\nu_2} ... \delta^{\mu_l}_{\nu_l}) (\gamma_{\mu_l ... \mu_1})_{\alpha\beta} (\gamma^{\nu_l ... \nu_1})_{\gamma\delta} \frac{1}{(l-1)!}$$

$$= \frac{1}{2^{n/2}} \frac{1}{k^2} \sum_{l=0}^{n} k^\mu k^\nu (\gamma_{\mu_l ... \mu_2 \mu})_{\alpha\beta} (\gamma^{\mu_l ... \mu_2}{}_\nu)_{\gamma\delta} \frac{1}{(l-1)!} \ . \qquad (6.7.26)$$

On the other hand, a Fierz rearrangement of the first term on the right-hand side of (6.7.25) yields [23]

$$\frac{1}{2k^2} (\gamma_5 \not{k})_{\alpha\gamma} (\gamma_5 \not{k})_{\beta\delta} = \frac{-1}{2k^2} (\gamma_5 \not{k})_{\alpha\gamma} (\not{k} \gamma_5)_{\delta\beta}$$

$$= \frac{1}{2^{n/2}} \frac{-1}{2k^2} \sum_{l=0}^{n} \frac{1}{l!} (\gamma_{\mu_1 ... \mu_l})_{\alpha\beta} (\not{k} \gamma_5 \gamma^{\mu_l ... \mu_1} \gamma_5 \not{k})_{\delta\gamma}$$

$$= \frac{1}{2^{n/2}} \frac{-1}{2k^2} \sum_{l=0}^{n} \frac{1}{l!} (\gamma_{\mu_1 ... \mu_l})_{\alpha\beta} k^2 (\gamma^{\mu_l ... \mu_1})_{\delta\gamma}$$

$$+ \frac{1}{2^{n/2}} \frac{-1}{2k^2} \sum_{l=0}^{n} \frac{(-1)^l}{(l-1)!} 2k^{\mu_1} (\gamma_{\mu_1 ... \mu_l})_{\alpha\beta} (\not{k} \gamma^{\mu_l ... \mu_2})_{\delta\gamma} \ . \qquad (6.7.27)$$

The first term is a Fierz rearrangement of $-\frac{1}{2k^2} k^2 \delta_{\alpha\gamma} \delta_{\delta\beta}$ and the second term become equal to (6.7.26) after using

$$(\not{k} \gamma^{\mu_l ... \mu_2})_{\delta\gamma} = (-)^{l-1} (\not{k} \gamma^{\mu_2 ... \mu_l})_{\gamma\delta} = (-)^{l-1} k_\nu (\gamma^{\nu \mu_2 ... \mu_l})_{\gamma\delta} \ . \qquad (6.7.28)$$

This proves the expression for the $\psi_{\alpha\beta}$ propagator.

The sum of the stress tensors of the AT fields in terms of bispinors is given by

$$T_{\mu\nu}(\psi) = \frac{1}{4} \psi_{\alpha\beta} \psi_{\gamma\delta} (\gamma_\mu \gamma_5)^{\alpha\gamma} (\gamma_\nu \gamma_5)^{\beta\delta} + (\mu \leftrightarrow \nu) \ . \qquad (6.7.29)$$

Note that the two types of indices α and β are propagated independently in (6.7.25) and do not get mixed by the interactions in (6.7.29).

We shall again calculate the anomaly in Euclidean space, but here we run into the problem that tensors which are selfdual in Minkowski space, no longer are selfdual in Euclidean space because the square of the duality operator $F \to {}^*F$ equals -1 in Euclidean space. To still be able to use Euclidean space we therefore complexify the AT in Minkowski space, and divide the final answer for the anomaly by a factor 2 to undo the complexification.

[23] The general formula reads $M_{\alpha\gamma} N_{\delta\beta} = \frac{1}{2^{n/2}} \sum_{l=0}^{n} \frac{1}{l!} (\gamma_{\mu_1 ... \mu_l})_{\alpha\beta} (N \gamma^{\mu_l ... \mu_1} M)_{\delta\gamma}$ and can be proven by taking the trace with $(\gamma^{\nu_l ... \nu_1})_{\delta\alpha}$ using $\frac{1}{2^{n/2}} \mathrm{Tr} \gamma_{\mu_1 ... \mu_l} \gamma^{\nu_l ... \nu_1} = \delta^{\nu_1}_{\mu_1} ... \delta^{\nu_l}_{\mu_l} \pm (l! - 1)$ permutations.

The duality operation from one field strength to the dual of another becomes $F \to i\,{}^*F$ in Euclidean space. It corresponds to multiplication of one of the indices of $\psi_{\alpha\beta}$ by γ_5, for example

$$\psi_{\alpha\beta} \to (\gamma_5)_\alpha{}^{\alpha'}\psi_{\alpha'\beta}\ . \tag{6.7.30}$$

(Recall that γ_5 is purely imaginary in Euclidean space). In particular the field strength with $2k+1$ indices is mapped into i times its own dual. Hence the matrix $\frac{1}{2}(1+\gamma_5)_\alpha{}^{\alpha'}$ projects this field strength onto to its selfdual part. Consider a Feynman graph with gravitational couplings to $T_{\mu\nu}(F)$ at all vertices, except at one vertex where one couples to a selfdual AT as $\frac{1}{2}h^{\mu\nu}T_{\mu\nu}\Big(\frac{1}{2}(F+i\,{}^*F)\Big)$. It corresponds in the bispinor approach to a loop with couplings to $T_{\mu\nu}(\psi)$ at all vertices, except at one vertex where one projects onto a chiral bispinor (which we denote by ψ_L). The stress tensor at this vertex reads

$$\begin{aligned} T_{\mu\nu}(\psi_L) \;&=\; \frac{1}{4}\Big(\frac{1+\gamma_5}{2}\Big)_\alpha{}^{\alpha'}\psi_{\alpha'\beta}\Big(\frac{1+\gamma_5}{2}\Big)_\gamma{}^{\gamma'}\psi_{\gamma'\delta}(\gamma_\mu\gamma_5)^{\alpha\gamma}(\gamma_\nu\gamma_5)^{\beta\delta} \\ &\quad +(\mu\leftrightarrow\nu)\ . \end{aligned} \tag{6.7.31}$$

The transformation rule δ_{sym} of the AT is, as in the spin 3/2 case, a sum of a covariant translation and a Lorentz transformation; the latter acts on the flat vector indices of the AT

$$\delta_{sym}(\xi)\tilde{F}_{m_1\ldots m_n} = \Big[\frac{1}{2}(D_\mu\xi^\mu+\xi^\mu D_\mu)+\delta_{lL}(D_{[m}\xi_{n]})\Big]\tilde{F}_{m_1\ldots m_n}\ . \tag{6.7.32}$$

In the bispinor approach this corresponds to

$$\begin{aligned} \delta_{sym}(\xi)\tilde{\psi}_{\alpha\beta} \;&=\; \frac{1}{2}(D_\mu\xi^\mu+\xi^\mu D_\mu)\tilde{\psi}_{\alpha\beta} \\ &\quad +\frac{1}{4}D_{[m}\xi_{n]}(\gamma^{mn})_\alpha{}^{\alpha'}\,\tilde{\psi}_{\alpha'\beta}+\frac{1}{4}D_{[m}\xi_{n]}(\gamma^{mn})_\beta{}^{\beta'}\,\tilde{\psi}_{\alpha\beta'} \end{aligned} \tag{6.7.33}$$

where the Lorentz transformation now acts both on the α and β indices. The last term can also be written as $-\frac{1}{4}D_{[m}\xi_{n]}\psi_{\alpha\beta'}(\gamma^{mn})^{\beta'}{}_\beta$ because spinor indices are raised and lowered by the charge conjugation matrix which is the unit matrix. Now we would like to compute the gravitational anomaly due to $2\delta_{sym}(\xi)$, which is precisely the transformation used both in the spin 1/2 and 3/2 cases. To rewrite this transformation in a useful form which will make the calculation easy, we derive again a lemma, this time applied to the bispinor instead of the spin 3/2 field. The lemma uses the regulator $\mathcal{R}=\tilde{\not{D}}\tilde{\not{D}}$, where the Dirac matrices γ^μ which contract D_μ act in α space (i.e. as matrix multiplication from the left). The lemma states that for the anomaly calculation $\delta_{cov}(\xi)$ equals $-2\delta_{lL}^{(\alpha-\text{space})}(D_{[m}\xi_{n]})$, where the Lorentz transformation $\delta_{lL}^{(\alpha-\text{space})}$ acts in

α-space and is analogous to the $\delta_{lL}^{(1/2)}$ acting in the spinor space of the gravitino. Thus with this regulator one finds the relation

$$2\delta_{sym}(\xi) = \delta_{cov}(\xi) + 2\delta_{lL}^{(\beta-\text{space})}(D_{[m}\xi_{n]}) \tag{6.7.34}$$

Then the transformation law of the bispinor density which produces the Jacobian can be written as

$$\begin{aligned}
\delta_{AW}\tilde{\psi}_{\alpha\beta} &= [\delta_{cov}(\xi) + 2\delta_{lL}^{(\beta-\text{space})}(D_{[m}\xi_{n]})]\tilde{\psi}_{\alpha\beta} \qquad (6.7.35)\\
&= \frac{1}{2}(D_\mu\xi^\mu + \xi^\mu D_\mu)_{\alpha\beta}{}^{\alpha'\beta'}\tilde{\psi}_{\alpha'\beta'} + 2\delta_\alpha{}^{\alpha'}\frac{1}{4}D_{[m}\xi_{n]}(\gamma^{mn})_\beta{}^{\beta'}\tilde{\psi}_{\alpha'\beta'} \ .
\end{aligned}$$

The covariant derivative $(D_\mu)_{\alpha\beta}{}^{\alpha'\beta'}$ contains spin connection terms which act on both spinor indices of $\psi_{\alpha\beta}$. (The Lorentz transformations can be transferred from $F_{m_1\ldots m_n}$ to $\psi_{\alpha\beta}$ because the Dirac matrices are Lorentz invariant tensors). The regulator is proportional to $\tilde{\not{D}}\tilde{\not{D}}$ where

$$\begin{aligned}
\tilde{\not{D}} &= g^{1/4}(\gamma^\mu)^\alpha{}_{\alpha''}\Big[\delta_{\alpha'}^{\alpha''}\partial_\mu\delta_{\beta'}^{\beta} + \frac{1}{4}\omega_{\mu mn}(\gamma^{mn})^{\alpha''}{}_{\alpha'}\,\delta_{\beta'}^{\beta}\\
&\qquad + \frac{1}{4}\omega_{\mu mn}(\gamma^{mn})^\beta{}_{\beta'}\,\delta_{\alpha'}^{\alpha''}\Big]g^{-1/4} \ .
\end{aligned} \tag{6.7.36}$$

The connection in the β sector can be treated as a Yang–Mills field, so adding anticommuting ghosts we write

$$\gamma^{mn} = c_\beta^*(\gamma^{mn})^\beta{}_{\beta'}c^{\beta'} \ , \quad \{c^\beta, c_\gamma^*\} = \delta_\gamma^\beta \qquad (\beta \text{ sector}) \tag{6.7.37}$$

This is analogous to the replacement of the internal symmetry generators $(T_a)^I{}_J$ which we discussed before.

The term in the α sector is treated as for the spin 1/2 case, hence in the α sector we set

$$\gamma^{mn} = 2\psi_1^a\psi_1^b \ , \quad \{\psi_1^a, \psi_1^b\} = \delta^{ab} \qquad (\alpha \text{ sector}) \ . \tag{6.7.38}$$

The regulator $\tilde{\not{D}}\tilde{\not{D}}$ leads to a term with $D_\mu D^\mu$ and a term with $\gamma^{\mu\nu}[D_\mu, D_\nu]$. The latter contains curvatures in the α sector and curvatures in the β sector

$$\begin{aligned}
\tilde{\not{D}}\tilde{\not{D}} &= g^{-\frac{1}{4}}D_\mu\sqrt{g}g^{\mu\nu}D_\nu g^{-\frac{1}{4}}\\
&\qquad + \psi_1^a\psi_1^b\Big(\frac{1}{2}R_{abcd}\psi_1^c\psi_1^d + \frac{1}{4}R_{abmn}c^*\gamma^{mn}c\Big) \ .
\end{aligned} \tag{6.7.39}$$

The operator D_μ is given by the expression inside the square brackets in (6.7.36).

The covariant translation in (6.7.35) yields a term $D_{[i}\xi_{j]}q^i\dot{q}^j$ in the α sector, while the extra Lorentz transformation of the β indices (whose parameter was $D_{[m}\xi_{n]}$) produces a term $D_{[m}\xi_{n]}c^*\gamma^{mn}c$ in the β sector. In both the α sector and the β sector we again encounter the combination

$$\frac{1}{4}R_{mnab}\psi_1^a\psi_1^b - D_{[m}\xi_{n]}\ . \tag{6.7.40}$$

The trace in the α sector leads to $q\dot{q}$ loops which produce a factor

$$\exp\frac{1}{2}\mathrm{tr}\ \ln\left(\frac{\tilde{R}/4}{\sinh\tilde{R}/4}\right)\ . \tag{6.7.41}$$

where the trace tr is over the vector indices of $\tilde{R}_{ij} = \tilde{R}_{ijab}\psi_1^a\psi_1^b$. This was discussed in section 6.3. The trace in the β sector produces a trace

$$\mathrm{Tr}\,\mathrm{e}^{F/2} = \mathrm{Tr}\ \exp\frac{1}{8}\psi_1^a\psi_1^b R_{abmn}\gamma^{mn} \tag{6.7.42}$$

where the trace Tr is over the spinor indices of γ^{mn}. This was discussed in section 6.2 (the antihermitian matrices $\frac{1}{2}\gamma^{mn}$ $(m < n)$ correspond to the antihermitian Yang–Mills generators T_α).

Putting these factors together, we find for the gravitational anomaly due to a selfdual real antisymmetric tensor field in n dimensions

$$\begin{aligned} An(\text{grav, AT}) \ =\ & \frac{(-i)^{\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}}\int\left(d^n x_0^i\sqrt{g(x_0)}\right)\left(\prod_{a=1}^{n}d\psi_{1,bg}^a\right) \\ & \left(-\frac{1}{4}\right)\mathrm{Tr}\ \exp\left(\frac{1}{8}\psi_1^a\psi_1^b\tilde{R}_{abmn}\gamma^{mn}\right) \\ & \exp\frac{1}{2}\mathrm{tr}\ln\left(\frac{\tilde{R}/4}{\sinh(\tilde{R}/4)}\right) \end{aligned} \tag{6.7.43}$$

The minus sign in the factor $-\frac{1}{4}$ is due to the fact that we are now computing a loop with bosonic bispinors instead of fermionic fields. The factor $\frac{1}{4}$ is due to the factor $\frac{1}{2}$ from the chiral projection operator $\frac{1}{2}(1+\gamma_5)$ which appears in the Jacobian, and the factor $\frac{1}{2}$ one needs to undo the complexification of the AT which was needed to be able to go to Euclidean space. Since the symmetrized trace of an odd number of Lorentz generators vanishes[24] the first factor only yields products of an even number of $\tilde{R}$ terms,

[24] From group theory we know that $SO(2n)$ has Casimir operators of rank $2, 4, ..., 2n-2, n$. However the trace over n generators in the spinor representation vanishes. In 10 dimensions the first factor yields a contributions with six $\tilde{R}$ factors (a hexagon graph), but not a contribution with five $\tilde{R}$ factors; for example, the product of five Lorentz generators with all indices different from each other is proportional to the trace of the chirality matrix γ_5, which vanishes.

and so does the second factor because $\frac{x}{\sinh x}$ is even in x. It follows that there is **only a gravitational anomaly in $\mathbf{n = 4k+2}$ dimensions**, as in the case of spin 1/2 and 3/2 fields.

The trace over the spinor indices in $\mathrm{Tr}\,\exp\left(\frac{1}{8}\tilde{R}_{mn}\gamma^{mn}\right)$ with $\tilde{R}_{mn} = R_{mnab}\psi_1^a\psi_1^b - 4D_{[m}\xi_{n]}$ can be rewritten as a trace over the vector indices of $\tilde{R}_{mn}$ as follows. We can skew diagonalize the real antisymmetric matrix $\tilde{R}_{mn}$ so that it attains the 2×2 block form

$$\tilde{R}_{mn} = \begin{pmatrix} & x_1 & & & & \\ -x_1 & & & & & \\ & & & x_2 & & \\ & & -x_2 & & & \\ & & & & \cdot & \\ & & & & & \cdot \end{pmatrix} \tag{6.7.44}$$

with real x_j. We then decompose the 2^{2k+1} dimensional spinor space as a direct product of $2k+1$ two-dimensional spinor spaces, and we can choose γ^{mn} such that γ^{12} acts nontrivially only in the first two-dimensional subspace, γ^{34} in the second two-dimensional subspace, etc. Then the exponent of the direct sum becomes the direct product of the exponents

$$\exp\frac{1}{8}\tilde{R}_{mn}\gamma^{mn} = \bigotimes_{l=1}^{n/2}\exp\frac{1}{4}\tilde{R}_{2l-1,2l}\gamma^{2l-1,2l}\ . \tag{6.7.45}$$

In each subspace the trace yields $2\cosh\tilde{R}_{2l-1,2l}$ since the square of $\gamma^{2l-1,2l}$ equals minus unity and $\tilde{R}^2_{mn}$ has $-x_j^2$ along the diagonal. Except for the factor 2^{2k+1} which is the dimension of the spinor space, this is the same result as one obtains from $\exp\mathrm{tr}\ln\cosh\tilde{R}$. Hence we can make the following replacement in the expression for the anomaly

$$\mathrm{Tr}\,\mathrm{e}^{\left(\frac{1}{8}\tilde{R}_{mn}\gamma^{mn}\right)} = 2^{2k+1}\exp\mathrm{tr}\ln\cosh\tilde{R}/4\ . \tag{6.7.46}$$

In the end, we need $2k+2$ factors $\tilde{R}$ because we need one factor with $D_{[m}\xi_{n]}$ and $2k+1$ factors $R_{mnab}\psi_1^a\psi_1^b$ to saturate the integral over the fermionic zero modes. We can then absorb a factor 2^{2k+2} into $\tilde{R}_{mn}$, which leads to another overall factor $\frac{1}{2}$. The overall factor is then $-\frac{1}{8}$.

Our final answer for the gravitational anomaly of a real selfdual AT is given by

$$\begin{aligned} An(\text{grav, AT}) \ &=\ \frac{(-i)^{\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}}\int\left(d^n x_0^i\sqrt{g(x_0)}\right)\left(\prod_{a=1}^{n}d\psi^a_{1,bg}\right) \\ &\qquad \left(-\frac{1}{8}\right)\exp\frac{1}{2}\mathrm{tr}\ln\left(\frac{\tilde{R}/2}{\tanh(\tilde{R}/2)}\right) \end{aligned} \tag{6.7.47}$$

where $\tilde{R} = R_{mnab}\psi_1^a\psi_1^b - 2(D_m\xi_n - D_n\xi_m)$. We need $\frac{1}{2}n+1$ factors of $\tilde{R}$ in n dimensions to saturate the Grassmann integral (one of the $\tilde{R}$ should yield the $D\xi$). The α index has yielded the usual result for spin 1/2 with the sinh, while the β index has yielded a similar result but with cosh. Together they yield the tanh. The prefactor $-\frac{1}{8}$ comes form the $\frac{1}{2}$ in $\frac{1}{2}(1+\gamma_5)$, from the fact that we consider real AT fields, and from the conversion of the trace over β spinor space into a trace over vector indices.

The reason β space gives a different result from α space can be traced to the fact that we acted with the operator $\frac{1}{2}(1+\gamma_5)$ only in the α space to project out the selfdual part of the AT fields. In the approach of ref. [1] this means that the fermions ψ_1^a have periodic boundary conditions whereas the fermions ψ_2^a are antiperiodic (see (2.4.7) and (2.4.9) and footnote 21).

6.8 Cancellation of gravitational anomalies in IIB supergravity

The gravitational anomaly for a complex chiral spin 1/2 field, a complex chiral spin 3/2 field, and a real selfdual antisymmetric tensor field are given by

$$\begin{aligned} An(\text{grav, spin } 1/2) &= \int \exp\frac{1}{2}\text{tr}\ln\left(\frac{\tilde{R}/4}{\sinh(\tilde{R}/4)}\right) \\ An(\text{grav, spin } 3/2) &= \int \left[\left(\text{tr}\, e^{\tilde{R}/2}\right) - 1\right]\exp\frac{1}{2}\text{tr}\ln\left(\frac{\tilde{R}/4}{\sinh(\tilde{R}/4)}\right) \\ An(\text{grav, AT}) &= \int -\frac{1}{8}\exp\frac{1}{2}\text{tr}\ln\left(\frac{\tilde{R}/2}{\tanh(\tilde{R}/2)}\right). \end{aligned} \tag{6.8.1}$$

where we recall that $\tilde{R} = R_{ijab}\psi_{1,bg}^a\psi_{1,bg}^b - 4D_{[i}\xi_{j]}$. The symbol $\int$ denotes the measure $\frac{(-i)^{n/2}}{(2\pi)^{n/2}}\int\prod_{i=1}^n dx_0^i\sqrt{g(x_0)}\prod_{i=a}^n d\psi_{1,bg}^a$.

As a first application we check that in 1+1 dimensions the gravitational anomaly for a complex chiral spin 1/2 field is equal to the gravitational anomaly of a real selfdual antisymmetric tensor (chiral boson[25]). This result is well-known in string theory where it is used in the calculation of the central charge [133]. For this purpose we need the term quadratic in $\tilde{R}$ (one of these $\tilde{R}$ yields the contribution proportional to $D_{[i}\xi_{j]}$). We find

[25] A selfdual antisymmetric tensor in 2 dimensions satisfies $\partial_\mu\varphi = \epsilon_{\mu\nu}\partial^\nu\varphi$ or $(\partial_0 + \partial_1)\varphi = 0$, and thus describes a left-moving (chiral) boson.

for a complex chiral spin 1/2 field

$$\exp\frac{1}{2}\mathrm{tr}\ln\left(\frac{\tilde{R}/4}{\sinh(\tilde{R}/4)}\right) = \exp\Big[-\frac{1}{2}\mathrm{tr}\ln\Big(1+\frac{1}{3!}(\tilde{R}/4)^2+...\Big)\Big]$$
$$= ... - \frac{1}{12}\mathrm{tr}(\tilde{R}/4)^2 + ... \qquad (6.8.2)$$

while for the chiral boson we obtain, using $\tanh x = x - \frac{1}{3}x^3 + \cdots$,

$$-\frac{1}{8}\exp\frac{1}{2}\mathrm{tr}\ln\left(\frac{\tilde{R}/2}{\tanh(\tilde{R}/2)}\right) = -\frac{1}{8}\exp\Big[-\frac{1}{2}\mathrm{tr}\ln\Big(1-\frac{1}{3}(\tilde{R}/2)^2+...\Big)\Big]$$
$$= ... - \frac{1}{48}\mathrm{tr}(\tilde{R}/2)^2 + ... \,. \qquad (6.8.3)$$

Clearly the anomalies are equal.

A less obvious case is IIB supergravity. This theory contains: a complex chiral spin 3/2 field, a complex antichiral spin 1/2 field, and a real five-index selfdual antisymmetric field strength. To check that the sum of these anomalies cancels, too, we must expand the formulas for the anomalies to sixth order in $\tilde{R}$. Let us simplify the notation and denote $\tilde{R}/4$ by y and $\mathrm{tr}\, y^n$ by t_n.

The spin 1/2 field yields

$$\begin{aligned} An(1/2) &= \exp\left[-\frac{1}{2}\mathrm{tr}\ln\left(1+\frac{1}{3!}y^2+\frac{1}{5!}y^4+\frac{1}{7!}y^6+...\right)\right] \\ &= \exp\left[-\frac{1}{12}t_2+\frac{1}{360}t_4-\frac{1}{5670}t_6+...\right] \\ &= 1+\left[-\frac{1}{12}t_2\right]+\left[\frac{1}{360}t_4+\frac{1}{288}t_2^2\right] \\ &\quad +\left[-\frac{1}{5670}t_6-\frac{1}{4320}t_2t_4-\frac{1}{10368}t_2^3\right]+... \,. \end{aligned} \qquad (6.8.4)$$

The spin 3/2 field yields

$$\begin{aligned} An(3/2) &= \left[\mathrm{tr}\left(1+2y^2+\frac{2}{3}y^4+\frac{4}{45}y^6+...\right)-1\right]An(1/2) \\ &= \left((n-1)+2t_2+\frac{2}{3}t_4+\frac{4}{45}t_6+...\right)An(1/2) \\ &= (n-1)+\left[\left(2-\frac{(n-1)}{12}\right)t_2\right] \\ &\quad +\left[\left(\frac{2}{3}+\frac{(n-1)}{360}\right)t_4+\left(-\frac{1}{6}+\frac{(n-1)}{288}\right)t_2^2\right] \\ &\quad +\left[\left(\frac{4}{45}-\frac{(n-1)}{5670}\right)t_6+\left(-\frac{1}{20}-\frac{(n-1)}{4320}\right)t_2t_4 \right. \end{aligned}$$

$$
+\left(\frac{1}{144}-\frac{(n-1)}{10368}\right)t_2^3\Big]+\ldots\ . \tag{6.8.5}
$$

The selfdual AT field yields

$$
\begin{aligned}
An(AT) &= -\frac{1}{8}\exp\left[-\frac{1}{2}\mathrm{tr}\ln\left(1-\frac{1}{3}4y^2+\frac{2}{15}16y^4-\frac{17}{315}64y^6+\ldots\right)\right] \\
&= -\frac{1}{8}\exp\left[\frac{2}{3}t_2-\frac{28}{45}t_4+\frac{1984}{2835}t_6+\ldots\right] \\
&= -\frac{1}{8}+\left[-\frac{1}{12}t_2\right]+\left[\frac{7}{90}t_4-\frac{1}{36}t_2^2\right] \\
&\quad+\left[-\frac{248}{2835}t_6+\frac{7}{135}t_2t_4-\frac{1}{162}t_2^3\right]+\ldots\ .
\end{aligned} \tag{6.8.6}
$$

One may check that for $n=10$ all terms of sixth order in $\tilde{R}$ (the terms with t_6, t_2t_4 and t_2^3) cancel in the following combination

$$
An(3/2)-An(1/2)+An(AT)=0\ . \tag{6.8.7}
$$

Indeed

$$
\begin{aligned}
&\left(\frac{4}{45}-\frac{9}{5670}\right)+\frac{1}{5670}-\frac{248}{2835}=0 \\
&\left(-\frac{1}{20}-\frac{9}{4320}\right)+\frac{1}{4320}+\frac{7}{135}=0 \\
&\left(\frac{1}{144}-\frac{9}{10368}\right)+\frac{1}{10368}-\frac{1}{162}=0\ .
\end{aligned} \tag{6.8.8}
$$

This corresponds to the cancellation of gravitational anomalies in type IIB supergravity [1].

6.9 Cancellation of anomalies in $N=1$ supergravity

As we mentioned in the introduction, Alvarez-Gaumé and Witten derived compact expressions for chiral and gravitational anomalies in any dimensions in 1983 [1]. Then they applied these formulas to IIB supergravity in $9+1$ dimensions where gravitational anomalies are present, and found that they cancel. We discussed this in the preceding section. They also applied these formulas to $N=1$ supergravity in $9+1$ dimensions coupled to Yang-Mills theory, but in this case the sum of all anomalies did not cancel, and they concluded that the $N=1$ theory is anomalous. Green and Schwarz [4] noted that even if anomalies do not seem to cancel, it is sometimes still possible to construct a local counterterm in the action whose variation cancels the anomalies. In such cases one has candidate anomalies which are not genuine anomalies. They indeed were able to construct such counterterms, but only for certain choices of the gauge group of the Yang-Mills theory, namely $SO(32)$ and $E_8 \times E_8$. Thus this constituted a double success: $N = 1$ supergravity was also non-anomalous, and in addition the gauge groups were determined. In this section we show that anomalies in $N = 1$ supergravity can indeed be canceled; this is straightforward and only the expressions of the gravitational anomalies which we obtained before are needed. It should be noted that Green and Schwarz also showed that in string theory the anomalies in the Yang-Mills sector (the open string sector) cancel. They considered the NSR string. There are in this case three sets of string loop diagrams to be computed: a planar graph, nonorientable graphs (graphs with an odd number of twists) and nonplanar graphs (graphs with an even number of twists). The last graphs do not produce anomalies, while the first two graphs contain anomalies which sum up to a complicated expression multiplied by a factor $(1+\frac{32\eta}{n})$, where $\eta = -1$ for $SO(n)$, $\eta = +1$ for $Usp(n)$ and $\eta = 0$ for $U(n)$ [115]. Thus also in string theory the anomalies of the open string cancel, but only for $SO(32)$. The cancellation of the Yang–Mills anomalies in $N = 1$ supergravity for the group $E_8 \times E_8$ corresponds in string theory to cancellation of anomalies of the heterotic string. The analysis of the closed string, which should lead to cancellations of anomalies involving external gravitons has never been worked out. (TRUE???)

The anomaly cancellation in the dual version of $N = 1$ supergravity (with a 6-form instead of a 2-form field B) was given in [116].

After this work on 9+1 dimensional supergravity, similar work was done in other models. For example, in 6 dimensions the authors of [117] studied cancellation of gravitational anomalies for supergravity coupled to several matter multiplets, and found several solutions. There are again $B \wedge R \wedge R$ counterterms [116]. There exist auxiliary fields for $N = 2$ supergravity in 6 dimensions [118], so one could in principle construct supergravity actions with Chern-Simons terms (using tensor calculus supergravity). More recently, anomaly cancellation on $K_3 \times S_1/Z_2$ has been discussed [119].

The field content of $N = 1$ supergravity coupled to $N = 1$ supersymmetric Yang–Mills theory is given by

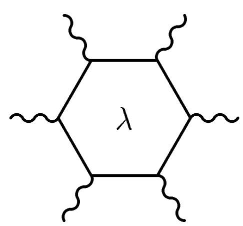
$$(e_\mu{}^m, \psi_{\mu L}, \chi_R, B_{\mu\nu}, \varphi) \quad \text{and} \quad (A_\mu^a, \lambda_L^a) \tag{6.9.1}$$

where $\psi_{\mu L}$ is a Majorana-Weyl (real chiral) gravitino, χ_R a real antichiral "dilatino", and $e_\mu{}^m$ is the real vielbein, $B_{\mu\nu}$ is a real antisymmetric tensor, while φ is the real dilaton. Further A_μ^a is the Yang-Mills field with gauge group G, and λ_L^a are real chiral "gauginos" (partners of the gauge fields). Note that the gauginos are in the same representation of G as the gauge fields, namely the adjoint representation. This is due to supersymmetry which requires that the fermionic partner of the gauge field be in the same representation of the gauge group G as the gauge field. The chirality of the gaugino is the same as that of the gravitino but opposite to that of the dilatino. Furthermore, all fermionic fields are real fields in Minkowski space, so we should add an extra factor $\frac{1}{2}$ to our formulas for anomalies because they were given for complex chiral fermions. However, since we shall require that anomalies cancel, we shall not keep these overall factors $\frac{1}{2}$. Ahead of time we mention that the antisymmetric tensor field $B_{\mu\nu}$, though not selfdual nor antiselfdual, will play a crucial role in the construction of counterterms, and precisely because the representation of the gauginos is the adjoint representation, it is possible to cancel the Yang-Mills anomalies.

For readers not familiar with supersymmetry and supergravity, we mention that the $N = 1$ in "$N = 1$ supergravity" refers to the fact that there is only one real chiral gravitino and one real chiral supersymmetry parameter. The IIB theory has two real chiral gravitinos which one often combines into one complex gravitino (as we did in the previous section). There is also a IIA supergravity theory with one real chiral and one real antichiral gravitino; this is a "vector theory" with one real nonchiral gravitino, which is free from anomalies. (One might call this theory $N = (1, 1)$ supergravity, and the previous one $N = (2, 0)$ supergravity, but this terminology is not common). The number of bosonic states matches the number of fermionic states, both for the $N = 1$ Yang-Mills theory and for the $N = 1$ supergravity theory: $8 = 8$ for the Yang-Mills theory and $(\frac{1}{2}8 \times 9 - 1) + \frac{1}{2}8 \times 7 + 1 = \frac{1}{2}(8 - 1) \times 16 + \frac{1}{2}16 = 64$ for the supergravity theory.

There are three sets of anomalies to be dealt with, which we now first briefly introduce:

I) **Purely Yang-Mills anomalies**. These are due to a hexagon loop with the gaugino λ in the loop coupled to external Yang-Mills fields.



Wiggly lines denote external Yang-Mills gauge fields. There are no loops with a dilatino or gravitino because these fields have no minimal couplings to the Yang-Mills fields. (There are nonminimal couplings of the form $\bar{\psi}^{\mu}\gamma^{\nu}\lambda F_{\mu\nu}$ but these do not lead to anomalies). We shall see that the counterterm which cancels these anomalies in the case of $G = SO(32)$ has the form

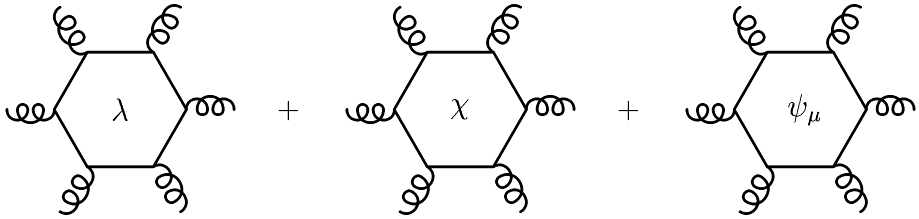
$$\Delta\mathcal{L}^{SO(32)} \sim B\,\mathrm{tr}F^4 + \omega_{3Y}^0\,\omega_{7Y}^0\ . \tag{6.9.2}$$

For $E_8 \times E_8$ the counterterm is different, and we shall construct it in appendix F. The counterterm is a ten-form which is integrated over ten-dimensional space. The symbols ω_k^0 denote Chern-Simons k-forms, and the subscript Y stands for Yang-Mills. The reasons that Chern-Simons actions appear has to do with the fact that the anomalies we derived before are **covariant anomalies**, whereas the counterterm which we shall construct cancel **consistent anomalies**. The latter are obtained by using the descent equations as we shall discuss, and the descent equations produce Chern-Simons terms. If the covariant anomalies cancel, then also the consistent anomalies cancel, and vice-versa. This is discussed in general articles on anomalies [2] and we refer to these articles for proofs. However, the fundamental anomalies are the consistent anomalies because they yield the variation of the effective action. The consistent anomalies can be constructed in two steps: first the general form is given by the descent equations, and then the coefficients are fixed by matching the leading term of the consistent and covariant anomalies (up to an overall constant [2]). Thus we shall cancel consistent anomalies by counterterms, while the covariant anomalies are merely a technical tool.

The hexagon graph is the first graph which can be anomalous (just like the triangle graph is the first graph which is anomalous in four spacetime dimensions). There are also polygon graphs with 7 vertices, 8 vertices, etc., but these graphs merely complete the leading expression from the hexagon graph. The complete consistent anomaly must satisfy the so-called consistency conditions which are so strong that if one knows the leading term of the consistent anomaly (corresponding to hexagon graphs), the other terms are completely fixed. (Because the transformation law $\delta A_\mu = \partial_\mu \Lambda + [A_\mu, \Lambda]$ is nonlinear in A_μ, one obtains relations between terms with different numbers of A fields). The leading term in the consistent anomaly is equal to the leading term in the covariant anomaly up to an overall factor $(\frac{d}{2}+1)^{-1}$, but since we are concerned with the question when anomalies cancel, we shall not keep track of this overall constant. Thus, if the leading terms in the one-loop anomaly cancel, all nonleading terms also cancel. We shall actually obtain directly the complete formulas for the consistent anomalies and the complete counterterms, so we shall not restrict ourselves to only the leading terms.

