GAUGE-FIELD THEORIES

Ulrich Mosel
Institut für Theoretische Physik, Universität Giessen
D–35392 Giessen, Germany

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Contents

Part I

Preliminaries

Chapter 1

UNITS AND METRIC

1.1 Units

It is customary in elementary particle physics and field theories to choose the system of units such that the resulting expressions assume a simple form. In particular, instead of working with the usual three mechanical units for length, mass and time one introduces instead three basic units for velocity, action and length.

The choice of the velocity of light, c, as the unit for the velocity implies that

$$c = 1. \tag{1.1}$$

This in turn means that in such a system length and time have the same dimensions and are equivalent units. With $c = 1$ the relativistic energy-momentum relation assumes the simple form

$$E^2 = p^2 + m^2. \tag{1.2}$$

Choosing next the unit of action, $\hbar$, such that

$$\hbar = 1 \tag{1.3}$$

connects the dimensions of mass, time and length. Since time and length are equivalent mass assumes the dimension of an inverse length; the same holds consequently for the momentum and the energy (equation 1.2). The choice of 1 fm $= 10^{-13}$ cm as a length unit, for example, leads to masses, momenta and energies all given in fm^{-1}. The transformation of the expressions back to the standard MKSA system can be achieved by multiplying all quantities with the appropriate combinations of $\hbar$ and c.

1.2 Metric and Notation

Spatial three-vectors in this book are written with an overhead arrow; in contrast, vectors in some internal space are distinguished by boldface print.

In relativistic expressions four-vectors are always written as

$$A^\mu = (A^0, \vec{A}) = (A^0, A^1, A^2, A^3) \quad . \tag{1.4}$$

Four-vectors with an upper index are called *contravariant* vectors. *Covariant* vectors, denoted by lower indices, are then defined by

$$A_\mu = g_{\mu\nu} A^\nu \tag{1.5}$$

with the *metric tensor*

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{1.6}$$

The metric tensor with upper indices is defined as the inverse of g. It is given by

$$g^{\mu\nu} = g_{\mu\nu} \; . \tag{1.7}$$

A scalar product of 2 four-vectors is then defined by

$$A \cdot B = A_\mu B^\mu = A^\mu B_\mu \; . \tag{1.8}$$

The invariant square of the four-vector is thus given by

$$A^2 = A_\mu A^\mu = A_0^2 - \vec{A}^2 \quad . \tag{1.9}$$

Here, as always in the text, the Einstein convention of summing over double indices, one lower and one upper, is used. The components of four-vectors are generally labeled by Greek indices, whereas roman indices are used to refer specifically to the last three (vector) components.

The space-time four-vector is

$$x^\mu = (x^0, \vec{x}) = (t, \vec{x}) \; , \tag{1.10}$$

and the four-momentum is given by

$$p^\mu = (E, \vec{p}) \; . \tag{1.11}$$

Finally, the components of the contravariant four-gradient are abbreviated by

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\vec{\nabla} \right) \ , \tag{1.12}$$

so that the four-momentum operator is given by

$$\hat{p}^\mu = \mathrm{i}\partial^\mu \ , \tag{1.13}$$

and the Lorentz-invariant *d'Alembert operator* by

$$\partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 = \Box \ . \tag{1.14}$$

Other important four-vectors are the current

$$j^\mu = \left(\rho, \vec{j} \right) \tag{1.15}$$

and the electromagnetic potential

$$A^\mu = \left(\phi, \vec{A} \right) \ . \tag{1.16}$$

Part II

Fundamentals of Field Theory

Chapter 2

CLASSICAL FIELDS

In this chapter a few essential facts of classical field theory will be discussed. It will first be shown how to derive the equations of motion of a field theory, for example the Maxwell equations of electrodynamics, from a Lagrangian. Secondly, the connection between symmetries of the Lagrangian and conservation laws will be discussed. This is one of the central points for all the theoretical developments in this book.

2.1 Equations of Motion

The equations of motion of classical mechanics can be obtained from a Lagrange function by using Hamilton's principle that the action for a given mechanical system is minimal for the physical space-time development of the system.

The equations of motion for fields that determine their space-time dependence can be obtained in an analogous way by identifying the field amplitudes at a coordinate $\vec{x}$ with the dynamical variables (coordinates) of the theory.

Let the functions that describe a field be denoted by

$$\Phi_\alpha(x) \quad \text{with} \quad x^\mu = (t, \vec{x}) \ . \tag{2.1}$$

The Lagrangian L of the system is expressed in terms of a Lagrange density $\mathcal{L}$, as follows:

$$L = \int \mathcal{L}(\Phi_\alpha, \partial_\mu \Phi_\alpha) d^3x \tag{2.2}$$

where the spatial integration is performed over the volume of the system.

The action S is then defined as usual by

$$S = \int_{t_0}^{t_1} L \, dt = \int_{\Omega} \mathcal{L}(\Phi_\alpha, \partial_\mu \Phi_\alpha) d^4x \tag{2.3}$$

with $d^4x = d^3x \; dt$. The, in general finite, space-time volume of the system is denoted by Ω.

As pointed out before the fields $\Phi_\alpha(x)$ play the same role as the generalized coordinates q_i in classical mechanics; the analogy here is such that the fields Φ_α correspond to the coordinates q and the points $\vec{x}$ to the classical indices i. The corresponding velocities are in a direct analogy given by the time derivative of $\Phi_\alpha : \partial_t \Phi_\alpha$. Lorentz covariance then requires that also the derivatives with respect to the first three coordinates appear; this explains the presence of the four-gradients $\partial_\mu \Phi_\alpha$ in eq. (2.3).

The action S is now varied with respect to the fields and their derivatives:

$$\begin{aligned} \Phi_\alpha(x) &\rightarrow \Phi'_\alpha(x) = \Phi_\alpha(x) + \delta\Phi_\alpha(x) \\ \partial_\mu \Phi_\alpha(x) &\rightarrow (\partial_\mu \Phi_\alpha(x))' = \partial_\mu \Phi_\alpha(x) + \delta(\partial_\mu \Phi_\alpha(x)) \quad . \end{aligned} \tag{2.4}$$

This yields

$$\begin{aligned} \delta S &= \int_{\Omega} \mathcal{L}(\Phi'_\alpha, (\partial_\mu \Phi_\alpha)') d^4x - \int_{\Omega} \mathcal{L}(\Phi_\alpha, \partial_\mu \Phi_\alpha) d^4x \\ &= \int \left(\frac{\partial \mathcal{L}}{\partial \Phi_\alpha} \delta\Phi_\alpha + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_\alpha)} \delta(\partial_\mu \Phi_\alpha) \right) d^4x \\ &= \int \left[\left(\frac{\partial \mathcal{L}}{\partial \Phi_\alpha} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_\alpha)} \right) \delta\Phi_\alpha \right. \\ &\qquad \left. + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_\alpha)} \delta\Phi_\alpha \right) \right] d^4x \quad . \end{aligned} \tag{2.5}$$

According to the Einstein-convention a summation over μ is implicitly contained in this expression.

The equations of motion are now obtained from the variational principle

$$\delta S = 0 \tag{2.6}$$

for arbitrary variations $\delta\Phi_\alpha$ under the constraint that

$$\delta\Phi_\alpha(t_0) = \delta\Phi_\alpha(t_1) = 0 \quad .$$

The last term in eq. (2.5) can be converted into a surface integral by using Gauss's law; for fields which are localized in space this surface integral vanishes. Since the variations $\delta\Phi_\alpha$ are arbitrary the condition $\delta S = 0$ leads to the equations of motion

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)}\right) - \frac{\partial\mathcal{L}}{\partial\Phi_\alpha} = 0 \quad . \tag{2.7}$$

The relativistic equivalence principle demands that these equations have the same form in every inertial frame of reference, i.e. that they are Lorentz-covariant. This is only possible if $\mathcal{L}$ is a Lorentz scalar, i.e. if it has the same functional dependence on the fields and their derivatives in each reference frame.

In a further analogy to classical mechanics, the canonical field momentum is defined as

$$\Pi_\alpha = \frac{\partial\mathcal{L}}{\partial\dot{\Phi}_\alpha} = \frac{\partial\mathcal{L}}{\partial(\partial_0\Phi_\alpha)} \quad . \tag{2.8}$$

From $\mathcal{L}$ and Π_α the Hamiltonian H is obtained as

$$H = \int \mathcal{H}\, d^3x = \int (\Pi_\alpha\dot{\Phi}_\alpha - \mathcal{L}) d^3x \quad . \tag{2.9}$$

The Hamiltonian H represents the energy of the field configuration.

2.1.1 Examples

2.1.1.1 Electrodynamics

The two inhomogeneous Maxwell equations[1]

$$\vec{\nabla}\cdot\vec{E} = \rho \tag{2.10}$$

$$\vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{j} \tag{2.11}$$

can be rewritten by introducing the four-vector field

$$A^\mu = (A^0, \vec{A}) \ , \tag{2.12}$$

defined via

$$\begin{aligned} \vec{B} &= \vec{\nabla}\times\vec{A} \\ \vec{E} &= -\vec{\nabla}A^0 - \frac{\partial\vec{A}}{\partial t} \quad . \end{aligned} \tag{2.13}$$

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[1] Here the Heaviside units are used with $c = 1$ and $\epsilon_0 = \mu_0 = 1$.

Note that the 2 homogeneous Maxwell equations are now automatically fulfilled.

In terms of A^μ the antisymmetric field tensor

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{2.14}$$

can be defined. With its help the equations of motion (2.10,2.11) can be written as

$$\partial_\mu F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^\mu} = j^\nu \quad , \tag{2.15}$$

with the four-current

$$j^\nu = (\rho, \vec{j}) \quad . \tag{2.16}$$

Equation (2.15) is the equation of motion for the field tensor or the four-vector field A_μ.

It is easy to show that eq. (2.15) can be obtained from the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\nu A_\nu \tag{2.17}$$

by using equation (2.7); the fields A^ν here play the role of the fields Φ_α in (2.7). We have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial A_\nu} &= -j^\nu \\ \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} &= -\frac{1}{4} 2(+F^{\mu\nu} - F^{\nu\mu}) = -F^{\mu\nu} \quad . \end{aligned} \tag{2.18}$$

The last step is possible because F is an antisymmetric tensor.

The equation of motion is therefore

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = -\frac{\partial F^{\mu\nu}}{\partial x^\mu} + j^\nu = 0 \quad , \tag{2.19}$$

in agreement with eq. (2.15). It is now easy to interpret the two terms in $\mathcal{L}$ (eq. (2.17)): the first one gives the Lagrangian for the free electromagnetic field, whereas the second describes the interaction of the field with charges and currents.

The two homogeneous Maxwell equations can also be expressed in terms of the field tensor by first introducing the *dual field tensor*

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \; ; \tag{2.20}$$

here $\varepsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita antisymmetric tensor which assumes the values +1 or -1 according to whether $(\mu\nu\rho\sigma)$ is an even or odd permutation of (0,1,2,3), and 0 otherwise. In terms of $\tilde{F}$ the homogeneous Maxwell equations read

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \ . \tag{2.21}$$

By definition (2.14) the field tensor is invariant under a gauge transformation

$$A^\mu \longrightarrow A^{\mu\prime} = A^\mu + \partial^\mu \phi \ . \tag{2.22}$$

The same then holds for the equation of motion (2.19). This freedom can be used to impose constraints on the four components of the vector field A^μ.

2.1.1.2 Massive vector fields

Vector fields in which – in contrast to the electromagnetic field - the field quanta are massive are described by the so-called Proca equation

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = j^\nu \ . \tag{2.23}$$

Operating on this equation with the four-divergence ∂_ν gives, because F is antisymmetric,

$$m^2 \partial_\nu A^\nu = \partial_\nu j^\nu \ . \tag{2.24}$$

For $m \neq 0$ and a conserved current, this reduces the equation of motion (2.23) to

$$\left(\Box + m^2\right) A^\nu = j^\nu \ ; \qquad \partial_\nu A^\nu = 0 \ . \tag{2.25}$$

Thus for massive vector fields the freedom to make gauge transformations on the vector field is lost. The gauge-condition reduces the degrees of freedom of the field from 4 to 3. The space-like components represent the physical degrees of freedom.

The Lagrangian that leads to (2.23) is given by

$$\mathcal{L} = -\frac{1}{4} F^2 + \frac{1}{2} m^2 A^2 - j \cdot A \ . \tag{2.26}$$

2.1.1.3 Klein-Gordon fields

A particularly simple example is provided by the so-called Klein-Gordon field ϕ that obeys the equation of motion

$$\left(\partial_\mu \partial^\mu + m^2\right) \phi = \left(\Box + m^2\right) \phi = 0 \ ; \tag{2.27}$$

such a field describes scalar particles, i.e. particles without intrinsic spin. The Lagrangian leading to (2.27) is given by

$$\mathcal{L} = \frac{1}{2}\left((\partial_\mu \phi)\, (\partial^\mu \phi) - m^2 \phi^2 \right) \ . \tag{2.28}$$

An interesting case occurs if we consider 2 independent real scalar fields, ϕ_1 and ϕ_2, with the same mass m. The total Lagrangian is then simply given by a sum over the Lagrangians describing the individual fields, i.e.

$$\mathcal{L} = \frac{1}{2}\left((\partial_\mu \phi_1)\, (\partial^\mu \phi_1) - m^2 \phi_1^2 \right) + \frac{1}{2}\left((\partial_\mu \phi_2)\, (\partial^\mu \phi_2) - m^2 \phi_2^2 \right) \ . \tag{2.29}$$

On the other hand, we can also construct from the 2 real fields ϕ_1 and ϕ_2 2 complex fields, namely

$$\phi = \frac{1}{\sqrt{2}} \left(\phi_1 + \mathrm{i}\phi_2 \right) \tag{2.30}$$

and its complex conjugate. In terms of these the Lagrangian (2.29) can be rewritten into

$$\mathcal{L} = (\partial_\mu \phi)^\dagger\, (\partial^\mu \phi) - m^2 \phi^\dagger \phi \ . \tag{2.31}$$

2.1.1.4 Dirac Fields

A particularly simple example is provided by the Dirac field Φ for which the equation of motion is just the Dirac equation

$$\left(\mathrm{i}\gamma^\mu \partial_\mu - m \right) \Psi = 0 \ , \tag{2.32}$$

where the γ_μ are the usual (4×4) matrices of Dirac theory (see appendix A.1).

The corresponding Lagrangian is given by

$$\mathcal{L} = \bar{\Psi} \left(i\gamma^\mu \partial_\mu - m \right) \Psi \quad . \tag{2.33}$$

This can be seen by identifying the fields Φ_α in eq. (2.7) with the four components of the Dirac spinor $\bar{\Psi} = \Psi^\dagger \gamma_0$. Since $\mathcal{L}$ does not depend on $\partial_\mu \bar{\Psi}$ the equation of motion is simply given by

$$\frac{\partial \mathcal{L}}{\partial \bar{\Psi}} = \left(\mathrm{i}\gamma^\mu \partial_\mu - m \right) \Psi = 0 \tag{2.34}$$

2.2 Symmetries and Conservation Laws

As in classical mechanics there is also in field theory a conservation law associated with each continuous symmetry of $\mathcal{L}$. The theorem which describes the connection between the invariance of the Lagrangian under a continuous symmetry transformation and the related conserved current is known as Noether's theorem. In the following this will be illustrated for different types of symmetries which then lead to the well known conservation laws.

2.2.1 Geometrical Space-Time Symmetries

In this section we investigate the consequences of translations in four-dimensional space-time, i.e. infinitesimal transformations of the form

$$x_\nu \to x'_\nu = x_\nu + \epsilon_\nu \quad , \tag{2.35}$$

where the infinitesimal shift ϵ_ν is independent of the coordinate x_ν. Under such transformations the change of $\mathcal{L}$ is given by

$$\delta\mathcal{L} = \epsilon_\nu \frac{\partial\mathcal{L}}{dx_\nu} = \epsilon_\nu \partial^\nu \mathcal{L} \quad , \tag{2.36}$$

since $\mathcal{L}$ is a scalar.

If now $\mathcal{L}$ is required to be form-invariant under translations, it does not explicitly depend on x_ν, so that the derivative ∂_ν in (2.36) acts only on the arguments of $\Phi(x_\nu)$ and $\partial_\nu\Phi(x_\nu)$. The change of $\mathcal{L}$ can therefore also be expressed as

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\Phi_\alpha}\delta\Phi_\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)}\delta(\partial_\mu\Phi_\alpha) \quad . \tag{2.37}$$

Since the variation and the differentiation are linear operations their order can be interchanged. We thus obtain

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\Phi_\alpha}\delta\Phi_\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)}\partial_\mu\left(\delta\Phi_\alpha\right) \quad . \tag{2.38}$$

Using the equation of motion (2.7) for rewriting the first term yields

$$\begin{aligned} \delta\mathcal{L} &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)}\right)\delta\Phi_\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)}\partial_\mu\left(\delta\Phi_\alpha\right) \\ &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)}\delta\Phi_\alpha\right) \quad . \end{aligned} \tag{2.39}$$

The changes of the fields Φ_α appearing here are given by

$$\delta\Phi_\alpha(x) = \epsilon_\nu \frac{\partial\Phi_\alpha}{\partial x_\nu} = \epsilon_\nu \partial^\nu \Phi_\alpha(x) \tag{2.40}$$

Inserting eqs. (2.40) into (2.39) yields

$$\delta\mathcal{L} = \epsilon_\nu \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)} \partial^\nu \Phi_\alpha \right) \quad . \tag{2.41}$$

Equating eqs. (2.41) and (2.36) finally gives

$$\partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)} \partial^\nu \Phi_\alpha - \mathcal{L}\, g^{\nu\mu} \right) = 0 \quad , \tag{2.42}$$

since the ϵ_ν are arbitrary. By defining the tensor $T^{\mu\nu}$ as

$$T^{\mu\nu} \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)} \partial^\nu \Phi_\alpha - \mathcal{L}\, g^{\mu\nu} \tag{2.43}$$

this equation reads

$$\partial_\mu T^{\mu\nu} = 0 \quad . \tag{2.44}$$

Relation (2.44) has the form of a continuity equation. Spatial integration over a finite volume yields

$$\frac{d}{dt} \left(\int\limits_V T^{0\mu}(x) d^3x \right) = - \int\limits_V \frac{\partial T^{i\mu}}{\partial x_i} d^3x = \oint_S \vec{S}^{(\mu)} \cdot \vec{n}\, dS \quad . \tag{2.45}$$

Here $\vec{n}$ is a unit vector vertical on the surface S pointing outwards and $\vec{S}^{(\mu)}$ is a three-vector:

$$\vec{S}^{(\mu)} = -(T_{1\mu}, T_{2\mu}, T_{3\mu}) \quad . \tag{2.46}$$

The surface integral on the rhs of eq. (2.45) is taken over the surface S of volume V. For localized fields it can be made to vanish by extending the volume towards infinity. It is then evident that the quantities

$$P^\mu = \int T^{0\mu} d^3x \tag{2.47}$$

are conserved. These are the components of the four-momentum of the field, as can be verified for the zeroth component,

$$\begin{aligned} P^0 &= \int T^{00}\, d^3x = \int \left(\frac{\partial\mathcal{L}}{\partial(\partial_0\Phi_\alpha)} \partial^0 \Phi_\alpha - \mathcal{L} \right) d^3x \\ &= \int (\Pi_\alpha \dot{\Phi}_\alpha - \mathcal{L})\, d^3x = H \quad , \end{aligned} \tag{2.48}$$

according to eqs. (2.9) and (2.10). The spatial components of the field momentum are

$$P^k = \int T^{0k}\, d^3x = \int \frac{\partial \mathcal{L}}{\partial(\partial_0 \Phi_\alpha)} \partial^k \Phi_\alpha \, d^3x \quad . \tag{2.49}$$

In classical mechanics the form invariance of the Lagrangian under rotations leads to the conservation of angular momentum. Analogously, in a relativistic field theory the form invariance of $\mathcal{L}$ under four-dimensional space-time rotations (Lorentz covariance) leads to the conservation of a quantity that is identified with the angular momentum of the field. To obtain the same form for the angular momentum as in classical mechanics it is essential that $T_{\nu\mu}$ is symmetric. This can always be achieved, because Eq. (2.44) does not define the tensor T uniquely. We can always add a term of the form $\partial_\lambda D^{\lambda\mu\nu}$, where $D^{\lambda\mu\nu}$ is a tensor antisymmetric in the indices λ and μ, such that T becomes symmetric.

Comparing (2.45) with (2.49) and assuming T to be symmetric we see that the normal components of the vectors $\vec{S}^{(\mu)}$ in eq. (2.46) describe the momentum flow through the surface S of the volume V and thus determine the "radiation pressure" of the field[2]. These properties allow us to identify $T_{\mu\nu}$ as the energy-momentum tensor of the field. For the Lagrangian (2.17) of electrodynamics $T_{\mu\nu}$ is just the well-known Maxwell's stress tensor.

As already mentioned at the beginning of this chapter these conservation laws are special cases of Noether's theorem, which can be summarized for the general case as follows:

> *Each continuous symmetry transformation that leaves the Lagrangian invariant is associated with a conserved current. The spatial integral over this current's zeroth component yields a conserved charge.*

2.2.2 Internal Symmetries

Relativistic field theories may contain conservation laws that are not consequences of space-time symmetries of the Lagrangian, but instead are connected with symmetries in the internal degrees of freedom such as, e. g. , isospin or charge.

We therefore now allow for a mixture of the different fields under the transformation

$$\Phi_\alpha(x) \to \Phi'_\alpha(x) = \mathrm{e}^{-\mathrm{i}\epsilon q_{\alpha\beta}} \Phi_\beta \quad , \tag{2.50}$$

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[2] More precisely, $S_k^{(\mu)}$ denotes the flux of the μ th component of the field momentum in the direction x_k.

where ϵ is an infinitesimal parameter and the $q_{\alpha\beta}$ are fixed c-numbers. We then have

$$\delta\Phi_\alpha(x) = \Phi'_\alpha(x) - \Phi_\alpha(x) = -\mathrm{i}\ \epsilon\ q_{\alpha\beta}\Phi_\beta(x) \quad . \tag{2.51}$$

The change of the Lagrangian is in general given by eq. (2.39)

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)} \delta\Phi_\alpha \right) \quad . \tag{2.52}$$

If $\mathcal{L}$ is invariant under this variation $\delta\Phi_\alpha$, then we have

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)} \delta\Phi_\alpha \right) = 0 \quad . \tag{2.53}$$

Eq. (2.53) is in the form of a continuity equation for the "current"

$$j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)} \delta\Phi_\alpha \frac{1}{\epsilon} \quad . \tag{2.54}$$

Inserting the field variations $\delta\Phi_\alpha$ yields for the current

$$j^\mu(x) = -\mathrm{i}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)} q_{\alpha\beta}\Phi_\beta \quad . \tag{2.55}$$

Eqs. (2.54) and (2.55) imply that the "charge"

$$Q = \int j^0(x)\, d^3x = -\mathrm{i}\int \frac{\partial\mathcal{L}}{\partial(\partial_0\Phi_\alpha)} q_{\alpha\beta}\Phi_\beta\, d^3x \tag{2.56}$$

of the system is conserved. The physical nature of these "charges" and "currents" has to remain open. It depends on the specific form of the symmetry transformation (2.50) and can be determined only by coupling the system to external fields.

Example: QED. To illustrate this conservation law, the theory of electromagnetic interactions is used as an example. However, in contrast to the considerations in section 2.1.1 we now consider a coupled system of a fermion field $\Psi(x)$ and the electromagnetic field $A_\mu(x)$ to determine the physical meaning of the conserved current. Together with a quantization procedure this theory is called Quantum Electrodynamics (QED). The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}\left[\mathrm{i}\gamma^\mu(\partial_\mu + \mathrm{i}eA_\mu) - m\right]\Psi \quad . \tag{2.57}$$

$\mathcal{L}$ contains a part that describes the free electromagnetic field (first term). The second term, to be discussed and explained in detail later (in chapt. 7), describes the fermion Lagrangian; it is obtained from the free particle Lagrangian of eq. (2.33) by replacing the derivative ∂_μ by the covariant derivative

$$D_\mu = \partial_\mu + \mathrm{i}eA_\mu \tag{2.58}$$

(minimal coupling).

The Lagrangian (2.57) is obviously invariant under a variation of the fermion fields of the form

$$\Psi \to \Psi' = \mathrm{e}^{-\mathrm{i}\epsilon e}\Psi \quad . \tag{2.59}$$

Comparison with eq. (2.50) gives $q_{\alpha\beta} = e\ \delta_{\alpha\beta}$ so that the conserved "current" given by eq. (2.55) is:

$$j_\mu(x) = e\ \bar{\Psi}\gamma_\mu\Psi \quad . \tag{2.60}$$

Note that this conserved current is exactly the quantity that couples to the electromagnetic field in eq. (2.57). This property allows one to identify the current (2.60) as the electromagnetic current of the electron fields.

Example: Complex scalar field. The Lagrangian for the case of a complex scalar field interacting with an electromagnetic field is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger (D^\mu\phi) - m^2\phi^\dagger\phi \ . \tag{2.61}$$

This Lagrangian is simply the sum of the free electromagnetic Lagrangian (2.17) and the Lagrangian for a complex scalar field (2.31), where in the latter again the derivative ∂_μ has been replaced – through minimal substitution – by the covariant derivative D_μ (2.58).

The Lagrangian (2.61) is obviously invariant under the phase transformations

$$\begin{aligned} \phi(x) &\longrightarrow e^{-\mathrm{i}\varepsilon e}\phi(x) \\ \phi^\dagger(x) &\longrightarrow e^{+\mathrm{i}\varepsilon e}\phi^\dagger(x) \end{aligned} \tag{2.62}$$

The conserved current connected with this invariance can be obtained from the definition (2.55)

$$\begin{aligned} j^\mu &= -\mathrm{i}\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}e\phi + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi^\dagger\right)}(-e)\phi^\dagger\right) \\ &= \mathrm{i}e\left(\phi^\dagger D^\mu\phi - \phi\left(D^\mu\phi\right)^\dagger\right) \ . \end{aligned} \tag{2.63}$$

The conserved charge is then given by

$$Q = \int d^3x\, j^0(x) = \mathrm{i}e \int d^3x\, \left(\phi^\dagger D^0 \phi - \phi \left(D^0 \phi \right)^\dagger \right) \ . \tag{2.64}$$

It is evident that the charge disappears if the field ϕ is real. It is furthermore remarkable that now the electromagnetic field A^μ appears in the conserved current (through the covariant derivative $D^\mu = \partial^\mu + \mathrm{i}eA^\mu$). Again the conserved current provides the coupling to the electromagnetic field.

Part III

Global Symmetries

Chapter 3

SYMMETRIES OF MESON AND BARYON SYSTEMS

In chapter 2 it was shown how geometrical and internal symmetries of the Lagrangian lead to conserved currents and charges. In this chapter we take the opposite approach of using the observations of conservation laws to deduce the required symmetries of the Lagrangian. These, together with a desire for simplicity, will allow us to formulate a Lagrangian.

To demonstrate this construction of a Lagrangian out of the underlying symmetry properties, we first consider the physics of strongly interacting particles, namely mesons and baryons. In this context we consider both as elementary particles. This is a very useful approximation for describing nuclear phenomena at low and medium energies where any possible substructure could not be resolved, but is certainly not valid for the description of high-energy processes.

Nevertheless, historically this approach has lead to the prediction of the substructure of the mesons and baryons. This will be discussed later in this book. After some general remarks about the underlying symmetries, we will start out with the pion-nucleon system as the simplest example for the variety of the known systems of mesons and baryons.

3.1 $U(1)$ Symmetry

The simplest transformations of the fields appearing in a Lagrangian are global one-parameter changes of their phases. These transformations (2.50) with $q_{\alpha\beta} = q\delta_{\alpha\beta}$ for different values of the only free parameter ϵ form a group, called $U(1)$. Invariance under $U(1)$ transformations is connected with charge conservation (cf. the discussion at the end of chapter 2) (remember that the term “charge” does not necessarily denote the electric charge; the nature of the charge can be determined only by coupling it to an external field).

The physical conservation laws connected with invariance of the Lagrangian under these transformations are those for the electric charge and the baryon and lepton numbers.

The conservation of the total electric charge of a closed system can be concluded from the electric neutrality of macroscopic matter. This yields very stringent tests for the conservation of charge with time. Furthermore, in all known reactions charge is also conserved; for example, in the β-decay of the neutron one finds for the relative change of charge: $\Delta q/q \leq 10^{-23}$.

Particles and resonance states that decay ultimately into protons are called baryons. These particles all have half-integer spins and masses higher than that of the nucleon. The baryons can only decay into other lower-mass baryons, and cannot disappear altogether; at present the lower limit for the half-life of the ultimate decay product, the proton, is $T_{1/2} \geq 10^{32}$ years. This empirical conservation can be expressed by introducing a baryon number $B = 1$ for the baryons, $B = -1$ for the antibaryons and $B = 0$ for all other particles. B is then conserved in all processes known up today.

To finish our brief discussion of $U(1)$-symmetries observed in nature, we discuss the application to the six leptons known at present (e, μ, τ and their neutrinos) together with their antiparticles. Like the baryons, these have all half-integer spin but interact only through the weak and the electromagnetic forces. Again, one observes that, if one ascribes a lepton number $L = 1$ to each generation of leptons, i.e. to each lepton and its neutrino separately, and $L = -1$ to their antiparticles, then the lepton number L is a conserved quantity for each generation.

All of these conservation laws are consequences of $U(1)$ symmetries of the Lagrangian that will be discussed in this section.

3.1.1 Properties of the Group $U(1)$

In chapter 2 it was shown that there exists a conserved Noether current connected with an invariance of $\mathcal{L}$ under the global phase transformation[1]

$$\Phi(x) \rightarrow \Phi'(x) = \mathrm{e}^{-\mathrm{i}\epsilon q}\Phi(x) \quad , \tag{3.1}$$

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[1] The following considerations hold for all types of fields, scalar, vector and Dirac.

where ϵ is an arbitrary constant and q a strength parameter for the conserved charge. The unitary transformations

$$U(\epsilon) = \mathrm{e}^{-\mathrm{i}\epsilon q} \quad , \tag{3.2}$$

with the fixed real constant q and labeled by the real parameter ϵ, form the elements of the so-called $U(1)$ group, the group of unitary transformations in one dimension (in general $U(n)$ is defined as the group of unitary matrices of dimension $n \times n$).

Before we start with the explicit construction of a $U(1)$-symmetric Lagrangian in the next section, we can already make some statements about its structure, based on the requirement of $U(1)$ invariance alone. A free Lagrangian that is required to lead to charge conservation can contain the fields only in such a form that it is invariant under the transformation (3.1). This can obviously be achieved only if $\mathcal{L}$ depends only on combinations of the form:

$$\bar{\Psi}(x) O \Psi(x) \tag{3.3a}$$

for fermions or

$$\Phi^\dagger O \Phi \tag{3.3b}$$

for bosons, where O is an operator acting in space and time or some internal space like, e.g. for the case of fermions, in the internal spinor space.

Quantummechanical consistency. It is instructive to consider whether after quantization the Noether charge $\hat{Q}$ is still a conserved quantity. Necessary for the quantum mechanical conservation law is that $\hat{Q}$ commutes with the Hamilton operator $\hat{H}$

$$\left[\hat{H}, \hat{Q}\right] = 0 \tag{3.4}$$

as follows directly from Heisenberg's equation of motion. Eq. (3.4) implies that the Hamiltonian operator $\hat{H}$ is invariant under the unitary transformation

$$\hat{H} \longrightarrow \hat{H}' = \mathrm{e}^{\mathrm{i}\epsilon\hat{Q}} \hat{H} \mathrm{e}^{-\mathrm{i}\epsilon\hat{Q}} \quad . \tag{3.5}$$

Since the classical Hamiltonian H of a physical system is always a $U(1)$-invariant functional of Φ and $\Phi^\dagger$ (see, e. g. eqs. (2.57) or (2.61)), the invariance (3.5) follows if the *field operators* transform as

$$\Phi \to \Phi' = \mathrm{e}^{\mathrm{i}\epsilon\hat{Q}} \Phi \mathrm{e}^{-\mathrm{i}\epsilon\hat{Q}} = \mathrm{e}^{-\mathrm{i}\epsilon q} \Phi \quad , \tag{3.6}$$

i. e. if the unitary transformation of Φ changes just the phase of Φ. Condition (3.6) leads immediately to

$$\left[\hat{Q}, \Phi\right] = -q\Phi \quad . \tag{3.7}$$

This relation shows that the consistency condition (3.6) implies that q has to be an eigenvalue of $\hat{Q}$ as can be seen by applying eq. (3.7) to the vacuum state

$$\begin{aligned} \hat{Q}\Phi|0> &= \Phi\hat{Q}|0> -q\Phi|0> \\ &= 0 - q\Phi|0> \quad . \end{aligned} \tag{3.8}$$

Essential for this result is the assumption that $\hat{Q}|0>= 0$, i.e. that the physical vacuum is uncharged.

Field quantization tells us that the state $\Phi|0>$ is indeed a one-antiparticle state; eq. (3.8) then tells us that the Noether charge is also conserved in a quantized theory if the parameter q appearing in the classical charge is an eigenvalue of the charge operator $\hat{Q}$. Turning this argument around, we could also postulate that the conservation law is quantummechanically consistent; this would then lead to eq. (3.8) and thus to the general result that every field operator creates an antiparticle and annihilates a particle. As an example we consider the charge operator for a charged scalar field ϕ

$$Q = \mathrm{i}q \int d^3x \, \left(\phi^\dagger \partial_0 \phi - \phi \partial_0 \phi^\dagger\right) \; . \tag{3.9}$$

By using equal-time commutation relations for the field operators we indeed find that (3.7) is fulfilled.

3.1.2 Structure of the Nucleon Lagrangian

As a first example we now construct a Lagrangian that describes free neutrons and protons as elementary particles. Since baryon-number is experimentally known to be conserved, we require the corresponding $U(1)_B$ symmetry from the outset. The Lagrangian is then that of free fermions for both the protons and the neutrons

$$\mathcal{L}_{pn} = \bar{\Psi}_p(\mathrm{i}\gamma^\mu\partial_\mu - M_p)\Psi_p + \bar{\Psi}_n(\mathrm{i}\gamma^\mu\partial_\mu - M_n)\Psi_n \quad . \tag{3.10}$$

The experimental values for the masses of proton and neutron are $M_p =$ 938.3 MeV and $M_n = 939.6$ MeV, respectively.

$\mathcal{L}_{pn}$ is obviously invariant under the $U(1)$ transformation

$$\begin{aligned} \Psi_p &\to \Psi'_p = \mathrm{e}^{-\mathrm{i}\epsilon q_p}\Psi_p, \qquad \bar{\Psi}_p \to \bar{\Psi}'_p = \mathrm{e}^{+\mathrm{i}\epsilon q_p}\bar{\Psi}_p \\ \Psi_n &\to \Psi'_n = \mathrm{e}^{-\mathrm{i}\epsilon q_n}\Psi_n, \qquad \bar{\Psi}_n \to \bar{\Psi}'_n = \mathrm{e}^{+\mathrm{i}\epsilon q_p}\bar{\Psi}_n \quad , \end{aligned} \tag{3.11}$$

since it depends only on bilinear field combinations of the form $\bar{\Psi}\Psi$. Assigning the value $q_n = q_p = 1$ to both the proton and neutron charge leads to conservation of the baryon number

$$B = \int \left(\Psi_p^\dagger \Psi_p + \Psi_n^\dagger \Psi_n\right) d^3x \quad ; \tag{3.12}$$

the corresponding symmetry operation is $U(1)_B$.

In addition to baryon number conservation, charge conservation also holds; in this case one chooses $q_p = 1$, $q_n = 0$, so that the transformation is applied to the proton field alone. The corresponding transformations

$$\begin{aligned} \Psi_p &\to \Psi'_p = \mathrm{e}^{-\mathrm{i}\epsilon}\Psi_p, \qquad \bar{\Psi}_p \to \bar{\Psi}'_p = \mathrm{e}^{+\mathrm{i}\epsilon}\bar{\Psi}_p \\ \Psi_n &\to \Psi'_n = \Psi_n, \qquad \bar{\Psi}_n \to \bar{\Psi}'_n = \bar{\Psi}_n \end{aligned}$$

form another group, $U(1)_Q$, with the conserved charge

$$Q = \int \Psi_p^\dagger \Psi_p \, d^3x \quad . \tag{3.13}$$

Both of these conservation laws hold in nature. Another one, that could be obtained by setting $q_p = 0$, $q_n = 1$, would lead to neutron number conservation that, however, is violated in nature.

The close agreement of the experimental values of neutron and proton masses suggests to view the small difference between them as the consequence of a perturbative interaction. Abbreviating the latter as $\mathcal{L}_{\mathrm{int}}$ allows us to rewrite the Lagrangian (3.10) as

$$\mathcal{L}_{pn} = \sum_{q=p,n} \left[\bar{\Psi}_q \left(\mathrm{i}\gamma^\mu \partial_\mu - M\right) \Psi_q\right] + \mathcal{L}_{\mathrm{int}} \quad , \tag{3.14}$$

where M is the common mass for proton and neutron in the absence of the perturbing interaction.

We now formally introduce a two-dimensional spinor

$$\Psi_N = \begin{pmatrix} \Psi_p \\ \Psi_n \end{pmatrix} \quad , \tag{3.15}$$

in terms of which the Lagrangian (3.14) can be written as

$$\begin{aligned}\mathcal{L}_{pn} &= \bar{\Psi}_N \left(\mathrm{i}\gamma^\mu \partial_\mu - M\right) \Psi_N + \mathcal{L}_{int} \\ &= \mathcal{L}_N + \mathcal{L}_{int} \quad . \end{aligned} \tag{3.16}$$

Here the Lagrangian has been split into two parts; $\mathcal{L}_N$ is that of the noninteracting, equal-mass protons and neutrons whereas $\mathcal{L}_{int}$ denotes the residual interaction responsible for the small mass difference between proton and neutron.

The form (3.15) for the state Ψ_N suggests to view proton and neutron as two different intrinsic states of the same particle, the nucleon, described by the Lagrangian $\mathcal{L}_N$. The new degree of freedom that distinguishes between proton and neutron is called isospin.

3.2 $SU(2)$ Symmetry

From eq. (3.14) it is evident that $\mathcal{L}_{pn}$ is invariant under an exchange of protons and neutrons, if the relatively small $\mathcal{L}_{int}$ is neglected. That the total strong interaction Lagrangian is indeed invariant under such a transformation is suggested by experimental observations, for example, the near-equality of the binding energy and spectra of the mirror nuclei ^{7}Li and ^{7}Be that differ by the change of one neutron into one proton. The structural similarity of these two nuclei indicates that proton-proton and neutron-neutron forces are nearly equal. Thus the strong interaction Lagrangian seems to be invariant under an exchange of protons and neutrons. Therefore, it can also not distinguish between any mixture of protons and neutrons. This implies that $\mathcal{L}_N$ of eq. (3.16) contains a further symmetry – beyond the simple $U(1)$ – that mixes the field components Ψ_p and Ψ_n. To discuss this symmetry is the purpose of this section.

3.2.1 Properties of the Group $SU(2)$

We consider a space- and time-independent general transformation of the two-dimensional spinor Ψ_N which was defined in (3.15)[2]

$$\Psi \rightarrow \Psi' = U\Psi \quad . \tag{3.17}$$

In its most general form U must be a (2×2)-dimensional matrix with space- and time-independent complex components. We will now look for such a transformation U that leaves $\mathcal{L}_N$ invariant.

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[2] We suppress here, as in the following, the subscript N of Ψ.

The fields Ψ' must also fulfill the anticommutation relations for fermions. This requirement implies that the transformation U must be unitary:

$$U^\dagger U = UU^\dagger = 1 \quad . \tag{3.18}$$

Because of this property one obtains for the determinant of the transformation matrix U

$$\det(U^\dagger)\det(U) = (\det(U))^*\det(U) = 1 \tag{3.19}$$

and thus

$$\det(U) = \mathrm{e}^{-\mathrm{i}\varphi} \qquad (\varphi \text{ real}) \quad . \tag{3.20}$$

Eq. (3.20) implies that U contains a simple $U(1)$ transformation. Since the $U(1)$ symmetry has also already been incorporated we can disregard it here by simply setting $\varphi = 0$. Then one has

$$\det(U) = 1 \quad . \tag{3.21}$$

Conditions (3.18) and (3.20) narrow the possible matrices U down to the unitary matrices with unit determinant, the so-called special unitary matrices. The set of all of these (2×2)-matrices forms a group, the Lie group $SU(2)$; the product of any two of the elements of this group is just the normal matrix multiplication.

Lie groups have the special property that their elements can be parametrized smoothly and analytically such that each element can be obtained by a continuous parameter change from the identity. In order to obtain the parametrization of the group elements it is, therefore, sufficient to consider an infinitesimal change of the spinor Ψ

$$\Psi' = (1 - i\xi)\Psi \tag{3.22}$$

where ξ is a (2×2)-dimensional matrix with infinitesimally small matrix elements and 1 is the (2×2)-dimensional unit matrix. The unitarity condition (3.18) then reads

$$\begin{aligned} U^\dagger U &= (1 + \mathrm{i}\xi^\dagger)(1 - \mathrm{i}\xi) \\ &= 1 + \mathrm{i}(\xi^\dagger - \xi) + \mathcal{O}(\xi^2) = 1 \quad . \end{aligned} \tag{3.23}$$

This can be fulfilled only if

$$\xi^\dagger = \xi \quad , \tag{3.24}$$

i.e. if ξ is a hermitian matrix. Furthermore, from

$$1 = \det(U) = 1 - \mathrm{i}\xi_{11} - \mathrm{i}\xi_{22} + \mathcal{O}(\xi^2) \tag{3.25}$$

follows

$$\mathrm{tr}(\xi) = 0 \quad . \tag{3.26}$$

Equations (3.24) and (3.26) show that ξ is a hermitean matrix with vanishing trace. Since its dimension is (2×2), it contains three independent elements. Such matrices can in general be written as a linear superposition of three linearly independent, hermitian, traceless basis-matrices τ_i $(i = 1, 2, 3)$

$$\xi = \boldsymbol{\epsilon} \cdot \frac{\boldsymbol{\tau}}{2} \tag{3.27}$$

with the real parameters

$$\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3)$$

and the matrices

$$\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3) \quad .$$

Here the ϵ_i are infinitesimal, real constants and the τ_i are the Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; , \quad \tau_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \; , \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \tag{3.28}$$

These obey the commutation relations

$$\left[\frac{\tau_i}{2}, \frac{\tau_j}{2} \right] = \mathrm{i}\epsilon_{ijk} \frac{\tau_k}{2} \tag{3.29}$$

where ϵ_{ijk} is the Levi-Civita tensor:

$$\epsilon_{ijk} = \begin{cases} 1 & i,j,k = 1,2,3 \quad \text{and cyclical} \\ -1 & i,j,k = 3,2,1 \quad \text{and cyclical} \\ 0 & \text{in all other cases} \end{cases} \quad . \tag{3.30}$$

The commutation relations (3.29) define the "algebra" of the group $SU(2)$, the elements of the tensor ϵ_{ijk} are called the "structure constants" of $SU(2)$ and the matrices $\tau_i/2$ are the matrix representations of the "generators" of $SU(2)$-transformations in isospin space, i.e. in the space spanned by the vectors Ψ_N (eq. (3.15)). They are normalized such that

$$\mathrm{tr}(\tau_l \tau_k) = 2\delta_{lk} \; . \tag{3.31}$$

Summarizing all these results, the most general infinitesimal symmetry transformation of the two component spinor Ψ is given by

$$\Psi \to \Psi' = \left(1 - \mathrm{i}\boldsymbol{\epsilon} \cdot \frac{\boldsymbol{\tau}}{2}\right) \Psi \quad .$$

For a finite parameter $\boldsymbol{\epsilon}$ this is easily generalized to

$$\Psi \to \Psi' = \mathrm{e}^{-\mathrm{i}\boldsymbol{\epsilon}\frac{\boldsymbol{\tau}}{2}}\Psi \quad . \tag{3.32}$$

Formally, this is a generalization of the global gauge transformation of the $U(1)$ type with, however, two major differences:

1. There are now 3 "phase angles" ϵ_i .
2. The exponent in (3.32) now contains three matrices τ_i that do not commute.

Thus, two $SU(2)$ transformations will in general not commute. Consequently $SU(2)$ is a non-Abelian group. The special (det = 1) unitary (2×2) matrices

$$U = \mathrm{e}^{-\mathrm{i}\boldsymbol{\epsilon}\boldsymbol{t}} \quad , \qquad \text{with} \quad \boldsymbol{t} = \boldsymbol{\tau}/2 \tag{3.33}$$

form the fundamental representation of the group $SU(2)$; the matrices t_i are the matrix representations of the generators. The matrix U acts on two-dimensional vectors, the isospinors.

Higher-dimensional representations. There exist also matrices of higher dimension than (2×2) that obey the same $SU(2)$ algebra (3.29). These form higher dimensional representations of the symmetry group. For example, in 3 dimensions such matrices are

$$\begin{aligned} t_1 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad t_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \\ t_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{aligned} \tag{3.34}$$

in a representation in which t_3 is diagonal. The elements of $SU(2)$ are then again given by eq. (3.33), with the t_i now defined by eq. (3.34).

Note that the generators t_i are not unique. All matrices that fulfill the commutation relations (3.29) are generators of $SU(2)$, though in a different representation. For example, the (3×3) matrices

$$(t_k)_{lm} = -\mathrm{i}\epsilon_{klm} \tag{3.35}$$

also obey the same commutation rules[3]. In the case of (3.35) the structure constants of $SU(2)$ define the matrix representation of the generators.

The $SU(2)$ transformation in a three-dimensional representation also acts on three-dimensional basis vectors. If we denote a vector in this space by $\mathbf{\Pi}$, then an infinitesimal transformation in the representation (3.35) is given by

$$\delta\Pi_l = -\mathrm{i}\left(\epsilon_k t_k \mathbf{\Pi}\right)_l = -\epsilon_k \epsilon_{klm} \Pi_m = (\boldsymbol{\epsilon} \times \mathbf{\Pi})_l \quad , \tag{3.36}$$

i.e. just by the transformation law of an ordinary vector under an infinitesimal spatial rotation. This is a consequence of the fact that the algebra (3.29) for the $SU(2)$ generators $\tau_i/2$ is exactly the same as that for the orbital angular momentum operators l_i. The latter are the generators of the rotation group in three dimensions, $SO(3)$, that act on three-dimensional ordinary vectors. The two groups are thus locally equivalent.

General definitions. In general, the algebra of a group is defined by the commutation relation of its generators

$$[Q_i, Q_j] = \mathrm{i} c_{ijk} Q_k$$

where the c_{ijk} are the structure constants of the group. Then the matrices q_k with

$$(q_k)_{lm} = -\mathrm{i} c_{klm}$$

are the generators of the "regular" or "adjoint" representation of the group. Thus (3.35) defines the regular representation of $SU(2)$. The dimension of this special representation is equal to the number of group generators, i.e. $d = 3$ for $SU(2)$.

The representations (3.34) and (3.35) are examples of so-called "irreducible" representations that cannot be put into block-diagonal form by a similarity transformation, i.e. by a transformation $U \to U' = SUS^{-1}$ where S is any nonsingular, invertible matrix acting on the basis vectors of U. If this is possible for a given representation, then it is said to be reducible.

In general, in an n-dimensional representation, the group elements U act on an n-dimensional vector space with n linearly independent basis vectors. If the group representation is irreducible, then these basis vectors are said to form a "multiplet" . If the Hamiltonian is invariant under the transformation U, then all states of a multiplet have the same energy; they differ by the

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[3] This can be shown by using the relation $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$.

eigenvalues of the diagonal generator, for example, by their value of t_3 in the irreducible representations (3.28) or (3.34).

Another important property of a Lie group is its rank, defined as the maximum number of simultaneously diagonal generators. In the set of the three Pauli matrices τ_i each matrix commutes only with itself; thus $SU(2)$ is a group of rank 1. From the members of an algebra of a Lie group of rank n one can, by combining the generators in a nonlinear form, construct n invariant operators, the so-called Casimir operators, which commute with all generators. For $SU(2)$, this Casimir operator is given by $\boldsymbol{\tau}^2 = \tau_1^2 + \tau_2^2 + \tau_3^2$. The Casimir operator is usually used to label the multiplets.

3.2.2 Application to the Pion-Nucleon System

The application of the considerations in the preceding section to physical particles is obtained if one can identify n particles with an n-dimensional multiplet of basis states. For example, in a theory with an $SU(2)$ invariant Hamiltonian this identification is possible if the n particles have approximately the same mass.

In this section we will now apply these symmetry considerations to the pion-nucleon system since this is one of the elementary and, at the same time, most relevant systems of nuclear physics. As already mentioned, the $SU(2)$ isospin symmetry is most evident in the charge independence of nuclear forces that can, for example, be observed in the binding energy of mirror nuclei. However, as will become clear from the discussions in this chapter, isospin symmetry goes further than charge independence, which is merely a consequence of it.

3.2.2.1 Symmetry of the Nucleon Lagrangian

We now apply some of the formal considerations of the last section to the case of the two-component nucleon spinor Ψ. Since

$$U = \mathrm{e}^{-\mathrm{i}\boldsymbol{\epsilon}\frac{\boldsymbol{\tau}}{2}} \tag{3.37}$$

is unitary, it is evident that the bilinear product $\bar{\Psi}\Psi$ in eq. (3.16) and thus the whole nucleon Lagrangian $\mathcal{L}_N$ is invariant under the $SU(2)$ transformation (3.37). The transformation U acts on the intrinsic degree of freedom that distinguishes between the two elements of the spinor Ψ, i.e. between proton and neutron. This internal degree of freedom is called isospin. The vectors $\boldsymbol{\epsilon}$ and $\boldsymbol{\tau}$ are then vectors in the isospin-space; they are called isovectors.

The conserved current corresponding to this $SU(2)$ symmetry can be obtained from expression (2.55) by generalizing it to the case, where $\boldsymbol{\epsilon}$ and $\boldsymbol{q}_{\alpha\beta}$ are vectors in the space of the internal degrees of freedom,

$$\boldsymbol{q}_{\alpha\beta} = \frac{\boldsymbol{\tau}_{\alpha\beta}}{2} \quad .$$

This yields for the conserved (isospin) current

$$\boldsymbol{j}_\mu = \bar{\Psi}\gamma_\mu \frac{\boldsymbol{\tau}}{2}\Psi \tag{3.38}$$

where $\boldsymbol{j}_\mu$ is now the μ th component of a four-vector in coordinate space and at the same time an isovector. The conserved charge is then another isovector,

$$\boldsymbol{T} = \int \boldsymbol{j}_0(x)\, d^3x = \int \Psi^\dagger(x)\frac{\boldsymbol{\tau}}{2}\Psi(x)\, d^3x \quad , \tag{3.39}$$

the *isospin*; Ψ is the two-dimensional spinor (3.15). Its third component can be written as

$$T_3 = \frac{1}{2}\int \left(\Psi_p^\dagger\Psi_p - \Psi_n^\dagger\Psi_n\right)\, d^3x \quad . \tag{3.40}$$

$\mathcal{L}_N$ is also invariant under the $U(1)_B$ transformation (3.11)

$$\Psi \to \Psi' = \mathrm{e}^{-\mathrm{i}\epsilon}\Psi \quad , \tag{3.41}$$

leading to the conservation of baryon number

$$B = \int \left(\Psi_p^\dagger\Psi_p + \Psi_n^\dagger\Psi_n\right)\, d^3x = \int \Psi^\dagger\Psi\, d^3x \quad . \tag{3.42}$$

Combining the symmetry operations (3.37) and (3.41), we see that there is also an invariance of $\mathcal{L}_N$ under the combined $U(1)_B\otimes$ $SU(2)$ transformation

$$\Psi \to \Psi' = \mathrm{e}^{-\mathrm{i}\epsilon\frac{(1+\tau_3)}{2}}\Psi \quad , \tag{3.43}$$

leading to the conservation of the charge

$$Q = \int \Psi^\dagger\frac{1+\tau_3}{2}\Psi\, d^3x = \int \Psi_p^\dagger\Psi_p\, d^3x \quad . \tag{3.44}$$

Q is obviously equal to the number of electrical charges of the nucleons. Combining eqs (3.40), (3.42) and (3.44) yields the relation

$$Q = \frac{1}{2}B + T_3 \quad . \tag{3.45}$$

Although we have derived these relations only for the free nucleon Lagrangian (3.16), the empirically well founded isospin conservation together with that for electric charge and baryon number indicates that the full Lagrangian for the strong interaction respects these symmetries. Thus (3.45) must hold for any system of nucleons with $U(1)$ and $SU(2)$ symmetries if the Lagrangian contains no derivative terms other than those in the kinetic energy (in this case the currents remain unchanged); it is based only on the symmetry operations (3.37), (3.41) and (3.43) and not on any specific symmetry properties of the Lagrangian.

