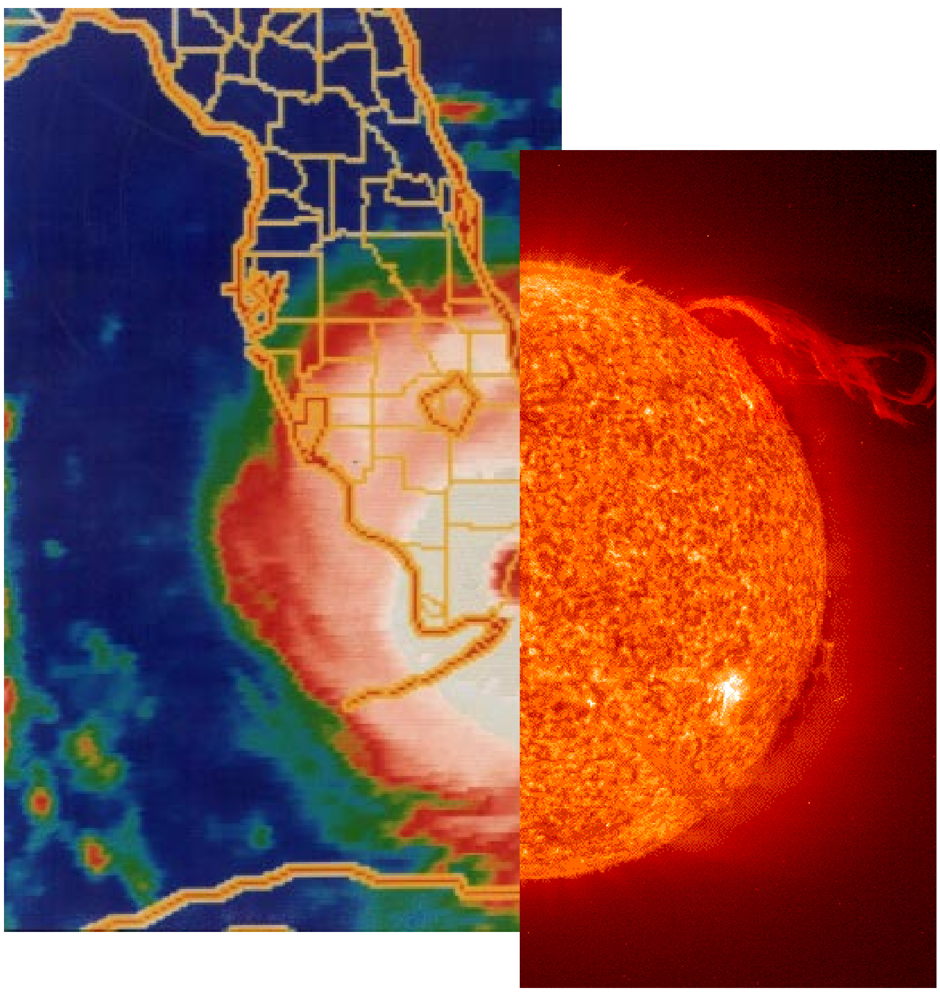
Conservation Laws

Benjamin Crowell



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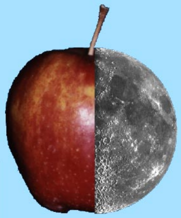

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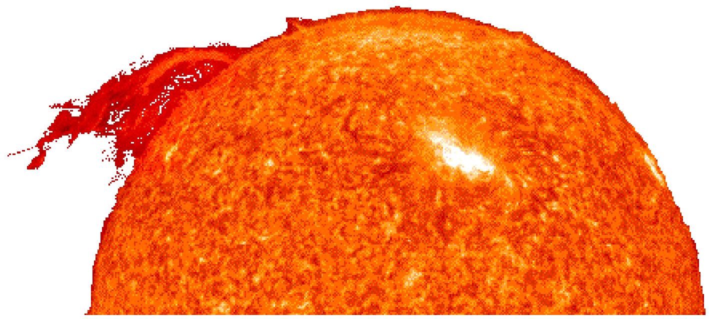
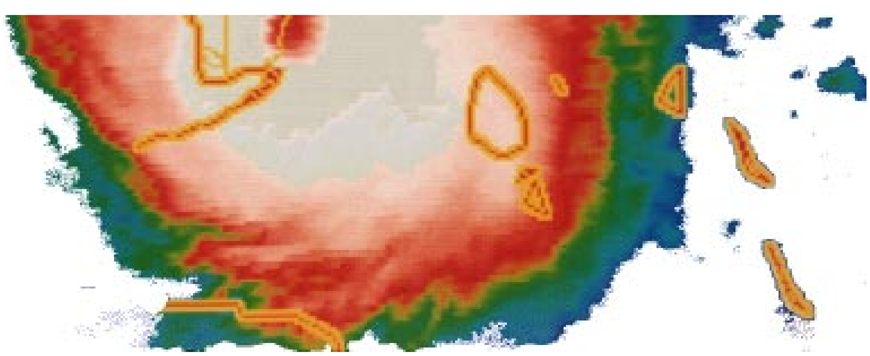
To Uri Haber-Schaim, John Dodge, Robert Gardner, and Edward Shore.



Brief Contents



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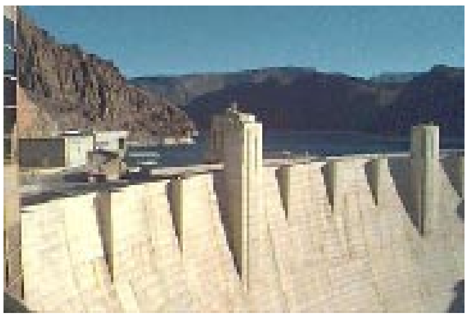
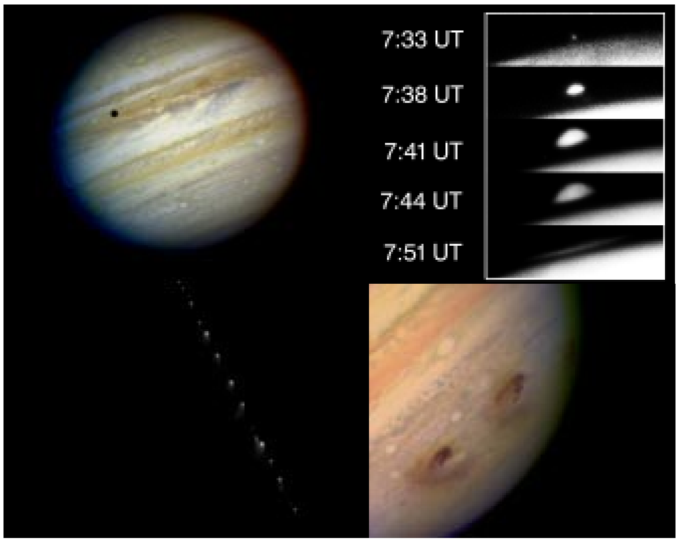


Note: See Simple Nature (www.lightandmatter.com/area1sn.html) for coverage of the following topics: thermodynamics, rigid-body rotation, and angular momentum in three dimensions.



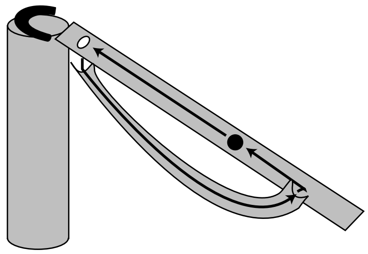
In July of 1994, Comet Shoemaker-Levy struck the planet Jupiter, depositing 7×10^{22} joules of energy, and incidentally giving rise to a series of Hollywood movies in which our own planet is threatened by an impact by a comet or asteroid. There is evidence that such an impact caused the extinction of the dinosaurs. Left: Jupiter's gravitational force on the near side of the comet was greater than on the far side, and this difference in force tore up the comet into a string of fragments. Two separate telescope images have been combined to create the illusion of a point of view just behind the comet. (The colored fringes at the edges of Jupiter are artifacts of the imaging system.) Top: A series of images of the plume of superheated gas kicked up by the impact of one of the fragments. The plume is about the size of North America. Bottom: An image after all the impacts were over, showing the damage done.

