CHAPTER 14
LAPLACE TRANSFORMS

14.1 *Introduction*

If y(x) is a function of x, where x lies in the range 0 to ∞, then the function $\bar{y}(p)$, defined by

$$\bar{y}(p) = \int_0^\infty e^{-px} y(x)dx \ , \qquad 14.1.1$$

is called the *Laplace transform* of $y(x)$. However, in this chapter, where we shall be applying Laplace transforms to electrical circuits, y will most often be a voltage or current that is varying with *time* rather than with "x". Thus I shall use t as our variable rather than x, and I shall use s rather than p (although it will be noted that, as yet, I have given no particular physical meaning to either p or to s.) Thus I shall define the Laplace transform with the notation

$$\bar{y}(s) = \int_0^\infty e^{-st} y(t)dt, \qquad 14.1.2$$

it being understood that t lies in the range 0 to ∞.

For short, I could write this as

$$\bar{y}(s) = \mathbf{L}\, y(t). \qquad 14.1.3$$

When we first learned differential calculus, we soon learned that there were just a few functions whose derivatives it was worth committing to memory. Thus we learned the derivatives of x^n, $\sin x$, e^x and a very few more. We found that we could readily find the derivatives of more complicated functions by means of a few simple rules, such as how to differentiate a product of two functions, or a function of a function, and so on. Likewise, we have to know only a very few basic Laplace transforms; there are a few simple rules that will enable us to calculate more complicated ones.

After we had learned differential calculus, we came across integral calculus. This was the inverse process from differentiation. We had to ask: What function would we have had to differentiate in order to arrive at this function? It was as though we were given the answer to a problem, and had to deduce what the question was. It will be a similar situation with Laplace transforms. We shall often be given a function $\bar{y}(s)$, and we shall want to know: what function $y(t)$ is this the Laplace transform of? In other words, we shall need to know the *inverse Laplace transform*:

$$y(t) = \mathbf{L}^{-1}\, \bar{y}(s). \qquad 14.1.4$$

We shall find that facility in calculating Laplace transforms and their inverses leads to very quick ways of solving some types of differential equations – in particular the types of differential equations that arise in electrical theory. We can use Laplace transforms to see the relations between varying current and voltages in circuits containing resistance, capacitance and inductance.

However, these methods are quick and convenient only if we are in constant daily practice in dealing with Laplace transforms with easy familiarity. Few of us, unfortunately, have the luxury of calculating Laplace transforms and their inverses on a daily basis, and they lose many of their advantages if we have to refresh our memories and regain our skills every time we may want to use them. It may therefore be asked: Since we already know perfectly well how to do AC calculations using complex numbers, is there any point in learning what just amounts to another way of doing the same thing? There is an answer to that. The theory of AC circuits that we developed in Chapter 13 using complex numbers to find the relations between current and voltages dealt primarily with *steady state conditions*, in which voltages and current were varying sinusoidally. It did not deal with the *transient* effects that might happen in the first few moments after we switch on an electrical circuit, or situations where the time variations are *not sinusoidal*. The Laplace transform approach will deal equally well with steady state, sinusoidal, non-sinusoidal and transient situations.

14.2 *Table of Laplace Transforms*

It is easy, by using equation 14.1.2, to derive all of the transforms shown in the following table, in which $t > 0$. (Do it!)

$y(t)$	$\bar{y}(s)$
1	$1/s$
t	$1/s^2$
$\dfrac{t^{n-1}}{(n-1)!}$	$1/s^n$
$\sin at$	$\dfrac{a}{s^2 + a^2}$
$\cos at$	$\dfrac{s}{s^2 + a^2}$
$\sinh at$	$\dfrac{a}{s^2 - a^2}$
$\cosh at$	$\dfrac{s}{s^2 - a^2}$
e^{at}	$\dfrac{1}{s-a}$

This table can, of course, be used to find inverse Laplace transforms as well as direct transforms. Thus, for example, $\mathbf{L}^{-1}\dfrac{1}{s-1} = e^t$. In practice, you may find that you are using it more often to find inverse transforms than direct transforms.