To determine the isospin T (defined via the eigenvalue $T(T+1)$ of the operator $\boldsymbol{T}^2$, which is just the Casimir operator of $SU(2)$) of a multiplet, we make use of the similarity of the group $SU(2)$ and the group of spatial rotations in three dimensions, $SO(3)$, whose generators are the angular momentum operators; both groups have the same algebra. A well known example for an $SO(3)$ symmetry is the hydrogen atom. There the spectrum consists of $2L+1$ dimensional multiplets labeled by the eigenvalues of the Casimir operator L^2; the states in each multiplet are labeled by L_3. Analogously, the n particles of the isospin multiplet have isospin T given via $n = 2T+1$ and can be distinguished via T_3.

In the case treated in this section proton and neutron are both members of the $T = 1/2$ multiplet, distinguished by $T_3 = +1/2$ and $T_3 = -1/2$, respectively. In an $SU(2)$-symmetric world they are thus two different states of the same particle, the nucleon.

The fundamental doublet of $SU(2)$ has $T = 1/2$ and $T_3 = \pm 1/2$. The corresponding antiparticles also form a doublet. Since the charge-conjugate spinor is given by

$$\Psi^C = \mathrm{i}\gamma^2\gamma^0\Psi^* \, , \tag{3.46}$$

the antiparticle spinor transforms under U^*. In $SU(2)$ we have

$$U^* = (-\mathrm{i}\tau_2)U(\mathrm{i}\tau_2) \tag{3.47}$$

because of $\vec{\tau}^* = -\tau_2\vec{\tau}\tau_2$. Note that the transformation U^* can be obtained from U by a similarity transformation; U and U^* are thus equivalent representations[4]. Therefore, we have

$$\Psi^{c\prime} = U^*\Psi^c = (-\mathrm{i}\tau_2)U(\mathrm{i}\tau_2)\Psi^c \tag{3.48}$$

and thus

$$(\mathrm{i}\tau_2\Psi^c)' = U(\mathrm{i}\tau_2\Psi^c) \tag{3.49}$$

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[4] "Equivalent" matrix representations of $SU(N)$ can be transformed into each other by means of a unitary transformation belonging to the same group.

Thus, with

$$\Psi = \begin{pmatrix} \Psi_p \\ \Psi_n \end{pmatrix} \quad , \tag{3.50}$$

also

$$\Psi^C = (-\mathrm{i}\tau_2) \begin{pmatrix} \Psi_p^C \\ \Psi_n^C \end{pmatrix} = \begin{pmatrix} -\Psi_n^C \\ \Psi_p^C \end{pmatrix} \tag{3.51}$$

forms an $SU(2)$ doublet.

In analogy to the nucleon the particles that transform under the representation (3.34) have isospin $T = 1$ and $T_3 = +1, 0, -1$. If this representation is realized in nature then we should find a triplet of particles with the same mass and different electric charges. The $SU(2)$ transformation U, when applied to such a triplet, simply amounts to a rotation of a vector in isospin space.

In order to construct many-particle states with good $SU(2)$ transformation properties, the single-particle states of $SU(2)$ multiplets can also be coupled to states of different multiplets (i.e. different isospin) by using the normal Clebsch-Gordan coefficients well known from angular-momentum coupling. This analogy arises again because the group $SO(3)$ has the same algebra as $SU(2)$. Thus, e.g., the $T = 1$ multiplet mentioned at the start of this chapter could be obtained by coupling two nucleons each with $T = 1/2$.

Quantummechanical consistency. The components $\hat{T}_i$ of the isospin *operator*, after field quantization, obey the same commutation relations as the Pauli-matrices $\tau_i/2$

$$\left[\hat{T}_i, \hat{T}_j\right] = \mathrm{i}\epsilon_{ijk}\hat{T}_k \quad . \tag{3.52}$$

This follows from the general relation

$$\begin{aligned} &\left[\int \Psi^\dagger(x)\hat{O}\Psi(x)\, d^3x \;,\; \int \Psi^\dagger(y)\hat{Q}\Psi(y) d^3y \right] \\ &\qquad = \int \Psi^\dagger(x) \left[\hat{O}\,,\hat{Q}\right] \Psi(x)\, d^3x \qquad (x_0 = y_0) \end{aligned} \tag{3.53}$$

where all the points x and y are taken at the same time and $\hat{O}$ and $\hat{Q}$ are operators acting on the spinor components. Equation (3.53) can be easily proven if canonical *anti*commutation relations for the fields Ψ and $\Psi^\dagger$

are used. Considering the group properties, the operators $\hat{T}_i$ are therefore equivalent to the Pauli matrices

$$t_i = \frac{\tau_i}{2} \quad .$$

They only act onto different basis states, the $\hat{T}_i$ on Hilbert space state vectors and the t_i on basis vectors in the space of the intrinsic isospin degree of freedom.

Generalizing relation (3.7) to the isospin-case ($\hat{\boldsymbol{O}} = \boldsymbol{\tau}/2$) gives

$$\left[\hat{\boldsymbol{T}}, \Psi(x)\right] = -\frac{\boldsymbol{\tau}}{2}\Psi(x) \quad . \tag{3.54}$$

This is nothing other than the relation

$$\mathrm{e}^{\mathrm{i}\boldsymbol{\epsilon}\hat{\boldsymbol{T}}}\Psi\mathrm{e}^{-\mathrm{i}\boldsymbol{\epsilon}\hat{\boldsymbol{T}}} = \mathrm{e}^{-\mathrm{i}\boldsymbol{\epsilon}\frac{\boldsymbol{\tau}}{2}}\Psi \tag{3.55}$$

for infinitesimal $\boldsymbol{\epsilon}$. The lhs of this equation is the transformed field operator. Thus $\hat{\boldsymbol{T}}$, the conserved "charge" operator, is in analogy to the $U(1)$ case, the generator of the $SU(2)$ representation acting in Hilbert space. This again shows that the correspondence principle is satisfied, since the field operator transforms in the same way as the classical field. Also, as in the $U(1)$ case (eq. (3.8)), eq. (3.54) implies that $\tau_i/2$ is the eigenvalue of $\hat{T}_i$

$$\hat{T}_i\Psi|0> - \Psi\hat{T}_i|0> = -\frac{\tau_i}{2}\Psi|0> \quad . \tag{3.56}$$

Essential for this result is again the assumption that the vacuum is "uncharged", i.e. carries zero isospin. Formally, this corresponds to the use of a normal product in the definition of the charge in eq. (3.39), which here has not been written explicitly in order to simplify the notation.

Eq. (3.55) implies that the classical and the quantum symmetry operations have the same effect. Thus, if the Hamiltonian $\hat{H}$ of the system is invariant under (3.55), then the $\hat{T}_i$ represent conserved quantities also after quantization. This implies that $\hat{H}$ commutes with all the $\hat{T}_i$; its $SU(2)$-transformed eigenstates then belong to the same energy as the original ones.

3.2.2.2 Symmetry of the Pion Lagrangian

The pion carries the quantum numbers $J^\pi = 0^-$ and is thus a pseudoscalar particle. It exists in three different charge states, with nearly equal mass ($m(\pi^+) = m(\pi^-) = 139.6$ MeV, $m(\pi^0) = 135.0$ MeV); it is natural to associate the somewhat higher mass of the two charged pions with the effect of the repulsive Coulomb interaction. We therefore assume that in a world in which only the strong π–N force acts the three pions would have equal mass. This suggests that the three pions can be assigned to the three states of a $T = 1$ triplet, the basis vectors of the irreducible representation (3.34) and that, correspondingly, the strong interaction Lagrangian is symmetric under $SU(2)$ transformations. The pions thus have to be represented by an isovector with components π^+, π^0 and π^-; these are charged, nonhermitean pseudoscalar fields. That these components indeed belong to the representation (3.34) we can see by inspecting the eigenvalues of t_3 in that representation.

From these three components we can construct the basis triplet of a cartesian representation

$$\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3) \tag{3.57}$$

where the three components π_1, π_2 and π_3 are hermitean, pseudoscalar fields. These are obtained from the physical fields by means of the transformation

$$\begin{aligned} \pi_1(x) &= \frac{1}{\sqrt{2}}(\pi^+(x) + \pi^-(x)) \\ \pi_2(x) &= \frac{\mathrm{i}}{\sqrt{2}}(\pi^+(x) - \pi^-(x)) \\ \pi_3(x) &= \pi^0(x) \quad . \end{aligned} \tag{3.58}$$

The transformation (3.58) is exactly the same as the one that converts the components of an irreducible spherical tensor of rank 1, an object with well defined transformation properties under spatial rotations, into those of a normal, cartesian vector. This should come as no surprise since, as already mentioned at the end of the last subsection, $SO(3)$ has the same algebra as $SU(2)$.

The Lagrangian for the pions is given by that for charged scalar fields (cf. eq. (2.31))

$$\mathcal{L}_\pi = \left((\partial^\mu \pi^+)(\partial_\mu \pi^-) - m^2 \pi^+ \pi^-\right) + \frac{1}{2}\left((\partial_\mu \pi^0)^2 - m^2(\pi^0)^2\right) \tag{3.59}$$

according to the general considerations in section 2.1.1.3. In terms of the cartesian isovector $\boldsymbol{\pi}$ (3.58) this can be rewritten as

$$\mathcal{L}_\pi = \frac{1}{2}\left[(\partial_\mu \boldsymbol{\pi})^2 - m^2 \boldsymbol{\pi}^2\right] \quad . \tag{3.60}$$

Written in this form, it is evident that $\mathcal{L}_\pi$ is $SU(2)$ symmetric since it contains only scalar products of the isovector $\boldsymbol{\pi}$ which are invariant under rotations in isospin space (eqs. (3.33), (3.37)).

The conserved current is

$$\boldsymbol{j}_\mu = -\mathrm{i}\,\partial_\mu \pi_k\,(\boldsymbol{t})_{kl}\,\pi_l = \boldsymbol{\pi} \times \partial_\mu \boldsymbol{\pi} \tag{3.61}$$

and the conserved charge is given by

$$\boldsymbol{T} = -\mathrm{i}\int \partial_t \pi_k\,(\boldsymbol{t})_{kl}\,\pi_l\,d^3x = \int (\boldsymbol{\pi} \times \boldsymbol{\Pi})\,d^3x \quad ; \tag{3.62}$$

$\boldsymbol{j}_\mu$ and $\boldsymbol{T}$ are in the cartesian, regular representation, both are vectors in isospin-space; $\boldsymbol{\Pi}$ is the conjugate field momentum.

The third component of $\boldsymbol{T}$ expressed by the multiplet states (3.57) is obtained as

$$T_3 = \mathrm{i}\int \left[(\partial_t \pi^+)\pi^- - (\partial_t \pi^-)\pi^+\right] d^3x \quad . \tag{3.63}$$

For the charge of the pions one has, from the $U(1)$ symmetry of (3.59),

$$Q = \mathrm{i}\int \left[(\partial_t \pi^+)\pi^- - (\partial_t \pi^-)\pi^+\right] d^3x \quad . \tag{3.64}$$

Comparing (3.64) and (3.63) gives

$$Q = T_3 \quad . \tag{3.65}$$

This relation can be viewed as a special case of eq. (3.45), since the pion has baryon number $B = 0$.

3.2.2.3 Symmetry of the Interaction

In this section we will now construct the Lagrangian for the pion-nucleon interaction, $\mathcal{L}_{\pi N}$. The first requirement of baryon number conservation means that $\mathcal{L}_{\pi N}$ can contain only bilinear forms in the nucleon field Ψ; in lowest order the coupling thus must read

$$\mathcal{L}_{\pi N} \sim \bar{\Psi}\hat{O}\Psi\,\hat{Q}\pi \quad , \tag{3.66}$$

where $\hat{O}$ is an operator that acts on the components of Ψ, and $\hat{Q}$ acts on the pion field. The latter can appear linearly in (3.66) since the pions carry no baryon number, so that single pions can be created or annihilated.

The structure of $\hat{O}$ can be obtained most easily by considering first interactions with the neutral pions only. Since π^0 is a pseudoscalar field (spin = 0, parity = −1) and the Lagrangian has to be a true scalar, $\bar{\Psi}\hat{O}\Psi$ itself must be a pseudoscalar, if $\hat{Q}$ is taken to be equal to 1 for simplicity. This can be achieved by an ansatz of the form (see eq. (A.31))[5]

$$\hat{O} = \gamma_5 \qquad \text{(pseudoscalar coupling)} \quad . \tag{3.67}$$

With this coupling, we obtain

$$\mathcal{L}_{\pi N} = \mathrm{i} g_1 \bar{\Psi}_n \gamma_5 \Psi_n \pi^0 + \mathrm{i} g_2 \bar{\Psi}_p \gamma_5 \Psi_p \pi^0 \tag{3.68}$$

where g_i is a real coupling strength. The factor i is necessary to make $\mathcal{L}_{\pi N}$ hermitian:

$$\left(\bar{\Psi}_n \gamma_5 \Psi_n \pi^0\right)^\dagger = \pi^0 \Psi_n^\dagger \gamma_5 \bar{\Psi}_n^\dagger = \pi^0 \Psi_n^\dagger \gamma_5 \gamma_0 \Psi_n = -\pi^0 \bar{\Psi}_n \gamma_5 \Psi_n \quad . \tag{3.69}$$

These considerations can be extended to the interaction with the charged pions. The requirement of both baryon number conservation and also charge conservation imposes immediately the constraint on $\mathcal{L}_{\pi N}$ that it has to contain combinations of the form

$$\bar{\Psi}_n \gamma_5 \Psi_p \pi^- \quad \text{or} \quad \bar{\Psi}_p \gamma_5 \Psi_n \pi^+$$

to be invariant under a global $U(1)$ transformation. That these terms conserve baryon number is evident since they contain only bilinear forms in the nucleon field. That they are also charge conserving follows from their invariance under the $U(1)_Q$ transformations

$$\begin{aligned} \Psi_p \to \Psi_p' = \mathrm{e}^{-\mathrm{i}\epsilon q} \Psi_p \quad &; \quad \bar{\Psi}_p \to \bar{\Psi}_p' = \mathrm{e}^{+\mathrm{i}\epsilon q} \bar{\Psi}_p \\ \Psi_n \to \Psi_n' = \Psi_n \quad &; \quad \bar{\Psi}_n \to \bar{\Psi}_n' = \bar{\Psi}_n \\ \pi^+ \to (\pi^+)' = \mathrm{e}^{-\mathrm{i}\epsilon q} \pi^+ \quad &; \quad \pi^- \to (\pi^-)' = \mathrm{e}^{+\mathrm{i}\epsilon q} \pi^- \quad . \end{aligned}$$

The complete interaction Lagrangian now reads

$$\begin{aligned} \mathcal{L}_{\pi N} \;=\; & \mathrm{i} g_1 \bar{\Psi}_n \gamma_5 \Psi_n \pi^0 + \mathrm{i} g_2 \bar{\Psi}_p \gamma_5 \Psi_p \pi^0 \\ & + \mathrm{i} g_3 \bar{\Psi}_n \gamma_5 \Psi_p \pi^- + \mathrm{i} g_4 \bar{\Psi}_p \gamma_5 \Psi_n \pi^+ \quad . \end{aligned} \tag{3.70}$$

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[5] Also possible is a coupling in which the derivative of the pion field appears ($\hat{Q} = \partial_\mu$). Since this is a pseudovector, it has to be multiplied by another pseudovector:

$$\mathcal{L}_{\pi N} \sim \bar{\Psi} \gamma_\mu \gamma_5 \Psi \, \partial^\mu \pi^0 \qquad \text{(pseudovector coupling)} \quad .$$

By introducing the matrices

$$\begin{aligned} \frac{\tau_+}{2} &= \frac{1}{2}(\tau_1 + \mathrm{i}\tau_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \frac{\tau_-}{2} &= \frac{1}{2}(\tau_1 - \mathrm{i}\tau_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \tau_0 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \tau_3 \end{aligned} \tag{3.71}$$

$\mathcal{L}_{\pi N}$ can be written in terms of the nucleon spinor Ψ as

$$\begin{aligned} \mathcal{L}_{\pi N} &= \mathrm{i}g_1\bar{\Psi}\gamma_5\frac{1}{2}\left(1-\tau_0\right)\Psi\pi^0 + \mathrm{i}g_2\bar{\Psi}\gamma_5\frac{1}{2}\left(1+\tau_0\right)\Psi\pi^0 \\ &+\mathrm{i}g_3\bar{\Psi}\gamma_5\frac{\tau_-}{2}\Psi\pi^- + \mathrm{i}g_4\bar{\Psi}\gamma_5\frac{\tau_+}{2}\Psi\pi^+ \quad . \end{aligned} \tag{3.72}$$

$\mathcal{L}_{\pi N}$ is, by construction, symmetric under $U(1)_B \otimes U(1)_Q$.

If now, in addition to charge and baryon number conservation, $SU(2)$ symmetry is also required for $\mathcal{L}_{\pi N}$, as suggested by the experimental fact of isospin conservation in the pion-nucleon interaction, then $\mathcal{L}_{\pi N}$ must be the scalar product of the isovector of the pion field with an isovector of the nucleon. Thus $\mathcal{L}_{\pi N}$ must have the structure

$$\mathcal{L}_{\pi N} = \mathrm{i}g\bar{\Psi}\gamma_5\frac{\boldsymbol{\tau}}{2}\Psi\cdot\boldsymbol{\pi} \quad , \tag{3.73}$$

with only one coupling constant, g. Since

$$\boldsymbol{\tau}\cdot\boldsymbol{\pi} = \frac{1}{\sqrt{2}}(\tau_+\pi^+ + \tau_-\pi^-) + \tau_0\pi^0 \quad ,$$

the Lagrangian of eq. (3.72) goes over into the form (3.73) for

$$g = -2g_1 = 2g_2 = \sqrt{2}g_3 = \sqrt{2}g_4 \quad .$$

There is then only one universal constant, g, that couples all pion fields to all states of the nucleon. This has the important consequence that the transition rates for seemingly quite different processes, like, e.g. , $p+p \to p+n+\pi^+$ and $n+n \to n+n+\pi^0$, are governed by one and the same coupling constant so that, if one rate has been measured, the other one can be predicted.

In summary, the interaction Lagrangian (3.73) is by construction invariant under baryon number and charge $U(1)$, isospin $SU(2)$, parity and

Lorentz transformations. In addition, it is also invariant under charge conjugation, i.e. particle-antiparticle conjugation. Its structure is that of a conserved current (cf. eq. (3.38)) multiplied with the pion field.

We have now the important result that the full pion-nucleon Lagrangian

$$\mathcal{L} = \mathcal{L}_N + \mathcal{L}_\pi + \mathcal{L}_{\pi N} \tag{3.74}$$

is $SU(2)$-invariant. As a consequence, not the isospin of nucleon and pion separately but instead the total isospin

$$\boldsymbol{T} = \boldsymbol{T}_N + \boldsymbol{T}_\pi = \int \Psi^\dagger \frac{\boldsymbol{\tau}}{2} \Psi \, d^3x - \mathrm{i} \int \partial_t \pi_k \, (\boldsymbol{t})_{kl} \, \pi_l \, d^3x \tag{3.75}$$

is conserved. The pion-nucleon coupling (3.73) does not contribute to $\boldsymbol{T}$ because it contains no field derivatives. Since $\mathcal{L}$ is obviously also $U(1)$ invariant, leading to baryon number conservation, it is also invariant under the combined $U(1) \otimes SU(2)$ transformation. Thus relation (3.45) linking charge, baryon-number and third component of isospin is also valid for the coupled system.

There is strong experimental evidence that $SU(2)$ is a good symmetry of the strong interactions so that the total isospin of any system of only strongly interacting particles is conserved [PER87]. If one takes into account additional fields that carry isospin, the expression (3.75) for $\boldsymbol{T}$ would contain contributions also from these other fields.

3.2.3 Structure of $SU(2)$ Multiplets

As already discussed in section 3.2.1, the basis states on which the isospin acts can be identified with particle multiplets. Since the algebra of $SU(2)$ is exactly that of the normal angular momentum in three-dimensional space it is evident that these multiplets can be characterized by the eigenvalues of $\boldsymbol{T}^2$; their individual states can be labeled by their eigenvalue of T_3. The mathematical reason for this is that $\boldsymbol{T}^2$ is the Casimir operator of the Lie group $SU(2)$. Since $SU(2)$ has only one Casimir operator, $\boldsymbol{T}^2$, it follows that besides $\boldsymbol{T}^2$ additionally only one generator, usually T_3, can be diagonalized simultaneously.

Starting from one state in a multiplet the others can be constructed by using the raising and lowering operators $T_+ = T_1 + \mathrm{i}T_2$ and $T_- = T_1 - \mathrm{i}T_2$. All the states within one multiplet have the same energy eigenvalue of the Hamiltonian, i.e. the same mass, if the underlying interaction is $SU(2)$-invariant. The $SU(2)$ transformation (3.56) transforms the states into others within the same multiplet of energetically degenerate states.

The product of two basis vectors of $SU(2)$ belonging to two multiplets (not necessarily different) does not in general form a basis vector of a specific multiplet but rather a superposition of basis vectors of other multiplets. For example, in the case of the pion-nucleon system treated in section 3.2.2 the nucleon forms an isospin doublet ($T = 1/2$) whereas the pions belong to an isospin triplet ($T = 1$). A simple nucleon-pion product state is then a superposition of states belonging to $T = 3/2$ and $T = 1/2$ multiplets. Their weight is given by the standard Clebsch-Gordan coefficients, well known from angular momentum coupling

$$|N\pi>_{T_3} = (|N> \otimes |\pi>)_{T_3} = \sum_T \left(\frac{1}{2}1T|T_{3N}T_{3\pi}T_3\right) |TT_3> \quad , \tag{3.76}$$

where $T_3 = T_{3N} + T_{3\pi}$ and the sum over T runs over the values $|T| \geq |T_3|$. Thus the product of a triplet and doublet can be reduced into the sum of a doublet and a quadruplet, symbolically written as

$$3 \otimes 2 = 4 \oplus 2 \quad .$$

This reduction of the product can also be performed graphically. In this method all states of the product are represented by points on the T_3 axis. For each state of the product representation we have

$$T_3 = T_{3,1} + T_{3,2} \quad , \tag{3.77}$$

so that a given point T_3 can in general be constructed in several different ways. The highest possible value of T_3, however, is unique. Applying a lowering operator to this highest state generates the next lower state, and so on.

In the pion-nucleon example discussed above the highest state is the one with $T_3 = 3/2$, and the next lower is the one with $T_3 = 1/2$, which can be constructed in two different ways (($T_{3\pi} = 1$) + ($T_{3N} = -1/2$) or ($T_{3\pi} = 0$) + ($T_{3N} = 1/2$)). One of these two states must belong to the $T = 3/2$ multiplet, the other is then assigned to the $T = 1/2$ multiplet. Graphically this looks as shown in Fig. 3.1.

The near equality of the masses within a given multiplet is a convincing evidence for the invariance of the strong interaction under $SU(2)$ transformations. The remaining mass splittings are, except for the pion, of the order of a few promille and can be ascribed to the electromagnetic interaction. The latter is true also for the pion; in this case the relative mass splitting is larger simply because of the exceptionally small mass of the pion.

Figure 3.1: Construction of the isospin $SU(2)$ multiplet for the pion-nucleon system. The circles around the $T_3 = \pm 1/2$ points denote double occupation of these members.

From all the considerations above it is obvious that there can exist several isospin multiplets with the same T but distinguished by other quantum numbers like charge, spin, parity, etc. Examples for such multiplets are given in Table I at the end of this book. There, besides spin and isospin, a new conserved quantum number called "strangeness" S is listed as well as the so-called "hypercharge" Y, which is defined as the sum of strangeness and baryon number B:

$$Y = S + B \quad . \tag{3.78}$$

The discovery of the particles with the new quantum number S and the implications for the symmetry properties of strong interactions will be discussed in the next section.

3.3 $SU(3)$ Symmetry

The particles with strangeness $S \neq 0$ in Table I, the so-called strange mesons and baryons ("hyperons"), all share one common (and at first sight astonishing) feature. They are on one hand copiously produced in, for example, pion nucleon interactions. This large production rate seems to indicate a strong interaction process. On the other hand, these particles have lifetimes of the order of 10^{-8} to 10^{-10} s. They thus live much longer than the typical strong interaction reaction time of 10^{-23} s. This long lifetime seems to indicate that these particles couple only weakly to non-strange hadrons, in direct contradiction to the strong interaction assumption based on the large production rates.

A way out of this seeming contradiction is offered by the hypothesis that there exists a new quantum number called strangeness that is conserved by the strong interactions. In this case, the production of strange particles is not hindered if it only happens pairwise such that the total strangeness is conserved. This is, for example, the case in the reaction

$$\pi^+ + n \rightarrow \Lambda + K^+ \quad , \tag{3.79}$$

where both particles on the lhs have $S = 0$, whereas the Λ has $S = -1$ and the K^+ has $S = +1$, so that the total S is conserved.

It is known today that the strong interaction and the Coulomb interaction both conserve strangeness; the weak interaction, however, violates it. Thus the decay of strange particles, like e.g.

$$\Lambda \rightarrow p + \pi^- \quad , \tag{3.80}$$

can proceed only via the weak interaction. This explains the long half life of these particles (for example, the process in (3.80) has $T_{1/2} = 2.6 \cdot 10^{-10}$ s).

Another implication of the new quantum number S can be seen in Table I, which shows that the relation (3.45)

$$Q = \frac{1}{2}B + T_3 \quad ,$$

which holds for pions and nucleons, is obviously not valid for the strange particles. For example, the K^0 has $T_3 = -1/2$, $B = Y - S = 0$ and $Q = 0$. Instead, the following, at this point purely empirical, relation holds for all particles in the table:

$$Q = \frac{1}{2}Y + T_3 \quad , \tag{3.81}$$

where Y is the hypercharge defined in (3.78). This is the so-called "Gell-Mann-Nishijima" relation, of which (3.45) is only a special case.

The observation of a conserved quantum number, the strangeness S, points to the existence of another, so far not treated, symmetry of the strong interaction Lagrangian which is — via Noether's theorem — responsible for it. This is the so-called $SU(3)$ flavor-symmetry, which will be discussed in the following sections.

3.3.1 Properties of the Group $SU(3)$

The group $SU(3)$ is defined as the group of (3×3) unitary matrices with unit determinant. As in chapter 3.2 for the $SU(2)$ case, the general matrix representation of a generator of $SU(3)$ of dimension (3×3) can be constructed by observing that a hermitian (3×3) matrix with vanishing trace contains 8 independent matrix elements. It can thus in general be represented as a linear superposition of 8 linearly independent traceless, hermitian basis matrices. These are usually taken to have the Gell-Mann form

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & \cdot \\ 1 & 0 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \quad , \quad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & \cdot \\ \mathrm{i} & 0 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \quad , \\
\lambda_3 &= \begin{pmatrix} 1 & 0 & \cdot \\ 0 & -1 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \quad , \quad \lambda_4 = \begin{pmatrix} 0 & \cdot & 1 \\ \cdot & \cdot & \cdot \\ 1 & \cdot & 0 \end{pmatrix} \quad , \\
\lambda_5 &= \begin{pmatrix} 0 & \cdot & -\mathrm{i} \\ \cdot & \cdot & \cdot \\ \mathrm{i} & \cdot & 0 \end{pmatrix} \quad , \quad \lambda_6 = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & 0 & 1 \\ \cdot & 1 & 0 \end{pmatrix} \quad , \\
\lambda_7 &= \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & 0 & -\mathrm{i} \\ \cdot & \mathrm{i} & 0 \end{pmatrix} \quad , \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad .
\end{aligned} \tag{3.82}$$

The dots here stand for zeros, they are used to accentuate the $SU(2)$ submatrices in the λ_i. Except for extra zeros, λ_1, λ_2 and λ_3 exhibit the structure of the isospin $SU(2)$ group which therefore forms a subgroup of $SU(3)$. The matrix λ_8, finally, commutes with λ_3 and thus has no analogue in $SU(2)$.

The matrices λ_k fulfill the following commutation relations:

$$\left[\frac{\lambda_i}{2}, \frac{\lambda_j}{2}\right] = \mathrm{i} f_{ijk} \frac{\lambda_k}{2} \quad , \qquad i, j, k = 1, 2, \ldots 8 \quad , \tag{3.83}$$

where the "structure constants" f_{ijk} are completely antisymmetric under exchange of any two indices (the f_{ijk} are given in Table 3.1). The relations (3.83) define the algebra of $SU(3)$; they are valid for all representations of the λ_i, also for those of a higher dimension than 3×3. In contrast to this the anticommutation relations

$$\{\lambda_i, \lambda_j\} = \frac{4}{3}\delta_{ij} + 2 d_{ijk} \lambda_k \quad , \tag{3.84}$$

with the completely symmetric coefficients d_{ijk} are special to the 3×3 representation; the values of the d_{ijk} are also given in Table 3.1. Another useful property is the normalization of these matrices according to

$$\mathrm{tr}(\lambda_k \lambda_l) = 2\delta_{lk} \quad . \tag{3.85}$$

Table 3.1: $SU(3)$ structure constants f_{ijk} and symmetric coefficients d_{ijk} (see eqs. (3.83), (3.84)).

ijk	f_{ijk}	ijk	d_{ijk}	ijk	d_{ijk}
123	1	118	$1/\sqrt{3}$	366	$-1/2$
147	$1/2$	146	$1/2$	377	$-1/2$
156	$-1/2$	157	$1/2$	448	$-1/(2\sqrt{3})$
246	$1/2$	228	$1/\sqrt{3}$	558	$-1/(2\sqrt{3})$
257	$1/2$	247	$-1/2$	668	$-1/(2\sqrt{3})$
345	$1/2$	256	$1/2$	778	$-1/(2\sqrt{3})$
367	$-1/2$	338	$1/\sqrt{3}$	888	$-1/\sqrt{3}$
458	$\sqrt{3}/2$	344	$1/2$		
678	$\sqrt{3}/2$	355	$1/2$		

Furthermore, one sees from the explicit representation of the λ_i in equation (3.82) that $\lambda_i (i = 1, 2, 3)$ and λ_8 commute with each other, although the λ_i do not commute among themselves. Thus any pair $(\lambda_i,\ \lambda_8)$ can be made simultaneously diagonal; in the representation (3.82) λ_3 is chosen for λ_i. This is in contrast to $SU(2)$ where all three operators τ_i commute only with themselves. As mentioned already in section 3.2.1, the number of elements in an algebra that commute with one another is called the "rank" of the group. While $SU(2)$ has the rank 1, $SU(3)$ has the rank 2 [LIC78].

It is in this context interesting to note that there exists a second matrix representation of the $SU(3)$ algebra in 3×3 dimensions that is obtained by taking the negative complex conjugate of the matrices λ_k

$$\lambda'_k = -\lambda^*_k \quad . \tag{3.86}$$

The λ'_k obey the same commutator algebra as the λ_k, but they are not equivalent to the original λ_k. "Equivalent" matrix representations of $SU(N)$ can be transformed into each other by means of a unitary transformation belonging to the same group. In $SU(3)$ no such transformation from λ_k to λ'_k exists. This is in contrast to $SU(2)$ where the corresponding change to

$$\tau'_k = -\tau^*_k$$

leads to an equivalent representation (cf. equation (3.47)). Because only τ_y is complex the transformation between the two representations is a rotation in isospin space by the angle π around the y-axis, i.e. an $SU(2)$ transformation.

For our purposes, the most relevant consequence of this difference is the classification of the antiparticles that, because of eq. (3.46), transform under U^*. In $SU(3)$ the antiparticles therefore form the basis vectors of a separate representation, whereas for $SU(2)$ they belong to an equivalent representation (see eq. (3.51)).

The general $SU(3)$ transformation now has the form

$$U = \mathrm{e}^{-\mathrm{i}\sum_{k=1}^{8} \epsilon_k \frac{\lambda_k}{2}} \quad , \tag{3.87}$$

so that the change of a field operator under an infinitesimal $SU(3)$ transformation is given by

$$\Phi \rightarrow \Phi' = (1 - \mathrm{i}\boldsymbol{\epsilon}\frac{\boldsymbol{\lambda}}{2})\Phi \quad . \tag{3.88}$$

Here $\boldsymbol{\epsilon}$ and $\boldsymbol{\lambda}$ are eight-component vectors. Since the matrices λ_k of (3.82) form the $SU(3)$ representation of lowest dimension it is evident that any non-singlet Φ in (3.88) must have at least three intrinsic degrees of freedom, i.e. must be at least a three-component vector.

Conserved charges. Supposing that $SU(3)$ is a good symmetry of the strong interaction Lagrangian, what would be the associated conservation law? Considering for the moment only the baryonic part of the full Lagrangian and assuming that derivatives of the fermion fields appear only in the kinetic energy, we see immediately that for the $SU(3)$ symmetry transformation (3.87) the conserved current is given by

$$j^k_\mu(x) = \bar{\Psi}(x)\gamma_\mu \frac{\lambda_k}{2}\Psi(x) \quad , \tag{3.89}$$

where k denotes the component in the $SU(3)$ space and μ gives the space-time component of the current operator. The conserved charge is given by

$$Q_k = \int j^k_0(x)\, d^3x = \int \Psi^\dagger \frac{\lambda_k}{2}\Psi\, d^3x \quad . \tag{3.90}$$

After quantization the operators $\hat{Q}_k$ are the Hilbert-space generators of $SU(3)$ and obey the same commutation relations as the λ_k, which are their matrix representations in the internal space. The transformation of the field operators is therefore given by

$$\Psi \rightarrow \Psi' = \mathrm{e}^{+\mathrm{i}\epsilon_k \hat{Q}_k}\Psi \mathrm{e}^{-\mathrm{i}\epsilon_k \hat{Q}_k} = \mathrm{e}^{-\mathrm{i}\epsilon_k \frac{\lambda_k}{2}}\Psi \tag{3.91}$$

which is in infinitesimal form equivalent to

$$\left[\hat{Q}_k, \Psi\right] = -\frac{\lambda_k}{2}\Psi \quad , \tag{3.92}$$

where the rhs of eq. (3.92) follows from inserting the explicit form of the operator $\hat{Q}_k$ (3.90) if anticommutation relations are used for the field operators.

As discussed in connection with the $U(1)$ and $SU(2)$ symmetries, the operators $\hat{Q}_k$ represent conserved quantities if the Hamilton operator $\hat{H}$ is invariant under the transformation (3.91), i.e. if it commutes with the generators $\hat{Q}_k$. As we have already discussed, we can make two of these generators simultaneously diagonal. In the Gell-Mann representation (3.82) the diagonal matrices are λ_3 and λ_8, so that $\hat{Q}_3$ and $\hat{Q}_8$ are chosen as diagonal generators.

The physical meaning of these two operators is expressed by the following new notation

$$\begin{aligned} T_3 &= \hat{Q}_3 \\ Y &= \frac{2}{\sqrt{3}}\hat{Q}_8 \quad . \end{aligned} \tag{3.93}$$

The identification of $\hat{Q}_3$ with T_3, the third component of the isospin, is motivated by the fact that λ_3 contains t_3 as a submatrix. In addition, Y appears as a new conserved quantum number. In the following discussion of the $SU(3)$ multiplets, Y will be shown to be just the hypercharge introduced empirically at the end of the $SU(2)$ section. Since baryon number is experimentally known to be conserved the strong interaction Lagrangian must also be $U(1)$ symmetric. The independently conserved operators B and Y can be combined to the strangeness

$$S = Y - B \quad ,$$

which is then also conserved.

Since isospin $SU(2)$ is a subgroup of $SU(3)$, all states of $SU(3)$ multiplets can also be assigned to $SU(2)$ multiplets and can thus be labeled in addition to y and t_3 also by t, the isospin[6]. For the fundamental multiplet of $SU(3)$, denoted by 3 in the following discussions, we find the eigenvalues of Y to be $\frac{1}{3}$, $\frac{1}{3}$, and $-\frac{2}{3}$, whereas those for T_3 are $+\frac{1}{2}$, $-\frac{1}{2}$ and 0. For the second fundamental triplet, denoted by $\bar{3}$, all these eigenvalues go over into their negative values because of eq. (3.86).

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[6] Small letters denote the eigenvalues of the corresponding operators.

3.3.2 Structure of $SU(3)$ Multiplets

For $SU(2)$ the multiplets can be labeled by the eigenvalues of the Casimir operator $\boldsymbol{T}^2$. $SU(2)$, being a group of rank 1, has only one such Casimir operator. $SU(3)$, on the other hand, is of rank 2 and has 2 Casimir operators. These, however, have no direct physical significance, so that generally in the case of $SU(3)$ a pair of numbers (p, q) is used to label the multiplets. The physical meaning of (p, q) will become transparent in the actual construction of the multiplets later in this section. States within a multiplet can be labeled by the eigenvalues of T_3 and Y, which are simultaneously diagonal in the representation (3.82), and by the eigenvalue of $\boldsymbol{T}^2$. A graphical representation of $SU(3)$ multiplets thus has to be two-dimensional.

The structure of the $SU(3)$ multiplets can be most easily discussed by introducing, instead of the generators λ_k or Q_k, the in general $(n \times n)$-dimensional matrices

$$F_k = \frac{1}{2}\lambda_k \quad , \tag{3.94}$$

that act on n-dimensional multiplets and obey the algebra

$$[F_i, F_j] = \mathrm{i} f_{ijk} F_k \quad . \tag{3.95}$$

In terms of these, the $SU(3)$ transformation is given by

$$U = \mathrm{e}^{-\mathrm{i}\boldsymbol{\epsilon}\boldsymbol{F}} \quad , \tag{3.96}$$

where $\boldsymbol{\epsilon}$ and $\boldsymbol{F}$ are eight-dimensional vectors.

For an explicit construction of the multiplets we now introduce the spherical representation of the operators F_k:

$$\begin{aligned} T_\pm = F_1 \pm \mathrm{i}F_2 \quad &; \quad U_\pm = F_6 \pm \mathrm{i}F_7 \; ; \\ V_\pm = F_4 \pm \mathrm{i}F_5 \quad &; \quad T_3 = F_3 \; ; \; Y = \frac{2}{\sqrt{3}}F_8 \\ U_3 = \frac{3}{4}Y - \frac{1}{2}T_3 \quad &; \quad V_3 = \frac{3}{4}Y + \frac{1}{2}T_3 \quad . \end{aligned} \tag{3.97}$$

Introducing these operators instead of the F_k serves the same purpose as in $SU(2)$ where the spherical representations of the generators act as raising or lowering operators (3.71) and can thus be used to construct a multiplet.

The same holds for $SU(3)$. The commutation relations of the operators (3.97) given in Table 3.2 show that $T_\pm$, $V_\pm$ and $U_\pm$ are raising or lowering operators within a multiplet for t_3 and y, the eigenvalues of T_3 and Y. For $T_\pm$, for example, this is very easy to see. From

$$[T_3, T_\pm] = \pm T_\pm \tag{3.98}$$

Table 3.2: Commutator relations for the $SU(3)$ operators $T_\pm$, $V_\pm$, $U_\pm$, T_3, and Y (see eqs (3.97)).

$$\begin{array}{rclrcl}
[T_3, T_\pm] & = & \pm T_\pm & [T_+, T_-] & = & 2T_3 \\
[U_3, U_\pm] & = & \pm U_\pm & [U_+, U_-] & = & 2U_3 \\
[V_3, V_\pm] & = & \pm V_\pm & [V_+, V_-] & = & 2V_3 \\
 & & & & & \\
[T_3, U_\pm] & = & \mp\frac{1}{2}U_\pm & [Y, U_\pm] & = & \pm U_\pm \\
[T_3, V_\pm] & = & \pm\frac{1}{2}V_\pm & [Y, V_\pm] & = & \pm V_\pm \\
 & & & & & \\
[T_+, V_-] & = & -U_- & [T_+, U_+] & = & V_+ \\
[U_+, V_-] & = & T_- & [U_+, V_+] & = & 0 \\
[T_3, Y] & = & 0 & [Y, T_\pm] & = & 0 \\
[T_+, V_+] & = & 0 & [T_+, U_-] & = & 0
\end{array}$$

follows immediately

$$\begin{aligned}
T_3 T_\pm |t_3, y> &= T_\pm T_3 |t_3, y> \pm T_\pm |t_3, y> \\
&= (t_3 \pm 1) T_\pm |t_3, y> \quad . \qquad (3.99)
\end{aligned}$$

Similarly, from

$$[T_3, V_\pm] = \pm\frac{1}{2}V_\pm \qquad (3.100)$$

we obtain directly

$$\begin{aligned}
T_3 V_\pm |t_3, y> &= V_\pm T_3 |t_3, y> \pm \frac{1}{2} V_\pm |t_3, y> \\
&= (t_3 \pm \frac{1}{2}) V_\pm |t_3, y> \quad . \qquad (3.101)
\end{aligned}$$

Not shown in these equations are all extra quantum numbers needed to uniquely specify a state in an $SU(3)$-symmetric theory (in $SU(3)$, 5 numbers are needed to specify a state: 2 Casimir Operators or (p,q), t_3, y and the eigenvalue of $\boldsymbol{T}^2$, represented by t). Using now the commutator

$$[Y, V_{\pm}] = \pm V_{\pm} \tag{3.102}$$

gives, on the other hand,

$$\begin{aligned} Y V_{\pm}|t_3, y> &= V_{\pm} Y |t_3, y> \pm V_{\pm}|t_3, y> \\ &= (y \pm 1) V_{\pm}|t_3, y> \quad . \end{aligned} \tag{3.103}$$

Comparison of (3.101) and (3.103) shows that $V_{\pm}$ raises (or lowers) the eigenvalue t_3 by 1/2 and the eigenvalue y by 1.

Analogously one can construct the action of the other operators. The result is given in Fig. 3.2 that illustrates the action of all the operators (3.97). All states of a given multiplet can now be constructed by successive applications of these operators on one given state of a multiplet. Since the states within a multiplet are labeled by the eigenvalues of the two operators T_3 and Y they have to be represented by points in the two-dimensional (t_3, y) plane. A certain multiplet is determined by specifying which points are occupied and how often.

The general shape of a multiplet-boundary in the (t_3, y) plane can be inferred from symmetry considerations. First, we know that $SU(2)$ is a subgroup of $SU(3)$. Therefore, within a given isospin multiplet all states from $-t_3$ to $+t_3$ (for fixed y) must belong to the $SU(3)$ multiplet. The boundary thus has to be symmetric under reflection at the y axis. Second, we see from the second and third line of Table 3.2 that $SU(3)$ has two more $SU(2)$ subgroups, connected with the so-called U-spin and V-spin. This leads to a symmetry of the boundary under reflection at the axes $U_3 = 3/4\, y - 1/2\, t_3 = 0$ and $V_3 = 3/4\, y + 1/2\, t_3 = 0$. We thus have three symmetry-axes and (because of the raising and lowering operators) six pairwise antiparallel directions available. This necessitates that the geometrical structure of the boundary of an $SU(3)$ multiplet has to be either a triangle or a hexagon in the (t_3, y) plane.

In the following few paragraphs we will now sketch how these multiplets can be constructed by exploiting the algebraic properties of the operators (3.97)[7] . This method uses the fact that the occupied (t_3, y) points are all connected by the operators (3.97). To construct their boundary one starts with the member of the multiplet with the highest possible value of t_3 which

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[7] The discussion here follows closely that of S. Gasiorowicz [GAS66] where also more details can be found.

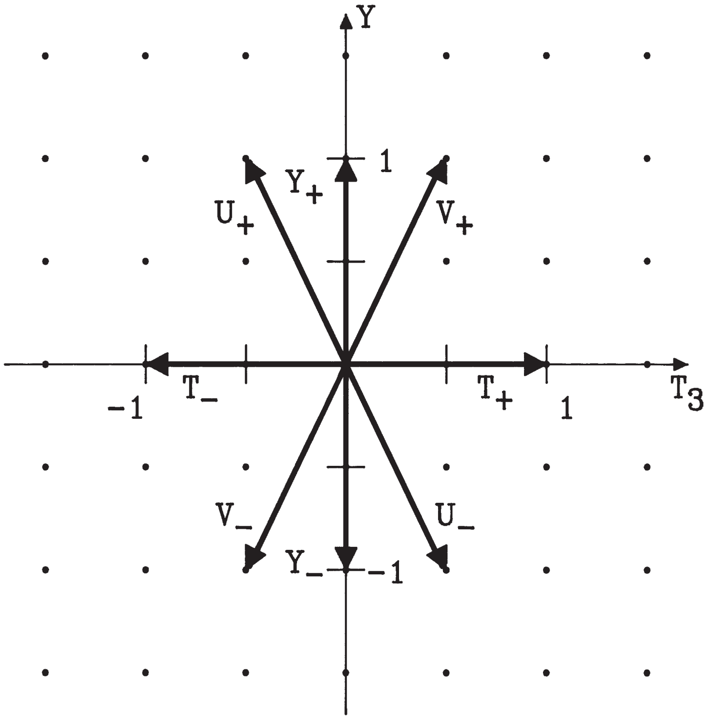

Figure 3.2: Action of the operators $T_\pm$, $U_\pm$, $V_\pm$, and $Y_\pm$ of eq. (3.97) in the isospin (T_3)-hypercharge (Y) plane.

we will call $|\max>$. We then have

$$V_+|\max>=T_+|\max>=U_-|\max>=0 \quad , \tag{3.104}$$

since all three operators act to increase t_3. We can now apply the operator V_- p-times on the state $|\max>$ until we reach the lower bound in y so that further application of V_- yields zero

$$(V_-)^{p+1}|\max>=0 \quad . \tag{3.105}$$

In this way, we have constructed the lower right boundary. It is essential for this result that the boundary of the multiplet sites in the (t_3, y) plane is convex [GAS66].

The preceding arguments show that the state $(V_-)^p|\max>$ is the last state of the multiplet that can be reached by repeated action of the operator V_-. We can now apply repeatedly the operator T_- to this state until after q-times another boundary of the multiplet has been reached

$$(T_-)^{q+1}(V_-)^p|\max>=0 \quad . \tag{3.106}$$

The remaining boundary of the $SU(3)$ multiplet can be constructed by repeated applications of the other operators. p successive applications of U_+, q steps of V_+, p steps of T_+ and finally q steps of U_- close the boundary.

For example, the lowest possible multiplets besides the singlet $(0,0)$ are given by $(1,0)$ or $(0,1)$, where the numbers stand for the two indices (p,q). Their geometrical structure is shown in Fig. 3.3. In this case, the possible values of (t_3, y) are given by the eigenvalues of the operators T_3 and Y (3.93) in the (3×3) representations. The representation 3 belongs to the λ matrices of (3.82), whereas $\bar{3}$ is connected with the generators of the second fundamental representation (3.86); this can be seen by inspecting the eigenvalues of λ_3' and λ_8' (see eq. (3.86)). Both representations act on triplet states; therefore $(1,0)$ is abbreviated by 3 and $(0,1)$ by $\bar{3}$. There are thus two triplets possible (cf. the discussion in section 3.3.1). This is so because $SU(3)$ is a group of rank 2 and is in contrast to $SU(2)$, which is a group of rank 1 [LIC78].

The multiplet $(1,1)$ has the structure shown in Fig. 3.4. Here the point $t_3 = y = 0$ deserves special attention. First, it can be reached by raising or lowering operators on several ways. With the help of the commutation relations one can, however, show that all these states $(t_3 = 0, y = 0)$ that have been reached on different ways are linear combinations of two basis vectors, namely $T_-|\max>$ and $V_-U_+|\max>$. Since there exist two linearly

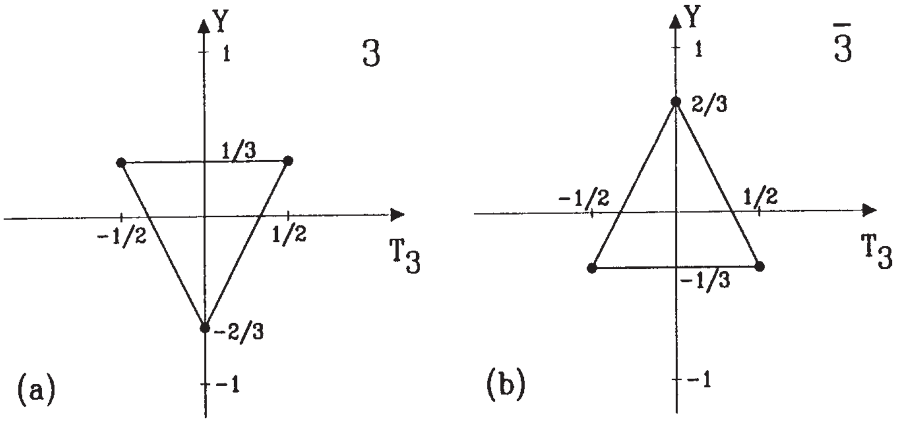

Figure 3.3: Fundamental triplets for (*a*) the 3 representation and (*b*) the $\bar{3}$ representation of $SU(3)$ in the isospin (T_3)-hypercharge (Y) plane.

independent linear combinations of these two states, the center point is doubly occupied and the multiplet $(1,1)$ forms an octet of states.

Since $SU(2)$ is a subgroup of $SU(3)$ states with the same y can also be assigned to isospin multiplets; one of the two center states then belongs to the $T=1$ triplet and the other one to the $T=0$ singlet.

The states in an octet transform in a particularly simple way because the eight-dimensional generators acting on them can be chosen to be in the regular representation

$$(F_k)_{lm} = -\mathrm{i} f_{klm} \quad , \tag{3.107}$$

where the f_{klm} are the structure constants of $SU(3)$. In this case the transformation law for the fields Ψ (eq. (3.92)) is given by

$$\delta\Psi_l = \mathrm{i}\epsilon_k \left[Q_k, \Psi\right]_l = -\epsilon_k f_{klm} \Psi_m \quad . \tag{3.108}$$

Here Ψ_l is a member of an octet. States in higher dimensional multiplets can be transformed only after the corresponding matrix representations of the generators λ_i have been constructed. There is, however, a more efficient method available for this problem; it is based on the construction of tensors with well defined transformation properties and will be discussed in section 4.2.

Figure 3.4: Baryon multiplets of $SU(3)$. (a) Octet of spin-1/2 baryons, (b) decuplet of spin-3/2 baryons.

3.3.3 Assignments of hadrons to $SU(3)$ multiplets.

The experimentally observed baryons and mesons can indeed be assigned to various $SU(3)$ multiplets. In these $SU(3)$ multiplets, all quantum numbers associated with other symmetries, like spin and parity, are the same. It is then found that the energetically lowest spin-1/2 baryons form an octet, whereas the spin-3/2 baryons can be assigned to a decuplet with $(p, q) = (3, 0)$ (see Fig. 3.4). The antiparticles of the baryon multiplet members have baryon number $B = -1$ and opposite intrinsic parity, as can be seen from eqs. (A.34) and (A.35), and thus form other multiplets.

A baryon singlet with $y = t_3 = 0$ and positive parity does not seem to exist. There are, however, higher-lying particles with negative parity, such as the $\Lambda(1520)$ with $J^\pi = 3/2^-$ that seem to be mainly $SU(3)$ singlet states.

The mesons can also be grouped into multiplets (see Fig. 3.5). The pseudoscalar mesons form the multiplet with lowest average mass, an octet. Here the antiparticles are not distinguished by any other quantum number, they are therefore members of the same multiplet as the corresponding particles; in particular, in the pseudoscalar octet the "center particles" η and π^0 are their own antiparticles.

