CHAPTER 15
MAXWELL'S EQUATIONS

15.1 *Introduction*

One of Newton's great achievements was to show that all of the phenomena of classical mechanics can be deduced as consequences of three basic, fundamental laws, namely Newton's laws of motion. It was likewise one of Maxwell's great achievements to show that all of the phenomena of classical electricity and magnetism – all of the phenomena discovered by Oersted, Ampère, Henry, Faraday and others whose names are commemorated in several electrical units – can be deduced as consequences of four basic, fundamental equations. We describe these four equations in this chapter, and, in passing, we also mention Poisson's and Laplace's equations. We also show how Maxwell's equations predict the existence of electromagnetic waves that travel at a speed of 3×10^8 m s^{-1}. This is the speed at which light is measured to move, and one of the most important bases of our belief that light is an electromagnetic wave.

Before embarking upon this, we may need a reminder of two mathematical theorems, as well as a reminder of the differential equation that describes wave motion.

The two mathematical theorems that we need to remind ourselves of are:

The surface integral of a vector field over a closed surface is equal to the volume integral of its divergence.

The line integral of a vector field around a closed plane curve is equal to the surface integral of its curl.

A function $f(x - \upsilon t)$ represents a function that is moving with speed υ in the positive x-direction, and a function $g(x + \upsilon t)$ represents a function that is moving with speed υ in the negative x-direction. It is easy to verify by substitution that $y = Af + Gy$ is a solution of the differential equation

$$\frac{d^2y}{dt^2} = \upsilon^2 \frac{d^2y}{dt^2}. \qquad 15.1.1$$

Indeed it is the most general solution, since f and g are quite general functions, and the function y already contains the only two arbitrary integration constants to be expected from a second order differential equation. Equation 15.1.1 is, then, the differential equation for a wave in one dimension. For a function $\psi(x, y, z)$ in three dimensions, the corresponding wave equation is

$$\ddot{\psi} = \upsilon^2 \nabla^2 \psi. \qquad 15.1.2$$

It is easy to remember which side of the equation υ^2 is on from dimensional considerations.

One last small point before proceeding – I may be running out of symbols! I may need to refer to *surface charge density*, a scalar quantity for which the usual symbol is σ. I shall also need to refer to *magnetic vector potential*, for which the usual symbol is **A**. And I shall need to refer to area, for which either of the symbols A or σ are commonly used – or, if the vector nature of area is to be emphasized, **A** or $\boldsymbol{\sigma}$. What I shall try to do, then, to avoid this difficulty, is to use **A** for magnetic vector potential, and $\boldsymbol{\sigma}$ for area, and I shall try to avoid using surface charge density in any equation. However, the reader is warned to be on the lookout and to be sure what each symbol means in a particular context.

15.2 *Maxwell's First Equation*

Maxwell's first equation, which describes the electrostatic field, is derived immediately from Gauss's theorem, which in turn is a consequence of Coulomb's inverse square law. Gauss's theorem states that the surface integral of the electrostatic field **D** over a closed surface is equal to the charge enclosed by that surface. That is

$$\int_{\text{surface}} \mathbf{D} \cdot \mathbf{d}\boldsymbol{\sigma} = \int_{\text{volume}} \rho \, dv. \qquad 15.2.1$$

Here ρ is the charge per unit volume.

But the surface integral of a vector field over a closed surface is equal to the volume integral of its divergence, and therefore

$$\int_{\text{volume}} \text{div}\mathbf{D} \, dv = \int_{\text{volume}} \rho \, dv. \qquad 15.2.2$$

Therefore $$\text{div}\, \mathbf{D} = \rho, \qquad 15.2.3$$

or, in the nabla notation, $$\nabla \cdot \mathbf{D} = \rho. \qquad 15.2.4$$

This is the first of Maxwell's equations.