These are really all the transforms that it is necessary to know – and they need not be committed to memory if this table is handy. For more complicated functions, there are rules for finding the transforms, as we shall see in the following sections, which introduce a number of theorems. Although I shall derive some of these theorems, I shall merely state others, though perhaps with an example. Many (not all) of them are straightforward to prove, but in any case I am more anxious to introduce their applications to circuit theory than to write a formal course on the mathematics of Laplace transforms.

After you have understood some of these theorems, you may well want to apply them to a number of functions and hence greatly expand your table of Laplace transforms with results that you will discover on application of the theorems.

14.3 *The First Integration Theorem*

The theorem is:

$$\mathbf{L}\int_0^t y(x)dx = \frac{\bar{y}(s)}{s}. \qquad 14.3.1$$

Before deriving this theorem, here's a quick example to show what it means. The theorem is most useful, as in this example, for finding an *inverse* Laplace transform. I.e. $\mathbf{L}^{-1}\dfrac{\bar{y}(s)}{s} = \int_0^t y(x)dx$.

Calculate

$$\mathbf{L}^{-1}\frac{1}{s(s-a)}.$$

Solution. From the table, we see that $\mathbf{L}^{-1}\dfrac{1}{s-a} = e^{at}$. The integration theorem tells us that $\mathbf{L}^{-1}\dfrac{1}{s(s-a)} = \int_0^t e^{ax}dx = (e^{at}-1)/a$. You should now verify that this is the correct answer by substituting this in equation 14.1.2 and integrating – or (and!) using the table of Laplace transforms.

The proof of the theorem is just a matter of integrating by parts. Thus

$$\mathbf{L}\int_0^t y(x)\,dx = \int_0^\infty\left(\int_0^t y(x)dx\right)e^{-st}dt = -\frac{1}{s}\int_0^\infty\left(\int_0^t y(x)dx\right)d\left(e^{-st}\right)$$

$$= \left[-\frac{1}{s}e^{-st}\int_0^t y(x)dx\right]_{t=0}^{\infty} + \frac{1}{s}\int_0^\infty e^{-st}y(t)\,dt.$$

The expression in brackets is zero at both limits, and therefore the theorem is proved.

14.4 *The Second Integration Theorem (Dividing a Function by t)*

This theorem looks very like the first integration theorem, but "the other way round". It is

$$\mathbf{L}\left(\frac{y(t)}{t}\right) = \int_s^\infty \bar{y}(x)dx. \qquad 14.4.1$$

I'll leave it for the reader to derive the theorem. Here I just give an example of its use. Whereas the first integration theorem is most useful in finding inverse transforms, the second integration theorem is more useful for finding direct transforms.

Example: Calculate $\mathbf{L}\left(\frac{\sin at}{t}\right)$.

This means calculate $\int_0^\infty \frac{e^{-st}\sin at}{t}dt.$

While this integral can no doubt be done, you may find it a bit daunting, and the second integration theorem provides an alternative way of doing it, resulting in an easier integral.

Note that the right hand side of equation 14.4.1 is a function of s, not of x, which is just a dummy variable. The function $\bar{y}(x)$ is the Laplace transform, with x as argument, of $y(t)$. In our particular case, $y(t)$ is sin at, so that, from the table, $\bar{y}(x) = \frac{a}{a^2 + x^2}$. The second integration theorem , then, tells us that $\mathbf{L}\left(\frac{\sin at}{t}\right) = \int_s^\infty \frac{a}{a^2+x^2}dx.$ This is a much easier integral. It is $\left[\tan^{-1}\left(\frac{x}{a}\right)\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \tan^{-1}\left(\frac{a}{s}\right).$ You may want to add this result to your table of Laplace integrals. Indeed, you may already want to expand the table considerably by applying both integration theorems to several functions.

14.5 *Shifting Theorem*

This is a very useful theorem, and one that is almost trivial to prove. (Try it!) It is

$$\mathbf{L}\left(e^{-at}y(t)\right) = \bar{y}(s+a). \qquad 14.5.1$$

For example, from the table, we have $\mathbf{L}(t) = 1/s^2$. The shifting theorem tells us that

$\mathbf{L}\left(te^{-at}\right) = 1/(s+a)^2$. I'm sure you will now want to expand your table even more. Or you may want to go the other way, and cut down the table a bit! After all, you know that $\mathbf{L}(1) = 1/s$. The shifting theorem, then, tells you that $\mathbf{L}(e^{at}) = 1/(s-a)$, so that entry in the table is superfluous! Note that you can use the theorem to deduce either direct or inverse transforms.

