Fermion Quantum Field Theory In Black-hole Spacetimes

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(ABSTRACT)

The need to construct a fermion quantum field theory in black-hole spacetimes is an acute one. The study of gravitational collapse necessitates the need of such. In this dissertation, we construct the theory of free fermions living on the static Schwarzschild black-hole and the rotating Kerr black-hole. The construction capitalises upon the fact that both black-holes are stationary axisymmetric solutions to Einstein's equation. A factorisability ansatz is developed whereby simple quantum modes can be found, for such stationary spacetimes with azimuthal symmetry. These modes are then employed for the purposes of a canonical quantisation of the corresponding fermionic theory. At the same time, we suggest that it may be impossible to extend a quantum field theory continuously across an event horizon. This split of a quantum field theory ensures the thermal character of the Hawking radiation. In our case, we compute and prove that the spectrum of neutrinos emitted from a black-hole via the Hawking process is indeed thermal. We also study fermion scattering amplitudes off the Schwarzschild black-hole.

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CHAPTER 1 : INTRODUCTION

The gravitational collapse of compact objects (white dwarfs, neutron stars) to form black-holes still remains much of a problem in modern physics [1]. A detailed description of such a collapse is still missing. In part, this is due to the extreme conditions found on these compact objects. Typically, a neutron star is between 1 to 3 solar masses, with a radius 10^{-5} of the solar radius, is of nuclear densities ($\leq 10^{15}\ g\ cm^{-3}$) and has a surface gravity 10^5 times solar. With surface gravities like these, general relativity is an integral part of the description of neutron stars. At the same time, the nuclear densities of neutron stars necessitates a quantum mechanical description of the neutron star matter. Indeed, the star is supported against collapse primarily by the quantum mechanical, neutron degeneracy pressure.

Depending on how one models the interior nuclear matter, neutron stars have a maximum density beyond which they are unstable with respect to gravitational collapse. For stable neutron stars, the extra mass needed to tip them over the stability limit can be acquired via accretion processes such as in binary X-ray systems. Once tipped over the stability limit, collapse is inevitable.

It is clear that the details of the collapse, is sensitive to the elementary particle physics relevant at each stage of the process. Indeed, there has been some debate as to the existence of quark stars which could be created during the collapse of neutron stars. In this sense, the gravitational collapse of compact objects, specifically neutron stars, can be used as a tool in the study of elementary particles in the regime of strong gravitational forces.

Furthermore, there are many interesting and deep theoretical questions that one can pose in this situation. For example, one may ask about the role that current algebra plays during gravitational collapse since after all, gravity couples to the energy-momentum tensor of all fields. Or one may ask about the implications of CP violation and CPT invariance on the collapsing matter.

Unfortunately, such a program of investigation is difficult to carry out. For one thing, the intractability of non-perturbative computations in realistic quantum field theories is prohibitive enough even in ordinary Minkowskian spacetime. Compounding this, is the presence of very strong gravitational fields which couples to the energy-momentum tensor of all fields, and thereby making general relativistic effects non-negligible.

However the situation is not entirely hopeless. For within the context of quantum field theory in curved spacetime [2], we may hope to gain some insight into the collapse process simply by quantising the fields about a black-hole background and using these quantum modes to study the detailed elementary particle physics of the problem. Of course, this approach is restricted to regimes where gravity is treated as a classical field and is useful only insofar as this semiclassical approximation is valid.

Since the primary matter fields are all fermionic in Nature, it is therefore of some importance to know how to build a fermion QFT in black-hole spacetimes. There has been some previous work in this area by some authors [3,4]. Unfortunately, most authors rely on the Newman-Penrose formalism which is not well adapted for computations in elementary particle physics. On the other hand, in [4], there is no systematic procedure employed in order to obtain the *simplest possible* mode solutions.

In this dissertation, I present a systematic approach to obtain fermion quantum modes in black-hole spacetimes. In particular, the method that I propose produces quantum modes which are analytically simple and have a direct physical interpretation. Moreover, I also show that by using these modes, we can duplicate Hawking's result on thermal radiation from black-holes [5], therefore increasing our confidence in them.

CHAPTER 2 : THE DIRAC EQUATION IN BLACK-HOLE SPACETIMES

Let us first introduce our notation. We will always work with a metric of signature $(+,-,-,-)$ and Greek indices will refer to the general world-index, whilst Latin indices refer to the flat Minkowskian tangent-space. Moreover, we take η_{ab} to always represent the Minkowskian metric and $g_{\mu\nu}$ to be the metric of curved spacetime. Our spinor conventions generally follow that of Itzykson and Zuber [6].

We begin with the Dirac equation in a general curved spacetime [7,8]. It can be written as, $(i\not{D} - m_0)\Psi = 0$ where m_0 is the bare fermion mass and $\not{D}$ is given in terms of the inverse vierbeins, $E_c{}^\mu$, and spin connection one-form, $\omega^a{}_b$,

$$\not{D} = \gamma^c E_c{}^\mu \partial_\mu + \gamma^c \Gamma_c \tag{2.1}$$

and where

$$\Gamma_c = \frac{1}{2} i (\omega_{ab})_c \Sigma^{ab} \tag{2.2}$$

with $\Sigma^{ab} = \frac{1}{4} i [\gamma^a, \gamma^b]$ as the spinor representation matrices of the Lorentz group [7]. Of course, the gamma matrices that we use, carry tangent space indices so that they take on the familiar flat-spacetime form. A glance at the above two equations reveals a fundamental difference between the usual Yang-Mills type coupling and gravitational couplings. The non-compactness of the Lorentz group (as compared to the SU(N) groups), is reflected in the spinor representation of the Lorentz generators; they turn out to be commutators of the gamma matrices. This means that in (2.1), the term $\gamma^c\Gamma_c$ contains products of three gamma matrices. Consequently further simplification may be obtained by multiplying out these matrices. Such a situation could never arise in the Yang-Mills case because the generators of the corresponding Lie algebra are not constructed from gamma matrices. Using the identity, $\gamma^a\gamma^b\gamma^c = \eta^{ab}\gamma^c - \eta^{ac}\gamma^b + \eta^{bc}\gamma^a + i\epsilon^{abcd}\gamma_d\gamma_5$, we find for the Dirac equation, upon simplification of (2.1),

$$i\gamma^c E_c{}^\mu \partial_\mu \Psi - \frac{1}{4} i(\omega_{ab})_c [\eta^{ca}\gamma^b - \eta^{cb}\gamma^a]\Psi + \frac{1}{4}\epsilon^{abcd}(\omega_{ab})_c\gamma_d\gamma_5\Psi - m_0\Psi = 0 \qquad (2.3)$$

The term involving the epsilon tensor is intimately connected to the *spin-tensor current* [9,10]. We will return to it later when we study the Kerr black-hole.

Although equation (2.3) appears rather formidable, it is actually not so in spacetimes which possess enough symmetries. These symmetries which are encoded in the spin-connection and the inverse vierbein can, and will be exploited when solving the Dirac equation in black-hole spacetimes.

In particular, since all black-holes are *stationary axisymmetric solutions* of Einstein's equation [11,12], it is therefore sufficient for us to focus on this class of spacetimes. The distinguishing feature of stationary axisymmetric spacetimes is that they possess a pair of commuting Killing vector fields which may be taken to be the time-like vector field $\frac{\partial}{\partial t}$ and the spacelike vector field $\frac{\partial}{\partial \phi}$ in a coordinate system where t denotes a temporal coordinate and ϕ denotes an azimuthal coordinate. Because of the high degree of symmetry, it is particularly advantageous to work in coordinate systems which manifestly reflects this symmetry. However, the price we pay for physical clarity, is the loss of manifest general covariance. In a very precise sense, we have made a convenient choice of gauge to find exact solutions and so we lose gauge invariance. This is inevitable when constructing exact solutions.

The key point to note is the very specific nature of axially symmetric solutions to Einstein's equation [11,12], which leads to a restricted form of the vierbein field, $e^a{}_\mu$. For example, an arbitrary axisymmetric spacetime (not necessarily a solution to Einstein's equation) has a metric tensor which may be written as,

$$\begin{aligned} g_{\mu\nu} dx^\mu dx^\nu &= g_{00}(x^1, x^2) dt^2 + 2 g_{03}(x^1, x^2) dt d\phi \\ &\quad + g_{33}(x^1, x^2) d\phi^2 \\ &\quad + g_{11}(x^1, x^2)(dx^1)^2 + g_{22}(x^1, x^2)(dx^2)^2 \\ &\quad + 2 g_{12}(x^1, x^2) dx^1 dx^2 \end{aligned}$$

if we choose coordinates so that $(x^0, x^3) = (t, \phi)$. For axisymmetric *solutions* to Einstein's equation, the g_{12} term may be omitted whilst the g_{11} term is directly related to the g_{22} term [11,12], thus achieving greater simplification. This is not surprising since the Einstein equation imposes further constraints on the general, axisymmetric, metric tensor which are over and above those due to the azimuthal symmtery alone. With this in mind, the vierbein field for an axisymmetric *solution* may be written as,

$$e^a{}_\mu = \begin{bmatrix} e^0{}_0 & 0 & 0 & e^3{}_0 \\ 0 & e^1{}_1 & 0 & 0 \\ 0 & 0 & e^2{}_2 & 0 \\ e^0{}_3 & 0 & 0 & e^3{}_3 \end{bmatrix} \tag{2.4}$$

where $e^a{}_\mu$ is a function of x^1 and x^2 alone. The inverse vierbein field, $E_a{}^\mu$ is also a funtion of x^1 and x^2 alone and may be written as the inverse to $e^a{}_\mu$. Furthermore, the components of the vierbein field in (2.4) also has to obey some constraints that are due to the special form of the metric tensor. It is easy to see from the conditions, $\eta_{ab} e^a{}_\mu e^b{}_\nu = g_{\mu\nu}$ and $g_{12} = 0$, that the vierbein components satisfy the following constraints :

$$\begin{aligned} g_{00} &= (e^0{}_0)^2 - (e^3{}_0)^2 \\ g_{03} &= (e^0{}_0)(e^0{}_3) - (e^3{}_0)(e^3{}_3) \\ g_{33} &= (e^0{}_3)^2 - (e^3{}_3)^2 \\ g_{11} &= -(e^1{}_1)^2 \\ g_{22} &= -(e^2{}_2)^2 \end{aligned} \tag{2.5}$$

Clearly, the requirements of symmetry places severe restrictions on the theory that we shall develop.

Solving The Dirac Equation In Axially Symmetric Spacetimes Using A Factorisability Ansatz

In this section we shall elaborate on how to solve (2.3) by using a factorisability ansatz. We will derive an integrability condition which we shall show, is satisfied in *any* coordinate system that reflects the full symmetry of the spacetime. In other words, the ansatz works *specially* for axially symmteric spacetimes; without the azimuthal symmetry, the integrability condition may not be satisfied. Also, it is important to note that this method does not require the axially symmetric spacetime to be asymptotically flat. Therefore it may even be applied to the Taub-NUT [13] spacetime.

We begin by imposing the following condition on Ψ in (2.3),

$$\Psi = f(x^1, x^2)\Phi \tag{2.6}$$

where the spinor, Φ, satisfies the *reduced* equation,

$$i\gamma^c E_c{}^\mu \partial_\mu \Phi + \frac{1}{4}\epsilon^{abcd}(\omega_{ab})_c \gamma_d \gamma_5 \Phi - m_0 \Phi = 0 \tag{2.7}$$

Then substitution of (2.6) into (2.3) and using (2.7) yields,

$$i\gamma^c E_c{}^\mu \partial_\mu f - \frac{1}{4} i(\omega_{ab})_c [\eta^{ca}\gamma^b - \eta^{cb}\gamma^a] f = 0 \tag{2.8}$$

We can derive an integrability condition for f if we can first remove the gamma matrices in equation (2.8). To this end, simply multiply (2.8) by γ^e on the left and take the matrix trace (i.e use gamma trace identities) on both sides of the equation. Noting the anti-symmetry of the spin-connection and relabelling indices, we find that

$$\eta^{ab}E_b{}^\mu\partial_\mu f - \frac{1}{2}(\omega^{ba})_b f = 0 \tag{2.9}$$

From [8], we know that $(\omega^b{}_a)_c = -E_a{}^\mu E_c{}^\lambda\partial_\lambda e^b{}_\mu + E_a{}^\mu e^b{}_\nu\Gamma^\nu{}_{\mu\lambda}E_c{}^\lambda$ where $\Gamma^\nu{}_{\mu\lambda}$ is the usual Christoffel symbol. Armed with this information and the fact that $\Gamma^\lambda{}_{\mu\lambda} = \partial_\mu \log[e] = \partial_\mu \log[\sqrt{-g}]$ (where e is the vierbein determinant), we can simplify (2.9) to obtain,

$$\partial_\mu \log[f] = -\frac{1}{2}E_b{}^\lambda\partial_\lambda e^b{}_\mu + \partial_\mu \log[e^{1/2}] \tag{2.10}$$

It is advantageous to define a new function, $h(x^1, x^2)$, such that $f = he^{1/2}$ so that the previous equation simplifies to

$$\partial_\mu \log[h] = -\frac{1}{2}E_b{}^\lambda\partial_\lambda e^b{}_\mu \tag{2.11}$$

Clearly the existence of h and the success of the factorisation ansatz depends on the *integrability* of (2.11). However, the integrability of (2.11) is not a priori guaranteed unless the vierbeins take on a very special form. That this is the case, is assured to us by the very specific nature of the most general, canonical form for an axially symmetric solution to the Einstein equation [11,12]. Let us see how this works.

First, when we evaluate (2.11) for various values of μ, and remembering that the vierbein and inverse vierbein depends only on x^1 and x^2, we get,

$$\partial_0 \log[h] \;=\; \partial_t \log[h] \;=\; 0$$

$$\begin{aligned}
\partial_3 \log[h] &= \partial_\phi \log[h] \;=\; 0 \\
\partial_1 \log[h] &= \partial_1 \log[(e^1{}_1)^{-1/2}] \\
&= \partial_1 \log[(E_1{}^1)^{1/2}] \\
\partial_2 \log[h] &= \partial_2 \log[(e^2{}_2)^{-1/2}] \\
&= \partial_2 \log[(E_2{}^2)^{1/2}]
\end{aligned}$$

Consequently we only need $e^1{}_1$ and $e^2{}_2$ to be appropriately related to each other so as to make the above equations integrable. But this is precisely the case with axially symmetric solutions of Einstein's equation [11,12]. And the factorisation ansatz works by this token.

Of course the reduced equation, (2.7), appears no less formidable than (2.3) but there is a simplification. In order to further simplify (2.7), we have to consider two distinct cases separately. These are the cases when the term, $\frac{1}{4}\epsilon^{abcd}(\omega_{ab})_c\gamma_d\gamma_5$, vanishes or otherwise. Obviously the case when this term vanishes is a lot easier to handle. In fact, when it does not vanish then the general problem is insoluble *except* when the fermion is *massless*. We briefly consider these cases separately below, leaving the detailed analysis to subsequent chapters.

Case One : $\frac{1}{4}\epsilon^{abcd}(\omega_{ab})_c\gamma_d\gamma_5 = 0$

In this case (2.7) becomes, $i\gamma^c E_c{}^\mu \partial_\mu \Phi - m_0 \Phi = 0$, an analytically simple equation. There is no further reduction necessary. This case corresponds to two physically important cases which we shall study - flat Minkowskian spacetime and the Schwarzschild spacetime.

Case Two : $\frac{1}{4}\epsilon^{abcd}(\omega_{ab})_c\gamma_d\gamma_5 \neq 0$

This the case that corresponds to the Kerr black-hole and Taub-NUT spacetime. In general, (2.7) is insoluble in this situation except when the fermion is massless. For then, the bispinor Φ is an eigenstate of γ_5 and is either left or right handed depending on which eigenvalue it corresponds to (± 1). In other words the four-dimensional representation of gamma matrices decomposes into the two dimensional representation of Pauli spin matrices. This means that the γ_5 becomes redundant and another factorisation is possible. We shall work this out in detail later on in the chapter on the Kerr black-hole.

CHAPTER 3 : MINKOWSKI SPACETIME IN SPHERICAL COORDINATES

In this chapter we shall solve the gravitationally coupled Dirac equation in Minkowski spacetime, in spherical coordinates [14]. Although Minkowski spacetime is flat, some of the results we obtain here will be used when we attack the Schwarzschild problem. Moreover the Minkowskian theory will serve as a nice consistency check when we set the black-hole parameters (mass and angular momentum) to zero - where we expect a "correspondence principle" to hold. In any case, far from the black-hole, the mode solutions for the Dirac equation should asymptotically reduce to those of the Minkowskian example. Hence the Minkowskian case is the best point to begin our investigation of the Dirac equation in black-hole spacetimes.

The Minkowskian line element in spherical coordinates reads as, $ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + r^2 \sin^2\theta d\phi^2)$ so that we may choose as basis one-forms, $\theta^0 = dt$, $\theta^1 = dr$, $\theta^2 = rd\theta$, $\theta^3 = r\sin\theta d\phi$. Using this set of basis one-forms and the formula, $2(\omega)_{ab} = \theta^c i_a i_b d\theta_c + i_b d\theta_a - i_a d\theta_b$, we can work out the spin connection $(\omega)_{ab}$. Thus we find that,

$$(\omega)_{01} = (\omega)_{02} = (\omega)_{03} = 0, (\omega)_{12} = \frac{1}{r}\theta^2, (\omega)_{13} = \frac{1}{r}\theta^3, (\omega)_{23} = \frac{\cot\theta}{r}\theta^3 \tag{3.1}$$

With the spin-connection determined as above, we can check that the term $\frac{1}{4}\epsilon^{abcd}(\omega_{ab})_c\gamma_d\gamma_5$

actually vanishes in this case. Therefore we may directly solve (2.11) to obtain a factorisation of the Dirac spinor, Ψ. But first, note that the vierbeins are given by $e^0{}_0 = 1, e^1{}_1 = 1, e^2{}_2 = r, e^3{}_3 = r\sin\theta$ and the inverse vierbeins are simply the inverse to this diagonal set. With this, it is easy to see from (2.10) and (2.11) that

$$f = e^{1/2} = r\sin^{1/2}\theta \tag{3.2}$$

In particular, this means that $\Psi = r\sin^{1/2}\theta\Phi$ where Φ solves

$$i\gamma^a E_a{}^\mu \partial_\mu \Phi - m_0\Phi = 0 \tag{3.3}$$

It is to this reduced equation that we now devote our attention. Define $\vec{\nabla}$ by $\vec{\nabla} = E_k{}^\mu \partial_\mu$ where $k = 1, 2, 3$. After some manipulations we get,

$$iE_0{}^\mu \partial_\mu \Phi = i\frac{\partial}{\partial t}\Phi = -i\vec{\alpha}\cdot\vec{\nabla}\Phi + m_0\beta\Phi \tag{3.4}$$

where $\vec{\alpha}$ and β are the Dirac matrices. But $\vec{\nabla} = E_k{}^\mu \partial_\mu = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$ so that the previous equation is nothing but the *free* Dirac equation in spherical coordinates. Here we see how the factorisation ansatz works; all dependence on the spin connection has been absorbed into the multiplicative factor f. As we shall see later, the same happens for black-hole spacetimes too. We now solve the free equation in detail [14].

