CHAPTER 6
MAGNETIC EFFECT OF AN ELECTRIC CURRENT

6.1 *Introduction*

Most of us are familiar with the more obvious properties of magnets and compass needles. A magnet, often in the form of a short iron bar, will attract small pieces of iron such as nails and paper clips. Two magnets will either attract each other or repel each other, depending upon their orientation. If a bar magnet is placed on a sheet of paper and iron filings are scattered around the magnet, the iron filings arrange themselves in a manner that reminds us of the electric field lines surrounding an electric dipole. All in all, a bar magnet has some properties that are quite similar to those of an electric dipole. The region of space around a magnet within which it exerts its magic influence is called a *magnetic field*, and its geometry is rather similar to that of the electric field around an electric dipole – although its *nature* seems a little different, in that it interacts with iron filings and small bits of iron rather than with scraps of paper or pith-balls.

The resemblance of the magnetic field of a bar magnet to the electric field of an electric dipole was sometimes demonstrated in Victorian times by means of a Robison Ball-ended Magnet, which was a magnet shaped something like this:

FIGURE VI.1

The geometry of the magnetic field (demonstrated, for example, with iron filings) then *greatly* resembled the geometry of an electric dipole field. Indeed it looked as though a magnet had two *poles* (analogous to, but not the same as, electric charges), and that one of them acts as a *source* for magnetic field lines (i.e. field lines diverge from it), and the other acts as a *sink* (i.e. field lines converge to it). Rather than calling the poles "positive" and "negative", we somewhat arbitrarily call them "north" and "south" poles, the "north" pole being the source and the "south" pole the sink. By experimenting with two or more magnets, we find that like poles repel and unlike poles attract.

We also observe that a freely-suspended magnet (i.e. a compass needle) will orient itself so that one end points approximately north, and the other points approximately south, and it is these poles that are called the "north" and "south" poles of the magnet.

Since unlike poles attract, we deduce (or rather William Gilbert, in his 1600 book *De Magnete, Magneticisque Corporibus, et de Magno Magnete Tellure*, deduced) that Earth itself acts as a giant magnet, with a south magnetic pole somewhere in the Arctic and a north magnetic pole in the Antarctic. The Arctic magnetic pole is at present in Bathurst Island in northern Canada and is usually marked in atlases as the "North Magnetic Pole",

though magnetically it is a *sink*, rather than a source. The Antarctic magnetic pole is at present just offshore from Wilkes Land in the Antarctic continent. The Antarctic magnetic pole is a *source*, although it is usually marked in atlases as the "South Magnetic Pole". Some people have advocated calling the end of a compass needle that points north the "north-seeking pole", and the other end the "south-seeking pole. This has much to commend it, but usually, instead, we just call them the "north" and "south" poles. Unfortunately this means that the Earth's magnetic pole in the Arctic is really a south magnetic pole, and the pole in the Antarctic is a north magnetic pole.

The resemblance of the magnetic field of a bar magnet to a dipole field, and the *very* close resemblance of a "Robison Ball-ended Magnet" to a dipole, with a point source (the north pole) at one end and a point sink (the south pole) at the other, is, however, deceptive.

In truth a magnetic field has *no* sources and *no* sinks. This is even expressed as one of Maxwell's equations, div $\mathbf{B} = 0$, as being one of the defining characteristics of a magnetic field. The magnetic lines of force always form closed loops. *Inside* a bar magnet (even inside the connecting rod of a Robison magnet) the magnetic field lines are directed from the south pole to the north pole. If a magnet, even a Robison magnet, is cut in two, we do not isolate two separate poles. Instead each half of the magnet becomes a dipolar magnet itself.

All of this is very curious, and matters stood like this until Oersted made an outstanding discovery in 1820 (it is said while giving a university lecture in Copenhagen), which added what may have seemed like an additional complication, but which turned out to be in many ways a great simplification. He observed that, if an electric current is made to flow in a wire near to a freely suspended compass needle, the compass needle is deflected. Similarly, if a current flows in a wire that is free to move and is near to a fixed bar magnet, the wire experiences a force at right angles to the wire.

From this point on we understand that a magnetic field is something that is *primarily associated with an electric current.* All the phenomena associated with magnetized iron, nickel or cobalt, and "lodestones" and compass needles are somehow secondary to the fundamental phenomenon that an electric current is always surrounded by a magnetic field. Indeed, Ampère speculated that the magnetic field of a bar magnet may be caused by many circulating current loops within the iron. He was right! – the little current loops are today identified with electron spin.

If the direction of the magnetic field is taken to be the direction of the force on the north pole of a compass needle, Oersted's observation showed that the magnetic field around a current is in the form of concentric circles surrounding the current. Thus in figure VI.2, the current is assumed to be going away from you at right angles to the plane of your computer screen (or of the paper, if you have printed this page out), and the magnetic field lines are concentric circles around the current,

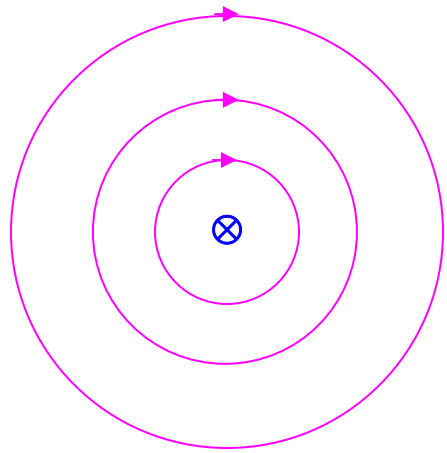


FIGURE VI.2

In the remainder of this chapter, we shall no longer be concerned with magnets, compass needles and lodestones. These may come in a later chapter. In the remainder of this chapter we shall be concerned with the magnetic field that surrounds an electric current.

6.2 *Definition of the Amp*

We have seen that an electric current is surrounded by a magnetic field; and also that, if a wire carrying a current is situated in an external magnetic field, it experiences a force at right angles to the current. It is therefore not surprising that two current-carrying wires exert forces upon each other.

More precisely, if there are two parallel wires each carrying a current in the same direction, the two wires will *attract* each other with a force that depends on the strength of the current in each, and the distance between the wires.