14.6 *A Function Times t^n*

I'll just give this one with out proof:

For n a positive integer, $$L(t^n y) = (-1)^n \frac{d^2\overline{y}}{ds^n}. \qquad 14.6.1$$

Example: What is $\mathbf{L}(t^2 e^{-t})$?

Answer: For $y = e^{-t}$, $\overline{y} = 1/(s+1)$. $\therefore L(t^2 e^{-t}) = 2/(s+1)^3$.

Before proceeding further, I strongly recommend that you now apply theorems 14.3.1, 14.4.1, 14.5.1 and 14.6.1 to the several entries in your existing table of Laplace transforms and greatly expand your table of Laplace transforms. For example, you can already add $(\sin at)/t$, te^{-at} and $t^2 e^{-t}$ to the list of functions for which you have calculated the Laplace transforms.

14.7 *Differentiation Theorem*

$$\mathbf{L}\left(\frac{d^n y}{dt^n}\right) = s^n\overline{y} - s^{n-1}y_0 - s^{n-2}\left(\frac{dy}{dt}\right)_0 - s^{n-3}\left(\frac{d^2y}{dt^2}\right)_0 - \ldots \quad \ldots - s\left(\frac{d^{n-2}y}{dt^{n-2}}\right)_0 - \left(\frac{d^{n-1}y}{dt^{n-1}}\right)_0. \qquad 14.7.1$$

This looks formidable, and you will be tempted to skip it – but don't, because it is essential! However, to make it more palatable, I'll point out that one rarely, if ever, needs derivatives higher than the second, so I'll re-write this for the first and second derivatives, and they will look much less frightening.

$$\mathbf{L}\dot{y} = s\overline{y} - y_0 \qquad 14.7.2$$

and $$\mathbf{L}\ddot{y} = s^2\overline{y} - sy_0 - \dot{y}_0. \qquad 14.7.3$$

Here, the subscript zero means "evaluated at $t = 0$".

Equation 14.7.2 is easily proved by integration by parts:

$$\overline{y} = \mathbf{L}y = \int_0^\infty ye^{-st}dt = -\frac{1}{s}\int_0^\infty y\,de^{-st} = -\frac{1}{s}\left[ye^{-st}\right]_0^\infty + \frac{1}{s}\int_{t=0}^\infty e^{-st}dy$$

$$= \frac{1}{s}y_0 + \frac{1}{s}\int \dot{y}dt = \frac{1}{s}y_0 + \frac{1}{s}\mathbf{L}\dot{y}. \qquad 14.7.4$$

$\therefore$ $$L\dot{y} = s\bar{y} - y_0. \qquad 14.7.5$$

From this, $L\ddot{y} = s\bar{\dot{y}} - \dot{y}_0 = sL\dot{y} - \dot{y}_0 = s(s\bar{y} - y_0) - \dot{y}_0 = s^2\bar{y} - sy_0 - \dot{y}_0.$ 14.7.6

Apply this over and over again, and you arrive at equation 14.7.1

14.8 *A First Order Differential Equation*

Solve $\dot{y} + 2y = 3te^{-t}$, with initial condition $y_0 = 0$.

If you are in good practice with solving this type of equation, you will probably multiply it through by e^{2t}, so that it becomes

$$\frac{d}{dt}\left(ye^{2t}\right) = 3te^{2t},$$

from which $$y = (t - \tfrac{1}{3})e^t + Ce^{-2t}.$$

(You can now substitute this back into the original differential equation, to verify that it is indeed the correct solution.)

With the given initial condition, it is quickly found that $C = \frac{1}{3}$, so that the solution is

$$y = te^t - \tfrac{1}{3}e^t + \tfrac{1}{3}e^{-2t}.$$

Now, here's the same solution, using Laplace transforms.