II) **Purely gravitational anomalies**. The counterpart of the purely Yang-Mills anomalies are the one-loop graphs with only external gravitons. Since all fields couple minimally to gravity, but only chiral fermions yield anomalies (there are no selfdual antisymmetric tensor fields in $N = 1$

supergravity), the graphs to be studied are the following

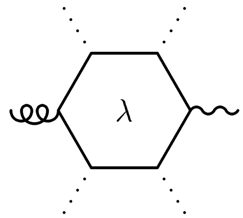


Curly lines denote gravitons. The counterterm which cancels these anomalies will be derived below and has the generic form

$$\Delta\mathcal{L}^{grav} \sim B(\mathrm{tr}R^4 + (\mathrm{tr}R^2)^2) + \omega^0_{3L}\,\omega^0_{7L}\ . \tag{6.9.3}$$

The subscript L stands for Lorentz. This counterterm is of course independent of the gauge group G, so it is the same for the $SO(32)$ theory and the $E_8 \times E_8$ theory. Its structure is very similar to the counterterm in the pure Yang-Mills case, but note that the we now need Lorentz Chern-Simons terms, instead of Yang-Mills Chern-Simons terms. In the $N = 1$ supergravity theory one encounters Yang-Mills Chern-Simons terms in the action and in the transformation rules, but no Lorentz Chern-Simons terms [120]. Thus in order to cancel anomalies one has to go beyond the minimal $N = 1$ supergravity theory. It is not known whether one can construct an extended supergravity theory with a finite number of fields which contains Lorentz Chern-Simons terms in the action. Most experts believe that this is not possible, and that by adding a Lorentz Chern-Simons term to minimal $N = 1$ supergravity, and adding further terms to obtain local supersymmetry, the answer is the full string effective action (whatever that means). Formally, Lorentz Chern-Simons terms are similar to Yang-Mills Chern-Simons terms. The only difference is that the Lorentz group $SO(9, 1)$ is noncompact, whereas we shall only consider compact Lie groups for the gauge fields. The anomaly cancellation is a local phenomenon (local in spacetime) at the perturbative (one-loop) level, so issues of compactness or noncompactness do not matter as far as anomaly cancellation is concerned. (We shall, however, sometimes perform partial integrations, so to be precise we should state that we restrict ourselves to manifolds without boundaries, such as compactified R^{10} space).

III) **Mixed anomalies**. The third and last class of anomalies are the mixed anomalies: hexagon graphs with at least one graviton and at least one gauge field.



We may distinguish the case with r gauge fields where $r = 1, 2, 3, 4, 5$. (The case $r = 0$ and $r = 6$ correspond to purely gravitational and purely Yang-Mills anomalies, and for given r the gravitons and the gauge fields may appear in any order). The structure of the counterterm which cancels these anomalies is of a form which we might expect in view of the counterterms previously given. For $SO(32)$ the counterterm has the following form

$$\begin{aligned}\Delta\mathcal{L}^{mixed,SO(32)} \;\sim\;& B(\mathrm{tr}F^2\mathrm{tr}R^2 + \omega_{3L}^{(0)}\,\omega_{3Y}^{(0)}(\mathrm{tr}F^2 + \mathrm{tr}R^2)\\ &+\omega_{3L}^{(0)}\,\omega_{7Y}^{(0)} + \omega_{7L}^{(0)}\,\omega_{3Y}^{(0)}\ .\end{aligned} \tag{6.9.4}$$

The counterterm for the $E_8 \times E_8$ case is again different. There are no counterterms of the form $\omega_{3L}^{(0)}\,\omega_{3L}^{(0)}(\mathrm{tr}F^2 + \mathrm{tr}R^2)$ or $\omega_{3Y}^{(0)}\,\omega_{3Y}^{(0)}(\mathrm{tr}F^2 + \mathrm{tr}R^2)$ because they vanish. (The 6-forms $\omega_{3L}^{(0)}\,\omega_{3L}^{(0)}$ and $\omega_{3Y}^{(0)}\,\omega_{3Y}^{(0)}$ vanish since interchanging two 3-forms yields one overall minus sign). Actually, we shall not separately construct the counterterm for the mixed anomalies but rather construct the whole counterterm at one fell swoop. The reason is that the field $B_{\mu\nu}$ transforms simultaneously into gauge fields and into gravitational fields; as we shall discuss

$$\delta_{gauge}B = \omega_{2Y}^{(1)} - \omega_{2L}^{(1)}\ . \tag{6.9.5}$$

Here $\omega_{2Y}^{(1)}$ and $\omega_{2L}^{(1)}$ are 2-forms which are constructed from the variation of $\omega_{3Y}^{(0)}$ and $\omega_{3L}^{(0)}$ as we shall discuss. It is clear that substituting $\delta_{gauge}B$ in the counterterms $\Delta\mathcal{L}^{grav}$ and $\Delta\mathcal{L}^{YM}$ yields variations which contain simultaneously gravitational fields and gauge fields ("mixed variations"). Rather than first constructing $\Delta\mathcal{L}^{YM}$ and $\Delta\mathcal{L}^{grav}$, and then using their mixed variations in the construction of $\Delta\mathcal{L}^{mixed}$, it is easier to construct the complete $\Delta\mathcal{L}^{total}$ at once.

However, to isolate the salient points where one finds the restrictions on the gauge group, we shall first construct $\Delta\mathcal{L}^{grav}$ and $\Delta\mathcal{L}^{YM}$ separately. Then, as we already said, we shall construct $\Delta\mathcal{L}^{total}$.

The covariant gauge anomalies for a complex spin 1/2 field were derived earlier

$$An_{YM} \;=\; \int \mathrm{Tr}\, e^{\frac{1}{2}\tilde{F}}$$

$$\frac{1}{2}\tilde{F} \;\equiv\; \frac{1}{2}F^{\alpha}_{ab}T_{\alpha}\psi^{a}_{1,bg}\psi^{b}1, bg + \eta^{\alpha}T_{\alpha} \; . \tag{6.9.6}$$

The symbol $\int$ was defined below (6.8.1). In 10 dimensions we need the terms with six $\tilde{F}$ (recall that we need terms linear in η^{α}, so in one curvature we must take the term $\eta^{\alpha}T_{\alpha}$; we are then left with five curvatures, i.e. a ten-from). Thus the anomaly is proportional to

$$An_{YM} \sim \mathrm{Tr}\,\tilde{F}^{6} \; . \tag{6.9.7}$$

(The precise coefficient in front does not concern us here; later when we construct $\Delta\mathcal{L}^{total}$ we shall be careful with coefficients). There are now two issues we must deal with:

(i) the relation between traces of expressions in the adjoint representation (in particular $\mathrm{Tr}\,\tilde{F}^{6}$) and traces in the vector representation (which we shall denote by the symbol tr).

(ii) the construction of consistent anomalies from the descent equations.

We now briefly discuss these issues, and then return to the construction of counterterms.

Traces in group theory

Consider the adjoint representation of $SO(n)$. (We begin with $SO(n)$ because this is the simplest example, but we shall also discuss the other groups). The carrier space (the space o which the group acts) is given by a "vector" $v_{kl} = -v_{lk}$ where $k, l = 1, ..., n$. The group $SO(n)$ acts on v_{kl} as follows

$$v_{kl} \to v'_{kl} = (\Omega v)_{kl} \;; \quad (\Omega v)_{kl} = \sum_{m<n} \Omega_{kl}{}^{mn} v_{mn} \; . \tag{6.9.8}$$

Thus the pair of indices $I = (k, l)$ with $k < l$ runs over $N = \frac{1}{2}n(n-1)$ values, and we can also write

$$v_I \to v'_I = \Omega_I{}^J v_J \;, \qquad I, J = 1, .., N \; . \tag{6.9.9}$$

On the other hand, the adjoint transformation can also be written in terms of the defining representation of $SO(n)$ (the $n \times n$ real orthogonal matrices denoted by $O_n{}^{n'}$). Namely

$$(\Omega v)_I = (\Omega v)_{kl} = \sum_{k', l'} O_k{}^{k'} O_l{}^{l'} v_{k'l'} = 2 \sum_{k'<l'} O_k{}^{k'} O_l{}^{l'} v_{k'l'} \; . \tag{6.9.10}$$

Note that we discuss here group elements for finite transformations, thus $O^T = O^{-1}$ and O itself is not antisymmetric, however v_{kl} is antisymmetric. We can then write the following relation between the adjoint and vector representation of $SO(n)$

$$\Omega_{kl}{}^{mn} = O_k{}^m O_l{}^n - O_k{}^n O_l{}^m \tag{6.9.11}$$

where $k < l$ and $m < n$. Readers who are not sure whether one should add a factor 1/2 or not, may check this relation for the case of $SO(3)$.

We need relations between traces of products of generators in the adjoint representation, and similar traces in the defining representation. They can all be derived by taking the trace of the group elements in (6.9.11). Namely, set $m = k$ and $n = l$, and sum over k and l from 1 to n. This yields

$$\begin{aligned}
\mathrm{Tr}\,\Omega &= \sum_I \Omega_I{}^I = \sum_{k<l} \Omega_{kl}{}^{kl} = \frac{1}{2}\sum_{k,\,l} \Omega_{kl}{}^{kl} \\
&= \frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{N}\Big(O_k{}^k O_l{}^l - O_k{}^l O_l{}^k\Big) \\
&= \frac{1}{2}\Big[(\mathrm{tr}\,O)^2 - \mathrm{tr}\,(O^2)\Big]\,. \qquad (6.9.12)
\end{aligned}$$

To find expressions for traces over elements in the enveloping Lie algebra (products of elements in the Lie algebra such as $\mathrm{Tr}\,\tilde{F}^6$), we write $\Omega = e^{\mathcal{A}}$ and $O = e^A$ where $\mathcal{A}$ lies in the adjoint representation and A lies in the vector representation of the Lie algebra of $SO(n)$. From (6.9.12) one finds (after multiplying by 2 to simplify the notation)

$$\begin{aligned}
2\mathrm{Tr}\,e^{\mathcal{A}} &= 2\mathrm{Tr}\Big(1 + \mathcal{A} + \frac{1}{2!}(\mathcal{A})^2 + \frac{1}{3!}(\mathcal{A})^3 + \cdots\Big) \\
&= \Big[\mathrm{tr}\,(1 + A + \frac{1}{2!}(A)^2 + \frac{1}{3!}(A)^3 + \cdots)\Big]^2 \\
&\quad -\Big[\mathrm{tr}\,(1 + 2A + \frac{1}{2!}(2A)^2 + \cdots)\Big]\,. \qquad (6.9.13)
\end{aligned}$$

Comparing terms with the same number of factors yields a hierarchy of relations for $SO(n)$

$$\begin{aligned}
2\,\mathrm{Tr}\,1 &= (\mathrm{tr}\,1)^2 - \mathrm{tr}\,1 = n(n-1) \\
\mathrm{Tr}\,\mathcal{A} &= 0 \qquad (\text{because } \mathrm{tr}\,A = 0) \\
\mathrm{Tr}\,\mathcal{A}^2 &= (n-2)\mathrm{tr}\,A^2 \\
\mathrm{Tr}\,\mathcal{A}^4 &= (n-8)\mathrm{tr}\,A^4 + 3(\mathrm{tr}\,A^2)^2 \\
\mathrm{Tr}\,\mathcal{A}^6 &= (n-32)\mathrm{tr}\,A^6 + 15(\mathrm{tr}\,A^2)(\mathrm{tr}\,A^4)\,. \qquad (6.9.14)
\end{aligned}$$

The first line gives the dimension of the adjoint representation.

For $Sp(n)$ one finds the same formulas, but with + signs instead of − signs because the adjoint representation for $Sp(n)$ is given by a tensor v_{kl} which is symmetric[26] ($v_{kl} = v_{lk}$). As we shall see, anomalies which are factorized (such as $\mathrm{tr}F^2\mathrm{tr}F^4$) can be canceled by counterterms, but

[26] The group $Sp(n)$ leaves the bilinear form $x^i\Omega_{ij}y^j$ $(i, j = 1, ..., n)$ invariant, where Ω_{ij}

non-factorized expressions (such as $\mathrm{tr}F^6$) can never be canceled. This immediately rules out the $Sp(n)$ groups, and of the $SO(n)$ groups only $SO(32)$ needs to be kept.

What about $SU(n)$, or the exceptional groups? For $SU(n)$ the carrier space for the adjoint representation in terms of the vector (= defining) representation is given by vectors ${v_i}^j$

$$(Uv)_i{}^j = U_i{}^{j;k}{}_l \, v_k{}^l = u_i{}^k (u^*)^j{}_l \, v_k{}^l \ . \tag{6.9.15}$$

This adjoint representation is obtained by taking the direct product of the $\boldsymbol{n}$ and $\boldsymbol{n}^*$ of $SU(n)$ and removing the trace[27], hence v is traceless, ${v_k}^k = 0$. Thus the dimension of the adjoint representation of $SU(n)$ is $n^2 - 1$. We have then

$$U_i{}^{j;k}{}_l = u_i{}^k (u^*)^j{}_l - \frac{1}{n}\delta_i^j \delta_l^k \tag{6.9.16}$$

and taking the trace we obtain

$$\mathrm{Tr}\, U = (\mathrm{tr}\, u)(\mathrm{tr}\, u^*) - 1 \ . \tag{6.9.17}$$

Setting $U = e^{\mathcal{A}}$ and $u = e^A$, with $\mathcal{A}$ in the adjoint representation of the Lie algebra of $SU(n)$, and A in the vector representation, leads to

$$\begin{aligned} &\mathrm{Tr}\left(1 + \mathcal{A} + \frac{1}{2!}(\mathcal{A})^2 + \frac{1}{3!}(\mathcal{A})^3 + \cdots\right) = \\ &\quad \mathrm{tr}\left(1 + A + \frac{1}{2!}(A)^2 + \cdots\right) \mathrm{tr}\left(1 + A + \frac{1}{2!}(A)^2 + \cdots\right)^* - 1 \ . \end{aligned} \tag{6.9.18}$$

Equating terms with the same number of generators yields

$$\begin{aligned} \mathrm{Tr}\, 1 &= n^2 - 1 \\ \mathrm{Tr}\, \mathcal{A} &= 0 \qquad\qquad (\text{because } \mathrm{tr}\, A = 0) \\ \mathrm{Tr}\, \mathcal{A}^2 &= n(\mathrm{tr}\, A^2 + \mathrm{tr}\, A^{*2}) \\ \mathrm{Tr}\, \mathcal{A}^3 &= n(\mathrm{tr}\, A^3 + \mathrm{tr}\, A^{*3}) \\ \mathrm{Tr}\, \mathcal{A}^4 &= n(\mathrm{tr}\, A^4 + \mathrm{tr}\, A^{*4}) + 6(\mathrm{tr}\, A^2)(\mathrm{tr}\, A^{*2}) \\ \mathrm{Tr}\, \mathcal{A}^6 &= n(\mathrm{tr}\, A^6 + \mathrm{tr}\, A^{*6}) + 15(\mathrm{tr}\, A^2)(\mathrm{tr}\, A^{*4}) \\ &\quad + 15(\mathrm{tr}\, A^4)(\mathrm{tr}\, A^{*2}) + 20(\mathrm{tr}\, A^3)(\mathrm{tr}\, A^{*3}) \ . \end{aligned} \tag{6.9.19}$$

---

is non-degenerate and antisymmetric $\Omega_{ij} = -\Omega_{ji}$. Then for infinitesimal transformations $x \to Mx$ one obtains $M^T\Omega + \Omega M = 0$. The matrices ΩM are symmetric, and define $Sp(n)$ (they are the generators of $Sp(n)$ in the defining representation). They act on v_{kl} just as in (6.9.10), but now (6.9.11) obtains a + sign instead of a − sign. Actually in supergravity and string theory one uses the group $Usp(n)$ which is the intersection of $U(n)$ and $Sp(n, C)$. It has the same symmetry properties as $Sp(n)$.

[27] One may identify each ${v_i}^j$ with an $n \times n$ matrix with only an entry in the i-th row and j-th column; if the sum of these matrices ${v_i}^j$ is a matrix v which is antihermitian then the transformation rule for v correspond to a commutator of v with the generators in the fundamental representation. Clearly the carrier space define by v is $n^2 - 1$ dimensional.

The first line gives the dimension of the adjoint representation. Because the generators A of $SU(n)$ in the fundamental representation are anti-hermitian $n \times n$ matrices and the trace is invariant under transposition, we may replace $\mathrm{tr}(A^*)^k$ by $(-)^k\,\mathrm{tr}A^k$. Hence, all $SU(n)$ groups must be rejected because the coefficient of the leading term $(2n)$ never vanishes. Also $U(n)$ must be rejected because the only difference with $SU(n)$ is the relation $\mathrm{Tr}\,I = n^2-1$ which becomes $\mathrm{Tr}\,I = n^2$ for $U(n)$.

Finally we consider the exceptional groups. Here the coefficient of the leading term sometimes vanishes identically, due to properties of the Casimir invariants C_k. Let us first list these Casimir invariants for the simple Lie groups

$$\begin{aligned} SU(n) &: C_2, C_3, \cdots, C_n \\ SO(2n+1) &: C_2, C_4, \cdots, C_{2n} \\ Sp(2n) &: C_2, C_4, \cdots, C_{2n} \\ SO(2n) &: C_2, C_4, \cdots, C_{2n-2}, C_n \\ G_2 &: C_2, C_6 \\ F_4 &: C_2, C_6, C_8, C_{12} \\ E_6 &: C_2, C_5, C_6, C_8, C_9, C_{12} \\ E_7 &: C_2, C_6, C_8, C_{10}, C_{12}, C_{14}, C_{18} \\ E_8 &: C_2, C_8, C_{12}, C_{14}, C_{18}, C_{20}, C_{24}, C_{30}\ . \end{aligned} \tag{6.9.20}$$

The Casimir operators C_k for a representation R are obtained by contracting a totally symmetric irreducible tensor in the adjoint representation $d^{a_1 \ldots a_k}$ with the generators in the representation R

$$C_k(R) = d^{a_1 \ldots a_k} T^{(R)}_{a_1} \cdots T^{(R)}_{a_k}\ . \tag{6.9.21}$$

By irreducible we mean that "traces" (contractions with lower-order invariant tensors) have been removed. The usual Casimir operator corresponds to the quadratic Casimir operator, with d^{ab} equal to the inverse of the Killing metric $g_{ab} = f_{pa}{}^q f_{qb}{}^p$. (According to the definition of semisimple groups, these groups have an invertible Killing metric). For example, for $SU(3)$ one has $C_2(R) = g^{ab} T^{(R)}_a T^{(R)}_b$ and $C_3(R) = d^{abc} T^{(R)}_a T^{(R)}_b T^{(R)}_c$, with d^{abc} the "d-symbols" which yield the chiral triangle anomalies in 4 dimensions[28]. For $SO(6)$ one has $C_2 = g^{ab} T_a T_b$, $C_3 = \epsilon^{ijklmn} T_{ij} T_{kl} T_{mn}$ (where T_{ij} with $i<j$ corresponds to T_a) and $C_4 = T_{ij} T^{jk} T_{kl} T^{li}$.

One can construct invariant tensors by taking traces over products of generators, $\mathrm{tr}(T^{(R)}_{a_1} \cdots T^{(R)}_{a_k}) \equiv T^{(R)}_{a_1 \ldots a_k}$. These $T^{(R)}_{a_1 \ldots a_k}$ are invariant ten-

[28] For $SU(n)$ we define the d symbols by $\{T^{(F)}_a, T^{(F)}_b\} = \frac{1}{n} g_{ab} + i d_{abc} g^{cd} T^{(F)}_d$, where (F) denotes the fundamental representation. One usually normalizes the generators such that $g_{ab} = -\delta_{ab}$. In that case $\mathrm{tr} T^{(F)}_a T^{(F)}_b = -\frac{1}{2}\delta_{ab}$.

sors. Again we may restrict our attention to $T^{(R)}_{a_1\ldots a_k}$ which are totally symmetric and irreducible. Since for given k there is only at most one such invariant tensor, we have $T^{(R)}_{a_1\ldots a_k} = T_k(R) d_{a_1\ldots a_k}$. For the quadratic Casimir operators one has $T^{(R)}_{ab} = T_2(R) g_{ab}$ where $T_2(R)$ are called the Dynkin labels. There is a simple relation between $C_2(R)$ and $T_2(R)$, obtained by tracing $C_2(R)$

$$\dim G \; T_2(R) = \dim R \; C_2(R) \tag{6.9.22}$$

where $\dim G$ denotes the number of generators of the gauge group, and $\dim R$ denotes the dimension of the representation R.

For certain representations of certain groups it may happen that for certain k the trace $\mathrm{tr} F^k$ does not contain only a term with the irreducible $d_{a_1\ldots a_k}$ but also products of terms with lower dimensional $d_{a_1\ldots a_l}$. Suppose this happens for two representations R_1 and R_2. Then one finds a relation of the form $\mathrm{tr}_1 F^k = a\, \mathrm{tr}_2 F^k + b\, (\mathrm{tr}_2 F^{k-m})(\mathrm{tr}_2 F^m) + \cdots$. In particular, for our purposes it will be crucial that the trace of F^6 in the adjoint representation, denoted by $\mathrm{Tr} F^6$, does not contain a term with the maximal Casimir invariant $d_{a_1\ldots a_6}$. As we now discuss in more detail, this means that $\mathrm{Tr} F^6$ factorizes, and factorization will permit the construction of counterterms which cancel anomalies. The trace $\mathrm{Tr} F^6$ can be written in terms Casimir invariants which are irreducible and totally symmetric invariant tensors in the adjoint representation. For example, for $SO(n)$ one should symmetrize $\mathrm{Tr}\, T_{a_1} \cdots T_{a_6}$ and remove traces to obtain C_6. (The Kronecker symbol δ^{ab} is an invariant tensor of $SO(n)$). For other groups one should subtract contractions with all invariant tensors if they exists. Then $\mathrm{Tr} F^6$ becomes a polynomial in the invariant tensors $d_{a_1 \cdots a_k}$ contracted with curvatures $F^{a_1} \ldots F^{a_k}$. Next note that $\mathrm{Tr} F^k$ contains a term with $d_{a_1 \cdots a_k}$ if the latter exists, but the coefficient of this term may vanish.

Consider now E_8. Since it has no Casimir invariants of rank less than 8 except the quadratic Casimir invariant, it follows that $\mathrm{Tr} F^6$ must factorize into a constant times $(\mathrm{Tr} F^2)^3$. For E_7 one will still be left with a term involving C_6 and for E_6, F_4 and G_2 the same situation holds. We analyze these groups in appendix F, but we mention here that their anomalies cannot be canceled: these groups must be rejected. So, only E_8 has a chance to be anomaly free. At this point we mention ahead of time that from an analysis of anomalies in the purely gravitational sector it will follow that the number of generators should be 496. Miraculously, this is the number of generators of $E_8 \times E_8$ and of $SO(32)$. So, our analysis of traces in the fundamental representation of the gauge groups has narrowed the choice of the gauge groups down to $E_8 \times E_8$ and $SO(32)$.

Actually there are two further solutions. One possibility is $[U(1)]^{496}$;

here anomalies can be canceled but since it not known how to construct a string theory which has this gauge group and also produces the Standard Model group $SU(3) \times SU(2) \times U(1)$ by some other mechanism, this case has received little attention. Another possibility is $E_8 \times [U(1)]^{248}$, but again no string theory is known which can accommodate this group, and we shall not pursue this possibility further.

Descent equations

We summarize here the construction of the consistent Yang-Mills and gravitational anomalies in n dimensions from invariant polynomials in $n+2$ dimensions by means of the descent equations. For proofs see [2] or [134]. The construction proceeds in 5 steps. Afterwards we shall give examples.

1. One starts from an invariant $(n+2)$-form I_{n+2} with n even, for example $\mathrm{tr}F^{\frac{n}{2}+1}$, or $\mathrm{tr}R^{\frac{n}{2}+1}$, or $\mathrm{tr}R^{\frac{n}{2}-1}\mathrm{tr}R^2$, or $\mathrm{tr}F^{\frac{n}{2}-1}\mathrm{tr}R^2$, etc. The curvatures are defined by $F = dA + AA$ and $R = d\omega + \omega\omega$. Any representation of the Yang-Mills generators can be used.

2. Since $dI_{n+2} = 0$ (easy to prove, using $dF = [F, A]$ and $dR = [R, \omega]$ and cyclicity of the trace), I_{n+2} is closed and therefore exact, $I_{n+2} = d\omega_{n+1}^{(0)}$ (closure implies exactness, at least locally in $n+2$ dimensions). The $(n+1)$-form $\omega_{n+1}^{(0)}$ is the Chern-Simons term.

3. The gauge variation of the Chern-Simons term is an exact form

$$\delta_{gauge}\,\omega_{n+1}^{(0)} = d\omega_n^{(1)} \qquad \text{where} \qquad \omega_n^{(1)} = \mathrm{tr}\Lambda d(...)\ . \tag{6.9.23}$$

The proof of this relation can be found in [2]. This shows that the Chern-Simons term is invariant under rigid (constant Λ) gauge transformations, and the Chern-Simons action (the integrated Chern-Simons term) is invariant under infinitesimal gauge transformations (if there are no boundaries).

4. The consistent anomaly G is given by

$$G(\Lambda) = (n+2)\int \omega_n^{(1)}\ . \tag{6.9.24}$$

Note that G is linear in the local gauge parameter $\Lambda(x)$. To indicate this we write $G(\Lambda)$. Because we can choose $\Lambda(x)$ nonvanishing only in a small region, the integral is always well defined. If one begins with a compact $n+1$ dimensional manifold B whose boundary is n-dimensional spacetime Σ, then the gauge variation of the Chern-Simons term in B is the consistent anomaly in Σ

$$\delta_{gauge}\int_B \omega_{n+1}^{(0)} = \int_\Sigma \omega_n^{(1)}\ . \tag{6.9.25}$$

5. The consistent anomaly $G(\Lambda)$ must satisfy the consistency conditions[29]

$$\delta_{gauge}(\Lambda_1)G(\Lambda_2) - \delta_{gauge}(\Lambda_2)G(\Lambda_1) = G([\Lambda_1, \Lambda_2]) \ . \tag{6.9.26}$$

The reason is that the consistent anomaly is the response of the effective action Γ under a gauge variation:

$$G(\Lambda) = \delta_{gauge}(\Lambda)\Gamma = \int dx \left(\delta_{gauge}(\Lambda) A^a_\mu(x)\right) \frac{\delta}{\delta A^a_\mu(x)} \Gamma \ . \tag{6.9.27}$$

The consistency conditions state that two ordinary derivatives $\frac{\delta}{\delta A^a_\mu}$ of the effective action Γ commute. Let us prove that G in (6.9.25) indeed satisfies the consistency conditions in (6.9.26). Imagine that the n-dimensional space Σ over which $G \equiv \int_\Sigma g$ is integrated is the boundary of a $(n+1)$-dimensional ball B. Then, using that $dg = \delta_{gauge}\, \omega^{(0)}_{n+1}$, we obtain

$$G(\Lambda) = \int_B dg(\Lambda) = \delta_{gauge}(\Lambda) \int_B \omega^{(0)}_{n+1} \ . \tag{6.9.28}$$

Since $[\delta_{gauge}(\Lambda_1), \delta_{gauge}(\Lambda_2)] = \delta_{gauge}([\Lambda_1, \Lambda_2])$, the consistency conditions are satisfied.

Let us now give two examples; these examples will be used in the construction of the counterterms.

Example 1: $d = 2$.

In this case one begins with $I_4 = \mathrm{tr}F^2$ (or $\mathrm{Tr}F^2$; it does not matter which representation one uses for the descent equations). Then the Chern-Simons form is

$$\omega^{(0)}_3 = \mathrm{tr}(FA - \frac{1}{3}A^3) \ . \tag{6.9.29}$$

Since $\delta_{gauge}\, \omega^{(0)}_3 = d(\mathrm{tr}\Lambda dA)$, as one readily verifies by using $\delta_{gauge} A = D\Lambda = d\Lambda + [A, \Lambda]$ and $\delta_{gauge} F = [F, \Lambda]$, we find

$$\omega^{(1)}_2 = \mathrm{tr}\Lambda dA \ . \tag{6.9.30}$$

So the consistent anomaly in 2 dimensions is

$$G(\Lambda) = 4 \int d^2x\, \mathrm{tr}\Lambda dA \ . \tag{6.9.31}$$

The consistency conditions reduce to

$$\int d^2x \left(\mathrm{tr}(\Lambda_2 dD\Lambda_1) - \mathrm{tr}(\Lambda_1 dD\Lambda_2)\right) = \int d^2x\, \mathrm{tr}([\Lambda_1, \Lambda_2] dA) \tag{6.9.32}$$

[29] In the BRST formalism one replaces Λ by a ghost c. Then the BRST variation removes the terms $G([\Lambda_1, \Lambda_2])$ on the right hand side and the consistency conditions reduce to the statement that the consistent anomaly must be BRST invariant: $QG(c) = 0$.

which is clearly true since $dd = 0$, and

$$\begin{aligned}
&\mathrm{tr}\Lambda_2 d[A,\Lambda_1] - \mathrm{tr}\Lambda_1 d[A,\Lambda_2] \\
&\quad = 2\mathrm{tr}[\Lambda_1,\Lambda_2]dA - \mathrm{tr}Ad[\Lambda_1,\Lambda_2] \\
&\quad = \mathrm{tr}[\Lambda_1,\Lambda_2]dA + d(\mathrm{tr}[\Lambda_1,\Lambda_2]A) \; . \qquad (6.9.33)
\end{aligned}$$

Example 2: $d = 6$.
We start from $I_8 = \mathrm{tr}F^4$. Then the Chern-Simons term is

$$\begin{aligned}
\omega_7^{(0)} &= \mathrm{tr}\Big[(dA)^3A + \frac{8}{5}(dA)^2A^3 + \frac{4}{5}dAA^2dAA + 2dAA^5 + \frac{4}{7}A^7\Big] \\
&= \mathrm{tr}\Big[F^3A - \frac{2}{5}F^2A^3 - \frac{1}{5}FA^2FA + \frac{1}{5}FA^5 - \frac{1}{35}A^7\Big] \; . \qquad (6.9.34)
\end{aligned}$$

One may check that $d\omega_7^{(0)} = \mathrm{tr}F^4$. To compute the gauge variation of this expression, all terms due to $\delta_{gauge}\, A = [A,\Lambda]$ and $\delta_{gauge}\, F = [F,\Lambda]$ cancel in the trace due to cyclicity, and one only needs to use $\delta_{gauge}\, A = d\Lambda$. One finds that

$$\delta_{gauge}\, \omega_7^{(0)} = \mathrm{tr}[F^3 d\Lambda + \cdots] = d\mathrm{tr}[\Lambda dAdAdA + \cdots] \; . \qquad (6.9.35)$$

Since $\delta_{gauge}\, \omega_7^{(0)} = d\omega_6^{(1)}$ one obtains

$$\omega_6^{(1)} = \mathrm{tr}\Lambda d(dAdAA + \cdots) \qquad (6.9.36)$$

and the consistent anomaly is now

$$G(\Lambda) = 8\int d^6x\, \mathrm{tr}\Lambda d(dAdAA + \cdots) \; . \qquad (6.9.37)$$

The reader may complete the terms denoted by ellipses, but we shall not need the explicit form of these terms.

Cancellation of pure gravitational anomalies

The covariant purely gravitational anomalies due to spin 3/2 gravitinos and spin 1/2 gauginos and a spin 1/2 dilatino were given in (6.8.4) – (6.8.6). The gaugino consists of $\dim G$ spin 1/2 fields, because gauginos are in the adjoint representation. The hexagon graphs correspond to terms with six curvatures, and extracting an overall minus sign we get

$$\begin{aligned}
An_{grav} &= (\dim G - 1)An(1/2) + An(3/2) \\
&= (\dim G - 1)\left[\frac{1}{5670}t_6 + \frac{1}{4320}t_2t_4 + \frac{1}{10368}t_2^3\right] \\
&\quad + \left[\left(-\frac{4}{45} + \frac{9}{5670}\right)t_6 + \left(\frac{1}{20} + \frac{9}{4320}\right)t_2t_4\right. \\
&\quad \left. + \left(-\frac{1}{144} + \frac{9}{10368}\right)t_2^3\right] \\
&= \frac{\dim G - 496}{5670}\,t_6 + \frac{\dim G + 224}{4320}\,t_2t_4 + \frac{\dim G - 64}{10368}\,t_2^3 \qquad (6.9.38)
\end{aligned}$$

where $t_n = \mathrm{tr}\, y^n$ and $y = \tilde{R}/4\pi$. (All fields are real, so we should add an overall factor 1/2, but at this point we are not interested in overall factors). As we explained before, products of traces have a chance of being canceled. Hence, since $\mathrm{tr}\tilde{R}^6$ is nonvanishing (it is easy to write down a 10×10 antisymmetric matrix A for which $\mathrm{tr}A^6$ is nonvanishing), we must restrict the number of generators $\dim G$ of the gauge group G by

$$\dim G = 496 \ . \tag{6.9.39}$$

The remaining terms then simplify considerably

$$An_{grav} = c\, \mathrm{tr}\tilde{R}^2 \Big[\mathrm{tr}\tilde{R}^4 + \frac{1}{4}(\mathrm{tr}\tilde{R}^2)^2\Big] \tag{6.9.40}$$

where the constant c ($c = 1/6$) does not interest us at this point. We want now to apply the descent equations to find the consistent anomaly and the counterterm, but for this we must first find the invariant 12-form I_{12} from which to start. One obtains I_{12} by replacing $\tilde{R}$ by R in An_{grav}; indeed, it will reproduce the leading terms in An_{grav} as we shall see. Thus,

$$I_{12} = c\, \mathrm{tr}R^2 \Big[\mathrm{tr}R^4 + \frac{1}{4}(\mathrm{tr}R^2)^2\Big] \ . \tag{6.9.41}$$

Since $dI_{12} = 0$, I_{12} itself is $d\omega^{(0)}_{11L}$ where

$$\begin{aligned} \omega^{(0)}_{11L} &= \alpha\, c\, \omega^{(0)}_{3L} \Big[\mathrm{tr}R^4 + \frac{1}{4}(\mathrm{tr}R^2)^2\Big] \\ &+ (1-\alpha) c\, \mathrm{tr}R^2 \Big[\omega^{(0)}_{7L} + \frac{1}{4}\omega^{(0)}_{3L}\mathrm{tr}R^2\Big] \ . \end{aligned} \tag{6.9.42}$$

As a check note that $d\omega^{(0)}_{3L} = \mathrm{tr}R^4$ and $d(\omega^{(0)}_{7L} + \frac{1}{4}\omega^{(0)}_{3L}\mathrm{tr}R^2)$ equals $\mathrm{tr}R^4 + \frac{1}{4}(\mathrm{tr}R^2)^2$, so we have a free parameter α in $\omega^{(0)}_{11L}$. We now show that as far as cancellation of anomalies by counterterms is concerned, any value of α can be taken.

Since the terms proportional to α are annihilated by d, they are d-exact. (They are given by $\alpha d(-\omega^{(0)}_{3L}\omega^{(0)}_{7L})$). Since any term dX in $\omega^{(0)}_{11L} = ... + dX$ will lead to a term $\delta_{gauge} X$ in the anomaly which can be removed by a counterterm $\Delta\mathcal{L} = -X$, any choice of α will be allowed. We shall impose Bose symmetry ($\alpha = 4/12$) because this will yield an expression for the consistent anomaly whose leading term agrees with the leading term of the covariant anomaly

$$\omega^{(0)}_{11L} = \frac{c}{12}\Big[4\omega^{(0)}_{3L}\Big(\mathrm{tr}R^4 + \frac{1}{4}(\mathrm{tr}R^2)^2\Big) + 8\mathrm{tr}R^2\Big(\omega^{(0)}_{7L} + \frac{1}{4}\omega^{(0)}_{3L}\mathrm{tr}R^2\Big)\Big] \ . \tag{6.9.43}$$

The gauge variation of this Chern-Simons term is then the following exact form

$$\begin{aligned}\delta_{gauge}\,\omega^{(0)}_{11L} &= d\omega^{(1)}_{10L} \\ &= \frac{4c}{12}d\omega^{(1)}_{2L}\Big(\mathrm{tr}R^4+\frac{1}{4}(\mathrm{tr}R^2)^2\Big) \\ &\quad +\frac{8c}{12}d\Big(\omega^{(1)}_{6L}\mathrm{tr}R^2+\frac{1}{4}\omega^{(1)}_{2L}(\mathrm{tr}R^2)^2\Big)\ . \end{aligned} \tag{6.9.44}$$

Hence the consistent anomaly (with all subleading terms included) reads

$$G_{cons} = c'\int dx\Big[\omega^{(1)}_{2L}\Big(\mathrm{tr}R^4+\frac{1}{4}(\mathrm{tr}R^2)^2\Big)+2\Big(\omega^{(1)}_{6L}+\frac{1}{4}\omega^{(1)}_{2L}\mathrm{tr}R^2\Big)\mathrm{tr}R^2\Big]\ . \tag{6.9.45}$$

Let us now compare this result with the covariant anomaly in (6.9.40) by expanding $\tilde{R}_{ij} = R_{ij} - 2(D_i\xi_j - D_j\xi_i)$ and taking the terms linear in $(D_i\xi_j - D_j\xi_i)$

$$G_{cov} = c''\int dx\Big[(\mathrm{tr}D\xi R)(\mathrm{tr}R^4+\frac{3}{4}(\mathrm{tr}R^2)^2)+2(\mathrm{tr}D\xi R^3)\mathrm{tr}R^2\Big]\ . \tag{6.9.46}$$

Since the leading term in $\omega^{(1)}_{2L}$ and $(\mathrm{tr}D\xi R)$ are ΛdA and $D\xi d\omega$, respectively, they agree if we note that $A=\omega$ for the Lorentz group, and identify Λ with $D\xi$. Similarly, $\omega^{(1)}_{6L}$ and $\mathrm{tr}D\xi R^3$ agree as far as the leading terms are concerned. Hence the consistent and covariant anomalies agree, and this was done by fixing α according to Bose symmetry.