Definition. One *amp* (also called an ampère) is that steady current which, flowing in each of two parallel wires of negligible cross-section one metre apart *in vacuo*, gives rise to a force between them of 2×10^{-7} newtons per metre of their length.

At last! We now know what an amp is, and consequently we know what a coulomb, a volt and an ohm are. We have been left in a state of uncertainty until now. No longer!

But you may ask: Why the factor 2×10^{-7}? Why not define an amp in such a manner that the force is 1 N m^{-1}? This is a good question, and its answer is tied to the long and tortuous history of units in electromagnetism. I shall probably discuss this history, and the various "CGS" units, in a later chapter. In brief, it took a long time to understand that electrostatics, magnetism and current electricity were all aspects of the same basic

phenomena, and different systems of units developed within each topic. In particular a so-called "practical" unit, the *amp* (defined in terms of the rate of deposition of silver from an electrolytic solution) became so entrenched that it was felt impractical to abandon it. Consequently when all the various systems of electromagnetic units became unified in the twentieth century (starting with proposals by Giorgi based on the metre, kilogram and second (MKS) as long ago as 1895) in the "Système International" (SI), it was determined that the fundamental unit of current should be identical with what had always been known as the ampère. (The factor 2, by the way, is not related to their being two wires in the definition.) The amp is the only SI unit in which any number other than "one" is incorporated into its definition, and the exception was forced by the desire to maintain the amp.

One last point before leaving this section. In the opening paragraph I wrote that "It is therefore not surprising that two current-carrying wires exert forces upon each other." Yet when I first learned, as a student, of the mutual attraction of two parallel electric currents, I was very astonished indeed. The reason why this is astonishing is discussed in Chapter 15 (Special Relativity) of the Classical Mechanics section of these notes.

6.3 *Definition of the Magnetic Field*

We are going to define the magnitude and direction of the magnetic field entirely by reference to its effect upon an electric current, without reference to magnets or lodestones. We have already noted that, if an electric current flows in a wire in an externally-imposed magnetic field, it experiences a force at right angles to the wire.

I want you to imagine that there is a magnetic field in this room, originating, perhaps, from some source outside the room. This need not entail a great deal of imagination, for there already *is* such a magnetic field – namely, Earth's magnetic field. I'll tell you that the field within the room is uniform, but I shan't tell you anything about either its magnitude or its direction.

You have a straight wire and you can pass a current through it. You will note that there is a force on the wire. Perhaps we can define the direction of the field as being the direction of this force. But this won't do at all, because the force is always at right angles to the wire no matter what its orientation! We do notice, however, that the *magnitude* of the force depends on the orientation of the wire; and there is one *unique orientation* of the wire in which it experiences *no force at all*. Since this orientation is unique, we choose to *define the direction of the magnetic field* as being parallel to the wire when the orientation of the wire is such that it experiences no force.

This leaves a two-fold ambiguity since, even with the wire in its unique orientation, we can cause the current to flow in one direction or in the opposite direction. We still have to resolve this ambiguity. Have patience for a few more lines.

As we move our wire around in the magnetic field, from one orientation to another, we notice that, while the direction of the force on it is always at right angles to the wire, the *magnitude* of the force depends on the orientation of the wire, being zero (by definition) when it is parallel to the field and greatest when it is perpendicular to it.

Definition. The intensity B (also called the flux density, or field strength, or merely "field") of a magnetic field is equal to the maximum force exerted per unit length on unit current (this maximum force occurring when the current and field are at right angles to each other).

The dimensions of B are $\dfrac{\mathrm{MLT}^{-2}}{\mathrm{LQT}^{-1}} = \mathrm{MT}^{-1}\mathrm{Q}^{-1}$.

Definition. If the maximum force per unit length on a current of 1 amp (this maximum force occurring, of course, when current and field are perpendicular) is 1 N m^{-1}, the intensity of the field is 1 *tesla* (T).

By definition, then, when the wire is parallel to the field, the force on it is zero; and, when it is perpendicular to the field, the force per unit length is IB newtons per metre.

It will be found that, when the angle between the current and the field is θ, the force per unit length, F', is

$$F' = IB\sin\theta. \qquad 6.3.1$$

In vector notation, we can write this as

$$\mathbf{F'} = \mathbf{I} \times \mathbf{B}, \qquad 6.3.2$$

where, in choosing to write $\mathbf{I} \times \mathbf{B}$ rather than $\mathbf{F'} = \mathbf{B} \times \mathbf{I}$, we have removed the two-fold ambiguity in our definition of the direction of $\mathbf{B}$. Equation 6.3.2 expresses the "right-hand rule" for determining the relation between the directions of the current, field and force.

6.4 *The Biot-Savart Law*

Since we now know that a wire carrying an electric current is surrounded by a magnetic field, and we have also decided upon how we are going to define the intensity of a magnetic field, we want to ask if we can calculate the intensity of the magnetic field in the vicinity of various geometries of electrical conductor, such as a straight wire, or a plane coil, or a solenoid. When we were calculating the *electric* field in the vicinity of various geometries of *charged bodies*, we started from *Coulomb's Law*, which told us what the field was at a given distance from a point charge. Is there something similar in electromagnetism which tells us how the *magnetic* field varies with distance from an *electric current*? Indeed there is, and it is called the *Biot-Savart Law.*

