Introduction to Path Integrals in Field Theory

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This manuscript on path integrals is based on lectures I have given at the University of Giessen. If you find any conceptual or typographical errors I would like to learn about them.
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Contents

Chapter 1

PATH INTEGRAL IN QUANTUM MECHANICS

In this starting chapter we introduce the concepts of propagators and path integrals in the framework of nonrelativistic quantum mechanics.

1.1 Propagator of the Schrödinger Equation

We start by considering a nonrelativistic particle in a one-dimensional potential $V(x)$. The Schrödinger equation reads

$$H\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) = \mathrm{i}\hbar\frac{\partial\psi(x,t)}{\partial t} \ . \tag{1.1}$$

This equation allows us to calculate the wavefunction $\psi(x,t)$ at a later time, if we know $\psi(x,t_0)$ at the time t_0. For further calculations we rewrite this equation into the following form

$$\left(\mathrm{i}\hbar\frac{\partial}{\partial t} - H\right)\psi(x,t) = 0 \ . \tag{1.2}$$

Next, we consider the function $K(x,t;x_i,t_i)$ which is defined as solution of the equation

$$\left(\mathrm{i}\hbar\frac{\partial}{\partial t} - H\right)K(x,t;x_i,t_i) = \mathrm{i}\hbar\delta(x-x_i)\delta(t-t_i) \ . \tag{1.3}$$

K is the "Green's function" of the Schrödinger equation (K is also often called "propagator") with the boundary condition

$$K(x,t_i;x_i,t_i) = \delta(x-x_i) \ . \tag{1.4}$$

We now show that if K is known (we will later discuss its explicit form), then the solution of the Schrödinger equation (1.2) can be written as

$$\Theta(t-t_i)\,\psi(x,t) = \int K(x,t;x_i,t_i)\,\psi(x_i,t_i)\,\mathrm{d}x_i \tag{1.5}$$

(Huygen's principle) with the stepfunction $\Theta(x) = 0$ for $x < 0$ and $\Theta(x) = 1$ for $x \geq 0$, implying

$$\psi(x,t) = \int K(x,t;x_i,t_i)\,\psi(x_i,t_i)\,\mathrm{d}x_i \quad \text{for } t \geq t_i \ . \tag{1.6}$$

The boundary condition $t \geq t_i$ is thus converted into a boundary condition for the Green's function in (1.5). Relation (1.5) can be proven by inserting the lhs into the Schrödinger equation

$$\begin{aligned} &\left(\mathrm{i}\hbar\frac{\partial}{\partial t} - H\right)\Theta(t-t_i)\,\psi(x,t) \\ &= \mathrm{i}\hbar\delta(t-t_i)\,\psi(x,t) + \Theta(t-t_i)\left(\mathrm{i}\hbar\frac{\partial}{\partial t} - H\right)\psi \ . \end{aligned} \tag{1.7}$$

On the other hand, we also have

$$\begin{aligned} &\left(\mathrm{i}\hbar\frac{\partial}{\partial t} - H\right)\int K(x,t;x_i,t_i)\,\psi(x_i,t)\,\mathrm{d}x_i \\ &= \mathrm{i}\hbar\int \delta(t-t_i)\,\delta(x-x_i)\,\psi(x_i,t)\,\mathrm{d}x_i \\ &= \mathrm{i}\hbar\delta(t-t_i)\,\psi(x,t) \ . \end{aligned} \tag{1.8}$$

Equating (1.7) and (1.8) gives immediately

$$\Theta(t-t_i)\left(i\hbar\frac{\partial}{\partial t} - H\right)\psi = 0 \ . \tag{1.9}$$

Thus the ψ defined by (1.5) is indeed a solution of the Schrödinger equation for all times $t \geq t_i$. Since

$$\psi(x,t) = \int K(x,t;x_i,t_i)\,\psi(x_i,t_i)\,\mathrm{d}x_i \qquad \text{for} \qquad t \geq t_i \ , \tag{1.10}$$

$K(x,t;x_i,t_i)$ is the probability amplitude for a transition from x_i, at time t_i, to the position x, at the later time t.

We can find an explicit form for the propagator, if the solutions of the stationary Schrödinger equation, $\varphi_n(x)$, and the corresponding eigenvalues,

E_n, are known. Since the φ_n form a complete system, K can certainly be expanded in this basis (for $t \geq t_i$)

$$K(x,t;x_i,t_i) = \sum_n a_n \varphi_n(x) \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_n t} \; . \tag{1.11}$$

Here the expansion coefficients obviously depend on x_i, t_i

$$a_n = a_n(x_i,t_i) \; . \tag{1.12}$$

Because of the initial condition $K(x,t_i;x_i,t_i) = \delta(x-x_i)$ we have

$$\delta(x-x_i) = \sum_n a_n(x_i,t_i)\varphi_n(x)\mathrm{e}^{-\frac{i}{\hbar}E_n t_i} \; . \tag{1.13}$$

The lhs is time-independent; thus we must have

$$a_n(x_i,t_i) = a_n(x_i)\mathrm{e}^{+\frac{\mathrm{i}}{\hbar}E_n t_i} \quad , \tag{1.14}$$

and consequently also

$$\delta(x-x_i) = \sum_n a_n(x_i)\,\varphi_n(x) \; . \tag{1.15}$$

This can be fulfilled by

$$a_n(x_i) = \varphi_n^*(x_i) \tag{1.16}$$

(closure relation). Thus we have a representation of $K(x,t;x_i,t_i)$ in terms of the eigenfunctions and eigenvalues of the underlying Hamiltonian

$$K(x,t;x_i,t_i) = \Theta(t-t_i)\sum_n \varphi_n^*(x_i)\,\varphi_n(x)\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_n(t-t_i)} \; . \tag{1.17}$$

In Dirac's bra and ket notation this result can also be written as

$$\begin{aligned} K(x,t;x_i,t_i) &= \sum_n \varphi_n^*(x_i)\,\varphi_n(x)\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_n(t-t_i)} \\ &= \sum_n \langle n|x_i\rangle \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_n(t-t_i)}\langle x|n\rangle \\ &= \sum_n \langle n|\mathrm{e}^{+\frac{\mathrm{i}}{\hbar}\hat{H}t_i}|x_i\rangle\langle x|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}t}|n\rangle \\ &= \langle x|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}(t-t_i)}|x_i\rangle \;\equiv\; \langle x|\hat{U}(t,t_i)\,|x_i\rangle \; . \end{aligned} \tag{1.18}$$

Thus the propagator is nothing else than the time development operator

$$\hat{U}(t,t_i) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}(t-t_i)} \tag{1.19}$$

in the x representation. It is also often written as

$$K(x,t;x_i,t_i) = \langle x|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}(t-t_i)}|x_i\rangle \equiv \langle xt|x_i t_i\rangle \tag{1.20}$$

.

The notation here is that of the Heisenberg representation of quantum mechanics. In this representation the physical state vectors are time-independent and the operators themselves carry all the time-dependence whereas this is just the opposite for the Schrödinger representation. For example, for the position operator $\hat{x}$ in the Schrödinger representation we obtain the time-dependent operator

$$x_{\mathrm{H}}(t) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\hat{H}t}\hat{x}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}t} \tag{1.21}$$

and

$$x_{\mathrm{H}}|xt\rangle = x|xt\rangle \tag{1.22}$$

with

$$|xt\rangle = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\hat{H}t}|x\rangle \ . \tag{1.23}$$

1.2 Propagator as Path Integral

We start by dividing the time-interval between t_i and t by inserting the time t_1. The wavefunction is first propagated until t_1 and then, in a second step, until t

$$\begin{aligned}\psi(x_1,t_1) &= \int K(x_1,t_1;x_i,t_i)\,\psi(x_i,t_i)\,\mathrm{d}x_i \\ \psi(x,t) &= \int K(x,t;x_1,t_1)\psi(x_1,t_1)\,\mathrm{d}x_1 \ .\end{aligned} \tag{1.24}$$

Taking these two equations together we get

$$\psi(x,t) = \int\int K(x,t;x_1,t_1)\,K(x_1,t_1;x_i,t_i)\,\psi(x_i,t_i)\,\mathrm{d}x_i\,\mathrm{d}x_1 \ . \tag{1.25}$$

Comparing this result with (1.5) yields

$$K(x,t;x_i,t_i) = \int K(x,t;x_1,t_1)\,K(x_1,t_1;x_i,t_i)\,\mathrm{d}x_1 \ . \tag{1.26}$$

We can thus view the transition from (x_i,t_i) to (x,t) as the result of a transition first from (x,t) to all possible intermediate points (x_1,t_1), which is then followed by a transition from these intermediate points to the endpoint. We could also say that the integration in (1.26) is performed over all possible

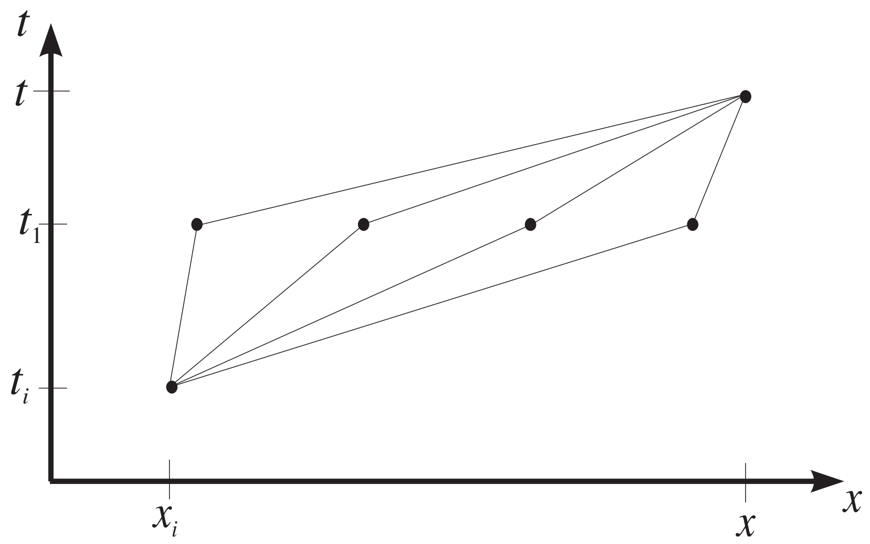


Figure 1.1: Possible paths from x_i to x, corresponding to (1.26).

paths between the points (x_i, t_i) and (x, t), which consist of two straight line segments with a bend at t_1. This is illustrated in Fig. (1.1).

We now subdivide the time interval further into $(n+1)$ equal parts of length $\Delta t = \eta$. We then have in direct generalization of the previous result

$$\begin{aligned} K(x,t;x_i,t_i) &= \int \ldots \int \mathrm{d}x_1\, \mathrm{d}x_2\, \ldots\, \mathrm{d}x_n \\ &\quad \times \left[K(x,t;x_n,t_n)\, K(x_n,t_n;x_{n-1},t_{n-1}) \ldots K(x_1,t_1;x_i,t_i)\right] \; . \end{aligned} \tag{1.27}$$

The integrals run here over all possible paths between (x_i, t_i) and (x, t) which consist of $(n+1)$ segments with boundaries that are determined by the time steps $t_i, t_1, \ldots, t_n, t$.

We now calculate the propagator for a small time interval $\Delta t = \eta$ from t_j to t_{j+1}. For this propagation we have according to (1.20)

$$\begin{aligned} K(x_{j+1},t_{j+1};x_j,t_j) &= \langle x_{j+1}|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}\eta}|x_j\rangle \\ &\cong \langle x_{j+1}|1 - \frac{\mathrm{i}}{\hbar}\hat{H}\eta|x_j\rangle \\ &= \delta(x_{j+1}-x_j) - \frac{\mathrm{i}}{\hbar}\eta\langle x_{j+1}|\hat{H}|x_j\rangle \\ &= \frac{1}{2\pi\hbar}\int \mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(x_{j+1}-x_j)}\,\mathrm{d}p - \frac{\mathrm{i}\eta}{\hbar}\langle x_{j+1}|\hat{H}|x_j\rangle \; . \end{aligned} \tag{1.28}$$

We now assume that $\hat{H}$ is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(\hat{x}) \qquad (\hat{p}, \hat{x} : \text{operators}) \; . \tag{1.29}$$

In this special case we can also bring the last term in (1.28) into an integral form. We have

$$\langle x_{j+1}|\hat{H}|x_j\rangle = \langle x_{j+1}|\frac{\hat{p}^2}{2m} + \hat{V}|x_j\rangle \; . \tag{1.30}$$

First, we consider the first summand

$$\begin{aligned}\langle x_{j+1}|\hat{p}^2|x_j\rangle &= \int \mathrm{d}p'\,\mathrm{d}p\langle x_{j+1}|p'\rangle\langle p'|\hat{p}^2|p\rangle\langle p|x_j\rangle \\ &= \int \mathrm{d}p'\,\mathrm{d}p\langle x_{j+1}|p'\rangle\delta(p'-p)p^2\langle p|x_j\rangle \\ &= \int \mathrm{d}p\langle x_{j+1}|p\rangle p^2\langle p|x_j\rangle \; .\end{aligned} \tag{1.31}$$

With the normalized momentum eigenfunctions

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}px} \tag{1.32}$$

we obtain

$$\langle x_{j+1}|\hat{p}^2|x_j\rangle = \frac{1}{2\pi\hbar}\int \mathrm{d}p\,p^2\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(x_{j+1}-x_j)} \; . \tag{1.33}$$

While there is an operator $\hat{p}$ on the lhs of this equation there are only numbers p on its rhs.

For the potential part an analogous transformation can be performed

$$\begin{aligned}\langle x_{j+1}|\hat{V}|x_j\rangle &= V\left(\frac{x_j+x_{j+1}}{2}\right)\delta\left(x_{j+1}-x_j\right) \\ &= \frac{1}{2\pi\hbar}\int \mathrm{d}p\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(x_{j+1}-x_j)}V(\overline{x}_j)\end{aligned} \tag{1.34}$$

with $\overline{x}_j = (x_j + x_{j+1})/2$. Again, on the lhs $\hat{V}$ is an operator, while the rhs of this equation contains no operators.

In summary, we have for the propagator over a time-segment η

$$\begin{aligned}K\left(x_{j+1},t_{j+1};x_j,t_j\right) &= \frac{1}{2\pi\hbar}\int \mathrm{d}p\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(x_{j+1}-x_j)} \\ &\quad -\frac{\mathrm{i}\eta}{\hbar}\left(\frac{1}{2\pi\hbar}\int \mathrm{d}p\,\frac{p^2}{2m}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(x_{j+1}-x_j)} + \frac{1}{2\pi\hbar}\int \mathrm{d}p\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(x_{j+1}-x_j)}V(\overline{x}_j)\right) \\ &= \frac{1}{2\pi\hbar}\int \mathrm{d}p\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(x_{j+1}-x_j)}\left(1-\frac{i\eta}{\hbar}H\left(p,\overline{x}_j\right)\right) \\ &\underset{\eta\to 0}{\longrightarrow} \frac{1}{2\pi\hbar}\int \mathrm{d}p_j\;\exp\left\{\frac{\mathrm{i}}{\hbar}\left[p_j\left(x_{j+1}-x_j\right)-\eta H\left(p_j,\overline{x}_j\right)\right]\right\} \; .\end{aligned} \tag{1.35}$$

In the last step we have renamed the integration variable into p_j to indicate that it may be viewed as the momentum of a classical particle moving from x_j to x_{j+1} between times t_j and t_{j+1}.

Note that the result (1.35) is valid also for more general Hamiltonians than (1.29); it holds as long as the p- and the x-dependent terms in the Hamiltonian are decoupled such that $H = H_0(p) + V(x)$.

We now insert (1.35) into (1.27) and obtain

$$
\begin{aligned}
&K\left(x, t; x_i, t_i\right) \qquad\qquad (1.36)\\
&= \lim_{n\to\infty} \int \prod_{j=1}^{n} \mathrm{d}x_j \int \prod_{j=0}^{n} \frac{\mathrm{d}p_j}{2\pi\hbar} \exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=0}^{n}\left[p_j\left(x_{j+1}-x_j\right) - \eta H\left(p_j, \overline{x}_j\right)\right]\right\} .
\end{aligned}
$$

(with $x_0 = x_i$ and $x_{n+1} = x$). In the limit $n \to \infty$ we obtain for the exponent

$$
\begin{aligned}
&\sum_{j=0}^{n}\left[p_j\left(x_{j+1}-x_j\right) - \eta H\left(p_j, \overline{x}_j\right)\right] \\
&\quad = \sum_{j=0}^{n} \eta \left(p_j \frac{x_{j+1}-x_j}{\eta} - H\left(p_j, \overline{x}_j\right)\right) \\
&\quad \underset{n\to\infty}{\longrightarrow} \int_{t_i}^{t} dt' \left[p\dot{x} - H(p, x)\right] . \qquad (1.37)
\end{aligned}
$$

With this result we rewrite (1.36) in symbolic form

$$
K\left(x, t; x_i, t_i\right) = \int \mathcal{D}x \int \mathcal{D}p \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{t_i}^{t} dt' \left[p\dot{x} - H(p,x)\right]} , \qquad (1.38)
$$

where $\mathcal{D}x$ stands for $\prod \mathrm{d}x_j$ and $\mathcal{D}p$ for $\prod \mathrm{d}p_j/(2\pi\hbar)$. The integrals here are limits of n-dimensional integrals over x and p for $n \to \infty$, they are integrals over all functions $x(t)$ and $p(t)$.

Equation (1.38) represents an important result. It allows to calculate the propagator and thus the solution of the Schrödinger equation in terms of a path integral over classical functions.

Even though the propagator (1.38) looks like a path integral over an exponential function of the action, this is in general not the case, because

$$
p\dot{x} - H(p, x) = \overline{L}(x, \dot{x}, p) \qquad (1.39)
$$

is not equal to the classical Lagrange function since p is not the canonical momentum, but just an integration variable. Therefore, in general one cannot express (1.38) in terms of the action.

1.3 Quadratic Hamiltonians

A further simplification of the path integral is possible for a special p-dependence of the Hamiltonian. If H depends at most quadratically on p, then the path integration over the momentum p can be performed. This will be discussed in the next 2 sections.

1.3.1 Cartesian metric

Even though we have made in the last section the special ansatz $H = p^2/2m + V$ (1.29) the result (1.38) is in general valid for every H of the form

$$H = f(p) + g(x) \ , \tag{1.40}$$

as we have already mentioned above. For the special case (1.29), in which H depends quadratically on p with constant coefficient (i.e. $f(p) \sim p^2$), we can further simplify the path-integral (1.36)

$$\begin{aligned} K(x,t;x_i,t_i) &= \lim_{n\to\infty} \int \prod_{j=1}^{n} \mathrm{d}x_j \prod_{j=0}^{n} \int \frac{\mathrm{d}p_j}{2\pi\hbar} \\ &\quad \times \exp\left\{ \frac{\mathrm{i}}{\hbar}\eta \sum_{j=0}^{n} \left(p_j \frac{x_{j+1}-x_j}{\eta} - \frac{p_j^2}{2m} - V(\overline{x}_j) \right) \right\} \ . \end{aligned} \tag{1.41}$$

Using the integral relation

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-ap^2+bp+c} \mathrm{d}p = \sqrt{\frac{\pi}{a}} \mathrm{e}^{\frac{b^2}{4a}+c} \ , \tag{1.42}$$

we obtain by performing the p-integration

$$\begin{aligned} K &= \lim_{n\to\infty} \left(\frac{m}{i2\pi\hbar\eta} \right)^{\frac{n+1}{2}} \\ &\quad \times \int \prod_{j=1}^{n} \mathrm{d}x_j \ \exp\left\{ \frac{\mathrm{i}}{\hbar}\eta \sum_{j=0}^{n} \left[\frac{m}{2} \left(\frac{x_{j+1}-x_j}{\eta} \right)^2 - V(\bar{x}_j) \right] \right\} \\ &= N \int \mathcal{D}x \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} \int_{t_i}^{t} dt' L(x,\dot{x}) \mathrm{d}t'} \ , \end{aligned} \tag{1.43}$$

with $L(x,\dot{x}) = \frac{m}{2}\dot{x}^2 - V(x)$. Thus in this special case ($H = p^2/2m + V$) K is given by

$$K(x,t;x_i,t_i) = N \int \mathcal{D}x \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S} \ , \tag{1.44}$$

where S is the classical action $S = \int L dt$. K has thus been reduced to a one-dimensional path-integral, which is only possible for Hamiltonians which are quadratic in p. This is a quite important result that we will use throughout all the following sections.

At this point we should realize that in going from (1.41) to (1.43) we have integrated an oscillatory integrand ($\mathrm{e}^{\mathrm{i} f(p)}$) over an infinite interval. This was only possible by a mathematical trick: in applying the Gaussian integration formula (1.42) we have in effect used the quantity $i\eta$ in (1.41) as if it were real. In other words: we have analytically continued expression (1.41) into the complex plane by setting the time interval $\eta \to \mathrm{i}\eta'$ with η' real and positive. Then (1.41) becomes a well-behaved Gaussian integral. After performing the integration we have then gone back to the original η. This analytical continuation is in general possible only if no singularities are encountered while going to the real variable η'. Also, one has to worry about phase ambiguities connected with the appearance of the squareroot of a complex number.

1.3.2 Non-Cartesian metric

The momentum integration is even possible for Hamiltonians of the form

$$H = \frac{1}{2m} f_1(x) p^2 + f_2(x) p + f_3(x) \ , \tag{1.45}$$

but here a problem arises because the canonical quantization of such a Hamiltonian is ambiguous. This is so because the classical coordinates and momenta commute, so that H can be brought into various forms that are classically all equal, but differ after quantization because the operators $\hat{p}$ and $\hat{x}$ do not commute. Even though the time development operator (1.28) could still be evaluated for any of these various forms, these would in general not lead to a path integral of the form (1.38).

We, therefore, turn the question around and ask:
Given a path integral

$$\int \mathcal{D}x \int \mathcal{D}p \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} \int\limits_{t_i}^{t} \mathrm{d}t' \, (p\dot{x} - H(p,x))} \tag{1.46}$$

with the classical Hamiltonian (1.53), can this still be identified as a propagator and, if so, for which Hamiltonian?

The answer to this question is given here without proof[1]:

$$\int \mathcal{D}x \int \mathcal{D}p \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int\limits_{t_i}^{t} \mathrm{d}t' \, (p\dot{x} - H(p,x))} = \langle x' | \mathrm{e}^{-\frac{i}{\hbar}(t-t_i)\hat{H}_W} | x \rangle \; . \tag{1.47}$$

Here $\hat{H}_W$ is the "Weyl-ordered" Hamiltonian

$$\begin{aligned} \hat{H}_W \;&=\; \frac{1}{8m}\left(\hat{p}^2 f_1(x) + 2\hat{p} f_1(x)\hat{p} + f_1(x)\hat{p}^2\right) \\ &\quad + \frac{1}{2}\left[\hat{p} f_2(x) + f_2(x)\hat{p}\right] + f_3(x) \; . \end{aligned} \tag{1.48}$$

In this form the momentum- and coordinate-dependent terms are symmetrized. Note that the path integral in (1.47) is well determined because all quantities on the rhs of this equation are classical, commuting quantities.

The lhs of (1.47) can be simplified by performing the integration over the momenta in

$$\begin{aligned} K \;&=\; \lim_{n\to\infty} \int \prod_{j=1}^{n} \mathrm{d}x_j \prod_{j=0}^{n} \frac{\mathrm{d}p_j}{2\pi\hbar} \\ &\quad \times \exp\left\{ \frac{\mathrm{i}}{\hbar} \sum_{j=0}^{n} \eta \left[p_j\,(x_{j+1} - x_j) - \left(f_1(x_j)\frac{p_j^2}{2m} + f_2(x_j)p_j + f_3(x_j) \right) \right] \right\} \; . \end{aligned} \tag{1.49}$$

Using the Gaussian integral formula (1.42) this gives

$$\begin{aligned} K \;&=\; \lim_{n\to\infty} \left(\frac{m}{i2\pi\hbar\eta}\right)^{\frac{n+1}{2}} \int \prod_{j=1}^{n} \mathrm{d}x_j \frac{1}{\sqrt{f_1(\overline{x}_j)}} \\ &\quad \times \exp\left\{ -\frac{1}{\hbar^2} \, \frac{\left[(x_{j+1} - x_j) - \eta f_2(\overline{x})\right]^2 2m\hbar}{4 f_1(\overline{x})\mathrm{i}\eta} - \frac{\mathrm{i}}{\hbar}\eta f_3(\overline{x}) \right\} \; . \end{aligned} \tag{1.50}$$

The exponent is

$$\begin{aligned} \{\ldots\} \;&=\; \frac{\mathrm{i}}{\hbar}\eta \left(\frac{m}{2} \frac{\left[\frac{x_{j+1}-x_j}{\eta} - f_2(\overline{x})\right]^2}{f_1(\overline{x})} - f_3(\overline{x}) \right) \\ &\underset{\eta\to 0}{\longrightarrow} \frac{\mathrm{i}}{\hbar}\eta \left(\frac{m}{2} \frac{\dot{x}^2 - 2\dot{x} f_2(\overline{x}) + f_2^2(\overline{x})}{f_1(\overline{x})} - f_3(\overline{x}) \right) \; . \end{aligned} \tag{1.51}$$

This last expression can be rewritten into a form that contains the Lagrangian. In order to see this we now consider the case that L is quadratic in

[1] The proof can be found in [LEE], p. 475

$\dot{x}$, but with an x-dependent mass, corresponding to a non-cartesian metric; in addition it can also contain a term linear in $\dot{x}$

$$L = \frac{m}{2} g_1(x)\dot{x}^2 + g_2(x)\dot{x} + g_3(x) \ . \tag{1.52}$$

A kinetic term of this form appears, for example, when rewriting the kinetic energy of a particle from cartesian into polar coordinates. In this case the corresponding classical Hamiltonian is given by

$$\begin{aligned} H &= p\dot{x} - L = (mg_1\dot{x} + g_2)\,\dot{x} - \frac{m}{2} g_1 \dot{x}^2 - g_2\dot{x} - g_3 \\ &= \frac{m}{2} g_1 \dot{x}^2 - g_3 \\ &= \frac{m}{2} g_1 \left(\frac{p - g_2}{mg_1} \right)^2 - g_3 \\ &= \frac{1}{2m}\frac{1}{g_1} p^2 - \frac{g_2}{mg_1} p + \frac{1}{2m}\frac{g_2^2}{g_1} - g_3 \\ &\equiv \frac{1}{2m} f_1(x) p^2 + f_2(x) p + f_3(x) \ . \end{aligned} \tag{1.53}$$

We therefore obtain with

$$f_1 = \frac{1}{g_1} \ , \quad f_2 = -\frac{g_2}{mg_1} \ , \quad f_3 = \frac{g_2^2}{2mg_1} - g_3 \tag{1.54}$$

the result

$$\{\ldots\} = \frac{\mathrm{i}}{\hbar}\eta \left(g_1 \frac{m}{2} \dot{x}^2 + g_2 \dot{x} + g_3 \right) \ . \tag{1.55}$$

We thus have for the complete propagator in this case

$$\begin{aligned} &K(x,t;x_i,t_i) \\ &= \lim_{n\to\infty} \left(\frac{m}{i2\pi\hbar\eta} \right)^{\frac{n+1}{2}} \int \prod_{j=1}^{n} \mathrm{d}x_j \sqrt{g_1(\overline{x}_j)} \sqrt{g_1(\overline{x}_0)} \, \exp\left\{ \frac{\mathrm{i}}{\hbar}\eta \sum_{j=1}^{n} L(\overline{x}_j, \dot{x}_j) \right\} \\ &= N \int \mathcal{D}x \sqrt{g_1(x)} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{t_i}^{t} L(x,\dot{x})\mathrm{d}t} \ . \end{aligned} \tag{1.56}$$

Thus, in this case the path integral is changed. The square root of a function that determines the metric of the system appears in the integrand. In the more general case of curvilinear coordinates an additional potential term appears in the Lagrangian [LEE].

1.4 Classical Interpretation

The simple form (1.44) for the path integral allows a very physical interpretation of the classical limit to quantum mechanics. For the classical path the variation of the action is, according to Hamilton's principle, equal to zero, i.e. the action is stationary

$$\delta S = \int_{t_i}^{t} L\left(q_{cl} + \delta q, \dot{q}_{cl} + \delta \dot{q}\right) \mathrm{d}t - \int_{t_i}^{t} L\left(q_{cl}, \dot{q}_{cl}, t\right) = 0 \; . \tag{1.57}$$

This implies that all the paths close to the classical path give about equal contributions to the path integral. For each path (C_1) somewhat more removed from the classical one there will also be another one, (C_2), whose action differs from that on C_1 just by $\pi\hbar$. Then we have

$$\mathrm{e}^{\frac{\mathrm{i}}{\hbar} S(C_1)} = \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S(C_2) + \mathrm{i}\pi} = -\mathrm{e}^{\frac{\mathrm{i}}{\hbar} S(C_2)} \; , \tag{1.58}$$

so that the contributions from these two paths cancel each other. Sizeable contributions to the path integral thus come only from paths close to the classical one

$$K\left(x, t; x_i, t_i\right) \sim \mathrm{e}^{\frac{\mathrm{i}}{\hbar} \int_{t_i}^{t} L(x_{cl}, \dot{x}_{cl}, t)\, \mathrm{d}t} \; . \tag{1.59}$$

This equation forms the basis for a semiclassical approximation. Quantum mechanics describes the fluctuations of the action around the classical path.

Chapter 2

PERTURBATION THEORY

In most cases the exact propagator cannot be calculated. Thus one has to resort to perturbation theory.

2.1 Perturbative Expansion

We thus assume that the unperturbed particle moves freely and that the perturbing interaction is given by $V(x,t)$. We furthermore assume that H has the special form $H = p^2/2m + V(x,t)$, so that the propagator is given by

$$K\left(x,t;x_i,t_i\right) = N \int \mathcal{D}x \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S} \tag{2.1}$$

with

$$S = \int\limits_{t_i}^{t} L(x,\dot{x}) \, \mathrm{d}t' = \int\limits_{t_i}^{t} \left(\frac{m}{2}\dot{x}^2 - V(x,t)\right) \, \mathrm{d}t \ . \tag{2.2}$$

Since the integrand here is a classical function, we have

$$\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S} = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int\limits_{t_i}^{t}\frac{m}{2}\dot{x}^2\,\mathrm{d}t} \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int\limits_{t_i}^{t}V(x,t)\,\mathrm{d}t} \ . \tag{2.3}$$

The second factor can now be expanded in powers of the potential, thus yielding a perturbative expansion of the action

$$\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int\limits_{t_i}^{t}V(x,t)\,\mathrm{d}t} \cong 1 - \frac{\mathrm{i}}{\hbar}\int\limits_{t_i}^{t} V(x,t)\,\mathrm{d}t - \frac{1}{2!}\frac{1}{\hbar^2}\left(\int\limits_{t_i}^{t} V(x,t)\,\mathrm{d}t\right)^2 + \ldots \ . \tag{2.4}$$

When we substitute this expansion into the expression (2.1) we obtain

$$
\begin{aligned}
K = N \int \mathcal{D}x \Bigg\{ & e^{\frac{i}{\hbar}S_0} \\
& \times \left[1 - \frac{i}{\hbar} \int\limits_{t_i}^{t} V(x,t)\,dt - \frac{1}{2!}\frac{1}{\hbar^2} \left(\int\limits_{t_i}^{t} V(x,t)\,dt \right)^2 + \ldots \right] \Bigg\} \\
= \; & K_0 + K_1 + K_2 + \ldots \; , \qquad (2.5)
\end{aligned}
$$

i.e. a sum ordered in powers of the interaction.

Free propagator. The free propagator K_0 is given by

$$
\begin{aligned}
K_0 &= N \int \mathcal{D}x \, e^{\frac{i}{\hbar}S_0} \\
&= \lim_{n\to\infty} \left(\frac{m}{2\pi i\hbar\eta} \right)^{\frac{n+1}{2}} \int\limits_{-\infty}^{+\infty} \prod_{j=1}^{n} dx_j \exp\left[\frac{i}{\hbar} \eta \sum_{j=0}^{n} \frac{m}{2} \left(\frac{x_{j+1}-x_j}{\eta} \right)^2 \right] \; . \qquad (2.6)
\end{aligned}
$$

This path-integral can be performed exactly. Similar to the Gaussian integral (1.42) the following integral relation, which can be proven by induction from n to $n+1$, holds also

$$
\begin{aligned}
I_n &= \int\limits_{-\infty}^{+\infty} dx_1 \ldots dx_n \\
&\quad \times \exp\left\{ i\lambda \left[(x_1-a)^2 + (x_2-x_1)^2 + \ldots + (b-x_n)^2 \right] \right\} \\
&= \left(\frac{i^n \pi^n}{(n+1)\lambda^n} \right)^{\frac{1}{2}} \exp\left(\frac{i\lambda}{n+1} (b-a)^2 \right) \; . \qquad (2.7)
\end{aligned}
$$

With this result we obtain for the free propagator

$$
\begin{aligned}
K_0 &= \lim_{n\to\infty} \left(\frac{m}{2\pi i\hbar\eta} \right)^{\frac{n+1}{2}} \left(\frac{i^n \pi^n}{(n+1)\left(\frac{m}{2\hbar\eta}\right)^n} \right)^{\frac{1}{2}} \qquad (2.8) \\
&\quad \times \exp\left(\frac{i}{n+1} \frac{m}{2\hbar\eta} (x-x_i)^2 \right) \; ,
\end{aligned}
$$

since $x_{n+1} = x$, $x_0 = x_i$. With $(n+1)\eta = t - t_i$ this becomes

$$
\begin{aligned}
K_0 &= \sqrt{\frac{m}{2\pi\hbar i\,(t-t_i)}}\, e^{i\frac{m(x-x_1)^2}{2\hbar(t-t_i)}} = \sqrt{\frac{m}{2\pi\hbar i\Delta t}} e^{i\frac{m\Delta x^2}{2\hbar\Delta t}} \\
&= K_0(\Delta x, \Delta t) \qquad (2.9)
\end{aligned}
$$

with $\Delta x = x - x_i$, $\Delta t = t - t_i$. This is the propagator of a free particle for $\Delta t \geq 0$; for $\Delta t < 0$ it has to be supplemented by the condition $K = 0$. Because of Galilei invariance and time homogeneity the free particle propagator depends only on the space- and time-distances.

Since a free particle has conserved momentum, it is advantageous to transform K_0 into the momentum representation

$$\begin{aligned} K_0(p, \Delta t) &= \frac{1}{\sqrt{2\pi\hbar}} \int \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} p \Delta x} K_0\left(\Delta x, \Delta t\right) \mathrm{d}\Delta x \\ &= \frac{1}{\sqrt{2\pi\hbar}} \sqrt{\frac{m}{2\pi\hbar \mathrm{i} \Delta t}} \int \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} p \Delta x} \mathrm{e}^{\mathrm{i} \frac{m}{2\hbar\Delta t} \Delta x^2} \; \mathrm{d}\Delta x \end{aligned} \tag{2.10}$$

We can now use again the integral relation (1.42) in the form

$$\int\limits_{-\infty}^{+\infty} \mathrm{e}^{-a\Delta x^2 + b\Delta x} \mathrm{d}\Delta x = \sqrt{\frac{\pi}{a}}\, \mathrm{e}^{\frac{b^2}{4a}} \tag{2.11}$$

(with $a = -\mathrm{i}m/(2\hbar\Delta t)$ and $b = -\mathrm{i}p/\hbar$) to write

$$\begin{aligned} K_0(p, \Delta t) &= \frac{1}{2\pi\hbar} \sqrt{\frac{m}{\mathrm{i}\Delta t}} \sqrt{\frac{\pi 2\hbar\Delta t}{-\mathrm{i}m}} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{p^2}{2m}\Delta t} \\ &= \frac{1}{\sqrt{2\pi\hbar}} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{p^2}{2m}\Delta t} \; . \end{aligned} \tag{2.12}$$

The boundary condition $t \geq t_i$ can be incorporated into this expression by multiplying with $\Theta(t - t_i)$ and using the relation

$$\Theta(\Delta t) = \frac{1}{2\pi\mathrm{i}} \int\limits_{-\infty}^{+\infty} \mathrm{d}\omega \, \frac{\mathrm{e}^{\mathrm{i}\omega\Delta t}}{\omega - \mathrm{i}\varepsilon} \quad (\varepsilon > 0) \; , \tag{2.13}$$

which follows directly from the residue theorem (if $\Delta t < 0$, then the loop integration has to be closed in the lower half-space thus missing the pole at $\omega = +\mathrm{i}\varepsilon$). Multiplying (2.12) with (2.13) gives

$$\begin{aligned} K_0\left(x, t; x_i, t_i\right) &= \frac{1}{\sqrt{2\pi\hbar}} \int \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p \Delta x} K_0(p, \Delta t) \Theta(\Delta t) \mathrm{d}p \\ &= \frac{1}{(2\pi)^2 \hbar \mathrm{i}} \int\limits_{-\infty}^{+\infty} \mathrm{d}p \, \mathrm{d}\omega \, \frac{\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\left(p\Delta x - \frac{p^2}{2m}\Delta t + \hbar\omega\Delta t\right)}}{\omega - \mathrm{i}\varepsilon} \; . \end{aligned} \tag{2.14}$$

We now put

$$E = \frac{p^2}{2m} - \hbar\omega \tag{2.15}$$

and obtain

$$K_0\,(x,t;x_i,t_i) = \frac{\mathrm{i}\hbar}{(2\pi\hbar)^2}\int \mathrm{d}p\,\mathrm{d}E\,\frac{\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p\Delta x - E\Delta t)}}{E-\frac{p^2}{2m}+\mathrm{i}\varepsilon} \tag{2.16}$$

(the "$-$" sign coming from the substitution is cancelled by another sign obtained by inverting the integration boundaries).

The integrand on the right-hand side of (2.16) is the free propagator in the momentum representation. Since p and E are independent variables in the integral, we see that propagation also takes place at energies $E \neq p^2/2m$. The classical dispersion relation does show up as a pole in the propagator.

First order propagator. We next determine the first-order propagator K_1. According to (2.5) it is given by

$$\begin{aligned} K_1 &= -\frac{\mathrm{i}}{\hbar}N\int \mathcal{D}x\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_0}\int_{t_i}^{t_f} V(x,t)\,\mathrm{d}t \\ &= -\frac{\mathrm{i}}{\hbar}\lim_{n\to\infty}\left(\frac{m}{2\pi\hbar\mathrm{i}\eta}\right)^{\frac{n+1}{2}} \\ &\quad\times\int_{-\infty}^{+\infty}\prod_{i=1}^{n}\mathrm{d}x_i\,\sum_{k=1}^{n}V\left(x_k,t_k\right)\eta\exp\left(\mathrm{i}\frac{m}{2\hbar\eta}\sum_{j=0}^{n}\left(x_{j+1}-x_j\right)^2\right)\;. \end{aligned} \tag{2.17}$$

Here the time-integral has been written as a sum. In a next step we now split the sum in the exponent into two pieces, one running from $j=0$ to $j=k-1$ and the other from $j=k$ to $j=n$ and separate the corresponding integrals. This gives

$$\begin{aligned} K_1 &= -\frac{\mathrm{i}}{\hbar}\lim_{n\to\infty}\sum_{k=1}^{n}\eta\int\mathrm{d}x_k\,\Bigg[V\left(x_k,t_k\right) \\ &\quad\times\left(N^{\frac{k}{2}}\int\mathrm{d}x_1\,\mathrm{d}x_2\,\cdots\,\mathrm{d}x_{k-1}\,\mathrm{e}^{\mathrm{i}\frac{m}{2\hbar\eta}\sum_{j=0}^{k-1}(x_{j+1}-x_j)^2}\right) \\ &\quad\times\left(N^{\frac{n-k+1}{2}}\int\mathrm{d}x_{k+1}\,\cdots\,\mathrm{d}x_n\mathrm{e}^{\mathrm{i}\frac{m}{2\hbar\eta}\sum_{j=k}^{n}(x_{j+1}-x_j)^2}\right)\Bigg] \end{aligned} \tag{2.18}$$

with

$$N \equiv \frac{m}{2\pi\hbar\mathrm{i}\eta}\;. \tag{2.19}$$

The term in the first bracket is nothing else than the propagator from t_i to t_k $(K_0(x_k, t_k; x_i, t_i))$, and that in the second bracket is that from t_k to t $(K_0(x, t; x_k, t_k))$. Thus we have

$$K_1(x_f, t_f; x_i, t_i) = -\frac{\mathrm{i}}{\hbar}\int\limits_{-\infty}^{+\infty} \mathrm{d}x \int\limits_{t_i}^{t_f} \mathrm{d}t\, K_0(x_f t_f; x, t)\, V(x,t) K_0(x, t; x_i, t_i) \ . \tag{2.20}$$

The time integral over the interval from t_i to t_f can be extended to ∞ by noting that

$$\begin{aligned} K_0(x, t; x_i, t_i) &= 0 \qquad \text{for } t < t_i \\ K_0(x_f, t_f; x, t) &= 0 \qquad \text{for } t_f < t \ . \end{aligned} \tag{2.21}$$

This gives

$$K_1(x_f, t_f; x_i, t_i) = -\frac{\mathrm{i}}{\hbar}\int\limits_{-\infty}^{+\infty} \mathrm{d}t \int\limits_{-\infty}^{+\infty} \mathrm{d}x\, K_0(x_f, t_f; x, t)\, V(x,t) K_0(x, t; x_i, t_i) \ . \tag{2.22}$$

Higher order propagators. Similarly, the higher order terms in the perturbation expansion (2.5) can be obtained. This yields finally

$$\begin{aligned} K(x_f, t_f; x_i, t_i) = \\ & K_0(x_f, t_f) - \frac{\mathrm{i}}{\hbar}\int K_0(x_f, t_f; x_1, t_1)\, V(x_1, t_1)\, K_0(x_1, t_1; x_i, t_i)\, \mathrm{d}x_1\, \mathrm{d}t_1 \\ & - \frac{1}{\hbar^2}\int \Big[K_0(x_f, t_f; x_1, t_1)\, V(x_1, t_1)\, K_0(x_1, t_1; x_2, t_2) \\ & \qquad \times V(x_2, t_2)\, K_0(x_2, t_2; x_i, t_i)\Big] \mathrm{d}x_1\, \mathrm{d}t_1\, \mathrm{d}x_2\, \mathrm{d}t_2 \\ & + \ \cdots \ . \end{aligned} \tag{2.23}$$

Note that the time-integrations in (2.23) are effectively time-ordered because of the implicit $\Theta(t_f - t_i)$ function in each of the propagators, leading to $t_f > t_1 > t_2 > \cdots > t_i$. This fact explains why, for example, the factor $1/(2!)$ in the second order term of (2.5) no longer appears in (2.23)

$$\begin{aligned} \frac{1}{2!}\left(\int \mathrm{d}t V(x,t)\right)^2 &= \frac{1}{2!}\int V(x,t)V(x,t')\,\mathrm{d}t\,\mathrm{d}t' \\ &= \frac{1}{2!}\int \left[V(x,t)V(x,t')\Theta(t'-t) + V(x,t)V(x,t')\Theta(t-t')\right]\,\mathrm{d}t\,\mathrm{d}t' \\ &= \int V(x,t)V(x,t')\Theta(t'-t)\ \mathrm{d}t\,\mathrm{d}t' \ . \end{aligned} \tag{2.24}$$

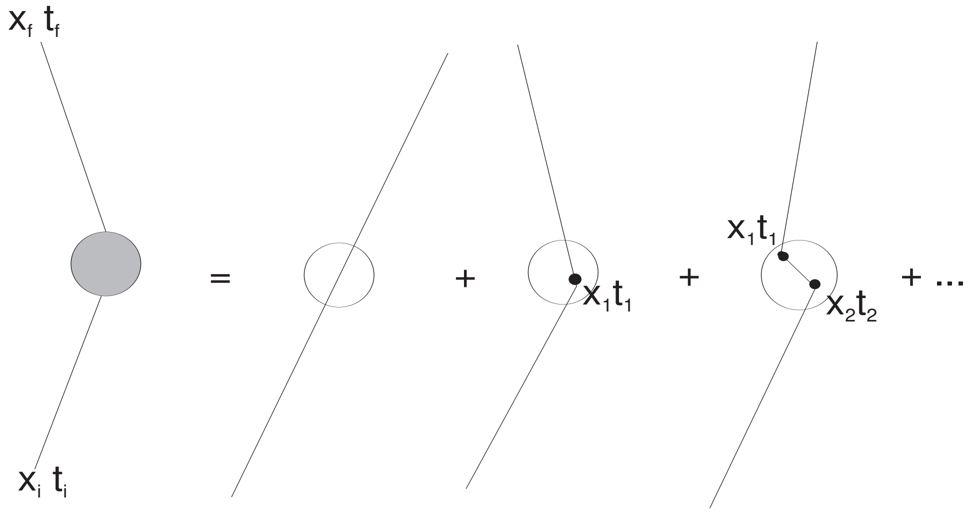


Figure 2.1: Born series for the propagator

The last line follows from a simple change of variables; the Θ function in it is absorbed into the propagator in (2.23).

Equation (2.23) is the Born series for the propagator; it can be represented graphically as shown in Fig. 2.1. A time axis runs here from bottom to top. The straight lines denote the free propagation of the particles (K_0) and the dots stand for the interaction vertices ($-\mathrm{i}V/\hbar$) (the circles mark the interaction range) and a space and time integration appears at each vertex.

Bethe-Salpeter equation. The expansion (2.23) can be formally summed. This can be seen as follows (in obvious shortterm notation)

$$\begin{aligned} K &= K_0 + K_0UK_0 + K_0UK_0UK_0 + \cdots \\ &= K_0 + K_0U\left(K_0 + K_0UK_0 + \cdots\right) \qquad \text{with } U = -\tfrac{\mathrm{i}}{\hbar}V \ . \end{aligned} \tag{2.25}$$

The expression in parentheses is just again K, so that we obtain the *Bethe-Salpeter* equation

$$K = K_0 + K_0UK \ . \tag{2.26}$$

The Bethe-Salpeter equation is an integral equation for the full interacting propagator K as can be seen most easily from its space-time representation

$$\begin{aligned} K\left(x_f,t_f;x_i,t_i\right) &= K_0\left(x_f,t_f;x_i,t_i\right) \\ &\quad -\frac{\mathrm{i}}{\hbar}\int K_0\left(x_f,t_f;x,t\right)V(x,t)K\left(x,t;x_i,t_i\right)\mathrm{d}x\,\mathrm{d}t \ . \end{aligned} \tag{2.27}$$

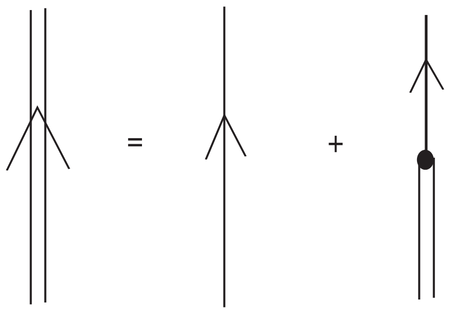


Figure 2.2: Bethe-Salpeter equation (2.29).

This equation can also be written in an equivalent form for the interacting wavefunction

$$\begin{aligned}
\Psi\left(x_f, t_f\right) &= \int K\left(x_f, t_f; x_i, t_i\right) \Psi\left(x_i, t_i\right) \mathrm{d}x_i \\
&= \int K_0\left(x_f, t_f; x_i, t_i\right) \Psi\left(x_i, t_i\right) \mathrm{d}x_i \\
&\quad - \frac{\mathrm{i}}{\hbar} \int K_0\left(x_f, t_f; x, t\right) V(x,t) K\left(x, t; x_i, t_i\right) \Psi\left(x_i, t_i\right) \mathrm{d}x_i \, \mathrm{d}x \, \mathrm{d}t \\
&= \int K_0\left(x_f, t_f; x_i, t_i\right) \Psi\left(x_i, t_i\right) \mathrm{d}x_i \qquad (2.28)\\
&\quad - \frac{\mathrm{i}}{\hbar} \int K_0\left(x_f, t_f; x, t\right) V(x,t) \Psi(x,t) \, \mathrm{d}x \, \mathrm{d}t \ .
\end{aligned}$$

This constitutes an integral equation for the unknown wavefunction $\Psi(x,t)$.

In the diagrammatic way we can also represent the Bethe-Salpeter equation (2.27)

$$\begin{aligned}
K\left(x_f, t_f; x_i, t_i\right) &= K_0\left(x_f, t_f; x_i, t_i\right) \qquad (2.29)\\
&\quad - \frac{\mathrm{i}}{\hbar} \int K_0\left(x_f, t_f; x, t\right) V(x,t) K\left(x, t; x_i, t_i\right) \mathrm{d}x \, \mathrm{d}t \ .
\end{aligned}$$

If we denote the so-called "dressed propagator" K, that includes all the effects of the interactions, by a double line

$$= K\left(x_2, t_2; x_1, t_1\right) \ , \qquad (2.30)$$

then this equation can be graphically represented as shown in Fig. 2.2.

2.2 Application to Scattering

Let us now apply the results of the last section to a scattering process. In this case the particle is free at $t=-\infty$, then undergoes the scattering interaction and then, at $t=+\infty$, is free again.

We treat this problem as usual by adiabatically switching on and off the interaction $V(x,t)$. The initial condition (for $t \to -\infty$) for the wavefunction then is

$$\Psi_{\mathrm{in}}(x,t) = N\mathrm{e}^{\mathrm{i}(\vec{k}_i\cdot\vec{x}-\omega_i t)} \ . \tag{2.31}$$

We choose here a box normalization with periodic boundary conditions so that $N=1/\sqrt{V}$. The scattering state that evolves from this incoming state is denoted by $\Psi^{(+)}(x,t)$. The superscript $(+)$ indicates that the state evolves forward in time, starting from Ψ_{in} at $t=-\infty$; it thus fulfills the boundary condition

$$\Psi^{(+)}(x,t\to-\infty) = \Psi_{\mathrm{in}}(x,t) \ . \tag{2.32}$$

In a scattering experiment one looks at $t\to+\infty$ for a free scattered particle with definite momentum; the corresponding final state is denoted by

$$\Psi_{\mathrm{out}}(x,t) = N\mathrm{e}^{\mathrm{i}(\vec{k}_f\cdot\vec{x}-\omega_f t)} \ . \tag{2.33}$$

The probability amplitude for the presence of Ψ_{out} in the scattered state $\Psi^{(+)}$ is given by

$$S_{fi} = \int \Psi^*_{\mathrm{out}}\left(x_f,t_f\right)\Psi^{(+)}\left(x_f,t_f\right)\mathrm{d}^3x_f \qquad \text{for} \qquad t_f\to\infty \ . \tag{2.34}$$

This is just the transition amplitude from the initial state i to the final state f (S-matrix). By introducing the scattering state $\Psi^{(-)}$ that evolves backwards in time, starting from Ψ_{out} at $t=+\infty$, we can write the S-matrix also as

$$S_{fi} = \int \Psi^{(-)^*}(x,t\to+\infty)\Psi^{(+)}(x,t\to+\infty)\,\mathrm{d}^3x \ . \tag{2.35}$$

Expansion (2.27) yields the Bethe-Salpeter equation for the scattering wavefunction

$$\begin{aligned}\Psi^{(+)}\left(x_f,t_f\right) &= \int K_0\left(x_f,t_f;x_i,t_i\right)\Psi_{\mathrm{in}}\left(x_i,t_i\right)\ \mathrm{d}^3x_i \\ &- \frac{\mathrm{i}}{\hbar}\int K_0\left(x_f,t_f;x,t\right)V(x,t)K(x,t;x_i,t_i)\Psi_{\mathrm{in}}(x_i,t_i)\,\mathrm{d}^3x_i\,\mathrm{d}^3x\,\mathrm{d}t \ .\end{aligned} \tag{2.36}$$

Inserting this into (2.34) gives for the S-matrix

$$\begin{aligned}S_{fi} \ &= \ \int \Psi^*_{\mathrm{out}}\left(x_f,t_f\right)K_0\left(x_f,t_f;x_i,t_i\right)\Psi_{\mathrm{in}}\left(x_i,t_i\right)\mathrm{d}^3x_i\mathrm{d}^3x_f \\ &\quad - \frac{\mathrm{i}}{\hbar}\int \Psi^*_{\mathrm{out}}\left(x_f,t_f\right)K_0\left(x_f,t_f;x,t\right)V(x,t) \\ &\qquad \times K(x,t;x_i,t_i)\Psi_{\mathrm{in}}\left(x_i,t_i\right)\mathrm{d}^3x_i\,\mathrm{d}^3x\,\mathrm{d}t\,\mathrm{d}^3x_f \ .\end{aligned} \tag{2.37}$$

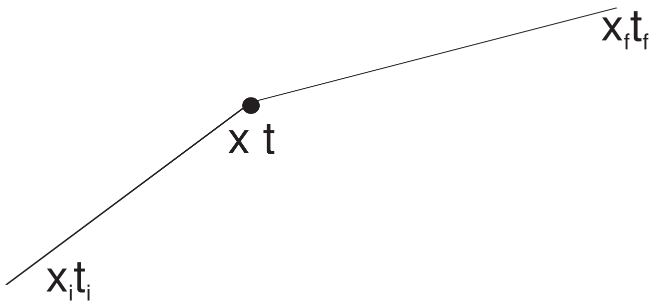


Figure 2.3: First order scattering diagram, corresponding to the second term in (2.41). Time runs from left to right.

Since Ψ_{in} is a plane wave, we know that

$$\phi\left(x_f, t_f\right) = \int K_0\left(x_f, t_f; x_i, t_i\right) \Psi_{\text{in}}\left(x_i, t_i\right) \mathrm{d}^3 x_i \tag{2.38}$$

is also a plane wave state, since K_0 is the free propagator so that no interaction takes place. ϕ is actually the same wavefunction as Ψ_{in}, only taken at a later time and space point

$$\phi\left(x_f, t_f\right) = N \mathrm{e}^{\mathrm{i}\left(\vec{k}_i \cdot \vec{x}_f - \omega_i t_f\right)} . \tag{2.39}$$

This means that the first integral in (2.37) can be easily evaluated

$$\int \Psi_{\text{out}}^*\left(x_f, t_f\right) \phi\left(x_f, t_f\right) \mathrm{d}^3 x_f = \delta^3\left(\vec{k}_f - \vec{k}_i\right) . \tag{2.40}$$

We thus have

$$\begin{aligned} S_{fi} = \delta^3\left(\vec{k}_f - \vec{k}_i\right) - \frac{\mathrm{i}}{\hbar} \int \mathrm{d}^3 x_f\, \mathrm{d}^3 x\, \mathrm{d}^3 x_i\, \mathrm{d}t \\ \times\left[\Psi_{\text{out}}^*\left(x_f, t_f\right) K_0\left(x_f, t_f; x, t\right) V(x, t) K(x, t; x_i, t_i) \Psi_{\text{in}}\left(x_i, t_i\right)\right] . \end{aligned} \tag{2.41}$$

The amplitude for a scattering process is given by the second term.

If K on the rhs of (2.41) is now represented by the Born series expansion of the Bethe-Salpeter equation (2.25) this amplitude can be graphically represented as shown in Fig. 2.3. This diagram can be translated into the amplitude just given by writing for each straight-line piece

$$x_1 t_1 \quad \text{——} \quad x_2 t_2 \quad = K_0\left(x_2, t_2; x_1, t_1\right) \tag{2.42}$$

and for each interaction vertex

$$\underset{\mathsf{xt}}{\bullet} \quad = -\frac{\mathrm{i}}{\hbar} V(x, t) + \text{ integration over } x, t . \tag{2.43}$$

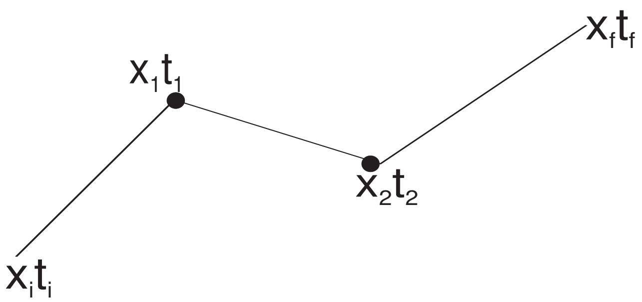


Figure 2.4: Second order scattering diagram, corresponding to (2.44).

The rules are completed by multiplying $\Psi_{\rm in}$ and $\Psi^*_{\rm out}$ at the corresponding sides of the diagram and then integrating over the spatial variables of these wavefunctions and over all intermediate times. The ordering of factors is such that in writing the various factors from left to right one goes *against* the flow of time in the figure.