The counterterm whose variation is equal to minus the consistent anomaly, is given by

$$\int \Delta\mathcal{L}_{grav} = c'\int[\alpha B(\mathrm{tr}R^4+\beta(\mathrm{tr}R^2)^2)+\gamma\omega^{(0)}_{3L}\omega^{(0)}_{7L}] \tag{6.9.47}$$

where the constants α, β and γ are still to be determined. To construct the gauge variation of this counterterm, one must first discuss how the 2-form B transforms.

In the $N=1$ supergravity theory coupled to Yang-Mills theory, the action contains the modified field strength $H = dB + \omega^{(0)}_{3Y}$. Supersymmetry requires this combination [120, 133], but we can rescue Yang-Mills gauge invariance by defining that $\delta_{gauge}\,B = -\omega^{(1)}_{2Y}$. We now extend the definition of $\delta_{gauge}\,B$ to include a term $\omega^{(1)}_{2L}$ because then we shall be able to cancel anomalies

$$\delta_{gauge}\,B = \omega^{(1)}_{2L} - \omega^{(1)}_{2Y}\ . \tag{6.9.48}$$

Although we shall not need it, let us mention that this suggests also to introduce a modified field strength

$$H = dB + \omega^{(0)}_{3Y} - \omega^{(0)}_{3L}\ . \tag{6.9.49}$$

It is invariant both under gauge transformations and under local Lorentz transformations. One then finds from $\int dH = 0$ over a compact space that $\int(\mathrm{tr}R^2 - \mathrm{tr}F^2) = 0$. This is used for Kaluza-Klein compactifications to Calabi-Yau manifolds.

With the gauge variation of B fixed, we can now construct the variation of the counterterm. It varies into the following purely gravitational expression

$$\delta\Delta\mathcal{L}_{grav} = c'[\alpha\omega^{(1)}_{2L}(\mathrm{tr}R^4 + \beta(\mathrm{tr}R^2)^2) + \gamma d\omega^{(1)}_{2L}\omega^{(0)}_{7L} + \gamma\omega^{(0)}_{3L}d\omega^{(1)}_{6L}] \quad (6.9.50)$$

where we used that $\delta_{gauge}B = \omega^{(1)}_{2L} + \cdots$. Partially integrating to make curvatures out of the Chern-Simons terms yields

$$\delta\Delta\mathcal{L}_{grav} = c'[\alpha\omega^{(1)}_{2L}(\mathrm{tr}R^4 + \beta(\mathrm{tr}R^2)^2) - \gamma(\omega^{(1)}_{2L}\mathrm{tr}R^4 - \omega^{(1)}_{6L}\mathrm{tr}R^2)] \quad (6.9.51)$$

Let us now compare this expression with the consistent anomaly in (6.9.46). Choosing $-\alpha + \gamma = 1$, $\gamma = -2$ and $\alpha\beta = -3/4$, the variation of the counterterm cancels the gravitational consistent anomaly. Hence, the purely gravitational anomalies can always be canceled by a suitable counterterm, as long as the gauge group G has 496 generators.

Cancellation of pure Yang-Mills anomalies

We now consider the opposite case, namely the pure Yang-Mills anomalies (anomalies due to Yang-Mills gauge transformations which only depend on Yang-Mills fields). The covariant Yang-Mills anomalies due to hexagon graphs with a gaugino in the loop are proportional to $\mathrm{Tr}\tilde{F}^6$. We already discussed that for $SO(32)$ the anomaly factorizes into $(\mathrm{tr}\tilde{F}^2)(\mathrm{tr}\tilde{F}^4)$ where the trace tr is now over the defining (vector) representation. Hence, we start from $I_{12} = \mathrm{tr}F^2\mathrm{tr}F^4$ and obtain the Chern-Simons term in 11 dimensions by extracting d

$$\omega^{(0)}_{11Y} = \alpha\omega^{(0)}_{3Y}\mathrm{tr}F^4 + (1-\alpha)\omega^{(0)}_{7Y}\mathrm{tr}F^2 \ . \quad (6.9.52)$$

Bose symmetry sets $\alpha = \frac{4}{12}$ and $(1-\alpha) = \frac{8}{12}$, but this time we keep α to see where it ends up. The consistent anomaly is then

$$G_{cons} = \int dx\Big[\alpha\omega^{(1)}_{2Y}\mathrm{tr}F^4 + (1-\alpha)\omega^{(1)}_{6Y}\mathrm{tr}F^2\Big] \ . \quad (6.9.53)$$

The counterterm is now of the form

$$\Delta\mathcal{L}_{YM} = aB(\mathrm{tr}F^4 + b(\mathrm{tr}F^2)^2) + c\omega^{(0)}_{3Y}\omega^{(0)}_{7Y} \ . \quad (6.9.54)$$

where the constants a, b and c are to be determined. Variation yields

$$\begin{aligned}\delta_{gauge}\Delta\mathcal{L}_{YM} &= [-a\omega^{(1)}_{2Y}(\mathrm{tr}F^4 + b(\mathrm{tr}F^2)^2) + c(d\omega^{(1)}_{2Y}\,\omega^{(0)}_{7Y} + \omega^{(0)}_{3Y}\,d\omega^{(1)}_{6Y})] \\ &= [-a\omega^{(1)}_{2Y}(\mathrm{tr}F^4 + b(\mathrm{tr}F^2)^2) - c\omega^{(1)}_{2Y}\mathrm{tr}F^4 + c\omega^{(1)}_{6Y}\mathrm{tr}F^2] \ . \quad (6.9.55)\end{aligned}$$

where we have used $\delta_{gauge}B = -\omega^{(1)}_{2Y}$. Hence, for $a+c=\alpha$, $c=\alpha-1$ and $b=0$, the Yang-Mills anomalies are also canceled. These equations have a solution for any α. Hence, the purely gauge (Yang-Mills) anomalies can be canceled for $SO(32)$. In appendix F we obtain the same result for $E_8 \times E_8$.

Cancellation of mixed anomalies: the complete counterterm

Finally, we consider mixed anomalies. In fact, as explained before, we consider all anomalies together. The covariant anomalies come from $An_{3/2}(\psi_\mu)$, $An_{1/2}(\chi)$ and $An_{1/2}(\lambda)$, but only $An_{1/2}(\lambda)$ depends on $\tilde{F}$. The purely gravitational anomalies are recorded in (6.8.4), (6.8.5) and (6.8.6). We need to multiply the result for $An_{1/2}$ in (6.8.4) by $\mathrm{Tr}e^{\tilde{F}}-1$ where the -1 refers to the dilatino. We absorb a factor $-\frac{i}{2\pi}$ into each $\tilde{R}$ and $\tilde{F}$. The total result for the terms which contribute to the anomaly in 10 dimensions reads

$$\begin{aligned}
&An_{3/2}(\psi_\mu) + An_{1/2}(\chi) + An_{1/2}(\lambda) = \\
&= \int dx \Big\{ \Big(-\frac{4}{45} + \frac{9}{5670} \Big) \mathrm{tr}(\tilde{R}/8\pi)^6 \\
&+ \Big(\frac{1}{20} + \frac{9}{4320} \Big) \mathrm{tr}(\tilde{R}/8\pi)^2 \mathrm{tr}(\tilde{R}/8\pi)^4 \\
&+ \Big(-\frac{1}{144} + \frac{9}{10368} \Big) (\mathrm{tr}(\tilde{R}/8\pi)^2)^3 \\
&+ \Big[1 + \frac{1}{12}\mathrm{tr}(\tilde{R}/8\pi)^2 + \frac{1}{360}\mathrm{tr}(\tilde{R}/8\pi)^4 + \frac{1}{288}(\mathrm{tr}(\tilde{R}/8\pi)^2)^2 \\
&\quad + \frac{1}{5670}\mathrm{tr}(\tilde{R}/8\pi)^6 + \frac{1}{4320}\mathrm{tr}(\tilde{R}/8\pi)^4 \mathrm{tr}(\tilde{R}/8\pi)^2 \\
&\quad + \frac{1}{10368}(\mathrm{tr}(\tilde{R}/8\pi)^2)^3 \Big] \\
&\times \Big[(\dim G - 1) - \frac{1}{2}\mathrm{Tr}(\tilde{F}/4\pi)^2 + \frac{1}{24}\mathrm{Tr}(\tilde{F}/4\pi)^4 \\
&\quad - \frac{1}{720}\mathrm{Tr}(\tilde{F}/4\pi)^6 \Big] \Big\} \,. \qquad (6.9.56)
\end{aligned}$$

The -1 in $(\dim G - 1)$ accounts for $An_{1/2}(\chi)$. The mixed anomalies involve $\mathrm{Tr}\tilde{F}^2$ and $\mathrm{Tr}\tilde{F}^4$. Substituting $\dim G = 496$, but not yet using any other properties of the gauge group, yields for the total covariant anomaly in ten dimensions

$$\begin{aligned}
An(total) \;=\; & \frac{1}{(4\pi)^6} \int dx \, \frac{1}{48} \Big[\frac{1}{8}\mathrm{tr}\tilde{R}^2 \mathrm{tr}\tilde{R}^4 + \frac{1}{32}(\mathrm{tr}\tilde{R}^2)^3 \\
& - \Big(\frac{1}{240}\mathrm{tr}\tilde{R}^4 + \frac{1}{192}(\mathrm{tr}\tilde{R}^2)^2 \Big) \mathrm{Tr}\tilde{F}^2
\end{aligned}$$

$$+\frac{1}{24}\mathrm{tr}\tilde{R}^2\mathrm{Tr}\tilde{F}^4-\frac{1}{15}\mathrm{Tr}\tilde{F}^6\Big] \ . \tag{6.9.57}$$

In order that the total set of anomalies can be canceled by a counterterm involving the field B, the anomaly must factorize as follows

$$An(total)=\frac{1}{(4\pi)^6}\int dx\,\frac{1}{48}(\mathrm{tr}\tilde{R}^2+a\mathrm{tr}\tilde{F}^2)X \tag{6.9.58}$$

where a is a constant, and X is a polynomial in $\tilde{R}$ and $\tilde{F}$. This is only possible if $\mathrm{Tr}\tilde{F}^6$ factorizes: it should be possible to write it as a linear combination of $\mathrm{Tr}\tilde{F}^4$ $\mathrm{Tr}\tilde{F}^2$ and $(\mathrm{Tr}\tilde{F}^2)^3$

$$\mathrm{Tr}\tilde{F}^6=b\,\mathrm{Tr}\tilde{F}^4\mathrm{Tr}\tilde{F}^2+c\,(\mathrm{Tr}\tilde{F}^2)^3 \ . \tag{6.9.59}$$

Note that so far we have only been dealing with traces Tr over the adjoint representation. Later we shall express the traces $\mathrm{Tr}F^4$ and $\mathrm{Tr}F^2$ for $SO(32)$ in terms of the traces $\mathrm{tr}F^4$ and $\mathrm{tr}F^2$ over the defining representation. For $E_8\times E_8$ all results will be given in the adjoint representation because in that case the adjoint representation is equal to the defining representation.

The purely gravitational and purely Yang-Mills terms can always be factorized when (6.9.59) holds, and then the anomaly must be of the form

$$\begin{aligned} An \ \sim \ & (\mathrm{tr}\tilde{R}^2+a\mathrm{Tr}\tilde{F}^2)\Big[\frac{1}{8}\mathrm{tr}\tilde{R}^4+\frac{1}{32}(\mathrm{tr}\tilde{R}^2)^2+d\,\mathrm{tr}\tilde{R}^2\mathrm{Tr}\tilde{F}^2 \\ & -\frac{b}{15a}\mathrm{Tr}\tilde{F}^4-\frac{c}{15a}(\mathrm{Tr}\tilde{F}^2)^2\Big] \ . \end{aligned} \tag{6.9.60}$$

Note that there appears a new constant in X, namely d.

In order that this formula also correctly reproduces the cross terms, the following conditions should be satisfied

$$\begin{aligned} R^4F^2 \text{ terms}: &\quad -\frac{1}{240}=\frac{a}{8} &&\Rightarrow \quad a=-\frac{1}{30} \\ R^2F^4 \text{ terms}: &\quad -\frac{1}{24}=-\frac{b}{15a} &&\Rightarrow \quad b=\frac{1}{48} \\ (R^2)^2F^2 \text{ terms}: &\quad -\frac{1}{192}=\frac{a}{32}+d &&\Rightarrow \quad d=-\frac{1}{240} \\ R^2(F^2)^2 \text{ terms}: &\quad 0=-\frac{c}{15a}+ad &&\Rightarrow \quad c=-\frac{1}{(120)^2} \ . \end{aligned} \tag{6.9.61}$$

Hence, anomaly cancellation is only possible if

$$\mathrm{Tr}\tilde{F}^6=\frac{1}{48}\mathrm{Tr}\tilde{F}^4\mathrm{Tr}\tilde{F}^2-\frac{1}{120^2}(\mathrm{Tr}\tilde{F}^2)^3 \ . \tag{6.9.62}$$

If this relation holds, factorization of the anomaly is possible, and the anomaly is given by

$$
\begin{aligned}
An \;=\; & \frac{1}{(4\pi)^6}\int dx\left(\mathrm{tr}\tilde{R}^2 - \frac{1}{30}\mathrm{Tr}\tilde{F}^2\right)\Big[\frac{1}{8}\mathrm{tr}\tilde{R}^4 + \frac{1}{32}(\mathrm{tr}\tilde{R}^2)^2 \\
& -\frac{1}{240}\mathrm{tr}\tilde{R}^2\mathrm{Tr}\tilde{F}^2 + \frac{1}{24}\mathrm{Tr}\tilde{F}^4 - \frac{1}{7200}(\mathrm{Tr}\tilde{F}^2)^2\Big] \; . \qquad (6.9.63)
\end{aligned}
$$

We now specialize to the case $SO(32)$ for which

$$
\begin{aligned}
&\mathrm{Tr}F^2 = 30\mathrm{tr}F^2 \\
&\mathrm{Tr}F^4 = 24\mathrm{tr}F^4 + 3(\mathrm{tr}F^2)^2 \\
&\mathrm{Tr}F^6 = 15\mathrm{tr}F^4\mathrm{tr}F^2 \; . \qquad (6.9.64)
\end{aligned}
$$

Expressing $\mathrm{Tr}F^6$ in terms of $\mathrm{Tr}F^4$ and $\mathrm{Tr}F^2$, one finds that (6.9.59) is satisfied. In terms of the traces over the fundamental representation (6.9.63) reduces to

$$
\begin{aligned}
I_{12} \;=\; & \frac{1}{(4\pi)^6}\int dx\left[\frac{1}{48}\Big(\mathrm{tr}\tilde{R}^2 - \mathrm{tr}\tilde{F}^2\Big)\right. \\
& \left.\Big(\mathrm{tr}\tilde{F}^4 - \frac{1}{8}\mathrm{tr}\tilde{F}^2\mathrm{tr}\tilde{R}^2 + \frac{1}{8}\mathrm{tr}\tilde{R}^4 + \frac{1}{32}(\mathrm{tr}\tilde{R}^2)^2\Big)\right] \; . \qquad (6.9.65)
\end{aligned}
$$

Note that no term with $((\mathrm{tr}\tilde{F}^2)^3$ is present. The consistent anomaly is thus

$$
\begin{aligned}
G \;\sim\; & \int dx\,\Big[\beta(\omega^{(1)}_{2Y} - \omega^{(1)}_{2L})\Big(\mathrm{tr}F^4 - \frac{1}{8}\mathrm{tr}F^2\mathrm{tr}R^2 + \frac{1}{8}\mathrm{tr}R^4 + \frac{1}{32}(\mathrm{tr}R^2)^2\Big) \\
& +(1-\beta)(\mathrm{tr}F^2 - \mathrm{tr}R^2)\Big(\omega^{(1)}_{6Y} - \frac{\alpha}{8}\omega^{(1)}_{2Y}\mathrm{tr}R^2 - \frac{(1-\alpha)}{8}\mathrm{tr}F^2\omega^{(1)}_{2L} \\
& +\frac{1}{8}\omega^{(1)}_{6L} + \frac{1}{32}\omega^{(1)}_{2L}\mathrm{tr}R^2\Big)\Big] \qquad (6.9.66)
\end{aligned}
$$

where α and β are free parameters.

We can now construct the counterterm. We have a two-parameter solution, depending on α and β, but using Bose symmetry we set $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$. Then the counterterm is given by

$$
\begin{aligned}
\Delta\mathcal{L}_{total} \;\sim\; & \frac{1}{3}B\Big[\mathrm{tr}F^4 + -\frac{1}{8}\mathrm{tr}F^2\mathrm{tr}R^2 + \frac{1}{8}\mathrm{tr}R^4 + \frac{1}{32}(\mathrm{tr}R^2)^2\Big] \\
& +\frac{2}{3}(\omega_{3Y} - \omega_{3L})X_7 \qquad (6.9.67)
\end{aligned}
$$

where $dX_7 = X_8$ with X_8 the coefficient of $\frac{1}{3}B$

$$
X_7 = \omega_{7Y} - \frac{1}{16}\omega_{3Y}\mathrm{tr}R^2 - \frac{1}{16}\mathrm{tr}F^2\omega_{3L} + \frac{1}{8}\omega_{7L} + \frac{1}{32}\omega_{3L}\mathrm{tr}R^2 \; . \qquad (6.9.68)
$$

Variation of this counterterm indeed cancels the consistent anomaly. Note that the transformation rule of B is fixed by requiring that anomalies cancel; it reads $\delta B = \omega_{2L}^{(1)} - \omega_{2Y}^{(1)}$, and for example $\delta B = \omega_{2L}^{(1)} + \omega_{2Y}^{(1)}$, would have made anomaly cancellation impossible.

6.10 The $SO(16) \times SO(16)$ string

As a last example of anomaly cancellation in a field theory we consider the $SO(16) \times SO(16)$ heterotic string theory. The massless sector leads to a field theory in 10 dimensions with gauge group $SO(16) \times SO(16)$ and chiral spin 1/2 fermions in the $\mathbf{16} \times \mathbf{16}$ vector representation of $SO(16) \times SO(16)$, and further antichiral spin 1/2 fermions in the $\mathbf{128} \times \mathbf{1}$ and the $\mathbf{1} \times \mathbf{128}$ spinor representations of $SO(16) \times SO(16)$. Hence there are again Yang-Mills anomalies and gravitational anomalies, and we shall apply the general formulas to check whether these anomalies cancel. The anomalies are contained in the expressions

$$\Big(\mathrm{tr}_{16\times 16}\, e^{-i\tilde{F}/4\pi} - \mathrm{tr}_{128\times 1}\, e^{-i\tilde{F}/4\pi} - \mathrm{tr}_{1\times 128}\, e^{-i\tilde{F}/4\pi}\Big) \\ \times \exp\left[\frac{1}{2}\mathrm{tr}\log\left(\frac{-i\tilde{R}/8\pi}{\sinh(-i\tilde{R}/8\pi)}\right)\right] \tag{6.10.1}$$

Denoting $\mathrm{tr}_{16\times 16} - \mathrm{tr}_{128\times 1} - \mathrm{tr}_{1\times 128}$ by Tr, the relevant terms are

$$\begin{aligned} An_{total} &= (\mathrm{Tr}I)An_{1/2} - \Big(\frac{1}{2}\mathrm{Tr}\tilde{F}^2\Big)\Big(\frac{1}{16}\Big)\Big(\frac{1}{360}\mathrm{Tr}\tilde{R}^4 + \frac{1}{288}(\mathrm{Tr}\tilde{R}^2)^2\Big) \\ &\quad + \Big(\frac{1}{24}\mathrm{Tr}\tilde{F}^4\Big)\Big(\frac{1}{48}\mathrm{Tr}\tilde{R}^2\Big) - \frac{1}{720}\mathrm{Tr}\tilde{F}^6 \ . \end{aligned} \tag{6.10.2}$$

First of all we note that $\mathrm{Tr}I = 256 - 128 - 128 = 0$, hence the purely gravitational anomalies cancel: $-32 = 16(-2)$. Also $\mathrm{Tr}F^2 = 0$ but to prove this relation, and deduce other relations for $\mathrm{Tr}F^4$ and $\mathrm{Tr}F^6$, we must first express $\mathrm{Tr}F^p$ into $\mathrm{tr}_{16}F^p$ where by $\mathrm{tr}_{16}F^p$ we mean the trace over the defining vector representation of $SO(16)$. We shall now first derive expressions for $\mathrm{Tr}F^2$, $\mathrm{Tr}F^4$ and $\mathrm{Tr}F^6$, and then return to the issue whether one can find a counterterm to cancel anomalies.

To begin with, consider the spinor representation of $SO(16)$, denoted by its dimension $\mathbf{128}$. We claim that

$$\mathrm{tr}_{128}F^2 = 16\mathrm{tr}_{16}F^2 \ . \tag{6.10.3}$$

To show this, we consider the generator A for a rotation in the x-y plane, corresponding to $A = \frac{1}{2}\gamma_1\gamma_2$ in the spinor representation

$$A_{16} = \begin{pmatrix} 0 & 1 & 0 & . \\ -1 & 0 & 0 & . \\ 0 & 0 & 0 & . \\ . & . & . & . \end{pmatrix}_{16\times 16} \qquad\qquad A_{128} = \frac{1}{2}\gamma^{12} \ . \tag{6.10.4}$$

Since chiral spinors of $SO(16)$ have $\frac{1}{2}256 = 128$ component, while $\mathrm{tr}A_{16}^2 = -2$, and $\mathrm{tr}A_{128}^2 = -\frac{1}{4}128$, we see that (6.10.3) holds: $-32 = 16(-2)$. Consider next $\mathrm{Tr}F^2$. It contains $\mathrm{tr}_{16\times 16}F^2 = 16(\mathrm{tr}_{16}F_1^2 + \mathrm{tr}_{16}F_2^2)$ and $\mathrm{tr}_{128\times 1}F^2 = 16\mathrm{tr}_{16}F_1^2$ and $\mathrm{tr}_{1\times 128}F^2 = 16\mathrm{tr}_{16}F_2^2$. Thus indeed $\mathrm{Tr}F^2 = 0$.

We turn to the expression $\mathrm{Tr}F^4$. It contains the following contributions

$$\mathrm{tr}_{16\times 16}F^4 = 16\mathrm{tr}_{16}F_1^4 + 16\mathrm{tr}_{16}F_2^4 + 6\mathrm{tr}_{16}F_1^2\mathrm{tr}_{16}F_2^2 \tag{6.10.5}$$

note that $F = F_1 \otimes I_2 + I_1 \otimes F_2$, hence F^4 contains cross terms)

$$\mathrm{tr}_{128\times 1}F^4 = 16\mathrm{tr}_{128}F_1^4 \ , \quad \mathrm{tr}_{1\times 128}F^4 = 16\mathrm{tr}_{128}F_2^4 \ . \tag{6.10.6}$$

To proceed we must express $\mathrm{tr}_{128}F^4$ into $\mathrm{tr}_{16}F^4$. We do this as follows: we assume

$$\mathrm{tr}_{128}F^4 = a\mathrm{tr}_{16}F^4 + b(\mathrm{tr}_{16}F^2)^2 \tag{6.10.7}$$

and evaluate these expression for two suitable generators. The first choice is obviously given by (6.10.4). The second choice of a suitable generator is the simultaneous rotation in the x-y and x-z plane

$$A'_{16} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}_{16\times 16} \qquad\qquad A'_{128} = \frac{1}{2}\gamma_1\gamma_2 + \frac{1}{2}\gamma^1\gamma^3 \ . \tag{6.10.8}$$

The first generator in (6.10.4) satisfies

$$(A_{128})^4 = \frac{1}{16} \ , \quad (A_{16})^2 = -I \ , \quad (A_{16})^4 = I \tag{6.10.9}$$

and (6.10.7) yields $\frac{1}{16}128 = 2a + 4b$. The second generator satisfies $(A'_{128})^2 = -\frac{1}{2}$, hence

$$(A'_{128})^4 = \frac{1}{4} \ , \quad (A'_{16})^2 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}_{16\times 16}$$

$$(A'_{16})^4 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}_{16\times 16} \ . \tag{6.10.10}$$

Then (6.10.7) yields $\frac{1}{4}128 = 8a + 16b$. From these relations one finds a and b

$$\mathrm{tr}_{128}F^4 = -8\mathrm{tr}_{16}F^4 + 6(\mathrm{tr}_{16}F^2)^2 \ . \qquad (6.10.11)$$

From here it is easy to obtain $\mathrm{Tr}F^4$ in terms of $\mathrm{tr}_{16}F^4$ and $\mathrm{tr}_{16}F^2$. Namely

$$\begin{aligned} \mathrm{Tr}F^4 &= \mathrm{tr}_{16\times 16}F^4 - \mathrm{tr}_{128}F_1^4 - \mathrm{tr}_{128}F_2^4 \\ &= 24\mathrm{tr}_{16}F_1^4 + 24\mathrm{tr}_{16}F_2^4 - 6(\mathrm{tr}_{16}F_1^2)^2 - 6(\mathrm{tr}_{16}F_2^2)^2 \\ &\quad +6\mathrm{tr}_{16}F_1^2\mathrm{tr}_{16}F_2^2 \ . \end{aligned} \qquad (6.10.12)$$

The last expression we need is the relation between $\mathrm{Tr}F^6$ and $\mathrm{tr}F^6$, $\mathrm{tr}F^4$ and $\mathrm{tr}F^2$. We claim that

$$\mathrm{Tr}F^6 = a\mathrm{tr}_{16}F^6 + b\mathrm{tr}_{16}F^4\mathrm{tr}_{16}F^2 + c(\mathrm{tr}_{16}F^2)^3 \qquad (6.10.13)$$

with $a = 16$, $b = -15$ and $c = \frac{15}{4}$. This relation follows again by postulating this expression and then evaluating it for 3 suitable generators. As such we take the rotation in the x-y plane in (6.10.4), the simultaneous rotation in the x-y and x-z planes given in (6.10.8), and finally the simultaneous rotation in the x-y, x-z and y-z planes

$$A''_{16} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}_{16\times 16} \qquad (A''_{16})^2 = \begin{pmatrix} -2 & 1 & -1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}_{16\times 16}$$

$$(A''_{16})^4 = \begin{pmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{pmatrix}_{16\times 16}$$

$$A''_{128} = \frac{1}{2}\gamma_1\gamma_2 + \frac{1}{2}\gamma^1\gamma^3 + \frac{1}{2}\gamma^2\gamma^3 \qquad (A''_{128})^4 = -\frac{3}{4} \ . \qquad (6.10.14)$$

One finds then the following 3 equations for a, b, c

$$\begin{aligned} (-\frac{1}{4})^3 128 &= a(-2) + b(-4) + c(-8) \\ (-\frac{1}{2})^3 128 &= a(-16) + b(-32) + c(-64) \\ (-\frac{3}{4})^3 128 &= a(-54) + b(-108) + c(-216) \ . \end{aligned} \qquad (6.10.15)$$

This yields indeed

$$\mathrm{tr}_{128}F^6 = 16\mathrm{tr}_{16}F^6 - 15\mathrm{tr}_{16}F^4\mathrm{tr}_{16}F^2 + \frac{15}{4}(\mathrm{tr}_{16}F^2)^3 \ . \qquad (6.10.16)$$

From here it is straightforward to evaluate $\mathrm{Tr}F^6$

$$\begin{aligned}
\mathrm{Tr}F^6 &= \mathrm{tr}_{16\times 16}F^6 - \mathrm{tr}_{128}F_1^6 - \mathrm{tr}_{128}F_2^6 \\
&= \Big\{16\mathrm{tr}_{16}F_1^6 + 15\mathrm{tr}_{16}F_1^4\mathrm{tr}_{16}F_2^2 + 15\mathrm{tr}_{16}F_1^2\mathrm{tr}_{16}F_2^4 + 16\mathrm{tr}_{16}F_2^6\Big\} \\
&\quad -\Big\{16\mathrm{tr}_{16}F_1^6 - 15\mathrm{tr}_{16}F_1^4\mathrm{tr}_{16}F_1^2 + \frac{15}{4}(\mathrm{tr}_{16}F_1^2)^3 \\
&\quad +16\mathrm{tr}_{16}F_2^6 - 15\mathrm{tr}_{16}F_2^4\mathrm{tr}_{16}F_2^2 + \frac{15}{4}(\mathrm{tr}_{16}F_2^2)^3\Big\} \\
&= 15(\mathrm{tr}_{16}F_1^4 + \mathrm{tr}_{16}F_2^4)(\mathrm{tr}_{16}F_1^2 + \mathrm{tr}_{16}F_2^2) \\
&\quad -\frac{15}{4}\Big[(\mathrm{tr}_{16}F_1^2)^3 + (\mathrm{tr}_{16}F_2^2)^3\Big] \qquad (6.10.17)
\end{aligned}$$

Note that the dangerous non-factorized terms $\mathrm{tr}_{16}F_1^6$ and $\mathrm{tr}_{16}F_2^6$ have canceled.

Finally we add up all contributions to the anomalies

$$An_{total} = (\frac{1}{24}\mathrm{Tr}\tilde{F}^4)(\frac{1}{48}\mathrm{Tr}\tilde{R}^2) - \frac{1}{720}\mathrm{Tr}\tilde{F}^6 \ . \qquad (6.10.18)$$

We expect this expression factorizes. Indeed does factorize

$$An_{total} = \frac{1}{24\times 48}\Big[\mathrm{Tr}\tilde{R}^2 - \mathrm{tr}_{16}\tilde{F}_1^2 - \mathrm{tr}_{16}\tilde{F}_2^2\Big]\mathrm{tr}_{16}\tilde{F}^4 \ . \qquad (6.10.19)$$

From here on, we follow the same path as before: we omit the twiddles in An_{total} to obtain I_{12}, extract an exterior derivative d to obtain the 11 dimensional Chern-Simons term, and vary it to obtain the integrand for the consistent anomaly. One expression for the latter is

$$\begin{aligned}
G &= \frac{1}{3}c\int dx\,(\omega_{2L}^{(1)} - \omega_{2Y_1}^{(1)} - \omega_{2Y_2}^{(1)})\mathrm{Tr}F^4 \\
&\quad +\frac{2}{3}c\int dx\,[\mathrm{tr}R^2 - \mathrm{tr}_{16}F_1^2 - \mathrm{tr}_{16}F_2^2]\omega_{6Y}^{(1)} \qquad (6.10.20)
\end{aligned}$$

where $\mathrm{Tr}F^4 = dX_7$ and $\delta X_7 = d\omega_{6Y}^{(1)}$. The last term is rewritten as $\frac{2}{3}c\int dx\,(\omega_{3L}^{(0)} - \omega_{3Y_1}^{(0)} - \omega_{3Y_2}^{(0)}\delta X_7$. The counterterm whose variation cancel G is then given by

$$\Delta\mathcal{L}_{total} = B\mathrm{Tr}F^4 - \frac{2}{3}(\omega_{3L}^{(0)} - \omega_{3Y_1}^{(0)} - \omega_{3Y_2}^{(0)})X_7 \ . \qquad (6.10.21)$$

Thus the gravitational, Yang-Mills and mixed anomalies of the Majorana-Weyl fermions in the $SO(16)\times SO(16)$ string can be canceled by a suitable counterterm. This is a nontrivial result because the string is not finite (there are infrared divergences due to dilaton tadpoles). However, it is modular invariant (large-diffeomorphism anomalies on the worldsheet cancel).

7

Trace anomalies from ordinary and susy quantum mechanics

We now turn to a second class of anomalies, namely the trace anomalies. These are anomalies in the local scale invariance of actions for scalar fields, spin 1/2 fields and certain vector and antisymmetric tensor fields (vectors in $n = 4$, antisymmetric tensors with two indices in $n = 6$, etc.). From a technical point of view, these anomalies are very interesting, because one needs higher loop graphs on the worldline to compute them. In fact, due to the β dependence of the measure of the quantum mechanical path integrals, $A = (2\pi\hbar\beta)^{-\frac{n}{2}}$, one needs $(\frac{n}{2}+1)$-loop calculations in quantum mechanics for the one-loop trace anomalies of n dimensional quantum field theories. Already in 2 dimensions one needs 2-loop graphs, and in 4 dimensions 3-loop graphs. Another interesting technical point regards the fermions. In the path integral they now have antiperiodic boundary conditions. Originally we devised a path integral approach in which fermions were still treated by an operator formalism, and in which actions are operator valued [24]. We shall instead present here a complete path integral approach, with ordinary actions, in which the fermions are described in the path integral by Grassmann fields. The results we get agree with the results in the literature for trace anomalies obtained by different methods (see [121], for example).

We shall separately discuss the anomalies for spin 0, spin 1/2 and spin 1 fields.

7.1 Trace anomalies for scalar fields in 2 and 4 dimensions

The classical action of a massless real scalar field φ in n dimensions, which we take for definiteness with Euclidean signature, is Weyl invariant after one adds a so-called improvement term to the action

$$S = \int d^n x \sqrt{g} \frac{1}{2} \left(g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \xi R \varphi^2 \right) \ , \qquad \xi = \frac{(n-2)}{4(n-1)}$$
$$\delta_W \varphi = \frac{1}{4}(2-n)\sigma(x)\varphi \ , \qquad \delta_W g_{\mu\nu} = \sigma(x) g_{\mu\nu} \ . \qquad (7.1.1)$$

The proof of the Weyl invariance is easy if one uses $\delta_W R = -\sigma R + (n-1)D^\mu D_\mu \sigma$, see appendix A. In the QFT path integral we integrate over $\tilde{\varphi} = g^{1/4}\varphi$ and then one obtains a functional Z of the metric

$$Z[g_{\mu\nu}] = \int [d\tilde{\varphi}] e^{-\frac{1}{\hbar} S[\tilde{\varphi}, g_{\mu\nu}]} \ . \qquad (7.1.2)$$

The field $\tilde{\varphi}$ transform under Weyl rescaling as $\delta_W \tilde{\varphi} = \frac{1}{2}\sigma\tilde{\varphi}$ in any dimension. Under an infinitesimal local scale transformation of the metric, combined with a compensating change of the integration variable $\tilde{\varphi} \to \tilde{\varphi} + \delta_W \tilde{\varphi}$ such that the action remains invariant, one obtains the following Jacobian

$$\begin{aligned} An_W(\text{spin } 0) &= \int d^n x \, \delta_W g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} Z[g] = \frac{1}{2\hbar} \int d^n x \, \sqrt{g} \sigma g^{\mu\nu} \langle T_{\mu\nu} \rangle \\ &= \lim_{\beta \to 0} \mathrm{Tr} \frac{\partial \delta_W \tilde{\varphi}}{\partial \tilde{\varphi}} e^{-\beta \mathcal{R}} = \lim_{\beta \to 0} \mathrm{Tr} \, \frac{1}{2} \sigma e^{-\beta \mathcal{R}} \ . \qquad (7.1.3) \end{aligned}$$

We defined the stress tensor by $T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} S$. Classically $S[g_{\mu\nu} + \delta_W g_{\mu\nu}, \tilde{\varphi} + \delta_W \tilde{\varphi}] = S[g_{\mu\nu}, \tilde{\varphi}]$, hence on-shell the trace of the classical stress tensor vanishes, $2\delta_W g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} S = \sqrt{g} \sigma T_\mu{}^\mu = 0$. At the quantum level, there is an anomaly, proportional to the regulated trace of the unit operator. The regulator $\mathcal{R}$ is fixed by requiring that it preserves Einstein invariance and reads

$$\mathcal{R} = -g^{-1/4} \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu g^{-1/4} - \xi R \ . \qquad (7.1.4)$$

This regulator can be derived from the algorithm of [75]: one adds a mass term $\sqrt{g} m^2 \varphi^2$ to the Weyl invariant action which preserves Einstein symmetry but breaks Weyl invariance, and one constructs $\mathcal{R}$ from the terms quadratic in quantum fields. The fact that one cannot write down a mass term which is simultaneously Einstein and Weyl invariant implies that there may be an anomaly in these symmetries. In fact, we shall see that an anomaly appears in even dimensions. We choose again to preserve the Einstein symmetry, thus locating the anomaly in the Weyl symmetry.