We take the Laplace transform of both sides of the original differential equation:

$$s\bar{y} + 2\bar{y} = 3\mathbf{L}(te^t) = \frac{3}{(s-1)^2}.$$

Thus $$\bar{y} = \frac{3}{(s+2)(s-1)^2}.$$

Partial fractions: $$\bar{y} = \frac{1}{3}\left(\frac{1}{s+2}\right) - \frac{1}{3}\left(\frac{1}{s-1}\right) + \frac{1}{(s-1)^2}.$$

Inverse transforms: $y = \frac{1}{3}e^{-2t} - \frac{1}{3}e^{t} + te^{t}.$

You will probably admit that you can follow this, but will say that you can do this at speed only after a great deal of practice with many similar equations. But this is equally true of the first method, too.

14.9 *A Second Order Differential Equation*

Solve $\ddot{y} - 4\dot{y} + 3y = e^{-t}$

with initial conditions $y_0 = 1, \quad \dot{y}_0 = -1.$

You probably already know some method for solving this equation, so please go ahead and do it. Then, when you have finished, look at the solution by Laplace transforms.

Laplace transform: $s^2\bar{y} - s + 1 - 4(s\bar{y} - 1) + 3\bar{y} = 1/(s+1).$

(My! Wasn't that fast!)

A little algebra: $$\bar{y} = \frac{1}{(s-3)(s-1)(s+1)} + \frac{s-5}{(s-3)(s-1)}.$$

Partial fractions: $$\bar{y} = \frac{1}{8}\left(\frac{1}{s-3} - \frac{2}{s-1} + \frac{1}{s+1}\right) + \frac{2}{s-1} - \frac{1}{s-3},$$

or $$\bar{y} = \frac{1}{8}\left(\frac{1}{s+1}\right) + \frac{7}{4}\left(\frac{1}{s-1}\right) - \frac{7}{8}\left(\frac{1}{s-3}\right).$$

Inverse transforms: $y = \frac{1}{8}e^{-t} + \frac{7}{4}e^{t} - \frac{7}{8}e^{3t}$, and you can verify that this is correct by substitution in the original differential equation.

So: We have found a new way of solving differential equations. If (but only if) we have a lot of practice in manipulating Laplace transforms, and have used the various manipulations to prepare a slightly larger table of transforms from the basic table given above, and we can go from t to s and from s to t with equal facility, we can believe that our new method can be both fast and easy.

But, what has this to do with electrical circuits? Read on.

14.10 *Generalized Impedance*

We have dealt in Chapter 13 with a sinusoidally varying voltage applied to an inductance, a resistance and a capacitance in series. The equation that governs the relation between voltage and current is

$$V = L\dot{I} + RI + Q/C. \qquad 14.10.1$$

If we multiply by C, differentiate with respect to time, and write I for $\dot{Q}$, this becomes just

$$C\dot{V} = LC\ddot{I} + RC\dot{I} + I. \qquad 14.10.2$$

If we suppose that the applied voltage V is varying sinusoidally (that is, $V = \hat{V}e^{j\omega t}$, or, if you prefer, $V = \hat{V}\sin\omega t$), then the operator d^2/dt^2, or "double dot", is equivalent to multiplying by $-\omega^2$, and the operator d/dt, or "dot", is equivalent to multiplying by $j\omega$. Thus equation 14.10.2 is equivalent to

$$j\omega CV = -LC\omega^2 I + jRC\omega I + I. \qquad 14.10.3$$

That is,

$$V = [R + jL\omega + 1/jC\omega)]I. \qquad 14.10.4$$

The complex expression inside the brackets is the now familiar impedance Z, and we can write

$$V = IZ. \qquad 14.10.5$$

But what if V is not varying *sinusoidally*? Suppose that V is varying in some other manner, perhaps not even periodically? This might include, as one possible example, the situation where V is constant and not varying with time at all. But whether or not V varying with time, equation 14.10.2 is still valid – except that, unless the time variation is sinusoidally, we cannot substitute $j\omega$ for d/dt. We are faced with having to solve the differential equation 10.4.2.