These rules are illustrated for a second-order scattering process in Fig. 2.4. In this figure the time runs as usual from t_i to t_f. The corresponding amplitude is given by

$$A^{(2)} = \int \mathrm{d}^3x_i\,\mathrm{d}^3x_f\,\mathrm{d}^3x_1\,\mathrm{d}^3x_2\,\mathrm{d}t_1\,\mathrm{d}t_2\Big[\Psi^*_{\rm out}(x_f,t_f)\,K_0(x_f,t_f;x_2,t_2)$$
$$\left(-\frac{\mathrm{i}}{\hbar}V(x_2,t_2)\right)K_0(x_2,t_2;x_1,t_1)\left(-\frac{\mathrm{i}}{\hbar}V(x_1,t_1)\right)K_0(x_1,t_1;x_i,t_i)\,\Psi_{\rm in}(x_i,t_i)\Big]\,. \tag{2.44}$$

We now evaluate the first order scattering amplitude (cf. (2.41))

$$A^{(1)} = -\frac{\mathrm{i}}{\hbar}\int\Big[\Psi^*_{\rm out}(x_f,t_f)\,K_0(x_f,t_f;x,t)\,V(x,t)K_0(x,t;x_i,t_i)$$
$$\times\,\Psi_{\rm in}(x_i,t_i)\Big]\,\mathrm{d}^3x\,\mathrm{d}t\,\mathrm{d}^3x_i\,\mathrm{d}^3x_f \tag{2.45}$$

somewhat further by using expression (2.12) for the free propagator (extended to three dimensions)

$$K_0(x',t';x,t) = \frac{1}{(2\pi\hbar)^3}\int \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\left(\vec{p}\cdot(\vec{x}'-\vec{x})-\frac{p^2}{2m}(t'-t)\right)}\mathrm{d}^3p\;\Theta(t'-t)\,. \tag{2.46}$$

Inserting this into the expression for $A^{(1)}$ gives

$$A^{(1)} = -\frac{i}{\hbar}\frac{1}{V}\frac{1}{(2\pi\hbar)^6}\int \mathrm{d}^3x\,\mathrm{d}t\,\mathrm{d}^3x_i\,\mathrm{d}^3x_f\,\mathrm{d}^3p\,\mathrm{d}^3q \tag{2.47}$$
$$\left(\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(\vec{p}_f\cdot\vec{x}_f-E_ft_f)}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\left(\vec{p}\cdot(\vec{x}_f-\vec{x})-\frac{p^2}{2m}(t_f-t)\right)}V(\vec{x},t)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\left(\vec{q}\cdot(\vec{x}-\vec{x}_i)-\frac{q^2}{2m}(t-t_i)\right)}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(\vec{p}_i\cdot\vec{x}_i-E_it_i)}\right)\,,$$

where the time-ordering $t_f > t > t_i$ is understood and we have abbreviated $E_i = p_i^2/(2m)$ and $E_f = p_f^2/(2m)$.

We integrate first over x_i and x_f. This yields

$$\begin{aligned}\int \mathrm{d}^3x_i &\longrightarrow (2\pi\hbar)^3\delta^3\left(\vec{p}_i - \vec{q}\right) \\ \int \mathrm{d}^3x_f &\longrightarrow (2\pi\hbar)^3\delta^3\left(\vec{p}_f - \vec{p}\right) \ .\end{aligned} \tag{2.48}$$

Performing next the integrals over p and q gives for the amplitude

$$\begin{aligned}A^{(1)} &= -\frac{\mathrm{i}}{\hbar V}\int \mathrm{d}^3x\,\mathrm{d}t \\ &\quad\times\left[\mathrm{e}^{+\frac{\mathrm{i}}{\hbar}E_f t_f}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\left(-\vec{p}_f\cdot\vec{x}-\frac{p_f^2}{2m}\left(t_f-t\right)\right)}V(x,t)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\left(\vec{p}_i\cdot\vec{x}-\frac{p_i^2}{2m}(t-t_i)\right)}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_i t_i}\right] \\ &= -\frac{\mathrm{i}}{\hbar V}\int \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\left(-\vec{p}_f\cdot\vec{x}+E_f t\right)}V(x,t)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(+\vec{p}_i\cdot x-E_i t)}\,\mathrm{d}^3x\,\mathrm{d}t \ .\end{aligned} \tag{2.49}$$

The time-integration is performed next. For that we assume that $V(x,t)$ acts only during a finite, but long time-interval ($[\frac{T}{2},\frac{T}{2}]$), in which it is time-independent; at the boundaries of the interval it is adiabatically, i.e. without any significant energy-transfer, turned on and off. We then obtain

$$\int \mathrm{d}t\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\left(E_f-E_i\right)t}V(x,t)\mathrm{d}t = \int_{-T/2}^{T/2} V(x)\mathrm{e}^{\mathrm{i}(\omega_f-\omega_i)t}\mathrm{d}t \underset{T\to\infty}{\longrightarrow} V(x)2\pi\delta\left(\omega_f-\omega_i\right) \ , \tag{2.50}$$

with $\omega_f - \omega_i = (E_f - E_i)/\hbar$. Thus $A^{(1)}$ becomes

$$A^{(1)} = -\frac{\mathrm{i}}{\hbar V}2\pi\delta\left(\omega_f-\omega_i\right)\int \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\left(\vec{p}_i-\vec{p}_f\right)\cdot\vec{x}}V(\vec{x},t)\,\mathrm{d}^3x \ . \tag{2.51}$$

This is the well-known lowest-order Born-approximation result.

2.2.1 Coulomb scattering

We now specialize to Coulomb scattering. We assume that the Coulomb-potential is adiabatically turned on and off and that it is shielded. Thus

$$V\left(\vec{x},t\right) = \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r}\mathrm{e}^{-\alpha r} \ . \tag{2.52}$$

We thus have to evaluate the integral

$$I = \int \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\left(\vec{p}_i-\vec{p}_f\right)\cdot\vec{x}}\frac{1}{r}\mathrm{e}^{-\alpha r}\mathrm{d}^3x \ . \tag{2.53}$$

Setting now

$$\vec{q} = \vec{p}_i - \vec{p}_f \tag{2.54}$$

and choosing polar coordinates with the z-axis parallel to $\vec{q}$ we have

$$\begin{aligned}
I &= \int \mathrm{e}^{\frac{\mathrm{i}}{\hbar}qr\cos\vartheta}\frac{1}{r}e^{-\alpha r}r^2\mathrm{d}r\sin\vartheta\mathrm{d}\vartheta\mathrm{d}\varphi \qquad (2.55)\\
&= 2\pi\int_0^{+\infty}\mathrm{d}r\; r\; e^{-\alpha r}\int_{-1}^{+1}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}qrx}\Delta x\\
&= 2\pi\int_0^{+\infty}\mathrm{d}r\; r\; e^{-\alpha r}\frac{\hbar}{\mathrm{i}qr}\left(\mathrm{e}^{\frac{\mathrm{i}}{\hbar}qr}-\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qr}\right)\\
&= 4\pi\int_0^{+\infty}\mathrm{d}r\frac{\hbar}{q}\sin\left(\frac{q}{\hbar}r\right)e^{-\alpha r}\\
&= \frac{4\pi\hbar}{q}\int_0^{+\infty}e^{-\alpha r}\sin\left(\frac{q}{\hbar}r\right)\mathrm{d}r\\
&= \frac{4\pi\hbar}{q}\frac{e^{-\alpha r}}{\alpha^2+\left(\frac{q}{\hbar}\right)^2}\left(-\alpha\sin\left(\frac{q}{\hbar}r\right)-\left(\frac{q}{\hbar}\right)\cos\left(\frac{q}{\hbar}r\right)\right)\Bigg|_0^\infty\\
&= \frac{8\pi\hbar}{q}\frac{\hbar^2}{\alpha^2+q^2}\frac{q}{\hbar}\\
&= \frac{8\pi\hbar^2}{\alpha^2+q^2}\quad\underset{\alpha\to 0}{\longrightarrow}\quad\frac{4\pi\hbar^2}{q^2}\;.
\end{aligned}$$

We thus finally have for the amplitude

$$\begin{aligned}
A &= -\frac{\mathrm{i}}{\hbar V}2\pi\delta\left(\omega_f-\omega_i\right)\frac{Ze^2}{4\pi\varepsilon_0}\frac{4\pi\hbar^2}{q^2} \qquad (2.56)\\
&= -\frac{\mathrm{i}}{\hbar V}\frac{2\pi Ze^2}{\varepsilon_0}\frac{1}{q^2}\delta\left(\omega_f-\omega_i\right)\;.
\end{aligned}$$

The cross section is obtained from this amplitude

$$d\sigma = \frac{1}{j_{\mathrm{in}}}|A|^2\frac{V\mathrm{d}^3k_f}{(2\pi)^3}\frac{1}{T} \tag{2.57}$$

where T is the duration of the interaction. The incoming current is

$$j_{\mathrm{in}} = \frac{1}{V}\frac{\hbar k_i}{m} \tag{2.58}$$

so that the differential cross section for scattering into the momentum range between $\vec{k}_f$ and $\vec{k}_f + \mathrm{d}^3k_f$ is

$$\frac{d\sigma}{\mathrm{d}^3k_f} = \frac{mV}{\hbar k_i}\frac{\hbar^2}{V^2}\frac{4\pi^2 Z^2 e^4}{\varepsilon_0^2}\frac{1}{q^4}\left|\delta\left(\omega_f - \omega_i\right)\right|^2\frac{V}{(2\pi)^3}\frac{1}{T} \; . \tag{2.59}$$

The square of the δ-function can be evaluated as follows

$$\begin{aligned}\left|\delta\left(\omega_f - \omega_i\right)\right|^2 &= \delta\left(\omega_f - \omega_i\right)\frac{1}{2\pi}\int\limits_{-T/2}^{T/2} \mathrm{e}^{\mathrm{i}\left(\omega_i - \omega_f\right)t}\mathrm{d}t \\ &= \delta\left(\omega_f - \omega_i\right)\frac{1}{2\pi}T \; .\end{aligned} \tag{2.60}$$

Inserting this yields for the cross section

$$\frac{d\sigma}{\mathrm{d}^3k_f} = \frac{\hbar}{4\pi^2}\;\frac{m}{k_i}\;\frac{Z^2e^4}{\varepsilon_0^2}\;\frac{1}{q^4}\delta\left(\omega_f - \omega_i\right) \; . \tag{2.61}$$

This gives for the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{Z^2e^4 m}{4\pi^2\varepsilon_0^2}\frac{\hbar}{k_i}\frac{1}{q^4}\int\delta\left(\omega_f - \omega_i\right)k_f^2\,\mathrm{d}k_f \; . \tag{2.62}$$

The integral appearing here can be evaluated

$$\begin{aligned}\int\delta\left(\omega_f - \omega_i\right)k_f^2\,dk_f &= \int\delta\left(\frac{\hbar k_f^2}{2m} - \frac{\hbar k_i^2}{2m}\right)k_f^2\,\mathrm{d}k_f \\ &= \int\delta\left(k_f - k_i\right)\frac{m}{\hbar k_f}k_f^2\,\mathrm{d}k_f = \frac{m}{\hbar}k_i \; .\end{aligned} \tag{2.63}$$

The differential cross section is then

$$\frac{d\sigma}{d\Omega} = \left(\frac{Ze^2 m}{2\pi\varepsilon_0}\right)^2\frac{1}{q^4} \; . \tag{2.64}$$

The momentum transfer squared is given by

$$q^2 = 4\hbar^2 k_i^2\sin^2\frac{\vartheta}{2} \; , \tag{2.65}$$

so that we finally obtain

$$\frac{d\sigma}{d\Omega} = \left(\frac{Ze^2}{8\pi\varepsilon_0 m v_i^2}\right)^2\frac{1}{\sin^4\frac{\vartheta}{2}} \; . \tag{2.66}$$

This is the well known Rutherford formula.

Chapter 3

GENERATING FUNCTIONALS

In this section we consider the transition amplitude in the presence of an external "source" $J(t)$, so that the Hamiltonian reads

$$H_J(x,p) = H(x,p) - \hbar J(t)x \ . \tag{3.1}$$

A classical example is that of a harmonic oscillator with externally driven equilibrium position $x_0(t)$. Its Hamiltonian is

$$\begin{aligned} H_{x_0} &= \frac{p^2}{2m} + \frac{1}{2}k(x - x_0(t))^2 \\ &= H - kx_0(t)x + \mathcal{O}(x_0^2) \ . \end{aligned} \tag{3.2}$$

With $\hbar J(t) = kx_0(t)$ and for small amplitudes x_0 we just have the form of (3.1). It is evident that the states of this system will change as time develops, because of its changing equilibrium position.

Suppose now that the system was in its groundstate at $t \to -\infty$ and that $x_0(t)$ acts only for a limited time period. We could then calculate the transition probability for the system to be in any excited state at $t \to +\infty$ by using the techniques developed in sections 2.1 and 2.2. There we saw (cf. (2.23) and (2.44)) that this probability is determined by matrixelements of time-ordered products of the interaction, i.e. of the operator $\hat{x}$ in the present case. In this chapter we will show that these matrixelements can all be generated once only the probability for the system to remain in its groundstate is known.

3.1 Groundstate to Groundstate Transitions

Generalizing the special example of the introduction we assume that the groundstate $|0\rangle$ of an arbitrary physical system is at first (at t_i) stationary and then changes under the influence of a source $J(t)$ of finite duration. We can then calculate the probability amplitude for the system to be still in its groundstate after the source J has acted (at t_f)

$$\langle 0t_f|0t_i\rangle_J \ . \tag{3.3}$$

Through the relation

$$\begin{aligned}\langle 0t_f|0t_i\rangle_J &= \int \mathrm{d}x_f\,\mathrm{d}x_i\,\langle 0|x_f\rangle\langle x_f t_f|x_i t_i\rangle_J\langle x_i|0\rangle \\ &= \int \mathrm{d}x_f\,\mathrm{d}x_i\,\varphi_0^*(x_f)\langle x_f t_f|x_i t_i\rangle_J\varphi_0(x_i) \\ &= \int \mathrm{d}x_f\,\mathrm{d}x_i\,\varphi_0^*(x_f,t_f)\langle x_f t_f|x_i t_i\rangle_J\varphi_0(x_i,t_i)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}E_0(t_i-t_f)}\end{aligned} \tag{3.4}$$

it is determined by the corresponding propagator

$$\langle x_f,t_f|x_i,t_i\rangle_J = \int\mathcal{D}x\int\mathcal{D}p\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int\limits_{t_i}^{t_f}[p\dot{x}-H(x,p)+\hbar J(t)x]\,\mathrm{d}t} \ . \tag{3.5}$$

The transition amplitude (3.4) is thus obviously a functional of the source $J(t)$.

In the following paragraphs we will now discuss this functional dependence for $t_f \to +\infty$ and $t_i \to -\infty$ and show that in this limit it is given by a path integral from arbitrary x_i to x_f. We will also show that it determines the needed transition matrixelements of $\hat{x}$ operators, i.e. the vacuum expectation values of time-ordered $\hat{x}$ operators.

We start by calculating the propagator of a system under the influence of a source, i.e. for a system described by (3.1). We first assume that the source is nonzero only for a limited time between t and t'

$$J(\tilde{t}) \neq 0 \qquad \text{for} \qquad t_i < t < \tilde{t} < t' < t_f \ . \tag{3.6}$$

We can then write for the transition amplitude

$$\langle x_f t_f|x_i t_i\rangle_J = \int \mathrm{d}x'\,\mathrm{d}x\,\langle x_f t_f|x't'\rangle\langle x't'|xt\rangle_J\langle xt|x_i t_i\rangle \ . \tag{3.7}$$

Note that the two outer propagators are taken with the sourceless Hamiltonian ($J=0$) because of condition (3.6). They are given by, e.g.,

$$\begin{aligned}\langle xt|x_i t_i\rangle &= \langle x|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}(t-t_i)}|x_i\rangle \\ &= \sum_n \varphi_n(x)\varphi_n^*(x_i)\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_n(t-t_i)} \ .\end{aligned} \tag{3.8}$$

Similarly we get

$$\begin{aligned} \langle x_f t_f | x' t' \rangle &= \langle x_f | \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}(t_f - t')} | x' \rangle \\ &= \sum_n \varphi_n(x_f)\varphi_n^*(x')\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_n(t_f - t')} \ . \end{aligned} \tag{3.9}$$

Here the φ_n are eigenstates of the Hamiltonian without a source. The dependence on t_i and t_f is now isolated.

Taking now the limits $t_i \to -\infty$ and $t_f \to +\infty$ is not straightforward because both times appear as arguments of oscillatory functions. These transitions are, therefore, now performed by a mathematical trick, namely by replacing the physical Minkowski time by a complex time τ

$$t \longrightarrow \tau = t\mathrm{e}^{-\mathrm{i}\delta} \qquad (0 \leq \delta \leq \pi/2) \ . \tag{3.10}$$

This corresponds to an analytic continuation; the real, four-dimensional Minkowski space can be viewed as a subspace of a four-dimensional complex space (or of a five-dimensional space with 3 real space-coordinates and 1 complex time-coordinate). The substitution (3.10) corresponds to a clockwise rotation of the time-axis out of the real subspace into the larger complex space. For $\delta = +\pi/2$ this corresponds to a rotation $t = +\infty \to \tau = -i\infty, t = -\infty \to \tau = +i\infty$. The whole operation is known as a *Wick rotation.* Physical results are obtained by evaluating the relevant expressions in the enlarged, complex space and then, at the end, rotating the result back to the real time-axis (which, of course, works only if no singularities are encountered on this way).

We now apply this Wick rotation to the expression

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_0 t_i}\langle xt | x_i t_i \rangle = \sum_n \varphi_n(x)\varphi_n^*\,(x_i)\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_n t} e^{\frac{\mathrm{i}}{\hbar}(E_n - E_0)t_i} \ . \tag{3.11}$$

and thus perform the limit $t_i \to -\infty$ in the complex time plane $(0 \leq \delta \leq \frac{\pi}{2})$

$$\begin{aligned} &\lim_{\tau_i \to -\infty \mathrm{e}^{-\mathrm{i}\delta}} \ \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_0\tau_i}\langle xt | x_i \tau_i \rangle \qquad (3.12) \\ &= \lim_{t_i \to -\infty} \sum_n \varphi_n(x)\varphi_n^*\,(x_i)\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_n t}\ \mathrm{e}^{+\frac{\mathrm{i}}{\hbar}(E_n - E_0)t_i(\cos\delta - \mathrm{i}\sin\delta)} \ . \end{aligned}$$

For $t_i \to -\infty$ the last factor vanishes, except for $n = 0$. Thus we get

$$\begin{aligned} &\lim_{\tau_i \to -\infty\ \mathrm{e}^{i\delta}} \ \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_0\tau_i}\langle xt | x_i \tau_i \rangle \qquad (3.13) \\ &= \varphi_0(x)\varphi_0^*(x_i)\ \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_0 t} = \varphi_0(x,t)\varphi_0^*(x_i) \ . \end{aligned}$$

Analogously we also obtain

$$\begin{aligned}
&\lim_{\tau_f \to +\infty\ \mathrm{e}^{-\mathrm{i}\delta}} \mathrm{e}^{+\frac{\mathrm{i}}{\hbar}E_0\tau_f} \langle x_f \tau_f | x' t' \rangle \\
&= \quad \varphi_0(x_f)\varphi_0^*(x')\ \mathrm{e}^{\frac{\mathrm{i}}{\hbar}E_0 t'} = \varphi_0^*(x',t')\varphi_0(x_f)\ .
\end{aligned} \tag{3.14}$$

These final expressions no longer contain τ_i or τ_f and can therefore be rotated back to the real physical times t_i and t_f without any change in the results. By means of the Wick rotation we have thus been able to make the expression (3.13) and (3.14) convergent; in this process we found that only the groundstate contributes to the transition probability.

We now insert these expressions into (3.7) and get

$$\begin{aligned}
&\lim_{\substack{\tau_i \to -\infty\ \mathrm{e}^{-\mathrm{i}\delta} \\ \tau_f \to +\infty \mathrm{e}^{-\mathrm{i}\delta}}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}E_0(\tau_f - \tau_i)} \langle x_f \tau_f | x_i \tau_i \rangle_J \\
&= \int \mathrm{d}x\, \mathrm{d}x'\ \varphi_0^*(x',t')\varphi_0(x_f)\ \langle x' t' | x t \rangle_J\ \varphi_0(x,t)\varphi_0^*(x_i) \\
&= \int \mathrm{d}x\, \mathrm{d}x'\ \varphi_0^*(x',t')\ \langle x' t' | x t \rangle_J\ \varphi_0(x,t)\ \varphi_0(x_f)\varphi_0^*(x_i) \\
&= \varphi_0(x_f)\varphi_0^*(x_i) \mathrm{e}^{\frac{\mathrm{i}}{\hbar}E_0(t'-t)}\ \langle 0t' | 0t \rangle_J\ .
\end{aligned} \tag{3.15}$$

In the last line (3.4) has been used. We obtain by Wick-rotating back

$$\langle 0t' | 0t \rangle_J\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}E_0(t'-t)} = \lim_{\substack{t_i \to -\infty \\ t_f \to +\infty}} \frac{\mathrm{e}^{\frac{\mathrm{i}}{\hbar}E_0(t_f - t_i)}}{\varphi_0^*(x_i)\varphi_0(x_f)} \langle x_f t_f | x_i t_i \rangle_J\ . \tag{3.16}$$

Equation (3.16) implies that the groundstate-to-groundstate transition amplitude is – up to a factor – given by a path integral from arbitrary x_i to arbitrary x_f and thus does not depend on these quantities, if only the corresponding times are taken to infinity. The times t and t' can also be taken to $\pm\infty$ so that we can define a functional of the function $J(t)$

$$W[J] = \langle 0 + \infty | 0 - \infty \rangle_J\ . \tag{3.17}$$

With

$$\langle x_f t_f | x_i t_i \rangle_{J=0} = \varphi_0^*(x_i)\varphi_0(x_f)\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_0(t_i - t_f)} \tag{3.18}$$

we can also define a normalized functional

$$Z[J] = \frac{W[J]}{W[0]} = \lim_{\substack{t_i \to -\infty \\ t_f \to +\infty}} \frac{\langle x_f t_f | x_i t_i \rangle_J}{\langle x_f t_f | x_i t_i \rangle_{J=0}} = \frac{\langle 0 + \infty | 0 - \infty \rangle_J}{\langle 0 + \infty | 0 - \infty \rangle_{J=0}}\ . \tag{3.19}$$

The functional $Z[J]$ describes the processes relative to the unperturbed ($J = 0$) time-development. The numerator in (3.19) is a transition amplitude and can therefore be written as a path integral

$$\begin{aligned} \langle x_f + \infty | x_i - \infty \rangle_J &= \int \mathcal{D}x \int \mathcal{D}p \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} \int\limits_{-\infty}^{+\infty} [p\dot{x} - H(p,x) + \hbar J(t) x] \, \mathrm{d}t} \\ &= N \int \mathcal{D}x \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} \int\limits_{-\infty}^{+\infty} [L(x,\dot{x}) + \hbar J(t) x] \, \mathrm{d}t} . \end{aligned} \tag{3.20}$$

In the normalized functional $Z[J]$ the (infinite) factor N cancels out because it is independent of J.

As discussed in Sect. 1.3 the last step in (3.20) is possible only when the Hamiltonian is quadratic in p. A complication arises if the kinetic term becomes coordinate dependent; this can, for example, happen in spherical polar coordinates. In this case a metric factor (see (1.56)) appears in the path integral.

To conclude this section we just want to note that instead of using the Wick-rotation to make the functional $W[J]$ well behaved we could also have added a small negative imaginary term $-\mathrm{i}\varepsilon x^2/2$ to the Hamiltonian. This would have given a negative imaginary part to the energies E_n that becomes larger with n and would thus have led to convergence. This becomes apparent by looking at expressions (3.8) and (3.9).

3.2 Functional Derivatives of Transition Amplitudes

In this section we will show that the gs to gs transition matrixelement of a time-ordered product of interaction operators – in this present case of the operators $\hat{x}$ – can be obtained as functional derivatives of the functional $W[J]$. We, therefore, first define the functional derivatives and then apply these definitions to $W[J]$.

3.2.1 Functional derivatives

Suppose that $F[f]$ is a functional of the function $f(x)$. The functional derivative of F is then defined by

$$\frac{\delta F[f(x)]}{\delta f(y)} = \lim_{\varepsilon \to 0} \frac{F\left[f(x) + \varepsilon \delta(x-y)\right] - F\left[f(x)\right]}{\varepsilon} . \tag{3.21}$$

For example, let us take

$$F[f] = \int_{-\infty}^{+\infty} f(x)\mathrm{d}x \ . \tag{3.22}$$

According to the definition just given we have

$$\begin{aligned} \frac{\delta F}{\delta f(y)} &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left\{\int\left[f(x)+\varepsilon\delta(x-y)\right]\,\mathrm{d}x - \int f(x)\,\mathrm{d}x\right\} \\ &= \int \delta(x-y)\,\mathrm{d}x = 1 \end{aligned} \tag{3.23}$$

A second example is given by

$$F[f] = \mathrm{e}^{\mathrm{i}\int f(x)x\,\mathrm{d}x} \tag{3.24}$$

$$\begin{aligned} \frac{\delta F}{\delta f(y)} &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\mathrm{e}^{\mathrm{i}\int[f(x)+\varepsilon\delta(x-y)]x\,\mathrm{d}x} - \int \mathrm{e}^{\mathrm{i}\int f(x)x\,\mathrm{d}x}\right) \\ &= \mathrm{i}y\mathrm{e}^{\mathrm{i}\int f(x)x\,\mathrm{d}x} \ . \end{aligned} \tag{3.25}$$

A general formula that we will need quite often is written as follows. Be $F[J]$ of the form

$$F[J] = \int \mathrm{d}x_1 \ldots \int \mathrm{d}x_n\, f\left(x_1,\ldots,x_n\right) J(x_1)J(x_2)\ldots J(x_n) \tag{3.26}$$

with f symmetric in all variables, then the functional derivative with respect to J has the form

$$\begin{aligned} \frac{\delta F[J]}{\delta J(x)} &= n\int \mathrm{d}x_1\,\mathrm{d}x_2\ldots\mathrm{d}x_{n-1}\, f(x_1,x_2,\ldots,x_{n-1},x) \\ &\qquad \times J(x_1)J(x_2)\ldots J(x_{n-1}) \ . \end{aligned} \tag{3.27}$$

In our later considerations the functional may also be defined by a power series expansion

$$\phi[J] = \sum_{n=1}^{\infty}\frac{1}{n!}\int \mathrm{d}x_1\ldots\mathrm{d}x_n\,\phi_n(x_1,\ldots,x_n)J(x_1)J(x_2)\ldots J(x_n) \ . \tag{3.28}$$

Then the functional derivative is given by

$$\left.\frac{\delta^k\phi[J]}{\delta J(y_1)\delta J(y_2)\ldots\delta J(y_k)}\right|_{J=0} = \frac{1}{k!}\sum_p \phi\left(y_{p_1},y_{p_2},\ldots,y_{p_k}\right) \ , \tag{3.29}$$

where the sum runs over all permutations $p_1, \ldots, p_k$ of the indices $1, \ldots, k$. If we assume that ϕ_k is a symmetric function under exchange of any of the coordinates $x_1, \ldots, x_k$, then the functional derivative is given by

$$\left.\frac{\delta^k \phi[J]}{\delta J(y_1)\delta J(y_2)\ldots\delta J(y_k)}\right|_{J=0} = \phi_k(y_1, \ldots, y_k)\ , \tag{3.30}$$

just as in a normal Taylor series.

3.2.2 Time-ordered products as functional derivatives

We now apply the considerations of the last section by calculating the functional derivative of the transition amplitude (3.5) with respect to J. In order to become familiar with functional derivatives we perform this derivative in detail (cf. (3.25)). In order to do so we go back to the definition of a path integral as a limit of a finite dimensional integral (see (1.36))

$$\begin{aligned} & K\left(x, t; x_i, t_i\right) \\ &= \lim_{n\to\infty} \int \prod_{j=1}^{n} \mathrm{d}x_j \int \prod_{j=0}^{n} \frac{\mathrm{d}p_j}{2\pi\hbar} \\ &\quad\times \exp\left(\frac{\mathrm{i}}{\hbar}\sum_{j=0}^{n}\left[p_j\left(x_{j+1}-x_j\right) - \eta H\left(p_j, \overline{x}_j\right) + \hbar J_j x_j\right]\right)\ . \end{aligned} \tag{3.31}$$

Using now the definition (3.21) we get

$$\frac{\delta\langle x_f t_f | x_i t_i\rangle_J}{\delta J(t_1)} \tag{3.32}$$

$$\begin{aligned} &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left\{\lim_{n\to\infty}\int\prod_{j=1}^{n}\mathrm{d}x_j\int\prod_{j=0}^{n}\frac{\mathrm{d}p_j}{2\pi\hbar}\right. \\ &\quad\times\left[\exp\left(\frac{\mathrm{i}}{\hbar}\sum_{j=0}^{n}\left\{F(x_j,p_j)+\hbar x_j\left[J_j+\varepsilon\delta(t_j-t_1)\right]\right\}\right)\right. \\ &\qquad\left.\left.-\exp\left(\frac{\mathrm{i}}{\hbar}\sum_{j=0}^{n}\left(F(x_j,p_j)+\hbar x_j J_j\right)\right)\right]\right\} \end{aligned} \tag{3.33}$$

$$= \lim_{n\to\infty}\int\prod_{j=1}^{n}\mathrm{d}x_j\int\prod_{j=0}^{n}\frac{\mathrm{d}p_j}{2\pi\hbar}\,\mathrm{i}x_1\,\exp\left(\frac{\mathrm{i}}{\hbar}\sum_{j=0}^{n}\left(F(x_j,p_j)+\hbar x_j J_j\right)\right)\ ,$$

which can be written as

$$\frac{\delta\langle x_f t_f | x_i t_i\rangle_J}{\delta J(t_1)} = \mathrm{i}\int\mathcal{D}x\int\mathcal{D}p\,x(t_1)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{t_i}^{t_f}[p\dot{x}-H(x,p)+\hbar J(t)x]\,\mathrm{d}t}\ . \tag{3.34}$$

The higher order functional derivatives yield

$$
\begin{aligned}
&\frac{\delta^n \langle x_f t_f | x_i t_i \rangle_J}{\delta J(t_1)\delta J(t_2)\dots\delta J(t_n)} \\
&= (\mathrm{i})^n \int \mathcal{D}x \int \mathcal{D}p\, x(t_1)x(t_2)\dots x(t_n) \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int\limits_{t_i}^{t_f}[p\dot{x}-H(x,p)+\hbar J(t)x]\,\mathrm{d}t} .
\end{aligned}
\tag{3.35}
$$

We now want to understand the physical significance and meaning of these derivatives. In order to do so, we go back to the definition of the propagator. There we had (in (1.27))

$$
\begin{aligned}
&\langle x_f, t_f | x_i t_i \rangle \\
&= \int \mathrm{d}x_1 \dots \mathrm{d}x_n \, \langle x_f t_f | x_n, t_n \rangle \langle x_n t_n | x_{n-1} t_{n-1} \rangle \dots \langle x_1 t_1 | x_i t_i \rangle .
\end{aligned}
\tag{3.36}
$$

Now, in (3.34), we have one factor more on the righthand side of this equation

$$
\begin{aligned}
&\int \mathrm{d}x_1 \dots \mathrm{d}x_n \, \langle x_f t_f | x_n t_n \rangle \dots \langle x_1 t_1 | x_i t_i \rangle x_1(t_1) \\
&\quad = \int \mathrm{d}x_1 \dots \mathrm{d}x_n \, \langle x_f t_f | x_n t_n \rangle \dots \langle x_1 t_1 | \hat{x}(t_1) | x_i t_i \rangle ;
\end{aligned}
\tag{3.37}
$$

the last step is possible because $|x_1\rangle$ is an eigenstate of the $\hat{x}$ operator with eigenvalue x_1. The last integral is obviously equal to

$$
\langle x_f t_f | \hat{x}(t_1) | x_i t_i \rangle ,
\tag{3.38}
$$

i.e. to a matrixelement of the position operator. We thus have

$$
\begin{aligned}
\left.\frac{\delta \langle x_f t_f | x_i t_i \rangle_J}{\delta J(t_1)}\right|_{J=0} &= \mathrm{i} \int \mathcal{D}x \int \mathcal{D}p\, x(t_1) \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int\limits_{t_i}^{t_f}[p\dot{x}-H(x,p)]\,\mathrm{d}t} \\
&= \mathrm{i} \langle x_f t_f | \hat{x}(t_1) | x_i t_i \rangle .
\end{aligned}
\tag{3.39}
$$

For the case that H is separable in x and p and quadratic in p, this relation reads

$$
\begin{aligned}
\left.\frac{\delta \langle x_f t_f | x_i t_i \rangle_J}{\delta J(t_1)}\right|_{J=0} &= \mathrm{i}N \int \mathcal{D}x\, x(t_1) \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int\limits_{t_i}^{t_f} L(x,\dot{x})\,\mathrm{d}t} \\
3mm] &= \mathrm{i} \langle x_f t_f | \hat{x}(t_1) | x_i t_i \rangle .
\end{aligned}
\tag{3.40}
$$

The functional derivative of the propagator with respect to the source thus gives the transition matrix element of the coordinate x.

Let us now consider the meaning of the higher derivatives. One might guess that

$$\left.\frac{\delta^n \langle x_f t_f | x_i t_i\rangle_J}{\delta J(t_1)\delta J(t_2)\ldots\delta J(t_n)}\right|_{J=0} = \mathrm{i}^n \langle x_f t_f|\hat{x}(t_1)\hat{x}(t_2)\ldots\hat{x}(t_n)|x_i t_i\rangle \ , \tag{3.41}$$

but this equation is not quite correct. We see this by considering explicitly the second derivative. We can proceed there exactly in the same way as for the first. We have for the rhs of (3.35) in the case of the second derivative

$$\begin{aligned}
\left.\frac{\delta^n \langle x_f t_f | x_i t_i\rangle_J}{\delta J(t_\alpha)\delta J(t_\beta)}\right|_{J=0} &= \mathrm{i}^2 \int \mathcal{D}x \int \mathcal{D}p \, x(t_\alpha)x(t_\beta)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int\limits_{t_i}^{t_f}[p\dot{x}-H(x,p)]\,\mathrm{d}t} \\
&= \mathrm{i}^2 \int \mathrm{d}x_i \ldots \mathrm{d}x_n \, \langle x_f t_f|x_n t_n\rangle \cdots \langle x_k t_k|\hat{x}(t_\alpha)|x_{k-1}t_{k-1}\rangle \\
&\qquad \cdots \langle x_l t_l|\hat{x}(t_\beta)|x_{l-1}t_{l-1}\rangle \cdots \langle x_1 t_1|x_i t_i\rangle \ .
\end{aligned} \tag{3.42}$$

Here we have assumed that $t_\alpha > t_\beta$, since each of the infinitesimal Green's functions propagates only forward in time. In this case (3.42) is indeed equal to

$$\mathrm{i}^2 \langle x_f t_f|\hat{x}(t_\alpha)\hat{x}(t_\beta)|x_i t_i\rangle \ . \tag{3.43}$$

However, if $t_\alpha < t_\beta$, then these two times would appear in a different ordering in the rhs of (3.42) and thus of the matrix element. We therefore have

$$\left.\frac{\delta^2 \langle x_f t_f | x_i t_i\rangle_J}{\delta J(t_1)\delta J(t_2)}\right|_{J=0} = \mathrm{i}^2 \langle x_f t_f|T\left[\hat{x}(t_1)\hat{x}(t_2)\right]|x_i t_i\rangle \ , \tag{3.44}$$

with the time-ordering operator T

$$T\left[\hat{x}(t_1)\hat{x}(t_2)\right] = \begin{cases} \hat{x}(t_1)\hat{x}(t_2) & t_1 > t_2 \\ \hat{x}(t_2)\hat{x}(t_1) & t_1 < t_2 \ . \end{cases} \tag{3.45}$$

The same reasoning leads to the following result for higher-order derivatives

$$\begin{aligned}
&\langle x_f t_f|T\left[\hat{x}(t_1)\hat{x}(t_2)\ldots\hat{x}(t_n)\right]|x_i t_i\rangle \\
&\quad = \left(\frac{1}{\mathrm{i}}\right)^n \left.\frac{\delta^n}{\delta J(t_1)J(t_2)\ldots\delta J(t_n)}\right|_{J=0} \langle x_f t_f|x_i t_i\rangle_J \ .
\end{aligned} \tag{3.46}$$

This is the generalization of (3.44).

The same relation holds also for the groundstate-to-groundstate matrixelements. We thus have[1]

$$
\begin{aligned}
&\frac{\langle 0t_f|T\left[x(t_1)x(t_2)\dots x(t_n)\right]|0t_i\rangle}{\langle 0t_f|0t_i\rangle} && (3.47)\\
&= \left(\frac{1}{\mathrm{i}}\right)^n \frac{\delta^n}{\delta J(t_1)\delta J(t_2)\dots\delta J(t_n)} \frac{\langle 0t_f|0t_i\rangle_J}{\langle 0t_f|0t_i\rangle}\Bigg|_{J=0}\\
&= \left(\frac{1}{\mathrm{i}}\right)^n \frac{\delta^n}{\delta J(t_1)\delta J(t_2)\dots\delta J(t_n)} Z[J]\Bigg|_{J=0} ,
\end{aligned}
$$

where the limit $t_i \to -\infty$, $t_f \to +\infty$ is understood and all the times $t_1, \dots, t_n$ lie in between these limits. This is a very important result. It shows that the groundstate-to-groundstate matrixelement of a time-ordered product of position operators can be obtained as a functional derivative of the functional $Z[J]$ defined in (3.19). $Z[J]$ is, therefore, called a *generating functional.* For the remainder of this book we will be concerned only with these generating functionals.

So far we have formulated the theory for one x-coordinate only. However, it is immediately clear that everything deduced so far is also valid in n dimensions.

[1]From now on, we drop the symbol ˆ over the operators.

Chapter 4

PATH INTEGRALS FOR SCALAR FIELDS

All the results obtained in the preceding chapters can be taken over into field theory remembering that fields play the role of the coordinates of the theory and the spatial locations $\vec{x}$ correspond to the indices of the classical coordinates[1].

In this chapter we thus apply the methods developed in chapter 3 to the case of scalar fields[2]. There we showed that the gs-transition amplitudes of time-ordered products of $\hat{x}$ operators could be obtained as functional derivatives of a generating functional. Taking this result over to field theory would imply that we can obtain the vacuum expectation values of time-ordered field operators by performing the derivatives on an appropriate functional. To discuss this functional is the main purpose of this chapter.

It is easy to see that these vacuum expectation values of time-ordered products of field operators play an important role in quantum field theory. Each field operator creates or annihilates particles and a time-ordered product of field operators can thus describe the probability amplitude for a physical process in which particles are created and annihilated. The quantitative information about such a process is contained in the S matrix which can be obtained from the vacuum expectation values of time-ordered field operators by means of the so-called reduction theorem which we will derive in a later chapter. The rest of this manuscript will therefore be concerned with calculating these expectation values and with developing perturbative methods for their determination when an exact calculation is not possible.

[1]The units, the metric and the notation used in this and the following chapters is explained in Appendix. A.

[2]The fundamentals of classical field theory are summarized in Appendix. B.

4.1 Generating Functional for Fields

We assume that the system is described by a Lagrangian of the form

$$\mathcal{L}\left(\phi, \partial_\mu \phi\right) = \frac{1}{2}\left(\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2 - V(\phi)\right) \ . \tag{4.1}$$

In order to obtain the functional $W[J]$ for fields we note that the fields play the role of the coordinates of the theory and that sums over the different coordinates have to be replaced by integrals over the space-time coordinates. In order to define more stringently what is actually meant by a path integral for fields we write it down here in detail for a free field Lagrangian ($V = 0$). In order to do so we bring it into a form as close as possible to the classical definition (1.41).

We first Fourier-expand the field

$$\phi(\vec{x}, t) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} q_{\vec{k}}(t) \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} \ . \tag{4.2}$$

For a real field ϕ we have

$$q^*_{-\vec{k}} = q_{\vec{k}} \ . \tag{4.3}$$

Inserting this expansion into the free Lagrangian gives

$$\begin{aligned} L &= \int \mathcal{L}\, \mathrm{d}^3x = \frac{1}{2}\int \left(\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2\right) \mathrm{d}^3x \\ &= \frac{1}{2V} \sum_{kk'} \int \left[q_k q_{k'}\left(\vec{k}\cdot\vec{k}' - m^2\right) + \dot{q}_k \dot{q}_{k'}\right] \mathrm{e}^{\mathrm{i}(\vec{k}+\vec{k}')\cdot\vec{x}}\, \mathrm{d}^3x \ . \end{aligned} \tag{4.4}$$

Integration over d^3x gives $V\delta^3(\vec{k}+\vec{k}')$ so that we have

$$L = \frac{1}{2}\sum_k \left(\dot{q}_k \dot{q}_{-k} - (\vec{k}^2 + m^2) q_k q_{-k}\right) \ . \tag{4.5}$$

Using (4.3), writing $q_k = X_k + \mathrm{i}Y_k$, and grouping then the coordinates X_k and $-\mathrm{i}Y_k$ into a new vector x_k gives

$$L = \frac{1}{2}\sum_k \left(\dot{x}_k^2 - \omega_k^2 x_k^2\right) \tag{4.6}$$

with $\omega_k = \sqrt{\vec{k}^2 + m^2}$.

In the general spirit of field theory we introduce a source density J which we also expand

$$J(\vec{x}, t) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} J_{\vec{k}}(t) \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} \ . \tag{4.7}$$

With this expansion we obtain for the source term

$$\int \mathrm{d}^3x\, J(x)\phi(x) = \sum_k J_k(t)x_k(t)\ . \tag{4.8}$$

The Lagrangian and the source term thus read formally just the same as for the classical systems treated earlier. We thus have for the vacuum-to-vacuum transition amplitude (cf. (3.17))

$$\begin{aligned} W[J] &= \lim_{\substack{t_f\to+\infty \\ t_i\to-\infty}} \langle 0t_f|0t_i\rangle_J \\ &= \lim_{\eta\to 0}\left(\frac{1}{2\pi\hbar\mathrm{i}\eta}\right)^{\frac{n+1}{2}} \int \prod_k \prod_{j=1}^{n} \mathrm{d}x_{kj}\ \mathrm{e}^{\mathrm{i}\eta\sum\limits_{l=0}^{n}(L_l+J_k x_{kl})}\ , \end{aligned} \tag{4.9}$$

with

$$L_l = L\left(\bar{x}_l, \dot{x}_l\right) = \frac{1}{2}\sum_k \left(\dot{x}_{kl}^2 - \omega_k^2 \bar{x}_{kl}^2\right)\ , \tag{4.10}$$

where the first index (k) denotes the coordinate and the second (j) the time interval. If we now identify the integration measure as

$$\mathcal{D}\phi = \lim_{\eta\to 0}\left(\frac{1}{2\pi\hbar\mathrm{i}\eta}\right)^{\frac{n+1}{2}} \prod_k \prod_{j=1}^{n} \mathrm{d}x_{kj}\ , \tag{4.11}$$

we can write the generating functional also as

$$\begin{aligned} W[J] &= \int \mathcal{D}\phi\ \exp\left(\mathrm{i}\int\left\{\frac{1}{2}\left[\partial_\mu\phi\partial^\mu\phi - \left(m^2 - \mathrm{i}\varepsilon\right)\phi^2\right] + J\phi\right\}\mathrm{d}^4x\right) \\ &= \int \mathcal{D}\phi\ \exp\left[\mathrm{i}\int\left(\mathcal{L}\left(\phi, \partial_\mu\phi\right) + J\phi + \mathrm{i}\frac{\varepsilon}{2}\phi^2\right)\mathrm{d}^4x\right]\ . \end{aligned} \tag{4.12}$$

Here the volume element d^4x is given by

$$\mathrm{d}^4x = \mathrm{d}^3x\,\mathrm{d}t\ . \tag{4.13}$$

Note that the coupling to the source in (4.12) is the same as it would be if J were a true external source. The term $\mathrm{i}\varepsilon\phi^2/2$ with positive ε has been introduced in an ad hoc manner to ensure the convergence of W when taking the fields to infinity with the understanding that ultimately ε has to be taken to 0.

The second line of (4.12) also gives the generating functional for an *interacting* theory with the interaction V included in $\mathcal{L}$.

Operator Representation. For completeness, we now derive an alternative expression for the generating functional $Z[J]$. We start by defining the *operator* functional

$$\hat{\mathcal{Z}}[J] = T\, \mathrm{e}^{\mathrm{i}\int J(x)\hat{\phi}(x)\,\mathrm{d}^4x} \tag{4.14}$$

where $\hat{\phi}$ is an operator! If we form the functional derivatives of $\hat{\mathcal{Z}}[J]$ we get

$$\left(\frac{1}{\mathrm{i}}\right)^n \frac{\delta^n \hat{\mathcal{Z}}}{\delta J(x_1)\ldots\delta J(x_n)} = T\left[\phi(x_1)\ldots\phi(x_n)\hat{\mathcal{Z}}[J]\right] \;, \tag{4.15}$$

so that, because of $\hat{\mathcal{Z}}[0] = 1$,

$$\begin{aligned}&\left(\frac{1}{\mathrm{i}}\right)^n \frac{\delta^n}{\delta J(x_1)\ldots\delta J(x_n)} \langle 0|\mathcal{Z}[J]|0\rangle|_{J=0} \\ &= \langle 0|T\left[\phi(x_1)\ldots\phi(x_n)\right]|0\rangle = \left(\frac{1}{\mathrm{i}}\right)^n \left.\frac{\delta^n Z}{\delta J(x_1)\ldots\delta J(x_n)}\right|_{J=0} \;.\end{aligned} \tag{4.16}$$

Thus all the functional derivatives of $\langle 0|\hat{\mathcal{Z}}[J]|0\rangle$ agree with those of $Z[J]$ at $J = 0$. The two expressions therefore have to be equal

$$Z[J] = \langle 0|\hat{\mathcal{Z}}[J]|0\rangle \;. \tag{4.17}$$

This is the generating functional of the interacting theory; the effect of the interactions is contained in the interacting field operator $\hat{\phi}$.

4.1.1 Normalized generating functional for free fields

The generating functional for a free scalar field theory plays a special role in the theory of path integrals. This is so because of two reasons: first, a free-field theory is the simplest possible field theory and, second, the effects of the interaction term $V(\phi)$ can often be described with the help of perturbation theory in which the functional of the full, interacting theory is expanded around that of the free theory.

The representation of the generating functional of a free field theory as a path integral is, however, still quite cumbersome for practical applications. For these it would be very desirable if we could factorize out the functional dependence on J in form of a normal integral; the remaining path integral would then disappear in the normalization.

In the following we will therefore separate the path integral (4.9) into 2 factors, one depending on J and the other one being an integral over ϕ, i.e. just a number. For this purpose, we first obtain through partial integration

$$\int \partial^\mu\phi\partial_\mu\phi \mathrm{d}^4x = -\int \phi\Box\phi\,\mathrm{d}^4x \tag{4.18}$$

and thus get the result

$$W[J] = \int \mathcal{D}\phi\, \mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4x \left[\frac{1}{2}\phi\left(\Box + m^2 - \mathrm{i}\varepsilon\right)\phi - J\phi\right]} . \tag{4.19}$$

The field ϕ here is an integration variable; it does not fulfill a Klein-Gordon equation.

We therefore introduce now a field ϕ_0 that does just that

$$\left[\Box + (m^2 - \mathrm{i}\varepsilon)\right] \phi_0(x) = J(x) . \tag{4.20}$$

J thus plays the role of a source to ϕ_0. We now take this field ϕ_0 as a reference field and expand ϕ around it, setting $\phi = \phi_0 + \phi'$. We thus obtain for the integrand in the exponent in (4.19)

$$\begin{aligned}
& \frac{1}{2}(\phi_0 + \phi')\,\Box\,(\phi_0 + \phi') + (\phi_0 + \phi')\left(m^2 - i\varepsilon\right)(\phi_0 + \phi') - J(\phi_0 + \phi') \\
= \;& \frac{1}{2}\phi'\left[\Box + \left(m^2 - i\varepsilon\right)\right]\phi' \\
& + \frac{1}{2}\phi_0\left[\Box + \left(m^2 - \mathrm{i}\varepsilon\right)\right]\phi_0 + \frac{1}{2}\phi_0\left[\Box + \left(m^2 - \mathrm{i}\varepsilon\right)\right]\phi' \\
& + \frac{1}{2}\phi'\left[\Box + \left(m^2 - \mathrm{i}\varepsilon\right)\right]\phi_0 - J\phi_0 - J\phi' .
\end{aligned} \tag{4.21}$$

The third and the fourth terms on the rhs give the same contribution when integrated over. The second term gives, according to (4.20), $\frac{1}{2}J\phi_0$. Collecting all terms we therefore have for the action in (4.19)

$$\begin{aligned}
S_J &= -\int \mathrm{d}^4x \left\{\frac{1}{2}\phi\left[\Box + \left(m^2 - \mathrm{i}\varepsilon\right)\right]\phi + J\phi\right\} \\
&= -\int \mathrm{d}^4x \left\{\frac{1}{2}\phi'\left[\Box + \left(m^2 - \mathrm{i}\varepsilon\right)\right]\phi' - J\phi' + \frac{1}{2}J\phi_0 \right. \\
&\qquad\qquad \left. + \phi'\left[\Box + \left(m^2 - \mathrm{i}\varepsilon\right)\right]\phi_0 - J\phi_0\right\} .
\end{aligned} \tag{4.22}$$

Here we can again apply (4.20) to obtain

$$S_J = -\int \mathrm{d}^4x \left\{\frac{1}{2}\phi'\left[\Box + \left(m^2 - \mathrm{i}\varepsilon\right)\right]\phi' - \frac{1}{2}J\phi_0\right\} \tag{4.23}$$

We are now very close to our aim to factorize out the J dependence. To do so we solve (4.20) by writing

$$\phi_0(x) = -\int D_F(x-y)J(y)\mathrm{d}^4y , \tag{4.24}$$

where D_F, the so-called *Feynman propagator*, fulfills the equation

$$\left[\Box + \left(m^2 - \mathrm{i}\varepsilon\right)\right] D_F(x) = -\delta^4(x) \tag{4.25}$$

in complete analogy to the nonrelativistic propagator K in chapter 1. Using

$$\delta^4(x) = \left(\frac{1}{2\pi}\right)^4 \int \mathrm{d}^4k\, \mathrm{e}^{\mathrm{i}kx} \tag{4.26}$$

we obtain for D_F

$$D_F(x-y) = \frac{1}{(2\pi)^4} \int \frac{\mathrm{e}^{-\mathrm{i}k(x-y)}}{k^2 - m^2 + i\varepsilon}\, \mathrm{d}^4k\ . \tag{4.27}$$

Substituting (4.24) into the action (4.23) gives

$$\begin{aligned} S_J &= -\int \mathrm{d}^4x \left\{\frac{1}{2}\phi' \left[\Box + \left(m^2 - i\varepsilon\right)\right] \phi' + \frac{1}{2}J(x) \int D_F(x-y)J(y)\, \mathrm{d}^4y\right\} \\ &= -\int \mathrm{d}^4x \left\{\frac{1}{2}\phi' \left[\Box + \left(m^2 - i\varepsilon\right)\right] \phi'\right\} \\ &\quad -\frac{1}{2}\int J(x)D_F(x-y)J(y)\, \mathrm{d}^4x\, \mathrm{d}^4y\ . \end{aligned} \tag{4.28}$$

The last term no longer depends on ϕ. The first part appears also in the denominator of the normalized generating function (4.3).

We thus obtain now

$$Z_0[J] = \frac{W[J]}{W[0]} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\int J(x)D_F(x-y)J(y)\, \mathrm{d}^4x\, \mathrm{d}^4y}\ . \tag{4.29}$$

This is the vacuum-to-vacuum transition amplitude for a free scalar field theory.

4.1.2 Feynman Propagator

The imaginary part of the mass, originally introduced to achieve convergence for the path integrals, appears here now in the denominator of the propagator D_F

$$D_F(x) = \frac{1}{(2\pi)^4} \int \frac{\mathrm{e}^{-\mathrm{i}kx}}{k^2 - m^2 + i\varepsilon}\, \mathrm{d}^4k\ . \tag{4.30}$$

It determines the position of the poles in D_F, which are, in the k_0-integration, at

$$k_0^2 = \vec{k}^2 + m^2 - i\varepsilon \tag{4.31}$$

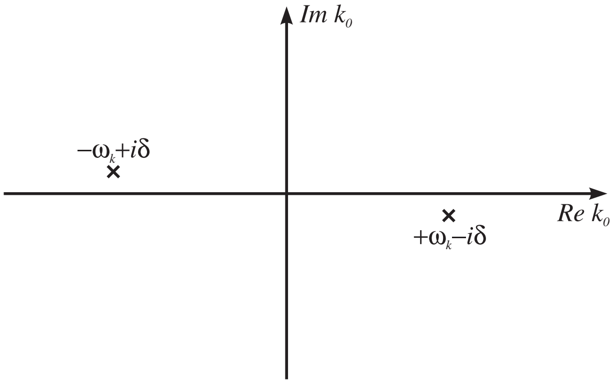


Figure 4.1: Location of the poles in the Feynman propagator.

or

$$k_0 = \pm\sqrt{\vec{k}^2 + m^2} \mp i\delta \ . \tag{4.32}$$

The poles are therefore located as indicated in Fig. 4.1. This implies that in the limit $\delta \to 0$ the integration path has to have the form shown in Fig. 4.2 where the poles are now on the real axis.

The location of the poles, originally introduced only in an ad-hoc way to achieve convergence of the path integrals, determines now the properties of the propagator of the free Klein-Gordon equation, the Feynman propagator.

This can be seen by rewriting the Feynman propagator D_F in the following form

$$\begin{aligned} D_F(x) &= \frac{1}{(2\pi)^4}\int \mathrm{d}^4k \, \frac{\mathrm{e}^{-\mathrm{i}kx}}{k^2 - m^2 + i\varepsilon} \qquad (4.33) \\ &= \frac{1}{(2\pi)^4}\int \mathrm{d}^3k \, \mathrm{d}k_0 \, \frac{\mathrm{e}^{-\mathrm{i}kx}}{k_0^2 - \vec{k}^2 - m^2 + i\varepsilon} \\ &= \frac{1}{(2\pi)^4}\int \mathrm{d}^3k \, \mathrm{d}k_0 \, \mathrm{e}^{-\mathrm{i}kx}\frac{1}{2\omega_k}\left(\frac{1}{k_0 - \omega_k + i\delta} - \frac{1}{k_0 + \omega_k - i\delta}\right) \end{aligned}$$

with $\omega_k^2 = \vec{k}^2 + m^2$. The path for the integration is as shown by the dashed line in the figure. Since the exponential contains a factor $\mathrm{e}^{-\mathrm{i}k_0 t}$, the path can be completed in the upper half plane for negative times and in the lower half for $t > 0$. Cauchy's theorem then gives

$$\begin{aligned} D_F(x) &= \frac{\mathrm{i}}{(2\pi)^3}\int \mathrm{d}^3k \, \frac{\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}}{2\omega_k}\left(-\Theta(-t)\mathrm{e}^{+i\omega_k t} - \Theta(t)\mathrm{e}^{-\mathrm{i}\omega_k t}\right) \\ &= -\frac{\mathrm{i}}{(2\pi)^3}\int \mathrm{d}^3k \, \frac{\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}}{2\omega_k}\left(\Theta(-t)\mathrm{e}^{+i\omega_k t} + \Theta(t)\mathrm{e}^{-\mathrm{i}\omega_k t}\right) \qquad (4.34) \end{aligned}$$

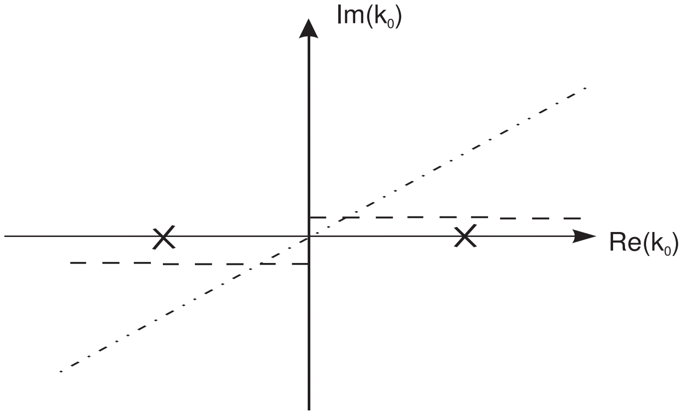


Figure 4.2: Integration path for the Feynman propagator (dashed line); Wick-rotated path for finite rotation angle (dash-dotted).

(in the second term here an extra "$-$" sign appears because of the negative direction of the contour integral).