To evaluate the anomaly we shall use the same quantum mechanical approach as for the chiral anomalies, and consider

$$An_W(\text{spin } 0) = \lim_{\beta \to 0} \mathrm{Tr} \, \sigma_S(x) e^{-\frac{\beta}{\hbar} \hat{H}}$$

$$\hat{H} \;=\; \frac{1}{2}g^{-1/4}p_i\sqrt{g}g^{ij}p_j g^{-1/4} - \frac{1}{2}\hbar^2\xi R \qquad (7.1.5)$$

where $\sigma_S(x) = \frac{1}{2}\sigma(x)$ is the product of the Weyl weight of $\tilde{\varphi}$ and the Weyl rescaling parameter.

Trace anomalies for scalars in 2 dimensions

To evaluate the trace (local scale) anomaly for a real scalar field in n dimensions, we use the transition element discussed in part one of this book. In terms of $\sigma_S = \frac{1}{2}\sigma$ we obtain

$$\begin{aligned} An_W(\text{spin } 0, n=2) &= \mathrm{Tr}\,\sigma_S(x)e^{-\frac{\beta}{\hbar}H} \\ &= \int \prod_{i=1}^{n} dx_0^i \sqrt{g(x_0)}\sigma_S(x_0)\langle x_0|e^{-\frac{\beta}{\hbar}H}|x_0\rangle \\ &= \int dx_0\sqrt{g(x_0)}\sigma_S(x_0)\frac{1}{(2\pi\beta\hbar)^{n/2}}\langle e^{-\frac{1}{\hbar}S^{(int)}}\rangle \;, \end{aligned} \qquad (7.1.6)$$

$$\begin{aligned} -\frac{1}{\hbar}S^{int} \;&=\; -\frac{1}{\beta\hbar}\int_{-1}^{0}\frac{1}{2}\Big\{g_{ij}(x_0+q) - g_{ij}(x_0)\Big\}(\dot{q}^i\dot{q}^j + b^ic^j + a^ia^j) \\ &\quad -\beta\hbar\int_{-1}^{0}\Big(\frac{1}{8}R + \frac{1}{8}g^{ij}\Gamma_{ik}{}^{l}\Gamma_{jl}{}^{k} - \frac{1}{2}\xi R\Big)d\tau \;. \end{aligned} \qquad (7.1.7)$$

One can use any regularization discussed in the first part of this book and for definiteness we have chosen time slicing, so that we have included the corresponding counterterm in the action. In $n=2$ dimensions the action for a scalar is Weyl invariant by itself, $\xi = 0$, which has enormous implications for string theory. We must extract the term proportional to $\beta\hbar$ from $\langle\exp(-\frac{1}{\hbar}S^{int})\rangle$ to cancel the factor $(\beta\hbar)^{-1}$ in the measure. Since propagators are proportional to $\beta\hbar$ while vertices are proportional to $(\beta\hbar)^{-1}$ or $\beta\hbar$, we need tree graphs with one vertex proportional to $\beta\hbar$, or graphs with one more propagator than vertices.

To facilitate the computation, we introduce normal coordinates in which the symmetrized derivatives $\partial_{(i}\partial_{j\ldots}\partial_l\Gamma_{mn)}{}^p$ at $x = x_0$ vanish. Then

$$\begin{aligned} g_{ij}(x_0+q) \;&=\; g_{ij}(x_0) - \frac{1}{3}R_{iklj}(x_0)q^kq^l - \frac{1}{6}D_mR_{iklj}(x_0)q^mq^kq^l \\ &\quad -R_{mniklj}(x_0)q^mq^nq^kq^l + \cdots \end{aligned} \qquad (7.1.8)$$

where

$$R_{mniklj} = \frac{1}{20}D_mD_nR_{iklj} + \frac{2}{45}R_{ikmp}R_{ljn}{}^{p}\;. \qquad (7.1.9)$$

All Riemann curvatures in this chapter are curvatures in terms of Christoffel symbols, and not spin connections. We refer to appendix A for definitions.

Only the 2-loop graph with the topology of the number 8, and the tree graph with the R and $\Gamma\Gamma$ vertex (but with $\Gamma^i_{jk}(x_0) = 0$ in normal coordinates) contribute. We find

$$An_W(\text{spin }0, n=2) = \int d^2x_0 \sqrt{g(x_0)}\sigma_S(x_0)\frac{1}{(2\pi\beta\hbar)}$$

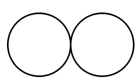
$$\int_{-1}^{0}\left[\Big(-\frac{1}{2}\frac{1}{\beta\hbar}\Big)\Big\langle -\frac{1}{3}R_{iklj}q^k q^l(\dot{q}^i\dot{q}^j + b^i c^j + a^i a^j)\Big\rangle + (-\beta\hbar)\frac{1}{8}R\right]d\tau$$

$$= \int \frac{d^2x}{2\pi}\sqrt{g(x)}\sigma_S(x)$$

$$\times\left[\frac{1}{6}\frac{1}{(\beta\hbar)^2}R_{iklj}\int_{-1}^{0}\langle q^k q^l(\dot{q}^i\dot{q}^j + b^i c^j + a^i a^j)\rangle d\tau - \frac{1}{8}R\right]. \qquad (7.1.10)$$

The propagators $\langle q^i(\sigma)q^j(\tau)\rangle = -\beta\hbar g^{ij}(z)\Delta(\sigma,\tau)$ with $\Delta(\sigma,\tau) = \sigma(\tau + 1)\theta(\sigma-\tau)+\tau(\sigma+1)\theta(\tau-\sigma)$ and $\langle a^i(\sigma)a^j(\tau)\rangle+\langle b^i(\sigma)c^j(\tau)\rangle = -\beta\hbar g^{ij}(z){}^{\bullet\bullet}\Delta(\sigma,\tau)$ with ${}^{\bullet\bullet}\Delta(\sigma,\tau) = \partial_\sigma^2\Delta(\sigma,\tau)$ help in canceling the factor $(\beta\hbar)^{-2}$.

The two-loop graph yields, using $R_{iklj}g^{kl} = -R_{ij}$, the following integral over equal-time propagators



$$= \frac{1}{6}R\int_{-1}^{0}[-\Delta({}^\bullet\Delta^\bullet + {}^{\bullet\bullet}\Delta) + {}^\bullet\Delta\,{}^\bullet\Delta]\,d\tau$$

$$= \frac{1}{6}R\int_{-1}^{0}\left(-\tau(\tau+1) + (\tau+\frac{1}{2})^2\right)d\tau = \frac{1}{24}R\ . \quad (7.1.11)$$

We used here time slicing, according to which ${}^\bullet\Delta^\bullet(\sigma,\tau) = 1 - \delta(\sigma,\tau)$ and ${}^{\bullet\bullet}\Delta(\sigma,\tau) = \delta(\sigma,\tau)$ where $\delta(\sigma,\tau)$ is a Kronecker delta at equal times $\sigma = \tau$. Furthermore, the $\theta(\sigma,\tau)$ in ${}^\bullet\Delta(\sigma,\tau) = \tau - \theta(\sigma,\tau)$ equals $\frac{1}{2}$ at equal time contractions. One obtains then the nonsingular integrals in (7.1.11). Other schemes give the same result. For example in mode regularization the $\Gamma\Gamma$ term, though different, again does not contribute since ${\Gamma_{ij}}^k(x_0) = 0$ in normal coordinates, and using the properties of $\Delta(\sigma,\tau)$ in mode regularization to partially integrate, see (3.3.3), one finds the same result.

Altogether one finds in terms of $\sigma_S = \frac{1}{2}\sigma$

$$An_W(\text{spin }0, n=2) = \int \frac{d^2x_0}{2\pi}\sqrt{g(x_0)}\sigma_S(x_0)\left(\frac{-1}{12}\right)R\ . \qquad (7.1.12)$$

Trace anomalies for scalars in 4 dimensions

For a real scalar field we now must evaluate

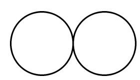
$$An_W(\text{spin }0, n=4) = \int d^4x\sqrt{g(x_0)}\sigma_S(x_0)\frac{1}{(2\pi\beta\hbar)^2}$$

$$\Big\langle \exp\Big[-\frac{1}{\beta\hbar}\frac{1}{2}\int_{-1}^{0}\Big(-\frac{1}{3}R_{iklj}(x_0)q^k q^l - \frac{1}{6}D_m R_{iklj}(x_0)q^m q^k q^l$$

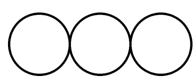
$$
-R_{mniklj}(x_0) q^m q^n q^k q^l + \dots \Big) (\dot{q}^i \dot{q}^j + b^i c^j + a^i a^j) d\tau
$$

$$
-\beta\hbar \int_{-1}^{0} \left(\frac{1}{8} R + \frac{1}{8} g^{ij} \Gamma_{ik}{}^{l} \Gamma_{jl}{}^{k} - \frac{1}{2} \xi R \right) d\tau \Big] \Big\rangle \tag{7.1.13}
$$

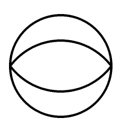
where $\xi = 1/6$. Clearly we now need the terms proportional to $(\beta\hbar)^2$ from $\langle \exp(-\frac{1}{\hbar} S^{int}) \rangle$. In particular, one finds a contribution from the $\Gamma\Gamma$ vertices, due to expanding both $\Gamma(x_0+q)$ into $q^m \partial_m \Gamma(x_0)$ and contracting with an equal-time loop. One must also expand the counterterm with the scalar curvature terms to order q^2, and contract the two q fields to a loop. Since there are no 3-point vertices in normal coordinates (because in normal coordinates $\Gamma_{ij}{}^{k}(x_0) = 0$), we do not need the 5-point vertices. However, one needs the 6-point vertices which yield "clover-leaf graphs", and 4-point vertices which yield "3-bubble graphs" and "eye-graphs". Finally, there are also disconnected diagrams: one-half of the square of the $n = 2$ result; however with $\xi = \frac{1}{6}$ in $n = 4$ their contribution cancels, as one may easily check from (7.1.7) and (7.1.11)

$\frac{1}{2!}\Big($  $+ \bullet \Big)^2 \; = \; 0 \; .$ (7.1.14)

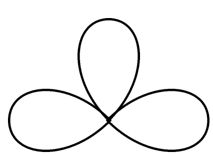
The other contributions, together with the graphs from which they are obtained, follow. First there are the contributions from two $Rqq(\dot{q}\dot{q}+bc+aa)$ vertices. These yield "three-bubble graphs"

 $= \frac{1}{72}(-\beta\hbar)^2 \left[-\frac{1}{6} R_{ij}^2 \right]$ (7.1.15)

and "eye-graphs"

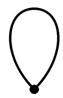
 $= \frac{1}{72}(-\beta\hbar)^2 \left[-\frac{1}{4} R_{ijkl}^2 \right] \; .$ (7.1.16)

Then there are the contributions from one $Rq^4(\dot{q}\dot{q}+bc+aa)$ vertex. They yield various "clover-leaf graphs"

 $= \frac{1}{72}(-\beta\hbar)^2 \left[\frac{3}{20} D^2 R + \frac{1}{10} R_{ijkl}^2 + \frac{1}{15} R_{ij}^2 \right] \; .$ (7.1.17)

The propagators in these graphs denote q or a, b, c ghost propagators.

Finally there are the one-loop graph contributions from the R and $\Gamma\Gamma$ vertices; using $\xi = 1/6$ they yield



$$= \left\{ -\frac{\beta\hbar}{24}\left(\frac{1}{2}D^2 R\right) - \frac{\beta\hbar}{8} g^{mn} g^{ij} \partial_m \Gamma_{ik}{}^l \partial_n \Gamma_{jl}{}^k \right\}$$
$$\times(-\beta\hbar)\left\{ \int_{-1}^{0} \Delta(\tau,\tau) d\tau \right\} . \tag{7.1.18}$$

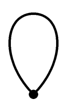
Since in Riemann normal coordinates

$$\begin{aligned} \partial_m \Gamma_{ik}{}^l &= \frac{1}{3}(\partial_m \Gamma_{ik}{}^l + \partial_i \Gamma_{km}{}^l + \partial_k \Gamma_{mi}{}^l) \\ &\quad + \frac{1}{3}(\partial_m \Gamma_{ik}{}^l - \partial_i \Gamma_{mk}{}^l) + \frac{1}{3}(\partial_m \Gamma_{ik}{}^l - \partial_k \Gamma_{im}{}^l) \\ &= \frac{1}{3} R_{mik}{}^l + \frac{1}{3} R_{mki}{}^l \end{aligned} \tag{7.1.19}$$

we obtain for (7.1.18), using $\int_{-1}^{0} \Delta(\tau,\tau) d\tau = -\frac{1}{6}$,

$$\left(-\frac{1}{6}\right)(\beta\hbar)^2 \left[\frac{1}{48} D^2 R + \frac{1}{72}(R_{mikl} + R_{mkil})(R^{milk} + R^{mlik})\right] . \tag{7.1.20}$$

Using the cyclic identity for the Riemann curvatures to obtain $R_{milk} R^{mlik} = \frac{1}{2}(R_{milk})^2$, the $R_{ijkl}{}^2$ terms acquire a factor $(-1-1/2-1/2+1/2) = -3/2$ and the one-loop graphs yield



$$= \frac{1}{72}(\beta\hbar)^2 \left[-\frac{1}{4} D^2 R + \frac{1}{4} R_{ijkl}^2\right] . \tag{7.1.21}$$

Adding all terms, the contributions from the eye-graphs cancel those from the $\Gamma\Gamma$ term, and the R_{ijkl}^2 terms only come from the clover-leaf graphs. We obtain

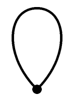
$$An_W(\text{spin } 0, n=4) = \int \frac{d^4 x}{(2\pi)^2} \sqrt{g(x)} \sigma_S(x) \left[a R_{ijkl}^2 + b R_{ij}^2 + c R^2 + d D^2 R\right] \tag{7.1.22}$$

with $\sigma_S = \sigma/2$ and

$$a = \frac{1}{720},\ b = -\frac{1}{720},\ c = 0,\ d = -\frac{1}{720} . \tag{7.1.23}$$

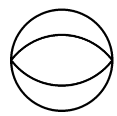
This is the correct result.

Let us now compare this calculation based on time slicing with the equivalent one in dimensional regularization (DR). In DR the counterterm is covariant, $V_{DR} = \frac{1}{8}R$, so that we should drop the $\Gamma\Gamma$ term the in last line of (7.1.13), but of course the Feynman graphs should be evaluated in DR. The counterterm graph needs no regularization and dropping the $\Gamma\Gamma$ term we get



$$= \frac{1}{72}(\beta\hbar)^2 \left[-\frac{1}{4}D^2 R\right] . \qquad (7.1.24)$$

All the other graphs give in DR the same contribution as in TS, except for the "eye-graph" which yield



$$= \frac{1}{72}(\beta\hbar)^2\, 3\, R_{ijkl}^2 K \qquad (7.1.25)$$

where

$$K = \int_{-1}^{0} d\tau \int_{-1}^{0} d\sigma \, \{\Delta^2[({}^\bullet\!\Delta^\bullet)^2 - (\Delta^{\bullet\bullet})^2] - 2\,\Delta\, {}^\bullet\!\Delta\, \Delta^\bullet\, {}^\bullet\!\Delta^\bullet + ({}^\bullet\!\Delta)^2(\Delta^\bullet)^2\} . \quad (7.1.26)$$

This graph should vanish because the $\Gamma\Gamma$ term is absent, and the "eye-graph" canceled the $\Gamma\Gamma$ in time slicing. Clearly the first three terms in K need regularization (also recall that in the sum of the first two terms divergences cancel). Using DR we get for the first three terms

$$\begin{aligned}
&\int d^{D+1}t \int d^{D+1}s \; \{\Delta^2[({}_\mu\Delta_\nu)^2 - (\Delta_{\nu\nu})(\Delta_{\mu\mu})] - 2\,\Delta({}_\mu\Delta)(\Delta_\nu)({}_\mu\Delta_\nu)\} \\
&= \int\int \; \{2\,\Delta(\Delta_\nu)^2({}_{\mu\mu}\Delta) - 4\,\Delta({}_\mu\Delta)(\Delta_\nu)({}_\mu\Delta_\nu)\} \\
&= \int\int \; \{4\,\Delta\; (\Delta_\nu)^2({}_{\mu\mu}\Delta) + 2\,({}_\mu\Delta)^2(\Delta_\nu)^2\} \\
&\to 4\int_{-1}^{0} d\tau\, \Delta(\Delta^\bullet)^2|_\tau + 2\int_{-1}^{0} d\tau \int_{-1}^{0} d\sigma\, ({}^\bullet\!\Delta)^2(\Delta^\bullet)^2 \; . \qquad (7.1.27)
\end{aligned}$$

We have twice integrated by parts in the first term of the first line so that the second term due to the ghosts is canceled. In the second line we have integrated by parts the derivative μ in $(\Delta_\nu)({}_\mu\Delta_\nu) = \frac{1}{2}\,{}_\mu(\Delta_\nu^2)$. Finally in the third line we have used that ${}_{\mu\mu}\Delta(t,s) = \delta^{D+1}(t,s)$, used the Dirac delta function in $D+1$ dimensions, and then removed the regulating parameter $D \to 0$. The final expression in the fourth line is then evaluated at $D = 0$. In fact the limiting values of the various functions (like $(\Delta^\bullet)|_\tau$) given as Fourier series have no ambiguities when multiplied together and can be safely used inside integrals. Adding to this result the last term in (7.1.26) and using (3.3.16 – 3.3.20) we get

$$K = 4\int_{-1}^{0} d\tau\, \Delta(\Delta^\bullet)^2|_\tau + 3\int_{-1}^{0} d\tau \int_{-1}^{0} d\sigma\, ({}^\bullet\!\Delta)^2(\Delta^\bullet)^2 = -\frac{1}{30} + \frac{1}{30} = 0 \, . \quad (7.1.28)$$

Thus the "eye diagrams" vanish and the total result for the anomaly is the same as obtained before in time slicing. We see that DR is computationally simpler, since one does not have to expand the noncovariant $\Gamma\Gamma$ counterterm in normal coordinates (which is rather laborious in higher dimensions). This simplification turns out to be quite useful for the calculation at the forth-loop order, needed to calculate the trace anomaly for a scalar in 6 dimensions. The QM path integral method [67, 68], again produces the expected result [122].

7.2 Trace anomalies for spin 1/2 fields in 2 and 4 dimensions

For complex spin 1/2 fields (Dirac fields) in n dimensions, the action

$$S = \int \sqrt{g}\bar{\psi}\gamma^m e_m{}^\mu D_\mu \psi d^n x\,; \qquad D_\mu\psi = \partial_\mu\psi + \frac{1}{4}\omega_\mu{}^{mn}\gamma_m\gamma_n\psi \qquad (7.2.1)$$

is locally scale invariant in any dimension under[1]

$$\delta_W\psi = \frac{1}{4}(1-n)\sigma(x)\psi, \qquad \delta_W e_m{}^\mu = -\frac{1}{2}\sigma(x)e_m{}^\mu\ . \qquad (7.2.2)$$

We use again an Euclidean signature so that ψ and $\bar{\psi}$ are independent complex Grassmann variables. So for fermions there is no improvement term.

We use $\tilde{\psi} = g^{1/4}\psi$ and $\tilde{\bar{\psi}} = g^{1/4}\bar{\psi}$ as integration variables in the path integral. In any dimension $\delta_W\tilde{\psi} = \frac{1}{4}\sigma(x)\tilde{\psi}$. We introduce the parameter

$$\sigma_F = -2\frac{1}{4}\sigma = -\frac{1}{2}\sigma \qquad (7.2.3)$$

which contains the Weyl weight of ψ and $\bar{\psi}$ and the minus sign for the fermionic Jacobian. One finds then

$$An_W(\mathrm{Dirac}) = \lim_{\beta\to 0}\mathrm{Tr}\,\sigma_F(x)e^{-\beta\mathcal{R}}\ . \qquad (7.2.4)$$

The regular $\mathcal{R}$ is now the square of the Dirac operator as in (6.1.7). Hence

$$\begin{aligned}\hat{H} &= \frac{1}{2}g^{-1/4}\pi_i\sqrt{g}g^{ij}\pi_j g^{-1/4} - \frac{1}{8}\hbar^2 R \\ \pi_i &= p_i - \frac{i\hbar}{2}\omega_{iab}\psi_1^a\psi_1^b\ . \end{aligned}\qquad (7.2.5)$$

To evaluate this anomaly, we will have to use fermions with antiperiodic boundary conditions.They yield the propagators obtained in previous

[1] The proof is easy: for constant σ the weights clearly cancel, while for local $\sigma(x)$ the terms with $\partial_\mu\sigma$ produced by $D_\mu\psi$ cancel if one uses (A.19).

chapters. The general formula for the spin 1/2 trace anomaly is

$$An_W(\text{Dirac}) = \int d^n x_0 \sqrt{g(x_0)} \sigma_F(x_0) \frac{2^{n/2}}{(2\pi\beta\hbar)^{n/2}} \langle e^{-\frac{1}{\hbar}S^{int}} \rangle \qquad (7.2.6)$$

with

$$\begin{aligned} -\frac{1}{\hbar}S^{int} &= -\frac{1}{\beta\hbar}\int_{-1}^{0} d\tau \, \frac{1}{2}\Big[g_{ij}(x_0+q) - g_{ij}(x_0)\Big](\dot{q}^i\dot{q}^j + b^i c^j + a^i a^j) \\ & -\frac{1}{\beta\hbar}\int_{-1}^{0} d\tau \, \frac{1}{2}\dot{q}^i \omega_{iab}\psi_1^a\psi_1^b \\ & -\beta\hbar \int_{-1}^{0} d\tau \, \frac{1}{8}\Big(g^{ij}\Gamma_{ik}{}^l\Gamma_{jl}{}^k + \frac{1}{2}g^{ij}\omega_{iab}\omega_j{}^{ab}\Big) \end{aligned} \qquad (7.2.7)$$

where the ψ_1^a are Majorana spinors with propagator $\langle \psi_1^a(\sigma)\psi_1^b(\tau)\rangle = \frac{1}{2}\beta\hbar\delta^{ab}\epsilon(\sigma-\tau)$. We have normalized the expectation value such that it yields unity when there are no interactions, $\langle 1 \rangle = 1$. The factor $2^{n/2}$ in the measure is needed since the trace over the fermionic states should yield $2^{n/2}$ (we consider here even n dimensions).

There is no term with the scalar curvature in the last line in (7.2.7) because the Riemann term from the Weyl reordering of the bosonic part of the Hamiltonian cancels with the R term from expanding $\not{D}\not{D}$, while for fermions there is no improvement term.

Trace anomalies for fermions in 2 dimensions

The trace anomaly for complex spin 1/2 fields is thus given by

$$\begin{aligned} &An_W(\text{Dirac}, n=2) = \int d^2 x_0 \sqrt{g(x_0)}\sigma_F(x_0)\frac{2}{2\pi\beta\hbar} \\ &\Big\langle \exp\Big[-\frac{1}{\beta\hbar}\int_{-1}^{0}\frac{1}{2}\left\{g_{ij}(x_0+q) - g_{ij}(x_0)\right\}\Big(\dot{q}^i\dot{q}^j + b^i c^j + a^i a^j\Big) d\tau \\ &-\frac{1}{\beta\hbar}\int_{-1}^{0}\frac{1}{2}\dot{q}^i\omega_{iab}\psi_1^a\psi_1^b d\tau - \beta\hbar\int_{-1}^{0}\frac{1}{8}\Big(\Gamma\Gamma + \frac{1}{2}\omega\omega\Big)d\tau\Big]\Big\rangle \end{aligned} \qquad (7.2.8)$$

The terms with ψ_1 do not contribute if one uses normal coordinates in which $\omega_{iab}(x_0) = 0$, because the graph with the topology of the number 8 with one q loop and one ψ_1 loop vanishes (ω_{iab} is traceless). So, also the spin 1/2 anomaly in $n = 2$ dimensions comes only from the purely bosonic sector, namely from the two-loop graph in (7.1.11) which yielded $\frac{1}{24}R$

$$An_W(\text{Dirac } n=2) = 2\int \frac{d^2x}{2\pi}\sqrt{g(x)}\sigma_F(x)\frac{1}{24}R \, . \qquad (7.2.9)$$

Hence, as well known from string theory, the trace anomaly for real spin 0 fields and complex spin 1/2 fields in two dimensions are equal (recall that $\sigma_F = -\sigma_S$).

Trace anomalies for fermions in 4 dimensions

Next we consider the trace anomaly for Dirac fermions in $n = 4$ dimensions. The expression for the anomaly to be evaluated is given by

$$An_W(\text{Dirac } n=4) = \int d^4x_0 \sqrt{g(x_0)}\sigma_F(x_0)\frac{2^2}{(2\pi\beta\hbar)^2}\langle e^{-\frac{1}{\hbar}S^{int}}\rangle \quad (7.2.10)$$

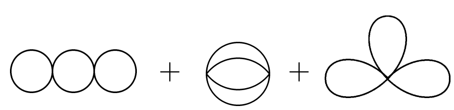
where as before the expectation value is unity if S^{int} vanishes, while the factor 2^2 yields the dimensions of the fermionic part of the Hilbert space. Using normal coordinates, in which $\partial_{(j}\omega_{i)ab}(x_0) = 0$, S^{int} reduces to

$$\begin{aligned} -\frac{1}{\hbar}S^{int} &= -\frac{1}{\beta\hbar}\int_{-1}^{0}\Big(-\frac{1}{3}R_{iklj}q^kq^l - \frac{1}{6}D_mR_{iklj}q^mq^kq^l \\ &\quad -R_{mniklj}q^mq^nq^kq^l + \dots\Big)(\dot{q}^i\dot{q}^j + b^ic^j + a^ia^j)d\tau \\ &\quad -\frac{1}{4}\frac{1}{\beta\hbar}\int_{-1}^{0}\Big(\dot{q}^iq^jR_{jiab}\left(\omega(x_0)\right)\psi_1{}^a\psi_1{}^b + \dots\Big)d\tau \\ &\quad -\frac{\beta\hbar}{8}\int_{-1}^{0}\Big(g^{ij}\Gamma_{ik}{}^l\Gamma_{jl}{}^k + \frac{1}{2}g^{ij}\omega_{iab}\omega_j{}^{ab}\Big)d\tau\ . \end{aligned} \quad (7.2.11)$$

Curvatures depending on $\Gamma_{ik}{}^l$ are denoted with indices i, j, k, l, while curvatures depending on ω_{iab} are denoted by R_{ijab}. Since only squares of each appear below and they only differ by a sign, one needs not be careful about the sign difference, but using a different notation helps to identify the corresponding Feynman graphs.

The total contribution to $\langle e^{-\frac{1}{\hbar}S^{int}}\rangle$ is a sum of the following terms:

(i) the contributions from the first two lines in (7.2.11). They are the sum of (7.1.15), (7.1.16) and (7.1.17)



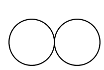
$$= \frac{(\beta\hbar)^2}{720}\left[-\frac{3}{2}R^2_{ijkl} - R^2_{ij} + \frac{3}{2}D^2R\right] \quad (7.2.12)$$

(ii) the contributions from the $\Gamma\Gamma$ and $\omega\omega$ terms; they yield

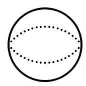
$$\begin{aligned} &= -\frac{\beta\hbar}{8}\left[\frac{1}{9}(-\frac{3}{2})R^2_{ijkl} + \frac{1}{8}R^2_{ijab}\right](-\beta\hbar)\Big(\int_{-1}^{0}\Delta(\tau,\tau)d\tau\Big) \\ &= \frac{-(\beta\hbar)^2}{192}R^2_{ijkl}\Big(-\frac{1}{6}\Big) \end{aligned} \quad (7.2.13)$$

(iii) disconnected graphs: one-half of the square of the 2-loop graphs in (7.1.14)

$$\frac{1}{2!}\Big(\quad\Big)^2 = \frac{(\beta\hbar)^2}{2!}\Big(-\frac{1}{24}R\Big)^2 \tag{7.2.14}$$



(iv) the contribution from the graphs with fermions



$$\begin{aligned} &= \frac{1}{32}\frac{1}{(\beta\hbar)^2} \\ &\quad\times\Big\langle \int_{-1}^{0}(\dot q^i q^j R_{jiab}\psi_1^a\psi_1^b)(\sigma)d\sigma \int_{-1}^{0}(\dot q^k q^l R_{lkcd}\psi_1^c\psi_1^d)(\tau)d\tau\Big\rangle \\ &= -\frac{(\beta\hbar)^2}{16}R_{ijab}^2\int_{-1}^{0}\int_{-1}^{0}({}^\bullet\Delta^\bullet\Delta - {}^\bullet\Delta\Delta^\bullet)\frac{1}{4}\epsilon^2(\sigma-\tau)d\sigma d\tau\ . \end{aligned} \tag{7.2.15}$$

Solid lines lines indicate scalars and dotted lines denote fermions. Other contractions vanish since R_{ijab} is traceless in ij and in ab. However, this graph vanishes, since ${}^\bullet\Delta\Delta^\bullet = \Delta$ and with time slicing ${}^\bullet\Delta^\bullet = 1-\delta(\sigma-\tau)$, Hence

$${}^\bullet\Delta^\bullet\Delta - {}^\bullet\Delta\Delta^\bullet = {}^\bullet\Delta^\bullet\Delta - \Delta = -\delta(\sigma-\tau)\Delta(\sigma,\tau) \tag{7.2.16}$$

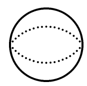
and $\delta(\sigma-\tau)\epsilon^2(\sigma-\tau)$ vanishes according to our rule that $\delta(\sigma-\tau)$ is a Kronecker delta. Hence, again no fermion loops contribute in time slicing.

The final result is

$$\begin{aligned} An_W(\text{Dirac}, n=4) &= \int\frac{d^4x}{(2\pi)^2}\sqrt{g(x)}\sigma_F(x)4\Big[\left(\frac{-3/2}{720}+\frac{1}{288}\right)R_{ijkl}^2 \\ &\qquad -\frac{1}{720}R_{ij}^2+\frac{1}{2!}\left(-\frac{1}{24}R\right)^2+\frac{1}{480}D^2R\Big] \\ &= \int\frac{d^4x}{(2\pi)^2}\sqrt{g(x)}\sigma_F(x)\left[-\frac{7}{1440}R_{ijkl}^2-\frac{1}{180}R_{ij}^2+\frac{1}{288}R^2+\frac{1}{120}D^2R\right] \end{aligned} \tag{7.2.17}$$

This is the correct result.

Let us check once more this final result by employing dimensional regularization (DR) instead of time slicing (TS). In DR the fermions do not modify the counterterm which came form the bosonic sector, and thus the last line of the action in (7.2.11) is now absent. As we have seen from the previous computation in TS, apart form the coefficient 2^2 (which is due to the normalization of the fermionic path integral) only the counterterm $\frac{1}{2}\omega\omega$ gave an additional contribution with respect to the anomaly of a scalar field with the coupling $\xi = \frac{1}{4}$. In DR the $\frac{1}{2}\omega\omega$ counterterm is absent and thus the extra contribution can only come from a nonvanishing fermionic loop. This fermionic loop is the one in (7.2.16) that was vanishing in TS



$$= -\frac{(\beta\hbar)^2}{16} R^2_{ijab} \int_{-1}^{0} d\tau \int_{-1}^{0} d\sigma \; [{}^{\bullet}\!\Delta^{\bullet}\, \Delta - {}^{\bullet}\!\Delta \Delta^{\bullet}]\, \Delta^2_{AF} \quad (7.2.18)$$

where we recall that all functions Δ and Δ_{AF} are functions of τ and σ (recall that Δ_{AF} is antisymmetric, $\Delta_{AF}(\tau,\sigma) = -\Delta_{AF}(\sigma,\tau)$, in fact $\Delta_{AF}(\tau,\sigma) = \frac{1}{2}\epsilon(\tau-\sigma)$ where $\epsilon(x)$ is the sign function $\epsilon(x) = \frac{x}{|x|}$).

We regulate the first term in (7.2.18) with DR. The second contribution in (7.2.18) does not need regularization and could be directly computed integrating the sums of the Fourier mode expansions defining the propagators, but we carry it along anyway. In order to apply DR we must generalize propagators and interactions as discussed in chapter 4. We obtain

$$\begin{aligned}
&\int_{-1}^{0} d\tau \int_{-1}^{0} d\sigma \; [{}^{\bullet}\!\Delta^{\bullet}\, \Delta - {}^{\bullet}\!\Delta \Delta^{\bullet}] \Delta^2_{AF} \quad \to \\
&\quad \to \quad \int d^{D+1}t \int d^{D+1}s \; \{({}_{\mu}\Delta_{\nu}\; \Delta - {}_{\mu}\Delta\, \Delta_{\nu})\, \mathrm{tr}[-\gamma^{\mu}\Delta_{AF}(t,s)\gamma^{\nu}\Delta_{AF}(s,t)] \\
&= \int d^{D+1}t \int d^{D+1}s \; \Delta_{\nu}\; \Delta\, \mathrm{tr}[2\, \underbrace{(\gamma^{\mu}\frac{\partial}{\partial t^{\mu}}\Delta_{AF}(t,s))}_{\delta^{D+1}(t,s)}\, \gamma^{\nu}\Delta_{AF}(s,t)] \\
&-2 \int d^{D+1}t \int d^{D+1}s \; {}_{\mu}\Delta\, \Delta_{\nu}\, \mathrm{tr}[-\gamma^{\mu}\Delta_{AF}(t,s)\gamma^{\nu}\Delta_{AF}(s,t)] \\
&= -2 \int d^{D+1}t \int d^{D+1}s \; {}_{\mu}\Delta\, \Delta_{\nu}\, \mathrm{tr}[-\gamma^{\mu}\Delta_{AF}(t,s)\gamma^{\nu}\Delta_{AF}(s,t)] \\
&\to -2 \int_{-1}^{0} d\tau \int_{-1}^{0} d\sigma \; {}^{\bullet}\!\Delta \Delta^{\bullet}\, \Delta^2_{AF} = \frac{1}{24} \qquad (7.2.19)
\end{aligned}$$

where in the second line we integrated by parts the μ derivative in ${}_{\mu}\Delta_{\nu}$, which when acting on fermions produces a delta functions ("equation of motion terms"). The delta function is integrated in $D+1$ dimensions and gives a vanishing contribution as $\Delta_{AF}(0) = 0$. The remaining terms are then computed at $D \to 0$, where ${}^{\bullet}\!\Delta \Delta^{\bullet} = \Delta$ and $\Delta^2_{AF} = \frac{1}{4}$.

This gives the same contribution as the $\omega\omega$ term in (7.2.13), and thus one obtains the correct answer for the trace anomaly also in dimensional regularization.

7.3 Trace anomalies for a vector field in 4 dimensions

The Maxwell action is Weyl invariant in 4 dimensions if one does not transform A_μ but only the metric. However, to quantize one must add a gauge fixing term and a ghost action, which themselves are not Weyl invariant. We compute the trace anomaly from the general formula $An = \mathrm{Tr}\,(\sigma_V\, e^{-\frac{\beta}{\hbar}\mathcal{R}_V}) + \mathrm{Tr}\,(\sigma_{gh}\, e^{-\frac{\beta}{\hbar}\mathcal{R}_{gh}})$ where the second term yields the contribution from the ghosts. The sum of the contributions from the Maxwell field and its ghosts will satisfy the Wess–Zumino consistency conditions, justifying to some extent this procedure of adding the contributions from Weyl noninvariant actions to compute the Weyl anomaly of a classically Weyl invariant system. The sum of the gauge fixing term and the ghost action is BRST exact, and as the vacuum is BRST invariant, one could rigorously justify our procedure[2]. This approach has been used before [121, 123], and we follow it here. We take as path integral variables $\tilde{A}_m = g^{1/4} e_m{}^\mu A_\mu$ and represent again the vector indices by ghosts, just as in the case of the Yang–Mills anomalies of section 7.2.

