CHAPTER 10
ELECTROMAGNETIC INDUCTION

10.1 *Introduction*

In 1820, Oersted had shown that an electric current generates a magnetic field. But can a magnetic field generate an electric current? This was answered almost simultaneously and independently in 1831 by Joseph Henry in the United States and Michael Faraday in Great Britain. Faraday constructed an iron ring, about six inches in diameter. He wound two coils of wire tightly around the ring; one coil around one half (semicircle) of the ring, and the second coil around the second half of the ring. The two coils were not connected to one another other than by sharing the same iron core. One coil (which I'll refer to as the "primary" coil") was connected to a battery; the other coil (which I'll refer to as the "secondary" coil) was connected to a galvanometer. When the battery was connected to the primary coil a current, of course, flowed through the primary coil. This current generated a magnetic field throughout the iron core, so that there was a magnetic field inside each of the two coils. As long as the current in the primary coil remained constant, there was no current in the secondary coil. What Faraday observed was that at the instant when the battery was connected to the primary, and during that brief moment when the current in the primary was rising from zero, a current momentarily flowed in the secondary – but only while the current in the primary was changing. When the battery was disconnected, and during the brief moment when the primary current was falling to zero, again a current flowed in the secondary (but in the opposite direction to previously). Of course, while the primary current was changing, the magnetic field in the iron core was changing, and Faraday recognized that a current was generated in the secondary *while the magnetic flux through it was changing.* The strength of the current depended on the resistance of the secondary, so it is perhaps more fundamental to note that when the magnetic flux through a circuit changes, an electromotive force (EMF) is generated in the circuit, and the faster the flux changes, the greater the induced EMF. Quantitative measurements have long established that:

While the magnetic flux through a circuit is changing, an EMF is generated in the circuit which is equal to the rate of change of magnetic flux $\dot{\Phi}_B$ through the circuit.

This is generally called "**Faraday's Law** of Electromagnetic Induction". A complete statement of the laws of electromagnetic induction must also tell us the *direction* of the induced EMF, and this is generally given in a second statement usually known as "Lenz's Law of Electromagnetic Induction", which we shall describe in Section 10.2. When asked, therefore, for the laws of electromagnetic induction, both laws must be given: Faraday's, which deals with the magnitude of the EMF, and Lenz's, which deals with its direction.

You will note that the statement of Faraday's Law given above, says that the induced EMF is not merely "proportional" to the rate of change of magnetic B-flux, but is *equal* to it. You will therefore want to refer to the dimensions of electromotive force (SI unit: volt) and of *B*-flux (SI unit: weber) and verify that $\dot{\Phi}_B$ is indeed dimensionally similar to EMF. This alone does not tell you the constant of proportionality between the induced EMF and $\dot{\Phi}_B$, though the constant is in

fact unity, as stated in Faraday's law. You may then ask: Is this value of 1 for the constant of proportionality between the EMF and $\dot{\Phi}_B$ an experimental value (and, if so, how close to 1 is it, and what is its currently determined best value), or is it expected theoretically to be exactly 1? Well, I suppose it has to be admitted that physics is an experimental science, so that from that point of view the constant has to be determined experimentally. But I shall advance an argument shortly to show not only that you would expect it to be exactly 1, but that the very phenomenon of electromagnetic induction is only to be expected from what we already knew (before embarking upon this chapter) about electricity and magnetism.

Incidentally, we recall that the SI unit for Φ_B is the weber (Wb). To some, this is not a very familiar unit and some therefore prefer to express Φ_B in T m^2. Yet again, consideration of Faraday's law tells us that a perfectly legitimate SI unit (which many prefer) for Φ_B is V s.

10.2 *Electromagnetic Induction and the Lorentz Force*

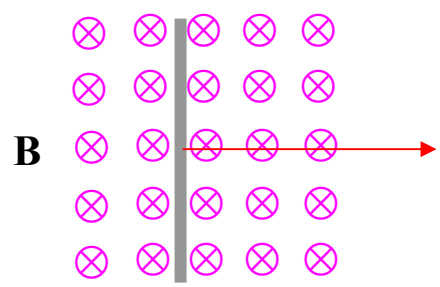


FIGURE X.1

Imagine that there is a uniform magnetic field directed into the plane of the paper (or your computer screen), as in figure X.1. Suppose there is a metal rod, as in the figure, and that the rod is being moved steadily to the right. We know that, within the metal, there are many free conduction electrons, not attached to any particular atom, but free to wander about inside the metal. As the metal rod is moved to the right, these free conduction electrons are also moving to the right and therefore they experience a Lorentz $q\,\mathbf{v} \times \mathbf{B}$ force, which moves them down (remember – electrons are negatively charged) towards the bottom end of the rod. Thus the movement of the rod through the magnetic field induces a potential difference across the ends of the rod. We have achieved electromagnetic induction, and, seen this way, there is nothing new: electromagnetic induction is nothing more than the Lorentz force on the conduction electrons within the metal.

You may speculate that, as an aircraft flies through Earth's magnetic field, a potential difference will be induced across the wingtips. You might try to imagine how you might set up an experiment to detect or measure this. You might also speculate that, as seawater flows up the English Channel, a potential difference is induced between England and France. You might also ask yourself: What if the rod were stationary, and the magnetic field were moving to the left? That's an interesting

discussion for lunchtime: Can you imagine the magnetic field moving to the left? Who's to say whether the rod or the field is moving?

If we were somehow to connect the ends of the rod in figure X.1 to a closed circuit, we might cause a current to flow – and we would then have made an electric generator. Look at figure X.2.

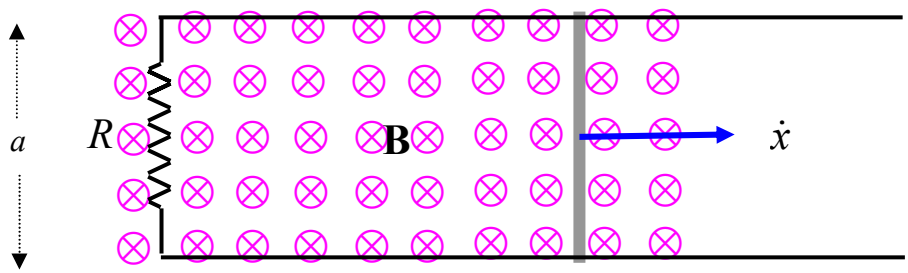


FIGURE X.2

We imagine that our metal bar is being pulled steadily to the right at speed υ, and that it is in contact with, and sliding smoothly without friction upon, two rails a distance a apart, and that the rails are connected via a resistance R. As a consequence, a current I flows in the circuit in the direction shown, counterclockwise. (The current is, of course, made up of negative conduction electrons moving clockwise.) Now the magnetic field will exert a force on the current in the rod. The force on the rod will be $a\,\mathbf{I} \times \mathbf{B}$; that is aIB acting to the left. In order to keep the rod moving steadily at speed υ to the right against this force, *work* will have to be done at a rate $aIB\upsilon$. The work will be dissipated in the resistance at a rate $I\,V$ where V is the induced EMF. Therefore the induced EMF is $Ba\upsilon$. But $a\upsilon$ is the rate at which the area of the circuit is increasing, and $Ba\upsilon$ is the rate at which the magnetic B-flux through the circuit is increasing. Therefore the induced EMF is *equal to* the rate of change of magnetic flux through the circuit. Thus we have predicted Faraday's law quantitatively merely from what we already know about the forces on currents and charged particles in a magnetic field.