It can now be shown that D_F propagates free fields with negative frequencies backwards in time and those with positive frequencies forwards. To demonstrate this we write

$$\begin{aligned} D_F(x-y) &= -\frac{\mathrm{i}}{(2\pi)^3}\int \mathrm{d}^3k\, \frac{\mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})}}{2\omega_k}\Big(\Theta(y_0-x_0)\mathrm{e}^{+i\omega_k(x_0-y_0)} \\ &\qquad + \Theta(x_0-y_0)\mathrm{e}^{-\mathrm{i}\omega_k(x_0-y_0)}\Big) \\ &= -\Theta(y_0-x_0)\frac{\mathrm{i}}{(2\pi)^3}\int \mathrm{d}^3k\,\frac{1}{2\omega_k}\mathrm{e}^{\mathrm{i}k(x-y)} \\ &\quad -\Theta(x_0-y_0)\frac{\mathrm{i}}{(2\pi)^3}\int \mathrm{d}^3k\,\frac{1}{2\omega_k}\mathrm{e}^{-\mathrm{i}k(x-y)}\ . \end{aligned} \tag{4.35}$$

Here we have changed $\vec{k} \to -\vec{k}$ in the first integral, which does not change its value under this substitution.

The integrands are products of orthogonal, normalized solutions of the Klein-Gordon equation. We can thus write

$$\begin{aligned} \mathrm{i}D_F(x-y) &= \Theta(y_0-x_0)\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\,\frac{1}{\sqrt{2\omega_k}}\mathrm{e}^{\mathrm{i}kx}\frac{1}{\sqrt{2\omega_k}}\mathrm{e}^{-\mathrm{i}ky} \\ &\quad +\Theta(x_0-y_0)\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\,\frac{1}{\sqrt{2\omega_k}}\mathrm{e}^{-ikx}\frac{1}{\sqrt{2\omega_k}}\mathrm{e}^{\mathrm{i}ky} \end{aligned}$$

$$
\begin{aligned}
&= \Theta(y_0 - x_0)\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\, \varphi_{\vec{k}}^{(-)*}(y)\varphi_{\vec{k}}^{(-)}(x) \\
&\quad + \Theta(x_0 - y_0)\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\, \varphi_{\vec{k}}^{(+)*}(y)\varphi_{\vec{k}}^{(+)}(x)
\end{aligned}
\tag{4.36}
$$

with

$$
\varphi_{\vec{k}}^{(\pm)}(x) = \frac{1}{\sqrt{2\omega_k}}\mathrm{e}^{\mp ikx} \ . \tag{4.37}
$$

Comparison with (1.17) shows that negative-frequency solutions are propagated backwards in time, and positive-frequency solutions forward. This particular behavior is a consequence of the location of the poles relative to the integration path. This location is fixed by the sign of the ε term which in turn was needed to achieve convergence for the generating functional.

4.1.3 Euclidean representation

In the preceding sections we have ensured convergence of the generating functional by introducing the ε-dependent term. In this subsection we discuss an alternative method that relies on the Wick rotation, i.e. the Euclidean representation of the propagator and the generating functional.

The real Euclidean space is obtained from Minkowski space by rotating the real axis in the x_0 plane by $\delta = -\pi/2$ into the negative imaginary axis (Wick rotation). We denote a space-time point in Euclidean space by x^{E}; it is related to the usual space-time point x in Minkowski space by

$$
x^{\mathrm{E}} = (\vec{x}, x_4) \qquad \text{with} \quad x_4 = \mathrm{i}x_0 = \mathrm{i}t \ . \tag{4.38}
$$

Under the Wick rotation $t \to -\mathrm{i}t$ and x_4 thus becomes real. With this definition we can extend the usual Minkowski-space definitions of volume element and space-time distance to Euclidean space

$$
\begin{aligned}
\mathrm{d}^4x^{\mathrm{E}} &\equiv \mathrm{d}^3x\,\mathrm{d}x_4 = \mathrm{d}^3x\,\mathrm{i}\mathrm{d}t = \mathrm{i}\mathrm{d}^4x \\
\mathrm{d}x_{\mathrm{E}}^2 &= \sum_{j=1}^{3}\mathrm{d}x_j^2 + x_4^2 = -\mathrm{d}x^2 \ .
\end{aligned}
\tag{4.39}
$$

We can define a corresponding Euclidean momentum space by requiring that $k^0x^0 = k_4^{\mathrm{E}}x_4^{\mathrm{E}}$; this condition ensures that a plane wave propagating forward in time does so both in Euclidean and in Minkowski space. We thus get

$$
k_{\mathrm{E}} = (\vec{k}, k_4) \qquad \text{with} \quad k_4 = -\mathrm{i}k_0 \ . \tag{4.40}
$$

This gives for the momentum space volume element and the energy-momentum distance in Euclidean space

$$\begin{aligned} \mathrm{d}^4k^{\mathrm{E}} &\equiv \mathrm{d}^3k\,\mathrm{d}k_4 = -\mathrm{d}^3k\,\mathrm{i}\mathrm{d}k_0 = -\mathrm{i}\mathrm{d}^4k \\ \mathrm{d}k_{\mathrm{E}}^2 &= \sum_{j=1}^{3} \mathrm{d}k_j^2 + \mathrm{d}k_4^2 = -\mathrm{d}k^2 \ . \end{aligned} \tag{4.41}$$

Combining both of these definitions yields for the typical exponent of a plane wave

$$\begin{aligned} kx = k_\mu x^\mu &= k_0 x^0 - k_j x_j = \mathrm{i}k_4(-\mathrm{i})x_4 - k_j x_j \\ &= k_4 x_4 - k_j x_j \neq k_{\mathrm{E}} x_{\mathrm{E}} \ . \end{aligned} \tag{4.42}$$

Note that this is not equal to the Euclidean scalar product. However, in Fourier transforms, where this expression often appears, we always have an integration over d^3k and can thus change $\vec{k} \to -\vec{k}$; thus in these Fourier integrals – and only there – we can replace kx by $k_{\mathrm{E}}x_{\mathrm{E}}$. Finally, the d'Alembert operator becomes

$$\Box = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^{\,2} = -\frac{\partial^2}{\partial x_4^2} - \vec{\nabla}^{\,2} = -\sum_{a=1}^{4} \frac{\partial^2}{\partial x_a^2} \equiv -\Box_{\mathrm{E}} \tag{4.43}$$

with eigenvalues $k^2 = k_0^2 - \vec{k}^2 = -(k_4^2 + \vec{k}^2) = -k_{\mathrm{E}}^2$.

The generating functional in its Euclidean representation thus becomes

$$W^{\mathrm{E}}[J] = \int \mathcal{D}\phi\, \mathrm{e}^{-\int \mathrm{d}^4x^{\mathrm{E}} \left\{ \frac{1}{2}\left[\left(\partial_\mu^{\mathrm{E}}\phi\right)^2 + m^2\phi^2 \right] - J\phi \right\}} \ , \tag{4.44}$$

with $\left(\partial_\mu^{\mathrm{E}}\phi\right)^2 = -\left(\partial_\mu\phi\right)^2$. Because x_4 is now real, $\left(\partial_\mu^{\mathrm{E}}\phi\right)^2$ is always positive and the exponent is negative definite; the integral thus converges and is well-defined. Since the exponent is furthermore quadratic, $W^{\mathrm{E}}[J]$ can be evaluated by using techniques that we will develop in the following chapter. Physical results are then obtained by rotating backwards after all integrations have been performed.

The Feynman propagator can now also be rewritten. For that purpose we choose a different path for the integration over k_0 after we have Wick-rotated the time axis. Instead of integrating along the real energy (k_0) axis we integrate along the imaginary energy axis. We then close the integration path in the right half of the k_0 plane for $t > 0$ and in the left half for $t < 0$. Since in this way the same poles are included as on the original path the value of the integral does not change. This fact makes this rotation in the k_0-space different from the one in the t-space where the imaginary time-axis has to be rotated back to the real one after the calculations in order to obtain physical results.

The Euclidean propagator then reads

$$\begin{aligned} D_F(x) &= -\frac{\mathrm{i}}{(2\pi)^4}\int \frac{\mathrm{e}^{-\mathrm{i}k_\mathrm{E}x_\mathrm{E}}}{k_\mathrm{E}^2+m^2}\,\mathrm{d}^4k^\mathrm{E} \\ &= -\frac{\mathrm{i}}{(2\pi)^4}\int \frac{\mathrm{e}^{-\mathrm{i}k_\mathrm{E}x_\mathrm{E}}}{\vec{k}^2+k_4^2+m^2}\,\mathrm{d}^4k^\mathrm{E}\ . \end{aligned} \tag{4.45}$$

Because k_4 is real, the integral no longer contains any poles on its integration path and is therefore well defined.

Chapter 5

EVALUATION OF PATH INTEGRALS

In this section we first derive relations for the evaluation of Gaussian path integrals, then apply these developments to the evaluation of the generating functional of the free scalar field theory and finally show how more general types of path integrals can approximately be reduced to the Gaussian form. In this reduction we find a systematic method for a semiclassical expansion in terms of the Planck constant $\hbar$. Finally, we briefly discuss methods for the numerical evaluation of path integrals.

5.1 Gaussian Integrals

In the evaluation of generating functionals we are often confronted with the evaluation of path integrals over exponential functions whose exponents are quadratic in the fields. It is thus useful to generalize the Gaussian integral formula

$$\int\limits_{-\infty}^{+\infty} \mathrm{e}^{-\frac{1}{2}ax^2} \mathrm{d}x = \sqrt{\frac{2\pi}{a}} \qquad (a > 0) \tag{5.1}$$

to this case. We develop these generalizations first for the coordinates x_i and then, later, generalize to fields.

From the definition of the path integral it is obvious that we have to consider products of such integrals

$$\int \exp\left(-\frac{1}{2}\sum_{k=1}^{n} a_k x_k^2\right) \mathrm{d}x_1\, \mathrm{d}x_2 \ldots \mathrm{d}x_n = \frac{\sqrt{(2\pi)^n}}{\prod_{k=1}^{n}\sqrt{a_k}} \,. \tag{5.2}$$

We next assume that the n numbers a_k are all positive and form the elements of a diagonal matrix $\mathbf{A}$. We thus have

$$\det(\mathbf{A}) = \prod_{k=1}^{n} a_k \tag{5.3}$$

and

$$\sum_{k=1}^{n} a_k x_k^2 = \sum_{k=1}^{n} x_k A_{kk} x_k = \mathbf{x}^T \mathbf{A} \mathbf{x} \ , \tag{5.4}$$

where $\mathbf{x}$ is a column vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \ .$$

Thus (5.2) becomes

$$\int \mathrm{e}^{-\frac{1}{2}\mathbf{x}^T \mathbf{A} \mathbf{x}} \, \mathrm{d}^n x = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det(\mathbf{A})}} \ . \tag{5.5}$$

So far, we have derived this equation only for a diagonal matrix $\mathbf{A}$. It is, however, valid for a more general class of matrices. This can be seen by noting that for each real, symmetric matrix $\mathbf{B}$ there exists a real, orthogonal matrix $\mathbf{O}$ that diagonalizes $\mathbf{B}$ to $\mathbf{A}$

$$\mathbf{O}^T \mathbf{B} \mathbf{O} = \mathbf{A} \qquad (\mathbf{O} \text{ real, orthogonal}) \tag{5.6}$$

or, equivalently

$$\mathbf{B} = \mathbf{O} \mathbf{A} \mathbf{O}^T \qquad (\mathbf{B} \text{ real, symmetric}) \ . \tag{5.7}$$

We thus get

$$\begin{aligned} \int \mathrm{e}^{-\frac{1}{2}\mathbf{y}^T \mathbf{B} \mathbf{y}} \, \mathrm{d}^n y &= \int \mathrm{e}^{-\frac{1}{2}\mathbf{y}^T \mathbf{O} \mathbf{A} \mathbf{O}^T \mathbf{y}} \, \mathrm{d}^n y = \int \mathrm{e}^{-\frac{1}{2}\mathbf{x}^T \mathbf{A} \mathbf{x}} \, \mathrm{d}^n x \\ &= \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det(\mathbf{A})}} = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det(\mathbf{B})}} \ , \end{aligned} \tag{5.8}$$

where we have substituted $\mathbf{y} = \mathbf{O}\mathbf{x}$ and have used the fact that the Jacobian of an orthogonal transformation is 1 (because $\det(\mathbf{O}) = 1$). The last step is possible because the determinant of a matrix is invariant under an orthogonal transformation. Equation (5.5) is thus valid also for general symmetric matrices with positive eigenvalues; it can also be shown to hold for complex matrices with positive real parts.

This result can also be extended to more general quadratic forms in the exponent. We assume that the exponential is given by

$$e^{-F(\mathbf{x})} = \mathrm{e}^{-\left(\frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x}+\mathbf{B}^T\mathbf{x}+C\right)} \tag{5.9}$$

where $\mathbf{A}$ is a real symmetric matrix, $\mathbf{B}$ is a vector and C a constant. We bring $F(\mathbf{x})$ into a quadratic form by writing

$$F(\mathbf{x}) = \frac{1}{2}(\mathbf{x}-\mathbf{x}_0)^T\mathbf{A}(\mathbf{x}-\mathbf{x}_0) + F(\mathbf{x}_0) \tag{5.10}$$

where $\mathbf{x}_0$ is given by

$$\mathbf{x}_0 = -\mathbf{A}^{-1}\mathbf{B} \tag{5.11}$$

and $F_0 = F(\mathbf{x}_0) = C - \frac{1}{2}\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$. Setting now $\mathbf{y} = \mathbf{x} - \mathbf{x}_0$ gives

$$\begin{aligned} \int \mathrm{e}^{-\left(\frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x}+\mathbf{B}^T\mathbf{x}+C\right)}\,\mathrm{d}^n x &= \int \mathrm{e}^{-\frac{1}{2}\mathbf{y}^T\mathbf{A}\mathbf{y}-F_0}\,\mathrm{d}^n y \\ &= \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det(\mathbf{A})}}\mathrm{e}^{\frac{1}{2}\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}-C} . \end{aligned} \tag{5.12}$$

We can generalize these formulas now to complex integration by noting that (5.1) can be squared and then be written as

$$\frac{\pi}{a} = \int \mathrm{e}^{-ax^2}\mathrm{d}x \int \mathrm{e}^{-ay^2}\mathrm{d}y = \int \mathrm{e}^{-a(x^2+y^2)}\mathrm{d}x\,\mathrm{d}y . \tag{5.13}$$

Introducing now the complex variable $z = x + \mathrm{i}y$ gives

$$\frac{\pi}{a} = \frac{1}{2\mathrm{i}}\int \mathrm{e}^{-az^*z}\mathrm{d}z^*\mathrm{d}z . \tag{5.14}$$

This can be generalized as before to many coordinates (by replacing orthogonal matrices by unitary ones). We obtain

$$\int \mathrm{e}^{-\mathbf{z}^\dagger\mathbf{A}\mathbf{z}}\,\mathrm{d}^n z^*\,\mathrm{d}^n z = \int \mathrm{e}^{-z_i^* A_{ij} z_j}\,\mathrm{d}^n z^*\,\mathrm{d}^n z = \frac{(2\pi\mathrm{i})^n}{\det(\mathbf{A})} , \tag{5.15}$$

where $\mathbf{A}$ is a Hermitian matrix with positive eigenvalues.

Another convenient, often used relation in this context is

$$\ln\det A = \mathrm{tr}\ln A , \tag{5.16}$$

most easily proven for diagonal matrices. With the help of this relation we have

$$\int \mathrm{e}^{-\mathbf{z}^\dagger\mathbf{A}\mathbf{z}}\,\mathrm{d}^n z^*\,\mathrm{d}^n z = (2\pi\mathrm{i})^n\mathrm{e}^{-\mathrm{tr}\ \ln A} . \tag{5.17}$$

5.2 Application to Scalar Field Theory

We now apply the Gaussian integration formulas of Sect. 5.1 to the path integral (4.19)

$$W[J] = \int \mathcal{D}\phi \, \mathrm{e}^{-\frac{\mathrm{i}}{2}\int \phi(\Box + m^2 - \mathrm{i}\varepsilon)\phi \, \mathrm{d}^4x} \mathrm{e}^{\mathrm{i}\int J\phi \, \mathrm{d}^4x} \; . \tag{5.18}$$

In order to make the matrix structure of the exponent more visible we use two-fold partial integration and write

$$\begin{aligned} W[J] &= \int \mathcal{D}\phi \, \mathrm{e}^{-\frac{\mathrm{i}}{2}\int \phi(x)\delta^4(x-y)(\Box_y + m^2 - \mathrm{i}\varepsilon)\phi(y) \, \mathrm{d}^4x \, \mathrm{d}^4y} \mathrm{e}^{\mathrm{i}\int J\phi \, \mathrm{d}^4x} \\ &= \int \mathcal{D}\phi \, \mathrm{e}^{-\frac{\mathrm{i}}{2}\int \phi(x)\left[(\Box_y + m^2 - \mathrm{i}\varepsilon)\delta^4(x-y)\right]\phi(y) \, \mathrm{d}^4x \, \mathrm{d}^4y} \mathrm{e}^{\mathrm{i}\int J\phi \, \mathrm{d}^4x} \\ &\to \int \mathcal{D}\phi \, \mathrm{e}^{-\int \left\{\frac{1}{2}\phi(x^{\mathrm{E}})\left[\left(-\Box_y^E + m^2\right)\delta(x^{\mathrm{E}} - y^{\mathrm{E}})\right]\phi(y^{\mathrm{E}}) \, \mathrm{d}^4y_{\mathrm{E}} - J\phi\right\} \mathrm{d}^4x_{\mathrm{E}}} \; . \end{aligned} \tag{5.19}$$

In the last step we have gone over to the Euclidean representation of the generating functional (cf. (4.44)), using $\Box^{\mathrm{E}} = -\Box$ and $\delta^4(x^{\mathrm{E}} - y^{\mathrm{E}}) = -\mathrm{i}\delta^4(x - y)$ (cf. Sect. 4.1.3).

The integrals appearing here are now of Gaussian type and can thus be integrated by using the expressions developed in the last section. We first identify the matrices A and B in the Gaussian integration formula (5.12) as

$$\begin{aligned} A^{\mathrm{E}}(x,y) &= -(\Box_y^E - m^2)\delta^4(x^{\mathrm{E}} - y^{\mathrm{E}}) \\ B(x) &= -J^T(x) \; . \end{aligned} \tag{5.20}$$

A^{E} is real with positive eigenvalues ($k_{\mathrm{E}}^2 + m^2 > 0$) as required by the derivation in section 5.1. It is also symmetric as can be seen by writing the d'Alembert operator in a discretized form, e.g.

$$\begin{aligned} \left.\frac{\mathrm{d}^2}{dx^2}\phi(x)\right|_{x_i} &= \lim_{h\to 0}\frac{1}{h^2}\left[\phi(x_i + h) - 2\phi(x_i) + \phi(x_i - h)\right] \\ &= A_{ij}\phi(x_j) \end{aligned} \tag{5.21}$$

with $\phi(x_i + h) = \phi(x_{i+1})$ etc. and

$$A_{ij} = \delta_{i,j-1} - 2\delta_{i,j} + \delta_{i,j+1} \; .$$

Thus the necessary conditions for the application of (5.8) are fulfilled.

Applying now (5.12) gives

$$W^{\mathrm{E}}[J] = \left[\det(-\Box^E + m^2)\delta(x^{\mathrm{E}} - y^{\mathrm{E}})\right]^{-\frac{1}{2}} \mathrm{e}^{\frac{1}{2}\int J(x^{\mathrm{E}})\left(A^{\mathrm{E}}(x^{\mathrm{E}}, y^{\mathrm{E}})\right)^{-1} J(y^{\mathrm{E}}) \, \mathrm{d}^4x^{\mathrm{E}} \mathrm{d}^4y^{\mathrm{E}}} \; . \tag{5.22}$$

The Wick rotation back to real times is easily performed by the transformation (4.43) $\Box^{\mathrm{E}} \to -\Box - \mathrm{i}\varepsilon$; it reintroduces the term $+\mathrm{i}\varepsilon$ to guarantee the proper treatment of the poles. In this case A becomes

$$A^{\mathrm{E}} \to A = (\Box + m^2 - \mathrm{i}\varepsilon)\mathrm{i}\delta(x-y) \ . \tag{5.23}$$

and the volume element

$$\mathrm{d}^4x^{\mathrm{E}}\mathrm{d}^4y^{\mathrm{E}} \to -\mathrm{d}^4x\mathrm{d}^4y \tag{5.24}$$

so that we finally obtain[1]

$$W[J] = \frac{1}{\sqrt{\det A}}\mathrm{e}^{-\frac{1}{2}\int J(x)\left[\mathrm{i}\left(\Box_y + m^2 + \mathrm{i}\varepsilon\right)\delta^4(x-y)\right]^{-1} J(y)\,\mathrm{d}^4x\,\mathrm{d}^4y} \ . \tag{5.25}$$

Here the determinant of an operator is understood as the determinant over a matrix representation of this operator, or, equivalently, as the product of the eigenvalues of the operator. Now we use that the inverse operator appearing in (5.25) is just the Feynman propagator. This can be seen by deriving the equation of motion for the inverse operator

$$\begin{aligned}
\delta^4(x-y) &= \int A(x,z)A^{-1}(z,y)\,\mathrm{d}^4z \\
&= \int \left[\mathrm{i}\left(\Box_z + m^2 - \mathrm{i}\varepsilon\right)\delta^4(x-z)\right] A^{-1}(z,y)\,\mathrm{d}^4z \\
&= \mathrm{i}\int \delta^4(x-z)\left(\Box_z + m^2 - \mathrm{i}\varepsilon\right) A^{-1}(z,y)\,\mathrm{d}^4z \\
&= \mathrm{i}\left(\Box_x + m^2 - \mathrm{i}\varepsilon\right) A^{-1}(x,y) \ .
\end{aligned} \tag{5.26}$$

This is just the defining equation for $-D_F$ (4.25), so that we have $A^{-1} = \mathrm{i}D_F$. For the normalized generating functional we thus obtain

$$Z_0[J] = \frac{W[J]}{W[0]} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\int J(x)D_F(x-y)J(y)\,\mathrm{d}^4x\,\mathrm{d}^4y} \ . \tag{5.27}$$

Equation (5.27) is the result derived earlier in section 4.1 (cf. (4.29)).

5.3 Stationary Phase Approximation

If the path integral in question is not that over a Gaussian, it can be approximately brought into a Gaussian form by using the so-called stationary phase

[1] Note that formally we could have obtained this also by using (5.12) with $A = \mathrm{i}(\Box_y + m^2 - \mathrm{i}\varepsilon)\delta^4(x-y)$, $B = -\mathrm{i}J, C = 0$.

or saddle point method. In this method one first looks for the stationary point of the exponent in the path integral. As explained earlier this will give a major contribution to the path integral. The remaining contributions are approximated by expanding the exponent around the stationary point.

We illustrate this method here for the case of a scalar field with selfinteractions. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\left[\partial^\mu\phi\partial_\mu\phi - m^2\phi^2\right] - V(\phi) \tag{5.28}$$

and the action

$$S[\phi, J] = \int \mathrm{d}^4x\,(\mathcal{L} + J\phi) \tag{5.29}$$

is a functional of the field ϕ and the source J. We next determine the stationary point by calculating the functional derivative

$$\left.\frac{\delta S[\phi, J]}{\delta\phi}\right|_{\phi_0} = 0\ . \tag{5.30}$$

This gives

$$\left(\Box + m^2\right)\phi_0 + V'(\phi_0) - J = 0\ , \tag{5.31}$$

because this is the classical equation of motion corresponding to the action $S[\phi, J]$. The stationary field is just the classical field; the corresponding classical action is

$$\begin{aligned} S[\phi_0, J] &= \int \mathrm{d}^4x\,\left[\frac{1}{2}\left[(\partial_\mu\phi_0)^2 - m^2\phi_0^2\right] - V(\phi_0) + J\phi_0\right] \\ &= -\int \mathrm{d}^4x\,\left[\frac{1}{2}\phi_0\left(\Box + m^2\right)\phi_0 + V(\phi_0) - J\phi_0\right]\ . \end{aligned} \tag{5.32}$$

We now expand $S[\phi, J]$ around this stationary field (cf. (3.28))

$$\begin{aligned} S[\phi, J] &= S[\phi_0, J] \\ &+ \frac{1}{2}\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\left.\frac{\delta^2 S}{\delta\phi(x_1)\delta\phi(x_2)}\right|_{\phi_0}[\phi(x_1) - \phi_0(x_1)]\,[\phi(x_2) - \phi_0(x_2)] + \cdots\ . \end{aligned} \tag{5.33}$$

The second functional derivative appearing here can be obtained by varying the first derivative, i.e. the negative of the left side of (5.31). We thus get from the definition (3.21)

$$\begin{aligned} \left.\frac{\delta^2 S}{\delta\phi(x_2)\delta\phi(x_1)}\right|_{\phi_0} &= -\left.\frac{\delta}{\delta\phi(x_2)}\left[(\Box_1 + m^2)\phi + V'(\phi) - J\right]_1\right|_{\phi_0} \\ &= -\left[\Box_1 + m^2 + V''(\phi_0)\right]_1\delta^4(x_2 - x_1) \end{aligned} \tag{5.34}$$

which is an operator. The index 1 here means that the corresponding expressions are to be taken at the point x_1.

The functional (5.32) is to be calculated at the fixed field ϕ_0. It can, therefore, be taken out of the path integral so that we finally obtain

$$
\begin{aligned}
W[J] &= \int \mathcal{D}\phi \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^4x\,(\mathcal{L}+J\phi)} \qquad (5.35)\\
&= \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[\phi_0,\hbar J]} \int \mathcal{D}\phi \, \exp\Big(-\frac{\mathrm{i}}{2\hbar}\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2 \\
&\times \Big\{[\phi(x_1)-\phi_0(x_1)]\Big\{\Big[\Box + m^2 + V''(\phi_0)\Big]_1 \delta^4(x_2-x_1)\Big\}\;[\phi(x_2)-\phi_0(x_1)]\Big\}\Big) \\
&\quad + \ldots\;. \qquad (5.36)
\end{aligned}
$$

In order to facilitate the following discussion we have put the unit of action, $\hbar$, explicitly into this expression by setting $\mathrm{i} \to \mathrm{i}/\hbar$ and $J \to \hbar J$.

The path integral remaining here is now in a Gaussian form. It can be evaluated after a Wick rotation, just as in the developments leading to (5.25). After a "coordinate transformation" $\phi \to \phi' = \phi - \phi_0$ and after scaling the fields by $\phi \to \sqrt{\hbar}\phi$ we get for the action integral (see (5.25))

$$
\Big\{\det\Big[\mathrm{i}\Big(\Box + m^2 + V''(\phi_0)\Big)\Big]\Big\}^{-\frac{1}{2}}\;. \qquad (5.37)
$$

We now use (5.16)

$$
\det A = \mathrm{e}^{\mathrm{tr}\;\ln A} \qquad (5.38)
$$

and write finally

$$
\begin{aligned}
W[J] &= \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^4x\,\left(\frac{1}{2}\phi_0(\Box+m^2)\phi_0 + V(\phi_0) - \hbar J\phi_0\right)}\mathrm{e}^{-\frac{1}{2}\mathrm{tr}\;\ln\left[\mathrm{i}\left(\Box+m^2+V''(\phi_0)\right)\right]} + \ldots \\
&= \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[\phi,J]} \qquad (5.39)
\end{aligned}
$$

with

$$
\begin{aligned}
S[\phi,J] &= -\int \mathrm{d}^4x\,\Big[\frac{1}{2}\phi_0\left(\Box + m^2\right)\phi_0 + V(\phi_0)\Big] + \hbar\int \mathrm{d}^4x\,J\phi_0 \\
&\quad + \frac{i}{2}\hbar\mathrm{tr}\;\ln\Big[\mathrm{i}\Big(\Box + m^2 + V''(\phi_0)\Big)\Big] + \mathcal{O}(\hbar)^2\;. \qquad (5.40)
\end{aligned}
$$

The first line is just the classical action $S[\phi_0, J]$. Expression (5.40) thus shows clearly that the saddle point approximation amounts to an expansion of the action in powers of $\hbar$. This is in accordance with the discussion in Sect. 1.4 that quantum mechanics describes the fluctuations of the action around the classical path.

We now perform a normalization with respect to the free case, i.e. to

$$W_0[0] = \mathrm{e}^{-\frac{1}{2}\mathrm{tr}\ \ln\left[\mathrm{i}\left(\Box+m^2\right)\right]}, \tag{5.41}$$

where the index 0 on W denotes $V = 0$. This gives

$$Z[J] = \frac{W[J]}{W_0[0]} = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_N[\phi,J]} \tag{5.42}$$

with

$$\begin{aligned} S_N[\phi, J] &= S[\phi_0, J] + \frac{\mathrm{i}}{2}\hbar\mathrm{tr}\ln\left(\frac{\Box + m^2 + V''(\phi_0)}{\Box + m^2}\right) + \mathcal{O}(\hbar^2) \\ &= S[\phi_0, J] + \frac{\mathrm{i}}{2}\hbar\mathrm{tr}\ln\left[1 + D_F V''(\phi_0)\right] + \mathcal{O}(\hbar^2)\ . \end{aligned} \tag{5.43}$$

This latter form suggests a perturbative treatment through an expansion of the logarithm ($\ln(1+x) = x - x^2/2 + x^3/3 - \ldots$). In an x-space representation the trace corresponds to an integration over x such that the initial and final space-time points in the propagator D_F are identical, i.e. to a closed loop integration. With higher orders in the expansion of the logarithm more and more vertices appear, but they are always located on this one closed loop. This loop expansion approximates the quantum mechanical behavior, whereas the perturbation treatment takes interactions into account.

5.4 Numerical Evaluation of Path Integrals

An alternative method for the evaluation of path integrals is that of direct numerical computation; with rapidly increasing computer power this method becomes more and more important nowadays.

5.4.1 Imaginary time method

The generating functional is in general given by

$$W[J] = \langle 0|\mathrm{e}^{-i(\hat{H}+J)(t_f - t_i)}|0\rangle \tag{5.44}$$

for $t_i \to -\infty$ and $t_f \to +\infty$, as we have seen in chapter 3. After a Wick rotation this becomes

$$W[J] = \lim_{\beta\to\infty}\langle 0|e^{-\beta(\hat{H}+J)}|0\rangle\ , \tag{5.45}$$

where β denotes the real Euclidean time.

It is immediately obvious that (5.45) also equals the groundstate expectation value of the statistical operator of quantum statistics if we identify β with the inverse temperature, i.e. $\beta = 1/T$.

Inserting the explicit definition (4.9) and performing the Wick rotation gives

$$W[J] = \lim_{n\to 0} \left(\frac{n+1}{2\pi\hbar\beta}\right)^{\frac{n+1}{2}} \int \prod_k \prod_{j=1}^{n} dx_{kj}\, \mathrm{e}^{-\frac{\beta}{n+1}\sum\limits_{l=0}^{n}(H_l + J_k x_{kl})} , \tag{5.46}$$

with

$$H_l = H_l(\bar{x}_l, \dot{x}_l) = \frac{1}{2}\sum_k \left[\dot{x}_{kl}^2 + \omega_k^2 \bar{x}_{kl}^2\right] + V(\bar{x}_{kl}) , \tag{5.47}$$

so that $W[J]$ can be written as

$$W[J] = \int \prod_k \prod_j dx_{kj}\, \rho(x_{kj}) \mathrm{e}^{-\frac{\beta}{n+1}\sum_l V(\bar{x}_{kl})} \tag{5.48}$$

with

$$\rho(\mathbf{x}) = \mathrm{e}^{-\frac{\beta}{n+1}\sum_l\left[\frac{1}{2}(\dot{x}_{kl}^2 + \omega_k^2\bar{x}_{kl}^2) + J_k x_{kl}\right]} . \tag{5.49}$$

The function $\rho(\mathbf{x})$ is of Gaussian shape and can, therefore, analytically be normalized into

$$P(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\int d\mathbf{x}\, \rho(\mathbf{x})} . \tag{5.50}$$

The multiple integrals appearing here can be evaluated by a Monte-Carlo technique which samples the integrand at a large number of points, where each 'point' really corresponds to a full path $x(t)$. Given a certain point $\mathbf{x} = x_1, \ldots, x_n$ one randomly chooses a new point $\mathbf{x}' = x_1', \ldots, x_n'$, often by just changing one single coordinate. One then evaluates

$$r = \frac{P(\mathbf{x}')}{P(\mathbf{x})} . \tag{5.51}$$

If r is larger than 1, the new point is accepted. If $r < 1$, on the other hand, then a random number ρ between 0 and 1 is picked. If $r > \rho$, then the new point is also accepted, otherwise it is rejected. This method is repeated until a large enough number of points is sampled. In this way the most important regions in $\mathbf{x}$ space are sampled, thus generating finally M accepted points $\mathbf{x}_m$. The integral is then approximated by

$$W[J] = \frac{1}{M}\sum_{m=1}^{M} \mathrm{e}^{-\frac{\beta}{n+1}\sum_{l=0}^{n} V((\bar{x}_{kl})_m)} . \tag{5.52}$$

The sampling algorithm just described is known after its inventor as the *Metropolis algorithm*; it plays an important role in numerical evaluations of statistical physics expressions.

In the present case of evaluating the generating functional for gs to gs transitions one has to choose a fixed value of β, i.e. the Euclidean time. The calculations then have to be performed for several values of β with a subsequent extrapolation to $\beta \to \infty$ (or $T \to 0$).

5.4.2 Real time formalism

The major difficulty in evaluating a path integral numerically stems from the oscillatory character of the integrand. The discretized form of $W[J]$ (see. (4.9)) can – in an obvious abbreviation – be written as the limit of a multidimensional integral

$$W = \int \mathrm{d}\mathbf{x}\, \mathrm{e}^{iS(\mathbf{x})} \ , \tag{5.53}$$

where $\mathbf{x}$ is a n-dimensional vector. Importance sampling such as the one just discussed in the last section cannot directly be used because there is no positive probability weight function in the integrand.

A weight function can, however, be inserted by a mathematical trick. Using (5.12) in the form

$$\int \mathrm{d}\mathbf{x}_0 \sqrt{\det(\mathbf{A}/2\pi)}\mathrm{e}^{-\frac{1}{2}(\mathbf{x}-\mathbf{x}_0)^T\mathbf{A}(\mathbf{x}-\mathbf{x}_0)} = 1 \tag{5.54}$$

we can write

$$W = \int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{x}_0 \sqrt{\det(\mathbf{A}/2\pi)}\mathrm{e}^{-\frac{1}{2}(\mathbf{x}-\mathbf{x}_0)^T\mathbf{A}(\mathbf{x}-\mathbf{x}_0)}\mathrm{e}^{iS(\mathbf{x})} \ . \tag{5.55}$$

The Gaussian factor under the integral ensures that values of $\mathbf{x}$ close to $\mathbf{x}_0$ will contribute the most to the integral. The function $S(\mathbf{x})$ can, therefore be expanded around $\mathbf{x}_0$

$$S(\mathbf{x}) = S(\mathbf{x}_0) + \mathbf{S}_1(\mathbf{x})(\mathbf{x}-\mathbf{x}_0) + \frac{1}{2}(\mathbf{x}-\mathbf{x}_0)^T\mathbf{S}_2(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0) + \ldots \tag{5.56}$$

with

$$\mathbf{S}_1(\mathbf{x}_0) = \left.\frac{\mathrm{d}S}{\mathrm{d}\mathbf{x}}\right|_{\mathbf{x}_0} \qquad \text{and} \qquad \mathbf{S}_2(\mathbf{x}_0) = \left.\frac{\mathrm{d}^2S}{\mathrm{d}\mathbf{x}^2}\right|_{\mathbf{x}_0} \ . \tag{5.57}$$

After inserting this expansion we can perform the x-integration and obtain from (5.12)

$$W = \int \mathrm{d}\mathbf{x}_0\, \mathrm{e}^{\mathrm{i}S(\mathbf{x}_0)} \left(\frac{\det(\mathbf{A})}{\det[\mathbf{A}-\mathrm{i}\mathbf{S}_2(\mathbf{x}_0)]}\right)^{\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}\mathbf{S}_1^T(\mathbf{x}_0)[\mathbf{A}-i\mathbf{S}_2(\mathbf{x}_0)]^{-1}\mathbf{S}_1(\mathbf{x}_0)} \ . \tag{5.58}$$

If we now use that (5.54) is still approximately valid even if $\mathbf{A}$ is a function of $\mathbf{x}_0$, we can choose

$$\mathbf{A} = \mathbf{A}(\mathbf{x}_0) = \mathrm{i}\mathbf{S}_2(\mathbf{x}_0) + c^{-1}\mathbf{1} \tag{5.59}$$

with $c > 0$ and obtain

$$W = \int d\mathbf{x}_0 \, \mathrm{e}^{\mathrm{i}S(\mathbf{x}_0)} \left[\det(1 - \mathrm{i}\mathbf{S}_2(\mathbf{x}_0)\mathbf{A}^{-1})\right]^{-\frac{1}{2}} \mathrm{e}^{-\frac{c}{2}\mathbf{S}_1^T(\mathbf{x}_0)\mathbf{S}_1(\mathbf{x}_0)} \, . \tag{5.60}$$

The function

$$P(\mathbf{x}_0) = \mathrm{e}^{-\frac{c}{2}\mathbf{S}_1^T(\mathbf{x}_0)\mathbf{S}_1(\mathbf{x}_0)} \tag{5.61}$$

thus provides a probability distribution for sampling the remaining integrand and the expression can be evaluated with the Monte-Carlo method discussed in the last section. We obtain, therefore,

$$W = \frac{1}{M}\sum_{i=1}^{M} \mathrm{e}^{iS(\mathbf{x}_i)} \left[\det(\mathbf{1} + \mathrm{i}c\mathbf{S}_2(\mathbf{x}_i)\right]^{-\frac{1}{2}} \, , \tag{5.62}$$

where the M points $\mathbf{x}_i$ are taken randomly from the distribution $P(\mathbf{x})$.

For large values of c the points with $S_1 \approx 0$ will contribute the most to W. This is just the stationarity condition discussed in section 5.3 which corresponds to the classical solution. Since only a few points close to this configuration will contribute significantly, a good sampling with good statistics can be obtained. On the other hand, for small c the probability distribution becomes broad and the statistics correspondingly worse; however, quantum mechanical effects beyond the one-loop approximation now start to contribute. Thus, by choosing the proper value of c quantum mechanics can be switched on.

Chapter 6

GREEN'S FUNCTIONS

In this chapter we introduce a fundamental quantity of field theory, the Green's function. As we will see later (in Chapt. 11), all the matrix elements of the S-matrix and therefore all physical transition rates can be expressed in terms of these Green's functions.

We first define the n-point Green's function by

$$G(x_1, x_2, \ldots, x_n) = \left(\frac{1}{\mathrm{i}}\right)^n \left. \frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\ldots\delta J(x_n)}\right|_{J=0} . \tag{6.1}$$

According to (3.47) this Green's function is equal to the normalized vacuum expectation value of a time ordered product of field-operators

$$\begin{aligned} G(x_1, x_2, \ldots, x_n) &= \frac{\langle 0|T\left[\phi(x_1)\phi(x_2)\ldots\phi(x_n)\right]|0\rangle}{\langle 0+\infty|0-\infty\rangle} \\ &= \frac{\displaystyle\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2)\ldots\phi(x_n) \mathrm{e}^{\mathrm{i}\int_{-\infty}^{+\infty} \mathrm{d}^4x \mathcal{L}}}{\displaystyle\int \mathcal{D}\phi\, \mathrm{e}^{\mathrm{i}\int_{-\infty}^{+\infty} \mathrm{d}^4x\mathcal{L}}} . \end{aligned} \tag{6.2}$$

The vacuum here is that of the full, interacting Hamiltonian.

We note that $G(x_1, \ldots, x_n)$ is a symmetric function of its arguments. Therefore, according to (3.30) the following relation holds

$$Z[J] = \sum_n \frac{1}{n!} \int dx_1 \ldots dx_n\, \mathrm{i}^n G(x_1, x_2, \ldots, x_n) J(x_1)J(x_2)\ldots J(x_n) . \tag{6.3}$$

Momentum representation. Very often it is advantageous to work in momentum space. In general the transformation of the Green's function into

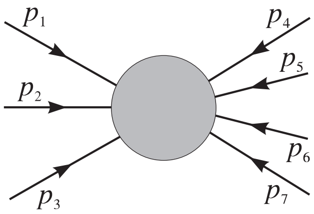


Figure 6.1: Momentum representation of the n-point function. Note that all momenta are pointing into the shaded interaction region.

the momentum-representation reads

$$\begin{aligned}&\int \mathrm{e}^{-i(p_1x_1+p_2x_2+\ldots+p_nx_n)}G_{\mathrm{c}}(x_1,x_2,\ldots,x_n)\,\mathrm{d}^4x_1\,\mathrm{d}^4x_2\ldots\mathrm{d}^4x_n\\ &= (2\pi)^4\delta^4(p_1+p_2+\ldots+p_n)\,G_{\mathrm{c}}(p_1,p_2,\ldots,p_n)\end{aligned} \tag{6.4}$$

The δ-function here reflects the momentum conservation due to translational invariance. This can be seen by performing pairwise transformations of two space-time points to their cm. point and their relative coordinate. If we then assume that G depends only on the latter, the integral over the cm. coordinate can be performed and yields the δ-function. Since we have a factor $\exp(+\mathrm{i}\vec{p}\cdot\vec{x})$ in the integrand in (6.4) we project out the components corresponding to $-\vec{p}$ from G_{c}. Thus, all the momenta are pointing from the x_i into the vertex (see Fig. 6.1).

6.1 Free Scalar Fields

We consider first the case of free fields. In this case the generating functional can be given analytically (4.29)

$$Z_0[J] = \mathrm{e}^{-\frac{\mathrm{i}}{2}\int J(x)D_F(x-y)J(y)\,\mathrm{d}^4x\,\mathrm{d}^4y}\ . \tag{6.5}$$

The first functional derivative vanishes at $J=0$ because the integral (6.5) is Gaussian. For the second functional derivative we obtain

$$\begin{aligned}\frac{\delta^2 Z_0[J]}{\delta J(x_1)J(x_2)} &= -\mathrm{i}D_F(x_1-x_2)\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\int J(x)D_F(x-y)J(y)\,\mathrm{d}^4x\,\mathrm{d}^4y}\\ &\quad+(-\mathrm{i})^2\int \mathrm{d}^4x\,\mathrm{d}^4y\,D_F(x_1-x)D_F(x_2-y)J(x)J(y)\\ &\quad\times\mathrm{e}^{-\frac{\mathrm{i}}{2}\int J(x)D_F(x-y)J(y)\,\mathrm{d}^4x\,\mathrm{d}^4y}\ ,\end{aligned} \tag{6.6}$$

so that we have

$$G(x_1, x_2) = -\frac{\delta^2 Z_0[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} = \mathrm{i}D_F(x_1 - x_2) \ . \tag{6.7}$$

Thus $Z_0[J]$ is a generating functional for $D_F(x-y)$.

6.1.1 Wick's theorem

The higher order derivatives can be most easily obtained by expanding (6.5)

$$\begin{aligned} Z_0[J] &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{\mathrm{i}}{2}\int J(x)D_F(x-y)J(y)\,\mathrm{d}x\,\mathrm{d}y\right)^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\left(-\frac{\mathrm{i}}{2}\right)^n \int \mathrm{d}x_1 \ldots \mathrm{d}x_{2k}\, D_{12}D_{34}\ldots D_{2n-12n}\, J_1 J_2 \ldots J_{2n} \end{aligned} \tag{6.8}$$

with the shorthand notation $D_{ij} = D_F(x_i - x_j)$, $J_k = J(x_k)$. Taking the $2k$-th functional derivative of Z_0 and using (6.1), (6.3) and (3.29) we obtain

$$\begin{aligned} G(x_1, x_2, \ldots, x_{2k}) &= \left(\frac{1}{\mathrm{i}}\right)^{2k} \frac{\delta^{2k} Z_0}{\delta J_1 \ldots \delta J_{2k}} \,|_{J=0} \\ &= \frac{(\mathrm{i})^k}{2^k k!}\sum_P D_{p_1p_2}\ldots D_{p_{2k-1}p_{2k}} \\ &= \mathrm{i}^k \sum_P D_{1p_{k+1}}\ldots D_{kp_{2k}} \ , \end{aligned} \tag{6.9}$$

where the last sum runs over all permutations of the k indices $p_{k+1}, \ldots, p_{2k}$ over the numbers $k+1, \ldots, 2k$. The factor in front of the first sum removes the doublecounting because of the symmetry $D_{p_2p_1} = D_{p_1p_2}$ (2^k) and because of the random order of factors under the sum $(k!)$. We note that $G = 0$ for odd k because the functional derivative has to be taken at $J = 0$.

Equation (6.9) states that the n-point function of a system of free bosons can be written as a properly normalized and symmetrized product of two-point functions. This is the so-called *Wick's theorem*.

As an example we consider the case $n = 4$. We then have

$$G(x_1, x_2, x_3, x_4) = -\frac{1}{8}\sum_{P\in S_4} D_{p_1p_2}D_{p_3p_4} \ . \tag{6.10}$$

Among the 24 terms in the sum, 12 are pairwise equal because the product of the two propagators commutes. Furthermore, the propagator $D_{p_1p_2}$ and $D_{p_2p_1}$ are pairwise equal. Thus, there are only $24 : 2 : 2 : 2 = 3$ essentially distinct terms in the sum; the factor 1/8 just takes care of all the others. We thus have

$$\begin{aligned} G(x_1, x_2, x_3, x_4) \quad = \quad & - D_F(x_1 - x_2) D_F(x_3 - x_4) \\ & - D_F(x_1 - x_3) D_F(x_2 - x_4) \\ & - D_F(x_1 - x_4) D_F(x_2 - x_3) \; . \end{aligned} \tag{6.11}$$

The first few n-point Green's functions are therefore – according to (6.7), (6.9) and (6.11) – given by

$$\begin{aligned} G(x_1) &= 0 \\ G(x_1, x_2) &= \frac{\langle 0 | T\left[\phi(x_1)\phi(x_2)\right] |0\rangle}{\langle 0|0\rangle} = \mathrm{i} D_F(x_1 - x_2) \\ G(x_1, x_2, x_3) &= 0 \\ G(x_1, x_2, x_3, x_4) &= \frac{\langle 0 | T\left[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\right] |0\rangle}{\langle 0|0\rangle} \\ &= - D_F(x_1 - x_2) D_F(x_3 - x_4) \\ & \quad - D_F(x_1 - x_3) D_F(x_2 - x_4) \\ & \quad - D_F(x_1 - x_4) D_F(x_2 - x_3) \\ & \text{etc.} \end{aligned} \tag{6.12}$$

Here $|0\rangle$ is the vacuum state of the free Hamiltonian because it was obtained as a functional derivative of the non-interacting functional Z_0 (6.8).

All n-point functions with odd n vanish. According to (6.2) this also means that, for example for $n = 3$,

$$\int \mathcal{D}\phi \, \phi(x_1)\phi(x_2)\phi(x_3) \mathrm{e}^{\mathrm{i} \int \mathrm{d}^4x \, \mathcal{L}} = 0 \; . \tag{6.13}$$

This explains now why in the stationary phase approximation of section 5.3 no odd terms in ϕ appeared.

6.1.2 Feynman rules

As in the classical case (cf. Sect. 2.2) we can again represent these results in a graphical way. The Feynman rules, that establish the connection between the algebraic and the graphical representation, are for the case of free fields still rather trivial.

They are given by

1) 

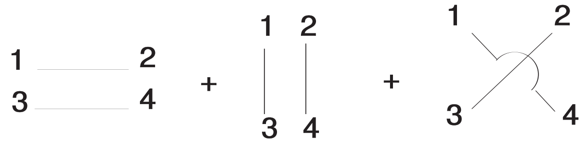
$= \mathrm{i} D_F(x - y)$

2)  $= iJ(x)$

3) Integration over all space-time coordinates

4) Each diagram has a factor that takes its symmetry into account. For example, if there is an integration over the endpoints x and y, these could be exchanged without changing the result; the symmetry factor is, correspondingly, 1/2.

With rule 1) we get, for example, for the fourpoint function, i.e. the two-particle Green's function

$G(x_1, x_2, x_3, x_4) =$  (6.14)

Each line connecting the two points x and y denotes the free propagator and gives a factor $\mathrm{i}D_F(x-y)$.

We now set

$$Z_0[J] = \mathrm{e}^{\mathrm{i}S_0[J]} \tag{6.15}$$

with

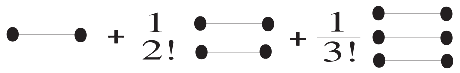
$$\mathrm{i}S_0[J] = -\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\; J(x)D_F(x-y)J(y) \tag{6.16}$$

By using rules 2), 3) and 4) we find immediately

$\mathrm{i}S_0 =$ (6.17)

Now we expand

$$\begin{aligned} Z_0[J] &= 1 + \left(-\frac{\mathrm{i}}{2}\right)\int J(x)D_F(x-y)J(y)\,\mathrm{d}^4x\,\mathrm{d}^4y \\ &\quad + \frac{1}{2!}\left(-\frac{\mathrm{i}}{2}\int J(x)D_F(x-y)J(y)\,\mathrm{d}^4x\,\mathrm{d}^4y\right)^2 + \cdots \\ &= 1 + \mathrm{i}S_0[J] + \frac{1}{2!}\left(\mathrm{i}S_0[J]\right)^2 + \frac{1}{3!}\left(\mathrm{i}S_0[J]\right)^3 + \ldots \end{aligned}$$

$= 1 +$  $+ \ldots\,.$ (6.18)

Summing these graphs gives formally

$Z_0[J] = \mathrm{e}^{\mathrm{i}S_0[J]} = \mathrm{e}$ (6.19) where $\mathrm{i}S_0$ is represented by a line with a dot at each end.

We now call all graphs that hang together connected graphs and all the others unconnected graphs. In our simple case here $S_0[J]$ is represented by only one connected graph (6.17), whereas $Z_0[J]$ – through the power expansion (6.18) of the exponential function – generates unconnected graphs as well. We can use this fact to define the *connected Green's functions* in terms of $S[J]$ by the relation

$$G_{\mathrm{c}}(x_1, \ldots, x_n) = \left(\frac{1}{\mathrm{i}}\right)^{n-1} \frac{\delta^n S}{\delta J(x_1) \ldots \delta J(x_n)}\bigg|_{J=0} . \tag{6.20}$$

In the simple case of a free field discussed here we get

$$G_{\mathrm{c}}(x_1, x_2) = \frac{1}{\mathrm{i}} \frac{\delta^2 S_0}{\delta J(x_1) \delta J(x_2)} \;|_{J=0} = \mathrm{i} D_{\mathrm{F}}(x_1 - x_2) \; ; \tag{6.21}$$

all higher Green's functions vanish.

6.1.3 Momentum representation

Taking the Fourier-transform of the two-point function (6.21) gives

$$\begin{aligned} &\int \mathrm{e}^{-\mathrm{i}(p_1 x_1 + p_2 x_2)} G_{\mathrm{c}}(x_1, x_2)\, \mathrm{d}^4 x_1\, \mathrm{d}^4 x_2 \qquad (6.22)\\ &= \int \left[\mathrm{e}^{-\mathrm{i}(p_1 x_1 + p_2 x_2)} \left(\frac{\mathrm{i}}{(2\pi)^4}\right) \int \mathrm{d}^4 q \, \frac{\mathrm{e}^{-\mathrm{i}q(x_1 - x_2)}}{q^2 - m^2 + \mathrm{i}\varepsilon}\right] \mathrm{d}^4 x_1 \mathrm{d}^4 x_2 \; . \end{aligned}$$

We first perform the integrations over x_1, x_2 and x and obtain for the rhs

$$(2\pi)^4 \delta^4(p_1 + p_2) G_{\mathrm{c}}(p_1, p_2) = (2\pi)^4 \delta^4(p_1 + p_2) \frac{\mathrm{i}}{p_1^2 - m^2 + \mathrm{i}\varepsilon} \; . \tag{6.23}$$

From this equation we can read off the momentum representation of the propagator. The momenta p_1 and p_2 are incoming momenta that point towards a vertex. Thus, the momentum representation of the free propagator is given by

$$G_{\mathrm{c}}(p, p' = -p) = \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon} \; , \tag{6.24}$$

pictured in Fig. 6.2. Note that here the second momentum appears with a negative sign. This is due to our notation to take all momenta as incoming (see Fig. 6.1).

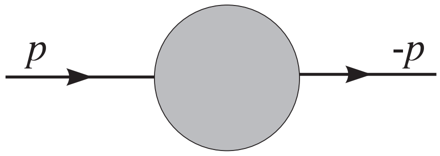


Figure 6.2: Two-point function of a scalar theory.

6.2 Interacting Scalar Fields

In this section we consider Lagrangians of the form

$$\mathcal{L} = \mathcal{L}_0 - V(\phi) \; , \tag{6.25}$$

where $\mathcal{L}_0$ is the free scalar Lagrangian (4.1) and V represents a selfinteraction of the field. For such a Lagrangian the generating functional for the n-point functions can no longer be given in closed form. In order to obtain the n-point function one has to resort to perturbative methods.

6.2.1 Perturbative expansion

According to our general considerations the functional for the Lagrangian (6.25) is given by

$$\begin{aligned} Z[J] &= \tilde{N} \int \mathcal{D}\phi \, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x \left(\mathcal{L}_0 - V + J\phi + \mathrm{i}\frac{\varepsilon}{2}\phi^2\right)} \\ &= \tilde{N} \int \mathcal{D}\phi \, \mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4x \, V(\phi)} \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x \left(\mathcal{L}_0 + J\phi + \mathrm{i}\frac{\varepsilon}{2}\phi^2\right)} \; . \end{aligned} \tag{6.26}$$

Here $\tilde{N}$ is just the inverse of the path integral for $J = 0$.

We now use the relation

$$\frac{1}{\mathrm{i}} \frac{\delta}{\delta J(y)} \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x \left(\mathcal{L}_0 + J\phi + \mathrm{i}\frac{\varepsilon}{2}\phi^2\right)} = \phi(y) \, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x \left(\mathcal{L}_0 + J\phi + \mathrm{i}\frac{\varepsilon}{2}\phi^2\right)} \; . \tag{6.27}$$

This relation, read from right to left, will also be true for any function $V(\phi)$, as can be seen by expanding V into a series in powers of ϕ. We thus have also

$$V\left[\phi(y)\right] \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x \left(\mathcal{L}_0 + J\phi + \mathrm{i}\frac{\varepsilon}{2}\phi^2\right)} = V\left[\frac{1}{\mathrm{i}} \frac{\delta}{\delta J(y)}\right] \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x \left(\mathcal{L}_0 + J\phi + \mathrm{i}\frac{\varepsilon}{2}\phi^2\right)} \tag{6.28}$$

and consequently, after exponentiation, also

$$\begin{aligned} & \mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4y \, V[\phi(y)]} \, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x \left(\mathcal{L}_0 + J\phi + \mathrm{i}\frac{\varepsilon}{2}\phi^2\right)} \\ &= \; \mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4y \, V\left(\frac{1}{\mathrm{i}} \frac{\delta}{\delta J(y)}\right)} \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x \left(\mathcal{L}_0 + J\phi + \mathrm{i}\frac{\varepsilon}{2}\phi^2\right)} \; . \end{aligned} \tag{6.29}$$

This relation allows us to take the V-dependent factor in (6.26) out of the path-integral. We thus have

$$Z[J] = \tilde{N} \mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4y\, V\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(y)}\right)} \int \mathcal{D}\phi\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\left(\mathcal{L}_0 + J\phi + \mathrm{i}\frac{\varepsilon}{2}\phi^2\right)} . \tag{6.30}$$

The last factor in (6.30) has been expressed in terms of the free two-particle propagator introduced in the last chapter. We thus have

$$\begin{aligned} Z[J] &= N \mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4z\, V\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(z)}\right)} \mathrm{e}^{-\frac{\mathrm{i}}{2}\int J(x) D_F(x-y) J(y)\, \mathrm{d}^4x\, \mathrm{d}^4y} \\ &= N \mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4x\, V\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)} \mathrm{e}^{\mathrm{i}S_0[J]} \\ &= N \mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4x\, V\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)} Z_0[J] . \end{aligned} \tag{6.31}$$

Expanding the exponential that contains the interaction V then gives the perturbative expansion for $Z[J]$

$$Z[J] = N \sum_{N=0}^{\infty} \frac{1}{N!} \left(-\mathrm{i}\int \mathrm{d}^4x\, V\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)\right)^N Z_0[J] . \tag{6.32}$$

The n-point function then follows from its definition (6.2)

$$\begin{aligned} &G(x_1, x_2, \ldots, x_n) \\ &= \frac{\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2)\ldots\phi(x_n) \sum_{N=0}^{\infty} \frac{1}{N!}\left(-\mathrm{i}\int V\, \mathrm{d}^4x\right)^N \mathrm{e}^{\mathrm{i}\int_{-\infty}^{+\infty} \mathrm{d}^4x\, \mathcal{L}_0}}{\int \mathcal{D}\phi \sum_{N=0}^{\infty} \frac{1}{N!}\left(-\mathrm{i}\int V\, \mathrm{d}^4x\right)^N \mathrm{e}^{\mathrm{i}\int_{-\infty}^{+\infty} \mathrm{d}^4x\, \mathcal{L}_0}} . \end{aligned} \tag{6.33}$$

The rhs of this equation involves the free Lagrangian $\mathcal{L}_0$ and thus a free propagator. By using (6.2) and the developments in Sect. 3.2.2 we can thus rewrite this equation also in terms of normalized vacuum expectation values

$$\begin{aligned} &\langle\tilde{0}|T\left[\phi(x_1)\phi(x_2)\ldots\phi(x_n)\right]|\tilde{0}\rangle \\ &= \frac{\langle 0|T\left[\phi(x_1)\ldots\phi(x_n)\sum_{N=0}^{\infty}\frac{1}{N!}\left(-\mathrm{i}\int V\, \mathrm{d}^4x\right)^N\right]|0\rangle}{\langle 0|T\left[\sum_{N=0}^{\infty}\frac{1}{N!}\left(-\mathrm{i}\int V\, \mathrm{d}^4x\right)^N\right]|0\rangle} ; \end{aligned} \tag{6.34}$$

for clarity, here $|\tilde{0}\rangle$ denotes the vacuum state of the full, interacting Hamiltonian, whereas $|0\rangle$ is that of the free Hamiltonian.