Define the orbital angular momentum operator to be $\vec{L} = -i\hat{r}\wedge\vec{\nabla}$ so that $-i\vec{\nabla} = -i\hat{r}\frac{\partial}{\partial r} - \frac{1}{r}\hat{r}\wedge\vec{L}$. Consequently, we get $-i\vec{\alpha}\cdot\vec{\nabla} = -i(\vec{\alpha}\cdot\hat{r})\frac{\partial}{\partial r} - \frac{1}{r}\vec{\alpha}\cdot(\hat{r}\wedge\vec{L})$. The term $\vec{\alpha}\cdot(\hat{r}\wedge\vec{L})$ can be simplified by using the identities $(\vec{\alpha}\cdot\vec{A})(\vec{\alpha}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\vec{\Sigma}\cdot(\vec{A}\wedge\vec{B})$ with $\vec{A} = \hat{r}$ and

$\vec{B} = \vec{L}$ and $\gamma_5\vec{\alpha} = \vec{\alpha}\gamma_5 = \vec{\Sigma}$. Of course, $\vec{\Sigma}$ is the usual spin matrix [6]. The simplification we need is given by, $i\vec{\alpha}\cdot(\hat{r}\wedge\vec{L}) = (\vec{\alpha}\cdot\hat{r})(\vec{\Sigma}\cdot\vec{L})$, because,

$$\begin{aligned} -i(\vec{\alpha}\cdot\vec{\nabla}) &= -i(\vec{\alpha}\cdot\hat{r})\frac{\partial}{\partial r} + \frac{i}{r}(\vec{\alpha}\cdot\hat{r})(\vec{\Sigma}\cdot\vec{L}) \\ &= -i(\vec{\alpha}\cdot\hat{r})\left[\frac{\partial}{\partial r} - \frac{1}{r}(\vec{\Sigma}\cdot\vec{L})\right] \\ &= -i(\vec{\alpha}\cdot\hat{r})\left[\frac{\partial}{\partial r} + \frac{1}{r} - \frac{1}{r}\beta^2(\vec{\Sigma}\cdot\vec{L}+1)\right] \end{aligned}$$

where we have used the fact that $\beta^2 = 1$. We can define a new operator, K, by

$$K = \beta(\vec{\Sigma}\cdot\vec{L}+1) = \beta(2\vec{S}\cdot\vec{L}+1) = \beta(\vec{J}^2 - \vec{L}^2 - \vec{S}^2 + 1) \tag{3.5}$$

where $\vec{S} = \frac{1}{2}\vec{\Sigma}$ is the spin operator and $\vec{J} = \vec{L}+\vec{S}$ is the total angular momentum operator. And therefore we may now write,

$$i\frac{\partial}{\partial t}\Phi = -i(\vec{\alpha}\cdot\hat{r})\left[\frac{\partial}{\partial r} + \frac{1}{r} - \frac{1}{r}\beta K\right]\Phi + m_0\beta\Phi = H_0 \tag{3.6}$$

where H_0 is the free Dirac Hamiltonian. To proceed from here, we require a complete set of commuting observables (CSCO) so that we may attempt a separation of variables in (3.6).

A Complete Set Of Commuting Observables

Finding a CSCO for (3.6) is quite easy because we are dealing with the free Dirac Hamiltonian in flat spacetime. As we show below, it is given by the set $\{H_0, J^3, \vec{J}^2, \mathcal{P}, K\}$ where $\mathcal{P} = \beta P$ is the parity operator acting on spinors and P is the parity operator acting on coordinates. The proof is constructed in several stages and exhaustive use is made of the list of identities satisfied by the Dirac matrices, as given in [14]. Furthermore, we shall employ the Dirac representation of the Dirac and gamma matrices.

H_0, J^3 and $\vec{J}^2$ mutually commutes

Represent $\vec{L}$ and H_0 by $L^i = -i\epsilon^{ijk}x_j\partial_k$ and $H_0 = -i\alpha^l\partial_l + m_0\beta$. Then,

$$\begin{aligned}
\left[L^i, H_0\right] &= \left[-i\epsilon^{ijk}x_j\partial_k, -i\alpha^l\partial_l + m_0\beta\right] \\
&= -\left[\epsilon^{ijk}x_j\partial_k, \alpha^l\partial_l\right] \\
&= -\epsilon^{ijk}\alpha^l\left[x_j\partial_k, \partial_l\right] \\
&= \epsilon^{ijk}\alpha_j\partial_k
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left[S^i, H_0\right] &= \left[\frac{1}{2}\Sigma^i, -i\alpha^j\partial_j + m_0\beta\right] \\
&= -\frac{i}{2}\gamma^5\left[\alpha^i, \alpha^j\right]\partial_j \\
&= -\frac{i}{2}\gamma^5\left(2i\epsilon^{ijk}\Sigma_k\partial_j\right) \\
&= -\epsilon^{ijk}\alpha_j\partial_k
\end{aligned}$$

where we have used the fact that $[\Sigma^i, \beta] = 0$. Thus $[L^i + S^i, H_0] = [J^i, H_0] = 0$ which means that H_0 commutes with J^3 and $\vec{J}^2$. And since $\vec{J}^2$ is a Casimir operator of the rotation group, the result follows.

H_0 commutes with K

Here we show that K commutes with H_0. The proof is slightly longer than the previous one.

$$\begin{aligned}[H_0, K] &= \left[-i(\vec{\alpha}\cdot\hat{r})\left(\frac{\partial}{\partial r}+\frac{1}{r}-\frac{1}{r}\beta K\right), K\right] \\ &= -\,i\,[(\vec{\alpha}\cdot\hat{r}), K]\left(\frac{\partial}{\partial r}+\frac{1}{r}-\frac{1}{r}\beta K\right)\end{aligned}$$

since $[\beta, K] = \left[\beta, \beta\left(2\vec{S}\cdot\vec{L}+1\right)\right] = \left[\beta, \vec{S}\right] = 0$. Therefore we only need to prove that $[(\vec{\alpha}\cdot\hat{r}), K] = 0$. To this end, we note that

$$\begin{aligned}[\vec{\alpha}\cdot\hat{r}, K] &= \left[\vec{\alpha}\cdot\hat{r}, \beta\left(2\vec{S}\cdot\vec{L}+1\right)\right] \\ &= -\,\beta(\vec{\alpha}\cdot\hat{r})\left(2\vec{S}\cdot\vec{L}+1\right) - \left(2\vec{S}\cdot\vec{L}+1\right)\beta(\vec{\alpha}\cdot\hat{r})\end{aligned}$$

since $\{\beta, \alpha^i\} = 0$. Hence we find that the commutator may be cast into the form, $[\vec{\alpha}\cdot\hat{r}, K] = -2\left\{\beta(\vec{\alpha}\cdot\hat{r}), \vec{S}\cdot\vec{L}\right\} - 2\beta(\vec{\alpha}\cdot\hat{r})$. On the other hand, we also have the identity $\{AB, C\} = A\{B, C\} - [A, C]B$ so that if we put $A = \beta$, $B = \vec{\alpha}\cdot\hat{r}$ and $C = \vec{S}\cdot\vec{L}$, we get $\left\{\beta(\vec{\alpha}\cdot\hat{r}), \vec{S}\cdot\vec{L}\right\} = \beta\left\{\vec{\alpha}\cdot\hat{r}, \vec{S}\cdot\vec{L}\right\}$. Using this result, we can simplify the commutator to obtain $[\vec{\alpha}\cdot\hat{r}, K] = -2\beta\left\{(\vec{\alpha}\cdot\hat{r}), \vec{S}\cdot\vec{L}\right\} - 2\beta(\vec{\alpha}\cdot\hat{r})$ so that we now require the anticommutator, $\left\{(\vec{\alpha}\cdot\hat{r}), \vec{S}\cdot\vec{L}\right\}$. To this end, put $\vec{\alpha}\cdot\hat{r} = \alpha_i\frac{x^i}{r}$ where $r = x^i x_i$ and $\vec{S}\cdot\vec{L} = -iS_j\epsilon^{jkl}x_k\partial_l$ in the anticommutator expression. After some simple manipulations, it can be shown that $\left\{(\vec{\alpha}\cdot\hat{r}), \vec{S}\cdot\vec{L}\right\} = i\epsilon^{ijk}S_i\alpha_j\frac{x_k}{r}$. But $4i\epsilon^{ijk}S_i = 2(\alpha^j\alpha^k - \delta^{jk})$ and $\{\alpha_j, \alpha_k\} = 2\delta_{jk}$ so that we finally obtain $\left\{(\vec{\alpha}\cdot\hat{r}), \vec{S}\cdot\vec{L}\right\} = -\vec{\alpha}\cdot\hat{r}$. With this, we see that $[(\vec{\alpha}\cdot\hat{r}), K] = 0$. In particular, this implies that H_0 commutes with K.

$\underline{J^i \text{ commutes with } K}$

We now prove that K commutes with the generators of the rotation group.

$$\begin{aligned}
\left[L^i, K\right] &= \left[L^i, \beta\left(2\vec{S}\cdot\vec{L}+1\right)\right] \\
&= 2\beta S_j \left[L^i, L^j\right] \\
&= 2\beta S_j i\epsilon^{ijk} L_k
\end{aligned}$$

On the other hand, it is trivial to verify that $[S^i, K] = -2\beta S_j i\epsilon^{ijk} L_k$ so that K commutes with J^i and hence with J^3 and $\vec{J}^2$. Finally, we show that $\mathcal{P}$ commutes with H_0, K and J^i.

$\underline{\mathcal{P} \text{ commutes with } H_0,\ K \text{ and } J^i}$

Put $\mathcal{P} = \beta P$. Then,

$$\begin{aligned}
[\mathcal{P}, H_0] &= \left[\mathcal{P}, -i\alpha^i\partial_i + m_0\beta\right] \\
&= \left[\beta P, -i\alpha^i\partial_i\right] \\
&= -i\left(\beta\alpha^i P\partial_i - \alpha^i\beta\partial_i P\right)
\end{aligned}$$

But $P\partial_i = -\partial_i P$ so that $[\mathcal{P}, H_0] = i\left\{\beta, \alpha^i\right\}\partial_i P = 0$ and hence $\mathcal{P}$ commutes with H_0. Next, we show that $\mathcal{P}$ commutes with J^i.

$$\begin{aligned}
\left[\mathcal{P}, J^i\right] &= \left[\beta P, J^i\right] \\
&= \left(\beta P J^i - J^i\beta P\right)
\end{aligned}$$

But $[P, J^i] = 0$ since J^i is a pseudovector. Thus $[\mathcal{P}, J^i] = [\beta, J^i]\, P = 0$ because $[\beta, J^i] = 0$.

The final step is to prove that $\mathcal{P}$ commutes with K. For this purpose, consider

$$
\begin{aligned}
[\mathcal{P}, K] &= \left[\beta P, \beta\left(2\vec{S}\cdot\vec{L}+1\right)\right] \\
&= \beta\left[\beta P, 2\vec{S}\cdot\vec{L}\right] \\
&= 0
\end{aligned}
$$

since $\vec{S}\cdot\vec{L}$ transforms as a scalar.

This completes our proof that $\left\{H_0, K, J^3, \vec{J}^2, \mathcal{P}\right\}$ forms a complete set of commuting operators. We are now ready to perform a separation of variables in (3.6).

Separation Of Variables

Let $\Phi_{m_j\kappa_j}$ be the simultaneous eigenstate of J^3, $\vec{J}^2$ and K. The eigenvalues corresponding to J^3 and $\vec{J}^2$ are well known and are given by,

$$
\begin{aligned}
\vec{J}^2\Phi_{m_j\kappa_j} &= j(j+1)\Phi_{m_j\kappa_j} \\
J^3\Phi_{m_j\kappa_j} &= m_j\Phi_{m_j\kappa_j}
\end{aligned}
$$

with $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ and $m_j = -j, \ldots, +j$ so that we only have to determine the eigenvalues corresponding to K. For this purpose, we consider K^2 and use the commutator identity, $[\beta, S^i] = 0$, to get

$$
K^2 = \beta^2\left(2\vec{S}\cdot\vec{L}+1\right)^2
$$

$$
\begin{aligned}
&= 4\left(\vec{S}\cdot\vec{L}\right)\left(\vec{S}\cdot\vec{L}\right)+4\left(\vec{S}\cdot\vec{L}\right)+1 \\
&= 4\left[\frac{1}{4}\vec{L}^2+\frac{i}{2}\vec{S}\cdot\left(\vec{L}\wedge\vec{L}\right)\right]+4\left(\vec{S}\cdot\vec{L}\right)+1
\end{aligned}
$$

where we have used the identity $\left(\vec{\sigma}\cdot\vec{A}\right)\left(\vec{\sigma}\cdot\vec{B}\right)=\vec{A}\cdot\vec{B}+i\vec{\sigma}\cdot\left(\vec{A}\wedge\vec{B}\right)$ and $\vec{S}=\frac{1}{2}\begin{bmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{bmatrix}$.

But $\vec{L}\wedge\vec{L}=i\vec{L}$ since $-i\epsilon^{ijk}L_iL_j=L^k$ so that

$$
\begin{aligned}
K^2 &= \left(\vec{L}+\vec{S}\right)^2-\vec{S}^2+1 \\
&= \vec{J}^2-\vec{S}^2+1
\end{aligned}
$$

But $\vec{S}^2=\frac{3}{4}$ so that $K^2=\vec{J}^2+\frac{1}{4}$ and hence if we set,

$$
\begin{aligned}
K\Phi_{m_j\kappa_j} &= -\kappa_j\Phi_{m_j\kappa_j} \\
\Rightarrow K^2\Phi_{m_j\kappa_j} &= \left[j(j+1)+\frac{1}{4}\right]\Phi_{m_j\kappa_j} \\
&= (j+\frac{1}{2})^2\Phi_{m_j\kappa_j} \\
&= (-\kappa_j)^2\Phi_{m_j\kappa_j}
\end{aligned}
$$

Therefore we conclude that the eigenvalue problem for K may be written as, $K\Phi_{m_j\kappa_j} = -\kappa_j\Phi_{m_j\kappa_j}$ with $-\kappa_j=\ldots,3,2,1,-1,-2,-3,\ldots$ and $-\kappa_j\neq 0$.

The explicit construction of the eigenfunction, $\Phi_{m_j\kappa_j}$, follows from the Clebsch-Gordon addition of angular momenta [14]. Because of this, it turns out that these eigenfunctions are also simultaneous eigenstates of parity, $\mathcal{P}$. Consequently we can define these eigenfunctions as

$$
\begin{aligned}
\Phi^{(+)}_{m_j\kappa_j=\mp(j+\frac{1}{2})} &= \begin{bmatrix} i\Psi^{m_j}_{j\mp\frac{1}{2}} \\ 0 \end{bmatrix} \\
\Phi^{(-)}_{m_j\kappa_j=\mp(j+\frac{1}{2})} &= \begin{bmatrix} 0 \\ i\Psi^{m_j}_{j\pm\frac{1}{2}} \end{bmatrix}
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi^{m_j}_{j-1/2} &= \frac{1}{\sqrt{2j}} \begin{bmatrix} \sqrt{j+m_j}\, Y^{m_j-1/2}_{j-1/2} \\ \sqrt{j-m_j}\, Y^{m_j+1/2}_{j-1/2} \end{bmatrix} \\
\Psi^{m_j}_{j+1/2} &= \frac{1}{\sqrt{2j+2}} \begin{bmatrix} \sqrt{j+1-m_j}\, Y^{m_j-1/2}_{j+1/2} \\ -\sqrt{j+1+m_j}\, Y^{m_j+1/2}_{j+1/2} \end{bmatrix}
\end{aligned}
$$

The Y_l^m are the usual spherical harmonics. With these definitions, it is trivial to prove that the Φ's are parity eigenstates with opposite eigenvalues. Furthermore, they form a complete and orthonormal set of bispinors over the sphere and satisfy the following orthonormality and completeness relations,

$$
\begin{aligned}
\int \Phi^{(+)\dagger}_{m_j\kappa_j} \Phi^{(+)}_{m_{j'}\kappa_{j'}}\, d\Omega = \int \Phi^{(-)\dagger}_{m_j\kappa_j} \Phi^{(-)}_{m_{j'}\kappa_{j'}}\, d\Omega &= \delta_{m_j m_{j'}} \delta_{\kappa_j\kappa_{j'}} \delta_{jj'} \\
\sum_{jm_j\kappa_j} \left[\Phi^{(+)}_{m_j\kappa_j}(\Omega) \otimes \Phi^{(+)\dagger}_{m_j\kappa_j}(\Omega') + \Phi^{(-)}_{m_j\kappa_j}(\Omega) \otimes \Phi^{(-)\dagger}_{m_j\kappa_j}(\Omega') \right] &= \delta(\Omega-\Omega')\, 1_{4\times4}
\end{aligned}
$$

The final task that we have to perform before we can achieve a separation of variables in (3.6), is the determination of the action of H_0 in (3.6) on these eigenfunctions. Only then can we perform a partial wave expansion leading to a separation of variables. To this end, we shall need the following identity which is not difficult to prove,

$$(\vec{\sigma} \cdot \hat{r})\, \Psi^{m_j}_{j \pm 1/2}(\hat{r}) \quad = \quad \Psi^{m_j}_{j \mp 1/2}(\hat{r})$$

Note the flip from $j \pm 1/2$ to $j \mp 1/2$. Armed with this identity, we can show that *in the Dirac representation*,

$$\begin{aligned} i\,(\vec{\alpha} \cdot \hat{r})\, \Phi^{(+)}_{m_j \kappa_j} &= -\,\Phi^{(-)}_{m_j \kappa_j} \\ i\,(\vec{\alpha} \cdot \hat{r})\, \Phi^{(-)}_{m_j \kappa_j} &= \Phi^{(+)}_{m_j \kappa_j} \end{aligned}$$

and we are now ready to perform a partial wave expansion. Set *each* partial wave of (3.6) to be the sum of two independent pieces as thus,

$$\psi_{m_j \kappa_j} = f^{+}_{m_j \kappa_j}(r,t)\Phi^{(+)}_{m_j \kappa_j} + f^{-}_{m_j \kappa_j}(r,t)\Phi^{(-)}_{m_j \kappa_j} \tag{3.7}$$

Noting that *in the Dirac representation* we have $\beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ then it is trivial to see that the following simplifications are true,

$$\begin{aligned} \beta K \Phi^{(+)}_{m_j \kappa_j} &= -\,\kappa_j \Phi^{(+)}_{m_j \kappa_j} \\ \beta K \Phi^{(-)}_{m_j \kappa_j} &= \kappa_j \Phi^{(-)}_{m_j \kappa_j} \\ m_0 \beta \Phi^{(+)}_{m_j \kappa_j} &= m_0 \Phi^{(+)}_{m_j \kappa_j} \\ m_0 \beta \Phi^{(-)}_{m_j \kappa_j} &= -\,m_0 \Phi^{(-)}_{m_j \kappa_j} \end{aligned}$$

With these, the partial wave (3.7) when substituted in (3.6) along with the time dependence,

$$\begin{aligned} f^{+}_{m_j \kappa_j}(r,t) &= e^{-iEt} a(r) \\ f^{-}_{m_j \kappa_j}(r,t) &= e^{-iEt} b(r) \end{aligned}$$

leads to the linear non-autonomous system,

$$\frac{d}{dr}\begin{bmatrix} a \\ \\ b \end{bmatrix} = \begin{bmatrix} -\frac{1+\kappa_j}{r} & m_0 + E \\ \\ m_0 - E & -\frac{1-\kappa_j}{r} \end{bmatrix}\begin{bmatrix} a \\ \\ b \end{bmatrix} \tag{3.8}$$

Equation (3.8) above represents the radial Dirac equation - the angular pieces having been separated as in (3.7). In the present Minkowskian case, the radial equation is amenable to an exact and explicit solution. This will be done shortly. On the other hand, as we shall see later on, the radial equation in a black-hole background is far more complicated. Only a qualitative study of the radial equation is possible in that situation. Because of this, we will also perform a simple qualitative analysis of (3.8) so that we may later on, compare, the qualitative behaviours of the radial equations in black-hole and flat spacetimes.

The radial equation, (3.8), may be separated into two second-order differential equations for $a(r)$ and $b(r)$. These equations must then be solved for the separate cases of when $\kappa_j < 0$ and $\kappa_j > 0$. It turns out that the second-order differential equations that we seek are of the form,

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) - \frac{\ell(\ell+1)}{r^2} + p^2\right] f(r) \quad = \quad 0$$

where $p^2 = E^2 - m_0^2 \geq 0$ is the momentum squared. Of course, the solutions to this equation that are regular at $r = 0$ exist only for $p \geq 0$ and *integral* $\ell \geq 0$. They are the spherical Bessel functions j_ℓ.

The equation for $a(r)$

The equation satisfied by $a(r)$ is given by,

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) - \frac{\kappa_j(\kappa_j+1)}{r^2} + \left(E^2 - m_0^2\right)\right] a(r) \quad = \quad 0$$

For $\kappa_j > 0$, the solutions to this equation are $\{j_{\kappa_j}(pr)\}$. On the other hand when $\kappa_j < 0$, some rearrangement of the equation is needed in order to make an identification with $\ell \geq 0$. Thus, rewrite

$$-\frac{\kappa_j(\kappa_j+1)}{r^2} \quad = \quad -\frac{(-\kappa_j-1)(-\kappa_j-1+1)}{r^2}$$

and since $-\kappa_j - 1 \geq 0$ for $\kappa_j < 0$, we can identify ℓ with $-\kappa_j - 1$. Hence the solutions for $\kappa_j < 0$ are $\{j_{-\kappa_j-1}(pr)\}$.

