CHAPTER 12
FORCED OSCILLATIONS

12.1 *More on Differential Equations*

In Section 11.4 we argued that the most general solution of the differential equation

$$ay'' + by' + cy = 0 \qquad 11.4.1$$

is of the form

$$y = Af(x) + Bg(x). \qquad 11.4.2$$

In this chapter we shall be looking at equations of the form

$$ay'' + by' + cy = h(x). \qquad 12.1.1$$

If you look back at the arguments that led to the conclusion that equation 11.4.2 is the most general solution of equation 11.4.1, you will be able to conclude that 11.4.2 is still a solution of equation 11.4.1, but it is not the only solution. There is another function that is a solution, so that the most general solution to equation 12.1.1 is of the form

$$y = Af(x) + Bg(x) + H(x). \qquad 12.1.2$$

The solution $H(x)$ is called the *particular integral*, while the part $Af(x) + Bg(x)$ is the *complementary function.* I shall be dealing in this chapter mainly with the particular integral, though we shall not entirely forget the complementary function.

This is a book on classical mechanics rather than on differential equations, so I am not going into how to obtain the particular integral $H(x)$ for a given $h(x)$. There are several ways of doing it; for those who know what they are and are in practice with them, Laplace transforms are among the more attractive methods. Some readers will already know how to do it. They will doubtless want to go back to equation 11.6.3 in the previous chapter and try their hand at finding the particular integral for that. Those who do not may be happy and content to take my word for the particular integral in the sections that follow, or perhaps at least to differentiate it to verify that it is indeed a solution.

12.2 *Forced Oscillatory Motion.*

We are thinking of a mass m attached to a spring of force constant k and subject to a damping force $b\dot{x}$, but also subject to a periodic sinusoidal force $\hat{F}\cos\omega t$. The equation of motion is

$$m\ddot{x} + b\dot{x} + kx = \hat{F}\cos\omega t, \qquad 12.2.1$$

or, if we divide by m:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 = \hat{f}\cos\omega t. \qquad 12.2.2$$

Here $\gamma = b/m$, $\omega_0^2 = k/m$, $\hat{f} = \hat{F}/m$. ω is the forcing angular frequency and ω_0 is the natural frequency of mass and spring in the absence of damping. One part of the general solution of equation 12.2.2 is the complementary function, which we have dealt with at length in Chapter 11. In this section I shall be interested in the particular integral. I shall not derive it here (those who are familiar with differential equations will be able to do so), but you should at least verify by differentiation and substitution that the following is a solution, and it is indeed the particular integral:

$$x = \frac{\hat{f}}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}\left[\left(\omega_0^2 - \omega^2\right)\cos\omega t + \gamma\omega\sin\omega t\right]. \qquad 12.2.3$$

This can also be written

$$x = \frac{\hat{f}}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right]^{\frac{1}{2}}}\cos(\omega t - \alpha), \qquad 12.2.4$$

where $\cos\alpha = \dfrac{\omega_0^2 - \omega^2}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right]^{\frac{1}{2}}}$, $\sin\alpha = \dfrac{\gamma\omega}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right]^{\frac{1}{2}}}$, $\tan\alpha = \dfrac{\gamma\omega}{\omega_0^2 - \omega^2}$. 12.2.5

The response frequency is the same as the forcing frequency, but there is a phase lag between x and F. Figure XII.1 shows α as a function of $\Omega = \omega/\omega_0$ for several different values of $\Gamma = \gamma/\omega_0$.

The particular integral can also be written

$$x = \hat{x}\cos(\omega t - \alpha), \qquad 12.2.6$$

where the *displacement amplitude* $\hat{x}$ varies with forcing frequency ω as

$$\hat{x} = \frac{\hat{f}}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right]^{\frac{1}{2}}}. \qquad 12.2.7$$