1 Conservation of Energy

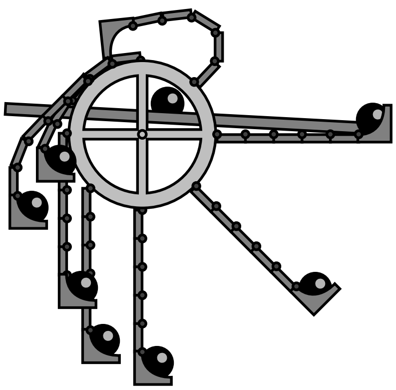
1.1 The Search for a Perpetual Motion Machine

Don't underestimate greed and laziness as forces for progress. Modern chemistry was born from the collision of lust for gold with distaste for the hard work of finding it and digging it up. Failed efforts by generations of alchemists to turn lead into gold led finally to the conclusion that it could not be done: certain substances, the chemical elements, are fundamental, and chemical reactions can neither increase nor decrease the amount of an element such as gold.

Now flash forward to the early industrial age. Greed and laziness have created the factory, the train, and the ocean liner, but in each of these is a boiler room where someone gets sweaty shoveling the coal to fuel the steam



(a) The magnet draws the ball to the top of the ramp, where it falls through the hole and rolls back to the bottom.



(b) As the wheel spins clockwise, the flexible arms sweep around and bend and unbend. By dropping off its ball on the ramp, the arm is supposed to make itself lighter and easier to lift over the top. Picking its own ball back up again on the right, it helps to pull the right side down.

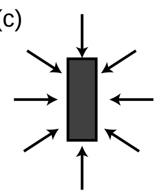
engine. Generations of inventors have tried to create a machine, called a perpetual motion machine, that would run forever without fuel. Such a machine is not forbidden by Newton's laws of motion, which are built around the concepts of force and inertia. Force is free, and can be multiplied indefinitely with pulleys, gears, or levers. The principle of inertia seems even to encourage the belief that a cleverly constructed machine might not ever run down.

The figures show two of the innumerable perpetual motion machines that have been proposed. The reason these two examples don't work is not much different from the reason all the others have failed. Consider machine (a). Even if we assume that a properly shaped ramp would keep the ball rolling smoothly through each cycle, friction would always be at work. The designer imagined that the machine would repeat the same motion over and over again, so that every time it reached a given point its speed would be exactly the same as the last time. But because of friction, the speed would actually be reduced a little with each cycle, until finally the ball would no longer be able to make it over the top.

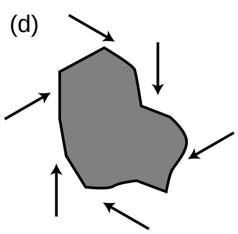
Friction has a way of creeping into all moving systems. The rotating earth might seem like a perfect perpetual motion machine, since it is isolated in the vacuum of outer space with nothing to exert frictional forces on it. But in fact our planet's rotation has slowed drastically since it first formed, and the earth continues to slow its rotation, making today just a little longer than yesterday. The very subtle source of friction is the tides. The moon's gravity raises bulges in the earth's oceans, and as the earth rotates the bulges progress around the planet. Where the bulges encounter land, there is friction, which slows the earth's rotation very gradually.

1.2 Energy

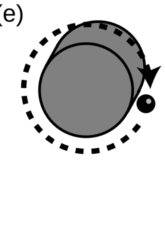
(c)



(d)



(e)



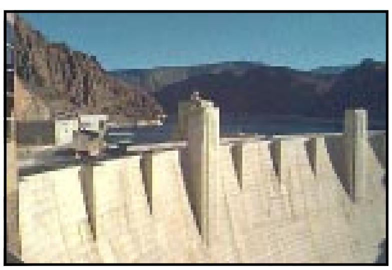
The analysis based on friction is somewhat superficial, however. One could understand friction perfectly well and yet imagine the following situation. Astronauts bring back a piece of magnetic ore from the moon which does not behave like ordinary magnets. A normal bar magnet, (c), attracts a piece of iron essentially directly toward it, and has no left- or right-handedness. The moon rock, however, exerts forces that form a whirlpool pattern around it, (d). NASA goes to a machine shop and has the moon rock put in a lathe and machined down to a smooth cylinder, (e). If we now release a ball bearing on the surface of the cylinder, the magnetic force whips it around and around at ever higher speeds. Of course there is some friction, but there is a net gain in speed with each revolution.

Physicists would lay long odds against the discovery of such a moon rock, not just because it breaks the rules that magnets normally obey but because, like the alchemists, they have discovered a very deep and fundamental principle of nature which forbids certain things from happening. The first alchemist who deserved to be called a chemist was the one who realized one day, "In all these attempts to create gold where there was none before, all I've been doing is shuffling the same atoms back and forth among different test tubes. The only way to increase the amount of gold in my laboratory is to bring some in through the door." It was like having some of your money in a checking account and some in a savings account. Transferring money from one account into the other doesn't change the total amount.

We say that the number of grams of gold is a *conserved* quantity. In this context, the word "conserve" does not have its usual meaning of trying not to waste something. In physics, a conserved quantity is something that you wouldn't be able to get rid of even if you wanted to. Conservation laws in physics always refer to a *closed system*, meaning a region of space with boundaries through which the quantity in question is not passing. In our example, the alchemist's laboratory is a closed system because no gold is coming in or out through the doors.

A similar lightbulb eventually lit up in the heads of the people who had been frustrated trying to build a perpetual motion machine. In perpetual motion machine (b) in the previous section, consider the motion of one of the balls. It performs a cycle of rising and falling. On the way down it gains speed, and coming up it slows back down. Having a greater speed is like having more money in your checking account, and being high up is like having more in your savings account. The device is simply shuffling funds back and forth between the two. Having more balls doesn't change anything fundamentally. Not only that, but friction is always draining off money into a third "bank account:" heat. The reason we rub our hands together when we're cold is that kinetic friction heats things up. The continual buildup in the "heat account" leaves less and less for the "motion account" and "height account," causing the machine eventually to run down.

These insights can be distilled into the following basic principle of physics:



The water behind the Hoover Dam has energy because of its position relative to the planet earth, which is attracting it with a gravitational force. Letting water down to the bottom of the dam converts that energy into energy of motion. When the water reaches the bottom of the dam, it hits turbine blades that drive generators, and its energy of motion is converted into electrical energy.

The Law of Conservation of Energy

It is possible to give a numerical rating, called energy, to the state of a physical system. The total energy is found by adding up contributions coming from characteristics of the system such as motion of objects in it, heating of the objects, and the relative positions of objects that interact via forces. The total energy of a closed system always remains constant. Energy cannot be created or destroyed, but only transferred into or out of a system.

The moon rock story violates conservation of energy because the rock-cylinder and the ball together constitute a closed system. Once the ball has made one revolution around the cylinder, its position relative to the cylinder is exactly the same as before, so the numerical energy rating associated with its position is the same as before. Since the total amount of energy must remain constant, it is impossible for the ball to have a greater speed after one revolution. If it had picked up speed, it would have more energy associated with motion, the same amount of energy associated with position, and a little more energy associated with heating through friction. There cannot be a net increase in energy.