With (6.33) we have thus achieved a remarkable result: (6.33) expresses the expectation value of the time-ordered product of field operators in the vacuum state of the full, interacting theory by a perturbative expansion over free field expectation values. This enables us to calculate G, and ultimately also the scattering matrix S (Chapt. 11), perturbatively up to any desired order in the interaction.

Connected Green's Functions. In (6.20) we have shown how to obtain the connected Green's functions as functional derivatives of the exponent of the generating functional. In the next few steps we, therefore, bring the exponent into a form that can easily be expanded. We start by inserting a factor $1 = \exp(+\mathrm{i}S_0)\exp(-\mathrm{i}S_0)$

$$\begin{aligned} \ln Z[J] &= \ln N + \ln\left(\mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4x\, V\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right)}\mathrm{e}^{\mathrm{i}S_0[J]}\right) \\ &= \ln N + \ln\left(\mathrm{e}^{-\mathrm{i}S_0}\mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4x V}\mathrm{e}^{\mathrm{i}S_0}\right) + \mathrm{i}S_0 \\ &= \ln N + \mathrm{i}S_0[J] + \ln\left[1 + \mathrm{e}^{-\mathrm{i}S_0[J]}\left(\mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4x\, V\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right)} - 1\right)\mathrm{e}^{\mathrm{i}S_0[J]}\right] . \end{aligned} \tag{6.35}$$

A perturbation theoretical treatment is now based on a Taylor expansion of the logarithm. For that purpose we abbreviate

$$\varepsilon[J] = \mathrm{e}^{-\mathrm{i}S_0[J]}\left(\mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4x\, V} - 1\right)\mathrm{e}^{\mathrm{i}S_0[J]} \tag{6.36}$$

and obtain

$$\begin{aligned} iS[J] &= \ln Z[J] = \ln N + \mathrm{i}S_0[J] + \ln(1+\varepsilon[J]) \\ &= \ln N + \mathrm{i}S_0[J] + \left(\varepsilon[J] - \frac{1}{2}\varepsilon^2[J] + \frac{1}{3}\varepsilon^3[J]\ldots\right) . \end{aligned} \tag{6.37}$$

Equation (6.37) represents an expansion of S in powers of the (for $V \to 0$) small quantity ε. In order to obtain a perturbative expansion in terms of the potential V we now rearrange the expansion (6.37). First we expand $\varepsilon[J]$ of (6.36) in terms of the strength of the interaction. This gives

$$\begin{aligned} \varepsilon[J] &= \mathrm{e}^{-\mathrm{i}S_0[J]}\left\{-\mathrm{i}\int \mathrm{d}^4x\, V\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right)\right. \\ &\quad \left. + \frac{1}{2!}\left[-\mathrm{i}\int \mathrm{d}^4x\, V\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right)\right]^2 + \cdots\right\}\mathrm{e}^{+\mathrm{i}S_0[J]} . \end{aligned} \tag{6.38}$$

We now insert this expression into (6.37) and obtain

$$\begin{aligned} \mathrm{i}S[J] &= \ln N + \mathrm{i}S_0[J] + \left(\varepsilon[J] - \frac{1}{2}\varepsilon^2[J] + \ldots\right) \\ &= \ln N + \mathrm{i}S_0[J] + \mathrm{e}^{-\mathrm{i}S_0[J]}\left\{-\mathrm{i}\int \mathrm{d}^4x\, V\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right)\right\}\mathrm{e}^{\mathrm{i}S_0[J]} \\ &\quad + \frac{1}{2!}\mathrm{e}^{-\mathrm{i}S_0[J]}\left\{\left[(-\mathrm{i})\int \mathrm{d}^4x\, V\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right)\right]^2\right\}\mathrm{e}^{\mathrm{i}S_0[J]} \end{aligned} \tag{6.39}$$

$$
\begin{aligned}
& -\frac{1}{2}\left\{\mathrm{e}^{-\mathrm{i}S_0[J]}\left[(-\mathrm{i})\int \mathrm{d}^4x\, V\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right)\right]\mathrm{e}^{\mathrm{i}S_0[J]}\right\}^2 + \mathcal{O}(V^3) \\
= \; & \ln N + \mathrm{i}S_0[J] + \mathrm{i}\tilde{S}_1[J] + \mathrm{i}\tilde{S}_2[J] - \frac{1}{2}\left(\mathrm{i}\tilde{S}_1[J]\right)^2 + \mathcal{O}(V^3) \;,
\end{aligned}
$$

with

$$
\begin{aligned}
\mathrm{i}S_0[J] &= -\frac{\mathrm{i}}{2}\int J(x)D_F(x-y)J(y)\,\mathrm{d}^4x\,\mathrm{d}^4y \\
\mathrm{i}\tilde{S}_1[J] &= \mathrm{e}^{-\mathrm{i}S_0[J]}(-\mathrm{i})\int \mathrm{d}^4x\, V\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right)\mathrm{e}^{+iS_0[J]} \\
\mathrm{i}\tilde{S}_2[J] &= \frac{1}{2!}\mathrm{e}^{-\mathrm{i}S_0[J]}\left[(-\mathrm{i})\int \mathrm{d}^4x\, V\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right)\right]^2\mathrm{e}^{+\mathrm{i}S_0[J]} \;. \qquad (6.40)
\end{aligned}
$$

Note that the term of second order in the interaction V receives contributions both from the linear and the quadratic term in the original expansion (6.37) in ε.

Equation (6.39) represents a perturbative expansion of S in powers of V. It can be used to generate a perturbative expansion for the Green's function for interacting systems for which a closed expression for S cannot be given.

Chapter 7

PERTURBATIVE ϕ^4 THEORY

In this chapter we apply the formalism developed in the preceding chapter to the so-called ϕ^4 theory whose Lagrangian is given by

$$\mathcal{L} = \mathcal{L}_0 - V(\phi) = \mathcal{L}_0 - \frac{g}{4!}\phi^4 \ . \tag{7.1}$$

Here g is a coupling constant. This ϕ^4 theory is a prototype of a field theory with selfinteractions. It serves as a didactical example which exhibits all phenomena of more complex field theories.

7.1 Perturbative Expansion of the Generating Function

Inserting the interaction of ϕ^4 theory (7.1) into (6.39) we obtain

$$\mathrm{i}S_0[J] = -\frac{\mathrm{i}}{2}\int \mathrm{d}^4z\, \mathrm{d}^4y\, J(z) D_F(z-y) J(y) \ , \tag{7.2}$$

$$\mathrm{i}\tilde{S}_1[J] = -\frac{\mathrm{i}g}{4!}\mathrm{e}^{-\mathrm{i}S_0[J]}\int \mathrm{d}^4x\, \frac{\delta^4}{\delta J^4(x)}\mathrm{e}^{\mathrm{i}S_0[J]} \tag{7.3}$$

and

$$\mathrm{i}\tilde{S}_2[J] = \frac{1}{2!}\left(\frac{-\mathrm{i}g}{4!}\right)^2 \mathrm{e}^{-\mathrm{i}S_0[J]}\int \mathrm{d}^4x\, \mathrm{d}^4y\, \frac{\delta^4}{\delta J^4(x)}\frac{\delta^4}{\delta J^4(y)}\, \mathrm{e}^{\mathrm{i}S_0[J]} \ . \tag{7.4}$$

For notational convenience in the following we now introduce the S_i defined by

$$\begin{aligned} \mathrm{i}\tilde{S}_1[J] &= -\frac{\mathrm{i}g}{4!}S_1[J] \\ \mathrm{i}\tilde{S}_2[J] &= \left(\frac{-\mathrm{i}g}{4!}\right)^2 S_2[J] \ . \end{aligned} \tag{7.5}$$

The generating function for the connected Green's functions in ϕ^4 theory reads then

$$\mathrm{i}S[J] = N + \mathrm{i}S_0[J] + \frac{-\mathrm{i}g}{4!}S_1[J] + \left(\frac{-\mathrm{i}g}{4!}\right)^2 S_2[J] - \frac{1}{2}\left(\frac{-\mathrm{i}g}{4!}S_1[J]\right)^2 + \ldots . \tag{7.6}$$

Since each of the $S_i[J]$ contains functional derivatives of the known $S_0[J]$ we can now evaluate the functional derivatives of $S[J]$ and obtain all the Green's functions.

First, we calculate the terms linear in g

$$\begin{aligned} S_1[J] &= \mathrm{e}^{-\mathrm{i}S_0[J]} \int \mathrm{d}^4x \, \frac{\delta^4}{\delta J^4(x)} \mathrm{e}^{+\mathrm{i}S_0[J]} \\ &= \mathrm{e}^{-\mathrm{i}S_0[J]} \int \mathrm{d}^4x \, \frac{\delta^4}{\delta J^4(x)} \mathrm{e}^{-\frac{\mathrm{i}}{2}\int J(z)D_F(z-y)J(y)\,\mathrm{d}^4z\,\mathrm{d}^4y} . \end{aligned} \tag{7.7}$$

The fourth functional derivative is most easily obtained by going to a discrete representation. Noting that

$$\begin{aligned} \frac{\partial^4}{\partial J_k^4} \mathrm{e}^{-\frac{\mathrm{i}}{2}J_i D_{ij} J_j} &= [-3D_{kk}D_{kk} + 6\mathrm{i}D_{kk}(DJ)_k(DJ)_k \\ &\quad + (DJ)_k(DJ)_k(DJ)_k(DJ)_k]\,\mathrm{e}^{-\frac{\mathrm{i}}{2}J_i D_{ij} J_j} \end{aligned} \tag{7.8}$$

we obtain ($k \hat{=} x$)

$$\begin{aligned} S_1[J] &= -3\int D_F(x-x)D_F(x-x)\,\mathrm{d}^4x \\ &\quad + 6\mathrm{i}\int D_F(y-x)D_F(x-x)D_F(x-z)J(y)J(z)\,\mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4z \\ &\quad + \int [D_F(x-y)D_F(x-z)D_F(x-v)D_F(x-w) \\ &\qquad \times J(y)J(z)J(v)J(w)\,\mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4v\,\mathrm{d}^4w\,\mathrm{d}^4z\big] . \end{aligned} \tag{7.9}$$

The first term has no sources and will, therefore, not contribute to any Green's function, the second term is quadratic in J and thus contributes to the two-point function and the last term here has the structure of a point interaction of four fields generated by independent sources at y, z, v, and w and thus contributes only to the four-point function.

7.1.1 Feynman rules

We can again represent these results in a graphical form by using the rules given in section 6.1.2. These were

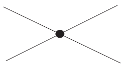
1) $\mathrm{i}D_F(x-y) =$ 

2) $iJ(x) =$ 

3) Integration over all space-time coordinates of external points

4) Symmetry factor for each diagram

We supplement these now for the interacting theory by the additional rules

5) $\dfrac{-\mathrm{i}g}{4!} =$ 

6) Integration $\int \mathrm{d}^4x$ for each loop.

If we represent the interacting connected functional $iS[J]$ by a double line

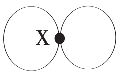
$$\mathrm{i}S[J] = \quad (7.10)$$

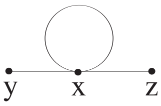
we can draw the graphs for

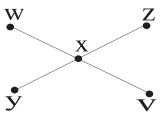
$$iS[J] = \ln N + iS_0[J] + \frac{-\mathrm{i}g}{4!}S_1[J] + \mathcal{O}(g^2) \tag{7.11}$$

as

 $= \ln N +$ (7.12)

$+\Big($ 

$+$ 

$+$ 

$\Big) + \mathcal{O}(g^2)$

The first graph on the right-hand side is again the zeroth order term (7.2), whereas the graphs in the parentheses represent all the terms of first order in the interaction (7.9). The first one of these describes a process without any external lines; this is a vacuum process that takes place regardless if physical particles are present or not. It constitutes a background to all physical processes. The second graph with the single loop describes a mass change due to the selfinteraction that we will discuss in the next section. The third graph, finally, describes a true interaction process.

The symmetry factors in these graphs are the factors in front of the integrals in (7.9); they can be obtained as follows. The first graph carries the factor 1/2 as explained in section 6.1.2.

To construct the vacuum graph we pick one of the 4 legs of the interaction vertex and then connect with any of the other three free legs; there are always 2 pairwise equal loops. Thus in total we get a weight of $4 \times 3/(2 \times 2) = 3$.

For the second term in parentheses in (7.12) we have four legs of the vertex to connect with the external line to y; this gives 4 possibilities. The external line to z can then still be connected with 3 remaining vertex legs. Since there is an exchange symmetry between y and z we get an additional factor 1/2 (as in S_0), so that the weight of this vertex becomes 6.

The last graph, finally, is obtained by joining one of the four legs of the vertex to one of the external points, say z. This generates 4 possibilities. Next we join any one of the 3 remaining free legs of the vertex to the external point y; there are obviously 3 ways to do this. The remaining 2 legs can be joined in 2 different ways with the two external points v and w. Thus, there are in total $4! = 24$ possibilities. However, since all the external points v,w,y and z can be exchanged without changing any of the physics (v,w,y and z are integration variables), there are also 4! identical terms so that the last graph in the parentheses in Fig. 7.12 carries the weight 1.

These weights (symmetry factors) have to be multiplied for each graph to the analytical expression obtained by following the rules given above for the translation of the pictorial representation into an analytical one. Indeed, using the symmetry factors just given and following the Feynman rules for the graphs (7.12) gives the expression (6.39) (together with (7.5) and (7.9)).

7.1.2 Vacuum contributions

We now consider the normalization term $\ln N$. Since we are working with normalized generating functionals, N is given by $W[0]^{-1}$

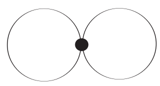
$$\begin{aligned} N^{-1} &= \int \mathcal{D}\phi \, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x \left(\mathcal{L}_0 - V + i\frac{\varepsilon}{2}\phi^2\right)} \\ &= \mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4x V\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)} \mathrm{e}^{-\frac{\mathrm{i}}{2}\int J(x) D_F(x-y) J(y) \mathrm{d}^4x \, \mathrm{d}^4y}\Big|_{J=0} \; . \end{aligned} \tag{7.13}$$

We can now treat this expression in exactly the same way as we just did for $S[J]$; the only change being that we have to take the final result at $J = 0$. This gives (see (7.11))

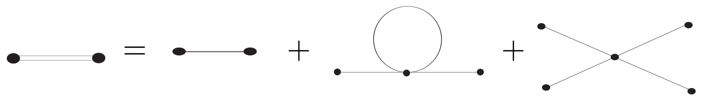
$$\ln N = -\mathrm{i}S[0] = -\mathrm{i}S_0[0] - \frac{(-\mathrm{i}g)}{4!} S_1[0] + \mathcal{O}(g^2) \; . \tag{7.14}$$

In the graphical representation this reads, using $S_0[0] = 0$ and $S_1[0] = -3D_F^2(0)\int \mathrm{d}^4x$ (cf. (7.9)),

$$\ln N = - \tag{7.15}$$



The normalization constant, or – in other words – the denominator of the generating functional, thus contains just the vacuum graph. Inserting (7.15) into (7.12) then removes the vacuum contribution giving, finally, for $Z[J]$ the graphical representation up to terms of first order in the interaction



(7.16)

Although we have shown here only for first-order coupling that the denominator in $Z[J]$ (see (4.3)) just removes the vacuum contributions, this is a general result that holds to all orders of perturbation theory.

7.2 Two-Point Function

The connected n-point function can now be obtained from its definition in (6.20)

$$G_c(x_1, \ldots, x_n) = \left(\frac{1}{\mathrm{i}}\right)^{n-1} \frac{\delta^n S}{\delta J(x_1) \ldots \delta J(x_n)}\bigg|_{J=0} . \tag{7.17}$$

In this section we work out the two-point function in lowest order of the coupling constant.

7.2.1 Terms up to $\mathcal{O}(g)$

Up to terms linear in the coupling strength we obtain from (6.39)

$$\begin{aligned} G_c(x_1, x_2) &= -\mathrm{i}\frac{\delta^2 S}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} \\ &= -\mathrm{i}\frac{\delta^2 S_0}{\delta J(x_1)\delta J(x_2)} - \frac{(-\mathrm{i}g)}{4!}\frac{\delta^2 S_1}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} + \mathcal{O}(g^2) . \end{aligned} \tag{7.18}$$

With $S_0[J]$ from (7.2) and $S_1[J]$ from (7.9) this gives for the two-point function

$$\begin{aligned} G_c(x_1, x_2) &= \mathrm{i}D_F(x_1 - x_2) \\ &\quad - \left(-\frac{\mathrm{i}g}{4!}\right) 12\mathrm{i} \int \mathrm{d}^4x D_F(x - x) D_F(x - x_1) D_F(x - x_2) + \mathcal{O}(g^2) \\ &= \mathrm{i}D_F(x_1 - x_2) - \frac{g}{2}\int \mathrm{d}^4x D_F(x_2 - x) D_F(x - x) D_F(x - x_1) \\ &\quad + \mathcal{O}(g^2) . \end{aligned} \tag{7.19}$$

This is the connected propagator, up to terms of $\mathcal{O}(g)$, of the interacting theory.

We can represent this equation in the following graphical form, where denotes the "dressed" propagator

(7.20)

with the rules developed above.

The weight factors for these diagrams have been explained at the end of section 6.1.2. Since we deal here with Green's functions with definite, fixed external points, the exchange symmetry factors must not be divided out here. Thus, the first diagram on the rhs in (7.20) carries the weight 1 and the second, the so-called tadpole diagram, the weight 12.

7.2.2 Momentum representation

At the start of Chapt. 6 we have introduced the momentum representation of the Green's function and in Sect. 6.1.3 we have already evaluated it for the free case. Here we now determine it for the ϕ^4 theory up to terms of order $\mathcal{O}(g)$.

Taking the Fourier-transform of the two-point function (7.19) gives

$$
\begin{aligned}
&\int \mathrm{e}^{-\mathrm{i}(p_1x_1+p_2x_2)} G_\mathrm{c}(x_1,x_2)\,\mathrm{d}^4x_1\,\mathrm{d}^4x_2 \qquad (7.21)\\
&= \int \left[\mathrm{e}^{-\mathrm{i}(p_1x_1+p_2x_2)} \left(\frac{\mathrm{i}}{(2\pi)^4}\right) \int \mathrm{d}^4q\, \frac{\mathrm{e}^{-\mathrm{i}q(x_1-x_2)}}{q^2-m^2+\mathrm{i}\varepsilon}\right] \mathrm{d}^4x_1\mathrm{d}^4x_2\\
&\quad -\frac{g}{2}\int \left[\mathrm{e}^{-\mathrm{i}(p_1x_1+p_2x_2)} \int \mathrm{d}^4x \left(\frac{1}{(2\pi)^4}\right)^3\right.\\
&\quad \times \left.\int \mathrm{d}^4q_1\,\mathrm{d}^4q_2\,\mathrm{d}^4q_3\, \frac{\mathrm{e}^{-\mathrm{i}q_2(x_2-x)}\mathrm{e}^{-\mathrm{i}q_1(x-x_1)}}{(q_1^2-m^2+\mathrm{i}\varepsilon)(q_2^2+m^2-i\varepsilon)(q_3^2-m^2+\mathrm{i}\varepsilon)}\right] \mathrm{d}^4x_1\,\mathrm{d}^4x_2\ .
\end{aligned}
$$

As in Sect. 6.1.3 we first perform the integrations over x_1, x_2 and x and obtain for the rhs

$$
\begin{aligned}
(2\pi)^4\delta^4(p_1+p_2)G_\mathrm{c}(p_1,p_2) &= (2\pi)^4\delta^4(p_1+p_2)\frac{\mathrm{i}}{p_1^2-m^2+\mathrm{i}\varepsilon}\\
&-\frac{g}{2}(2\pi)^4\delta^4(p_1+p_2)\frac{1}{p_1^2-m^2+\mathrm{i}\varepsilon}\frac{1}{p_2^2-m^2+\mathrm{i}\varepsilon}
\end{aligned}
$$

$$\times \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{q^2 - m^2 + \mathrm{i}\varepsilon} \,. \tag{7.22}$$

This equation is used to read off the momentum representation of the propagator (see (6.4)). The momenta p_1 and p_2 are incoming momenta that point towards a vertex. Thus, the momentum representation of (7.19) is given by

$$\begin{aligned} G_{\mathrm{c}}(p,p' = -p) &= \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon} \\ &+ S \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon} \left(\frac{-\mathrm{i}g}{4!} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathrm{i}}{q^2 - m^2 + \mathrm{i}\varepsilon} \right) \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon} \end{aligned} \tag{7.23}$$

where the symmetry factor is $S = 12$. Equation (7.23) gives the momentum representation of the two-point function up to terms of order g. The first term on the rhs gives the free propagator (6.24) already obtained in Sect. 6.1.3, whereas the second term gives the contribution of the interaction to this two-point function.

Selfenergy. With the abbreviation

$$\Sigma = \frac{g}{2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathrm{i}}{q^2 - m^2 + \mathrm{i}\varepsilon} \tag{7.24}$$

we can write the two-point function (7.23) as

$$\begin{aligned} G_{\mathrm{c}} &= G_0 + G_0 \frac{\Sigma}{\mathrm{i}} G_0 = G_0 \left(1 + \frac{\Sigma}{\mathrm{i}} G_0 \right) \approx G_0 \left(1 - \frac{\Sigma G_0}{\mathrm{i}} \right)^{-1} \\ &= \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon} \frac{1}{1 - \Sigma \frac{1}{p^2 - m^2 + \mathrm{i}\varepsilon}} \\ &= \frac{\mathrm{i}}{p^2 - m^2 - \Sigma + \mathrm{i}\varepsilon} \,. \end{aligned} \tag{7.25}$$

Here we have consistently kept terms up to $\mathcal{O}(g)$. The quantity Σ appears like an additional mass term in the final result. It is, therefore, called a *selfenergy* and the second graph on the rhs in (7.20) is called a selfenergy insertion.

7.2.2.1 Momentum space Feynman rules

The Feynman rules for (7.23) are now

1) each line gives a factor $\dfrac{\mathrm{i}}{q^2 - m^2 + \mathrm{i}\varepsilon}$.

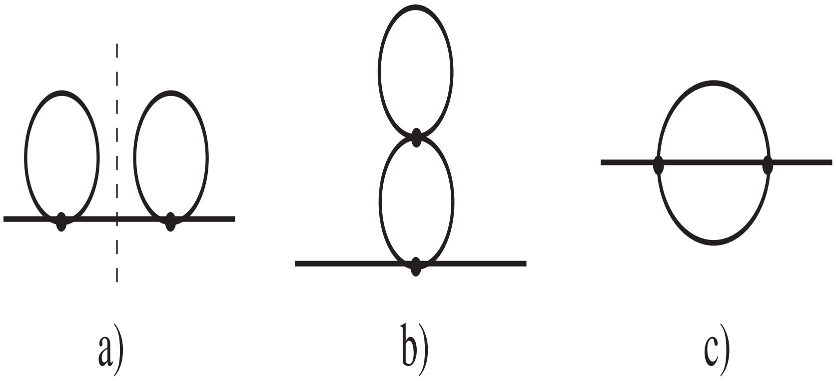


Figure 7.1: Feynman graphs for the two-point function up to $\mathcal{O}(g^2)$

.

2) each vertex gives a factor $\dfrac{-\mathrm{i}g}{4!}$.

3) there is four-momentum conservation for the sum of all momenta flowing into a vertex.

4) each internal line gives an integration $\displaystyle\int \frac{\mathrm{d}^4q}{(2\pi)^4}$.

5) to each diagram a weight factor has to be multiplied as explained above.

7.2.3 Terms up to $\mathcal{O}(g^2)$

With the help of the Feynman rules we can now construct the corresponding Feynman graphs up to terms of $\mathcal{O}(g^2)$. For the two-point function these are given in Fig. 7.1. The two-point function for the graph (a) in Fig. 7.1 is given by

$$\begin{aligned} G_a(p,-p) &= S_a \frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon}\left(\frac{-\mathrm{i}g}{4!}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\mathrm{i}}{q^2-m^2+\mathrm{i}\varepsilon}\right) \\ &\times \frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon}\left(\frac{-\mathrm{i}g}{4!}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\mathrm{i}}{q^2-m^2+\mathrm{i}\varepsilon}\right)\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon}\,, \end{aligned} \tag{7.26}$$

S_a is the symmetry factor; it is simply given by the product of the corresponding factors for the one-loop graphs, $S_a = 12\cdot 12 = 144$.

It is evident from Fig. 7.1a, as well as from its algebraic representation in (7.26), that the graph can be cut into two parts, each representing a first order process. This reflects the appearance of the last term in (6.39) that is simply the square of the first order term. Such a graph that falls apart into 2 unconnected parts, if one internal line is cut, is called *one-particle-reducible*; otherwise it is *one-particle-irreducible (1PI)*.

In order to facilitate the following discussions we introduce now the 1PI Green's function $\Gamma(p_1, p_2, \ldots, p_n)$, sometimes also called *connected proper vertex function*, which describes only 1PI graphs and in which the propagators for the external lines are missing. The 1PI n-point function is, therefore, given by

$$\begin{aligned}&\Gamma(p_1, p_2, \ldots, p_n) \\ &\quad = G^{-1}(p_1, -p_1) G^{-1}(p_2, -p_2) \ldots G^{-1}(p_n, -p_n) G_c(p_1, p_2, \ldots, p_n) \;.\end{aligned} \tag{7.27}$$

The free 1PI 2-point function is then simply given by

$$\Gamma(p, -p) = p^2 - m^2 \;. \tag{7.28}$$

The 1PI part of the Green's function for the graph 7.1a reads

$$\begin{aligned}\Gamma_a(p, -p) &= G^{-1}(p, -p) G^{-1}(-p, p) G_{\mathrm{a}}(p, -p) \\ &= S_a \frac{-\mathrm{i}g}{4!} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathrm{i}}{q^2 - m^2 + \mathrm{i}\varepsilon}\end{aligned} \tag{7.29}$$

with $S_a = 144$.

We then get for the graph in Fig. 7.1b, which is one-particle irreducible,

$$\begin{aligned}\Gamma_b(p, -p) &= S_b \left(\frac{-\mathrm{i}g}{4!}\right)^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathrm{d}^4 u}{(2\pi)^4} \frac{\mathrm{i}}{q^2 - m^2 + \mathrm{i}\varepsilon} \frac{\mathrm{i}(2\pi)^4 \delta^4(q-u)}{u^2 - m^2 + \mathrm{i}\varepsilon} \\ &\quad \times \int \frac{\mathrm{d}^4 r}{(2\pi)^4} \frac{\mathrm{i}}{r^2 - m^2 + \mathrm{i}\varepsilon}\end{aligned} \tag{7.30}$$

with $S_b = 6 \cdot 4! = 144$. For the graph in Fig. 7.1c (also 1PI) we finally obtain

$$\begin{aligned}\Gamma_c(p, -p) &= S_c \left(\frac{-\mathrm{i}g}{4!}\right)^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathrm{d}^4 r}{(2\pi)^4} \frac{\mathrm{d}^4 s}{(2\pi)^4} \delta(p - (q + r + s)) \\ &\quad \times \frac{\mathrm{i}}{q^2 - m^2 + \mathrm{i}\varepsilon} \frac{\mathrm{i}}{r^2 - m^2 + \mathrm{i}\varepsilon} \frac{\mathrm{i}}{s^2 - m^2 + \mathrm{i}\varepsilon} \;,\end{aligned} \tag{7.31}$$

with $S_c = 4 \cdot 4! = 96$.

7.3 Four-Point Function

It is easy to see that the three-point function vanishes for the model considered here since the third functional derivative (see (7.17)) of the action (6.39) at $J = 0$ vanishes.

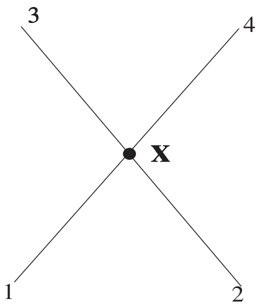


Figure 7.2: Feynman graph for the four-point function.

7.3.1 Terms up to $\mathcal{O}(g)$

The four-point function up to terms of $\mathcal{O}(g)$ is given by

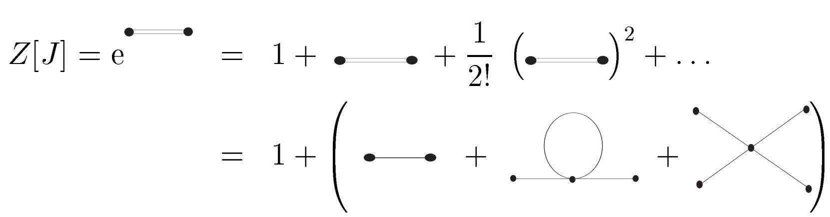
$$\begin{aligned} G_c(x_1, x_2, x_3, x_4) &= \left(\frac{1}{\mathrm{i}}\right)^3 \left.\frac{\delta^4 S}{\delta J(x_1)\ldots\delta J(x_4)}\right|_{J=0} \qquad (7.32) \\ &= \mathrm{i}\left.\frac{\delta^4 S_0}{\delta J(x_1)\ldots\delta J(x_4)}\right|_{J=0} + \frac{(-\mathrm{i}g)}{4!}\left.\frac{\delta^4 S_1}{\delta J(x_1)\ldots\delta J(x_4)}\right|_{J=0} \\ &\quad + \mathcal{O}(g^2) \end{aligned}$$

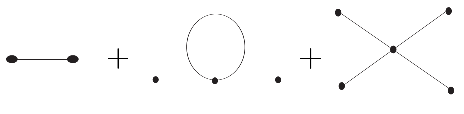
where S is given by (7.6) and Fig. 7.1. Because G_c involves the fourth derivative with respect to the source and S_0 depends on J only quadratically (see (7.2)), only the last term of S_1 in (7.9) can contribute to the Green's function. Thus we get

$$G_c(x_1, \ldots, x_4) = -\mathrm{i}g \int \mathrm{d}^4x D_F(x-x_1) D_F(x-x_2) D_F(x-x_3) D_F(x-x_4) \ . \qquad (7.33)$$

In momentum space this is simply the product of the four propagators (6.24) times the factor $-\mathrm{i}g$. The corresponding Feynman graph is given in Fig. 7.2. It carries the symmetry factor 4!, corresponding to a symmetry under permutation of all external legs.

Unconnected graphs. From the connected graphs calculated so far, we could reconstruct also the unconnected graphs. Up to terms linear in the coupling constant g we get in symbolic notation

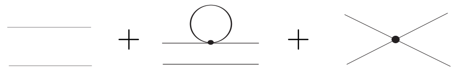


$$+\frac{1}{2!}\Bigg($$

$$\Bigg)^2$$

$$+\mathcal{O}(g^2)\ . \tag{7.34}$$

The four-point function is generated by diagrams with four external legs (each external leg corresponds to a factor J in the generating functional), because G is given by a fourth functional derivative. Therefore, only the last diagram in the second line of (7.34) and the square of the first term in the last line can contribute to the four-point function in this order.

We thus have for the four-point function up to terms linear in the coupling

$$G(x_1, x_2, x_3, x_4) = \tag{7.35}$$


In both of the first two diagrams the two particles just move by each other, without interaction. These unconnected graphs thus do not contribute to any interaction processes.

7.3.2 Terms up to $\mathcal{O}(g^2)$

.

In order to become more familiar with Feynman graphs, we now construct the connected four-point function up to terms of order g^2 in a graphical way. This four-point function is shown in Fig. 7.3.

The first line in Fig. 7.3 gives the four-point function just constructed, with a symmetry factor $S_1 = 4!$. The four diagrams on the second line are just the basic vertices with self-energy insertions on each of the external legs; these insertions carry the extra symmetry factor $S_2 = 12$, as we have seen in section 7.1.1. The last three diagrams in the third line are of a new topological structure. They represent modifications of the basic interaction vertex through the insertion of internal lines.

Each one of these graphs has the same external lines. We thus have an overall factor for all graphs

$$S_1 \prod_{k=1}^{4} \frac{\mathrm{i}}{p_k^2 - m^2}\ , \tag{7.36}$$

with $S_1 = 24$ so that the full four-point function is given by

$$G_{\mathrm{c}}(p_1, p_2, p_3, p_4) = S_1 \prod_{k=1}^{4} \frac{\mathrm{i}}{p_k^2 - m^2}\,(G_1 + G_2 + G_3)\ , \tag{7.37}$$

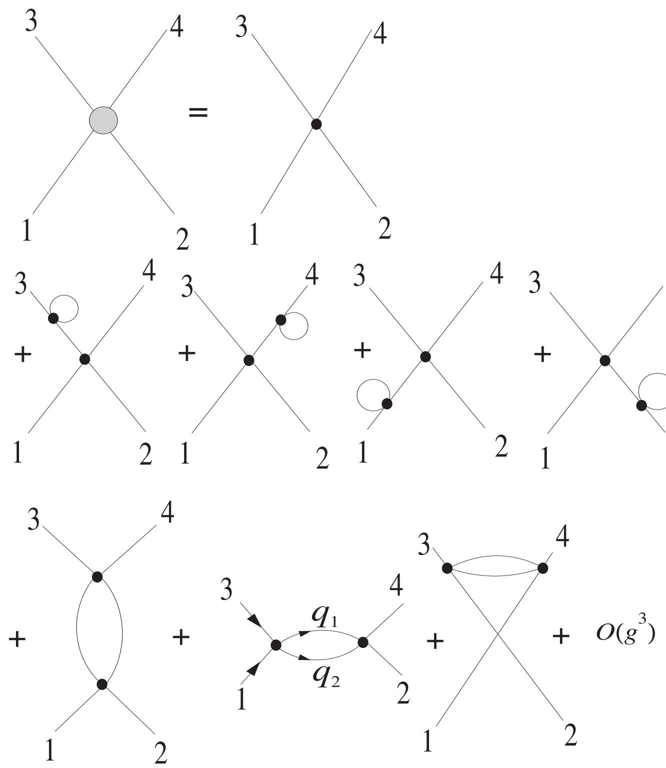


Figure 7.3: Four-point function of ϕ^4 theory up to terms $\sim g^2$.

where G_i denotes the contribution from the i-th line in Fig. 7.3 without the external line symmetry factor.

The first basic vertex then just gives the contribution

$$G_1 = \frac{-\mathrm{i}g}{4!} \ . \tag{7.38}$$

The graphs of the second line have one loop in addition. We thus have

$$G_2 = (\frac{-\mathrm{i}g}{4!})^2 S_2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathrm{i}}{q^2 - m^2} \sum_{l=1}^{4} \frac{\mathrm{i}}{p_l^2 - m^2} \tag{7.39}$$

for their contribution, with $S_2 = 12$. The internal momenta p_l here are those between the loop and the four-point vertex. Since the loop carries no momentum away they are the same as the corresponding incoming momenta.

The three graphs of the third line, finally, have two internal lines, which are, however, related by energy - and momentum conservation at the incoming vertex. They give

$$G_3 = (\frac{-\mathrm{i}g}{4!})^2 S_3 \int \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \frac{\mathrm{d}^4 q_2}{(2\pi)^4} \frac{\mathrm{i}}{q_1^2 - m^2} \frac{\mathrm{i}}{q_2^2 - m^2} \tag{7.40}$$

$$\times \quad (2\pi)^4 \sum_{kl} \delta^4(q_1 + q_2 - (p_k + p_l))$$

where the last sum runs only over the pairs of numbers (1,2), (1,3) and (1,4), i.e. the external legs at one of the vertices in each of the three graphs. The δ-function appears because the net momentum running into the dressed vertex has to be zero (see (6.4)).

The symmetry factor S_3 can, for example for the middle graph of Fig. 7.3, be obtained as follows. The external leg 1 can be connected with the left vertex in 4 different ways; the same holds for the leg 2 with the right vertex. Once these connections (4×4 possibilities) have been done, each of the 3 free legs of the left vertex can be connected with the external leg 3; the same holds for the connections of the right vertex to the external point 4 (3×3 possibilities). Finally, each of the remaining 2 legs of the left vertex can be connected to the right vertex; for the remaining leg there is no freedom left (2 possibilities). Thus, the symmetry factor for the last 3 graphs in Fig. 7.3 is

$$S_3 = \frac{(4 \cdot 4)(3 \cdot 3)2}{S_1} = \frac{(4!)^2}{2S_1} = \frac{4!}{2} \; . \tag{7.41}$$

Chapter 8

DIVERGENCES IN n-POINT FUNCTIONS

Many of the expressions obtained in the preceding sections for two- and four-point functions are actually ill-defined because they diverge, as we will show in this section. We start with a rather general discussion of divergences in ϕ^4 theory and then evaluate explicitly the two- and four-point functions.

To illustrate the divergence of the Green's functions obtained we consider, as an example, the two-point function

$$\begin{aligned} G_c(p,-p) &= \frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon} \\ &\quad + \left(-\mathrm{i}\frac{g}{2}\right)\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon}\mathrm{i}D_F(0)\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon}\ . \end{aligned} \tag{8.1}$$

The loop contribution between the two Feynman propagators in the second term on the rhs is given by

$$\Sigma = \frac{g}{2}\mathrm{i}D_F(0) = \frac{g}{2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\,\frac{\mathrm{i}}{q^2-m^2+\mathrm{i}\varepsilon}\ . \tag{8.2}$$

The integral here diverges: after integrating over q_0 we obtain integrals of the form (cf. section 4.1.2)

$$\int\frac{\mathrm{d}^3q}{\sqrt{\vec{q}^{\,2}+m^2}}\ . \tag{8.3}$$

By introducing an upper bound Λ for the integral over $|\vec{q}|$ and then taking $\Lambda\to\infty$ we see that the integral diverges as Λ^2; this is called a quadratic divergence. Because this divergence happens here for large q one also speaks of a quadratic ultraviolet divergence.

Another way to see the degree of divergence is the so-called "power-counting": There are 4 powers of q in the integration measure, but only two powers of q in the denominator of (8.2); the degree of divergence is then given by the net power of q.

8.1 Power Counting

The power-counting just illustrated for the case of the tadpole diagram can be generalized to any Feynman graphs with an arbitrary number of loops. In order to see this we consider a theory with an interaction $\sim \phi^p$ in n dimensions. Since each loop contributes according to the Feynman rules an integral $\int \mathrm{d}^n q$ to the total expression and since each internal propagator gives a power q^{-2}, we have for the degree of divergence D in a diagram with L loops and I internal lines

$$D = nL - 2I \ . \tag{8.4}$$

Note that here each loop has also at least 1 internal line. For example, the tadpole diagram has $L = 1$ and $I = 1$, giving $D = 2$ in four dimensions. $D > 0$ clearly diverges, $D = 0$ corresponds to a logarithmic divergence, and $D < 0$ seems to be convergent.

If a graph has V interaction vertices, then the total number of lines in ϕ^p theory is pV, since each vertex has p legs. These legs can be either external or internal lines. If they are internal, they count twice because each internal line originates and disappears at a vertex. Thus we have

$$pV = E + 2I \ , \tag{8.5}$$

where E is the number of external lines. In addition, the number of loops, L, is related to the number of vertices, V, by

$$L = I - V + 1 \ . \tag{8.6}$$

Combining equations (8.5) and (8.6) with (8.4) allows us to eliminate L and I to obtain

$$D = n + \left(\frac{n(p-2)}{2} - p\right) V - \left(\frac{p}{2} - 1\right) E \ . \tag{8.7}$$

The degree of divergence of a connected graph in this ϕ^p theory in n dimensions thus depends on the number of external legs and, in general, also on the number of vertices. In the perturbative treatment of field theory derived in Chap. 6.2 we have seen that the order of perturbation theory directly equals the number of vertices, V, in a Feynman diagram. Thus, the degree of divergence becomes larger and larger with increasing order of a perturbative

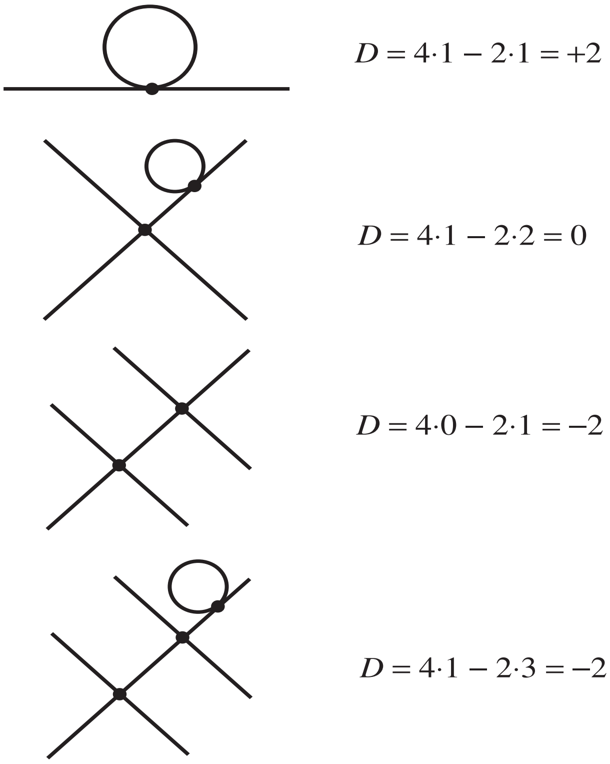


Figure 8.1: Examples for graphs with different degree of divergence. On the right equation (8.4) is illustrated.

treatment, if the factor of V in (8.7) is positive. On the other hand, D is independent of this order if that factor is zero and D becomes even smaller when going to higher orders of perturbation theory if the factor is negative. Thus the perturbative treatment leads to a finite number of divergent terms if and only if

$$\frac{n(p-2)}{2} - p \leq 0 \ . \tag{8.8}$$

If this factor is zero, then the total number of divergent diagrams in a perturbative expansion can still be infinite, but in each order of perturbation theory only the same finite number of divergent diagrams appears. In this case the theory is called *renormalizable*. If the factor is negative, then even the total number of divergent terms is finite; in this case the theory is called *superrenormalizable*. In both cases one can add a finite number of so-called counter terms to the Lagrangian that just remove these divergences.

In order to become familiar with this power counting for ϕ^4 theory in four dimensions we give three examples in Fig. 8.1. According to our earlier considerations we expect that $D \geq 0$ diverges. However, this does not guarantee that graphs with $D < 0$ actually converge. This is illustrated by the lowest example, in Fig. 8.1, that has $L = 1, I = 3$, and thus $D = 4\cdot 1 - 2\cdot 3 = -2$, but diverges, because of the loop on the externals legs. Only when each possible subgraph has also $D < 0$, then the whole expression converges (Weinberg's Theorem).Note that in the physical case $n = 4$ and $p = 4$ D does not depend on V, i.e. it is independent of the order of perturbation theory.

8.2 Dimensional Regularization

We now evaluate the divergent integrals by a modern technique called dimensional regularization. This technique starts by considering the whole theory in n dimensions.

We first consider the dimensions in the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left[\partial^\mu\phi\partial_\mu\phi - m^2\phi^2\right] - \frac{g}{4!}\phi^4 \ . \tag{8.9}$$

Since the action

$$S = \int \mathcal{L}\,\mathrm{d}^n x \tag{8.10}$$

is a dimensionless quantity (in units in which $\hbar = 1$), $\mathcal{L}$ must have the dimension ℓ^{-n} (ℓ is a length). The kinetic energy term in $\mathcal{L}$ thus has also dimension ℓ^{-n} and – since $[\partial_\mu] = \ell^{-1}$ – we get $[\phi] = \ell^{1-n/2}$. Any potential term of the form $g\phi^p$ must have the same dimension as the Lagrangian, i.e. ℓ^{-n} ; we thus get $[g] = \ell^{-n}/\ell^{p-pn/2} = \ell^{-(n+p(1-\frac{n}{2}))}$. The mass term ($p = 2$) thus has $[m^2] = \ell^{-2}$ in n dimensions, as it should. Note that the dimension of g is just the factor of the number of vertices, V, in (8.7). Thus, if the dimension of g is that of a negative or zeroth power of ℓ, then the theory is superrenormalizable or renormalizable, respectively.

These considerations show that for $n = 4$

$$[g] = \ell^{p-4} \ ; \tag{8.11}$$

i.e. in four dimensions the coupling constant of a ϕ^4 theory is dimensionless and the theory is, therefore, renormalizable; in fact, only theories with $p \leq 4$ are renormalizable. Therefore, according to the discussion in the preceding section the ϕ^4 theory contains only a finite number of divergent vertices. In $n \neq 4$ dimensions, however, this is no longer true.

If we want to keep the coupling constant dimensionless in order to ensure renormalizability also in n dimensions we have to modify the ϕ^4 term in the Lagrangian (6.40) such that an additional factor absorbs the dimension

$$\mathcal{L} = \mathcal{L}_0 - \frac{g}{4!}\mu^{4-n}\phi^4 \ . \tag{8.12}$$

Here μ is an arbitrary quantity with the dimension of a mass. Since $[\mathcal{L}] = \ell^{-n}$ and $[\phi^4] = \ell^{4-2n}$ we have

$$[g\mu^{4-n}] = \ell^{n-4} \tag{8.13}$$

and thus, with $[\mu] = \ell^{-1}$, g is indeed dimensionless.

8.2.1 Two-point function

We now calculate the term Σ coming from the tadpole diagram. To do so we first go to n dimensions so that (8.2) becomes

$$\Sigma = \frac{g}{2}\mu^{4-n} \int \frac{\mathrm{d}^n q}{(2\pi)^n} \frac{\mathrm{i}}{q^2 - m^2 + \mathrm{i}\varepsilon} \ . \tag{8.14}$$

This integral can be obtained analytically by going into a space of n-dimensional polar coordinates (see Appendix. C). The result is

$$\Sigma = \frac{g\mu^{4-n}}{2(2\pi)^n} m^{n-2} \pi^{\frac{n}{2}} \Gamma\left(1 - \frac{n}{2}\right) \ . \tag{8.15}$$

The divergence of this expression is now manifest, since the Γ-function has poles at 0 and the negative integers, and thus also for $n = 4$.

We now expand Γ around this pole. For that purpose we write

$$\Gamma\left(1 - \frac{n}{2}\right) = \Gamma\left(-1 + \frac{\varepsilon}{2}\right) \tag{8.16}$$

with $\varepsilon = 4 - n$ and expand in powers of ε

$$\Gamma\left(-1 + \frac{\varepsilon}{2}\right) = -\frac{2}{\varepsilon} - 1 + \gamma + \mathcal{O}(\varepsilon) \ , \tag{8.17}$$

where γ is the Euler-Mascheroni constant ($\gamma = 0.577$). We thus get for n close to 4

$$\begin{aligned} \Sigma &= \frac{g\mu^{\varepsilon}}{2(2\pi)^{4-\varepsilon}} m^{2-\varepsilon} \pi^{\frac{4-\varepsilon}{2}} \left(-\frac{2}{\varepsilon} - 1 + \gamma + \mathcal{O}(\varepsilon)\right) \qquad (8.18) \\ &= \frac{gm^2}{32\pi^2} \left(\frac{4\pi\mu^2}{m^2}\right)^{\frac{\varepsilon}{2}} \left(-\frac{2}{\varepsilon} - 1 + \gamma + \mathcal{O}(\varepsilon)\right) \ . \end{aligned}$$

We now use

$$x^{\varepsilon} = e^{\varepsilon \ln x} \xrightarrow{\varepsilon \to 0} 1 + \varepsilon \ln x \tag{8.19}$$

and obtain

$$\begin{aligned}
\Sigma &\overset{\varepsilon\to 0}{\longrightarrow} \frac{gm^2}{32\pi^2}\left[1+\frac{\varepsilon}{2}\ln\left(\frac{4\pi\mu^2}{m^2}\right)\right]\left(-\frac{2}{\varepsilon}-1+\gamma+\mathcal{O}(\varepsilon)\right) \\
&= \frac{gm^2}{32\pi^2}\left[-\frac{2}{\varepsilon}-1+\gamma-\ln\left(\frac{4\pi\mu^2}{m^2}\right)\right]+\mathcal{O}(\varepsilon) \\
&= -\frac{gm^2}{16\pi^2}\frac{1}{\varepsilon}-\frac{gm^2}{32\pi^2}\left[1-\gamma+\ln\left(\frac{4\pi\mu^2}{m^2}\right)\right]+\mathcal{O}(\varepsilon)\ . \qquad (8.20)
\end{aligned}$$

We have thus split Σ into two parts. The first one is clearly diverging as n approaches 4. The second term is finite and depends on the arbitrary mass μ. Note that Σ is independent of p. In next higher order the same is true for the selfenergy contribution of Fig. 7.1b on page 80 whereas that of Fig. 7.1c depends quadratically on p [Ramond].

8.2.2 Four-point function

In this section we will now evaluate the four-point function. By looking at Fig. 7.3 on page 84 we expect that the four graphs in the second line just contribute again to the selfenergy. On the other hand, we expect that the three diagrams in the lowest line can all graphically be contracted such that they contain only one interaction point with possibly modified interaction strength.

As an example, we now evaluate the middle graph in the last line of Fig. 7.3. According to the Feynman rules the second order contribution to the 1PI vertex is (see (7.40))

$$\Gamma(p_1,p_2,p_3,p_4)=\left(\frac{-ig}{4!}\mu^{4-n}\right)^2 S_3S_1\int\frac{\mathrm{d}^4q}{(2\pi)^4}\,\frac{\mathrm{i}}{q^2-m^2}\,\frac{\mathrm{i}}{(p-q)^2-m^2} \qquad (8.21)$$

with $p=p_1+p_3$. The symmetry factor is $S_1S_3=12\cdot 4!$ (7.40).

The two denominators in the integrand can be combined into one by a mathematical trick due to Feynman that is very often used for the evaluation of such expressions. This trick starts from the elementary integral relation

$$\int_a^b\frac{\mathrm{d}x}{x^2}=-\frac{1}{x}\Big|_a^b=-\frac{1}{b}+\frac{1}{a}=\frac{b-a}{ab}\ . \qquad (8.22)$$

By setting now

$$x=az+b(1-z) \qquad (8.23)$$

we obtain

$$\int_a^b \frac{\mathrm{d}x}{x^2} = (a-b)\int_1^0 \frac{\mathrm{d}z}{\left[az+b(1-z)\right]^2} \ . \tag{8.24}$$

Combining (8.22) and (8.24) gives

$$\frac{1}{ab} = \int_0^1 \frac{\mathrm{d}z}{\left[az+b(1-z)\right]^2} \ . \tag{8.25}$$

We now apply this to the integrand in (8.21) and obtain

$$\begin{aligned} \frac{1}{q^2-m^2}\,\frac{1}{(p-q)^2-m^2} &= \int_0^1 \frac{\mathrm{d}z}{\left\{(q^2-m^2)z+\left[(p-q)^2-m^2\right](1-z)\right\}^2} \\ &= \int_0^1 \frac{\mathrm{d}z}{\left[q^2-2pq(1-z)+p^2(1-z)-m^2\right]^2} \ . \end{aligned} \tag{8.26}$$

The first three terms in the denominator can be combined by substituting

$$q' = q - p(1-z) \ . \tag{8.27}$$

This gives for the denominator

$$\begin{aligned} & q^2-2pq(1-z)+p^2(1-z)-m^2 \\ &= \left[q-p(1-z)\right]^2 - m^2 - p^2(1-z)^2 + p^2(1-z) \\ &= q'^2 - m^2 - p^2 z(z-1) \ . \end{aligned} \tag{8.28}$$

The 1PI vertex now reads in n dimensions

$$\Gamma(p_1,p_2,p_3,p_4) = \frac{1}{2}g^2(\mu^2)^{4-n}\int \frac{\mathrm{d}^n q'}{(2\pi)^n}\int_0^1 \mathrm{d}z\,\frac{1}{\left[q'^2-m^2-sz(z-1)\right]^2} \tag{8.29}$$

with $s = p^2 = (p_1+p_3)^2$.

We now interchange the order of integration and evaluate the integral over q' first with the help of (C.11) in Appendix C

$$\begin{aligned} & \int \frac{\mathrm{d}^n q'}{(2\pi)^n}\,\frac{1}{\left[q'^2-m^2-sz(1-z)\right]^2} \\ &= \frac{\mathrm{i}}{(2\pi)^n}\left[m^2+sz(1-z)\right]^{\frac{n-4}{2}}\,\pi^{\frac{n}{2}}\,\frac{\Gamma\left(2-\frac{n}{2}\right)}{\Gamma(2)} \ . \end{aligned} \tag{8.30}$$

The four-point function thus reads

$$\begin{aligned}\Gamma(p_1,p_2,p_3,p_4) &= \frac{1}{2}g^2(\mu^2)^{4-n}\mathrm{i}\frac{\pi^{\frac{n}{2}}}{(2\pi)^n}\Gamma\left(2-\frac{n}{2}\right)\\ &\times\int\limits_0^1 \mathrm{d}z\,\left[m^2+sz(1-z)\right]^{\frac{n-4}{2}}\ . \end{aligned} \tag{8.31}$$

We now introduce again $\varepsilon = 4-n$. This gives

$$\Gamma(p_1,p_2,p_3,p_4) = \frac{1}{2}g^2\mu^\varepsilon\mathrm{i}\frac{1}{16\pi^2}\Gamma\left(\frac{\varepsilon}{2}\right)\int\limits_0^1\left[\frac{m^2+sz(1-z)}{4\pi\mu^2}\right]^{-\frac{\varepsilon}{2}}\mathrm{d}z\ . \tag{8.32}$$

Now the four-point function is in a form that allows to take the limit $\varepsilon \to 0$. Using again

$$x^\varepsilon = e^{\varepsilon\ln x} \to 1+\varepsilon\ln x \tag{8.33}$$

and

$$\Gamma\left(\frac{\varepsilon}{2}\right) = \frac{2}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon) \tag{8.34}$$

gives

$$\begin{aligned}\Gamma(p_1,p_2,p_3,p_4) &= \frac{\mathrm{i}g^2\mu^\varepsilon}{32\pi^2}\left[\frac{2}{\varepsilon}-\gamma++\mathcal{O}(\varepsilon)\right]\\ &\times\left[1-\frac{\varepsilon}{2}\int\limits_0^1\ln\left(\frac{m^2+sz(1-z)}{4\pi\mu^2}\right)\,\mathrm{d}z\right]\\ &= \frac{\mathrm{i}g^2\mu^\varepsilon}{16\pi^2}\frac{1}{\varepsilon} - \frac{\mathrm{i}g^2\mu^\varepsilon}{32\pi^2}\left[\gamma+\int\limits_0^1\ln\left(\frac{m^2+sz(1-z)}{4\pi\mu^2}\right)\,\mathrm{d}z\right]\\ &+\mathcal{O}(\varepsilon) \qquad \text{with} \qquad s=(p_1+p_3)^2\ . \end{aligned} \tag{8.35}$$

Here the four-point function has been separated into a divergent part and a convergent term (for $\varepsilon \to 0$). The integral appearing is a function of p, m, and μ; the latter dependence remains even when $\varepsilon \to 0$.

So far, we have only calculated the middle graph in the last line of Fig. 7.3 on page 84. It is evident, however, that the result can be directly taken over also to the other 2 graphs in the last line by taking for p the appropriate total momentum at one of the vertices. We, therefore, introduce now the three Lorentz-invariant Mandelstam variables

$$\begin{aligned} s &= (p_1+p_3)^2 = p^2\\ t &= (p_1+p_2)^2\\ u &= (p_1+p_4)^2\ , \end{aligned} \tag{8.36}$$

with the property $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$, which represent the total squared four-momentum at the vertex that involves p_1 in each of the graphs in the last line of Fig. 7.3.

We thus get for the sum of all three diagrams in the last line of Fig. 7.3

$$\begin{aligned}\Gamma(p_1, p_2, p_3, p_4) &= \frac{3\mathrm{i}g^2\mu^\varepsilon}{16\pi^2}\frac{1}{\varepsilon} \\ &\quad - \frac{\mathrm{i}g^2\mu^\varepsilon}{32\pi^2}\left[3\gamma + F(s, m, \mu) + F(t, m, \mu) + F(u, m, \mu)\right] \ .\end{aligned} \tag{8.37}$$

Here $F(s, m, \mu)$ denotes the integral in (8.35).