But we have just learned a neat new way of solving differential equations of this type. We can take the Laplace transform of each side of the equation. Thus

$$C\overline{\dot{V}} = LC\overline{\ddot{I}} + RC\overline{\dot{I}} + \bar{I}. \qquad 14.10.6$$

Now we are going to make use of the differentiation theorem, equations 14.7.2 and 14.7.3.

$$C(s\bar{V} - V_0) = LC(s^2\bar{I} - sI_0 - \dot{I}_0) + RC(s\bar{I} - I_0) + \bar{I}. \qquad 14.10.7$$

Let us suppose that, at $t = 0$, V_0 and I_0 are both zero – i.e. before $t = 0$ a switch was open, and we close the switch at $t = 0$. Furthermore, since the circuit contains inductance, the current cannot change instantaneously, and, since it contains capacitance, the voltage cannot change instantaneously, so the equation becomes

$$\overline{V} = (R + Ls + 1/Cs)\bar{I}. \qquad 14.10.8$$

This is so regardless of the form of the variation of V: it could be sinusoidal, it could be constant, or it could be something quite different. This is a *generalized Ohm's law*. The *generalized impedance* of the circuit is $R + Ls + \frac{1}{Cs}$. Recall that in the complex number treatment of a steady-state sinusoidal voltage, the complex impedance was $R + jL\omega + \frac{1}{jC\omega}$.

To find out how the current varies, all we have to do is to take the inverse Laplace transform of

$$\bar{I} = \frac{\overline{V}}{R + Ls + 1/(Cs)}. \qquad 14.10.9$$

We look at a couple of examples in the next sections.

14.11 *RLC Series Transient*

A battery of constant EMF V is connected to a switch, and an R, L and C in series. The switch is closed at time $t = 0$. We'll first solve this problem by "conventional" methods; then by Laplace transforms. The reader who is familiar with the mechanics of damped oscillatory motion, such as is dealt with in Chapter 11 of the Classical Mechanics notes of this series, may have an advantage over the reader for whom this topic is new – though not necessarily so!

"Ohm's law" is
$$V = Q/C + RI + L\dot{I}, \qquad 14.11.1$$

or
$$LC\ddot{Q} + RC\dot{Q} + Q = CV. \qquad 14.11.2$$

Those who are familiar with this type of equation will recognize that the general solution (complementary function plus particular integral) is

$$Q = Ae^{\lambda_1 t} + Be^{\lambda_2 t} + CV, \qquad 14.11.3$$

where
$$\lambda_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \text{ and } \lambda_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}. \qquad 14.11.4$$

(Those who are not familiar with the solution of differential equations of this type should not give up here. Just go on to the part where we do this by Laplace transforms. You'll soon be streaking ahead of your more learned colleagues, who will be struggling for a while.)

Case I. $\dfrac{R^2}{4L^2} - \dfrac{1}{LC}$ is positive. For short I'm going to write equations 14.11.4 as

$$\lambda_1 = -a + k \quad \text{and} \quad \lambda_2 = -a - k. \qquad 14.11.5$$

Then
$$Q = Ae^{-(a-k)t} + Be^{-(a+k)t} + CV \qquad 14.11.6$$

and, by differentiation with respect to time,

$$I = -A(a-k)e^{-(a-k)t} - B(a+k)e^{-(a+k)t}. \qquad 14.11.7$$

At $t = 0$, Q and I are both zero, from which we find that

$$A = -\frac{(a+k)CV}{2k} \quad \text{and} \quad B = \frac{(a-k)CV}{2k}. \qquad 14.11.8$$

Thus
$$Q = \left[-\left(\frac{a+k}{2k}\right)e^{-(a-k)t} + \left(\frac{a-k}{2k}\right)e^{-(a+k)t} + 1\right]CV \qquad 14.11.9$$

and
$$I = \left[\left(\frac{a^2-k^2}{2k}\right)\left(e^{-(a-k)t} - e^{-(a+k)t}\right)\right]CV. \qquad 14.11.10$$

On recalling the meanings of a and k and the sinh function, and a little algebra, we obtain

$$I = \frac{V}{Lk}e^{-at}\sinh kt. \qquad 14.11.11$$

Exercise: Verify that this equation is dimensionally correct. Draw a graph of $I : t$. The current is, of course, zero at $t = 0$ and ∞. What is the maximum current, and when does it occur?