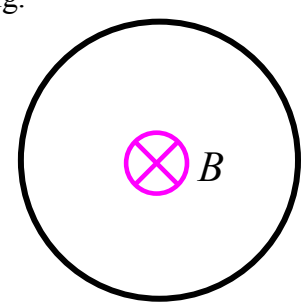
10.3 *Lenz's Law*

We can now address ourselves to the *direction* of the induced EMF. From our knowledge of the Lorentz force $q\,\mathbf{v} \times \mathbf{B}$ we see that the current flows counterclockwise, and that this results in a force on the rod that is in the opposite direction to its motion. But, even if we did not know this law, or had forgotten the formula, or if we didn't understand a vector product, we could see that this must be so. For, suppose that we move the rod to the right, and that, as a consequence, there will be a force also the right. Then the rod moves faster, and the force to the right is greater, and the rod moves yet faster, and so on. The rod would accelerate indefinitely, for the expenditure of no work. No – this cannot be right. The direction of the induced EMF must be such as to *oppose* the change of flux that causes it. This is merely a consequence of conservation of energy, and it can be stated as **Lenz's Law**:

When an EMF is induced in a circuit as a result of changing magnetic flux through the circuit, the direction of the induced EMF is such as to *oppose* the change of flux that causes it.

In our example of Section 10.2, we increased the magnetic flux through a circuit by increasing the area of the circuit. There are other ways of changing the flux through a circuit. For example, in figure X.3, we have a circular wire and a magnetic field perpendicular to the plane of the circle, directed into the plane of the drawing.

FIGURE X.3



We could increase the magnetic flux through the coil by increasing the strength of the field rather than by increasing the area of the coil. The rate of increase of the flux would then be $A\dot{B}$ rather than $\dot{A}B$. We could imagine increasing B, for example by moving a magnet closer to the coil, or by moving the coil into a region where the magnetic field was stronger; or, if the magnetic field is generated by an electromagnet somewhere, by increasing the current in the electromagnet. One way or another, we increase the strength of the field through the coil. An EMF is generated in the coil equal to the rate of change of magnetic flux, and consequently a current flows in the coil. In which direction does this induced current flow? It flows in such a direction as to *oppose* the increase in B that causes it. That is, the current flows counterclockwise in the coil. If this were not so, and the induced current were clockwise, this would still further increase the flux through the coil, and the current would increase further, and the flux would increase further, and so on. A runaway increase in the current and the field would result, and energy would not be conserved.

If we were in *decrease* the strength of the field through the coil, a current would flow clockwise in the coil – i.e. in such a sense as to tend to increase the field – i.e. to oppose the decrease in field that we are trying to impose. It may well occur to you at this stage that it is impossible to increase the current in a circuit instantaneously, and it takes a finite time to establish a new level of current. This is correct – a point to which we shall return later, when indeed we shall calculate just how long it does take.

Another way in which we could change the magnetic flux through a coil would be to *rotate* the coil in a magnetic field. For example, in figure X.4a, we see a magnetic field directed to the right, and a coil whose normal is perpendicular to the field. There is no magnetic flux through the coil. If we now rotate the coil, as in figure X.4b, the flux through the coil will increase, an EMF will be induced in the coil, equal to the rate of increase of flux, and a current will flow. The current will flow in a direction such that the magnetic moment of the coil will be as shown, which will result in an opposition to our imposed rotation on the coil, and the current will flow in the direction indicated by the symbols $\odot$ and $\otimes$.

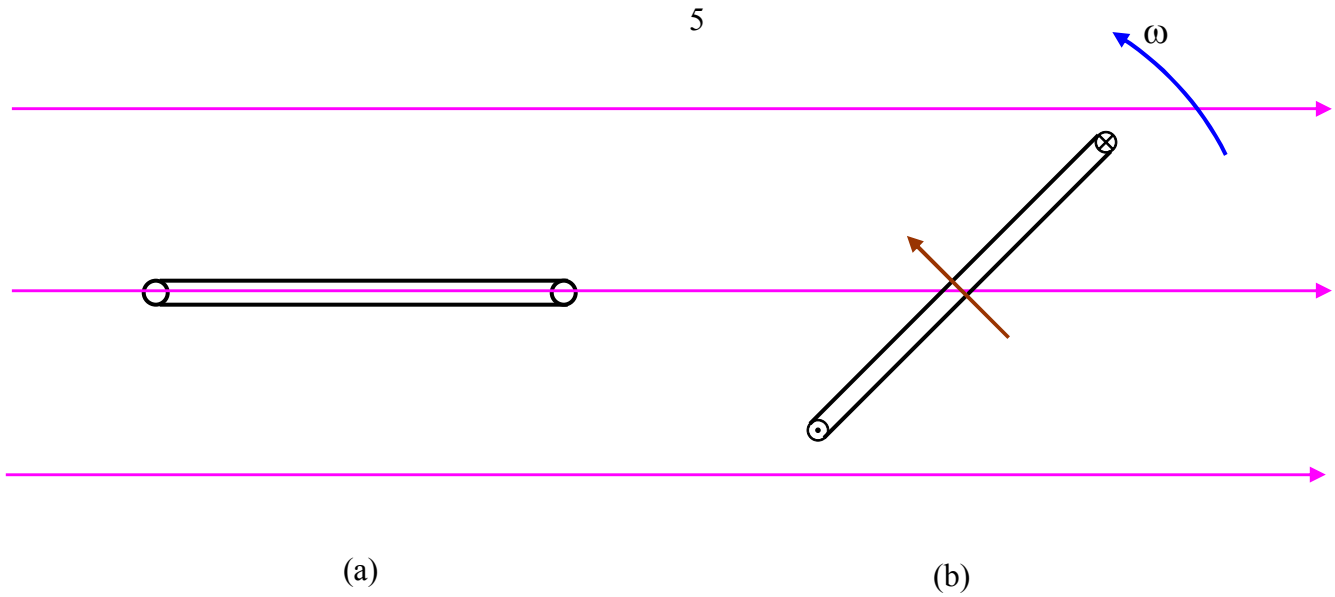


FIGURE X.4

If the flux through a coil changes at a rate $\dot{\Phi}_B$, and if the coil is not just a single turn but is made of N turns, the induced EMF will be $\dot{\Phi}_B$ *per turn*, so that the induced EMF in the coil as a whole will be $N\dot{\Phi}_B$.