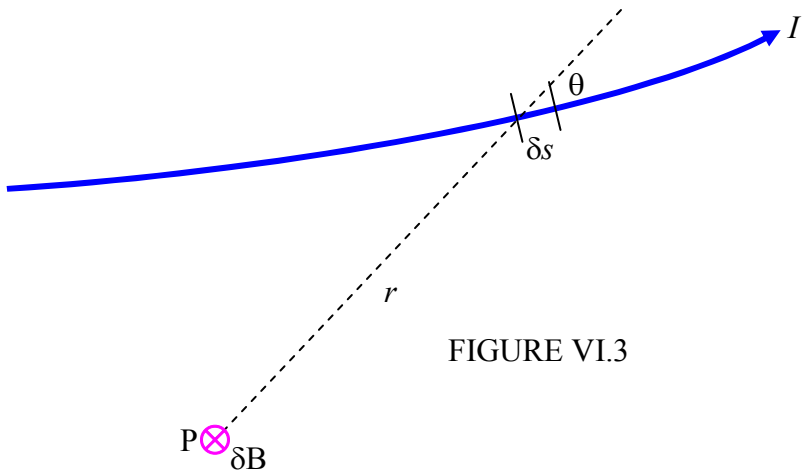


FIGURE VI.3

Figure VI.3 shows a portion of an electrical circuit carrying a current I. The Biot-Savart Law tells us what the contribution δB is at a point P from an elemental portion of the electrical circuit of length δs at a distance r from P, the angle between the current at δs and the radius vector from P to δs being θ. The Biot-Savart Law tells us that

$$\delta B \propto \frac{I\,\delta s \sin\theta}{r^2}. \qquad 6.4.1$$

This law will enable us, by integrating it around various electrical circuits, to calculate the total magnetic field at any point in the vicinity of the circuit.

But – can I prove the Biot-Savart Law, or is it just a bald statement from nowhere? The answer is neither. I cannot prove it, but nor is it merely a bald statement from nowhere. First of all, it is a not unreasonable guess to suppose that the field is proportional to I and to δs, and also inversely proportional to r^2, since δs, in the limit, approaches a point source. But you are still free to regard it, if you wish, as speculation, even if reasonable speculation. Physics is an experimental science, and to that extent you cannot "prove" anything in a mathematical sense; you can experiment and measure. The Biot-Savart law enables us to calculate what the magnetic field ought to be near a straight wire, near a plane circular current, inside a solenoid, and indeed near any geometry you can imagine. So far, after having used it to calculate the field near millions of conductors of a myriad shapes and sizes, the predicted field has always agreed with experimental measurement. Thus the Biot-Savart law is *likely* to be true – but you are perfectly correct in asserting that, no matter how many magnetic fields it has correctly predicted, there is always the chance that, some day, it will predict a field for some unusually-shaped circuit that disagrees with what is measured. All that is needed is *one* such example, and the law is disproved. You may, if you wish, try and discover, for a Ph.D. project, such a circuit; but I would not recommend that you spend your time on it!

There remains the question of what to write for the constant of proportionality. We are free to use any symbol we like, but, in modern notation, we symbol we use is $\frac{\mu_0}{4\pi}$. Why

the factor 4π? The inclusion of 4π gives us what is called a "rationalized" definition, and it is introduced for the same reasons that we introduced a similar factor in the constant of proportionality for Coulomb's law, namely that it results in the appearance of 4π in spherically-symmetric geometries, 2π in cylindrically-symmetric geometries, and no π where the magnetic field is uniform. Not everyone uses this definition, and this will be discussed in a later chapter, but it is certainly the recommended one.

In any case, the Biot-Savart Law takes the form

$$\delta B = \frac{\mu_0}{4\pi} \frac{I\,\delta s \sin\theta}{r^2}. \qquad 6.4.2$$

The constant μ_0 is called the *permeability of free space*, "free space" meaning a vacuum. The subscript allows for the possibility that if we do an experiment in a medium other than a vacuum, the permeability may be different, and we can then use a different subscript, or none at all. In practice the permeability of air is very little different from that of a vacuum, and hence I shall normally use the symbol μ_0 for experiments performed in air, unless we are discussing measurement of very high precision.

From equation 6.4.2, we can see that the SI units of permeability are T m A^{-1} (tesla metres per amp). In a later chapter we shall come across another unit – the *henry* – for a quantity (*inductance*) that we have not yet described, and we shall see then that a more convenient unit for permeability is H m^{-1} (henrys per metre) – but we are getting ahead of ourselves.

What is the numerical value of μ_0? I shall reveal that in the next chapter.

Exercise. Show that the *dimensions* of permeability are MLQ^{-2}. This means that you may, if you wish, express permeability in units of kg m C^{-2} – although you may get some queer looks if you do.

Thought for the Day.

The sketch shows two current elements, each of length δs, the current being the same in each but in different directions. Is the force on one element from the other equal but opposite to the force on the other from the one? If not, is there something wrong with Newton's third law of motion? Discuss this over lunch.

6.5 *Magnetic Field Near a Long, Straight, Current-carrying Conductor*

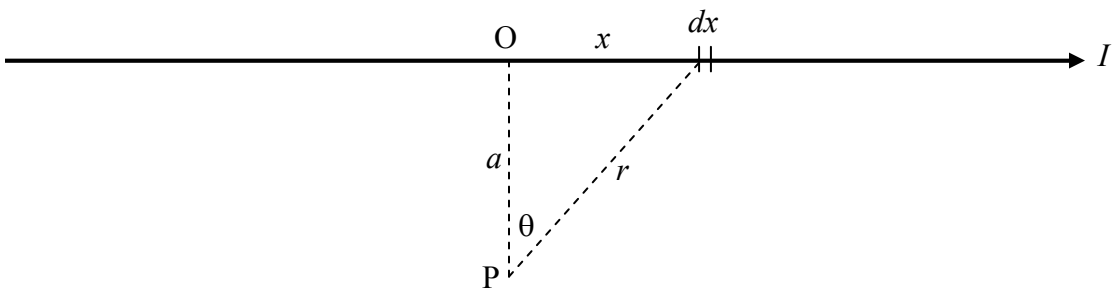


FIGURE VI.4

Consider a point P at a distance a from a conductor carrying a current I (figure VI.4). The contribution to the magnetic field at P from the elemental length dx is

$$dB = \frac{\mu}{4\pi} \cdot \frac{I\,dx \sin\theta}{r^2}. \qquad 6.5.1$$

Here I have omitted the subscript zero on the permeability to allow for the possibility that the wire is immersed in a medium in which the permeability is not the same as that of a vacuum. (The permeability of liquid oxygen, for example, is slightly greater than that of free space.) The direction of the field at P is into the plane of the "paper" (or of your computer screen).