In addition, a similar pseudoscalar meson exists, the η', which forms a pseudoscalar singlet and is mixed with the octet state η that carries the

Figure 3.5: Meson multiplets of $SU(3)$. (a) Octet of pseudoscalar mesons with $J^{\pi} = 0^{-}$; (b) octet of vector mesons with $J^{\pi} = 1^{-}$.

same quantum numbers. The same is true for the vector mesons where the isoscalar vector meson ω is mixed with the ϕ-meson. Thus there is really a nonet of states present instead of a well-defined octet and singlet. This fact cannot be understood in the present framework and will be discussed again later (in chapter 8). It is also worthwhile to note here explicitly that the lowest dimensional multiplet of $SU(3)$, the triplet, is realized neither in the meson sector nor the baryon sector. The absence of just the fundamental triplet of $SU(3)$ is at this point a surprising experimental fact that demands an explanation.

Another fact that cannot be understood in the present framework is the origin of the Gell-Mann-Nishijima formula (3.81). It says that for all particles, mesons and baryons, in $SU(3)$ multiplets the electrical charge Q is empirically given by

$$Q = \frac{1}{2}Y + T_3 \quad .$$

It is thus natural to define the charge operator $\hat{Q}$ by this relation, thus expressing it in terms of the operators for hypercharge and third component of isospin.

In analogy to the considerations in section 3.2.2, where an $SU(2)$ symmetric Lagrangian for mesons and baryons was constructed, the same can be done here for a $SU(3)$ symmetric Lagrangian. In such a Lagrangian, all baryons of a certain $SU(3)$ multiplet are coupled to all mesons in an $SU(3)$

multiplet with one additional coupling constant [GAS66]. Thus an $SU(3)$ symmetric theory yields very definite predictions for the baryon-meson interactions that can be checked experimentally through various rules for the relative intensities of reactions involving baryons or mesons of the same multiplet. These rules — together with the masses — are actually used to assign the mesons and baryons ot the multiplets discussed above but they also show that the $SU(3)$ symmetry is not nearly as perfect as $SU(2)$. This is also indicated in the mass splittings within the multiplets that are discussed in the next section.

Finally, it should be mentioned that heavier particles with "charm", a new, approximately conserved quantum number, have also been detected. This quantum number emerges only in the next higher symmetry group, $SU(4)$. The experimentally observed mass-splitting in the $SU(4)$ multiplets is significantly larger than in $SU(3)$, so that the $SU(4)$ symmetry breaking components of the strong interaction are obviously much larger than those breaking $SU(3)$. Thus $SU(4)$ does not appear to be a useful symmetry. This is even more so for the flavor symmetry group $SU(5)$ which would encompass the newly discovered particles with "beauty", another conserved quantum number.

3.3.4 $SU(3)$ Symmetry-Breaking

Symmetry of the strong interaction Hamiltonian under $SU(3)$ transformation implies that all states of a multiplet should have the same mass. However, for the baryons this is only fulfilled within about 40 per cent. For the pseudoscalar mesons the symmetry breaking is even worse; here the energies range from 135 MeV (π^0) to 549 MeV (η). Thus, the $SU(3)$ symmetry breaking is significantly larger than that of $SU(2)$ and the assignment of particles to $SU(3)$ multiplets may be difficult.

The mere size of the $SU(3)$ symmetry breaking indicates that it cannot be attributed to some weaker interaction but must be due to components of the strong interaction itself. We thus assume that the strong interaction can be split into a $SU(3)$ invariant part H_0 and a part H' that explicitly breaks the symmetry

$$H_{st} = H_0 + H' \quad .$$

The possible form of H' can be obtained from the following considerations. Within an isospin multiplet the masses are equal (except for the small splitting due to the Coulomb force). Therefore, H' must commute with T_3. The only generator of $SU(3)$ that commutes with T_3 is Y, thus suggesting a form for H',

$$H' = a + bY \tag{3.109}$$

(T_3 cannot appear linearly since such a term would break the very good $SU(2)$ symmetry). Since H' of eq. (3.109) does not account for the mass-difference between the Λ^0 and the Σ in the baryon octet (Λ^0 belongs to an isospin singlet, Σ to the $T = 1$ triplet), H' is supplemented by quadratic terms

$$H' = a + bY + cT^2 + dY^2 \quad , \tag{3.110}$$

where a, b, c and d are constants. With $d = -c/4$, this is the Gell-Mann-Okubo mass formula that was used by Gell-Mann 1964 to order the baryon resonances into a decuplet and to predict the mass of the Ω^-. Equation (3.110) is good within a few per cent for the baryons but does not work well for the mesons.

Chapter 4

QUARKS

In chapter 4 it was shown how the observed strongly interacting particles can be grouped into multiplets according to the relevant symmetry group. In particular, the $SU(3)$ classification of mesons and baryons was shown there to account quite well for many observed features of these particles and their interactions. However, a number of problems that arise with such a scheme were set aside in section 3.3. First of all, we were not able to explain why in particular the baryons appear in octets and decuplets, whereas the mesons can be grouped into nonets, which seemed to be mixtures of singlets and octets. Further, the question arose why the fundamental $SU(3)$ triplet does not seem to be realized in nature.

In this chapter we will take up these questions again by assuming that mesons and baryons are bound states of particles that belong to the fundamental triplets of $SU(3)$.

4.1 Construction of $SU(3)$ Multiplets

The starting point for the considerations in this chapter is the observation that the mesons and baryons can be grouped into $SU(3)$ singlets, octets and decuplets, whereas particles that could be identified with the fundamental $SU(3)$ triplets do not seem to occur in nature. The decisive link between these two facts is provided by the assumption that mesons and baryons are (strongly) bound many-body states of particles that belong to the fundamental triplets. This assumption finds some support in the possibility to construct the various $SU(3)$ multiplets by coupling these fundamental triplets.

As already discussed in chapter 3.3, the fundamental representation acts on a triplet with $(p, q) = (1, 0)$ or $(0, 1)$, denoted by 3 or $\bar{3}$, respectively. Both representations are not equivalent (see discussion at the end of section 3.2.2.1). They transform under

$$3: \quad U = \mathrm{e}^{-\mathrm{i}\epsilon^l \frac{\lambda^l}{2}} \tag{4.1a}$$

or (see equation (3.46))

$$\bar{3}: \quad U^* = \mathrm{e}^{+\mathrm{i}\epsilon^l \frac{\lambda^l}{2}^*} \quad , \tag{4.1b}$$

respectively. If a fermion field q transforms under U, then the antiparticle field $\bar{q} = \gamma_2 q^*$ (see equation (A.34)) will transform under U^*, which is just the matrix representation of the second $SU(3)$ transformation $\bar{U}$. Therefore the states of the triplet $\bar{3}$ are the antiparticles to those of the triplet 3.

To each of the different states in both triplets we now assign particles, which we call "quarks" for the 3 representation and "antiquarks" for the $\bar{3}$. These are at this stage purely hypothetical hadronic building blocks; their quantum numbers are given for each of the three "flavors" in Table 4.1.

The assignment of the quantum numbers T_3 and Y is dictated by the eigenvalues of the Gell-Mann matrices λ_3 and λ_8 ($T_3 = \lambda_3/2$, $Y = \lambda_8/\sqrt{3}$); the electrical charge Q is assigned according to the empirical Gell-Mann-Nishijima relation (3.81). The masses of all three quarks have to be the same, so that the three states of the triplet have the same energy, if we require the $SU(3)$ symmetry of the Lagrangian. The assignment of the baryon number B will be justified later; it is necessary to obtain the correct baryon numbers for the hadrons. The strangeness S, is then assigned according to the general relation $S = Y - B$. In addition, it is assumed that these quarks are fermions, i.e. spin-1/2 particles; the justification for this assumption will become clear in the following sections.

In the graphical representation of section 3.3, the fundamental triplets have the structure shown in Fig. 4.1.

All higher multiplets of $SU(3)$ can now be constructed from the fundamental multiplets 3 and $\bar{3}$ by appropriate vector coupling methods (Clebsch-Gordan coefficients of $SU(3)$). Instead of this formal approach, here again, as in section 3.3.2, a graphical construction method will be used. This method exploits the fact that Y and T_3 are conserved quantities under an $SU(3)$ transformation and that in a direct product of both multiplets they are additive:

$$\begin{aligned} Y &= Y_1 + Y_2 \\ T_3 &= T_{3,1} + T_{3,2} \quad , \end{aligned}$$

flavor	T_3	Y	Q	B	S
up (u)	$+1/2$	$+1/3$	$+2/3$	$1/3$	0
down (d)	$-1/2$	$+1/3$	$-1/3$	$1/3$	0
strange (s)	0	$-2/3$	$-1/3$	$1/3$	-1

Table 4.1: Quantum numbers for the three quark flavors. The antiquarks $\bar{u}$, $\bar{d}$ and $\bar{s}$ have opposite signs for all quantum numbers.

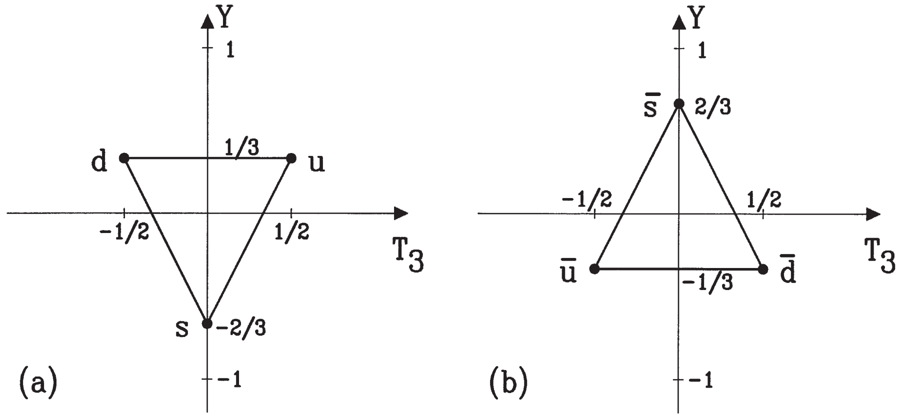

Figure 4.1: The fundamental triplets for (*a*) the quarks and (*b*) the antiquarks in the isospin-hypercharge plane.

where Y and T_3 are the hypercharge and the third isospin component of the product state. By using these relations, all possible basis vectors of any product representation can be generated. These then have to be decomposed into the multiplets, i.e. the basis vectors of irreducible representations, that have been discussed in section 3.3.2.

The graphical construction method based on this relation is illustrated here for the physically relevant cases $3 \otimes \bar{3}$ and $3 \otimes 3 \otimes 3$. This will show that baryons and mesons can be understood as three quark and quark-antiquark systems, respectively.

4.1.1 Construction of the Representation $3 \otimes \bar{3}$

The construction of the product space starts by attaching to each state of the fundamental triplet 3 the antitriplet $\bar{3}$ such that each state of the original triplet forms the center of the attached antitriplet. The corner points of the antitriplets then obey by construction the additive relation for Y and T_3.

As can be seen in Fig. 4.2, an octet and a singlet are generated in this way since the center point $(Y, T_3) = (0, 0)$ is threefold degenerate. Thus one has

$$3 \otimes \bar{3} = 8 \oplus 1 \quad . \tag{4.2}$$

The product states formed by the product $3 \otimes \bar{3}$ thus have been decomposed into a sum of multiplets of $SU(3)$. This is the analogue of the Clebsch-Gordon series, which is well known in the theory of angular momentum. The decomposition (4.2) implies for the symmetry operator U that its matrix representation for the $3 \otimes \bar{3}$ product space can be brought into block diagonal form where the dimensions of the special unitary submatrices are (8×8) and (1×1).

Since the states of $\bar{3}$ are the antiparticles of 3, all particles belonging to the $3 \otimes \bar{3}$ representation must have vanishing baryon number. We may thus be tempted to identify the nine states constructed with the states of the observed meson nonet. Because the $SU(3)$-symmetry-breaking component of the strong interaction mixes the two multiplets, which agree in all other quantum numbers such as baryon number, spin and parity, such an identification explains naturally why the $SU(3)$ meson octets always appear mixed with a singlet, thus forming a nonet.

The observation that mesons appear only with angular momentum $J = 0$ and $J = 1$ suggests to assign the spin 1/2 to the quarks, if one assumes a relative orbital angular momentum $L = 0$ for the lowest energy states. The parity of the meson state must then be negative, since fermions and antifermions always have opposite parities. This simple theory thus predicts

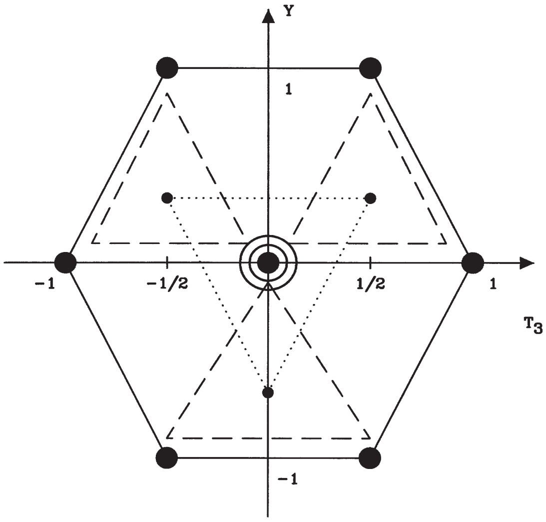

Figure 4.2: Graphical construction of the $SU(3)$ octet and singlet from the two fundamental triplets 3 and $\bar{3}$ of $SU(3)$: $3 \otimes \bar{3} = 8 \oplus 1$. The triple occupation of the center point is indicated by the two circles around it. The dashed $\bar{3}$ representations centered around each corner point of 3 are drawn on a slightly smaller scale to increase the visibility in this plot.

the existence of both pseudoscalar (0^-) and vector (1^-) meson multiplets. Both are indeed observed in nature (see the discussion in section 3.3.2).

4.1.2 Construction of the Representation $3 \otimes 3 \otimes 3$

In section 4.1.1 we discussed how the mesons may be understood as quark-antiquark ($q\bar{q}$) pairs. Since baryons also form $SU(3)$ multiplets, it is tempting to also try to explain the structure of these in terms of elementary building blocks, the quarks. Since baryons are spin-1/2 particles it is clear that one needs an odd number of quarks for such a construction. The first representation with this property is $3 \otimes 3 \otimes 3$, which will now be constructed. This construction of baryons out of 3 quarks fixes the baryon number of q to 1/3.

For that purpose, we first construct $3 \otimes 3$. This yields, with the same construction method as in section 4.1.1,

$$3 \otimes 3 = 6 \oplus \bar{3} \quad , \tag{4.3}$$

which is shown in Fig. 4.3. In contrast to $3 \otimes \bar{3}$, here no singlet appears.

In a next step we have now to construct $3 \otimes 6$, since $3 \otimes 3 \otimes 3$ has been reduced to

$$3 \otimes 3 \otimes 3 = 3 \otimes (6 \oplus \bar{3}) = (3 \otimes 6) \oplus (3 \otimes \bar{3}) \tag{4.4}$$

and $3 \otimes \bar{3}$ is already known from section 4.1.1. The construction of $3 \otimes 6$ yields the multiplet shown in Fig. 4.4.

Obviously, a decuplet is generated. In addition, some points are occupied twofold, the center point even threefold. One of the states at the center has to belong to an isospin triplet of the decuplet, the other two belong to an octet. We thus have

$$3 \otimes 6 = 10 \oplus 8 \quad . \tag{4.5}$$

Combining this decomposition with (4.4) and using (4.2) finally gives

$$3 \otimes 3 \otimes 3 = (3 \otimes 6) \oplus (3 \otimes \bar{3}) = 10 \oplus 8 \oplus 8 \oplus 1 \quad . \tag{4.6}$$

Thus the basis states of the product representation can be reduced into a sum over states of a decuplet, two octets and a singlet.

That the octet appears twice is due to the fact that in the coupling of three particles one and the same final state can be reached in two different ways. The two representations are connected by a unitary transformation. Thus, in their physical content, the octets constructed above are equivalent.

All the particles that belong to the multiplets on the rhs of equation (4.6) can only have either spin 1/2 or spin 3/2, in agreement with experiment, if

Figure 4.3: Construction of the $SU(3)$ sextet and antitriplet from the two fundamental triplets 3 and 3 dotted and dashed lines: $3 \otimes 3 = 6 \oplus \bar{3}$. Multiple occupations of individual points are again indicated by circles. As in Fig. 4.2, the dashed inner 3 representations are drawn on a slightly smaller scale to increase the visibility in this plot.

Figure 4.4: Construction of the $SU(3)$ decuplet and octet from the triplet (dashed lines) and sextet (dotted lines): $3 \otimes 6 = 10 \oplus 8$. The dashed inner 3 representations are not drawn to scale to increase the visibility.

the individual quarks carry the spin 1/2 (cf. the discussion in section 4.1.1). It is furthermore clear that the baryon number $B = 1$ for the nucleon can be obtained if each of the quarks carries the baryon number $B = 1/3$; the charge of the baryons obeys the Gell-Mann-Nishijima relation (3.81) if each of the quarks obeys it, because charge is an additive quantum number. This fixes the quark charges given in table 4.1. Thus the observed states of the baryons can indeed be generated by a coupling of three quarks.

The masses of all three quarks have to be the same in an $SU(3)$ symmetric world. Turning this argument round we can conclude from the empirical observation that $SU(2)$ is a very good symmetry of the strong interaction whereas $SU(3)$ is significantly broken that the masses of *up* and *down* quarks have to be very similar, whereas that of the *strange* quark must be significantly heavier.

Thus all the observed states of mesons and baryons can be understood as being composed of ensembles of quarks and antiquarks. This is a remarkable success of the quark picture. However, also new problems arise in such a description. One of them is why the spin-1/2 baryons appear only in an octet and the spin-3/2 baryons only in a decuplet. Furthermore, it is quite unknown at this point why other quark states such as qq, $qq\bar{q}$ or $qqqq$ are not seen in nature. A possible explanation would be in terms of the quark-quark interactions that could make these states energetically very unfavorable. This possibility will be discussed together with our present understanding of the $q - q$ interactions in chapter 9.

4.2 State Vectors for the Multiplets

In section 4.3 it was shown how $SU(3)$ multiplets can be constructed. These multiplets form the basis vectors of the irreducible, unitary representations of the $SU(3)$ group. For explicit couplings of multiplets, or actual transformations of the higher multiplets, first the higher-dimensional representations of the generators of $SU(3)$ have to be constructed, a cumbersome procedure.

There is, however, an alternative method based on the introduction of tensors, whose transformation properties are determined by the three-dimensional representation of $SU(3)$ alone and which are therefore very useful tools for explicit calculations in $SU(3)$. In particular, the coupling of the fundamental triplets to higher-dimensional multiplets, discussed in the preceding section, with a graphical method, can easily be performed in this formalism.

4.2.1 Tensor Algebra

The explicit structure of the composite baryon and meson wavefunctions can be obtained most easily by using tensor methods. These methods start from the fact that the $SU(3)$ triplet written as a vector

$$q = \begin{pmatrix} q^1 \\ q^2 \\ q^3 \end{pmatrix} \equiv \begin{pmatrix} u \\ d \\ s \end{pmatrix} \tag{4.7}$$

containing the three quarks is transformed under $SU(3)$ as

$$q' = Uq \quad , \tag{4.8}$$

where U is the fundamental 3×3 matrix representation 3 of the $SU(3)$ symmetry operation. The matrix U is written as U^i_j so that (4.8) reads in components

$$q^{i\prime} = U^i_j q^j \quad . \tag{4.9}$$

In the language of group theory, this transformation property defines an irreducible tensor of rank $(1, 0)$.

The states that transform according to the complex conjugate representation $\bar{3}$ are labelled by $\bar{q}_i$. They transform under the $SU(3)$ transformation with $\lambda'_k = -\lambda^*_k$ (see equation (4.1b)), so that the transformation law for them is

$$\bar{q}'_i = \left(U^*\right)^i_j \bar{q}_j = \left(U^\dagger\right)^j_i \bar{q}_j \quad . \tag{4.10}$$

The $\bar{q}_i$ form an irreducible tensor of rank (0, 1); physically they describe the antiquarks (see beginning of section 8.1).

In general a tensor of rank (m, n) is defined as an object $T^{i_1\cdots i_m}_{j_1\cdots j_n}$ that transforms under $SU(3)$ as

$$T^{i_1\cdots i_m}_{j_1\cdots j_n} \to \left(T^{i_1\cdots i_m}_{j_1\cdots j_n}\right)' = U^{i_1}_{k_1}\cdots U^{i_m}_{k_m} \left(U^\dagger\right)^{l_1}_{j_1} \cdots \left(U^\dagger\right)^{l_n}_{j_n} T^{k_1\cdots k_m}_{l_1\cdots l_n} \quad , \tag{4.11}$$

where U is the 3 representation of the group. Special cases of these tensors are the q^i and $\bar{q}_j$ defined above.

A tensor that is essential for our present purpose of constructing the quark wave functions of mesons and baryons explicitly is the Kronecker δ-Symbol, δ^i_j. It is an "invariant" (or "isotropic") tensor of rank $(1, 1)$ because its numerical values do not change under an $SU(3)$ transformation

$$\left(\delta^i_j\right)' = U^i_k \left(U^\dagger\right)^l_j \delta^k_l = U^i_k \left(U^\dagger\right)^k_j = \delta^i_j. \tag{4.12}$$

This follows from the unitarity $UU^\dagger = 1$ of the group $SU(3)$.

Another invariant tensor is the completely antisymmetric Levi-Civita tensor ϵ_{ijk}. One has

$$(\epsilon_{ijk})' = \left(U^\dagger\right)^\alpha_i \left(U^\dagger\right)^\beta_j \left(U^\dagger\right)^\gamma_k \epsilon_{\alpha\beta\gamma} = \det(U^\dagger)\epsilon_{ijk} = \epsilon_{ijk} \quad , \tag{4.13}$$

since U describes a special unitary transformation. Equation (4.13) not only shows that ϵ_{ijk} is a tensor of rank $(0,3)$, but also that it is invariant. The same holds for ϵ^{ijk}, which is a tensor of rank $(3,0)$.

New tensors can be formed by adding tensors with the same rank or by multiplying tensors that can also have different ranks. The product of a tensor T of rank (m,n) with $\tilde{T}$ of rank $(\tilde{m},\tilde{n})$

$$C^{i_1\cdots i_m k_1\cdots k_{\tilde{m}}}_{j_1\cdots j_n l_1\cdots l_{\tilde{n}}} = T^{i_1\cdots i_m}_{j_1\cdots j_n} \cdot \tilde{T}^{k_1\cdots k_{\tilde{m}}}_{l_1\cdots l_{\tilde{n}}} \quad ,$$

can easily be shown to be itself a tensor of rank $(m+\tilde{m}, n+\tilde{n})$.

For our present purpose it is further interesting that the rank of a tensor can be lowered by contraction with either the δ- or the ϵ-tensor. For example, if T^i_j is a tensor of rank (1, 1), then its contraction with the δ-tensor, which just equals its trace,

$$T^i_j \delta^j_i = T^i_i = \mathrm{tr}(T)$$

transforms as

$$\begin{aligned} \left(T^i_j \delta^j_i\right)' &= U^i_k \left(U^\dagger\right)^l_j U^j_m \left(U^\dagger\right)^n_i T^k_l \delta^m_n \\ &= \delta^n_k \delta^l_m T^k_l \delta^m_n \\ &= T^k_l \delta^l_k = T^l_l = \mathrm{tr}(T) \quad . \end{aligned} \tag{4.14}$$

Thus the tensor T of rank $(1,1)$ has been reduced to a tensor of rank $(0,0)$, i.e. a scalar. In general, contraction of a tensor of rank (m,n) with δ yields a new tensor of rank $(m-1, n-1)$. The contraction can in this case run over anyone of the lower and anyone of the upper indices of the tensor.

Similarly, one can contract, for example, a tensor of rank $(3,0)$ with the ϵ-tensor (4.13). This yields

$$\begin{aligned} \left(T^{ijk}\epsilon_{ljk}\right)' &= U^i_p U^j_q U^k_r \left(U^\dagger\right)^s_l \left(U^\dagger\right)^t_j \left(U^\dagger\right)^u_k T^{pqr}\epsilon_{stu} \\ &= U^i_p \left(U^\dagger\right)^s_l \delta^t_q \delta^u_r T^{pqr}\epsilon_{stu} \\ &= U^i_p \left(U^\dagger\right)^s_l T^{pqr}\epsilon_{sqr} \quad . \end{aligned} \tag{4.15}$$

Thus, the contraction

$$C^i_l = T^{ijk}\epsilon_{ljk} \tag{4.16}$$

of the tensor T of rank $(3, 0)$ with the tensor ϵ of rank $(0, 3)$ transforms as a tensor C of rank $(1, 1)$. In general, a tensor of rank (m, n) contracted with the tensor ϵ_{ijk} will yield a new tensor of rank $(m-2, n+1)$. Likewise, contracting with the tensor ϵ^{ijk} of rank $(3, 0)$ yields a new one with rank $(m+1, n-2)$. In the former case the contraction can run over any two of the upper indices, in the latter case over any two of the lower indices of the tensor.

If by contraction of a tensor of rank (m, n) with either the δ- or the ϵ-tensor a new nonvanishing tensor of rank(p, q) with $(p + q < m + n)$ can be formed, then the original tensor is called reducible. Otherwise it is irreducible. From (4.15) it is evident that tensors that are completely symmetric under exchange of any pair of upper or lower indices can no more be contracted with the ϵ-tensor. If in addition their trace (contraction with the δ tensor) vanishes they are irreducible. These conditions imply that not all components of irreducible tensors are linearly independent.

At the beginning of this section we defined a tensor and its rank via the $SU(3)$ transformation properties. The investigation of tensors and their properties is thus closely related to the corresponding representation of the group. If one picks out d linearly independent components of the tensor T and orders them into a d-dimensional vector then this vector will transform under a $(d\times d)$-dimensional representation of the symmetry group. If the original tensor is irreducible then the group representation also is irreducible and the d linearly independent tensor components, to be identified with the physical particles, form a multiplet.

Working with tensors has the advantage that the transformation behavior of even the higher-rank tensors is determined by the matrix U^i_j, i.e. the fundamental representation. Thus, higher-dimensional matrix-representations of U never have to be constructed. By assigning the physical states of a multiplet to the linearly independent elements of an irreducible tensor, we fix the transformation properties of the former in an easy way. This will be especially applied to the physically relevant products $q^i\bar{q}_j$ and $q^iq^jq^k$, which transform respectively as

$$q^i\bar{q}_j \to \left(q^i\bar{q}_j\right)' = U^i_k \left(U^\dagger\right)^l_j q^k\bar{q}_l \tag{4.17a}$$

$$q^iq^jq^k \to \left(q^iq^jq^k\right)' = U^i_l U^j_m U^k_n q^l q^m q^n \tag{4.17b}$$

as follows directly from equations (4.9) and (4.10), respectively. Comparison with the general definition (4.11) shows that the two products of equation (4.17) are $SU(3)$-tensors of rank $(1,1)$ and rank $(3,0)$, respectively. These can be decomposed by contraction into irreducible tensors and thus into the states of the physical multiplets. This reduction is the main point of the following section.

4.2.2 Hadron Multiplets

Meson multiplets. For the contraction of $q\bar{q}$ with δ, we have

$$S = q^i \bar{q}_j \delta_i^j = q^i \bar{q}_i \quad . \tag{4.18}$$

According to the discussion after equation (4.14), S is a tensor of rank $(0,0)$, i.e. a scalar. S thus describes a flavor singlet.

The tensor (4.17a) can obviously be decomposed in the following way:

$$q^i \bar{q}_j = \left(q^i \bar{q}_j - \frac{1}{3} q^k \bar{q}_k \delta_j^i \right) + \frac{1}{3} q^k \bar{q}_k \delta_j^i \quad . \tag{4.19}$$

Here the first term in parentheses cannot be contracted any more because it has vanishing trace

$$\left(q^i \bar{q}_j - \frac{1}{3} q^k \bar{q}_k \delta_j^i \right) \delta_i^j = \left(q^i \bar{q}_i - \frac{1}{3} q^k \bar{q}_k \delta_i^i \right) = 0 \quad . \tag{4.20}$$

It is, therefore, an irreducible tensor of rank $(1,1)$. The second term in equation (4.19) is just the singlet (4.18), so that equation (4.19) describes the well known decomposition

$$3 \otimes \bar{3} = 8 \oplus 1.$$

That part of the meson tensor which transforms according to the octet representation

$$M_j^i = q^i \bar{q}_j - \frac{1}{3} q^k \bar{q}_k \delta_j^i \tag{4.21}$$

is explicitly given by

$$M = \begin{pmatrix} \frac{1}{3}\left(2u\bar{u} - d\bar{d} - s\bar{s}\right) & u\bar{d} & u\bar{s} \\ d\bar{u} & \frac{1}{3}\left(2d\bar{d} - u\bar{u} - s\bar{s}\right) & d\bar{s} \\ s\bar{u} & s\bar{d} & \frac{1}{3}\left(2s\bar{s} - u\bar{u} - d\bar{d}\right) \end{pmatrix} . \tag{4.22}$$

This is one of the nicest features of the tensor method: the tensor of rank $(1,1)$ that describes the important octet is just an ordinary matrix.

The physical particles of the meson octet can now be expressed in terms of the linearly independent components of M. Because of their unique isospin content, the non-diagonal elements of M can directly be identified with members of the octet simply by comparing the isospin and strangeness properties of the former with those of the latter.

One obtains for the *pseudoscalar* mesons

$$\begin{array}{llll} M^1_2 & = \pi^+ = u\bar{d} \qquad & M^2_1 & = \pi^- = d\bar{u} \\ M^1_3 & = K^+ = u\bar{s} & M^3_1 & = K^- = s\bar{u} \\ M^2_3 & = K^0 = d\bar{s} & M^3_2 & = \bar{K}^0 = s\bar{d} \quad . \end{array} \tag{4.23}$$

Because M is traceless, only two of its diagonal components are independent; these must therefore represent linear combinations of the π^0 and η mesons, the two remaining members of the octet of pseudoscalar mesons. The structure of these two particles can again be determined by taking into account the isospin symmetry: the π^0 has to be the missing member of the pion isotriplet with $t_3 = 0$. This determines the π^0 state uniquely as follows[1]:

$$\pi^0 = \frac{1}{\sqrt{2}}\left(u\bar{u} - d\bar{d}\right) \sim T_- \pi^+$$

From this flavor structure it is evident why the π^0 is its own antiparticle whereas the K^0 has a distinctly different antiparticle $\bar{K}^0$. The requirement of orthogonality to the π^0 together with the condition $\mathrm{tr}(M) = 0$ then gives

$$\eta = \frac{1}{\sqrt{6}}\left(2s\bar{s} - u\bar{u} - d\bar{d}\right) .$$

These two mesons can therefore be identified in the diagonal elements of M in the following way:

$$\begin{aligned} M^1_1 &= \frac{1}{\sqrt{2}}\pi^0 - \frac{1}{\sqrt{6}}\eta \quad , \\ M^2_2 &= -\frac{1}{\sqrt{2}}\pi^0 - \frac{1}{\sqrt{6}}\eta \quad , \\ M^3_3 &= \frac{2}{\sqrt{6}}\eta \quad . \end{aligned} \tag{4.24}$$

---

[1] The "−" sign in the π^0 state occurs because of the $SU(2)$ transformation properties of the charge-conjugate states (equation (3.51)) that contain an extra "−" sign.

The structure of the pseudoscalar-meson singlet is given by eq. (4.18). It is identified with the η' meson, observed at a mass of 958 MeV. The $SU(3)$-breaking component of the strong interaction actually mixes the $t_3 = 0$, $y = 0$ mesons η and η', but their mixing angle is quite small [PER87], so that these two mesons can in a reasonable first approximation be assumed to be pure $SU(3)$ multiplet members.

An analogous identification of the quark structure can be performed for the *vector* mesons which differ only by their spin, i.e. a non-$SU(3)$ quantum number, from the pseudoscalar ones. Thus the vector, isovector ρ has the same flavor-structure as the pseudoscalar, isovector π, and the K^* mesons agree in their flavor content with the K mesons. The eighth member of the octet, called here ϕ_8, and the $SU(3)$ singlet, called here ϕ_0, are strongly mixed with a mixing angle of $\approx 35°$, i.e. $\sin\theta = 1/\sqrt{3}$ [PER87]. This leads to an "ideal" mixing for the observed vector mesons ω and ϕ in the sense that all the strangeness is concentrated in the ϕ meson:

$$\begin{aligned} \phi &= \tfrac{1}{\sqrt{3}}\phi_0 + \sqrt{\tfrac{2}{3}}\phi_8 &&= s\bar{s} \\ \omega &= \sqrt{\tfrac{2}{3}}\phi_0 - \tfrac{1}{\sqrt{3}}\phi_8 &&= \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) \quad . \end{aligned} \tag{4.25}$$

Baryon Multiplets. In section 4.1.2 it was shown how the observed baryon multiplets can be understood by combining three quarks with each other. In this subsection we now decompose the 3-quark tensor $q^i q^j q^k$ into its irreducible constituents that describe the physical multiplets. This decomposition corresponds to the decomposition $3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$ achieved graphically in section 8.1.2.

For this purpose we first apply the contraction (4.16) to the product $q^i q^j q^k$ (4.17b). This yields

$$C^i_l = q^i q^j q^k \epsilon_{ljk}.$$

C is a tensor of rank $(1,1)$. It is, however, not yet irreducible, because its trace does not vanish. It can be made irreducible by subtracting one third of its trace from the diagonal elements

$$B^i_l = q^i q^j q^k \epsilon_{ljk} - \frac{1}{3}\delta^i_l q^m q^j q^k \epsilon_{mjk} \quad . \tag{4.26}$$

B is an irreducible tensor of rank $(1,1)$; it forms an $SU(3)$ octet. Note that B is of mixed symmetry; it is antisymmetric under exchange of quarks 2 and 3, but has no specific symmetry under exchange of quark 1 with the others.

Alternatively, we could have defined another octet in which quark 1 and 2 are antisymmetrized; this corresponds to the appearance of the second octet in the group-reduction (4.6).

The identification of the elements of the baryon tensor (4.26) with the physical baryons can again be made as in the case of the meson octet by comparing isospin and hypercharge. One obtains for the spin-1/2 baryons in the octet

$$\begin{aligned}
B_1^1 &= \frac{1}{\sqrt{2}}\Sigma^0 - \frac{1}{\sqrt{6}}\Lambda^0 \\
B_2^1 &= \Sigma^+ \qquad\qquad B_3^1 = p \\
B_1^2 &= \Sigma^- \qquad\qquad B_3^2 = n \\
B_2^2 &= -\frac{1}{\sqrt{2}}\Sigma^0 - \frac{1}{\sqrt{6}}\Lambda^0 \\
B_1^3 &= \Xi^- \qquad\qquad B_2^3 = \Xi^0 \\
B_3^3 &= \frac{2}{\sqrt{6}}\Lambda^0 \quad .
\end{aligned} \tag{4.27}$$

Two other irreducible tensors can be formed out of the product $q^i q^j q^k$. First, we can form the completely symmetric tensor

$$D^{ijk} = \sum_P q^\alpha q^\beta q^\gamma \quad , \tag{4.28}$$

where the sum runs over all permutations (α, β, γ) of the indices (i, j, k). By construction D is completely symmetric in any pair of indices and thus cannot be contracted; it is, therefore, irreducible. Although D has 27 components, because of the symmetrization (4.28) it contains only 10 independent elements. D can thus be identified with the decuplet of spin-3/2 baryons.

Finally, we can also form a singlet state out of qqq

$$\begin{aligned}
S &= \sum_P (-)^P q^\alpha q^\beta q^\gamma \\
&= q^i q^j q^k \epsilon_{ijk} \quad ,
\end{aligned} \tag{4.29}$$

where the sum runs over all permutations (α, β, γ) of the indices (i, j, k) and the sign $(-)^P$ is positive or negative, depending on whether the number of pairwise exchanges necessary to build up the permutation P is even or odd. This $SU(3)$ singlet state is thus completely antisymmetric under exchange of any two particles. It is remarkable that there is no low-lying singlet state in nature; all the particles that can possibly be assigned to the $SU(3)$ singlet, such as the $\Lambda(1520)$, have negative parity and lie at energies higher than those of the decuplet. In the next section we will discuss the origin of this fact.

In all of these constructions it has been assumed from the outset that the flavor $SU(3)$ is an exact symmetry of the strong interactions, so that the hadrons within a given multiplet all have the same mass. In particular, for the $SU(3)$ triplet this means that also the masses of all three quarks, u, d and s, have to be equal. As we have discussed in section 4.3.3, however, the observed $SU(3)$-multiplets reveal a significant mass-splitting. By assuming that this breaking of the $SU(3)$ symmetry has its origin in different masses for the three quarks, spectroscopic studies of hadron states [LEE81] lead to the conclusion that

$$m_u = m_d \cong 10\,\mathrm{MeV}, \qquad m_s \cong 250\,\mathrm{MeV} \quad . \tag{4.30}$$

The larger mass of the strange quark reflects the observed $SU(3)$ mass splitting with consistently higher masses for the strange hadrons. If the $SU(3)$ symmetry is already broken at the quark level then it is natural that the members of the higher-dimensional $SU(3)$ multiplets cannot all have the same mass.

In 1974 new metastable mesons were detected that do not fit into any of the $SU(3)$ ordering schemes. This discovery has necessitated the introduction of heavier quarks, namely the charmed and the bottom quarks. Their masses are $m_c \approx 1.5\,\mathrm{GeV}$ and $m_b \approx 4.2\,\mathrm{GeV}$. Both quarks manifest themselves in the appearance of mesons and their excited states with the structure $c\bar{c}$ or $b\bar{b}$. Systems of the structure $c\bar{c}$ are generically called "charmonium", the so-called J/ψ at 3.1 GeV being the lowest-lying state, whereas $b\bar{b}$ states are denoted by "upsilon" with the lowest-lying state at 9.5 GeV. The two quarks carry new quantum numbers, charm and beauty (or "bottom"), with all $SU(3)$ quantum numbers being equal to zero. On theoretical grounds, which suggest a correspondence between the observed three (lepton, neutrino) families (e, ν_e), (μ, ν_μ), (τ, ν_τ) and the quark families (u, d), (s, c) and $(b, ?)$ (see the discussion in chapter 11), a still heavier quark, the so-called *top* quark, was predicted to exist as the partner of the b quark . In 1994 in an experiment at Fermilab this particle has indeed been found with a mass of about 174 GeV.

4.3 Color Degree of Freedom

The state vectors listed above for the mesons and baryons contain only the flavor part. For a construction of the total state vector space- and spin-parts also have to be attached. If we assume that the quarks are bound in some potential well then the energetically lowest configuration corresponds to all quarks being in a relative s-state, i.e. in a spatially symmetric state. This immediately implies that for the mesons only the spin parity combinations $J^\pi = 0^-$ and 1^- (negative parity because of the $q\bar{q}$ composition) and for the baryons only $J^\pi = 1/2^+$ and $3/2^+$ are possible.

The spin-3/2 baryons, however, pose an immediate problem. Their three-quark wavefunction is obviously completely symmetric in its spinpart, flavorpart, and spacepart, thus violating the Pauli principle.

A solution to this problem lies in the experimental observation that reactions that proceed via an intermediate $q\bar{q}$ pair (like, e.g., $\pi^0 \to 2\gamma$, $e^+e^- \to$ hadrons) point to the existence of an additional degree of freedom of the quarks that can assume three different values [PER87]. It is suggestive to associate also with this new degree of freedom, called "color", a new symmetry of the strong-interaction Lagrangian. We thus assume that the three color degrees of freedom transform as an $SU(3)$ triplet and that the Lagrangian of the strong interaction is invariant under this color $SU(3)$ transformation.

All the hadrons can then be assigned to color $SU(3)$ multiplets that can be constructed in exactly the same way as discussed in the preceding section for the flavor degree of freedom. However, because free quarks, and in general quark ensembles of the type discussed at the end of section 4.1, are not observed in nature it is further postulated that the baryons occur only as color singlets. The color part of the baryon wavefunction can then be written as

$$C_b = q^c q^d q^e \epsilon_{cde} \quad , \tag{4.31}$$

where c, d and e are color indices. That the totally antisymmetric state (4.31) indeed describes a singlet follows from our considerations of the $SU(3)$ flavor case in equation (4.29).

Similarly, we have to postulate also that the mesons appear only in color singlet states since, if all color combinations were possible, the number of mesons would have to be three times larger than is observed. The color part of their wavefunctions is thus given by

$$C_m = q^c \bar{q}_c \quad , \tag{4.32}$$

which is analogous to the scalar tensor in flavor space in equation (4.18).

The introduction of this new degree of freedom and in particular the structure of the color states (4.31) and (4.32), immediately solves some of the puzzles mentioned earlier. For example, the meson octet is found with both spins, whereas the baryon multiplets appear each with only one well determined spin, the octet with $J = 1/2$ and the decuplet with $J = 3/2$. This different behavior can now be understood as being due to the different character of meson- and baryon quark-wavefunctions under particle exchange.

Mesons are composed of one quark and one antiquark, i.e. distinguishable particles, so that no particular exchange symmetry has to be required. On the other hand, the three-quark wavefunctions describing the baryons must be completely antisymmetric under exchange of any two quarks[2]. A spin-3/2 wavefunction is obviously symmetric in its spin part. Since the color part is antisymmetric (color singlet), and since it is reasonable to assume that the three quarks making up the baryon occupy a spatially symmetric state (the s-state) in some potential well, the flavor part of the wavefunction must be totally symmetric, as it is the case for the decuplet (see equation (4.28)). This explains why the spin-3/2 baryons appear only in the $SU(3)$ decuplet. For the spin-1/2 baryons, on the other hand, the colorpart and spaceparts are antisymmetric and symmetric, respectively, whereas the spinpart is of mixed symmetry. It can, therefore, only be combined with a flavorpart also of mixed symmetry, so that the flavor-spin product is symmetric. This leaves for spin-1/2 baryons only the flavor octet (equation (4.26)) as a possibility. As a further consequence, if the three quarks occupy a spatially symmetric groundstate, the flavor singlet state (4.29) for baryons is now forbidden by the Pauli principle, since both the colorpart and the flavorpart of the total wavefunction are antisymmetric, so that the spin wavefunction would also have to be completely antisymmetric. Such a spin state of three quarks, however, does not exist. Thus baryonic flavor singlets can appear only for higher angular momenta and therefore have a higher energy.

Summarizing, one can understand the structure of all hadrons by postulating that all of them are color singlet states. This postulate immediately rules out the observation of exotic states such as qq or even an individual quark q and yields the correct flavor $SU(3)$ and spinassignments for the baryons and mesons. This postulate must have a dynamical origin, which means that it must be due to the color interactions between quarks and antiquarks. These forces have to be such that color singlets are energeti-

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[2] The explicit form of the total wavefunction is constructed in Appendix B.

cally strongly favored. There is an analogue to such a situation in classical electrodynamics. There the electrostatic force binds charged protons and electrons into neutral atoms. Likewise, for the quarks a force is needed that binds colored quarks into color singlets.

Chapter 5

CHIRAL SYMMETRY

A symmetry of the strong interaction that is less well known but plays an important role in particle physics is the chiral symmetry. It is the purpose of this chapter to present a major experimental result that reflects this symmetry. Although chiral symmetry is a property of the strong interaction, we will see that it can be indirectly observed in a weak interaction process, the β-decay. Finally we will discuss the underlying group structure of the symmetry.

5.1 Phenomenology of β-decay

Since one of the most important experimental hints for the chiral symmetry of strong interactions is obtained from the β-decay we will first very briefly discuss the necessary properties of the classical theory of β-decay.

Leptonic β-decay. In the classical theory of β-decay, the interaction part of the Lagrangian for the purely leptonic β-decay process, e.g. the decay of the muon,

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu \tag{5.1}$$

is given by the coupling of currents

$$\mathcal{L}_l = \frac{4G_F}{\sqrt{2}} j^{\lambda +} j_\lambda^- \tag{5.2}$$

with the "charged" current

$$j_\lambda^- = \frac{1}{2} \sum_\ell \bar{\ell} \gamma_\lambda (1 - \gamma_5) \nu_\ell \quad . \tag{5.3a}$$

Here ℓ stands for the spinor of the lepton field and ν_ℓ for that of the corresponding neutrino, the sum runs over all three leptons (e, μ, τ). The other transition current is given by

$$j_\lambda^+ = \frac{1}{2} \sum_\ell \bar{\nu}_\ell \gamma_\lambda (1 - \gamma_5) \ell \quad . \tag{5.3b}$$

Formally one can also write the currents, for example j_λ^+, as

$$j_\lambda^+ = \sum_\ell \bar{L}_\ell \gamma_\lambda t_+ L_\ell \quad , \tag{5.4}$$

with the two-dimensional lepton spinor

$$L_\ell = \begin{pmatrix} \nu_{\ell L} \\ \ell_L \end{pmatrix} = \begin{pmatrix} \nu_\ell \\ \frac{1}{2}(1 - \gamma_5)\ell \end{pmatrix} \quad .$$

Here $t_+ = \tau_+/2$ is the normal isospin raising operator (eq. (3.71)) and ℓ_L is the left-handed part of the lepton spinor (note that the neutrino is only left-handed so that $\nu_L = \nu$). The intrinsic degree of freedom associated with the two fields in the doublet L is called the 'weak isospin'.

The appearance of the combination $(1 + \gamma_5)$ implies that the interaction contains vector as well as pseudovector contributions and thus does not conserve parity. The factor $(1 + \gamma_5)$ in front of the neutrino field operator reflects the fact that only left-handed neutrinos are coupled to other particles via the weak interaction. This holds under the assumption that the neutrinos are exactly massless (see appendix A). In front of the massive electron field (as well as μ, τ), this factor leads to a polarization of the electrons involved in weak interaction processes like, e.g., β-decay.

For the specific decay (5.1) of a muon into a muon-neutrino, an electron-antineutrino, and an electron, one obtains for the relevant interaction

$$\mathcal{L}_\mu = \frac{G_F}{\sqrt{2}} \bar{\nu}_\mu \gamma^\lambda (1 - \gamma_5) \mu \, \bar{e} \gamma_\lambda (1 - \gamma_5) \nu_e \quad . \tag{5.5}$$

This Lagrangian mediates the μ-decay. The effective coupling constant for the μ-decay is

$$G_\mu = \frac{G_F}{\sqrt{2}} \quad .$$

Semileptonic β-decay. In analogy to the leptonic decay, the β-decay of hadrons can also be treated. Now the interaction Lagrangian is

$$\mathcal{L}_H = \frac{4G_F}{\sqrt{2}} \left(J^{\lambda +} j_\lambda^- + J^{\lambda -} j_\lambda^+ \right) \quad , \tag{5.6}$$

where J denotes the hadron current and j the lepton current. The hadronic current is not known a priori, but has to be constructed from case to case. For the neutron decay, for example, one has found

$$J^{\lambda +} = \frac{1}{2} \bar{p} \gamma^\lambda (g_V - g_A \gamma_5) n \ , \tag{5.7}$$

so that the relevant part of the Lagrangian becomes

$$\mathcal{L}_n = \frac{G_F}{\sqrt{2}} \bar{p} \gamma^\lambda (g_V - g_A \gamma_5) n \, \bar{e} \gamma_\lambda (1 - \gamma_5) \nu_e \quad . \tag{5.8}$$

Here e, p and n stand for the electron, the proton and the neutron fields. The hadron current has a form that is very similar to that of the leptons but contains the constants g_A and g_V which have to be determined by experiment. Their presence is an indication for the composite structure of the hadrons.Equation (5.8) can also be written as

$$\mathcal{L}_n = \frac{G_F}{\sqrt{2}} \left(g_V V_\lambda - g_A A_\lambda \right) \bar{e} \gamma^\lambda (1 - \gamma_5) \nu_e \quad , \tag{5.9}$$

with the vector current

$$V_\lambda = \bar{p} \gamma_\lambda n = \bar{N} t_+ \gamma_\lambda N \tag{5.10}$$

and the axial vector current

$$A_\lambda = \bar{p} \gamma_\lambda \gamma_5 n = \bar{N} t_+ \gamma_\lambda \gamma_5 N \quad , \tag{5.11}$$

with the nucleon spinor N. Equation (5.9) exhibits the $V - A$ form of the nuclear β-decay interaction.

In a non relativistic approximation one can see that the axial vector A_λ mediates transitions with $\Delta J = 0, 1$, whereas the vector current is responsible for only $\Delta J = 0$ transitions, both without a parity change. The former flip the spin of the nucleon (Gamow-Teller transitions), whereas the latter conserve the spin direction (Fermi transitions).

Experimentally, one finds for the coupling constants

$$g_V = 0.98 \quad , \quad g_A / g_V = 1.25 \quad , \text{and} \quad G_F = 1.03 \cdot 10^{-5} / m_p^2 \quad , \tag{5.12}$$

where m_p is the proton rest mass. It is quite remarkable that G_F has the same value both for the leptonic and for the semileptonic decay and is thus a universal constant.

This universality of the coupling constant suggests that both interactions, the purely leptonic and the semileptonic, can be unified into one single interaction Lagrangian,

$$\mathcal{L}_W = \frac{4G_F}{\sqrt{2}} \left[\left(J^{\lambda +} + j^{\lambda +} \right) \left(J^-_\lambda + j^-_\lambda \right) \right] \quad , \tag{5.13}$$

which describes the pointlike interaction of charged currents, i. e. of currents that change the charge of the corresponding particle by one unit. Note that this Lagrangian describes not only weak decays of leptons and hadrons, but also other interaction processes such as (ν, e) scattering events.

This is the universal form of the weak interaction as it was generally accepted about fourty years ago. Since then two new observations have been made that do not fit into the theory described so far. One is the discovery of strangeness-changing weak decays of strange particles (like e.g. , $K^+ \to \mu^+\nu_\mu$ $(\Delta S = 1)$). The other is the existence of weak interaction processes which are not connected with a change of charge (like, e.g. $\nu_e e \to \nu_e e$) and are thus mediated by "neutral currents". Both processes are described in more detail in chapter 12.

Equations (5.5) and (5.9) contain point interactions of four fermion fields which violate the unitarity constraints on scattering cross sections at large energies, since the zero range of the interactions enables infinite momentum transfers [QUI83]. This situation can be improved by the introduction of very heavy intermediate particles, the W bosons, that transmit the interaction through boson exchange and lead to a finite, though very small, range of the interaction. To combine such bosons with the observed universality of the coupling constant in a renormalizable theory, that is free of such unitarity violations, is the essence of the so-called standard model of electroweak interactions [AIT82], [QUI83], discussed in chapter 12.

5.2 Current Conservation in Strong Interactions

5.2.1 Vector Current Conservation

The fact that the semileptonic vector coupling constant g_V for the β-decay of hadrons is very close to 1 and thus nearly the same as the one for purely leptonic decays is at first sight very astonishing. Nucleons are in contrast to leptons strongly interacting particles. We could therefore have expected that

Figure 5.1: Feynman diagrams illustrating the possible renormalization of the axial coupling constant g_A due to strong interactions. The broken lines illustrate the exchange of strongly interacting pions, the wavy line the intermediate boson of the weak interactions W^-.

the two coupling strengths are quite different owing to different renormalizations of the coupling constants. This is schematically indicated in Fig. 5.1, which shows some of the processes that contribute to the renormalization of the coupling constant in the case of the hadronic decay.

In 1958 Feynman and Gell-Mann have pointed an ingenious way out of this difficulty. Their explanation rests on the understanding of an analogous situation in quantum electrodynamics; there the charge renormalization is independent of particle type. The essential prerequisite for this result is the property of electrical current conservation, i.e. the existence of a $U(1)_Q$ symmetry of the Lagrangian. Therefore, starting from the same unrenormalized charges for, e.g. , the strongly interacting proton and the weakly interacting positron, we also arrive at the same physical charges for these particles after renormalization, regardless of their different other interactions.