Examples

Dropping a rock: The rock loses energy because of its changing position with respect to the earth. Nearly all that energy is transformed into energy of motion, except for a small amount lost to heat created by air friction.

Sliding in to home base: The runner's energy of motion is nearly all converted into heat via friction with the ground.

Accelerating a car: The gasoline has energy stored in it, which is released as heat by burning it inside the engine. Perhaps 10% of this heat energy is converted into the car's energy of motion. The rest remains in the form of heat, which is carried away by the exhaust.

Cruising in a car: As you cruise at constant speed in your car, all the energy of the burning gas is being converted into heat. The tires and engine get hot, and heat is also dissipated into the air through the radiator and the exhaust.

Stepping on the brakes: All the energy of the car's motion is converted into heat in the brake shoes.

Discussion Question

Hydroelectric power (water flowing over a dam to spin turbines) appears to be completely free. Does this violate conservation of energy? If not, then what is the ultimate source of the electrical energy produced by a hydroelectric plant?

1.3 A Numerical Scale of Energy

Energy comes in a variety of forms, and physicists didn't discover all of them right away. They had to start somewhere, so they picked one form of energy to use as a standard for creating a numerical energy scale. (In fact the history is complicated, and several different energy units were defined before it was realized that there was a single general energy concept that deserved a single consistent unit of measurement.) One practical approach is to define an energy unit based on heating water. The SI unit of energy is the joule, J, (rhymes with "cool"), named after the British physicist James Joule. One Joule is the amount of energy required in order to heat 0.24 g of water by 1°C. The number 0.24 is not worth memorizing.

Note that heat, which is a form of energy, is completely different from temperature, which is not. Twice as much heat energy is required to prepare two cups of coffee as to make one, but two cups of coffee mixed together don't have double the temperature. In other words, the temperature of an object tells us how hot it is, but the heat energy contained in an object also takes into account the object's mass and what it is made of.

Later we will encounter other quantities that are conserved in physics, such as momentum and angular momentum, and the method for defining them will be similar to the one we have used for energy: pick some standard form of it, and then measure other forms by comparison with this standard. The flexible and adaptable nature of this procedure is part of what has made conservation laws such a durable basis for the evolution of physics.

Example: heating a swimming pool

Question: If electricity costs 3.9 cents per MJ (1 MJ = 1 megajoule = 10^6 J), how much does it cost to heat a 26000-gallon swimming pool from 10°C to 18°C?

Solution: Converting gallons to cm^3 gives

$$26000\ \text{gallons} \times \frac{3780\ \text{cm}^3}{1\ \text{gallon}} = 9.8\text{x}10^7\ \text{cm}^3.$$

Water has a density of 1 gram per cubic centimeter, so the mass of the water is $9.8\text{x}10^7$ g. One joule is sufficient to heat 0.24 g by 1°C, so the energy needed to heat the swimming pool is

$$1\ \text{J} \times \frac{9.8\times10^7\ \text{g}}{0.24\ \text{g}} \times \frac{8°C}{1°C} = 3.3\text{x}10^9\ \text{J}$$

$$= 3.3\text{x}10^3\ \text{MJ} \quad .$$

The cost of the electricity is ($3.3\text{x}10^3$ MJ)($0.039/MJ)=$130.

Example: Irish coffee

Question: You make a cup of Irish coffee out of 300 g of coffee at 100°C and 30 g of pure ethyl alcohol at 20°C. One Joule is enough energy to produce a change of 1°C in 0.42 g of ethyl alcohol (i.e. alcohol is easier to heat than water). What temperature is the final mixture?

Solution: Adding up all the energy after mixing has to give the same result as the total before mixing. We let the subscript i stand for the initial situation, before mixing, and f for the final situation, and use subscripts c for the coffee and a for the alcohol. In this notation, we have

$$\text{total initial energy} = \text{total final energy}$$

$$E_{ci}+E_{ai} = E_{cf}+E_{af} \quad .$$

We assume coffee has the same heat-carrying properties as water. Our information about the heat-carrying properties of the two substances is stated in terms of the change in energy required for a certain change in temperature, so we rearrange the equation to express everything in terms of energy differences:

$$E_{af}-E_{ai} = E_{ci}-E_{cf} \quad .$$

Using the given ratios of temperature change to energy change, we have

$$E_{ci}-E_{cf} = (T_{ci}-T_{cf})(m_c)/(0.24\ \text{g})$$

$$E_{af}-E_{ai} = (T_{af}-T_{ai})(m_a)/(0.42\ \text{g})$$

Setting these two quantities to be equal, we have

$$(T_{af}-T_{ai})(m_a)/(0.42\ \text{g}) = (T_{ci}-T_{cf})(m_c)/(0.24\ \text{g}) \ .$$

In the final mixture the two substances must be at the same temperature, so we can use a single symbol $T_f=T_{cf}=T_{af}$ for the two quantities previously represented by two different symbols,

$$(T_f-T_{ai})(m_a)/(0.42\ \text{g}) = (T_{ci}-T_f)(m_c)/(0.24\ \text{g}) \ .$$

Solving for T_f gives

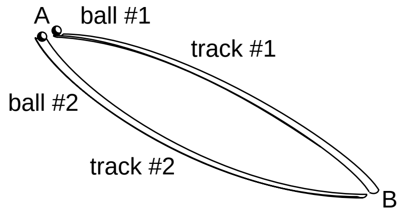
$$T_f = \frac{T_{ci}\frac{m_c}{.24} + T_{ai}\frac{m_a}{.42}}{\frac{m_c}{.24} + \frac{m_a}{.42}}$$

$$= 96°C.$$

Once a numerical scale of energy has been established for some form of energy such as heat, it can easily be extended to other types of energy. For instance, the energy stored in one gallon of gasoline can be determined by putting some gasoline and some water in an insulated chamber, igniting the gas, and measuring the rise in the water's temperature. (The fact that the apparatus is known as a "bomb calorimeter" will give you some idea of how dangerous these experiments are if you don't take the right safety precautions.) Here are some examples of other types of energy that can be measured using the same units of joules:

type of energy	**example**
chemical energy released by burning	About 50 MJ are released by burning 1 kg of gasoline.
energy required to break an object	When a person suffers a spiral fracture of the thighbone (a common type in skiing accidents), about 2 J of energy go into breaking the bone.
energy required to melt a solid substance	7 MJ are required to melt 1 kg of tin.
chemical energy released by digesting food	A bowl of Cheerios with milk provides us with about 800 kJ of usable energy.
raising a mass against the force of gravity	Lifting 1.0 kg through a height of 1.0 m requires 9.8 J.
nuclear energy released in fission	1 kg of uranium oxide fuel consumed by a reactor releases 2×10^{12} J of stored nuclear energy.

It is interesting to note the disproportion between the megajoule energies we consume as food and the joule-sized energies we expend in physical activities. If we could perceive the flow of energy around us the way we perceive the flow of water, eating a bowl of cereal would be like swallowing a bathtub's worth of energy, the continual loss of body heat to one's environment would be like an energy-hose left on all day, and lifting a bag of cement would be like flicking it with a few tiny energy-drops. The human body is tremendously inefficient. The calories we "burn" in heavy exercise are almost all dissipated directly as body heat.



Forms of Energy Discovered in Recent Times

Einstein showed that mass itself could be converted to and from energy, according to his celebrated equation $E=mc^2$, in which c is the speed of light. We thus speak of mass as simply another form of energy, and it is valid to measure it in units of joules. The mass of a 15-gram pencil corresponds to about 1.3×10^{15} J. The issue is largely academic in the case of the pencil, because very violent processes such as nuclear reactions are required in order to convert any significant fraction of an object's mass into energy. Cosmic rays, however, are continually striking you and your surroundings and converting part of their energy of motion into the mass of newly created particles. A single high-energy cosmic ray can create a "shower" of millions of previously nonexistent particles when it strikes the atmosphere. Einstein's theories are discussed in book 6 of this series.