Of the diagrams in Fig. 7.3 only the first and the three last ones are 1PI graphs. The four diagrams in the second line are all 1P reducible; they just differ by the propagators on the external legs that are left out when we consider the 1PI four-point function. The complete 1PI four-point function is, therefore, given by

$$\begin{aligned}\Gamma(p_1, p_2, p_3, p_4) &= -\mathrm{i}g\mu^\varepsilon + \frac{3\mathrm{i}g^2\mu^\varepsilon}{16\pi^2}\frac{1}{\varepsilon} \\ &\quad - \frac{ig^2\mu^\varepsilon}{32\pi^2}\left[3\gamma + F(s, m, \mu) + F(t, m, \mu) + F(u, m, \mu)\right] \\ &= -\,\mathrm{i}g\mu^\varepsilon\left\{\left(1 - \frac{3g}{16\pi^2}\frac{1}{\varepsilon}\right)\right. \\ &\quad \left. + \frac{g}{32\pi^2}\left[3\gamma + F(s, m, \mu) + F(t, m, \mu) + F(u, m, \mu)\right]\right\} \ .\end{aligned} \tag{8.38}$$

Equation (8.38) gives the effective, "dressed" interaction vertex. By comparison with the free 1PI four-point function, its forms suggests to absorb all effects of the loop graphs into a new coupling constant

$$g \longrightarrow g(1 + \delta g(s, t, u, m, \varepsilon, \mu)) \ . \tag{8.39}$$

This so-called *running coupling constant* depends on s, t, u, m and μ and separates into a divergent and a finite term.

8.3 Renormalization

In the preceding two subsections we have seen that both the 1PI two-point and the 1PI four-point functions have (for $n \to 4$) both divergent and finite contributions from higher order diagrams. In this section we now show how to handle these divergences. In doing so we assume that the parameters of the Lagrangian assume their physical, measured values, i.e. m is the physical mass and g the physical coupling constant.

We start by looking at the structure of the most general two-point function. When we go to terms that depend on g^2 and higher orders of the interactions we always encounter 1P reducible graphs, such as the one in Fig. 7.1a. The general structure of the (reducible) two-point function is of the form

$$\begin{aligned} G_c(p,-p) &= G_0(p,-p) + G_0(p,-p)\frac{\Sigma}{\mathrm{i}}G_0(p,-p) \\ &\quad + G_0(p,-p)\frac{\Sigma}{\mathrm{i}}G_0(p,-p)\frac{\Sigma}{\mathrm{i}}G_0(p,-p) + \dots \ , \end{aligned} \tag{8.40}$$

where Σ is the so-called proper self-energy. Note that (8.40) is not an expansion in terms of the coupling constant, but instead in terms of irreducible diagrams.

Equation (8.40) can be summed and gives

$$\begin{aligned} G_c(p,-p) &= G_0(p,-p)\left(1 + \frac{\Sigma}{\mathrm{i}}G_0(p,-p) + \frac{\Sigma}{\mathrm{i}}G_0(p,-p)\frac{\Sigma}{\mathrm{i}}G_0(p,-p) + \dots\right) \\ &= G_0(p,-p)\left(1 - \frac{\Sigma}{\mathrm{i}}G_0(p,-p)\right)^{-1} \\ &= \left(G_0^{-1}(p,-p) - \frac{\Sigma}{\mathrm{i}}\right)^{-1} \\ &= \frac{\mathrm{i}}{p^2 - m^2 - \Sigma + \mathrm{i}\varepsilon} \ . \end{aligned} \tag{8.41}$$

The self-energy Σ is in general a function of the momentum p of the particle. We can therefore expand $\Sigma(p^2)$ around an arbitrary point, e.g. around $p^2 = 0$

$$\Sigma(p^2) = \Sigma(0) + \Sigma_1 p^2 + \Sigma_2(p^2) \ . \tag{8.42}$$

Here

$$\Sigma_1 = \frac{\partial \Sigma}{\partial p^2}\Big|_{p^2=0} \qquad \text{and} \qquad \Sigma_2(0) = 0 \ . \tag{8.43}$$

In (8.42) only the first two terms of the expansion of Σ have been written out explicitly; $\Sigma_2(p^2)$ denotes the whole remainder of the expansion. Σ, being the selfenergy insertion of a two point function, is quadratically divergent. Consequently, Σ_1 as the first derivative of Σ with respect to p^2 is logarithmically divergent and Σ_2 as a second derivative is convergent.

Inserting this expansion into (8.41) gives for the propagator

$$G_c(p,-p) = \frac{\mathrm{i}}{p^2 - m^2 - \Sigma(0) - \Sigma_1 p^2 - \Sigma_2(p^2) + \mathrm{i}\varepsilon} \ . \tag{8.44}$$

The pole of this propagator should be at the physical mass and its residuum should be i. However, this is in general not the case, if we start with the observable, physical mass in the Lagrangian, because the selfinteractions contribute to the self-energy.

Counter terms. In order to get the pole to the correct, physical location we therefore have to change the mass in the Lagrangian by adding a so-called counterterm to it

$$\mathcal{L}_{cm} = -\frac{1}{2}\delta m^2 Z\phi^2 \qquad \text{with} \qquad Z = \frac{1}{1-\Sigma_1} \ . \tag{8.45}$$

Z is usually called "wavefunction" or "field renormalization constant" for reasons that will become obvious a little later (see (8.57). Note that this counterterm has exactly the same form as the mass term in the original Lagrangian.

We determine the unknown δm^2 by the requirement

$$m^2 + \delta m^2 Z + \Sigma(0) + \Sigma_2(m^2) = (1-\Sigma_1)m^2 \tag{8.46}$$

With the counterterm added, the propagator then becomes

$$\begin{aligned} G(p,-p) &= \frac{1}{1-\Sigma_1}\frac{\mathrm{i}}{p^2 - m^2 - Z\tilde{\Sigma}_2(p^2) + \mathrm{i}\varepsilon} \\ \text{with} \qquad & \tilde{\Sigma}_2(p^2) = \Sigma_2(p^2) - \Sigma_2(m^2) \ . \end{aligned} \tag{8.47}$$

Since $\tilde{\Sigma}_2(m^2) = 0$ by definition, this propagator has the correct pole at the physical mass m. Its residue, however, is – instead of being simply i –

$$\frac{\mathrm{i}}{1-\Sigma_1} = \mathrm{i}Z \ . \tag{8.48}$$

This deficiency can now be cured by adding another, additional counterterm

$$\mathcal{L}_{c\phi} = \frac{1}{2}(Z-1)\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) \tag{8.49}$$

to the Lagrangian. Then the propagator becomes

$$\begin{aligned} G(p,-p) &= \frac{1}{1-\Sigma_1}\frac{-\mathrm{i}}{p^2 - m^2 - Z\tilde{\Sigma}_2(p^2) + (Z-1)(p^2-m^2) + \mathrm{i}\varepsilon} \\ &= \frac{\mathrm{i}}{p^2 - m^2 - \tilde{\Sigma}_2(p^2) + \mathrm{i}\varepsilon} \ . \end{aligned} \tag{8.50}$$

This propagator has the pole at the correct, physical mass m and the correct residue i.

By adding the given counterterms we have thus removed the divergent quantities $\Sigma(0)$ and Σ_1 from the propagator. The one remaining quantity $\Sigma_2(p^2)$ involves a second derivative of the selfenergy with respect to p^2 and is convergent. Note that the counterterms all have the structure of terms already present in the original Lagrangian. If this is the case, in general a theory is called renormalizable.

8.3.1 Renormalization of ϕ^4 Theory

We now specify all of these general considerations to the example of ϕ^4 theory. As we have discussed in Sect. 8.2 the ϕ^4 theory is renormalizable, i.e. only a finite number of elementary vertices diverges. These are just the terms we have calculated in the last two subsections, namely the selfenergy contribution (8.20) and the vertex function (8.35).

For the selfenergy we have, up to one-loop diagrams, (cf. (8.20)),

$$\Sigma(p^2) = -\frac{gm^2}{16\pi^2}\frac{1}{\varepsilon} - \frac{gm^2}{32\pi^2}\left[1-\gamma+\ln\left(\frac{4\pi\mu^2}{m^2}\right)\right] + \mathcal{O}(\varepsilon) \ , \tag{8.51}$$

i.e. $\Sigma_1 = \Sigma_2 = 0$ and $Z = 1$. Equation (8.46) reduces to

$$\delta m^2 = -\Sigma(0) = -\Sigma \ . \tag{8.52}$$

Thus in this one-loop approximation, there is no field renormalization, but already in the two-loop approximation we would get $\Sigma_1 \neq 0$ because the graph Fig. 7.1c is p-dependent. To be general we, therefore, keep the factor Z in the following expressions.

The corresponding Lagrangian, including all the counterterms reads now in one-loop approximation

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\left[(\partial^\mu\phi\partial_\mu\phi - m^2\phi^2\right] - \frac{g}{4!}\phi^4 - \frac{1}{2}\delta m^2 Z\phi^2 \\ &\quad + \frac{1}{2}(Z-1)\left[\partial^\mu\phi\partial_\mu\phi - m^2\phi^2\right] \\ &= \frac{Z}{2}\left[(\partial_\mu\phi)^2 - m^2\phi^2\right] - \frac{1}{2}\delta m^2 Z\phi^2 - \frac{g\mu^\varepsilon}{4!}\phi^4 \ . \end{aligned} \tag{8.53}$$

This Lagrangian leads to the correct physical behaviour for the 2-point Green's function with the physical mass m and the proper residue. The fields ϕ, in terms of which the propagator is defined, are thus the physical fields, which include already the effects of selfinteractions.

So far we have not taken the change of the coupling constant due to higher order loop diagrams into account. In (8.38) we have indeed already seen that also the interaction vertex gets modified due to higher order loop corrections. The structure there was such that the coupling constant g was effectively replaced by a running coupling constant $g(1+\delta g(s,t,u,m,\varepsilon,\mu))$. If we again want to have the physical, observable coupling constant in our Lagrangian we must get rid of the modification by a proper counter term. We, therefore, define a vertex renormalization constant Z_g by

$$Z_g = (1+\delta g)^{-1}\big|_r \ , \tag{8.54}$$

where r denotes an arbitrary renormalization point (s, t, u), often taken to be the point $r = (0, 0, 0)$. We can then introduce the additional counter term

$$\mathcal{L}_{cv} = -\frac{g\mu^{\varepsilon}}{4!}\left(Z_g - 1\right)\phi^4 \ . \tag{8.55}$$

In this way we can ensure that the coupling constant g in the Lagrangian has its physical, observable value at $r = 0$.

The Lagrangian with all the counterterms now reads

$$\mathcal{L} = \frac{Z}{2}\left[(\partial_\mu\phi)^2 - m^2\phi^2\right] - \frac{1}{2}\delta m^2 Z\phi^2 - \frac{g\mu^{\varepsilon}}{4!}Z_g\phi^4 \ . \tag{8.56}$$

By introducing the “bare field” ϕ_0 and the “bare mass” m_0 by

$$\begin{aligned} \phi_0 &= \sqrt{Z}\phi \\ m_0^2 &= m^2 + \delta m^2 \end{aligned} \tag{8.57}$$

and a “bare coupling constant” by

$$g_0 = g\mu^{\varepsilon}\frac{Z_g}{Z^2} \ , \tag{8.58}$$

then the whole Lagrangian can be expressed in terms of bare quantities only

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu\phi_0)^2 - m_0^2\phi_0^2\right] - \frac{g_0}{4!}\phi_0^4 \ . \tag{8.59}$$

This bare Lagrangian has the same form as the original one, because all the counter terms had the same form as terms already appearing in the original Lagrangian. As a consequence it leads to finite physical quantities in all orders of perturbation theory. If this is the case, then the theory is well defined, i.e. it is said to be *renormalizable*. The bare Lagrangian is really considered to be the ‘true’ Lagrangian of the theory because it leads only to finite physical quantities.

In the preceding considerations we have chosen 2 arbitrary renormalization points; for the selfenergy we have expanded Σ around the point $p^2 = 0$ (cf. (8.42)) and for the vertex we have chosen the arbitrary point r (cf. (8.54)). If we insist that the coupling constant g is the measured, physical coupling, we thus have to specify at which kinematical point the measurement is to be done. This arbitrariness in the renormalization point corresponds to the fact that the divergent expressions for the selfenergy and the vertex all separate into a divergent and a finite term. Any scheme, that removes the infinite terms, will lead to finite expressions for the physical observables, independent of what it does with the finite contributions.

In the dimensional renormalization discussed here all the counter terms and renormalization factors also depend on the arbitrary mass μ, even after $\varepsilon \to 0$ and after the infinite terms have been removed. This μ was originally introduced to keep the coupling constant g dimensionless also for $n \neq 4$. It is obvious from our considerations above that – for specified renormalization points at which the comparison to experiment is performed – choosing different values for μ will lead to different values for the wavefunction renormalization Z and the coupling constant g. This arbitrariness, however, is nothing else than the arbitrariness we have already encountered in the choice of the renormalization point. The physics is thus independent of μ. This observation is the starting point for the development of the renormalization group method.

It is essential to realize that even in a massless theory, which contains no dimensional scale parameters, μ introduces a scale that can be determined experimentally by measuring, for example, the coupling constant at a specified renormalization point. Thus, at the quantum level, a bare Lagrangian is not enough to specify a theory, but a renormalization scheme must be added that introduces necessarily a scale into the theory.

Chapter 9

GREEN'S FUNCTIONS FOR FERMIONS

The main idea in using path integrals is to express quantum mechanical transition amplitudes by integrals over classical fields. These classical fields were so far taken to be commuting numbers. Such a formalism can, however, not "know" about the Pauli principle. For example, with the formalism developed so far, a fermion could be propagated to a point in configuration space which is already occupied. In reality this propagation is Pauli-forbidden. For the description of fermions it is, therefore, necessary to extend the theory developed so far such that the Pauli principle is taken into account. This can be achieved by using an anticommuting algebra for the classical fields, the so-called Grassmann algebra.

9.1 Grassmann Algebra

We define the n generators $\epsilon_i, \ldots, \epsilon_n$ of an n-dimensional Grassmann algebra by the anticommutation relations

$$\{\epsilon_i, \epsilon_j\} \equiv \epsilon_i \epsilon_j + \epsilon_j \epsilon_i = 0 \ . \tag{9.1}$$

Let us now consider series expansions in these variables. Since

$$\epsilon_1^2 = \epsilon_2^2 = \ldots = \epsilon_n^2 = 0 \ , \tag{9.2}$$

because of (9.1), any series in ϵ_i must have the form

$$\begin{aligned} \phi(\epsilon) &= \phi_0 + \sum_i \phi_1(i)\epsilon_i + \sum_{i<j} \phi_2(i,j)\epsilon_i\epsilon_j \\ &\quad + \sum_{i<j<k} \phi_3(i,j,k)\epsilon_i\epsilon_j\epsilon_k + \ldots + \phi_n(1,2,\ldots n)\epsilon_1\epsilon_2\ldots\epsilon_n \ . \end{aligned} \tag{9.3}$$

Here the ϕ_i are ordinary, commuting c-numbers. Note that the expansion actually terminates because of (9.2). Every element of the algebra can be written in the form (9.3); thus, the various powers of ϵ in this expansion constitute a basis of the algebra.

If we choose the function ϕ_i to be totally antisymmetric in all their variables, then we can also write

$$\begin{aligned}\phi(\epsilon) &= \phi_0 + \sum_i \phi_1(i)\epsilon_i + \frac{1}{2!}\sum_{ij}\phi_2(i,j)\epsilon_i\epsilon_j \\ &\quad + \frac{1}{3!}\sum_{ijk}\phi_3(i,j,k)\epsilon_i\epsilon_j\epsilon_k + \dots \; . \end{aligned} \tag{9.4}$$

From now on we can assume that the ϕ_i are indeed antisymmetric since any symmetric part of ϕ would not contribute anyway.

As a consequence of this relation any analytical function in 1 dimension has the form

$$\phi^{(1)}(\epsilon) = \phi_0 + \phi_1\epsilon \; , \tag{9.5}$$

and in 2 dimensions

$$\phi^{(2)}(\epsilon) = \phi_0 + \phi_1(1)\epsilon_1 + \phi_1(2)\epsilon_2 + \phi_2(1,2)\epsilon_1\epsilon_2 \; . \tag{9.6}$$

These relations imply for the Taylor expansion of a Gaussian

$$\mathrm{e}^{-\epsilon^2} = 1 \tag{9.7}$$

and

$$\mathrm{e}^{-\epsilon_i\epsilon_j} = 1 - \epsilon_i\epsilon_j \; . \tag{9.8}$$

9.1.1 Derivatives

Since functions of Grassmann variables can, because of (9.2), be at most linear in ε, the derivative operator is defined as an algebraic operation that, when applied to a Grassmann variable, simply replaces that variable by a 1; when applied to a product the variable first has to be commuted to a position next to the derivative before it can be removed. We then get, for example,

$$\begin{aligned}\left\{\frac{\partial}{\partial\epsilon_j}, \epsilon_i\right\}\epsilon_1 &= \frac{\partial}{\partial\epsilon_j}(\epsilon_i\epsilon_1) + \epsilon_i\frac{\partial\epsilon_1}{\partial\epsilon_j} \\ &= \frac{\partial\epsilon_i}{\partial\epsilon_j}\epsilon_1 - \frac{\partial\epsilon_1}{\partial\epsilon_j}\epsilon_i + \epsilon_i\frac{\partial\epsilon_1}{\partial\epsilon_j} = \delta_{ij}\epsilon_1 \; . \end{aligned} \tag{9.9}$$

The "$-$" sign in the second line appears, because the factor ϵ_1 must be first brought to the left before the derivative can act.

This relation is valid in general. We thus can define the derivative also by the anticommutation relation

$$\left\{\frac{\partial}{\partial\epsilon_j},\epsilon_i\right\}=\delta_{ij}\ . \tag{9.10}$$

This relation can be used to obtain the derivative of a function $\phi(\epsilon)$. Applying the lhs of (9.10) to ϕ, which can always be written as polynomial (see (9.4)), yields

$$\begin{aligned}\left\{\frac{\partial}{\partial\epsilon_j},\epsilon_i\right\}\phi(\epsilon) &= \frac{\partial}{\partial\epsilon_j}\left[\epsilon_i\phi(\epsilon)\right]+\epsilon_i\frac{\partial}{\partial\epsilon_j}\phi(\epsilon) \\ &= \delta_{ij}\phi(\epsilon)+\left[\frac{\partial}{\partial\epsilon_j}\left(\phi_0-\sum_k\phi_1(k)\epsilon_k+\frac{1}{2!}\sum_{kl}\phi_2(k,l)\epsilon_k\epsilon_l\right.\right. \\ &\quad\left.\left.-\ \frac{1}{3!}\sum_{klm}\phi_3(k,l,m)\epsilon_k\epsilon_l\epsilon_m+\dots\right)\right]\epsilon_i+\epsilon_i\frac{\partial}{\partial\epsilon_j}\phi(\epsilon)\ .\end{aligned} \tag{9.11}$$

Here all the terms in $\phi(\epsilon)$ with an odd number of $\epsilon's$ have changed sign when they were brought to the left so that the derivative can act on them. With

$$\frac{\partial}{\partial\epsilon_j}\epsilon_k\epsilon_l=\delta_{kj}\epsilon_l-\delta_{lj}\epsilon_k \tag{9.12}$$

and

$$\frac{\partial}{\partial\epsilon_j}\epsilon_k\epsilon_l\epsilon_m=\delta_{kj}\epsilon_l\epsilon_m-\delta_{lj}\epsilon_k\epsilon_m+\delta_{mj}\epsilon_k\epsilon_l \tag{9.13}$$

we get

$$\begin{aligned}\left\{\frac{\partial}{\partial\epsilon_j},\epsilon_i\right\}\phi(\epsilon) &= \delta_{ij}\phi(\epsilon)+\left(-\phi_1(j)+\frac{1}{2!}\sum_l\phi_2(j,l)\epsilon_l-\frac{1}{2!}\sum_k\phi_2(k,j)\epsilon_k\right. \\ &\quad-\ \frac{1}{3!}\sum_{lm}\phi_3(j,l,m)\epsilon_l\epsilon_m+\frac{1}{3!}\sum_{km}\phi_3(k,j,m)\epsilon_k\epsilon_m \\ &\quad\left.-\ \frac{1}{3!}\sum_{kl}\phi_3(k,l,j)\epsilon_k\epsilon_l+\dots\right)\epsilon_i+\epsilon_i\frac{\partial}{\partial\epsilon_j}\phi_(\epsilon)\ .\end{aligned} \tag{9.14}$$

Because of (9.10) this has to be equal to $\delta_{ij}\,\phi(\epsilon)$. Thus we must have

$$\frac{\partial}{\partial\epsilon_j}\phi(\epsilon)=\phi_1(j)+\sum_k\phi_2(j,k)\epsilon_k+\frac{1}{2!}\sum_{kl}\phi_3(j,k,l)\epsilon_k\epsilon_l+\dots\ . \tag{9.15}$$

This gives the derivative of a general function $\phi(\epsilon)$.

The second derivatives can also be defined

$$\begin{aligned}
\frac{\partial}{\partial\epsilon_i}\frac{\partial}{\partial\epsilon_j}\phi(\epsilon) &= \phi_2(j,i) \\
&\quad + \frac{1}{2!}\sum_l \phi_3(j,i,l)\epsilon_l - \frac{1}{2!}\sum_k \phi_3(j,k,i)\epsilon_k + \ldots \\
&= \phi_2(j,i) + \sum_l \phi_3(j,i,l)\epsilon_l + \ldots\ . \qquad (9.16)
\end{aligned}$$

Because the ϕ_i are antisymmetric this gives immediately

$$\frac{\partial^2}{\partial\epsilon_i\partial\epsilon_j}\phi(\epsilon) + \frac{\partial^2}{\partial\epsilon_j\partial\epsilon_i}\phi(\epsilon) = 0\ , \qquad (9.17)$$

or

$$\left\{\frac{\partial}{\partial\epsilon_i}, \frac{\partial}{\partial\epsilon_j}\right\} = 0\ . \qquad (9.18)$$

Thus we have, in particular,

$$\frac{\partial^2}{\partial\epsilon_i^2} = 0\ . \qquad (9.19)$$

This relation implies that there is no inverse to differentiation. This can be seen by multiplying the defining equation for the inverse

$$\frac{\partial}{\partial\varepsilon}\left(\frac{\partial}{\partial\varepsilon}\right)^{-1} = 1 \qquad (9.20)$$

from the left by $\frac{\partial}{\partial\varepsilon}$. This gives

$$\frac{\partial}{\partial\varepsilon}\frac{\partial}{\partial\varepsilon}\left(\frac{\partial}{\partial\varepsilon}\right)^{-1} = 0\cdot\left(\frac{\partial}{\partial\varepsilon}\right)^{-1} = \frac{\partial}{\partial\varepsilon} \qquad (9.21)$$

and thus the inverse does not exist.

9.1.2 Integration

Integration in the space of Grassmann variables can, therefore, be defined only in an operational sense. This is achieved by the following relations

$$\int \mathrm{d}\epsilon_i = 0 \qquad (9.22)$$

$$\int \mathrm{d}\epsilon_i\,\epsilon_i = 1\ . \qquad (9.23)$$

Here $\mathrm{d}\epsilon$ is not an infinitesimal interval and, in particular, not a member of the Grassmann algebra of the ϵ_i. Integration over Grassmann variables has thus the same effect as differentiation. The integral in (9.23) is the only nonvanishing integral over functions of ϵ_i. It has the property of translational invariance

$$\int \mathrm{d}\epsilon\, \phi^{(1)}(\epsilon) = \int \mathrm{d}\epsilon\, \phi^{(1)}(\epsilon+\alpha) = \phi_1 \tag{9.24}$$

with the nomenclature of (9.5); α is also a Grassmann number.

For Grassmann variables we can also define a δ-function. In 1 dimension we have

$$\begin{aligned} -\int \mathrm{d}\epsilon'(\epsilon-\epsilon')\phi(\epsilon') &= -\int \mathrm{d}\epsilon'(\epsilon-\epsilon')(\phi_0+\phi_1\epsilon') \\ &= \phi_1\epsilon+\phi_0 = \phi(\epsilon)\ . \end{aligned} \tag{9.25}$$

We can thus identify the δ function as

$$\delta(\epsilon-\epsilon') = -(\epsilon-\epsilon')\ ; \tag{9.26}$$

note that $\delta(\epsilon-\epsilon')$ is an odd function. There is also a Fourier-representation for the δ-function

$$\delta(\epsilon-\epsilon') = -(\epsilon-\epsilon') = \int \mathrm{d}\zeta\, [1-\zeta(\epsilon-\epsilon')] = \int \mathrm{d}\zeta\, \mathrm{e}^{-\zeta(\epsilon-\epsilon')}\ . \tag{9.27}$$

9.1.2.1 Multiple integrals

In multiple integrals the integration variable first has to be commuted to a position next to the integration measure. For example, we get

$$\begin{aligned} \int \mathrm{d}\epsilon_i\, \mathrm{d}\epsilon_j\, \mathrm{e}^{-\epsilon_i\epsilon_j} &= \int \mathrm{d}\epsilon_i\, \mathrm{d}\epsilon_j (1-\epsilon_i\epsilon_j) \\ &= 0 - \int \mathrm{d}\epsilon_i\, \mathrm{d}\epsilon_j\, \epsilon_i\epsilon_j = +\int \mathrm{d}\epsilon_i \left(\int \mathrm{d}\epsilon_j\, \epsilon_j\right)\epsilon_i \\ &= \int \mathrm{d}\epsilon_i\, \epsilon_i = 1\ . \end{aligned} \tag{9.28}$$

We now determine the Jacobian J that appears when we perform a linear transformation

$$\epsilon' = O\epsilon \tag{9.29}$$

of the integration variables. J is defined by

$$\int \mathrm{d}\epsilon'_1 \ldots \mathrm{d}\epsilon'_n\, \epsilon'_1 \ldots \epsilon'_n = \int \mathrm{d}\epsilon_1 \ldots \mathrm{d}\epsilon_n\, J (O\epsilon)_1 (O\epsilon)_2 \ldots (O\epsilon)_n \tag{9.30}$$

On one hand, we have

$$\int d\epsilon_1' \ldots d\epsilon_n' \, \epsilon_1' \ldots \epsilon_n' = \int d\epsilon_1 \ldots d\epsilon_n \, \epsilon_1 \ldots \epsilon_n \ , \tag{9.31}$$

because of (9.23). On the other hand, from the definition of J we also have

$$\begin{aligned}\int d\epsilon_1' \ldots d\epsilon_n' \, \epsilon_1' \ldots \epsilon_n' &= \int d\epsilon_1 \ldots d\epsilon_n \, J(O\epsilon)_1(O\epsilon)_2 \ldots (O\epsilon)_n \\ &= \int d\epsilon_1 \ldots d\epsilon_n \, J O_{1\alpha} O_{2\beta} \ldots O_{n\nu} \, \epsilon_\alpha \epsilon_\beta \ldots \epsilon_\nu \ .\end{aligned} \tag{9.32}$$

Note that this equality holds for general matrices O. The product of the Grassmann variables $\epsilon_\alpha, \ldots, \epsilon_\nu$ vanishes, if any of the indices appears twice. Thus only the permutations of $1, \ldots, n$ survive. We can therefore bring each of them into the ordered form by writing

$$\epsilon_\alpha \epsilon_\beta \ldots \epsilon_\nu = (-)^P \epsilon_1 \ldots \epsilon_n \ , \tag{9.33}$$

where $(-)^P$ is the sign of the permutation. We thus have

$$\begin{aligned}\int d\epsilon_1' \ldots d\epsilon_n' \, \epsilon_1' \ldots \epsilon_n' &= \int d\epsilon_1 \ldots d\epsilon_n \, J \sum_P (-)^P O_{1\alpha} \ldots O_{n\nu} \epsilon_1 \ldots \epsilon_n \\ &= \int d\epsilon_1 \ldots d\epsilon_n \, J \epsilon_1 \ldots \epsilon_n \det(O) \ .\end{aligned} \tag{9.34}$$

Comparison with (9.31) gives

$$J = [\det(O)]^{-1} \ . \tag{9.35}$$

Note that this is the inverse of the ordinary result!

Gaussian integrals. We next consider the Gaussian integral

$$I(n) = \int d\epsilon_1 \ldots d\epsilon_n \, e^{-\epsilon^T M \epsilon} \tag{9.36}$$

where M is an *antisymmetric* $(n \times n)$ matrix with ordinary c-number elements and ϵ is a vector with n elements. Expanding the exponential function we obtain

$$e^{-\epsilon^T M \epsilon} = 1 - \epsilon^T M \epsilon + \frac{1}{2!} \left(\epsilon^T M \epsilon\right)^2 - \cdots \ . \tag{9.37}$$

For simplicity let us consider the case $n = 2$ first. We have for $I(2)$

$$\begin{aligned}\int d\epsilon_1 d\epsilon_2 e^{-\epsilon^T M \epsilon} &= \int d\epsilon_1 \, d\epsilon_2 \left[1 - \epsilon^T M \epsilon + \frac{1}{2} \left(\epsilon^T M \epsilon\right)^2 - \ldots\right] \\ &= 0 - \int d\epsilon_1 \, d\epsilon_2 \, \epsilon^T M \epsilon + \frac{1}{2} \int d\epsilon_1 \, d\epsilon_2 \left(\epsilon^T M \epsilon\right)^2 - \ldots \\ &\equiv -I_1(2) + I_2(2) - \ldots\end{aligned} \tag{9.38}$$

The second integral is

$$\begin{aligned} I_2(2) &= \int \mathrm{d}\epsilon_1\, \mathrm{d}\epsilon_2\, \left(\epsilon^T M \epsilon\right)^2 \\ &= \int \mathrm{d}\epsilon_1\, \mathrm{d}\epsilon_2\, \epsilon_i M_{ij} \epsilon_j\, \epsilon_k M_{kl} \epsilon_l \ . \end{aligned} \tag{9.39}$$

The indices i, j, k and l all have to be different, because otherwise the product of the $\epsilon's$ vanishes. This, however, is impossible because we have only 2 different variables; the integral $I_2(2)$ thus vanishes. The same holds for all higher order terms in (9.38).

For the first integral we obtain

$$\begin{aligned} I_1(2) &= \int \mathrm{d}\epsilon_1\, \mathrm{d}\epsilon_2\, \epsilon^T M \epsilon = \int \mathrm{d}\epsilon_1\, \mathrm{d}\epsilon_2\, \epsilon_i M_{ij} \epsilon_j \\ &= \int \mathrm{d}\epsilon_1\, \mathrm{d}\epsilon_2\, \left(\epsilon_1 M_{11} \epsilon_1 + \epsilon_1 M_{12} \epsilon_2 + \epsilon_2 M_{21} \epsilon_1 + \epsilon_2 M_{22} \epsilon_2\right) \\ &= \int \mathrm{d}\epsilon_1\, \mathrm{d}\epsilon_2\, \left(\epsilon_1 M_{12} \epsilon_2 + \epsilon_2 M_{21} \epsilon_1\right) \ , \end{aligned} \tag{9.40}$$

because the integrands of the other two terms contain squares of Grassmann variables which vanish. We therefore have (because M is antisymmetric)

$$I_1(2) = -M_{12} + M_{21} = -2M_{12} = -2\sqrt{\det(M)} \ . \tag{9.41}$$

and thus

$$I(2) = -I_1(2) = +2\sqrt{\det(M)} = 2^{\frac{2}{2}}\sqrt{\det(M)} \ . \tag{9.42}$$

For 3 variables we obtain immediately

$$\int \mathrm{d}\epsilon_1\, \mathrm{d}\epsilon_2\, \mathrm{d}\epsilon_3\, \mathrm{e}^{-\epsilon^T M \epsilon} = 0 \ , \tag{9.43}$$

because the exponent is quadratic in ϵ and, therefore, the series expansion involves only even powers. The term of second order in ϵ gives 0, because of $\int \mathrm{d}\epsilon = 0$; the term of fourth order also vanishes, because, with only three variables, it has to contain a square of one of them, which vanishes.

A similar reasoning leads one to the conclusion that with four variables only the term of fourth order in ϵ can contribute. Thus we have

$$\begin{aligned} I(4) &= \int \mathrm{d}\epsilon_1\, \mathrm{d}\epsilon_2\, \mathrm{d}\epsilon_3\, \mathrm{d}\epsilon_4\, \mathrm{e}^{-\epsilon^T M \epsilon} \\ &= \frac{1}{2} \int \mathrm{d}\epsilon_1 \ldots \mathrm{d}\epsilon_4 \left(\epsilon^T M \epsilon\right)^2 \\ &= \frac{1}{2} \int \mathrm{d}\epsilon_1 \ldots \mathrm{d}\epsilon_4\, \epsilon_i M_{ij} \epsilon_j\, \epsilon_k M_{kl} \epsilon_l \\ &= \frac{1}{2} \sum_{ijkl=1}^{4} M_{ij} M_{kl} \int \mathrm{d}\epsilon_1 \ldots \mathrm{d}\epsilon_4\, \epsilon_i \epsilon_j \epsilon_k \epsilon_l \ . \end{aligned} \tag{9.44}$$

Here only the terms with all four indices different can contribute. We thus get $4! = 24$ terms. Of these 24 terms many are equal to each other: the antisymmetry $M_{ij} = -M_{ji}$ gives a factor 2^2, the symmetry $M_{ij}M_{kl} = M_{kl}M_{ij}$ another factor 2, so that only 3 essentially different terms remain. We then have

$$\begin{aligned} I(4) &= 4M_{12}M_{34}\int \mathrm{d}\epsilon_1 \ldots \mathrm{d}\epsilon_4\, \epsilon_1\epsilon_2\epsilon_3\epsilon_4 \\ &\quad + 4M_{13}M_{24}\int \mathrm{d}\epsilon_1 \ldots \mathrm{d}\epsilon_4\, \epsilon_1\epsilon_3\epsilon_2\epsilon_4 \\ &\quad + 4M_{14}M_{23}\int \mathrm{d}\epsilon_1 \ldots \mathrm{d}\epsilon_4\, \epsilon_1\epsilon_4\epsilon_2\epsilon_3 \\ &= 4\left(M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23}\right) \\ &= 2^{\frac{4}{2}}\sqrt{\det(M)} \end{aligned} \tag{9.45}$$

The last step can be seen by explicitly expanding the (4×4) determinant of M in terms of its elements in the first row.

The result just obtained holds in general

$$\int \mathrm{d}\epsilon_1 \ldots \mathrm{d}\epsilon_n\, \mathrm{e}^{-\epsilon^T M \epsilon} = \begin{cases} 2^{\frac{n}{2}}\sqrt{\det(M)} & \text{for} \quad n \text{ even} \\ 0 & \qquad\quad\; n \text{ odd} \end{cases} . \tag{9.46}$$

The proof is given in Appendix D. Equation (9.46) is the equivalent of (5.5) for commuting numbers. Note that here the squareroot of the determinant appears in the numerator whereas for commuting numbers it is in the denominator.

The other integral relations for commuting variables, derived in section 5.1, can also be obtained for Grassmann variables. One gets

$$\int \mathrm{d}\epsilon_1 \ldots \mathrm{d}\epsilon_n\, \mathrm{e}^{-\frac{1}{2}\epsilon^T M \epsilon + \eta_i \epsilon_i} = \sqrt{\det(M)}\, \mathrm{e}^{\frac{1}{2}\eta^T M^{-1}\eta} , \tag{9.47}$$

where the η_i are also Grassmann variables with

$$\{\eta_i, \eta_j\} = 0 = \{\eta_i, \epsilon_j\} \tag{9.48}$$

(there is no factor $2^{(2n)/2}$ here because the exponent contains an extra factor 1/2). Equation (9.47) corresponds directly to (5.5).

We now enlarge the algebra of the ϵ_n by adding the conjugate elements. Here conjugation is defined by the properties

$$\begin{aligned} (\epsilon_i)^* &= \epsilon_i^* \\ (\epsilon_i^*)^* &= \epsilon_i \\ (\lambda\epsilon_i)^* &= \lambda^*\epsilon_i^* \\ (\epsilon_1\epsilon_2\cdots\epsilon_n)^* &= \epsilon_n^*\epsilon_{n-1}^*\cdots\epsilon_1^* . \end{aligned} \tag{9.49}$$

The latter condition ensures that $\epsilon^*\epsilon$ is real. The n Grassmann generators $\epsilon_1^*, \ldots, \epsilon_n^*$ are then combined with the n generators $\epsilon_1, \ldots, \epsilon_n$ into a $2n$-dimensional algebra with generators $\kappa_1, \ldots, \kappa_{2n}$ such that

$$\{\kappa_1, \ldots, \kappa_{2n}\} = \{\epsilon_1^*, \epsilon_1, \epsilon_2^*, \epsilon_2, \ldots, \epsilon_n^*, \epsilon_n\} \ . \tag{9.50}$$

All the κ_i anticommute with each other. With this definition we evaluate a Gaussian integral containing also conjugate Grassmann numbers

$$Z \sim \int \mathrm{d}\epsilon_1^* \mathrm{d}\epsilon_1 \mathrm{d}\epsilon_2^* \mathrm{d}\epsilon_2 \ldots \mathrm{d}\epsilon_n^* \mathrm{d}\epsilon_n \, \mathrm{e}^{-\epsilon^\dagger M \epsilon} \ . \tag{9.51}$$

The matrix M in (9.51) can now be a general, complex matrix without any specific symmetries. In contrast to the c-number integrals discussed in section 5.1 it does not have to have special properties since convergence of the integrals is guaranteed here by the Grassmann algebra. In order to be able to apply the results obtained above (see (9.46)) to this integral we now enlarge the general $(n \times n)$-dimensional matrix M into an antisymmetric $(2n \times 2n)$-dimensional matrix N, so that the exponent in (9.51) can be rewritten

$$\epsilon^\dagger M \epsilon = \sum_{i,j=1}^{n} \epsilon_i^* M_{ij} \epsilon_j = \frac{1}{2} \sum_{k,l=1}^{2n} \kappa_k N_{kl} \kappa_l = \frac{1}{2} \kappa^T N \kappa \ . \tag{9.52}$$

For example, for the case $n = 2$ the matrix N reads

$$N = \begin{pmatrix} 0 & M_{11} & 0 & M_{12} \\ -M_{11} & 0 & -M_{21} & 0 \\ 0 & M_{21} & 0 & M_{22} \\ -M_{12} & 0 & -M_{22} & 0 \end{pmatrix} \ . \tag{9.53}$$

One then gets in general

$$\det(N) = [\det(M)]^2 \ . \tag{9.54}$$

Using now (9.46) gives

$$Z \sim \int \mathrm{d}\kappa_1 \ldots \mathrm{d}\kappa_{2n} \, \mathrm{e}^{-\frac{1}{2}\kappa^T N \kappa} = \sqrt{\det(N)} \ , \tag{9.55}$$

and with (9.54) we finally get

$$Z \sim \int \mathrm{d}\epsilon_1^* \mathrm{d}\epsilon_1 \ldots \mathrm{d}\epsilon_n^* \mathrm{d}\epsilon_n \, \mathrm{e}^{-\epsilon^\dagger M \epsilon} = \det(M) \ . \tag{9.56}$$

This result can also easily be obtained for a diagonal matrix M. Even though the integral is a $2n$-dimensional one, the matrix M on the rhs is only n-dimensional. Note again that the matrix M here is a general complex matrix.

A slightly more general form follows when we add linear terms in the exponent and write an expressively imaginary M. Then we obtain

$$\int \prod_n d\epsilon_n^* d\epsilon_n \, \mathrm{e}^{-\mathrm{i}\epsilon^\dagger M\epsilon + i\eta^\dagger \epsilon + \mathrm{i}\epsilon^\dagger \eta} = \det(\mathrm{i}M)\, \mathrm{e}^{\mathrm{i}\eta^\dagger M^{-1}\eta} \; . \tag{9.57}$$

In all these formulas for Grassmann integrals the determinant appears in the numerator in contrast to the Gaussian integrals over ordinary numbers (5.15).

9.2 Green's Functions for Fermions

In this section we now apply the Grassmann algebra to a path integral description of fermions. We do so by introducing classical Grassmann fields ψ for the fermions. These appear in the continuum limit when the number of generators n of the Grassmann algebra goes to infinity. The relations corresponding to (9.1),(9.10) and (9.23) are, respectively,

$$\begin{aligned} \{\psi(x), \psi(y)\} &= 0 \\ \left\{\frac{\delta}{\delta\psi(x)}, \psi(y)\right\} &= \delta(x-y) \\ \int d\psi(x) &= 0 \\ \int d\psi(x)\, \psi(x) &= 1 \; . \end{aligned} \tag{9.58}$$

With these fields we can construct path integrals of the type

$$Z \sim \int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi \, \mathrm{e}^{\mathrm{i}\int \bar{\psi} S^{-1}\psi \, \mathrm{d}^4x} \tag{9.59}$$

where the $\bar{\psi}$ and ψ are Grassmann fields.

9.2.1 Generating Functional for Fermions

The Lagrangian for free Dirac fields is

$$\mathcal{L} = \bar{\Psi}\left(\mathrm{i}\gamma_\mu \partial^\mu - m\right)\Psi \tag{9.60}$$

where Ψ and $\bar{\Psi}$ are to be considered as independent fields. They are now taken to be elements of an infinite-dimensional Grassmann algebra. The path integral is then – in direct generalization of our results for boson fields – given by (for the free case)

$$Z_0[\eta, \bar{\eta}] = N \int \mathcal{D}\bar{\Psi}\, \mathcal{D}\Psi \, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x \left[\bar{\Psi}(\mathrm{i}\gamma_\mu\partial^\mu - m)\Psi + \bar{\eta}(x)\Psi(x) + \bar{\Psi}(x)\eta(x)\right]} \; . \tag{9.61}$$

Here N is the normalization factor, equal to the inverse of the path integral without sources. The functions $\eta(x)$ and $\bar{\eta}(x)$ are four-component source functions corresponding to the classical fields $\bar{\Psi}$ and Ψ, respectively. They are also taken to be Grassmann fields that anticommute among themselves as well as with the fields Ψ and $\bar{\Psi}$.

$$\begin{aligned} \{\psi(x), \psi(x')\} = \{\psi(x), \bar{\psi}(x')\} = \{\bar{\psi}(x), \bar{\psi}(x')\} = 0 \\ \{\eta(x), \eta(x')\} = \{\eta(x), \bar{\eta}(x')\} = \{\bar{\eta}(x), \bar{\eta}(x')\} = 0 \\ \{\eta(x), \psi(x')\} = \{\eta(x), \bar{\psi}(x')\} = \{\bar{\eta}(x), \psi(x')\} = 0 \ . \end{aligned} \tag{9.62}$$

While (9.61) is an obvious generalization of our earlier results obtained in Chap. 4 for scalar fields, its actual (numerical) evaluation may not seem to be straightforward, because of the Grassmann nature of the fields. In order to define more stringently what is actually meant by the path integral in (9.61) we can proceed as in chapter 4.1 for scalar fields and Fourier-expand the fields. This is carried through in detail in [LEE]. For our present purpose it is enough to mention that the fields Ψ and $\bar{\Psi}$ can be expanded in terms of Grassmann numbers ϵ_α and $\bar{\epsilon}_\alpha$ such that

$$\begin{aligned} \Psi(x) &= \sum_\alpha \phi_\alpha(\vec{x}) \epsilon_\alpha(t) \\ \bar{\Psi}(x) &= \sum_\alpha \bar{\phi}_\alpha(\vec{x}) \bar{\epsilon}_\alpha(t) \ , \end{aligned} \tag{9.63}$$

where the ϕ_α are $\vec{x}$-dependent Dirac spinors ($u(\vec{p}_\alpha, s_\alpha)\mathrm{e}^{i\vec{p}_\alpha \cdot \vec{x}}$ or $v(\vec{p}_\alpha, s_\alpha)\mathrm{e}^{-\mathrm{i}\vec{p}_\alpha \cdot \vec{x}}$, for positive and negative energies, respectively, see [LEE], p. 517-521).

Equation (9.63) just amounts to an $\vec{x}$-dependent change of the basis from $\epsilon_\alpha, \bar{\epsilon}_\alpha$ to the Grassmann numbers Ψ and $\bar{\Psi}$. The fermionic action then reads in terms of the Grassmann numbers (with $\not{k} \equiv \gamma_\mu k^\mu$)

$$\begin{aligned} S &= \int \bar{\Psi} \left(\mathrm{i} \ \not{\partial} - m\right) \Psi \mathrm{d}^4x \\ &= \sum_{\alpha\beta} \int \bar{\phi}_\alpha \bar{\epsilon}_\alpha \left(\mathrm{i} \ \not{\partial} - m\right) \phi_\beta \epsilon_\beta \, \mathrm{d}^4x \ , \end{aligned} \tag{9.64}$$

and the path-integral integration measure is really

$$\begin{aligned} \mathcal{D}\bar{\Psi} \, \mathcal{D}\Psi &= \prod_k \prod_j \mathrm{d}\bar{\Psi}(x_k, t_j) \, \mathrm{d}\Psi(x_k, t_j) \\ &\sim \prod_\alpha \prod_j \mathrm{d}\bar{\epsilon}_\alpha(t_j) \, \mathrm{d}\epsilon_\alpha(t_j) \ . \end{aligned} \tag{9.65}$$

Here the first index (α) denotes the Grassmann generator, the second (j) the time interval. The last step here was possible because of the orthonormality of the functions ϕ_α and $\bar{\phi}_\alpha$.

9.2.1.1 Feynman propagator for Fermions

In the case of free boson fields we succeeded in writing the generating functional as a term involving space-time integrals over the sources (see (4.29) and (5.27)). This had the advantage that the two-point function could be directly read off from that expression. We achieve here the same for fermions by using (9.57). This gives

$$\begin{aligned} Z_0[\eta, \bar{\eta}] &= N \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x(\mathcal{L}+\bar{\eta}\Psi+\bar{\Psi}\eta)} \\ &= N \det(\mathrm{i}M)\, \mathrm{e}^{\mathrm{i}\bar{\eta}M^{-1}\eta} \;, \end{aligned} \tag{9.66}$$

where the matrix M is obtained by identifying

$$\begin{aligned} -\mathrm{i}\epsilon^\dagger M \epsilon \quad &\longleftrightarrow \quad \mathrm{i}\int \mathrm{d}^4x \left[\bar{\Psi}(x)(\mathrm{i}\,\partial\!\!\!/ - m)\Psi(x)\right] \\ &= \quad -\mathrm{i}\int \mathrm{d}^4x\, \mathrm{d}^4y\, \bar{\Psi}(x)\left[(\mathrm{i}\,\partial\!\!\!/_y + m)\delta^4(x-y)\right]\Psi(y) \end{aligned} \tag{9.67}$$

after partial integration. Since $N = Z_0[0,0]$, we have $N = [\det(\mathrm{i}M)]^{-1}$. We thus get

$$Z_0[\eta, \bar{\eta}] = \mathrm{e}^{\mathrm{i}\int \bar{\eta}(x)M^{-1}(x,y)\eta(y)\mathrm{d}^4x\,\mathrm{d}^4y} \;. \tag{9.68}$$

The inverse operator appearing here can be obtained as at the end of section (5.2)

$$\begin{aligned} \int M(x,z)M^{-1}(z,y)\,\mathrm{d}^4z &= \int \left[(\mathrm{i}\,\partial\!\!\!/^z + m)\,\delta^4(x-z)\right] M^{-1}(z,y)\,\mathrm{d}^4z \\ &= -\int \delta^4(x-z)\,(\mathrm{i}\,\partial\!\!\!/^z - m)\,M^{-1}(z,y)\,\mathrm{d}^4z \\ &= -\,(\mathrm{i}\,\partial\!\!\!/^x - m)\,M^{-1}(x,y) \stackrel{!}{=} \delta^4(x-y)\;. \end{aligned} \tag{9.69}$$

The last equation is fulfilled by

$$M^{-1}(x,y) = -\,(\mathrm{i}\,\partial\!\!\!/ + m)\,D_\mathrm{F}(x-y)\;, \tag{9.70}$$

since

$$\begin{aligned} -(\mathrm{i}\,\partial\!\!\!/ - m)(\mathrm{i}\,\partial\!\!\!/ + m)D_\mathrm{F}(x-y) &= +\,(\Box + m^2)D_\mathrm{F}(x-y) \\ &= -\,\delta^4(x-y)\;, \end{aligned} \tag{9.71}$$

because of (4.25). Here we have used $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$.

We therefore have now the desired result. For free fields the normalized generating functional is

$$Z_0[\eta,\bar\eta] = \mathrm{e}^{-\mathrm{i}\int \bar\eta(x) S_{\mathrm{F}}(x-y)\eta(y)\,\mathrm{d}^4x\mathrm{d}^4y}\ , \tag{9.72}$$

with

$$S_{\mathrm{F}}(x-y) = (\mathrm{i}\ \not\partial + m) D_{\mathrm{F}}(x-y) \tag{9.73}$$

being the propagator of the Dirac equation. Note that S_{F} is a (4×4) matrix because of the Dirac matrixstructure of γ_μ.

For S_{F} we can also find an integral representation

$$\begin{aligned} S_{\mathrm{F}}(x-y) &= (\mathrm{i}\ \not\partial + m)\frac{1}{(2\pi)^4}\int \mathrm{d}^4k\, \frac{\mathrm{e}^{-\mathrm{i}k(x-y)}}{k^2-(m-\mathrm{i}\epsilon)^2} \\ &= \frac{1}{(2\pi)^4}\int \mathrm{d}^4k\, \mathrm{e}^{-\mathrm{i}k(x-y)}\frac{\not k + m}{k^2-(m-i\epsilon)^2} \\ &= \frac{1}{(2\pi)^4}\int \mathrm{d}^4k\, \mathrm{e}^{-\mathrm{i}k(x-y)}\frac{1}{\not k - (m-i\epsilon)}\ . \end{aligned} \tag{9.74}$$

Note that $S_{\mathrm{F}}(x-y)$ is not symmetric under exchange $x \leftrightarrow y$, while $D_{\mathrm{F}}(x-y)$ is. However, it is antisymmetric under an operation that corresponds in Dirac theory to simultaneous $x \leftrightarrow y$ exchange and hermitean conjugation since

$$\gamma_0 S_{\mathrm{F}}(y-x)^\dagger \gamma_0 = S_{\mathrm{F}}(x-y)\ , \tag{9.75}$$

which can easily be derived with the help of the relation

$$\gamma_0\gamma_\mu^\dagger\gamma_0 = \gamma_\mu\ . \tag{9.76}$$

9.2.2 Green's Functions

The n-point functions for fermions can now be obtained, as in the case of bosons, as functional derivatives of the functional $Z[J]$. For this purpose we modify the definition (6.1) such that the presence of two fermion fields, Ψ and $\bar\Psi$, is taken into account. We therefore define the two-point function in analogy to (6.1) as

$$G_{\alpha\beta}(x_1,x_2) = \left. -\ \frac{\delta^2\, Z[\eta,\bar\eta]}{\delta\eta_\beta(x_2)\delta\bar\eta_\alpha(x_1)}\right|_{\eta=\bar\eta=0}\ . \tag{9.77}$$

Using (9.66) for the generating functional of the free theory we obtain

$$G_{\alpha\beta}(x_1,x_2) = \frac{\int \mathcal{D}\bar\Psi\mathcal{D}\Psi\ \Psi_\alpha(x_1)\bar\Psi_\beta(x_2)\mathrm{e}^{\mathrm{i}\int\mathcal{L}\,\mathrm{d}^4x}}{\int \mathcal{D}\bar\Psi\mathcal{D}\Psi\ \mathrm{e}^{\mathrm{i}\int\mathcal{L}\,\mathrm{d}^4x}} \tag{9.78}$$

Here α and β are Dirac-Spinor indices. This expression can, as for bosons, be written as an expectation value over a time-ordered product

$$G(x_1,x_2) = \frac{\langle 0|T\left[\Psi(x_1)\bar{\Psi}(x_2)\right]|0\rangle}{\langle 0|0\rangle} \tag{9.79}$$

where, now for fermions, we have

$$T\left[\Psi(x_1)\bar{\Psi}(x_2)\right] = \begin{cases} \Psi(x_1)\bar{\Psi}(x_2) & \text{for} \quad t_1 > t_2 \\ -\bar{\Psi}(x_2)\Psi(x_1) & \phantom{\text{for}} \quad t_2 > t_1 \end{cases} . \tag{9.80}$$

Note the extra minus sign compared to the boson case. This sign appears because of the Grassmann nature of the fermion fields when we reorder the fields as in the developments leading to (3.46).

We could have obtained the two-point function also from (9.72). This gives

$$G_{\alpha\beta}(x_1,x_2) = \mathrm{i}[S_{\mathrm{F}}(x_1-x_2)]_{\alpha\beta} . \tag{9.81}$$

The propagator of our fermionic theory is thus just the inverse of the operator appearing between the fields $\bar{\Psi}$ and Ψ in the Lagrangian

$$\begin{aligned} \mathcal{L} &= \bar{\Psi}(\mathrm{i}\,\partial\!\!\!/ - m)\Psi \\ G(x,y) &= \mathrm{i}(\mathrm{i}\,\partial\!\!\!/ - m)^{-1} = \mathrm{i}S_{\mathrm{F}}(x-y) . \end{aligned} \tag{9.82}$$

For boson fields this was also the case (see the discussion at the end of section 5.2)

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\phi(\Box + m^2)\phi \\ G(x,y) &= -\,\mathrm{i}(\Box + m^2)^{-1} = \mathrm{i}D_{\mathrm{F}}(x-y) . \end{aligned} \tag{9.83}$$

We can thus read off the propagator directly from the Lagrangian.[1]

[1]The factor 1/2 in the Lagrangian for the bosons is a special feature of the uncharged field and would not be there if we were working with charged fields.

Chapter 10

INTERACTING FIELDS

So far we have treated only the case of free fermion fields. In the case of an interacting theory, expression (6.31) can directly be taken over for the case of fermions so that we have

$$Z[\eta,\bar\eta] = N\,\mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4x\, V\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta\eta},\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar\eta}\right)}\,\mathrm{e}^{-\mathrm{i}\int \bar\eta(x)S_\mathrm{F}(x-y)\eta(y)\mathrm{d}^4x\,\mathrm{d}^4y}\ . \tag{10.1}$$

Here the interaction V is defined by the Lagrangian of the interacting theory

$$\mathcal{L} = \mathcal{L}_0 - V(\bar\Psi,\Psi)\ . \tag{10.2}$$

If the theory involves both boson and fermion fields and their couplings, i.e. if the Lagrangian is given by

$$\mathcal{L} = \mathcal{L}_0^\Psi + \mathcal{L}_0^\phi - V_1(\bar\Psi,\Psi) - V_2(\phi) - V_\mathrm{int}(\bar\Psi,\Psi,\phi)\ , \tag{10.3}$$

then $Z[J,\eta,\bar\eta]$ is given simply by (cf. (6.31))

$$\begin{aligned} Z[J,\eta,\bar\eta] \;=\; & N\,\mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4x\, V_\mathrm{int}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta\eta},\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar\eta},\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right)} \\ & \times \mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4x\, V_1\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta\eta},\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar\eta}\right)}\mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4x\, V_2\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right)} \\ & \times \mathrm{e}^{-\mathrm{i}\int \bar\eta(x)S_\mathrm{F}(x-y)\eta(y)\mathrm{d}^4x\,\mathrm{d}^4y} \\ & \times \mathrm{e}^{-\frac{\mathrm{i}}{2}\int J(x)D_\mathrm{F}(x-y)J(y)\mathrm{d}^4x\,\mathrm{d}^4y}\ . \end{aligned} \tag{10.4}$$

This functional can generate all the n-point functions of the interacting theory and these can again be graphically represented in the form of Feynman diagrams.

10.1 Feynman Rules

The Feynman rules of Sect. 7.1 can be extended by the following rules (in momentum space)

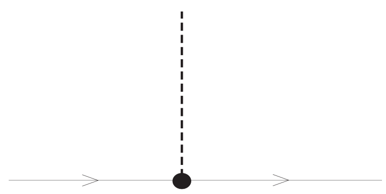


Figure 10.1: Fermion-boson vertex (see (10.6)). The solid line denotes the fermion, the dashed one the boson.

1) each internal fermion line gives a factor

$$\frac{\mathrm{i}}{\not{k} - (m - \mathrm{i}\epsilon)} \ . \tag{10.5}$$

The fermion lines carry now an arrow that points into the direction into which the fermion's *charge* flows. This is a consequence of the fact that the fermion propagator is no longer symmetric in its arguments (cf. (9.75)) so that we have to give the direction of motion explicitly. For fermions we deal with two fields, ψ and $\bar{\psi}$, that carry different charges. In a Feynman graph, in which time runs from bottom to top upwards, particle lines are always upwards directed whereas antiparticle lines move downwards. In a loose way of speaking the fermion lines move from $\bar{\psi}$ to ψ, because $\bar{\psi}$ creates particles and ψ annihilates them. Since the opposite holds for antiparticles their lines run downwards, reflecting the fact that the Feynman propagator propagates them backwards in time.