10.4 *Ballistic Galvanometer and the Measurement of Magnetic Field*

A galvanometer is similar to a sensitive ammeter, differing mainly in that when no current passes through the meter, the needle is in the middle of the dial rather than at the left hand end. A galvanometer is used not so much to *measure* a current, but rather to *detect* whether or not a current is flowing, and in which direction. In the *ballistic* galvanometer, the motion of the needle is undamped, or as close to undamped as can easily be achieved. If a small quantity of electricity is passed through the ballistic galvanometer in a time that is short compared with the period of oscillation of the needle, the needle will jerk from its rest position, and then swing to and fro in lightly damped harmonic motion. (It would be simple harmonic motion if it could be completely undamped.) The amplitude of the motion, or rather the extent of the first swing, depends on the quantity of electricity that was passed through the galvanometer. It could be calibrated, for example, by discharging various capacitors through it, and making a table or graph of amplitude of swing versus quantity of electricity passed.

Now, if we have a small coil of area A, N turns, resistance R, we could place the coil perpendicular to a magnetic field B, and then connect the coil to a ballistic galvanometer. Then, suddenly (in a time that is short compared with the oscillation period of the galvanometer), remove the coil from the field (or rotate it through 90°) so that the flux through the coil goes from AB to zero. While the

flux through the coil is changing, and EMF will be induced, equal to $NA\dot{B}$, and consequently a current will flow momentarily through the coil of magnitude

$$I = NA\dot{B}/(R + r), \qquad 10.4.1$$

where r is the resistance of the galvanometer. Integrate this with respect to time, with initial condition $Q = 0$ when $t = 0$, and we find for the total quantity of electricity that flows through the galvanometer

$$Q = NAB/(R + r). \qquad 10.4.2$$

Since Q can be measured from the amplitude of the galvanometer motion, the strength of the magnetic field, B is determined.

I mentioned that the ballistic galvanometer differs from that of an ordinary galvanometer or ammeter in that its motion is *undamped*. The motion of the needle in an ordinary ammeter is damped, so that the needle doesn't swing violently whenever the current is changed, and so that the needle moves promptly and purposefully towards its correct position. How is this damping achieved?

The coil of a moving-coil meter is wound around a small aluminium frame called a *former*. When the current through the ammeter coil is changed, the coil – and the former – swing round; but a current is induced in the former, which gives the former a magnetic moment in such a sense as to *oppose* and therefore dampen the motion. The resistance of the former is made just right so that critical damping is achieved, so that the needle reaches its equilibrium position in the least time without overshoot or swinging. The little aluminium former does not look as if it were an important part of the instrument – but in fact its careful design is very important!

10.5 *AC Generator*

This and the following section will be devoted to *generators* and *motors*. I shall not be concerned with – and indeed am not knowledgeable about – the engineering design or practical details of real generators or motors, but only with the scientific principles involved. The "generators" and "motors" of this chapter will be highly idealized abstract concepts bearing little obvious resemblance to the real things. Need an engineering student, then, pay any attention to this? Well, of course, all real generators and motors obey and are designed around these very scientific principles, and they wouldn't work unless their designers and builders had a very clear knowledge and understanding of the basic principles.

The rod sliding on rails in a magnetic field described in Section 10.2 in fact was a D.C. (direct current) generator. I now describe an A.C. (alternating current) generator.

In figure X.5 we have a magnetic field **B**, and inside the field we have a coil of area **A** (yes – area is a vector) and N turns. The coil is being physically turned counterclockwise by some outside agency at an angular speed ω radians per second. I am not concerned with who, what or how it is

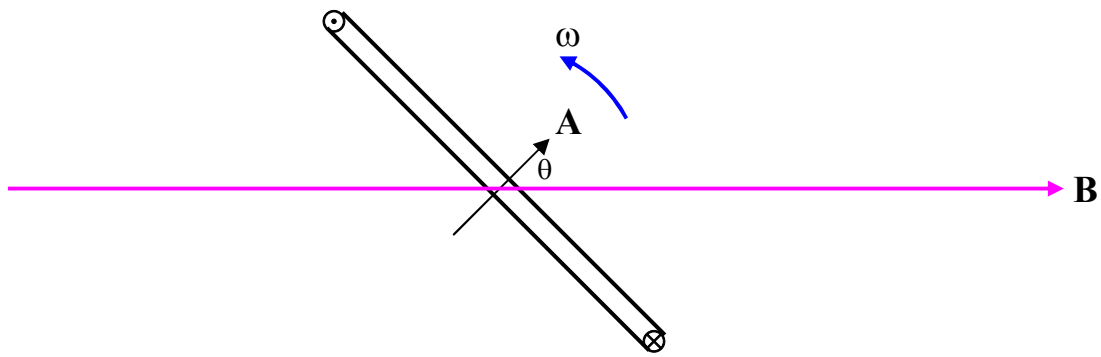


FIGURE X.5

being physically turned. For all I know, it might be turned by a little man turning a hand crank, or by a steam turbine driven by a coil- or oil-burning plant, or by a nuclear reactor, or it might be driven by a water turbine from a hydroelectric generating plant, or it might be turned by having something rubbing against the rim of your bicycle wheel. All I am interested in is that it is being mechanically turned at an angular speed ω. As the coil turns, the flux through it changes, and a current flows through the coil in a direction such that the magnetic moment generated for the coil is in the direction indicated for the area **A** in figure X.5, and also indicated by the symbols ⊙ and ⊗. This will result in an opposition to rotation of the coil; whoever or whatever is causing the coil to rotate will experience some opposition to his efforts and will have to do work. You can also deduce the direction of the induced current by considering the direction of the Lorentz force on the electrons in the wire of the coil.

At the instant illustrated in figure X.5, the flux through the coil is $AB\cos\theta$, or $AB\cos\omega t$, if we assume that $\theta = 0$ at $t = 0$. The rate of change of flux through the coil at this instant is the time derivative of this, or $-AB\,\omega\sin\omega t$. The magnitude of the induced EMF is therefore

$$V = NAB\omega\sin\omega t = \hat{V}\sin\omega t, \qquad 10.5.1$$

where $\hat{V}$ (" V-peak") is the peak or maximum EMF, given by

$$\hat{V} = NAB\omega. \qquad 10.5.2$$

Are you surprised that the peak EMF is proportional to N ? To A ? To B ? To ω ? Verify that $NAB\,\omega$ has the correct dimensions for $\hat{V}$.

The peak EMF occurs when the flux through the coil is *changing* most rapidly; this occurs when $\theta = 90^{\circ}$, at which time the coil is horizontal and the flux through it is zero.