15.3 *Poisson's and Laplace's Equations*

Equation 15.2.4 can be written $\nabla \cdot \mathbf{E} = \rho / \varepsilon$. where ε is the permittivity. But **E** is minus the potential gradient; i.e. $\mathbf{E} = -\nabla V$. Therefore,

$$\nabla^2 V = -\rho / \varepsilon. \qquad 15.3.1$$

This is *Poisson's equation.* At a point in space where the charge density is zero, it becomes

$$\nabla^2 V = 0, \qquad 15.3.2$$

which is generally known as *Laplace's equation.* Thus, regardless of how many charged bodies there may be an a place of interest, and regardless of their shape or size, the potential at any point can be calculated from Poisson's or Laplace's equations. Courses in differential equations commonly discuss how to solve these equations for a variety of *boundary conditions* – by which is meant the size, shape and location of the various charged bodies and the charge carried by each. It perhaps just needs to be emphasized that Poisson's and Laplace's equations apply only for *static* fields.

15.4 *Maxwell's Second Equation*

Unlike the electrostatic field, magnetic fields have no sources or sinks, and the magnetic lines of force are closed curves. Consequently the surface integral of the magnetic field over a closed surface is zero, and therefore

$$\text{div}\,\mathbf{B} = 0\,, \qquad 15.4.1$$

or, in the nabla notation

$$\nabla \cdot \mathbf{B} = 0. \qquad 15.4.2$$

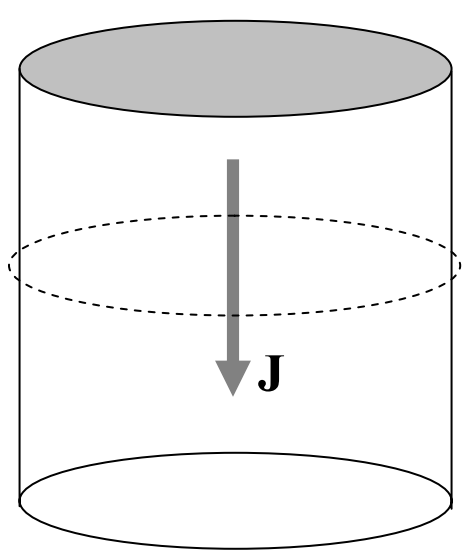
This is the second of Maxwell's equations.

15.5 *Maxwell's Third Equation*

This is derived from Ampère's theorem, which is that the line integral of the magnetic field **H** around a closed circuit is equal to the enclosed current.

Now there are two possible components to the "enclosed" current, one of which is obvious, and the other, I suppose, could also be said to be "obvious" once it has been pointed out! Let's deal with the immediately obvious one first, and look at figure XV.1.

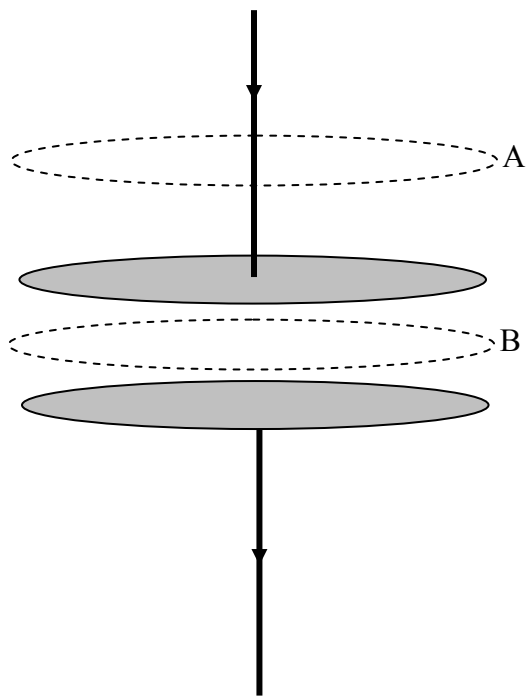
FIGURE XV.1



In figure XV.1, I am imagining a metal cylinder with current flowing from top to bottom (i.e. electrons flowing from bottom to top. It needn't be a metal cylinder, though. It could just be a volume of space with a stream of protons moving from top to bottom. In any case, the current density (which may vary with distance from the axis of the cylinder) is **J**, and the total current enclosed by the dashed circle is the integral of **J** throughout the cylinder. In a more general geometry, in which **J** is not necessarily perpendicular to the area of interest, and indeed in which the area need not be planar, this would be $\int \mathbf{J} \cdot \mathbf{d\sigma}$.