The classical Maxwell action

$$\mathcal{L} = -\frac{1}{4}\sqrt{g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \tag{7.3.1}$$

is Weyl invariant under $\delta_W g_{\mu\nu} = \sigma(x) g_{\mu\nu}$ and $\delta_W A_\mu = 0$. For definiteness we use again an Euclidean signature. The field $\tilde{A}_m = g^{1/4} e_m{}^\mu A_\mu$ transforms as

$$\delta_W \tilde{A}_m = \frac{1}{2}\sigma \tilde{A}_m \qquad \text{in} \quad n = 4\ . \tag{7.3.2}$$

As gauge fixing term we use the Fermi-Feynman term in curved space $\mathcal{L} = -\frac{1}{2}\sqrt{g}(D^\mu A_\mu)^2$ with $D^\mu A_\mu = g^{\mu\nu}(\partial_\mu A_\nu - \Gamma_{\mu\nu}{}^\rho A_\rho)$. The gauge fixed action then becomes

$$\mathcal{L} = -\frac{1}{2}\sqrt{g}(D_\mu A_\nu)(D_\rho A_\sigma) g^{\mu\rho} g^{\nu\sigma} + \frac{1}{2} A_\mu R^{\mu\nu} A_\nu \tag{7.3.3}$$

where $R_{\mu\nu}$ is defined in appendix A. As regulator in the space with $\tilde{A}_m$ we obtain

$$(\mathcal{R}_V)^m{}_n = (g^{-1/4} D_\mu \sqrt{g} g^{\mu\nu} D_\nu g^{-1/4})^m{}_n + R^m{}_n \tag{7.3.4}$$

where both D_μ and D_ν only contain a spin connection for the vector index of $\tilde{A}_m$

$$(D_\mu)^m{}_n = \partial_\mu \delta^m_n + \omega_\mu{}^m{}_n\ . \tag{7.3.5}$$

[2] This situation reminds one of the quantization of susy Yang–Mills theories, where in x-space gauge-fixing and ghost terms break susy. One can use an extension of the Batalin-Vilkovisky method to derive Ward identities which treat susy and gauge symmetry on a par [124].

The Hamiltonian for the corresponding quantum mechanical model is given by

$$\begin{aligned} \hat{H}_V &= \frac{1}{2} g^{-1/4} \pi_i \sqrt{g} g^{ij} \pi_j g^{-1/4} - \frac{\hbar^2}{2} c^*_m R^m{}_n c^n \\ \pi_i &= p_i - i\hbar (c^*_m \, \omega_i{}^m{}_n c^n) \ . \end{aligned} \tag{7.3.6}$$

where the vector ghosts satisfy the equal-time canonical commutation relations $\{c^n, c^*_m\} = \delta^n_m$. We continue to denote the indices for the internal vector space by $m, n, p, q, ...$ and the indices for the coordinates and momenta in the QM model by $i, j, k, l, ...$ This is useful for keeping track of the various contributions, but there is of course no intrinsic difference between both kinds of indices.

In the path integral we need the Hamiltonian in Weyl ordered form. However, as explained in section 7.2 where we evaluated the traces over the ghosts, **one should not Weyl order the ghosts**, rather products of $c^*\omega c$ correspond to products of matrices. One obtains

$$H_V = \left(\frac{1}{2} g^{ij} \hat{\pi}_i \hat{\pi}_j\right)_W + \frac{\hbar^2}{8}(R + g^{ij}\Gamma^k_{il}\Gamma^l_{jk}) - \frac{\hbar^2}{2}(c^*_m R^m{}_n c^n) \ . \tag{7.3.7}$$

The first three terms are the same as for a scalar, except that π contains $c^*\omega c$ terms. We repeat that one should not Weyl order the terms proportional to $(c^*\omega c)$ and $(c^*\omega c)(c^*\omega c)$.

The anomaly comes from a trace over the space of $\tilde{A}_m$ and a trace over the space of the ghosts. We first discuss the former. The contribution to the anomaly from $\tilde{A}_m$ reads

$$An(\tilde{A}_m) = \mathrm{Tr}\, \sigma_V \, e^{-\frac{\beta}{\hbar}\hat{H}_V} = \int d^4x_0 \sqrt{g(x_0)} \sigma_V(x_0) \frac{1}{(2\pi\beta\hbar)^2} \, \langle e^{-\frac{1}{\hbar} S^{int}_V} \rangle \tag{7.3.8}$$

where $\sigma_V = \frac{1}{2}\sigma$ and

$$\begin{aligned} -\frac{1}{\hbar} S^{int}_V &= -\frac{1}{\beta\hbar} \int_{-1}^0 \frac{1}{2}[g_{ij}(x_0+q) - g_{ij}(x_0)](\dot{q}^i\dot{q}^j + b^i c^j + a^i a^j)\, d\tau \\ &\quad - \int_{-1}^0 \dot{q}^i (c^*_m \, \omega_i{}^m{}_n c^n)\, d\tau + \frac{\beta\hbar}{2} \int_{-1}^0 c^*_m R^m{}_n c^n \, d\tau \\ &\quad - \frac{\beta\hbar}{8} \int_{-1}^0 \left[R + g^{ij}\Gamma_{ik}{}^l \Gamma_{jl}{}^k\right] d\tau \qquad . \end{aligned} \tag{7.3.9}$$

We need all graphs of order $(\beta\hbar)^2$ to cancel the factor $(\beta\hbar)^{-2}$ in the Feynman measure. The q and a, b, c propagators are of order $\beta\hbar$, but the $\langle c^m c^*_n \rangle$ ghost propagator is $\beta\hbar$-independent

$$\begin{aligned} \langle q^i(\sigma) q^j(\tau) \rangle &= -\beta\hbar g^{ij}(x_0)\Delta(\sigma, \tau) \\ \langle c^m(\sigma) c^*_n(\tau) \rangle &= \delta^m_n \theta(\sigma - \tau) \ . \end{aligned} \tag{7.3.10}$$

As already explained in section (7.2) we only get trees for the ghosts, and integration over the Grassmann variables at the front and at the back of the tree leads to a trace over the indices at the ends of the tree.

Using Riemann normal coordinates, the expansion of $g_{ij}(x_0+q)-g_{ij}(x_0)$ contains terms with 2,3,4... q-fields, leading to 4,5,6... point functions for the q fields and a, b, c ghosts. The contributions from this term alone were already determined when we calculated the trace anomaly of a scalar in 4 dimensions. There are no contributions from the product of this vertex and the second vertex to order $(\beta\hbar)^2$ because the Riemann tensor is traceless, but products of the first and last vertex lead to disconnected graphs whose contribution is proportional to the product of the two-loop graph for a scalar in $n=2$ with the scalar curvature.

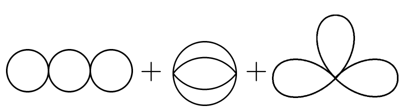
The next vertex is the $\dot{q}(c^*\omega c)$ vertex. In a frame were ${\omega_i}^m{}_n(x_0)=0$, and ${\partial_{(j}\omega_{i)}}^m{}_n(x_0)=0$, it becomes

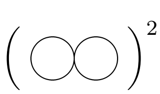
$$-\int_{-1}^{0} q^i\dot{q}^j(c^*_m\frac{1}{2}R_{ij}{}^m{}_n c^n)\,d\tau\ . \qquad (7.3.11)$$

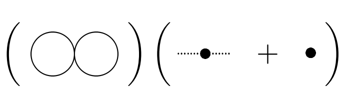
The square of this vertex yields an R^2_{ijmn} term. To order $(\beta\hbar)^2$ there are no contributions from the product of this vertex with the vertices in the last line of (7.3.9).

Finally, in the last line of S^{int} the (c^*Rc) vertex can either be squared to yield an R^2_{mn} term, or the R_{mn} inside (c^*Rc) can be expanded to second order in q to yield a D^2R term, or it can be multiplied with the $-\frac{1}{8}(R+\Gamma\Gamma)$ terms to yield a R^2 term. The $-\frac{1}{8}(R+\Gamma\Gamma)$ terms can be squared to to yield an R^2 term, or the R and $\Gamma\Gamma$ terms can be expanded to second order to yield a D^2R and an R^2_{ijkl} term.

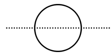
The contributions proportional to $(\hbar\beta)^2$ are as follows[3]

 $= 4\left[-\frac{1}{480}R^2_{ijkl}-\frac{1}{720}R^2_{ij}+\frac{1}{480}D^2R\right]$

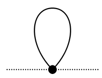
 $= \frac{1}{2!}\,4\left(\frac{1}{24}R\right)^2$

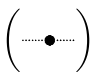
 $= \left(\frac{1}{24}R\right)\left(\frac{1}{2}R-\frac{4}{8}R\right)=0$

[3] Recall that one obtains an integral $\int d\chi_{gh}d\bar{\eta}_{gh}P^{gh}_{\bar{\eta},\chi}$ while the transition element contains a factor $e^{\bar{\eta}_{gh}\chi^{gh}}$. For the interactions which are independent of the ghost fields the factor $\bar{\eta}_{gh}\chi^{gh}$, obtained by expanding $e^{\bar{\eta}_{gh}\chi^{gh}}$, saturates the integral over $d\chi_{gh}$ and $d\bar{\eta}_{gh}$, and one obtains a factor $\mathrm{Tr}\,I=4$.

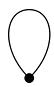
 $= -\dfrac{1}{48} R^2_{ijmn}$

 $= \dfrac{1}{8} R^2_{mn}$

 $= \dfrac{1}{24} D^2 R$

 $\left(\bullet \right) = -\dfrac{1}{16} R^2$

$$\left(\bullet \right)^2 = 4\,\frac{1}{2!} \left(\frac{1}{8} R\right)^2 = \frac{1}{32} R^2$$

 $= 4 \left[-\dfrac{1}{96} D^2 R + \dfrac{1}{288} R^2_{ijkl} \right] \qquad (7.3.12)$

Dotted lines indicate ghosts, and external ghosts are traced over. To evaluate the fourth diagram we used that $\int\int d\sigma d\tau\, ({}^\bullet\Delta^\bullet\Delta - {}^\bullet\Delta\Delta^\bullet)\theta = -\int\int d\sigma d\tau\, \delta(\sigma - \tau)\Delta(\sigma-\tau)\theta(\sigma-\tau) = -\frac{1}{2}\int d\tau\, \tau(\tau+1) = \frac{1}{12}$.

The QFT Faddeev-Popov ghosts, denoted by B and C, contribute, too. Their action reads

$$\mathcal{L}(ghosts) = \sqrt{g}\, B g^{\mu\nu} D_\mu D_\nu C\ . \tag{7.3.13}$$

The regulator which follows from this action is the same as for scalar fields but without improvement term. Under rigid scale transformation one has in 4 dimensions $\delta_W B = -\frac{1}{2}\sigma B$ and idem for C. Defining $\tilde{B} = g^{1/4} B$ and $\tilde{C} = g^{1/4} C$ one has $\delta_W \tilde{B} = \frac{1}{2}\sigma\tilde{B}$ and $\delta_W \tilde{C} = \frac{1}{2}\sigma\tilde{C}$. Defining a parameter σ_{gh} which takes into account the minus sign in the Jacobian for both ghosts

$$\sigma_{gh} = (-2)\frac{1}{2}\sigma \tag{7.3.14}$$

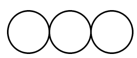
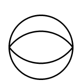
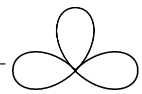
we obtain

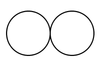
$$\begin{aligned} An(ghosts) &= \mathrm{Tr}\, \sigma_{gh}\, e^{-\frac{\beta}{\hbar}\mathcal{R}_{gh}} \\ &= \int d^4x_0 \sqrt{g(x_0)}\sigma_{gh}(x_0) \frac{1}{(2\pi\beta\hbar)^2} \langle e^{-\frac{1}{\hbar}S^{int}_{gh}} \rangle \end{aligned} \tag{7.3.15}$$

where

$$\begin{aligned} -\frac{1}{\hbar} S^{int}_{gh} &= -\frac{1}{\beta\hbar}\int_{-1}^{0} \frac{1}{2}[g_{ij}(x_0+q) - g_{ij}(x_0)](\dot{q}^i\dot{q}^j + b^i c^j + a^i a^j)\, d\tau \\ &- \frac{\beta\hbar}{8}\int_{-1}^{0} \left[R + g^{ij}\Gamma_{ik}{}^{l}\Gamma_{jl}{}^{k} \right] d\tau\ . \end{aligned} \tag{7.3.16}$$

The contributions of the ghosts are not multiplied by a factor 4 because there is no integration over internal symmetry ghosts (but since $\sigma_{gh} = (-2)\frac{1}{2}\sigma$, the ghosts still subtract two degrees of freedom from $\tilde{A}_m$). One finds to order $(\beta\hbar)^2$ the following contributions from the Faddeev–Popov ghosts to the trace anomaly

 $+$  $+$  $= \left[-\frac{1}{480}R_{ijkl}^2 - \frac{1}{720}R_{ij}^2 + \frac{1}{480}D^2 R\right]$

$\Big($  $+ \; \bullet \Big)^2 = \frac{1}{2!}\left(\frac{1}{24}R - \frac{1}{8}R\right)^2 = \frac{1}{288}R^2$

 $= -\frac{1}{96}D^2 R + \frac{1}{288}R_{ijkl}^2$ (7.3.17)

Adding the contributions of the ghosts to the contributions of the vector, not forgetting the factor -2 from $\sigma_{gh}(x_0) = -2\sigma_V(x_0)$, one obtains the total result

$$
\begin{aligned}
&An_W(\text{Maxwell}, n=4) = \qquad\qquad (7.3.18)\\
&= \int \frac{d^4x}{(2\pi)^2}\sqrt{g(x)}\sigma_V(x)\left[-\frac{13}{720}R_{ijkl}^2 + \frac{11}{90}R_{ij}^2 - \frac{5}{144}R^2 + \frac{1}{40}D^2 R\right] .
\end{aligned}
$$

This is the correct result. The coefficient of $D^2 R$ is scheme dependent (a counterterm R^2 can change it) but we find agreement with DeWitt (second reference in [8]), while Duff finds a coefficient $-\frac{1}{60}$.

7.4 String inspired approach to trace anomalies

8
Conclusions and Summary

Appendix A
Riemann curvatures

To define the Riemann curvatures in terms of a connection $\Gamma_{ij}{}^k$ or spin connection $\omega_i{}^a{}_b$, we begin with the "vielbein postulate"

$$D_i e_j{}^a \equiv \partial_i e_j{}^a - \Gamma_{ij}{}^k e_k{}^a + \omega_i{}^a{}_b e^b{}_j = 0 \ . \tag{A.1}$$

This equation has a geometrical meaning. Consider a vector field $v^i(x)$ with curved indices, and use the vielbein field $e_i{}^a(x)$ to construct a corresponding vector field $v^a(x) \equiv v^i(x)e_i{}^a(x)$ with flat indices. Parallel transport of $v^i(x)$ along a distance Δx^j with an arbitrary connection field $\Gamma_{ij}{}^k(x)$ leads to a vector field $\tilde{v}^i(x+\Delta x)$ at $x+\Delta x$ defined by $\tilde{v}^i(x+\Delta x) = v^i(x) - v^k(x)\Delta x^j \Gamma_{jk}{}^i(x)$. Similarly parallel transport of $v^a(x)$ along the distance Δx^j with an arbitrary connection field $\omega_i{}^a{}_b(x)$ yields a vector field $\tilde{v}^a(x+\Delta x)$ at $x+\Delta x$ defined by $\tilde{v}^a(x+\Delta x) = v^a(x) - \Delta x^j \omega_j{}^a{}_b(x) v^b(x)$. The vielbein postulate in (A.1) states that $\tilde{v}^i(x)$ and $\tilde{v}^a(x)$ are related to each other the same way as $v^i(x)$ and $v^a(x)$ are related, $\tilde{v}^a(x+\Delta x) \equiv \tilde{v}^i(x+\Delta x) e_i{}^a(x+\Delta x)$. Indeed, expansion to first order in Δx reproduces (A.1). Thus there is only one vector field which one can write as $v^i(x)$ or as $v^a(x)$, and only one connection, which is given by $\Gamma_{ij}{}^k(x)$ if one writes the vector field as $v^i(x)$, or given by $\omega_i{}^a{}_b(x)$ if one uses $v^a(x)$ to represent the vector field. In other words, the operations of parallel transport and conversion from curved to flat indices (or vice-versa) commute.

We next require that length is preserved by parallel transport. The square of the length of a vector v^a is by definition $v^a \delta_{ab} v^b$ in Euclidean space (or $v^a \eta_{ab} v^b$ in Minkowski space). Length is preserved if and only if $\omega_{iab} \equiv \omega_i{}^c{}_b \delta_{ca}$ is antisymmetric in ab. This we shall always assume to be the case. For a vector v^i we define the square of the length by $v^i g_{ij} v^j$ where the metric is related to the vielbein by $g_{ij} = e_i{}^a e_j{}^b \delta_{ab}$ in Euclidean space (and $g_{ij} = e_i{}^a e_j{}^b \eta_{ab}$ in Minkowski space). The connection $\Gamma_{ij}{}^k$ preserves length if it satisfies $\Gamma_{ij;k} + \Gamma_{ik;j} - \partial_i g_{jk} = 0$ (with $\Gamma_{ij;k} \equiv \Gamma_{ij}{}^l g_{lk}$), which is equivalent to the antisymmetry of ω_{iab} if the vielbein postulate holds.

The covariant derivatives of vectors are proportional to the difference of the original vector field and the parallel transported vector field

$$\begin{aligned} v^i(x+\Delta x) - \tilde{v}^i(x+\Delta x) &= \Delta x^j D_j v^i(x) \\ \Rightarrow \quad D_j v^i &= \partial_j v^i + \Gamma_{jk}{}^i v^k \\ v^a(x+\Delta x) - \tilde{v}^a(x+\Delta x) &= \Delta x^j D_j v^a(x) \\ \Rightarrow \quad D_j v^a &= \partial_j v^a + \omega_j{}^a{}_b v^b \ . \end{aligned} \tag{A.2}$$

By requiring that $D_j(w_i v^i) = \partial_j(w_i v^i) = D_j(w_a v^a)$ one also derives that $D_j w_i = \partial_j w_i - \Gamma_{ji}{}^k w_k$ and $D_j w_a = \partial_j w_a + \omega_{ja}{}^b w_b$.

From (A.1) we can express $\omega_i{}^a{}_b$ in terms of $\Gamma_{ij}{}^k$ and $e_i{}^a$. If $\Gamma_{ij}{}^k$ has an antisymmetric piece $\frac{1}{2}(\Gamma_{ij}{}^k - \Gamma_{ji}{}^k) = T_{ij}{}^k$, this piece is called the torsion tensor. If there is no torsion, $\Gamma_{ij}{}^k$ is the usual Christoffel symbol

$$\left\{ \begin{matrix} k \\ i\,j \end{matrix} \right\} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \tag{A.3}$$

as one easily shows by using $D_i g_{jk} = 0$. Torsion preserves length if $T_{ijl} \equiv T_{ij}{}^k g_{kl}$ is totally antisymmetric. However, until (A.13) we do not make the assumption that torsion is absent.

From $[D_i, D_j] e_k^a = 0$ one finds

$$[D_i, D_j] e_k^a = -R_{ijk}{}^l(\Gamma) e_l^a + R_{ij}{}^a{}_b(\omega) e_k{}^b = 0 \tag{A.4}$$

where evidently

$$R_{ijk}{}^l(\Gamma) \ = \ \partial_i \Gamma_{jk}{}^l + \Gamma_{im}{}^l \Gamma_{jk}{}^m - (i \leftrightarrow j) \tag{A.5}$$

$$R_{ij}{}^a{}_b(\omega) \ = \ \partial_i \omega_j{}^a{}_b + \omega_i{}^a{}_c \omega_j{}^c{}_b - (i \leftrightarrow j) \ . \tag{A.6}$$

Hence

$$R_{ijk}{}^a(\Gamma) = R_{ij}{}^a{}_k(\omega) \ . \tag{A.7}$$

The variation of a curvature is the covariant derivative of the variation of the corresponding connection

$$\begin{aligned} \delta R_{ijk}{}^l(\Gamma) &= D_i \delta\Gamma_{jk}{}^l - D_j \delta\Gamma_{ik}{}^l \\ \delta R_{ij}{}^a{}_b(\omega) &= D_i \delta\omega_j{}^a{}_b - D_j \delta\omega_i{}^a{}_b \end{aligned} \tag{A.8}$$

where

$$\begin{aligned} D_i \delta\Gamma_{jk}{}^l &= \partial_i \delta\Gamma_{jk}{}^l - \Gamma_{ij}{}^m \delta\Gamma_{mk}{}^l - \Gamma_{ik}{}^m \delta\Gamma_{jm}{}^l + \Gamma_{im}{}^l \delta\Gamma_{jk}{}^m \\ D_i \delta\omega_j{}^a{}_b &= \partial_i \delta\omega_j{}^a{}_b - \Gamma_{ij}{}^m \delta\omega_m{}^a{}_b + \omega_i{}^a{}_c \delta\omega_j{}^c{}_b + \omega_{ib}{}^c \delta\omega_j{}^a{}_c \ . \end{aligned} \tag{A.9}$$

From (A.6) it follows that $R_{ijab}(\omega)$ is antisymmetric in its last two indices. Then (A.7) can also be written as

$$R_{ijka}(\Gamma) = -R_{ijka}(\omega) \tag{A.10}$$

The Ricci tensor is defined by

$$R_{ij} = R_{ikj}{}^{k}(\Gamma) = R_{ik}{}^{k}{}_{j}(\omega) \tag{A.11}$$

and the scalar curvature is given by

$$R = g^{ij} R_{ij} \ . \tag{A.12}$$

From now on in this appendix we assume that $\Gamma_{ij}{}^{k}$ is the Christoffel symbol. At the linearized level $g_{ij} \simeq \eta_{ij} + h_{ij}$, and

$$\begin{aligned} R^{\text{lin}}_{ijkl}(\Gamma) &= \frac{1}{2}(\partial_i\partial_k h_{jl} + \partial_j\partial_l h_{ik} - \partial_j\partial_k h_{il} - \partial_i\partial_l h_{jk}) \qquad &(\text{A.13})\\ R^{\text{lin}}_{ij} &= \frac{1}{2}(-\partial_i h_j - \partial_j h_i + \partial_i\partial_j h + \Box h_{ij}) &(\text{A.14}) \end{aligned}$$

where $h_i = \partial^j h_{ij}$, $h = \eta^{ij} h_{ij}$ while $\Box = \eta^{ij}\partial_j\partial_j$. Clearly

$$R^{\text{lin}} = -\partial^i h_i + \Box h = (-\partial^i\partial^j + \partial^k\partial_k\eta^{ij})h_{ij} \ . \tag{A.15}$$

In section 2.6 we need the full nonlinear expression of R in terms of $\Box g \equiv g^{ij}g^{kl}\partial_k\partial_l g_{ij}$, $\partial^j g_j \equiv g^{ik}g^{jl}\partial_k\partial_l g_{ij}$, $\partial_k g \equiv g^{ij}\partial_k g_{ij}$, $g^i \equiv g^{ij}g^{kl}\partial_k g_{lj}$, and $g_k \equiv g^{ij}\partial_i g_{jk}$. Straightforward evaluation yields

$$\begin{aligned} R &= \frac{1}{2}g^{ik}g^{jl}(\partial_i\partial_k g_{jl} - \partial_i\partial_l g_{jk} - \partial_j\partial_k g_{il} + \partial_j\partial_l g_{ik}) + (\partial g)(\partial g) \text{ terms}\\ &= \Box g - \partial^j g_j - \frac{3}{4}\left(\partial_k g_{ij}\right)^2 + \frac{1}{2}\left(\partial_i g_{jk}\right)\partial_j g_{ik} + \frac{1}{4}\left(\partial_j g\right)^2 - \left(\partial_j g\right) g^j + g_j^2 \ . \end{aligned} \tag{A.16}$$

In the section 2.6 we also need the full nonlinear expression for the Riemann curvature in the form $R_{ikjl}(\Gamma) = \frac{1}{2}(\partial_i\partial_j g_{kl} + 3 \text{ terms}) + (\partial g)^2$ terms. Straightforward evaluation yields

$$\begin{aligned} R_{ikjl}(\Gamma) &= \frac{1}{2}(\partial_i\partial_j g_{kl} - \partial_i\partial_l g_{kj} - \partial_k\partial_j g_{il} + \partial_k\partial_l g_{ij})\\ &\quad + (\Gamma_{ij}{}^{m}\Gamma_{kl}{}^{n} - \Gamma_{kj}{}^{m}\Gamma_{il}{}^{n})g_{mn} \ . \end{aligned} \tag{A.17}$$

As a check we note that the expression on the right-hand side has all the symmetries of the left-hand side: antisymmetry in each pair, symmetry under pair exchange, and the cyclic identity.

For the calculations of trace anomalies we need to know how the scalar curvature transforms under local Weyl rescalings

$$\begin{aligned} \delta_W \, g_{ij} &= \sigma(x) g_{ij} \\ \delta_W \, e_i{}^a &= \frac{1}{2}\sigma(x) e_i{}^a \ . \end{aligned} \tag{A.18}$$

Using the vielbein postulate and (A.3), one finds

$$\begin{aligned} \delta_W \, \omega_i{}^a{}_b &= \delta_W (\Gamma_{ij}{}^k \, e_k{}^a e_b{}^j) - \frac{1}{2}\delta^a_b \partial_i \sigma \\ &= \frac{1}{2}(e_i{}^a e_b{}^j - e_{ib} e^{ja}) \partial_j \sigma \ . \end{aligned} \tag{A.19}$$

Substitution into (A.8) yields then in n dimensions

$$\delta_W \, R = -\sigma R + (n-1) D^i D_i \sigma \ . \tag{A.20}$$

Appendix B
Weyl ordering of bosonic operators

In this appendix we discuss the concept of Weyl ordering. For a detailed account, see for example [34].

To evaluate matrix elements of the form $M = \langle z|\hat{O}|y\rangle$ with $\langle z|$ and $|y\rangle$ eigenstates of the position operator $\hat{x}^i$ and $\hat{O}(\hat{x},\hat{p})$ an arbitrary operator, we first insert a complete set of momentum eigenstates $I = \int |p\rangle\langle p| d^n p$

$$M = \int \langle z|\hat{O}|p\rangle\langle p|y\rangle \, d^n p \ . \tag{B.1}$$

It is then very convenient to *rewrite* $\hat{O}$ as a Weyl-ordered operator $\hat{O}_W$, because, as we shall prove, one can then replace in $\hat{O}_W(\hat{x},\hat{p})$ the operators $\hat{x}^i$ and $\hat{p}_j$ by the c-number values $\frac{1}{2}(z^i + y^i)$ and p_j, respectively. If we denote the corresponding function of $\frac{1}{2}(z^i + y^i)$ and p_i by $O_W\left((z+y)/2, p\right)$, we can prove the following

$$\textbf{Theorem}: \qquad M = \int \langle z|p\rangle O_W\left((z+y)/2, p\right)\langle p|y\rangle \, d^n p \ . \tag{B.2}$$

We must clearly first define what the Weyl ordering is, and in particular how to construct the operator $\hat{O}_W$ from a given operator $\hat{O}$. In this construction we shall also need the notion of a symmetrized operator $\hat{O}_S$. We shall show that in general an operator $\hat{O}$ can be rewritten as a sum of the corresponding symmetrized operator $\hat{O}_S$ and more terms

$$\hat{O} = \hat{O}_S + more = \hat{O}_W \ . \tag{B.3}$$

As the notation indicates, the operator $\hat{O}_W$ is equal to the original operator $\hat{O}$, but it is written in such a way that $\hat{x}$ and $\hat{p}$ appear symmetrically.

Let us first give a few examples. The operator $\hat{x}\hat{p}$ can clearly be written as the sum of $\frac{1}{2}(\hat{x}\hat{p}+\hat{p}\hat{x})$ and $\frac{1}{2}(\hat{x}\hat{p}-\hat{p}\hat{x})$. The latter term is equal to $\frac{1}{2}i\hbar$.

The former term is the symmetrized form of $\hat{x}\hat{p}$, so $(\hat{x}\hat{p})_S \equiv \frac{1}{2}(\hat{x}\hat{p}+\hat{p}\hat{x})$. We then have

$$\hat{x}\hat{p} = (\hat{x}\hat{p})_S + \frac{1}{2}i\hbar \equiv (\hat{x}\hat{p})_W \ , \qquad \hat{p}\hat{x} = (\hat{p}\hat{x})_S - \frac{1}{2}i\hbar \equiv (\hat{p}\hat{x})_W \ . \tag{B.4}$$

Clearly $(\hat{x}\hat{p})_S$ is equal to $(\hat{p}\hat{x})_S$, but $(\hat{x}\hat{p})_W$ is not equal to $(\hat{p}\hat{x})_W$. As a second example consider the operator $\hat{x}\hat{x}\hat{p}\hat{p}$. Its symmetrized form, as we shall discuss below, can be written as

$$(\hat{x}\hat{x}\hat{p}\hat{p})_S = \frac{1}{4}\hat{x}\hat{x}\hat{p}\hat{p} + \frac{1}{2}\hat{p}\hat{x}\hat{x}\hat{p} + \frac{1}{4}\hat{p}\hat{p}\hat{x}\hat{x} \ . \tag{B.5}$$

One may then check that

$$\hat{x}\hat{x}\hat{p}\hat{p} - (\hat{x}\hat{x}\hat{p}\hat{p})_S = 2i\hbar(\hat{x}\hat{p})_S - \frac{1}{2}\hbar^2 \ . \tag{B.6}$$

So in this example we have

$$\hat{x}\hat{x}\hat{p}\hat{p} = (\hat{x}\hat{x}\hat{p}\hat{p})_S + 2i\hbar(\hat{x}\hat{p})_S - \frac{1}{2}\hbar^2 \equiv (\hat{p}\hat{x}\hat{x}\hat{p})_W \ . \tag{B.7}$$

The term $2i\hbar(\hat{x}\hat{p})_S - \frac{1}{2}\hbar^2$ corresponds to the term denoted by *more* in (B.3). So, as an operator $(\hat{x}\hat{x}\hat{p}\hat{p})_W$ is equal to $\hat{x}\hat{x}\hat{p}\hat{p}$, but for our purposes it is useful to use the fundamental commutation relations to write $\hat{x}\hat{x}\hat{p}\hat{p}$ in such a way that all $\hat{x}$ and $\hat{p}$ appear symmetrically, and if this is the case we call this expression the Weyl ordered form.

In general, we call an operator $\hat{A}(\hat{x},\hat{p})$ symmetrized, if all operator $\hat{x}^i$ and $\hat{p}_j$ appear in all possible orderings with equal weights. The symmetrized form of monomials are produced by the formula

$$\begin{aligned}&(\alpha^i\hat{p}_i + \beta_j\hat{x}^j)^N = \sum_{m_i,n_j} N! \prod_{i,j} \frac{1}{\Pi\, m_i!\, \Pi\, n_j!}(\alpha^i)^{m_i}(\beta_j)^{n_j}\Big((\hat{p}_i)^{m_i}(\hat{x}^j)^{n_j}\Big)_S \\ &\sum m_i + \sum n_j = N\end{aligned} \tag{B.8}$$

or, equivalently,

$$\begin{aligned}&N!\prod_{i,j}(\hat{p}_i^{m_i}(\hat{x}^j)^{n_j})_S = \prod_{i,j}\Big(\frac{\partial}{\partial\alpha^i}\Big)^{m_i}\,\Big(\frac{\partial}{\partial\beta_j}\Big)^{n_j}(\alpha^i\hat{p}_i + \beta_j\hat{x}^j)^N \\ &\sum m_i + \sum n_j = N \ .\end{aligned} \tag{B.9}$$

We shall discuss this result by first considering one pair of variables $\hat{x}$ and $\hat{p}$.

For one set of operators $\hat{p}$ and $\hat{x}$ one has (omitting hats for notational simplicity)

$$\begin{aligned} (xp)_S &= \frac{1}{2}(xp + px) = (px)_S \\ (x^2p)_S &= \frac{1}{3}(x^2p + xpx + px^2) \\ &= \frac{1}{4}(x^2p + 2xpx + px^2) \\ &= \frac{1}{2}(x^2p + px^2) \ . \end{aligned} \tag{B.10}$$

To derive the last two relations from the first one, one may repeatedly use the basic identity $2xpx = x^2p + px^2$. More generally

$$\begin{aligned} (x^n p)_S &= \frac{1}{n+1}\sum_{l=0}^{n} x^{n-l}px^l \\ &= \frac{1}{2^n}\sum_{l=0}^{n} \binom{n}{l} x^{n-l}px^l \\ &= \frac{1}{2}(x^n p + px^n) \ . \end{aligned} \tag{B.11}$$

To prove these relations one may combine the terms with $x^{n-l}px^l$ and x^lpx^{n-l} and move the p's past the x^l (or past the x^{n-l}). The commutators then cancel again. (As is clear from (B.11), when n is even, there is a term for which $n - l = l$ which one should first split into two terms. For example, for $n = 2$ one first rewrites $2xpx$ as $xpx + xpx$ and then the commutators $[x,p]x + x[p,x]$ cancel again). Of course, by the same argument one also has

$$(xp^2)_S = \frac{1}{3}(xp^2 + pxp + p^2x) = \frac{1}{4}(xp^2 + 2pxp + p^2x) = \frac{1}{2}(xp^2 + p^2x) \ . \tag{B.12}$$

The next term with two p's in this series is

$$\begin{aligned} (x^2p^2)_S &= \frac{1}{6}\begin{pmatrix} x^2p^2 & + & xpxp & + & xp^2x & + & p^2x^2 \\ & + & pxpx & + & px^2p & & \end{pmatrix} \\ &= \frac{1}{4}(x^2p^2 + 2xp^2x + p^2x^2) \\ &= \frac{1}{4}(x^2p^2 + 2px^2p + p^2x^2) \ . \end{aligned} \tag{B.13}$$

Note that in the second line, the p^2 are kept together, while in the third line the x^2 are kept together. We shall achieve this for all cases which follow below. To obtain this result, we wrote the second term in the first line as $\frac{1}{12}(x\underline{px}p + xp\underline{xp})$ and used the $[x, p]$ commutation relations for the underlined operators; similarly for the term $\frac{1}{6}pxpx$. Then we used that in the remainder $-xp^2x + px^2p = -x[p^2, x] + [p, x^2]p = 0$.

In a similar manner one shows that

$$\begin{aligned}
(x^3p^2)_S &= \binom{5}{2}^{-1} [x^3p^2 + ...] \quad \text{(10 terms)} \\
&= \frac{1}{8}(x^3p^2 + 3x^2p^2x + 3xp^2x^2 + p^2x^3) \quad \text{(keeping } p^2 \text{ together)} \\
&= \frac{1}{4}(x^3p^2 + 2px^3p + p^2x^3) \quad \text{(keeping } x^3 \text{ together)} \ . \qquad \text{(B.14)}
\end{aligned}$$

(We obtained the last line by writing the term $3x^2p^2x$ as $\frac{2}{3}$ times $3(p^2x^3 + [x^2, p^2]x)$ plus $\frac{1}{3}$ times $3(x^3p^2 + x^2[p^2, x])$, and similarly for $3xp^2x^2$, since in this way the commutators cancel).