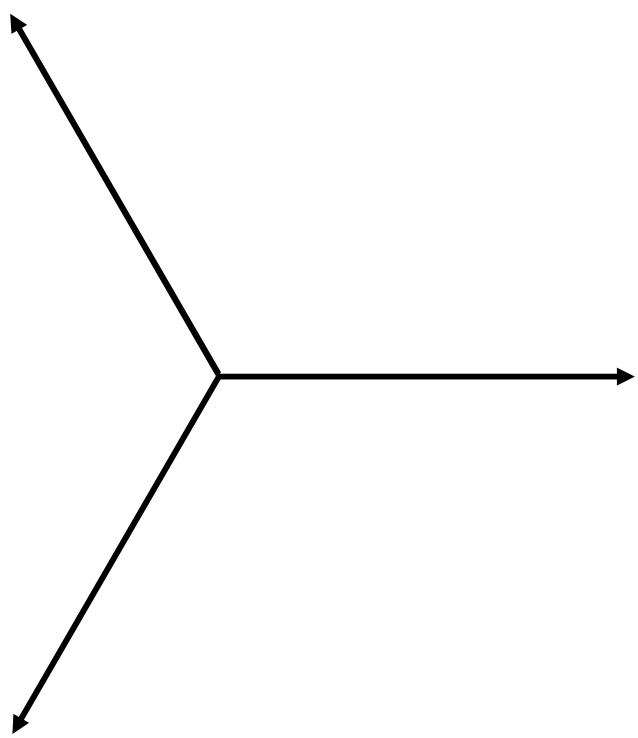
The leads from the coil can be connected to an external circuit via a pair of *slip rings* through which they can deliver current to the circuit.

The actual physical design of a generator is beyond the scope of this chapter and indeed of my expertise, though all depend on the physical principles herein described. In the "design" (such as it is) that I have described, the coil in which the EMF is induced is the *rotor* while the magnet is the *stator* – but this need not always be the case, and indeed designs are perfectly possible in which the magnet is the rotor and the coil the stator. In my design, too, I have assumed that there is but one coil – but there might be several in different planes. For example, you might have three coils whose planes make angles of 120° with each other. Each then generates a sinusoidal voltage, but the phase of each differs by 120° from the phases of the other two. This enables the delivery of power to three circuits. In a common arrangement these three circuits are not independent, but each is connected to a common line. The EMF in this common line is then made up three sine waves differing in phase by 120° :

$$V = \hat{V}[\sin \omega t + \sin(\omega t + 120^o) + \sin(\omega t + 240^o)]. \qquad 10.5.3$$

There are several ways in which you can see what this is like. For example, you could calculate this expression for numerous values of t and plot the function out as a graph. Or you could expand the expressions $V = \sin(\omega t + 120^o)$ and $V = \sin(\omega t + 240^o)$, and gather the various terms together to see what you get. (I recommend trying this.) Or you could simply add the three components in a *phase diagram*:

FIGURE X.6



It then becomes obvious that the sum is *zero*, and this line is the *neutral* line, the other three being *live* lines.

10.6 *AC Power*

When a current I flows through a resistance R, the rate of dissipation of electrical energy as heat is $I^2 R$. If an alternating potential difference $V = \hat{V}\sin\omega t$ is applied across a resistance, then an alternating current $I = \hat{I}\sin\omega t$ will flow through it, and the rate at which energy is dissipated as heat will also change periodically. Of interest is the *average* rate of dissipation of electrical energy as heat during a complete cycle of period $P = 2\pi/\omega$.

Let W = instantaneous rate of dissipation of energy, and $\overline{W}$ = average rate over a cycle of period $P = 2\pi/\omega$. Then

$$\overline{W}P = \int_0^P W dt = R\int_0^P I^2 dt = R\hat{I}^2\int_0^P \sin^2\omega t dt$$

$$= \tfrac{1}{2}R\hat{I}^2\int_0^P (1 - \cos 2\omega t)dt = \tfrac{1}{2}R\hat{I}^2[1 - \tfrac{1}{2\omega}\sin 2\omega t]_0^{P=2\pi/\omega} = \tfrac{1}{2}R\hat{I}^2. \qquad 10.6.1$$

Thus
$$\overline{W} = \tfrac{1}{2}R\hat{I}^2. \qquad 10.6.2$$

The expression $\tfrac{1}{2}\hat{I}^2$ is the mean value of I^2 over a complete cycle. Its square root $\hat{I}/\sqrt{2} = 0.707\hat{I}$ is the *root mean square* value of the current, I_{RMS}. Thus the average rate of dissipation of electrical energy is

$$\overline{W} = RI_{\text{RMS}}^2. \qquad 10.6.3$$

Likewise, the RMS EMF (pardon all the abbreviations) over a complete cycle is $\hat{V}/\sqrt{2}$.

Often when an AC current or voltage is quoted, it is the RMS value that is meant rather than the peak value. I recommend that in writing or conversation you always *make it explicitly clear* which you mean.

Also of interest is the *mean* induced voltage $\overline{V}$ over half a cycle. (Over a full cycle, the mean voltage is, of course, zero.) We have

$$\overline{V}P/2 = \int_0^{P/2} V dt = \hat{V}\int_0^{P/2}\sin\omega t dt = \frac{\hat{V}}{\omega}[\cos\omega t]_{\frac{P}{2}=\frac{\pi}{2}}^{0}$$

$$= \frac{\hat{V}}{\omega}(1 - \cos\pi) = \frac{2\hat{V}}{\omega}. \qquad 10.6.4$$

Remembering that $P = 2\pi/\omega$, we see that

$$\hat{V} = \frac{2\overline{V}}{\pi} = 0.6366\hat{V} = \frac{2\sqrt{2}V_{\text{RMS}}}{\pi} = 0.9003V_{\text{RMS}}. \qquad 10.6.5$$

10.7 *Linear Motors and Generators*

Most (but not all!) real motors and generators are, of course, rotary. In this section I am going to describe highly idealized and imaginary linear motors and generators, only because the geometry is simpler than for rotary motors, and it is easier to explain certain principles. We'll move on the rotary motors afterwards.

In figure X.7 I compare a motor and a generator. In both cases there is supposed to be an external magnetic field (from some external magnet) directed away from the reader. A metal rod is resting on a pair of conducting rails.