We need to express this in terms of one variable, and we'll choose θ. We can see that $r = a\sec\theta$ and $x = a\tan\theta$, so that $dx = a\sec^2\theta\, d\theta$. Thus equation 6.5.1 becomes

$$dB = \frac{\mu I}{4\pi a}\sin\theta\, d\theta. \qquad 6.5.2$$

Upon integrating this from $-\pi/2$ to $+\pi/2$ (or from 0 to $\pi/2$ and then double it), we find that the field at P is

$$B = \frac{\mu I}{2\pi a}. \qquad 6.5.3$$

Note the 2π in this problem with cylindrical symmetry.

6.6 *Field on the Axis of a Plane Circular Current-carrying Coil*

I strongly recommend that you compare and contrast this derivation and the result with the treatment of the electric field on the axis of a charged ring in Section 1.6.4 of Chapter 1. Indeed I am copying the drawing from there and then modifying it as need be.

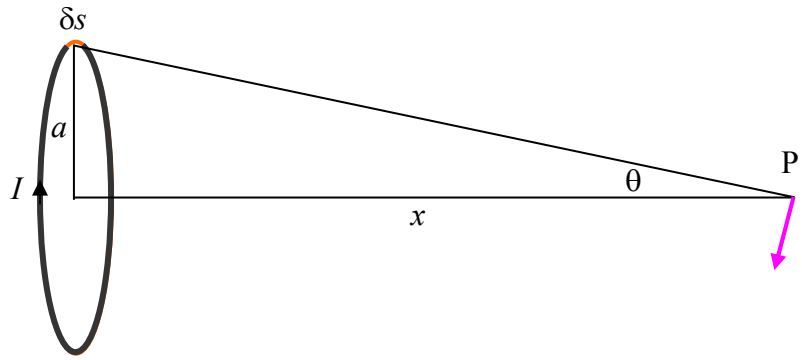


FIGURE VI.5

The contribution to the magnetic field at P from an element δs of the current is $\dfrac{\mu I\,\delta s}{4\pi(a^2 + x^2)}$ in the direction shown by the coloured arrow. By symmetry, the total component of this from the entire coil perpendicular to the axis is zero, and the only component of interest is the component along the axis, which is $\dfrac{\mu I\,\delta s}{4\pi(a^2 + x^2)}$ times $\sin\theta$.

The integral of δs around the whole coil is just the circumference of the coil, $2\pi a$, and if we write $\sin\theta = \dfrac{a}{(a^2 + x^2)^{1/2}}$, we find that the field at P from the entire coil is

$$B = \frac{\mu I a^2}{2(a^2 + x^2)^{3/2}}, \qquad 6.6.1$$

or N times this if there are N turns in the coil. At the centre of the coil the field is

$$B = \frac{\mu I}{2a}. \qquad 6.6.2$$

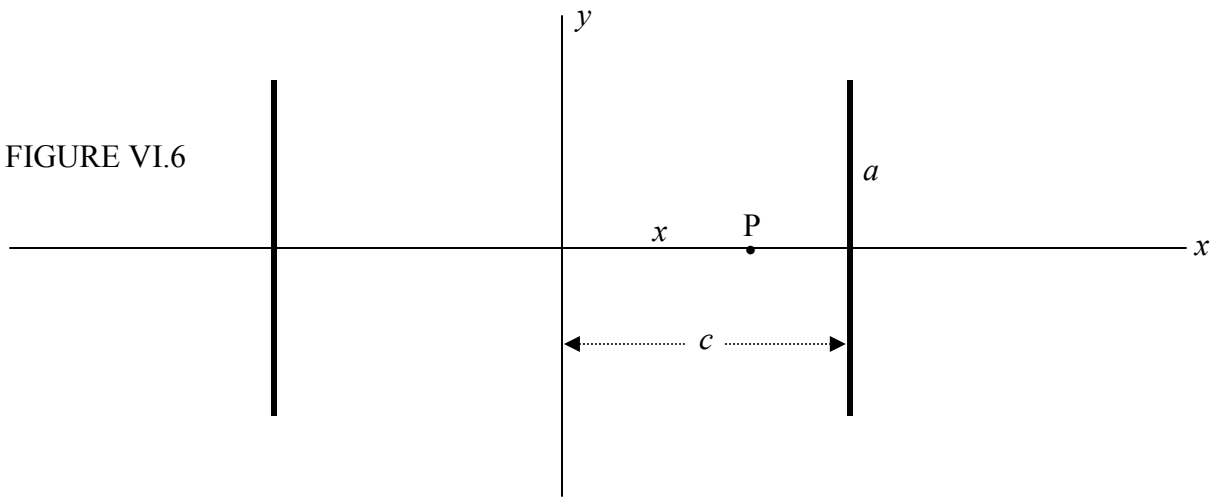
The field is greatest at the centre of the coil and it decreases monotonically to zero at infinity. The field is directed to the left in figure IV.5.

The field at a point off-axis is much more difficult to calculate. I may try it a little later.

6.7 *Helmholtz Coils*

Let us calculate the field at a point halfway between two identical parallel plane coils. If the separation between the coils is equal to the radius of one of the coils, the arrangement is known as "Helmholtz coils", and we shall see why they are of particular interest. To

FIGURE VI.6



begin with, however, we'll start with two coils, each of radius a, separated by a distance $2c$.

There are N turns in each coil, and each carries a current I.

The field at P is

$$B = \frac{\mu N I a^2}{2}\left(\frac{1}{[a^2 + (c-x)^2]^{3/2}} + \frac{1}{[a^2 + (c+x)^2]^{3/2}}\right). \qquad 6.7.1$$

At the origin ($x = 0$), the field is

$$B = \frac{\mu N I a^2}{(a^2 + c^2)^{3/2}}. \qquad 6.7.2$$

(What does this become if $c = 0$? Is this what you'd expect?)