In a coupled theory a new class of diagrams can appear because of the coupling of fermion and boson fields. For example, a coupling term of the form

$$V_{\text{int}} = g\bar{\Psi}\Psi\phi = g\bar{\Psi}_\alpha\Psi_\alpha\phi \ , \tag{10.6}$$

the so-called *Yukawa* coupling, will generate vertices of the form shown in Fig. 10.1. We, therefore, have a new Feynman rule dealing with fermion-boson vertices:

2) each boson-fermion vertex gives a factor $-\mathrm{i}g$. If the coupling term involves other bilinear forms of the Dirac spinors, e.g. $\bar{\Psi}\gamma_\mu\Psi$, then an additional factor γ_μ would appear at each vertex.

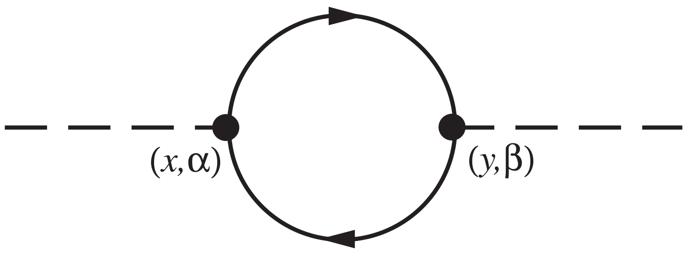


Figure 10.2: Fermion loop contribution to the boson propagator.

10.1.1 Fermion Loops

A change in the rules appears when we consider fermion loops, e.g. the graph shown in Fig. 10.2, where the solid line denotes the fermions and the dashed line the bosons. Such a term is obviously of second order in the fermion-boson coupling constant (indicated by the two vertices) and contributes to the bosonic 2-point function. It can thus be generated by the second order term in the expansion of (10.4), properly generalized to fermion fields. Because the graph in Fig. 10.2 is one-particle irreducible only the true second order term ($\sim \tilde{S}_2$ in (6.39),(6.40)) can contribute. This term has the structure

$$\begin{aligned} \mathrm{i}S[\eta,\bar\eta,J] &= \dots \\ &+ \frac{1}{2}\mathrm{e}^{-\mathrm{i}S_0[\eta,\bar\eta,J]}\left[(-\mathrm{i})\int \mathrm{d}^4x\, V_{\mathrm{int}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta\eta(x)},\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar\eta(x)},\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)\right]^2 \mathrm{e}^{+iS_0[\eta,\bar\eta,J]} \\ &+ \dots\ . \end{aligned} \tag{10.7}$$

Here $S_0[\eta,\bar\eta,J]$ is given by

$$\begin{aligned} \mathrm{i}S_0[\eta,\bar\eta,J] &= -\frac{\mathrm{i}}{2}\int \mathrm{d}^4v\,\mathrm{d}^4w\, J(v)D_{\mathrm{F}}(v-w)J(w) \\ &\quad -\mathrm{i}\int \mathrm{d}^4x'\,\mathrm{d}^4y'\,\bar\eta(x')S_{\mathrm{F}}(x'-y')\eta(y')\ . \end{aligned} \tag{10.8}$$

We now remember that only those terms in the generating functional can contribute to the boson propagator in Fig. 10.2 that are of second order in the bosonic current J and of zeroth order in the fermionic currents η and $\bar\eta$. Therefore, the only contributing term in (10.7) with the interaction (10.6) is

$$\begin{aligned} \mathrm{i}S_2^{\mathrm{loop}} &= -\frac{(-\mathrm{i}g)^2}{2}\mathrm{e}^{-\mathrm{i}S_0} \\ &\quad\times\int \mathrm{d}^4x\,\mathrm{d}^4y\,\frac{\delta^6}{\delta\eta_\alpha(x)\delta\bar\eta_\alpha(x)\delta J(x)\delta\eta_\beta(y)\delta\bar\eta_\beta(y)\delta J(y)}\mathrm{e}^{+\mathrm{i}S_0} \\ &= \frac{(-\mathrm{i}g)^2}{2}\mathrm{e}^{-\mathrm{i}S_0}\int \mathrm{d}^4x\,\mathrm{d}^4y\int \mathrm{d}^4v\,\mathrm{d}^4w \end{aligned} \tag{10.9}$$

$$\times \left[D_{\mathrm{F}}(x-v)J(v)D_{\mathrm{F}}(y-w)J(w)\frac{\delta^4}{\delta\eta_\alpha(x)\delta\bar{\eta}_\alpha(x)\delta\eta_\beta(y)\delta\bar{\eta}_\beta(y)}\right]\mathrm{e}^{+\mathrm{i}S_0} \ ,$$

after performing the derivative with respect to J.

We now concentrate on the fermionic derivative (for notational convenience we drop here the index F for the propagators)

$$\begin{aligned}
&\frac{\delta^4}{\delta\eta_\alpha(x)\delta\bar{\eta}_\alpha(x)\delta\eta_\beta(y)\delta\bar{\eta}_\beta(y)}\mathrm{e}^{-\mathrm{i}\int\bar{\eta}(x')S(x'-y')\eta(y')\,\mathrm{d}^4x'\,\mathrm{d}^4y'} \\
&= \frac{\delta^3}{\delta\eta_\alpha(x)\delta\bar{\eta}_\alpha(x)\delta\eta_\beta(y)}(-\mathrm{i})\int S_{\beta\gamma}(y-y')\eta_\gamma(y')\,\mathrm{d}^4y'\,\mathrm{e}^{-\mathrm{i}\int\cdots} \\
&= \frac{\delta^2}{\delta\eta_\alpha(x)\delta\bar{\eta}_\alpha(x)}\Bigg[\Bigg(-\mathrm{i}S_{\beta\beta}(y-y) \\
&\quad -\int S_{\beta\gamma}(y-y')\eta_\gamma(y')\,\mathrm{d}^4y'\int\bar{\eta}_\delta(x')S_{\delta\beta}(x'-y)\,\mathrm{d}^4x'\Bigg)\mathrm{e}^{-\mathrm{i}\int\cdots}\Bigg] \\
&= \frac{\delta}{\delta\eta_\alpha(x)}\Bigg[\Bigg(-S_{\beta\beta}(y-y)\int S_{\alpha\gamma}(x-y')\eta_\gamma(y')\,\mathrm{d}^4y' \\
&\quad \oplus\int S_{\beta\gamma}(y-y')\eta_\gamma(y')\,\mathrm{d}^4y'\,S_{\alpha\beta}(x-y)+\mathcal{O}(\eta\bar{\eta})\Bigg)\mathrm{e}^{-\mathrm{i}\int\cdots}\Bigg]\ . \qquad (10.10)
\end{aligned}$$

Here the symbol $\oplus$ denotes a 'plus' sign that is due to the Grassmann nature of the source functions and would have been opposite, if we had worked with bosons instead of fermions. The terms denoted by $\mathcal{O}(\eta\bar{\eta})$ stand for expressions that involve products of the two source functions and would thus be linear in η or $\bar{\eta}$ after the last functional derivative has been taken. Since the n-point function is given by a functional derivative at vanishing source, these terms do not contribute to the 2-point function we are after.

Performing now the last derivative gives for the expression (10.10)

$$\begin{aligned}
&= [-S_{\beta\beta}(y-y)S_{\alpha\alpha}(x-x)\oplus S_{\beta\alpha}(y-x)S_{\alpha\beta}(x-y)+\mathcal{O}(\eta\bar{\eta})] \\
&\quad \times\mathrm{e}^{-\mathrm{i}\int\bar{\eta}(x')S_{\mathrm{F}}(x'-y')\eta(y')\,\mathrm{d}^4x'\,\mathrm{d}^4y'}\ . \qquad (10.11)
\end{aligned}$$

The sign of the second term would have been opposite, if we had worked with bosons.

We now combine this result with (10.9). Since we are interested in particular in the fermion loop diagram shown above, we need to construct a bosonic 2-point function with no external fermion lines. Therefore, only those terms of the fermionic part (10.11) can contribute to this graph that contain no sources η or $\bar{\eta}$. Disregarding the vacuum contributions (first term in (10.11)) we thus have as the only term of (10.11) that contributes to the loop diagram

$$\oplus\mathrm{tr}\left[S_{\mathrm{F}}(y-x)S_{\mathrm{F}}(x-y)\right]\ . \qquad (10.12)$$

Inserting this result into (10.9) gives for the relevant part of S_2

$$\begin{aligned} \mathrm{i}S_2^{\mathrm{loop}} &= \oplus \frac{(-\mathrm{i}g)^2}{2} \int \mathrm{d}^4x\, \mathrm{d}^4y\, \mathrm{d}^4v\, \mathrm{d}^4w\, \mathrm{tr}\left[S_\mathrm{F}(y-x)S_\mathrm{F}(x-y)\right] \\ &\qquad \times D_\mathrm{F}(x-v)D_\mathrm{F}(y-w)J(v)J(w) \end{aligned} \tag{10.13}$$

and, correspondingly, for the 2-point function

$$\begin{aligned} G^{\mathrm{loop}}_{boson}(x_1,x_2) &= -\frac{\delta^2(\mathrm{i}S_2^{\mathrm{loop}})}{\delta J(x_1)\delta J(x_2)} \qquad (10.14) \\ &= \ominus\,(-\mathrm{i}g)^2 \int \mathrm{d}^4x\, \mathrm{d}^4y\, D_\mathrm{F}(x_1-x)\mathrm{tr}\left[S_\mathrm{F}(x-y)S_\mathrm{F}(y-x)\right] D_\mathrm{F}(y-x_2)\ . \end{aligned}$$

When we retrace this detailed calculation we find that the positive sign in (10.12) and thus the negative sign in (10.14) is due to the fact that η and $\bar{\eta}$ are Grassmann fields.

Thus, we get the additional Feynman rule, which holds in general,

3) Any fermionic loop graph is associated with an additional $(-)$ sign and a trace over the Dirac indices.

10.2 Wick's Theorem

Mixed n-point functions. In (6.9) we had expressed the n-point function for bosons $G(x_1,\ldots,x_n)$ in terms of a symmetrized product of $n/2$ 2-point functions (Wick's theorem). The same can now be done also for the general theory that involves bosons and fermions.

First, it is clear that boson and fermion fields commute; the corresponding field operators act in different Hilbert spaces so that the groundstate expectation value separates into a product of vacuum expectation values over fermion and boson operators separately, e.g.

$$\langle 0|\Psi(1)\phi(2)\phi(3)\bar{\Psi}(4)|0\rangle = \langle 0|\Psi(1)\bar{\Psi}(4)|0\rangle\langle 0|\phi(2)\phi(3)|0\rangle\ . \tag{10.15}$$

We, therefore, need to formulate here Wick's theorem only for the fermions.

The proof proceeds just as in section 6.1 for bosons. We start with the general definition of the n-point Green's function for fermions

$$\begin{aligned} & G_{\alpha,\beta,\ldots,2\nu}(x_1,x_2,\ldots,x_{2n}) \qquad (10.16) \\ &= \left(\frac{1}{\mathrm{i}}\right)^{2n} \left.\frac{\delta^{2n}Z}{\delta\eta_{2\nu}(x_{2n})\cdots\delta\eta_{\nu+1}(x_{n+1})\delta\bar{\eta}_\nu(x_n)\cdots\delta\bar{\eta}_\alpha(x_1)}\right|_{\eta=\bar{\eta}=0}\ . \end{aligned}$$

Note that the Green's function now depends also on the Dirac spinor-indices, in addition to the space-time coordinates. For notational convenience we will omit these indices in the following discussions.

As in (6.2) for bosons, the Green's function can be written as a vacuum expectation value of a time-ordered product of field operators

$$\begin{aligned} G(x_1, x_2, \ldots, x_{2n}) & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (10.17) \\ = \ & \frac{\int \mathcal{D}\bar\Psi\, \mathcal{D}\Psi\, \Psi(x_n) \cdots \Psi(x_1) \bar\Psi(x_{2n}) \cdots \bar\Psi(x_{n+1})\, \mathrm{e}^{\mathrm{i}S}}{\int \mathcal{D}\bar\Psi\, \mathcal{D}\Psi\, \mathrm{e}^{\mathrm{i}S}} \\ = \ & \langle 0| T \left[\Psi(x_n)\Psi(x_{n-1}) \cdots \Psi(x_1)\, \bar\Psi(x_{2n}) \cdots \bar\Psi(x_{n+1}) \right] |0\rangle \ , \end{aligned}$$

where S denotes the action. The special ordering of fields in (10.17) is chosen such that there is no additional phase present. This can be seen as follows: In calculating the derivatives with respect to $\bar\eta$ no phase appears, because first the derivative is anticommuted through all field operators already pulled down from the exponential, then acts without a further sign change on the exponential and then is anticommuted back, thus creating the same phase backwards as on the way forward. The extra phase $(-)$ that appears from the derivative with respect to η due to the necessary reordering in the exponent is cancelled by the same phase caused by anticommuting the derivative operator $\delta/\delta\eta$ with all the Grassmann fields Ψ and $\bar\Psi$, that were already pulled down, and by reordering them.

Thus, the n-point Green's function, originally introduced as a functional derivative of a generating functional, can in general also be defined as the vacuum expectation value of a time-ordered product of field operators $\Phi(x)$ that can describe either boson or fermion fields

$$G(x_1, x_2, \ldots, x_{2n}) = \langle 0| T \left[\Phi(x_n)\Phi(x_{n-1}) \cdots \Phi(x_1)\, \Phi(x_{2n}) \cdots \Phi(x_{n+1}) \right] |0\rangle \ . \tag{10.18}$$

The only difference between boson and fermion fields appears in the reordering caused by the T operator; here a "$(-)$" sign appears whenever a pair of fermions changes its order whereas there is no such sign when the boson operators are reordered.

Wick's theorem for Fermions. We now derive a simple method, known as Wick's Theorem that we encountered for bosons already in section 6.1, for the evaluation of a vacuum expectation value of a time-ordered product of free field operators. In that case we can use the form (9.68) of the generating functional for the evaluation of the functional derivative. Performing first the functional derivatives with respect to $\bar{\eta}$ on that expression gives[1]

$$
\begin{aligned}
&\frac{\delta^{2n}}{\delta\eta(x_{2n})\cdots\delta\eta(x_{n+1})\delta\bar{\eta}(x_n)\cdots\delta\bar{\eta}(x_1)}\,\mathrm{e}^{-\mathrm{i}\int\bar{\eta}(x)S_{\mathrm{F}}(x-y)\eta(y)\,\mathrm{d}^4x\,\mathrm{d}^4y}\Big|_{\eta=\bar{\eta}=0} \\
&= \frac{\delta^{n}}{\delta\eta(x_{2n})\cdots\delta\eta(x_{n+1})}\Bigg((-\mathrm{i})^n\int S_{\mathrm{F}}(x_n-y)\eta(y)\,\mathrm{d}^4y \\
&\quad\times\int S_{\mathrm{F}}(x_{n-1}-y)\eta(y)\,\mathrm{d}^4y\cdots\int S_{\mathrm{F}}(x_1-y)\eta(y)\,\mathrm{d}^4y \\
&\quad\times\mathrm{e}^{-\mathrm{i}\int\bar{\eta}(x)S_{\mathrm{F}}(x-y)\eta(y)\,\mathrm{d}^4x\,\mathrm{d}^4y}\Bigg)\Bigg|_{\eta=\bar{\eta}=0}\ .
\end{aligned}
\tag{10.19}
$$

Now the differentiation with respect to η is performed. Since the whole expression has to be taken at $\eta=\bar{\eta}=0$, only the prefactors of the exponential can contribute. This gives

$$
= (-\mathrm{i})^n\sum_P(-)^P S_{\mathrm{F}}(x_1-x_{p_{2n}})S_{\mathrm{F}}(x_2-x_{p_{2n-1}})\dots S_{\mathrm{F}}(x_n-x_{p_{n+1}})\ , \tag{10.20}
$$

where the sum over P runs over all permutations of the indices $n+1,\dots,2n$. The sign of the permutations is such that $(-)^P=+$, if always the outermost operators in (10.17) are combined into the 2-point function, i.e. if $(1,2n)$, $(2,2n-1)\cdots(n,n+1)$ are combined. The Dirac indices, that have not been written down here, have to be combined in the same way.

Inserting this result into the definition of the Green's function in (10.16) gives

$$
\begin{aligned}
G(x_1,x_2,\dots,x_{2n}) &= \left(\frac{1}{\mathrm{i}}\right)^{2n}\frac{\delta^{2n}Z}{\delta\eta(x_{2n})\dots\delta\eta(x_{n+1})\delta\bar{\eta}(x_n)\dots\delta\bar{\eta}(x_1)} \\
&= \mathrm{i}^n\sum_P(-)^P S_{\mathrm{F}}(x_1-x_{p_{2n}})\cdots S_{\mathrm{F}}(x_n-x_{p_{n+1}})\ .
\end{aligned}
\tag{10.21}
$$

Equation (10.21) is Wick's theorem for fermions. Written in a slightly different form it reads

$$
\begin{aligned}
&\langle 0|T\left[\Psi(x_n)\cdots\Psi(x_1)\,\bar{\Psi}(x_{2n})\cdots\bar{\Psi}(x_{n+1}\right]|0\rangle \\
&= \mathrm{i}^n\sum_P(-)^P\langle 0|T\left[\Psi(x_1)\bar{\Psi}(x_{p_{2n}})\right]|0\rangle\cdots\langle 0|T\left[\Psi(x_n)\bar{\Psi}(x_{p_{n+1}})\right]|0\rangle\ .
\end{aligned}
\tag{10.22}
$$

This form also exhibits clearly the important consequence of using anti-commuting Grassmann fields for the fermions. The exchange of any two of

[1]To facilitate the notation the Dirac indices are not written down here.

the coordinates appearing in the fields Ψ or of any 2 in the fields $\bar{\Psi}$ gives a "$-$" sign to the Green's function. In particular, the 2-particle, 4-point Green's function has the property

$$\begin{aligned} G_{\alpha\beta\gamma\delta}(x_1, x_2, y_1, y_2) &= -G_{\beta\alpha\gamma\delta}(x_2, x_1, y_1, y_2) = -G_{\alpha\beta\delta\gamma}(x_1, x_2, y_2, y_1) \\ &= G_{\beta\alpha\delta\gamma}(x_2, x_1, y_2, y_1) \ . \end{aligned} \tag{10.23}$$

This implies that for G not to vanish we must have $x_1 \neq x_2$ and $y_1 \neq y_2$, if all the Dirac-indices are the same, thus reflecting the Pauli principle.

Wick's theorem for fermions (10.22) has a form that is analogous to that for bosons (6.9). The essential difference is the appearance of the sign of the permutation that reflects the antisymmetric of fermionic states. In addition the degeneracy factor $1/2^n$ is missing here because for fermions we have no longer a symmetry under exchange: $S_\mathrm{F}(x-y) \neq S_\mathrm{F}(y-x)$ (see (9.75)).

10.3 Removal of Degrees of Freedom: Yukawa Theory

In Sect. 10 we have first met a coupled fermion-boson theory with a simple Yukawa coupling. Its Lagrangian reads

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\phi\Box\phi - V(\phi) + \bar{\psi}\left(\mathrm{i}\ \partial\!\!\!/ - m_\mathrm{F}\right)\psi - g\bar{\psi}\psi\phi \\ &= -\frac{1}{2}\phi\Box\phi - V(\phi) + \bar{\psi}\left(\mathrm{i}\ \partial\!\!\!/ - m_\mathrm{F} - g\phi\right)\psi \ . \end{aligned} \tag{10.24}$$

The term $V(\phi)$ may contain both the mass term and further selfinteraction terms. Since the coupling constant g here is dimensionless we expect that this theory is renormalizable (cf. the discussions in Sect. 8.2). The second line of (10.24) shows that the Yukawa coupling contributes the fermion mass and, for $m_\mathrm{F} = 0$, can even generate the whole mass of a fermion. This mechanism plays an important role in the theory of electroweak interactions.

In the preceding sections we have discussed the perturbative treatment of such a Lagrangian. Here we consider now an alternative treatment that "integrates out" the fermionic degrees of freedom. This method is based on the fact that the path integral (10.25) is quadratic in the fermion fields and, therefore, of Gaussian form since the generating functional is given by

$$Z[\eta, \bar{\eta}, J] = N \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}\phi \exp\left[\mathrm{i}\int \left(\mathcal{L}[\bar{\psi}, \psi, \phi] + J\phi + \bar{\eta}\psi + \bar{\psi}\eta\right)\,\mathrm{d}^4x\right] \ . \tag{10.25}$$

The path integral over the fermionic degrees of freedom can thus be performed. According to (9.57) and (9.66) this gives

$$Z[\eta,\bar\eta,J] = N\int \mathcal{D}\phi\, \det(M)\, \mathrm{e}^{\mathrm{i}\bar\eta M^{-1}\eta} \mathrm{e}^{-\mathrm{i}\int\left[\frac{1}{2}\phi\Box\phi - V(\phi) - J\phi\right]\mathrm{d}^4x} \tag{10.26}$$

with (see (9.67))

$$M = \mathrm{i}\,\not\partial - m^* \qquad \text{with} \qquad m^* = m_{\mathrm{F}} + g\phi\ ; \tag{10.27}$$

the factor 'i' before M in the determinant has been absorbed into the normalization constant N. Note that the fermion determinant M is a functional of ϕ. Equation (10.26) is completely equivalent to (10.25).

We now concentrate on the bosonic degrees of freedom by setting η and $\bar\eta$ to zero. In doing so we give up the possibility to calculate any n-point function with fermionic external legs because these are obtained as functional derivatives with respect to η. Our procedure thus amounts to keeping *only* fermionic *loops* in our theory for the scalar field ϕ. More specifically, as we will see below, our procedure keeps fermionic effects only on the one-loop level.

Using now (5.38)

$$\det(M) = \mathrm{e}^{\mathrm{tr}\ln M} \tag{10.28}$$

gives

$$Z[J] = N\int \mathcal{D}\phi\, \mathrm{e}^{-\mathrm{i}\int\left[\frac{1}{2}\phi\Box\phi + V(\phi) - J\phi\right]\mathrm{d}^4x + \mathrm{tr}\ln M}\ . \tag{10.29}$$

Equation (10.29) shows that now formally all the fermion degrees of freedom have been removed. Alternatively, we could also say that we are now working with a new, effective action for the bosonic sector

$$S_{\mathrm{B}} = -\int\left(\phi\frac{1}{2}\Box\phi + V(\phi)\right)\mathrm{d}^4x - \mathrm{i}\,\mathrm{tr}\ln\left(\mathrm{i}\,\not\partial - m^*\right)\ . \tag{10.30}$$

The trace here has to be performed over Dirac indices and over space-time coordinates (or, equivalently, in momentum space). The corresponding effective Lagrangian density reads

$$\mathcal{L}_{\mathrm{B}} = -\frac{1}{2}\phi\Box\phi - V(\phi) - \mathrm{i}\,\mathrm{Tr}\ln\left(\mathrm{i}\,\not\partial - m^*(\phi)\right)\ , \tag{10.31}$$

where we have used

$$\mathrm{tr} = \mathrm{Tr}\int \mathrm{d}^4x\ ; \tag{10.32}$$

Tr is a trace only over the Dirac (and possibly other internal) indices. In this form, which is still exact, the original Lagrangian (10.24) is said to be *bosonized*.

10.3.1 Perturbative Expansion

We now develop a perturbative treatment of this bosonized Lagrangian that uses an expansion around the groundstate of the bosonic sector.

The true groundstate (vacuum) of the bosonic sector is, of course, determined by the potential $V(\phi)$, but it may not coincide with the minimum of this potential. Nevertheless, we may take here the constant, translationally invariant field ϕ_0 which minimizes V as our reference field. In order to normalize the action, and thus the physics, to this state we now subtract the action belonging to ϕ_0. This gives (with $m_0^* = m_\mathrm{F} + g\phi_0$)

$$S_\mathrm{B} = -\int \left(\frac{1}{2}\phi\Box\phi - V(\phi)\right) \mathrm{d}^4x - \mathrm{i}\,\mathrm{tr}\ln\left(\mathrm{i}\,\partial\!\!\!/ - m^*\right) + \mathrm{i}\,\mathrm{tr}\ln\left(\mathrm{i}\,\partial\!\!\!/ - m_0^*\right) \ , \tag{10.33}$$

where a constant term has been dropped. Next we use the identity

$$\mathrm{tr}\ln\left(\mathrm{i}\,\partial\!\!\!/ - m^*\right) = \mathrm{tr}\ln\left(\mathrm{i}\,\partial\!\!\!/ - m^*\right)^\dagger = \mathrm{tr}\ln\left(-\mathrm{i}\,\partial\!\!\!/ - m^*\right) \tag{10.34}$$

which holds because of the hermiticity of the Dirac operator $\mathrm{i}\,\partial\!\!\!/ - m^*$; here we remember from our discussion in Sect. 5.2 that the determinant of an operator is just the product of its eigenvalues. We thus get

$$\begin{aligned} \mathrm{tr}\ln\left(\mathrm{i}\,\partial\!\!\!/ - m^*\right) &= \frac{1}{2}\mathrm{tr}\ln\left[\left(\mathrm{i}\,\partial\!\!\!/ - m^*\right)\left(-\mathrm{i}\,\partial\!\!\!/ - m^*\right)\right] \\ &= \frac{1}{2}\mathrm{tr}\ln\left(\Box + m^{*2} - \mathrm{i}g\,\partial\!\!\!/\phi\right) \ . \end{aligned} \tag{10.35}$$

Thus the effective bosonic action becomes

$$\begin{aligned} S_\mathrm{B} &= -\int \left(\frac{1}{2}\phi\Box\phi + V(\phi)\right) \mathrm{d}^4x \qquad (10.36) \\ &\quad -\frac{\mathrm{i}}{2}\mathrm{tr}\ln\left(\Box + m^{*2} - \mathrm{i}g\left(\partial\!\!\!/\phi\right)\right) + \frac{\mathrm{i}}{2}\mathrm{tr}\ln\left(\Box + m_0^{*2}\right) \\ &= -\int \left(\frac{1}{2}\phi\Box\phi + V(\phi)\right) \mathrm{d}^4x - \frac{\mathrm{i}}{2}\mathrm{tr}\ln\left(1 + \frac{m^{*2} - m_0^{*2} - \mathrm{i}g\,\partial\!\!\!/\phi}{\Box + m_0^{*2}}\right) \end{aligned}$$

and the Lagrangian density now reads

$$\mathcal{L}_\mathrm{B} = -\frac{1}{2}\phi\left(\Box + m_B^2\right)\phi - \frac{\mathrm{i}}{2}\mathrm{Tr}\ln\left(1 - \frac{m^{*2} - m_0^{*2} - \mathrm{i}g\,\partial\!\!\!/\phi}{-\Box - m_0^{*2}}\right) \ . \tag{10.37}$$

So far, no approximations have been made and the theory is still exact; note that the reference field is in principle arbitrary.

The action is now in a form that is suitable for a perturbative treatment. We thus expand the logarithm ($-\ln(1-x) = x + \frac{x^2}{2} + \ldots$) in (10.36) and obtain

$$-\mathrm{tr}\ln(\ldots) = \mathrm{tr}\sum_{n=1}^{\infty}\frac{1}{n}\left(G_0 W\right)^n \tag{10.38}$$

with

$$G_0 = \frac{1}{-\Box - m_0^{*2}} \qquad \text{and} \qquad W = m^{*2} - m_0^{*2} - \mathrm{i}g\ \partial\!\!\!/\phi\ . \tag{10.39}$$

Note that G_0 is nothing else than the Feynman propagator (cf. (4.25)). The trace in (10.38) indicates that each term in the expansion contains one closed fermion loop (with n vertices W coupling to the field ϕ). Equation (10.38) is therefore called the *one-loop expansion*, it represents a perturbative expansion of the effective action. Integrating out the fermion degrees of freedom has thus led to an effective theory that takes all fermionic one-loop diagrams into account. The expansion is expected to converge the more rapidly the closer the actual field ϕ is to the chosen field ϕ_0 and the smaller the gradients of ϕ are.

Our aim in the following developments is to write the trace term as an integral over a space-time density which could then be interpreted as a correction term to the Lagrangian density, thus yielding an effective Lagrangian which contains the effects of the fermions on the one-loop level. In order to be able to do so we have to separate the terms under the trace into a product of operators that are local in x and p. In this case the trace can be separated

$$\begin{aligned} \mathrm{tr}\left[F(\hat{p})G(\hat{x})\right] &= \mathrm{Tr}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\langle p|F(\hat{p})G(\hat{x})|p\rangle \\ &= \mathrm{Tr}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int\frac{\mathrm{d}^4p'}{(2\pi)^4}\int\mathrm{d}^4x\int\mathrm{d}^4x' \\ &\qquad \times\langle p|F(\hat{p})|p'\rangle\langle p'|x'\rangle\langle x'|G(\hat{x})|x\rangle\langle x|p\rangle \\ &= \mathrm{Tr}\int\frac{\mathrm{d}^4p}{(2\pi)^4}F(p)\int\mathrm{d}^4x\, G(x)\ . \end{aligned} \tag{10.40}$$

The expressions in (10.38) are, however, not in a form that allows direct application of (10.40), because the field ϕ, and thus also W, is dependent on x and the propagator G_0 acts on it. We thus first reorder each term such that all propagators are moved to the left. To do so we start from the identity

$$WG_0 = G_0W + G_0\left[G_0^{-1}, W\right]G_0 \tag{10.41}$$

and iterate this equality by applying it again to the quantity $W' = \left[G_0^{-1}, W\right]$

$$W'G_0 = G_0W' + G_0\left[G_0^{-1}, W'\right]G_0\ . \tag{10.42}$$

Combining now (10.41) and (10.42) gives

$$WG_0 = G_0W + G_0^2\left[G_0^{-1}, W\right] + G_0^2\left[G_0^{-1}, \left[G_0^{-1}, W\right]\right]G_0 \; . \tag{10.43}$$

This procedure can be continued and yields the operator identity

$$WG_0 = G_0W + G_0^2\left[G_0^{-1}, W\right] + G_0^3\left[G_0^{-1}, \left[G_0^{-1}, W\right]\right] + \dots \; . \tag{10.44}$$

The basic commutator appearing here is given by

$$\left[G_0^{-1}, W\right] = [-\Box, W] = -\Box W - 2\left(\partial^\mu W\right)\partial_\mu \; . \tag{10.45}$$

Since W contains a derivative of the field this term is of order $\mathcal{O}((\partial_\mu\phi)^3)$ and the next higher order term in (10.44) would be of order $\mathcal{O}((\partial_\mu\phi)^4)$. Equation (10.44) thus represents an expansion in terms of gradients of the field. We also see that it contains increasing powers of G_0 so that we expect to encounter only a finite number of ultraviolet, i.e. for large momenta, divergent terms.

We now use the identity (10.44) to rewrite the expansion (10.38). Up to terms of second order in W the expansion reads

$$\begin{aligned} -\mathrm{tr}\ln&\left(1 - \frac{m^{*2} - m_0^{*2} - \mathrm{i}g\;\partial\!\!\!/\phi}{-\Box - m_0^{*2}}\right) \\ &= \mathrm{tr}(G_0W) + \frac{1}{2}\mathrm{tr}(G_0WG_0W) + \mathcal{O}(W^3) \; . \end{aligned} \tag{10.46}$$

Using (10.44) we get

$$\begin{aligned} -\mathrm{tr}\ln(\dots) &= \mathrm{tr}(G_0W) + \frac{1}{2}\mathrm{tr}(G_0WG_0W) + \mathcal{O}(W^3) \qquad (10.47) \\ &= \mathrm{tr}(G_0W) + \frac{1}{2}\mathrm{tr}(G_0^2W^2) + \frac{1}{2}\mathrm{tr}\left(G_0^3\left[G_0^{-1}, W\right]W\right) + \mathcal{O}(W^3) \; . \end{aligned}$$

Note that all the terms on the rhs have the momentum dependence and the x-dependence separated so that – according to (10.40) – they can be written as an integral over the momentum times a spatial integral over a Lagrangian density.

We now discuss the terms in the expansion (10.47) and start with the term $\mathrm{tr}(G_0W)$. This is explicitly given by

$$\mathrm{tr}(G_0W) = \mathrm{tr}\left(\frac{1}{-\Box - m_0^{*2}}\left(m^{*2} - m_0^{*2} - \mathrm{i}g\;\partial\!\!\!/\phi\right)\right) \; . \tag{10.48}$$

Since

$$\mathrm{tr}\,\gamma_\mu = 0 \tag{10.49}$$

and

$$m^{*2} - m_0^{*2} = 2m_{\mathrm{F}} g\,(\phi - \phi_0) + g^2\left(\phi^2 - \phi_0^2\right) \tag{10.50}$$

we get

$$\begin{aligned}\mathrm{tr}(G_0 W) &= 2m_{\mathrm{F}} g\,\mathrm{tr}\left(\frac{1}{-\Box - m_0^{*2}}\,(\phi - \phi_0)\right) \\ &\quad + g^2 \mathrm{tr}\left(\frac{1}{-\Box - m_0^{*2}}\left(\phi^2 - \phi_0^2\right)\right) \;. \end{aligned} \tag{10.51}$$

The trace operation can now be performed. This gives

$$\begin{aligned}\mathrm{tr}(G_0 W) &= +8m_{\mathrm{F}} g \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\,\frac{1}{k^2 - m_0^{*2}} \int \mathrm{d}^4 x\,(\phi(x) - \phi_0) \\ &\quad + 4g^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\,\frac{1}{k^2 - m_0^{*2}} \int \mathrm{d}^4 x \left(\phi^2(x) - \phi_0^2\right) \;, \end{aligned} \tag{10.52}$$

where we have used (10.40) with the trace over the Dirac indices $\mathrm{Tr} = 4$. This term is now in the desired form, i.e. it is given by a space-time integral over fields. However, the momentum-integral appearing here actually diverges quadratically as power counting shows us.

Also the next term in (10.47) diverges. It is given by

$$\begin{aligned}\mathrm{tr}(G_0^2 W^2) &= g^2 \mathrm{Tr} \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\,\frac{1}{\left(k^2 - m_0^{*2}\right)^2} \\ &\quad \times \int \mathrm{d}^4 x \left[2m_{\mathrm{F}}\,(\phi(x) - \phi_0) - \mathrm{i}\,\partial\!\!\!/\phi(x) + g\left(\phi^2(x) - \phi_0^2\right)\right]^2 \end{aligned} \tag{10.53}$$

and diverges logarithmically.

The next term in (10.47)

$$\mathrm{tr}\left(G_0^3\left[G_0^{-1}, W\right] W\right) = \mathrm{tr}\left[\left(\frac{1}{-\Box - m_0^{*2}}\right)^3 \left(-\Box W - 2(\partial^\mu W)\partial_\mu\right) W\right] \tag{10.54}$$

and all higher order terms are clearly ultraviolet convergent because they contain higher and higher powers of the propagator G_0.

This means that we have only two divergent terms ((10.52) and (10.53)) in the expansion (10.38). These can be removed by introducing proper counter terms. We can thus define a finite, renormalized effective Lagrangian density in the boson sector of our theory

$$\mathcal{L}_{\mathrm{B}}(x) = -\frac{1}{2}\phi(x)\left(\Box + m_B^2\right)\phi(x)$$

$$
\begin{aligned}
&-\frac{\mathrm{i}}{2}\mathrm{Tr}\ln\left(1-\frac{m^{*2}(x)-m_0^{*2}-\mathrm{i}g\ \partial\!\!\!/\phi(x)}{-\Box-m_0^{*2}}\right)\\
&-\mathrm{i}g\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{1}{k^2-m_0^{*2}}\\
&\qquad\times\left[4m_{\mathrm{F}}\left(\phi(x)-\phi_0\right)-2g\left(\phi^2(x)-\phi_0^2\right)\right]\\
&-\mathrm{i}g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{1}{\left(k^2-m_0^{*2}\right)^2}\\
&\qquad\times\left[2m_{\mathrm{F}}\left(\phi(x)-\phi_0\right)-\mathrm{i}\ \partial\!\!\!/\phi(x)+g\left(\phi^2(x)-\phi_0^2\right)\right]^2\\
&+\mathcal{O}(W^3)\ , \qquad (10.55)
\end{aligned}
$$

where Tr denotes a trace over the Dirac indices only and where it is understood that we have to use the expansion (10.46) for the Tr ln term. Performing a linear shift in the field ϕ shows that the counter terms all have the structure of terms already present in the original boson Lagrangian. The Yukawa theory is thus indeed renormalizable.

Chapter 11

S-MATRIX AND GREEN'S FUNCTIONS

In the preceding chapters we have discussed at some length how to calculate the n-point functions, i.e. the Green's functions, of a field theory. In this chapter we now derive the so-called reduction theorem that provides a connection between these Green's functions and observable transition rates.

11.1 Scattering Matrix

The transition rate of any quantum process is determined by the so-called S-matrix. We define the S-matrix as the probability amplitude for a process that leads from an ingoing state $|\alpha, \text{in}\ \rangle$ to an outgoing state $|\beta,\ \text{out}\ \rangle$. The particles are assumed to move freely in these asymptotic states outside the range of the interaction; both of these states can therefore be characterized by giving the momenta of all participating particles and possibly other quantum numbers as well all of which are denoted by α and β. The S-matrix is thus given by

$$S_{\beta\alpha} = \langle \beta, \text{out} | \alpha, \text{in} \rangle \ . \tag{11.1}$$

We can then also introduce an operator $\hat{S}$ that transforms in-states (bras) into out-states (bras)

$$\langle \beta, \text{out} | = \langle \beta, \text{in} | \hat{S} \tag{11.2}$$

so that we have

$$S_{\beta\alpha} = \langle \beta, \text{in} | \hat{S} | \alpha, \text{in} \rangle \ . \tag{11.3}$$

$\hat{S}$ is a unitary operator. This can be seen by taking the hermitean conjugate of (11.2) and writing

$$\langle \beta, \text{in} | \hat{S}\hat{S}^\dagger | \alpha, \text{in} \rangle = \langle \beta, \text{out} | \alpha, \text{out} \rangle = \delta_{\alpha,\beta} \ . \tag{11.4}$$

Thus we have

$$\hat{S}\hat{S}^\dagger = \hat{S}^\dagger\hat{S} = 1. \tag{11.5}$$

The field operators transform in the standard way under the unitary transformation S

$$\phi_{\text{out}} = \hat{S}^\dagger\hat{\phi}_{\text{in}}\hat{S}\ . \tag{11.6}$$

In Sect. 2.2 we have expressed the matrix element $S_{\beta\alpha}$ in terms of wavefunctions. There we showed that S (see (2.34)) could be written as

$$S_{\beta\alpha} = \int \Psi_\beta^{(-)*}(\vec{x}, t \to +\infty)\Psi_\alpha^{(+)}(\vec{x}, t \to +\infty)\,\mathrm{d}^3x\ . \tag{11.7}$$

Here $\Psi^{(+)}$ was a wavefunction that fulfilled an "in" boundary condition, i.e. it evolved forward in time, starting from an incoming free plane wave at $t \to -\infty$. $\Psi^{(-)}$, on the other hand, was a wavefunction with an "out" condition which describes a free plane wave for $t \to +\infty$. We thus have

$$\begin{aligned} \Psi_\alpha^{(+)}(\vec{x}, t \to -\infty) &= \phi_{\text{in},\alpha}(\vec{x}, t) \\ \Psi_\alpha^{(-)}(\vec{x}, t \to +\infty) &= \phi_{\text{out},\alpha}(\vec{x}, t)\ , \end{aligned} \tag{11.8}$$

where ϕ_{in} and ϕ_{out} describe the free asymptotic fields, i.e. plane waves.

11.2 Reduction Theorem

We will prove the reduction theorem here using the methods of canonical field quantization. We, therefore, first summarize the basic principles of this method in the next subsection. The final result, the reduction theorem, is valid in general and can therefore be used also in a path-integral based approach.

11.2.1 Canonical field quantization

Here and in the following we work in the interaction representation in which the physical states are propagated in time only by the interaction Hamiltonian and the time-development of the operators is determined by the free Hamiltonian. The field operators are obtained by quantizing the fields ϕ. This is done in the usual way by going into the normal mode expansion (see the discussion at the start of section 4.1)

$$\phi(x) = \frac{1}{\sqrt{V}}\sum_{\vec{k}}\frac{1}{\sqrt{2\omega_k}}\left(a_{\vec{k}}(t)\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} + a_{\vec{k}}^\dagger(t)\,\mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}}\right) \tag{11.9}$$

with $\omega_k = \sqrt{\vec{k}^2 + m^2}$. The inverse Fourier transformation is given by

$$\begin{aligned} a^\dagger_{\vec{k}}(t) &= \frac{-\mathrm{i}}{\sqrt{2\omega_k V}} \int_V \mathrm{d}^3x \left(\mathrm{e}^{+\mathrm{i}(\vec{k}\cdot\vec{x}-\omega t)} \partial_t \phi(\vec{x},t) + \mathrm{i}\omega_k \mathrm{e}^{+\mathrm{i}(\vec{k}\cdot\vec{x}-\omega t)} \phi(\vec{x},t) \right) \\ a_{\vec{k}}(t) &= \frac{\mathrm{i}}{\sqrt{2\omega_k V}} \int_V \mathrm{d}^3x \left(\mathrm{e}^{-\mathrm{i}(\vec{k}\cdot\vec{x}-\omega t)} \partial_t \phi(\vec{x},t) - \mathrm{i}\omega_k \mathrm{e}^{-\mathrm{i}(\vec{k}\cdot\vec{x}-\omega t)} \phi(\vec{x},t) \right) . \end{aligned} \tag{11.10}$$

We next impose on the numbers $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ the quantization conditions

$$\begin{aligned} \left[a_{\vec{k}}(t), a^\dagger_{\vec{k}'}(t) \right] &= \delta_{\vec{k},\vec{k}'} \\ \left[a_{\vec{k}}(t), a_{\vec{k}'}(t) \right] &= \left[a^\dagger_{\vec{k}}(t), a^\dagger_{\vec{k}'}(t) \right] = 0 . \end{aligned} \tag{11.11}$$

These are the same as the ones used in the well known algebraic treatment of the harmonic oscillators for the normal field modes. The time-dependence of these operators is obviously harmonic, because we are working in the interaction representation

$$\begin{aligned} a(t) &= a(0)\mathrm{e}^{+\mathrm{i}\omega t} \\ a^\dagger(t) &= a^\dagger(0)\mathrm{e}^{-i\omega t} . \end{aligned} \tag{11.12}$$

Using the commutator relations (11.11) we can obtain also the commutator relations for the fields themselves. These are

$$\begin{aligned} \left[\dot{\phi}(\vec{x},t), \phi(\vec{x}',t) \right] &= -\mathrm{i}\delta^3(\vec{x}-\vec{x}') \\ \left[\dot{\phi}(\vec{x},t), \dot{\phi}(\vec{x}',t) \right] &= 0 \\ \left[\phi(\vec{x},t), \phi(\vec{x}',t) \right] &= 0 . \end{aligned} \tag{11.13}$$

Noting that $\dot{\phi} = \Pi$ (see Appendix. B), where Π is the canonical field momentum, we see that these are just the canonical commutator relations of quantum mechanics for space and momentum coordinates.

11.2.2 Derivation of the reduction theorem

The algebra (11.9) produces field quanta; the $a^\dagger$ are the creation and the a the annihilation operators. We can use this fact to write for the S matrix element (11.1)

$$S_{\beta\alpha} = \langle \beta, \mathrm{out} | \alpha, \mathrm{in} \rangle = \langle \beta, \mathrm{out} | a^\dagger_{\mathrm{in}}(k) | \alpha - k, \mathrm{in} \rangle , \tag{11.14}$$

where we have assumed that the in-state $|\alpha, \mathrm{in}\rangle$ contained a free particle with momentum k ; $|\alpha - k, \mathrm{in}\rangle$ is then that in state in which just this particle is missing. We can further write this as

$$\begin{aligned} \langle\beta, \mathrm{out}|\alpha, \mathrm{in}\rangle &= \langle\beta, \mathrm{out}|a^\dagger_{\mathrm{out}}(k)|\alpha - k, \mathrm{in}\rangle \\ &\quad + \langle\beta, \mathrm{out}|a^\dagger_{\mathrm{in}}(k) - a^\dagger_{\mathrm{out}}(k)|\alpha - k, \mathrm{in}\rangle \ . \end{aligned} \tag{11.15}$$

In the first term $\langle\beta, \mathrm{out}|a^\dagger_{\mathrm{out}}(k) = \langle\beta - k, \mathrm{out}| = 0$, if $\langle\beta, \mathrm{out}|$ does not contain a particle with momentum k.

We now rewrite the creation operator into a more compact form

$$a^\dagger_{\vec{k}} = \frac{-\mathrm{i}}{\sqrt{2\omega_k V}} \int \mathrm{d}^3x \, \mathrm{e}^{+\mathrm{i}(\vec{k}\cdot\vec{x} - \omega t)} \overset{\leftrightarrow}{\partial}_t \ \phi(\vec{x}, t) \tag{11.16}$$

with

$$f(t) \overset{\leftrightarrow}{\partial}_t \ \phi(\vec{x}, t) = f\frac{\partial\phi}{\partial t} - \frac{\partial f}{\partial t}\phi \ . \tag{11.17}$$

With this expression we can get the S-matrix element into the form

$$\begin{aligned} &\langle\beta, \mathrm{out}\,|\alpha, \mathrm{in}\rangle \\ &= \langle\beta - k, \mathrm{out}|\alpha - k, \mathrm{in}\rangle \\ &\quad - \frac{\mathrm{i}}{\sqrt{2\omega_k V}} \langle\beta, \mathrm{out}| \int \mathrm{d}^3x \, \mathrm{e}^{+\mathrm{i}(\vec{k}\cdot\vec{x} - \omega t)} \overset{\leftrightarrow}{\partial}_t \left[\phi_{\mathrm{in}}(x) - \phi_{\mathrm{out}}(x)\right] |\alpha - k, \mathrm{in}\rangle \ . \end{aligned} \tag{11.18}$$

For only 2 particles in the *in* and *out* states the first term represents a single particle transition matrixelement. In this case it can obviously only contribute if both particles do not change their energy and momentum, i.e. if $|\mathrm{out}\rangle$ and $|\mathrm{in}\rangle$ are identical. It is then just the forward scattering amplitude.

The rhs of (11.18) is time-independent. We can see this by calculating the time derivative of its integrand

$$\partial_t \left(f(x) \overset{\leftrightarrow}{\partial}_t \phi_{\mathrm{in}}(x)\right) = f\partial_t^2\phi_{\mathrm{in}} - \left(\partial_t^2 f\right)\phi_{\mathrm{in}} \ . \tag{11.19}$$

Here $f(x) = \mathrm{e}^{-\mathrm{i}kx}$ and ϕ_{in} both solve the free Klein-Gordon equation. We therefore have

$$\partial_t \left(f(x) \overset{\leftrightarrow}{\partial}_t \ \phi_{\mathrm{in}}(x)\right) = f\vec{\nabla}^2\,\phi_{\mathrm{in}} - (\vec{\nabla}^2\,f)\phi_{\mathrm{in}} \ . \tag{11.20}$$

The integral over this expression vanishes after twofold partial integration of one of the terms. The same holds, of course, for the term involving ϕ_{out} in (11.18).

The rhs of (11.18) is thus indeed time-independent. We can, therefore, take it at any time and in particular also at $t \to \pm\infty$. Then we can replace the *in* and *out* fields at these times by the limits of the field $\phi(x)$ (see (11.8)). This gives for the S-matrix element

$$\begin{aligned}\langle\beta,\mathrm{out}|\alpha,\mathrm{in}\rangle &= \langle\beta-k,\mathrm{out}|\alpha-k,\mathrm{in}\rangle \\ &+ \lim_{t\to+\infty}\frac{\mathrm{i}}{\sqrt{2\omega_k V}}\langle\beta,\mathrm{out}|\int \mathrm{d}^3x\,\mathrm{e}^{-ikx}\ \overleftrightarrow{\partial}_t\ \phi(x)|\alpha-k,\mathrm{in}\rangle \\ &- \lim_{t\to-\infty}\frac{\mathrm{i}}{\sqrt{2\omega_k V}}\langle\beta,\mathrm{out}|\int \mathrm{d}^3x\,\mathrm{e}^{-ikx}\ \overleftrightarrow{\partial}_t\ \phi(x)|\alpha-k,\mathrm{in}\rangle\ . \end{aligned}\tag{11.21}$$

We now write this expression in a covariant form by using

$$\begin{aligned}&\left(\lim_{t\to+\infty}-\lim_{t\to-\infty}\right)\int \mathrm{d}^3x\,\left(f(x)\ \overleftrightarrow{\partial}_t\ \phi\right) \\ &= \int_{-\infty}^{+\infty}\mathrm{d}t\int \mathrm{d}^3x\,\frac{\partial}{\partial t}\left(f(x)\ \overleftrightarrow{\partial}_t\ \phi\right) \\ &= \int \mathrm{d}^4x\,\left[f\partial_t^2\phi-\left(\partial_t^2 f\right)\phi\right] \\ &= \int \mathrm{d}^4x\,\left[f\partial_t^2-\left((\vec{\nabla}^2-m^2)f\right)\right]\phi\ . \end{aligned}\tag{11.22}$$

Note that here ϕ is an interacting field, since we integrate now over all times. Thus, in contrast to (11.19) this integral does not vanish. Twofold partial integration in the second term on the rhs allows us now to roll the Laplace operator from f over to ϕ. This gives

$$\int \mathrm{d}^4x\,\left(f\partial_t^2\,\phi-(\partial_t^2 f)\phi\right)=\int \mathrm{d}^4x\,f(x)(\Box+m^2)\,\phi(x)\ .\tag{11.23}$$

We thus have

$$\begin{aligned}&\langle\beta,\mathrm{out}|\alpha,\mathrm{in}\rangle = \\ &\qquad\langle\beta-k,\mathrm{out}|\alpha-k,\mathrm{in}\rangle \\ &\qquad+\frac{\mathrm{i}}{\sqrt{2\omega_k V}}\langle\beta,\mathrm{out}|\int \mathrm{d}^4x\,\mathrm{e}^{-ikx}(\Box+m^2)\,\phi(x)|\alpha-k,\mathrm{in}\rangle \\ &= \langle\beta-k,\mathrm{out}|\alpha-k,\mathrm{in}\rangle \\ &\qquad+\frac{\mathrm{i}}{\sqrt{2V\omega_k}}\int \mathrm{d}^4x\,\mathrm{e}^{-\mathrm{i}kx}(\Box+m^2)\langle\beta,\mathrm{out}|\phi(x)|\alpha-k,\mathrm{in}\rangle\ . \end{aligned}\tag{11.24}$$

In this expression we have removed one particle from the in state.

We now continue by removing one particle with the momentum k' from the out state by going through exactly the same steps as before. Disregarding the first term in (11.24), that contributes only to forward scattering, we get

$$\begin{aligned}\langle\beta,\mathrm{out}|\phi(x)|\alpha-k,\mathrm{in}\rangle &= \langle\beta-k',\mathrm{out}|a_{\mathrm{out}}(k')\phi(x)|\alpha-k,\mathrm{in}\rangle \\ &= \langle\beta-k',\mathrm{out}|\phi(x)a_{\mathrm{in}}(k')|\alpha-k,\mathrm{in}\rangle \\ &\quad+\langle\beta-k',\mathrm{out}|a_{\mathrm{out}}(k')\phi(x)-\phi(x)a_{\mathrm{in}}(k')|\alpha-k,\mathrm{in}\rangle\ . \end{aligned}\tag{11.25}$$

We next replace again the annihilation operators by the corresponding field operators as in (11.16) (here we have to take the hermitian conjugate operator) and obtain

$$\begin{aligned}
&\langle \beta - k', \mathrm{out}|\phi(x)|\alpha - k - k', \mathrm{in}\rangle \\
&+ \frac{\mathrm{i}}{\sqrt{2\omega_{k'}V}}\langle \beta - k', \mathrm{out}| \int \mathrm{d}^3x' \left\{ \mathrm{e}^{\mathrm{i}k'x'} \overleftrightarrow{\partial}_{t'} \left[\phi_{\mathrm{out}}(x')\phi(x) \right.\right. \\
&\qquad \left.\left. - \phi(x)\phi_{\mathrm{in}}(x')\right] \right\} |\alpha - k, \mathrm{in}\rangle \ .
\end{aligned} \tag{11.26}$$

Taking now the limits $t' \to \pm\infty$ gives, as above, for this expression

$$\begin{aligned}
& \langle \beta - k', \mathrm{out}|\phi(x)|\alpha - k - k', \mathrm{in}\rangle \\
+ & \frac{\mathrm{i}}{\sqrt{2\omega_{k'}V}}\langle \beta - k', \mathrm{out}| \int \mathrm{d}^4x' \frac{\partial}{\partial t'} \left(\mathrm{e}^{\mathrm{i}k'x'} \overleftrightarrow{\partial}_{t'} T\left[\phi(x')\phi(x)\right] \right) |\alpha - k, \mathrm{in}\rangle \\
= & \ \ldots \\
& + \frac{\mathrm{i}}{\sqrt{2\omega_{k'}V}}\langle \beta - k', \mathrm{out}| \int \mathrm{d}^4x' \left(\mathrm{e}^{\mathrm{i}k'x'} \partial_{t'}^2 \, T\left[\phi(x')\phi(x)\right] \right. \\
& \qquad \left. - \left(\partial_{t'}^2 \, \mathrm{e}^{-\mathrm{i}k'x'}\right) T\left[\phi(x')\phi(x)\right] \right) |\alpha - k, \mathrm{in}\rangle \ .
\end{aligned} \tag{11.27}$$

We now use again (11.23) and obtain

$$\begin{aligned}
= & \ \ldots \\
& \frac{\mathrm{i}}{\sqrt{2\omega_{k'}V}} \\
& \times \langle \beta - k', \mathrm{out}| \int \mathrm{d}^4x' \, \mathrm{e}^{\mathrm{i}k'x'} (\Box' + m^2) T\left[\phi(x')\phi(x)\right] |\alpha - k, \mathrm{in}\rangle \ .
\end{aligned} \tag{11.28}$$

Combining this result with (11.24) finally gives (neglecting the forward scattering amplitude)

$$\begin{aligned}
& \langle \beta, \mathrm{out}|\alpha, \mathrm{in}\rangle \\
= & \ \frac{\mathrm{i}}{\sqrt{2\omega_k V}} \frac{\mathrm{i}}{\sqrt{2\omega_{k'}V}} \int \mathrm{d}^4x \, \mathrm{d}^4x' \, \mathrm{e}^{\mathrm{i}k'x'} \, \mathrm{e}^{-\mathrm{i}kx} \\
& \times \langle \beta - k', \mathrm{out}|(\Box' + m^2)(\Box + m^2) T\left[\phi(x')\phi(x)\right] |\alpha - k, \mathrm{in}\rangle \\
= & \ \mathrm{i}^2 \int \mathrm{d}^4x \, \mathrm{d}^4x' \, f_{k'}^*(x') f_k(x) (\Box' + m^2)(\Box + m^2) \\
& \qquad \times \langle \beta - k', \mathrm{out}| T\left[\phi(x')\phi(x)\right] |\alpha - k, \mathrm{in}\rangle
\end{aligned} \tag{11.29}$$

with

$$f_k(x) = \frac{1}{\sqrt{2\omega_k V}} \, \mathrm{e}^{-\mathrm{i}kx}.$$

This reduction can obviously be continued on both sides until we have (with n particles in the out state and m particles in the in state)

$$\begin{aligned} S_{\beta\alpha} &= \langle\beta,\mathrm{out}|\alpha,\mathrm{in}\rangle \qquad (11.30) \\ &= \mathrm{i}^{m+n}\int\prod_{i=1}^{m}\mathrm{d}^4x_i\int\prod_{j=1}^{n}\mathrm{d}^4x'_j\, f^*_{k'_j}(x'_j)f_{k_i}(x_i) \\ &(\Box'_j+m^2)(\Box_i+m^2)\langle 0|T\left[\phi(x'_1)\phi(x'_2)\ldots\phi(x'_n)\phi(x_1)\phi(x_2)\ldots\phi(x_m)\right]|0\rangle\ . \end{aligned}$$

This is the so-called *Reduction Theorem* that enables us to express the S-matrix in terms of the $(n+m)$-point Green's function.

The action of the Klein-Gordon operator on the Green's function is

$$(\Box_i+m^2)G(x_1,x_2,\ldots,x_i,\ldots,x_n) = -\delta^4(x_i-x_v)\ , \qquad (11.31)$$

if the point x_i lies on an external line. This is so because in this case the full Green's function contains the free two-point function from the point x_i in to the vertex x_v as a factor (see, for example, sections 6.2 and 6.3). The integration over x_i in (11.30) can then be performed. In effect, each external-line two-point function is thus replaced by a free particle wavefunction $f_k(x)$.