If we now introduce dimensionless quantities

$$\Omega = \frac{\omega}{\omega_0}, \quad \Gamma = \frac{\gamma}{\omega_0}, \quad \hat{X} = \frac{\hat{x}}{\hat{f}/\omega_0^2}, \qquad 12.2.8$$

equations 12..2.5 and 12.2.7 become

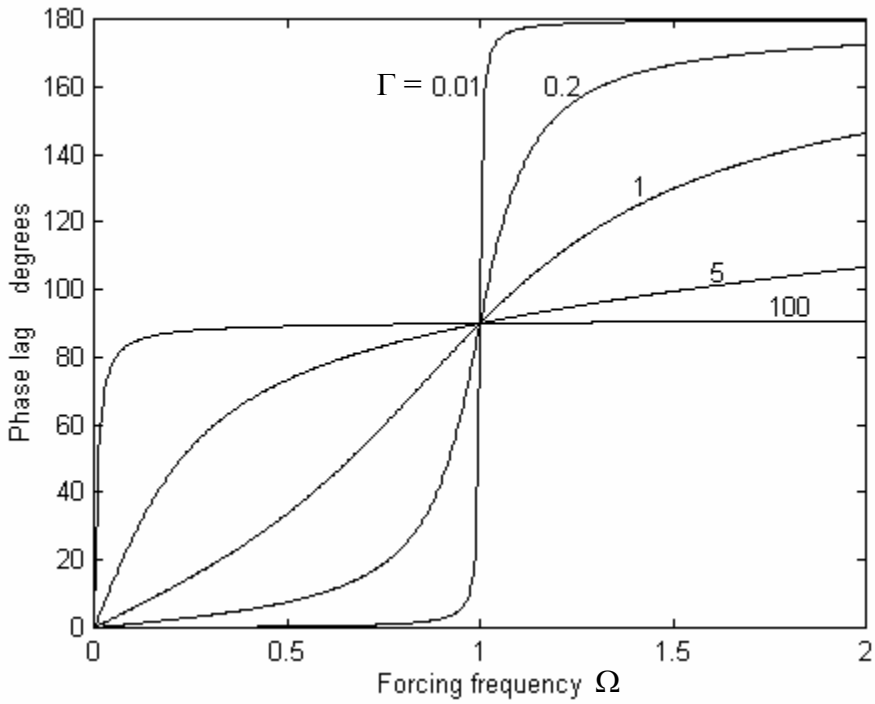
$$\tan\alpha = \frac{\Gamma\Omega}{1-\Omega^2} \qquad 12.2.9$$

and

$$\hat{X} = \frac{1}{\left[\left(1-\Omega^2\right)^2 + \Gamma^2\Omega^2\right]^{\frac{1}{2}}}. \qquad 12.2.10$$

The phase lag α and the displacement amplitude $\hat{X}\left(=\hat{x}/(\hat{f}/\omega_0^2)\right)$ are shown as a function of forcing frequency for various values of the damping constant in figures XII.1 and 2.