Case II. $\dfrac{R^2}{4L^2} - \dfrac{1}{LC}$ is zero. In this case, those who are in practice with differential equations will obtain for the general solution

$$Q = e^{\lambda t}(A + Bt) + CV, \qquad 14.11.12$$

where
$$\lambda = -R/(2L), \qquad 14.11.13$$

from which
$$I = \lambda(A + Bt)e^{\lambda t} + Be^{\lambda t}. \qquad 14.11.14$$

After applying the initial conditions that Q and I are initially zero, we obtain

$$Q = \left[1 - \left(1 - \frac{Rt}{2L}\right)e^{-RT/(2L)}\right] \qquad 14.11.15$$

and

$$I = \frac{V}{L}te^{-Rt/(2L)}. \qquad 14.11.16$$

As in case II, this starts and ends at zero and goes through a maximum, and you may wish to calculate what the maximum current is and when it occurs.

Case III. $\frac{R^2}{4L^2} - \frac{1}{LC}$ is negative. In this case, I am going to write equations 14.11.4 as

$$\lambda_1 = -a + j\omega \text{ and } \lambda_2 = -a - j\omega, \qquad 14.11.17$$

where

$$a = \frac{R}{2L} \text{ and } \omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2}. \qquad 14.11.18$$

All that is necessary, then, is to repeat the analysis for Case I, but to substitute $-\omega^2$ for k^2 and $j\omega$ for k, and, provided that you know that $\sinh j\omega t = j\sin\omega t$, you finish with

$$I = \frac{V}{L\omega}e^{-at}\sin\omega t. \qquad 14.11.19$$

This is lightly damped oscillatory motion.

Now let us try the same problem using Laplace transforms. Recall that we have a V in series with an R , L and C, and that initially Q, I and $\dot{I}$ are all zero. (The circuit contains capacitance, so Q cannot change instantaneously; it contains inductance, so I cannot change instantaneously.)

Immediately, automatically and with scarcely a thought, our first line is the generalized Ohm's law, with the Laplace transforms of V and I and the generalized impedance:

$$\bar{V} = [R + Ls + 1/(Cs)]\bar{I}. \qquad 14.11.20$$

Since V is constant, reference to the very first entry in your table of transforms shows that $\bar{V} = V/s$, and so

$$\bar{I} = \frac{V}{s[R + Ls + 1/(Cs)]} = \frac{V}{L(s^2 + bs + c)}, \qquad 14.11.21$$

where

$$b = R/L \text{ and } c = 1/(LC). \qquad 14.11.22$$

Case I. $b^2 > 4c$.

$$\bar{I} = \frac{V}{L}\left(\frac{1}{(s-\alpha)(s-\beta)}\right) = \frac{V}{L}\left(\frac{1}{\alpha-\beta}\right)\left(\frac{1}{s-\alpha} - \frac{1}{s-\beta}\right). \qquad 14.11.23$$

Here, of course, $2\alpha = -b + \sqrt{b^2 - 4c}$ and $2\beta = -b - \sqrt{b^2 - 4c}$ 14.11.24

On taking the inverse transforms, we find that

$$I = \frac{V}{L}\left(\frac{1}{\alpha-\beta)}\right)(e^{\alpha t} - e^{\beta t}). \qquad 14.11.25$$

From there it is a matter of routine algebra (do it!) to show that this is exactly the same as equation 14.11.11.

In order to arrive at this result, it wasn't at all necessary to know how to solve differential equations. All that was necessary was to understand generalized impedance and to look up a table of Laplace transforms.

Case II. $b^2 = 4c$.

In this case, equation 14.11.21 is of the form

$$\bar{I} = \frac{V}{L}\cdot\frac{1}{(s-\alpha)^2}, \qquad 14.11.26$$

where $\alpha = -\frac{1}{2}b$. If you have dutifully expanded your original table of Laplace transforms, as suggested, you will probably already have an entry for the inverse transform of the right hand side. If not, you know that the Laplace transform of t is $1/s^2$, so you can just apply the shifting theorem to see that the Laplace transform of $t\,e^{\alpha t}$ is $1/(s-\alpha)^2$. Thus

$$I = \frac{V}{L}te^{\alpha t} \qquad 14.11.27$$

which is the same as equation 14.11.16.