The connection between the S-matrix and the Green's function as expressed by the reduction theorem has been derived here only for scalar fields, but it is valid in general. The only formal difference is that the Klein-Gordon operator $\Box+m^2$ has to be replaced by the corresponding free-field operator. Furthermore, in the case of fermions the time-ordered product always contains an equal number of operators ψ and $\bar{\psi}$. It is also important to note that the reduction theorem has been derived here starting from the method of canonical quantization. However, even for fields where this method runs into difficulties, like, e.g., the gauge fields to be treated in the next chapter, the reduction theorem holds in the form given above.

Note that in the reduction theorem (11.30) the information about the interaction of the particles is contained in the (interacting) field operators ϕ. The n-point function appearing there is therefore that of the interacting theory! Using the techniques developed in Sect. 7 we can then expand this interacting Green's function in terms of the free Green's functions and in terms of powers of the interaction. With the help of Wick's theorem the free n-point functions can then all be expressed as combinations of the much simpler, analytically known 2-point functions.

11.3 Functional form of the scattering operator

In this section we will show that the scattering operator $\hat{S}$ can also be written in a functional form as

$$\hat{S} =: \mathrm{e}^{\mathrm{i}\int \hat{\phi}_{\mathrm{in}}(x)(\Box+m^2)\frac{\delta}{\delta J(x)}\mathrm{d}^4x} : Z[J]\big|_{J=0} \tag{11.32}$$

Here the : : symbol denotes the so-called normal product and the operator $(\Box + m^2)\frac{\delta}{\delta J(x)}$ acts only on Z[J]. The normal-ordered product is defined in such a way that all operators in it are reordered so that all the annihilation operators are moved to the right. This reordering takes place without a sign change for boson fields and with a sign-change for each pairwise exchange for fermion fields. Using (4.17) equation (11.32) can also be written as

$$\hat{S} =: \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, \hat{\phi}_{\mathrm{in}}(x)(\Box+m^2)\frac{\delta}{\delta J(x)}} : \langle 0|T\, \mathrm{e}^{\mathrm{i}\int J(x)\hat{\phi}(x)\,\mathrm{d}^4x}|0\rangle\, \big|_{J=0} \ . \tag{11.33}$$

The matrix elements of the operator (11.32) indeed agree with (11.30). This can be seen by considering again a matrixelement with n particles in the *out* states and m particles in the *in* state. In the expansion of the exponential in (11.32) only that term can contribute that contains exactly $m+n$ powers of the fields. We thus have

$$\begin{aligned}\langle n, \mathrm{out}|\hat{S}|m, \mathrm{in}\rangle &= \mathrm{i}^{m+n} \int \prod_{i=1}^{m} \mathrm{d}^4x_i \int \prod_{j=m+1}^{m+n} \mathrm{d}^4x_j \\ \times\ & \frac{1}{(m+n)!}\langle n| : \phi_{\mathrm{in}}(x_1)\ldots\phi_{\mathrm{in}}(x_{n+m}) : |m\rangle \\ \times\ & (\Box_1 + m^2)(\Box_2 + m^2)\ldots(\Box_{m+n} + m^2)\,(\mathrm{i})^{m+n}\, G(x_1, x_2, \ldots, x_{m+n}) \ .\end{aligned} \tag{11.34}$$

Here the *in* field operator $\phi_{\mathrm{in}}(x)$ can simply be replaced by operators of the free field. The normal product reorders the expansion (11.34) such that all the annihilation operators are on the right. Since each field contains two independent sums over positive and negative energy eigenstates, respectively, we have in total 2^{m+n} operator products; of these only the term with n creation operators and m annihilation operators can contribute. This gives with the expansion (11.9)

$$\begin{aligned}&\langle n| : \phi_{\mathrm{in}}(x_1)\ldots\phi_{\mathrm{in}}(x_{m+n}) : |m\rangle \\ &= \frac{(m+n)!}{m!\,n!}\langle n|\prod_{i=1}^{m} f_{k_i}(x_i)a_i^\dagger \prod_{j=m+1}^{m+n} f^*_{k_j}(x_j)a_j|m\rangle \ .\end{aligned} \tag{11.35}$$

The degeneracy factor in front of the matrix element follows from the binomial expansion of the individual terms in the normal mode expansion $(f_k a_k + f_k^* a_k^\dagger)^{m+n}$. Integration over the x_i and x_j just gives an extra degeneracy factor $m!n!$. Taking this result together with (11.34) is the same as (11.30).

Chapter 12

PATH INTEGRALS FOR GAUGE FIELDS

Special problems arise when gauge theories are quantized. The reason is quite easy to see in the framework of path integrals. If we consider a field A_μ, the path integral would be naively written down as

$$\int \mathcal{D}A\, \mathrm{e}^{\mathrm{i}S[A_\mu]} \tag{12.1}$$

where the action S is a functional of A_μ. The integration measure stands for a straightforward generalization of the measure used for scalar fields

$$\mathcal{D}A \sim \prod_{\vec{x}} \prod_{j=1}^{n} \prod_{\mu=1}^{4} \mathrm{d}A_\mu(\vec{x}, t_j) \tag{12.2}$$

in the limit $n \to \infty$ (cf. (4.11)); the measure so defined is gauge invariant. S must be invariant under gauge transformations since these cannot change any of the physics. This implies that for a fixed field $\bar{A}_\mu$ there are infinitely many other fields, connected to $\bar{A}_\mu$ by gauge transformations, that all give the same contribution to S. Thus the path integral as written down here cannot converge.

While this problem of the gauge-equivalent potentials appears for both Abelian and non-Abelian gauge field theories, there is another difficulty that is specific for non-Abelian theories. The path integral in the form given above, in which an integration only over the fields appears, was shown in chapter 1 to be valid only for quadratic Hamiltonians with constant coefficients in front of the kinetic term. This, however, is no longer the case for a non-Abelian gauge theory, where the field tensor F depends not only on the derivatives of the fields, but also on the fields themselves. Effectively this leads to a kinetic energy term in which the coefficient of the momentum-dependent terms depends on the fields themselves. When considering the nonrelativistic analogon of this case in section 1.3.2 we have started from the Hamiltonian formulation of the path integral. There we have found that a new factor involving the squareroot of the coordinate dependent coefficient appears under the path integral (see (1.48)). It is therefore natural that in the treatment of non-Abelian gauge field theories we would also have to start from the Hamiltonian representation of the path integral (1.35) (see [Lee] for a detailed treatment). At the expense of some mathematical rigor, however, the concept of a path integral over the fields only can be maintained and this is what we are going to do in this chapter.

12.1 Quantum Electrodynamics

We start our considerations with pointing out the difficulties that arise already when we want to apply our developments to the case of Abelian gauge theories like Quantum Electrodynamics (QED). These difficulties arise because the gauge conditions reduce the number of degrees of freedom; the caveats concerning the integration over the field momenta do not apply here because the kinetic energy is still quadratic in the field momenta. The difficulties become clear when we try to apply the Gaussian integration rules to path integrals over the electromagnetic field.

For the electromagnetic field the free Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \tag{12.3}$$

and is clearly invariant under a gauge transformation

$$A_\mu \longrightarrow A'_\mu = A_\mu + \partial_\mu \Lambda \ . \tag{12.4}$$

Here $\Lambda = \Lambda(x)$ is an arbitrary differentiable function of space-time. The corresponding equation of motion for the fields A_μ is

$$\begin{aligned} \partial_\nu F^{\nu\mu} = \partial_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu) &= (\delta^\mu_\nu \Box - \partial_\nu \partial^\mu) A^\nu = 0 \\ \longrightarrow (g_{\mu\nu}\Box - \partial_\mu \partial_\nu) A^\nu &= 0 \ , \end{aligned} \tag{12.5}$$

which is also gauge invariant. If we now fix a gauge, e.g. the Lorentz gauge $\partial_\mu A^\mu = 0$, we obtain the free wave equation for A_μ

$$\Box A_\mu = 0 \ , \tag{12.6}$$

which is no longer gauge invariant.

We now proceed to obtain the propagator for the electromagnetic field. As mentioned at the end of section 9.2.2 the propagator of the theory could be read off from the term in the Lagrangian that is quadratic in the field. The Lagrangian (12.3) can indeed also be converted into a form quadratic in the fields by writing

$$\begin{aligned}\int \mathcal{L}\,\mathrm{d}^4x &= -\frac{1}{4}\int \mathrm{d}^4x\,(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \\ &= -\frac{1}{4}\int \mathrm{d}^4x\,[\,(\partial^\mu A^\nu)(\partial_\mu A_\nu) - (\partial^\mu A^\nu)(\partial_\nu A_\mu) \\ &\qquad - (\partial^\nu A^\mu)(\partial_\mu A_\nu) + (\partial^\nu A^\mu)(\partial_\nu A_\mu)]\ .\end{aligned} \tag{12.7}$$

By partial integration we obtain

$$\begin{aligned}\int \mathcal{L}\,\mathrm{d}^4x &= \frac{1}{4}\int \mathrm{d}^4x\,(A^\mu \Box A_\mu - 2A^\mu \partial_\nu \partial_\mu A^\nu + A^\mu \Box A_\mu) \\ &= \frac{1}{2}\int \mathrm{d}^4x\,A^\mu\,(g_{\mu\nu}\Box - \partial_\mu\partial_\nu)\,A^\nu\ .\end{aligned} \tag{12.8}$$

According to our normal procedure (cf. (9.83)) we thus expect to obtain the propagator of the electromagnetic field as

$$D^{-1}_{\mu\nu} \sim g_{\mu\nu}\Box - \partial_\mu\partial_\nu\ . \tag{12.9}$$

The problem arises when we realize that the operator D does not exist. We see this by applying its inverse (12.9) to an arbitrary four-gradient $\partial^\nu G$

$$D^{-1}_{\mu\nu}\,\partial^\nu G = (\Box\partial_\mu - \partial_\mu\Box)\,G = 0\ . \tag{12.10}$$

Thus, D^{-1} has a zero eigenvalue and, in a loose way of speaking, its inverse D is infinite. This infinity is linked to that in the path integral over gauge fields discussed at the start of this chapter. This can be seen by a straightforward application of (5.12) which gives the equivalent of (5.25) with $\det(A) = 0$ in the denominator.[1]

The physical origin of this infinity is caused by the integration over gauge-equivalent potentials, as we have discussed at the start of this chapter. Therefore, we expect that one way to overcome this difficulty is to fix a gauge, for example by imposing the Lorentz gauge condition

$$\partial^\nu A_\nu = 0\ . \tag{12.11}$$

If we then consider only potentials that fulfill this condition and integrate only over them, we expect the path integral to be well-behaved.

[1] Remember here that the determinant of an operator is given by a product of its eigenvalues.

This expectation is based on the fact that we can find a Lagrangian that incorporates the gauge condition (12.11) at the price of loosing its manifest gauge invariance

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\lambda}\left(\partial_\mu A^\mu\right)^2 \ . \tag{12.12}$$

This Lagrangian, that leads directly to the wave equation (12.6) as we will show below, contains a so-called gauge-fixing term as a quadratic constraint, coupled in by means of a Lagrange-multiplier ($\frac{1}{2\lambda}$). This additional term could be considered as a potential that becomes the more repulsive the more the actual potential A_μ differs from the Lorentz-gauge (12.9); the constraint thus drives the system towards that gauge.

We can thus either start from a manifestly covariant Lagrangian, derive the equations of motion from it and then impose a suitable gauge condition or we can, alternatively, incorporate the gauge condition into the Lagrangian, at the price of giving up its gauge invariance, and then obtain the gauge-fixed equations of motion directly from it. The physics must obviously be the same in both methods.

Performing the steps leading to (12.8) once again, we find that (12.12) is equivalent to the Lagrangian

$$\mathcal{L} = \frac{1}{2}A^\mu\left(g_{\mu\nu}\Box - \left(1 - \frac{1}{\lambda}\right)\partial_\mu\partial_\nu\right)A^\nu \tag{12.13}$$

which leads to the same action. The equations of motion following from (12.13) are

$$\left(g_{\mu\nu}\Box - \left(1 - \frac{1}{\lambda}\right)\partial_\mu\partial_\nu\right)A^\nu = 0 \ . \tag{12.14}$$

Here A^ν is still a field unconstrained by gauge conditions (i.e. it does not fulfill $\partial_\mu A^\mu = 0$!).

Reading again – as at the end of Chap. 9 on p. 113 – the inverse propagator off from the Lagrangian (12.13) gives

$$D^{-1}_{\mu\nu} = g_{\mu\nu}\Box - \left(1 - \frac{1}{\lambda}\right)\partial_\mu\partial_\nu \tag{12.15}$$

with the momentum space representation

$$D^{-1}_{\mu\nu}(k) = -g_{\mu\nu}k^2 + \left(1 - \frac{1}{\lambda}\right)k_\mu k_\nu \ . \tag{12.16}$$

This operator no longer has a zero eigenvalue as we can easily see by performing again the steps in (12.10). The propagator D, defined by $D^{-1}_{\mu\nu}D^{\nu\lambda} = \delta^\lambda_\mu$, thus exists, in momentum space it is given by[2]

$$D_{\mu\nu}(k) = -\frac{1}{k^2}\left(g_{\mu\nu} + (\lambda - 1)\frac{k_\mu k_\nu}{k^2}\right) . \tag{12.17}$$

This is the propagator for the electromagnetic field. The generating functional can then be obtained as usual (cf. (5.12))

$$Z[J] = \mathrm{e}^{-\frac{\mathrm{i}}{2}\int J^\mu(x) D_{\mu\nu}(x-y) J^\nu(y)\, \mathrm{d}^4x \mathrm{d}^4y} , \tag{12.18}$$

and the two-point function is, as usual (see (9.82), (9.83)), found to be

$$G_{\mu\nu}(x,y) = \mathrm{i}D_{\mu\nu}(x-y) \tag{12.19}$$

The constant λ in (12.17) is arbitrary, but note that the propagator does not exist for $\lambda \to \infty$; this is just the case of the unconstrained Lagrangian discussed earlier in this section (cf. (12.3) and (12.12)). Choosing $\lambda = 1$ gives the so-called *Feynman gauge*, whereas $\lambda = 0$ describes the *Landau-gauge*. The Feynman gauge is the one that we are used to from classical electrodynamics; in it the equations of motion for the electromagnetic field (12.14) read

$$\Box A_\nu = 0 . \tag{12.20}$$

This is the equation of motion that we obtained earlier for the Lorentz gauge.

Now that we have constructed a propagator for the electromagnetic field, we can write down the generating functional as

$$Z[J] = \int \mathcal{D}A\, \mathrm{e}^{\mathrm{i}\int(\mathcal{L} + \mathcal{L}_{GF} + J_\mu A^\mu)\mathrm{d}^4x} \tag{12.21}$$

where $\mathcal{D}A = \prod_{\mu=1}^4 \mathcal{D}A_\mu$ and $\mathcal{L}_{GF}$ is the gauge-fixing term

$$\mathcal{L}_{GF} = -1/(2\lambda)(\partial_\mu A^\mu)^2 . \tag{12.22}$$

We will see a little later, in section 12.2.2, that this functional can actually be rewritten into a form that exhibits explicitly the integration over the gauge degree of freedom.

The path integral (12.21) does not suffer from the problems mentioned at the start of this section. Now the matrix in the quadratic term of $\mathcal{L}$ (12.13) can be inverted and the propagator exists. Furthermore, all fields that differ from a specific field just by a gauge transformation that leads out of the covariant Lorentz gauge have a higher action and thus contribute less to the

[2]The denominator k^2 from here on should really read $k^2 - i\varepsilon$ in order to ensure the proper boundary conditions for the propagator.

path integral. The action itself is gauge-independent, whereas the integral over the gauge-fixing term and the source terms do depend on the gauge. Thus, the generating functional itself is gauge dependent and, consequently, also the Green's functions obtained from it by functional derivatives. All of the gauges corresponding to the various values of λ, however, lead to the same physics because in calculating a transition amplitude $D_{\mu\nu}$ will couple to conserved currents only which fulfill $\partial_\mu j^\mu = 0 \to k_\mu j^\mu(k) = 0$. As a consequence the second, λ-dependent term in the parentheses in (12.17) always drops out and the physically observable transition amplitudes are all gauge independent.

12.2 Path Integrals for Non-Abelian Gauge Fields

After giving a short summary of the relevant formulas for non-Abelian gauge-field theories we formulate in this section the path integral for such theories and obtain from the generating functional the Green's functions and Feynman rules.

12.2.1 Non-abelian gauge fields

In non-Abelian gauge field theories one demands invariance of the Lagrangian under the transformation of the particle fields (see [MOSEL], chapter 7)

$$\psi(x) \to \psi'(x) = U(x)\psi(x) \ . \tag{12.23}$$

The particle fields are described by N-component spinors ψ where the components represent the internal degrees of freedom. Here U is a unitary, local gauge transformation

$$U(x) = \mathrm{e}^{-\mathrm{i}\epsilon^k(x)T^k} \tag{12.24}$$

where the T^k are generators of the Lie group $SU(N)$. They are Hermitian so that the Lie group is unitary and have vanishing trace so that the generated Lie algebra is special, i.e. all group elements are represented by $(N \times N)$ matrices that have unit determinant and form themselves a Lie algebra with the defining commutation relations

$$[T^l, T^m] = \mathrm{i}f^{lmn}T^n \qquad (l = 1, 2, \ldots N^2 - 1) \ . \tag{12.25}$$

The generators are normalized such that

$$\mathrm{tr}\left(T^l T^m\right) = \frac{1}{2}\delta^{lm} \tag{12.26}$$

in the fundamental representation, the f^{lmn} in (12.25) are the completely antisymmetric structure constants of the group. Another important matrix representation of the generators is the so-called *regular representation*

$$\left(T^j\right)^{kl} = -\mathrm{i} f^{jkl} \; . \tag{12.27}$$

Invariance under the transformation (12.23) can be achieved only if the derivative ∂_μ is replaced by the so-called covariant derivative

$$D_\mu = \partial_\mu + \mathrm{i} g T^l A^l_\mu \tag{12.28}$$

where g is the coupling constant and A^l_μ is a vector (gauge) field; the superscript l labels the internal degrees of freedom. This "minimal substitution" fixes the interaction between particle and gauge fields. The gauge transformation of these fields must then be given by

$$A_\mu \to A^U_\mu = U A_\mu U^{-1} - \frac{\mathrm{i}}{g} U \partial_\mu U^{-1} \; , \tag{12.29}$$

if the Lagrangian in the particle sector is required to be invariant. Here we have introduced the fields A_μ in (12.29) as contractions of the generators with the fields A^k_μ

$$A_\mu = T^k A^k_\mu \; . \tag{12.30}$$

The change of A under an infinitesimal gauge transformation is given by

$$\begin{aligned} \delta A^j_\mu &= -\epsilon^l f^{ljk} A^k_\mu + \frac{1}{g}\partial_\mu \epsilon^j \\ &= \mathrm{i}\epsilon^l \left(T^k A^k_\mu\right)^{jl} + \frac{1}{g}\partial_\mu \epsilon^j \\ &= \frac{1}{g}\left(D_\mu \epsilon\right)^j \end{aligned} \tag{12.31}$$

where we have used the regular representation of $SU(N)$, i.e. $\left(T^j\right)^{kl} = -\mathrm{i} f^{jkl}$.

Also for non-Abelian gauge fields the free Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4} F^l_{\mu\nu} F^{l\mu\nu} \; , \tag{12.32}$$

but now the field tensor also carries the quantum numbers of the internal gauge group and must have a more complicated structure

$$F^l_{\mu\nu} = \partial_\mu A^l_\nu - \partial_\nu A^l_\mu - g f^{lmn} A^m_\mu A^n_\nu \tag{12.33}$$

in order to be invariant under the gauge transformation (12.29). Gauge invariance also requires that the gauge fields are massless, since a mass term of the form $m^2 A \cdot A$ would clearly violate gauge invariance.

The field tensor $F^l_{\mu\nu}$ can also be contracted with the generators to form a scalar in the intrinsic space ($F_{\mu\nu} = F^l_{\mu\nu} T^l$). These scalars change in the gauge-transformation just as under any unitary transformation

$$F_{\mu\nu} \to F^U_{\mu\nu} = U F_{\mu\nu} U^{-1} \ . \tag{12.34}$$

The complete 'generic' Lagrangian of a non-Abelian gauge field theory then reads

$$\mathcal{L} = -\frac{1}{4} F^l_{\mu\nu} F^{l\,\mu\nu} + \sum_f \left[\bar{\psi}_f (\mathrm{i}\gamma^\mu D_\mu - m_f) \psi_f \right] \quad . \tag{12.35}$$

The sum over f runs over all Fermions.

Equation (12.32) together with (12.33) show that the free gauge field is selfinteracting, with terms $\sim A^3$ and $\sim A^4$ in $\mathcal{L}$. These terms are generated by the non-Abelian piece of $F_{\mu\nu}$ and are absent for an Abelian theory ($f^{lmn} = 0$). Notice also that the coupling constant g, determining the interaction strength, is the same for gauge field–gauge field and gauge field–particle interactions.

For the case of QED $U(x) = \mathrm{e}^{-\mathrm{i}\epsilon(x)}$, $T^k = 1$ and $f^{lmn} = 0$; thus U is Abelian. Indeed (12.29) then reduces to the well known gauge transformation of electrodynamics

$$A_\mu \to A'_\mu = A_\mu - \frac{1}{g} \partial_\mu \epsilon \tag{12.36}$$

and also the field tensor assumes its well known form. The case of QED is thus contained as a special case in the developments in this section.

An example for the relevance of a non-Abelian gauge field theory is provided by Quantum Chromodynamics (QCD), the theory of the strong interactions between quarks. Here the relevant internal degree of freedom of the quarks is color and the symmetry group is $SU(3)$. Another example is provided by the theory of the electroweak interaction where the gauge group is $U(1) \times SU(2)$; in this case the relevant degrees of freedom are the electrical charge ($U(1)$) and the weak isospin ($SU(2)$). Both of them will be discussed in some more detail in Chapt. 13.

12.2.2 Path integrals

We now want to develop a path integral formulation for these non-Abelian fields. We start again with the naive expression for a path integral over gauge fields

$$Z[0] = \int \mathcal{D}A \, \mathrm{e}^{\mathrm{i}S[A]} \,. \tag{12.37}$$

Here the integration measure $\mathcal{D}A$ stands for

$$\mathcal{D}A = \prod_{\vec{x}} \prod_{j} \prod_{l=1}^{N^2-1} \prod_{\mu=1}^{4} \mathrm{d}\, A^l_\mu(\vec{x}, t_j) \tag{12.38}$$

(cf. (4.11)). This integral, of course, diverges as in the case of the Abelian theory because, as discussed in the last section, the path integral runs over all field configurations, i.e. both over essentially distinct ones and those that differ only by gauge transformations from each other.

In the following we will now develop a method to split this path integral up into two factors, one containing the divergent part and the other one being convergent. The divergent part will be seen to be an infinite constant that drops out when we work with the normalized functional. The divergence is here – as in the Abelian case – connected with the presence of zero eigenvalues of the matrix in the quadratic term of the gauge field action. It is just those zero eigenvalues that we have to factor out by integrating only over those fields that correspond to nonzero eigenvalues.

We start by dividing up the total configuration space of all possible $A_\mu(x)$ into equivalence classes. Each of these equivalence classes contains all the possible fields A^U_μ that can be obtained from an initial field $A_\mu(x)$ by gauge transformations. The integrand of (12.37) is constant along the fields A^U_μ, i.e. in an equivalence class. This means that the path integral (12.37) is proportional to an infinite constant which is just the "volume" of the gauge group U. The physically relevant part of the path integral then just runs over the essentially distinct fields $A_\mu(x)$. This is the part we want to factor out.

With this aim in mind we now define a functional integration over the elements of the symmetry group $SU(N)$ and denote the integration measure by $\mathcal{D}U$; in Fig. 12.1 this integration over the gauge group corresponds to an integration along the solid lines. This integration measure is gauge invariant

$$\mathcal{D}U'' = \mathcal{D}(U'U) = \mathcal{D}U \tag{12.39}$$

because of the group-property of the transformations U; this can also be seen by noting that we integrate over *all* gauge transformations U. Because of the group character of $SU(N)$ it often suffices to consider only infinitesimal gauge transformations

$$U(x) \approx 1 - \mathrm{i}\epsilon^l(x) T^l \quad \text{with} \quad \epsilon \ll 1 \,. \tag{12.40}$$

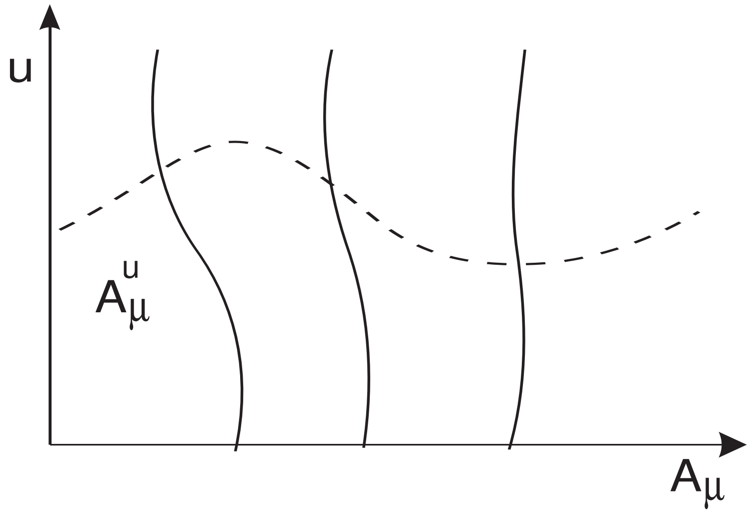


Figure 12.1: Lines of constant integrands. The horizontal axis contains the physically distinct fields, the vertical one the gauge-transformation degree of freedom. The solid lines depict the fields within a given equivalence class; the dashed line represents the gauge condition $F(A_\mu^U) = 0$.

In this case the measure can be written as

$$\mathcal{D}U = \prod_{\vec{x},j} \prod_{l=1}^{N^2-1} d\epsilon^l(\vec{x}, t_j) \tag{12.41}$$

in a discrete space-time representation. For the measure over infinitesimal transformations in (12.41) we have

$$U'' = U'U \approx (1 - \mathrm{i}\epsilon'^l T^l)(1 - \mathrm{i}\epsilon^l T^l) \approx 1 - \mathrm{i}(\epsilon'^l + \epsilon^l)T^l \tag{12.42}$$

so that the Jacobian from ϵ to $\epsilon'' = \epsilon' + \epsilon$ is

$$\det\left(\frac{\delta\epsilon^l}{\delta\epsilon''^k}\right) = 1 \ . \tag{12.43}$$

Thus the measure (12.41) is indeed gauge-invariant. To factor out an integral over $\mathcal{D}U$ is the aim in the following paragraphs.

With this integration measure we now consider the integral

$$\mathcal{M}^{-1}[A^l_\mu] = \int \mathcal{D}U \, \delta[F^l(A^{k^U}_{\ \mu})] \tag{12.44}$$

Here $F^l(A_\mu^k) = 0$ is a gauge-fixing condition, i.e. a function of the fields A_μ that vanishes for some field A_μ (for example, $F^l(A_\mu) = \partial^\mu A_\mu^l$). The superscripts l, k stand for internal degrees of freedom of the symmetry group $SU(N)$; for notational convenience, we will write them out only when necessary. The δ-functional in (12.44) is really a product of Dirac δ-functions at each space-time point $(\vec{x}, t_j)$. A^U is the gauge-transformed field. The condition $F(A_\mu^U) = 0$ then defines a surface that crosses that of each equivalence class (dashed line in Fig. 12.1); we assume that for an initial field A_μ this is the case for exactly one group element.[3]

The integral (12.44) is gauge invariant. This can be seen by writing the Jacobian as an integral and starting from a gauge-field $A^{U'}$

$$\mathcal{M}^{-1}[A_\mu^{U'}] = \int \mathcal{D}U \, \delta\left[F\left(\left(A_\mu^{U'}\right)^U\right)\right] . \tag{12.45}$$

We now change the variables from U to $U'' = UU'$ and get, because the integration measure is gauge-invariant,

$$\mathcal{M}^{-1}[A_\mu^{U'}] = \int \mathcal{D}U'' \, \delta\left[F\left(A_\mu^{U''}\right)\right] = \mathcal{M}^{-1}[A_\mu] , \tag{12.46}$$

since U'' is just an integration variable that could be renamed into U. $\mathcal{M}^{-1}$ is thus indeed gauge-invariant.

We can now transform the original path integral (12.37) by inserting the 1 from (12.44). This gives

$$Z[0] = \int \mathcal{D}A \, e^{iS[A]} = \int \mathcal{D}A \, \underbrace{\mathcal{M}[A_\mu] \int \mathcal{D}U \, \delta\left[F\left(A_\mu^U\right)\right]}_{=1} \, e^{iS[A]} \tag{12.47}$$

with the action

$$S = \int d^4x \, \mathcal{L} . \tag{12.48}$$

Since $\mathcal{M}$ and $\mathcal{L}$ are gauge invariant and $\int \mathcal{D}A = \int \mathcal{D}A^U$, we can first replace the fields A in (12.47) by A^U and then change the notation $A^U \to A$. As a consequence, the integrand no longer depends on U and we can take the integral over U out of the remaining integral

$$Z[0] = \left(\int \mathcal{D}U\right) \int \mathcal{D}A \, \mathcal{M}[A_\mu] \, \delta\left[F\left(A_\mu\right)\right] \, e^{iS} . \tag{12.49}$$

[3] As Gribov has pointed out in non-Abelian gauge field theories this is not true in some gauges, for example, in the Coulomb gauge. Thus, in these gauges one also has to remove the Gribov copies from the path integration. However, since these copies appear only at large field strengths we can neglect them in a perturbative treatment around a classical field configuration.

This functional determines our theory in the $F(A_\mu) = 0$ gauge. The integral

$$\int \mathcal{D}U \tag{12.50}$$

is just the infinite factor we wanted to pull out from the expression.

Since this factor is canceled when normalizing the functional, we can from now on work with the generating functional

$$Z[J] = \int \mathcal{D}A\, \mathcal{M}[A_\mu]\, \delta\left[F^l\left(A_\mu^k\right)\right]\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, (\mathcal{L}+J_\mu^l A^{l\,\mu})} \ . \tag{12.51}$$

This functional can be used for a perturbation theoretical treatment because it is finite. This was not possible for the original functional (12.37) which diverges because of the integration over physically equivalent gauge fields.

12.3 Faddeev-Popov Determinant

It now remains to evaluate $\mathcal{M}^{-1}[A_\mu]$. In order to understand the meaning of the integral in (12.44) better and to be able to actually evaluate it we write it for n discretized space coordinates[4]

$$\mathcal{M}_n^{-1} = \int \mathrm{d}\epsilon_1 \mathrm{d}\epsilon_2 \ldots \mathrm{d}\epsilon_n\, \delta^n[F(A_\mu^\epsilon)] \tag{12.52}$$

with $\epsilon_n = \epsilon(x_n)$. In order to evaluate this integral we now change the integration variables from $\mathrm{d}\epsilon_n$ to $\mathrm{d}F$. This gives

$$\mathrm{d}\epsilon_1 \mathrm{d}\epsilon_2 \ldots \mathrm{d}\epsilon_n \ \det M_{ij} = \mathrm{d}F(x_1)\, \mathrm{d}F(x_2) \ldots \mathrm{d}F(x_n) \ , \tag{12.53}$$

where $\det M$ is the Jacobian for this transformation of the integral measure. We thus get

$$\mathcal{M}_n^{-1} = \underbrace{\int \mathrm{d}F(x_1)\, \mathrm{d}F(x_2) \ldots \mathrm{d}F(x_n)\, \delta^n[F]}_{=1} \left(\det M_{ij}\right)^{-1} \ . \tag{12.54}$$

The integral is identically equal to 1 so that $\mathcal{M}_n$ is just the Jacobian

$$\mathcal{M}_n = \det M_{ij} = \det \left(\frac{\delta F(x_i)}{\delta \epsilon(x_j)}\right)_{F=0} \ . \tag{12.55}$$

We now generalize these equations to the continuum case. Since $\mathcal{M}[A_\mu]$ is gauge invariant we can always choose a field A_μ that fulfills the condition

[4] To facilitate the notation we drop here the group indices.

$F(A_\mu) = 0$. This allows us to replace the condition $F = 0$ for the derivative in (12.55) by the simpler condition $U = 1$. We thus have to consider only infinitesimal gauge transformations

$$U \approx 1 - \mathrm{i}\epsilon^l T^l \tag{12.56}$$

for the evaluation of the derivative. Under such an infinitesimal gauge transformation U the gauge condition $F = 0$ will change into

$$F^m(A_\mu^U(x)) = \int \mathrm{d}^4y \; M^{ml}(x,y)\varepsilon^l(y) \tag{12.57}$$

with

$$M^{ml}(x,y) = \frac{\delta F^m\left(A_\mu^U(x)\right)}{\delta\epsilon^l(y)} \,|_{\epsilon=0} \;\; . \tag{12.58}$$

We thus get for $\mathcal{M}^{-1}$ at a field close to one fulfilling the gauge condition

$$\mathcal{M}^{-1}[A_\mu] = \int \mathcal{D}U\,\delta\left[F^m\left(A_\mu^U\right)\right] = \int \mathcal{D}U\,\delta\left[\int \mathrm{d}^4y\, M^{ml}(x,y)\epsilon^l(y)\right] \;. \tag{12.59}$$

For these integrations close to $U = 1$ we can use the integration measure (12.41) and obtain (see (12.55))

$$\mathcal{M}^{-1}[A_\mu] = \left[\det M^{ml}(x,y)\right]^{-1} = \left[\det\left(\frac{\delta F^m(x)}{\delta\epsilon^l(y)}\right)_{F=0}\right]^{-1} \;. \tag{12.60}$$

The determinant here, the so-called *Faddeev-Popov (FP) determinant*, has to be calculated with respect to both the space-time indices (x, y) and the $N^2 - 1$ $SU(N)$ group indices (m, l). Note that the FP determinant depends in general on the gauge fields A_μ^k.

We can now replace $\mathcal{M}[A_\mu]$ in (12.51) by the Faddeev-Popov determinant (12.60) and obtain as the generating functional

$$Z[J] = \int \mathcal{D}A \,\det\left(M^{ml}(x,y)\right)\, \delta[F^l(A_\mu)]\, \mathrm{e}^{\mathrm{i}\int \mathcal{L}_J\, \mathrm{d}^4x} \tag{12.61}$$

with $\mathcal{L}_J = \mathcal{L} + J \cdot A$. This generating functional is the central result of this chapter. The presence of the FP-determinant in (12.61) is a typical feature of non-Abelian gauge field theories.

Abelian Theory. This can be seen by analyzing M for the special case of an Abelian gauge field theory. Indeed, in an Abelian theory, like e.g.

QED, the FP determinant is not essential, since we obtain from the definition (12.58)

$$M(x,y) = \left.\frac{\delta F\left(A_\mu^U(x)\right)}{\delta\epsilon(y)}\right|_{F=0} = \left.\frac{\delta F\left(A_\mu(x)+\frac{1}{g}\partial_\mu\epsilon(x)\right)}{\delta\epsilon(y)}\right|_{\epsilon=0} . \tag{12.62}$$

Choosing, for example, the covariant gauge $\partial_\mu A^\mu = 0$ gives

$$M(x,y) = \frac{1}{g}\frac{\Box\delta\epsilon(x)}{\delta\epsilon(y)} = \frac{1}{g}\Box\delta^4(x-y) . \tag{12.63}$$

Thus, M is independent of A_μ, and therefore its determinant can be pulled out of the path integral in (12.61), changing only the normalization. The generating functional thus reduces to

$$Z[J] = \int \mathcal{D}A\,\delta\left[F(A_\mu)\right]\mathrm{e}^{\mathrm{i}\int d^4x\,\mathcal{L}_J} . \tag{12.64}$$

In this form the functional contains the free Lagrangian without any gauge-fixing terms. It is nevertheless finite because of the presence of the δ-functional.

In non-Abelian theories, to be discussed in the following subsection, the FP determinant in general depends on the fields and thus cannot be pulled out of the path integral. The deeper reason for this difference of Abelian and non-Abelian theories lies in the different structure of the field tensors in both theories. Due to the presence of the selfinteraction term in the non-Abelian field tensor (12.33) the kinetic part of the field energy becomes field-dependent and fields and conjugate momenta no longer decouple in the Hamiltonian. This is then exactly the situation that we have encountered already in Sect. 1.3.2 in the framework of nonrelativistic quantum mechanics where a coordinate dependent term also led to an additional factor in the path integral (cf. (1.56)).

12.3.1 Explicit forms of the FP determinant

For a non-Abelian field the first term in the gauge transformation (12.29) has a more complicated form so that it will in general contribute an A_μ-dependent factor. We start by again expanding $F(A_\mu^U)$ around a field that fulfills the gauge condition, i.e. $F^m(A_\mu) = 0$ and $U \approx 1$ (cf. (12.57)). Under $A_\mu \to A_\mu^U = A_\mu + \delta A_\mu$ the condition F changes into

$$\begin{aligned} F^m(A_\mu^U) &= 0 + \int \mathrm{d}^4z\,\frac{\delta F^m\left(A_\mu(x)\right)}{\delta A_\mu^k(z)}\delta A_\mu^k(z) \\ &= \frac{1}{g}\int \mathrm{d}^4z\,\frac{\delta F^m\left(A_\mu(x)\right)}{\delta A_\mu^k(z)}\left(D_\mu^z\epsilon(z)\right)^k \end{aligned} \tag{12.65}$$

with the covariant derivative

$$D_\mu = \partial_\mu + \mathrm{i}gT^n A_\mu^n \ . \tag{12.66}$$

Here we have used (12.31). Taking now the derivative according to (12.58) we get

$$\begin{aligned} M^{ml}(x,y) &= \left.\frac{\delta F^m\left(A_\mu(x)\right)}{\delta\epsilon^l(y)}\right|_{\epsilon=0} \\ &= \left.\frac{1}{g}\int \mathrm{d}^4 z\, \frac{\delta F^m\left(A_\mu(x)\right)}{\delta A_\mu^k(z)} D_\mu^{kl}(z)\delta^4(z-y)\right|_{F=0} \ . \end{aligned} \tag{12.67}$$

This is a more general form for the FP-matrix. Its actual value depends on the gauge used.

Lorentz gauge. The Lorentz gauge is given by

$$\partial^\mu A_\mu^n = 0 \ . \tag{12.68}$$

The FP-matrix is then obtained from (12.67) as

$$M^{ml}(x,y) = \left.\frac{1}{g}\int \mathrm{d}^4 z\, \frac{\delta\left(\partial^\lambda A_\lambda^m(x)\right)}{\delta A_\mu^k(z)} D_\mu^{kl}(z)\delta^4(z-y)\right|_{F=0} \tag{12.69}$$

$$= \left. -\frac{1}{g}\int \mathrm{d}^4 z\, \delta^{mk}\partial_z^\mu \delta^4(x-z) D_\mu^{kl}(z)\delta^4(z-y)\right|_{F=0} \ .$$

By partial integration we obtain

$$\begin{aligned} M^{ml}(x,y) &= \left.\frac{1}{g}\int \mathrm{d}^4 z\, \delta^{mk}\delta^4(x-z)\partial_z^\mu D_\mu^{kl}(z)\delta^4(z-y)\right|_{F=0} \\ &= \left.\frac{1}{g}\left(\partial_x^\mu D_\mu^{ml}(x)\delta^4(x-y)\right)\right|_{F=0} \ . \end{aligned} \tag{12.70}$$

Since we have to evaluate this expression for $F^m = \partial^\mu A_\mu^m = 0$ we can interchange the order of the two derivatives and obtain

$$M^{ml}(x,y) = \frac{1}{g}\left(D_\mu^{ml}\partial_x^\mu \delta^4(x-y)\right) \ . \tag{12.71}$$

For the case of QED this expression indeed reduces to the form already found earlier (see (12.63)). This is so because in this case $\delta A_\mu = \frac{1}{g}\partial_\mu \epsilon$ in (12.65) and thus in this case instead of D_μ simply ∂_μ appears in (12.71).

Axial gauge. It is interesting to note that the FP-determinant is independent of the fields A_μ not only for Abelian gauge theories, but also in a non-Abelian theory, if the so-called axial gauge is chosen. This axial gauge is defined by the constraint

$$n \cdot A = 0 \; , \tag{12.72}$$

where n is a fixed four-vector. In this case we obtain from (12.67)

$$\begin{aligned} M^{ml}(x,y) &= \frac{1}{g}\int \mathrm{d}^4 z\, \frac{\delta\left(n^\lambda A^m_\lambda(x)\right)}{\delta A^k_\mu(z)} D^{kl}_\mu(z)\delta^4(z-y)\Big|_{F=0} \\ &= \frac{1}{g}\delta^{mk} n^\mu \int \mathrm{d}^4 z\, \delta^4(x-z)\left(\delta^{kl}\partial^z_\mu + \mathrm{i}g\,(T^n)^{kl} A^n_\mu(z)\right)\delta^4(z-y)\Big|_{F=0} \\ &= \frac{1}{g}\delta^{ml} n^\mu \partial^x_\mu \delta^4(x-y) \; , \end{aligned} \tag{12.73}$$

because $F = n^\mu A^n_\mu = 0$. Thus in this case the FP-determinant is again independent of A_μ so that it can, as in the Abelian case, be pulled out of the path integral and put into the normalization.

12.4 Gauge Fixing and Feynman Rules

For a derivation of the perturbation theory rules, i.e. the Feynman rules, the functional (12.61) is still not easy to handle because of the presence of the gauge-fixing δ-functional. As a guideline for a simplification we can use the easier example of an Abelian Theory, treated in Sect. 12.1. There we had succeeded in (12.21) to write down a meaningful generating functional that did not contain such a δ-functional, but instead a gauge-fixing term in the Lagrangian.

Taking this result as guideline we, therefore, in the following paragraphs reformulate $Z[J]$ into a more tractable form. To do this we consider somewhat more specialized gauge conditions

$$F^m(A_\mu(x)) - C^m(x) = 0 \; , \tag{12.74}$$

where $C^m(x)$ is an arbitrary function of space-time. Note that for given A, C^m and F^m this condition determines a gauge-transformation. The definition of $\mathcal{M}[A]$ changes correspondingly

$$\mathcal{M}^{-1}[A_\mu] = \int \mathcal{D}U\, \delta\left[F^m(A^U_\mu(x)) - C^m(x)\right] \; . \tag{12.75}$$

Suppose, we looked at a field for which the argument of the δ-functional with given $C^m(x)$ vanishes and then looked at this argument again, but with a different function $\tilde{C}^m(x)$, so that the argument no longer vanishes. We could then find a gauge transformation that makes the argument of the δ-functional zero again. Since the rest of the integral is invariant under gauge transformations (and we integrate anyway over *all* gauge transformations) the value of the integral does not change. Thus, the FP determinant $\mathcal{M}$ does not change under this replacement and is even independent of the specific function $C^m(x)$. Then, obviously, also

$$Z[J] = \int \mathcal{D}A\, \mathcal{M}[A_\mu]\, \delta\left[F^m(A_\mu) - C^m(x)\right]\, \mathrm{e}^{\mathrm{i}\int \mathcal{L}_J \mathrm{d}^4x} \tag{12.76}$$

(with $\mathcal{L}_J = \mathcal{L} + J \cdot A$) is independent of $C(x)$. We are thus free to change the integrand by a weighting factor

$$\mathrm{e}^{-\frac{\mathrm{i}}{2\lambda}\int [C^m(x)]^2 \mathrm{d}^4x} \tag{12.77}$$

and integrate functionally over the function C; this will only change the normalization of the integral. We then get

$$Z[J] = \int \mathcal{D}C \int \mathcal{D}A\, \mathcal{M}[A_\mu]\, \delta\left[F^m(A_\mu) - C^m(x)\right]\, \mathrm{e}^{\mathrm{i}\int \left(\mathcal{L}_J - \frac{1}{2\lambda}(C(x))^2\right) \mathrm{d}^4x}\ . \tag{12.78}$$

Now the functional integration over C can be performed since C appears in the δ-functional. This gives

$$Z[J] = \int \mathcal{D}A\, \mathcal{M}[A_\mu]\, \mathrm{e}^{\mathrm{i}\int \left(\mathcal{L}_J - \frac{1}{2\lambda}[F^m(A_\mu)]^2\right) \mathrm{d}^4x}\ . \tag{12.79}$$

In this form $\mathcal{L}$ has again picked up a 'penalty potential' (see discussion after (12.12) on p. 140)). This generating functional is finite due to the presence of the gauge fixing term and can, therefore, be used for a derivation of the Feynman rules for perturbation theory. As we have seen in the last section for the case of QED the divergence of the original path integral was linked to the zero eigenvalues of the matrix in the quadratic part of the gauge-field action so that no one-particle propagator could be defined. The vanishing eigenvalues were connected to the infinite integration over the gauge transformation degree of freedom. Here now this infinity has been removed by fixing the gauge and the propagators are finite.

While everything else in the integrand of the path integral (12.79) is gauge invariant, the source term and the gauge-fixing terms break this invariance. Thus, the specific rules of the perturbation theory, e.g. the Feynman rules, will depend on the gauge. In particular also the Green's functions obtained as functional derivatives of $Z[J]$ are not gauge invariant. This, however, is no problem since these Green's functions only appear at intermediate stages of the calculation. The S matrix elements, which are physical quantities, are then gauge invariant again. We have seen an example for that in the last section where we discussed the QED propagators; in calculating the S matrix the gauge-dependent propagators are always contracted with conserved currents and the gauge-dependent terms then drop out.

12.4.1 Ghost fields

We now continue with the discussion of the general non-Abelian case (12.79). The trick now is to use (9.57) to express the Faddeev-Popov determinant by another path integral over (hypothetical) Grassmann fields $\eta(x)$. We have

$$\det(\mathrm{i}M) = \int \mathcal{D}\bar{\eta}\,\mathcal{D}\eta\,\mathrm{e}^{-\mathrm{i}\int \bar{\eta}^m(x)M^{ml}(x,y)\eta^l(y)\,\mathrm{d}^4x\mathrm{d}^4y}\ . \tag{12.80}$$

Here the Grassmann fields η and $\bar{\eta}$ carry the indices of the internal symmetry group $SU(N)$. The generating functional of the theory then becomes (cf. (12.79))

$$\begin{aligned} Z[J] &= \int \mathcal{D}A\ \det(M)\,\mathrm{e}^{\mathrm{i}\int\left(\mathcal{L}_J - \frac{1}{2\lambda}[F^m(A_\mu)]^2\right)\mathrm{d}^4x} \\ &= \int \mathcal{D}A\,\mathcal{D}\bar{\eta}\,\mathcal{D}\eta\,\mathrm{e}^{\mathrm{i}\int\left(\mathcal{L}_J - \frac{1}{2\lambda}[F^m(A_\mu)]^2 - \int \bar{\eta}M\eta\,\mathrm{d}^4y\right)\mathrm{d}^4x}\ . \end{aligned} \tag{12.81}$$

If we remember that the FP determinant owes its existence to the gauge fixing condition, then we see that the introduction of the fields $\eta(x)$ amounts to a dynamical treatment of the gauge invariance. We therefore expect that this treatment generates also new Feynman graphs involving the fields $\eta(x)$.

In order to be able to calculate their Green's functions we now supplement this generating functional with sources also for the so-called "ghost field" η and obtain for the generating functional of the full theory in the gauge-ghost sector

$$Z[J,\xi,\bar{\xi}] = \int \mathcal{D}A\,\mathcal{D}\bar{\eta}\,\mathcal{D}\eta\,\mathrm{e}^{\mathrm{i}\int \mathcal{L}_J^{\mathrm{eff}}\,\mathrm{d}^4x} \tag{12.82}$$

with

$$\begin{aligned} \mathcal{L}_J^{\mathrm{eff}}(x) = \mathcal{L}(x) &- \frac{1}{2\lambda}[F^m(A_\mu)]^2 \\ &- \int \bar{\eta}(x)M\eta(y)\,\mathrm{d}^4y + J_\mu^n(x)A^{n\mu}(x) + \bar{\xi}^n(x)\eta^n(x) + \bar{\eta}^n(x)\xi^n(x)\ . \end{aligned} \tag{12.83}$$

The second term here is the gauge-fixing term whereas the third introduces the fictitious "ghost field". The last three terms are the source terms for the gauge fields and the ghost fields, respectively. As stressed earlier the gauge-fixing term breaks the gauge-invariance and thus causes the path integral to be well defined.

The ghost field η is clearly unphysical. It is a Lorentz-scalar, but anti-commuting field and has thus the "wrong" spin-statistics relation.

As we have pointed out above, there is no need to introduce a ghost field when we are dealing with an Abelian gauge theory. In this case M is independent of the gauge field A_μ and the ghost-field η is totally decoupled from A_μ; it can, therefore, be left out from the beginning in this case. It is worthwhile to remember that the same situation is encountered in a non-Abelian theory if we work in the axial gauge. In this gauge, however, the propagator is rather complicated because it involves explicitly the (arbitrary) vector n_μ.

12.4.2 Feynman Rules

Equations (12.82) and (12.83) give the final result for the generating functional of a non-Abelian gauge-field theory. The generating functional now has the same form as the one obtained earlier for QED (cf. (12.21)), except for the presence of the ghosts which are a typical feature of non-Abelian gauge field theories. If the system under study contains also physical fermions, then the Lagrangian would have to be supplemented by a Fermion term along the lines developed in Chap. 9 together with the proper source terms. The fermion-gauge field coupling is mandated by local gauge invariance (cf. (12.28)) and appears through the covariant derivative.

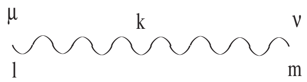
Since the generating functional (12.82) is finite and free of the overcounting of gauge fields discussed at the start of this chapter it can be taken as a starting point to derive the Feynman rules for non-Abelian gauge field theories.

Gauge field couplings. For the gauge field and its couplings we obtain the Feynman-rules by isolating the selfinteraction terms in $\mathcal{L}$. We get

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F^l_{\mu\nu}F^{l\mu\nu} \\
&= -\frac{1}{4}\left(\partial_\mu A^l_\nu - \partial_\nu A^l_\mu - gf^{lmn}A^m_\mu A^n_\nu\right)\left(\partial^\mu A^{l\nu} - \partial^\nu A^{l\mu} - gf^{lop}A^{o\mu}A^{p\nu}\right) \\
&= -\frac{1}{4}\left(\partial_\mu A^l_\nu - \partial_\nu A^l_\mu\right)\left(\partial^\mu A^{l\nu} - \partial^\nu A^{l\mu}\right) \\
&\quad + \frac{1}{2}gf^{lmn}A^m_\mu A^n_\nu\left(\partial^\mu A^{l\nu} - \partial^\nu A^{l\mu}\right) \\
&\quad - \frac{1}{4}g^2 f^{lmn}A^m_\mu A^n_\nu f^{lop}A^{o\mu}A^{p\nu} \ . \qquad (12.84)
\end{aligned}$$

The first term alone just looks like the Lagrangian of an Abelian field. It is the only term quadratic in the fields, so that the gauge boson propagator is just that of the photon found in (12.17), except for the additional SU(N) labels. We thus have for the gauge boson two-point function (cf. (12.19))

1)



$$= \mathrm{i}D^{lm}_{\mu\nu} = \frac{-\mathrm{i}\delta^{lm}}{k^2 + \mathrm{i}\epsilon}\left(g_{\mu\nu} + (\lambda - 1)\frac{k_\mu k_\nu}{k^2}\right) \ . \tag{12.85}$$

Note that D has to be diagonal in the intrinsic quantum numbers because $\mathcal{L}$ involves a trace over the group indices.

The second and third line in (12.84) give the cubic self-coupling terms of the gauge field. The cubic term

$$\frac{1}{2} g f^{lmn} A^m_\mu A^n_\nu \left(\partial^\mu A^{l\nu} - \partial^\nu A^{l\mu}\right) = g f^{lmn} A^m_\mu A^n_\nu \, \partial^\mu A^{l\nu} \tag{12.86}$$

can be represented by the graph shown in Fig. 12.2. Its Feynman rules can be obtained by considering the generating functional of lowest order in the coupling constant g. Quite analogous to the situation in the ϕ^4 model (cf. section 7.1, (7.9)) the only term of Z that contributes to the three-point Green's function is that of order $\mathcal{O}(J^3)$; it is obtained by just replacing the fields by those generated by the external source. This gives

$$\begin{aligned} Z[J] \; = \; & -\mathrm{i} g f^{lmn} \int \Bigg[\int D^{mk}_{\mu\lambda}(x - y') J^{k\lambda}(y') \mathrm{d}^4 y' \int D^{nk}_{\nu\lambda}(x - y') J^{k\lambda}(y') \mathrm{d}^4 y' \\ & \times \partial^\mu \int D^{lk\,\nu\lambda}(x - y') J^k_\lambda(y') \mathrm{d}^4 y' \\ & + \text{ terms of lower order in } J \Bigg] \mathrm{d}^4 x \\ & \times \exp\left(-\frac{\mathrm{i}}{2} \int J^{j\kappa}(x') D^{jk}_{\kappa\lambda}(x' - y') J^{k\lambda}(y') \mathrm{d}^4 x' \, \mathrm{d}^4 y'\right) \ . \end{aligned} \tag{12.87}$$

Here the Latin superscripts refer to the intrinsic $SU(N)$ degrees of freedom, the Greek subscripts and superscripts are those of the Lorentz group. The three-point Green's function can now be obtained by differentiating $Z[J]$

$$G^{hij}_{\pi\rho\sigma}(x_1, x_2, x_3) = \left(\frac{1}{\mathrm{i}}\right)^3 \left. \frac{\delta^3 Z}{\delta J^{h\pi}(x_1) \delta J^{i\rho}(x_2) \delta J^{j\sigma}(x_3)} \right|_{J=0} \ . \tag{12.88}$$

This gives

$$G^{hij}_{\pi\rho\sigma}(x_1, x_2, x_3)$$

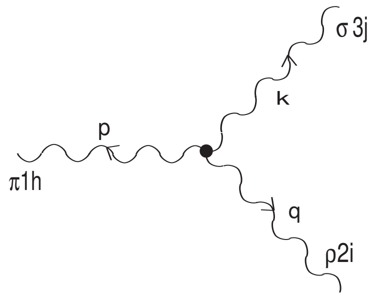


Figure 12.2: Cubic gauge coupling term. The first symbols at each gluon line give the Lorentz indices, the second number the lines and the third denote the $SU(N)$ labels.

$$
\begin{aligned}
= \quad & \mathrm{i} g f^{lmn} \int \mathrm{d}^4 x \Big\{ \mathrm{i} D^{mj}_{\mu\sigma}(x - x_3) \mathrm{i} D^{ni}_{\nu\rho}(x - x_2) \partial^\mu \mathrm{i} D^{lh\,\nu}_{\pi}(x - x_1) \\
& \qquad + \text{permutations of external legs} \Big\}. \qquad (12.89)
\end{aligned}
$$

Equation (12.89) becomes in momentum space (with $G = \mathrm{i}D$)

$$
\begin{aligned}
G^{hij}_{\pi\rho\sigma}(p, q, k) &= \mathrm{i} g f^{lmn} \Big(G^{mj}_{\mu\sigma}(k) G^{ni}_{\nu\rho}(q) (-\mathrm{i} p^\mu) G^{lh\,\nu}_{\pi}(p) \\
&\qquad + \text{permutations of external legs} \Big) \qquad (12.90) \\
&= g f^{hji} \Big(p^\mu g^{\nu\lambda} G^{jj}_{\mu\sigma}(k) G^{ii}_{\nu\rho}(q) G^{hh}_{\lambda\pi}(p) \\
&\qquad + \text{permutations of external legs} \Big) \; .
\end{aligned}
$$

In the Feynman gauge we obtain the gauge field three-point function

$$
\begin{aligned}
G^{hij}_{\pi\rho\sigma}(p, q, k) &= g f^{hij} \frac{-\mathrm{i}}{k^2} \frac{-\mathrm{i}}{p^2} \frac{-\mathrm{i}}{q^2} \qquad (12.91) \\
&\quad \times \Big((k - q)_\pi \, g_{\rho\sigma} + (q - p)_\sigma \, g_{\rho\pi} + (p - k)_\rho \, g_{\pi\sigma} \Big) \; .
\end{aligned}
$$

From this expression the vertex factor can be read off. We then have the rules

2) The vertex factor for the *three-point vertex* is

$$
g f^{hij} \Big((k - q)_\pi \, g_{\rho\sigma} + (q - p)_\sigma \, g_{\rho\pi} + (p - k)_\rho \, g_{\pi\sigma} \Big) \; . \qquad (12.92)
$$

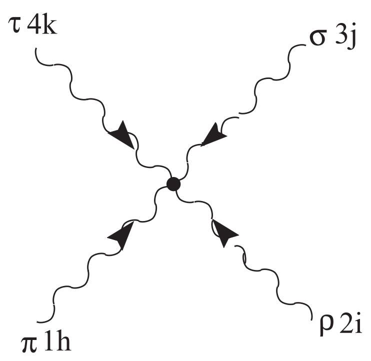


Figure 12.3: Quartic gauge coupling graph.