Even today, when the energy concept is relatively mature and stable, a new form of energy has been proposed based on observations of distant galaxies whose light began its voyage to us billions of years ago. Astronomers have found that the universe's continuing expansion, resulting from the Big Bang, has not been decelerating as rapidly in the last few billion years as would have been expected from gravitational forces. They suggest that a new form of energy may be at work.

Example: You take the high road and I'll take the low road.

Question: The figure shows two ramps which two balls will roll down. Compare their final speeds, when they reach point B. Assume friction is negligible.

Solution: Each ball loses some energy because of its decreasing height above the earth, and conservation of energy says that it must gain an equal amount of energy of motion (minus a little heat created by friction). The balls lose the same amount of height, so their final speeds must be equal.

It's impressive to note the complete impossibility of solving this problem using only Newton's laws. Even if the shape of the track had been given mathematically, it would have been a formidable task to compute the balls' final speed based on vector addition of the normal force and gravitational force at each point along the way.

How New Forms of Energy Are Discovered

Textbooks often give the impression that a sophisticated physics concept was created by one person who had an inspiration one day, but in reality it is more in the nature of science to rough out an idea and then gradually refine it over many years. The idea of energy was tinkered with from the early 1800s on, and new types of energy kept getting added to the list.

To establish the existence of a new form of energy, a physicist has to

(1) show that it could be converted to and from other forms of energy; and

(2) show that it related to some definite measurable property of the object, for example its temperature, motion, position relative to another object, or being in a solid or liquid state.

For example, energy is released when a piece of iron is soaked in water, so apparently there is some form of energy already stored in the iron. The release of this energy can also be related to a definite measurable property of the chunk of metal: it turns reddish-orange. There has been a chemical change in its physical state, which we call rusting.

Although the list of types of energy kept getting longer and longer, it was clear that many of the types were just variations on a theme. There is an obvious similarity between the energy needed to melt ice and to melt butter, or between the rusting of iron and many other chemical reactions. The topic of the next chapter is how this process of simplification reduced all the types of energy to a very small number (four, according to the way I've chosen to count them).

It might seem that if the principle of conservation of energy ever appeared to be violated, we could fix it up simply by inventing some new type of energy to compensate for the discrepancy. This would be like balancing your checkbook by adding in an imaginary deposit or withdrawal to make your figures agree with the bank's statements. Step (2) above guards against this kind of chicanery. In the 1920s there were experiments that suggested energy was not conserved in radioactive processes. Precise measurements of the energy released in the radioactive decay of a given type of atom showed inconsistent results. One atom might decay and release, say, 1.1×10^{-10} J of energy, which had presumably been stored in some mysterious form in the nucleus. But in a later measurement, an atom of exactly the same type might release 1.2×10^{-10} J. Atoms of the same type are supposed to be identical, so both atoms were thought to have started out with the same energy. If the amount released was random, then apparently the total amount of energy was not the same after the decay as before, i.e. energy was not conserved.

Only later was it found that a previously unknown particle, which is very hard to detect, was being spewed out in the decay. The particle, now called a neutrino, was carrying off some energy, and if this previously unsuspected form of energy was added in, energy was found to be conserved after all. The discovery of the energy discrepancies is seen with hindsight as being step (1) in the establishment of a new form of energy, and the discovery of the neutrino was step (2). But during the decade or so between step (1) and step (2) (the accumulation of evidence was gradual), physicists had the admirable honesty to admit that the cherished principle of conservation of energy might have to be discarded.

Self-Check

How would you carry out the two steps given above in order to establish that some form of energy was stored in a stretched or compressed spring?

Discussion Question

I'm not making this up. XS Energy Drink has ads that read like this: *All the "Energy" ... Without the Sugar! Only 8 Calories!* Comment on this.

1.4 Kinetic Energy

The technical term for the energy associated with motion is *kinetic energy*, from the Greek word for motion. (The root is the same as the word "cinema" for motion picture, and in French the term for kinetic energy is énergie cinématique.) To find how much kinetic energy is possessed by a given moving object, we must convert all its kinetic energy into heat energy, which we have chosen as the standard reference type of energy. We could do this, for example, by firing projectiles into a tank of water and measuring the increase in temperature of the water as a function of the projectile's mass and velocity. Consider the following data from a series of three such experiments:

m (kg)	**v** (m/s)	**energy** (J)
1.00	1.00	0.50
1.00	2.00	2.00
2.00	1.00	1.00

(1) A spring-loaded toy gun can cause a bullet to move, so the spring is capable of storing energy and then converting it into kinetic energy. (2) The amount of energy stored in the spring relates to amount of compression, which can be measured with a ruler.

Comparing the first experiment with the second, we see that doubling the object's velocity doesn't just double its energy, it quadruples it. If we compare the first and third lines, however, we find that doubling the mass only doubles the energy. This suggests that kinetic energy is proportional to mass and to the square of velocity, $KE \propto mv^2$, and further experiments of this type would indeed establish such a general rule. The proportionality factor equals 0.5 because of the design of the metric system, so the kinetic energy of a moving object is given by

$$KE = \frac{1}{2}mv^2 \quad .$$

The metric system is based on the meter, kilogram, and second, with other units being derived from those. Comparing the units on the left and right sides of the equation shows that the joule can be reexpressed in terms of the basic units as $kg \cdot m^2/s^2$.

Students are often mystified by the occurrence of the factor of 1/2, but it is less obscure than it looks. The metric system was designed so that some of the equations relating to energy would come out looking simple, at the expense of some others, which had to have inconvenient conversion factors in front. If we were using the old British Engineering System of units in this course, then we'd have the British Thermal Unit (BTU) as our unit of energy. In that system, the equation you'd learn for kinetic energy would have an inconvenient proportionality constant, $KE=(1.29 \times 10^{-3})mv^2$, with KE measured in units of BTUs, v measured in feet per second, and so on. At the expense of this inconvenient equation for kinetic energy, the designers of the British Engineering System got a simple rule for calculating the energy required to heat water: one BTU per degree Fahrenheit per gallon. The inventor of kinetic energy, Thomas Young, actually defined it as $KE=mv^2$, which meant that all his other equations had to be different from ours by a factor of two. All these systems of units work just fine as long as they are not combined with one another in an inconsistent way.

Example: energy released by a comet impact

Question: Comet Shoemaker-Levy, which struck the planet Jupiter in 1994, had a mass of roughly 4×10^{13} kg, and was moving at a speed of 60 km/s. Compare the kinetic energy released in the impact to the total energy in the world's nuclear arsenals, which is 2×10^{19} J. Assume for the sake of simplicity that Jupiter was at rest.

Solution: Since we assume Jupiter was at rest, we can imagine that the comet stopped completely on impact, and 100% of its kinetic energy was converted to heat and sound. We first convert the speed to mks units, $v=6 \times 10^4$ m/s, and then plug in to the equation $KE = \frac{1}{2}mv^2$ to find that the comet's kinetic energy was roughly 7×10^{22} J, or about 3000 times the energy in the world's nuclear arsenals.

Is there any way to derive the equation $KE = \frac{1}{2}mv^2$ mathematically from first principles? No, it is purely empirical. The factor of 1/2 in front is definitely not derivable, since it is different in different systems of units. The proportionality to v^2 is not even quite correct; experiments have shown deviations from the v^2 rule at high speeds, an effect that is related to Einstein's theory of relativity. Only the proportionality to *m* is inevitable. The whole energy concept is based on the idea that we add up energy contributions from all the objects within a system. Based on this philosophy, it is logically necessary that a 2-kg object moving at 1 m/s have the same kinetic energy as two 1-kg objects moving side-by-side at the same speed.