[Gosh – what could be quicker and easier than that!?]

Case III. $b^2 < 4c$.

This time, we'll complete the square in the denominator of equation 14.11. 21:

$$\bar{I} = \frac{V}{L}\cdot\frac{1}{(s+\frac{1}{2}b)^2 + (c-\frac{1}{4}b^2)} = \frac{V}{L\omega}\frac{\omega}{(s+\frac{1}{2}b)^2 + \omega^2}, \qquad 14.11.28$$

where I have introduced ω with obvious notation.

On taking the inverse transform (from our table, with a little help from the shifting theorem) we obtain

$$I = \frac{V}{L\omega}\cdot e^{-\frac{1}{2}bt}\sin\omega t, \qquad 14.11.29$$

which is the same as equation 14.11.19.

With this brief introductory chapter to the application of Laplace transforms to electrical circuitry, we have just opened a door by a tiny crack to glimpse the potential great power of this method. With practice, it can be used to solve complicated problems of many sorts with great rapidity. All we have so far is a tiny glimpse. I shall end this chapter with just one more example, in the hope that this short introduction will whet the reader's appetite to learn more about this technique.

14.12 *Another Example*

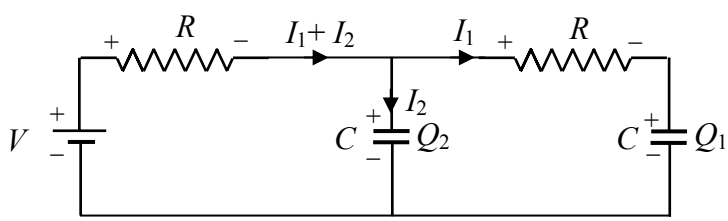

FIGURE XIV.1

The circuit in figure XIV.1 contains two equal resistances, two equal capacitances, and a battery. The battery is connected at time $t = 0$. Find the charges held by the capacitors after time t.

Apply Kirchhoff's second rule to each half:

$$(\dot{Q}_1 + \dot{Q}_2)RC + Q_2 = CV, \qquad 14.12.1$$

and

$$\dot{Q}_1 RC + Q_1 - Q_2 = 0. \qquad 14.12.2$$

Eliminate Q_2:

$$R^2C^2\ddot{Q}_1 + 3RCQ_1 + Q_1 = CV. \qquad 14.12.3$$

Transform, with Q_1 and $\dot{Q}_1$ initially zero:

$$(R^2C^2s^2 + 3RCs + 1)\overline{Q_1} = \frac{CV}{s}. \qquad 14.12.4$$

I.e.
$$R^2C\overline{Q_1} = \frac{1}{s(s^2 + 3as + a^2)}\cdot V, \qquad 14.12.5$$

where
$$a = 1/(RC). \qquad 14.12.6$$

That is
$$R^2C\overline{Q_1} = \frac{1}{s(s + 2.618a)(s + 0.382a)}V. \qquad 14.12.7$$

Partial fractions:
$$R^2C\overline{Q_1} = \left[\frac{1}{s} + \frac{0.1708}{s+2.618a} - \frac{1.1708}{s+0.382a}\right]\frac{V}{a^2}. \qquad 14.12.8$$

That is,
$$\overline{Q_1} = \left[\frac{1}{s} + \frac{0.1708}{s+2.618a} - \frac{1.1708}{s+0.382a}\right]CV. \qquad 14.12.8$$

Inverse transform:
$$Q_1 = \left[1 + 0.1708e^{-2.618t/(RC)} - 1.1708e^{-0.382t/(RC)}\right] \qquad 14.12.9$$

The current can be found by differentiation.

I leave it to the reader to eliminate Q_1 from equations 14.12.1 and 2 and hence to show that

$$Q_2 = \left[1 - 0.2764e^{-2.618/(RC)} - 0.7236e^{-0.382/(RC)}\right]. \qquad 14.12.10$$