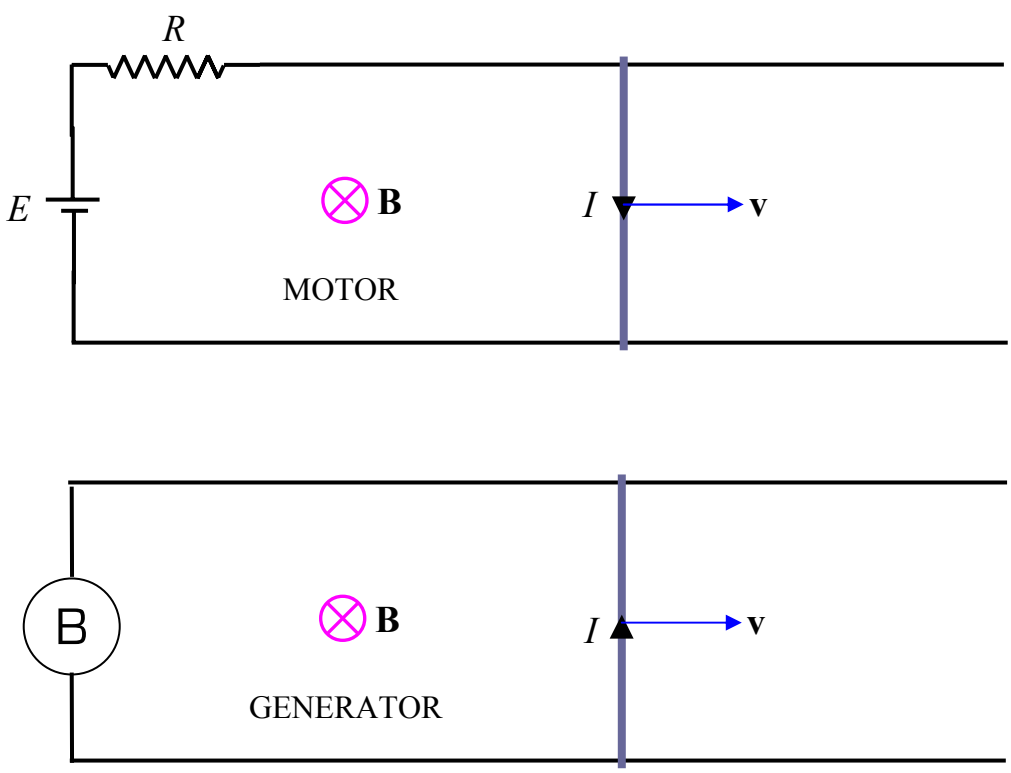


FIGURE X.7

In the motor, a battery is connected in the circuit, causing a current to flow clockwise around the circuit. The interaction between the current and the external magnetic field produces a force on the rod, moving it to the right.

In the generator, the rod is moved to the right by some externally applied force, and a current is induced counterclockwise. If the B inside the circle represents a light bulb, a current will flow through the bulb, and the bulb will light up.

Let us suppose that the rails are smooth and frictionless, and suppose that, in the motor, the rod isn't pulling any weight. That is to say, suppose that there is no mechanical load on the motor. How fast will the rod move? Since there is a force moving the rod to the right, will it continue to accelerate indefinitely to the right, with no limit to its eventual speed? No, this is not what happens. When the switch is first closed and the rod is stationary, a current will flow, given by $E = IR$, where E is the EMF of the battery and R is the total resistance of the circuit. However, when the rod has reached a speed υ, the area of the circuit is increasing at a rate $a\upsilon$, and a *back EMF* (which opposes the EMF of the battery), of magnitude $a\upsilon B$ is induced, so the net EMF in the circuit is now $E - a\upsilon B$ and the current is correspondingly reduced according to

$$E - a\upsilon B = IR. \qquad 10.7.6$$

Eventually the rod reaches a limiting speed of $E/(aB)$, at which point no further current is being taken from the battery, and the rod (sliding as it is on frictionless rails with no mechanical load) then obeys Newton's first law of motion – namely it will continue in its state of uniform motion, because no forces are no acting upon it.

Problem 1. Show that the speed increases with time according to

$$\upsilon = \frac{E}{aB}\left(1 - \exp\left(-\frac{(aB)^2}{mR}\right)t\right), \qquad 10.7.7$$

where m is the mass of the rod.

Problem 2. Show that the time for the rod to reach half of its maximum speed is

$$t_{1/2} = \frac{mR\ln 2}{(aB)^2}. \qquad 10.7.8$$

Problem 3. Suppose that $E = 120$ V, $a = 1.6$ m, $m = 1.92$ kg and $R = 4\ \Omega$. If the rod reaches a speed of 300 m s^{-1} in 300 s, what is the strength of the magnetic field?

I'll give solutions to these problems at the end of this section. Until then – no peeking.

In a frictionless *rotary* motor, the situation would be similar. Initially the current would be E/R, but, when the motor is rotating with angular speed ω, the average back EMF is $2NAB\omega/\pi$ (recall

equation 10.5.5), and by the time this has reached the EMF of the battery, the frictionless, loadless coil carries on rotating at constant angular speed, taking no current from the battery.

Now let's go back to our linear motor consisting of a metal rod lying on two rails, but this time suppose that there is some mechanical resistance to the motion. This could be either because there is friction between the rod and the rails, or perhaps the rod is dragging a heavy weight behind it, or both. One way or another, let us suppose that the rod is subjected to a constant force F towards the left. As before, the relation between the current and the speed is given by equation 10.5.6, but, when a steady state has been reached, the electromagnetic force aIB pulling the rod to the right is equal to the mechanical load F dragging the rod to the left. That is, $E - a\upsilon B = IR$ and $F = a I B$. If we eliminate I between these two equations, we obtain

$$E - a\upsilon B = \frac{FR}{aB}, \qquad 10.7.9$$

or

$$\upsilon = \frac{E}{a B} - \frac{R}{(a B)^2} F. \qquad 10.7.10$$

This equation, which relates the speed at which the motor runs to the mechanical load, is called the *motor performance characteristic*. In our particular motor, the performance characteristic shows that the speed at which the motor runs decreases steadily as the load is increased., and the motor runs to a grinding halt for a load equal to $a B E / R$. (Verify that this has the dimensions of force.) The current is then E/R. This current may be quite large. If you physically prevent a real motor from turning by applying a mechanical torque to it so large that the motor cannot move, a large current will flow through the coil – large enough to heat and possibly fuse the coil. You will hear a sharp crack and see a little puff of smoke.

If we multiply equation 10.7.6 by I, we obtain

$$E I = a I B\upsilon + I^2 R, \qquad 10.7.11$$

or

$$E I = F\upsilon + I^2 R. \qquad 10.7.12$$

This shows that the power produced by the battery goes partly into doing external mechanical work, and the remainder is dissipated as heat in the resistance. Restrain the motor so that $\upsilon = 0$, and *all* of that $E I$ goes into $I^2 R$.

If you were physically to move the rod to the right at a speed faster than the equilibrium speed, the back EMF becomes greater than the battery EMF, and current flows back into the battery. The device is then a generator rather than a motor.

The nature of the performance characteristic varies with the details of motor design. You may not want a motor whose speed decreases so drastically with load. You may have to decide in advance what sort of performance characteristic you want the motor to have, depending on what tasks you want it to perform, and then you have to design the motor accordingly. We shall mention some possibilities in the next section.

Now – the promised solutions to the problems.

Solution to Problem 1.

When the speed of the rod is υ, the net EMF in the circuit is $E - aB\upsilon$, so the current is $(E - aB\upsilon)/R$, and so the force on the rod will be $aB(E - aB\upsilon)/R$ and the acceleration $d\upsilon/dt$ will be $aB(E - aB\upsilon)/(mR)$. The equation of motion is therefore

$$\frac{d\upsilon}{E - aB\upsilon} = \frac{aB}{mR}dt. \qquad 10.7.13$$

Integration, with $\upsilon = 0$ when $t = 0$, gives the required equation 10.7.7.