The equation for $b(r)$

The equation satisfied by $b(r)$ is given by,

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) - \frac{(-\kappa_j)(-\kappa_j+1)}{r^2} + \left(E^2 - m_0^2\right)\right] b(r) \quad = \quad 0$$

For $\kappa_j > 0$, a slight rearrangement is needed. Rewrite

$$-\frac{(-\kappa_j)(-\kappa_j+1)}{r^2} \quad = \quad -\frac{(\kappa_j-1)(\kappa_j-1+1)}{r^2}$$

since $\kappa_j - 1 \geq 0$ when $\kappa_j > 0$. Therefore the solutions when $\kappa_j > 0$ are $\{j_{\kappa_j-1}(pr)\}$. In the case when $\kappa_j < 0$, no rearrangement of terms is needed as $-\kappa_j > 0$. And in that case the acceptable solutions are $\{j_{-\kappa_j}(pr)\}$.

Altogether if we denote the a solutions by $\{j_{a_{\kappa_j}}\}$ and the b solutions by $\{j_{b_{\kappa_j}}\}$, then we have that

$$a_{\kappa_j} = \begin{cases} \kappa_j & : \quad \kappa_j > 0 \\ -\kappa_j - 1 & : \quad \kappa_j < 0 \end{cases}$$

$$b_{\kappa_j} = \begin{cases} \kappa_j - 1 & : \quad \kappa_j > 0 \\ -\kappa_j & : \quad \kappa_j < 0 \end{cases}$$

Notice that for all values of $\kappa_j = \ldots, -3, -2, -1, 1, 2, 3, \ldots$, the inequality $a_{\kappa_j} \neq b_{\kappa_j}$ holds. This means that for each value of κ_j, the radial part of each partial wave decomposes into two linearly independent pieces corresponding to the two Φ's, just as in(3.7). Therefore the partial wave expansion for an arbitrary solution to the gravitationally coupled Dirac equation in Minkowski spacetime and in spherical coordinates, is thus given by

$$\Psi = e^{-iEt} r \sin^{1/2}\theta \times \sum_{jm_j\kappa_j} \left[A_{jm_j\kappa_j} j_{a_{\kappa_j}}(pr)\, \Phi^{(+)}_{m_j\kappa_j}(\Omega) + B_{jm_j\kappa_j} j_{b_{\kappa_j}}(pr)\, \Phi^{(-)}_{m_j\kappa_j}(\Omega) \right] \tag{3.9}$$

where $A_{jm_j\kappa_j}$ and $B_{jm_j\kappa_j}$ are complex Fourier coefficients. Notice that we have included the $r\sin^{1/2}\theta$ factor on account of the factorisation ansatz. Also, we note that we have recovered the usual spectrum of the free Dirac operator since either $E \leq -m_0$ or $E \geq m_0$.

Qualitative Analysis Of The Radial Equation

In this section, we perform a simple qualitative analysis of the radial Dirac equation - (3.8).

In the present Minkowskian case, the analysis is particularly simple. However the methods we employ here may be extended to black-hole spacetimes without much modification. From (3.8), we define the matrix $\mathbf{F}(\mathbf{r})$ by,

$$\mathbf{F}(\mathbf{r}) = \begin{bmatrix} -\frac{1+\kappa_j}{r} & m_0 + E \\ m_0 - E & -\frac{1-\kappa_j}{r} \end{bmatrix}$$

and we note that it is continuous on the domain $r \in [0, \infty)$. By the Fundamental Existence Theorem of differential equations [15], we are thus assured of the existence of a unique solution on $r \in [0, \infty)$. On the other hand, the Wronskian for the system (3.8) is given by the Abel-Liouville formula as

$$\begin{aligned} W(r) &= W(r_0) \exp\left[\int_{r_0}^{r} Tr\mathbf{F}(\mathbf{s})\ ds\right] \\ &= W(r_0)\left(\frac{r_0}{r}\right)^2 \end{aligned}$$

Thus if $W(r_0) \neq 0$ for some positive r_0, then $W(r) \neq 0$ for all r. This means that we have 2 linearly independent solutions to the system. In particular, fixing free-particle boundary conditions at $r \to \infty$ and demanding regularity at $r = 0$ freezes out the exact solutions that we previously obtained. To see this better, we recall that from Sturm-Liouville theory [15], the second-order differential equation,

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) - \frac{\ell(\ell+1)}{r^2} + p^2\right] f(r) = 0$$

has two linearly independent solutions - the spherical Bessel functions, $\{j_\ell\}$ which is regular at the origin and the spherical Neumann functions, $\{n_\ell\}$ which diverges at the origin. Our boundary conditions restricts us to the spherical Bessel functions. Indeed, for the Sturm-Liouville problem with solutions $y_1(x)$ and $y_2(x)$,

$$\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right] + Q(x)y \;\; = \;\; 0$$

the Wronskian of the solutions (not to be confused with the Wronskian of a 2×2 system) may be defined as $W[y_1(x), y_2(x)] = \begin{vmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{vmatrix}$ and it satisfies the relation

$$W[y_1(x), y_2(x)]P(x) \;\; = \;\; \text{constant.}$$

In our case, it is clear that $P = r^2$ and $Q = -\ell(\ell+1) + p^2r^2$ and it is gratifying to know that this relation is indeed satisfied since $W[j_\ell(pr), n_\ell(pr)]r^2 = \frac{1}{p} = \text{constant}$.

Normalisation And Inner Product

Before we proceed with the quantisation of the Dirac theory in the gravitationally coupled Minkowskian case, we need to define a sensible inner product and normalisation for the mode solutions that we have found. First of all, we recognise that the solution space $\{j_\ell\}$ does not constitute an L^2 Hilbert space; indeed, the standard normalisation for the spherical Bessel functions are,

$$\begin{aligned}
\int_0^\infty r^2 j_\ell(kr) j_{\ell'}(k'r)\; dr &= \frac{\pi}{2k^2}\delta(k-k')\delta_{\ell\ell'} \\
\int_0^\infty k^2 j_\ell(kr) j_{\ell'}(kr')\; dk &= \frac{\pi}{2r^2}\delta(r-r')\delta_{\ell\ell'}
\end{aligned}$$

as it should be since our modes represent free spherical waves. Consequently we can only hope to get delta-function normalisations. With this in mind, we recall that the standard normalisation and inner-product for spinors is given by,

$$\begin{aligned} \|\Psi\|^2 &= \int \Psi^\dagger \Psi \; d(vol)_{3-space} \\ \langle \Psi_1, \Psi_2 \rangle &= \int \Psi_1^\dagger \Psi_2 \; d(vol)_{3-space} \end{aligned}$$

In the cases that we will be dealing with, the bispinor Ψ will always have the form $\Psi = e^{1/2} h(r,\theta)\Phi$ as per the factorisation ansatz that we have discussed in Chapter 2. Consequently, in a very precise sense, all information regarding the quantum state is carried in the bispinor Φ with the $he^{1/2}$ factor as "excess baggage". Therefore, we define the integration measure for the inner-product as,

$$d\mu = d(vol)_{3-space} \; h^{-2}(r,\theta)e^{-1} \tag{3.10}$$

so that we have the following simplification,

$$\begin{aligned} \langle \Psi_1, \Psi_2 \rangle &= \int \Psi_1^\dagger \Psi_2 \; d\mu \\ &= \int \left(he^{1/2}\Phi_1\right)^\dagger \left(he^{1/2}\Phi_2\right) \; d\mu \\ &= \int \Phi_1^\dagger \Phi_2 \; d(vol)_{3-space} \end{aligned}$$

Clearly this approach to the inner-product may be generalised to the cases when the underlying spacetime is a black-hole. Having determined the inner-product of choice and with the help of (3.9) plus the completeness relation satisfied by the spinor harmonics, it is easy to see that the correct, normalised eigenstate of the full Dirac Hamiltonian is given by

$$\Psi_{Ejm_j\kappa_j} = e^{-iEt} r \sin^{1/2}\theta \times \frac{1}{\sqrt{2}} \left[j_{a_{\kappa_j}}(pr)\, \Phi^{(+)}_{m_j\kappa_j}(\Omega) + j_{b_{\kappa_j}}(pr)\, \Phi^{(-)}_{m_j\kappa_j}(\Omega) \right] \tag{3.11}$$

and is normalised according to,

$$\begin{aligned} \left\langle \Psi_{Ejm_j\kappa_j}, \Psi_{E'j'm_{j'}\kappa_{j'}} \right\rangle &= \int \Psi^{\dagger}_{Ejm_j\kappa_j} \Psi_{E'j'm_{j'}\kappa_{j'}}\, d\mu \\ &= e^{i(E-E')t} \frac{\pi}{2p^2} \delta(p-p') \delta_{jj'} \delta_{m_j m_{j'}} \delta_{\kappa_j \kappa_{j'}} \end{aligned}$$

But $p^2 = E^2 - m_0^2$ so that $\frac{dp}{dE} = \frac{E}{p}$ and $\delta\left[p(E) - p'(E')\right] = \frac{\delta(E-E')}{\left|\frac{dp}{dE}\right|}$ and thus,

$$\left\langle \Psi_{Ejm_j\kappa_j}, \Psi_{E'j'm_{j'}\kappa_{j'}} \right\rangle = \frac{\pi}{2\,|E|\,p} \delta(E-E') \delta_{jj'} \delta_{m_j m_{j'}} \delta_{\kappa_j \kappa_{j'}} \tag{3.12}$$

From this normalisation of the eigenstates, we can immediately read off the *spectral measure* (density of states), $\rho(E)dE = \frac{2|E|p}{\pi}$, and verify the spectral expansion,

$$\int \sum_{jm_j\kappa_j} \Psi_{Ejm_j\kappa_j}(\vec{r}) \otimes \Psi^{\dagger}_{Ejm_j\kappa_j}(\vec{r'})\ \rho(E)dE = \delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')$$

We are now ready to perform a quantisation of the theory that we have developed so far.

Quantisation

In order to quantise the theory that we have developed, it is necessary to specify positive and negative frequency modes with respect to the timelike direction, t. But because the bispinor $\Phi^{(+)}_{m_j\kappa_j}$ only has upper components whereas $\Phi^{(-)}_{m_j\kappa_j}$ only has lower components, we may define the required normalised modes as,

$$
\begin{aligned}
u_{jm_j\kappa_j} &= \sqrt{\frac{2}{\pi}} j_{a_{\kappa_j}}(pr)\Phi^{(+)}_{m_j\kappa_j}(\Omega)\, pr\sin^{1/2}\theta \qquad \text{(positive frequency)} \\
v_{jm_j\kappa_j} &= \sqrt{\frac{2}{\pi}} j_{b_{\kappa_j}}(pr)\Phi^{(-)}_{m_j\kappa_j}(\Omega)\, pr\sin^{1/2}\theta \qquad \text{(negative frequency)}
\end{aligned}
$$

The normalisation of these modes are chosen so that,

$$
\begin{aligned}
\int d\mu\ \overline{u}_{jm_j\kappa_j}(p)u_{j'm_{j'}\kappa_{j'}}(p') &= \delta_{jj'}\delta_{m_jm_{j'}}\delta_{\kappa_j\kappa_{j'}}\delta(p-p') \\
\int d\mu\ \overline{v}_{jm_j\kappa_j}(p)v_{j'm_{j'}\kappa_{j'}}(p') &= -\,\delta_{jj'}\delta_{m_jm_{j'}}\delta_{\kappa_j\kappa_{j'}}\delta(p-p')
\end{aligned}
$$

where the bar over the spinors denotes Dirac adjoints i.e $\overline{\psi} = \psi^\dagger\gamma^0$. With this, we proceed to expand a wave packet, ψ, as follows

$$
\psi = \int_0^\infty dp \sum_{jm_j\kappa_j} \left[e^{-i\omega_p t}u_{jm_j\kappa_j}(p)b_{jm_j\kappa_j}(p) + e^{i\omega_p t}v_{jm_j\kappa_j}(p)d^\dagger_{jm_j\kappa_j}(p)\right] \tag{3.13}
$$

where $\omega_p = \sqrt{p^2+m_0^2}$. In order to quantise the theory, the coefficients $b_{jm_j\kappa_j}(p)$ and $d^\dagger_{jm_j\kappa_j}(p)$ must be elevated to be operators. And (3.13) must then be interpreted as the quantum expansion of the *field operator*, ψ.

If we accept this implementation of the quantum principle, the next stage would be to postulate the appropriate algebra for these operators. To this end, we follow the usual prescription and adopt the creation and annihilation operator algebra for these operators,

$$
\left\{b_{jm_j\kappa_j}(p), b^\dagger_{j'm_{j'}\kappa_{j'}}(p')\right\} = \left\{d_{jm_j\kappa_j}(p), d^\dagger_{j'm_{j'}\kappa_{j'}}(p')\right\}
$$

$$
\begin{aligned}
&= \delta_{jj'}\delta_{m_j m_{j'}}\delta_{\kappa_j\kappa_{j'}}\delta(p-p') && (3.14)\\
\left\{b_{jm_j\kappa_j}(p), d_{j'm_{j'}\kappa_{j'}}(p')\right\} &= \left\{b_{jm_j\kappa_j}(p), d^\dagger_{j'm_{j'}\kappa_{j'}}(p')\right\} \\
&= 0 && (3.15)
\end{aligned}
$$

Using this algebra and (3.13), it is in fact rather easy to prove that the equal time anti-commutation relation for the field operator ψ is indeed satisfied,

$$
\{\psi(r,\theta,\phi), \psi(r',\theta',\phi')\}_{\text{Equal Time}} = \delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi') \qquad (3.16)
$$

The particle concept is introduced by defining the appropriate vacuum state to be the state $|0\rangle$ satisfying $b_{jm_j\kappa_j}(p)\,|0\rangle = d_{jm_j\kappa_j}(p)\,|0\rangle = 0$. These states are particle states of definite angular momenta and energy. One can proceed further to define the propagator etc, but we shall refrain from doing that here. Suffice it to say that we have a well-defined quantum field theory of Dirac particles in the gravitationally coupled Minkowski spacetime, in spherical coordinates. In the next chapter, we shall construct a similar theory for the Schwarzschild black-hole.

CHAPTER 4 : THE DIRAC EQUATION IN SCHWARZSCHILD SPACETIME

In this chapter, we shall construct a fermion quantum field theory living on the black-hole, Schwarzschild, spacetime. Our approach will be different from the one in [4] because we will systematically employ the factorisability ansatz as discussed in Chapter 2. The case of the Schwarzschild black-hole is an important one; it being the "hydrogen atom" of black-hole studies. The spherical symmetry of the Schwarzschild black-hole implies that the angular components of the solutions to the Dirac equation are precisely the same spinor harmonics used in the last chapter. This is a very useful simplification as only the radial wavefunctions are different from the Minkowskian case. It turns out, as we shall see later on, that the radial wavefunction cannot be expressed as a closed-form, analytic solution. However this is not a hindrance to explicit calculations. Indeed, we shall compute the spectrum of neutrino emission from Schwarzschild black-holes via the Hawking process, as well as perform a partial-wave analysis of fermion scattering amplitudes using the solutions presented here, in later chapters.

We begin our analysis with the Schwarzschild line element written in Schwarzschild coordinates,

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

Observe that in the correspondence limit, $\frac{2M}{r} \ll 1$, we ought to recover the Minkowskian results. In particular, this should also apply to the solutions of the Dirac equation. From the line element, we are motivated to choose as basis one-forms, the following set : $\theta^0 = \left(1 - \frac{2M}{r}\right)^{1/2} dt$, $\theta^1 = \left(1 - \frac{2M}{r}\right)^{-1/2} dr$, $\theta^2 = rd\theta$ and $\theta^3 = r \sin\theta d\phi$. Notice that this basis of one-forms is only valid outside the event horizon at $r = 2M$ because of the coordinate singularities encountered there. In fact, it is impossible to define a singularity free basis of one-forms when using the Schwarzschild coordinates. This is a non-trivial issue as we shall see later on. We need the spin-connection, and so by using the formula $2\omega_{ab} = \theta^c i_a i_b d\theta_c + i_b d\theta_a - i_a d\theta_b$, we compute the spin connection one-form to be,

$$\begin{aligned}
(\omega)_{01} &= \frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1/2} \theta^0 \\
(\omega)_{02} &= (\omega)_{03} \;=\; 0, \\
(\omega)_{12} &= \frac{1}{r}\left(1 - \frac{2M}{r}\right)^{1/2} \theta^2, \\
(\omega)_{13} &= \frac{1}{r}\left(1 - \frac{2M}{r}\right)^{1/2} \theta^3, \\
(\omega)_{23} &= \frac{\cot\theta}{r}\theta^3
\end{aligned} \tag{4.1}$$

and with this, it is easy to verify that $\frac{1}{4}\epsilon^{abcd}(\omega_{ab})_c\gamma_d\gamma_5 = 0$. We still need to specify a vierbein set before we can apply (2.11) for the factorisation ansatz. To this end, our task is almost trivial since the metric is diagonal; we choose the diagonal set ${e^0}_0 = \left(1 - \frac{2M}{r}\right)^{1/2}$, ${e^1}_1 = \left(1 - \frac{2M}{r}\right)^{-1/2}$, ${e^2}_2 = r$ and ${e^3}_3 = r\sin\theta$ so that $e = r^2 \sin\theta$. Then (2.10) and (2.11) yields

$$f = \left(1 - \frac{2M}{r}\right)^{1/4} r \sin^{1/2}\theta \tag{4.2}$$

Consequently the factorisation ansatz reduces our task to one of solving the reduced equation, $i\gamma^c E_c{}^\mu \partial_\mu \Phi - m_0\Phi = 0$ where $\Psi = f\Phi$ solves the full Dirac equation in Schwarzschild spacetime. This is pretty much the same as in the Minkowskian case. In fact, the spherical symmetry of Schwarzschild spacetime implies that a separation of variables as in the Minkowski case, applies identically to the present situation. Indeed, if we define $\vec{\nabla} = E_k{}^\mu \partial_\mu = \hat{r}\left(1 - \frac{2M}{r}\right)^{1/2} \frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$, then just as (3.6), we may write

$$\begin{aligned} H_0\Phi = i\left(1 - \frac{2M}{r}\right)^{-1/2} \frac{\partial}{\partial t}\Phi = \\ - i(\vec{\alpha}\cdot\hat{r})\left[\left(1 - \frac{2M}{r}\right)^{1/2}\frac{\partial}{\partial r} + \frac{1}{r} - \frac{1}{r}\beta K\right]\Phi + m_0\beta\Phi \end{aligned} \tag{4.3}$$

Following the Minkowskian example, we choose to work in the Dirac representation of gamma and Dirac matrices, and we set each partial wave of (4.3) to be

$$\psi_{m_j\kappa_j} = f^+_{m_j\kappa_j}(r,t)\Phi^{(+)}_{m_j\kappa_j} + f^-_{m_j\kappa_j}(r,t)\Phi^{(-)}_{m_j\kappa_j} \tag{4.4}$$

When (4.4) is substituted into (4.3) along with the time dependence

$$\begin{aligned} f^+_{m_j\kappa_j}(r,t) &= e^{-i\omega t}A(r) \\ f^-_{m_j\kappa_j}(r,t) &= e^{-i\omega t}B(r) \end{aligned}$$

we get the radial Dirac equation - this time on a genuine black-hole background,

$$\frac{d}{dr}\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{1 - \frac{2M}{r}}\begin{bmatrix} -\frac{1+\kappa_j}{r}\left(1 - \frac{2M}{r}\right)^{1/2} & m_0\left(1 - \frac{2M}{r}\right)^{1/2} + \omega \\ m_0\left(1 - \frac{2M}{r}\right)^{1/2} - \omega & -\frac{1-\kappa_j}{r}\left(1 - \frac{2M}{r}\right)^{1/2} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} \tag{4.5}$$

Clearly (4.5) exhibits features which are far more complicated than those of (3.8) - yet it reduces to (3.8) in the correspondence limit when $\frac{2M}{r} \ll 1$. In the next section, we shall present a detailed study of this equation.