Energy and relative motion

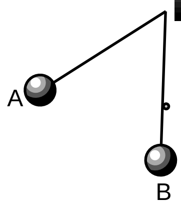
Although I mentioned Einstein's theory of relativity above, it's more relevant right now to consider how conservation of energy relates to the simpler Galilean idea, which we've already studied, that motion is relative. Galileo's Aristotelian enemies (and it is no exaggeration to call them enemies!) would probably have objected to conservation of energy. After all, the Galilean idea that an object in motion will continue in motion indefinitely in the absence of a force is not so different from the idea that an object's kinetic energy stays the same unless there is a mechanism like frictional heating for converting that energy into some other form.

More subtly, however, it's not immediately obvious that what we've learned so far about energy is strictly mathematically consistent with the principle that motion is relative. Suppose we verify that a certain process, say the collision of two pool balls, conserves energy as measured in a certain frame of reference: the sum of the balls' kinetic energies before the collision is equal to their sum after the collision. (In reality we'd need to add in other forms of energy, like heat and sound, that are liberated by the collision, but let's keep it simple.) But what if we were to measure everything in a frame of reference that was in a different state of motion? A particular pool ball might have less kinetic energy in this new frame; for example, if the new frame of reference was moving right along with it, its kinetic energy in that frame would be zero. On the other hand, some other balls might have a greater kinetic energy in the new frame. It's not immediately obvious that the total energy before the collision will still equal the total energy after the collision. After all, the equation for kinetic energy is fairly complicated, since it involves the square of the velocity, so it would be surprising if everything still worked out in the new frame of reference. It *does* still work out. Homework problem 13 in this chapter gives a simple numerical example, and the general proof is taken up in ch. 4, problem 15 (with the solution given in the back of the book).

Discussion Questions

A. Suppose that, like Young or Einstein, you were trying out different equations for kinetic energy to see if they agreed with the experimental data. Based on the meaning of positive and negative signs of velocity, why would you suspect that a proportionality to mv would be less likely than mv^2?

B. The figure shows a pendulum that is released at A and caught by a peg as it passes through the vertical, B. To what height will the bob rise on the right?



Discussion question B.

1.5 Power

A car may have plenty of energy in its gas tank, but still may not be able to increase its kinetic energy rapidly. A Porsche doesn't necessarily have more energy in its gas tank than a Hyundai, it is just able to transfer it more quickly. The rate of transferring energy from one form to another is called *power*. The definition can be written as an equation,

$$P = \frac{\Delta E}{\Delta t} \quad ,$$

where the use of the delta notation in the symbol ΔE has the usual notation: the final amount of energy in a certain form minus the initial amount that was present in that form. Power has units of J/s, which are abbreviated as watts, W (rhymes with "lots").

If the rate of energy transfer is not constant, the power at any instant can be defined as the slope of the tangent line on a graph of E versus t. Likewise ΔE can be extracted from the area under the P-versus-t curve.

Example: converting kilowatt-hours to joules

Question: The electric company bills you for energy in units of kilowatt-hours (kilowatts multiplied by hours) rather than in SI units of joules. How many joules is a kilowatt-hour?

Solution:

1 kilowatt-hour = (1 kW)(1 hour) = (1000 J/s)(3600 s) = 3.6 MJ.

Example: human wattage

Question: A typical person consumes 2000 kcal of food in a day, and converts nearly all of that directly to heat. Compare the person's heat output to the rate of energy consumption of a 100-watt lightbulb.

Solution: Looking up the conversion factor from calories to joules, we find

$$\Delta E = 2000\text{ kcal} \times \frac{1000\text{ cal}}{1\text{ kcal}} \times \frac{4.18\text{ J}}{1\text{ cal}} = 8\text{x}10^6\text{ J}$$

for our daily energy consumption. Converting the time interval likewise into mks,

$$\Delta t = 1\text{ day} \times \frac{24\text{ hours}}{1\text{ day}} \times \frac{60\text{ min}}{1\text{ hour}} \times \frac{60\text{ s}}{1\text{ min}} = 9\text{x}10^4\text{ s} \quad .$$

Dividing, we find that our power dissipated as heat is 90 J/s = 90 W, about the same as a lightbulb.

It is easy to confuse the concepts of force, energy, and power, especially since they are synonyms in ordinary speech. The table on the following page may help to clear this up:

	force	energy	power
conceptual definition	A force is an interaction between two objects that causes a push or a pull. A force can be defined as anything that is capable of changing an object's state of motion	Heating an object, making it move faster, or increasing its distance from another object that is attracting it are all examples of things that would require fuel or physical effort. There is a numerical way of measuring all these kinds of things using a single unit of measurement, and we describe them all as forms of energy.	Power is the rate at which energy is transformed from one form to another or transferred from one object to another.
operational definition	A spring scale can be used to measure force.	If we define a unit of energy as the amount required to heat a certain amount of water by a 1°C, then we can measure any other quantity of energy transferring it into heat in water and measuring the temperature increase.	Measure the change in the amount of some form of energy possessed by an object, and divide by the amount of time required for the change to occur.
scalar or vector?	vector – has a direction in space which is the direction in which it pulls or pushes	scalar – has no direction in space	scalar – has no direction in space
unit	newtons (N)	joules (J)	watts (W) = joules/s
Can it run out? Does it cost money?	No. I don't have to pay a monthly bill for the meganewtons of force required to hold up my house.	Yes. We pay money for gasoline, electrical energy, batteries, etc. because they contain energy.	More power means you are paying money at a higher rate. A 100-W lightbulb costs a certain number of cents per hour.
Can it be a property of an object?	No. A force is a relationship between two interacting objects. A home-run baseball doesn't "have" force.	Yes. What a home-run baseball has is kinetic energy, not force.	Not really. A 100-W lightbulb doesn't "have" 100 W. 100 J/s is the rate at which it converts electrical energy into light.

Summary

Selected Vocabulary

energy .... A numerical scale used to measure the heat, motion, or other properties that would require fuel or physical effort to put into an object; a scalar quantity with units of joules (J).

power .... The rate of transferring energy; a scalar quantity with units of watts (W).

kinetic energy .... The energy an object possesses because of its motion.

heat .... The energy that an object has because of its temperature. Heat is different from temperature because an object with twice as much mass requires twice as much heat to increase its temperature by the same amount.

temperature .... What a thermometer measures. Objects left in contact with each other tend to reach the same temperature. Cf. heat. As discussed in more detail in chapter 2, temperature is essentially a measure of the average kinetic energy per molecule.

Notation

E .... energy

J .... joules, the SI unit of energy

KE .... kinetic energy

P .... power

W .... watts, the SI unit of power; equivalent to J/s

Other Notation and Terminology to be Aware of

Q *or* ΔQ .... the amount of heat transferred into or out of an object

K *or* T .... alternative symbols for kinetic energy, used in the scientific literature and in most advanced textbooks

thermal energy .... Careful writers make a distinction between heat and thermal energy, but the distinction is often ignored in casual speech, even among physicists. Properly, thermal energy is used to mean the total amount of energy possessed by an object, while heat indicates the amount of thermal energy transferred in or out. The term heat is used in this book to include both meanings.

Summary

Heating an object, making it move faster, or increasing its distance from another object that is attracting it are all examples of things that would require fuel or physical effort. There is a numerical way of measuring all these kinds of things using a single unit of measurement, and we describe them all as forms of *energy*. The SI unit of energy is the Joule. The reason why energy is a useful and important quantity is that it is always conserved. That is, it cannot be created or destroyed but only transferred between objects or changed from one form to another. Conservation of energy is the most important and broadly applicable of all the laws of physics, more fundamental and general even than Newton's laws of motion.