FIGURE XII.1

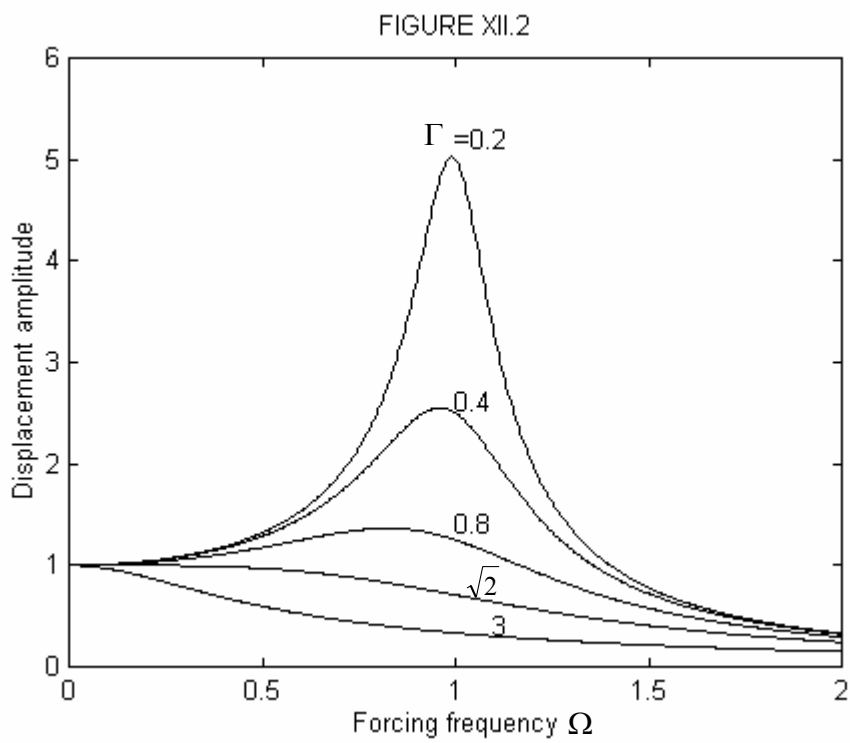


A common misunderstanding is that the displacement amplitude is greatest when the forcing frequency is equal to the undamped frequency ω_0. That this is far from the case is immediately obvious from a glance at figure XII.2. We can find the frequency that results in the greatest displacement amplitude by maximizing equation 12.2.7 or 12.2.10. This is most easily achieved by minimizing the square of the denominator. Let D be the square of the denominator of equation 12.2.10, and let $W = \Omega^2$ and $G = \Gamma^2$. Then $D = (1-W)^2 + GW$, which is greatest for $W = 1-\frac{1}{2}G$, or, provided $\gamma < \sqrt{2}\omega_0$,

$$\omega = \sqrt{\omega_0^2 - \tfrac{1}{2}\gamma^2}. \qquad 12.2.11$$

This is less not only than ω_0, but also less than ω'. For the frequency given by equation 12.2.11, the displacement amplitude will be

$$\hat{x}_{\max} = \frac{\hat{f}}{\gamma\left(\omega_0^2 - \frac{1}{4}\gamma^2\right)^{\frac{1}{2}}}. \qquad 12.2.12$$



The locus of the maxima in figure XII.2 is found by eliminating γ from equation 12.2.7 and $d\hat{x}/d\omega = 0$, which gives

$$\hat{x}_{\max} = \frac{\hat{f}}{\sqrt{\omega_0^4 - \omega^4}}. \qquad 12.2.13$$

The solution given by equations 12.2.7 and 12.2.8 is the *particular integral*. As pointed out in section 1 of this chapter, the complete solution is the sum of the particular integral and the *complementary function*, the latter being the unforced solutions of chapter 11. The particular integral represents the *steady state* solution, whereas the complementary function, which dies out with time, is a *transient solution.* When a mechanical oscillation is started, or when an alternating current electric circuit is first switched on, the solution is the sum of transient and steady state parts, the former more or less rapidly dying away. Often when an electric fuse blows, the overload is caused by the large, but temporary, amplitude of the transient part of the solution.

Equations 12.2.7 and 8 give the displacement of the system as a function of time. Differentiation with respect to time gives the velocity as a function of time. Thus:

$$\dot{x} = \upsilon = -\hat{\upsilon}\sin(\omega t - \alpha), \qquad 12.2.14$$

where

$$\hat{\upsilon} = \frac{\hat{f}\omega}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right]^{\frac{1}{2}}} \qquad 12.2.15$$

is the *velocity amplitude.* In dimensionless units, this can be written

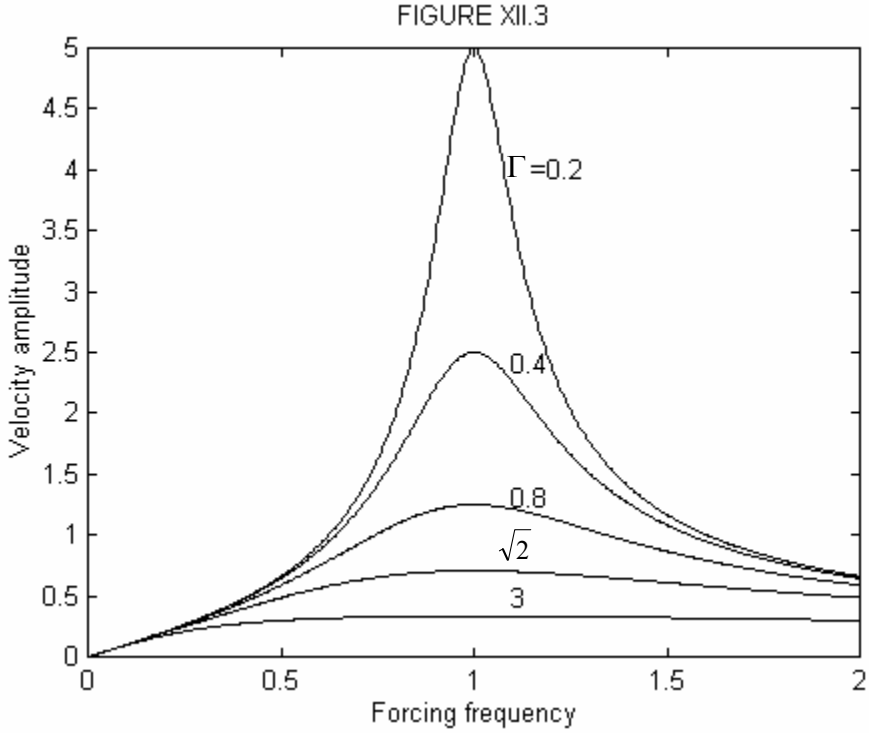
$$\hat{V} = \frac{\Omega}{\left[\left(1 - \Omega^2\right)^2 + \Gamma^2\Omega^2\right]^{\frac{1}{2}}}, \qquad 12.2.16$$