3) The *four-point vertex* in Fig. 12.3 gives similarly the factor

$$\begin{aligned} -\mathrm{i}g^2 \big[\quad & f^{his} \quad f^{jks} \left(g_{\pi\sigma} g_{\rho\tau} - g_{\pi\tau} g_{\rho\sigma}\right) \\ + \quad & f^{hjs} \quad f^{kis} \left(g_{\pi\tau} g_{\rho\sigma} - g_{\pi\rho} g_{\sigma\tau}\right) \\ + \quad & f^{hks} \quad f^{ijs} \left(g_{\pi\rho} g_{\sigma\tau} - g_{\pi\sigma} g_{\rho\tau}\right) \big] \ . \end{aligned} \tag{12.93}$$

Fermion-gauge coupling. If there are also particles present, then the gauge bosons couple to them through the minimal substitution (12.28)

$$D_\mu = \partial_\mu + \mathrm{i}g T^l A^l_\mu \tag{12.94}$$

and the Feynman rules for these couplings can directly be obtained from our considerations in Chaps. 6.2 and 9. In particular, any fundamental fermion field couples to the gauge fields according to the rule

4) The fermion-gauge vertex (Fig. 12.4) for a gauge boson with Dirac index λ and group index l is given by:

$$-\mathrm{i}g \left(\gamma_\lambda\right)_{\beta\alpha} \left(T^l\right)_{fi} \tag{12.95}$$

Ghost couplings. We now discuss the rules for handling those terms in the Lagrangian that involve the ghost field.

The ghost-part of the Lagrangian reads (cf. (12.83) and (12.71))

$$\begin{aligned} Z_{\text{ghost}}[\xi, \bar{\xi}] &= \int \mathcal{D}\bar{\eta}\, \mathcal{D}\eta\, \mathrm{e}^{-\mathrm{i}\int (\bar{\eta} M \eta) \mathrm{d}^4x\, \mathrm{d}^4y + \mathrm{i}\int (\bar{\xi}\eta + \bar{\eta}\xi)\, \mathrm{d}^4x} \\ &= \int \mathcal{D}\bar{\eta}\, \mathcal{D}\eta \end{aligned}$$

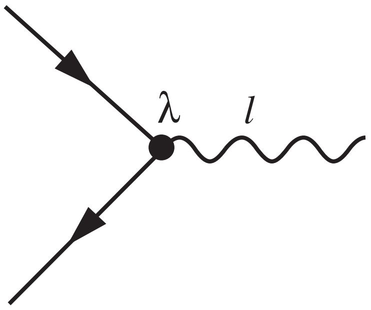


Figure 12.4: Fermion-gauge boson vertex. The fermion line is represented by a solid line.

$$
\begin{aligned}
&\times \exp\left\{ -\frac{\mathrm{i}}{g}\int \left[\bar{\eta}^m(x)\left(D_\mu^{ml}(x)\partial_x^\mu \delta^4(x-y)\right)\eta^l(y)\right]\mathrm{d}^4x\,\mathrm{d}^4y \right.\\
&\qquad\qquad \left. +\ \text{source terms}\right\}\\
=\ &\int \mathcal{D}\bar{\eta}\,\mathcal{D}\eta\\
&\times \exp\left\{ -\frac{\mathrm{i}}{g}\int \left[\bar{\eta}^m(x) D_\mu^{ml}\partial^\mu \eta^l(x)\right]\mathrm{d}^4x +\ \text{source terms}\right\}\\
=\ &\int \mathcal{D}\bar{\eta}\,\mathcal{D}\eta\\
&\times \exp\left\{ -\frac{\mathrm{i}}{g}\int \left[\bar{\eta}^m(x)\left(\delta^{ml}\partial_\mu + \mathrm{i}g\,(T^n)^{ml}A_\mu^n(x)\right)\partial^\mu\eta^l(x)\right]\mathrm{d}^4x \right.\\
&\qquad\qquad \left. +\ \text{source terms}\right\} . \qquad (12.96)
\end{aligned}
$$

Using the regular representation (12.27) for the generators we get for the exponent

$$
\begin{aligned}
&-\frac{\mathrm{i}}{g}\int \bar{\eta}^m(x)\Box\eta^m(x)\,\mathrm{d}^4x + \int \bar{\eta}^m(x)\left(-\mathrm{i}f^{nml}\right)A_\mu^n(x)\,\partial^\mu\eta^l(x)\,\mathrm{d}^4x\\
=\ &-\frac{\mathrm{i}}{g}\int \bar{\eta}^m(x)\Box\eta^m(x)\,\mathrm{d}^4x + \mathrm{i}f^{lmn}\int \bar{\eta}^m(x)A_\mu^n(x)\,\partial^\mu\eta^l(x)\,\mathrm{d}^4x\ .
\end{aligned}
$$

After rescaling the ghost field ($\eta \to \sqrt{g}\eta$) we are thus left with the following ghost contribution to the generating functional

$$
\begin{aligned}
&Z_{\text{ghost}}[\xi,\bar{\xi}] \qquad (12.97)\\
=\ &\int \mathcal{D}\bar{\eta}\,\mathcal{D}\eta\ \exp\left\{-\mathrm{i}\int \mathrm{d}^4x\ \left[\bar{\eta}^l\Box\eta^l - g f^{lmn}A_\mu^n\,\bar{\eta}^m\,\partial^\mu\eta^l + \bar{\xi}^l\eta^l + \bar{\eta}^l\xi^l\right]\right\}\ .
\end{aligned}
$$

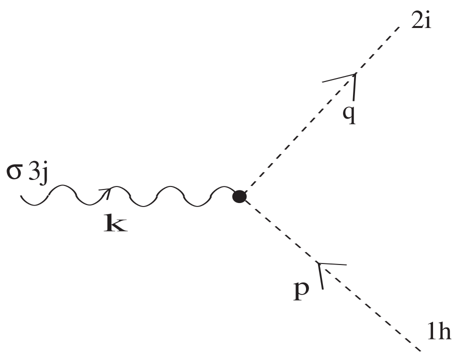
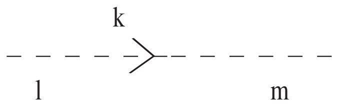


Figure 12.5: Ghost-gauge coupling. The wavy line denotes a gauge boson whereas the dashed line denotes the ghost. The first symbol on the gauge field line (σ) denotes the Lorentz index, the second just numbers the particle and the third (j) gives the SU(N) label.

Note that the ghost field couples only to the gauge field and not to any of the other fields, like e.g. fermions, in the theory. Also, we remember that the ghost fields η and $\bar{\eta}$ cannot appear on external lines because the ghosts follow the wrong spin-statistics relation. This ensures that the ghosts are integrated out so that the FP determinant is generated.

The Feynman rules for the ghost fields can now be read off from (12.97).

6) The ghost propagator is given by



$$= \delta^{lm} \frac{\mathrm{i}}{k^2 + \mathrm{i}\epsilon} \, . \tag{12.98}$$

Note that the ghost line carries an arrow; this reflects the presence of ghosts and antighosts, because the FP matrix is in general not hermitean. The notation is then such that particles always run with the time axis from bottom to top and antiparticles run against it. More loosely speaking, the ghost line runs from $\bar{\eta}$ to η.

Finally, according to (12.97) the ghost can couple to the gauge field as depicted in Fig. 12.5. The vertex factor is obtained by considering the ghost-ghost-gauge boson three-point function. As for the gauge three-point function just discussed it can be obtained by considering the lowest order (in the gauge-ghost coupling) expression

$$
\begin{aligned}
Z[\xi,\bar{\xi},J] &= -\mathrm{i}\int \mathrm{d}^4x \left(-gf^{lmn}\right)\frac{1}{\mathrm{i}}\frac{\delta}{\delta J^{n\mu}(x)}\,\frac{1}{\mathrm{i}}\frac{\delta}{\delta\xi^m(x)}\,\partial^\mu\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\xi}^l(x)}\,\mathrm{e}^{\mathrm{i}S_0[\xi,\bar{\xi},J]} \\
&= -\mathrm{i}gf^{lmn}\int\Bigg[\int D^{nk}_{\mu\nu}(x-y')J^{k\nu}(y')\,\mathrm{d}^4y'\int\bar{\xi}^k D^{km}(x-y')\,\mathrm{d}^4y' \\
&\qquad\times\partial^\mu\int D^{lk}(x-y')\xi^k(y')\,\mathrm{d}^4y' \\
&\qquad+\ \text{terms of lower order in } J\Bigg]\,\mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}S_0[\xi,\bar{\xi},J]}
\end{aligned}
\tag{12.99}
$$

This generating functional determines the following rule for the gauge boson-ghost coupling vertex:

7) The gauge boson-ghost vertex (Fig. 12.5) has to be associated with a factor

$$gf^{hij}q_\sigma \tag{12.100}$$

where q is the outgoing momentum of the ghost. The four-momentum conservation ($p+k=q$) is understood here; it is not written explicitly.

In addition, we have, as for real, physical fermions, the rule

8) All ghost loops carry a sign (-1).

We also have to remember that ghosts cannot appear on external lines.

Chapter 13

Examples for Gauge Field Theories

The Feynman rules developed in the preceding chapter are of a 'generic' nature: they apply to all non-Abelian gauge field theories. The particular physics content of a gauge field theory is determined by specifying the symmetry group $SU(N)$.

In the preceding chapter we have referred to Quantum Electrodynamics (QED), Quantum Chromodynamics (QCD) and the theory of electroweak Interactions. In this chapter we, very briefly, summarize the physical contents and the Lagrangian of these theories.

13.1 Quantum Electrodynamics

Quantum Electrodynamics is the most simple gauge field theory. It describes the interactions between electrons and the electromagnetic field, i.e. the Coulomb and Lorentz forces. Its ingredients are discussed at the end of appendix B and in Sects. 12.1 and 12.2.1. It is based on the $U(1)$ gauge group and is thus an Abelian theory. Its Lagrangian is given in (B.56) in appendix B.

13.2 Quantum Chromodynamics

Quantum Chromodynamics is the theory of the strong interactions. Its foundations, first discussed in the 60's, lie in the observation that the observed hadron spectrum can be understood in terms of quarks, mesons being composed of quark-antiquark pairs and baryons of three quarks; the quarks carry spin 1/2 and a so-called *flavor* quantum number. For certain baryons, such as the Δ^{++}, the total wavefunction, i.e. the product of space-part, spin-part and flavor-part, one obtains in such a scheme (*flavor SU(3)*) a completely symmetric wavefunction for the three quarks, in glaring contradiction to the Pauli principle. This problem was solved by the introduction of a new, additional intrinsic degree of freedom for the quarks, called *color*, in which the quarks can differ (for a detailed discussion see [MOSEL]). The Pauli principle then requires a completely antisymmetric wavefunction in this new intrinsic space. In experiments that produce mesons, i.e. quark-antiquark pairs, through lepton-annihilation the number of colors has been found to be three. The completely antisymmetric color state has therefore been identified with the completely antisymmetric singlet state of a *color SU(3) group*. Gauging this group has led to the non-Abelian theory of *Quantum Chromodynamics* (QCD).

The Lagrangian of QCD is then very simply given by

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} F^c_{\mu\nu} F^{c\mu\nu} + \sum_f \left[\bar{q}_f (\mathrm{i}\gamma^\mu D_\mu - m_f) q_f \right] \quad . \tag{13.1}$$

Here the fundamental triplets of $SU(3)_C$ are given by the quark spinors

$$q_f = \begin{pmatrix} q_r^f \\ q_g^f \\ q_b^f \end{pmatrix} \quad , \tag{13.2}$$

which transform according to the fundamental (= lowest-dimensional) representation of $SU(3)_C$. The indices r, g and b stand for the three colors (*red, green* and *blue*), the index f for the flavor. The covariant derivative in (13.1) is given by (cf. (12.28))

$$D_\mu = \partial_\mu + \mathrm{i}g \frac{\lambda^c}{2} G^c_\mu \quad . \tag{13.3}$$

Here the eight matrices λ^c are the so-called *Gell-Mann* matrices, the generators of $SU(3)$, and the fields G^c represent the eight gauge-fields, here called the *gluon* fields, which form an $SU(3)$ octett.

The field tensors $F^c_{\mu\nu}$ have the general form of a non-Abelian theory (12.33). The gluons thus also interact among themselves. One of the predictions of this theory is, therefore, the existence of so-called *glueballs*, that consist only of gauge-field quanta.

The theory, furthermore, has the property of *asymptotic freedom*, i.e. its running coupling constant (cf. the discussion at the end of Sect. 8.2.2) becomes larger with decreasing gluon momentum. This leads to a quark-quark potential that at small distances is Coulomb-like, but at large distances increases linearly, and thus provides an explanation for the fact that free quarks and gluons do not seem to exist, i.e. that they are confined inside the hadrons.

13.3 Electroweak Interactions

While the Lagrangian of QCD has the rather simple, generic form (13.1), that of the electroweak interactions is considerably more complicated. Its foundations go back to the early seventies when a gauge-field theory was found that comprised both 'classical' Quantum Electrodynamics and the phenomena of weak interactions, responsible, for example, for the nuclear β-decay.

This theory is based on a product of two gauge-groups, one an Abelian $U(1)$ and the other one a non-Abelian $SU(2)$. For the latter the left-handed parts of the observed fermions, quarks and leptons,

$$\psi_{\mathrm{L}} = \frac{1}{2}\,(1-\gamma_5)\,\psi \tag{13.4}$$

are assumed to form doublets

$$\Psi_k = \begin{pmatrix} \nu_{k\mathrm{L}} \\ e_{k\mathrm{L}} \end{pmatrix} \ \text{or}\ \begin{pmatrix} u_{k\mathrm{L}} \\ d'_{k\mathrm{L}} \end{pmatrix} \tag{13.5}$$

for the left-handed fields of the kth family of leptons and quarks. The right-handed parts

$$\Psi_k = e_{k\mathrm{R}} \ \text{or}\ q_{k\mathrm{R}} \tag{13.6}$$

are assumed to form $SU(2)$ singlets. In (13.5) we have used generic abbreviations

$$\begin{pmatrix} \nu_k \\ e_k \end{pmatrix} \ \text{for}\ \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix} \ \text{or}\ \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix} \tag{13.7}$$

and

$$\begin{pmatrix} u_k \\ d'_k \end{pmatrix} \ \text{for}\ \begin{pmatrix} u \\ d' \end{pmatrix}, \begin{pmatrix} c \\ s' \end{pmatrix} \ \text{or}\ \begin{pmatrix} t \\ b' \end{pmatrix} \quad . \tag{13.8}$$

The $U(1)$ gauge group is very similar to QED, but it is not connected with the electric charge, but instead a so-called *weak hypercharge* $y = 2q - 2t_3$ where q is the electrical charge of the fermion and t_3 its *weak isospin* $\pm\frac{1}{2}$, depending on the position of the particle in the two-component spinors (13.7), (13.8), just as in the usual Pauli-spinors.

The complete Lagrangian of this theory is then given by

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{4}\mathbf{G}^{\mu\nu}\cdot\mathbf{G}_{\mu\nu} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ & + M_W^2 W_\mu^\dagger W^\mu \left(1+\frac{\varphi}{v}\right)^2 + \frac{1}{2}M_Z^2 Z_\mu Z^\mu \left(1+\frac{\varphi}{v}\right)^2 \\ & + \sum_k \bar{\Psi}_k \mathrm{i}\gamma^\mu D_\mu \Psi_k + \mathcal{L}_\mathrm{Y} \\ & + \frac{1}{2}\left(\partial^\mu\varphi\right)\left(\partial_\mu\varphi\right) - \frac{1}{2}m_H^2\varphi^2\left(1+\frac{\varphi}{v}+\frac{1}{4}\left(\frac{\varphi}{v}\right)^2\right) \quad .\end{aligned} \tag{13.9}$$

Since there are now two gauge groups present, we also have two gauge fields appearing in the Lagrangian: $\mathbf{G}_{\mu\nu}$ for the non-Abelian $SU(2)_\mathrm{L}$ and $F_{\mu\nu}$ for the Abelian $U(1)_\mathrm{Y}$. Thus, also the covariant derivative contains both of these fields with two different coupling constants

$$D_\mu \Psi_k = \left(\partial_\mu + \mathrm{i}g t^l W_\mu^l + \mathrm{i}g' \frac{1}{2} y_k B_\mu\right)\Psi_k \ , \tag{13.10}$$

where y_k is the hypercharge quantum number of Ψ_k and the t^l are the generators of $SU(2)$. The three fields W^l and the single field B are the gauge fields for $SU(2)_\mathrm{L}$ and $U(1)_\mathrm{Y}$, resp. By linearly combining them one obtains the 2 physical fields W_μ and $W_\mu^\dagger$, the field Z_μ as well as the photon field A_μ. With the covariant derivative (13.10) the first term in the third line describes massless fermions interacting with the gauge fields in the standard way.

The gauge fields W and B normally have to be massless, since an explicit mass term breaks gauge invariance. Since these fields transmit the weak interaction the latter would then be long ranged. This, however, presents an immediate problem because the weak interaction, since Fermi's theory, is known to be very short-ranged. A way out of this difficulty is provided by the so-called *Higgs* mechanism. The main ingredient of this mass generation mechanism is the existence of a scalar Higgs field φ, yet to be discovered, with non-linear self-couplings (last line in (13.9); these self-couplings lead to a non-vanishing vacuum field. The second ingredient is a Yukawa coupling of the gauge-fields to this Higgs field that generates their masses M_W and M_Z in the same way as described in Sect. 10.3. Similarly, the term $\mathcal{L}_\mathrm{Y}$ in (13.9), generates masses for all the fermions through a Yukawa coupling to the Higgs field.

Appendix A

Units and Metric

A.1 Units

It is customary in elementary particle physics and field theories to choose the system of units such that the resulting expressions assume a simple form. In particular, instead of working with the usual three mechanical units for length, mass and time one introduces instead three basic units for velocity, action and length.

The choice of the velocity of light, c, as the unit for the velocity implies that

$$c = 1 \; . \tag{A.1}$$

This in turn means that in such a system length and time have the same dimensions and are equivalent units. With $c = 1$ the relativistic energy–momentum relation assumes the simple form

$$E^2 = p^2 + m^2 \; . \tag{A.2}$$

Choosing next the unit of action, $\hbar$, such that

$$\hbar = 1 \tag{A.3}$$

connects the dimensions of mass, time and length.

Since time and length are equivalent, mass assumes the dimension of an inverse length; the same holds consequently for the momentum and the energy (A.2). The choice of 1 fm $= 10^{-13}$ cm as a length unit, for example, leads to masses, momenta and energies all given in fm^{-1}. Another often-used unit for energy is that of 1 MeV, i.e. the energy that an electron acquires when it is accelerated by the voltage of 1 MV. The transformation of the expressions back to the standard MKSA system can be achieved by multiplying all quantities with the appropriate combinations of $\hbar$ and c. A particularly useful relation for this conversion is

$$\hbar c \approx 197.3\,\mathrm{MeV\ fm}\ . \tag{A.4}$$

A.2 Metric and Notation

All vectors, both ordinary three-component vectors and those in some internal space, are distinguished by boldface italic print in this book.

In relativistic expressions four-vectors are always written as

$$A^\mu = \left(A_0, \vec{A}\right) = \left(A^0, A^1, A^2, A^3\right) \tag{A.5}$$

Four-vectors with a superscript are called *contravariant vectors. Covariant vectors*, denoted by subscripts, are then defined by

$$A_\mu = g_{\mu\nu} A^\nu \tag{A.6}$$

with the *metric tensor* $g_{\mu\nu}$ that reads in Cartesian coordinates in Minkowski space

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{A.7}$$

The metric tensor with superscripts is used to raise the indices of four-vectors; it is thus the inverse of g and is given by

$$g^{\mu\nu} = g_{\mu\nu}\ . \tag{A.8}$$

A scalar product of 2 four-vectors is then defined by

$$A \cdot B = A_\mu B^\mu = A^\mu B_\mu = A_0 B_0 - \vec{A} \cdot \vec{B} \tag{A.9}$$

if A^μ and B^μ are defined by

$$A^\mu = (A^0, \vec{A})\ , \qquad B^\mu = (B^0, \vec{B})\ . \tag{A.10}$$

The invariant square of the four-vector is thus given by

$$A^2 = A_\mu A^\mu = A_0^2 - \vec{A}^2\ . \tag{A.11}$$

Here, as always in the text, the Einstein convention of summing over double indices, one lower and one upper, is used. The components of four-vectors are generally labeled by Greek indices, whereas roman indices are used to refer specifically to the last three (vector) components.

The space-time four-vector is

$$x^\mu = (x^0, \vec{x}) = (t, \vec{x}) \; , \tag{A.12}$$

and the four-momentum is given by

$$p^\mu = (E, \vec{p}) \; . \tag{A.13}$$

Finally, the components of the contravariant four-gradient are abbreviated by

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\vec{\nabla} \right) \; , \tag{A.14}$$

so that the four-momentum operator is given by

$$\hat{p}^\mu = \mathrm{i}\partial^\mu \; , \tag{A.15}$$

and the Lorentz-invariant *d'Alembert operator* by

$$\partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 = \Box \; . \tag{A.16}$$

Other important four-vectors are the current

$$j^\mu = (\rho, \vec{j}) \tag{A.17}$$

and the electromagnetic potential

$$A^\mu = \left(\phi, \vec{A} \right) \; . \tag{A.18}$$

Appendix B

Classical Fields

In this chapter a few essential facts of classical field theory will be discussed. It will first be shown how to derive the equations of motion of a field theory, for example the Maxwell equations of electrodynamics, from a Lagrangian. Second, the connection between symmetries of the Lagrangian and conservation laws will be discussed. This is one of the central points for all the theoretical developments in this book.

B.1 Equations of Motion

The equations of motion of classical mechanics can be obtained from a Lagrange function by using Hamilton's principle that the action for a given mechanical system is stationary for the physical space–time development of the system.

The equations of motion for fields that determine their space–time dependence can be obtained in an analogous way by identifying the field amplitudes at a coordinate $\vec{x}$ with the dynamical variables (coordinates) of the theory.

Let the functions that describe the fields be denoted by

$$\Phi_\alpha(x) \quad \text{with} \quad x^\mu = (t, \vec{x}) \ , \tag{B.1}$$

where α labels the various fields appearing in a theory. The Lagrangian L of the system is expressed in terms of a Lagrange density $\mathcal{L}$, as follows:

$$L = \int \mathcal{L}(\Phi_\alpha, \partial_\mu \Phi_\alpha) \, \mathrm{d}^3x \tag{B.2}$$

where the spatial integration is performed over the volume of the system. The action S is then defined as usual by

$$S = \int\limits_{t_0}^{t_1} L \, \mathrm{d}t = \int\limits_{\Omega} \mathcal{L}(\Phi_\alpha, \partial_\mu \Phi_\alpha) \, \mathrm{d}^4x \tag{B.3}$$

with $\mathrm{d}^4x = \mathrm{d}^3x\,\mathrm{d}t$. The, in general finite, space–time volume of the system is denoted by Ω.

As pointed out before the fields $\Phi_\alpha(x)$ play the same role as the generalized coordinates q_i in classical mechanics; the analogy here is such that the fields Φ_α correspond to the coordinates q and the points $\vec{x}$ and α to the classical indices i. The corresponding velocities are given in a direct analogy by the time derivatives of $\Phi_\alpha : \partial_t\Phi_\alpha$. Lorentz covariance then requires that also the derivatives with respect to the first three coordinates appear; this explains the presence of the four-gradients $\partial_\mu\Phi_\alpha$ in (B.2).

In order to derive the field equations from the action S by Hamilton's principle, we now vary the fields and their derivatives

$$\begin{aligned} \Phi_\alpha &\to \Phi'_\alpha = \Phi_\alpha + \delta\Phi_\alpha \\ \partial_\mu\Phi_\alpha &\to (\partial_\mu\Phi_\alpha)' = \partial_\mu\Phi_\alpha + \delta(\partial_\mu\Phi_\alpha) \ . \end{aligned} \tag{B.4}$$

This yields

$$\begin{aligned} \delta\mathcal{L} &= \mathcal{L}(\Phi'_\alpha, (\partial_\mu\Phi_\alpha)') - \mathcal{L}(\Phi_\alpha, \partial_\mu\Phi_\alpha)\,\mathrm{d}^4x \\ &= \frac{\partial\mathcal{L}}{\partial\Phi_\alpha}\delta\Phi_\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)}\delta(\partial_\mu\Phi_\alpha) \\ &= \frac{\partial\mathcal{L}}{\partial\Phi_\alpha}\delta\Phi_\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)}\partial_\mu(\delta\Phi_\alpha) \ . \end{aligned} \tag{B.5}$$

According to the Einstein convention a summation over μ is implicitly contained in this expression. In going from the second to the third line differentiation and variation could be commuted because both are linear operations.

The equations of motion are now obtained from the variational principle

$$\begin{aligned} \delta S &= \int_\Omega \left[\left(\frac{\partial\mathcal{L}}{\partial\Phi_\alpha} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)}\right)\delta\Phi_\alpha + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)}\delta\Phi_\alpha\right)\right]\mathrm{d}^4x \\ &= 0 \end{aligned} \tag{B.6}$$

for arbitrary variations $\delta\Phi_\alpha$ under the constraint that

$$\delta\Phi_\alpha(t_0) = \delta\Phi_\alpha(t_1) = 0 \ ,$$

where t_0 and t_1 are the time-like boundaries of the four-volume Ω. The last term in (B.6) can be converted into a surface integral by using Gauss's law; for fields which are localized in space this surface integral vanishes if the surface is moved out to infinity. Since the variations $\delta\Phi_\alpha$ are arbitrary the condition $\delta S = 0$ leads to the equations of motion

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)}\right) - \frac{\partial\mathcal{L}}{\partial\Phi_\alpha} = 0 \ . \tag{B.7}$$

The relativistic equivalence principle demands that these equations have the same form in every inertial frame of reference, i.e. that they are Lorentz covariant. This is only possible if $\mathcal{L}$ is a Lorentz scalar, i.e. if it has the same functional dependence on the fields and their derivatives in each reference frame. In a further analogy to classical mechanics, the canonical field momentum is defined as

$$\Pi_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_\alpha} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \Phi_\alpha)} \; . \tag{B.8}$$

From $\mathcal{L}$ and Π_α the Hamiltonian H is obtained as

$$H = \int \mathcal{H} \, \mathrm{d}^3x = \int (\Pi_\alpha \dot{\Phi}_\alpha - \mathcal{L}) \, \mathrm{d}^3x \; . \tag{B.9}$$

The Hamiltonian H represents the energy of the field configuration.

B.1.1 Examples

The following sections contain examples of classical field theories and their formulation within the Lagrangian formalism just introduced. We start out with the probably best-known case of classical electrodynamics, then generalize it to a treatment of massive vector fields and then move on to a discussion of classical Klein–Gordon and Dirac fields that will play a major role in the later chapters of this book.

Electrodynamics

The best-known classical field theory is probably that of electrodynamics, in which the Maxwell equations are the equations of motion. The two homogeneous Maxwell equations allow us to rewrite the fields in terms of a four-potential

$$A^\mu = (A^0, \vec{A}) \; , \tag{B.10}$$

defined via

$$\begin{aligned} \vec{B} &= \vec{\nabla} \times \vec{A} \; , \\ \vec{E} &= -\vec{\nabla} A^0 - \frac{\partial \vec{A}}{\partial t} \; . \end{aligned} \tag{B.11}$$

Note that the 2 homogeneous Maxwell equations are now automatically fulfilled. The the two inhomogeneous Maxwell equations[1]

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \rho \; , \\ \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} &= \vec{j} \end{aligned} \tag{B.12}$$

[1] Here the Heaviside units are used with $c = 1$ and $\epsilon_0 = \mu_0 = 1$.

can be rewritten as

$$\partial_\mu F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^\mu} = j^\nu \quad , \tag{B.13}$$

with the four-current

$$j^\nu = (\rho, \vec{j}) \tag{B.14}$$

and the antisymmetric field tensor

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \ . \tag{B.15}$$

Equation (B.13) is the equation of motion for the field tensor or the four-vector field A_μ.

It is easy to show that (B.13) can be obtained from the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\nu A_\nu \tag{B.16}$$

by using (B.7); the fields A^ν here play the role of the fields Φ_α in (B.7). We have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial A_\nu} &= -j^\nu \ , \\ \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} &= -\frac{1}{4} 2(+F^{\mu\nu} - F^{\nu\mu}) = -F^{\mu\nu} \ . \end{aligned} \tag{B.17}$$

The last step is possible because F is an antisymmetric tensor.

The equation of motion is therefore

$$\frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = -\frac{\partial F^{\mu\nu}}{\partial x^\mu} + j^\nu = 0 \quad , \tag{B.18}$$

in agreement with (B.13). It is now easy to interpret the two terms in $\mathcal{L}$ (B.16): the first one gives the Lagrangian for the free electromagnetic field, whereas the second describes the interaction of the field with charges and currents.

The two homogeneous Maxwell equations can also be expressed in terms of the field tensor by first introducing the *dual field tensor*

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \ ; \tag{B.19}$$

here $\epsilon^{\mu\nu\rho\sigma}$ is the Levi–Civita antisymmetric tensor which assumes the values $+1$ or -1 according to whether $(\mu\nu\rho\sigma)$ is an even or odd permutation of

(0,1,2,3), and 0 otherwise. In terms of $\tilde{F}$ the homogeneous Maxwell equations read

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \; . \tag{B.20}$$

By definition the field tensor (B.15) is invariant under a *gauge transformation*

$$A^\mu \longrightarrow A^{\mu\prime} = A^\mu + \partial^\mu \phi \; . \tag{B.21}$$

The same then holds for the equation of motion (B.18). This freedom can be used to impose constraints on the four components of the vector field A^μ. In addition, for free fields this gauge freedom can be used, for example, to set the 0th component of the four-potential equal to zero. Thus, a free electromagnetic field has only two degrees of freedom left.

Massive Vector Fields

Vector fields in which – in contrast to the electromagnetic field – the field quanta are massive are described by the so-called Proca equation:

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = j^\nu \; . \tag{B.22}$$

Operating on this equation with the four-divergence ∂_ν gives, because F is antisymmetric,

$$m^2 \partial_\nu A^\nu = \partial_\nu j^\nu \; . \tag{B.23}$$

For $m \neq 0$ and a conserved current, this reduces the equation of motion (B.22) to

$$\left(\Box + m^2\right) A^\nu = j^\nu \; ; \qquad \partial_\nu A^\nu = 0 \; . \tag{B.24}$$

Thus for massive vector fields the freedom to make gauge transformations on the vector field is lost. The condition of vanishing four-divergence of the field reduces the degrees of freedom of the field from 4 to 3. The space-like components represent the physical degrees of freedom.

The Lagrangian that leads to (B.22) is given by

$$\mathcal{L} = -\frac{1}{4} F^2 + \frac{1}{2} m^2 A^2 - j \cdot A \; . \tag{B.25}$$

Klein–Gordon Fields

A particularly simple example is provided by the so-called Klein–Gordon field ϕ that obeys the equation of motion

$$\left(\partial_\mu \partial^\mu + m^2\right) \phi = \left(\Box + m^2\right) \phi = 0 \; ; \tag{B.26}$$

such a field describes scalar particles, i.e. particles without intrinsic spin. The Lagrangian leading to (B.26) is given by

$$\mathcal{L} = \frac{1}{2}\left(\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - m^2\phi^2\right) \tag{B.27}$$

since we have

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} = \partial^\mu \phi \tag{B.28}$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi\ . \tag{B.29}$$

An interesting case occurs if we consider two independent real scalar fields, ϕ_1 and ϕ_2, with the same mass m. The total Lagrangian is then simply given by a sum over the Lagrangians describing the individual fields, i.e.

$$\mathcal{L} = \frac{1}{2}\left(\left(\partial_\mu \phi_1\right)\left(\partial^\mu \phi_1\right) - m^2\phi_1^2\right) + \frac{1}{2}\left(\left(\partial_\mu \phi_2\right)\left(\partial^\mu \phi_2\right) - m^2\phi_2^2\right)\ . \tag{B.30}$$

On the other hand, we can also construct two complex fields from the two real fields ϕ_1 and ϕ_2, namely

$$\phi = \frac{1}{\sqrt{2}}\left(\phi_1 + \mathrm{i}\phi_2\right) \tag{B.31}$$

and its complex conjugate. In terms of these the Lagrangian (B.30) can be rewritten to

$$\mathcal{L} = \left(\partial_\mu \phi\right)^*\left(\partial^\mu \phi\right) - m^2\phi^*\phi\ . \tag{B.32}$$

Dirac Fields

A particularly simple example is provided by the Dirac field Ψ for which the equation of motion is just the Dirac equation

$$\left(\mathrm{i}\gamma^\mu \partial_\mu - m\right)\Psi = 0\ , \tag{B.33}$$

where the γ_μ are the usual (4×4) matrices of Dirac theory. Ψ itself is a (4×1) matrix of four independent fields, a so-called *spinor*.

The corresponding Lagrangian is given by

$$\mathcal{L} = \bar{\Psi}\left(\mathrm{i}\gamma^\mu \partial_\mu - m\right)\Psi\ . \tag{B.34}$$

This can be seen by identifying the fields Φ_α in (B.7) with the four components of the Dirac spinor $\bar{\Psi} = \Psi^\dagger\gamma_0$. Since $\mathcal{L}$ does not depend on $\partial_\mu \bar{\Psi}$ the equation of motion is simply given by

$$\frac{\partial \mathcal{L}}{\partial \bar{\Psi}} = \left(\mathrm{i}\gamma^\mu \partial_\mu - m\right)\Psi = 0\ . \tag{B.35}$$

B.2 Symmetries and Conservation Laws

As in classical mechanics there is also in field theory a conservation law associated with each continuous symmetry of $\mathcal{L}$. The theorem which describes the connection between the invariance of the Lagrangian under a continuous symmetry transformation and the related conserved current is known as Noether's theorem. In the following, this will be illustrated for different types of symmetries which then lead to the well-known conservation laws.

The common expression in the arguments to follow is the change of the Lagrangian density under a change of the fields and their derivatives (see (B.4)). According to (B.5) and the Lagrange equations of motion (B.7) this change is given by

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu \Phi_\alpha)} \delta\Phi_\alpha \right) \quad . \tag{B.36}$$

B.2.1 Geometrical Space–Time Symmetries

In this section we investigate the consequences of translations in four-dimensional space–time, i.e. infinitesimal transformations of the form

$$x_\nu \to x'_\nu = x_\nu + \epsilon_\nu \quad , \tag{B.37}$$

where the infinitesimal shift ϵ_ν is independent of the coordinate x_ν. Under such transformations the change of $\mathcal{L}$ is given by

$$\delta\mathcal{L} = \epsilon_\nu \frac{\partial\mathcal{L}}{\partial x_\nu} = \epsilon_\nu \partial^\nu \mathcal{L} \quad , \tag{B.38}$$

since $\mathcal{L}$ is a scalar.

If now $\mathcal{L}$ is required to be form-invariant under translations, it does not explicitly depend on x_ν. In this case, $\delta\mathcal{L}$ is also given by (B.36). The changes of the fields Φ_α appearing there are for the space–time translation considered here given by

$$\delta\Phi_\alpha = \epsilon_\nu \frac{\partial\Phi_\alpha}{\partial x_\nu} = \epsilon_\nu \partial^\nu \Phi_\alpha \ . \tag{B.39}$$

Inserting (B.39) into (B.36) yields

$$\delta\mathcal{L} = \epsilon_\nu \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu \Phi_\alpha)} \partial^\nu \Phi_\alpha \right) \quad . \tag{B.40}$$

Equating (B.40) and (B.38) finally gives

$$\partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu \Phi_\alpha)} \partial^\nu \Phi_\alpha - \mathcal{L}\ g^{\mu\nu} \right) = 0 \quad , \tag{B.41}$$

since the ϵ_ν are arbitrary. By defining the tensor $T^{\mu\nu}$ as

$$T^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_\alpha)} \partial^\nu \Phi_\alpha - \mathcal{L}\, g^{\mu\nu} \tag{B.42}$$

this equation reads

$$\partial_\mu T^{\mu\nu} = 0\ . \tag{B.43}$$

Relation (B.43) has the form of a continuity equation. Spatial integration over a finite volume yields

$$\frac{d}{dt}\left(\int_V T^{0\mu}(x)\mathrm{d}^3x\right) = -\int_V \frac{\partial T^{i\mu}}{\partial x^i}\,\mathrm{d}^3x = -\oint_S \vec{S}^{(\mu)} \cdot \vec{n}\,\mathrm{d}S\ . \tag{B.44}$$

Here $\vec{n}$ is a unit vector vertical on the surface S pointing outwards and $\vec{S}^{(\mu)}$ is a three-vector:

$$\vec{S}^{(\mu)} = (T^{1\mu}, T^{2\mu}, T^{3\mu})\ . \tag{B.45}$$

The surface integral on the rhs of (B.44) is taken over the surface S of volume V. For localized fields it can be made to vanish by extending the volume towards infinity. It is then evident that the quantities

$$P^\mu = \int T^{0\mu}\mathrm{d}^3x \tag{B.46}$$

are conserved. These are the components of the four-momentum of the field, as can be verified for the zeroth component,

$$\begin{aligned} P^0 &= \int T^{00}\,\mathrm{d}^3x = \int\left(\frac{\partial\mathcal{L}}{\partial(\partial_0\Phi_\alpha)}\partial^0\Phi_\alpha - \mathcal{L}\right)\mathrm{d}^3x \\ &= \int(\Pi_\alpha\dot{\Phi}_\alpha - \mathcal{L})\,\mathrm{d}^3x = H \quad , \end{aligned} \tag{B.47}$$

according to (B.8) and (B.9). The spatial components of the field momentum are

$$P^k = \int T^{0k}\,\mathrm{d}^3x = \int \frac{\partial\mathcal{L}}{\partial(\partial_0\Phi_\alpha)}\partial^k\Phi_\alpha\,\mathrm{d}^3x\ . \tag{B.48}$$

$T^{\mu\nu}$, as defined in (B.42), has no specific symmetry properties. It can, however, always be made symmetric in its Lorentz-indices because (B.43) does not define the tensor T uniquely. We can always add a term of the form $\partial_\lambda D^{\lambda\mu\nu}$, where $D^{\lambda\mu\nu}$ is a tensor antisymmetric in the indices λ and μ, such that T becomes symmetric.[2]

[2]In classical mechanics the form invariance of the Lagrangian under rotations leads to the conservation of angular momentum. Analogously, in a relativistic field theory the form invariance of $\mathcal{L}$ under four-dimensional space–time rotations (Lorentz covariance) leads to the conservation of a quantity that is identified with the angular momentum of the field. To obtain the same form for the angular momentum as in classical mechanics it is essential that $T_{\mu\nu}$ is symmetric.

Comparing (B.44) with (B.48) and assuming T to be symmetric we see that the normal components of the vectors $\vec{S}^{(\mu)}$ in (B.45) describe the momentum flow through the surface S of the volume V and thus determine the "radiation pressure" of the field.[3] These properties allow us to identify $T_{\mu\nu}$ as the energy-momentum tensor of the field. For the Lagrangian (B.16) of electrodynamics $T_{\mu\nu}$ is just the well-known Maxwell's stress tensor.

As already mentioned at the beginning of this chapter these conservation laws are special cases of Noether's theorem, which can be summarized for the general case as follows:

> *Each continuous symmetry transformation that leaves the Lagrangian invariant is associated with a conserved current. The spatial integral over this current's zeroth component yields a conserved charge.*

B.2.2 Internal Symmetries

Relativistic field theories may contain conservation laws that are not consequences of space–time symmetries of the Lagrangian, but instead are connected with symmetries in the internal degrees of freedom such as, e.g., isospin or charge.

We therefore now allow for a mixture of the different fields under the transformation

$$\Phi_\alpha(x) \to \Phi'_\alpha(x) = \mathrm{e}^{-\mathrm{i}\varepsilon q_{\alpha\beta}}\Phi_\beta \ , \tag{B.49}$$

where ϵ is an infinitesimal parameter and the $q_{\alpha\beta}$ are fixed c-numbers. We then have

$$\delta\Phi_\alpha(x) = \Phi'_\alpha(x) - \Phi_\alpha(x) = -\mathrm{i}\varepsilon q_{\alpha\beta}\Phi_\beta(x) \ . \tag{B.50}$$

The change of the Lagrangian is given by (B.36)

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)}\delta\Phi_\alpha\right) \ . \tag{B.51}$$

If $\mathcal{L}$ is invariant under this variation $\delta\Phi_\alpha$, then we have

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)}\delta\Phi_\alpha\right) = 0 \ . \tag{B.52}$$

Equation (B.52) is in the form of a continuity equation for the "current"

$$j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)}\delta\Phi_\alpha\frac{1}{\varepsilon} \ . \tag{B.53}$$

[3] More precisely, $S^{k(\mu)}$ denotes the flux of the μth component of the field momentum in the direction x^k.

Inserting the field variations $\delta\Phi_\alpha$ yields for the current

$$j^\mu(x) = -\mathrm{i}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)}q_{\alpha\beta}\Phi_\beta \ . \tag{B.54}$$

Equations (B.52) and (B.54) imply that the "charge"

$$Q = \int j^0(x)\,\mathrm{d}^3x = -\mathrm{i}\int \frac{\partial\mathcal{L}}{\partial(\partial_0\Phi_\alpha)}q_{\alpha\beta}\Phi_\beta\,\mathrm{d}^3x \tag{B.55}$$

of the system is conserved. The physical nature of these "charges" and "currents" has to remain open. It depends on the specific form of the symmetry transformation (B.49) and can be determined only by coupling the system to external fields.

Example: Quantum Electrodynamics

To illustrate this conservation law, the theory of electromagnetic interactions is used as an example. However, in contrast to the considerations in Sect. B.1.1 we now consider a coupled system of a fermion field $\Psi(x)$ and the electromagnetic field $A_\mu(x)$ to determine the physical meaning of the conserved current. Together with a quantization procedure this theory is called Quantum Electrodynamics (QED). The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}\left[\mathrm{i}\gamma^\mu(\partial_\mu + \mathrm{i}eA_\mu) - m\right]\Psi \ . \tag{B.56}$$

$\mathcal{L}$ contains a part that describes the free electromagnetic field (first term). The second term describes the fermion Lagrangian; it is obtained from the free particle Lagrangian of (B.34) by replacing the derivative ∂_μ by the covariant derivative

$$D_\mu = \partial_\mu + \mathrm{i}eA_\mu \tag{B.57}$$

(minimal coupling). Here e is the electron's charge ($e = -|e|$).

The Lagrangian (B.56) is obviously invariant under a variation of the fermion fields of the form

$$\Psi \to \Psi' = \mathrm{e}^{-\mathrm{i}\epsilon e}\Psi \ . \tag{B.58}$$

Comparison with (B.49) gives $q_{\alpha\beta} = e\,\delta_{\alpha\beta}$ so that the conserved "current" given by (B.54) is:

$$j_\mu(x) = e\,\bar{\Psi}\gamma_\mu\Psi \ . \tag{B.59}$$

Note that this conserved current is exactly the quantity that couples to the electromagnetic field in (B.56). This property allows one to identify the current (B.59) as the electromagnetic current of the electron fields.

Example: Complex Scalar Field

The Lagrangian for the case of a complex scalar field interacting with an electromagnetic field is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \left(D_\mu \phi\right)^* \left(D^\mu \phi\right) - m^2 \phi^* \phi \ . \tag{B.60}$$

This Lagrangian is simply the sum of the free electromagnetic Lagrangian (B.16) and the Lagrangian for a complex scalar field (B.32), where in the latter again the derivative ∂_μ has been replaced – through minimal substitution – by the covariant derivative D_μ (B.57).

The Lagrangian (B.60) is obviously invariant under the phase transformations

$$\begin{aligned} \phi(x) &\longrightarrow \mathrm{e}^{-\mathrm{i}\varepsilon e} \phi(x) \\ \phi^*(x) &\longrightarrow \mathrm{e}^{+\mathrm{i}\varepsilon e} \phi^*(x) \end{aligned} \tag{B.61}$$

The conserved current connected with this invariance can be obtained from the definition (B.54)

$$\begin{aligned} j^\mu &= -\mathrm{i} \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} e\phi + \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi^*\right)} (-e) \phi^* \right) \\ &= \mathrm{i} e \left(\phi^* D^\mu \phi - \phi \left(D^\mu \phi\right)^* \right) \ . \end{aligned} \tag{B.62}$$

The conserved charge is then given by

$$Q = \int \mathrm{d}^3 x \, j^0(x) = \mathrm{i} e \int \mathrm{d}^3 x \, \left(\phi^* D^0 \phi - \phi \left(D^0 \phi \right)^* \right) \ . \tag{B.63}$$

It is evident that the charge disappears if the field ϕ is real. It is furthermore remarkable that now the electromagnetic field A^μ appears in the conserved current (through the covariant derivative $D^\mu = \partial^\mu + \mathrm{i} e A^\mu$). Again the conserved current provides the coupling to the electromagnetic field.

Appendix C

RENORMALIZATION INTEGRALS

In Chapt. 8 we encountered divergent integrals in the calculation of higher-order two- and fourpoint functions. To evaluate these integrals analytically in n dimensional Minkowski space is the purpose of this appendix.

The typical integral to be evaluated reads

$$I = \int \mathrm{d}^n q \, \frac{1}{(q^2 - m^2)^l} \ . \tag{C.1}$$

with one timelike and $n-1$ spacelike coordinates.

We now perform a Wick rotation and evaluate the integral in d-dimensional Euclidean space. Using the transformations (4.40),(4.41)

$$q_n = -\mathrm{i} q_0 \qquad \mathrm{d}^n q_{\mathrm{E}} = -\mathrm{i}\mathrm{d}^n q \ , \tag{C.2}$$

where q_0 is the timelike coordinate, and going to real q_n we obtain the Euclidean integral

$$I = (-)^l \mathrm{i} \int \mathrm{d}q_{\mathrm{E}} \, q^{n-1} \, d\Omega_n \, \frac{1}{(q_{\mathrm{E}}^2 + m^2)^l} \tag{C.3}$$

with

$$q_{\mathrm{E}}^2 = \sum_{i=1}^{n-1} q_i^2 \ . \tag{C.4}$$

We thus have for the integral[1]

$$I = (-)^l 2\pi \mathrm{i} \int \mathrm{d}\Omega_n \int\limits_0^\infty q^{n-1} \mathrm{d}q \frac{1}{(q^2 + m^2)^l} \ . \tag{C.5}$$

[1]In order to simplify the notation we are no longer denoting the Euclidean vectors by the subscript E.

The integral over the solid angle can be analytically performed and yields[2]

$$\int \mathrm{d}\Omega_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \ . \tag{C.7}$$

so that we have for the integral

$$I = (-)^l \mathrm{i} \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty q^{n-1} \mathrm{d}q \frac{1}{(q^2+m^2)^l} \ . \tag{C.8}$$

The remaining integral can be evaluated with the help of Euler's β function

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2\int_0^\infty \mathrm{d}t\, t^{2x-1}(1+t^2)^{-x-y} \ , \tag{C.9}$$

(see Abramowitz (6.2.1)); we obtain

$$\int \frac{q^{n-1}}{(q^2+m^2)^l} \mathrm{d}q = \frac{1}{2} m^{n-2l} B(\frac{n}{2}, l-\frac{n}{2}) = \frac{1}{2} m^{n-2l} \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(l-\frac{n}{2}\right)}{\Gamma(l)} \ . \tag{C.10}$$

Thus the complete integral is now given by

$$I = (-)^l \mathrm{i} \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \frac{1}{2} m^{n-2l} \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(l-\frac{n}{2}\right)}{\Gamma(l)} = (-)^l \mathrm{i} \pi^{\frac{n}{2}} m^{n-2l} \frac{\Gamma\left(l-\frac{n}{2}\right)}{\Gamma(l)} \ . \tag{C.11}$$

This expression no longer depends on the special metric; it can thus analytically be continued back into Minkowski space. In this expression, furthermore, the dependence of the integral on the dimension n is explicit; it can also analytically be continued to the physical case $n = 4$.

[2] This can be seen by considering the n-th power of the Gaussian integral

$$\begin{aligned} \left(\sqrt{\pi}\right)^n &= \left(\int_0^\infty \mathrm{d}x\, \mathrm{e}^{-x^2}\right)^n = \int \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n \mathrm{e}^{-\sum_{k=1}^n x_k^2} \\ &= \int \mathrm{d}\Omega_n \int_0^\infty x^{n-1} \mathrm{e}^{-x^2} \mathrm{d}x = \int \mathrm{d}\Omega_n \frac{1}{2} \int_0^\infty \mathrm{d}(x^2)\left(x^2\right)^{\frac{n-2}{2}} \mathrm{e}^{-x^2} \\ &= \int \mathrm{d}\Omega_n \frac{1}{2} \Gamma\left(\frac{n}{2}\right) \ . \end{aligned} \tag{C.6}$$

Appendix D

GRASSMANN INTEGRATION FORMULA

A more general proof than the specific examples given in section (9.1.2.1) uses the fact that any antisymmetric $(n \times n)$ dimensional matrix M can be brought into a block-diagonal form through an orthogonal transformation

$$M \longrightarrow M' = O^T M O \tag{D.1}$$

$$M' = \begin{pmatrix} 0 & M_1 & 0 & 0 & \cdots & 0 & 0 \\ -M_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & M_2 & \cdots & \vdots & \vdots \\ 0 & 0 & -M_2 & 0 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & M_n \\ 0 & 0 & 0 & 0 & \cdots & -M_n & 0 \end{pmatrix} \tag{D.2}$$

for n even (for odd n the matrix M' has one more row and one more column with all zeros, so that its determinant vanishes). The form of M' shows clearly that

$$\det(M) = \det(M') = M_1^2 M_2^2 \cdots M_n^2 \ . \tag{D.3}$$

We thus get

$$\begin{aligned} I(n) &= \int \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_n \, e^{-\eta^T M \eta} = \int \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_n \, e^{-\eta^T (O M' O^T) \eta} \\ &= \int \mathrm{d}\epsilon_1 \ldots \mathrm{d}\epsilon_n \, J \, e^{-\epsilon^T M' \epsilon} \end{aligned} \tag{D.4}$$

with $\epsilon = O^T \eta$ and J being the Jacobian. In our special case here, O is orthogonal and therefore, $[\det(O)]^{-1} = J = 1$.

We can now continue with the evaluation of $I(n)$ in (D.4) and get

$$I(n) = \int \mathrm{d}\epsilon_1 \dots \mathrm{d}\epsilon_n \, (-)^{\frac{n}{2}} \frac{1}{\left(\frac{n}{2}\right)!} \left(\epsilon^T M' \epsilon\right)^{\frac{n}{2}} \ , \tag{D.5}$$

because only that term in the expansion of the exponental can contribute in a n-dimensional integral that has exactly n factors of ϵ_i. In detail, we have

$$I(n) = \frac{(-)^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!} \int \mathrm{d}\epsilon_1 \dots \mathrm{d}\epsilon_n \left(\epsilon_\alpha M'_{\alpha\beta} \epsilon_\beta\right)^{\frac{n}{2}} \ . \tag{D.6}$$

Since M' has the special, block-diagonal form given above (D.2), we have $M'_{\alpha\beta} = 0$, except for $\alpha = 2\alpha', \beta = \alpha - 1$ or $\alpha = 2\alpha' - 1, \beta = \alpha + 1$. Again, there can be no two equal $\epsilon's$ under the integral, if this integral is to be nonzero. We thus have

$$\begin{aligned} I(n) &= \frac{(-)^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!} \int \mathrm{d}\epsilon_1 \dots \mathrm{d}\epsilon_n \Big(\epsilon_{2\alpha'} M'_{2\alpha',2\alpha'-1} \epsilon_{2\alpha'-1} \\ &\qquad\qquad - \epsilon_{2\alpha'-1} M'_{2\alpha'-1,2\alpha'} \epsilon_{2\alpha'}\Big)^{\frac{n}{2}} \\ &= \frac{2^{\frac{n}{2}}(-)^n}{\left(\frac{n}{2}\right)!} \int \mathrm{d}\epsilon_1 \dots \mathrm{d}\epsilon_n \left(\epsilon_{2\alpha'-1} M'_{2\alpha'-1,2\alpha'} \epsilon_{2\alpha'}\right)^{\frac{n}{2}} \ , \end{aligned} \tag{D.7}$$

since M' is antisymmetric. Here the index α' still runs from 1 to $n/2$; n is even so that the phase disappears.

We now perform the exponentiation

$$\begin{aligned} I(n) &= \frac{2^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!} \int \mathrm{d}\epsilon_1 \dots \mathrm{d}\epsilon_n \underbrace{\sum_{\alpha'} \sum_{\beta'} \dots \sum_{\nu'}}_{n\,\mathrm{sums}} \epsilon_{2\alpha'-1} \epsilon_{2\alpha'} \cdots \epsilon_{2\nu'-1} \epsilon_{2\nu'} \\ &\qquad \times M'_{2\alpha'-1,2\alpha'} M'_{2\beta'-1,2\beta'} \cdots M'_{2\nu'-1,2\nu'} \ . \end{aligned} \tag{D.8}$$

The product of the factors $M'_{2\alpha'-1,2\alpha'} \cdots M'_{2\nu'-1,2\nu'}$ just gives $\sqrt{\det(M')}$. Since the ϵ_i appear always pairwise, they can be commuted to normal ordering $(\epsilon_n \cdots \epsilon_1)$ without sign-change; the sums then give $(n/2)!$ times the same result. Thus

$$\begin{aligned} I(n) &= 2^{\frac{n}{2}} \int \mathrm{d}\epsilon_1 \dots \mathrm{d}\epsilon_n \, \epsilon_n \cdots \epsilon_1 \sqrt{\det(M')} \\ &= 2^{\frac{n}{2}} \sqrt{\det(M')} \ . \end{aligned} \tag{D.9}$$

The last step is possible because we have for each blockmatrix $M_{\beta\alpha}$ separately

$$\begin{aligned}\det(M_{\beta\alpha}) &= -\left(M_{2\alpha-1,2\alpha}\right)\left(+M_{2\alpha,2\alpha-1}\right) \\ &= +\left(M_{2\alpha-1,2\alpha}\right)^2 \ ,\end{aligned} \tag{D.10}$$

and for $\det(M')$

$$\det(M') = \prod_\alpha \det(M'_{\beta\alpha}) \ . \tag{D.11}$$

Using (D.3) we thus get the desired result

$$\int \mathrm{d}\epsilon_1 \ldots \mathrm{d}\epsilon_n \, e^{-\epsilon^T M \epsilon} = 2^{\frac{n}{2}} \sqrt{\det(M)} \ . \tag{D.12}$$

Equation (D.12) corresponds to (5.5) for the boson case. Note that here – in contrast to the bosonic case – the determinant appears in the numerator!

Appendix E

BIBLIOGRAPHY

We list here a number of textbooks that provide for more information on the topics treated in this manuscript.

Introductory texts

1. J.J. Sakurai, *Modern Quantum Mechanics*, Addison-Wesley, Redwood City, 1985; contains a very nice introduction into the non-relativistic path integral formalism.

2. U. Mosel, *Fields, Symmetries, and Quarks*, McGraw-Hill, Hamburg 1989; contains a short introduction into fundamentals of field theories and gauge field theories. 2nd revised and enlarged edition: Springer, Heidelberg, March 1999

3. T.D. Lee, *Particle Physics and Introduction to Field Theory*, Harwood, Chur, 1981; very physical treatment of field theory and high-energy phenomenology, somewhat annoying typesetting.

4. L.H. Ryder, *Quantum Field Theory*, Cambridge University Press, 1985; very didactical presentation of modern field theories.

5. C. Itzykson and J-B. Zuber, *Quantum Field Theory*, McGraw-Hill, New York, 1985; nearly comprehensive book on field theory, didactically not very well done.

6. M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley, Reading, 1995; also didactically very good, rather comprehensive book on quantum field theory. The new standard book on this topic!

Specialized Literature

1. Ta-Pei Cheng and Ling-Fong Li, *Gauge Theory of elementary particle physics*, Clarendon Press, Oxford, 1984; quite comprehensive book, well-structured, easy to read, but not always complete in its arguments and derivations.

2. P. Ramond, *Field Theory, a modern Primer*, Addison-Wesley, Redwood City, 1990; very nice introduction to field theories, also modern aspects, builds entirely on path integral formalism.

3. P.H. Frampton, *Gauge Field Theories*, Benjamin-Cummings, Menlo Park, 1987; bad typesetting, not easy to read.

4. S. Pomorski, *Gauge Field Theories*, Cambridge University Press, 1987; not very didactical, but quite deep.

5. D. Bailin and A. Love, *Introduction to Gauge Field Theory*, 2nd ed., Hilger, Bristol, 1994; very comprehensive treatment of gauge field theories based on path integral methods from the start, didactical, contains modern aspects of field theory beyond Standard Model.