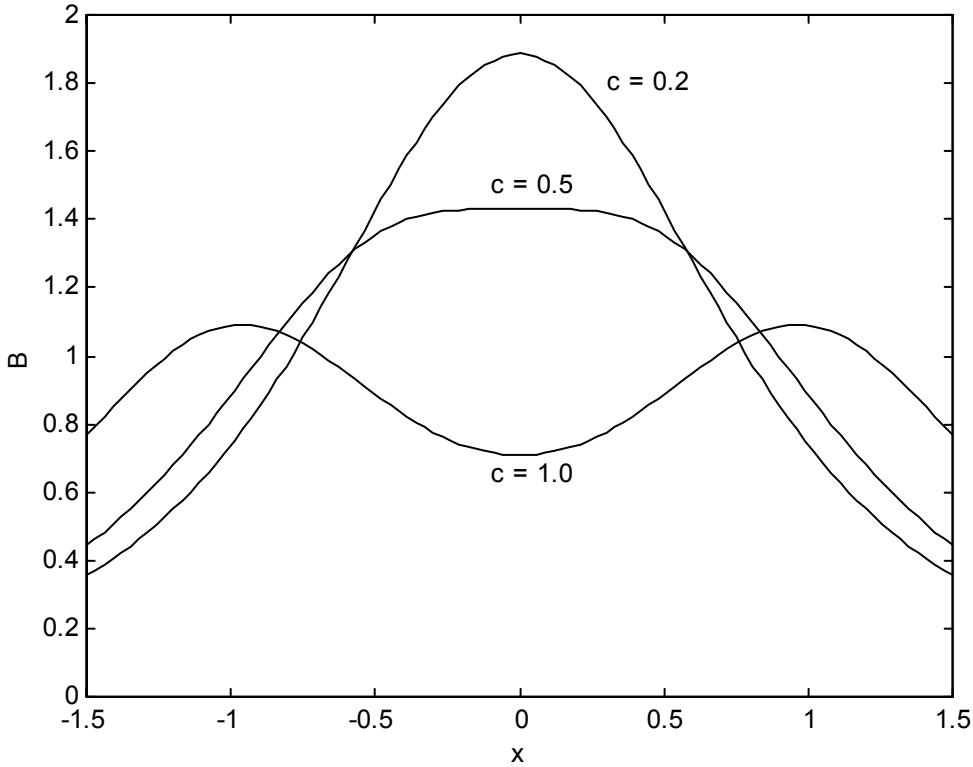
If we express B in units of $\mu NI/(2a)$ and c and x in units of a, equation 6.7.1 becomes

$$B = \frac{1}{[1 + (c - x)^2]^{3/2}} + \frac{1}{[1 + (c + x)^2]^{3/2}}. \qquad 6.7.4$$

Figure VI.7 shows the field as a function of x for three values of c. The coil separation is $2c$, and distances are in units of the coil radius a. Notice that when $c = 0.5$, which means that the coil separation is equal to the coil radius, the field is uniform over a large range, and this is the usefulness of the Helmholtz arrangement for providing a uniform field. If you are energetic, you could try differentiating equation 6.7.4 twice with respect to x and show that the second derivative is zero when $c = 0.5$.

For the Helmholtz arrangement the field at the origin is $\frac{8\sqrt{5}}{25}\cdot\frac{\mu NI}{a} = \frac{0.7155\mu NI}{a}$.

FIGURE VI.7



6.8 *Field on the Axis of a Long Solenoid*

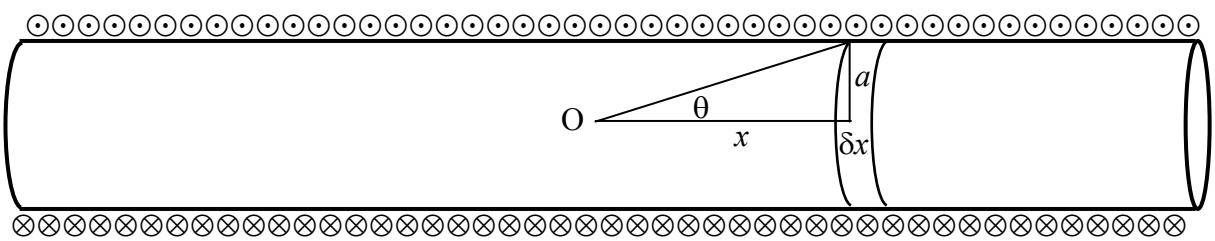


FIGURE VI.8

The solenoid, of radius a, is wound with n turns per unit length of a wire carrying a current in the direction indicated by the symbols ⊗ and ⊙. At a point O on the axis of

the solenoid the contribution to the magnetic field arising from an elemental ring of width δx (hence having $n\,\delta x$ turns) at a distance x from O is

$$\delta B = \frac{\mu n \delta x\, I\, a^2}{2(a^2 + x^2)^{3/2}} = \frac{\mu n I}{2a} \cdot \frac{a^3\, \delta x}{(a^2 + x^2)^{3/2}} \cdot \qquad 6.8.1$$

This field is directed towards the right.

Let us express this in terms of the angle θ.

We have $x = a \cot\theta$, $\delta x = -a \csc^2\theta\, \delta\theta$, and $\dfrac{a^3}{(a^2 + x^2)^{3/2}} \sin^3\theta$. Equation 6.8.1 becomes

$$\delta B = -\tfrac{1}{2}\mu n I \sin\theta\, \delta\theta. \qquad 6.8.2$$

If the solenoid is of infinite length, to find the field from the entire infinite solenoid, we integrate from $\theta = \pi/2$ to 0 and double it. Thus

$$B = -\mu n I \int_{\pi/2}^{0} \sin\theta\, d\theta. \qquad 6.8.3$$

Thus the field on the axis of the solenoid is

$$B = \mu n I. \qquad 6.8.4$$

This is the field on the *axis* of the solenoid. What happens if we move away from the axis? Is the field a little greater as we move away from the axis, or is it a little less? Is the field a maximum on the axis, or a minimum? Or does the field go through a maximum, or a minimum, somewhere between the axis and the circumference? We shall answer these questions in section 6.11.

6.9 *The Magnetic Field H*

If you look at the various formulas for the magnetic field B near various geometries of conductor, such as equations 6.5.3, 6.6.2, 6.7.1, 6.8.4, you will see that there is always a μ on the right hand side. It is often convenient to define a quantity $H = B/\mu$. Then these equations become just

$$H = \frac{I}{2\pi a}, \qquad 6.9.1$$

$$H = \frac{I}{2a}, \qquad 6.9.2$$

$$H = \frac{NIa^2}{2}\left(\frac{1}{[a^2 + (c-x)^2]^{3/2}} + \frac{1}{[a^2 + (c+x)^2]^{3/2}}\right), \qquad 6.9.3$$

$$H = nI. \qquad 6.9.10$$

It is easily seen from any of these equations that the SI units of H are A m^{-1}, or amps per metre, and the dimensions are $QT^{-1}M^{-1}$.