In general

$$\begin{aligned}
(x^np^2)_S &= \frac{2}{(n+1)(n+2)} \sum_{l,m=0}^{n} x^{n-l-m}px^lpx^m \\
&= \frac{1}{2^n} \sum_{l=0}^{n} \binom{n}{l} x^{n-l}p^2x^l \\
&= \frac{1}{4}(x^np^2 + 2px^np + p^2x^n) \ . \qquad \text{(B.15)}
\end{aligned}$$

The most general formula for one pair of canonical variables is then

$$\begin{aligned}
(x^mp^r)_S &= \frac{1}{2^m} \sum_{l=0}^{m} \binom{m}{l} x^{m-l}p^rx^l \quad \text{(keeping } p^r \text{ together)} \\
&= \frac{1}{2^r} \sum_{k=0}^{r} \binom{r}{k} p^{r-k}x^mp^k \quad \text{(keeping } x^m \text{ together)} \ . \quad \text{(B.16)}
\end{aligned}$$

Consider now the matrix element $M = \langle z|(\hat{x}^m\hat{p}^r)_S|y\rangle$ where we reinstated the hats. Inserting a complete set of p-states and using the symmetrized expression with p^r kept together one finds

$$\begin{aligned}
M &= \int \langle z| \frac{1}{2^m} \sum_{l=0}^{m} \binom{m}{l} \hat{x}^{m-l}\hat{p}^r|p\rangle\langle p|\hat{x}^l|y\rangle \, d^np \\
&= \int \langle z|p\rangle \frac{1}{2^m} \sum_{l=0}^{m} \binom{m}{l} z^{m-l}y^lp^r\langle p|y\rangle \, d^np \\
&= \int \langle z|p\rangle \left(\frac{z+y}{2}\right)^m p^r\langle p|y\rangle \, d^np \ . \qquad \text{(B.17)}
\end{aligned}$$

This is enough to prove the theorem in (B.2) because any Weyl ordered operator is a sum of symmetrized terms (see for example (B.7)). In particular, for any Weyl-ordered hamiltonian operator $\hat{H}_W(\hat{x}, \hat{p})$ we find the midpoint rule

$$\langle z|\hat{H}_W(\hat{x}, \hat{p})|y\rangle = \int \langle z|p\rangle H_W\left(\frac{1}{2}(z+y), p\right) \langle p|y\rangle\, d^n p \ . \tag{B.18}$$

Consider now as a particular case the operator

$$\hat{H} = \frac{1}{2} g^{-1/4} p_i g^{1/2} g^{ij} p_j g^{-1/4} \ . \tag{B.19}$$

To write it in Weyl-ordered form we first simplify this expression by moving p_i to the left and p_j to the right. The result is

$$\begin{aligned} \hat{H} &= \frac{1}{2}\left(p_i - \frac{1}{4} i\hbar \partial_i \ln g\right) g^{-1/4} g^{1/2} g^{ij} g^{-1/4} \left(p_j + \frac{1}{4} i\hbar \partial_j \ln g\right) \\ &= \frac{1}{2} p_i g^{ij} p_j + \frac{\hbar^2}{8} \partial_i \left(g^{ij} \partial_j \ln g\right) + \frac{\hbar^2}{32} g^{ij} (\partial_i \ln g)(\partial_j \ln g) \ . \end{aligned} \tag{B.20}$$

The first term is not yet Weyl-ordered, hence we rewrite it using its symmetrized form, keeping the x operators in $g^{ij}(x)$ together. The symmetrized form of $\frac{1}{2} p_i g^{ij} p_j$ is

$$\frac{1}{2}(p_i g^{ij} p_j)_S = \frac{1}{8}(p_i p_j g^{ij} + 2 p_i g^{ij} p_j + g^{ij} p_i p_j) \ . \tag{B.21}$$

The difference between $\frac{1}{2} p_i g^{ij} p_j$ and its symmetrized form is given by

$$\begin{aligned} \frac{1}{2}(p_i g^{ij} p_j) - \frac{1}{2}(p_i g^{ij} p_j)_S &= \frac{1}{8} p_i [g^{ij}, p_j] + \frac{1}{8}[p_i, g^{ij}] p_j \\ &= \frac{1}{8}[p_i, [g^{ij}, p_j]] = \frac{\hbar^2}{8} \partial_i \partial_j g^{ij} \ . \end{aligned} \tag{B.22}$$

Hence, $\hat{H}$ reads in Weyl-ordered form

$$\begin{aligned} \hat{H} &= \frac{1}{2}(p_i g^{ij} p_j)_S + \frac{\hbar^2}{8}\left[\partial_i \partial_j g^{ij} + \partial_i (g^{ij} \partial_j \ln g) + \frac{1}{4} g^{ij} (\partial_i \ln g)(\partial_j \ln g)\right] \\ &= \frac{1}{2}(p_i g^{ij} p_j)_S + \frac{\hbar^2}{8}(\partial_i \partial_j g^{ij} + g^{-1/4} \partial_i (g^{1/4} g^{ij} \partial_j \ln g)) \ . \end{aligned} \tag{B.23}$$

The last two terms can be written in terms of Christoffel symbols and the scalar curvature. To find the coefficient of R we evaluate the leading terms of the form $\partial_i \partial_j g_{kl}$ and find with (A.15)

$$-\frac{\hbar^2}{8} g^{ik} (\partial_i \partial_j g_{kl}) g^{jl} + \frac{\hbar^2}{8} g^{ij} g^{kl} \partial_i \partial_j g_{kl} + \cdots = \frac{\hbar^2}{8} R + \cdots \ . \tag{B.24}$$

The final result reads

$$\hat{H} = \frac{1}{2}(p_i g^{ij} p_j)_S + \frac{\hbar^2}{8}(R + g^{ij}\Gamma^l_{ik}\Gamma^k_{jl}) = \hat{H}_W \ . \tag{B.25}$$

An easy way to check the coefficient of the Christoffel term is to consider the one-dimensional case where R vanishes. On the other hand, an easy way to check the coefficient of the term with R is to go to a frame where $\partial_i g_{jk}$ vanishes at a given point. However, the fact that $\hat{H}$ is of the form given in (B.25) only follows from an explicit computation.

Appendix C
Weyl ordering of fermionic operators

In this appendix we extend the Weyl ordering of bosonic canonical variables discussed in the previous appendix to the case of fermionic canonical variables.

Consider operators $O(\hat{\psi}^a, \hat{\psi}_b^\dagger)$ depending on fermionic canonical variables $\hat{\psi}^a$ and $\hat{\psi}_b^\dagger$ $(a, b = 1, n)$ which satisfy

$$\{\hat{\psi}^a, \hat{\psi}^b\} = 0 \; ; \quad \{\hat{\psi}^a, \hat{\psi}_b^\dagger\} = \hbar\delta_b^a \; ; \quad \{\hat{\psi}_a^\dagger, \hat{\psi}_b^\dagger\} = 0 \; . \tag{C.1}$$

We define the antisymmetric ordering, which we still denote by a subscript S, as the ordering which results if one expands $(\alpha_a\hat{\psi}^a + \beta^b\hat{\psi}_b^\dagger)^N$ where α_a and β^b are Grassmann variables. Hence

$$\prod_{a,b} N! \left((\hat{\psi}^a)^{m_a}(\hat{\psi}_b^\dagger)^{n_b}\right)_S = \prod_{a,b}\left(\frac{\partial}{\partial\alpha_a}\right)^{m_a}\left(\frac{\partial}{\partial\beta^b}\right)^{n_b}(\alpha_a\hat{\psi}^a + \beta^b\hat{\psi}_b^\dagger)^N \tag{C.2}$$

where $N = \Sigma m_a + \Sigma n_b$ and the order in which the $\hat{\psi}^a$ and $\hat{\psi}_b^\dagger$ appear on the left-hand side is the same as the order in which the $\frac{\partial}{\partial\alpha_a}$ and $\frac{\partial}{\partial\beta^b}$ appear on the right-hand side. For example

$$(\hat{\psi}^a\hat{\psi}_b^\dagger)_S = \frac{1}{2}(\hat{\psi}^a\hat{\psi}_b^\dagger - \hat{\psi}_b^\dagger\hat{\psi}^a) \tag{C.3}$$

and

$$\begin{aligned}
(\hat{\psi}^a\hat{\psi}^b\hat{\psi}_c^\dagger)_S &= \frac{1}{6}(\hat{\psi}^a\hat{\psi}^b\hat{\psi}_c^\dagger - \hat{\psi}^b\hat{\psi}^a\hat{\psi}_c^\dagger - \hat{\psi}^a\hat{\psi}_c^\dagger\hat{\psi}^b + \hat{\psi}^b\hat{\psi}_c^\dagger\hat{\psi}^a \\
&\quad + \hat{\psi}_c^\dagger\hat{\psi}^a\hat{\psi}^b - \hat{\psi}_c^\dagger\hat{\psi}^b\hat{\psi}^a) \\
&= \frac{1}{3}\Big(\hat{\psi}^a\hat{\psi}^b\hat{\psi}_c^\dagger - \frac{1}{2}\hat{\psi}^a\hat{\psi}_c^\dagger\hat{\psi}^b + \frac{1}{2}\hat{\psi}^b\hat{\psi}_c^\dagger\hat{\psi}^a + \hat{\psi}_c^\dagger\hat{\psi}^a\hat{\psi}^b\Big) \\
&= \frac{1}{2}(\hat{\psi}^a\hat{\psi}^b\hat{\psi}_c^\dagger + \hat{\psi}_c^\dagger\hat{\psi}^a\hat{\psi}^b)
\end{aligned}$$

$$= \frac{1}{4}(\hat{\psi}^a\hat{\psi}^b\hat{\psi}_c^\dagger - \hat{\psi}^a\hat{\psi}_c^\dagger\hat{\psi}^b + \hat{\psi}^b\hat{\psi}_c^\dagger\hat{\psi}^a + \hat{\psi}_c^\dagger\hat{\psi}^a\hat{\psi}^b) \ . \qquad \text{(C.4)}$$

As in the bosonic case one has for any function $f(\hat{\psi})$

$$(\psi_a^\dagger\psi_b^\dagger f(\psi))_S = \frac{1}{4}\Big(\psi_a^\dagger\psi_b^\dagger f(\psi) \pm \psi_a^\dagger f(\psi)\psi_b^\dagger \mp \psi_b^\dagger f(\psi)\psi_a^\dagger + f(\psi)\psi_a^\dagger\psi_b^\dagger\Big) \qquad \text{(C.5)}$$

where the upper (lower) signs apply when $f(\psi)$ is commuting (anticommuting). For any function $g(\hat{\psi}^\dagger)$ one has the equivalent results

$$(\hat{\psi}^a\hat{\psi}^b g(\psi^\dagger))_S = \frac{1}{4}\Big(\hat{\psi}^a\hat{\psi}^b g(\hat{\psi}^\dagger) \pm \hat{\psi}^a g(\hat{\psi}^\dagger)\hat{\psi}^b \mp \hat{\psi}^b g(\psi^\dagger)\hat{\psi}^a + g(\psi^\dagger)\hat{\psi}^a\psi^b\Big). \qquad \text{(C.6)}$$

Let us now write the $N = 2$ supersymmetric Hamiltonian in Weyl-ordered form. The action in Minkowski time reads (see appendix D)

$$L = \frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j + \frac{i}{2}\psi_\alpha^a(\dot{\psi}_\alpha^a + \dot{x}^i\omega_i{}^a{}_b\psi_\alpha^b) + \frac{1}{8}R_{abcd}(\omega)\psi_\alpha^a\psi_\alpha^b\psi_\beta^c\psi_\beta^d \qquad \text{(C.7)}$$

where summation over $\alpha = 1, 2$ and $\beta = 1, 2$ is understood, $a = 1, \ldots, n$, and $i = 1, \ldots, n$, and $\omega_i{}^a{}_b(x)$ is the spin connection. The curvature $R_{abcd}(\omega)$ is defined in appendix A.

The momentum p_i conjugate to x^i is given by

$$p_i = g_{ij}\dot{x}^j + \frac{i}{2}\omega_{iab}\psi_\alpha^a\psi_\alpha^b \ . \qquad \text{(C.8)}$$

Further, one has for a fixed value of a the relation $\frac{1}{2}\psi_\alpha^a\dot{\psi}_\alpha^a = \bar{\psi}_a\dot{\psi}^a +$ *total derivative*, with $\psi^a = (\psi_1 + i\psi_2)/\sqrt{2}$ and $\bar{\psi}_a = (\psi_1 - i\psi_2)/\sqrt{2}$. Then, using left derivatives to define anticommuting canonically conjugate momenta

$$p(\psi)_a = \frac{\partial}{\partial\dot{\psi}^a}S = -i\bar{\psi}_a \quad \Rightarrow \quad \{\bar{\psi}_a, \psi^b\} = \hbar\delta_a^b \ . \qquad \text{(C.9)}$$

The classical Hamiltonian then reads

$$\begin{aligned} H &= \dot{x}^i p_i + \dot{\psi}^a p(\psi)_a - L \\ &= g^{ij}p_i\left(p_j - \frac{i}{2}\omega_{jab}\psi_\alpha^a\psi_\alpha^b\right) + i\bar{\psi}^a\dot{\psi}^a \\ &\quad -\frac{1}{2}g^{ij}\left(p_i - \frac{i}{2}\omega_{iab}\psi_\alpha^a\psi_\alpha^b\right)\left(p_j - \frac{i}{2}\omega_{jcd}\psi_\beta^c\psi_\beta^d\right) - i\bar{\psi}^a\dot{\psi}^a \\ &\quad -\frac{i}{2}g^{ij}\left(p_j - \frac{i}{2}\omega_{jab}\psi_\alpha^a\psi_\alpha^b\right)\omega_{icd}\psi_\beta^c\psi_\beta^d - \frac{1}{8}R_{abcd}\psi_\alpha^a\psi_\alpha^b\psi_\beta^c\psi_\beta^d \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}g^{ij}\left(p_i - \frac{i}{2}\omega_{iab}\psi^a_\alpha\psi^b_\alpha\right)\left(p_j - \frac{i}{2}\omega_{jcd}\psi^c_\beta\psi^d_\beta\right) \\
&\quad -\frac{1}{8}R_{abcd}\psi^a_\alpha\psi^b_\alpha\psi^c_\beta\psi^d_\beta \ .
\end{aligned} \tag{C.10}
$$

In terms of ψ^a and $\bar{\psi}^a$ this becomes

$$
\begin{aligned}
H &= \frac{1}{2}g^{ij}(p_i - i\omega_{iab}\bar{\psi}^a\psi^b)(p_j - i\omega_{jcd}\bar{\psi}^c\psi^d) \\
&\quad -\frac{1}{2}R_{abcd}\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d \ .
\end{aligned} \tag{C.11}
$$

We now define the corresponding Hamiltonian operator. We fix the operator ordering by requiring that $\hat{H}$ be general coordinate (Einstein) invariant and locally Lorentz invariant, namely it should commute with the generators of general coordinate and local Lorentz transformations. To achieve this, the same factors $g^{\pm 1/4}$ as in the bosonic case are needed. The $N = 2$ Hamiltonian operator thus becomes

$$
\begin{aligned}
\hat{H} &= \frac{1}{2}g^{-1/4}\left(p_i - i\omega_{iab}\bar{\psi}^a\psi^b\right)g^{1/2}g^{ij}\left(p_j - i\omega_{jcd}\bar{\psi}^c\psi^d\right)g^{-1/4} \\
&\quad -\frac{1}{2}R_{abcd}\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d
\end{aligned} \tag{C.12}
$$

where $\{\psi^a, \bar{\psi}_b\} = \hbar\delta^a_b$.

To write this operator in Weyl-ordered form, i.e. to *rewrite* it such that all canonical variables appear symmetrized or antisymmetrized, we note that the two-fermion terms are already antisymmetrized, since $\{\bar{\psi}_a, \psi^b\}$ is proportional to δ^b_a and ω_{iab} is traceless. For the same reason we can write the four-fermion term in $\hat{H}$ as follows

$$
\begin{aligned}
\hat{H}(\psi^4) &= -\frac{1}{2}(g^{ij}\omega_{iab}\omega_{jcd} + R_{abcd}(\omega))\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d \\
&= -\frac{1}{16}(g^{ij}\omega_{iab}\omega_{jcd} + R_{abcd}(\omega))\{[\bar{\psi}^a, \psi^b], [\bar{\psi}^c, \psi^d]\}
\end{aligned} \tag{C.13}
$$

We now first prove the following

$$
\textbf{Lemma}: \quad \frac{1}{8}\{[\bar{\psi}^a, \psi^b], [\bar{\psi}^c, \psi^d]\} - (\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d)_S = \frac{\hbar^2}{4}\delta^{ad}\delta^{bc} \ . \tag{C.14}
$$

Proof: $(\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d)_S = -(\bar{\psi}^a\bar{\psi}^c\psi^b\psi^d)_S$. Next rewrite this term, once keeping $\psi^b\psi^d$ together and once keeping $\bar{\psi}^a\bar{\psi}^c$ together

$$
(\bar{\psi}^a\bar{\psi}^c\psi^b\psi^d)_S = \frac{1}{8}\left[\begin{array}{l} \bar{\psi}^a\bar{\psi}^c\psi^b\psi^d + \bar{\psi}^a\psi^b\psi^d\bar{\psi}^c - \bar{\psi}^c\psi^b\psi^d\bar{\psi}^a + \psi^b\psi^d\bar{\psi}^a\bar{\psi}^c \\ +\bar{\psi}^a\bar{\psi}^c\psi^b\psi^d + \psi^b\bar{\psi}^a\bar{\psi}^c\psi^d - \psi^d\bar{\psi}^a\bar{\psi}^c\psi^b + \psi^b\psi^d\bar{\psi}^a\bar{\psi}^c \end{array}\right]
$$

Adding this to

$$\frac{1}{8}\{[\bar{\psi}^a,\psi^b],[\bar{\psi}^c,\psi^d]\}$$
$$= \frac{1}{8}\left[\begin{array}{l} +\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d - \psi^b\bar{\psi}^a\bar{\psi}^c\psi^d - \bar{\psi}^a\psi^b\psi^d\bar{\psi}^c + \psi^b\bar{\psi}^a\psi^d\bar{\psi}^c \\ +\bar{\psi}^c\psi^d\bar{\psi}^a\psi^b - \psi^d\bar{\psi}^c\bar{\psi}^a\psi^b - \bar{\psi}^c\psi^d\psi^b\bar{\psi}^a + \psi^d\bar{\psi}^c\psi^b\bar{\psi}^a \end{array}\right]$$

one finds by combining corresponding pairs of terms

$$\frac{\hbar}{8}(\bar{\psi}^a\delta^{bc}\psi^d + 0 + 0 + \psi^b\delta^{ad}\bar{\psi}^c + \bar{\psi}^c\delta^{ad}\psi^b + 0 + 0 + \psi^d\delta^{bc}\bar{\psi}^a) = \frac{\hbar^2}{4}\delta^{ad}\delta^{bc}\,.$$

It follows that

$$\begin{aligned} \hat{H}(\psi^4) &= \hat{H}(\psi^4)_S - \frac{\hbar^2}{8}\delta^{ad}\delta^{bc}(g^{ij}\omega_{iab}\omega_{jcd} + R_{abcd}(\omega)) \\ &= \hat{H}(\psi^4)_S + \frac{\hbar^2}{8}(g^{ij}\omega_{iab}\omega_j{}^{ab} - R)\ . \end{aligned} \tag{C.15}$$

The terms in $\hat{H}$ without fermions yield back the Hamiltonian of the bosonic model obtained in (B.25)

$$\hat{H}(no\ \psi) = \frac{1}{2}(g^{ij}p_ip_j)_S + \frac{\hbar^2}{8}(R + g^{ij}\Gamma^k_{il}\Gamma^l_{jk})\ . \tag{C.16}$$

The two-fermion terms are also Weyl-ordered in the sector with x and p

$$\begin{aligned} &-\frac{i}{4}g^{-1/4}p_ig^{1/4}g^{ij}\omega_{jcd}\psi^c_\beta\psi^d_\beta - \frac{i}{4}g^{1/4}\omega_{iab}\psi^a_\alpha\psi^b_\alpha g^{ij}p_jg^{-1/4} \\ &\quad = -\frac{i}{4}\{p_i, g^{ij}\omega_{jcd}\psi^c_\beta\psi^d_\beta\}\ . \end{aligned} \tag{C.17}$$

We therefore conclude that rewriting (C.12) in Weyl-ordered form yields

$$\begin{aligned} \hat{H}(N=2) &= \left(\frac{1}{2}g^{ij}\pi_i\pi_j - \frac{1}{2}R_{abcd}(\omega)\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d\right)_S \\ &\quad + \frac{\hbar^2}{8}g^{ij}(\Gamma^l_{ik}\Gamma^k_{jl} + \omega_{iab}\omega_j{}^{ab}) \qquad \text{(C.18)} \\ \pi_i &= p_i - i\omega_{iab}\bar{\psi}^a\psi^b\ . \end{aligned}$$

We note that the terms with R from the bosonic sector and the $(\psi)^4$ sector cancel.

One can even achieve a formulation of the $N=2$ model where the $\Gamma\Gamma$ and $\omega\omega$ terms cancel. Although the choice of ψ^a as basic variable is very suitable (since it yields as kinetic term simply $i\psi^a_\alpha\dot{\psi}^a_\alpha$) one can make a different choice, namely ψ^i with curved index i. Since $\psi^a = e^a_i\psi^i$, the action becomes $\frac{i}{2} g_{ij} \psi^i_\alpha (\dot{\psi}^j_\alpha + \dot{x}^k \Gamma^j_{kl} \psi^l_\alpha)$. We used the "vielbein postulate" $\partial_i e^a_j - \Gamma_{ij}{}^k e^a_k + \omega_i{}^a{}_b e^b_j = 0$ of (A.1). The momentum conjugate to $\psi^i \equiv (\psi^i_1 + i\psi^i_2)/\sqrt{2}$ is then $p(\psi)_j = -i\psi^\dagger_j \equiv -ig_{ij}(\psi^j_1 - i\psi^j_2)/\sqrt{2}$ and the canonical commutation relations become

$$\{\psi^i, \psi^\dagger_j\} = \delta^i_j \hbar \ . \tag{C.19}$$

(Note that the Jacobi identity for $(p_j(\psi), \psi^i, \bar{\psi}_k)$ is still satisfied because the bracket of ψ^i and $\bar{\psi}_k$ is a constant). The conjugate momentum of x^i becomes $p_i = g_{ij}\dot{x}^j + \frac{i}{2}\Gamma_{il;k}\psi^k_\alpha \psi^l_\alpha$. The Hamiltonian now reads

$$\begin{aligned} \hat{H} &= \frac{1}{2} g^{-1/4} \left(p_i - \frac{i}{2} \Gamma_{il;k} \psi^k_\alpha \psi^l_\alpha \right) g^{ij} g^{1/2} \left(p_j - \frac{i}{2} \Gamma_{jn;m} \psi^m_\beta \psi^n_\beta \right) g^{-1/4} \\ &\quad - \frac{1}{2} R_{abcd}(\omega) \bar{\psi}^a \psi^b \bar{\psi}^c \psi^d \end{aligned} \tag{C.20}$$

and the four-fermi terms now read

$$-\frac{1}{2} \left(g^{ij} \Gamma_{il;k} \Gamma_{jn;m} + R_{klmn}(\omega) \right) \bar{\psi}^k \psi^l \bar{\psi}^m \psi^n \ . \tag{C.21}$$

The antisymmetrization of $\hbar^2 \bar{\psi}^k \psi^l \bar{\psi}^m \psi^n$ yields a factor $\frac{1}{4} g^{kn} g^{lm}$ and this produces a term

$$-\frac{\hbar^2}{8} (g^{ij} \Gamma^k_{il} \Gamma^l_{jk} + R) \ . \tag{C.22}$$

Adding (C.16) and (C.22), all $\hbar^2$ terms now cancel.

For practical calculations, the choice of ψ^a as basis variable is preferred, even at the expense of the extra $\hbar^2$ terms.

Appendix D
Nonlinear susy sigma models and d = 1 superspace

As explained in the introduction, for the computation of anomalies in d-dimensional quantum field theories which themselves need not be supersymmetric field theories, supersymmetric $d = 1$ nonlinear σ-models (a particular class of supersymmetric quantum mechanical models) are needed. One can write down these models in "t space", beginning with a kinetic terms $g_{ij}(\varphi)\partial_t\varphi^i\partial_t\varphi^j$ for the bosonic fields $\varphi^i(t)$ and finding further terms with fermionic $\psi^i(t)$, by using the so-called Noether method. This is instructive if one is interested in the structure of the theory, in particular the leading terms of the action and transformation rules. To obtain the complete answer, the Noether method is somewhat cumbersome (although it always yields the complete answer if such a complete answer exists). We shall begin by using this Noether method for the models we are interested in; the procedure becomes clear along the way, and has pedagogical value. Then, however, we shall follow the superspace approach, and obtain the complete answer at once. The superspace method is less intuitive, but once one has understood the overall structure of a theory, the superspace approach gives complete answers while avoiding the tedious labour of the Noether approach. The Noether method and the superspace method are complementary. We end by deriving the $N = 1$ and $N = 2$ models.

The Noether method. In the Noether method we start with a bosonic and fermionic kinetic term

$$L^{kin} = \frac{1}{2} g_{ij}(\varphi)\left[\dot{\varphi}^i\dot{\varphi}^j + i\psi^i\dot{\psi}^j\right] . \qquad \text{(D.1)}$$

By a dot we indicate a d/dt derivative (we are in Minkowskian time), and $\varphi^i(t)$ is a real function. The anticommuting $\psi^i(t)$ are by definition real under hermitian conjugation, and we added the factor i in order that the action be hermitian

$$(i\psi^i\dot{\psi}^j)^\dagger = -i\dot{\psi}^j\psi^i = +i\psi^i\dot{\psi}^j \ . \tag{D.2}$$

For superspace applications it is natural that φ^i and ψ^i have the same index i, because then one can construct a superfield $\varphi^i + i\theta\psi^i$. However, one can use either ψ^i or $\psi^a = e_i^a\psi^i$, where $e_i^a(\varphi)$ are the vielbein fields which are square roots of the metric, $g_{ij} = e_i^a e_j^b \eta_{ab}$ (in target space). At the end of this appendix we shall mention the changes which occur if one uses ψ^a, but for now we use ψ^i.

Under supersymmetry transformations, φ^i should transform into ψ^i, and vice-versa. From the action we see that the dimension of φ^i, denoted by $[\varphi^i]$, differs from that of ψ^i by one-half the dimension of t (which has by definition minus one, $[t] = -1$). Namely: $2[\varphi] - 2[t] = 2[\psi] - [t]$. Since the action is dimensionless (if $\hbar = 1$), we find $[\varphi] = -1/2$ and $[\psi] = 0$. It follows that if $\delta\varphi^i \sim \epsilon\psi^i$, then $[\epsilon] = -1/2$. Consequently, in $\delta\psi^i \sim \epsilon\varphi^i$ we need a derivative to make the dimensions come out right: $\delta\psi^i \sim \epsilon\dot{\varphi}^i$. We do not consider terms with $\dot{\epsilon}$ because we consider at this point rigid susy models which have by definition a constant ϵ. Since ψ^i and φ^i are real, also ϵ must be real or purely imaginary. Choosing ϵ to be real we need a factor i in $\delta\varphi^i \sim i\epsilon\psi^i$. To obtain $\delta\varphi^i = -i\epsilon\psi^i$, one can scale ϵ appropriately. Then $\delta\psi^i = \beta\dot{\varphi}^i\epsilon$, and the value $\beta = 1$ will be shown to follow by requiring invariance of the action, or closure of the supersymmetry algebra. Hence, we assume the following transformation rules

$$\delta\varphi^i = -i\epsilon\psi^i \ , \ \delta\psi^i = \beta\dot{\varphi}^i\epsilon \ , \qquad (\beta = 1) \ . \tag{D.3}$$

The parameter ϵ is constant. We could study locally supersymmetric theories (supergravity theories[1]) with a local parameter $\epsilon(t)$ and a gauge field for supersymmetry, but we shall only need rigidly supersymmetric theories with constant ϵ.

We now begin the Noether procedure. We vary L using $\delta\varphi^i$ and $\delta\psi^i$ given above, and find

$$\begin{aligned} \delta L(\text{kin}) \ &= \ \frac{1}{2}(\partial_k g_{ij})(-i\epsilon\psi^k)\left(\dot{\varphi}^i\dot{\varphi}^j + i\psi^i\dot{\psi}^j\right) \\ &+ \ \frac{1}{2}g_{ij}\left[2\dot{\varphi}^i\frac{d}{dt}(\,-i\epsilon\psi^j) + i\beta(\dot{\varphi}^i\epsilon)\dot{\psi}^j + i\beta\psi^i\frac{d}{dt}(\dot{\varphi}^j\epsilon)\right] . \end{aligned} \tag{D.4}$$

In the action, this expression is integrated over t, and if we partially integrate the last term to remove double derivatives, all $\dot{\varphi}\dot{\psi}$ terms cancel

[1] A discussion of supergravity in quantum mechanics is given in [125]. One can couple φ and ψ to the supergravity gauge fields (the vielbein and the gravitino), but no gauge action for supergravity itself exists in $d = 1$ dimensions.

if $\beta = 1$. We are left with the variation

$$\delta L(\text{kin}) = \Big(-\frac{i}{2}\partial_k g_{ij}\Big)\epsilon\Big(\psi^k\dot\varphi^i\dot\varphi^j + i\psi^k\psi^i\dot\psi^j - \dot\varphi^k\psi^i\dot\varphi^j\Big) \tag{D.5}$$

where we used that $\beta = 1$ and ϵ and ψ^i anticommute: $\epsilon\psi^i = -\psi^i\epsilon$. There are no terms with $\dot\epsilon$ because ϵ is constant.

We now observe that the combination $-i\epsilon\psi^k$ can be written as $\delta\varphi^k$ and so one might be tempted to write the first term in δL(kin) as $[\frac{1}{2}\partial_k g_{ij}]\delta\varphi^k\dot\varphi^i\dot\varphi^j$. However, this is just how we found this term, so we would go backwards. We can also consider another combination, for example $\dot\varphi^i\epsilon$, and replace it by $\delta\psi^i$. We shall choose the latter alternative for reasons to become clear. Our aim is to find a new term in the action, L(extra), such that δL (extra) $= -\delta L$(kin). Then, obviously, the t-integral of $L = L(\text{kin}) + L(\text{extra})$ is invariant. We claim that a solution is

$$L\ (\text{extra})\ = \frac{i}{2}(\partial_k g_{ij})(\psi^i\dot\varphi^j\psi^k)\ . \tag{D.6}$$

To verify this claim, note that we need not vary $\partial_k g_{ij}(\varphi)$, because it would yield $(\partial_l\partial_k g_{ij})\ (-i\epsilon\psi^l)$ and since $\psi^l\psi^k$ is antisymmetric in l, k, this variation would vanish. The variation of ψ^i cancels the $\psi^k\dot\varphi^i\dot\varphi^j$ in (D.5). The variation of $\dot\varphi^j$ in L (extra) cancels the $\psi\psi\dot\psi$ term in δL (kin). Finally, the variations of ψ^k in L (extra) cancel the $\dot\varphi^k\psi^i\dot\varphi^j$ term in δL (kin).

Hence, $I \equiv \int L dt$ with

$$L = \frac{1}{2}g_{ij}\dot\varphi^i\dot\varphi^j + \frac{i}{2}\psi^i(g_{ij}\dot\psi^j + \partial_k g_{ij}\dot\varphi^j\psi^k) \tag{D.7}$$

is invariant under the transformation laws in (D.3). We can rewrite the action in a way which has a geometrical meaning

$$\psi^i(g_{ij}\dot\psi^j + \partial_k g_{ij}\dot\varphi^j\psi^k) = \psi^i g_{ij}(\dot\psi^j + \dot\varphi^k\Gamma_{kl}{}^j\psi^l) \equiv \psi^i g_{ij}\frac{D}{Dt}\psi^j \tag{D.8}$$

where $\Gamma_{kl}{}^j$ is the Christoffel symbol. The derivative $\frac{D}{Dt}\psi^j$ transforms as a contravariant vector under general diffeomorphisms $x^i \to x^i + \xi^i(x)$, as we shall shortly discuss.

To obtain the Noether current for supersymmetry, we let ϵ become a local parameter, and repeat the evaluation of δL. The terms proportional to $\dot\epsilon$ then yield the Noether current [56]. From (D.4) we obtain one such term, namely when the $\frac{d}{dt}$ in the first term inside the square brackets hits the ϵ. The last term in (D.4) was partially integrated, so it does not yield a $\dot\epsilon$ term. Another term with $\dot\epsilon$ might seem to come from varying $\dot\varphi^j$ in (D.6), but this contribution vanishes as it is proportional to $\partial_k g_{ij}\psi^i\psi^j$. Hence, we find that the Noether current for supersymmetry is proportional to

$$j_N = g_{ij}\dot\varphi^i\psi^j\ . \tag{D.9}$$

It is now clear that the other way to proceed mentioned under (D.5), namely replacing $-i\epsilon\psi^k$ by $\delta\varphi^k$ in $\delta L(\text{kin})$, would not have worked. It would have led to an L (extra) $= \frac{1}{2}\partial_k g_{ij}\varphi^k\dot{\varphi}^i\dot{\varphi}^j$ but whereas variation of φ^k would have canceled the first term in δL_{kin} (by construction), the other variations would not have been canceled. One might have expected this, since a coordinate φ^k is not a tensor in general relativity (in contrast to $\dot{\varphi}^k$ or $\delta\varphi^k$), so that the action should not contain undifferentiated φ's (except in $g_{ij}(\varphi)$).

The difficult step was to find the correct L (extra). One might have guessed this result by noting that the terms in parentheses in (D.7) form a covariant derivative

$$\psi^i(g_{ij}\dot{\psi}^j + \partial_k g_{ij}\dot{\varphi}^j\psi^k) = \psi_j\frac{D}{Dt}\psi^j \ , \qquad \psi_j = g_{ji}\psi^i$$
$$\frac{D}{Dt}\psi^j = \dot{\psi}^j + \dot{\varphi}^k\Gamma_{kl}{}^j\psi^l \ , \quad \Gamma_{kl}{}^j = \frac{1}{2}g^{jm}\Big[\partial_k g_{lm} + \partial_l g_{km} - \partial_m g_{kl}\Big] \ . \tag{D.10}$$

The covariant derivative $\frac{D}{Dt}\psi^j$ is indeed a contravariant vector under infinitesimal general coordinate transformation in spacetime (because $\Gamma_{kl}{}^j$ is a connection: $\delta\Gamma_{kl}{}^j = \partial_k\partial_l\xi^j + \ldots$)

$$\varphi^i \ \to \ \varphi^i + \xi^i(\varphi) \ , \quad \psi^i \ \to \ \psi^i + \frac{\partial\xi^i}{\partial\varphi^j}\psi^j$$
$$\frac{D}{Dt}\psi^j \ \to \ \frac{D}{Dt}\psi^j + \frac{\partial\xi^j}{\partial\varphi^k}\,\frac{D}{Dt}\psi^k \ . \tag{D.11}$$

A less insightful, but still correct way to obtain the result in (D.7) would have been to write down all possible candidates for L (extra), with arbitrary coefficients, and then fixing these coefficients by requiring that, up to partial integrations, δL (extra) $= -\delta L$ (kin). However, one would like to have a method which guarantees success even if one is not clever enough to use such tricks or patient enough to do much algebra, and such a method is the superspace method.

The superspace method. In $d = 1$ $N = 1$ superspace, one has one bosonic coordinate which we call t (because $d = 1$), and one fermionic coordinate θ (because $N = 1$). The θ's are Grassmann variables, $\{\theta, \theta\} = 2\theta\theta = 0$, and by definition θ is real, $(\theta)^\dagger = \theta$. In this superspace, we consider superfields, i.e., fields depending on t and θ.

We consider superfields with an index, $\phi^i(t,\theta)$. Expanding in powers of θ, there are only two terms since $\theta^2 = 0$, and we define

$$\phi^i(t,\theta) = \varphi^i(t) + i\beta\theta\psi^i(t) \tag{D.12}$$

where β is a real constant to be fixed later. The factor i is again needed in order that $\phi^i(t,\theta)$ be real, and we require that $\phi^i(t,\theta)$ be real because $\varphi^i(t)$ is real.

We shall now obtain the transformation law of $\phi^i(t,\theta)$ under supersymmetry and construct an invariant action involving superfields ϕ^i by using an approach called the "coset method" which is also used in more complicated cases (the $d=4, N=1$ or $N=2$ cases in particular). This approach starts from a superalgebra which is at the basis of the whole approach. In our case we shall use the superalgebra of supersymmetric quantum mechanics, with Hamiltonian H and supersymmetry generator Q

$$\{Q,Q\} = 2H \ , \qquad [Q,H] = 0 \ . \tag{D.13}$$

In fact, $[Q,H]=0$ follows from $\{Q,Q\} = 2Q^2 = 2H$. We shall assume that H and Q are hermitian, $(H)^\dagger = H$ and $(Q)^\dagger = Q$. Given any superalgebra, one first *deduces* how supercoordinates transform under supersymmetry. Then one finds how superfields transform under supersymmetry.