Turning this argument round, one may understand the universal value for the weak interaction coupling constant g_V by assuming that, much as in the case just discussed, all weakly interacting particles have the same unrenormalized "weak charge" and that this charge is conserved in the strong interaction. As in the case of QED, the renormalized charges then also have to be the same regardless of the quite different interactions that, e.g., a baryon and a lepton experience.

We know that the conserved current related to this charge has to include V_λ of eq. (5.10). The special form of V_λ therefore suggests the identification of the weak hadronic vector current with the conserved isospin current. This suggestion goes much further than the original ansatz for $\mathcal{L}_W$ since the conserved isospin vector current contains contributions from many different fields:

$$\begin{aligned} \boldsymbol{V}_\lambda &= \bar{N}\gamma_\lambda \boldsymbol{t}\, N - \mathrm{i}\,\partial_\lambda \pi_\alpha \boldsymbol{t}_{\alpha\beta}\pi_\beta \\ &\qquad + \text{contributions of other particles} \quad . \end{aligned} \tag{5.14}$$

Replacing the vector part of the nucleon current J_λ in $\mathcal{L}_W$ by this conserved isospin vector current leads to definite predictions, with fixed coupling constants, for other hadronic decays which have been verified by experiment.

The assumption that the weak hadronic vector current is conserved and can be identified with the isospin current is known as the conserved vector current (CVC) hypothesis. In mathematical form, the conservation of $\boldsymbol{V}_\lambda$ in the strong interaction can be formulated as

$$\partial^\lambda \boldsymbol{V}_\lambda = 0 \quad , \tag{5.15}$$

with $\boldsymbol{V}_\lambda$ given by eq. (5.14). The corresponding symmetry-operation is

$$U = \mathrm{e}^{-\mathrm{i}\boldsymbol{\epsilon}\boldsymbol{t}} \quad . \tag{5.16}$$

All the considerations concerning the CVC hypothesis have rested on the assumption that $g_V = 1$. The small deviation of g_V from 1 (see section 5.1) is experimentally well established and is understood as being due to the presence of a strangeness-changing component in the weak hadronic current. This will be discussed in detail in chapter 12. Since this is not essential for the discussion here, we neglect it for the moment.

In conclusion, we should note that we have found a connection between an invariance property of the strong interaction (isospin symmetry) and the weak-interaction decay process. Via the CVC hypothesis properties of weak decays, which themselves do not conserve isospin, can be used to deduce that isospin is conserved in the strong interaction.

It is also essential to realize that the conserved vector current is intimately connected with electromagnetic current. The latter contains an isoscalar and an isovector part ($Q = \frac{1}{2}B + T_3$). An $SU(2)$ transformation of the isovector part generates just the conserved vector current of the weak interactions. This is a first hint that the weak and the electromagnetic interaction are related to each other.

5.2.2 Axial Vector Current Conservation

For the axial current, the coupling constant g_A is not equal to 1, so that $G_F \cdot g_A$ is larger than that for purely leptonic decays. However, g_A is still reasonably close to 1. This could be the consequence of an approximate symmetry of the strong interaction that — if it were exact — leads to axial vector current conservation, in analogy to the arguments in the last section. The fact that $g_A \neq 1$ shows that this symmetry must be broken in nature so that the axial vector current is only partially conserved.

A very good estimate for the strength of this symmetry breaking can be obtained by studying the weak decay of the pion,

$$\pi^- \to \mu^- + \bar{\nu}_\mu \quad .$$

The matrix element that describes this decay in a time-dependent perturbation theory is given by

$$< f|\mathcal{H}_{\text{eff}}|i> = < \mu\bar{\nu}|\mathcal{H}_{\text{eff}}|\pi> \quad , \tag{5.17}$$

where $|\pi>$ is the initial one-pion state and $|\mu\bar{\nu}>$ denotes the final state after the decay. $\mathcal{H}_{\text{eff}}$ is the relevant effective interaction Hamiltonian density

$$\mathcal{H}_{\text{eff}} = -J^{\lambda +} j_\lambda^- \quad ,$$

where J again denotes the hadron and j the lepton current. Inserting this into eq. (5.17) gives

$$\begin{aligned} < f|\mathcal{H}_{\text{eff}}|i> &= -< 0|J^{\lambda +}|\pi>< \mu\bar{\nu}|j_\lambda^-|0> \\ &= -< 0|J^{\lambda +}|\pi>< \mu\bar{\nu}|\frac{1}{2}\bar{\mu}\gamma_\lambda(1+\gamma_5)\nu_\mu|0> \quad . \end{aligned} \tag{5.18}$$

In this transition matrix element only the axial vector part A_λ of J_λ^+ can contribute because of the pseudoscalar nature of the pion.

Since the explicit form of A_λ is not known, we will try to deduce the general structure of the matrix element from quite general symmetry considerations. First, the requirement of Lorentz covariance of the decay rate demands that the matrix element of A_λ transforms as a four-vector. Since the only four-vector that can appear in the problem is the momentum of the pion, k, we have

$$< 0|J_\lambda^+|\pi> = < 0|A_\lambda^i(x)|\pi^j> = -\mathrm{i}\delta^{ij} f_\pi k_\lambda \frac{1}{\sqrt{2E_\pi V}} \mathrm{e}^{-\mathrm{i}kx} \quad . \tag{5.19}$$

Here the upper indices on A and π specify the isospin degree of freedom and k_λ the pion momentum, while f_π is a proportionality constant, the so-called "pion decay constant". This is related to the half-life of the pion and is experimentally determined to $f_\pi \approx 93$ MeV.

The appearance of the plane wave function in (5.19) is a consequence of translational invariance. If $|0>$ and $|\pi>$ are eigenstates of the four-momentum operator p_μ, with momenta 0 and k_μ, respectively, then we have in general

$$<0|A_\lambda(x)|\pi>=<0|A_\lambda(0)|\pi>\mathrm{e}^{-\mathrm{i}kx} \quad . \tag{5.20}$$

This follows from a relativistic generalization of the Heisenberg equation of motion

$$[p_\mu, A_\lambda(x)] = -\mathrm{i}\partial_\mu A_\lambda \quad . \tag{5.21}$$

Here the p_μ are the Hilbert space operators for the four-momentum of the field[1]. Taking the matrix element of (5.21) gives

$$<0|p_\mu A_\lambda(x) - A_\lambda(x)p_\mu|\pi>= -\mathrm{i}\partial_\mu <0|A_\lambda(x)|\pi> \quad . \tag{5.22a}$$

Since $|0>$ and $|\pi>$ are supposed to be eigenstates of p_μ with eigenvalues 0 and k_μ we get

$$-k_\mu <0|A_\lambda(x)|\pi>= -\mathrm{i}\partial_\mu <0|A_\lambda(x)|\pi> \quad . \tag{5.22b}$$

Integrating (5.22b) gives (5.20) and thus explains the exponential function in (5.19). The factor $1/\sqrt{2E_\pi V}$ appears in the normal mode expansion of the pion field and is a consequence of its normalization.

In order to bring eq. (5.19) into the form of a conservation law, the four-divergence of (5.19) is taken. We obtain

$$\begin{aligned} <0|\partial^\lambda A^i_\lambda(x)|\pi^j> &= \delta^{ij}k^\lambda k_\lambda f_\pi \frac{1}{\sqrt{2E_\pi V}}\mathrm{e}^{-\mathrm{i}kx} \\ &= \delta^{ij} f_\pi m_\pi^2 \frac{1}{\sqrt{2E_\pi V}}\mathrm{e}^{-\mathrm{i}kx} \quad , \end{aligned} \tag{5.23}$$

since

$$k^\lambda k_\lambda = E_\pi^2 - \vec{k}^2 = m_\pi^2 \quad .$$

Equation (5.23) shows that the axial current is not conserved. Its divergence differs from zero just by a term proportional to the square of the pion mass. On the other hand, the mass of the pion ($m_\pi \approx 140$ MeV) is small on the normal hadronic mass scale ($M_N \approx 1$ GeV); this explains the observed partial conservation of the axial current (PCAC).

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[1] Eq. (5.21) follows from the translational transformation of the field operator $A_\lambda(x)$

$$A_\lambda(0) = \mathrm{e}^{-\mathrm{i}xp}A_\lambda(x)\mathrm{e}^{\mathrm{i}xp} = A_\lambda(x) - \mathrm{i}\,[x_\mu p^\mu, A_\lambda(x)] \quad .$$

On the other hand, we also have by direct Taylor expansion,

$$A_\lambda(0) = A_\lambda(x) - x_\mu \partial^\mu A_\lambda(x)\big|_{x=0} \quad .$$

Comparing the last two expressions for $A_\lambda(0)$ yields relation (5.21).

5.3 Chiral Symmetry Group

The question now arises as to which symmetry of the Lagrange density of the strong interaction determines the observed partial conservation of the axial vector current. It is the purpose of this section to find and discuss the properties of this symmetry, the so-called "chiral symmetry", first for the fermions and then for the mesons.

5.3.1 Chiral Symmetry Transformations for the Fermions

By construction of the conserved currents it is easy to see that the axial current for the nucleons,

$$A_\mu^k = \mathrm{i}\bar{N}\gamma_\mu\gamma_5\frac{\tau^k}{2}N + \ldots \qquad , \tag{5.24}$$

is conserved if the strong interaction is invariant under the transformation

$$U = \mathrm{e}^{-\mathrm{i}\boldsymbol{\eta}\frac{\boldsymbol{\tau}}{2}\gamma_5} \quad , \tag{5.25}$$

where $\boldsymbol{\tau}$ acts on the nucleon spinors and $\boldsymbol{\eta}$ is a constant isovector.

From the discussions of chirality in appendix A.2, it is clear that the Lagrangian of a system of massless fermions indeed has this symmetry. Of course, since real nucleons are massive there is still a difficulty in understanding the observed PCAC.

The solution to this puzzle lies in the observation that the physical states of a system need not necessarily have the same symmetry as the Lagrangian. We will illustrate the consequences of this fact with a simple, but very important model, the σ-model, in section 6.2. In this chapter, however, we will try to explore some more general properties of the two conserved currents (CVC and PCAC), and especially investigate the underlying symmetry group.

The comparison of the chiral transformations (5.25) with the $SU(2)$ isospin transformations (3.33) shows that also the set of transformations (5.25) forms an $SU(2)$ symmetry group since the extra factor γ_5 acts only on the Dirac spinor. Because of the related conserved currents we will in the following distinguish the two transformations by denoting (3.33) as $SU(2)_V$ and (5.25) as $SU(2)_A$.

According to Noether's theorem, discussed in chapter 2.2.2, the conserved currents belonging to the $SU(2)_V$ and the $SU(2)_A$ symmetry are the vector current,

$$V_\mu^k = \bar{\Psi}\gamma_\mu\frac{\tau^k}{2}\Psi \quad , \tag{5.26}$$

and the axial vector current,

$$A_\mu^k = \bar{\Psi}\gamma_\mu\gamma_5\frac{\tau^k}{2}\Psi \quad , \tag{5.27}$$

respectively. Here Ψ denotes a two component isospinor. The conserved charges and generators of the symmetry groups are then

$$\begin{aligned} Q_V^k &= \int V_0^k d^3x = \int \Psi^\dagger \frac{\tau^k}{2}\Psi \; d^3x && (5.28)\\ Q_A^k &= \int A_0^k d^3x = \int \Psi^\dagger\gamma_5 \frac{\tau^k}{2}\Psi \; d^3x \quad . && (5.29)\end{aligned}$$

The vector charges $\boldsymbol{Q}_V$ are scalars but the axial charges $\boldsymbol{Q}_A$ are pseudoscalars because they involve γ_5. Both can be obtained from the combined transformation

$$U = \mathrm{e}^{-\mathrm{i}(\boldsymbol{\epsilon}+\boldsymbol{\eta}\gamma_5)\frac{\boldsymbol{\tau}}{2}} \quad . \tag{5.30}$$

One can now generate an algebra from the charges $\boldsymbol{Q}_V$ and $\boldsymbol{Q}_A$ and finds

$$\begin{aligned} \left[Q_V^k, Q_V^l\right] &= \mathrm{i}\epsilon^{klm}Q_V^m \\ \left[Q_V^k, Q_A^l\right] &= \mathrm{i}\epsilon^{klm}Q_A^m \\ \left[Q_A^k, Q_A^l\right] &= \mathrm{i}\epsilon^{klm}Q_V^m \end{aligned} \tag{5.31}$$

where ϵ^{klm} are the elements of the completely antisymmetric Levi-Civita tensor, which are just the structure constants of $SU(2)$.

These commutation relations (5.31) can be decoupled by defining linear combinations of the generators

$$\begin{aligned} Q_R^k &= \frac{1}{2}\left(Q_V^k + Q_A^k\right) \\ Q_L^k &= \frac{1}{2}\left(Q_V^k - Q_A^k\right) \end{aligned} \tag{5.32}$$

with the following properties:

$$\begin{aligned} \left[Q_R^k, Q_R^l\right] &= \mathrm{i}\epsilon^{klm}Q_R^m \\ \left[Q_L^k, Q_L^l\right] &= \mathrm{i}\epsilon^{klm}Q_L^m \\ \left[Q_L^k, Q_R^l\right] &= 0 \quad . \end{aligned} \tag{5.33}$$

Thus, the operators Q_L^k and Q_R^k ($k = 1, 2, 3$) are the generators of two separate $SU(2)$ groups, $SU(2)_L$ and $SU(2)_R$, which commute with each other. Their combined symmetry group is, therefore, the cartesian product $SU(2)_R \times SU(2)_L$ which is called the chiral symmetry group. Its representations can be labeled by giving the eigenvalues of $\boldsymbol{Q}_L^2$ and $\boldsymbol{Q}_R^2$.

In parallel to the transition from $\boldsymbol{Q}_A$, $\boldsymbol{Q}_V$ to the commuting generators $\boldsymbol{Q}_L$, $\boldsymbol{Q}_R$ a chiral symmetry transformation (5.30) can thus be expressed in terms of the commuting matrices $\boldsymbol{\tau}_L$, $\boldsymbol{\tau}_R$ via

$$U = \mathrm{e}^{-\mathrm{i}[(\boldsymbol{\epsilon}-\boldsymbol{\eta})\boldsymbol{\tau}_L + (\boldsymbol{\epsilon}+\boldsymbol{\eta})\boldsymbol{\tau}_R]} \quad , \tag{5.34a}$$

$$\boldsymbol{\tau}_L = \frac{1}{2}(1-\gamma_5)\frac{\boldsymbol{\tau}}{2} \quad ; \quad \boldsymbol{\tau}_R = \frac{1}{2}(1+\gamma_5)\frac{\boldsymbol{\tau}}{2} \quad , \tag{5.34b}$$

where $\boldsymbol{\tau}_L$ and $\boldsymbol{\tau}_R$ obey the commutation relations (5.33). Consequently, if Ψ_L and Ψ_R are basis vectors of each the two $SU(2)$ transformations $SU(2)_L$ and $SU(2)_R$ (see appendix A.2), then they transform as

$$\Psi_{L,R} \rightarrow \Psi'_{L,R} = \mathrm{e}^{-\mathrm{i}(\boldsymbol{\epsilon}\pm\boldsymbol{\eta})\frac{\boldsymbol{\tau}}{2}}\Psi_{L,R} \equiv U_{L,R}\Psi_{L,R} \quad , \tag{5.35}$$

where the upper sign belongs to the spinor Ψ_R and the lower sign to the spinor Ψ_L.

The meaning of the indices L and R becomes clear when we note that in the case of massless fermions (appendix A.2) the transformations generated by $\boldsymbol{Q}_L(\sim (1-\gamma_5))$ act only on left-handed particles, whereas the elements of $SU(2)_R$ ($\boldsymbol{Q}_R \sim (1+\gamma_5)$) transform only right-handed particles.

A very important case which will be discussed later in this chapter is where the full chiral $SU(2)_L \times SU(2)_R$ symmetry is broken in the physical states of the system so that only a normal $SU(2)$ isospin symmetry ($\equiv SU(2)_V$) survives; this latter is a subgroup of the full $SU(2)_R \times SU(2)_L$. In this case the physical particles belong to $SU(2)_V$ multiplets rather than to multiplets of the full group. In the fermionic sector this is the case for all the massive baryons that belong to isospin multiplets and cannot be classified via their handedness.

5.3.2 Chiral Symmetry Transformations for the Mesons

Since the Lagrangians that we will be concerned with often contain fermions and mesons, it is necessary to know also the transformation properties of the meson fields under chiral symmetry transformations.

In order to find these, we first consider a (2×2) dimensional isospin matrix of the form

$$W = \Psi_L \Psi_R^\dagger \quad . \tag{5.36}$$

Since we have expressed this matrix as a dyadic product of two-dimensional spinors, we can easily write down its transformation properties under the chiral symmetry group

$$\begin{aligned} W \to W' &= U_R W U_L^\dagger \\ &= \mathrm{e}^{-\mathrm{i}(\boldsymbol{\epsilon}-\boldsymbol{\eta})\frac{\boldsymbol{\tau}}{2}} W \mathrm{e}^{+\mathrm{i}(\boldsymbol{\epsilon}+\boldsymbol{\eta})\frac{\boldsymbol{\tau}}{2}} \quad , \end{aligned} \tag{5.37}$$

with $U_{L,R}$ from eq. (5.35).

Since W is a matrix of dimension (2×2) in the isospin degrees of freedom, obtained from two isospin $-1/2$ objects, it can in general be expanded as a superposition of a two-dimensional unit matrix and the three Pauli-matrices τ_i

$$W = \sigma + \mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{\pi} \quad , \tag{5.38}$$

where σ and $\boldsymbol{\pi}$ are isoscalar and isovector fields, respectively. For infinitesimal $\boldsymbol{\epsilon}$ and $\boldsymbol{\eta}$ the general transformation (5.37) then reads

$$\begin{aligned} W' &= W - \mathrm{i}\boldsymbol{\epsilon}\left[\frac{\boldsymbol{\tau}}{2}, W\right] + \mathrm{i}\boldsymbol{\eta}\left\{\frac{\boldsymbol{\tau}}{2}, W\right\} \\ &= W - \mathrm{i}\boldsymbol{\epsilon}\left[\frac{\boldsymbol{\tau}}{2}, \mathrm{i}\boldsymbol{\pi}\cdot\boldsymbol{\tau}\right] + \mathrm{i}\boldsymbol{\eta}\left\{\frac{\boldsymbol{\tau}}{2}, \sigma + \mathrm{i}\boldsymbol{\pi}\cdot\boldsymbol{\tau}\right\} \\ &= W + \mathrm{i}\boldsymbol{\tau}(\boldsymbol{\epsilon}\times\boldsymbol{\pi}) + \mathrm{i}(\boldsymbol{\eta}\cdot\boldsymbol{\tau})\sigma - \boldsymbol{\eta}\cdot\boldsymbol{\pi} \\ &\equiv \sigma' + \mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{\pi}' \quad . \end{aligned} \tag{5.39}$$

Even though this transformation has been obtained here for a W constructed from fermion fields it must also hold for any mesonic W of the form (5.38) whose change under a chiral transformation is given by (5.37).

For a pure isospin transformation, which follows for $\boldsymbol{\eta} = 0$, we find the following transformed fields:

$$\begin{aligned} \sigma' &= \sigma \\ \boldsymbol{\pi}' &= \boldsymbol{\pi} + \boldsymbol{\epsilon}\times\boldsymbol{\pi} \quad . \end{aligned} \tag{5.40}$$

This shows that indeed $\boldsymbol{\pi}$ transforms as a vector under $SU(2)_V$ (see eq. (3.36)), whereas σ remains invariant.

For $\boldsymbol{\epsilon} = 0$, on the other hand, we have

$$\begin{aligned} \sigma' &= \sigma - \boldsymbol{\eta}\cdot\boldsymbol{\pi} \\ \boldsymbol{\pi}' &= \boldsymbol{\pi} + \boldsymbol{\eta}\sigma \end{aligned} \tag{5.41}$$

which corresponds to the axial transformation $SU(2)_A$. The transformations (5.41) and (5.40) conserve the Euclidean norm of the four-component vector $(\boldsymbol{\pi}, \sigma)$ since one has

$$\sigma'^2 + \boldsymbol{\pi}'^2 = W^\dagger W = \sigma^2 + \boldsymbol{\pi}^2 \quad . \tag{5.42}$$

This property reflects the fact that the chiral symmetry group has the same algebra as the group of rotations in four-dimensional Euclidean space, SO(4).

Equation (5.41) reads in vector form

$$\delta \begin{pmatrix} \boldsymbol{\pi} \\ \sigma \end{pmatrix} = -\mathrm{i}\boldsymbol{\eta} \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\pi} \\ \sigma \end{pmatrix} \quad . \tag{5.43}$$

If a meson Lagrangian is invariant under this transformation then the conserved current is given by

$$A_\mu^k = \sigma \partial_\mu \pi^k - \pi^k \partial_\mu \sigma \quad . \tag{5.44}$$

Here π^k is the k th component of the isovector $\boldsymbol{\pi}$. The conserved charges are thus

$$Q_A^k = \int \left(\sigma \dot{\pi}^k - \pi^k \dot{\sigma} \right) d^3x \quad . \tag{5.45}$$

They are the analogues of (5.29) in the mesonic vector. Together with the usual vector charges for the isospin-triplet $\boldsymbol{\pi}$ and the singlet σ, they obey the commutation relations (5.31).

In all the preceding discussions, the four fields $\boldsymbol{\pi}$ and σ have been assumed to form an $SU(2) \times SU(2)$ multiplet. These four mesons should thus have the same mass. In nature, however, for the mesons the full symmetry is also broken to the pure isospin symmetry, as we can conclude from the absence of a scalar particle with the same mass as the pion. We therefore have three pion fields with equal mass forming an isospin triplet, and an isoscalar field σ with a mass different from that of the pions.

Chapter 6

SPONTANEOUS SYMMETRY BREAKING

The preceding chapter has shown how the observed partial conservation of the axial isospin current can be understood as the consequence of the approximate chiral symmetry of the strong interaction Lagrangian. A problem that has been ignored so far, however, is the nonzero mass of the nucleons so that their Lagrangian is at first sight clearly not chirally invariant. A way out of this dilemma is offered by making a careful distinction between the symmetries of the Lagrangian and those of the actual state of the physical system. The discussion of the consequences of this distinction is the purpose of this chapter.

6.1 Goldstone Theorem

In every case discussed so far, in which invariance of the Lagrangian under a continuous symmetry has led to the conservation of a charge, the vacuum of the theory possessed the same symmetry, so that

$$\mathrm{e}^{-\mathrm{i}\epsilon Q}|0>=|0>\to Q|0>=0 \quad , \tag{6.1}$$

where Q is a generator of the symmetry group under consideration. It has become customary to call this realization of the symmetry the “Wigner mode”. Here, as in the following considerations, we define as vacuum $|0>$ that state of the system for which

$$<0|H|0>=\min \quad . \tag{6.2}$$

In the following we will now consider the case that $\mathcal{L}$ is invariant under a certain symmetry operation, but $|0>$ is not. The standard classical example for this situation is that of the ferromagnet. The spin-spin interactions in this system are, clearly, rotationally invariant, but below the Curie temperature the actual ground-state of the ferromagnet, in which all spins point in one direction, is not. Thus the rotational symmetry of the Hamiltonian of the system is "spontaneously" broken down to cylindrical symmetry in which one direction is marked by the direction of the magnetization. There are actually infinitely many other ground states in which all the spins point together into different directions; all of these states are degenerate in energy. In a finite system, one of these degenerate configurations would tunnel through to the other configurations, so that the true groundstate of the finite ferromagnet would be a linear superposition of the degenerate states. In an infinite system, however, this tunneling cannot take place because the potential barriers between the different configurations get infinitely large; the different groundstates and the excitations on top of them are thus completely decoupled, and they form different Hilbert spaces. Each of these groundstates is not rotationally symmetric and the groundstate expectation value of the total spin, i.e. the generator of the spatial rotation, does not vanish.

Exactly the same situation holds in quantum field theories. If the vacuum is not invariant under the symmetry operation characterized by the generator Q, then

$$Q|0> \neq 0 \quad . \tag{6.3}$$

In this case one speaks of a "hidden" symmetry or of a realization of the symmetry in the "Goldstone mode". Indeed, $Q|0>$ would generate a state in the other "domain" that does not belong to the Hilbert space of physical states built on $|0>$ and to which, therefore, tunneling cannot occur. In the following subsection we will now discuss the physical consequences of a hidden symmetry.

6.1.1 Goldstone Bosons

We now assume that $O(y)$ is some field operator whose commutator with the operator Q of eq. (6.3) does not vanish; $O(y)$ is thus not invariant under the symmetry operation generated by Q

$$[Q, O(y)] = O'(y) \quad . \tag{6.4a}$$

Since Q does not annihilate the vacuum (see eq. (6.3)) the transformed field operator has a nonzero vacuum expectation value,

$$< 0|O'(y)|0 > \neq 0 \quad . \tag{6.4b}$$

Q is a generator of the symmetry under question and thus simultaneously a conserved charge operator (see discussions in chapter 4). One therefore has for the corresponding current j_μ,

$$\partial^\mu j_\mu(x) = 0 \tag{6.5}$$

and

$$< 0|[Q, O(y)]|0 > = < 0|[\int d^3x \; j_0(x), O(y)]|0 > \quad . \tag{6.6}$$

Combining (6.5) and (6.6) gives

$$\begin{aligned} & \frac{\partial}{\partial x_0} \int d^3x < 0|\left[j_0(x), O(y)\right]|0 > \\ = & \; -\int d^3x < 0|\left[\vec{\nabla} \cdot \vec{j}(x), O(y)\right]|0 > \quad . \end{aligned} \tag{6.7}$$

The volume integral on the rhs of eq. (6.7) can be converted into a surface integral (by Gauss's law). Since $\vec{j}$ and O are local operators and the separation between a point x on the surface and y can be made arbitrarily large by choosing a large enough volume, the commutator vanishes. Then

$$\frac{\partial}{\partial x_0} < 0|O'(y)|0 > = 0 \quad . \tag{6.8}$$

Thus the nonvanishing vacuum expectation value (6.4b) is independent of time.

It can now be further evaluated by inserting a complete set of states between the operators Q and O:

$$\begin{aligned} < 0|O'(y)|0 > &= < 0|[Q, O(y)]|0 > \\ &= \int d^3x < 0|[j_0(x), O(y)]|0 > \\ &= \int d^3x \sum_n \{< 0|j_0(x)|n >< n|O(y)|0 > \\ &\qquad - < 0|O(y)|n >< n|j_0(x)|0 >\} \quad . \end{aligned} \tag{6.9}$$

Using now relation (5.20) for the matrix elements involving j_0 in eq. (6.9), we have

$$\begin{aligned} < 0|O'(y)|0 > &= \int d^3x \sum_n \Big[< 0|j_0(0)|n >< n|O(y)|0 > \mathrm{e}^{-\mathrm{i}p_n x} \\ &\qquad - < 0|O(y)|n >< n|j_0(0)|0 > \mathrm{e}^{+\mathrm{i}p_n x}\Big] \quad . \end{aligned}$$

The spatial integration on the right hand side can now be performed. It yields

$$
\begin{aligned}
<0|O'(y)|0> &= V \sum_n \delta^3(\vec{p}_n) \Big[<0|j_0(0)|n><n|O(y)|0> \mathrm{e}^{+\mathrm{i}E_n t} \\
&\qquad - <0|O(y)|n><n|j_0(0)|0> \mathrm{e}^{-\mathrm{i}E_n t} \Big] \\
&= \mathrm{const} \neq 0 \quad . \qquad (6.10)
\end{aligned}
$$

The δ-function in (6.10) limits the states $|n>$ to those with vanishing momentum $\vec{p}_n = 0$ and thus to states with $E_n = m_n$. These states have a harmonic time-dependence. Since, however, the whole expression (6.10) has to be time-independent, the matrix elements of j_0 for all massive states have to vanish

$$
<0|j_0(0)|n> = 0 \quad \text{for } m_n \neq 0 \quad . \qquad (6.11)
$$

Since, furthermore, the expression (6.10) is not equal to zero, there must be states in the sum for which $m_n = 0$ and $<0|j_0|n><n|O(y)|0> \neq 0$.

The existence of a nonvanishing vacuum expectation value of a generator of a symmetry group thus leads necessarily to the existence of massless particles. These carry the quantum numbers of the generator, since in (6.10) $<0|j_0|n>$ can be nonvanishing only if the quantum numbers of $|n>$ agree with those of $j_0|0>$, i.e. if the corresponding quanta carry the quantum numbers of j_0. Since j_μ can be either a vector or an axial-vector current, j_0 carries zero *internal* angular momentum; the related massless particles are consequently scalar bosons. The existence of these massless "Goldstone-bosons", whenever a symmetry is realized in the Goldstone mode, is the essence of the Goldstone theorem.

In the following we will now illustrate how one can use experimental information to obtain knowledge of the mode in which a symmetry is realized. To facilitate the arguments we assume that the axial vector current is strictly conserved by the strong interactions

$$
\frac{\partial A^k_\mu}{\partial x_\mu} = 0 \quad , \qquad (6.12)
$$

so that the axial charges Q^k_A are constants of motion and therefore commute with the strong interaction Hamiltonian

$$
[H_{st}, Q_A] = 0 \quad . \qquad (6.13)
$$

This means that with the state of any physical hadron $|h>$, $Q_A|h>$ is also an eigenstate of H_{st}

$$\begin{aligned} H_{st}|h> &= M_h|h> \\ Q_A H_{st}|h> &= H_{st}(Q_A|h>) = M_h(Q_A|h>) \quad . \end{aligned} \tag{6.14}$$

Equation (6.14) shows that $Q_A|h>$ is an eigenstate of H_{st} with the same mass as the original hadron. Since A_μ is an axial vector, $Q_A|h>$ has negative internal parity. Equation (6.14), therefore, predicts that with every hadron state there exists also a multiplet of states with the same mass but different parity. This prediction is clearly in contradiction to experiment[1].

The only way out of this glaring contradiction is to postulate the existence of massless, pseudoscalar bosons π that form continuum states with the hadrons such as $|h\pi>$. The actual existence of such particles, the physical pions with their, although nonzero, on a hadronic scale very small mass, together with the observed partial axial vector current conservation, is then an indication that in nature the chiral symmetry is realized in the Goldstone mode and not in the Wigner mode. That the pion mass is not exactly zero is related to the experimental fact that the axial current is only partially conserved. This relation has been discussed in section 5.2.2 where it was shown that the square of the pion mass is directly connected to the nonconservation of the axial current (see eq. (5.23)), thus supporting the identification of the pion as a Goldstone boson.

6.1.2 Spontaneous Breaking of a Global non-Abelian Symmetry

As an explicit example for the consequences of the Goldstone theorem we consider here first a theory that involves only two complex scalar boson fields ϕ_1 and ϕ_2 that form an $SU(2)$-doublet

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad . \tag{6.15}$$

The model Lagrangian is given by

$$\begin{aligned} \mathcal{L} &= (\partial^\mu \phi_1)(\partial_\mu \phi_1) + (\partial^\mu \phi_2)(\partial_\mu \phi_2) - V\left(|\phi_1|^2 + |\phi_2|^2\right) \\ &= \left(\partial^\mu \phi^\dagger\right)(\partial_\mu \phi) - V\left(\phi^\dagger \phi\right) \end{aligned} \tag{6.16}$$

---

[1] The same operations as above with the isospin vector charge Q_V would have yielded states with the same parity belonging to the original isospin multiplet, in agreement with experiment.

with

$$V\left(\phi^\dagger\phi\right) = \frac{1}{2}\mu^2\phi^\dagger\phi + \frac{1}{2}\lambda^2\left(\phi^\dagger\phi\right)^2 \quad . \tag{6.17}$$

Here, in order to ensure the stability of the vacuum of the theory, we require $\lambda^2 > 0$.

For $\mu^2 > 0$ this Lagrangian describes a system of interacting particles with mass $\mu/\sqrt{2}$. The energetically lowest state of this theory, i. e. the groundstate or vacuum, is the space-time independent state with $\phi = 0$. This state exhibits all the symmetries ($U(1) \times SU(2)$) of the Lagrangian.

On the other hand, if we choose $\mu^2 < 0$, then the minimum of the potential is given by

$$\left(\phi^\dagger\phi\right)_0 = \left(|\phi_1|^2 + |\phi_2|^2\right)_0 = -\frac{1}{2}\frac{\mu^2}{\lambda^2} \equiv \frac{v^2}{2} \quad . \tag{6.18}$$

In this case the vacuum state is infinitely degenerate; all the states that lie on a circle in the (ϕ_1, ϕ_2) plane with radius $v/\sqrt(2)$ fulfill eq. (6.18). All these degenerate states are connected by an $SU(2)$ transformation.

The Lagrangian $\mathcal{L}$ of eq. (6.16) is invariant under a global $U(1)$ transformation as well as under a global $SU(2)$ transformation acting on the doublet ϕ. At the minimum (6.18) we can, therefore, choose ϕ in the following form

$$< 0|\phi|0 >= \phi_0 = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \tag{6.19}$$

with $v = |\mu|/|\lambda|$. To set the first component to zero is possible by a proper choice of the coordinate system or — in other words — by an $SU(2)$ transformation ; note here that the vacuum field, that minimizes the energy of the system, is x-independent. v can be chosen to be real by exploiting the $U(1)$-symmetry; this choice ensures that the vacuum is uncharged.

The Goldstone theorem discussed in section 6.1 predicts that massless Goldstone bosons appear whenever the vacuum is not annihilated by a generator of the symmetry group under which $\mathcal{L}$ is invariant. In the present case, the full symmetry group of the Lagrangian is

$$G = U(1) \otimes SU(2) \tag{6.20}$$

and thus contains four generators[2]. The vacuum state (6.19) does not exhibit all of these symmetries anymore. For example, the vacuum state (eq. (6.19)) is not invariant under an $SU(2)$ transformation. More specifically, it is not

---

[2] Note that the theory also contains four fields since ϕ_1 and ϕ_2 are complex.

annihilated by $\tau_1/2$, $\tau_2/2$ and $(1-\tau_3)/2$; $(1+\tau_3)/2$, on the other hand, does annihilate (6.19). In the vacuum state the full symmetry group G is thus spontaneously broken down to

$$U = \mathrm{e}^{-\mathrm{i}\epsilon q} = \mathrm{e}^{-\mathrm{i}\epsilon(1+\tau_3)/2} \qquad (6.21)$$

which corresponds to a normal Abelian $U(1)$ symmetry; U leaves the vacuum (6.19) invariant. From the discussion in section 6.1 we thus expect to find three massless Goldstone bosons in our theory and one massive boson (compare the discussion of the linear σ-model in section 6.2).

This expectation is borne out by an explicit expansion of $\mathcal{L}$ around the vacuum point

$$\mathcal{L} = \left(\partial^\mu \phi'\right)^\dagger \left(\partial_\mu \phi'\right) - V\left(\phi', \phi_0\right) , \qquad (6.22)$$

with

$$\phi' = \phi - \phi_0 \quad .$$

We now set

$$\phi' = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_1 + \mathrm{i}\varphi_2 \\ \varphi_3 + \mathrm{i}\varphi_4 \end{pmatrix} \qquad (6.23)$$

and rewrite V in terms of the real fields φ_i

$$\begin{aligned} V(\phi', \phi_0) &= +\frac{1}{2}\mu^2 \left(\phi_0 + \phi'\right)^\dagger \left(\phi_0 + \phi'\right) \\ &\quad + \frac{\lambda^2}{2}\left[\left(\phi_0 + \phi'\right)^\dagger \left(\phi_0 + \phi'\right)\right]^2 \\ &= +\frac{1}{2}\mu^2 \phi'^\dagger \phi' + \frac{\lambda^2}{2}\left(\phi_0^\dagger \phi' + \phi'^\dagger \phi_0 + \phi'^\dagger \phi' + \phi_0^\dagger \phi_0\right)^2 \\ &\quad + \text{const} \qquad (6.24) \\ &= +\frac{1}{4}\mu^2 \left(\varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2\right) + \frac{\lambda^2}{2}(\ldots) + \mathcal{O}(\varphi_i^3) + \text{const} \quad . \end{aligned}$$

There are no terms linear in ϕ' because we have effectively expanded V around its minimum ϕ_0. The dotted expression in parentheses gives up to terms quadratic in ϕ'

$$\begin{aligned} (\ldots) &= \left(\phi_0^\dagger \phi'\right)^2 + \left(\phi'^\dagger \phi_0\right)^2 + 2\left(\phi_0^\dagger \phi'\right)\left(\phi'^\dagger \phi_0\right) + \\ &\quad + 2\left(\phi'^\dagger \phi'\right)\phi_0^\dagger \phi_0 + \text{interaction terms} + \text{const} \\ &= \frac{v^2}{4}\left(\varphi_3^2 - \varphi_4^2\right) + \frac{v^2}{4}\left(\varphi_3^2 - \varphi_4^2\right) + \frac{v^2}{2}\left(\varphi_3^2 + \varphi_4^2\right) \\ &\quad + \frac{v^2}{2}\left(\varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2\right) + \mathcal{O}(\varphi_i^3) + \text{const} \quad . \end{aligned}$$

Summing all terms we obtain with $v^2\lambda^2 = -\mu^2$

$$V(\phi) = -\frac{\mu^2}{2}\varphi_3^2 + \mathcal{O}(\varphi_i^3) + \text{const} \quad . \tag{6.25}$$

Since $\mu^2 < 0$ we see that the field φ_3 has become massive; the three other fields have all become massless. This result is obviously in agreement with our earlier expectations based on the Goldstone theorem.

Expressed in terms of the field φ_i the Lagrangian (6.22) is no longer manifestly $SU(2)$ invariant; the $SU(2)$ invariance of (6.16) is now hidden.

6.2 Linear σ-Model

A well known model, which exhibits two important symmetries of strong interactions (chiral symmetry and $SU(2)$) amd serves as an example for spontaneous symmetry breaking both in particle and nuclear physics, is the so-called "linear σ-model". It is the purpose of this chapter to discuss its Lagrangian and to demonstrate the symmetry breaking and its consequences for the particle spectrum of this model.

6.2.1 Lagrangian and Symmetries of the σ-Model

The model describes the interaction of massless fermions with pseudoscalar π- and scalar σ-mesons. Its Lagrangian is given by

$$\begin{aligned} \mathcal{L} \;=\; & \bar{\Psi}\mathrm{i}\gamma^\mu\partial_\mu\Psi - g\bar{\Psi}(\sigma + \mathrm{i}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi})\Psi \\ & +\frac{1}{2}\left(\partial_\mu\boldsymbol{\pi}\right)^2 + \frac{1}{2}\left(\partial_\mu\sigma\right)^2 - \frac{1}{2}\mu^2\left(\boldsymbol{\pi}^2 + \sigma^2\right) \\ & -\frac{1}{4}\lambda^2\left(\boldsymbol{\pi}^2 + \sigma^2\right)^2 \quad . \end{aligned} \tag{6.26}$$

The first term describes massless, free fermions, the second their linear coupling to the fields σ and $\boldsymbol{\pi}$, whereas the third and fourth terms give the kinetic energy of the $\boldsymbol{\pi}$- and σ-field and the fifth term their mass, assumed to be equal for both mesons. The last term contains the nonlinear meson interaction, which is the important ingredient responsible for the spontaneous breaking of the symmetry in this model, as we will see later. For the theory to have a stable ground state we must obviously have $\lambda^2 > 0$, since otherwise the potential energy would not be bounded from below.

The field Ψ represents an $SU(2)_V$ isospin doublet, $\boldsymbol{\pi}$ an isovector ($SU(2)_V$-triplet) and σ a $SU(2)_V$ singlet field. With respect to the total $SU(2) \times$

$SU(2)$ chiral symmetry group, the σ and $\boldsymbol{\pi}$ fields form the meson quadruplet and thus transform as discussed in section 5.3.2.

For later interpretations it is useful to point out that the mass and interaction terms in (6.26) can also be written as

$$\begin{aligned} V(\sigma,\boldsymbol{\pi}) &= \frac{1}{2}\mu^2\left(\boldsymbol{\pi}^2+\sigma^2\right)+\frac{1}{4}\lambda^2\left(\boldsymbol{\pi}^2+\sigma^2\right)^2 \\ &= \frac{1}{4}\lambda^2\left(\boldsymbol{\pi}^2+\sigma^2-f^2\right)^2+\text{const} \quad , \end{aligned} \tag{6.27}$$

with $f^2=-\mu^2/\lambda^2$.

In the following discussion, we will treat the Lagrangian (6.26) in the semiclassical or mean-field approximation, which means that we replace the field operators for the σ and $\boldsymbol{\pi}$ field by their expectation values. These then behave as classical fields.

Vector $SU(2)$ Invariance of $\mathcal{L}$. The first manifest symmetry of $\mathcal{L}$ is that of $SU(2)_V$ invariance. Thus $\mathcal{L}$ is invariant under the "vector $SU(2)$ transformation"

$$U=\mathrm{e}^{-\mathrm{i}\boldsymbol{\epsilon t}} \quad , \quad \boldsymbol{t}=\frac{\boldsymbol{\tau}}{2} \quad .$$

The corresponding field changes are (see eqs (5.40))

$$\begin{aligned} \delta\Psi &= -\mathrm{i}\epsilon\frac{\boldsymbol{\tau}}{2}\Psi \\ \delta\pi &= \boldsymbol{\epsilon}\times\boldsymbol{\pi} \qquad \delta\sigma=0 \quad . \end{aligned} \tag{6.28}$$

As a consequence of the $SU(2)_V$ invariance, the isospin vector current

$$\boldsymbol{V}_\mu(x)=\bar{\Psi}\gamma_\mu\frac{\boldsymbol{\tau}}{2}\Psi-\mathrm{i}\partial_\mu\pi_\alpha\boldsymbol{t}_{\alpha\beta}\pi_\beta \tag{6.29}$$

is conserved.

Axial $SU(2)$ Invariance of $\mathcal{L}$. The Lagrangian (6.26) is also invariant under the "axial $SU(2)_A$ transformation" (5.25, 5.41). Its generators are given by eqs. (5.29) and (5.45) for the fermion and meson sectors, respectively. The corresponding field changes are (eqs (5.25, 5.41))

$$\begin{aligned} \delta\Psi &= -\mathrm{i}\boldsymbol{\eta}\frac{\boldsymbol{\tau}}{2}\gamma_5\Psi \\ \delta\Psi^\dagger &= +\mathrm{i}\boldsymbol{\eta}\Psi^\dagger\gamma_5\frac{\boldsymbol{\tau}}{2} \quad \rightarrow \quad \delta\bar{\Psi}=\delta\Psi^\dagger\gamma_0=-\mathrm{i}\boldsymbol{\eta}\bar{\Psi}\gamma_5\frac{\boldsymbol{\tau}}{2} \quad ; \\ \delta\boldsymbol{\pi} &= \boldsymbol{\eta}\sigma \quad ; \quad \delta\sigma=-\boldsymbol{\eta}\cdot\boldsymbol{\pi} \quad . \end{aligned} \tag{6.30}$$

The kinetic energy of the fermions is obviously invariant under this transformation. The same holds for all the other terms as will be shown now term by term.

The change of the meson kinetic energy is

$$\begin{aligned}\delta\left[(\partial_\mu\boldsymbol{\pi})^2+(\partial_\mu\sigma)^2\right] &= 2\left\{(\partial_\mu\boldsymbol{\pi})\cdot(\partial_\mu\delta\boldsymbol{\pi})+(\partial_\mu\sigma)\,(\partial_\mu\delta\sigma)\right\}\\ &= 2\left\{(\partial_\mu\boldsymbol{\pi})\cdot(\boldsymbol{\eta}\partial_\mu\sigma)-(\partial_\mu\sigma)\,(\boldsymbol{\eta}\cdot\partial_\mu\boldsymbol{\pi})\right\}=0 \quad . \qquad (6.31a)\end{aligned}$$

Analogously, it is easy to see also that the interaction term between the $\boldsymbol{\pi}$- and the σ-field as well as the meson mass term is invariant under the transformation (6.30). The fact that $\sigma^2+\boldsymbol{\pi}^2$ is invariant illustrates that the transformation (6.30) belongs to the group of rotations in the four-dimensional Euclidean space in which $(\boldsymbol{\pi},\sigma)$ is a vector (see discussion in 5.3.2).

The invariance of the fermion-meson interaction is easiest to see by rewriting it in terms of the fields Ψ_L,Ψ_R and W (eqs. (A.40,A.41), (5.36)). We obtain

$$\begin{aligned}\bar\Psi(\sigma+\mathrm{i}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi})\Psi &= \bar\Psi_R(\sigma+\mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{\pi})\Psi_L+\bar\Psi_L(\sigma-\mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{\pi})\Psi_R\\ &= \bar\Psi_R W\Psi_L+\bar\Psi_L W^\dagger\Psi_R \quad . \qquad (6.31b)\end{aligned}$$

Using the transformation property (5.37) for W and (5.35) for the Ψ fields shows directly the invariance of this interaction term.

Thus $\mathcal{L}$ (eq. (6.26)) is indeed invariant under the transformation (6.30). Connected with this invariance is the conserved current (see eqs (5.27, 5.44))

$$\boldsymbol{A}_\mu=\bar\Psi\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\Psi+(\sigma\partial_\mu\boldsymbol{\pi}-\boldsymbol{\pi}\partial_\mu\sigma) \quad . \qquad (6.32)$$

This conserved current is an axial vector, while the corresponding charges are pseudoscalars.

Finally it is worthwhile mentioning explicitly that $\mathcal{L}$ does not contain a fermion mass term. Such a term would indeed violate the invariance of the Lagrangian under the chiral transformation ($SU(2)_A$), eq. (5.25)

$$\begin{aligned}\delta(\bar\Psi\Psi) &= (\delta\bar\Psi)\Psi+\bar\Psi\delta\Psi\\ &= -\mathrm{i}\boldsymbol{\epsilon}\bar\Psi\gamma_5\frac{\boldsymbol{\tau}}{2}\Psi-\mathrm{i}\boldsymbol{\epsilon}\bar\Psi\frac{\boldsymbol{\tau}}{2}\gamma_5\Psi\neq 0 \quad . \qquad (6.33)\end{aligned}$$

The invariance of $\mathcal{L}$ under chiral transformations, therefore, holds only for massless fermions, as already discussed at the start of section 5.3.1.

6.2.2 Particle Spectrum of the σ-Model

For $\mu^2 > 0$ the meson potential energy has a minimum at $\boldsymbol{\pi}^2 + \sigma^2 = 0$ and the corresponding ground state is

$$< \sigma >_{\text{vac}} = 0 \qquad , \qquad < \pi >_{\text{vac}} = 0 \quad . \tag{6.34}$$

This shows that the vacuum state of the theory itself is $SU(2) \times SU(2)$-invariant. Thus the chiral symmetry here is manifest; it is realized in the Wigner mode.

Since the vacuum expectation values vanish in the mean field approximation, the particle masses can be read off from the terms quadratic in the fields in the Lagrangian. One sees that in this case ($\mu^2 > 0$) the linear σ-model describes two massless fermions coupled in a chirally invariant way to the massive mesons σ and $\boldsymbol{\pi}$ which have the same mass μ. This mass degeneracy reflects that all four mesons belong to the same $SU(2) \times SU(2)$ multiplet. For the fermions the helicity is conserved separately for both the left-handed and the right-handed fermions; there are, therefore, two sorts of distinguishable, massless fermions.

If the isospin-doublet Ψ is assumed to describe the nucleon, then this situation is obviously unphysical, since nucleons are massive; furthermore, there exists no scalar partner to the pseudoscalar pion with the same mass.

In the following we will, therefore, consider the case $\mu^2 < 0$. Now the potential has no longer a minimum at $\boldsymbol{\pi}^2 + \sigma^2 = 0$ but has instead the form shown in Fig. 6.1 (with $\lambda^2 > 0$ required by overall stability).

The minimum now lies in a ring at $\boldsymbol{\pi}^2 + \sigma^2 = f^2$ as can be seen most easily from eq. (6.27). The ground state of the theory is thus given by

$$< \pi >_{\text{vac}} = 0 \qquad ; \qquad < \sigma >_{\text{vac}} = f = +\sqrt{-\frac{\mu^2}{\lambda^2}} = \text{const.} \tag{6.35}$$

The vanishing expectation value of the pion field is necessary to obtain a vacuum state with positive parity, whereas the choice of the sign in $< \sigma >_{\text{vac}}$ is arbitrary.

In contrast to the case $\mu^2 > 0$, we now have a vacuum which is not itself invariant under the $SU(2)_A$ transformation (6.30). Therefore the chiral symmetry is said to be spontaneously broken in the ground state and thus realized in the Goldstone mode. Although the ground state is no longer $SU(2)_A$ invariant, note that the axial current (6.32) and therefore the charges are still conserved, since $\mathcal{L}$ is invariant under the transformation.

This becomes more transparent if we look at the usual operator relations for the conserved charges already well known from the considerations in

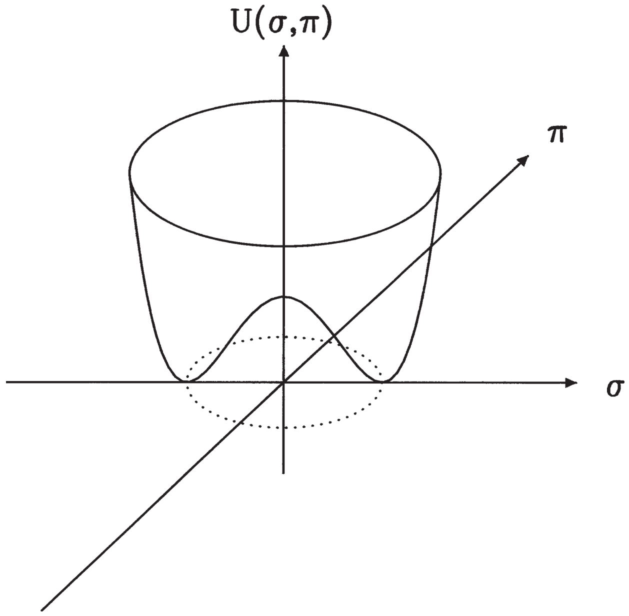

Figure 6.1: Illustration of the "mexican hat" potential of eq. (6.27) for $\mu^2 < 0$.

chapter 4. We know that the conserved charges corresponding to the current (6.32)

$$\boldsymbol{Q}_A = \int \Psi^\dagger \gamma_5 \frac{\boldsymbol{\tau}}{2} \Psi d^3x + \int \left(\sigma \partial_t \boldsymbol{\pi} - \boldsymbol{\pi} \partial_t \sigma\right) d^3x \tag{6.36}$$

are the Hilbert-space generators of the chiral symmetry. Thus the equivalent of eq. (6.30) for the meson fields is in operator form

$$\left[Q_A^i, \pi^j\right] = -\mathrm{i}\delta_{ij}\sigma \tag{6.37}$$

and can be verified using the commutation relations for meson fields. Taking the vacuum expectation value of this relation gives

$$< 0| \left[Q_A^i, \pi^i\right] |0 >= -\mathrm{i} < 0|\sigma|0 >\neq 0 \quad . \tag{6.38}$$

The vacuum expectation value of the commutator does not vanish, so that we must have

$$Q_A^i|0 >\neq 0 \qquad (i = 1, 2, 3) \quad . \tag{6.39}$$

Thus the condition for the Goldstone theorem to hold is fulfilled.