Solution to Problem 2.

Just put $\upsilon = \dfrac{E}{2aB}$ in equation 10.7.7 and solve for t. Verify that the expression has the dimensions of time.

Solution to Problem 3.

Put the given numbers into equation 10.7.7 to get

$$B = \tfrac{1}{4}(1 - e^{-100B^2}) \qquad 10.7.14$$

and solve this for B. (Nice and easy. But if you are not experienced in solving equations such as this, the Newton-Raphson process is described in Chapter 1 of the Celestial Mechanics notes of this series. This equation would be good practice.) I make the answer 0.0440 tesla.

10.8 *Rotary Motors*

Most real motors, of course, are rotary motors, though all of the principles described for our highly idealized linear motor of the previous section still apply.

Current is fed into a coil (known as the *armature)* via a *split-ring commutator* and the coil therefore develops a magnetic moment. The coil is in a magnetic field, and it therefore experiences a torque. (Figure X.5) The coil rotates and soon its magnetic moment vector will be parallel to the field and there would be no further torque – except that, at that instant, the split-ring commutator reverses the direction of the current in the coil, and hence reverses the direction of the magnetic moment. Thus the coil continues to rotate until, half a period later, its new magnetic moment again lines up with the magnetic field, and the commutator again reverses the direction of the moment.

As in the case of the linear motor, the coil reaches a maximum angular speed, which depends on the mechanical load (this time a torque). and the relation between the maximum angular speed and the torque is the motor performance characteristic.

Also, as with a generator, there may be several coils (with a corresponding number of sections in the commutator), and it is also possible to design motors in which the armature is the stator and the magnet the rotor – but I am not particularly knowledgeable about the detailed engineering designs of real motors – except that all of them depend upon the same scientific principles.

In all of the foregoing, it has been assumed that the magnetic field is constant, as if produced by a permanent magnet. In real motors, the field is generally produced by an *electromagnet*. (Some types of iron retain their magnetism permanently unless deliberately demagnetized. Others become magnetized only when placed in a strong magnetic field such as produced by a solenoid, and they lose most of their magnetization as soon as the magnetizing field is removed.)

The field coils may be wound in series with the armature coil (a series-wound motor) or in parallel with it (a shunt-wound motor), or even partly in series and partly in parallel (a compound-wound motor). Each design has it own performance characteristic, depending on the use for which it is intended.

With a single coil rotating in a magnetic field, the induced back EMF varies periodically, the average value being, as we have seen, $2NAB\omega/\pi$. In practice the coil may be wound around many slots placed around the perimeter of a cylindrical core every few degrees, and there are a corresponding number of sections in the split-ring commutator. The back EMF is then less variable than with a single coil, and, although the formula $2NAB\omega/\pi$ is no longer appropriate, the back EMF is still proportional to $B\omega$. We can write the average back EMF as $KB\omega$, where the *motor constant K* depends on the detailed geometry of a particular design.

Shunt-wound Motor. In the shunt-wound motor, the field coil is wound in parallel to the armature coil. In this case, the back EMF generated in the armature does not affect the current in the field coil, so the motor operates rather as previously described for a constant field. That is, the motor performance characteristic, giving the equilibrium angular speed in terms of the mechanical load (torque, τ) is given by

$$\omega = \frac{E}{KB} - \frac{R}{(KB)^2}\tau. \qquad 10.8.1$$

Here, R is the armature resistance. In practice, there may be a variable resistance (rheostat) in series with the field coil, so that the current through the field coil – and hence the field strength – can be changed.

Series-wound Motor. The field coil is wound in series with the armature, and the motor performance characteristic is rather different that for the shunt-wound motor. If the magnet core does not saturate, then, to a linear approximation, the field is proportional to the current, and the back EMF is proportional to the product of the current I and the angular speed ω - so let's say that the back EMF is $kI\omega$. We then have

$$E - kI\omega = IR, \qquad 10.8.2$$

where E is the externally applied EMF (from a battery, for example) and R is the total resistance of field coil plus armature.

Multiply both sides by I:

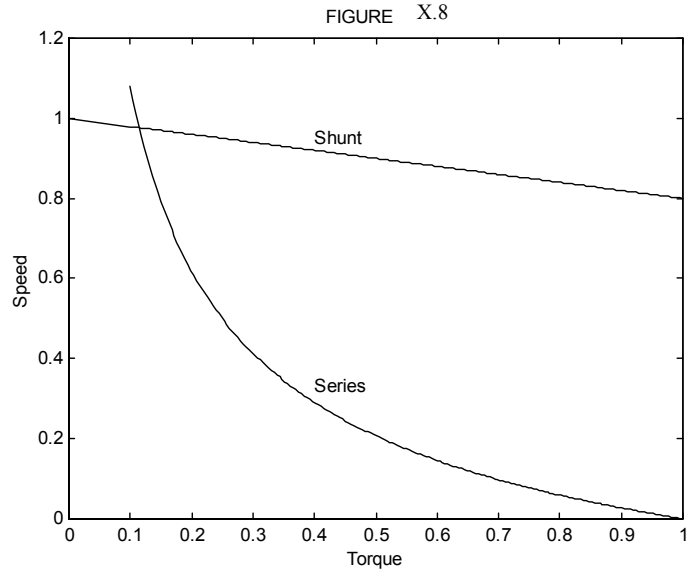
$$EI - kI^2\omega = I^2R. \qquad 10.8.3$$

The term EI is the power supplied by the battery and I^2R is the power dissipated as heat. Thus the rate of doing mechanical work is $kI^2\omega$, which shows that the torque exerted by the motor is $\tau = kI^2$. If we now substitute $\sqrt{\tau / k}$ for I in equation 10.8.2, we obtain the motor performance characteristic – i.e. the relation between ω and τ:

$$\omega = \frac{E}{\sqrt{k\tau}} - \frac{R}{k}. \qquad 10.8.4$$

In figure X.8 we show the performance characteristics, in arbitrary units, for shunt- and series-wound motors, based in our linear analysis, which assumes in both cases no saturation of the electromagnet iron core. The maximum possible torque in both cases is the torque that makes $\omega = 0$ in the corresponding performance characteristic, namely KBE/R for the shunt-wound motor and kE^2/R for the series-wound motor. The latter goes to infinity for zero load. This does not happen in practice, because we have made some assumptions that are not real (such as no saturation of the magnet core, and also there can never be literally zero load), but nevertheless the analysis is sufficient to show the general characteristics of the two types.

FIGURE X.8



The characteristics of the two may be combined in a compound-wound motor, depending on the intended application. For example, a tape-recorder requires constant speed, whereas a car starter requires a high starting torque.