Now for the less obvious component to the "enclosed current". See figure XV.2.

FIGURE XV.2



In figure XV.2, I imagine two capacitor plates in the process of being charged. There is undoubtedly a current flowing in the connecting wires. There is a magnetic field at A, and the line integral of the field around the upper dotted curve is undoubtedly equal to the enclosed current. The current is equal to the rate at which charge is being built up on the plates. Electrons are being deposited on the lower plate and are leaving the upper plate. There is also a magnetic field at B (it doesn't suddenly stop!), and the field at B is just the same as the field at A, which is equal to the rate at which charge is being built up on the plates. The charge on the plates (which may not be uniform, and indeed won't be while the current is still flowing or if the plates are not infinite in extent) is equal to the integral of the charge density times the area. And the charge density on the plates, by Gauss's theorem, is equal to the electric field **D** between the plates. Thus the current is equal to the integral of $\dot{\mathbf{D}}$ over the surface of the plates. Thus the line integral of **H** around either of the dashed closed loops is equal to $\int \dot{\mathbf{D}} \cdot \mathbf{d\sigma}$..

In general, both types of current (the obvious one in which there is an obvious flow of charge, and the less obvious one, where the electric field is varying because of a real flow of charge elsewhere) contributes to the magnetic field, and so Ampère's theorem in general must read

$$\int_{\text{loop}} \mathbf{H}\cdot\mathbf{ds} \;=\; \int_{\text{area}} (\dot{\mathbf{D}}+\mathbf{J})\cdot\mathbf{d}\sigma. \qquad 15.5.1$$

But the line integral of a vector field around a closed plane curve is equal to the surface integral of its curl, and therefore

$$\int_{\text{area}} \mathbf{curl\,H}\cdot\mathbf{d}\sigma \;=\; \int_{\text{area}} (\dot{\mathbf{D}}+\mathbf{J})\cdot\mathbf{d}\sigma. \qquad 15.5.2$$

Thus we arrive at:

$$\mathbf{curl\,H} \;=\; \dot{\mathbf{D}} + \mathbf{J}, \qquad 15.5.3$$

or, in the nabla notation,

$$\nabla\times\mathbf{H} \;=\; \dot{\mathbf{D}} + \mathbf{J}. \qquad 15.5.4$$

This is the third of Maxwell's equations.

15.6 *The Magnetic Equivalent of Poisson's Equation*

This deals with a static magnetic field, where there is no electrostatic field or at least any electrostatic field is indeed static – i.e. not changing. In that case $\mathbf{curl\,H} = \mathbf{J}$. Now the magnetic field can be derived from the curl of the magnetic vector potential, defined by the two equations

$$\mathbf{B} \;=\; \mathbf{curl\,A} \qquad 15.6.1$$

and

$$\text{div}\,\mathbf{A} \;=\; 0. \qquad 15.6.2$$

(See Chapter 9 for a reminder of this.) Together with $\mathbf{H} = \mathbf{B}/\mu$ (μ = permeability), this gives us

$$\mathbf{curl\;curl\,A} \;=\; \mu\mathbf{J}. \qquad 15.6.3$$

If we now remind ourselves of the jabberwockian-sounding vector differential operator equivalence

$$\mathbf{curl\;curl} \;\equiv\; \mathbf{grad}\;\text{div} \;-\; \text{nabla-squared}, \qquad 15.6.4$$

together with equation 15.6.2, this gives us

$$\nabla^2\mathbf{A} \;=\; -\mu\mathbf{J}. \qquad 15.6.5$$

I don't know if this equation has any particular name, but it plays the same role for static magnetic fields that Poisson's equation plays for electrostatic fields. No matter what the distribution of currents, the magnetic vector potential at any point must obey equation 15.6.5

15.7 *Maxwell's Fourth Equation*

This is derived from the laws of electromagnetic induction.