The Radial Dirac Equation In Schwarzschild Spacetime

The radial Dirac equation in Schwarzschild spacetime, (4.5), stands in stark contrast to the similar equation but in Minkowski spacetime - (3.8). Much of the analysis that we will perform can only be of a qualitative nature, since (4.5) is analytically complicated.

The first thing we should note about the radial equation, is that it is singular at $r = 2M$ and is well-defined only for $r > 2M$. This is directly related to our choice of coordinates and we will discuss this point later. Define the matrix,

$$\mathbf{F}(\mathbf{r}) = \frac{1}{1-\frac{2M}{r}} \begin{bmatrix} -\frac{1+\kappa_j}{r}\left(1-\frac{2M}{r}\right)^{1/2} & m_0\left(1-\frac{2M}{r}\right)^{1/2}+\omega \\ m_0\left(1-\frac{2M}{r}\right)^{1/2}-\omega & -\frac{1-\kappa_j}{r}\left(1-\frac{2M}{r}\right)^{1/2} \end{bmatrix}$$

where it is continuous on $r > 2M$. This means that solutions to the radial equation exist and are continuous, on this semi-infinite interval. Furthermore, the Wronskian of the system, $W(r)$, is given by the Abel-Liouville formula as,

$$\begin{aligned} W(r) &= W(r_0)e^{\int_{r_0}^{r} Tr\mathbf{F}(\mathbf{s})\, ds} \\ &= \left[r - M + \sqrt{(r-M)^2 - M^2}\right]^{-2} \end{aligned}$$

so that $W(r) = M^{-2}$ at the event horizon and is non-vanishing for all $r \geq 2M$. This implies that the solutions to (4.5) are linearly independent for all $r \geq 2M$. Notice that $W(r)$ becomes imaginary when $r < 2M$ - a fact that is related to the choice of coordinates.

We can convert (4.5) into a pair of complicated second-order differential equations for $A(r)$ and $B(r)$ via a simple, albeit tedious, elimination process. The equation for $A(r)$, for example, may be written as

$$
\begin{aligned}
&\left(1-\frac{2M}{r}\right)\frac{d^2A}{dr^2} \\
&\quad+\left\{\frac{2M}{r^2}+\frac{2}{r}\left(1-\frac{2M}{r}\right)^{1/2}-\frac{\left(1-\frac{2M}{r}\right)^{1/2}}{m_0\left(1-\frac{2M}{r}\right)^{1/2}+\omega}\frac{m_0M}{r^2}\right\}\frac{dA}{dr} \\
&\quad-\left\{\frac{1+\kappa_j}{r^2}\left[\left(1-\frac{2M}{r}\right)^{1/2}-\frac{M}{r}\left(1-\frac{2M}{r}\right)^{-1/2}+\frac{\frac{m_0M}{r}}{m_0\left(1-\frac{2M}{r}\right)^{1/2}+\omega}\right]\right. \\
&\quad\left.+\frac{m_0^2\left(1-\frac{2M}{r}\right)-\omega^2}{\left(1-\frac{2M}{r}\right)}-\frac{1-\kappa_j{}^2}{r^2}\right\}A \;=\; 0
\end{aligned}
\tag{4.6}
$$

which is a messy equation indeed. However, it is easy to check that the potential term (the last term which multiplies $A(r)$) is bounded from below on $2M \leq r \leq \infty$ so that the radial wavefunctions are meaningful there. The preceding equation is of the form $a_0(r)\frac{d^2y}{dr^2}+a_1(r)\frac{dy}{dr}+a_2(r)y=0$ which can be put into the self-adjoint Sturm-Liouville format if the following two conditions are met, (i) a_0, a_1 and a_2 are continuous on the domain of the problem and (ii) $a_0 \neq 0$ throughout the domain of the problem. The conversion to a Sturm-Liouville problem can be effected by the multiplication factor $\frac{1}{a_0}\exp\left[\int \frac{a_1}{a_0}\,dr\right]$ to obtain, $\frac{d}{dr}\left[P(r)\frac{dy}{dr}\right]+Q(r)y=0$ with $P(r)=\exp\left[\int \frac{a_1}{a_0}\,dr\right]$ and $Q(r)=\frac{a_2}{a_0}P(r)$. In our case, $a_0(r)=1-\frac{2M}{r}$ vanishes at $r=2M$ so that the preceding transformation will yield a *singular Sturm-Liouville problem* [15] since $P(r)$ vanishes at the end-point, $r = 2M$. In particular, we have that

$$P(r) \;=\; \frac{\left(1-\frac{2M}{r}\right)\left[r-M+\sqrt{(r-M)^2-M^2}\right]^2}{m_0\left(1-\frac{2M}{r}\right)^{1/2}+\omega}$$

which clearly shows that $P(r = 2M) = 0$ and $P(r) \to \frac{4r^2}{m_0+\omega}$ when $\frac{2M}{r} \ll 1$.

To gain more information regarding the behaviour of the radial wave functions, we need to check that it obeys the correspondence principle; it must behave like the Minkowskian radial wave functions far away from the black-hole. To this end, we return to (4.5) and put $\frac{2M}{r} \ll 1$. Then we note that in this limit,

$$\begin{aligned}\left(1-\frac{2M}{r}\right)^{-1/2} &= 1+\frac{M}{r}+O\left(\frac{M^2}{r^2}\right)\\ \left(1-\frac{2M}{r}\right)^{-1} &= 1+\frac{2M}{r}+O\left(\frac{M^2}{r^2}\right)\end{aligned}$$

so that (4.5) reduces to,

$$\frac{d}{dr}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -\frac{1+\kappa_j}{r}\left(1+\frac{M}{r}\right) & m_0+\omega+\frac{M}{r}(m_0+2\omega) \\ m_0-\omega+\frac{M}{r}(m_0-2\omega) & -\frac{1-\kappa_j}{r}\left(1+\frac{M}{r}\right) \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}$$

and upon elimination of $B(r)$ (dropping terms of $O\left(\frac{M^2}{r^2}\right)$ and higher) we get,

$$\begin{aligned}\frac{d^2A}{dr^2}+\frac{2}{r}\frac{dA}{dr}-\frac{\kappa_j(\kappa_j+1)}{r^2}A+(\omega^2-m_0^2)A \;=&\\ -\frac{2M}{r^2}\frac{dA}{dr}+\frac{2M}{r}\frac{\kappa_j(\kappa_j+1)}{r^2}A+\frac{M}{r}(m_0^2-4\omega^2)A&\end{aligned}$$

which is nothing more than the spherical Bessel equation on the left hand side plus inhomogeneous terms on the right hand side. This clearly demonstrates that the Schwarzschild radial wave functions are asymptotic to the Minkowskian ones, far away from the black-hole. Furthermore, there is an obvious one-to-one correspondence between the Schwarzschild radial wave functions and the Minkowskian wave functions. The κ_j dependent term on the right hand side ensures this.

Consequently we expect there to be two linearly independent solutions to the Schwarzschild radial equation; $\mathcal{Z}_{a_{\kappa_j}}(pr)$ and $\mathcal{Z}_{b_{\kappa_j}}(pr)$ are the acceptable solutions that are *regular* at the event horizon and are aymptotic to $j_{a_{\kappa_j}}(pr)$ and $j_{b_{\kappa_j}}(pr)$, respectively, and $\mathcal{Q}_\ell$ - which *diverges* at the event horizon. This is very much like the spherical Bessel and spherical Neumann functions in the Minkowskian case with $\omega^2 = p^2 + m_0^2$.

In fact, we can evaluate the Wronskian of $\mathcal{Z}_\ell$ and $\mathcal{Q}_\ell$ *exactly* by using the formula,

$$W[\mathcal{Z}_\ell, \mathcal{Q}_\ell]P(r) = (\text{constant})$$

to find that

$$W[\mathcal{Z}_\ell, \mathcal{Q}_\ell] = (\text{constant})\frac{m_0\left(1-\frac{2M}{r}\right)^{1/2} + \omega}{\left(1-\frac{2M}{r}\right)\left[r - M + \sqrt{(r-M)^2 - M^2}\right]^2}$$

When $\frac{2M}{r} \to 0$, $W[\mathcal{Z}_\ell(pr), \mathcal{Q}_\ell(pr)] \to W[j_\ell(pr), n_\ell(pr)] = \frac{1}{pr^2} = \frac{1}{r^2\sqrt{\omega^2 - m_0^2}}$ and hence since,

$$\lim_{\frac{2M}{r}\to 0} P(r) = \frac{4r^2}{m_0 + \omega}$$

we see that the constant is given by

$$(\text{constant}) = \frac{4}{m_0 + \omega}\frac{1}{p}$$

Finally, we obtain

$$
\begin{aligned}
W[\mathcal{Z}_\ell(pr), \mathcal{Q}_\ell(pr)] \; &= \\
&\frac{1}{p}\frac{4}{m_0+\omega}\frac{m_0\left(1-\frac{2M}{r}\right)^{1/2}+\omega}{\left(1-\frac{2M}{r}\right)\left[r-M+\sqrt{(r-M)^2-M^2}\right]^2}
\end{aligned}
$$

which diverges as $\left(1-\frac{2M}{r}\right)^{-1}$ as $r \to 2M$. For the purpose of computing scattering amplitudes later on, we shall require the asymptotic behaviour of $\mathcal{Z}_\ell(pr)$ near the event horizon, $r = 2M$. For this purpose, we need the *tortoise coordinate*, $r^* = r + 2M \log\left|\frac{r}{2M} - 1\right|$ which is such that its derivative operator of first and second order may be written as,

$$
\begin{aligned}
\frac{d}{dr^*} &= \left(1-\frac{2M}{r}\right)\frac{d}{dr} \\
\frac{d^2}{dr^{*2}} &= \left(1-\frac{2M}{r}\right)^2\frac{d^2}{dr^2} + \left(1-\frac{2M}{r}\right)\frac{2M}{r^2}\frac{d}{dr}.
\end{aligned}
$$

Multiplying the second order differential equation for $A(r)$, (4.6), by an extra factor of $\left(1-\frac{2M}{r}\right)$ and simplifying using the preceding relations yields,

$$
\begin{aligned}
\frac{d^2A}{dr^{*2}} + \omega^2 A + \left(1-\frac{2M}{r}\right)^{3/2}\{\text{regular terms at } r=2M\}\frac{dA}{dr} \\
+ \left(1-\frac{2M}{r}\right)^{1/2}\{\text{regular terms at } r=2M\}\,A \;&=\; 0
\end{aligned}
$$

showing that for the regular solution *only*, $\mathcal{Z}_\ell(pr)$, the asymptotic behaviour at $r = 2M$ is,

$$
\mathcal{Z}_\ell(pr) \sim e^{\pm i\omega r^*} \qquad \text{as } r \to 2M \tag{4.7}
$$

Of course, this analysis does not apply to the solution which diverges at the event horizon; for there, we cannot drop the A and $\frac{dA}{dr}$ terms in the preceding differential equation, as the event horizon is approached.

Normalisation And Measure

Having elucidated the properties of the acceptable radial wave functions, we can now write down the most general solution to the full Dirac equation, that is regular at the event horizon,

$$\begin{aligned}\Psi &= e^{-i\omega t}\left(1-\frac{2M}{r}\right)^{1/4} r\sin^{1/2}\theta \\ &\times \sum_{jm_j\kappa_j}\left[A_{jm_j\kappa_j}\mathcal{Z}_{a_{\kappa_j}}(pr)\Phi^{(+)}_{m_j\kappa_j}(\Omega)+B_{jm_j\kappa_j}\mathcal{Z}_{b_{\kappa_j}}(pr)\Phi^{(-)}_{m_j\kappa_j}(\Omega)\right]\end{aligned} \tag{4.8}$$

where we have inserted the factor of $\left(1-\frac{2M}{r}\right)^{1/4} r\sin^{1/2}\theta$ on account of the factorisation ansatz. Clearly this solution approaches the Minkowskian one when $\frac{2M}{r}\ll 1$ as required by the correspondence principle. In particular, the spectrum of the free Dirac operator is preserved by virtue of the relation, $\omega^2 = p^2 + m_0^2$.

Following the method outlined in the Minkowskian case, we can define the inner product for the solution space, as follows

$$\langle\Psi_1,\Psi_2\rangle = \int \Psi_1^\dagger\Psi_2 \; d\mu$$

where $d\mu = d(vol)_{3-space}h^{-2}e^{-1}$ so that

$$\begin{aligned}\langle\Psi_1,\Psi_2\rangle &= \int \Phi_1^\dagger\Phi_2 \; d(vol)_{3-space} \\ &= \int \Phi_1^\dagger\Phi_2 \;\; r^2\sin\theta\left(1-\frac{2M}{r}\right)^{-1/2} dr\wedge d\theta\wedge d\phi\end{aligned}$$

where $\Psi = f\Phi = he^{1/2}\Phi$ as discussed in Chapter 2. With this in mind, we *postulate* the following normalisation for the radial wave function which is to be interpreted strictly in the sense of the theory of distributions [16],

$$\begin{aligned}\int_{2M}^{\infty} \mathcal{Z}_{\ell}(kr)\mathcal{Z}_{\ell'}(k'r)\; r^2 \left(1-\frac{2M}{r}\right)^{-1/2} dr &= \frac{\pi}{2k^2}\delta(k-k')\delta_{\ell\ell'} \\ \int_0^{\infty} \mathcal{Z}_{\ell}(kr)\mathcal{Z}_{\ell'}(kr')\; k^2 dk &= \frac{\pi}{2r^2}\left(1-\frac{2M}{r}\right)^{-1/2}\delta(r-r')\delta_{\ell\ell'}\end{aligned}$$

just as for the normalisation of the spherical Bessel function. Using this normalisation, we can define the properly normalised eigenstate of the full Dirac Hamiltonian as,

$$\begin{aligned}\Psi_{\omega j m_j \kappa_j} &= e^{-i\omega t}\left(1-\frac{2M}{r}\right)^{1/4} r\sin^{1/2}\theta \\ &\times\frac{1}{\sqrt{2}}\left[\mathcal{Z}_{a_{\kappa_j}}(pr)\Phi^{(+)}_{m_j\kappa_j}(\Omega)+\mathcal{Z}_{b_{\kappa_j}}(pr)\Phi^{(-)}_{m_j\kappa_j}(\Omega)\right] \qquad (4.9)\end{aligned}$$

and where

$$\left\langle \Psi_{\omega j m_j \kappa_j}, \Psi_{\omega' j' m_{j'} \kappa_{j'}}\right\rangle = \frac{\pi}{2|\omega|p}\delta(\omega-\omega')\delta_{jj'}\delta_{m_j m_{j'}}\delta_{\kappa_j\kappa_{j'}} \qquad (4.10)$$

so that we may read off the spectral measure (density of states), $\rho(\omega)d\omega = \frac{2|\omega|p}{\pi}d\omega$, and obtain the spectral expansion,

$$\int\sum_{j m_j \kappa_j}\Psi_{\omega j m_j \kappa_j}(\vec{r})\otimes\Psi^{\dagger}_{\omega j m_j \kappa_j}(\vec{r'})\;\rho(\omega)d\omega = \delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi').$$

We are now ready to quantise the Dirac theory in Schwarzschild spacetime.

Quantisation

Just as in the Minkowskian case, we must first define positive and negative frequency modes. But this is easy since we can use the Minkowskian theory as a precedent. Indeed, we choose the following positive ($u_{jm_j\kappa_j}$) and negative ($v_{jm_j\kappa_j}$) frequency modes :

$$
\begin{aligned}
u_{jm_j\kappa_j}(p) &= \sqrt{\frac{2}{\pi}} p \mathcal{Z}_{a_{\kappa_j}}(pr) \Phi^{(+)}_{m_j\kappa_j}(\Omega) \left(1 - \frac{2M}{r}\right)^{1/4} r \sin^{1/2}\theta \\
v_{jm_j\kappa_j}(p) &= \sqrt{\frac{2}{\pi}} p \mathcal{Z}_{b_{\kappa_j}}(pr) \Phi^{(-)}_{m_j\kappa_j}(\Omega) \left(1 - \frac{2M}{r}\right)^{1/4} r \sin^{1/2}\theta
\end{aligned}
$$

As before, these modes are normalised as

$$
\begin{aligned}
\int \overline{u}_{jm_j\kappa_j}(p) u_{j'm_{j'}\kappa_{j'}}(p')\, d\mu &= \delta(p-p')\delta_{jj'}\delta_{m_jm_{j'}}\delta_{\kappa_j\kappa_{j'}} \\
\int \overline{v}_{jm_j\kappa_j}(p) v_{j'm_{j'}\kappa_{j'}}(p')\, d\mu &= -\,\delta(p-p')\delta_{jj'}\delta_{m_jm_{j'}}\delta_{\kappa_j\kappa_{j'}}
\end{aligned}
$$

so that we can use them to define the quantum field operator,

$$
\psi = \int_0^\infty dp \sum_{jm_j\kappa_j} \left[e^{-i\omega_p t} u_{jm_j\kappa_j}(p) b_{jm_j\kappa_j}(p) + e^{i\omega_p t} v_{jm_j\kappa_j}(p) d^\dagger_{jm_j\kappa_j}(p) \right]
$$

where $b_{jm_j\kappa_j}(p)$ and $d^\dagger_{jm_j\kappa_j}(p)$ are operators satisfying the usual fermionic creation and annihilation operator algebra (see equations (3.13,14,15)). From here, it is not difficult to prove that ψ also satisfies the usual equal time anti-commutation relation (see equation (3.16)). Of course, we can also define the vacuum and other particle states exactly in the same fashion as per the Minkowskian case - and a well defined quantum field theory has been constructed.

Beyond The Event Horizon

In the previous sections, we have repeatedly observed that our choice of coordinates restricts the construction of the Dirac theory, to regions outside the event horizon ($r > 2M$). Here we will discuss the issue of extending the theory across the horizon.

First of all, in order to describe a Hamiltonian evolution of quantum modes, we need to define a time-like Killing vector field. Or put in simple terms, we need to define the notion of a temporal parameter. Of course, this can usually be done *locally* on the spacetime manifold of interest and as such, poses no problem insofar as the *local* evolution of quantum modes is concerned. However, the structure of general relativistic spacetimes [17,18] does not guarantee the existence of a *global* time-like Killing vector field. And even if such vector fields exist, it is in general unclear how we should select a particular one. Moreover, there is also the problem of interpretation since the coordinate representation of such vector fields may be very complicated and may involve the spatial coordinates as well. In short, the Hamiltonian evolution of quantum modes is not guaranteed globally. Such is the situation with the Schwarzschild black-hole - it has no global time-like Killing vector fields. In part, this is due to the event horizon which acts as an obstruction to extending the time-like vector field that we have chosen, into the horizon. Consequently, we cannot extend the Dirac theory across the event horizon, using Schwarzschild coordinates. The time-like Killing vector fields that we obtain from this choice of coordinates can only describe the temporal evolution of quantum modes, outside the horizon.

But what about other coordinate systems ? These other systems all involve an *analytic extension* [18] of the manifold in question. On the one hand, we have the Kruskal-Szekeres maximal analytic extension, and on the other hand, we have the Eddington-Finkelstein extension [17,18,19]. Unfortunately, the Kruskal-Szekeres extension includes what we consider to be a *non-physical* white-hole. This is, of course, a personal judgement call. But until such time when there is sufficient evidence to take this maximal extension seriously, it will be prudent to exclude the region which constitutes the white-hole. And we are left with the Eddington-Finkelstein extension, which fortunately, does not include extraneous features. In this case there are, strictly speaking, two coordinate systems in question; the ingoing and outgoing Eddington-Finkelstein systems [18]. Because they are so closely related, it suffices to study just one of them. Since we are particularly interested in gravitational collapse, we shall select the ingoing Eddington-Finkelstein system.

The ingoing Eddington-Finkelstein line element, regular at the horizon reads as,

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt'^2 - \frac{4M}{r} dt' dr - \left(1 + \frac{2M}{r}\right) dr^2 - r^2 d\Omega^2$$

and it can be obtained from the Schwarzschild line element by the analytic continuation, $t' = t + 2M \log(r - 2M)$. Observe that it is *not* of the form expected of a stationary axisymmetric solution due to the g_{01} term, $-\frac{4M}{r} dt' dr$. This indicates that the gravitational field within the horizon is dynamical - there are no static observers within the horizon. Actually, this fact can be deduced directly from the Schwarzschild metric and so, it should come as no surprise. What happens then to our factorisability ansatz ? The answer is that it fails to apply here. However, the investigation of the Dirac theory on the Eddington-Finkelstein extended manifold, is still a worthwhile pursuit. At the very least, we can learn the effects of an event horizon on the structure of quantum field theory.