Of course the magnetic field, whether represented by the quantity B or by H, is a vector quantity, and the relation between the two representations can be written

$$\mathbf{B} = \mu\mathbf{H}. \qquad 6.9.11$$

In an isotropic medium **B** and **H** are parallel, but in an anisotropic medium they are not parallel (except in the directions of the eigenvectors of the permeability tensor), and permeability is a tensor. This was discussed in section 1.7.1 with respect to the equation $\mathbf{D} = \varepsilon\mathbf{E}$.

6.10 *Flux*

Recall from Section 1.8 that we defined two extensive scalar quantities $\Phi_E = \iint \mathbf{E} \cdot d\mathbf{A}$ and $\Phi_D = \iint \mathbf{D} \cdot d\mathbf{A}$, which I called the *E*-flux and the *D*-flux. In an entirely similar manner I can define the *B*-flux and *H*-flux of a magnetic field by

$$\Phi_B = \iint \mathbf{B} \cdot d\mathbf{A} \qquad 6.10.1$$

and

$$\Phi_H = \iint \mathbf{H} \cdot d\mathbf{A}. \qquad 6.10.2$$

The SI unit of Φ_B is the tesla metre-squared, or T m^2, also called the *weber* Wb.

A summary of the SI units and dimensions of the four fields and fluxes might not come amiss here.

E	V m^{-1}	$MLT^{-2}Q^{-1}$
D	C m^{-2}	$L^{-2}Q$
B	T	$MT^{-1}Q^{-1}$
H	A m^{-1}	$L^{-1}T^{-1}Q$
Φ_E	V m	$ML^3T^{-2}Q^{-1}$
Φ_D	C	Q
Φ_B	Wb	$ML^2T^{-1}Q^{-1}$
Φ_H	A m	$LT^{-1}Q$

6.11 *Ampère's Theorem*

In Section 1.9 we introduced Gauss's theorem, which is that the total normal component of the D-flux through a closed surface is equal to the charge enclosed within that surface. Gauss's theorem is a consequence of Coulomb's law, in which the electric field from a point source falls off inversely as the square of the distance. We found that Gauss's theorem was surprisingly useful in that it enabled us almost immediately to write down expressions for the electric field in the vicinity of various shapes of charged bodies without going through a whole lot of calculus.

Is there perhaps a similar theorem concerned with the magnetic field around a current-carrying conductor that will enable us to calculate the magnetic field in its vicinity without going through a lot of calculus? There is indeed, and it is called *Ampère's Theorem*.

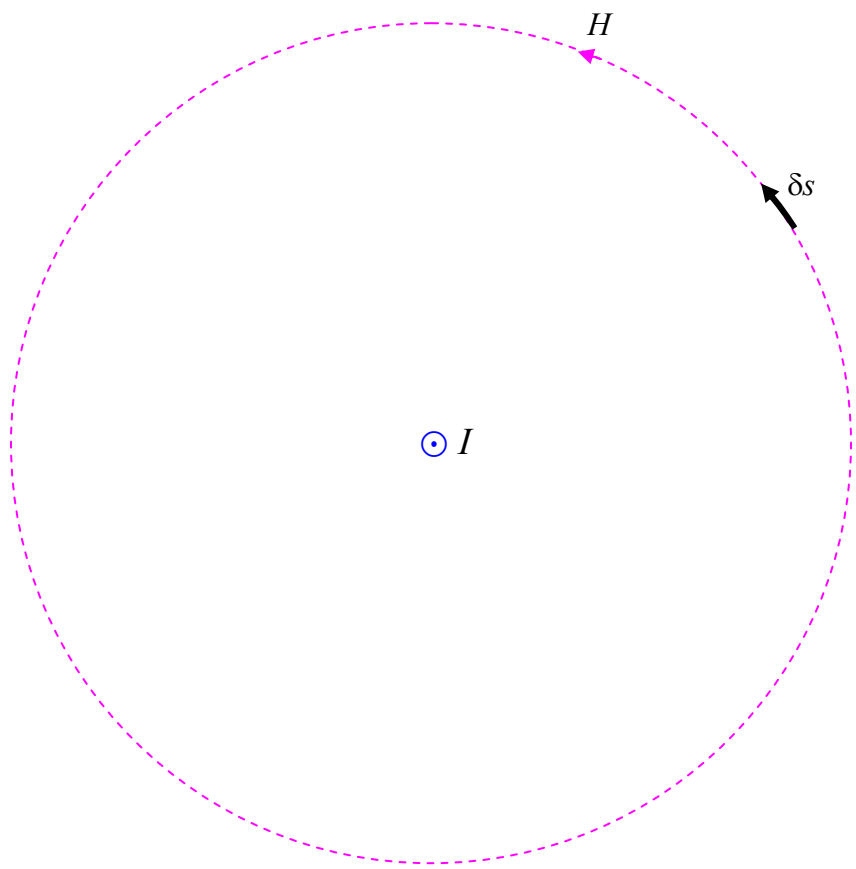


FIGURE VI.9

In figure VI.9 there is supposed to be a current I coming towards you in the middle of the circle. I have drawn one of the magnetic field lines – a dashed line of radius r. The strength of the field there is $H = I/(2\pi r)$. I have also drawn a small elemental length ds on

the circumference of the circle. The line integral of the field around the circle is just H times the circumference of the circle. That is, the line integral of the field around the circle is just I. Note that this is independent of the radius of the circle. At greater distances from the current, the field falls off as $1/r$, but the circumference of the circle increases as r, so the product of the two (the line integral) is independent of r.