The massless Goldstone bosons predicted by this general theorem can now indeed be identified by determining the particle spectrum of the σ-model. For this purpose the Lagrangian is expanded around its true vacuum state (6.35), since the point $\boldsymbol{\pi}^2 + \sigma^2 = 0$ is now a maximum of the energy and thus no longer stable. By setting

$$\sigma = f + \sigma'$$

we obtain

$$\begin{aligned}
\mathcal{L} &= \bar{\Psi}\mathrm{i}\gamma^\mu\partial_\mu\Psi - g\bar{\Psi}(f + \sigma' + \mathrm{i}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi})\Psi \\
&\quad +\frac{1}{2}\left(\partial_\mu\boldsymbol{\pi}\right)^2 + \frac{1}{2}\left(\partial_\mu\sigma'\right)^2 - \frac{1}{2}\mu^2\left(\boldsymbol{\pi}^2 + \sigma'^2 + 2f\sigma' + f^2\right) \\
&\quad -\frac{1}{4}\lambda^2\left(\boldsymbol{\pi}^2 + \sigma'^2 + 2f\sigma' + f^2\right)^2 \\
&= \bar{\Psi}\left(\mathrm{i}\gamma^\mu\partial_\mu - gf\right)\Psi - \mathrm{i}g\bar{\Psi}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}\Psi \\
&\quad +\frac{1}{2}\left(\partial_\mu\boldsymbol{\pi}\right)^2 + \frac{1}{2}\left(\partial_\mu\sigma'\right)^2 + \mu^2\sigma'^2 - g\bar{\Psi}\sigma'\Psi \\
&\quad -\lambda^2 f\sigma'\left(\boldsymbol{\pi}^2 + \sigma'^2\right) - \frac{1}{4}\lambda^2\left(\boldsymbol{\pi}^2 + \sigma'^2\right)^2 + \text{const.}
\end{aligned} \tag{6.40}$$

The chiral symmetry of the original Lagrangian is now hidden since $\mathcal{L}$ considered as a functional of $\boldsymbol{\pi}$ and σ' is no longer chirally invariant; only the $SU(2)_V$ isospin symmetry is still manifestly present.

The particle spectrum after this spontaneous symmetry breaking can be read off from (6.40). First we note that the fermion has acquired a mass $m = gf$. The σ-meson mass, on the other hand, is now $m_\sigma = \sqrt{-2\mu^2}$ and the pion finally appears as the Goldstone boson with mass zero. The appearance of three massless Goldstone bosons (the pions form an isospin triplet) follows because three of the original symmetries (generated by $Q_A^i, i = 1, 2, 3$) are spontaneously broken in the vacuum state (eq. (6.39)).

The mass of the σ-meson in the Goldstone mode is directly related to the restoring force for small oscillations about the new minimum $\sigma = f, \boldsymbol{\pi} = 0$. One obtains by Taylor-expanding the potential $V(\sigma, \boldsymbol{\pi})$ (eq. (6.27)) around this minimum

$$\begin{aligned} V(\sigma, 0) &= \text{const.} + \frac{1}{2} \left. \frac{\partial^2 V}{\partial \sigma^2} \right|_f (\sigma - f)^2 + \mathcal{O}(\sigma'^3) \\ &= \text{const.} + \lambda^2 f^2 \sigma'^2 + \mathcal{O}(\sigma'^3) \\ &= \text{const.} - \mu^2 \sigma'^2 + \mathcal{O}(\sigma'^3) \\ &\cong \text{const.} + \frac{1}{2} m_\sigma^2 \sigma'^2 + \mathcal{O}(\sigma'^3) \quad . \end{aligned} \tag{6.41}$$

Analogously, the pion mass vanishes because $V(\sigma, \boldsymbol{\pi})$ has no restoring force when going from the vacuum configuration with $\boldsymbol{\pi} = 0$ to $\boldsymbol{\pi} \neq 0$.

This particle spectrum with massive fermions and pseudoscalar mesons with vanishing mass is very close to the one realized in nature. Together with the observed PCAC, this shows again that the chiral $SU(2) \times SU(2)$ symmetry is realized in nature in the Goldstone mode. The explicit example of the σ-model demonstrates a mechanism by which the axial current can be conserved in the presence of massive fermions.

In the model considered so far the pions are massless because an exact $SU(2) \otimes SU(2)$ symmetry of the original Lagrangian (6.26) has been assumed. The experimentally observed small, but non zero, mass of the pion can be generated in this model by adding a term $\mathcal{L}'$ to the Lagrangian (6.26) that breaks the chiral symmetry explicitly

$$\mathcal{L}' = +\eta\sigma \qquad (\eta > 0) \tag{6.42}$$

with constant η. This term specifies a direction in the $(\boldsymbol{\pi}, \sigma)$ space and thus breaks the $SO(4)$ invariance (or, equivalently, the $SU(2) \otimes SU(2)$ invariance). With this symmetry-breaking term the axial vector current is no longer conserved, in either the Goldstone or in the Wigner mode. For $\mu^2 < 0$ the minimum of the potential lies now at $\boldsymbol{\pi} = 0$ and $\sigma = f'$; the latter is obtained as a solution of the equation

$$\eta = \lambda^2 f'(f'^2 - f^2) \quad .$$

If we now expand the potential energy of the meson fields around this new minimum we obtain, with $\sigma = f' + \sigma'$,

$$V(\sigma, \boldsymbol{\pi}) = \frac{1}{2}\lambda^2(f'^2 - f^2)\boldsymbol{\pi}^2 + \frac{1}{2}\lambda^2(3f'^2 - f^2)\sigma'^2 + U_{\text{int}}(\sigma', \boldsymbol{\pi}) \tag{6.43}$$

where U_{int} contains higher order interactions of σ' and $\boldsymbol{\pi}$. The meson masses can be read off directly from (6.43):

$$m_\pi^2 = \lambda^2(f'^2 - f^2) = \frac{\eta}{f'} \tag{6.44a}$$

$$m_\sigma^2 = \lambda^2(3f'^2 - f^2) \quad . \tag{6.44b}$$

The nucleon mass is now

$$m = gf' \quad . \tag{6.45}$$

Equation (6.44a) shows that the pion has now acquired a mass that is directly proportional to the explicit symmetry breaking strength η in (6.42).

Connected with this result is that the axial vector current is no longer conserved. The change of $\mathcal{L}'$ under a chiral transformation is given by

$$\delta\mathcal{L}' = +\eta\delta\sigma = -\eta\boldsymbol{\eta}\cdot\boldsymbol{\pi}$$

whereas that of the total Lagrangian is (see eq. (2.53))

$$\delta(\mathcal{L} + \mathcal{L}') = \boldsymbol{\eta}\cdot\partial^\mu \boldsymbol{A}_\mu \quad .$$

Since $\mathcal{L}$ is invariant, one obtains with $\delta\mathcal{L} = 0$

$$\partial^\mu \boldsymbol{A}_\mu = -\eta\boldsymbol{\pi} \quad . \tag{6.46}$$

The axial current conservation is thus violated just by a term proportional to the mass of the pion field. Inserting eq. (6.44a) into (6.46), we obtain

$$\partial^\mu \boldsymbol{A}_\mu = -f' m_\pi^2 \boldsymbol{\pi} \quad . \tag{6.47}$$

Equation (6.47) constitutes the so-called PCAC (Partially Conserved Axial Current) hypothesis by which — independently of the present model — the divergence of the axial current is linked to the square of the pion mass and vanishes in the limit $m_\pi^2 \to 0$. It can be used to define a pion field operator in terms of the divergence of the axial current. Comparison with eq. (5.23) shows that $f' = f_\pi$.

This identification, together with eq. (6.45), yields the remarkable relation

$$m = g f_\pi \quad , \tag{6.48}$$

which links the nucleon mass to the pion decay constant. It is remarkable because it connects the mass of a strongly interacting particle, the nucleon, to the strength of the weak decay of the pion. The coupling constant g in relation (6.48) is that between pions and nucleons; it is empirically known to be $g^2/4\pi \approx 14$. The pion decay constant has the value $f_\pi = 93$ MeV; relation (6.48) is then fulfilled only to about 30 per cent. If, however, quantum effects are taken into account then instead of (6.48) the so-called Goldberger-Treimann relation

$$m g_A = g f_\pi$$

is obtained, which is empirically fulfilled within an accuracy of about 10 per cent [ITZ85].

6.2.3 Nonlinear σ-Model

The σ-field that appears in the Lagrangian (6.26) describes a scalar meson with a typical mesonic mass of the order of a few 100 MeV. Such a scalar particle does not seem to exist in nature. One way out of this problem is to assume that the σ-meson describes in some effective way the effects of 2π-exchange interactions, although then its role as the chiral partner of the pion is not obvious.

There is, however, also another way out of this difficulty. This is based on the observation that for large "stiffness parameters" λ in the potential (6.27)

$$V(\sigma, \boldsymbol{\pi}) = \frac{1}{4}\lambda^2 \left(\boldsymbol{\pi}^2 + \sigma^2 - f_\pi^2\right)^2 + \text{const.}$$

the fields σ and $\boldsymbol{\pi}$ are more and more forced on the "chiral circle"

$$\sigma^2 + \boldsymbol{\pi}^2 = f_\pi^2 \quad . \tag{6.49}$$

In the limit $\lambda \to \infty$, the Lagrangian of the so-called "nonlinear σ-model" is in the mesonic sector alone given by

$$\mathcal{L}_m = \frac{1}{2}\left(\partial_\mu \boldsymbol{\pi}\right)^2 + \frac{1}{2}\left(\partial_\mu \sigma\right)^2 \tag{6.50}$$

with the subsidiary condition

$$\sigma^2 + \boldsymbol{\pi}^2 = f_\pi^2 = \text{const} \quad .$$

In this case the two boson fields can be expressed by the "chiral angle" θ and the unit vector $\hat{\boldsymbol{\pi}}$ in isospin space via

$$\begin{aligned} \sigma &= f_\pi \cos\theta(x) \quad , \\ \boldsymbol{\pi} &= f_\pi \hat{\boldsymbol{\pi}} \sin\theta(x) \quad . \end{aligned} \qquad (6.51)$$

We can combine the two fields (6.51) as usual (see section 5.3.2) into the chiral field U (see eq. (5.38)) by setting

$$\begin{aligned} U(x) &= \frac{1}{f_\pi}\left[\sigma(x) + \mathrm{i}\gamma_5 \boldsymbol{\tau} \cdot \boldsymbol{\pi}(x)\right] = \cos\theta(x) + \mathrm{i}\gamma_5 \boldsymbol{\tau} \cdot \hat{\boldsymbol{\pi}} \sin\theta(x) \\ &= \mathrm{e}^{\mathrm{i}\gamma_5 \theta(x) \boldsymbol{\tau} \cdot \hat{\boldsymbol{\pi}}} \quad . \end{aligned} \qquad (6.52)$$

Compared to eq. (5.39) an extra factor γ_5 has been inserted here in order to take care of the pseudoscalar nature of the pion. U thus now also acts in the Dirac spinor-space. In terms of this field U the condition (6.49) can be expressed as $U^\dagger U = 1$.

By using (6.51) we obtain now for the Lagrangian (6.50)

$$\mathcal{L}_m = +\frac{1}{2} f_\pi^2 \left[(\partial_\mu \hat{\boldsymbol{\pi}})^2 \sin^2\theta + (\partial_\mu \theta)^2 \right] = \frac{1}{16} f_\pi^2 \,\mathrm{tr}\left((\partial^\mu U)^\dagger (\partial_\mu U) \right) \qquad (6.53)$$

by tracing out the spin and isospin degrees of freedom (this gives the extra factor 1/8). Equation (6.53) incorporates the condition $U^\dagger U = 1$. The total Lagrangian of the nonlinear σ-model can now be written as

$$\mathcal{L} = \frac{1}{16} f_\pi^2 \,\mathrm{tr}\left((\partial_\mu U)^\dagger (\partial_\mu U) \right) + \bar{\Psi} \left(\mathrm{i}\gamma^\mu \partial_\mu - g f_\pi U \right) \Psi \quad . \qquad (6.54)$$

Since the σ-field no longer appears explicitly in the Lagrangian it is evident that the chiral symmetry is realized in the Goldstone mode in this model; this can also be seen by noting that the pion is massless and that the σ-field $\sigma = (f_\pi^2 - \boldsymbol{\pi}^2)^{\frac{1}{2}}$ has a nonvanishing value.

Part IV

Local Gauge Symmetries

Chapter 7

GAUGE FIELD THEORIES

The symmetries discussed in the preceding chapters were all global symmetries. For example, the $SU(3)$ symmetry of the strong interactions discussed in section 3.3 implies that the physics of an interacting system of mesons and baryons is invariant under a global rotation in the isospin-hypercharge space, i.e. a rotation by the same angle at each point in space and time. In this chapter we will now discuss the implications of the symmetry under the much more general local transformations in which the rotation angle can vary from point to point in space and time.

7.1 Conserved Currents in QED

The usual way to introduce the coupling of charged particles to an electromagnetic field A_μ in quantum theory is by the so-called "minimal coupling" in which the four-momentum is modified

$$p_\mu \to p_\mu - q\ A_\mu \tag{7.1a}$$

or, equivalently,

$$\partial_\mu \to \partial_\mu + \mathrm{i}q\ A_\mu \tag{7.1b}$$

where q is the charge of the particle. The combination on the rhs of eq. (7.1b) is often called the "covariant derivative", D_μ:

$$D_\mu = \partial_\mu + \mathrm{i}q\ A_\mu \quad . \tag{7.2}$$

The same minimal coupling is also applied in quantum field theory. For example, the Lagrangian for a free charged fermion field

$$\mathcal{L}_0 = \bar{\Psi}(\mathrm{i}\gamma^\mu \partial_\mu - m)\Psi \quad ,$$

becomes in the presence of an electromagnetic field A_μ

$$\begin{aligned}\mathcal{L}_F &= \bar{\Psi}(\mathrm{i}\gamma^\mu D_\mu - m)\Psi \\ &= \bar{\Psi}(\mathrm{i}\gamma^\mu \partial_\mu - m)\Psi - q\bar{\Psi}\gamma^\mu\Psi\, A_\mu \\ &= \mathcal{L}_0 + \mathcal{L}_{\mathrm{int}} \quad . \end{aligned} \tag{7.3}$$

Note that the structure of the interaction term is uniquely determined by the minimal coupling.

In order to obtain the full Lagrangian for the coupled system of fermion and electromagnetic field one has to add to $\mathcal{L}_F$ the Lagrange density of the free electromagnetic field (eq. (2.17)),

$$\mathcal{L}_{\mathrm{em}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad , \tag{7.4}$$

with the field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad . \tag{7.5}$$

The equations of motion for the electromagnetic field A_μ that follow from the Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\mathrm{em}} + \mathcal{L}_{\mathrm{int}}$$

are then given by

$$\partial^\mu F_{\mu\nu} = \partial^\mu\partial_\mu A_\nu - \partial_\nu(\partial^\mu A_\mu) = j_\nu \quad , \tag{7.6}$$

with the conserved current

$$j_\nu = q\bar{\Psi}\gamma_\nu\Psi \quad . \tag{7.7}$$

Thus the conserved Noether current (7.7) for free fermions provides also the coupling to the electromagnetic field (see eq. 7.6).

For bosons there is an interesting modification of this statement because here the current is modified by the interactions. This can be seen by looking at the Lagrangian for free charged bosons

$$\mathcal{L}_0 = (\partial^\mu\Phi)^\dagger\,(\partial_\mu\Phi) - m^2\Phi^\dagger\Phi \tag{7.8}$$

and the corresponding conserved current (eq. (2.63))

$$j_\mu(x) = \mathrm{i}q\left(\Phi^\dagger\partial_\mu\Phi - \Phi\partial_\mu\Phi^\dagger\right) \quad . \tag{7.9}$$

Replacing the derivatives in the Lagrangian (7.8) by the covariant derivative (7.2) yields

$$\begin{aligned}\mathcal{L} &= (D^\mu\Phi)^\dagger (D_\mu\Phi) - m^2\Phi^\dagger\Phi \\ &= \mathcal{L}_0 - \mathrm{i}q\left(\Phi^\dagger\partial_\mu\Phi - \Phi\partial_\mu\Phi^\dagger\right)A^\mu + q^2A^2\Phi^\dagger\Phi \quad . \end{aligned} \tag{7.10}$$

The conserved Noether current for this Lagrangian is

$$j_\mu(x) = \mathrm{i}q\left(\Phi^\dagger D_\mu\Phi - \Phi\,(D_\mu\Phi)^\dagger\right) \quad . \tag{7.11}$$

This is just the Noether current (7.9) for the free Lagrangian with the derivatives ∂_μ replaced by the covariant derivatives D_μ. In contrast to the fermion case the Noether current of the interacting boson system is thus different from the one of the free boson system. However, again the current (7.11) is identical to the one acting as the source term in the field equations given by the second term in the Lagrange equations of motion,

$$j_\mu(x) = -\frac{\partial\mathcal{L}}{\partial A^\mu} \quad , \tag{7.12}$$

as can be directly seen by differentiating eq. (7.10). The coupling is thus provided by the Noether current for the interacting system (eq. (7.11)).

The fact that the conserved current (eqs. (7.7) and (7.11)) appears as the source term on the rhs of the wave equation for the electromagnetic field allows one to interpret j_ν as the electromagnetic current. The current thus plays an interesting dual role: on one hand it is the source of the electromagnetic field and on the other it is a conserved quantity due to the global $U(1)$ symmetry of the Lagrangian. This is a very important feature; its consequence is that the electric charge is not modified in the presence of strong and weak interactions as discussed in section 5.2.1 in connection with the vector coupling constant g_V.

The presented coupling of charged particles to an electromagnetic field via the covariant derivative was first introduced because of the analogy to the classical theory. There it is known that the minimal coupling leads to the right electromagnetic forces between charged particles. In the following we will show that this specific form of the coupling also leads to a locally gauge-invariant Lagrangian. By turning this argument round it will then be shown that the more abstract and aesthetic concept of local gauge invariance itself can be used to derive the coupling term. The importance of this concept will become clear in section 7.3 since it can be easily generalized to derive the interaction for cases where we have no analogous classical theory to compare with.

7.2 Local Abelian Gauge Invariance

From classical electrodynamics it is known that the vector potential is not uniquely determined. A transformation

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \frac{1}{q}\partial_\mu \epsilon(x) \quad , \tag{7.13}$$

where $\epsilon(x)$ is an arbitrary function of space and time, leaves the field tensor $F_{\mu\nu}$, and therefore the electric and magnetic fields, invariant. This is the so-called "gauge-invariance" of classical electrodynamics.

At first sight it is not evident that this gauge invariance also holds if the interaction between fermions and the electromagnetic field is included. $\mathcal{L}_0$ and $\mathcal{L}_{\text{em}}$ are obviously invariant under the transformation (7.13), but $\mathcal{L}_{\text{int}}$ (7.3) changes by

$$\delta\mathcal{L}_{\text{int}} = -\bar{\Psi}\gamma^\mu\Psi\,\partial_\mu\epsilon \quad . \tag{7.14}$$

The solution to this problem is, however, also well known from quantum mechanics: together with A_μ we have to change the particle field by a "local" phase transformation according to

$$\Psi(x) \rightarrow \Psi'(x) = \mathrm{e}^{-\mathrm{i}\epsilon(x)}\Psi(x) \quad . \tag{7.15}$$

This causes a change of $\mathcal{L}_0$ by

$$\delta\mathcal{L}_0 = +\bar{\Psi}\gamma^\mu\Psi\partial_\mu\,\epsilon \quad , \tag{7.16}$$

which exactly cancels $\delta\mathcal{L}_{\text{int}}$ in eq. (7.14).

The transformation (7.15) is a *local* phase transformation of the field $\Psi(x)$. It differs from the global phase transformations discussed so far by the fact that the constant parameter ϵ in the $U(1)$-transformations has been replaced by the (space, time)-dependent function $\epsilon(x)$. The simultaneous transformation of the fermion fields Ψ (eq. (7.15)) and the electromagnetic field A_μ (eq. (7.13)) is called a "local gauge transformation". QED is thus a locally gauge invariant theory. Its "gauge field" is the electromagnetic field A_μ; its field quanta, the so-called "gauge-bosons", are the photons.

Invariance of QED under the transformations (7.13) and (7.15) implies physically that the theory is invariant under an arbitrary space- and time-dependent change of the phases of the charged fields.

This invariance is due to the coupling via the covariant derivative D_μ. It follows because of the essential property that $D_\mu\Psi$ is affected by local gauge transformations in exactly the same way as $\partial_\mu\Psi$ is by the corresponding global one. For constant ϵ the transformed expression reads

$$(\partial_\mu\Psi(x))' = e^{-\mathrm{i}\epsilon}\partial_\mu\Psi(x) \quad , \quad \epsilon = \text{const.} \ , \tag{7.17a}$$

and for x-dependent $\epsilon(x)$

$$\begin{aligned} (D_\mu\Psi(x))' &= \left(\partial_\mu + \mathrm{i}qA'_\mu(x)\right)\Psi'(x) \\ &= \left(\partial_\mu + \mathrm{i}qA_\mu(x) + \mathrm{i}\partial_\mu\epsilon(x)\right) e^{-\mathrm{i}\epsilon(x)}\Psi(x) \\ &= e^{-\mathrm{i}\epsilon(x)}\left(\partial_\mu + \mathrm{i}qA_\mu(x)\right)\Psi(x) \\ &= e^{-\mathrm{i}\epsilon(x)}D_\mu\Psi(x) \quad . \end{aligned} \tag{7.17b}$$

Comparison of eqs. (7.17a) and (7.17b) proves the statement made above. This fact suggests a simple method of transforming a globally gauge invariant Lagrangian into a locally gauge invariant one simply by replacing everywhere the normal derivatives by the covariant ones.

Since the $U(1)$ transformations form an Abelian group, QED is said to be invariant under an Abelian local gauge transformation. Turning these considerations now around by requiring local Abelian gauge invariance for a system of free fermions, i.e. invariance under the transformation (7.15), we see immediately that there has to exist a massless vector field $A_\mu(x)$, the “gauge field”, that couples to the fermions via the covariant derivative, i.e. in a well-defined way. Furthermore, this field must change together with the fermion field as specified in eq. (7.13). The electromagnetic field $A_\mu(x)$ can thus be understood as a gauge field whose coupling to the fermions is specified by the requirement of local gauge invariance.

Essential for this result is that the mass of the gauge boson of QED, the photon, is zero[1]. A mass term in the Lagrangian of the form (see eq. (2.26))

$$\frac{1}{2}m^2A^\mu A_\mu$$

is obviously not invariant under the gauge transformation (7.13) and would spoil the gauge invariance of the whole theory.

7.3 Non-Abelian Gauge Fields

At the end of section 7.2 it has been shown how the generalization of the global phase transformation ($U(1)$-symmetry) to a local one and the requirement of local gauge invariance leads directly to a well-defined theory for

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[1] The present experimental upper limit on the photon restmass is $3 \cdot 10^{-33}$ MeV.

the interaction of fermions and bosons with the electromagnetic field. It is, therefore, suggestive to try to generalize the other global phase symmetries discussed in chapter 4 to local ones and to generate interactions of isospin currents (for $SU(2)$) or $SU(3)$ currents with the corresponding gauge fields by the gauge invariance requirement.

For the $SU(2)$ case, this approach was actually tried in 1954 by Yang and Mills with the hope of thus constructing a theory of the strong interactions. An insurmountable obstacle at that time was the fact that the gauge field has to be massless — as was discussed before for QED and is also true in general (see below) — so that the force between nucleons would have been infinitely long-ranged, in glaring contrast to the short-range nature of nucleon-nucleon forces.

More recently, however, the non-Abelian gauge field theories have had an impressive revival, mainly owing to the realization that masses of gauge field quanta can be generated by spontaneous local symmetry breaking (the so-called "Higgs mechanism"). This is similar to the spontaneous global symmetry breaking discussed in chapter 6 and will be discussed in detail in chapter 8. We will, therefore, in the following paragraphs develop the general theory of the non-Abelian gauge fields.

7.3.1 Lagrangian for Non-Abelian Gauge Field Theories

We consider a system of fermions whose free Lagrangian is globally $SU(N)$ symmetric. The particle field is described by an N-component spinor and thus belongs to a fundamental representation of the group. The symmetry operation is given by

$$U(x) = \mathrm{e}^{-\mathrm{i}\epsilon^l T^l} \tag{7.18}$$

(compare with eqs. (3.33), (3.96)). The T^l are the $N^2 - 1$ generators of $SU(N)$ ($T^l = \frac{1}{2}\tau^l$ for $SU(2)$, $T^l = \frac{1}{2}\lambda^l$ for $SU(3)$). They obey the commutation relation

$$\left[T^l, T^m\right] = \mathrm{i} f^{lmn} T^n \quad (l = 1, 2, \ldots, N^2 - 1) \quad , \tag{7.19}$$

where the f^{lmn} are the antisymmetric structure constants of the group $SU(N)$. They are normalized such that

$$\mathrm{tr}\left(T^l T^m\right) = \frac{1}{2}\delta^{lm} \quad . \tag{7.20}$$

This normalization is possible because the matrix $T^l T^m$ can always be diagonalized in the space of the intrinsic degrees of freedom by a proper choice of basis vectors.

The global $SU(N)$-symmetry is now made local. This necessitates the appearance of a gauge field, W_μ. This vector field has N^2-1 internal components, each of them belonging to one of the N^2-1 generators of $SU(N)$. The gauge transformation reads now

$$\Psi'(x) = \mathrm{e}^{-\mathrm{i}\epsilon^l(x)T^l}\Psi(x) \quad . \tag{7.21}$$

As in the Abelian case of QED, the globally symmetric free fermion Lagrangian is made locally gauge invariant by replacing the normal derivative ∂_μ by the covariant derivative

$$D_\mu = \partial_\mu + \mathrm{i}gT^lW^l_\mu \tag{7.22}$$

where g is the coupling constant; this is a generalization of (7.2). The Lagrangian is now

$$\mathcal{L} = \bar\Psi\left(\mathrm{i}\gamma^\mu D_\mu - m\right)\Psi + \mathcal{L}_W \quad . \tag{7.23}$$

Here $\mathcal{L}_W$ is the Lagrangian of the free N^2-1 vector fields W^l_μ.

The requirement of local gauge invariance of (7.23) determines the changes of W^l_μ under the gauge transformation. These can be obtained by considering the requirement that the kinetic energy part in (7.23) has to be invariant under the transformation $\Psi \to \Psi' = U(x)\Psi$

$$\bar\Psi'\gamma^\mu D'_\mu\Psi' = \bar\Psi\gamma^\mu U^{-1}D'_\mu\left(U\Psi\right) \overset{!}{=} \bar\Psi\gamma^\mu D_\mu\Psi \quad . \tag{7.24}$$

This yields the condition

$$U^{-1}D'_\mu U\,\Psi = D_\mu\Psi \quad . \tag{7.25}$$

Explicitly, this reads

$$U^{-1}\left(\partial_\mu + \mathrm{i}gW'_\mu\right)\left(U\Psi\right) = \left(\partial_\mu + \mathrm{i}gW_\mu\right)\Psi \quad , \tag{7.26}$$

where for convenience the $(N\times N)$ matrix

$$W_\mu \equiv T^lW^l_\mu \tag{7.27}$$

has been introduced[2]. Equation (7.26) yields the necessary transformation law for W_μ,

$$\begin{aligned} W'_\mu &= UW_\mu U^{-1} + \frac{\mathrm{i}}{g}\left(\partial_\mu U\right)U^{-1} \\ &= UW_\mu U^{-1} - \frac{\mathrm{i}}{g}U\left(\partial_\mu U^{-1}\right) \quad , \end{aligned} \tag{7.28a}$$

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[2] The quantity W_μ is an $(N\times N)$ matrix acting on the N components of the fermion spinor Ψ and it is the μ-th component of a four-vector.

where the second line follows from differentiating $UU^{-1} = 1$. The corresponding transformation law for W^l_μ can be obtained by multiplying both sides of eq. (7.28a) with T^l. Taking the trace and using (7.20), we obtain for an infinitesimal transformation (considering only terms up to $\mathcal{O}(\epsilon)$)

$$\begin{aligned} \delta W^l_\mu &= -\epsilon^k f^{klm} W^m_\mu + \frac{1}{g}\partial_\mu \epsilon^l \\ &= \frac{1}{g}\left(\partial_\mu \delta^{lk} + \mathrm{i}\left(T^m W^m_\mu\right)^{lk}\right)\epsilon^k \\ &= \frac{1}{g}\left(D_\mu \epsilon\right)^l \end{aligned} \tag{7.28b}$$

It is remarkable that the change of the field W^l_μ is thus given simply by the covariant derivative of $\epsilon(x)$.

If we now take T^k in the regular representation ($(T^k)^{lm} = -\mathrm{i} f^{klm}$, cf. the discussion after eq. (3.36)) then the change of the gauge fields can also be written as

$$\delta W^l_\mu = -\mathrm{i}\epsilon^k \left(T^k W_\mu\right)^l + \frac{1}{g}\partial_\mu \epsilon^l \ . \tag{7.29}$$

This equation shows that under a *global* transformation with constant ϵ the gauge fields W^l_μ transform according to the regular representation of $SU(N)$ (cf. eq. (3.108)). The corresponding gauge bosons, the field quanta of W^l_μ, therefore form themselves an $(N^2 - 1)$-dimensional multiplet of $SU(N)$; they must be bosons because they belong to a vector field with angular momentum 1.

Equation (7.28a) contains the case of an Abelian gauge symmetry as a special case. In the case of QED, for example, the four-potential A_μ plays the role of W_μ and one obviously has

$$[A_\mu, A_\nu] = 0 \tag{7.30}$$

and $T^l = 1$. Then eq. (7.28a) becomes

$$\begin{aligned} A'_\mu(x) &= \mathrm{e}^{-\mathrm{i}\epsilon(x)} A_\mu(x) \mathrm{e}^{\mathrm{i}\epsilon(x)} + \frac{1}{g}\partial_\mu \epsilon(x) \\ &= A_\mu(x) + \frac{1}{g}\partial_\mu \epsilon(x) \end{aligned} \tag{7.31}$$

in agreement with eq. (7.13).

What we have shown up to now is that the fermionic part of the Lagrangian is locally gauge invariant if the gauge field is changed according to eq.

(7.28b). What is still missing is the form of $\mathcal{L}_W$ and the proof that $\mathcal{L}_W$ is also invariant under the transformation (7.28b). In analogy to electrodynamics we make the ansatz

$$\mathcal{L}_W = -\frac{1}{4} F^{l\mu\nu} F^l_{\mu\nu} = -\frac{1}{4} \mathbf{F}^{\mu\nu} \cdot \mathbf{F}_{\mu\nu} \tag{7.32}$$

where the dot product denotes a scalar product in the internal $SU(N)$ space[3]

In order to facilitate this proof we first rewrite $\mathcal{L}_W$ into a more convenient form by using the normalization condition (7.20).

$$\begin{aligned}
\mathcal{L}_W &= -\frac{1}{4} F^{l\mu\nu} F^l_{\mu\nu} = -\frac{1}{2} \sum_{l,m=1}^{N^2-1} F^{l\mu\nu} \mathrm{tr}\left(T^l T^m\right) F^m_{\mu\nu} \\
&= -\frac{1}{2} \sum_{l,m=1}^{N^2-1} \sum_{i,j=1}^{N} F^{l\mu\nu} \left(T^l\right)_{ij} \left(T^m\right)_{ji} F^m_{\mu\nu} \\
&= -\frac{1}{2} \sum_{i,j=1}^{N} \left(F^{\mu\nu}\right)_{ij} \left(F_{\mu\nu}\right)_{ji} \\
&= -\frac{1}{2} \mathrm{tr}\left(F^{\mu\nu} F_{\mu\nu}\right)
\end{aligned} \tag{7.33}$$

with $F_{\mu\nu} = T^l F^l_{\mu\nu}$.

The trace of a matrix is invariant under unitary transformations. For the invariance of $\mathcal{L}_W$ it is therefore sufficient to show that

$$F'_{\mu\nu} = U F_{\mu\nu} U^{-1} \quad . \tag{7.34}$$

This relation determines the functional dependence of $F_{\mu\nu}$ on the gauge fields W^l_μ. Since their transformation involves the structure constants of $SU(N)$ (see eq. (7.28b)) it is natural to expect that the field tensor $F_{\mu\nu}$ depends on W_μ in a more complicated way than in the Abelian case. As we will show below, eq. (7.34) is satisfied by an ansatz of the form

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu W_\nu - \partial_\nu W_\mu + \mathrm{i} g \left[W_\mu, W_\nu\right] \\
&\text{or in components} \\
F^l_{\mu\nu} &= \partial_\mu W^l_\nu - \partial_\nu W^l_\mu - g f^{lmn} W^m_\mu W^n_\nu \quad .
\end{aligned} \tag{7.35}$$

Again, for the electromagnetic case ($f^{lmn} = 0$) this reduces to

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

---

[3] From now on vectors in internal space are denoted by boldfaced symbols.

since in the Abelian case the commutator term (eq. (7.30)) vanishes. The additional term, quadratic in W, that appears in eq. (7.35) for the non-Abelian gauge field theory will lead to terms $\sim W^3$ and $\sim W^4$ in $\mathcal{L}_W$ (eq. (7.33)). Since free bosonic field theories contain only terms $\sim W^2$ (the mass-term) these new terms, which are present only in non-Abelian theories, obviously describe self-interactions of the gauge-fields.

The $F_{\mu\nu}$ given in eq. (7.35) indeed transforms according to (7.34) as will be shown now. For this purpose we first derive the expression for the derivative of the transformed W_ν

$$\begin{aligned} \partial_\mu W'_\nu &= (\partial_\mu U) W_\nu U^{-1} + U(\partial_\mu W_\nu) U^{-1} + U W_\nu \left(\partial_\mu U^{-1}\right) \\ &\quad - \frac{\mathrm{i}}{g}(\partial_\mu U)\left(\partial_\nu U^{-1}\right) - \frac{\mathrm{i}}{g} U \left(\partial_\mu \partial_\nu U^{-1}\right) \quad . \end{aligned} \tag{7.36}$$

Here eq. (7.28a) has been used. Using the relation

$$\left(\partial_\mu U^{-1}\right) U + U^{-1}(\partial_\mu U) = 0 \quad , \tag{7.37}$$

that follows from $U^{-1}U = 1$, the first three terms in (7.36) can be rewritten into the form

$$\begin{aligned} & U U^{-1}(\partial_\mu U) W_\nu U^{-1} + U(\partial_\mu W_\nu) U^{-1} + U W_\nu U^{-1} U \left(\partial_\mu U^{-1}\right) \\ &\quad = -U\left(\partial_\mu U^{-1}\right) U W_\nu U^{-1} + U(\partial_\mu W_\nu) U^{-1} + U W_\nu U^{-1} U \left(\partial_\mu U^{-1}\right) \\ &\quad = U(\partial_\mu W_\nu) U^{-1} - \left[U\left(\partial_\mu U^{-1}\right), U W_\nu U^{-1}\right] \quad . \end{aligned} \tag{7.38}$$

The resulting expression for the derivative of the gauge field (7.36) is now introduced into the definition (7.35). We obtain

$$\begin{aligned} F'_{\mu\nu} &= \partial_\mu W'_\nu - \partial_\nu W'_\mu + \mathrm{i}g\left[W'_\mu, W'_\nu\right] \\ &= U(\partial_\mu W_\nu) U^{-1} - \left[U\left(\partial_\mu U^{-1}\right), U W_\nu U^{-1}\right] \\ &\quad - U(\partial_\nu W_\mu) U^{-1} + \left[U\left(\partial_\nu U^{-1}\right), U W_\mu U^{-1}\right] \\ &\quad - \frac{\mathrm{i}}{g}\left((\partial_\mu U)\left(\partial_\nu U^{-1}\right) - (\partial_\nu U)\left(\partial_\mu U^{-1}\right)\right) \\ &\quad + \mathrm{i}g\left[U W_\mu U^{-1} - \frac{\mathrm{i}}{g} U\left(\partial_\mu U^{-1}\right), U W_\nu U^{-1} - \frac{\mathrm{i}}{g} U\left(\partial_\nu U^{-1}\right)\right] \\ &= U(\partial_\mu W_\nu - \partial_\nu W_\mu) U^{-1} + \mathrm{i}g U [W_\mu, W_\nu] U^{-1} \\ &= U F_{\mu\nu} U^{-1} \quad . \end{aligned} \tag{7.39}$$

Here extensive use has been made of eq. (7.37). Equation (7.39) is the desired result. Thus the Lagrangian $\mathcal{L}_W$ for the gauge field is indeed gauge-invariant.

As in the Abelian case this result holds only if $\mathcal{L}_W$ does not contain a mass term. Such a term of the form (cf. (2.26))

$$\frac{1}{2}m^2 W^{l\mu} W^l_\mu$$

cannot be invariant under the transformation (7.28a).

For our discussion in the next subsection it is essential to notice that the coupling constant g in the expression for the field tensor (7.35) has to be the same as the one appearing in the change of the gauge field (and therefore also as the one in the covariant derivative (7.22)), since otherwise the cancellation of terms necessary for the result (7.39) could not be achieved.

The locally gauge-invariant complete Lagrangian is, therefore, for a $SU(N)$ symmetry given by (7.23)

$$\begin{aligned}
\mathcal{L} &= \bar{\Psi}\left(\mathrm{i}\gamma^\mu D_\mu - m\right)\Psi - \frac{1}{4}F^{l\mu\nu}F^l_{\mu\nu} \\
&= \bar{\Psi}\left(\mathrm{i}\gamma^\mu\left(\partial_\mu + \mathrm{i}gW_\mu\right) - m\right)\Psi - \frac{1}{4}F^{l^{\mu\nu}}F^l_{\mu\nu} \\
&= \bar{\Psi}\left(\mathrm{i}\gamma^\mu\partial_\mu - m\right)\Psi - \frac{1}{4}F^{l^{\mu\nu}}F^l_{\mu\nu} - g\bar{\Psi}\gamma^\mu T^l \Psi W^l_\mu \quad . \qquad (7.40)
\end{aligned}$$

The last term provides the coupling of the fermions to the gauge field by means of the current

$$j_\mu(x) = g\bar{\Psi}\gamma_\mu T^l \Psi \quad . \qquad (7.41)$$

This is again the conserved Noether current corresponding to the invariance of the free fermion Lagrangian (first term in (7.40)) under the global $SU(N)$ transformation U.

As in classical electrodynamics, one has the freedom to choose a gauge that simplifies specific calculations as much as possible. The freedom to choose a convenient gauge lies here – as in QED – in the functions $\epsilon^l(x)$ appearing in the transformation (7.21). Often used gauges are

1.the time-axial gauge: $W_0(x) = 0$ for all x ;
2.the space-axial gauge: $W_k(x) = 0$ for all x ;
3.the Coulomb gauge: $\partial_i W_i(x) = 0$ for all x ;
4.the covariant gauge: $\partial_\mu W_\mu(x) = 0$ for all x .

7.3.2 Properties of Non-Abelian Gauge Field Theories

The consequences of the non-Abelian character of this gauge field theory, compared with an Abelian one like QED, are most evident in the equation of motion for the gauge field:

$$\partial^\mu F^l_{\mu\nu} = -g f^{lmn} W^{m\,\mu} F^n_{\mu\nu} - g\bar{\Psi}\gamma_\nu T^l \Psi \quad . \tag{7.42}$$

For $f^{lmn} = 0$ (Abelian case) and $T^l = 1$ this goes over into Maxwell's equations of QED (eq. (2.15)). However, in the non-Abelian case an additional source term appears that is even present for a vanishing fermion source: the first term on the rhs of (7.42) depends only on the gauge fields and contains terms of second and third order in these and their derivatives. This means physically that the gauge field quanta interact among themselves: they are "charged". The coupling constant for those self interactions is the same g as that for the coupling to the particle field; the self interactions can be traced back to the appearance of the commutator term in the field tensor (7.35) that is proportional to g.

If the original globally $SU(N)$-symmetric Lagrangian, before "gauging", contained other particle fields and their couplings, it could be made locally invariant in exactly the same way as described above, namely by introducing one common gauge field and replacing the normal derivatives in the kinetic energy terms by the covariant ones. Since for non-Abelian gauge symmetries the coupling constant appears also in the expression for the field tensor $F_{\mu\nu}$, all the different particle fields have to couple with the same coupling constant g to the gauge field. This fact is known as the universality of the interaction; the non-Abelian gauge field couples with an universal strength to all particle fields.

Necessary for this universality is that the symmetry group cannot be decomposed into a direct product of groups. If this were the case, on the other hand, then the generators of one subgroup, although not commuting among themselves, do commute with the generators of the other subgroups. Gauging each one of these independent symmetries will introduce separate gauge fields and coupling constants into the theory. For example, the Lagrangian (7.40) contains in addition to $SU(N)$ also a global $U(1)$ symmetry. If the latter were to be made local then we would have to introduce another gauge field and another coupling constant.

It is worthwhile to stress explicitly that universality is contained only in non-Abelian gauge field theories; it is not present in the Abelian case. For example, in QED, where the $U(1)$ symmetry is gauged, the different particles could have quite different electrical charges, since in this case $\mathcal{L}_W$

does not contain the coupling constant and is therefore invariant under the transformation (7.31) with arbitrary g. That the electrical charges in nature are quantized and that, for example, the charge of the proton is exactly the same as that of the positron can therefore not be understood on the basis of the $U(1)$ symmetry; it may be the consequence of some higher non-Abelian gauge group.

The first attempt to describe physical interactions by non-Abelian gauge field theories was made by Yang and Mills in 1954. They tried to generate the strong interactions by "gauging" the $SU(2)$ isospin symmetry. This attempt had to be given up because a theory with "charged" (i. e. isospin-carrying) nucleons and massless gauge bosons leads to infinitely ranged forces between the nucleons, in obvious contrast to the observed properties of the strong interactions. Only much later was it realized that a theory of the strong interactions should involve the basic building blocks of hadrons and not the hadrons themselves. A successful gauge field theory involving these elementary particles is Quantum Chromodynamics (QCD) in which the $SU(3)$ symmetry in an internal space (the color space) is gauged.

An essential feature of gauge field theories is their renormalizability, i.e. the possibility to derive meaningful, well defined predictions of the theory. This renormalizability and the universality of the coupling constant of non-Abelian gauge field theories have important implications for the theory of weak interactions. In section 5.2.1 we have argued that the observed universal value of the weak vector coupling strength can be understood as a consequence of the $SU(2)$ isospin current conservation. What was needed there was the assumption also that the unrenormalized weak charges are the same for all particles. This requirement suggests that the weak interactions might be mediated by non-Abelian gauge fields, which automatically lead to universal coupling constants as just discussed.

Such an idea has indeed recently had remarkable successes in the so-called standard model of the electroweak interactions ([QUI83], [HAL84], [HUA82]) which will be discussed in detail in chapter 12. In this model the weak and the Coulomb interaction are generated by gauging the weak isospin $SU(2)_W$ and the weak hypercharge $U(1)_Y$. By spontaneous symmetry breaking, here called the "Higgs mechanism", masses are generated for the gauge bosons that transmit the weak interaction (the $W^{\pm}$ and Z-bosons), whereas the gauge bosons responsible for the electromagnetic interaction (the photons) are left massless. The Higgs mechanism thus offers a reconciliation of the gauge invariance of the Lagrangian with the requirement of a very short range of the interaction.

Chapter 8

SPONTANEOUS SYMMETRY BREAKING IN GAUGE-FIELD THEORIES

In this chapter we construct a number of models that exhibit the mechanism of spontaneous symmetry breaking for gauge field theories. These models do not yet contain "real" particles but they exhibit the important Higgs mechanism that appears when a local symmetry is spontaneously broken.

8.1 Higgs Mechanism

In preparation for the full theory we consider here now a *locally* gauge invariant Lagrangian that allows spontaneous symmetry breaking. We restrict to $U(1)$ symmetry.

We consider a complex scalar field ϕ

$$\phi = \frac{1}{\sqrt{2}} \left(\phi_1 + \mathrm{i}\phi_2 \right) \quad , \tag{8.1}$$

which is coupled to a vector field (e.g. the electromagnetic field). As discussed in section 7.1 this coupling can be achieved by replacing everywhere normal derivatives by the covariant derivative

$$\mathcal{L} = \left(D^\mu \phi \right)^\dagger \left(D_\mu \phi \right) - V(\phi^\dagger \phi) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \quad , \tag{8.2}$$

where V is a potential given by

$$V\left(\phi^\dagger\phi\right) = \frac{1}{2}\mu^2\phi^\dagger\phi + \frac{1}{2}\lambda^2\left(\phi^\dagger\phi\right)^2 \tag{8.3}$$

with $\lambda^2 > 0$ (cf. the model in section 6.1.2). $F_{\mu\nu}$ is the field tensor

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \quad . \tag{8.4}$$

The Lagrangian (8.2) is globally $U(1)$-invariant, it is also locally $U(1)$-invariant under the gauge transformation (7.13), (7.15)

$$\begin{aligned} \phi(x) &\to \phi'(x) = \mathrm{e}^{-\mathrm{i}\epsilon(x)}\phi(x) \\ B_\mu(x) &\to B'_\mu(x) = B_\mu(x) + \frac{1}{q}\partial_\mu\epsilon(x). \end{aligned} \tag{8.5}$$

We consider the case with spontaneous symmetry breaking, $\mu^2 < 0$. In this case the vacuum state, i.e. the state that minimizes the energy, is given by a space-time independent state with

$$\left(\phi^\dagger(x)\phi(x)\right)_0 = -\frac{\mu^2}{2\lambda^2} \equiv \frac{v^2}{2} = \text{const.} \quad , \tag{8.6}$$

just as in the example treated in section 6.1.1. This vacuum state is infinitely degenerate. We chose as the physical vacuum the real, constant field

$$\phi_0 = \frac{v}{\sqrt{2}} \quad \text{with} \quad v = \sqrt{-\frac{\mu^2}{\lambda^2}} \quad \text{real} \ . \tag{8.7}$$

The other degenerate vacua differ only by a constant phase ϵ from this vacuum state. The vacuum is thus not annihilated by the symmetry operation and we therefore expect the appearance of Goldstone bosons, at least in a globally symmetric theory.

In an already familiar step we now expand the field ϕ around ϕ_0 by setting

$$\begin{aligned} \phi &= \frac{v}{\sqrt{2}} + \phi' = \frac{v}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(\varphi_1 + \mathrm{i}\varphi_2\right) \\ &\cong \frac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\varphi_2/v}\left(v + \varphi_1\right) \quad . \end{aligned} \tag{8.8}$$

The exponential, "polar" form of ϕ represents the variations of ϕ around the physical vacuum in terms of a phase $\varphi_2(x)/v$ and an absolute value φ_1.

Next, the Lagrangian is expressed in terms of φ_1 and φ_2

$$\begin{aligned}
\mathcal{L} &= \left[(\partial^\mu - \mathrm{i}qB^\mu)\,\phi^\dagger\right]\left[(\partial_\mu + \mathrm{i}qB_\mu)\,\phi\right] - \frac{1}{2}\mu^2\phi^\dagger\phi - \frac{1}{2}\lambda^2\left(\phi^\dagger\phi\right)^2 \\
&\quad - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\
&= \frac{1}{2}\left[(\partial^\mu - \mathrm{i}qB^\mu)\,(v+\varphi_1-\mathrm{i}\varphi_2)\right]\left[(\partial_\mu + \mathrm{i}qB_\mu)\,(v+\varphi_1+\mathrm{i}\varphi_2)\right] \\
&\quad - \frac{1}{4}\mu^2\,(v+\varphi_1-\mathrm{i}\varphi_2)\,(v+\varphi_1+\mathrm{i}\varphi_2) \\
&\quad - \frac{1}{8}\lambda^2\left[(v+\varphi_1-\mathrm{i}\varphi_2)\,(v+\varphi_1+\mathrm{i}\varphi_2)\right]^2 \\
&\quad - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad .
\end{aligned} \tag{8.9}$$

Collecting all terms up to those quadratic in the fields we obtain

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\left[(\partial_\mu\varphi_1)\,(\partial_\mu\varphi_1) + \mu^2\varphi_1^2\right] \\
&\quad + \frac{1}{2}\,(\partial_\mu\varphi_2)\,(\partial_\mu\varphi_2) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + qvB^\mu\partial_\mu\varphi_2 \\
&\quad + \frac{1}{2}q^2v^2B^\mu B_\mu + \ldots \quad .
\end{aligned} \tag{8.10}$$

The φ_1 field has now acquired a mass $m = \sqrt{-\mu^2}$, φ_2 seems to be massless (the Goldstone boson), but is coupled to B_μ, and B_μ itself has become massive. The terms that are not written out explicitly contain terms of order $\mathcal{O}(\varphi^3)$ and $\mathcal{O}(\varphi^4)$ as well as $\mathcal{O}(B\varphi^2)$ and $\mathcal{O}(B^2\varphi^2)$.

To get a clearer insight into the physics contained in eq. (8.10) we would like to get rid of the coupling between the field B_μ and the field φ_2. This can be achieved here in a particularly simple way that suggests itself by looking at the polar representation of ϕ in eq. (8.8). Since the model-Lagrangian considered here is gauge-invariant, the phase factor $\exp(\mathrm{i}\varphi_2/v)$ in (8.8) can be removed by an appropriate gauge transformation:

$$\phi \to \phi' = \mathrm{e}^{-\mathrm{i}\varphi_2(x)/v}\phi \quad . \tag{8.11}$$

This corresponds to a variation along the degree of freedom that distinguishes the various vacua of the theory. As we have seen earlier, oscillations in this degree of freedom are just the Goldstone bosons.

Eq. (8.11) effectively removes these excitations so that we expect that no massless particles should be present in our model-spectrum. This is indeed borne out as we can see by inserting the gauged fields

$$\phi' = \frac{1}{\sqrt{2}} (v + \varphi_1) \tag{8.12}$$

and consequently also

$$B'_\mu = B_\mu + \frac{1}{qv} \partial_\mu \varphi_2 \tag{8.13}$$

into the Lagrangian (8.2). This yields

$$\begin{aligned} \mathcal{L} \;=\; & \frac{1}{2} \left[(\partial^\mu \varphi_1)(\partial_\mu \varphi_1) + \mu^2 \varphi_1^2 \right] \\ & - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} q^2 v^2 B^\mu B_\mu \\ & + \frac{1}{2} q^2 B^\mu B_\mu \left(\varphi_1^2 + 2v\varphi_1 \right) - \frac{1}{8} \lambda^2 \varphi_1^4 \\ & - \frac{1}{2} \lambda^2 v \varphi_1^3 + \text{const} \quad . \end{aligned} \tag{8.14}$$

In this Lagrangian indeed the field φ_2 has vanished. The vector field B_μ and the φ_1 field have remained massive with masses $m = qv$ and $\sqrt{-\mu^2}$, respectively.

The missing degree of freedom, the Goldstone boson φ_2, has been "converted" into the longitudinal component of the massive vector field B_μ. The gauge in which the field φ_2 disappears, is known as the "U-gauge". The quanta of the remaining φ_1-field are the so-called "Higgs bosons", they are scalars.

The Lagrangian (8.14) shows no indication of the original gauge symmetry, primarily because of the appearance of the photon mass which masks the original gauge-invariance. The coupling to the Higgs boson thus provides a mechanism to make a gauge-field, in the present case an Abelian one, massive.

8.2 Spontaneous Breaking of a Local non-Abelian Symmetry

The only case that is left to consider in our model building is that of a local non-Abelian symmetry.

We thus consider now the Lagrangian (6.16) with the potential (6.17) which exhibits a global $U(1) \otimes SU(2)$ symmetry. If we first gauge the $SU(2)$ symmetry we know from our general considerations of gauge fields in chapter

7 that we have to introduce an isotriplet of gauge-fields W_μ^k to make the theory gauge-invariant. As before ϕ is an iso-doublet of charged boson fields so that the Lagrangian reads

$$\mathcal{L} = (D^\mu\phi)^\dagger (D_\mu\phi) - V(\phi^\dagger\phi) - \frac{1}{4}G^{\mu\nu}G_{\mu\nu} \quad , \tag{8.15}$$

with the potential

$$V(\phi^\dagger\phi) = \frac{1}{2}\mu^2\phi^\dagger\phi + \frac{1}{2}\lambda^2\left(\phi^\dagger\phi\right)^2 \quad , \tag{8.16}$$

the covariant derivative

$$D_\mu\phi = \partial_\mu\phi + \mathrm{i}gW_\mu\phi \quad \text{and} \quad (D_\mu\phi)^\dagger = \partial_\mu\phi^\dagger - \mathrm{i}g\phi^\dagger W_\mu \quad , \tag{8.17}$$

(cf. eq. (7.22)) and the field tensor

$$G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + \mathrm{i}g\left[W_\mu, W_\nu\right] \tag{8.18}$$

with (cf. eq. (7.35))

$$W_\mu = \frac{\tau^l}{2}W_\mu^l \quad , \tag{8.19}$$

and

$$G_{\mu\nu} = \frac{\tau^l}{2}G_{\mu\nu}^l \quad . \tag{8.20}$$

The dot product in eq. (8.15) denotes a scalar product in $SU(2)$ space.