Heating an object requires a certain amount of energy per degree of temperature and per unit mass, which depends on the substance of which the object consists. Heat and temperature are completely different things. Heat is a form of energy, and its SI unit is the joule (J). Temperature is not a measure of energy. Heating twice as much of something requires twice as much heat, but double the amount of a substance does not have double the temperature.

The energy that an object possesses because of its motion is called kinetic energy. Kinetic energy is related to the mass of the object and the magnitude of its velocity vector by the equation

$$KE = \frac{1}{2}mv^2 \quad .$$

Power is the rate at which energy is transformed from one form to another or transferred from one object to another,

$$P = \frac{\Delta E}{\Delta t} \quad .$$

The SI unit of power is the watt (W).

Homework Problems

1. Energy is consumed in melting and evaporation. Explain in terms of conservation of energy why sweating cools your body, even though the sweat is at the same temperature as your body.

2. Can kinetic energy ever be less than zero? Explain. [Based on a problem by Serway and Faughn.]

3. Estimate the kinetic energy of an Olympic sprinter.

4✓. You are driving your car, and you hit a brick wall head on, at full speed. The car has a mass of 1500 kg. The kinetic energy released is a measure of how much destruction will be done to the car and to your body. Calculate the energy released if you are traveling at (a) 40 mi/hr, and again (b) if you're going 80 mi/hr. What is counterintuitive about this, and what implication does this have for driving at high speeds?

5✓. A closed system can be a bad thing — for an astronaut sealed inside a space suit, getting rid of body heat can be difficult. Suppose a 60-kg astronaut is performing vigorous physical activity, expending 200 W of power. If none of the heat can escape from her space suit, how long will it take before her body temperature rises by 6°C (11°F), an amount sufficient to kill her? Assume that the amount of heat required to raise her body temperature by 1°C is the same as it would be for an equal mass of water. Express your answer in units of minutes.

6. All stars, including our sun, show variations in their light output to some degree. Some stars vary their brightness by a factor of two or even more, but our sun has remained relatively steady during the hundred years or so that accurate data have been collected. Nevertheless, it is possible that climate variations such as ice ages are related to long-term irregularities in the sun's light output. If the sun was to increase its light output even slightly, it could melt enough ice at the polar icecaps to flood all the world's coastal cities. The total sunlight that falls on the ice caps amounts to about $1\text{x}10^{16}$ watts. Presently, this heat input to the poles is balanced by the loss of heat via winds, ocean currents, and emission of infrared light, so that there is no net melting or freezing of ice at the poles from year to year. Suppose that the sun changes its light output by some small percentage, but there is no change in the rate of heat loss by the polar caps. Estimate the percentage by which the sun's light output would have to increase in order to melt enough ice to raise the level of the oceans by 10 meters over a period of 10 years. (This would be enough to flood New York, London, and many other cities.) Melting 1 kg of ice requires $3\text{x}10^{3}$ J.

7S. A bullet flies through the air, passes through a paperback book, and then continues to fly through the air beyond the book. When is there a force? When is there energy?

S A solution is given in the back of the book.
✓ A computerized answer check is available.
★ A difficult problem.
∫ A problem that requires calculus.

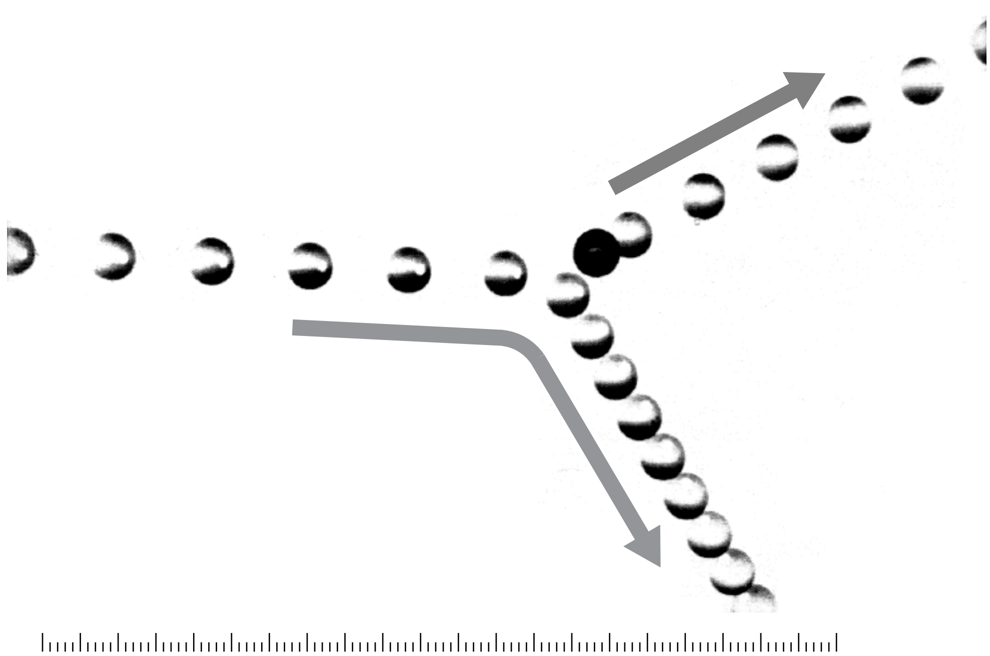
8 S. Experiments show that the power consumed by a boat's engine is approximately proportional to third power of its speed. (We assume that it is moving at constant speed.) (a) When a boat is crusing at constant speed, what type of energy transformation do you think is being performed? (b) If you upgrade to a motor with double the power, by what factor is your boat's crusing speed increased?

9 S. Object A has a kinetic energy of 13.4 J. Object B has a mass that is greater by a factor of 3.77, but is moving more slowly by a factor of 2.34. What is object B's kinetic energy?

10. The moon doesn't really just orbit the Earth. By Newton's third law, the moon's gravitational force on the earth is the same as the earth's force on the moon, and the earth must respond to the moon's force by accelerating. If we consider the earth in moon in isolation and ignore outside forces, then Newton's first law says their common center of mass doesn't accelerate, i.e. the earth wobbles around the center of mass of the earth-moon system once per month, and the moon also orbits around this point. The moon's mass is 81 times smaller than the earth's. Compare the kinetic energies of the earth and moon.

11 S. My 1.25 kW microwave oven takes 126 seconds to bring 250 g of water from room temperature to a boil. What percentage of the power is being wasted? Where might the rest of the energy be going?

12. The multiflash photograph below shows a collision between two pool balls. The ball that was initially at rest shows up as a dark image in its initial position, because its image was exposed several times before it was struck and began moving. By making measurements on the figure, determine whether or not energy appears to have been conserved in the collision. What systematic effects would limit the accuracy of your test? [From an example in PSSC Physics.]



13. This problem is a numerical example of the imaginary experiment discussed at the end of section 1.4 regarding the relationship between energy and relative motion. Let's say that the pool balls both have masses of 1.00 kg. Suppose that in the frame of reference of the pool table, the cue ball moves at a speed of 1.00 m/s toward the eight ball, which is initially at rest. The collision is head-on, and as you can verify for yourself the next time you're playing pool, the result of such a collision is that the incoming ball stops dead and the ball that was struck takes off with the same speed originally possessed by the incoming ball. (This is actually a bit of an idealization. To keep things simple, we're ignoring the spin of the balls, and we assume that no energy is liberated by the collision as heat or sound.) (a) Calculate the total initial kinetic energy and the total final kinetic energy, and verify that they are equal. (b) Now carry out the whole calculation again in the frame of reference that is moving in the same direction that the cue ball was initially moving, but at a speed of 0.50 m/s. In this frame of reference, both balls have nonzero initial and final velocities, which are different from what they were in the table's frame. [See also homework problem 15 in ch. 4.]