Faraday's and Lenz's laws of electromagnetic induction tell us that the E.M.F. induced in a closed circuit is equal to minus the rate of change of *B*-flux through the circuit. The E.M.F. around a closed circuit is the line integral of **E .ds** around the circuit, where **E** is the electric field. The line integral of **E** around the closed circuit is equal to the surface integral of its curl. The rate of change of *B*-flux through a circuit is the surface integral of $\dot{\mathbf{B}}$. Therefore

$$\mathbf{curl}\,\mathbf{E} = -\dot{\mathbf{B}}, \qquad 15.7.1$$

or, in the nabla notation,

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}. \qquad 15.7.2$$

This is the fourth of Maxwell's equations.

15.8 *Summary of Maxwell's and Poisson's Equations*

Maxwell's equations:

$$\nabla \cdot \mathbf{D} = \rho. \qquad 15.8.1$$

$$\nabla \cdot \mathbf{B} = 0. \qquad 15.8.2$$

$$\nabla \times \mathbf{H} = \dot{\mathbf{D}} + \mathbf{J}. \qquad 15.8.3$$

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}. \qquad 15.8.4$$

Sometimes you may see versions of these equations with factors such as 4π or c scattered liberally throughout them. If you do, my best advice is to white them out with a bottle of erasing fluid, or otherwise ignore them. I shall try to explain in Chapter 16 where they come from. They serve no scientific purpose, and are merely conversion factors between the many different systems of units that have been used in the past.

Poisson's equation for the potential in an electrostatic field:

$$\nabla^2 V = -\rho/\varepsilon. \qquad 15.8.5$$

The equivalent of Poisson's equation for the magnetic vector potential on a static magnetic field:

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}. \qquad 15.8.6$$

15.9 *Electromagnetic Waves*

Maxwell predicted the existence of electromagnetic waves, and these were generated experimentally by Hertz shortly afterwards. In addition, the predicted speed of the waves was 3×10^8 m s^{-1}, the same as the measured speed of light, showing that light is an electromagnetic wave.

In an isotropic, homogeneous, nonconducting, uncharged medium, where the permittivity and permeability are scalar quantities, Maxwell's equations can be written

$$\nabla \cdot \mathbf{E} = 0. \qquad 15.9.1$$

$$\nabla \cdot \mathbf{H} = 0. \qquad 15.9.2$$

$$\nabla \times \mathbf{H} = \varepsilon \dot{\mathbf{E}}. \qquad 15.9.3$$

$$\nabla \times \mathbf{E} = -\mu \dot{\mathbf{H}}. \qquad 15.9.4$$

Take the **curl** of equation 15.9.3, and make use of equation 15.6.4:

$$\mathbf{grad}\, div\ \mathbf{H} - \nabla^2 \mathbf{H} = \varepsilon \frac{\partial}{\partial t} \mathbf{curl}\, \mathbf{E}. \qquad 15.9.5$$

Substitute for *div* **H** and **curl E** from equations 15.9.2 and 15.9.4 to obtain

$$\nabla^2 \mathbf{H} = \varepsilon\mu \ddot{\mathbf{H}}. \qquad 15.9.6$$

Comparison with equation 15.1.2 shows that this is a wave of speed $1/\sqrt{\varepsilon\mu}$. (Verify that this has the dimensions of speed.)

In a similar manner the reader should easily be able to derive the equation

$$\nabla^2 \mathbf{E} = \varepsilon\mu \ddot{\mathbf{E}}. \qquad 15.9.7$$

In a vacuum, the speed is $1/\sqrt{\varepsilon_0 \mu_0}$. With $\mu_0 = 4\pi \times 10^{-7}$ H m^{-1} and $\varepsilon_0 = 8.854 \times 10^{-12}$ F m^{-1}, this comes to 2.998×10^8 m s^{-1}.