10.9 *The Transformer*

Two coils are wound on a common iron core. The primary coil is connected to an AC (alternating current) generator of (RMS) voltage V_1. If there are N_1 turns in the primary coil, the primary current will be proportional to V_1 / N_1 and, provided the core is not magnetically saturated, the magnetic field will also be proportional to this. The voltage V_2 induced in the secondary coil (of N_2 turns) will be proportional to N_2 and to the field, and so we have

$$\frac{V_2}{V_1} = \frac{N_2}{N_1} . \qquad 10.9.1$$

We shall give a more detailed analysis of the transformer in a later chapter. However, one aspect which can be noted here is that the rapidly-changing magnetic field induces *eddy currents* in the iron core, and for this reason the core is usually constructed of thin laminated sheets (or sometimes wires) insulated from each other to reduce these energy-wasting eddy currents. Sometimes these laminations vibrate a little unless tightly bound together, and this is often responsible for the "hum" of a transformer.

10.10 *Mutual Inductance*

Consider two coils, not connected to one another, other than being *close together in space*. If the current changes in one of the coils, so will the magnetic field in the other, and consequently an EMF will be induced in the second coil. **Definition:** The ratio of the EMF V_2 induced in the second coil to the rate of change of current $\dot{I}_1$ in the first is called the *coefficient of mutual inductance M* between the two coils:

$$V_2 = M\dot{I}_1. \qquad 10.10.1$$

The dimensions of mutual inductance can be found from the dimensions of EMF and of current, and are readily found to be ML^2Q^{-2}.

Definition: If an EMF of one volt is induced in one coil when the rate of change of current in the other is 1 amp per second, the coefficient of mutual inductance between the two is 1 *henry*, H.

Mental Exercise: If the current in coil 1 changes at a rate $\dot{I}_1$, the EMF induced in coil 2 is $M\dot{I}_1$. Now ask yourself this: If the current in coil 2 changes at a rate $\dot{I}_2$, is it true that the EMF induced in coil 1 will be $M\dot{I}_2$? (The answer is "yes" – but you are not excused the mental effort required to convince yourself of this.)

Example: Suppose that the primary coil is an infinite solenoid having n_1 turns per unit length wound round a core of permeability μ. Tightly would around this is a plain circular coil of N_2 turns. The solenoid and the coil wrapped tightly round it are of area A. We can calculate the mutual inductance of this arrangement as follows. The magnetic field in the primary is $\mu n_1 I$ so the flux through each coil is $\mu n_1 AI$. If the current changes at a rate $\dot{I}$, flux will change at a rate $\mu nA\dot{I}$, and the EMF induced in the secondary coil will be $\mu n_1 N_2 A\dot{I}$. Therefore the mutual inductance is

$$M = \mu n_1 N_2 A. \qquad 10.10.2$$

Several points:

1. Verify that this has the correct dimensions.

2 If the current in the solenoid changes in such a manner as to cause an increase in the magnetic field towards the right, the EMF induced in the secondary coil is such that, if it were connected to a closed circuit so that a secondary current flows, the direction of this current will produce a magnetic field towards the left – i.e. such as to oppose the rightward increase in B.

3. Because of the little mental effort you made a few minutes ago, you are now convinced that, if you were to change the current in the plane coil at a rate $\dot{I}$, the EMF induced in the solenoid would be $M\dot{I}$, where M is given by equation 10.10.2.

4. Equation 10.10.2 is the equation for the mutual inductance of the system, provided that the coil and the solenoid are *tightly coupled*. If the coil is rather loosely draped around the solenoid, or if the solenoid is not infinite in length, the mutual inductance would be rather less than given by equation 10.10.2. It would be, in fact, $k\mu n_1 N_2 A$, where k, a dimensionless number between 0 and 1, is the *coupling coefficient.*

5. While we have hitherto expressed permeability in units of tesla metres per amp (T m A^{-1}) or some such combination, equation 10.10.2 shows that permeability can equally well be (and usually is) expressed in henrys per metre, H m^{-1}. Thus, we say that the permeability of free space is $\mu_0 = 4\pi \times 10^{-7}$ H m^{-1}.

Exercise: A plane coil of 10 turns is tightly wound around a solenoid of diameter 2 cm having 400 turns per centimetre. The relative permeability of the core is 800. Calculate the mutual inductance. (I make it 0.126 H.)

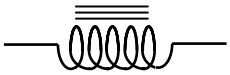
10.11 *Self Inductance*

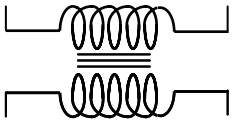
In this section we are dealing with the self inductance of a *single coil* rather than the mutual inductance between two coils. If the current through a single coil changes, the magnetic field inside that coil will change; consequently a back EMF will be induced in the coil that will oppose the change in the magnetic field and indeed will oppose the change of current. **Definition:** The ratio of

the back EMF to the rate of change of current is the *coefficient of self inductance L*. If the back EMF is 1 volt when the current changes at a rate of one amp per metre, the coefficient of self inductance is 1 henry.

Exercise: Show that the coefficient of self inductance (usually called simply the "inductance") of a long solenoid of length l and having n turns per unit length is $\mu n^2 Al$, where I'm sure you know what all the symbols stand for. Put some numbers in for an imaginary solenoid of your own choosing, and calculate its inductance in henrys.

The circuit symbol for inductance is

If a coil has an iron core, this may be indicated in the circuit by 

The symbol for a transformer is 

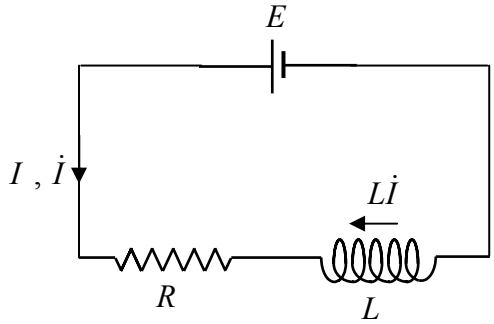
Finally, don't confuse self-inductance with self-indulgence.

10.12 *Growth of Current in a Circuit Containing Inductance*

It will have occurred to you that if the growth of current in a coil results in a back EMF which opposes the increase of current, current cannot change instantaneously in a circuit that contains inductance. This is correct. (Recall also that the potential difference in a circuit cannot change instantaneously in a circuit containing capacitance. Come to think of it, it is hardly possible for the capacitance or inductance of any circuit to be exactly zero; any real circuit must have some capacitance and inductance, even if very small.)

Consider the circuit of figure X.9. A battery of EMF E is in series with a resistance and an inductance. (A coil or solenoid or any inductor in general will have both inductance and resistance, so the R and the L in the figure may belong to one single item.) We have to be very careful about *signs* in what follows.