where

$$\hat{V} = \frac{\hat{\upsilon}}{\hat{f}/\omega_0}. \qquad 12.2.17$$

This is illustrated in figure XII.3.

FIGURE XII.3



It is left to the reader to show that the velocity amplitude is greatest and equal to $\hat{f}/\gamma$ when $\omega = \omega_0$.

We have now found the phase lag and the displacement and velocity amplitudes as a function of forcing frequency, but I must now try the reader's patience one step further for the most important part of the analysis, which really must not be skipped. Damping of oscillatory motion implies that some of the mechanical energy (which, in an undamped system, alternates between kinetic and potential energy) is lost - or, rather, that it is dissipated as heat. This happens if the damping is caused by the oscillator being immersed in a viscous fluid, or if it is caused by the repeated expansion and compression of a spring, or, in an electric circuit, by the dissipation of heat in the resistive part of the circuit. We aim now to find the rate at which the mechanical energy is dissipated as heat.

We return to the equation of motion:

$$m\ddot{x} + b\dot{x} + kx = \hat{F}\cos\omega t, \qquad 12.2.1$$

and multiply each side by $\dot{x}$:

$$m\dot{x}\ddot{x} + b\dot{x}^2 + kx\dot{x} = \dot{x}\hat{F}\cos\omega t. \qquad 12.2.18$$

Introduce the total mechanical energy:

$$E = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2. \qquad 12.2.19$$

The instantaneous rate of change of E is $m\dot{x}\ddot{x} + kx\dot{x}$, while the instantaneous rate at which F does work is $\dot{x}\hat{F}\cos\omega t$. The difference (see equation 12.2.18), $b\dot{x}^2$, is therefore the rate at which work is being dissipated as heat, which, of course, is zero if $b = 0$.

The average of $b\dot{x}^2$ over a complete period is

$$\overline{b\dot{x}^2} = \frac{b\hat{f}^2\omega^2}{\left(\omega_0^2 - \omega^2\right) + \gamma^2\omega^2}\overline{\sin^2\left(\omega t - \alpha\right)}, \qquad 12.2.20$$

where the bars denote the average value over a period. But $\overline{\sin^2\left(\omega t - \alpha\right)} = \tfrac{1}{2}$, so the average rate at which work is being dissipated as heat, for which I shall use the symbol $\dot{Q}$, is

$$\dot{Q} = \frac{b\hat{f}^2\omega^2}{2\left[\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right]}. \qquad 12.2.21$$

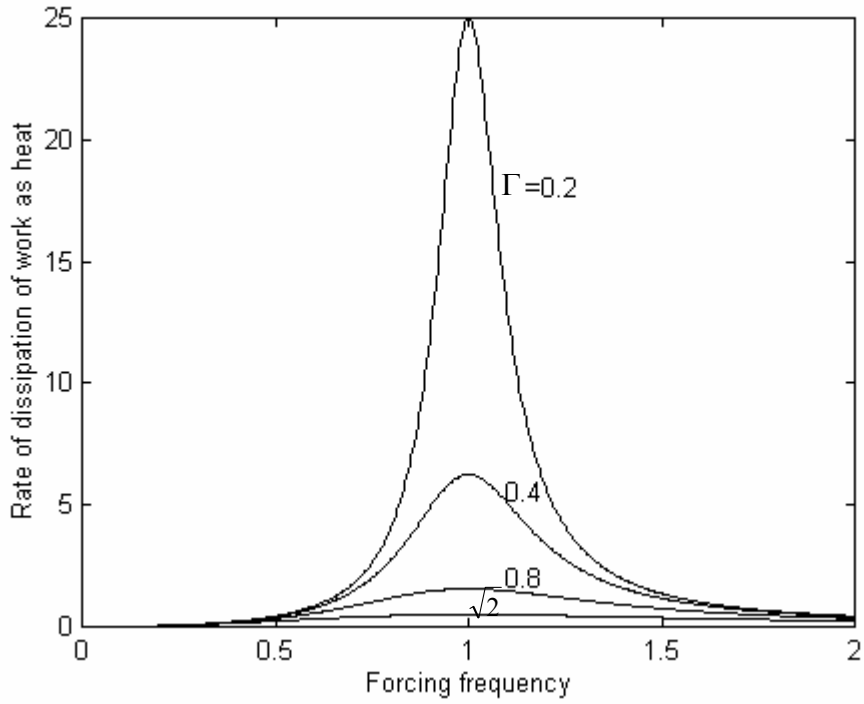
The reader should check that the right hand side has the dimensions of rate of dissipation of energy and hence the SI unit of watts.