Do these forms of energy have anything in common?

2 Simplifying the Energy Zoo

Variety is the spice of life, not of science. The figure shows a few examples from the bewildering array of forms of energy that surrounds us. The physicist's psyche rebels against the prospect of a long laundry list of types of energy, each of which would require its own equations, concepts, notation, and terminology. The point at which we've arrived in the study of energy is analogous to the period in the 1960's when a half a dozen new subatomic particles were being discovered every year in particle accelerators. It was an embarrassment. Physicists began to speak of the "particle zoo," and it seemed that the subatomic world was distressingly complex. The particle zoo was simplified by the realization that most of the new particles being whipped up were simply clusters of a previously unsuspected set of more fundamental particles (which were whimsically dubbed quarks, a made-up word from a line of poetry by James Joyce, "Three quarks for Master Mark.") The energy zoo can also be simplified, and it is the purpose of this chapter to demonstrate the hidden similarities between forms of energy as seemingly different as heat and motion.

2.1 Heat is Kinetic Energy

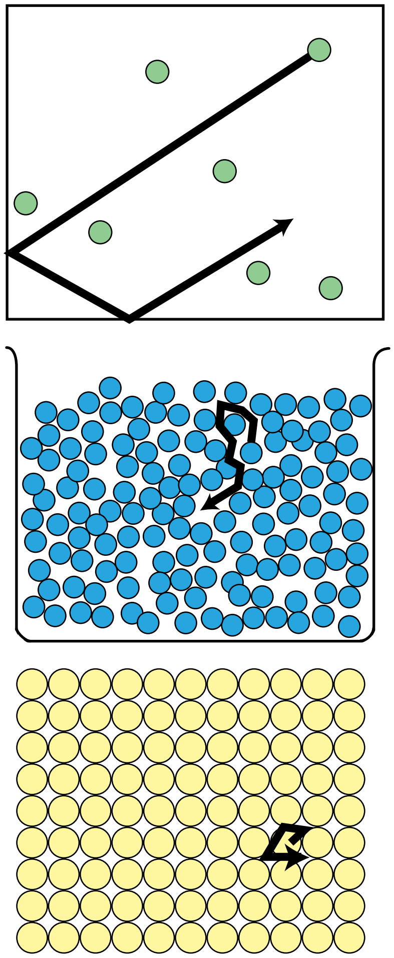
What is heat really? Is it an invisible fluid that your bare feet soak up from a hot sidewalk? Can one ever remove all the heat from an object? Is there a maximum to the temperature scale?

The theory of heat as a fluid seemed to explain why colder objects absorbed heat from hotter ones, but once it became clear that heat was a form of energy, it began to seem unlikely that a material substance could transform itself into and out of all those other forms of energy like motion or light. For instance, a compost pile gets hot, and we describe this as a case where, through the action of bacteria, chemical energy stored in the plant cuttings is transformed into heat energy. The heating occurs even if there is no nearby warmer object that could have been leaking "heat fluid" into the pile.

An alternative interpretation of heat was suggested by the theory that matter is made of atoms. Since gases are thousands of times less dense than solids or liquids, the atoms (or clusters of atoms called molecules) in a gas must be far apart. In that case, what is keeping all the air molecules from settling into a thin film on the floor of the room in which you are reading this book? The simplest explanation is that they are moving very rapidly, continually ricocheting off of the floor, walls, and ceiling. Though bizarre, the cloud-of-bullets image of a gas did give a natural explanation for the surprising ability of something as tenuous as a gas to exert huge forces. Your car's tires can hold it up because you have pumped extra molecules into them. The inside of the tire gets hit by molecules more often than the outside, forcing it to stretch and stiffen.

The outward forces of the air in your car's tires increase even further when you drive on the freeway for a while, heating up the rubber and the air inside. This type of observation leads naturally to the conclusion that hotter matter differs from colder in that its atoms' random motion is more

A vivid demonstration that heat is a form of motion. A small amount of boiling water is poured into the empty can, which rapidly fills up with hot steam. The can is then sealed tightly, and soon crumples. This can be explained as follows. The high temperature of the steam is interpreted as a high average speed of random motions of its molecules. Before the lid was put on the can, the rapidly moving steam molecules pushed their way out of the can, forcing the slower air molecules out of the way. As the steam inside the can thinned out, a stable situation was soon achieved, in which the force from the less dense steam molecules moving at high speed balanced against the force from the more dense but slower air molecules outside. The cap was put on, and after a while the steam inside the can began to cool off. The force from the cooler, thin steam no longer matched the force from the cool, dense air outside, and the imbalance of forces crushed the can.



Random motion of atoms in a gas, a liquid, and a solid.

rapid. In a liquid, the motion could be visualized as people in a milling crowd shoving past each other more quickly. In a solid, where the atoms are packed together, the motion is a random vibration of each atom as it knocks against its neighbors.

We thus achieve a great simplification in the theory of heat. Heat is simply a form of kinetic energy, the total kinetic energy of random motion of all the atoms in an object. With this new understanding, it becomes possible to answer at one stroke the questions posed at the beginning of the section. Yes, it is at least theoretically possible to remove all the heat from an object. The coldest possible temperature, known as absolute zero, is that at which all the atoms have zero velocity, so that their kinetic energies, $KE = \frac{1}{2}mv^2$, are all zero. No, there is no maximum amount of heat that a certain quantity of matter can have, and no maximum to the temperature scale, since arbitrarily large values of v can create arbitrarily large amounts of kinetic energy per atom.

The kinetic theory of heat also provides a simple explanation of the true nature of temperature. Temperature is a measure of the amount of energy per molecule, whereas heat is the total amount of energy possessed by all the molecules in an object.

There is an entire branch of physics, called thermodynamics, that deals with heat and temperature and forms the basis for technologies such as refrigeration. Thermodynamics is discussed in more detail in my calculus-based book **Simple Nature**, and I have provided here only a brief overview of the thermodynamic concepts that relate directly to energy, glossing over at least one point that would be dealt with more carefully in a thermodynamics course: it is really only true for a gas that all the heat is in the form of kinetic energy. In solids and liquids, the atoms are close enough to each other to exert intense electrical forces on each other, and there is therefore another type of energy involved, the energy associated with the atoms' distances from each other. Strictly speaking, heat energy is defined not as energy associated with random motion of molecules but as any form of energy that can be conducted between objects in contact, without any force.

2.2 Potential Energy: Energy of Distance or Closeness

We have already seen many examples of energy related to the distance between interacting objects. When two objects participate in an attractive noncontact force, energy is required to bring them farther apart. In both of the perpetual motion machines that started off the previous chapter, one of the types of energy involved was the energy associated with the distance between the balls and the earth, which attract each other gravitationally. In the perpetual motion machine with the magnet on the pedestal, there was also energy associated with the distance between the magnet and the iron ball, which were attracting each other.

The opposite happens with repulsive forces: two socks with the same type of static electric charge will repel each other, and cannot be pushed closer together without supplying energy.

In general, the term *potential energy,* with algebra symbol *PE,* is used for the energy associated with the distance between two objects that attract or repel each other via a force that depends on the distance between them. Forces that are not determined by distance do not have potential energy associated with them. For instance, the normal force acts only between objects that have zero distance between them, and depends on other factors besides the fact that the distance is zero. There is no potential energy associated with the normal force.

The skater has converted all his kinetic energy into potential energy on the way up the side of the pool.
Photo by J.D. Rogge, www.sonic.net/~shawn.