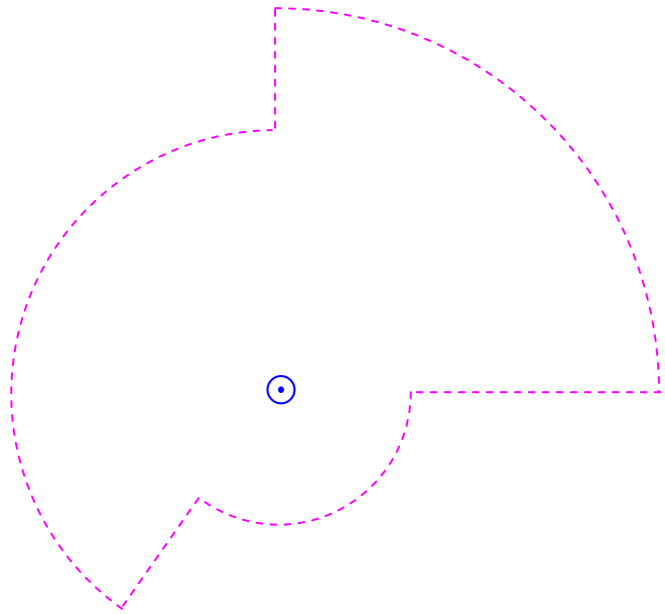


FIGURE VI.10

Consequently, if I calculate the line integral around a circuit such as the one shown in figure VI.10, it will still come to just I. Indeed it doesn't matter what the shape of the path is. The line integral is $\int \mathbf{H} \cdot \mathbf{ds}$. The field **H** at some point is perpendicular to the line joining the current to the point, and the vector **ds** is directed along the path of integration, and $\mathbf{H} \cdot \mathbf{ds}$ is equal to H times the component of **ds** along the direction of **H**, so that, regardless of the length and shape of the path of integration:

The line integral of the field **H** *around any closed path is equal to the current enclosed by that path.*

This is *Ampère's Theorem*.

So now let's do the infinite solenoid again. Let us calculate the line integral around the rectangular amperian path shown in figure VI.11. There is no contribution to the line integral along the vertical sides of the rectangle because these sides are perpendicular to the field, and there is no contribution from the top side of the rectangle, since the field there is zero (if the solenoid is infinite). The only contribution to the line integral is along the bottom side of the rectangle, and the line integral there is just Hl, where l is the length

of the rectangle. If the number turns of wire per unit length along the solenoid is n, there will be nl turns enclosed by the rectangle, and hence the current enclosed by the rectangle is nlI, where I is the current in the wire. Therefore by Ampère's theorem, $Hl = nlI$, and so $H = nI$, which is what we deduced before rather more laboriously. Here H is the strength of the field at the position of the lower side of the rectangle; but we can place the rectangle at any height, so we see that the field is nI <u>anywhere inside the solenoid</u>. That is, the field inside an infinite solenoid is <u>uniform</u>.

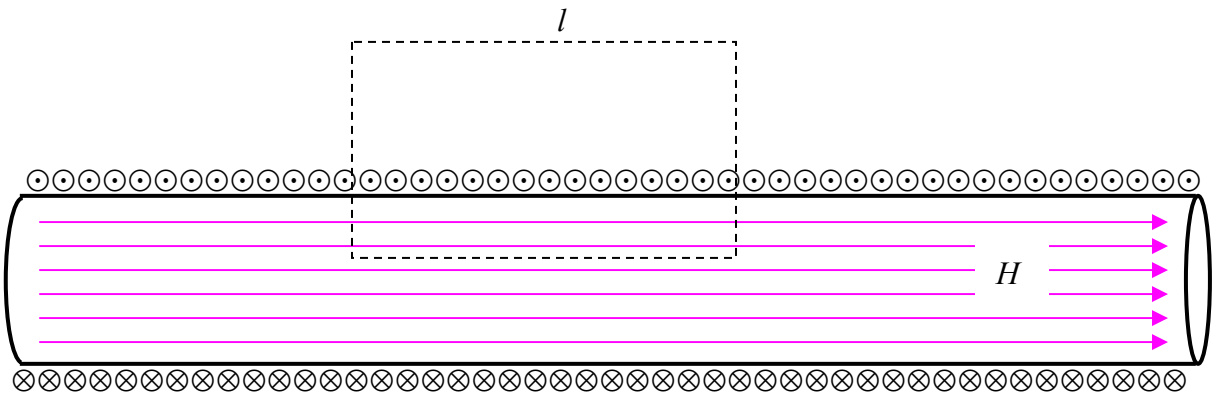


FIGURE VI.11

It is perhaps worth noting that Gauss's theorem is a consequence of the inverse square diminution of the electric field with distance from a point charge, and Ampère's theorem is a consequence of the inverse first power diminution of the magnetic field with distance from a line current.

Example.
Here is an example of the calculation of a line integral (figure VII.12)