There is a legitimate temptation to search for another coordinate system which somehow, diagonalises, the Eddington-Finkelstein metric. But we must be careful. Firstly, a coordinate transformation like that must satisfy various smoothness and differentiability conditions. Otherwise, we could just end up with the same analytic continuation that takes us back to the diagonal Schwarzschild metric - and that, to say the least, would be hilarious indeed ! Unfortunately, it can be shown that it is impossible to find a coordinate transformation that will diagonalise the Eddington-Finkelstein metric. Therefore the issue of reducing the Eddington-Finkelstein line element to a diagonal form is a hopelessly lost cause. It simply cannot be done. We now present a short proof of this strange fact.

Begin with the Eddington-Finkelstein metric written in matrix format,

$$g_{\mu\nu} = \begin{bmatrix} 1 - \frac{2M}{r} & -\frac{2M}{r} & 0 & 0 \\ -\frac{2M}{r} & -(1 + \frac{2M}{r}) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{bmatrix}$$

Let us look for a coordinate transformation that could possibly diagonalise this metric tensor. We shall later show that such a coordinate transformation does not exist. It is clear that we only need to consider those transformations that diagonalise the block,

$$\begin{bmatrix} (1 - \frac{2M}{r}) & -\frac{2M}{r} \\ -\frac{2M}{r} & -(1 + \frac{2M}{r}) \end{bmatrix}$$

whose determinant is -1 - so that the block is of $SO(1,1)$ type. Therefore we are looking for a transformation whose Jacobian acts on the previous block by a similarity transformation, reducing it to a diagonal form. And the method makes itself clear from the meaning of the previous statement. We first look for its eigenvalues and they are,

$$\begin{aligned}
\lambda_+(r) &= \sqrt{1+\frac{4M^2}{r^2}}-\frac{2M}{r} > 0 \\
\lambda_-(r) &= -\sqrt{1+\frac{4M^2}{r^2}}-\frac{2M}{r} < 0.
\end{aligned}$$

Having found the eigenvalues of of the block matrix, we proceed to write down the action of J on the block matrix,

$$J^t \begin{bmatrix} (1-\frac{2M}{r}) & -\frac{2M}{r} \\ -\frac{2M}{r} & -(1+\frac{2M}{r}) \end{bmatrix} J = \begin{bmatrix} \lambda_+(R) & 0 \\ 0 & \lambda_-(R) \end{bmatrix}$$

where J^t is the transpose of J and r and t are functions of R, T. But J can be written as,

$$J = \begin{bmatrix} \frac{\partial t'}{\partial T} & \frac{\partial t'}{\partial R} \\ \frac{\partial r}{\partial T} & \frac{\partial r}{\partial R} \end{bmatrix}$$

so that we only have to show that the elements of J constitute a non-integrable system. For this purpose, we need the eigenvectors of the 2×2 block. And they are,

$$\begin{aligned}
v_{\lambda_+} &= \frac{1}{\sqrt{2}\left[\frac{4M^2}{r^2}+1+\sqrt{\frac{4M^2}{r^2}+1}\right]^{1/2}} \begin{bmatrix} -\sqrt{\frac{4M^2}{r^2}+1}-1 \\ \frac{2M}{r} \end{bmatrix} \\
v_{\lambda_-} &= \frac{1}{\sqrt{2}\left[\frac{4M^2}{r^2}+1-\sqrt{\frac{4M^2}{r^2}+1}\right]^{1/2}} \begin{bmatrix} \sqrt{\frac{4M^2}{r^2}+1}-1 \\ \frac{2M}{r} \end{bmatrix}
\end{aligned}$$

From elementary linear algebra, we thus find that

$$J^t = \begin{bmatrix} \frac{\frac{-2M}{r}}{\sqrt{2}\left[\frac{4M^2}{r^2}+1-\sqrt{\frac{4M^2}{r^2}+1}\right]^{1/2}} = \frac{\partial T}{\partial t'} & \frac{\sqrt{\frac{4M^2}{r^2}+1}-1}{\sqrt{2}\left[\frac{4M^2}{r^2}+1-\sqrt{\frac{4M^2}{r^2}+1}\right]^{1/2}} = \frac{\partial T}{\partial r} \\ \frac{\frac{2M}{r}}{\sqrt{2}\left[\frac{4M^2}{r^2}+1+\sqrt{\frac{4M^2}{r^2}+1}\right]^{1/2}} = \frac{\partial R}{\partial t'} & \frac{\sqrt{\frac{4M^2}{r^2}+1}+1}{\sqrt{2}\left[\frac{4M^2}{r^2}+1+\sqrt{\frac{4M^2}{r^2}+1}\right]^{1/2}} = \frac{\partial R}{\partial r} \end{bmatrix}$$

and where $\det J = -1$ and $J^{-1} = J^t$ - by virtue of the fact that J is of $SO(1,1)$ type. And now its clear that the elements of J constitute a non-integrable system. Therefore there is *no* such transformation, $R(r,t')$ and $T(r,t')$, that leads to a diagonalisation of the Eddington-Finkelstein metric.

Given this fact, the Dirac equation then requires a suitable vierbein set to be chosen. Since we want our theory to make sense across the event horizon, we must choose a set that is regular everywhere except, possibly, at the singularity. To this end, let the vierbein set be written as

$$e^a{}_\mu = \begin{bmatrix} A(r,t') & C(r,t') & 0 & 0 \\ B(r,t') & D(r,t') & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{bmatrix}$$

so that the constraint, $e^a{}_\mu e^b{}_\nu \eta_{ab} = g_{\mu\nu}$ translates into the following equations,

$$\begin{aligned} A^2 - C^2 &= 1 - \frac{2M}{r} \\ B^2 - D^2 &= -\left(1 + \frac{2M}{r}\right) \\ 2(AB - CD) &= -\frac{4M}{r}. \end{aligned}$$

We also need to make a choice of a time-like vector field. Suppose that we want the vector to be of the form ${E_0}^\mu \partial_\mu = {E_0}^0 \partial_0$ (motivated by the Dirac equation) - in other words without any spatial derivatives such as ∂_1. Then from the vierbein and hence, inverse vierbein, matrices, we have to set $C = 0$. From the 3 constraint equations, it then follows that

$$\begin{aligned} A &= \left(1 - \frac{2M}{r}\right)^{1/2} \\ B &= \frac{-\frac{2M}{r}}{\left(1 - \frac{2M}{r}\right)^{1/2}} \\ D &= \left(1 - \frac{2M}{r}\right)^{-1/2} \end{aligned}$$

showing that we get a vierbein set that is singular at the horizon. This is not what we want as it will lead to severe problems in trying to extend the Dirac theory into the horizon. Hence any conceivable time-like vector field that we choose must contain a ∂_1 term. Looking back at the Eddington-Finkelstein metric, we see that the coefficient of the dr^2 term is $\left(1 + \frac{2M}{r}\right)$ - a quantity that is completely regular except at the singularity. We are thus motivated to choose ${e^1}_1 = D = \left(1 + \frac{2M}{r}\right)^{1/2}$. In which case we find that the vierbein and inverse vierbein components are,

$$\begin{aligned} B &= 0 \\ C &= \frac{\frac{2M}{r}}{\left(1 + \frac{2M}{r}\right)^{1/2}} \\ A &= \left(1 + \frac{2M}{r}\right)^{-1/2} \\ {E_0}^0 &= \left(1 + \frac{2M}{r}\right)^{1/2} \end{aligned}$$

$$
\begin{aligned}
E_0{}^1 &= -\frac{2M}{r}\left(1+\frac{2M}{r}\right)^{-1/2} \\
E_1{}^0 &= 0 \\
E_1{}^1 &= \left(1+\frac{2M}{r}\right)^{-1/2}
\end{aligned}
$$

Notice that $g_{\mu\nu}E_0{}^\mu E_0{}^\nu = 1$ so that $E_0{}^\mu$ is a globally defined time-like vector field. This is a necessary behaviour if we want to extend the theory into the horizon. Also, the basis one-forms may be chosen to be,

$$
\begin{aligned}
\theta^0 &= \left(1+\frac{2M}{r}\right)^{-1/2} dt' \\
\theta^1 &= \frac{2M}{r}\left(1+\frac{2M}{r}\right)^{-1/2} dt' + \left(1+\frac{2M}{r}\right)^{1/2} dr \\
\theta^2 &= r d\theta \\
\theta^3 &= r\sin\theta d\phi
\end{aligned}
$$

and the spin connection may be computed to be,

$$
\begin{aligned}
\omega_{01} &= \frac{M}{r^2}\left(1+\frac{2M}{r}\right)^{-3/2}\theta^0 + \frac{2M}{r^2}\left(1+\frac{M}{r}\right)\left(1+\frac{2M}{r}\right)^{-3/2}\theta^1 \\
\omega_{02} &= \frac{-2M}{r^2}\left(1+\frac{2M}{r}\right)^{-1/2}\theta^2 \\
\omega_{03} &= \frac{-2M}{r^2}\left(1+\frac{2M}{r}\right)^{-1/2}\theta^3 \\
\omega_{12} &= \frac{1}{r}\left(1+\frac{2M}{r}\right)^{-1/2}\theta^2 \\
\omega_{13} &= \frac{1}{r}\left(1+\frac{2M}{r}\right)^{-1/2}\theta^3 \\
\omega_{23} &= \frac{\cot\theta}{r}\theta^3
\end{aligned}
$$

Notice that *all* the quantities that we have defined so far are regular everywhere except at the singularity. At this point, we make use of (2.2) to find Γ_c (because we cannot follow the factorisation procedure),

$$\begin{aligned}
\Gamma_0 &= \frac{-M}{r^2}\left(1+\frac{2M}{r}\right)^{-3/2}(\frac{1}{2}\alpha^1) \\
\Gamma_1 &= \frac{-2M}{r^2}\left(1+\frac{M}{r}\right)\left(1+\frac{2M}{r}\right)^{-3/2}\left(\frac{1}{2}\alpha^1\right) \\
\Gamma_2 &= \frac{2M}{r^2}\left(1+\frac{2M}{r}\right)^{-1/2}\left(\frac{1}{2}\alpha^2\right)+\frac{1}{r}\left(1+\frac{2M}{r}\right)^{-1/2}\left(\frac{1}{2}i\Sigma^3\right) \\
\Gamma_3 &= \frac{2M}{r^2}\left(1+\frac{2M}{r}\right)^{-1/2}\left(\frac{1}{2}\alpha^3\right)-\frac{1}{r}\left(1+\frac{2M}{r}\right)^{-1/2}\left(\frac{1}{2}i\Sigma^2\right) \\
&\quad +\frac{\cot\theta}{r}\left(\frac{1}{2}i\Sigma^1\right).
\end{aligned}$$

Having all the ingredients ready, we then write down the Dirac eqn as

$$\begin{aligned}
i\left[\left(1+\frac{2M}{r}\right)^{1/2}\frac{\partial}{\partial t'}-\frac{2M}{r}\left(1+\frac{2M}{r}\right)^{-1/2}\frac{\partial}{\partial r}\right]\Psi &= -i\vec{\alpha}\cdot\vec{\nabla}\Psi+m_0\beta\Psi \\
&\quad -iU(r)\Psi
\end{aligned} \tag{4.11}$$

where $U(r)$ is the matrix function

$$\begin{aligned}
U(r) &= \frac{\frac{2M}{r^2}-\frac{M}{r^2}\frac{1+\frac{M}{r}}{1+\frac{2M}{r}}}{\left(1+\frac{2M}{r}\right)^{1/2}}-\left[\frac{1}{2}\frac{M}{r^2}\left(1+\frac{2M}{r}\right)^{-3/2}+\frac{1}{r}\left(1+\frac{2M}{r}\right)^{-1/2}\right]\alpha^1 \\
&\quad -\left[\frac{1}{2}\frac{\cot\theta}{r}\right]\alpha^2.
\end{aligned} \tag{4.12}$$

The equations written above reveal much about the behaviour of the Dirac theory inside the horizon. For one thing the interaction potential, $U(r)$, is in some sense, time dependent. This is because the time-like vector field, ${E_0}^{\mu}$, contains spatial terms which are functions of

r. Hence the mode solutions of the Dirac equation will most certainly be time dependent too. Therefore it is unclear as to whether the theory described by (4.11) follows a Hamiltonian evolution in the canonical sense. Unfortunately, in cases like this, we are unawares of any quantisation procedure that applies.

The same can be said for the outgoing Eddington-Finkelstein extension. We take this result to be a strong indication that it is *impossible* to find an extension of the quantum field theory into the horizon. This should not be surprising. There are other indications that also hints at this. In the next chapter, we shall discuss the famous result of Hawking - thermal emission from black-holes. There we shall present arguments that also support this conclusion.

CHAPTER 5 : THERMAL NEUTRINO EMISSION FROM THE SCHWARZSCHILD BLACK-HOLE

In the previous chapter, we have suggested that it may be impossible to extend the quantum field theory into the event horizon in a continuous manner linking the exterior of the horizon to its interior. In support of this hypothesis, we present in this chapter, the famous thermal Hawking radiation from black-holes [2,5]. However, instead of scalar fields, we shall use the machinery of the previous chapter to show that *neutrino* emission from black-holes is thermal. And we shall follow closely the calculations and discussions in [2,5].

Since the neutrino is massless according to our present understanding, we set m_0 to be zero in (4.6). Then as $r \to \infty$, the solutions to (4.6) may be approximated by $e^{\pm i\omega r^*}$. And since $p = \omega$ in the massless case, we may thus write for any *incoming* wave packet at infinity,

$$\psi_{\mathrm{INC}} = \int_0^\infty d\omega \sum_{jm_j\kappa_j} \left[e^{-i\omega v} u_{\omega jm_j\kappa_j} b_{\omega jm_j\kappa_j} + e^{i\omega v} v_{\omega jm_j\kappa_j} d^\dagger_{\omega jm_j\kappa_j} \right] \tag{5.1}$$

where $v = t + r^*$ and where,

$$\begin{aligned} u_{\omega jm_j\kappa_j} &= \sqrt{\frac{2}{\pi}}\omega \left(1 - \frac{2M}{r}\right)^{1/4} r \sin^{1/2}\theta \frac{1-\gamma^5}{2} \Phi^{(+)}_{m_j\kappa_j} \\ v_{\omega jm_j\kappa_j} &= \sqrt{\frac{2}{\pi}}\omega \left(1 - \frac{2M}{r}\right)^{1/4} r \sin^{1/2}\theta \frac{1-\gamma^5}{2} \Phi^{(-)}_{m_j\kappa_j} \end{aligned}$$

are the *left handed* neutrino wave functions. We shall study what happens to these modes in the case of a gravitationally collapsing star under spherical collapse, to form a Schwarzschild black-hole. This case is slightly different from an eternal black-hole insofar as the boundary conditions at infinity are very different in the two cases. Furthermore, the metric inside the collapsing body is not given by the Schwarzschild metric. And because we don't know how to compute the back-reaction of the spacetime metric on the collapsing matter, we shall also neglect the matter-gravitational coupling. The collapse process is best depicted by using the following Penrose diagram.

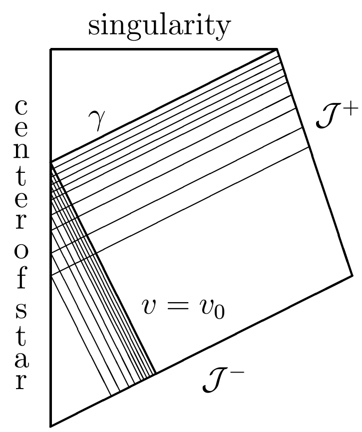

Fig.(1) The fate of neutrino waves during gravitational collapse

Since the neutrinos are massless, they travel along *null geodesics*. Past infinity is represented by $\mathcal{J}^-$ and future infinity by $\mathcal{J}^+$. The *rays* arriving from past infinity are blue shifted within the interior of the star, bounced of the center, and are tremendously red-shifted when they leave the star (over compensation). However there is a point during the collapse process for which rays arriving later than, all fall into the singularity. This last ray which can escape to future infinity marks the formation of the event horizon, and is indicated by γ in the diagram above and is formed at the coordinate *time*, $v = v_0$. But because of the dense piling up of the rays near γ, we only need to figure out the relationship between the incoming rays close to $v = v_0$ and the outgoing rays that reach future infinity, in order to describe the late time asymptotics of the escaping radiation.

The *outgoing* radiation at future infinity may be written in similar fashion as (5.1),

$$\psi_{\mathrm{OUT}} = \int_0^\infty d\omega' \sum_{j' m_{j'} \kappa_{j'}} \left[e^{-i\omega' u} s_{\omega' j' m_{j'} \kappa_{j'}} a_{\omega' j' m_{j'} \kappa_{j'}} + e^{i\omega' u} w_{\omega' j' m_{j'} \kappa_{j'}} c^\dagger_{\omega' j' m_{j'} \kappa_{j'}} \right] \tag{5.2}$$

where $u = t - r^*$, $s_{\omega' j' m_{j'} \kappa_{j'}}$ and $w_{\omega' j' m_{j'} \kappa_{j'}}$ are the outgoing modes. The incoming and outgoing modes may be related to each other via a Bogolubov transformation as follows,

$$e^{-i\omega' u} s_{j' m_{j'} \kappa_{j'}} = \int_0^\infty d\omega \sum_{j m_j \kappa_j} \left[\alpha_{\omega' \omega} e^{-i\omega v} u_{\omega j m_j \kappa_j} + \beta_{\omega' \omega} e^{i\omega v} v_{\omega j m_j \kappa_j} \right]. \tag{5.3}$$

In the next section, we shall derive some elementary properties of the so-called Bogolubov transformation, using a simplified notation.

The Bogolubov Transformation

The business of Bogolubov transformations is best written down in a very condensed and simplified notation. Let $u_i = e^{-i\omega v} u_{\omega j m_j \kappa_j}$, $v_i = e^{i\omega v} v_{\omega j m_j \kappa_j}$, $s_i = e^{-i\omega' u} s_{\omega' j' m_{j'} \kappa_{j'}}$ and $w_i = e^{i\omega' u} w_{\omega' j' m_{j'} \kappa_{j'}}$. Then with the *Hermitian* inner product that we have defined, and using a symbolic Kronecker delta to represent the delta function as well as other Kronecker deltas, we see that

$$\langle u_i, u_j \rangle = \delta_{ij}$$

$$
\begin{aligned}
\langle u_i, v_j \rangle &= 0 \\
\langle v_i, v_j \rangle &= -\delta_{ij} \\
\langle s_i, s_j \rangle &= \delta_{ij} \\
\langle s_i, w_j \rangle &= 0 \\
\langle w_i, w_j \rangle &= -\delta_{ij}
\end{aligned}
$$

If repeated indices now represent a generalised summation and integration, then (5.3) may be written as

$$
s_i = \alpha_{ij} u_j + \beta_{ij} v_j \tag{5.4}
$$

so that

$$
\begin{aligned}
\langle u_k, s_i \rangle &= \alpha_{ik} \\
\langle v_k, s_i \rangle &= -\beta_{ik}.
\end{aligned}
$$

Now set $w_i = \gamma_{ij} u_j + \delta_{ij} v_j$, then,

$$
\begin{aligned}
\langle w_i, s_j \rangle &= 0 \\
&= \langle w_i, \alpha_{jk} u_k + \beta_{jk} v_k \rangle \\
&= \overline{\gamma}_{ij} \alpha_{jk} - \overline{\delta}_{ij} \beta_{jk}
\end{aligned}
$$

and consequently we obtain,

$$
w_i = \overline{\beta}_{ij} u_j + \overline{\alpha}_{ij} v_j. \tag{5.5}
$$

These identities when taken together imply, in particular, that

$$\begin{aligned} u_i &= \overline{\alpha}_{ij} s_j - \beta_{ij} w_j \qquad (5.6)\\ v_i &= -\overline{\beta}_{ij} s_j + \alpha_{ij} w_j. \qquad (5.7)\end{aligned}$$

These equations takes care of the relationships that exist between the Bogolubov coefficients. However there is more to be studied. For not only do we require the Bogolubov coefficients, we also require a knowledge of how the various operators that appear in (5.1) and (5.2) are related to each other. To this end, we consider the operatorial equality

$$\begin{aligned}\sum_i u_i b_i + v_i d_i^\dagger &= \sum_j s_j a_j + w_j c_j^\dagger \\ &= \sum_j a_j \left(\sum_k \alpha_{jk} u_k + \beta_{jk} v_k \right) + c_j^\dagger \left(\sum_k \overline{\beta}_{jk} u_k + \overline{\alpha}_{jk} v_k \right) \\ &= \sum_k \left(\sum_j a_j \alpha_{jk} + c_j^\dagger \overline{\beta}_{jk} \right) u_k + \left(\sum_j a_j \beta_{jk} + c_j^\dagger \overline{\alpha}_{jk} \right) v_k\end{aligned}$$

from which we obtain the relation,

$$\begin{aligned} b_k &= \sum_j \alpha_{jk} a_j + \overline{\beta}_{jk} c_j^\dagger \qquad (5.8)\\ d_k^\dagger &= \sum_j \beta_{jk} a_j + \overline{\alpha}_{jk} c_j^\dagger. \qquad (5.9)\end{aligned}$$

Conversely, it can similarly be shown that

$$\begin{aligned} a_k &= \sum_j \overline{\alpha}_{jk} b_j - \overline{\beta}_{jk} d_j^\dagger \qquad (5.10)\\ c_k^\dagger &= \sum_j -\beta_{jk} b_j + \alpha_{jk} d_j^\dagger. \qquad (5.11)\end{aligned}$$

The Bogolubov coefficients also obey another non-trivial identity; plugging (5.8) into the anti-commutation relation obeyed by b_i and using the anti-commutation relations satisfied by a_i and c_i, yields

$$\sum_k \overline{\alpha}_{jk}\alpha_{ki} + \beta_{jk}\overline{\beta}_{ki} \;=\; \delta_{ij} \tag{5.12}$$

Of course, in the case of bosons obeying Bose statistics (e.g. the scalar field), then we would have been forced to use *commutation* relations and there would be a minus instead of a plus, on the left hand side of the preceding equation. This is all we need to know about Bogolubov transformations in order to derive the thermal spectrum of neutrino emission from black-holes.