In dimensionless units, in which $\dot{Q}^* = \dot{Q}/\left(b\hat{f}^2/(2\omega_0)\right)$, this can be written

$$\dot{Q}^* = \frac{\Omega^2}{\left(1-\Omega^2\right)^2 + \Gamma^2\Omega^2}. \qquad 12.2.22$$

This is illustrated in figure XII.4. The reader can easily prove that the rate at which work is dissipated as heat is greatest when the forcing frequency is equal to ω_0.

FIGURE XII.4



Summary.

Phase lag:

$$\cos\alpha = \frac{\omega_0^2 - \omega^2}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right]^{\frac{1}{2}}}, \quad \sin\alpha = \frac{\gamma\omega}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right]^{\frac{1}{2}}}, \quad \tan\alpha = \frac{\gamma\omega}{\omega_0^2 - \omega^2}. \qquad 12.2.5$$

Displacement amplitude:

$$\hat{x} = \frac{\hat{f}}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right]^{\frac{1}{2}}}. \qquad 12.2.7$$

Velocity amplitude:

$$\hat{\upsilon} = \frac{\hat{f}\omega}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right]^{\frac{1}{2}}} \qquad 12.2.15$$

Rate of dissipation of work as heat:

$$\dot{Q} = \frac{m\gamma\hat{f}^2\omega^2}{2\left[\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right]}. \qquad 12.2.21$$

In terms of dimensionless variables,

Phase lag:

$$\tan\alpha = \frac{\Gamma\Omega}{1-\Omega^2} \qquad 12.2.9$$

Displacement amplitude:

$$\hat{X} = \frac{1}{\left[\left(1-\Omega^2\right)^2 + \Gamma^2\Omega^2\right]^{\frac{1}{2}}}. \qquad 12.2.10$$

Velocity amplitude:

$$\hat{V} = \frac{\Omega}{\left[\left(1-\Omega^2\right)^2 + \Gamma^2\Omega^2\right]^{\frac{1}{2}}}, \qquad 12.2.16$$

Rate of dissipation of work as heat:

$$\dot{Q}^* = \frac{\Omega^2}{\left(1-\Omega^2\right)^2 + \Gamma^2\Omega^2}. \qquad 12.2.22$$

12.3 *Electrical Analogue*

Suppose that an alternating potential difference $E = \hat{E}\sin\omega t$ is applied across an *LCR* circuit. We refer to equation 11.6.3, and we see that the equation that governs the charge on the capacitor is

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = \hat{E}\sin\omega t. \qquad 12.3.1$$

We can differentiate both sides with respect to time, and divide by *L*, and hence see that the current is given by

$$\ddot{I} + \frac{R}{L}\dot{I} + \frac{1}{LC}I = \frac{\hat{E}\omega}{L}\cos\omega t. \qquad 12.3.2$$

We can compare this directly with equation 12.2.2, so that we have

$$\gamma = \frac{R}{L}, \quad \omega_0^2 = \frac{1}{\sqrt{LC}}, \quad \hat{f} = \frac{\hat{E}\omega}{L}. \qquad 12.3.3$$

Then, by comparison with equation 12.2.5, we see that *I* will lag behind *E* by α, where

$$\tan\alpha = \frac{\frac{R\omega}{L}}{\frac{1}{LC} - \omega^2} = \frac{R}{\frac{1}{C\omega} - L\omega}. \qquad 12.3.4$$

This is just what we obtain from the more familiar complex number approach to alternating current circuits.