FIGURE X.9



When the circuit is closed (by a switch, for example) a current flows in the direction shown. by an arrow, which also indicates the direction of the *increase* of current. An EMF $L\dot{I}$ is induced in the opposite direction to $\dot{I}$. Thus, Ohm's law, or, if your prefer, Kirchhoff's second rule, applied to the circuit (watch the signs carefully) is

$$E - IR - L\dot{I} = 0. \qquad 10.12.1$$

Hence:

$$\int_0^I \frac{dI}{\frac{E}{R} - I} = \frac{R}{L}dt. \qquad 10.12.2$$

Warning: Some people find an almost irresistible urge to write this as $\int_0^I \frac{dI}{I - \frac{E}{R}} = -\frac{R}{L}dt.$

Don't!

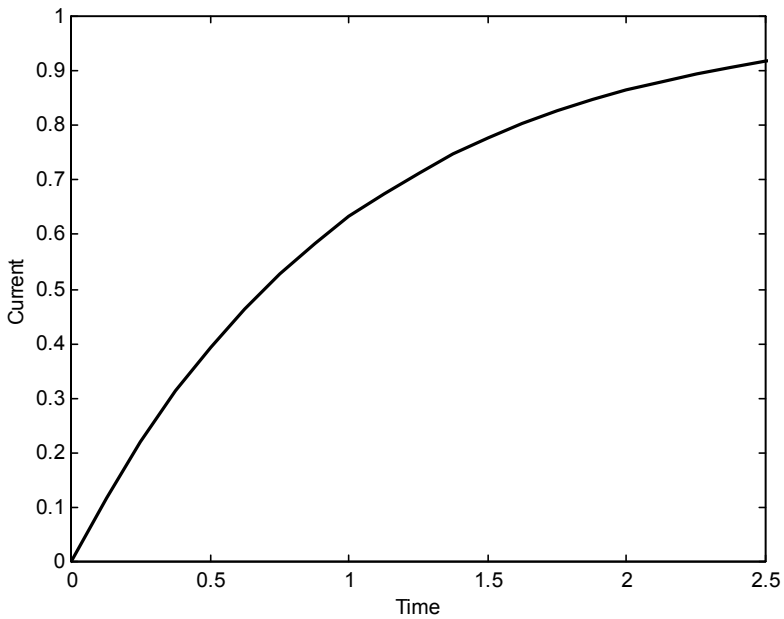
You can anticipate that the left hand side is going to be a logarithm, so make sure that the denominator is positive. You may recall a similar warning when we were charging and discharging a capacitor through a resistance.

Integration of equation 10.12.2 results in the following equation for the growth of the current with time:

$$I = \frac{E}{R}\left(1 - e^{-(R/L)t}\right). \qquad 10.12.3$$

Thus the current asymptotically approaches its ultimate value of E/R, reaching 63% (i.e. $1 - e^{-1}$) of its ultimate value in a time L/R. In figure X.10, the current is shown in units of E/R, and the time in units of L/R. You should check that L/R, which is called the *time constant* of the circuit, has the dimensions of time.

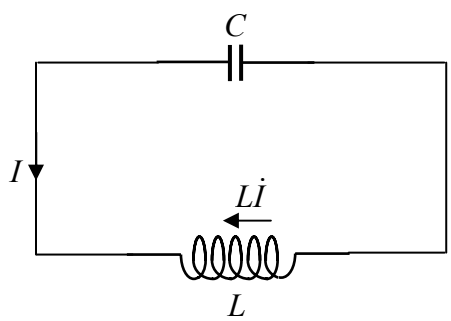
FIGURE X.10



10.13 *Discharge of a Capacitor through an Inductance*

The circuit is shown in figure X.11, and, once again, it is important to take care with the signs.

FIGURE X.11



If $+Q$ is the charge on the left hand plate of the capacitor at some time (and $-Q$ the charge on the right hand plate) the current I in the direction indicated is $-\dot{Q}$ and the potential difference across the plates is Q/C. The back EMF is in the direction shown, and we have

$$\frac{Q}{C} - L\dot{I} = 0, \qquad 10.13.1$$

or

$$\frac{Q}{C} + L\ddot{Q} = 0. \qquad 10.13.2$$

This can be written $$\ddot{Q} = -\frac{Q}{LC}, \qquad 10.13.3$$

which is simple harmonic motion of period $2\pi\sqrt{LC}$. (verify that this has dimensions of time.) Thus energy sloshes to and fro between storage as charge in the capacitor and storage as current in the inductor.

If there is resistance in the circuit, the oscillatory motion will be damped, the charge and current eventually approaching zero. But, even if there is no resistance, the oscillation does not continue for ever. While the details are beyond the scope of this chapter, being more readily dealt with in a discussion of electromagnetic radiation, the periodic changes in the charge in the capacitor and the current in the inductor, result in an oscillating electromagnetic field around the circuit, and in the generation of an electromagnetic wave, which carries energy away at a speed of $\sqrt{1/(\mu_0\varepsilon_0)}$. Verify that this has the dimensions of speed, and that it has the value 2.998×10^8 m s^{-1}. The motion in the circuit is damped just as if there were a resistance of $\sqrt{\mu_0/\varepsilon_0}$ in the circuit. Verify that this has the dimensions of resistance and that it has a value of 376.7 Ω. This effective resistance is called the *impedance of free space.*

10.14 *Discharge of a Capacitor through an Inductance and a Resistance*

This results in *damped* oscillatory motion. It is discussed in detail in Chapter 11 of the Classical Mechanics part of these notes, especially Section 11.6

10.15 *Energy Stored in an Inductance*

During the growth of the current in an inductor, at a time when the current is i and the rate of increase of current is $\dot{i}$, there will be a back EMF $L\dot{i}$. The rate of doing work against this back EMF is then $Li\dot{i}$. The work done in time dt is $Li\dot{i}dt = Lidi$, where di is the increase in current in time dt. The total work done when the current is increased from 0 to I is

$$L\int_0^I idi = \tfrac{1}{2}LI^2, \qquad 10.15.1$$

and this is the energy stored in the inductance. (Verify the dimensions.)

10.16 *Energy Stored in a Magnetic Field*

Recall your derivation (Section 10.11) that the inductance of a long solenoid is $\mu n^2 Al$. The energy stored in it, then, is $\frac{1}{2}\mu n^2 AlI^2$. The volume of the solenoid is Al, and the magnetic field is $B = \mu n I$, or $H = nI$. Thus we find that the energy stored per unit volume in a magnetic field is

$$\frac{B^2}{2\mu} = \tfrac{1}{2}BH = \tfrac{1}{2}\mu H^2 . \qquad 10.16.2$$

In a vacuum, the energy stored per unit volume in a magnetic field is $\frac{1}{2}\mu_0 H^2$ - even though the vacuum is absolutely empty!

Equation 10.16.2 is valid in any isotropic medium, including a vacuum. In an anisotropic medium, **B** and **H** are not in general parallel – unless they are both parallel to a crystallographic axis. More generally, in an anisotropic medium, the energy per unit volume is $\frac{1}{2}\mathbf{B}\bullet\mathbf{H}$.

Verify that the product of *B* and *H* has the dimensions of energy per unit volume.