Thermal Neutrino Flux

To compute the late time asymptotic radiation of neutrinos from the newly formed black-hole, we must be able to compute the vacuum expectation value of the neutrino field in the outgoing vacuum. In other words, we define the outgoing vacuum as $a_i\,|0\rangle_{\mathrm{out}} = c_i\,|0\rangle_{\mathrm{out}} = 0$ and ask how many incoming modes are there in this outgoing vacuum. But this is easily answered since,

$$\begin{aligned}\langle 0,\mathrm{out}|\sum_k b_k^\dagger b_k\,|0,\mathrm{out}\rangle \;&=\; Tr\beta^\dagger\beta \\ &\sim\; |\beta|^2\end{aligned}$$

where we have used (5.8) and the usual notions of creation and annihilation operators. Consequently, we only need to determine $\beta_{\omega'\omega}$ in (5.3). This in turn requires the overlap of the wave functions in (5.1) and (5.2), which translates into a knowledge of $u = t - r^*$ as a function of $v = t + r^*$. Now how do we find out what this relationship is ? The simplest way to proceed, is by matching the interior metric of the star to the exterior metric, across the surface of the star. And because we are dealing with a spherically symmetric situation, the matching problem essentially reduces to a two-dimensional problem involving only the radial and temporal coordinates. Therefore, we shall use the notation and terminology of the two-dimensional physics and follow closely the method as outlined in [2].

Let the two-dimensional, radial and temporal metric outside the star be

$$ds^2 = C(r)dudv$$

where $u = t - r^* - R_0^*$, $v = t + r^* - R_0^*$, $r^* = \int C^{-1}\, dr$ and $R_0^* =$ some constant. We demand that $C \to 1$ and $\frac{dC}{dr} \to 0$ as $r \to \infty$. There is an event horizon at some value of r for which $C = 0$. In our case, $C(r) = 1 - \frac{2M}{r}$, vanishes at $r = 2M$ and models the Schwarzschild black-hole.

Inside the collapsing star, we take the metric to be

$$ds^2 = A(U, V)dUdV$$

where A is an arbitrary smooth, non-singular function, $U = \tau - r + R_0$ and $V = \tau + r - R_0$. Here, τ is the proper time as measured on the surface of the collapsing star and R_0 is some constant. From these expressions, we deduce that the star is at rest before $\tau = 0$

and assume that its surface shrinks along the world-line $r = R(\tau)$ for $\tau > 0$. Furthermore, these coordinates have been specially chosen so that $\tau = t = 0$ at the onset of collapse and $u = v = U = V = 0$ at the surface of the star.

We need to reflect the null rays at the center of the star (see Fig. (1)). And for this purpose we can choose the boundary condition that the neutrino waves must vanish at $r = 0$ i.e. $\psi(r = 0) = 0$. Now let the transformation linking the exterior and interior coordinates be given by,

$$\begin{aligned} U &= \alpha(u) \\ v &= \beta(V). \end{aligned}$$

At the center of the star where $r = 0$, $U = U_0 = \tau + R_0$ and $V = V_0 = \tau - R_0$. Hence we see that the center of the star lies on the line $V = U - 2R_0$. We need solutions of the massless Dirac equation that vanishes along this line. But this is easy, given the preceding transformation equations. Consider $\beta(V) = \beta(U - 2R_0) = \beta\left[\alpha(u) - 2R_0\right]$ and set

$$\psi_{\mathrm{INC}} + \psi_{\mathrm{OUT}} \sim \left[e^{-i\omega v} - e^{-i\omega\beta[\alpha(u)-2R_0]}\right] u_{\omega j m_j \kappa_j} + [c.c.] v_{\omega j m_j \kappa_j}.$$

These modes clearly vanish at the center of the star *by construction*. In particular, we see that the simple incoming wave, $e^{-i\omega v}$, has been converted by the process of collapse into a complicated outgoing wave. This establishes the mode relation,

$$e^{-i\omega v} = e^{-i\omega\beta[\alpha(u)-2R_0]}. \tag{5.13}$$

Intuitively, we expect the complicated phase factor, $\beta\left[\alpha(u)-2R_0\right]$, to reduce to a steadily escalating redshift as the surface of the star approaches the event horizon. In the asymptotic limit, this redshift is all that matters. But to determine the form of this redshift, we perform the matching of the interior and exterior metrics across the surface of the star, $r = R(\tau)$. To do this, we note that

$$\begin{aligned} dV - dU &= 2dr \\ dv - du &= 2dr^{*} \\ &= 2C^{-1}dr \quad \text{so that} \\ dV - dU &= C(dv - du). \end{aligned} \tag{5.14}$$

On the other hand, we also have the matching condition,

$$AdVdU = Cdvdu. \tag{5.15}$$

From (5.14) and using the definition of U and V plus the fact that $r = R(\tau)$ on the surface of the star, we obtain the following equation satisfied at the surface of the star,

$$\frac{dU}{dV} = \frac{1-\dot{R}}{1+\dot{R}} \tag{5.16}$$

where $\dot{R} = \frac{dR}{d\tau}$. Using the last three equations and after some simple algebra, it can be shown that,

$$\alpha' = \frac{dU}{du} = C\left(1-\dot{R}\right)\left\{\left[AC\left(1-\dot{R}^2\right)+\dot{R}^2\right]^{1/2}-\dot{R}\right\}^{-1} \tag{5.17}$$

$$\beta' = \frac{dv}{dV} = C^{-1}\left(1+\dot{R}\right)^{-1}\left\{\left[AC\left(1-\dot{R}^2\right)+\dot{R}^2\right]^{1/2}+\dot{R}\right\} \tag{5.18}$$

Now as the surface of the star approaches the event horizon, $C \to 0$ and in this limit (with the help of l'Hospital's rule),

$$\lim_{C\to 0} \alpha' = \frac{\dot{R}-1}{2\dot{R}} C \tag{5.19}$$

$$\lim_{C\to 0} \beta' = \frac{\dot{R}-1}{2\dot{R}} A. \tag{5.20}$$

We may Taylor expand $R(\tau)$ in the preceding formulas as $R(\tau) = R_h + \nu(\tau_h - \tau) + O\left[(\tau_h - \tau)^2\right]$ where R_h is the value of R at the horizon, τ_h is the value that τ takes at the horizon and $-\nu = \dot{R}(\tau_h)$. We may also Taylor expand C and this takes the form, $C = C(r = R_h) + \frac{dC}{dr}_{r=R_h}(r - R_h) +$ higher order terms. But $C(r = R_h) = 0$ by definition and $\frac{dC}{dr}_{r=R_h} = 2\kappa$ where κ is the surface gravity of the black-hole. Thus, $C \approx 2\kappa(R - R_h)$ near the horizon. Plugging these inputs into (5.19) yields

$$\kappa du = \frac{1}{\nu+1}\frac{1}{\tau_h - \tau} dU$$

and using $U = \tau - R + R_0$ in iterating $\tau_h - \tau$ gives,

$$\kappa du = \frac{dU}{\tau_h - U - R_h + R_0}$$

so that we finally obtain the asymptotic relation,

$$U = e^{-\kappa u} + \text{constant}. \tag{5.21}$$

From (5.20), a similar treatment with A being approximately constant gives

$$v = AV\frac{\nu+1}{2\nu} + \text{constant}. \tag{5.22}$$

We now know the relationship between the exterior and interior coordinates - at least in the asymptotic limit when the event horizon is approached. Plugging (5.21) and (5.22) in the mode relation in (5.13) yields,

$$e^{-i\omega v} \;=\; e^{-i\omega[ce^{-\kappa u}+d]} \tag{5.23}$$

where c and d are constants. However, for the purpose of computing Bogolubov coefficients, it is convenient to take the surface of integration to lie in the "in" region. Physically this corresponds to selecting modes that are *standard out going waves at* $\mathcal{J}^+$ but which become complicated functions of v on $\mathcal{J}^-$. So all we need to do is functionally invert the mode relation given above.

Set $d = v_0$, corresponding to the latest advanced ray γ. Then,

$$u \;=\; -\frac{1}{\kappa}\log\left(\frac{v_0 - v}{c}\right) - \frac{1}{\kappa}\log(-1).$$

For $\alpha_{\omega'\omega}$ we take $\log(-1) = \pi i$ to obtain,

$$e^{-i\omega u} \;=\; e^{i\frac{\omega}{\kappa}\log\left(\frac{v_0-v}{c}\right)} e^{\frac{\pi\omega}{\kappa}} \quad \text{for } v < v_0 \text{ and } 0 \text{ otherwise} \tag{5.24}$$

where the last condition comes from the fact that all rays arriving later than γ strike the singularity, and do not escape to $\mathcal{J}^+$. For $\beta_{\omega'\omega}$ we make use of the relation,

$$e^{i\omega' u} w_{\omega' j m_j \kappa_j} \;=\; \int_0^\infty d\omega \sum_{j m_j \kappa_j} \left[e^{-i\omega v}\overline{\beta}_{\omega'\omega} u_{\omega j m_j \kappa_j} + e^{i\omega v}\overline{\alpha}_{\omega'\omega} v_{\omega j m_j \kappa_j}\right]$$

but this time, we take $\log(-1) = -\pi i$ (because of the complex conjugation of β) to obtain the equation,

$$e^{i\omega u} = e^{-i\frac{\omega}{\kappa}\log\left(\frac{v_0-v}{c}\right)} e^{\frac{\pi\omega}{\kappa}} \quad \text{for } v < v_0 \text{ and } 0 \text{ otherwise} \tag{5.25}$$

This shows consistency in our approach. Thus by Fourier transform, we find that

$$\begin{aligned} \alpha_{\omega'\omega} &= \int_{-\infty}^{\infty} \frac{dv}{2\pi} e^{i\omega v} e^{i\frac{\omega'}{\kappa}\log\left(\frac{v_0-v}{c}\right)} e^{\frac{\pi\omega'}{\kappa}} \quad \text{for } v < v_0 \text{ and } 0 \text{ otherwise} \\ &= \int_{-\infty}^{v_0} \frac{dv}{2\pi} e^{i\omega v} e^{i\frac{\omega'}{\kappa}\log\left(\frac{v_0-v}{c}\right)} e^{\frac{\pi\omega'}{\kappa}} \quad \text{and similarly} \qquad (5.26) \\ \overline{\beta}_{\omega'\omega} &= \int_{-\infty}^{v_0} \frac{dv}{2\pi} e^{i\omega v} e^{-i\frac{\omega'}{\kappa}\log\left(\frac{v_0-v}{c}\right)} e^{\frac{\pi\omega'}{\kappa}} \qquad (5.27) \end{aligned}$$

What is left now, is the evaluation of the above integrals and the proof that the resulting neutrino emission emanating from the newly formed black-hole is, indeed, thermal. To this end the substitution, $x = \frac{v_0-v}{c}$, in (5.26) and (5.27) gives

$$\begin{aligned} \alpha_{\omega'\omega} &= \frac{c}{2\pi} e^{\frac{\pi\omega'}{\kappa}} e^{i\omega v_0}\, \Gamma\left(1 + i\frac{\omega'}{\kappa}\right) \frac{1}{(i\omega c)^{1+i\frac{\omega'}{\kappa}}} \qquad (5.28) \\ \overline{\beta}_{\omega'\omega} &= \frac{c}{2\pi} e^{\frac{\pi\omega'}{\kappa}} e^{i\omega v_0}\, \Gamma\left(1 - i\frac{\omega'}{\kappa}\right) \frac{1}{(i\omega c)^{1-i\frac{\omega'}{\kappa}}}. \qquad (5.29) \end{aligned}$$

And since $\overline{\Gamma(1+iy)} = \Gamma(1-iy)$, we see that

$$\frac{|\alpha_{\omega'\omega}|}{|\overline{\beta}_{\omega'\omega}|} = e^{\frac{\pi\omega'}{\kappa}} \tag{5.30}$$

so that by (5.12) we obtain

$$|\beta_{\omega'\omega}|^2 = \frac{1}{e^{\frac{2\pi\omega'}{\kappa}} + 1} \tag{5.31}$$

which is the total neutrino flux emitted per unit frequency interval. The total flux is thus,

$$Tr\beta^\dagger\beta = \int_0^\infty d\omega \, \frac{1}{e^{\frac{2\pi\omega}{\kappa}} + 1}$$

which clearly shows its *thermal* character due to the Fermi-Dirac nature of the distribution. The temperature of the distribution being $\frac{\kappa}{2\pi}$. Of course, this result was obtained by ignoring the effects of back-scattering of the incoming neutrino waves. In actuality, the last equation would be modified by the presence of a *gray body factor*, $\Gamma_{\omega j m_j \kappa_j}$, in the numerator of the integrand as thus

$$Tr\beta^{\dagger}\beta \;=\; \int_0^{\infty} \, d\omega \, \frac{\Gamma_{\omega j m_j \kappa_j}}{e^{\frac{2\pi\omega}{\kappa}}+1} \tag{5.32}$$

This ends our derivation of the thermal neutrino emission from a Schwarzschild black-hole. In the next chapter, we shall focus on fermion scattering amplitudes off the same black-hole - thereby allowing us to compute the gray body factor.

Partitioning Of The Quantum Field Theory By The Horizon

In the previous chapter we suggested that it may be impossible to extend the quantum field theory continuously across the event horizon. The thermal Hawking radiation from black-holes seem to support this idea. For one thing, if it is possible to continuously extend the quantum field theory across the event horizon, then one expects there to be non-trivial correlation functions between field operators localised within and exterior to the horizon. Such correlations would be an obstruction to the thermal character of the Hawking flux. Indeed, the thermal nature of the flux is exactly preserved only when the quantum field theories are disjoint. For in that situation, the partition function of the theory would be a direct product between the "exterior partition function" and the *trace* over all "interior states". That, of course, will lead to a thermal structure for the Hawking flux.

The inclusion of "back-reaction effects" should not alter this view. After all, we can treat the back-reaction in similar fashion as the waves stirred on the surface of a pond, after a stone is thrown into it. Indeed, the "information loss paradox" [20] may very well be explained in that manner. The information carried by an infalling particle is, in fact, radiated away through a back-reaction mechanism which we have hitherto, been unable to compute. Surely such a mechanism will depend on a quantum field theory in which gravitation cannot be treated semi-classically. In such a scenario, the event horizon will presumably fluctuate but in no way will it affect the idea that quantum field theories are split across the horizon. Nowadays, a lot of effort has been spent in trying to use superstring theories [21,22] - which are candidate theories of everything, in order to derive a suitable back-reaction mechanism. Unfortunately, due to the non-perturbative nature of the problem, little progress has been made.

CHAPTER 6 : FERMION SCATTERING AMPLITUDES OFF A SCHWARZSCHILD BLACK-HOLE

In the last chapter, we mentioned that the thermal Hawking spectrum has to be convoluted with a gray body factor, $\Gamma_{\omega j m_j \kappa_j}$ which measures the effect of back-scattering of the incoming neutrino waves. This raises the issue of fermion scattering amplitudes off black-holes. For one thing, it is clear that we cannot expect analytic results. After all, even the radial wave functions are only qualitatively known. Furthermore, the advent of high-speed computers has made the need for closed-form, analytic solutions secondary.

The method that we shall adhere to can be found in many texts but shall follow closely the discussion in [23]. It is the method of partial waves. We shall determine the scattering and absorption cross sections via a partial wave analysis of these amplitudes. Hence our results will be expressed in terms of phase shifts and so on. And of course, these phase shifts are without an analytic, closed form expression. To compute these phase shifts, we will need to do some asymptotics and matching of wave functions [23]. How can we do this ? First, we note that we are dealing with a long-range force. Thus we may truncate the range of the force to a finite albeit large radial coordinate, R, where the force field may be assumed to vanish beyond. The accuracy of our solutions increases as $R \to \infty$. A choice of

R splits the domain $2M < R < \infty$ into two regions and due consideration must be given to the behaviour of the wave functions in the separate regions. Thus,

For $r > R > 2M$

This is the *exterior* region and the wave function here, Ψ_{ext}, has the asymptotics,

$$\Psi_{ext} \sim \Psi_{inc} + \Psi_{sca} \quad \text{as } r \to \infty \tag{6.1}$$

where the right hand side is a sum of the incident and scattered waves.

For $2M < r < R$

This is the *interior* region and the wave function, Ψ_{int}, must be asymptotic to the transmitted wave at the horizon,

$$\Psi_{int} \sim \Psi_{trans} \quad \text{as } r \to 2M \tag{6.2}$$

and finally, we have a matching condition to be satisfied by the wave function at $r = R$.

For $r = R$

Here the interior wave function must match with the exterior one, so as to provide continuity of the wave function.

$$\Psi_{int} = \Psi_{ext} \quad \text{at } r = R \tag{6.3}$$

The wave functions written in the above three equations are all of the form

$$\Psi_j \;=\; \left(1-\frac{2M}{r}\right)^{1/4} r\sin^{1/2}\theta\Phi_j$$

so that only a knowledge of Φ is sufficient. Since we truncate the gravitational field of the black-hole at $r = R$, we take Φ_{ext}, Φ_{inc} and Φ_{sca} to satisfy the Minkowskian Dirac equation. This essentially means that we neglect terms of order $\frac{2M}{r}$ for $r > R$.