To deduce how the supercoordinates x and θ transform, one takes a group element of the form

$$g(t,\theta) = e^{itH+\theta Q} \tag{D.14}$$

where $g^\dagger = g^{-1}$ (at least formally because $(\theta Q)^\dagger = Q^\dagger\theta^\dagger = Q\theta = -\theta Q$). We multiply from the left by a group element

$$h(\alpha,\epsilon) = e^{i\alpha H+\epsilon Q} \tag{D.15}$$

and we work to linear order in α and ϵ ("h near the origin"). Since the product can again be written as an exponent, with multiple commutators of ϵH and ϵQ in the exponent, we have

$$h(\alpha,\epsilon)g(t,\theta) = g(t+\delta t, \theta+\delta\theta) \tag{D.16}$$

where δt and $\delta\theta$ are linear in α and ϵ. Using the Baker-Campbell-Hausdorff formula for the bosonic objects $i\alpha H, \epsilon Q, itH$ and θQ we get

$$e^{i\alpha H+\epsilon Q}\ e^{itH+\theta Q} \ = e^{i(t+\alpha)H+(\theta+\epsilon)Q+\frac{1}{2}[\epsilon Q,\theta Q]} \ . \tag{D.17}$$

No further commutators are needed, since

$$[\epsilon Q,\theta Q] = -\epsilon\theta\{Q,Q\} = -2\epsilon\theta H \tag{D.18}$$

which commutes with all generators (H and Q). Hence

$$\delta t = \alpha + i\epsilon\theta \ , \qquad \delta\theta = \epsilon \ . \tag{D.19}$$

To check that this forms indeed a representation of the superalgebra, we rewrite this as

$$\delta t = [t, i\alpha H + \epsilon Q] \ , \qquad \delta\theta = [\theta, i\alpha H + \epsilon Q] \tag{D.20}$$

and by equating (D.19) and (D.20) we find the generators H and Q in the supercoordinate representation

$$H = i\frac{\partial}{\partial t} \ , \ Q = -\frac{\partial}{\partial\theta} - i\theta\frac{\partial}{\partial t} \ . \tag{D.21}$$

As one may check, one indeed has a representation of the superalgebra in terms of differential operators. For example

$$\{Q, Q\} = 2H \ . \tag{D.22}$$

We now *declare* $\phi^i(t,\theta)$ to be scalar superfields. By this we mean the same as, for example, in ordinary quantum mechanics: the transformation of the fields is induced by the transformation of the (super) coordinates

$$\phi'^i(t', \theta') = \phi^i(t, \theta) \ . \tag{D.23}$$

Putting $\phi' = \phi + \delta\phi$ and $(t', \theta') = (t, \theta) + (\delta t, \delta\theta)$, we obtain

$$\begin{aligned}
\delta\phi &= \Big(- \delta t\frac{\partial}{\partial t} - \delta\theta\frac{\partial}{\partial\theta}\Big)\phi \\
&= \Big[- \alpha\frac{\partial}{\partial t} - \epsilon\Big(\frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial t}\Big)\Big]\phi \\
&= [i\alpha H + \epsilon Q, \phi] \ .
\end{aligned} \tag{D.24}$$

From (D.20) and (D.24) we see that coordinates and fields transform contragradiently, again a well-known result from ordinary quantum mechanics (and just a consequence of the definition $\phi'(t', \theta') = \phi(t, \theta)$).

We have now obtained the supersymmetry generator

$$Q = -\Big(\frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial t}\Big) \ . \tag{D.25}$$

Let us check whether the components $\varphi^i(t)$ and $\psi^i(t)$ transform as in (D.3).

$$\begin{aligned}
\epsilon Q\phi^i &= -\epsilon\Big(\frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial t}\Big)\Big(\varphi^i(t) + i\beta\theta\psi^i(t)\Big) \\
&= \delta\varphi^i(t) + i\beta\theta\delta\psi^i(t) \ .
\end{aligned} \tag{D.26}$$

Equating terms with and without θ yields

$$\begin{aligned}
\delta\varphi^i(t) &= -\epsilon i\beta\psi^i(t) \\
i\beta\theta\delta\psi^i(t) &= -\epsilon i\theta\dot{\varphi}^i(t) = i\theta\epsilon\dot{\varphi}^i \ .
\end{aligned} \tag{D.27}$$

It is clear that for $\beta = 1$ we retrieve the transformation rules of $\varphi^i(t)$ and $\psi^i(t)$ which we obtained in (D.3). Hence, we have obtained so far the following results

$$\begin{aligned} \phi^i &= \varphi^i(t) + i\theta\psi^i(t) \\ \epsilon Q &= -\epsilon\Big(\frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial t}\Big) \\ \delta\phi^i &= [\epsilon Q, \phi^i] \ . \end{aligned} \tag{D.28}$$

For the construction of invariant actions it is useful to have an operator which anticommutes with Q. Such an operator is given by

$$D = \frac{\partial}{\partial\theta} - i\theta\frac{\partial}{\partial t} \ . \tag{D.29}$$

It is formally obtained by the same steps as Q, but using right multiplication for h and g. Since left and right multiplication commute, $(h_1 g)h_2 = h_1(gh_2)$, it follows that Q and D anticommute. Let us check explicitly that $\{D, Q\} = 0$. This follows by writing out all terms as

$$\begin{aligned} -\{D, Q\} &= \Big\{\frac{\partial}{\partial\theta} - i\theta\frac{\partial}{\partial t} \ , \ \frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial t}\Big\} \\ &= \Big\{\frac{\partial}{\partial\theta} \ , \ i\theta\Big\}\frac{\partial}{\partial t} + \Big\{-i\theta \ , \ \frac{\partial}{\partial\theta}\Big\}\frac{\partial}{\partial t} = 0 \ . \end{aligned} \tag{D.30}$$

We can now at once write down a set of invariant actions for ϕ^i. We claim that for any m and n the following action is invariant under (D.28)

$$I = \int dt d\theta \Big(D^m\phi^i\Big)\Big(D^n\phi^j\Big) g_{ij}(\phi) \ . \tag{D.31}$$

To see why $\delta I = 0$, note that $\delta g_{ij}(\phi) = [\epsilon Q, g_{ij}(\phi)]$, because of (D.28), and

$$\delta(D^m\phi^i) \equiv D^m\delta\phi^i = D^m\Big(\epsilon Q\phi^i\Big) = \Big(\epsilon Q(D^m\phi^i)\Big) \tag{D.32}$$

where we used in the last step that $\epsilon QD = D\epsilon Q$. Hence

$$\delta\Big(D^m\phi^i D^n\phi^j \ g_{ij}(\phi)\Big) = \Big(\epsilon Q[D^m\phi^i D^n\phi^j g_{ij}(\phi)]\Big) \ . \tag{D.33}$$

(We write $\delta\phi^i = (\epsilon Q\phi^i)$ to indicate that the differential operator Q acts on ϕ^i but not beyond ϕ^i. We could equivalently write $\delta\phi^i = [\epsilon Q, \phi^i]$). Now

$$\int dt d\theta \ \epsilon QL = 0 \tag{D.34}$$

for any L, because the $\frac{d}{dt}$ in Q yields zero (assuming, as we always do, that all functions vanish at $t = \pm\infty$), while the $\partial/\partial\theta$ in Q also gives zero due to the following property.

Theorem :
$$\int d\theta \, \frac{\partial}{\partial\theta} \, f(\theta) = 0 \quad \text{for any } f(\theta) \ . \tag{D.35}$$

The proof of this theorem is trivial: $f(\theta) = f_0 + \theta f_1$ and $\int d\theta\theta = 1$ but $\int d\theta = 0$, so $\int d\theta \frac{\partial}{\partial\theta} \, f(\theta) = \int d\theta f_0 = 0$.

We now use the superspace formalism to construct the $N = 1$ and $N = 2$ nonlinear sigma models which play a central role in the second part of this book.

The N = 1 model. This model is used for the calculation of chiral anomalies of spin 1/2 fields. We have seen that for any m and n the action $I = \int dt d\theta \, (D^m \phi^i)(D^n \phi^j) g_{ij}(\phi)$ is a supersymmetric action. Which m and n should we take? We want an action which contains, to begin with, the kinetic term $\frac{1}{2} g_{ij}(\varphi)\dot{\varphi}^i\dot{\varphi}^j$. Since $D = \partial/\partial\theta - i\theta\partial/\partial t$ satisfies

$$\begin{aligned} D^2 &= -i\frac{\partial}{\partial t} \ , \ iD^2\phi^i = \dot{\varphi}^i + i\theta\dot{\psi}^i \\ D\phi^i &= \left(\frac{\partial}{\partial\theta} - i\theta\frac{\partial}{\partial t}\right)\left(\varphi^i + i\theta\psi^i\right) = i\psi^i - i\theta\dot{\varphi}^i \end{aligned} \tag{D.36}$$

we see that for $m = 1, \ n = 2$ (or $m = 2, \ n = 1$) we obtain this kinetic term. So we take as action

$$\begin{aligned} I &= \alpha \int dt d\theta (D\phi^i)(D^2\phi^j) g_{ij}(\phi) \\ &= \alpha \int dt d\theta \Big[i\psi^i - i\theta\dot{\varphi}^i\Big]\Big[-i\dot{\varphi}^j + \theta\dot{\psi}^j\Big] \\ &\quad \Big[g_{ij}(\varphi) + i\theta\psi^k \, \partial_k g_{ij}(\varphi)\Big] \end{aligned} \tag{D.37}$$

where α is a constant we shall soon fix. To obtain a nonzero result for the θ integral, we need only the terms proportional to θ. There are only three such terms, and we find

$$L = -\alpha\dot{\varphi}^i\dot{\varphi}^j g_{ij} - i\alpha\psi^i\dot{\psi}^j g_{ij} - i\alpha\psi^i\dot{\varphi}^j\psi^k\partial_k g_{ij}(\varphi) \ . \tag{D.38}$$

(In obtaining this result, we moved θ to the left of all ψ functions, which causes some minus signs). For $\alpha = -\frac{1}{2}$, this is indeed the action of (D.7) obtained from the Noether method, but now the last term comes out automatically. In Euclidean space, we find, putting $t = -it_E$

$$L_E = \frac{1}{2} g_{ij}\dot{\varphi}^i\dot{\varphi}^j + \frac{1}{2}\psi^i g_{ij}\left(\dot{\psi}^j + \dot{\varphi}^l\Gamma^j_{lk}\psi^k\right) \ . \tag{D.39}$$

Another way to see that one needs the combination $D^2\phi^i D\phi^j$ in the action is to use dimensional arguments. Since $[dt] = -1, [d\theta] = +1/2$ (note that $[\theta] = -1/2$ but $\int d\theta\theta = 1), [D] = [\partial/\partial\theta] = 1/2$ and $[\phi^i] = [\varphi^i] = -1/2$ (because $\int dt\dot{\varphi}^i\dot{\varphi}^j$ should be dimensionless), we see that the action in (D.31) is dimensionless provided $-1+1/2+(m+n)\frac{1}{2}+2(-1/2) = 0$, hence $m+n = 3$. For $m = 0$ and $m = 3$ one obtains $(D^3\phi^i)\phi^j g_{ij}(\phi)$ which is not a tensor in target space, hence only $(D^2\phi^i)(D\phi^j)g_{ij}(\phi)$ is allowed.

One can write down a supersymmetric extension of the $A_i(x)\dot{x}^i$ coupling, namely $\int d\theta A_i(\phi)(D\phi^j)$. One can also try to add a potential term $V(\phi)$ to the action. Then

$$\int dtd\theta\, V(\phi) = \int dtd\theta \left[V(\varphi)+i\theta\psi^k\partial_k V\right] = \int dt\, i\psi^k(t)\partial_k V(\varphi)\ . \quad \text{(D.40)}$$

It follows that the resulting potential is fermionic, so not very interesting for most applications. However, the next model allows useful potentials. **The N = 2 model.** This model is used for the calculation of chiral anomalies of selfdual antisymmetric tensor fields. If one uses a $d = 1,\ N = 2$ superspace approach with coordinates $t, \theta, \bar{\theta} \equiv (\theta)^\dagger$, where of course $\{\theta, \bar{\theta}\} = 0$, then a suitable action is

$$\begin{aligned} I &= \int dtd\theta d\bar{\theta}(D\phi^i)(\bar{D}\phi^j)g_{ij}(\phi) \\ D &= \left(\frac{\partial}{\partial\bar{\theta}} - i\theta\frac{\partial}{\partial t}\right), \qquad \bar{D} = \left(\frac{\partial}{\partial\theta} - i\bar{\theta}\frac{\partial}{\partial t}\right) \\ Q &= -\left(\frac{\partial}{\partial\bar{\theta}} + i\theta\frac{\partial}{\partial t}\right), \quad \bar{Q} = -\left(\frac{\partial}{\partial\theta} + i\bar{\theta}\frac{\partial}{\partial t}\right). \end{aligned} \quad \text{(D.41)}$$

The susy of this action follows immediately from the observation that D and $\bar{D}$ anticommute with Q and $\bar{Q}$. One could also choose a real basis, with $D_+ = \frac{\partial}{\partial\theta^+} - i\theta^+\partial_t$ and $D_- = \partial/\partial\theta^- - i\theta^-\partial_t$ where $\theta^+ + i\theta^- = \sqrt{2}\theta$ and $\theta^+ - i\theta^- = \sqrt{2}\bar{\theta}$. Then $\sqrt{2}D = D_+ + iD_-$ and $\sqrt{2}\bar{D} = D_+ - iD_-$, and the action is written as

$$\int dtd\theta^+ d\theta^-\ g_{ij}D_+\phi^i D_-\phi^j\ . \quad \text{(D.42)}$$

The underlying superalgebra is now

$$\begin{aligned} &\{Q, \bar{Q}\} = -\{D, \bar{D}\} = 2H \\ &[H, D] = [H, \bar{D}] = [H, Q] = [H, \bar{Q}] = 0 \\ &\{Q, Q\} = \{\bar{Q}, \bar{Q}\} = \{D, D\} = \{\bar{D}, \bar{D}\} = 0\ . \end{aligned} \quad \text{(D.43)}$$

In the coordinate representation, $H = i\frac{\partial}{\partial t}$. Again $[H, D] = [H, \bar{D}] = [H, Q] = [H, \bar{Q}] = 0$ follows from the definition $2H = \{Q, \bar{Q}\}$. Most importantly, D and $\bar{D}$ anticommute with Q and $\bar{Q}$. The reader may apply the coset formalism discussed above to derive (D.41).

Dimensional arguments reveal that one needs one D and one $\bar{D}$ in the action (two D's would yield zero as $D\phi^i$ and $D\phi^j$ anticommute). Putting

$$\phi^i = \varphi^i + i\theta\bar{\psi}^i + i\bar{\theta}\psi^i + \bar{\theta}\theta F^i \tag{D.44}$$

one finds after integration over θ and $\bar{\theta}$

$$\begin{aligned} L &= g_{ij}(\dot{\varphi}^i\dot{\varphi}^j + F^iF^j) + ig_{ij}\Big(\psi^i\frac{D}{Dt}\bar{\psi}^j + \bar{\psi}^j\frac{D}{Dt}\psi^i\Big) \\ &\quad -(\partial_k\partial_l g_{ij})(\psi^i\bar{\psi}^j\psi^k\bar{\psi}^l) - 2\bar{\psi}^i\Gamma_{ij}{}^l\psi^jF_l \ . \end{aligned} \tag{D.45}$$

The transformation rules $\delta\phi = [\bar{\epsilon}Q,\phi] + [\epsilon\bar{Q},\phi]$ preserve the reality of ϕ and yield

$$\begin{aligned} \delta\varphi^i &= -i\bar{\epsilon}\psi^i - i\epsilon\bar{\psi}^i \ , \qquad \delta F^i = -\bar{\epsilon}\dot{\psi}^i + \epsilon\dot{\bar{\psi}}^i \\ \delta\psi^i &= \dot{\varphi}^i\epsilon + iF^i\epsilon \ , \qquad \delta\bar{\psi}^i = -\dot{\varphi}^i\bar{\epsilon} + iF^i\bar{\epsilon} \ . \end{aligned} \tag{D.46}$$

Substituting the algebraic field equation $F^i = \bar{\psi}^j\Gamma_{jk}{}^i\psi^k$, the fields ψ^i and $\bar{\psi}^i$ transform as follows

$$\begin{aligned} \delta(\epsilon)\psi^i + \delta(\epsilon)\varphi^j\Gamma_{jl}\psi^l &= \dot{\varphi}^i\epsilon \\ \delta(\bar{\epsilon})\bar{\psi}^i + \delta(\bar{\epsilon})\varphi^j\Gamma_{jl}\bar{\psi}^l &= \dot{\varphi}^i\bar{\epsilon} \ . \end{aligned} \tag{D.47}$$

The left-hand sides transform covariantly (as contravariant vectors) under general coordinate transformation $\varphi^i \to \varphi^i + \xi^i(\varphi)$. The left-hand side defines a covariant variation, similar to a covariant derivative, see (D.10).

The terms with $\partial_k g_{lm}$ covariantize the ψ derivatives. Eliminating F^i one obtains in the action $(\bar{\psi}\Gamma\psi)^2$ terms which covariantize $\partial_k\partial_l g_{ij}$ to a full Riemann tensor (see appendix A). This yields, adding an overall factor 1/2, and decomposing $\psi = (\psi_1 + i\psi_2)/\sqrt{2}$ and $\bar{\psi} = (\psi_1 - i\psi_2)/\sqrt{2}$,

$$\begin{aligned} L &= \frac{1}{2}g_{ij}\dot{\varphi}^i\dot{\varphi}^j + \frac{i}{2}g_{ij}\psi^i_\alpha\left(\dot{\psi}^j_\alpha + \dot{\varphi}^k\Gamma^j_{kl}\psi^l_\alpha\right) \\ &\quad -\frac{1}{8}R_{ijkl}(\Gamma)\psi^i_\alpha\psi^j_\alpha\psi^k_\beta\psi^l_\beta \end{aligned} \tag{D.48}$$

with $R_{ijk}{}^l(\Gamma) = \partial_i\Gamma_{jk}{}^l + \Gamma^l_{im}\Gamma^m_{jk} - (i \leftrightarrow j)$. To cast the 4-fermion term in this form, one may use the cyclic identity for the Riemann tensor. Note that $R_{ijk}{}^a(\Gamma) = R_{ij}{}^a{}_k(\omega)$. In Euclidean space $L_E = \frac{1}{2}g_{ij}\dot{\varphi}^i\dot{\varphi}^j + \frac{1}{2}g_{ij}\psi^i_\alpha\frac{D}{Dt}\psi^j_\alpha + \frac{1}{8}R_{abcd}\psi^a_\alpha\psi^b_\alpha\psi^c_\beta\psi^d_\beta$.

The potential term which we write as $(-i)W(\phi)$ to make it real now yields

$$\int dt d\theta d\bar{\theta}\; W(\phi) = \int dt\Big[F^k\partial_kW(\varphi) - \psi^i\bar{\psi}^j\partial_i\partial_jW(\varphi)\Big] \ . \tag{D.49}$$

Eliminating F^k by its field equation

$$F^k = g^{kl}\partial_l W \tag{D.50}$$

yields in t-space a positive definite bosonic potential

$$V = \frac{1}{2} g^{ij}\partial_i W(\varphi)\partial_j W(\varphi) \ . \tag{D.51}$$

Finally, we come back to our promise to discuss the changes that occur if one uses ψ^a instead of ψ^i. We begin with the $N = 1$ model. We recall that $\psi^a = e_i^a(\varphi)\psi^i$. The transformation rule of ψ^a becomes

$$\delta\psi^a = e_i^a \delta\psi^i + (\delta e_i^a)\psi^i = e_i^a \dot{\varphi}^i \epsilon + \delta\varphi^j \partial_j e_i^a \psi^i \ . \tag{D.52}$$

Next we use the vielbein postulate

$$\partial_j e_i^a = -\omega_j{}^a{}_b e_i^b + \Gamma_{ji}{}^k e_k^a \ . \tag{D.53}$$

Because $\delta\varphi^j$ contains ψ^j, the Christoffel term cancels when inserted into (D.52), and we find

$$\delta\psi^a + \delta\varphi^j \omega_j{}^a{}_b \psi^b = e_i^a \dot{\varphi}^i \epsilon \ . \tag{D.54}$$

The left hand side looks like a covariant derivative (with $\frac{d}{dt}$ replaced by δ) and is indeed locally Lorentz covariant, just as the right-hand side. For curved indices one would expect

$$\delta\psi^i + \delta\varphi^j \Gamma_{jk}{}^i \psi^k = \dot{\varphi}^i \epsilon \tag{D.55}$$

but the term with Christoffel symbol cancels, see (D.3).

In the action for the $N = 1$ model we have found the covariant derivative, see (D.8),

$$g_{ij}\psi^i \Big(\frac{D}{dt}\psi^j\Big) = g_{ij}\psi^i \Big(\dot{\psi}^j + \dot{\varphi}^l \Gamma_{lk}{}^j \psi^k\Big) \ . \tag{D.56}$$

It is straightforward to check that if one replaces the curved ψ^i by flat ψ^a one finds the corresponding covariant derivatives

$$\psi^a \Big(\frac{D}{dt}\psi^a\Big) = \psi^a \Big(\dot{\psi}^a + \dot{\varphi}^j \omega_j{}^a{}_b \psi^b\Big) \ . \tag{D.57}$$

For the $N = 2$ model one finds covariant derivatives of ψ and $\bar{\psi}$ (both for curved or flat indices), and in the transformation rules one finds now pullback terms both in the flat and curved case. The reason one finds now also for curved indices a covariantizing term in $\delta\psi^i$ is that this term now is of the form $\bar{\psi}\Gamma\psi$ instead of $\psi\Gamma\psi$.

Appendix E
Nonlinear susy sigma models for internal symmetries

In the main text, we were led to conjecture the existence of an extension of the usual $N = 1$ nonlinear supersymmetric σ-model for quantum mechanics which contains a term proportional to

$$c_A^* \, (T_I)^A{}_B \, c^B \, A_i{}^I(\varphi) \; . \tag{E.1}$$

Here the antihermitian matrices T_I generate a Lie algebra in the same representation as the spin 1/2 fields of the original spacetime quantum field theory, while $c_A^*(t)$ and $c^B(t)$ are now anticommuting functions, which we call antighost and ghost, respectively. In this section we shall construct this extension.

The usual $N = 1$ susy nonlinear σ-model in 0+1 dimensional Minkowski time is given by

$$L = \frac{1}{2} g_{ij}(\varphi)\dot{\varphi}^i\dot{\varphi}^j + \frac{i}{2}\psi^a\left(\dot{\psi}^a + \dot{\varphi}^k \omega_k{}^a{}_b \psi^b\right) \; . \tag{E.2}$$

We introduced vielbein fields $e_i{}^a(\varphi)$ satisfying $e_i{}^a e_j{}^b \eta_{ab} = g_{ij}$ and defined $\psi^a = e_i{}^a \psi^i$. Then (E.2) follows from (D.6) and (D.7) by using "the vielbein postulate" as in (D.57)

$$\partial_i e_j{}^a - \Gamma_{ij}{}^k e_k{}^a + \omega_i{}^a{}_b e^b{}_j = 0 \; . \tag{E.3}$$

Hermiticity requires the factor i. If we had chosen ψ^i instead of ψ^a to work with, there would have also been a metric g_{ij} in front of the fermionic terms.

We now require that the quantum mechanical Hamiltonian H should be a representation of the regulator $(\not{D})(\not{D})$ with $\not{D} = \gamma^a e_a{}^i(\varphi) D_i$ and

$$D_i = \left(\partial_i + \frac{1}{4}\omega_{iab}\gamma^a\gamma^b + A_i{}^I T_I\right) \; . \tag{E.4}$$

We have absorbed the Yang–Mills coupling constant in ${A_i}^I$. In the main text we have seen that the covariant derivative is proportional to $g_{ij}\dot{x}^j$. Here we denote $\dot{x}^i$ by $\dot{\varphi}^i$, hence we anticipate that the conjugate momentum for φ^i should be given by

$$p_i = g_{ij}\dot{\varphi}^j + \frac{i}{2}\omega_{iab}\psi^a\psi^b + iA_i^I\Big(c^*T_Ic\Big) \tag{E.5}$$

where the ghost field c^* is the hermitian conjugate of c, so $c^* = (c)^*$. Inverting this relation we obtain

$$\begin{aligned}\dot{\varphi}^i &= g^{ij}\left(p_j - \frac{i}{2}\omega_{jab}\psi^a\psi^b - iA_j^Ic^*T_Ic\right)\\ &= -ig^{ij}\left(\hbar\frac{\partial}{\partial\varphi^j} + \frac{1}{2}\omega_{jab}\psi^a\psi^b + A_j^Ic^*T_Ic\right) .\end{aligned} \tag{E.6}$$

To obtain the correct anticommutator for c^* and c after canonical quantization, $\{c^A, c^*_B\} = \hbar\delta^A_B$, we add the kinetic term $ic^*_A\dot{c}^A$. The normalization of this kinetic term is such that $p_A = \frac{\partial}{\partial\dot{c}^A}L$ satisfies the quantum anticommutator $\{p_A, c^B\} = -\hbar i{\delta_A}^B$. Furthermore from (E.2) we find $\{\psi^a(t), \psi^b(t)\} = \hbar\delta^{ab}$. Hence we must construct an action which leads to (E.5).

These considerations lead us to consider the following action

$$\begin{aligned}L &= \frac{1}{2}\left(g_{ij}\dot{\varphi}^i\dot{\varphi}^j + i\psi^a(\dot{\psi}^a + \dot{\varphi}^k{\omega_k}^a{}_b\psi^b)\right)\\ &+ ic^*_A\dot{c}^A + i\dot{\varphi}^kA_k^I(\varphi)(c^*T_Ic) .\end{aligned} \tag{E.7}$$

Since the T_I are antihermitian and $c^* = (c)^*$, the action is hermitian. As it turns out, this action is not yet supersymmetric. To find the terms which complete it, we shall analyze how it transforms. We already know that the c-independent part is invariant, so we study the c-dependent terms.

We begin by varying the φ fields in $L(c)$, using the known susy rule for $\delta\varphi$

$$\begin{aligned}\delta\varphi^i &= -i\epsilon\psi^i , \qquad \delta\psi^i = \dot{\varphi}^i\epsilon\\ \psi^a &= \psi^i{e_i}^a(\varphi) , \quad {e_i}^a{e_j}^b\delta_{ab} = g_{ij} .\end{aligned} \tag{E.8}$$

We find for this variation

$$\delta L(c) = \left\{i\frac{d}{dt}\Big(-i\epsilon\psi^k\Big){A_k}^I(\varphi) + i\dot{\varphi}^k\Big(\partial_l{A_k}^I\Big)\Big(-i\epsilon\psi^l\Big)\right\}\Big(c^*T_Ic\Big) . \tag{E.9}$$

Partially integrating the first term, we produce an ordinary curl of the vector field ${A_k}^I$, plus a t-derivative of c^*Tc. The latter we can cancel by suitable susy laws for $c^*{}_A$ and c^B, such that they combine also to a t-derivative of c^*Tc. Clearly

$$\begin{aligned} \delta c^A &= +i\epsilon\psi^k A_k{}^I (T_I)^A{}_B c^B \\ \delta c^*{}_A &= -i\epsilon\psi^k A_k{}^I c^*{}_B (T_I)^B{}_A \end{aligned} \tag{E.10}$$

does the job since, using a partial integration, we obtain

$$\begin{aligned} \delta\Big(ic^*\dot{c}\Big) &= ic^*\frac{d}{dt}\Big(i\epsilon\psi^k\ A_k{}^I T_I c\Big) + i\Big(-i\epsilon\psi^k A_k{}^I c^* T_I\Big)\dot{c} \\ &= \epsilon\psi^k A_k{}^I \frac{d}{dt}\Big(c^* T_I c\Big) \end{aligned} \tag{E.11}$$

which indeed cancels the terms in (E.9) proportional to $\frac{d}{dt}(c^*T_Ic)$ obtained by partially integrating the first term in (E.9).

The remainder reads

$$\delta L(c) = \dot{\varphi}^k \epsilon\psi^l \Big(\partial_l A_k{}^I - \partial_k A_l{}^I\Big)\ \Big(c^* T_I c\Big)\ . \tag{E.12}$$

Substituting the rules for $\delta c^*{}_A$ and δc^B into the last term of (E.7) yields a commutator of two T_I matrices and completes the ordinary curl in (E.12) to a nonabelian curl. Hence, at this point we have

$$\begin{aligned} \delta L(c) &= \dot{\varphi}^k(\epsilon\psi^l) F_{lk}{}^I (c^* T_I c) \\ F_{lk}{}^I &= \partial_l A_k{}^I - \partial_k A_l{}^I + f^I{}_{JK} A_l{}^J A_k{}^K\ . \end{aligned} \tag{E.13}$$

To cancel this $\delta L(c)$, we observe that the combination $\dot{\varphi}^k\epsilon$ is equal to $\delta\psi^k$. (As in appendix D, we could also consider the combination $\epsilon\psi^l$ as coming from $\delta\varphi^l$, but this would lead to bare φ^l in the new term in the action, which we already saw does not work). Hence, we add the following extra term to the action.

$$L(\text{extra}) = \frac{1}{2}\psi^k\psi^l F_{kl}{}^I \Big(c^* T_I c\Big)\ . \tag{E.14}$$

(We need the factor 1/2 since the variation of ψ^k and ψ^l both give a $\epsilon\dot{\varphi}$ term).

We have at this point canceled all variations of $L(c)$, but we still have to vary F and c^*Tc in L(extra). These variations cancel by themselves, due to the Bianchi identity for $F_{kl}{}^I$ as we now demonstrate. The variations of the φ fields in $F_{kl}{}^I$ produce $(\partial_i F_{kl}{}^I)(-i\epsilon\psi^i)$. Variation of c^*T_Ic yields $i\epsilon\psi^i A_i{}^J c^*[T_I, T_J]c$. These two variations combine into

$$\begin{aligned} \delta L(\text{extra}) &= \frac{1}{2}\psi^k\psi^l\Big(-i\epsilon\psi^i\Big)\Big[\partial_i F_{kl}{}^K + f^K{}_{JI} A_i{}^J F_{kl}{}^I \Big(c^* T_K c\Big)\Big] \\ &= -\frac{i\epsilon}{2}\Big(\psi^i\psi^k\psi^l\Big)\Big(D_i F_{kl}{}^K\Big)\ \Big(c^* T_K c\Big) \end{aligned} \tag{E.15}$$

where $D_i F_{jk} = \partial_i F_{jk} + [A_i, F_{jk}]$ for $F_{ij} = F_{ij}{}^I T_I$. This indeed vanishes since $\psi^i \psi^k \psi^l$ is totally antisymmetric, while $D_i F_{kl}$ + 2 cyclic terms = 0 due to the Bianchi identity.

We conclude that

$$L(\text{ghost}) = ic^*_A \dot{c}^A + \left(i\dot{\varphi}^i A_i{}^I + \frac{1}{2}\psi^i \psi^j F_{ij}{}^I \right) \left(c^*{}_A (T_I)^A{}_B c^B \right) \qquad \text{(E.16)}$$

is supersymmetric by itself under

$$\begin{aligned} \delta c^A &= i\epsilon \psi^i A_i{}^I (T_I)^A{}_B c^B \ , \quad \delta c^*{}_A = -i\epsilon \psi^i A_i{}^I c^*{}_B (T_I)^B{}_A \\ \delta \varphi^i &= -i\epsilon \psi^i \ , \qquad\qquad\qquad\quad \delta \psi^i = \dot{\varphi}^i \epsilon \ . \end{aligned} \qquad \text{(E.17)}$$

The susy Noether charge Q for the model consisting of the sum of (E.2) and (E.16) is unchanged. To see this, we repeat the analysis of varying L(ghost) in (E.16), this time with local $\epsilon = \epsilon(t)$. Nowhere do we pick up an $\dot{\epsilon}$ term, and hence L(ghost) is even locally supersymmetric. Therefore $\delta L(\text{total}) = -i\dot{\epsilon} g_{ij} \dot{\varphi}^i \psi^j$ as before, and

$$Q = g_{ij} \dot{\varphi}^i \psi^j = \psi^a e_{ai} \dot{\varphi}^i \ . \qquad \text{(E.18)}$$

We can write the ghost part of the action in a more covariant form as

$$\begin{aligned} L(\text{ghost}) &= \left[ic^*{}_A \frac{D}{Dt} c^A + \frac{1}{2}\psi^i \psi^j c^* F_{ij} c \right] \\ \frac{D}{Dt} c^A &= \frac{d}{dt} c^A + \dot{\varphi}^i (A_i c)^A \ , \ A_i = A_i{}^I \ T_I \ . \end{aligned} \qquad \text{(E.19)}$$

In this form, the local Yang-Mills (gauge) invariance of the action under

$$\delta A_i{}^I = \partial_i \lambda^I + f^I{}_{JK} A_i{}^J \lambda^K \quad \text{with} \quad \partial_i \lambda^I \equiv \frac{\partial}{\partial \varphi^i} \lambda^I(\varphi) \qquad \text{(E.20)}$$

and $\delta c^A = -\lambda^I (T_I)^A{}_B c^B, \delta c^*{}_A = c^*{}_B (T_I)^B{}_A \lambda^I$ becomes manifest. For example, the $c^* F c$ term varies into

$$\lambda^K \Big(- c^* [T_J, T_K] c F^J + f^I{}_{JK} F^J c^* T_I c \Big) \qquad \text{(E.21)}$$

which clearly vanishes. Furthermore, the covariant derivative $\frac{D}{Dt} c^A$ transforms indeed like c^A itself.

The Yang-Mills symmetry could have been used to anticipate the term with A_i in the ghost action, but the term with F_{ij} (a so-called Pauli term) is typical for supersymmetry and is not required by Yang-Mills symmetry. In a similar way, it follows that the action is invariant under general coordinate transformations and local Lorentz transformations in target space, with $\delta \varphi^i = \xi^i(\varphi)$, $\delta \dot{\varphi}^i = (\partial \xi^i(\varphi)/\partial \varphi^j) \dot{\varphi}^j, \delta \psi^a = \lambda^a{}_b(\varphi) \psi^b, \delta g_{ij} = \partial_i \xi^k g_{kj} + (i \leftrightarrow j)$.

Appendix F
Gauge anomalies for exceptional groups

In the main text we showed that gravitational and gauge anomalies cancel in 10 dimensions for $N = 1$ supergravity coupled to Yang-Mills theory if the gauge group G is either $SO(32)$ or $E_8 \times E_8$. None of the other "classical groups" ($SO(n)$, $SU(n)$ and $Sp(n)$) was allowed. We now complete this analysis by discussing the exceptional groups, namely G_2, F_4, E_6, E_7, and E_8. As we have shown in the main text, cancellation of gravitational anomalies allows only Lie groups with 496 generators. There are clearly many products of simple Lie algebras with this number of generators. In particular there are semisimple Lie algebras with one or more exceptional groups as simple factors. However, we can at once rule out these exceptional groups if we study one-loop hexagon graphs with 6 gauge fields all belonging to the same exceptional Lie group, and if factorization of the kind discussed in the main text does not occur. In 4 dimensions gauge anomalies are proportional to the symmetrized trace of 3 generators, $d_{abc}(R) = \mathrm{tr}(T_a\{T_b, T_c\})$ where T_a are the generators of the gauge group in a representation R, and thus real or pseudoreal representations do not carry anomalies [126]. A representation R can only carry an anomaly if $d_{abc}(R)$ is nonvanishing, and this is only possible if there exists a cubic Casimir operator for the group. In 10 dimensions gauge anomalies are proportional to the symmetrized trace of 6 generators, and then real or pseudoreal representations can carry anomalies. Now anomalies can only be present if there exists a sixth-order Casimir operator for the group, and even if it exists, it may still happen that for a particular representation the value of the sixth-order Casimir operator is zero, or factorizes into a product of lower-dimensional Casimir operators.

What follows is amusing group theory. Readers who are somewhat rusty in their G_2 or F_4 may brush up their knowledge of exceptional groups by working their way through the discussions below.