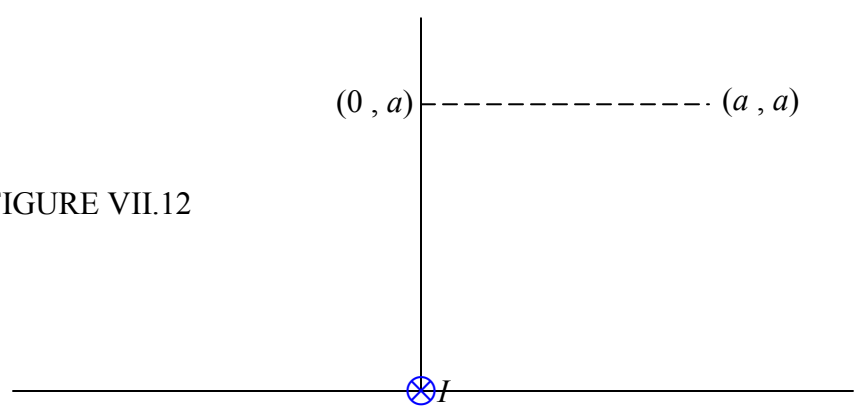
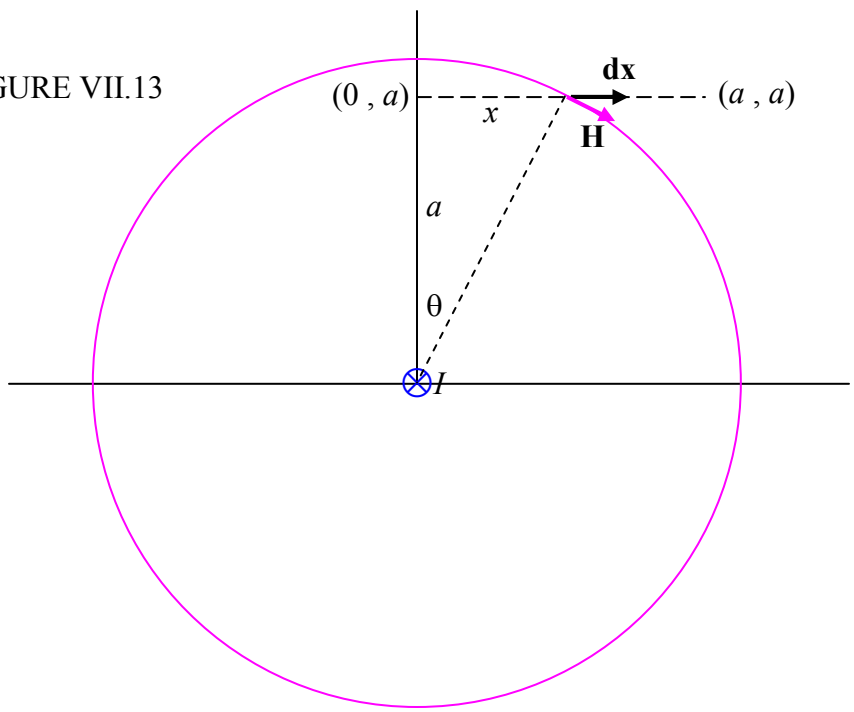


FIGURE VII.12

An electric current I flows into the plane of the paper at the origin of coordinates. Calculate the line integral of the magnetic field along the straight line joining the points $(0\ ,\ a)$ and $(a\ ,\ a)$.

In figure VII.13 I draw a (circular) line of force of the magnetic field **H**, and a vector **dx** where the line of force crosses the straight line of interest.

FIGURE VII.13

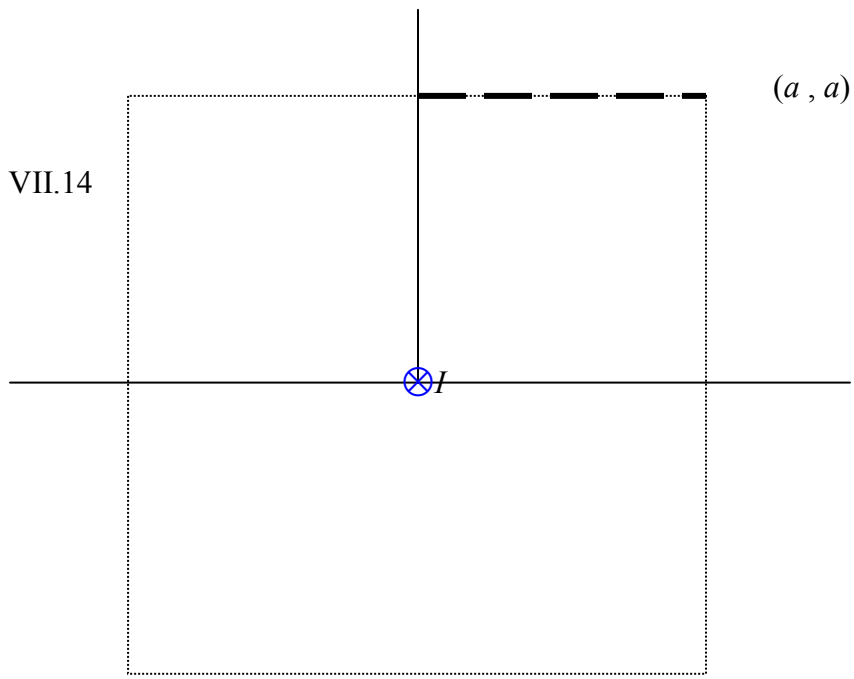


The line integral along the elemental length dx is $\mathbf{H}\ .\ \mathbf{dx} = H\ dx\ \cos\theta$. Here $H = \dfrac{I}{2\pi(a^2 + x^2)^{1/2}}$ and $\cos\theta = \dfrac{a}{(a^2 + x^2)^{1/2}}$, and so the line integral along **dx** is $\dfrac{aI\ dx}{2\pi(a^2 + x^2)}$. Integrate this from $x = 0$ to $x = a$ and you will find that the answer is $I/8$.

Figure VII.14 shows another method. The line integral around the square is, by Ampère's theorem, I, and so the line integral an eighth of the way round is $I/8$.

You will probably immediately feel that this second method is much the better and very "clever". I do not deny this, but it is still worthwhile to study carefully the process of line integration in the first method.

FIGURE VII.14



Another Example.

A straight cylindrical metal rod (or a wire for that matter) of radius a carries a current I. At a distance r from the axis, the magnetic field is clearly $I/(2\pi r)$ if $r > a$. But what is the magnetic field *inside* the rod at a distance r from the axis, $r < a$?

FIGURE VII.15

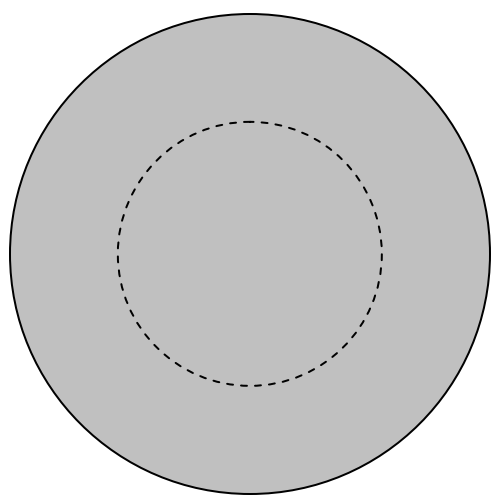


Figure VII.15 shows the cross-section of the rod, and I have drawn an amperian circle of radius r. If the field at the circumference of the circle is H, the line integral around the circle is $2\pi rH$. The current enclosed within the circle is Ir^2/a^2. These two are equal, and therefore $H = Ir/(2\pi a^2)$.