The Incident Wave

For the incident wave, we take a wave packet formed by component plane waves travelling along the positive z axis and with the spin pointing "up". Following the spinor conventions of [6], we choose to write

$$\Phi_{inc} \;=\; I(p)e^{-i\omega t}e^{ipr\cos\theta}u^{(1)}(m_0, p\hat{z}) \tag{6.4}$$

where $I(p)$ is the momentum profile function and it is square-integrable. A partial wave expansion yields,

$$\begin{aligned}\Phi_{inc} \;=\; & I(p)e^{-i\omega t}\int_0^\infty dp' \sum_{jm_j\kappa_j} \\ & \left[A_{jm_j\kappa_j} j_{a_{\kappa_j}}(p'r)\Phi^{(+)}_{m_j\kappa_j}(\Omega) + B_{jm_j\kappa_j} j_{b_{\kappa_j}}(p'r)\Phi^{(-)}_{m_j\kappa_j}(\Omega)\right]\end{aligned} \tag{6.5}$$

where

$$A_{jm_j\kappa_j} \;=\; \int_0^\infty dr\; r^2 j_{a_{\kappa_j}}(p'r)\int d\Omega\; e^{ipr\cos\theta}\Phi^{(+)\dagger}_{m_j\kappa_j}u^{(1)}(m_0, p\hat{z}) \tag{6.6}$$

$$B_{jm_j\kappa_j} \;=\; \int_0^\infty dr\; r^2 j_{b_{\kappa_j}}(p'r)\int d\Omega\; e^{ipr\cos\theta}\Phi^{(-)\dagger}_{m_j\kappa_j}u^{(1)}(m_0, p\hat{z}). \tag{6.7}$$

We can evaluate these Fourier coefficients directly by making use of the explicit representation of $\Phi^{(+)}_{m_j\kappa_j}$ and $\Phi^{(-)}_{m_j\kappa_j}$, as well as the following information,

$$\begin{aligned}
e^{ipr\cos\theta} &= \sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell}j_{\ell}(pr)\sqrt{\frac{4\pi}{2\ell+1}}\;Y_{\ell}^{m=0} \\
u^{(1)}(m_0,p\hat{z}) &= \left[\begin{array}{c} \left(\frac{\omega+m_0}{2m_0}\right)^{1/2}\varphi^{(1)}(m_0,0) \\ \\ \frac{p}{[2m_0(\omega+m_0)]^{1/2}}\varphi^{(1)}(m_0,0) \end{array}\right]
\end{aligned}$$

where the spinor conventions are from [6]. The final results of the computation are :

$$\begin{aligned}
&A_{jm_j\kappa_j=-(j+1/2)} = \\
&\quad (-1)i^{j+1/2}\frac{\pi}{2p^2}\delta(p-p')\sqrt{4\pi}\sqrt{j+m_j}\left(\frac{\omega+m_0}{2m_0}\right)^{1/2}\delta_{m_j-1/2,0} \qquad (6.8)\\
&A_{jm_j\kappa_j=(j+1/2)} = \\
&\quad (-1)i^{j+3/2}\frac{\pi}{2p^2}\delta(p-p')\sqrt{4\pi}\sqrt{j+1-m_j}\left(\frac{\omega+m_0}{2m_0}\right)^{1/2}\delta_{m_j-1/2,0} \qquad (6.9)\\
&B_{jm_j\kappa_j=-(j+1/2)} = \\
&\quad i^{j+1/2}\frac{\pi}{2p^2}\delta(p-p')\sqrt{4\pi}\sqrt{j+1-m_j}\frac{p}{[2m_0(\omega+m_0)]^{1/2}}\delta_{m_j-1/2,0} \qquad (6.10)\\
&B_{jm_j\kappa_j=(j+1/2)} = \\
&\quad i^{j-1/2}\frac{\pi}{2p^2}\delta(p-p')\sqrt{4\pi}\sqrt{j+m_j}\frac{p}{[2m_0(\omega+m_0)]^{1/2}}\delta_{m_j-1/2,0} \qquad (6.11)
\end{aligned}$$

Notice the delta functions in these formulas - they are well taken care of by the integral over p' in (6.5). It is for this reason that we have chosen to work with the *wave packet* as defined by (6.5).

The Scattered Wave

First of all, we observe that the interactions do not contain any term that might cause a spin-flip in the outgoing wave. Therefore we only need to consider an outgoing wave with a spin-polarisation that is identical to the incident wave. Hence we set,

$$\Phi_{sca} = e^{i\omega t}\frac{e^{ipr}}{r}f(p,\Omega)u^{(1)}(m_0,p\hat{r}) \tag{6.12}$$

Obviously, the scattering cross-section is encoded within $f(p,\Omega)$ which we shall determine via a partial wave analysis. This is just ordinary quantum mechanics.

The Exterior Solution

The exterior solution may be chosen to be

$$\begin{aligned}\Phi_{ext} &= e^{-i\omega t}\int_0^\infty dp' \sum_{jm_j\kappa_j} \\ &\quad \left[C_{jm_j\kappa_j}b_{a_{\kappa_j}}(p'r)\Phi^{(+)}_{m_j\kappa_j} + D_{jm_j\kappa_j}b_{b_{\kappa_j}}(p'r)\Phi^{(-)}_{m_j\kappa_j}\right]\end{aligned} \tag{6.13}$$

where since there is no regularity condition to be satisfied, we can allow for the presence of a singular piece in b_ℓ at $r=0$, and a phase shift to be determined later on,

$$\begin{aligned}b_\ell &= e^{i\delta_\ell}\left[\cos\delta_\ell j_\ell - \sin\delta_\ell n_\ell\right] \\ &\sim \frac{1}{2ipr}\left[(-1)^{\ell+1}e^{-ipr} + e^{2i\delta_\ell}e^{ipr}\right] \quad \text{as } r\to\infty.\end{aligned}$$

Of course, this description of the exterior wave function is incomplete because we need to specify the Fourier coefficients in (6.13). Now how can we do this computation ? The answer is very simple. Since $\Phi_{ext} \sim \Phi_{inc} + \Phi_{sca}$ as $r \to \infty$, we can match the incoming parts, e^{-ipr}, of the respective asymptotic expressions of Φ_{inc} with Φ_{ext}. Hence we recall that,

$$
\begin{aligned}
j_{j-1/2} &\sim \frac{1}{2ipr}\left[(-1)^{j+1/2}e^{-ipr} + e^{ipr}\right] \quad \text{as } r \to \infty \\
j_{j+1/2} &\sim \frac{1}{2ipr}\left[(-1)^{j+3/2}e^{-ipr} + e^{ipr}\right] \quad \text{as } r \to \infty
\end{aligned}
$$

so that by plugging into (6.5) and (6.13) and by comparing the incoming parts, we obtain

$$
\begin{aligned}
C_{jm_j\kappa_j} &= I(p)A_{jm_j\kappa_j} \\
D_{jm_j\kappa_j} &= I(p)B_{jm_j\kappa_j}.
\end{aligned}
$$

This fixes the exterior wave function to be

$$
\begin{aligned}
\Phi_{ext} &= e^{-i\omega t}I(p)\int_0^\infty dp' \sum_{jm_j\kappa_j} \\
&\left[A_{jm_j\kappa_j} b_{a_{\kappa_j}}(p'r)\Phi^{(+)}_{m_j\kappa_j} + B_{jm_j\kappa_j} b_{b_{\kappa_j}}(p'r)\Phi^{(-)}_{m_j\kappa_j}\right] \qquad (6.14)
\end{aligned}
$$

in which the Fourier coefficients are explicitly known.

Computation Of The Phase Shifts

We are now ready to express $f(p, \Omega)$ as a partial wave expansion and also compute the required phase shifts. From (6.1) and using the necessary asymptotic expressions, we find that

$$
\Phi_{sca} \sim \int_0^\infty dp'\ I(p)e^{-i\omega t}\frac{e^{ip'r}}{r} \sum_{jm_j\kappa_j}
$$

$$\left[A_{jm_j\kappa_j}\frac{1}{2ip'}\left[e^{2i\delta_{a_{\kappa_j}}}-1\right]\Phi^{(+)}_{m_j\kappa_j}+B_{jm_j\kappa_j}\frac{1}{2ip'}\left[e^{2i\delta_{b_{\kappa_j}}}-1\right]\Phi^{(-)}_{m_j\kappa_j}\right]$$

as $r \to \infty$ and where the phase shifts, δ_ℓ are from (6.13) and (6.14). Noting that $A_{jm_j\kappa_j}$ and $B_{jm_j\kappa_j}$ contains delta functions, and using (6.12) as well as the identity,

$$u^{(1)^\dagger}(m_0, p\hat{r})u^{(1)}(m_0, p\hat{r}) = \frac{\omega}{m_0}$$

we obtain the partial wave expansion,

$$\begin{aligned} f(p,\Omega) &= \frac{m_0}{\omega}\sum_{jm_j\kappa_j}\int_0^\infty dp'\ I(p) \\ &\left[A_{jm_j\kappa_j}\frac{1}{2ip}\left[e^{2i\delta_{a_{\kappa_j}}}-1\right]u^{(1)^\dagger}(m_0,p\hat{r})\Phi^{(+)}_{m_j\kappa_j}\right. \\ &\left.+B_{jm_j\kappa_j}\frac{1}{2ip}\left[e^{2i\delta_{b_{\kappa_j}}}-1\right]u^{(1)^\dagger}(m_0,p\hat{r})\Phi^{(-)}_{m_j\kappa_j}\right]. \end{aligned} \tag{6.15}$$

This is clearly a well-defined expression since the delta functions in $A_{jm_j\kappa_j}$ and $B_{jm_j\kappa_j}$, takes care of the integration over p'. Hence we only need to determine the phase shifts. To this end, we merely have to match the interior wave function to the exterior wave function at $r = R$. Set the interior wave function to be

$$\begin{aligned} \Phi_{int} &= e^{-i\omega t}\int_0^\infty dp'\ \sum jm_j\kappa_j \\ &\left[E_{jm_j\kappa_j}\mathcal{Z}_{a_{\kappa_j}}(p'r)\Phi^{(+)}_{m_j\kappa_j}+F_{jm_j\kappa_j}\mathcal{Z}_{b_{\kappa_j}}(p'r)\Phi^{(-)}_{m_j\kappa_j}\right] \end{aligned} \tag{6.16}$$

so that by the continuity of the wave function at $r = R$, we get

$$E_{jm_j\kappa_j}\mathcal{Z}_{a_{\kappa_j}}(p'r) = I(p)A_{jm_j\kappa_j}b_{a_{\kappa_j}}(p'r) \tag{6.17}$$

$$F_{jm_j\kappa_j}\mathcal{Z}_{b_{\kappa_j}}(p'r) = I(p)B_{jm_j\kappa_j}b_{b_{\kappa_j}}(p'r). \tag{6.18}$$

By matching of the logarithmic derivatives of the radial wave functions at $r = R$, we then obtain the phase shifts as the ratio of Wronskians,

$$\tan \delta_\ell \;=\; \left. \frac{W\left[\mathcal{Z}_\ell, j_\ell\right]}{W\left[\mathcal{Z}_\ell, n_\ell\right]} \right|_{r=R} \qquad \ell = a_{\kappa_j}, b_{\kappa_j}. \tag{6.19}$$

Equation (6.19) can be evaluated numerically for various values of R. Of course, the larger R is, the better the approximation. We have not done this however. But presumably, there will be an asymptotic value for which the ratio in (6.19) approaches, as $R \to \infty$. Such a limit would represent the "actual" phase shift. Moreover, if an analytic expression is needed, then we must find an approximation of $\mathcal{Z}_\ell$. This can be achieved by many approximation methods.

Absorption

First recall that $\mathcal{Z}_\ell \to e^{\pm i\omega r*}$ as $r \to 2M$. Hence the *ingoing* transmitted wave has the asymptotic form,

$$\Phi_{trans} \;=\; e^{-i\omega t} e^{-ipr*} \alpha(p, \Omega) u^{(1)}(m_0, -p\hat{r}). \tag{6.20}$$

By comparison with (6.16) and following the same steps leading to (6.15), we then obtain

$$\begin{aligned} \alpha(p, \Omega) \;=\; & \frac{m_0}{\omega} \int_0^\infty dp' \sum_{j m_j \kappa_j} \\ & \left[E_{j m_j \kappa_j} u^{(1)^\dagger}(m_0, -p\hat{r}) \Phi^{(+)}_{m_j \kappa_j} + F_{j m_j \kappa_j} u^{(1)^\dagger}(m_0, -p\hat{r}) \Phi^{(-)}_{m_j \kappa_j} \right] \end{aligned} \tag{6.21}$$

which expresses the fermionic absorption characteristics of the black-hole, with respect to known parameters.

Cross-sections, Currents, Unitarity And The Grey Body Factor

The fermion current can be expressed as

$$\begin{aligned} J^\mu &= \overline{\Psi}\gamma^\mu\Psi \\ &= h^2 e\overline{\Phi}\gamma^\mu\Phi \\ &= h^2 e j^\mu \\ &= h^2 e E_a{}^\mu j^a \end{aligned}$$

where j^μ is covariantly conserved i.e. $\nabla_\mu j^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}j^\mu\right] = 0$. To see this, we only need recall the measure used to define integration of fermion bilinears. Clearly it is sufficient to study j^a - as the other factors are fixed. We shall construct the currents for the incident, scattered and absorbed waves - thereby deducing the respective cross-sections. To this end, it is easy to see that the current of the incident wave is,

$$\begin{aligned} \vec{j}_{inc} &= I^2(p) u^{(1)^\dagger}(m_0, p\hat{z})\gamma^0\vec{\gamma}u^{(1)}(m_0, p\hat{z}) \\ &= I^2(p)\frac{p}{m_0}\hat{z}. \end{aligned}$$

On the other hand, from (6.12), we discover that

$$\vec{j}_{sca} = \frac{|f(p,\Omega)|^2}{r^2}\frac{p}{m_0}\hat{r}$$

and since $f(p,\Omega) \propto I(p)$, it then follows that the elastic scattering cross-section is given by

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)_{elastic} &= \frac{|\vec{j}_{sca}|}{|\vec{j}_{inc}|}r^2 \\ &= \frac{|f(p,\Omega)|^2}{I^2(p)}. \end{aligned} \tag{6.22}$$

Similarly, the absorption cross-section is found to be

$$\left(\frac{d\sigma}{d\Omega}\right)_{absorption} = \frac{|\alpha(p,\Omega)|^2}{I^2(p)}. \tag{6.23}$$

The expressions for the cross-sections are, of course, explicitly known from the previous sections. We have successfully defined these cross-sections in terms of a partial wave expansion and computable phase shifts. What remains is to verify that these processes are indeed *unitary*. Our strategy is simple - to prove unitarity, we shall simply establish the Optical Theorem of Bohr, Peierls and Placzek [23]. If the Optical Theorem holds, then we can be certain that the process is unitary. In order to prove the Optical Theorem, we begin by an application of Gauss' theorem to the covariant conservation of current,

$$\begin{aligned}\int d^4x\sqrt{-g}\,\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}j^\mu\right] &= \int d^3S_a\sqrt{-g}j^a \\ &= 0\end{aligned}$$

and since $\partial_0\left(\sqrt{-g}j^0\right)=0$, we find that for a large spherical surface (sphere at infinity), S,

$$\int_S dS\;\hat{r}\cdot\vec{j} = 0 \tag{6.24}$$

where $dS\;\hat{r} = d\vec{S}$ is a unit normal vector of surface element. In our case, it is $\vec{j}_{ext}$ that is applied to the previous result. And for a sufficiently large surface, $\vec{j}_{ext} = \vec{j}_{inc} + \vec{j}_{sca} + \vec{j}_{int}$, where $\vec{j}_{int}$ is a current purely due to the interference between the incident and scattered waves. In particular,

$$\vec{j}_{inc} = I^2(p)\frac{p}{m_0}\hat{z}$$

$$
\begin{aligned}
\vec{j}_{sca} &= \left(\frac{d\sigma}{d\Omega}\right)_{elastic} \frac{I^2(p)}{r^2}\frac{p}{m_0}\hat{r} \\
\vec{j}_{int} &= \frac{I(p)}{r}\frac{p}{m_0}(1+\cos\theta) Re\left[e^{ipr(1-\cos\theta)} f(p,\Omega)\right].
\end{aligned}
$$

The contribution of the incident current vanishes by symmetry. It is only the current of the scattered wave and the interference term that contributes. The integral,

$$
\begin{aligned}
\int d\Omega\ r^2 \vec{j}_{int}\cdot\hat{r} &= I^2(p)\frac{p}{m_0}\frac{1}{r}(2\pi r^2) \\
&\quad \times Re\int_{-1}^{1} d(\cos\theta)\ (1+\cos\theta) e^{ipr(1-\cos\theta)}\frac{f(p,\Omega)}{I(p)} \\
&= |\vec{j}_{inc}| 4\pi r Re\left[\frac{f(p,\theta=0)/I(p)}{-ipr}\right] \\
&= -|\vec{j}_{inc}|\frac{4\pi}{p} Im\frac{f(p,\theta=0)}{I(p)}
\end{aligned}
$$

has been evaluated in [23] and thus we finally obtain the Optical Theorem of Bohr, Peierls and Placzek as

$$
\sigma_{elastic} = \frac{4\pi}{p} Im\frac{f(p,\theta=0)}{I(p)}. \tag{6.25}
$$

Of course, we must remember that $f(p,\theta=0)$ contains a factor of $I(p)$ so that the last result actually does not contain the momentum profile function. This proves that the process we are studying is indeed a unitary process.

As for the gray body factor, mentioned in the last chapter, we can simply use the differential absorption cross-section, with all momenta expressed as a function of energy, ω. Consequently, the thermal spectrum of the neutrino emission is modulated by this absorption cross-section.

CHAPTER 7 : FERMIONS IN KERR AND TAUB-NUT SPACETIMES

In the past several chapters, we have successfully constructed a theory of free fermions living in a Schwarzschild spacetime. While this by itself is an interesting exercise, in astrophysical situations we will usually have to deal with a rotating but uncharged, black-hole. Hence it is absolutely essential for us to develop a quantum field theory of fermions living in the Kerr spacetime. Again, we shall make use of our factorisation ansatz to help us construct the fermion quantum field theory. The Kerr metric when written in Boyer-Lindquist coordinates, reads as

$$\begin{aligned} ds^2 &= \frac{\Delta - a^2\sin^2\theta}{\rho^2}dt^2 - \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - \Delta a^2\sin^2\theta\right]d\phi^2 \\ &+ \frac{2a\sin^2\theta}{\rho^2}\left[r^2+a^2-\Delta\right]dt d\phi - \rho^2 d\theta^2 - \frac{\rho^2}{\Delta}dr^2 \end{aligned}$$

where $\Delta = r^2 + a^2 - 2Mr$ and $\rho^2 = r^2 + a^2\cos^2\theta$. It is clear that the Kerr metric is of the stationary, axisymmetric type. Hence we can choose a vierbein set as follows,

$$e^a{}_\mu = \begin{bmatrix} A(r,\theta) & 0 & 0 & C(r,\theta) \\ 0 & \frac{\rho}{\sqrt{\Delta}} & 0 & 0 \\ 0 & 0 & \rho & 0 \\ B(r,\theta) & 0 & 0 & D(r,\theta) \end{bmatrix} \tag{7.1}$$

so that $e = \sqrt{-g} = \rho^2\sin\theta$. In particular, this vierbein set determines the choice of basis one-forms because $\theta^a = e^a{}_\mu dx^\mu$. In particular, we have

$$
\begin{aligned}
e^0{}_0 &= A \\
e^0{}_3 &= B \\
e^3{}_0 &= C \\
e^3{}_3 &= D \\
e^1{}_1 &= \frac{\rho}{\sqrt{\Delta}} \quad \text{and} \\
e^2{}_2 &= \rho
\end{aligned}
$$

so that our basis one-forms are

$$
\begin{aligned}
\theta^0 &= Adt + Bd\phi \\
\theta^1 &= \frac{\rho}{\sqrt{\Delta}} dr \\
\theta^2 &= \rho d\theta \\
\theta^3 &= Cdt + Dd\phi.
\end{aligned}
$$

The constraint equations that we derive from $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$ are :

$$
\begin{aligned}
A^2 - C^2 &= \frac{\Delta - a^2 \sin^2\theta}{\rho^2} \\
B^2 - D^2 &= -\frac{\sin^2\theta}{\rho^2}\left[(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta\right] \\
2\left[AB - CD\right] &= \frac{2a\sin^2\theta}{\rho^2}\left[r^2 + a^2 - \Delta\right].
\end{aligned}
\tag{7.2}
$$