If also the $U(1)$ symmetry is gauged we have to introduce an additional Abelian gauge field B_μ so that the covariant deriative becomes

$$D_\mu = \partial_\mu + \mathrm{i}gW_\mu + \mathrm{i}\frac{g'}{2}B_\mu \quad . \tag{8.21}$$

Note that the coupling strength g' is different from g since universality applies only if one and the same symmetry is gauged, as we have discussed in section 7.3.2.

The full Lagrangian now reads

$$\mathcal{L} = (D^\mu\phi)^\dagger (D_\mu\phi) - V(\phi^\dagger\phi) - \frac{1}{4}\mathbf{G}^{\mu\nu}\cdot\mathbf{G}_{\mu\nu} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \, , \tag{8.22}$$

where now

$$D_\mu\phi = \partial_\mu\phi + \left(\mathrm{i}g\frac{1}{2}\tau^l W_\mu^l + \mathrm{i}g'\frac{1}{2}B_\mu\right)\phi, \tag{8.23}$$

and $F_{\mu\nu}$ is the field tensor of the Abelian gauge field B_μ

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \quad . \tag{8.24}$$

If $\mu^2 > 0$ then we are dealing with a standard Yang-Mills theory with a particle spectrum of two massive scalar mesons of mass μ, three massless gauge bosons W_μ and one massless gauge boson B_μ.

If $\mu^2 < 0$, however, then the original $SU(2)$ symmetry can be spontaneously broken in the vacuum state. The vacuum is – as in section 6.1.2 – chosen as

$$< 0|\phi|0 >= \phi_0 = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} . \tag{8.25}$$

Degenerate with this special state are infinitely many others that can be obtained from (8.25) by an $SU(2)$ transformation.

We now expand the "Higgs field" around this minimum by writing as in section 6.1.1

$$\phi = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} + \begin{pmatrix} \varphi_1 + \mathrm{i}\varphi_2 \\ \varphi_3 + \mathrm{i}\varphi_4 \end{pmatrix} \quad . \tag{8.26}$$

Since we are dealing with a non-Abelian theory the second term can be generated from the first by a local $SU(2)$ transformation so that we can write

$$\phi(x) = \mathrm{e}^{-\mathrm{i}\epsilon^l(x)\frac{\tau^l}{2}} \begin{pmatrix} 0 \\ (v + \varphi(x))/\sqrt{2} \end{pmatrix} \quad , \tag{8.27}$$

since the space-time dependence of ϵ allows us to generate every possible isodoublet spinor in this way. Note the similarity of this expression to eq. (8.8). The transformation (8.27) contains four real fields, three $\epsilon^l(x)$ and one $\varphi(x)$. Since the theory is gauge invariant, its vacuum-energy does not change under a gauge transformation involving the three fields $\epsilon^l(x)$. We thus expect three Goldstone bosons which will be affected by the Higgs mechanism.

As in the last section we can now go the U-gauge by performing a straightforward gauge transformation. We obtain

$$\phi = \begin{pmatrix} 0 \\ (v + \varphi(x))/\sqrt{2} \end{pmatrix} \quad . \tag{8.28}$$

The gauge freedom is thus used to convert ϕ into an isospin-down spinor at each space-time point.

This field can now be inserted into the Lagrangian (8.22).

$$\begin{aligned}
\mathcal{L} &= \left[\partial^\mu\phi^\dagger - \phi^\dagger\left(\mathrm{i}gW^\mu + \mathrm{i}\frac{g'}{2}B^\mu\right)\right]\left[\left(\partial_\mu + \mathrm{i}gW_\mu + \mathrm{i}\frac{g'}{2}B_\mu\right)\phi\right] \\
&\quad - V\left(\phi^\dagger\phi\right) - \frac{1}{4}\mathbf{G}^{\mu\nu}\cdot\mathbf{G}_{\mu\nu} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\
&= \left(\partial^\mu\phi\right)^\dagger\left(\partial_\mu\phi\right) - \mathrm{i}g\phi^\dagger W^\mu\left(\mathrm{i}gW_\mu\right)\phi \\
&\quad - \left(\mathrm{i}g\phi^\dagger W^\mu\right)\left(\mathrm{i}\frac{g'}{2}B_\mu\right)\phi - \left(\mathrm{i}\frac{g'}{2}B^\mu\right)\phi^\dagger\left(\mathrm{i}gW_\mu\right)\phi \\
&\quad - \left(\mathrm{i}\frac{g'}{2}B^\mu\right)\phi^\dagger\left(\mathrm{i}\frac{g'}{2}B_\mu\right)\phi - \frac{1}{2}\mu^2\phi^\dagger\phi \\
&\quad - \frac{1}{2}\lambda^2\left(\phi^\dagger\phi\right)^2 - \frac{1}{4}\mathbf{G}^{\mu\nu}\cdot\mathbf{G}_{\mu\nu} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \qquad (8.29) \\
&= \frac{1}{2}\left(\partial^\mu\varphi\right)\left(\partial_\mu\varphi\right) - \frac{1}{4}\mu^2v^2\left(1+\frac{\varphi}{v}\right)^2 - \frac{1}{8}\lambda^2v^4\left(1+\frac{\varphi}{v}\right)^4 \\
&\quad - \frac{1}{4}\mathbf{G}^{\mu\nu}\cdot\mathbf{G}_{\mu\nu} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\
&\quad + \frac{g^2}{4}W^{l\mu}W^k_\mu\left(\phi^\dagger\tau^l\tau^k\phi\right) + \frac{gg'}{4}W^{l\mu}B_\mu\left(\phi^\dagger\tau^l\phi\right) \\
&\quad + \frac{gg'}{4}B^\mu W^l_\mu\left(\phi^\dagger\tau^l\phi\right) + \frac{g'^2}{8}B^\mu B_\mu\left(v+\varphi\right)^2 \ .
\end{aligned}$$

Since τ^l and τ^k anticommute for $l \neq k$, the terms with $l \neq k$ in $\phi^\dagger\tau^l\tau^k\phi$ disappear. Furthermore we have

$$\phi^\dagger\tau^{1,2}\phi = 0 \quad \text{and} \quad \phi^\dagger\tau^3\phi = -\frac{(v+\varphi)^2}{2} \quad . \qquad (8.30)$$

We can thus write for the three terms proportional to g^2 and gg', respectively,

$$\begin{aligned}
&\frac{g^2}{8}\left(W^{1\mu}W^1_\mu + W^{2\mu}W^2_\mu\right)(v+\varphi)^2 + \frac{g^2}{8}W^{3\mu}W^3_\mu(v+\varphi)^2 \\
&+ \frac{gg'}{4}B^\mu W^3_\mu(v+\varphi)^2 \quad . \qquad (8.31)
\end{aligned}$$

The total Lagrangian (8.29) thus reads

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\left[\left(\partial^\mu\varphi\right)\left(\partial_\mu\varphi\right) + \mu^2\varphi^2\right] \\
&\quad - \frac{1}{4}\mathbf{G}^{\mu\nu}\cdot\mathbf{G}_{\mu\nu} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}
\end{aligned}$$

$$\begin{aligned}
&+\frac{g^2}{8}v^2\left(W^{1\mu}W^1_\mu + W^{2\mu}W^2_\mu\right) \\
&+\frac{1}{8}v^2\left(gW^{3\mu} - g'B^\mu\right)\left(gW^3_\mu - g'B_\mu\right) \qquad (8.32) \\
&+ \text{higher order terms} + \text{const.} \quad .
\end{aligned}$$

Here again $\lambda^2 v^2 = -\mu^2$ has been used; the higher order terms all contain products of 3 and more fields.

In eq. (8.32) the fields W^3 and B are mixed. In order to disentangle them we now introduce the *Weinberg angle* ϑ_W by the definition

$$\tan\vartheta_W = \frac{g'}{g} \tag{8.33}$$

and write, accordingly,

$$\begin{aligned}
gW^3_\mu - g'B_\mu &= g\left(W^3_\mu - \frac{g'}{g}B_\mu\right) \\
&= g\left(W^3_\mu - \tan\vartheta_W B_\mu\right) \\
&= \frac{g}{\cos\vartheta_W}\left(\cos\vartheta_W W^3_\mu - \sin\vartheta_W B_\mu\right) \\
&\equiv \frac{g}{\cos\vartheta_W} Z_\mu \qquad (8.34)
\end{aligned}$$

with the new field

$$Z_\mu = \cos\vartheta_W W^3_\mu - \sin\vartheta_W B_\mu \quad . \tag{8.35}$$

If we also introduce the 'orthogonal' field

$$A_\mu = \sin\vartheta_W W^3_\mu + \cos\vartheta_W B_\mu \ , \tag{8.36}$$

the fields W^3_μ and B_μ decouple in (8.32). Using the inverse relations

$$W^3_\mu = \cos\vartheta_W Z_\mu + \sin\vartheta_W A_\mu \tag{8.37}$$

and

$$B_\mu = -\sin\vartheta_W Z_\mu + \cos\vartheta_W A_\mu \tag{8.38}$$

we can now rewrite the kinetic energy terms for W^3 and B in terms of A and Z. This gives

$$\mathcal{L} = \frac{1}{2}\left((\partial^\mu\varphi)(\partial_\mu\varphi) + \mu^2\varphi^2\right) \tag{8.39}$$

$$
\begin{aligned}
&-\frac{1}{4}\sum_{i=1}^{2}\left(\left(\partial^\mu W^{i\nu}-\partial^\nu W^{i\mu}\right)\left(\partial_\mu W^i_\nu-\partial_\nu W^i_\mu\right)-\frac{1}{2}g^2v^2W^{i\mu}W^i_\mu\right)\\
&-\frac{1}{4}\left(\left(\partial^\mu Z^\nu-\partial^\nu Z^\mu\right)\left(\partial_\mu Z_\nu-\partial_\nu Z_\mu\right)-\frac{1}{2}\left(g^2+g'^2\right)v^2Z^\mu Z_\mu\right)\\
&-\frac{1}{4}\left(\partial^\mu A^\nu-\partial^\nu A^\mu\right)\left(\partial_\mu A_\nu-\partial_\nu A_\mu\right)+\text{higher order terms}\quad .
\end{aligned}
$$

The higher order terms contain all the couplings of the fields.

The mass-spectrum of our model can now be read off from the Lagrangian (8.39). We have one massive scalar field, φ, with mass $\sqrt{-\mu^2}$. This represents the so-called Higgs particle. In addition, there are three massive vector particles — W^1, W^2 and Z — with masses

$$
\begin{aligned}
W^1_\mu &: \quad m_{W^1}=g\frac{v}{2}\equiv M_W\\
W^2_\mu &: \quad m_{W^2}=M_W\\
Z_\mu &: \quad m_Z=\sqrt{g^2+g'^2}\,\frac{v}{2}=\frac{M_W}{\cos\vartheta_W}\equiv M_Z
\end{aligned}
\tag{8.40}
$$

and one massless vector particle, A_μ. There are 9 degrees of freedom in the 3 massive gauge fields, 2 in the massless one and 1 in the Higgs field, giving a total of 12. This equals the number of degrees of freedom before symmetry breaking: 4 in the iso-doublet ϕ, 6 in the massless gauge-bosons W^l_μ, 2 in the massless gauge-boson B_μ.

As discussed earlier, in section 6.1.1, the generators $\tau_1/2$, $\tau_2/2$ and $(1-\tau_3)/2$ do not annihilate the vacuum, in other words, the unitary symmetry operations involving them do *not* leave the vacuum state (8.25) invariant. Only $(1+\tau_3)/2$ annihilates the vacuum state, so that we have

$$
\mathrm{e}^{-\mathrm{i}\epsilon(1+\tau_3)/2}\phi_0=\phi_0 \tag{8.41}
$$

as the symmetry operation that leaves the vacuum invariant and thus causes the masslessness of A_μ. This suggests that we could have gauged

$$
U=\mathrm{e}^{-\mathrm{i}\epsilon(1+\tau_3)/2} \tag{8.42}
$$

to obtain the theory of QED alone. We now see that the Higgs mechanism gives a mass to the Goldstone bosons, corresponding to the first three generators just listed, by "eating up" three of the four Higgs fields.

Finally, in order to exhibit the couplings to the gauge fields we rewrite the covariant derivative appearing in our Lagrangian in terms of the fields

A_μ and Z_μ. This gives

$$\begin{aligned}
D_\mu &= \partial_\mu + \mathrm{i}gW_\mu + \mathrm{i}\frac{g'}{2}B_\mu = \partial_\mu + \mathrm{i}\frac{g}{2}\tau^l W^l_\mu + \mathrm{i}\frac{g'}{2}B_\mu \\
&= \partial_\mu + \mathrm{i}\frac{g}{2}\sum_{k=1}^{2}\tau^k W^k_\mu + \mathrm{i}\frac{g}{2}\tau^3 W^3_\mu + \mathrm{i}\frac{g'}{2}B_\mu \\
&= \partial_\mu + \mathrm{i}\frac{g}{2}\sum_{k=1}^{2}\tau^k W^k_\mu + \mathrm{i}\frac{g}{2}\tau^3\left(\cos\vartheta_W Z_\mu + \sin\vartheta_W A_\mu\right) \\
&\quad + \mathrm{i}\frac{g'}{2}\left(-\sin\vartheta_W Z_\mu + \cos\vartheta_W A_\mu\right) \\
&= \partial_\mu + \mathrm{i}g\sin\vartheta_W \frac{1}{2}\left(1+\tau_3\right)A_\mu \qquad (8.43) \\
&\quad + \mathrm{i}\frac{g}{\cos\vartheta_W}\left(\frac{\tau_3}{2} - \frac{1}{2}\left(1+\tau_3\right)\sin^2\vartheta_W\right)Z_\mu + \ldots \quad .
\end{aligned}$$

The massless vector field A_μ is thus coupled to the conserved $SU(2)$ charge-operator $(1+\tau_3)/2$ (cf. eq. (3.44)). The coupling constant is in this case

$$\tilde{e} \equiv g\sin\vartheta_W \quad . \qquad (8.44)$$

At this point, of course, we have not included any physical particles in our model except for the scalar Higgs field ϕ and the gauge fields. The major challenge is now to connect this model with the "real world".

8.3 Summary of the Higgs Mechanism

In chapter 6 we have discussed how a Goldstone boson appears whenever a global symmetry of the Lagrangian is spontaneously broken or hidden. More specifically, we have shown how k massless, scalar particles, the Goldstone bosons, appear when k symmetries of $\mathcal{L}$ are not also symmetries of the groundstate of the theory.

If the theory is also locally symmetric then there exist massless gauge bosons. If such a gauge-field theory also exhibits the phenomenon of spontaneous symmetry breaking then the massless gauge bosons, that transmit the interaction, "eat up" the Goldstone bosons and become themselves massive. This is the Higgs mechanism. Only those gauge-fields that correspond to an unbroken symmetry will remain massless.

Part V

Electroweak Interaction

Chapter 9

WEAK INTERACTIONS OF QUARKS AND LEPTONS

In chapter 5 we discussed the phenomenology of weak interactions and Fermi's theory for the weak interaction Langrangian. This discussion involved only nucleons and leptons. If quarks are really the constituents of nucleons then a theory of weak interactions should involve interactions between these quarks. To develop such a theory is the purpose of this chapter.

Since the weak interaction does not act on the color degree of freedom,i.e. the charge of the strong interaction, it is diagonal in color space. Formally, this implies a sum over the color index of the quark spinors in all interaction terms. In order to simplify the notation, this sum is not explicitly written out in any of the expressions of this chapter.

9.1 Phenomenological Introduction

In this section we first add two experimental observations to the properties of weak interactions discussed in chapter 5. These could not be described by Fermi's theory and thus point to the need to generalize the classical theory of weak interaction.

9.1.1 Strangeness Changing Weak Decays

After the discovery of strange particles it has turned out that the theory as described in section 5.1 cannot be complete because e.g. flavor-changing weak decays, like that of the kaon

$$K^+ \to \mu^+ + \nu_\mu \quad , \tag{9.1}$$

cannot be described by it. In this process the strangeness changes by one unit and this possibility is so far not contained in the interaction. Formally, it can be introduced by generalizing the hadronic current J_λ to a superposition of two terms, one strangeness-conserving and one strangeness-changing

$$J_\lambda = \cos\theta_c \, J_\lambda^{(0)} + \sin\theta_c \, J_\lambda^{(1)} \quad , \tag{9.2}$$

where the superscripts denote the strangeness change and θ_c is the so-called *Cabibbo angle.*

As discussed earlier the weak interaction really acts on the quarks and the strangeness-conserving vector current is in analogy to eq. (5.8) given by

$$J_\lambda^+ = \mathrm{i}\bar{u}\gamma_\lambda(1+\gamma_5)d \tag{9.3}$$

with the d-quark replaced by a linear combination of d and s quark

$$d \longrightarrow \cos\theta_c \, d + \sin\theta_c \, s \ . \tag{9.4}$$

We thus obtain for the vector current

$$\begin{aligned} J_\lambda^+ &= \mathrm{i}\bar{u}\gamma_\lambda(1+\gamma_5)\,(\cos\theta_c \, d + \sin\theta_c \, s) \\ &= \cos\theta_c \, \mathrm{i}\bar{u}\gamma_\lambda(1+\gamma_5)d + \sin\theta_c \, \mathrm{i}\bar{u}\gamma_\lambda(1+\gamma_5)s \\ &\equiv \cos\theta_c \, J_\lambda^{+(0)} + \sin\theta_c \, J_\lambda^{+(1)} \quad . \end{aligned} \tag{9.5}$$

The new elementary process introduced by this generalization is thus the decay

$$s \to u + l + \bar{\nu}_l \quad , \tag{9.6}$$

mediated by the term $\sim J_\lambda^+ j_\lambda^-$ in eq. (5.6), which makes the kaon decay discussed above possible.

Since the Cabibbo angle appears only in hadronic processes it leads to a small difference between the coupling strengths for leptonic and semileptonic decays. The comparison of both processes actually serves to determine the Cabibbo angle ($\cos\theta_c = g_V = 0.98$); whereas for the former the coupling is given by G_F, for the latter it is $G_F \cos\theta_c \approx 0.98 G_F$ (see eq. (5.12). The appearance of the Cabibbo angle could be understood by assuming that the physical d and s quarks, i.e. the eigenstates of the total energy-operator, are not the objects that couple via the weak interaction: what couples is the linear combination (9.4).

9.1.2 Neutral Currents

In 1973 a new weak interaction process was discovered when it was found that neutrinos could be scattered off electrons and nucleons. Since the charge (zero) of the neutrino does not change during this process the scattering cannot be described by the interaction (5.11), which contains only charged currents, but necessitates the introduction of new interaction terms involving electrically neutral currents. It was also found that the rates for neutral current and charged current events were of comparable magnitude. Thus, the coupling strengths for both processes had to be about equal.

When more experiments on neutral current interactions involving hadrons became available, it was also found that there existed only $\Delta S = 0$ events, i.e. events without a strangeness change [PER87]. This poses an immediate problem since with the mixed d-quark of eq. (9.4) one obtains together with a $\Delta S = 0$ current automatically also a $\Delta S = 1$ contribution.

9.2 Intermediate Vector Bosons

We have briefly mentioned at the end of section 5.1 that Fermi's theory runs into difficulties in the description of high-energy processes where it violates unitarity constraints on the scattering amplitudes. This is actually easy to see. For example, for the reaction $\nu_\mu + e^- \to \mu^- + \nu_e$ (see Fig. 9.1) the total, angle-integrated cross section is given by [SCH88] [1]

$$\sigma \approx \frac{G_F^2}{\pi} s \qquad (\text{for } s \gg m_\mu^2) \quad ,$$

which exceeds already for energies $\sqrt{s} > 700$ GeV the bound imposed by unitarity for an s-wave scatterer such as the point-interaction of Fermi's theory

$$\sigma \leq \frac{4\pi}{s} \quad .$$

We have already mentioned at the end of section 5.1 that this problem arises mainly because a point-like interaction, which is constant in momentum space, enables infinite momentum transfers. A cure for this problem may therefore lie in the introduction of very heavy vector mesons that transmit the interaction.

---

[1] s is the square of the total four-momentum of both particles ($s = (p_1 + p_2)^2$ where p_1 and p_2 are the four-momenta of the two reaction partners, thus in the c.m. system $s =$ square of the total energy).

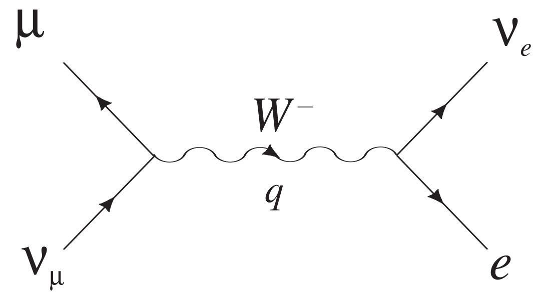

Figure 9.1: Feynman diagram for neutrino electron scattering.

We thus now take a closer look at the idea of introducing such heavy intermediate vector mesons into the theory of weak interaction. As we have discussed in section 5.1 the general form of the weak interaction Lagrangian $\mathcal{L}_W$ is that of a current-current interaction; e.g. for the neutrino-electron scattering $\mathcal{L}_W$ is given by (5.8)

$$\begin{aligned}\mathcal{L}_W^n(x) &= -\frac{G_F}{\sqrt{2}}\bar{\nu}_e\gamma_\lambda\left(1+\gamma_5\right)e\,\bar{e}\gamma_\lambda\left(1+\gamma_5\right)\nu_e \\ &= \frac{4G_F}{\sqrt{2}}j_\lambda^+(x)j_\lambda^-(x) \quad . \qquad (9.7)\end{aligned}$$

The last version of $\mathcal{L}_W^n$ is reminiscent of the current-current interaction in electrodynamics where

$$\mathcal{L}_{em}(x) = \frac{\mathrm{e}^2}{4\pi}\int d^3x'\frac{1}{|\vec{x}-\vec{x}'|}\vec{j}(\vec{x})\cdot\vec{j}(\vec{x}') \quad . \qquad (9.8)$$

Formally, by replacing

$$\frac{\mathrm{e}^2}{4\pi}\frac{1}{|\vec{x}-\vec{x}'|} \longrightarrow \frac{4G_F}{\sqrt{2}}\delta^3(\vec{x}-\vec{x}') \qquad (9.9)$$

we obtain (9.7) from (9.7). The weak interaction (9.7) thus describes a zero range interaction.

This latter can also be written as the potential resulting from the exchange of a very heavy meson W of mass m

$$V\left(|\vec{x}-\vec{x}'|\right) = \frac{g^2}{4\pi}\frac{1}{|\vec{x}-\vec{x}'|}\mathrm{e}^{-m|\vec{x}-\vec{x}'|} \quad . \qquad (9.10)$$

Here g is the coupling constant corresponding to the charge e in the Coulomb case. Taking the limit $m \to \infty$ and using the general relation

$$\lim_{m\to\infty} \frac{\mathrm{e}^{-m|\vec{x}|}}{|\vec{x}|} \longrightarrow \frac{4\pi}{m^2}\delta^3(\vec{x}) \tag{9.11}$$

yields

$$\begin{aligned} \mathcal{L}_W &= \int d^3x'\, V(|\vec{x}-\vec{x}'|) j_\lambda^+(\vec{x}) j_\lambda^-(\vec{x}') \\ &\overset{m\to\infty}{\rightarrow} \frac{g^2}{m^2} J_\lambda^+(x) j_\lambda^-(x) \end{aligned} \tag{9.12}$$

and thus

$$\frac{4G_F}{\sqrt{2}} = \frac{g^2}{m^2} \quad . \tag{9.13}$$

Assuming, guided by the similarity of the current-current interaction to that in electrodynamics, $g^2 \approx \mathrm{e}^2$, and taking for G_F the value given in section 5.1

$$G_F \approx 1 \cdot 10^{-5}/m_p^2 \tag{9.14}$$

gives

$$m \approx \mathcal{O}(100)\ \mathrm{GeV} \quad , \tag{9.15}$$

corresponding to a range of $d \approx 1/m \approx 2{\cdot}10^{-3}$ fm, which would be consistent with all experimental observations.

- That eq. (9.10) really describes the potential due to the exchange of a heavy meson can be seen by deriving the propagator, i. e. the momentum-space representation of the potential, for vector-bosons. Such bosons obey the Proca-equation (2.26)

  $$\partial^\nu \partial_\nu W_\mu - \partial_\mu \partial^\nu W_\nu + m^2 W_\mu = j_\mu \quad , \tag{9.16}$$

  where j_μ now constitutes the source term. By taking the divergence of both sides of eq. (9.16) we get

  $$m^2 \partial^\mu W_\mu = \partial^\mu j_\mu \quad . \tag{9.17}$$

  Therefore, eq. (9.16) becomes

  $$\begin{aligned} \partial^\nu \partial_\nu W_\mu + m^2 W_\mu &= j_\mu + \partial_\mu \partial^\nu W_\nu \\ &= j_\mu + \partial_\mu \left(\frac{1}{m^2} \partial^\nu j_\nu \right) \\ &= \left(\delta_{\mu\nu} + \frac{1}{m^2} \partial_\mu \partial^\nu \right) j_\nu \quad . \end{aligned} \tag{9.18}$$

> The component W_μ thus fulfills a Klein-Gordon equation with a modified source-term. For the Klein-Gordon-equation the momentum-space representation of the propagator[2] reads
>
> $$D(q) = \frac{\mathrm{i}}{q^2 - m^2 + \mathrm{i}\epsilon} \tag{9.19}$$
>
> so that the propagator for the Proca equation assumes the form
>
> $$D_{\mu\nu}(q) = \frac{-\mathrm{i}}{q^2 - m^2 + \mathrm{i}\epsilon}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{m^2}\right) \quad . \tag{9.20}$$
>
> In normal space this propagator indeed leads to a potential of the form (9.10).

With the propagator (9.20) we are now prepared to calculate again the cross section for the inverse muon decay,

$$\nu_\mu + e \to \mu + \nu_e \quad ,$$

mentioned at the beginning of this section. In a diagrammatic representation this process can be represented by the Feynman graph shown in Fig. 9.1; the calculation of this graph is given in detail in [SCH88,QUI83]. For $q^2 \ll m^2$ the propagator reduces to a constant in momentum space, i.e. a point-like interaction, so that the low-energy behavior of this theory is that of Fermi's theory. For $q^2 \gg m^2$, however, the interaction is cut off and the high-energy behaviour of the cross-section improves significantly [SCH88, QUI83].

These considerations show that Fermi's theory of weak interactions could be considered as the limiting case of a theory in which,like in electrodynamics, the forces and interactions are mediated by the exchange of vector mesons. These mesons must fulfill at least one obvious requirement: they have to exist both in charged and in neutral varieties since both charged- and neutral-current weak processes exist. Furthermore, these mesons must be quite heavy as just discussed.

Unfortunately, the unitarity problem reappears as soon as processes are considered in which the vector mesons appear as outgoing particles as, for example, in $\nu_e + \bar{\nu}_e \to W^+ + W^-$. Contrary to the corresponding QED process of annihilation ($e^- + e^+ \to \gamma + \gamma$), the cross section here diverges again quadratically with the c.m. energy; this divergence is entirely due to the longitudinal polarization degree of freedom of the massive W-bosons, which does not exist for the massless photon in the corresponding QED process [QUI83].

---

[2] Remember that the propagator (= Green's function) equals the potential of a point charge localized at $x = 0$.

The higher order diagrams for this scattering process will all diverge with even higher powers of s because of the term $q_\mu q_\nu$ in the vector meson propagator that introduces new divergences in each order of perturbation theory. The model formulated so far is thus clearly not renormalizable.

Some guidance for a possible solution of this puzzle may be obtained by looking at the process $e^+ + e^- \to \gamma + \gamma$ which also involves two outgoing vector particles. Here the transition amplitude stays finite because of a cancellation of divergent terms in the direct and the crossed diagram. Although here, of course, no longtitudinal polarizations appear because of gauge invariance this cancellation mechanisms nevertheless gives a hint that also for the weak theory convergence could possibly be achieved if a crossed diagram, this time necessarily with a different – still to be introduced – massive particle appears.

9.3 Fundamentals of a Theory of Weak Interactions

In the preceding section we have modeled the theory of weak interactions after the theory of quantum electrodynamics, an Abelian gauge field theory. That the coupling constants between, and therefore the "weak charges" of, hadrons and leptons are the same suggests that there is a non-Abelian gauge field theory underlying the weak interactions since such theories possess the property of universality as we have discussed in section 7.3.2.

We will therefore now proceed to construct a non-Abelian gauge-field theory for the weak interaction. In doing so we are immediately faced with two seemingly contradictory facts:

1. In gauge field theories the field quanta that transmit the interactions must be massless (cf. our discussion in chapter 7). Furthermore, the theory cannot involve massive vector mesons because then it would violate unitarity at very high energies and would not be renormalizable.
2. The phenomenology of β-decay indicates an extremely short-ranged interaction that requires a very large mass for the transmitting vector bosons. The presence of charged and neutral currents requires the existence of these vector bosons in three charge states.

The resolution of this problem is provided by the phenomenon of spontaneous symmetry breaking discussed in chapters 6 and 8. There it was shown how this mechanism could be used to generate masses for interacting particles while simultaneously respecting the *global* symmetries of the Lagrangian that required massless particles. The same phenomenon is needed here. In chapter 8 we have shown that several new features emerge when the spontaneous symmetry breaking takes place in non-Abelian gauge field theories. These results we will use in the next chapter to construct a theory of weak interactions, based on a non-Abelian gauge field theory.

Chapter 10

WEAK INTERACTIONS OF LEPTONS

We will now use the knowledge gained from the model-building in chapter 8 to construct a theory of the weak and the electromagnetic interactions of physical particles. In this first chapter we will consider only the interactions between leptons, restricting for simplicity ourselves first to a treatment of the electron and its neutrino.

10.1 Leptonic Multiplets and Interactions

In section 8.2 we have seen that a locally symmetric $SU(2) \otimes U(1)$ gauge field theory that allows for spontaneous symmetry breaking leads to desirable properties in the Higgs and gauge boson sector. It is, therefore, tempting to use this model as a basis for the theoretical description of weak interactions between the leptons.

If we want to maintain the mentioned symmetry in the presence of leptons we have to specify the transformation behavior of the leptons, in other words, we have to assign the leptons to $SU(2)$ multiplets. As guidance for this assignment we can use the observation discussed in chapter 5 that in the phenomenological theory of weak interactions the charged currents can be expressed in terms of the $SU(2)$ generators $t_{\pm}$ acting on the lepton spinor

$$L_e = \begin{pmatrix} \nu_e \\ e \end{pmatrix} \quad . \tag{10.1}$$

We, therefore, assume that L_e forms the $SU(2)$ doublet of the theory. This can obviously be correct only if electron and neutrino have the same mass, which they do not have in nature. We postpone this problem here, keeping in mind that our theory can be correct only at very high energies $\gg m_e$.

There is still another problem with the ansatz (10.1). According to our present knowledge the neutrino is massless and appears only in the left-handed state. Thus we have

$$\nu_L = \frac{1}{2}(1-\gamma_5)\nu = \nu \quad . \tag{10.2}$$

An $SU(2)$ rotation would not change the helicity so that only a left-handed electron could be produced. We thus have to assume that the $SU(2)$ doublet is not described by (10.1) but instead by

$$L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(1-\gamma_5)\nu \\ \frac{1}{2}(1-\gamma_5)e \end{pmatrix} . \tag{10.3}$$

Since a right-handed neutrino does not exist, the right-handed part of the electron is then assumed to form an $SU(2)$ singlet

$$R = e_R = \frac{1}{2}(1+\gamma_5)e \quad . \tag{10.4}$$

Note that

$$e = e_L + e_R \quad . \tag{10.5}$$

The Lagrangian can now be written down in analogy to the considerations in section 8.2 by adding the properly gauged term for the fermions. It then reads

$$\begin{aligned} \mathcal{L} &= \bar{L}\mathrm{i}\gamma^\mu D_\mu L + \bar{R}\mathrm{i}\gamma^\mu D_\mu R + (D^\mu\phi)^\dagger(D_\mu\phi) \\ &\quad - V(\phi^\dagger\phi) - \frac{1}{4}\mathbf{G}^{\mu\nu}\cdot\mathbf{G}_{\mu\nu} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \end{aligned} \tag{10.6}$$

with

$$V(\phi^\dagger\phi) = \frac{1}{2}\mu^2\phi^\dagger\phi + \frac{1}{2}\lambda^2(\phi^\dagger\phi)^2 \quad . \tag{10.7}$$

This Lagrangian describes a system of fermions (L, R) and self-interacting Higgs particles (ϕ) in the presence of $SU(2)\otimes U(1)$ gauge fields.

Before we now come to a discussion of the interaction Lagrangian generated by the covariant derivatives we briefly discuss the symmetries of the vacuum state which we take as in section 8.2 as

$$\phi_0 = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \quad . \tag{10.8}$$

for $\mu^2 < 0$. It is obvious that this state is not invariant under any $SU(2)$ transformation nor is it invariant under the group $U(1)_Y$ since the generators of both groups do not annihilate the vacuum. However, a special combination of generators does annihilate ϕ_0

$$\left(\frac{1}{2}y + t_3\right)\phi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\phi_0 = 0 \quad . \tag{10.9}$$

The full symmetry group has thus spontaneously been broken down to the gauge group of electromagnetism, $U(1)_{em}$, generated by the transformation

$$U_{em} = \mathrm{e}^{-\mathrm{i}\epsilon(x)(y+\tau_3)/2} = \mathrm{e}^{-\mathrm{i}\epsilon(x)q} \quad , \tag{10.10}$$

where q is the electrical charge of the field acted upon by U. This is just the symmetry group of QED; the corresponding gauge boson will remain massless whereas the other three acquire a mass due to the Higgs mechanism. We will, therefore, later identify the massless gauge bosons with the photons.

Since we have introduced two different fermion multiplets, L and R, the Lagrangian is invariant under the two different $U(1)$ transformations acting separately on L and on R. Gauging those would generate two massless gauge bosons, contrary to experiment. We therefore — rather arbitrarily — assume the existence of a common $U(1)$ symmetry

$$\begin{aligned} L &\rightarrow \mathrm{e}^{-\mathrm{i}y_L\epsilon}L \\ R &\rightarrow \mathrm{e}^{-\mathrm{i}y_R\epsilon}R, \end{aligned} \tag{10.11}$$

where y_R and y_L are the charges (fixed numbers) to be determined later and the parameters ϵ are the same in both transformations. Similarly, we act with the $U(1)$ transformation on the Higgs field ϕ

$$\phi \rightarrow \mathrm{e}^{-\mathrm{i}y_\phi\epsilon}\phi \quad . \tag{10.12}$$

The generating charge operator connected to this symmetry is then

$$\begin{aligned} Y &= y_L \int L^\dagger L\, d^3x + y_R \int R^\dagger R\, d^3x \\ &+ y_\phi\, \mathrm{i} \int \left(\phi^\dagger \frac{\partial\phi}{\partial t} - \frac{\partial\phi^\dagger}{\partial t}\phi\right) d^3x \end{aligned} \tag{10.13}$$

and the y_L, y_R and y_ϕ are, as usual, the eigenvalues of Y.

Since we work with a common $U(1)$ symmetry there is also only one common gauge field connected with it. The covariant derivative therefore reads

$$D_\mu = \partial_\mu + \mathrm{i}g\frac{\tau^l}{2}W^l_\mu + \mathrm{i}g'\frac{y_f}{2}B_\mu \tag{10.14}$$

(see eq. 8.15); it defines the coupling between the leptons and the gauge bosons. Note here the extra factor y_f in the last term that depends on the field on which D_μ acts. This has to be introduced because, when gauging the Abelian $U(1)$ symmetry, the various fields, in this case L, R and ϕ, can couple with different coupling strengths. We thus have to allow for different $g's$, or, in other words, for different factors y_f, in $D_\mu L$, $D_\mu R$ and $D_\mu \phi$. The factors $g' y_f$ give just the various conserved charges connected with the $U(1)$ symmetry.

These factors can be obtained from considering the covariant derivative D_μ in the lepton sector

$$D_\mu L = (\partial_\mu + \mathrm{i} g \frac{\tau^l}{2} W_\mu^l + \mathrm{i} g' \frac{y_L}{2} B_\mu) L \quad . \tag{10.15}$$

Rewriting this in terms of the fields Z_μ and A_μ (cf. eqs. 8.37, 8.38) yields

$$D_\mu L = \left(\partial_\mu + \mathrm{i} g \sin \vartheta_W \frac{1}{2} \left(y_L + \tau_3 \right) A_\mu \right) L + \ldots \quad . \tag{10.16}$$

Thus, if we are aiming at identifying the massless field A_μ with the electromagnetic field, then we have to identify the factor in front of A_μ with the electric charge of the field involved. In other words, here a relation between charge-number and isospin has to hold

$$q = \frac{1}{2} y_L + t_3 \quad , \tag{10.17}$$

with $t_3 = \tau_3/2$. Eq. (10.17) is formally the same as the Gell-Mann-Nishijima relation known from $SU(3)$ (cf. eq. (3.81)). This similarity has led to the name "weak hypercharge" for y which, however, appears here as a consequence of $SU(2) \otimes U(1)$, and not of $SU(3)$. We now assume that this relation holds not only for the $SU(2)$ doublet L, but also for the singlet R and the Higgs doublet ϕ, and that it reflects a general connection between conserved operators. This result is a direct consequence of the fact that we have taken $SU(2)_L \otimes U(1)_Y$ as the relevant local symmetry group of our problem.

Relation (10.17) allows us to determine the weak hypercharges from the known charges and weak-isospin quantum numbers. For the left-handed doublet we obtain

$$\begin{array}{llll} \nu_e : & t_3 = +\frac{1}{2}; & q = 0; & y = -1 \\ e_L : & t_3 = -\frac{1}{2}; & q = -1; & y = -1 \end{array} \quad .$$

The y quantum numbers must, of course, be the same for both left-handed leptons because their spinor is affected by $U(1)_Y$ as a whole.

In exactly the same way y can be assigned for the right-handed electron field e_R which forms an $SU(2)$ singlet

$$e_R : t_3 = 0; \quad q = -1; \quad y = -2 \quad .$$

The Higgs field forms a weak $SU(2)$ doublet just like L; its isospin-components must therefore have $t_3 = \pm 1/2$. Since the particles corresponding to the field ϕ have not been observed so far, we do not know their charges. However, if we require that the physical vacuum of the theory (eq. (10.8) should be uncharged, then the lower component of the $SU(2)$ doublet ϕ must have $q = 0$. This gives, using eq. (10.17), for the Higgs field

$$\begin{array}{llll} \phi_1 : & t_3 = +\frac{1}{2}; & q = +1; & y = +1 \\ \phi_2 : & t_3 = -\frac{1}{2}; & q = 0; & y = +1 \end{array} \quad .$$

We now return to a discussion of the Lagrangian (10.6) which can be written as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int} + \mathcal{L}_{Higgs} \quad . \tag{10.18}$$

Here $\mathcal{L}_0$ is the Lagrangian describing free leptons and free gauge fields, $\mathcal{L}_{int}$ contains all the interactions between leptons and gauge fields and $\mathcal{L}_{Higgs}$ contains all terms involving the Higgs field ϕ. According to the general philosophy of gauge field theories all interactions between the physical particles and the gauge fields are generated by the covariant derivatives. These are given by

$$\begin{aligned} D_\mu L &= \left(\partial_\mu + \mathrm{i} g \frac{\tau^l}{2} W_\mu^l - \mathrm{i} g' \frac{1}{2} B_\mu \right) L \\ D_\mu R &= \left(\partial_\mu - \mathrm{i} g' B_\mu \right) R \\ D_\mu \phi &= \left(\partial_\mu + \mathrm{i} g \frac{\tau^l}{2} W_\mu^l + \mathrm{i} g' \frac{1}{2} B_\mu \right) \phi \quad . \end{aligned} \tag{10.19}$$

There is no term $\sim g$ in $D_\mu R$ because of $\tau^l R = 0$ since R is an $SU(2)$ singlet.

The absence of a right-handed neutrino is automatically accounted for by this theory, if this neutrino is also assigned to an $SU(2)$ singlet. Using (10.17) we obtain with $q = 0, t_3 = 0$ also $y = 0$ so that a singlet right-handed neutrino would couple neither to the $U(1)$ field nor to the $SU(2)$ field.

We now turn to an explicit calculation of the interaction terms generated by the covariant derivatives. The interaction Lagrangian is given by

$$\begin{aligned} \mathcal{L}_{int} &= \bar{L} \gamma^\mu \left(-g \frac{\tau^l}{2} W_\mu^l + g' \frac{1}{2} B_\mu \right) L + \bar{R} \gamma^\mu (g' B_\mu) R \\ &= -\frac{g}{2} \bar{L} \gamma^\mu \tau^l W_\mu^l L + \frac{g'}{2} \bar{L} \gamma^\mu B_\mu L + g' \bar{R} \gamma^\mu B_\mu R \quad . \end{aligned} \tag{10.20}$$

With

$$\begin{aligned}
\bar{L}\tau^1 L &= (\bar{\nu} \quad \bar{e})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \nu \\ e \end{pmatrix} \\
&= (\bar{\nu} \quad \bar{e})\begin{pmatrix} e \\ \nu \end{pmatrix} = \bar{\nu}e + \bar{e}\nu \quad , \\
\bar{L}\tau^2 L &= (\bar{\nu} \quad \bar{e})\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} \nu \\ e \end{pmatrix} \\
&= (\bar{\nu} \quad \bar{e})\begin{pmatrix} -\mathrm{i}e \\ \mathrm{i}\nu \end{pmatrix} = -\mathrm{i}(\bar{\nu}e - \bar{e}\nu) \\
\bar{L}\tau^3 L &= \bar{\nu}\nu - \bar{e}e
\end{aligned} \tag{10.21}$$

(the index L in the spinor components has been dropped for convenience) we obtain

$$\begin{aligned}
\mathcal{L}_{int} = & -\frac{g}{2}\left[\left(\bar{\nu}_L\gamma^\mu e_L + \bar{e}_L\gamma^\mu\nu_L\right)W^1_\mu - \mathrm{i}\left(\bar{\nu}_L\gamma^\mu e_L - \bar{e}_L\gamma^\mu\nu_L\right)W^2_\mu\right] \\
& - \frac{g}{2}\left(\bar{\nu}_L\gamma^\mu\nu_L - \bar{e}_L\gamma^\mu e_L\right)W^3_\mu \\
& + \frac{g'}{2}\left(\bar{\nu}_L\gamma^\mu\nu_L + \bar{e}_L\gamma^\mu e_L\right)B_\mu \\
& + g'\bar{e}_R\gamma^\mu e_R B_\mu \quad .
\end{aligned} \tag{10.22}$$

We now rewrite this Lagrangian in such a form that the physically relevant fields, to which the various currents couple, become obvious

$$\begin{aligned}
\mathcal{L}_{int} = & -\frac{g}{2}\left(\bar{\nu}_L\gamma^\mu e_L\right)\left(W^1_\mu - \mathrm{i}W^2_\mu\right) \\
& - \frac{g}{2}\left(\bar{e}_L\gamma^\mu\nu_L\right)\left(W^1_\mu + \mathrm{i}W^2_\mu\right) \\
& - \frac{1}{2}\left(\bar{\nu}_L\gamma^\mu\nu_L\right)\left(gW^3_\mu - g'B_\mu\right) \\
& + \frac{1}{2}\left(\bar{e}_L\gamma^\mu e_L\right)\left(gW^3_\mu + g'B_\mu\right) \\
& + g'\bar{e}_R\gamma^\mu e_R B_\mu \quad .
\end{aligned} \tag{10.23}$$

This form of $\mathcal{L}_{int}$ shows clearly that the "charged current" $\bar{\nu}_L\gamma_\mu e_L$ does not couple to W^1_μ and W^2_μ, but instead to the charged field

$$W_\mu = \frac{1}{\sqrt{2}}\left(W^1_\mu - \mathrm{i}W^2_\mu\right) \quad . \tag{10.24}$$

W_μ is a field of positively charged W-bosons, $W_\mu^\dagger$ that of negatively charged ones. Note that these two combinations also form the $t_3 = \pm 1$ members of a $t = 1$ iso-triplet (see the discussion following eq. (7.28b)).

We next perform the transformation (8.35), (8.36) from the fields W_μ^3 and B_μ to the fields A_μ and Z_μ

$$\begin{aligned} gW_\mu^3 - g'B_\mu &= \frac{g}{\cos\vartheta_W} Z_\mu \\ gW_\mu^3 + g'B_\mu &= \frac{g}{\cos\vartheta_W}\left[2\cos\vartheta_W \sin\vartheta_W A_\mu + \left(1 - 2\sin^2\vartheta_W\right) Z_\mu\right] \ . \end{aligned} \tag{10.25}$$

This transformation gives

$$\begin{aligned} \mathcal{L}_{int} &= -\frac{g}{\sqrt{2}}\left(\bar\nu_L\gamma^\mu e_L W_\mu + h.c.\ \right) \\ &\quad - \frac{1}{2}\frac{g}{\cos\vartheta_W}\left(\bar\nu_L\gamma^\mu\nu_L\right) Z_\mu \\ &\quad + \frac{1}{2}\frac{g}{\cos\vartheta_W}\left(\bar e_L\gamma^\mu e_L\right) \\ &\qquad \times \left[2\sin\vartheta_W\cos\vartheta_W\, A_\mu + \left(1 - 2\sin^2\vartheta_W\right) Z_\mu\right] \\ &\quad + g\tan\vartheta_W\left(\bar e_R\gamma^\mu e_R\right)\left(-\sin\vartheta_W\, Z_\mu + \cos\vartheta_W A_\mu\right) \\ &= -\frac{g}{\sqrt{2}}\left(\bar\nu_L\gamma^\mu e_L W_\mu + h.c.\ \right) \\ &\quad + g\sin\vartheta_W\left(\bar e_L\gamma^\mu e_L + \bar e_R\gamma^\mu e_R\right) A_\mu \\ &\quad - \frac{1}{2}\frac{g}{\cos\vartheta_W}\left(\bar\nu_L\gamma^\mu\nu_L\right) Z_\mu \\ &\quad + g\frac{1 - 2\sin^2\vartheta_W}{2\cos\vartheta_W}\left(\bar e_L\gamma^\mu e_L\right) Z_\mu - g\frac{\sin^2\vartheta_W}{\cos\vartheta_W}\left(\bar e_R\gamma^\mu e_R\right) Z_\mu \ . \end{aligned} \tag{10.26}$$

Note that because the fields W_μ^3 and B_μ are uncharged also Z_μ and A_μ carry zero electrical charge.

We remember now that the spontaneous symmetry breaking process made Z_μ massive, whereas A_μ remained massless, corresponding to the existence of one generator of the $SU(2)_L \otimes U(1)_Y$ group that leaves the vacuum invariant $(1/2y + t_3)$. The massless vector field A_μ is now identified with the electromagnetic field. With this identification the neutrino-current should couple with zero charge to A_μ, whereas the electron should couple with the charge −e. Indeed, no term coupling the electrically neutral neutrino and the electromagnetic field appears in (10.26).

The electron current terms coupling to A_μ read

$$g\sin\vartheta_W\left(\bar e_L\gamma_\mu e_L + \bar e_R\gamma_\mu e_R\right) A^\mu \quad . \tag{10.27}$$

The sum of left-handed and right-handed currents gives

$$\begin{aligned}\bar{e}_L\gamma_\mu e_L + \bar{e}_R\gamma_\mu e_R &= \frac{1}{4}\Big[e^\dagger(1-\gamma_5)\gamma_0\gamma_\mu(1-\gamma_5)e \\ &\quad + e^\dagger(1+\gamma_5)\gamma_0\gamma_\mu(1+\gamma_5)e\Big] \\ &= \bar{e}\gamma_\mu e \quad , \end{aligned} \tag{10.28}$$

so that the electromagnetic coupling term in (10.25) finally reads

$$\mathrm{e}\,(\bar{e}\gamma_\mu e)\,A^\mu \quad , \tag{10.29}$$

i.e. just like the normal current-field coupling of QED, if we identify the electric charge with

$$\mathrm{e} = g\sin\vartheta_W \quad . \tag{10.30}$$

Next, the terms coupling to the field Z_μ are rewritten. We obtain for these terms (eq. (10.26))

$$\begin{aligned} & -\frac{1}{2}\frac{g}{\cos\vartheta_W}\Big[\,(\bar{\nu}_L\gamma_\mu\nu_L) \\ & - g\left(1-2\sin^2\vartheta_W\right)(\bar{e}_L\gamma_\mu e_L) + 2g\sin^2\vartheta_W\,(\bar{e}_R\gamma_\mu e_R)\Big]\,Z^\mu \\ = \; & -\frac{g}{2\cos\vartheta_W}\frac{1}{4}\Big[\nu^\dagger(1-\gamma_5)\gamma_0\gamma_\mu(1-\gamma_5)\nu \\ & - \left(1-2\sin^2\vartheta_W\right)e^\dagger(1-\gamma_5)\gamma_0\gamma_\mu(1-\gamma_5)e \\ & + 2\sin^2\vartheta_W\,e^\dagger(1+\gamma_5)\gamma_0\gamma_\mu(1+\gamma_5)e\Big]\,Z^\mu \\ = \; & -\frac{g}{4\cos\vartheta_W}\Big[\bar{\nu}\gamma_\mu(1-\gamma_5)\nu - \left(1-2\sin^2\vartheta_W\right)\bar{e}\gamma_\mu(1-\gamma_5)e \\ & + 2\sin^2\vartheta_W\,\bar{e}\gamma_\mu(1+\gamma_5)e\Big]\,Z^\mu \quad . \end{aligned} \tag{10.31}$$

Collecting all the various terms, i.e. the charged current interaction from the first line of eq. (10.26), the electron's electromagnetic current from eq. (10.29) and the neutral current from eq. (10.31), the total interaction Lagrangian $\mathcal{L}_{int}$ of eq. (10.26) becomes

$$\begin{aligned}\mathcal{L}_{int} &= -\frac{g}{\sqrt{2}}\left(\frac{1}{2}\bar{\nu}\gamma_\mu(1-\gamma_5)eW^\mu + h.c.\right) \\ &\quad + \mathrm{e}\,\bar{e}\gamma_\mu e A^\mu \\ &\quad - \frac{g}{\cos\vartheta_W}\left[\frac{1}{2}\bar{\nu}\gamma_\mu\nu - \frac{1}{4}\left(1-2\sin^2\vartheta_W\right)\bar{e}\gamma_\mu(1-\gamma_5)e \right. \\ &\quad\quad \left. + \frac{1}{2}\sin^2\vartheta_W\bar{e}\gamma_\mu(1+\gamma_5)e\right]Z^\mu \quad . \end{aligned} \tag{10.32}$$

Here we have used $\nu_L = \frac{1}{2}(1-\gamma_5)\nu = \nu$.

We now write this interaction Lagrangian in a more compact form by defining

$$l_1 = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \quad \text{and} \quad l_2 = e_R \quad . \tag{10.33}$$

With the help of l_ℓ we can define the *charged current*

$$j_\mu^+(x) = \frac{1}{2}\bar{\nu}\gamma_\mu(1-\gamma_5)e = \sum_{\ell=1}^{2} \bar{l}_\ell \gamma_\mu t_+ l_\ell \quad , \tag{10.34}$$

the electron's *electromagnetic current*

$$\begin{aligned} j_\mu^{el} &= \bar{e}\gamma_\mu e = -\sum_\ell \bar{l}_\ell \gamma_\mu \left(\frac{1}{2}y + t_3\right) l_\ell \\ &= -\sum_\ell \bar{l}_\ell \gamma_\mu q l_\ell \quad , \end{aligned} \tag{10.35}$$

and the *neutral current*

$$\begin{aligned} j_\mu^0 &= \frac{1}{2}\bar{\nu}\gamma_\mu\nu - \frac{1}{4}(1-2\sin^2\vartheta_W)\bar{e}\gamma_\mu(1-\gamma_5)e \\ &\quad + \frac{1}{2}\sin^2\vartheta_W \bar{e}\gamma_\mu(1+\gamma_5)e \\ &= \sum_\ell \bar{l}_\ell \gamma_\mu \left(t_3 - \left(\frac{1}{2}y + t_3\right)\sin^2\vartheta_W\right) l_\ell \\ &= \sum_\ell \bar{l}_\ell \gamma_\mu \left(t_3 - q\sin^2\vartheta_W\right) l_\ell \quad . \end{aligned} \tag{10.36}$$

Here q is the charge operator in the internal space (10.17) and the sums run both over the lefthanded and the righthanded lepton multiplets.