G_2

Do anomalies cancel for G_2? This group has 14 generators, and the fundamental representation (at the same time the defining representation) is the **7**. [1] We consider the maximal subgroup $SU(3)$. The **14** of G_2 decomposes under $SU(3)$ into $\mathbf{8}+\mathbf{3}+\mathbf{3}^*$. In fact, the **7** of G_2 is most easily defined by first decomposing it under $SU(3)$ as $\mathbf{7}\to\mathbf{3}+\mathbf{3}^*+\mathbf{1}$ and then defining the action of $SU(3)$ (with parameters $\lambda^\alpha{}_\beta$) and the extra generators (with parameters $\bar\sigma_\alpha$ and σ^α) on the $\mathbf{7}=\{x^\alpha,\bar x_\alpha,y\}$ in manifestly $SU(3)$ covariant form [127]

$$\begin{aligned}
\delta x^\alpha &= \lambda^\alpha{}_\beta x^\beta+\frac{1}{\sqrt2}\,\epsilon^{\alpha\beta\gamma}\bar\sigma_\beta\bar x_\gamma+\sigma^\alpha y\\
\delta\bar x_\alpha &= \bar\lambda_\alpha{}^\beta\bar x_\beta+\frac{1}{\sqrt2}\,\epsilon_{\alpha\beta\gamma}\sigma^\beta x^\gamma+\bar\sigma_\alpha y\\
\delta y &= -a(\bar\sigma_\alpha x^\alpha+\sigma^\alpha\bar x_\alpha)\quad (a=1\text{ see below})\ .
\end{aligned}\tag{F.1}$$

We defined $\bar x_\alpha=(x^\alpha)^*$, $\bar\lambda_\alpha{}^\beta=(\lambda^\alpha{}_\beta)^*$, and $\bar\sigma_\alpha=(\sigma^\alpha)^*$, and y is real. By rescaling of y and σ^α we can achieve that only a has to be fixed. The dimensions of representations and the number of parameters always refer to real quantities. For example, the **7** has seven real dimensions ($x+\bar x$, $-i(x-\bar x)$, and the real y), and there are **14** real parameters ($\lambda+\bar\lambda$, $-i(\lambda-\bar\lambda)$, $\sigma+\bar\sigma$, and $-i(\sigma-\bar\sigma)$). The anti-hermiticity of the $SU(3)$ generators $\lambda^\alpha{}_\beta$ requires that λ be equal to $-\lambda^\dagger$, namely $\lambda^\beta{}_\alpha=-\bar\lambda_\alpha{}^\beta$, and furthermore they are traceless, $\lambda^\alpha{}_\alpha=0$.

These transformations form a closed algebra. For the $SU(3)$ commutators one has

$$[\delta(\lambda_2),\delta(\lambda_1)]=\delta([\lambda_1,\lambda_2])\tag{F.2}$$

while [2]

$$[\delta(\lambda),\delta(\sigma)]=\delta(\sigma'=-\lambda\sigma)\ .\tag{F.3}$$

Finally,

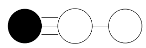
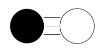
$$\begin{aligned}
[\delta(\sigma_2),\delta(\sigma_1)] &= \delta(\lambda'^\alpha{}_\beta)+\delta\Big(\sigma'^\alpha=\sqrt2\epsilon^{\alpha\beta\gamma}\bar\sigma_{1\beta}\bar\sigma_{2\gamma}\Big)\\
\lambda'^\alpha{}_\beta &= \frac32(\sigma_1^\alpha\bar\sigma_{2\beta}-\sigma_2^\alpha\bar\sigma_{1\beta})-\frac12\delta^\alpha_\beta\big((\bar\sigma_1\cdot\sigma_2-\bar\sigma_2\cdot\sigma_1)\big)\ .
\end{aligned}\tag{F.4}$$

[1] For any group the defining representation is the representation one uses to define the group, whereas the fundamental representation has by definition the property that its tensor products yield all other representations. Thus for $SO(n)$ with odd n the vector representation is the defining representation, but the spinor representation is the fundamental representation.

[2] To prove (F.3) one needs the "Schouten identity" $\lambda^\alpha{}_\beta\epsilon^{\beta\gamma\delta}=\lambda^\gamma{}_\beta\epsilon^{\alpha\beta\delta}+\lambda^\delta{}_\beta\epsilon^{\alpha\gamma\beta}$ and the anti-hermiticity relation $\lambda^\alpha{}_\beta=-\bar\lambda_\beta{}^\alpha$. The former follows by antisymmetrizing the 4 contravariant indices $\alpha,\beta,\gamma,\delta$.

Closure of the (σ, σ) commutator fixes $a = 1$, and this defines then the group G_2 as well as the defining representation **7**.

The decomposition of the generators of G_2 into the **8** of $SU(3)$ and the remaining **6** is not a symmetric decomposition (because of the term with $\delta(\sigma'^{\alpha})$ in (F.4)). A maximally noncompact version of an algebra always yields a symmetric decomposition, and has the property that the number of noncompact generators minus the number of compact generators is equal to its rank ($8 - 6 = 2$ for G_2). Such a decomposition is also called a Cartan decomposition [128]. The decomposition into the subgroup $SU(3)$ and six coset generators is not a Cartan decomposition because the subgroup $SU(3)$ does not have 6 generators. For G_2 a Cartan decomposition is for example into the generators of $SU(2) \times SU(2)$ and the $(\mathbf{2}, \mathbf{4})$ coset generators. The **7** decomposes under $SU(2) \times SU(2)$ into $(\mathbf{1}, \mathbf{3}) + (\mathbf{2}, \mathbf{2})$ [129]. One can find the maximal regular subalgebras of G_2 from its extended Dynkin diagram [130]

$G'_2 =$  $\qquad G_2 =$  (F.5)

Deleting one dot, one finds the maximal (rank 2) regular subalgebras $SU(3)$, $SU(2) \times SU(2)$ and of course G_2 itself.

Since G_2 has only two Casimir operators, C_2 and C_6, we know that

$$\mathrm{Tr} F^6 = \alpha \mathrm{tr} F^6 + \beta (\mathrm{tr} F^2)^3 \tag{F.6}$$

with α and β to be computed. The trace Tr is over the adjoint representation **14**, while the trace tr is over the fundamental **7**. If it would turn out that α vanishes, anomaly cancellation is possible. To compute α and β we choose two particular generators of the subgroup $SU(3)$, namely λ_3 and λ_8, and evaluate the trace relation (F.6) on each of them. We specify $SU(3)$ by the following generators λ_k normalized to $\mathrm{tr}\lambda_k\lambda_l = -\frac{1}{2}\delta_{kl}$

$$\lambda_1 = -\frac{i}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = -\frac{i}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_3 = -\frac{i}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_4 = -\frac{i}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = -\frac{i}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = -\frac{i}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = -\frac{i}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = -\frac{i}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} . \tag{F.7}$$

The first three generators λ_1, λ_2 and λ_3 define an $SU(2)$ subgroup of $SU(3)$, while (λ_4, λ_5) and (λ_7, λ_6) form doublets under this $SU(2)$. [3]

$$\begin{aligned}
&[\lambda_3, \lambda_1] = \lambda_2 \ ; \qquad [\lambda_3, \lambda_2] = -\lambda_1 \ ; \qquad [\lambda_3, \lambda_4] = \frac{1}{2}\lambda_5 \\
&[\lambda_3, \lambda_5] = -\frac{1}{2}\lambda_4 \ ; \ \ [\lambda_3, \lambda_6] = -\frac{1}{2}\lambda_7 \ ; \quad [\lambda_3, \lambda_7] = \frac{1}{2}\lambda_6 \\
&[\lambda_8, \lambda_4] = \frac{\sqrt{3}}{2}\lambda_5 \ ; \ \ [\lambda_8, \lambda_5] = -\frac{\sqrt{3}}{2}\lambda_4 \ ; \ \ [\lambda_8, \lambda_6] = \frac{\sqrt{3}}{2}\lambda_7 \\
&[\lambda_8, \lambda_7] = -\frac{\sqrt{3}}{2}\lambda_6 \ .
\end{aligned} \tag{F.8}$$

We now evaluate the trace Tr on the $\mathbf{8}+\mathbf{3}+\mathbf{3}^*$, and the trace tr on the $\mathbf{3}+\mathbf{3}^*+\mathbf{1}$. We find for the generator λ_3 that λ_3^2 is diagonal on all states. It is then easy to obtain

$$\mathrm{Tr}(\lambda_3)^6 = (-2 - \frac{1}{16}) + (\frac{-2-2}{64}) = -2 - \frac{1}{8} \tag{F.9}$$

$$\mathrm{tr}(\lambda_3)^6 = -\frac{4}{64} \qquad \mathrm{tr}(\lambda_3)^2 = -1 \tag{F.10}$$

where we used that $(\lambda_3)^2$ vanishes on y and equals $-\frac{1}{4}I$ on two of the three states in $\mathbf{3}$ and $\mathbf{3}^*$. As a check one may rederive these relations using (6.9.19) for $SU(3)$. Hence from λ_3 we learn that $-\frac{17}{8} = \alpha(-\frac{1}{16}) + \beta(-1)$.

Similarly we find for the generator λ_8

$$\begin{aligned}
&\mathrm{Tr}(\lambda_8)^6 = (-\frac{3}{4})^3 4 - (\frac{1}{12})^3(66+66) \\
&\mathrm{tr}(\lambda_8)^6 = (-\frac{1}{12})^3(66+66) \qquad \mathrm{tr}(\lambda_8)^2 = -\frac{1}{12}(6+6) \ .
\end{aligned} \tag{F.11}$$

Hence λ_8 tells us that $-\frac{27}{16} - \frac{11}{144} = \alpha(-\frac{11}{144}) + \beta(-1)$. There is no solution of both equations with $\alpha = 0$, and since α is nonzero, $\mathrm{Tr}F^6$ does not factorize for G_2. Hence, this group produces gauge anomalies which cannot be canceled by a counterterm.

F_4

Next we consider the group F_4. It has 52 generators, the defining representation is the **26**, and $SO(9)$ is a maximal regular subalgebra. (A regular subalgebra H of G has roots which are a subset of the roots of G, and Cartan generators which are a linear combination of the Cartan generators of G. If it is maximal, the rank of H is equal to the rank of G).

[3] An $SO(3)$ subgroup of $SU(3)$ is generated by λ_4, λ_5 and $\frac{1}{2}\lambda_3 + \frac{1}{2}\sqrt{3}\lambda_8$.

Spinors ψ^α of $SO(9)$ have 16 components, and $SO(9)$ has 36 generators, so we expect that under $SO(9)$ the **26** and the **52** decompose as follows

$$\begin{aligned}\mathbf{26} &\to \mathbf{9}+\mathbf{16}+\mathbf{1}\\ \mathbf{52} &\to \mathbf{36}+\mathbf{16}\ .\end{aligned} \tag{F.12}$$

So the **26** consists of a vector v^i, a spinor ψ^α and a scalar s, all real. We define F_4 by its action on the **26** [127]. Manifest $SO(9)$ covariance allows only

$$\begin{aligned}\delta v^i &= \Lambda^i{}_j v^j + \lambda^\alpha \Gamma^i_{\alpha\beta}\psi^\beta\\ \delta\psi^\alpha &= \frac{1}{4}\Lambda^{ij}(\Gamma_{ij})^\alpha{}_\beta\psi^\beta + v^i(\Gamma_i)^\alpha{}_\beta\lambda^\beta + s\lambda^\alpha\\ \delta s &= a\lambda^\alpha\psi_\alpha \quad (a=3,\ \text{see below})\ .\end{aligned} \tag{F.13}$$

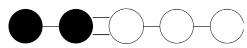
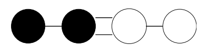
Orthogonal groups leave $y^i\delta_{ij}x^j$ invariant, hence $\delta_{ik}\Lambda^k{}_j + \Lambda^k{}_i\delta_{kj} = 0$. Then $\Lambda_{ij} = -\Lambda_{ji}$ where $\Lambda_{ij} = \delta_{ik}\Lambda^k{}_j$ are the 36 real parameters of $SO(9)$, and λ^α are the 16 extra real parameters. Again only a has to be fixed. Since the Dirac matrices (with $\{\Gamma_i,\Gamma_j\} = 2\delta_{ij}$) in 9 Euclidean dimensions can be taken to be real and symmetric 16×16 matrices [56, 107], the **26** is real. In fact, the charge conjugation matrix is the unit matrix since $C\Gamma^i = \Gamma^{i,T}C$ for $C=1$. (In odd dimensions one has either $C\Gamma^i = -\Gamma^{i,T}C$ or $C\Gamma^i = +\Gamma^{i,T}C$, but not both possibilities. Here, clearly one has the $+$ sign). Because $C=1$, the matrices $\Gamma^i_{\alpha\beta}$ and $(\Gamma^i)^\alpha{}_\beta$ in (F.13) are the same.

To show that (F.13) defines a Lie algebra, we must show that the commutators close. Of course the subalgebra of $SO(9)$ closes, $[\delta(\Lambda_2),\delta(\Lambda_1)] = \delta([\Lambda_1,\Lambda_2])$. Also $[\delta(\Lambda),\delta(\lambda)]$ is, as expected, equal to $\delta(\lambda' = -\frac{1}{4}\Lambda^{jk}\Gamma_{jk}\lambda)$. The crucial question is whether $[\delta(\lambda_2),\delta(\lambda_1)]$ closes. On s and v^i one easily establishes that this commutator is equal to an $SO(9)$ rotation with composite parameter $\Lambda'_{ij} = 2\lambda_1\Gamma_{ij}\lambda_2$. On ψ a Fierz rearrangement yields

$$[\delta(\lambda_2),\delta(\lambda_1)]\psi = \frac{1}{16}(\lambda_2\Gamma^i O^I\Gamma_i\lambda_1 + a\lambda_2 O^I\lambda_1)O_I\psi - (1\leftrightarrow 2)\ . \tag{F.14}$$

Only $O^I \sim \Gamma^{jkl}$ and $O^I\sim\Gamma^{jk}$ contribute (the rest, I and Γ^i and Γ^{ijkl}, are symmetric matrices); the contribution of the former cancels if one chooses $a=3$, and then the contribution of the latter yield the correctly normalized $SO(9)$ rotation. Hence, (F.13) defines a Lie algebra, namely F_4.

The decomposition of the generators of F_4 into generators of $SO(9)$ and coset generators is not a Cartan decomposition because that would require 24 subgroup generators and 28 coset generators; however, it still is a symmetric decomposition (because two λ transformations only produce an $SO(9)$ rotation). Hence, not every symmetric decomposition is a Cartan decomposition. (The reverse is, however, true: every Cartan decomposition is a symmetric decomposition). Of course, 24 is the number of generator of $SU(5)$, but 28 is not the sum of the dimensions of a set of irreducible representations of $SU(5)$ (except if one has many $SU(5)$ singlets). In fact, we may use a so-called extended Dynkin diagram [130] to find all maximal regular subalgebra of F_4

$F_4' =$  $\qquad F_4 =$  (F.15)

Deleting any point gives the following set of maximal regular subalgebras of F_4 [4]

$$SO(9); \quad SU(2) \times SU(4); \quad SU(3) \times SU(3); \quad Sp(6) \times SU(2)\ . \qquad \text{(F.16)}$$

The last one yields the Cartan decomposition $F_4/Sp(6) \times SU(2)$, and the **28** dimensional coset is indeed a representation of $Sp(6) \times SU(2)$, namely $\mathbf{28} = (\mathbf{14}, \mathbf{2})$ (where **14** is the antisymmetric symplectic-traceless representation $t^{ij} = -t^{ji}$ of $Sp(6)$).

The group F_4 has 4 Casimir operators (because it has rank 4), namely C_2, C_6, C_8, C_{12}. So, as in the case of G_2, we know that

$$\mathrm{Tr}F^6 = \alpha\, \mathrm{tr}F^6 + \beta\, (\mathrm{tr}F^2)^3 \qquad \text{(F.17)}$$

and the issue is whether $\alpha = 0$.

We need two particular generators of $SO(9)$ to be able to fix α and β. As such, we take a rotation in the 1-2 plane of R^9, and a simultaneous rotation in the 1-2 plane and the 1-3 plane. These generators have the following form in the defining representation of $SO(9)$

$$T_I = \begin{pmatrix} 0 & 1 & 0 & . \\ -1 & 0 & 0 & . \\ 0 & 0 & 0 & . \\ . & . & . & . \end{pmatrix}_{9\times 9} \qquad T_{II} = \begin{pmatrix} 0 & 1 & 1 & . \\ -1 & 0 & 0 & . \\ -1 & 0 & 0 & . \\ . & . & . & . \end{pmatrix}_{9\times 9} \qquad . \qquad \text{(F.18)}$$

In the spinor representation of $SO(9)$ they are given by $\frac{1}{2}\gamma^1\gamma^2$ and $\frac{1}{2}\gamma^1\gamma^2 + \frac{1}{2}\gamma^1\gamma^3$, respectively.

We evaluate the trace Tr on the **36** and **16** of $SO(9)$, and the trace tr on the **9**, **16** and **1** of $SO(9)$. One obtains, using (6.9.14) for the **36** of $SO(9)$

$$\begin{aligned} \mathrm{Tr}(T_I)^6 &= -14 - \frac{16}{64} \\ \mathrm{tr}(T_I)^6 &= -2 - \frac{16}{64} = -\frac{9}{4}\ , \quad \mathrm{tr}(T_I)^2 = -2 - \frac{16}{4} = -6\ . \end{aligned} \qquad \text{(F.19)}$$

[4] Actually, it is known [129] that for the exceptional groups there are 5 regular subalgebras obtained this way which are not maximal, and one of these exceptions is for F_4 where $SU(2) \times SU(4)$ is contained in $SO(9)$.

Hence the first equation for α and β is $-\frac{57}{4} = \alpha(-\frac{9}{4}) + \beta(-216)$.

The computation of the traces with T_{II} is simplified by noting that for the vector representation

$$T_{II}^2 = \begin{pmatrix} -2 & 0 & 0 & . \\ 0 & -1 & 0 & . \\ 0 & 0 & -1 & . \\ . & . & . & 0 \end{pmatrix}_{9\times 9} \tag{F.20}$$

while for the spinor representation $T_{II}^2 = -\frac{1}{2} I_{16\times 16}$. Now one finds, again using (6.9.14) to compute $\mathrm{Tr} T_{II}^6$ on the adjoint representation **36**,

$$\begin{aligned} \mathrm{Tr}(T_{II})^6 &= -23(-10) + 15(-4)6 - \frac{16}{8} = -132 \\ \mathrm{tr}(T_{II})^6 &= -10 - \frac{16}{8} = -12 \ , \quad \mathrm{tr}(T_{II})^2 = -4 - \frac{16}{2} \ . \end{aligned} \tag{F.21}$$

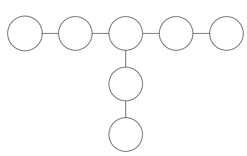
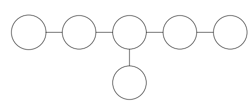
Hence the second equation for α and β is $-132 = \alpha(-12) + \beta(-12)$. Again, there is no solution to (F.17) with $\alpha = 0$, hence also for F_4 gauge anomalies do not cancel.

$\boldsymbol{E_6}$

The group E_6 has 78 generators, and the fundamental representation is the **27**. Suitable subgroups and the corresponding decompositions are [129]

$$\begin{aligned} SO(10)\times U(1): \quad & \mathbf{78} \to \mathbf{45}(0) + \mathbf{16}(-3) + \mathbf{16}^*(3) + \mathbf{1}(0) \\ & \mathbf{27} \to \mathbf{16}(1) + \mathbf{10}(-2) + \mathbf{1}(4) \\ Usp(8): \quad & \mathbf{78} \to \mathbf{36} + \mathbf{42} \\ & \mathbf{27} \to \mathbf{27} \\ F_4: \quad & \mathbf{78} \to \mathbf{52} + \mathbf{26} \\ & \mathbf{27} \to \mathbf{26} + \mathbf{1} \end{aligned} \tag{F.22}$$

None of these is a maximal regular subalgebra, as one may deduce from the extended Dynkin diagram for E_6

$E_6' =$  $E_6 =$  (F.23)

In the literature the coset $E_6/Usp(8)$ has been studied in detail [137, 138], so let us choose $Usp(8)$ as the subgroup of E_6. The corresponding decomposition of E_6 is a Cartan decomposition. The group E_6 acts on the **27** as

$$\delta z^{\alpha\beta} = \Lambda^{\alpha}{}_{\gamma} z^{\gamma\beta} + \Lambda^{\beta}{}_{\gamma} z^{\alpha\gamma} + \Sigma^{\alpha\beta}{}_{\gamma\delta} z^{\gamma\delta} \tag{F.24}$$

where $z^{\alpha\beta} = -z^{\beta\alpha}$ with $\alpha, \beta = 1, 8$ and $z^{\alpha\beta}\Omega_{\alpha\beta} = 0$ with $\Omega_{\alpha\beta}$ the metric of $Sp(8)$. The generators of $Usp(n)$ in the defining representation are the antihermitian matrices $\Lambda^{\alpha}{}_{\beta}$ preserving $z^{\alpha}\Omega_{\alpha\beta}y^{\beta}$. Hence $\Lambda^{\alpha}{}_{\beta} = -(\Lambda^{\beta}{}_{\alpha})^*$ and $\Lambda^{\gamma}{}_{\alpha}\Omega_{\gamma\beta} \equiv \Lambda_{\beta\alpha} = -\Omega_{\alpha\gamma}\Lambda^{\gamma}{}_{\beta} = \Lambda_{\alpha\beta}$. This shows that there are 36 generators. For $n = 2$ one easily checks that $Usp(2) = SU(2)$, but for higher n the dimension of $SU(n)$ is larger then that of $Usp(n)$. Furthermore, $\Sigma_{\alpha\beta\gamma\delta} \equiv \Sigma^{\alpha'\beta'}{}_{\gamma\delta}\Omega_{\alpha'\alpha}\Omega_{\beta'\beta}$ is totally antisymmetric, traceless with respect to $\Omega^{\alpha\beta}$, and satisfies the reality condition $(\Sigma^{\alpha\beta\gamma\delta})^* = \Sigma_{\alpha\beta\gamma\delta}$ where $\Sigma^{\alpha\beta\gamma\delta}\Omega_{\alpha\alpha'}\Omega_{\delta\delta'} \equiv \Sigma_{\alpha'\beta'\gamma'\delta'}$ with $\Omega^{\alpha\beta}\Omega_{\gamma\beta} = \delta^{\alpha}_{\gamma}$. The $\Sigma^{\alpha\beta}{}_{\gamma\delta}$ are the 42 generators of the coset $E_6/Usp(8)$.

There are 3 Casimir operators with rank ≤ 6, namely C_2, C_5 and C_6, but only C_2 and C_6 play a role in the decomposition of $\mathrm{Tr}F^6$

$$\mathrm{Tr}F^6 = a\,\mathrm{tr}F^6 + b\,(\mathrm{tr}F^2)^3\ . \tag{F.25}$$

We leave the proof that a is nonvanishing as an exercise. (Hint: two suitable 8×8 matrices in the defining representation of $Usp(8)$ are the matrices with $i\sigma_3$ or $i\sigma_1$ in the first 2×2 block, and for $\mathrm{Tr}T^6$ over the **36** one may use (6.9.14) with + signs instead of − signs).

E_7

The group E_7 has 133 generators, and $SU(8)$ is a maximal subgroup. The fundamental representation is the **56**, spanned by antisymmetric x^{ij} and $x_{ij} = (x^{ij})^*$ with $i, j = 1, 8$. E_7 is defined by its action on the **56** as follows [131]

$$\begin{aligned}\delta x^{ij} &= \Lambda^{i}{}_{k}x^{kj} + \Lambda^{j}{}_{k}x^{ik} + \frac{1}{4!}\epsilon^{ijklmnop}\Sigma_{mnop}\bar{x}_{kl} \\ \delta\bar{x}_{ij} &= \bar{\Lambda}_{i}{}^{k}\bar{x}_{kj} + \bar{\Lambda}_{j}{}^{k}\bar{x}_{ik} + \Sigma_{ijkl}x^{kl}\ .\end{aligned} \tag{F.26}$$

As before $\bar{x}_{ij} = (x^{ij})^*$ and $\bar{\Lambda}_i{}^k = (\Lambda^i{}_k)^*$. The $\Lambda^i{}_k$ yield the 63 real parameters of $SU(8)$, hence they are antihermitian and traceless, $\bar{\Lambda}_i{}^k = -\Lambda^k{}_i$, and $\Lambda^i{}_i = 0$, while the Σ_{ijkl} are totally antisymmetric and satisfy the selfduality relation $({}^*\Sigma)^{ijkl} \equiv (\Sigma_{ijkl})^* = \frac{1}{4!}\epsilon^{ijklmnop}\Sigma_{mnop}$, yielding the remaining 70 real parameters of E_7. Thus under $SU(8)$ the adjoint representation of E_7 decomposes as follows: **133** $\rightarrow$ **63**+**70**. Furthermore, the **56** of E_7 decomposes into the **28**+**28*** of $SU(8)$ as is clear from (F.26). (If we only allow real group parameters, the **56** remains irreducibe under $SU(8)$). This definition of E_7 in terms of $SU(8)$ resembles the definition of G_2 in terms of $SU(3)$, but $E_7/SU(8)$ yields a Cartan decomposition.

The relevant Casimir operators are in this case C_2 and C_6, so

$$\mathrm{Tr}F^6 = a\,\mathrm{tr}F^6 + b\,(\mathrm{tr}F^2)^3\ . \tag{F.27}$$

where the trace "Tr" is over the adjoint representation of E_7 and the trace "tr" is over the fundamental representation of E_7.

Decomposing the adjoint and fundamental representation of E_7 with respect to $SU(8)$, and denoting the resulting $SU(8)$ representations by their dimensionality we find from (F.27)

$$\mathrm{Tr}_{\mathbf{63}}F^6 + \mathrm{tr}_{\mathbf{70}}F^6 \;=\; 2a\mathrm{tr}_{\mathbf{28}}F^6 + 8b(\mathrm{tr}_{\mathbf{28}}F^2)^3 \;. \tag{F.28}$$

We choose again suitable generators for F, namely any linear combination A of the 7 independent generators which are diagonal (and imaginary) in the fundamental representation $\mathbf{8}$ of $SU(8)$. We denote these entries by a_i with $i = 1, 8$, and have then the constraint $\sum a_i = 0$.

The trace of A^6 in the adjoint representation of $SU(8)$ follows from (6.9.19)

$$\mathrm{Tr}_{\mathbf{63}}A^6 = 16\sum a_i^6 + 30(\sum a_i^2)(\sum a_j^4) - 20(\sum a_i^3)^2 \;. \tag{F.29}$$

All sums run over $1 \le i \le 8$ and $1 \le j \le 8$.

For the trace of A^6 in the $\mathbf{70}$ of $SU(8)$ we use that states are labeled by $i < j < k < l$[5] and A is represented on these states by $(a_i + a_j + a_k + a_l)$. We rewrite the restricted sum as a combination of unrestricted sums

$$\begin{aligned}\sum_{i<j<k<l}(a_i + a_j + a_k + a_l)^6 = \frac{1}{24}\Big[&\sum_{i,j,k,l}(a_i + a_j + a_k + a_l)^6\\ &-6\sum_{i=j,k,l}(2a_i + a_k + a_l)^6 + 3\sum_{i=j,k=l}(2a_i + 2a_k)^6\\ &+8\sum_{i=j=k,l}(3a_i + a_l)^6 - 6\sum_{i=j=k=l}(4a_i)^6\Big] \;.\end{aligned} \tag{F.30}$$

As a check one may verify that the number of terms on the left hand side is equal to that on the right hand side. This expression can be expanded into terms with $\sum a_i^6$, $(\sum a_i^4)(\sum a_j^2)$ and $(\sum a_i^2)^3$. Since A depends on

[5] As the 70 independent components of Σ_{ijkl} we can take the real parts R_{ijkl} and the imaginary parts I_{ijkl} with i, j, k, l running from 1 to 7. The selfduality relation expresses R_{ijk8} and I_{ijk8} in terms of these 70 components. The matrix A maps R_{ijkl} into $-(\alpha_i + \alpha_j + \alpha_k + \alpha_l)I_{ijkl}$, and I_{ijkl} into $(\alpha_i + \alpha_j + \alpha_k + \alpha_l)R_{ijkl}$, where the diagonal entries $a_1, .., a_8$ of A, satisfying $a_1 + \cdots + a_8 = 0$, have been written as $a_j = i\alpha_j$ with real α_j. Then A^2 is separately diagonal on the 35 states R_{ijkl} and the 35 states I_{ijkl}. The trace of A^{2p} becomes

$$\begin{aligned}\mathrm{tr}_{\mathbf{70}}A^{2p} \;&=\; (-2)\sum_{1\le i<j<k<l\le 7}(\alpha_i + \alpha_j + \alpha_k + \alpha_l)^{2p}\\ &=\; \sum_{1\le i<j<k<l\le 8}(a_i + a_j + a_k + a_l)^{2p}\end{aligned}$$

because the terms with an index α_8 gives the same contribution as the terms without an index α_8.

7 arbitrary parameters, there is enough resolving power to treat these invariants as independent.

For the traces of A^6 and A^2 over **28** of $SU(8)$ one finds easily

$$\begin{aligned} \mathrm{tr}_{\mathbf{28}} A^6 &= \frac{1}{2}\Big[\sum_{i,j}(a_i+a_j)^6 - \sum_{i=j}(2a_i)^6\Big] \\ \mathrm{tr}_{\mathbf{28}} A^2 &= \frac{1}{2}\Big[\sum_{i,j}(a_i+a_j)^2 - \sum_{i=j}(2a_i)^2\Big] \ . \end{aligned} \tag{F.31}$$

Finally we collect all terms with $\sum a_i^6$. There are no terms of this kind proportional to b, and our aim is to show that a is nonvanishing. Thus we assume that a vanishes, and show that this leads to a contradiction, namely we show that the sum of all terms in $\mathrm{Tr}_{133}A^6$ which are proportional to $\sum a_i^6$ does not vanish. From the **63** we find a coefficient 16, while the **70** yields the following coefficient

$$\begin{aligned} &\frac{1}{24}\Big[4 - 6(2^6+2) + 3(2^6+2^6) + 8(3^6+1) - 6(4^6)\Big] \\ &= \frac{1}{24}\Big[-3\cdot 2^7 + 3\cdot 2^7 + 2^3\cdot 3^6 - 3\cdot 2^{13}\Big] = 3^5 - 2^{10} \ . \end{aligned} \tag{F.32}$$

Clearly a is nonvanishing, and hence also the group E_7 leads to gauge anomalies in ten dimensions.

$\boldsymbol{E_8 \times E_8}$

We mentioned in the main text that also for the gauge group $E_8 \times E_8$ all anomalies in 10-dimensional simple supergravity coupled to Yang-Mills theory can be canceled by a counterterm. We give here the details.

From (6.9.57) we see that we must factorize $\mathrm{Tr}\tilde{F}^6$ in order to have a chance to cancel the anomalies. As we already discussed in the main text, for E_8 $\mathrm{Tr}\tilde{F}^6$ factorizes. In facts, also $\mathrm{Tr}\tilde{F}^4$ factorizes. Consequently, also for $E_8 \times E_8$ these traces factorize. This guarantees that the anomaly factorizes. We first prove this factorization and then explicitly construct the counterterm. Hence $E_8 \times E_8$ is as good a candidate as $SO(32)$.

For E_8 we have the following relations

$$\begin{aligned} \mathrm{Tr}\, F^6 &= \frac{1}{7200}(\mathrm{Tr}\, F^2)^3 \\ \mathrm{Tr}\, F^4 &= \frac{1}{100}(\mathrm{Tr}\, F^2)^2 \ . \end{aligned} \tag{F.33}$$

To derive these relations, we consider a particular generator A of E_8 and compute $\mathrm{Tr}A^6, \mathrm{Tr}A^4$ and $\mathrm{Tr}A^2$ separately. To find a suitable generator of E_8, we note that $SO(16)$ is a maximal regular subalgebra of E_8, and the adjoint representation **248** of E_8 decomposes under this $SO(16)$ into the adjoint **120** of $SO(16)$ and the spinor representation **128** of $SO(16)$. We then act with A separately onto the **120** part and the **128** part of **248**. Let the $SO(16)$ generator be a rotation in the x-y plane of R^{16}; then the spinor representation[6] is given by the 128×128 matrix $\frac{1}{2}\gamma^1\gamma^2$. So the generator A lies in the $SO(16)$ subgroup of E_8, and its representation in the vector representation and in the spinor representation is given by

$$A_{vector} = \begin{pmatrix} 0 & 1 & 0 & . \\ -1 & 0 & 0 & . \\ 0 & 0 & 0 & . \\ . & . & . & . \end{pmatrix}_{16\times 16} \qquad A_{spinor} = \left(\frac{1}{2}\gamma^1\gamma^2\right)_{128\times 128} \tag{F.34}$$

From applying (6.9.14) to $SO(16)$ we find $\mathrm{Tr}A^2 = (n-2)\mathrm{tr}A^2_{vector} = 14(-2)$ on the **120**, and from $(\frac{1}{2}\gamma^1\gamma^2)^2 = -\frac{1}{4}I$ one obtains $\mathrm{Tr}A^2_{spinor} = -\frac{1}{4}128$ on the **128**. Together $\mathrm{Tr}A^2 = -60$ for E_8. For $\mathrm{Tr}A^4$ one finds in the same way $\mathrm{Tr}A^4 = 14(+2) + \frac{1}{16}128 = 36$. Finally, $\mathrm{Tr}A^6 = 14(-2) - \frac{1}{64}128 = -30$. With these results one obtains (F.33).

For $E_8 \times E_8$ one has $\mathrm{Tr}\, F^n = \dim E_8\,(\mathrm{Tr_I}\, F^n + \mathrm{Tr_{II}}\, F^n)$ where $\mathrm{Tr_I} F^n$ refers to the trace in the first E_8, and $\mathrm{Tr_{II}}\, F^n$ to the second E_8. Furthermore, $\dim E_8$ equal 248. Then

$$\mathrm{Tr}\, F^6 = \frac{\dim E_8}{7200}\Big[(\mathrm{Tr_I}\, F^2)^3 + (\mathrm{Tr_{II}}\, F^2)^3\Big]\ . \tag{F.35}$$

This can be factorized using $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

$$\begin{aligned} \mathrm{Tr}\, F^6 \;&=\; \frac{\dim E_8}{7200}\Big[\mathrm{Tr_I}\, F^2 + \mathrm{Tr_{II}}\, F^2\Big] \\ &\quad \times\Big[(\mathrm{Tr_I}\, F^2)^2 - (\mathrm{Tr_I}\, F^2)(\mathrm{Tr_{II}}\, F^2) + (\mathrm{Tr_{II}}\, F^2)^2\Big]\ . \end{aligned} \tag{F.36}$$

For $\mathrm{Tr}F^4$ one finds

$$\begin{aligned} \mathrm{Tr}\, F^4 \;&=\; \dim E_8\Big(\mathrm{Tr_I}\, F^4 + \mathrm{Tr_{II}}\, F^4\Big) \\ &=\; \frac{\dim E_8}{100}\Big[(\mathrm{Tr_I}\, F^2)^2 + (\mathrm{Tr_{II}}\, F^2)^2\Big]\ . \end{aligned} \tag{F.37}$$

Finally, one has of course

$$\mathrm{Tr}\, F^2 \;=\; \dim E_8\Big(\mathrm{Tr_I}\, F^2 + \mathrm{Tr_{II}}\, F^2\Big)\ . \tag{F.38}$$

[6] The Dirac matrices in 16 dimensions are 256×256 matrices, which can be chosen to be block off-diagonal. Then $\frac{1}{2}\gamma^1\gamma^2$ is block diagonal with one 128×128 block for chiral spinors and the other 128×128 block for antichiral spinors. These two spinor representations are inequivalent, just as in the case of $SO(8)$. In the text we mean by $\frac{1}{2}\gamma^1\gamma^2$ one of these 128×128 blocks.

The first question to be settled is whether the factorization in (6.9.62) holds for $E_8 \times E_8$. Substitution of (F.35), (F.37) and (F.38) into (6.9.62) shows that it does hold. The counterterm is now constructed the same way as for $SO(32)$, see section 6.9. The 12-form from which the descent equation start is given by (6.9.63). Note that the traces $\mathrm{Tr}F^n$ in this formula refer to traces over $E_8 \times E_8$. The counterterm is then given by

$$\Delta\mathcal{L} = BX_8 + \Big(\mathrm{tr}R^2 - \frac{1}{30}\mathrm{Tr}F^2\Big)X_7 \tag{F.39}$$

where $dX_7 = X_8$ and

$$X_8 = \frac{1}{8}\mathrm{tr}R^4 + \frac{1}{32}(\mathrm{tr}R^2)^2 - \frac{1}{240}\mathrm{tr}R^2\mathrm{Tr}F^2 + \frac{1}{24}\mathrm{Tr}F^4 - \frac{1}{7200}(\mathrm{Tr}F^2)^2 \ . \tag{F.40}$$

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