We have seen that it may be impossible to extend the quantum field theory, continuously across an event horizon. Therefore, in searching for a suitable vierbein set, it is not necessary to impose any continuity condition across the horizons present in the Kerr spacetime. In view of this, we shall first compute the spin connection - and then look for a vierbein set that results in the greatest simplification. After a long and tedious calculation, we first define the Wronskian of partial derivatives and determinant as

$$
\begin{aligned}
W[f,g]_r &= f\frac{\partial g}{\partial r} - g\frac{\partial f}{\partial r} \\
W[f,g]_\theta &= f\frac{\partial g}{\partial \theta} - g\frac{\partial f}{\partial \theta} \\
\Xi &= AD - BC
\end{aligned}
$$

so that the spin connection may be written as :

$$
\begin{aligned}
\omega_{01} &= \left[\frac{\partial A}{\partial r}D - \frac{\partial B}{\partial r}C\right]\frac{\sqrt{\Delta}}{\rho}\frac{1}{\Xi}\theta^0 \\
&+\frac{1}{2}W[A,B]_r\frac{\sqrt{\Delta}}{\rho}\frac{1}{\Xi}\theta^3 + \frac{1}{2}W[C,D]_r\frac{\sqrt{\Delta}}{\rho}\frac{1}{\Xi}\theta^3 \\
\omega_{02} &= \left[\frac{\partial A}{\partial \theta}D - \frac{\partial B}{\partial \theta}C\right]\frac{1}{\rho}\frac{1}{\Xi}\theta^0 \\
&+\frac{1}{2}W[A,B]_\theta\frac{1}{\rho}\frac{1}{\Xi}\theta^3 + \frac{1}{2}W[C,D]_\theta\frac{1}{\rho}\frac{1}{\Xi}\theta^3 \\
\omega_{03} &= -\frac{1}{2}W[A,B]_r\frac{\sqrt{\Delta}}{\rho}\frac{1}{\Xi}\theta^1 - \frac{1}{2}W[A,B]_\theta\frac{1}{\rho}\frac{1}{\Xi}\theta^2 \\
&+\frac{1}{2}W[C,D]_r\frac{\sqrt{\Delta}}{\rho}\frac{1}{\Xi}\theta^1 + \frac{1}{2}W[C,D]_\theta\frac{1}{\rho}\frac{1}{\Xi}\theta^2 \\
\omega_{12} &= \frac{a^2\sin 2\theta}{2\rho^3}\theta^1 + \frac{r\sqrt{\Delta}}{\rho^3}\theta^2 \\
\omega_{13} &= \left[\frac{\partial D}{\partial r}A - \frac{\partial C}{\partial r}B\right]\frac{\sqrt{\Delta}}{\rho}\frac{1}{\Xi}\theta^3 \\
&-\frac{1}{2}W[C,D]_r\frac{\sqrt{\Delta}}{\rho}\frac{1}{\Xi}\theta^0 - \frac{1}{2}W[A,B]_r\frac{\sqrt{\Delta}}{\rho}\frac{1}{\Xi}\theta^0 \\
\omega_{23} &= \left[\frac{\partial D}{\partial \theta}A - \frac{\partial C}{\partial \theta}B\right]\frac{1}{\rho}\frac{1}{\Xi}\theta^3
\end{aligned}
$$

$$-\frac{1}{2}W[C,D]_\theta\frac{1}{\rho}\frac{1}{\Xi}\theta^0-\frac{1}{2}W[A,B]_\theta\frac{1}{\rho}\frac{1}{\Xi}\theta^0$$

Using this result as an input, we can compute the term $\frac{1}{4}\epsilon^{abcd}(\omega_{ab})_c\gamma_d\gamma_5$ to find that,

$$\begin{aligned}\frac{1}{4}\epsilon^{abcd}(\omega_{ab})_c\gamma_d\gamma_5 &= \frac{1}{4}\left\{W[A,B]_\theta-W[C,D]_\theta\right\}\frac{1}{\rho}\frac{1}{\Xi}\gamma_1\gamma_5\\ &-\frac{1}{4}\left\{W[A,B]_r-W[C,D]_r\right\}\frac{\sqrt{\Delta}}{\rho}\frac{1}{\Xi}\gamma_2\gamma_5.\end{aligned}\tag{7.3}$$

It is easy to see that this term *cannot vanish* without contradicting the constraint equations in (7.2). Thus for all consistent choices of a vierbein set, the term $\frac{1}{4}\epsilon^{abcd}(\omega_{ab})_c\gamma_d\gamma_5$ is non-vanishing. What is the meaning of this ? For one thing, the naive factorisation ansatz will only reduce the Dirac equation to the form as given in (2.7). But this term, as was pointed out earlier, is intimately related to the spin-tensor current [9,10]. In theories with torsion, the term $S^{abc}=\overline{\Psi}\frac{1}{4}\epsilon^{abcd}\gamma_d\gamma_5\Psi$ is the spin-tensor current obtained from the Dirac Lagrangian by variation with respect to the spin connection, and is the source of torsion. Hence in theories with torsion, the term $\frac{1}{4}\epsilon^{abcd}(\omega_{ab})_c\gamma_d\gamma_5$ represents a true dynamical interaction. However, we are dealing with torsionless manifolds. So to what extent is this term representative of a true interaction ? Could it, for example, mean that the fermions can actually "see" the rotation of the Kerr black-hole via a Lense-Thirring precession of the spin axis ? We must be very cautious. After all, this term does not even transform as a tensor. There is a way to find out if there is such a thing as a gravitationally induced spin-precession. In the gauge theory analogy, the spin-precession term is clearly established when we compute the square of the Dirac operator, $\not{D}^2$. In the case of torsionless gravitation however, the square of the Dirac operator is given by $\not{D}^2 = g_{\mu\nu}D^\mu D^\nu + \frac{1}{4}R$ where R is the scalar curvature. There is no term that could possibly represent a Lense-Thirring precession of the fermion spin axis. Hence we are forced to conclude that the $\frac{1}{4}\epsilon^{abcd}(\omega_{ab})_c\gamma_d\gamma_5$ term does not represent a physical effect in the case of vanishing torsion. This means that there ought to be a way to remove this term from (2.7). It turns out that we can remove this term from (2.7) in the *massless* case via another factorisation process. In the next section we will see how this works out.

The Factorisation Ansatz For The Kerr Metric

In order to reduce the Dirac equation to the form of (2.7), we will have to first apply the usual factorisation ansatz that we have relied upon, in the previous chapters. From (7.1) and (2.11), we see that

$$f \;=\; he^{1/2} \;=\; \left(\frac{\sqrt{\Delta}}{\rho}\right)^{1/2} \rho \sin^{1/2}\theta. \tag{7.4}$$

(2.7) itself suggests a situation when we can remove the offending term. Suppose the fermions are massless. Then Φ will be an eigenstate of γ^5. In that case, the γ_5 in the offending term is redundant because we have a reduction to a two dimensional representation of the Pauli spin matrices. Indeed, let us see how this works. Define, left and right handed bispinors as follows (using an obvious notation),

$$\gamma^5\Phi_L \;=\; -\Phi_L$$

$$\gamma^5\Phi_R \;=\; \Phi_R$$

so that they satisfy the projection equations,

$$\begin{aligned} P_L\Phi_L &= \frac{1-\gamma^5}{2}\Phi_L = \Phi_L \\ P_R\Phi_L &= \frac{1+\gamma^5}{2}\Phi_L = 0 \\ P_L\Phi_R &= \frac{1-\gamma^5}{2}\Phi_R = 0 \\ P_R\Phi_R &= \frac{1+\gamma^5}{2}\Phi_R = \Phi_R. \end{aligned}$$

Then it is easy to see that for *any* two-component spinor, χ, we can construct a left and right handed bispinor, *in the Dirac representation*, as follows :

$$\begin{aligned} \Phi_L &= \begin{bmatrix} \chi \\ -\chi \end{bmatrix} \\ \Phi_R &= \begin{bmatrix} \chi \\ \chi \end{bmatrix} \end{aligned}$$

and (2.7) breaks down into two separate equations for the left and right handed massless bispinors. As neutrinos are all observed to be left handed in nature, we shall only consider the left handed case. In fact, the equation is given by

$$i\gamma^c E_c{}^\mu \partial_\mu \Phi_L - \frac{1}{4}\epsilon^{abcd}(\omega_{ab})_c\gamma_d\Phi_L = 0. \tag{7.5}$$

We shall try out another factorisation. Set

$$\Phi_L = g(r,\theta)\chi_L \quad \text{where} \tag{7.6}$$

$$\chi_L = \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} \tag{7.7}$$

and where ζ is a two component spinor. We assume that χ_L satisfies,

$$i\gamma^c E_c{}^\mu \partial_\mu \chi_L \quad = \quad 0 \tag{7.8}$$

so that $g(r,\theta)$ must satisfy the constraint,

$$i\eta^{dc} E_c{}^\mu \partial_\mu \log g - \frac{1}{4}\epsilon^{abcd}(\omega_{ab})_c \quad = \quad 0. \tag{7.9}$$

What remains is to choose a vierbein set. To this end, we follow Unruh [4] and set

$$\begin{aligned} A &= \frac{\sqrt{\Delta}}{\rho} \\ B &= -\frac{\sqrt{\Delta}}{\rho} a \sin^2\theta \\ C &= -\frac{a\sin\theta}{\rho} \\ D &= \frac{r^2+a^2}{\rho}\sin\theta \end{aligned} \tag{7.10}$$

so that we obtain,

$$\frac{1}{4}\epsilon^{abcd}(\omega_{ab})_c\gamma_d \quad = \quad -\frac{1}{2}\frac{\sqrt{\Delta}a\cos\theta}{\rho^3}\gamma_1 - \frac{1}{2}\frac{ar\sin\theta}{\rho^3}\gamma_2. \tag{7.11}$$

Quite nicely, it turns out that (7.9) has a solution in this case. In fact, g is given by

$$g(r,\theta) \quad = \quad e^{\frac{i}{2}\arctan\left(\frac{a}{r}\cos\theta\right)}. \tag{7.12}$$

Hence we have successfully reduced the Dirac equation to the simpler equation, (7.8). In particular, defining $\vec{\nabla}$ in the usual manner, and working in the Dirac representation, (7.8) yields the following equation for the two component spinor ζ :

$$i\left(\frac{r^2+a^2}{\rho\sqrt{\Delta}}\right)\frac{\partial}{\partial t}\zeta \quad = \quad i\vec{\sigma}\cdot\vec{\nabla}\zeta - i\frac{a}{\rho\sqrt{\Delta}}\frac{\partial}{\partial\phi}\zeta. \tag{7.13}$$

Clearly we need to choose an ansatz that will lead to a separation of variables. This can be done quite easily. Because we have azimuthal symmetry, we are led to consider an ansatz for ζ that is simply

$$\zeta = \begin{bmatrix} u \\ \\ v \end{bmatrix} e^{-i\omega t} e^{-im\phi} \tag{7.14}$$

with u and v as functions to be determined. Of course, azimuthal symmetry and single-valuedness fixes m to be an integer. Substitution into (7.13) results in a pair of equations for u and v. These equations are coupled - and we shall subsequently uncouple them.

$$i\sqrt{\Delta}\frac{\partial}{\partial r}v + \frac{\partial}{\partial \theta}v = \left[-\omega a \sin\theta - \frac{m}{\sin\theta} + \frac{ma + r^2 + a^2}{\sqrt{\Delta}}\right] u \tag{7.15}$$

$$i\sqrt{\Delta}\frac{\partial}{\partial r}u - \frac{\partial}{\partial \theta}u = \left[\omega a \sin\theta + \frac{m}{\sin\theta} + \frac{ma + r^2 + a^2}{\sqrt{\Delta}}\right] v \tag{7.16}$$

To uncouple these equations, we consider the following linear combinations of u and v,

$$Z_+ = u + v = R_1(r)S_1(\theta)$$

$$Z_- = u - v = R_2(r)S_2(\theta)$$

which upon substitution into (7.15) and (7.16) yields

$$i\sqrt{\Delta}\frac{dR_1}{dr} - \frac{ma + r^2 + a^2}{\sqrt{\Delta}}R_1 = \lambda R_2 \tag{7.17}$$

$$i\sqrt{\Delta}\frac{dR_2}{dr} + \frac{ma + r^2 + a^2}{\sqrt{\Delta}}R_2 = \xi R_1 \tag{7.18}$$

$$\frac{dS_1}{d\theta} + \left(\omega a \sin\theta + \frac{m}{\sin\theta}\right) S_1 = \xi S_2 \tag{7.19}$$

$$\frac{dS_2}{d\theta} - \left(\omega a \sin\theta + \frac{m}{\sin\theta}\right) S_2 = \lambda S_1. \tag{7.20}$$

Here, λ and ξ are the constants that appear when we apply a separation of variables. By considering the effect of the transformation, $\theta \to -\theta$ on these equations, it is easy to deduce the following fact.

$$\lambda = \xi \tag{7.21}$$

$$S_1(\theta) = S_2(-\theta). \tag{7.22}$$

Thus the equations for a massless fermion living on the Kerr spacetime, is indeed separable. In particular, since the neutrino equation is linear, we can choose to expand any neutrino wave packet in the following manner :

$$\begin{aligned} \Psi_L &= e^{-i\omega t} e^{\frac{i}{2}\arctan\left(\frac{a}{r}\cos\theta\right)} \left(\frac{\sqrt{\Delta}}{\rho}\right)^{1/2} \rho \sin^{1/2}\theta \sum_{m,\lambda} \\ &\left[A_{m\lambda} R_{1_{m\lambda}}(r) \Phi^{(+)}_{m\lambda}(\theta) e^{-im\phi} + B_{m\lambda} R_{2_{m\lambda}}(r) \Phi^{(-)}_{m\lambda}(\theta) e^{-im\phi}\right] \end{aligned} \tag{7.23}$$

where the spinorial analogues of the Legendre function, $\Phi^{(\pm)}_{m\lambda}(\theta)$, are given by

$$\Phi^{(+)}_{m\lambda}(\theta) = \begin{bmatrix} S_{1_{m\lambda}}(\theta) \\ 0 \\ -S_{1_{m\lambda}}(\theta) \\ 0 \end{bmatrix} \tag{7.24}$$

$$\Phi^{(-)}_{m\lambda}(\theta) = \begin{bmatrix} 0 \\ S_{2_{m\lambda}}(\theta) \\ 0 \\ -S_{2_{m\lambda}}(\theta) \end{bmatrix}. \tag{7.25}$$

We need not proceed any further from here. The manner in which we should quantise these modes is transparent - given our efforts in the previous chapters. What is interesting is the fact that the neutrino equation is amenable to an exact treatment in the Kerr spacetime, to begin with. Furthermore, the modes that we obtain are quite simple - given the intricacies of the Kerr metric. In the next section, we shall develop a similar theory for Taub-NUT spacetime.

Taub-NUT Spacetime

The Taub-NUT solution to Einstein's equation, represents a gravitational instanton and it has a horizon [24,25]. What is interesting about this metric is that its not asymptotically flat, though it is of stationary, axisymmetric type. Here we shall briefly present the results of our computation. The calculations are identical to the case of the Kerr metric and, again, solutions exist only for *massless* fermions.

The Taub-NUT metric is given by

$$\begin{aligned} ds^2 &= \left[1 - \frac{2Mr + N^2}{r^2 + N^2}\right]\left[dt + 4N\sin^2\frac{\theta}{2}d\phi\right]^2 \\ &\quad - \left[1 - \frac{2Mr + N^2}{r^2 + N^2}\right]^{-1} dr^2 - \left(r^2 + N^2\right)\left[d\theta^2 + \sin^2\theta d\phi^2\right]. \end{aligned}$$

If we define,

$$\begin{aligned} \Delta &= r^2 - 2Mr \\ \rho^2 &= r^2 + N^2 \end{aligned}$$

then we can rewrite the metric in a way that is similar to the Kerr metric viz,

$$
\begin{aligned}
ds^2 &= \frac{\Delta}{\rho}dt^2 + 8\frac{\Delta}{\rho^2}N\sin^2\frac{\theta}{2}dt d\phi \\
&\quad - \left[\rho^2\sin^2\theta - 16N^2\sin^4\frac{\theta}{2}\frac{\Delta}{\rho^2}\right]d\phi^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2.
\end{aligned}
$$

Choosing a vierbein set like that in (7.1) with

$$
\begin{aligned}
A &= \frac{\sqrt{\Delta}}{\rho} \\
B &= 4N\frac{\sqrt{\Delta}}{\rho}\sin^2\frac{\theta}{2} \\
C &= 0 \\
D &= \rho\sin\theta \\
e^1{}_1 &= \frac{\rho}{\sqrt{\Delta}} \\
e^2{}_2 &= \rho
\end{aligned}
$$

and following *exactly* identical steps as in the case of the Kerr metric, yields,

$$
f = he^{1/2} = \left(\frac{\sqrt{\Delta}}{\rho}\right)^{1/2}\rho\sin^{1/2}\theta \tag{7.26}
$$

$$
\frac{1}{4}\epsilon^{abcd}(\omega_{ab})_c\gamma_d = \frac{1}{2}N\frac{\sqrt{\Delta}}{\rho^3}\gamma_1 \tag{7.27}
$$

$$
g(r,\theta) = e^{\frac{i}{2}\arctan\frac{r}{N}}. \tag{7.28}
$$

And choosing a similar ansatz as (7.14) leads to the pair of coupled equations,

$$
i\sqrt{\Delta}\frac{\partial}{\partial r}v + \frac{\partial}{\partial\theta}v = \left[-\frac{m}{\sin\theta} + 2\omega N\tan\frac{\theta}{2} + \frac{\omega(r^2+N^2)}{\sqrt{\Delta}}\right]u \tag{7.29}
$$

$$
i\sqrt{\Delta}\frac{\partial}{\partial r}u - \frac{\partial}{\partial\theta}u = \left[\frac{m}{\sin\theta} - 2\omega N\tan\frac{\theta}{2} + \frac{\omega(r^2+N^2)}{\sqrt{\Delta}}\right]v \tag{7.30}
$$

which can be solved in exactly the same fashion as (7.14). We shall not do this here though. After all, the motivation for studying the Taub-NUT case was to prove that the factorisation ansatz does indeed, apply even to the non-asymptotically flat spacetimes that are of stationary, axisymmetric type. It just so happens that, interestingly enough, the Taub-NUT solution is also a gravitational instanton.

Thus in this chapter, we have shown how a fermion quantum field theory may be constructed on the Kerr and Taub-NUT spacetimes. Unfortunately, the only cases in which we can explicitly construct such a theory, are the cases in which the fermions are massless. At present, we are unawares of any successful attempt at a construction of a similar theory for *massive* fermions. Even in the Newman-Penrose approach [3], the Dirac equation is separable only in the case of massless fermions.

CHAPTER 8 : CONCLUSIONS AND SPECULATIONS

In the previous chapters, we have constructed a quantum field theory of free fermions in black-hole spacetimes. We have also suggested that a quantum field theory is necessarily split into two disjoint components across an event horizon. This will ensure the thermal character of the Hawking radiation, but it would say nothing as regards to the information loss paradox. We believe that the solution to the information loss problem lies in an hitherto unknown back-reaction mechanism.

Much work still remains though. For one thing, our work has been based on the factorisation ansatz which is valid only for stationary axisymmetric spacetimes. The factorisation leaves behind an *effective* operator governing the dynamics which is manifestly non-unitary. It is the total wave function *inclusive* of the factored pieces which obeys a unitary evolution. How can we make sense of this ? It is not clear. Can we do without the factorisation ansatz ? Again, it is far from clear.

Secondly, there is the issue of pragmatic applications. Are the wave functions that we have found, suitable for computations in semi-classical gravitational collapse ? We believe so. Certainly the Schwarzschild wave functions are suitable for practical calculations in gravitational collapse. It would be nice if one can build a theory of quarks and leptons using these wave functions. Then it will be possible to do high-energy particle physics even in the strong gravitational field of a black-hole. This we believe to be possible and we are in fact working on such matters at present.

Thirdly there is the issue of unifying Einstein's theory of gravitation with a theory of matter. Clearly if gravity is united in some specific way to a theory of matter (e.g. supergravity, superstrings), then the gauge degree of freedom that we have in choosing a vierbein set will be dynamical. In other words, the dynamics of such a theory will force upon us a fixed choice of vierbeins. In which case, there is little we can do in terms of computational power. What are the implications of this ? It remains to be seen - as there has been little progress in this direction.

And finally, we are left to speculate on the role of torsion in fermionic theories in curved spacetime. Unfortunately, theories with torsion suffer from one serious technical drawback. Because of the Cartan structure equations [7], the spin connection is largely undetermined until we specify the torsion tensor. However the torsion tensor itself is given by the matter equations derived from the matter Lagrangian and requires the spin connection. Hence theories with torsion are coupled in a highly non-trivial manner. How can we break this chain of circular argument ? We are not sure. It appears to us that much needs to be done in the way of making suitable approximations. But that is the crux of the matter........

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