Note that there are axial currents hidden in (10.34) and (10.36). This becomes obvious when evaluating the effect of the t_3-operator, which acts differently on the left-handed and right-handed fermions, in the sum of a left-handed doublet and a right-handed singlet

$$\begin{aligned} \sum_{\ell=L,R} \bar{l}_\ell t_3 \gamma_\mu l_\ell &= (\bar{\nu}_L \quad \bar{e}_L) t_3 \gamma_\mu \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} + \bar{e}_R t_3 \gamma_\mu e_R \\ &= (\bar{\nu}_L \quad \bar{e}_L) t_3 \gamma_\mu \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \\ &= \frac{1}{2}\left(\bar{\nu}\gamma_\mu\nu - \bar{e}_L\gamma_\mu e_L\right) \quad . \end{aligned} \tag{10.37}$$

The axial operators γ_5 are contained in the left-handed fields.

With the currents (10.34), (10.35), and (10.36) we can write

$$\begin{aligned}\mathcal{L}_{int} &= \mathrm{e}\left(j^{el}_\mu A^\mu - \frac{1}{\sqrt{2}\sin\vartheta_W}\left(j^+_\mu W^\mu + h.c.\right)\right.\\ &\left.\quad - \frac{1}{\sin\vartheta_W\cos\vartheta_W} j^0_\mu Z^\mu\right) \quad . \end{aligned} \tag{10.38}$$

This is the central expression of the theory developed so far. It contains an interaction between a charged current and the charged vector mesons W and $W^\dagger$, the electromagnetic interaction and a new term that was not present in Fermi's theory and that describes the coupling of a neutral current to an uncharged boson field Z_μ. We will discuss various physical processes mediated by these interactions in the following section.

10.1.1 Lepton Masses

There is a final problem to solve, namely that of the electron mass, which so far had to be assumed to vanish so that we could form the basic isodoublet consisting of neutrino and electron. We now show that also the electron mass can be generated by spontaneous symmetry breaking thus respecting all the symmetries of the Lagrangian.

This can be achieved by simply adding a Yukawa-coupling term of the form

$$\mathcal{L}_Y = -g_e\left(\bar{R}\phi^\dagger L + \bar{L}\phi R\right) \quad . \tag{10.39}$$

This interaction has all the desired symmetries. Since (with $U = U_{SU(2)}$)

$$\begin{aligned} URU^{-1} &= R \\ ULU^{-1} &= \mathrm{e}^{-\mathrm{i}\epsilon^l\frac{\tau^l}{2}} L \\ U\phi U^{-1} &= \mathrm{e}^{-\mathrm{i}\epsilon^l\frac{\tau^l}{2}}\phi \quad , \end{aligned} \tag{10.40}$$

we have

$$U\bar{R}\phi^\dagger L U^{-1} = \bar{R}\phi^\dagger \mathrm{e}^{+\mathrm{i}\epsilon^l\frac{\tau^l}{2}}\mathrm{e}^{-\mathrm{i}\epsilon^l\frac{\tau^l}{2}} L = \bar{R}\phi^\dagger L \quad . \tag{10.41}$$

In the same way we can show the $U(1)$ invariance ($U = U_{U(1)}$)

$$U\bar{R}\phi^\dagger L U^{-1} = \bar{R}\mathrm{e}^{-2\mathrm{i}\epsilon}\phi^\dagger\mathrm{e}^{+\mathrm{i}\epsilon}\mathrm{e}^{+\mathrm{i}\epsilon} L = \bar{R}\phi^\dagger L \tag{10.42}$$

using the correct charges for each field.

From the vacuum value for the Higgs field ϕ we now obtain

$$\mathcal{L}_Y = -g_e \frac{v}{\sqrt{2}} \bar{e}e \left(1 + \frac{\varphi}{v}\right) \quad , \tag{10.43}$$

from which we can read off the mass of the electron as

$$m_e = g_e \frac{v}{\sqrt{2}} \quad . \tag{10.44}$$

10.1.2 Generalization to other Leptons

The generalization of the theory to the other known leptons μ and τ and their neutrinos is straightforward. These can be incorporated by defining spinor representations also for the other families of leptons such that the left-handed particles form weak $SU(2)$ doublets and the right-handed particles form $SU(2)$ singlets

$$l_\ell = \begin{pmatrix} \nu_{\ell L} \\ \ell_L \end{pmatrix} \quad \text{or} \quad l_\ell = \ell_R \quad , \tag{10.45}$$

and summing over all leptons $\ell = e$, μ and τ in all the currents. The total Lagrangian is then

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_H + \sum_{leptons} \mathcal{L}_Y + \sum_{leptons} \mathcal{L}_{int} \quad , \tag{10.46}$$

where $\mathcal{L}_0$ describes the free fermions and the free gauge bosons, $\mathcal{L}_H$ the Higgs field and its interactions, $\mathcal{L}_Y$ the mass-generating Yukawa-terms for all the leptons and $\mathcal{L}_{int}$ finally the interactions between the leptons and the gauge bosons.

10.2 Types of Interactions

In this section we will discuss all the various interactions contained in the theory that we have developed so far. For this purpose we first write the total Lagrangian of the model considered so far by collecting the various terms from (8.29), (8.39), (8.40), (10.43) and (10.44).

$\mathcal{L}$ is given by

$$\begin{aligned} \mathcal{L} \quad = \quad & -\frac{1}{4}\mathbf{G}^{\mu\nu} \cdot \mathbf{G}_{\mu\nu} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ & + M_W^2 W_\mu^\dagger W^\mu \left(1 + \frac{\varphi}{v}\right)^2 + \frac{1}{2}M_Z^2 Z_\mu Z^\mu \left(1 + \frac{\varphi}{v}\right)^2 \end{aligned}$$

$$
\begin{aligned}
&+\bar{e}\gamma_\mu\partial^\mu e-m_e\bar{e}e\left(1+\frac{\varphi}{v}\right)+\bar{\nu}\gamma_\mu\partial^\mu\nu\\
&+\frac{1}{2}\left(\partial_\mu\varphi\right)\left(\partial^\mu\varphi\right)-\frac{1}{2}m_H^2\varphi^2\left(1+\frac{\varphi}{v}+\frac{1}{4}\left(\frac{\varphi}{v}\right)^2\right)\\
&-\frac{\mathrm{e}}{\sqrt{2}\sin\vartheta_W}\left(\frac{1}{2}\bar{\nu}\gamma_\mu(1-\gamma_5)eW^\mu+h.c.\right)\\
&+\mathrm{e}\,\bar{e}\gamma_\mu eA^\mu \qquad\qquad (10.47)\\
&-\frac{\mathrm{e}}{\sin\vartheta_W\cos\vartheta_W}\left(\frac{1}{4}\bar{\nu}\gamma_\mu(1-\gamma_5)\nu\right.\\
&\left.-\frac{1-2\sin^2\vartheta_W}{4}\bar{e}\gamma_\mu(1-\gamma_5)e+\frac{1}{2}\sin^2\vartheta_W\bar{e}\gamma_\mu(1+\gamma_5)e\right)Z^\mu
\end{aligned}
$$

with the Higgs boson mass $m_H=\sqrt{-\mu^2}$, the Weinberg angle

$$\cos\vartheta_W=\frac{M_W}{M_Z}\quad, \tag{10.48}$$

and the vacuum field

$$v=\frac{2M_W}{g}=\frac{2M_W}{\mathrm{e}}\sin\vartheta_W\quad. \tag{10.49}$$

Here only the electron and the electron's neutrino field are written out explicitly; the other leptons can be treated analogously. The parameters of this theory are thus

$$\mathrm{e},\sin\vartheta_W,m_e,M_W,m_H\quad. \tag{10.50}$$

The various interactions contained in this Lagrangian are

1. an interaction between the gauge bosons W_μ, Z_μ and A_μ contained in the nonlinear parts of the field tensor $G_{\mu\nu}$, (Fig. 10.1),
2. interactions between the gauge bosons W_μ and Z_μ and the Higgs field φ that are directly proportional to the square of the masses M_W and M_Z, respectively, (Fig. 10.2),
3. selfinteractions of the Higgs field φ, (lowest row in Fig. 10.2),
4. an interaction between the electron and the Higgs field that is proportional to the electron's mass, (Fig. 10.2),
5. an electromagnetic interaction between the electron-current and the electromagnetic field, (Fig. 10.3, upper left corner),

Figure 10.1: Couplings between gauge bosons. All these couplings originate in the kinetic terms involving the tensors $\mathbf{G}_{\mu\nu}$ and $F_{\mu\nu}$.

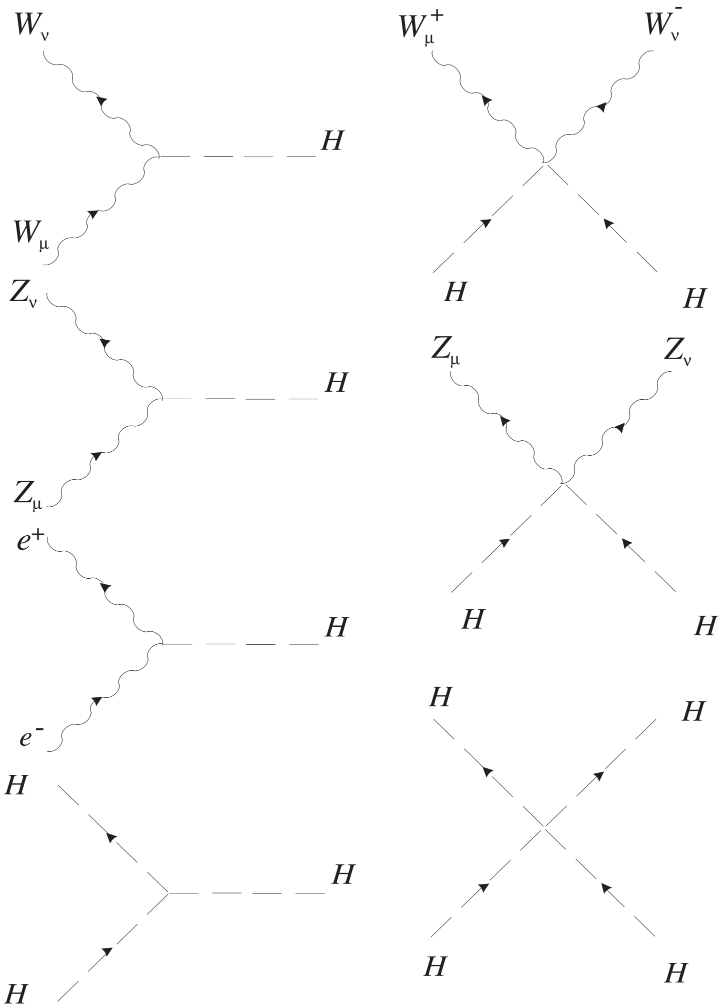

Figure 10.2: Couplings involving the Higgs boson.

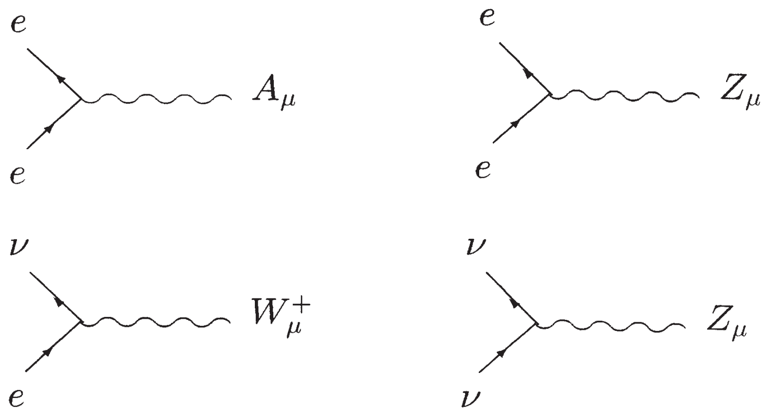

Figure 10.3: Coupling between fermions and gauge bosons.

6. and finally a new form of interaction that involves neutral currents coupling to the neutral gauge boson Z_μ with a strength that is fixed by the other processes. Notice also that this term contains couplings of the right-handed and of the left-handed component of the electron with a well defined ratio (cf. eq. (10.38)), (Fig. 10.3).

Higgs particles are proportional to m_e or $M^2_{W,Z}$, where m_e is the mass of the electron and M the mass of the gauge boson.

The last term in the Lagrangian (10.47) is represented by the graphs in the right column of Fig. 10.3. It involves, as already briefly mentioned above, a neutral current, which is charge-conserving, and does not appear in the classical Fermi theory of β-decay.

This neutral current contributes, for example, also to the neutrino-anti-neutrino annihilation discussed at the end of section 9.2. It can be shown [QUI83] that its contribution exactly cancels the diverging parts of the charged current graphs and thus leads to finite cross sections. Indeed, one can show [ITZ85] that this theory is then also renormalizable.

10.3 Parameters of the Lagrangian

In this section we will discuss how the parameters of the Lagrangian (10.47) can be determined.

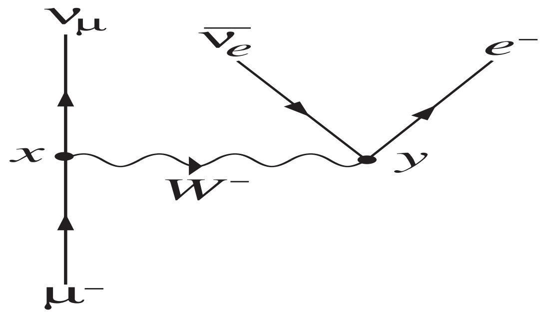

Figure 10.4: Lowest order Feynman graph for μ-decay.

10.3.1 Charged Current Experiments

In this section we determine the parameters (10.50) of the Lagrangian (10.47) by making contact with Fermi's theory of the weak interaction, which is empirically highly successful in its description of low-energy weak interaction processes. For this purpose we consider a process involving a charged current such as $\bar{\nu}\gamma_\mu(1-\gamma_5)e$. Such a current occurs, e.g. , in the muon decay, discussed in section 5.1, represented by the Feynman graph shown in Fig. 10.4.

Since this is a process of *second order* in the coupling, we obtain the S-matrix element as

$$\begin{aligned} S_{fi} &= \frac{(-\mathrm{i})^2}{2!} < \nu e| \int d^4x \int d^4y\, T\left(\mathcal{H}_{int}(x)\mathcal{H}_{int}(y)\right)|\nu e> \\ &= \left(\frac{g}{\sqrt{2}}\right)^2 \\ &\quad \times < f| \int d^4x \int d^4y\, j^{+\lambda}(x)\frac{(-\mathrm{i})}{(2\pi)^4}\int d^4q\, \frac{\mathrm{e}^{\mathrm{i}q(x-y)}}{q^2-M_W^2} j_\lambda^-(y)|i> \end{aligned} \qquad (10.51)$$

with the charged currents

$$\begin{aligned} j_\lambda^+(x) &= \frac{1}{2}\bar{\nu}_\mu\gamma_\lambda(1-\gamma_5)\mu \\ j_\lambda^-(x) &= \frac{1}{2}\bar{e}\gamma_\lambda(1-\gamma_5)\nu_e \quad . \end{aligned} \qquad (10.52)$$

For processes at low energies which involve only small momentum transfers

$$q^2 \ll M_W^2$$

this goes over into

$$S_{fi} = -\mathrm{i}\left(\frac{g}{\sqrt{2}}\right)^2 \frac{1}{M_W^2} \int d^4x\, j^{+\lambda}(x) j_\lambda^-(x) \quad . \tag{10.53}$$

In Fermi's theory the S-matrix element for the same process would read (now in *first order* perturbation theory)

$$S_{fi} = -\mathrm{i}\frac{4G_F}{\sqrt{2}} < f| \int d^4x\, j^{+\lambda}(x) j_\lambda^-(x) |i> \quad , \tag{10.54}$$

so that by choosing

$$\frac{4G_F}{\sqrt{2}} = \left(\frac{g}{\sqrt{2}}\right)^2 \frac{1}{M_W^2} \tag{10.55}$$

we would obtain the same transition rate as with the second order S-matrix element (10.53).

Since G_F is experimentally well determined, the ratio $(g/M_W)^2$ is thus known. Note that the "weak" interaction appears to be weak only because the true coupling constant is divided by the square of the large mass M_W. With

$$G_F = 1.03 \cdot 10^{-5}/m_p^2(\hbar c)^3 \tag{10.56}$$

we have

$$\left(\frac{g}{M_W}\right)^2 = \frac{8G_F}{\sqrt{2}} = 6.6 \cdot 10^{-5}(\hbar c)^3 \text{ GeV}^{-2} \quad . \tag{10.57}$$

This equation also explains quite naturally why Fermi's coupling constant G_F carries a dimension whereas, for example, the electromagnetic coupling constant is dimensionless.

Using the same arguments as employed above for the charged current now also for the neutral currents gives for the interaction Lagrangian

$$\begin{aligned} \mathcal{L}_{int} &= -\left(\frac{4G_F M_W^2}{\sqrt{2}}\right)^{\frac{1}{2}} \left(j_\mu^+ W^\mu + h.c.\right) \\ &\quad + \mathrm{e} j_\mu^{el} A^\mu \\ &\quad - \left(\frac{8G_F M_Z^2}{\sqrt{2}}\right)^{\frac{1}{2}} j_\mu^0 Z^\mu \end{aligned} \tag{10.58}$$

by using relation (10.55) between G_F and g. For the effective Lagrangian, valid at low energies, we get

$$\mathcal{L}_{eff} = -\frac{4G_F}{\sqrt{2}} \left(j^{+\mu} j_\mu^- + 2j^{0\mu} j_\mu^0\right) \tag{10.59}$$

10.3.2 Neutral current experiments

The Lagrangian (10.38) contains in addition to the known weak and electromagnetic interactions in its last line the coupling of a neutral current, i.e. a current in which the two field operators carry the same charge, to the uncharged massive boson field Z_μ.

Historically speaking, the presence of the neutral currents was first observed in 1973 in experiments such as

$$\bar{\nu}_\mu + e^- \rightarrow \bar{\nu}_\mu + e^- \quad .$$

The observation of this interaction impressively verified the prediction of the theory of electroweak interactions developed so far, since the process just mentioned can take place only if there are neutral currents.

We will now discuss one experimental evidence for the presence of this interaction, namely the asymmetry observed in the reaction

$$e^+ + e^- \rightarrow \mu^+ + \mu^-$$

that was observed in experiments at PETRA, the positron-electron storage ring at DESY, in 1982-1985. This process is dominated by the electromagnetic coupling, but contains also a contribution from the neutral current part of our Lagrangian. The relevant Feynman graphs are shown in Fig. 10.5.

For low energies in the entrance channel this process is dominated by the electromagnetic interaction (left) because of the large mass of the Z-boson that appears in the boson-propagator (9.20). The resulting angular distribution is symmetric around 90^0 in the cm-system because the interaction is parity-conserving.

At higher energies, however, where the momentum transfer becomes comparable to the Z-boson mass, also the right-hand graph, involving the exchange of the Z-boson, becomes relevant so that the total transition rate is given by the coherent sum over both amplitudes. The ensuing interference term between both amplitudes is not parity-conserving. Consequently, the resulting angular distribution exhibits an asymmetry around 90^0 from which $\sin^2\vartheta_W$ can be determined. One finds

$$\sin^2\vartheta_W = 0.226 \ . \tag{10.60}$$

Note that with this value for the Weinberg angle ($\sin^2\vartheta_W \approx 0.25$) the neutral current is a nearly pure axial vector.

Figure 10.5: Feynman graphs for $e^+ + e^- \to \mu^+ + \mu^-$.

Using

$$\mathrm{e}^2 = g^2 \sin^2 \vartheta_W \tag{10.61}$$

we can now determine

$$g^2 = \frac{\mathrm{e}^2}{\sin^2 \vartheta_W} = 0.63 \tag{10.62}$$

and (from (10.55))

$$\begin{aligned} M_W &= 77.7 \text{ GeV} \\ M_Z = \frac{M_W}{\cos \vartheta_W} &= 88.6 \text{ GeV} \quad . \end{aligned} \tag{10.63}$$

The actual experimental values for these masses, determined in $p\bar{p}$ reactions are $M_W = 80$ GeV and $M_Z = 91$ GeV, the differences to the estimates given above are due to radiative corrections. Thus the values obtained from quite different reactions are consistent with each other.

This consistency allows one to determine also the vacuum field value (10.8) of the Higgs field

$$v = \frac{2M_W}{g} \simeq 245 \text{ GeV} \tag{10.64}$$

as well as the Yukawa-coupling constant g_e that generates the electron mass

$$g_e = \frac{m_e \sqrt{2}}{v} = 3 \cdot 10^{-6} \quad . \tag{10.65}$$

Note that the mass of the Higgs particle, $m_H = \sqrt{-\mu^2}$, nowhere enters into any of these parameters. It thus remains completely undetermined by experiment.

As has been discussed in section 10 all parameters, except for the Higgs mass, can be determined from low-energy phenomena. The Higgs mass can only be estimated. An upper limit is usually obtained by noting that the theory outlined so far works very well "on the tree level", i.e. by assuming that all fields can be treated as classical fields; this corresponds to a neglect of all loop diagrams. This can be true only if the Higgs mass is not too large since otherwise, for example, the selfcoupling of the Higgs field ϕ, which is proportional to the square of the Higgs mass m_H (see eq. (10.31), would become too strong, so that a theory that replaces (in this case) $< 0|\phi^4|0 >$ by $< 0|\phi|0 >^4$ would not work. An estimate for the range of validity of such an approximation gives upper limits of a few hundred GeV for the Higgs mass [QUI83]. Similarly, a lower limit can be obtained by noting that for too low Higgs masses m_H the vacuum fluctuations would become very large so that one could no longer work with a fixed value v for the vacuum expectation value of the Higgs field. Both considerations together place the Higgs mass somewhere in the range 50 GeV $\lesssim m_H \lesssim$ 1000 GeV. Because the couplings of the other particles in the theory to the Higgs particle are always proportional to the square of their masses, the Higgs particle is expected to decay into the heaviest particles that are energetically accessible.

Chapter 11

WEAK INTERACTIONS OF HADRONS

So far we have considered only electroweak interactions between the leptons. In this chapter we will now extend this discussion to processes involving hadrons.

This extension is achieved by considering the electroweak interactions of quarks, the building blocks of the known hadrons. The task then is to find the relevant currents that couple to the fields W_μ, Z_μ and A_μ.

11.1 Hadronic multiplets

It is easy to guess these currents by considering, for example, the β-decay of the neutron

$$n \to p + e^- + \bar{\nu} \quad . \tag{11.1}$$

In quark language this corresponds to a change of one down-quark into an up-quark

$$d \to u + e^- + \bar{\nu} \quad , \tag{11.2}$$

which could be mediated by a current of the form

$$j_\mu^+ = \bar{u}\gamma_\mu \Gamma d \tag{11.3}$$

for the quarks; Γ here is some operator acting on the Dirac indices. In our discussions of quark models for the hadrons we will see that

$$\Gamma = \frac{1}{2}(1 - \gamma_5) \tag{11.4}$$

explains the observed properties such as the axial coupling form factor of the nucleon. Here we use eq. (11.4) because with it the quark currents have the same structure as those of the leptons.

The close similarity of the hadron current (11.3) with that for the leptons (10.34) suggests to introduce a new left-handed $SU(2)$-doublet made up of the up- and the down-quark (we postpone the introduction of the heavier quark flavors for a little while). We thus define a left-handed doublet of quarks

$$q_L = \begin{pmatrix} \frac{1}{2}(1-\gamma_5)u \\ \frac{1}{2}(1-\gamma_5)d \end{pmatrix} = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \quad (y = \frac{1}{3}) \tag{11.5}$$

with the weak hypercharge $y = 1/3$, as required by the relation between charge q, isospin t_3 and weak hypercharge y in eq. (10.17): $q = 1/2y + t_3$.

The two right-handed up- and down-quarks are then assigned to an $SU(2)$-singlet each

$$\begin{aligned} q_R^u &= \frac{1}{2}(1+\gamma_5)u = u_R \quad (y = \frac{4}{3}) \\ q_R^d &= \frac{1}{2}(1+\gamma_5)d = d_R \quad (y = -\frac{2}{3}) \quad . \end{aligned} \tag{11.6}$$

The hypercharge assignments given here are again necessitated by the general relation between charge, isospin and t_3, i.e. $q = y/2 + t_3$. Compared to the leptonic case, where the neutrino was purely left-handed, there are now two right-handed singlets instead of just one.

Again, as for the leptons, we assume that the quarks are originally massless and acquire their mass only due to spontaneous symmetry breaking.

With these two new fields we can go through exactly the same considerations as before for the leptons. The interaction between the quarks and the gauge-fields is thus obtained by simply replacing ν and e by u and d, respectively, in the interaction Lagrangian (10.32).

We thus obtain, by following exactly the same steps as in section 10.1 for the leptons, a theory that treats the quarks on the same basis as the leptons. This yields an interaction Lagrangian as in (10.38)

$$\begin{aligned} \mathcal{L}_{int} &= \mathrm{e}\left(j_\mu^{el} A^\mu - \frac{1}{\sqrt{2}\sin\vartheta_W}\left(j_\mu^+ W^\mu + h.c.\right)\right. \\ &\qquad \left. - \frac{1}{\sin\vartheta_W \cos\vartheta_W} j_\mu^0 Z^\mu\right) \end{aligned} \tag{11.7}$$

with the currents (see eqs. (10.34), (10.35), (10.36))

$$j_\mu^+ = \sum_k \bar{\Psi}_k \gamma_\mu t_+ \Psi_k \tag{11.8}$$

$$j_\mu^{el} = -\sum_k \bar{\Psi}_k \gamma_\mu q \Psi_k \tag{11.9}$$

$$j_\mu^0 = \sum_k \bar{\Psi}_k \gamma_\mu \left(t_3 - q \sin^2 \vartheta_W\right) \Psi_k \quad , \tag{11.10}$$

The Ψ_k describe either the leptons (10.45) or the newly introduced quark multiplets

$$\Psi_k = q_L \quad \text{or} \quad \Psi_k = q_R^u, \quad \text{or} \quad \Psi_k = q_R^d \quad . \tag{11.11}$$

The sums thus run over all leptons and quarks; q is the charge defined in eq. (10.17).

The generalization of the purely leptonic interaction to one that includes hadrons, as written down above, has been quite straightforward. What is, however, missing in our formulation so far, is the inclusion of strangeness-changing weak transitions.

It seems natural to incorporate these into the theory by using instead of the d-quark everywhere the linear combination

$$d' = d \cos \vartheta_c + s \sin \vartheta_c \quad , \tag{11.12}$$

where ϑ_c is the Cabibbo angle ($\cos \vartheta_c = 0.98$). If this replacement is performed, the weak neutral current (11.10) becomes

$$\begin{aligned} j_\mu^0 &= (\bar{u}_L \quad \bar{d}'_L)\, \gamma_\mu (t_3 - q \sin^2 \vartheta_W) \begin{pmatrix} u_L \\ d'_L \end{pmatrix} + \bar{d}'_R \gamma_\mu (t_3 - q \sin^2 \vartheta_W) d'_R \\ &\quad + \ldots \\ &= \cos \vartheta_c \sin \vartheta_c \Big[(\bar{d}_L \gamma_\mu s_L + \bar{s}_L \gamma_\mu d_L)\, (-\frac{1}{2} + \frac{1}{3} \sin^2 \vartheta_W) \\ &\quad + \frac{1}{3} \sin^2 \vartheta_W (\bar{s}_R d_R + \bar{d}_R s_R) \Big] + \ldots \quad . \end{aligned} \tag{11.13}$$

The model thus predicts strangeness-changing neutral current interactions.

However, such decays, like, e.g. ,

$$K^+ \to \pi^+ \nu \bar{\nu} \quad (\bar{s} \to \bar{d} \nu \bar{\nu}) \quad , \tag{11.14}$$

that are mediated by the term above (remember: $K^+ \sim u\bar{s}, \pi^+ \sim u\bar{d}$, see chapter 4), have not been observed ($\Gamma_{rel} < 10^{-9}$). As a general result of many experimental studies it can be concluded that strangeness-changing neutral currents very probably do not exist [PER87]. The hadronic extension of our electroweak theory developed so far (the Weinberg-Salam model) can, therefore, not be correct.

An elegant way out of this problem was proposed by Glashow et al. in 1970. These authors also allowed a mixture for the s-quark

$$\begin{aligned} s' &= -\sin\vartheta_c\, d + \cos\vartheta_c\, s \\ d' &= \cos\vartheta_c\, d + \sin\vartheta_c\, s \end{aligned} \tag{11.15}$$

Note that the Cabibbo-rotated s' and d' are obtained from the original quarks s and d by an orthogonal transformation. In order to be able to assign the s' quark to an $SU(2)$ multiplet Glashow et al. introduced a fourth quark, the "charmed"quark, with charge +2/3 e. This, at that time unobserved, quark was combined with the strange quark into another $SU(2)_L$ doublet.

Following this proposal we now introduce the following basic quark multiplets into the theory

$$\begin{aligned} q_L^u = \begin{pmatrix} u_L \\ d'_L \end{pmatrix}; \quad q_R^u &= u_R; \quad q_R^d = d'_R \\ q_L^c = \begin{pmatrix} c_L \\ s'_L \end{pmatrix}; \quad q_R^c &= c_R; \quad q_R^s = s'_R \quad . \end{aligned} \tag{11.16}$$

With this ansatz the terms involving the strange and the down quarks in the neutral current (11.10) go over into

$$\begin{aligned} & \bar{d}'_L\gamma_\mu\left(t_3 - q\sin^2\vartheta_W\right) d'_L + \bar{d}'_R\gamma_\mu\left(t_3 - q\sin^2\vartheta_W\right) d'_R \\ + \; & \bar{s}'_L\gamma_\mu\left(t_3 - q\sin^2\vartheta_W\right) s'_L + \bar{s}'_R\gamma_\mu\left(t_3 - q\sin^2\vartheta_W\right) s'_R \\ = \; & \sin\vartheta_c\cos\vartheta_c\left[\bar{s}_L\gamma_\mu\left(-\frac{1}{2}\right) d_L + \sin^2\vartheta_W \bar{s}_L\gamma_\mu\left(\frac{1}{3}\right) d_L\right] \\ - \; & \sin\vartheta_c\cos\vartheta_c\left[\bar{s}_L\gamma_\mu\left(-\frac{1}{2}\right) d_L + \sin^2\vartheta_W\, \bar{s}_L\gamma_\mu\left(\frac{1}{3}\right) d_L\right] + \ldots \quad . \end{aligned} \tag{11.17}$$

The strangeness-changing part of the neutral current is thus exactly cancelled by the corresponding term that originates in the s-quark mixture (11.15). Essential for this result is obviously that d' and s' are related to d and s by an orthogonal transformation.

So far we have included the two lightest quark families. As we have mentioned in section 4.2 there is experimental evidence for a fifth quark, the so-called bottom quark, with a mass of $m_b \approx 4.3\,\text{GeV}$ and a charge of $q = -1/3e$. This suggests to introduce a third quark family

$$\begin{pmatrix} t \\ b' \end{pmatrix} \tag{11.18}$$

into the summation over fermions in eq. (12.1). For symmetry reasons the existence of a sixth quark, the so-called top quark, was postulated here. Obviously, it should have charge $q = 2/3e$; in 1994 this particle was indeed observed at Fermilab with a mass of $m_t \approx 174\,GeV$.

In analogy with the $s - d$ mixing we must also allow for a mixing of t and b quarks with a possibly different Cabibbo angle. In fact, we can also, more generally, allow for a mixing of all the 'down' quarks in the form

$$d_i' = \sum_j V_{ij} d_j \qquad (i, j = 1, 2, 3) \quad , \tag{11.19}$$

where V is the so-called *Kobayashi-Maskawa matrix.* The physical quark-states, i. e. the eigenstates of the strong interaction Hamiltonian, are thus not the ones that interact weakly. This is manifest in the definition of Ψ in terms of the Cabibbo-rotated fields d', s', and b' in eq. (12.2).

11.1.1 Hadron Masses

When we were treating only one massive fermion, the electron in eq. (10.43), we generated its mass by a Yukawa-coupling between the right-handed and the left-handed electron and the Higgs particle. The same procedure can be followed here for the down-quarks. A difference arises, however, for the up-quark that stems from the fact that the "up-lepton", the neutrino, exists only as a left-handed, massless particle, so that it did not appear in the term $\mathcal{L}_Y$. This is, of course, not the case for the quarks where also the up-quark is massive and thus has to be contained in $\mathcal{L}_Y$. This implies that the term generating the mass for the lower member of the quark-doublet, the d-quark, can be written down in complete analogy to the one generating the mass for the lower member of the lepton-doublet, the electron. The up-quark requires the introduction of a new term so that $\mathcal{L}_Y$ still has the desired $SU(2)_L \otimes U(1)_Y$ symmetry.

We therefore write

$$\begin{aligned} \mathcal{L}_Y &= -\sum_{quarks} \left(g_k^d \bar{d}_{kR} \phi^\dagger \Psi_{kL} + h.c. \right) \qquad (11.20) \\ &\quad - \sum_{quarks} \left(g_k^u \bar{u}_{kR} \phi^{c\dagger} \Psi_{kL} + h.c. \right) \quad . \end{aligned}$$

Here d and u are again generic notations for d, s, b and u, c, t quarks, respectively.

The last term is the new one. It contains the charge-conjugate Higgs field ϕ^c instead of ϕ itself. This is necessary for $U(1)$ invariance reasons as can be seen immediately by noting that $\phi_i^c = \phi_i^\dagger$ carries a charge opposite to that of ϕ ($y_{\phi^c} = -1$).

The new term also respects $SU(2)$ invariance because with ϕ also ϕ^c is an $SU(2)$ doublet. This can be seen as follows. ϕ is an $SU(2)$-doublet made up of one charged and one uncharged field. It thus has the same isospin structure as, for example, the nucleon spinor made up of proton and neutron. For that we have shown (eq. (3.51)) that the charge-conjugate spinor also represents an $SU(2)$-doublet. The same holds here; also

$$\phi^c = (-\mathrm{i}\tau_2)\begin{pmatrix} \phi_1^c \\ \phi_2^c \end{pmatrix} = \begin{pmatrix} -\phi_2^c \\ \phi_1^c \end{pmatrix} = \begin{pmatrix} -\phi_2^\dagger \\ \phi_1^\dagger \end{pmatrix} \quad . \tag{11.21}$$

is an $SU(2)$ doublet. The charge-conjugate field ϕ_i^c is simply given by the Hermitian conjugate of ϕ_i.

Chapter 12

ELECTROWEAK LAGRANGIAN

In this chapter we summarize the results of the two preceding chapters by first writing down the complete Lagrangian for systems of leptons and quarks with electroweak interactions. We then investigates in some detail the symmetry properties of this Lagrangian under parity, charge-conjugation and time reversal.

12.1 Lagrangian of Electroweak Interactions

We can now write down the Lagrangian of the complete model describing the electroweak interactions of leptons and hadrons

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}\mathbf{G}^{\mu\nu}\cdot\mathbf{G}_{\mu\nu} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ &\quad + M_W^2 W_\mu^\dagger W^\mu\left(1+\frac{\varphi}{v}\right)^2 + \frac{1}{2}M_Z^2 Z_\mu Z^\mu\left(1+\frac{\varphi}{v}\right)^2 \\ &\quad + \sum_k \bar{\Psi}_k \mathrm{i}\gamma^\mu D_\mu \Psi_k + \mathcal{L}_Y \\ &\quad + \frac{1}{2}\left(\partial^\mu\varphi\right)\left(\partial_\mu\varphi\right) - \frac{1}{2}m_H^2\varphi^2\left(1+\frac{\varphi}{v}+\frac{1}{4}\left(\frac{\varphi}{v}\right)^2\right) \quad .\end{aligned} \tag{12.1}$$

The fermion spinor Ψ_k is given by the $SU(2)$ doublets

$$\Psi_k = \begin{pmatrix}\nu_{kL} \\ e_{kL}\end{pmatrix} \text{ or } \begin{pmatrix}u_{kL} \\ d'_{kL}\end{pmatrix} \tag{12.2}$$

for the left-handed fields of the k-th family of leptons and quarks and by $SU(2)$ singlets for the right-handed partners. In eq. (12.2) we have used generic abbreviations

$$\begin{pmatrix} \nu_k \\ e_k \end{pmatrix} \text{ for } \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix} \text{ or } \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix} \tag{12.3}$$

and

$$\begin{pmatrix} u_k \\ d'_k \end{pmatrix} \text{ for } \begin{pmatrix} u \\ d' \end{pmatrix}, \begin{pmatrix} c \\ s' \end{pmatrix} \text{ or } \begin{pmatrix} t \\ b' \end{pmatrix} \quad . \tag{12.4}$$

The covariant derivatives, that define the interactions between the fermions and the gauge bosons, are

$$D_\mu \Psi_k = \left(\partial_\mu + \mathrm{i} g t^l W^l_\mu + \mathrm{i} g' \frac{1}{2} y_k B_\mu \right) \Psi_k \ , \tag{12.5}$$

where y_k is, as usual, the hypercharge quantum number of Ψ_k and the t^l are the generators of $SU(2)$. $\mathcal{L}_Y$, finally, generates the fermionic masses.

$\mathcal{L}_Y$ contains the mass-generating interactions between the fermion fields and the Higgs field.

The Lagrangian (12.1) exhibits only one remaining symmetry, the $U(1)_Y$ symmetry: $U = \mathrm{e}^{-\mathrm{i}\frac{\varepsilon}{2}(1+\tau_3)}$. This is just the symmetry of QED, to which the original symmetry $SU(2) \otimes U(1)$ is broken down.

The Yukawa-term (10.43,11.20) reads written out

$$\begin{aligned} \mathcal{L}_Y \ = \ & -(g_e \bar{e}_R e_L + g_\mu \bar{\mu}_R \mu_L + g_\tau \bar{\tau}_R \tau_L \\ & + g_d \bar{d}_R d_L + g_s \bar{s}_R s_L + g_b \bar{b}_R b_L \\ & + g_u \bar{u}_R u_L + g_c \bar{c}_R c_L + g_t \bar{t}_R t_L + h.c.) \frac{v}{\sqrt{2}} \left(1 + \frac{\varphi}{v} \right) \quad . \end{aligned} \tag{12.6}$$

Since the interaction Lagrangian contains the Cabibbo-rotated fields d', s' and b' instead of the mass-eigenstates d, s, and b we now introduce those into the Lagrangian $\mathcal{L}_Y$. At the same time we introduce the masses

$$m_k = g_k \frac{v}{\sqrt{2}} \tag{12.7}$$

for all the fermions and obtain

$$\mathcal{L}_Y \ = \ \Bigg[-(\bar{e} \quad \bar{\mu} \quad \bar{\tau}) \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}$$

$$
\begin{aligned}
&- \begin{pmatrix} \bar{u} & \bar{c} & \bar{t} \end{pmatrix} \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} \begin{pmatrix} u \\ c \\ t \end{pmatrix} \qquad (12.8) \\
&- \begin{pmatrix} \bar{d}' & \bar{s}' & \bar{b}' \end{pmatrix} \left(V^{-1}\right)^{\dagger} \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} V^{-1} \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} \Bigg] \left(1 + \frac{\varphi}{v}\right) .
\end{aligned}
$$

Here V is the mixing matrix defined in equation (11.19)

$$d'_i = \sum_j V_{ij} d_j = (Vd)_i \quad .$$

The Kobayashi-Maskawa (KM) matrix appears thus in the mass term of the electroweak Lagrangian. This implies that the quarks that enter into the electroweak Lagrangian are not the physical mass-eigenstates. The latter are the eigenstates of the mass operator that contains in particular the strong interaction; in the absence of the mixing weak interactions these states would be stable.

12.2 CP Invariance of Electroweak Interactions

In this section we now discuss properties of the Kobayashi-Maskawa (KM) matrix and the influence of its structure on conservation laws in electroweak interactions.

12.2.1 Unitarity of the KM-matrix

That V is unitary is absolutely essential for the absence of flavor-changing neutral currents from the theory. This can be seen by writing out in detail the neutral current (11.10)

$$j^0_\mu = \sum_k \bar{\Psi}_k \gamma_\mu \left(t_3 - q \sin^2 \vartheta_W\right) \Psi_k \quad . \qquad (12.9)$$

We obtain for the down-quark part, i. e. the only one that can possibly lead to strangeness-changing transitions

$$\sum_{k_d} \bar{d}'_k \gamma_\mu \left(-\frac{1}{2}\frac{1-\gamma_5}{2} + \frac{1}{3}\sin^2 \vartheta_W\right) d'_k \qquad (12.10)$$

where the sum runs only over the down quarks. This term can also be written as

$$\begin{aligned} &\sum_{k_d,l_d} \bar{d}_l V^\dagger_{lk}\gamma_\mu \left(-\frac{1}{2}\frac{1-\gamma_5}{2}+\frac{1}{3}\sin^2\vartheta_W\right) V_{kl} d_l \\ &\quad = \sum_{k,down} \bar{d}_k\gamma_\mu \left(-\frac{1}{2}+\frac{1}{3}\sin^2\vartheta_W\right) d_k \quad , \end{aligned} \tag{12.11}$$

because V is unitary. There are thus no flavor-changing neutral currents for the down quarks if V is unitary.

Now we show that V also must be unitary [LEE81], if we insist on a lepton-quark symmetry for the weak currents. This can be seen as follows: the masses of d, s, and b quarks are input parameters of our theory; they are generated by interactions other than the weak ones considered here. This implies that as far as the weak interactions are concerned their masses can be taken to be all the same. Then the physics of electroweak interactions should be invariant under unitary transformations of the fields, and thus also under the transformations

$$\begin{pmatrix} d'' \\ s'' \\ b'' \end{pmatrix} = U_d \begin{pmatrix} d \\ s \\ b \end{pmatrix} ; \quad \begin{pmatrix} u'' \\ c'' \\ t'' \end{pmatrix} = U_u \begin{pmatrix} u \\ c \\ t \end{pmatrix} \quad . \tag{12.12}$$

In order to preserve the anticommutator relations of the quark fields, the transformations $U_{d,u}$ have to be unitary.

The charged current (11.8) reads for the quarks

$$\begin{aligned} j^+_{q\mu} &= \sum_{quarks} \bar{\Psi}_k\gamma_\mu t_+ \Psi_k = \frac{1}{2}\sum_k \bar{u}_k\gamma_\mu (1-\gamma_5)\, d'_k \\ &= \frac{1}{2}\sum_k \bar{u}_k\gamma_\mu (1-\gamma_5)\,(Vd)_k \quad , \end{aligned} \tag{12.13}$$

where d' stands for the Kobayashi-Maskawa rotated down quarks. This current can be brought into a form that is very symmetric to that for the leptons by writing $V = U_u^\dagger U_d$, i.e. by assuming implicitly that V is unitary

$$j^+_{q\mu} = \frac{1}{2}\sum_k \bar{u}''_k\gamma_\mu (1-\gamma_5)\, d''_k \quad . \tag{12.14}$$

We now reverse the arguments just given. We thus require that the theory be formally symmetric between leptons and quarks, i.e. that it leads to the charged current (12.13). Then the arguments just given show that this can be achieved only if V is unitary.

The fields d', s' and b' are defined only up to a phase. Thus the physics has to stay the same under the unitary transformation

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} \to \begin{pmatrix} e^{i\chi_1} d' \\ e^{i\chi_2} s' \\ e^{i\chi_3} b' \end{pmatrix} = U_\chi \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} \quad . \tag{12.15}$$

The same is true for the up-quarks. Correspondingly, the matrix V in the current term (12.13) can be transformed according to

$$V \to U_\varphi^\dagger V U_\chi \tag{12.16}$$

with

$$U_\varphi = \begin{pmatrix} e^{i\varphi_1} & 0 & 0 \\ 0 & e^{i\varphi_2} & 0 \\ 0 & 0 & e^{i\varphi_3} \end{pmatrix} \quad , \tag{12.17}$$

$$U_\chi = \begin{pmatrix} e^{i\chi_1} & 0 & 0 \\ 0 & e^{i\chi_2} & 0 \\ 0 & 0 & e^{i\chi_3} \end{pmatrix} \quad , \tag{12.18}$$

without changing any of the physics. The most general V thus contains 9 arbitrary phases. Of these only 5 (= no. of quarks - 1) are linearly independent, for example, the ones between (u, d), (u, s), (u, b), (c, d) and (t, d); once we have chosen these, the other four phases are fixed. Thus 5 of the 9 independent parameters of V are arbitrary phases, which can be removed by a suitable transformation (12.16).

The unitary Kobayashi-Maskawa matrix depends on 9 real parameters (9 complex matrix elements and 9 constraints from unitarity). After removal of the 5 phases just mentioned the remaining 4 parameters are usually chosen as ϑ_{12}, ϑ_{13}, ϑ_{23} and δ_{13} where the indices {1,2,3} label the generations and all angles ϑ_{ij} lie between 0 and $\pi/2$ [LEE81] whereas the phase δ_{13} lies in the range $0 - 2\pi$. In terms of these quantities V is written as

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix} \tag{12.19}$$

with

$$c_{ij} = \cos\vartheta_{ij} \quad , \quad s_{ij} = \sin\vartheta_{ij} \quad . \tag{12.20}$$

With these choices V is unitary as can be checked explicitly.

The numerical values of the absolute magnitude of the matrix elements are roughly [PDG90]

$$|V| \approx \begin{pmatrix} 0.975 & 0.221 & 0.004 \\ 0.221 & 0.974 & 0.044 \\ 0.011 & 0.044 & 0.999 \end{pmatrix} \quad . \tag{12.21}$$

Finally, we would like to note that only the down quarks are mixed by the KM-matrix. This is actually very reasonable since an additional mixing of the up-quarks would give no new physics because all the $SU(2)$-doublets can anyway be connected and mixed by an arbitrary $SU(2)$ transformation.

12.2.2 K^0 decay and CP-violation

The structure of the Kobayashi-Maskawa matrix is intimately connected with the problem of CP-violation observed in neutral kaon-decay. We will therefore first discuss qualitatively the evidence for CP-violation in K^0-decay [PER87].

In chapter 3.3 we have discussed that the flavor-$SU(3)$ octet of pseudoscalar mesons contains two uncharged K mesons, K^0 and $\bar{K}^0$. The quantum numbers of all the K-mesons are summarized in the following table ($K^0 = d\bar{s}, \bar{K}^0 = \bar{d}s$). The charge assignments follow the usual Gell-Mann Nishijima formula ($Y = B + S$)

t_3 S	$-1/2$	$+1/2$
$+1$	K^0	K^+
-1	K^-	$\bar{K}^0$

Table 12.1: Isospin and strangeness quantum numbers of K mesons.

$$q = t_3 + \frac{1}{2} y \quad . \tag{12.22}$$

The two neutral K^0 and $\bar{K}^0$ mesons are not identical because they differ in their strangeness; they are particle and antiparticle.

Strangeness conservation also puts stringent limitations on their production. For example, in a nucleon-nucleon collision the K^0 can be produced together with one hyperon

$$\begin{array}{lcccccc} & p & + & n & \rightarrow & K^0 & +\Lambda & +p \\ S & 0 & & 0 & & +1 & -1 & 0 \\ B & 1 & & 1 & & 0 & 1 & 1 \end{array} \quad , \tag{12.23}$$

whereas the $\bar{K}^0$ can be produced only together with a charged K or an antihyperon

$$\begin{array}{lccccccc} & p & + & n & \rightarrow & \bar{K}^0 & +K^+ & +n & +n \\ S & 0 & & 0 & & -1 & +1 & 0 & 0 \\ B & 1 & & 1 & & 0 & 0 & 1 & 1 \end{array} \tag{12.24}$$

$$\begin{array}{lcccccccc} & p & + & n & \rightarrow & \bar{K}^0 & +\bar{\Lambda} & +p & +n & +n \\ S & 0 & & 0 & & -1 & +1 & 0 & 0 & 0 \\ B & 1 & & 1 & & 0 & -1 & 1 & 1 & 1 \end{array} \quad . \tag{12.25}$$

The threshold energies for these reactions are quite different. For (12.23) it is $T_p = 1.59$ GeV, for (12.24) $T_p = 3.45$ GeV and for (12.25) $T_p = 9.52$ GeV (all in the laboratory system). One can thus easily produce pure K^0's by choosing an appropriate beam energy below the $\bar{K}_0$ threshold (3.45 GeV).

K-mesons carry strangeness and can thus decay only via the strangeness-changing component of the weak interactions. Empirically, the most probable decay mode both for the K^0 and the $\bar{K}^0$ meson is that into two pions

$$K^0, \bar{K}^0 \rightarrow \left\{ \begin{array}{cc} \pi^+\pi^- & 69\% \\ \pi^0\pi^0 & 31\% \end{array} \right. \quad . \tag{12.26}$$

Since the weak interaction contains strangeness-changing components ($\Delta S = 1$) it can in a second-order process connect K^0 and $\bar{K}^0$.

Since K^0 and $\bar{K}^0$ are both members of the octet of pseudoscalar mesons, their internal parity is negative

$$\begin{aligned} P|K^0> &= -|K^0> \\ P|\bar{K}^0> &= -|\bar{K}^0> \quad . \end{aligned} \tag{12.27}$$

Similarly, it is obvious from the quark content of the neutral K-mesons (eq. (4.23), $K^0 : d\bar{s}, \bar{K}^0 : \bar{d}s$) that we have

$$\begin{aligned} C|K^0> &= |\bar{K}^0> \\ C|\bar{K}^0> &= |K^0> \quad , \end{aligned} \tag{12.28}$$

where C is the charge conjugation operator. We thus have for the combined operation

$$\begin{aligned} CP|K^0> &= -|\bar{K^0}> \\ CP|\bar{K^0}> &= -|K^0> \quad . \end{aligned} \tag{12.29}$$

We now make the assumption that all interactions in nature, including the weak one, are invariant under CP transformations. Because the weak interaction mixes K^0 and $\bar{K^0}$ in second order of the weak coupling the physical particles cannot be these two mesons, but they must instead be mixtures of them that are CP eigenstates.

We can easily construct such eigenstates of the CP symmetry with opposite eigenvalues

$$\begin{aligned} |K^0_L> &= \frac{1}{\sqrt{2}}\left(|K^0> + |\bar{K^0}>\right) \\ |K^0_S> &= \frac{1}{\sqrt{2}}\left(|K^0> - |\bar{K^0}>\right) \end{aligned} \tag{12.30}$$

with

$$\begin{aligned} CP|K^0_S> &= +|K^0_S> \\ CP|K^0_L> &= -|K^0_L> \quad . \end{aligned} \tag{12.31}$$

The meaning of the indices S and L will become clear in a little while.

We now consider the decay modes of the CP eigenstates. We first see that the two CP-eigenstates $|K^0_S>$ and $|K^0_L>$ have quite different decay properties. $|K^0_S>$ still decays dominantly into two pions

$$|K^0_S> \to \pi^0\pi^0, \pi^+\pi^- \quad . \tag{12.32}$$

Since the pions are pseudoscalar particles we have

$$\begin{aligned} CP|\pi^+> &= -|\pi^-> \\ CP|\pi^0> &= -|\pi^0> \quad . \end{aligned} \tag{12.33}$$

The two-pion states (12.26) have both $CP = +1$, since because of angular momentum conservation the two pions must be in an $\ell = 0$ state. If CP is a conserved quantum number this decay is, however, not possible for the state $|K^0_L>$, which has $CP = -1$, . Instead, $|K^0_L>$ can decay only into at least 3 pions or into pions and leptons

$$|K^0_L> \to \pi^0\pi^0\pi^0, \quad \pi^+\pi^-\pi^0, \quad \pi^\pm e^\mp \nu, \quad \pi^\pm e^\mp \nu \quad . \tag{12.34}$$

The 3π states appearing here have $CP = -1$ (for $3\pi^0$ always, because their relative wavefunction has to be totally symmetric. For $\pi^0\pi^+\pi^-$ this holds only if all three pions are in a relative s-state; the decays from higher ℓ states are hindered, however, by the centrifugal barrier).

In the 3π decay mode a larger portion of the available decay energy is tied up in the mass of the pions. Therefore, the available phase-space factor is reduced compared to that for 2π-decay and thus the decay rate of the K_L is expected to be lower than that of the K_S.

This is exactly what is seen in experiments. Instead of the two particles K^0 and $\bar{K}^0$ one observes a long-lived K_L^0 ($\tau \approx 5.2 \cdot 10^{-8}$ *sec*) and a short-lived K_S^0 ($\tau \approx 0.9 \cdot 10^{-10}$ *sec*). Thus, the combinations (12.30) are indeed realized in nature and form the physical states.

The great discovery, in 1964, was that in spite of the arguments given above the K_L^0 also decays into $\pi^+\pi^-$ with a branching ratio of 0.2 %. This is a direct proof that CP is not conserved, since $CP(\pi^+\pi^-) = +1$, but $CP(K_L^0) = -1$. A quite general theorem states that any local, covariant field theory quantized with the correct spin-statistics connection has to conserve CPT [BJO65]. Thus CP violation implies also a violation of the time invariance T.

12.2.3 CP-invariance of the electroweak interaction

The question now arises what the origin for the observed CP-violation is. To answer this question we consider the behavior of the weak interaction Lagragian under CP transformations. For the transformation of the fields appearing in $\mathcal{L}$ we have (see appendix A, eqs. (A.33), (A.35))

$$\begin{aligned} \Psi(x) &\rightarrow \mathrm{i}\gamma_0\gamma_2 \left(\Psi^\dagger(-\vec{x},t)\right)^T \\ \varphi(x) &\rightarrow \varphi(-\vec{x},t) \quad . \end{aligned} \tag{12.35}$$

The transformation of the fermion fields is the normal one for Dirac spinors given by relations (A.23) and (A.35); that of the Higgs field φ is evident because the Higgs field φ represents an uncharged scalar field.

The transformation of the gauge fields is fixed by that of the fermion field because both are connected through the equation of motion (7.42) for the gauge fields, which contains a fermionic source term. We therefore consider here its transformation properties

$$\begin{aligned} &(CP) j_\mu^l(\vec{x},t)(CP)^\dagger = (CP)\Psi_\alpha^\dagger \left(\gamma_0\gamma_\mu\right)_{\alpha\beta} T^l \Psi_\beta (CP)^\dagger \\ &= \quad -\Psi_\gamma^a(-\vec{x},t) \left(\gamma_2\gamma_0\right)_{\gamma\alpha} \left(\gamma_0\gamma_\mu\right)_{\alpha\beta} \left(T^l\right)^{ab} \left(\gamma_0\gamma_2\right)_{\beta\delta} \Psi_\delta^{b\,\dagger}(-\vec{x},t) \end{aligned}$$

$$= \quad (T^l)^{ab}\Psi^{b\,\dagger}_{\delta}(-\vec{x},t)\,(\gamma_2\gamma_\mu\gamma_0\gamma_2)_{\gamma\delta}\,\Psi^a_\gamma(-\vec{x},t) \quad . \tag{12.36}$$

The minus sign at the start of the third line arises because of the anticommutation relations between the fermion field operators. Here T^l is a generator of the internal symmetry group ($T^l = \tau^l/2$ for $SU(2)$, $T^l = 1$ for $U(1)$). The upper indices a and b refer to the internal degrees of freedom. We now use

$$(\gamma_2\gamma_\mu\gamma_0\gamma_2)^T = \gamma_2\gamma_0\,(\gamma_\mu)^T\,\gamma_2 \quad = \quad \begin{cases} -\gamma_0\gamma_\mu & \text{for } \mu = 0 \\ \gamma_0\gamma_\mu & \text{for } \mu = i = 1,2,3 \end{cases} \quad ,$$

which holds because γ_2 and γ_0 are symmetric and γ_1 and γ_3 are antisymmetric, and get

$$\begin{aligned} (CP)j^l_\mu(\vec{x},t)(CP)^\dagger &= \left(T^l\Psi^\dagger_\delta(-\vec{x},t)\right)^a(\gamma_\mu\gamma_0)_{\delta\gamma}\,\Psi^a_\gamma(-\vec{x},t) \\ &= \begin{cases} +\bar{\Psi}(-\vec{x},t)\gamma_i(T^l)^T\Psi(-\vec{x},t) & \text{for } \mu = i = 1,2,3 \\ -\bar{\Psi}(-\vec{x},t)\gamma_0(T^l)^T\Psi(-\vec{x},t) & \text{for } \mu = 0 \quad . \end{cases} \end{aligned} \tag{12.37}$$

This transformation behavior of the source term under $SU(2)$ ($T^l = \tau^l/2$) and $U(1)$ ($T^l = 1$) transformations can be used to obtain the transformations properties of the fields W^l_μ and B^l_μ from the equation of motion (eq. (7.42)) for these fields. With $(T^l)^T = (-)^{l+1}T^l$ we obtain

$$(CP)W^l_\mu(\vec{x},t)(CP)^\dagger \quad = \quad \begin{cases} (-)^{l+1}W^l_i(-\vec{x},t) & \text{for } \mu = i = 1,2,3 \\ (-)^l W^l_0(-\vec{x},t) & \text{for } \mu = 0 \quad . \end{cases} \tag{12.38}$$

The transformation of the $U(1)$ field is similarly given by

$$(CP)B_\mu(\vec{x},t)(CP)^\dagger = \begin{cases} B_i(-\vec{x},t) & \text{for } \mu = i = 1,2,3 \\ -B_0(-\vec{x},t) & \text{for } \mu = 0 \end{cases} \quad . \tag{12.39}$$

Equations (12.38) and (12.39) define also the transformation properties of the physical fields W_μ, A_μ and Z_μ. We obtain

$$\begin{aligned} (CP)W_\mu(\vec{x},t)(CP)^\dagger &= \begin{cases} W^\dagger_i(-\vec{x},t) & \text{for } \mu = i = 1,2,3 \\ -W^\dagger_0(-\vec{x},t) & \text{for } \mu = 0 \end{cases} \\ (CP)Z_\mu(\vec{x},t)(CP)^\dagger &= \begin{cases} Z_i(-\vec{x},t) & \text{for } \mu = i = 1,2,3 \\ -Z_0(-\vec{x},t) & \text{for } \mu = 0 \end{cases} \quad . \end{aligned} \tag{12.40}$$

The electromagnetic field A_μ transforms in exactly the same way as Z_μ because both are obtained by a linear transformation from W^3_μ and B_μ.

Under the transformations (12.35), (12.38), and (12.39) the free Lagrangian is obviously invariant because it contains all the fields quadratically. We therefore need to consider only the interaction Lagrangian (11.7)

$$\begin{aligned}\mathcal{L}_{int} &= \mathrm{e}\left(j^{el}_\mu A^\mu - \frac{1}{\sqrt{2}\sin\vartheta_W}\left(j^+_\mu W^\mu + j^-_\mu W^{\mu\dagger}\right)\right. \\ &\quad \left. - \frac{1}{\sin\vartheta_W\cos\vartheta_W} j^0_\mu Z^\mu\right) \quad . \end{aligned} \tag{12.41}$$

With the transformation laws just given, the electromagnetic current ($T^l = 1$) and the neutral current ($T^3 = \tau^3/2$) terms are invariant under CP-transformations. Essential for this result is that for these cases the current operators contain the same fermion flavor on both sides. In particular, this implies that the KM matrix V appears twice

$$\bar{d}'\gamma_\mu d' = \bar{d}V^\dagger\gamma_\mu V d = \bar{d}\gamma_\mu d \quad .$$

Thus, because of the unitarity of the KM-matrix, V drops out of the current term, reflecting the absence of flavor-changing neutral currents.

We consider next that term of $\mathcal{L}_{int}$ that contains the charged current (11.8)

$$j^+_\mu = \sum_k \bar{\Psi}_k\gamma_\mu t_+\Psi_k \quad . \tag{12.42}$$

The isospin raising operator has a profound impact on the transformation properties of this current. It couples down- and up-quarks, so that the KM-matrix V appears only once in it, in contrast to the electromagnetic and the neutral currents. We therefore have, for example for $i = 1, 2, 3$,

$$\begin{aligned}(CP)j^+_i(\vec{x},t)(CP)^\dagger &= (CP)\sum_k \bar{u}_k\gamma_i(Vd)_k(CP)^\dagger \\ &= -\sum_{kk'}\bar{d}_{k'}(-\vec{x},t)\gamma_i u_k(-\vec{x},t)V_{kk'} \\ &= -\sum_k\left(\bar{d}(-\vec{x},t)V^T\right)_k\gamma_i u_k(-\vec{x},t) \quad , \end{aligned} \tag{12.43}$$

where the "-" sign in the second and third line originates in the behavior of an $SU(2)$ spinor under charge conjugation (see eq. (3.51). The structure of this current thus changes under the CP transformation.

The only way to achieve CP invariance of the charged-current part of $\mathcal{L}_{int}$ is then to require that

$$(CP)j_i^+(\vec{x},t)(CP)^\dagger = j_i^-(-\vec{x},t) \quad , \tag{12.44}$$

i.e. that the current goes over into its hermitian conjugate. If this were the case, then the combination $j_i^+W_i + j_i^-W_i^\dagger$ would be invariant because W_i changes into $W_i^\dagger$ under the CP transformation (see eq. (12.40.

Explicitly, we have

$$j_i^- = \sum_k \bar{\Psi}_k \gamma_i t_- \Psi_k = \sum_k \left(\bar{d}V^\dagger\right)_k \gamma_i u_k \quad . \tag{12.45}$$

For (12.45) and (12.43) to be equal we obviously have the condition

$$V^T = V^\dagger \quad , \tag{12.46}$$

or, equivalently,

$$V = V^* \quad . \tag{12.47}$$

The same condition is obtained also for the product of the zeroth components of the currents and fields; here the extra sign in the transformed current is neutralized by that in the transformed field.

If condition (12.47) is fulfilled, i. e. if the KM matrix is real, the theory of electroweak interactions is CP conserving. Vice versa, the weak-interaction is by itself CP-violating if V is complex. This is the case if the phase δ in the representation (12.19) of the KM matrix is nonzero.

There is now — after some recent experiments on K^0 decay — some indication that the observed 2π-decay of the K_L^0 is indeed due to the CP violating part of the KM matrix and not to some new "Superweak Interaction" [PDG90, p. VII.90]. A definite answer of the question if the strength of CP-violating parts of the electroweak theory, if any, is large enough to explain the observed decay rates of the K^0 is, however, not yet possible since this would require a quantitatively reliable description of the kaons in a quark-picture.

12.3 The Standard Model

The so-called Standard Model is obtained by combining the model of electroweak interactions with the theory of strong interactions, QCD. The full symmetry group of the Lagrangian is then

$$G = SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \quad , \tag{12.48}$$

where $SU(3)_C$ is the color $SU(3)$ group (see chapter 9). After spontaneous symmetry breaking the symmetry group of the electroweak interactions is reduced to

$$SU(2)_L \otimes U(1)_Y \longrightarrow U_{em}(1) \quad , \tag{12.49}$$

where $U_{em}(1)$ is the usual $U(1)$ group of classical *QED*. The reduction happens at energies that are smaller than about 100 GeV, i.e. the mass-scale of the W and Z bosons.

Only at energies higher than this natural border will the interactions exhibit features of the full symmetry group. Such high energies may be present in a very hot environment. Since the masses of the gauge bosons and the electron were generated by spontaneous symmetry breaking one expects – from analogy with other phase-transitions – that there is a critical temperature above which the original symmetry of the Lagrangian is restored. This is indeed what happens also here since at large temperatures T the field starts to fluctuate around $\phi = \phi_0$. At high enough temperatures we have then $< \phi >= v = 0$ so that as a direct consequence

$$M_W = M_Z = m_e = 0 \tag{12.50}$$

since all three masses are proportional to v. The state $< \phi >= 0$ is then manifestly invariant under $SU(2)_L \otimes U(1)_Y$ and all the symmetries are restored. In this regime then the weak and the electromagnetic interaction are both long-ranged and of equal effective strength.

In this chapter we have seen how remarkably successfull the standard model is in explaining quantitatively many electroweak phenomena. However, this success also has its price. In its present form the standard model contains the following parameters

1. 6 quark masses
2. 3 lepton masses : m_e, m_μ, and m_τ
3. 2 boson masses : M_W and m_H
4. 3 coupling constants : e, $\sin\vartheta_W$, and the color coupling constant g
5. 4 parameters of the Kobayashi-Maskawa matrix : ϑ_1, ϑ_2, ϑ_3 and δ .

This is a total of 18 parameters, too many for a truly fundamental theory! In addition, the left-right asymmetry is aesthetically not appealing. Furthermore it should be clear that the physical nature of the Higgs mechanism is really undetermined by experiment. In the present context it has been introduced merely to generate the masses of the gauge bosons by means of spontaneous symmetry breaking. It is, therefore, at present nothing more than a parametrization of an as yet unknown physical process. To find out if there exists an actual Higgs particle is one of the major challenges of today's research in this field.

Appendix A

Solutions of the free Dirac Equation

A.1 Properties of free Dirac states

The free Dirac equation

$$(\vec{\alpha}\cdot\hat{\vec{p}}+\beta m)\Psi = \mathrm{i}\frac{\partial \Psi}{\partial t} \tag{A.1}$$

is solved by the ansatz

$$\Psi(x,t) = w(p,s)\, e^{-\mathrm{i}px} \quad , \tag{A.2}$$

where $w(p,s)$ is a four-component spinor of c-numbers (the physical significance of the additional quantum number s is explained below). The eigenvalues are

$$E = \pm\sqrt{\vec{p}^{\,2}+m^2} \equiv \pm E_p \quad (E_p \geq 0) \quad . \tag{A.3}$$

Following the notation of Lee [LEE81], we denote the spinors w to positive energies by u, those to negative energies by v. These obey the following equations:

$$(\vec{\alpha}\cdot\vec{p}+\beta m)\left\{\begin{array}{c} u(\vec{p},s) \\ v(-\vec{p},s)\end{array}\right. = E_p \left\{\begin{array}{c} u(\vec{p},s) \\ -v(-\vec{p},s)\end{array}\right. \quad . \tag{A.4}$$

For simplicity, the vector arrows over $\vec{p}$ in the spinors will be dropped in the following equations and in the main text ($\vec{p}$ is here a vector with c-number components and not an operator). Note that $v(p,s)$ obeys the equation[1]

$$(\vec{\alpha}\cdot\vec{p}-\beta m)v(p,s) = E_p\, v(p,s) \quad . \tag{A.5}$$

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[1] The spinors u and v used here are identical with those of Bjorken and Drell [BJO65].

The Dirac equation can also be written in a covariant form by introducing the γ matrices

$$\gamma^0 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \beta\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} . \tag{A.6}$$

These obey the algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} , \tag{A.7}$$

so that

$$\gamma_0^2 = +1, \quad \gamma_k^2 = -1 . \tag{A.8}$$

In terms of these matrices the Dirac equation (A.1) reads

$$(\mathrm{i}\gamma^\mu\partial_\mu - m)\,\Psi = 0 . \tag{A.9}$$

Helicity. The meaning of the second quantum number s becomes clear by the explicit construction of an operator that commutes with the Hamiltonian of the free Dirac equation and thus represents a conserved observable,

$$\left[\vec{\Sigma}\cdot\vec{p}\;,\;\vec{\alpha}\cdot\vec{p}+\beta m\right] = 0 . \tag{A.10}$$

The (4×4) spin matrix $\vec{\Sigma}$ is built from the normal (2×2) Pauli spin matrices via

$$\vec{\Sigma} = \begin{bmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{bmatrix}$$

Therefore the eigenvalues of $\vec{\Sigma}\cdot\vec{p}$ can be used to specify the physical states of a free Dirac particle in addition to the momentum

$$\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}\left\{\begin{array}{l} u(p,s) \\ v(-p,s) \end{array}\right. = 2s\left\{\begin{array}{l} u(p,s) \\ v(-p,s) \end{array}\right. \tag{A.11}$$

with $s = \pm 1/2$. The quantity s is called “helicity”; it describes the projection of the particle’s spin onto its momentum direction.

Normalization. The equations determining the adjoint spinors $u^\dagger$ and $v^\dagger$ are obtained by first taking the adjoint of equations (A.4, A.5), using the fact that $\vec{\alpha}$ and β are hermitean:

$$\begin{aligned} u^\dagger(p,s)(\vec{\alpha}\cdot\vec{p}+\beta m) &= E_p\; u^\dagger(p,s) \\ v^\dagger(p,s)(\vec{\alpha}\cdot\vec{p}-\beta m) &= E_p\; v^\dagger(p,s). \end{aligned} \tag{A.12}$$

Next, $\beta^2 = 1$ is used to obtain

$$\begin{aligned} \bar{u}(p,s)(\beta\vec{\alpha}\cdot\vec{p}+m) &= E_p\bar{u}(p,s)\beta \\ \bar{v}(p,s)(\beta\vec{\alpha}\cdot\vec{p}-m) &= E_p\bar{v}(p,s)\beta. \end{aligned} \tag{A.13}$$

Multiplying equations (A.13) from the right with β and using the anticommutation relation $\{\beta,\alpha_i\} = 0$ finally gives

$$\begin{aligned} \bar{u}(p,s)(-\vec{\alpha}\cdot\vec{p}+\beta m) &= E_p\bar{u}(p,s) \\ \bar{v}(p,s)(\vec{\alpha}\cdot\vec{p}+\beta m) &= -E_p\bar{v}(p,s) \quad . \end{aligned} \tag{A.14}$$

Since $u(p,s)$ and $v(-p,r)$ are eigenvectors of the same hermitean matrix (see equation (A.4)) with different eigenvalues the following relations must hold for arbitrary s, r :

$$\begin{aligned} u^\dagger(p,s)v(-p,r) &= 0 \\ v^\dagger(p,r)u(-p,s) &= 0\ . \end{aligned} \tag{A.15}$$

Another orthogonality relation follows if $u(p,s)$ and $u(p,r)$ or $v(p,s)$ and $v(p,r)$ are eigenvectors of the helicity-operator (A.11) with $r \neq s$:

$$u^\dagger(p,s)u(p,r) = v^\dagger(p,s)v(p,r) = 0 \quad \text{for } r \neq s. \tag{A.16}$$

The normalization of the spinors is achieved by requiring

$$\begin{aligned} u^\dagger(p,s)u(p,s) &= \frac{E_p}{m} \quad , \\ v^\dagger(p,s)v(p,s) &= \frac{E_p}{m} \quad . \end{aligned} \tag{A.17}$$

This normalization condition is Lorentz-covariant because both sides of (A.17) represent the zeroth component of a four-vector, on the lhs the probability current and on the rhs the four-momentum.

The corresponding relations for the adjoint spinors $\bar{u}$ and $\bar{v}$ can be obtained by manipulating equations (A.4) and (A.10). Multiplying the first of equations (A.4) from the left with $\bar{u}$ and the first of equations (A.14) from the right with u and summing both gives

$$\bar{u}(p,s)u(p,r) = \frac{m}{E_p}u^\dagger(p,s)u(p,r) = \delta_{rs} \quad . \tag{A.18}$$

The analogous operation for the second of equations (A.4) and (A.14) yields

$$\bar{v}(p,s)v(p,r) = -\frac{m}{E_p}v^\dagger(p,s)v(p,r) = -\delta_{rs} \quad . \tag{A.19}$$

Similarly, one obtains

$$\bar{u}(p,s)v(p,r) = \bar{v}(p,s)u(p,r) = 0 \quad . \tag{A.20}$$

For convenience all these relations are summarized here:

$$\begin{array}{rclrcl} u^\dagger(p,s)u(p,r) & = & \delta_{rs}\dfrac{E_p}{m} & \bar{u}(p,s)u(p,r) & = & \delta_{rs} \\ \\ v^\dagger(p,s)v(p,r) & = & \delta_{rs}\dfrac{E_p}{m} & \bar{v}(p,s)v(p,r) & = & -\delta_{rs} \\ \\ u^\dagger(p,s)v(-p,r) & = & 0 & \bar{u}(p,s)v(p,r) & = & 0 \quad . \end{array} \tag{A.21}$$

Explicit representation. Explicitly, the free spinors are given by

$$u(p,s) = N \begin{pmatrix} \varphi(p,s) \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\varphi(p,s) \end{pmatrix} \tag{A.22a}$$

$$v(p,s) = N \begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi(p,s) \\ \chi(p,s) \end{pmatrix} . \tag{A.22b}$$

Here φ and χ are two-component spinors and the normalization constant is determined by equation (A.17),

$$N = \sqrt{\frac{E_p+m}{2m}} \quad , \tag{A.23}$$

if the two-spinors φ and χ are themselves normalized to 1. The ansatz of equations (A.22) can be verified by inserting it into equation (A.4).

The two spinors φ and χ in equations (A.22a – A.22b) have to fulfill conditions that stem from the helicity eigenvalue equation (A.11). These are

$$\begin{aligned} \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\varphi(p,s) &= 2s\ \varphi(p,s) \\ \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\chi(p,s) &= -2s\ \chi(p,s) \quad . \end{aligned} \tag{A.24}$$

Here $\vec{\sigma}$ represents the normal (2×2) Pauli spin matrices. The helicity operator on the lhs is given by

$$\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} = \frac{1}{|\vec{p}|}\begin{pmatrix} p_z & p_x - \mathrm{i}p_y \\ p_x + \mathrm{i}p_y & -p_z \end{pmatrix} \tag{A.25}$$

in a representation in which σ_z is diagonal. Note that in this representation the components of φ and χ are in general complex; they also still depend on $\vec{p}$. If the z-axis is chosen parallel to the momentum $\vec{p}$ then the helicity operator (A.25) is diagonal with eigenvalues $+1$ and -1 (helicity representation); in this case the two-component spinors φ and χ are the usual Pauli spinors.

Charge conjugation. We will now briefly establish a connection between the spinors to positive and negative energy, u and v. Taking the complex conjugate of (A.24) yields

$$\frac{\vec{\sigma}^* \cdot \vec{p}}{|\vec{p}|} \varphi^*(p,s) = 2s\ \varphi^*(p,s) \quad . \tag{A.26}$$

Using

$$\vec{\sigma}^* = (\mathrm{i}\sigma_y)\vec{\sigma}(\mathrm{i}\sigma_y) \tag{A.27}$$

and $\sigma_y^2 = 1$ and multiplying equation (A.26) from the left with $\mathrm{i}\sigma_y$ transforms equation (A.26) into

$$\frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|} (\mathrm{i}\sigma_y)\varphi^*(p,s) = -2s(\mathrm{i}\sigma_y)\varphi^*(p,s) \quad . \tag{A.28}$$

Comparison of (A.28) with (A.24) shows that $(\mathrm{i}\sigma_y)\varphi^*(p,s)$ and $\chi(p,s)$ obey the same eigenvalue equation (the same is true also for the Dirac equation (A.4)). Since they are both normalized, they have to be identical up to a phase that can be chosen to be 1:

$$\chi(p,s) = (\mathrm{i}\sigma_y)\varphi^*(p,s) \quad . \tag{A.29}$$

For the full four-spinor this condition reads

$$v(p,s) = \mathrm{i}\gamma^2 u^*(p,s) \quad . \tag{A.30}$$

Equations (A.29) and (A.30) thus connect spinors to positive and negative energy.

Since $v(p,s)$ solves equation (A.4) with negative energy and momentum, the corresponding full spinor (A.2) can be written as

$$\Psi^C(x,t) = v(p,s)\mathrm{e}^{\mathrm{i}px} \ . \tag{A.31}$$

This "charge conjugate" solution of the Dirac equation is related to the solution with positive energy and momentum,

$$\Psi(x,t) = u(p,s)\mathrm{e}^{-\mathrm{i}px} \quad , \tag{A.32}$$

via equation (A.30) as follows:

$$\begin{aligned} \Psi^C(x,t) &= \mathrm{i}\gamma^2 u^*(p,s)\mathrm{e}^{-\mathrm{i}px} \\ &= \mathrm{i}\gamma^2\Psi^*(x) = \mathrm{i}\gamma^2\left(\Psi^\dagger(x)\right)^T = \mathrm{i}\gamma^2\gamma^0(\bar{\Psi}(x))^T \quad . \end{aligned} \tag{A.33}$$

The term "charge-conjugate" becomes plausible when observing that the $U(1)$ phase transformation for Ψ is

$$\Psi \to \Psi' = \mathrm{e}^{-\mathrm{i}\epsilon q}\Psi \quad ,$$

whereas, because of (A.34), for Ψ^C it is

$$\Psi^C \to \Psi'^C = \mathrm{e}^{+\mathrm{i}\epsilon q}\Psi^C \tag{A.34}$$

so that the conserved charges associated with these fields differ by a sign. Thus, if Ψ describes a fermion of mass m and charge q, the charge conjugate solution describes an otherwise identical fermion with charge $-q$.

Parity. Since γ_0 anticommutes with all γ_k we can easily see from equation (A.1) that with $\Psi(\vec{x},t)$ also

$$P\ \Psi(\vec{x},t) = \gamma_0\ \Psi(-\vec{x},t) \tag{A.35}$$

fulfills the Dirac equation. (A.35) gives the parity transformation of the Dirac theory. An immediate consequence of this transformation property is that the antifermions, described by (A.34), have a parity opposite to that of the corresponding fermion.

Fermion Bilinears. Finally, we add without proof a list of the most important expressions that are bilinear in the fermion and their transformation properties. These operators can be obtained by using the Lorentz transformation properties of Dirac spinors.

$$\begin{array}{lll}
\text{scalar} & \bar{\Psi}'(x')\Psi'(x') & = \bar{\Psi}(x)\Psi(x) \\[2em]
\text{vector} & \bar{\Psi}'(x')\gamma_\mu\Psi'(x') & = a_{\mu\nu}\bar{\Psi}(x)\gamma^\nu\Psi(x) \\[2em]
\text{pseudoscalar} & \bar{\Psi}'(x')\gamma_5\Psi'(x') & = \det(a_{\mu\nu})\bar{\Psi}(x)\gamma_5\Psi(x) \\[2em]
\text{pseudovector} & \bar{\Psi}'(x')\gamma_5\gamma^\mu\Psi'(x') & = \det(a_{\mu\nu})a_{\mu\nu} \\
 & & \quad \times\bar{\Psi}(x)\gamma_5\gamma^\nu\Psi(x) \\[2em]
\begin{array}{c}\text{antisymmetric} \\ \text{tensor}\end{array} & \bar{\Psi}'(x')\sigma_{\kappa\lambda}\Psi'(x') & = a_{\kappa\mu}a_{\lambda\nu}\bar{\Psi}(x)\sigma^{\mu\nu}\Psi(x) \\[1em]
 & \text{with } \ \sigma_{\kappa\lambda} = \frac{1}{2}[\gamma_\kappa, \gamma_\lambda] &
\end{array} \tag{A.36}$$

Here $a_{\mu\nu}$ is the Lorentz transformation matrix and $x'_\mu = a_{\mu\nu}x^\nu$.

A.2 Massless fermions

For a massless fermion the Dirac equation reads

$$\mathrm{i}\gamma^\mu\partial_\mu\Psi = \gamma^\mu p_\mu\Psi = 0 \ . \tag{A.37}$$

We now introduce the matrix

$$\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{A.38}$$

which anticommutes with all γ^μ

$$\left\{\gamma^5, \gamma^\mu\right\} = 0, \qquad \left(\gamma^5\right)^2 = 1 \ . \tag{A.39}$$

Multiplying (A.37) with shows that with Ψ also $\gamma_5\Psi$ is a solution of the Dirac equation for a massless particle; γ_5 is called the “chirality operator. Thus, also

$$\Psi_L = \frac{1}{2}\left(1 - \gamma_5\right)\Psi \tag{A.40}$$

and

$$\Psi_R = \frac{1}{2}\left(1+\gamma_5\right)\Psi \tag{A.41}$$

are solutions; for a stationary state they all belong to the same eigenvalue as Ψ. These new states are eigenstates to γ_5

$$\begin{aligned} \gamma_5\Psi_L &= -\Psi_L & \text{(A.42)} \\ \gamma_5\Psi_R &= +\Psi_R \ . & \text{(A.43)} \end{aligned}$$

Multiplying eq. (A.37) with $\gamma^5\gamma^0 = -\mathrm{i}\gamma^1\gamma^2\gamma^3$ gives, on the other hand,

$$\vec{\Sigma}\cdot\hat{\vec{p}}\Psi = \gamma^5 p^0 \Psi \ , \tag{A.44}$$

because of $\gamma^5\gamma^0\gamma^i = \Sigma_i$. Since for a massless particle we have $p^0 = |\vec{p}|$, this equation gives

$$\vec{\Sigma}\cdot\frac{\hat{\vec{p}}}{|\vec{p}|} = \gamma^5\Psi \ . \tag{A.45}$$

Thus, for a massless particle the chirality equals the helicity. Note, that while all particle states can be chosen to be eigenstates of helicity, only massless particles can be in eigenstates of chirality. While the former can be changed by going from one Lorentz frame to another, the latter is an invariant property of the state. If the Hamiltonian describing the particle commutes with the chirality operator γ_5, then the chirality is a conserved quantum number.

The spinors Ψ_L and Ψ_R have only 2 independent components, as one can easily show. Massless particles could thus be described by a two-component spinor theory; the equation of motion governing their evolution is called the 'Weyl' equation.

There is strong experimental evidence that neutrinos are massless and have always chirality -1, i.e. for them spin and momentum are antiparallel, they are lefthanded; antineutrinos, on the other hand, occur only righthanded, i.e. with chirality +1. This implies that the operators responsible for the production, annihilation and interaction of neutrinos can contain only lefthanded neutrino currents

$$\begin{aligned} j_L^\mu &= \Psi_L\gamma^\mu\Psi_L = \frac{1}{4}\left(\left(1-\gamma_5\right)\Psi\right)^\dagger\gamma^0\gamma^\mu\left(1-\gamma_5\right)\Psi \\ &= \frac{1}{4}\bar{\Psi}\left(1+\gamma_5\right)\gamma^\mu\left(1-\gamma_5\right)\Psi \\ &= \frac{1}{2}\bar{\Psi}\gamma^\mu\left(1-\gamma_5\right)\Psi \\ &= \frac{1}{2}\left(V^\mu - A^\mu\right) \ . \end{aligned} \tag{A.46}$$

In the last line the lefthanded current is written as a difference of vector and axial vector current ($V - A$ form).

The lefthanded spinor remains lefthanded under charge-conjugation

$$(\Psi_L)^C = \mathrm{i}\gamma^2\gamma^0\Psi_L^* = \mathrm{i}\gamma^2\gamma^0\frac{1}{2}\left(1-\gamma_5\right)\Psi^* = \left(\Psi^C\right)_L \; ; \qquad \text{(A.47)}$$

this is understandable since charge conjugation does not affect spin or momentum of a particle. However, the field *operators* transform differently under C, as one can directly see from their normal mode expansion

$$\begin{aligned}
\left(\hat{\Psi}_L\right)^C &= \frac{1}{V}\sum_{\vec{p}}\frac{1}{2E}\left(a_R u_R(p,s)e^{-ipx} + a_L^\dagger v_L(p,s)e^{ipx}\right)^C \\
&= \frac{1}{V}\sum_{\vec{p}}\frac{1}{2E}\left(a_R^\dagger \mathrm{i}\gamma^2\gamma^0 u_R^*(p,s)e^{ipx} + a_L \mathrm{i}\gamma^2\gamma^0 v_L^*(p,s)e^{-ipx}\right)^C \\
&= \frac{1}{V}\sum_{\vec{p}}\frac{1}{2E}\left(a_R^\dagger v(p,s)e^{ipx} + a_L u(p,s)e^{-ipx}\right) \\
&= \hat{\Psi}_R \; . \qquad \text{(A.48)}
\end{aligned}$$

In the last step eq. (A.29) has been used.

TABLE I

Table of Hadronic Particle Properties

The following is a table of the properties of energetically low-lying hadrons. Given for each particle are the spin J, isospin T, parity π, strangeness S, hypercharge Y and mass and mean lifetime (or, equivalently, full width for short-lived particles). All numbers are taken from the 1990 Review of Particle Properties [PDG90] that contains listings of the properties of all known particles.

(a) **Baryon Octet**

Particle	$(\mathrm{J,T})^\pi$	S	Y	Mass/MeV	mean life/s
p	$(\frac{1}{2},\frac{1}{2})^+$	0	1	938.3	stable
n	$(\frac{1}{2},\frac{1}{2})^+$	0	1	939.6	889
Σ^+	$(\frac{1}{2},1)^+$	-1	0	1189.4	0.8×10^{-10}
Σ^0	$(\frac{1}{2},1)^+$	-1	0	1192.6	7.4×10^{-20}
Σ^-	$(\frac{1}{2},1)^+$	-1	0	1197.4	1.5×10^{-10}
Ξ^0	$(\frac{1}{2},\frac{1}{2})^+$	-2	-1	1314.9	2.9×10^{-10}
Ξ^-	$(\frac{1}{2},\frac{1}{2})^+$	-2	-1	1321.3	1.6×10^{-10}
Λ	$(\frac{1}{2},0)^+$	-1	0	1115.6	2.6×10^{-10}

(b) **Baryon Decuplet**

Since these states have very short lifetimes of the order 10^{-23} s, here the widths are given instead. The four members of the $T = 3/2$ multiplet all have very similar masses and widths; therefore the Δ has been listed only once.

Particle	$(J,T)^{\pi}$	S	Y	Mass/MeV	Full width/MeV
Δ	$(\frac{3}{2},\frac{3}{2})^+$	0	1	1232.0	115
$\Sigma(1385)^+$	$(\frac{3}{2},1)^+$	-1	0	1382.8	36
$\Sigma(1385)^0$	$(\frac{3}{2},1)^+$	-1	0	1383.7	36
$\Sigma(1385)^-$	$(\frac{3}{2},1)^+$	-1	0	1387.2	39
$\Xi(1530)^0$	$(\frac{3}{2},\frac{1}{2})^+$	-2	-1	1531.8	9
$\Xi(1530)^-$	$(\frac{3}{2},\frac{1}{2})^+$	-2	-1	1535.0	10
Ω^-	$(\frac{3}{2},0)^+$	-3	-2	1672.4	8×10^{-12}

(c) **Pseudoscalar** Mesons (meson octet and singlet)

Because of their quark-antiquark structure, all mesons contain their antiparticles in the same multiplet.

Particle	(J,T)$^{\pi}$	S	Y	Mass/MeV	mean life/s
K^+, K^-	$(0, \frac{1}{2})^-$	±1	±1	493.7	1.2×10^{-8}
$K^0, \bar{K}^0$	$(0, \frac{1}{2})^-$	±1	±1	497.7	$9\times 10^{-11} - 5.2\times 10^{-8}$
π^+, π^-	$(0, 1)^-$	0	0	139.6	2.6×10^{-8}
π^0	$(0, 1)^-$	0	0	135.0	8.4×10^{-17}
η	$(0, 0)^-$	0	0	548.8	6.3×10^{-19}
meson singlet					
$\eta'(958)$	$(0, 0)^-$	0	0	958.0	2.7×10^{-21}

(*d*) **Vector Mesons** (meson nonet)

Here the two isoscalars belonging to the vector meson singlet and octet are strongly mixed. The resulting physical particles are the ω and ϕ. Again the antiparticles belong to the same multiplet. They are obtained from the particles listed by inverting the signs of S and Y. Thus the K^* contributes four and the ρ three different mesons to the multiplet.

Particle	(J,T)$^{\pi}$	S	Y	Mass/MeV	Full width/MeV
$K^*(892)$	$(1, \frac{1}{2})^-$	+1	+1	891.8	49.8
$\rho(770)$	$(1, 1)^-$	0	0	768.3	149.1
$\omega(783)$	$(1, 0)^-$	0	0	782.0	8.4
$\phi(1020)$	$(1, 0)^-$	0	0	1019.4	4.4

TOPICAL BIBLIOGRAPHY

The literature in this topical bibliography is ordered according to topics. It is mostly restricted to review articles and textbooks that offer good sources for further reading. Not all of these references are actually quoted in the text.

General Textbooks on Quantum Field Theory (chapters 1-3)

J. D. Bjorken and S. D. Drell, Relativistic Quantum Mechanics, McGraw-Hill, New York, 1964 [BJO64]

J. D. Bjorken and S. D. Drell, Relativistic Quantum Fields, McGraw-Hill, New York, 1965 [BJO65]

C. Itzykson and J. B. Zuber, Quantum Field Theory, McGraw-Hill, New York, 1985 [ITZ85]

T. D. Lee, Particle Physics and Introduction to Field Theory, Harwood, Chur, 1981, chapters 1-6 [LEE81]

J. J. Sakurai, Advanced Quantum Mechanics, Addison Wesley, Reading, 1967 [SAK67]

S. Schweber, An Introduction to Relativistic Quantum Field Theory, Harper & Row, New York, 1961 [SCH61]

Symmetries (chapters 4, 8)

Lie groups

H. Georgi, Lie Algebras in Particle Physics, Benjamin/Cummings, Reading, 1982 [GEO82]

T. D. Lee, Particle Physics and Introduction to Field Theory, Harwood, Chur, 1981 [LEE81]

D. B. Lichtenberg, Unitary Symmetry and Elementary Particles, 2nd ed., Academic, New York, 1978 [LIC78]

Elementary Particle Physics

S. Gasiorowicz, Elementary Particle Physics, John Wiley, New York, 1966 [GAS66]

O. Nachtmann, Elementarteilchenphysik, Phänomene und Konzepte, Vieweg, Braunschweig, 1986 [NAC86]

Particle Data Group, Review of Particle Properties, Phys. Lett. 170B (1986) [PDG86]

D. H. Perkins, Introduction to High Energy Physics, 3rd ed., Addison Wesley, London, 1987 [PER87]

Chiral Symmetry (chapters 5, 6)

I. J. R. Aitchison, An Informal Introduction to Gauge Field Theories, Cambridge University Press, Cambridge, 1982 [AIT82]

J. Bernstein, 'Spontaneous Symmetry Breaking, Gauge Theories, the Higgs Mechanism and all that', Rev. Mod. Phys. 46, 7 (1974) [BER74]

D. K. Campbell, 'Field Theory, Chiral Symmetry, and Pion-Nucleus Interactions', in: Nuclear Physics with Heavy-Ions and Mesons, Vol. 2, *p.* 549, North Holland, Amsterdam, 1978 [CAM78]

B. W. Lee, Chiral Dynamics, Gordon and Breach, New York, 1972 [LEE72]

Gauge Symmetries (chapters 7, 9, 10)

I. J. R. Aitchison, An Informal Introduction to Gauge Field Theories, Cambridge University Press, Cambridge, 1982 [AIT82]

M. Creutz, Quarks, Gluons and Lattices, Cambridge University Press, Cambridge, 1983 [CRE83]

F. Halzen, Quarks and Leptons: An Introductory Course in Modern Particle Physics, Wiley, New York, 1984 [HAL84]

K. Huang, Quarks, Leptons & Gauge Fields, World Scientific, Singapore, 1982 [HUA82]

C. Quigg, Gauge Theories of the Strong, Weak and Electromagnetic Interactions, Benjamin/Cummings, Reading, 1983 [QUI83]

P. Schmüser, Feynman Graphen und Eichtheorien für Experimentalphysiker, Lecture Notes in Physics, vol. 295, Springer, Berlin, (1988) [SCH88]

Quark Gluon Plasma

J. Cleymans, R. V. Gavai and E. Suhoven, 'Quarks and Gluons at High Temperatures and Densities', Phys. Rep. 130, 217 (1986) [CLE86]

M. Gyulassy, 'Introduction to QCD Thermodynamics and the Quark-Gluon Plasma', Progress Part. Nucl. Phys. 15, (1986) [GYU86]

L. McLerran, 'The Physics of the Quark-Gluon Plasma', Rev. Mod. Phys. 58, 1021 (1986) [MCL86]

B. Müller, 'The Physics of the Quark-Gluon Plasma', Lecture Notes in Physics, Vol. 225, Berlin, 1985 [MUE85]

Quarks (chapter 8)

F. E. Close, An Introduction to Quarks and Partons, Academic Press, London, 1979 [CLO79]

S. Gasiorowicz, Elementary Particle Physics, John Wiley, New York, 1966 [GAS66]

D. B. Lichtenberg, Unitary Symmetry and Elementary Particles, 2nd ed., Academic, New York, 1978 [LIC78]

Bag Models (chapter 11)

R. F. Alvarez-Estrada, F. Fernandez, J. L. Sanchez-Gomez, and V. Vento, 'Models of Hadron Structure Based on Quantum Chromodynamics', Lecture Notes in Physics, Vol. 259, Springer, Berlin, 1986 [ALV86]

F. E. Close, An Introduction to Quarks and Partons, Academic Press, London, 1979 [CLO79]

T. De Grand, R. L. Jaffe, K. Johnson, and J. Kiskis, 'Masses and other Parameters of the Light Hadrons', Phys. Rev. D12, 1060, (1975) [DEG75]

C. De Tar and J. F. Donoghue, 'Bag Models of Hadrons', Ann. Rev. Nucl. Part. Sci. 33, 235 (1983) [DET83]

K. Johnson, 'The M.I.T. Bag Model', Acta Phys. Pol. B6, 865 (1975) [JOH75]

G. Plunien, B. Müller, and W. Greiner, 'The Casimir Effect', Phys. Rep. 134, 87 (1986) [PLU86]

A. W. Thomas, 'Chiral Symmetry and the Bag Model: A New Starting Point for Nuclear Physics', Adv. Nucl. Phys. 13, 1 (1984) [THO84]

C. W. Wong, 'An Overview of Quarks in Nuclei', Phys. Rep. 136, 3 (1986) [WON86]

Soliton Models (chapter 12)

G. Adkins, C. R. Nappi, and E. Witten, 'Static Properties of Nucleons in the Skyrme Model', Nucl. Phys. B228, 552 (1983) [ANW83]

M. K. Banerjee, W. Broniowski, and T. D. Cohen, 'A Chiral Quark Soliton Model', in: "Chiral Solitons", ed. K.-F. Liu, World Scientific, Singapore, 1987 [BAN87]

G. E. Brown and M. Rho, 'Towards a Basis in QCD for Nuclear Physics', Comm. Nucl. Part. Phys. 15, 245 (1986) [BRO86]

T. D. Cohen and W. Broniowski, 'Semiclassical Projection of Hedgehog Models with Quarks', Phys. Rev. D34, 3472 (1986) [COH86]

R. Goldflam and L. Wilets, 'Soliton Bag Model', Phys. Rev. D25, 1951 (1982) [GOL82]

G. Holzwarth and B. Schwesinger, 'Baryons in the Skyrme Model', Reports on Progress in Physics 49, 825 (1986) [HOL86]

A. D. Jackson and M. Rho, 'Baryons as Chiral Solitons', Phys. Rev. Lett. 51, 751 (1983) [JAC83]

A. D. Jackson, D. E. Kahana, L. Vepstas, H. Verschelde, and W. Wüst, 'Baryon Observables in the Chiral Bag Model', Nucl Phys. A462, 661 (1987) [JAC87]

S. Kahana and G. Ripka, 'Baryon Density of Quarks Coupled to a Chiral Field', Nucl. Phys. A429, 462 (1984) [KAH84]

T. D. Lee, Particle Physics and Introduction to Field Theory, Harwood, Chur, 1981, chapter 20 [LEE81]

G. Ripka, 'Quark-Loop and Bag Models', Proc. Int. School of Intermediate Energy Physics, Verona, 1985 [RIP85]

T. H. R. Skyrme, 'A Unified Theory of Mesons and Baryons', Nucl. Phys. 31, 556 (1962) [SKY62]

L. Wilets, Quark Models of Hadronic Interactions, Lectures at the Advanced Course in Theoretical Physics, Cape Town, 1984 [WIL84]

L. Wilets, 'The Non-Topological Soliton Bag', in: Chiral Solitons, ed. K.F. Liu, World Scientific, Singapore 1987 [WIL87]

E. Witten, 'Global Aspects of Current Algebra', Nucl. Phys. B223, 422 (1983); 'Current Algebra Baryons, and Quark Deconfinement', Nucl. Phys. B223, 433 (1983) [WIT83]

I. Zahed and G. E. Brown, 'The Skyrme Model', Phys. Rep. 142, 1 (1986) [ZAB86]

Bibliography

[AIT82] I. J. R. Aitchison, An Informal Introduction to Gauge Field Theories, Cambridge University Press, Cambridge, 1982

[ANW83] G. Adkins, C. R. Nappi, and E. Witten, 'Static Properties of Nucleons in the Skyrme Model', Nucl. Phys. B228, 552 (1983)

[ALV86] R. F. Alvarez-Estrada, F. Fernandez, J. L. Sanchez-Gomez, and V. Vento, Models of Hadron Structure Based on Quantum Chromodynamics, Lecture Notes in Physics, Vol. 259, Springer, Berlin, 1986

[BAN87] M. K. Banerjee, W. Broniowski, and T. D. Cohen, 'A Chiral Quark Soliton Model', in: "Chiral Solitons", ed. K.-F. Liu, World Scientific, Singapore, 1987

[BER74] J. Bernstein, 'Spontaneous Symmetry Breaking, Gauge Theories, the Higgs Mechanism and all that', Rev. Mod. Phys. 46, 7 (1974)

[BJO64] J. D. Bjorken and S. D. Drell, Relativistic Quantum Mechanics, McGraw-Hill, New York 1964

[BJO65] J. D. Bjorken and S. D. Drell, Relativistic Quantum Fields, McGraw-Hill, New York 1965

[BRO86] G. E. Brown and M. Rho, 'Towards a Basis in QCD for Nuclear Physics', Comm. Nucl. Part. Phys. 15, 245 (1986)

[CAM78] D. K. Campbell, 'Field Theory, Chiral Symmetry, and Pion-Nucleus Interactions', in: Nuclear Physics with Heavy-Ions and Mesons, Vol. 2, *p.* 549, North Holland, Amsterdam, 1978

[CLE86] J. Cleymans, R. V. Gavai and E. Suhoven, 'Quarks and Gluons at High Temperatures and Densities', Phys. Rep. 130, 217 (1986)

[CLO79] F. E. Close, An Introduction to Quarks and Partons, Academic Press, London, 1979

[COH86] T. D. Cohen and W. Broniowski, 'Semiclassical Projection of Hedgehog Models with Quarks', Phys. Rev. D34, 3472 (1986)

[CRE83] M. Creutz, Quarks, Gluons and Lattices, Cambridge University Press, Cambridge, 1983

[DEG75] T. De Grand, R. L. Jaffe, K. Johnson, and J. Kiskis, 'Masses and other Parameters of the Light Hadrons', Phys. Rev. D12, 2060 (1975)

[DET83] C. De Tar and J.F. Donoghue, 'Bag Models of Hadrons', Ann. Rev. Nucl. Part. Sci. 33, 235 (1983)

[GAS66] S. Gasiorowicz, Elementary Particle Physics, Wiley, New York, 1966

[GEO82] H. Georgi, Lie Algebras in Particle Physics, Benjamin-Cummings, Reading, 1982

[GOL82] R. Goldflam and L. Wilets, 'Soliton Bag Model', Phys. Rev. D25, 1951 (1982)

[GYU86] M. Gyulassy, 'Introduction to QCD Thermodynamics and the Quark-Gluon Plasma', Progress Part. Nucl. Phys. 15, (1986)

[HAL84] F. Halzen, Quarks and Leptons: An Introductory Course in Modern Particle Physics, Wiley, New York, 1984

[HOL86] G. Holzwarth and B. Schwesinger, 'Baryons in the Skyrme Model', Reports on Progress in Physics 49, 825 (1986)

[HUA82] K. Huang, Quarks, Leptons & Gauge Fields, World Scientific, Singapore, 1982

[ITZ85] C. Itzykson and J.B. Zuber, 'Quantum Field Theory', McGraw-Hill, New York, 1985

[JAC83] A. D. Jackson and M. Rho, 'Baryons as Chiral Solitons', Phys. Rev. Lett. 51, 751 (1983)

[JAC87] A. D. Jackson, D. E. Kahana, L. Vepstas, H. Verschelde, and W. Wüst, 'Baryon Observables in the Chiral Bag Model', Nucl. Phys. A462, 661 (1987)

[JOH75] K. Johnson, 'The M.I.T. Bag Model', Acta Phys. Pol. B6, 865 (1975)

[KAH84] S. Kahana and G. Ripka, 'Baryon Density of Quarks Copuled to a Chiral Field', Nucl. Phys. A 429, 462 (1984)

[LEE72] B. W. Lee, Chiral Dynamics, Gordon and Breach, New York 1972

[LEE81] T. D. Lee, Particle Physics and Introduction to Field Theory, Harwood, Chur, 1981

[LIC78] D. B. Lichtenberg, Unitary Symmetry and Elementary Particles, 2nd ed., Academic, New York, 1978

[MCL86] L. McLerran, 'The Physics of the Quark-Gluon Plasma', Rev. Mod. Phys. 58, 1021 (1986)

[MUE85] B. Müller, 'The Physics of the Quark-Gluon Plasma', Lecture Notes in Physics, Vol. 225, Springer, Berlin, 1985

[NAC86] O. Nachtmann, Elementarteilchenphysik, Phänomene und Konzepte, Vieweg, Braunschweig, 1986

[PDG90] Particle Data Group, 'Review of Particle Properties', Phys. Lett. B239 (1990)

[PER87] D. H. Perkins, Introduction to High Energy Physics, 3rd ed., Addison Wesley, London, 1987

[PLU86] G. Plunien, B. Müller, and W. Greiner, 'The Casimir Effect', Phys. Rep. 134, 87 (1986)

[QUI83] C. Quigg, Gauge Theories of the Strong, Weak and Electromagnetic Interactions, Benjamin/Cummings, Reading 1983

[RIN80] P. Ring and P. Schuck, The Nuclear Many-Body Problem, Springer, New York, 1980

[RIP85] G. Ripka, 'Quark-Loop and Bag Models', Proc. Int. School of Intermediate Energy Physics, Verona, 1985

[SAK67] J. J. Sakurai, Advanced Quantum Mechanics, Addison Wesley, Reading, 1967

[SCH61] S. Schweber, An Introduction to Relativistic Quantum Field Theory, Harper & Row, New York, 1961

[SCH88] P. Schmüser, Feynman Graphen und Eichtheorien für Experimentalphysiker, Lecture Notes in Physics, vol. 295, Springer, Berlin, 1988

[SKY62] T. H. R. Skyrme, ‘A Unified Field Theory of Mesons and Baryons’, Nucl. Phys. 31, 556 (1962)

[THO84] A. W. Thomas, ‘Chiral Symmetry and the Bag Model: A New Starting Point for Nuclear Physics’, Adv. Nucl. Phys. 13, 1 (1984)

[WIL84] L. Wilets, Quark Models of Hadronic Interactions, Lectures at the Advanced Course in Theoretical Physics, Cape Town, 1984

[WIL87] L. Wilets, ‘The Non-Topological Soliton Bag’, in: “Chiral Solitons”, ed. K. F. Liu, World Scientific, Singapore, 1987

[WIT83] E.Witten, ‘Global Aspects of Current Algebra’, Nucl. Phys. B223, 422 (1983); ‘Current Algebra, Baryons, and Quark Deconfinement’, Nucl. Phys. B223, 433 (1983)

[WON86] C. W. Wong, ‘An Overview of Quarks in Nuclei’, Phys. Rep. 136, 3 (1986)

[ZAB86] I. Zahed and G. E. Brown, ‘The Skyrme Model’, Phys. Rep. 142, 1 (1986)

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