The following are some commonplace examples of potential energy:

gravitational potential energy: The skateboarder in the photo has risen from the bottom of the pool, converting kinetic energy into gravitational potential energy. After being at rest for an instant, he will go back down, converting PE back into KE.

magnetic potential energy: When a magnetic compass needle is allowed to rotate, the poles of the compass change their distances from the earth's north and south magnetic poles, converting magnetic potential energy into kinetic energy. (Eventually the kinetic energy is all changed into heat by friction, and the needle settles down in the position that minimizes its potential energy.)

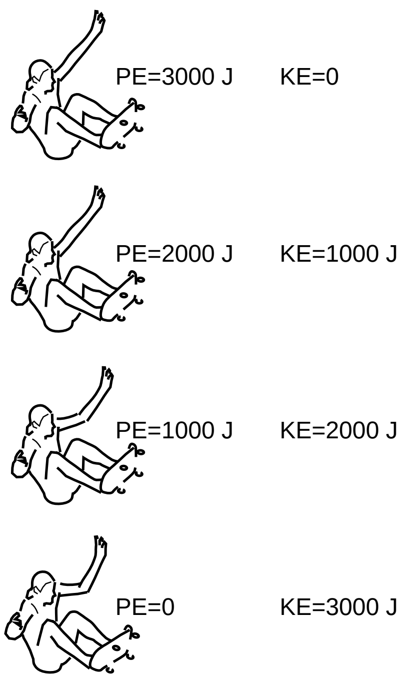
electrical potential energy: Socks coming out of the dryer cling together because of attractive electrical forces. Energy is required in order to separate them.

potential energy of bending or stretching: The force between the two ends of a spring depends on the distance between them, i.e. on the length of the spring. If a car is pressed down on its shock absorbers and then released, the potential energy stored in the spring is transformed into kinetic and gravitational potential energy as the car bounces back up.

I have deliberately avoided introducing the term potential energy up until this point, because it tends to produce unfortunate connotations in the minds of students who have not yet been inoculated with a careful description of the construction of a numerical energy scale. Specifically, there is a tendency to generalize the term inappropriately to apply to any situation where there is the "potential" for something to happen: "I took a break from digging, but I had potential energy because I knew I'd be ready to work hard again in a few minutes."

An Equation for Gravitational Potential Energy

All the vital points about potential energy can be made by focusing on the example of gravitational potential energy. For simplicity, we treat only vertical motion, and motion close to the surface of the earth, where the gravitational force is nearly constant. (The generalization to the three dimensions and varying forces is more easily accomplished using the concept of work, which is the subject the next chapter.)



As the skater free-falls, his PE is converted into KE. (The numbers would be equally valid as a description of his motion on the way up.)

To find an equation for gravitational PE, we examine the case of free fall, in which energy is transformed between kinetic energy and gravitational PE. Whatever energy is lost in one form is gained in an equal amount in the other form, so using the notation ΔKE to stand for $KE_f - KE_i$ and a similar notation for PE, we have

$$\Delta KE = -\Delta PE_{grav} \quad . \qquad (1)$$

It will be convenient to refer to the object as falling, so that PE is being changed into KE, but the math applies equally well to an object slowing down on its way up. We know an equation for kinetic energy,

$$KE = \frac{1}{2}mv^2 \quad , \qquad (2)$$

so if we can relate v to height, y, we will be able to relate ΔPE to y, which would tell us what we want to know about potential energy. The y component of the velocity can be connected to the height via the constant acceleration equation

$$v_f^2 = v_i^2 + 2a\Delta y \quad , \qquad (3)$$

and Newton's second law provides the acceleration,

$$a = F/m \quad , \qquad (4)$$

in terms of the gravitational force.

The algebra is simple because both equation (2) and equation (3) have velocity to the second power. Equation (2) can be solved for v^2 to give $v^2 = 2KE/m$, and substituting this into equation (3), we find

$$2KE_f/m = 2KE_i/m + 2a\Delta y \quad .$$

Making use of equations (1) and (4) gives the simple result

$$\Delta PE_{grav} = -F\Delta y \quad .$$

[change in gravitational PE resulting from a change in height Δy; F is the gravitational force on the object, i.e. its weight; valid only near the surface of the earth, where F is constant]

Example: dropping a rock
Question: If you drop a 1-kg rock from a height of 1 m, how many joules of KE does it have on impact with the ground? (Assume that any energy transformed into heat by air friction is negligible.)
Solution: If we choose the y axis to point up, then F_y is negative, and equals –(1 kg)(g)=-9.8 N. A decrease in y is represented by a negative value of Δy, Δy=–1 m, so the change in potential energy is –(–9.8 N)(–1 m)≈–10 J. (The proof that newtons multiplied by meters give units of joules is left as a homework problem.) Conservation of energy says that the loss of this amount of PE must be accompanied by a corresponding increase in KE of 10 J.

It may be dismaying to note how many minus signs had to be handled correctly even in this relatively simple example: a total of four. Rather than depending on yourself to avoid any mistakes with signs, it is better to check whether the final result make sense physically. If it doesn't, just reverse the sign.

Although the equation for gravitational potential energy was derived by imagining a situation where it was transformed into kinetic energy, the equation can be used in any context, because all the types of energy are freely convertible into each other.

Example: Gravitational PE converted directly into heat
Question: A 50-kg firefighter slides down a 5-m pole at constant velocity. How much heat is produced?
Solution: Since she slides down at constant velocity, there is no change in KE. Heat and gravitational PE are the only forms of energy that change. Ignoring plus and minus signs, the gravitational force on her body equals mg, and the amount of energy transformed is

$$(mg)(5 \text{ m}) = 2500 \text{ J} \quad .$$

On physical grounds, we know that there must have been an increase (positive change) in the heat energy in her hands and in the flagpole.

Here are some questions and answers about the interpretation of the equation $\Delta PE_{grav} = -F\Delta y$ for gravitational potential energy.

Question: In a nutshell, why is there a minus sign in the equation?
Answer: It is because we increase the PE by moving the object in the *opposite* direction compared to the gravitational force.

Question: Why do we only get an equation for the *change* in potential energy? Don't I really want an equation for the potential energy itself?
Answer: No, you really don't. This relates to a basic fact about potential energy, which is that it is not a well defined quantity in the absolute sense. Only changes in potential energy are unambiguously defined. If you and I both observe a rock falling, and agree that it deposits 10 J of energy in the dirt when it hits, then we will be forced to agree that the 10 J of KE must have come from a loss of 10 joules of PE. But I might claim that it started with 37 J of PE and ended with 27, while you might swear just as truthfully that it had 109 J initially and 99 at the end. It is possible to pick some specific height as a reference level and say that the PE is zero there, but it's easier and safer just to work with changes in PE and avoid absolute PE

altogether.

Question: You referred to potential energy as the energy that *two* objects have because of their distance from each other. If a rock falls, the object is the rock. Where's the other object?
Answer: Newton's third law guarantees that there will always be two objects. The other object is the planet earth.

Question: If the other object is the earth, are we talking about the distance from the rock to the center of the earth or the distance from the rock to the surface of the earth?
Answer: It doesn't matter. All that matters is the change in distance, Δy, not y. Measuring from the earth's center or its surface are just two equally valid choices of a reference point for defining absolute PE.

Question: Which object contains the PE, the rock or the earth?
Answer: We may refer casually to the PE of the rock, but technically the PE is a relationship between the earth and the rock, and we should refer to the earth and the rock together as possessing the PE.

Question: How would this be any different for a force other than gravity?
Answer: It wouldn't. The derivation was derived under the assumption of constant force, but the result would be valid for any other situation where two objects interacted through a constant force. Gravity is unusual, however, in that the gravitational force on an object is so nearly constant under ordinary conditions. The magnetic force between a magnet and a refrigerator, on the other hand, changes drastically with distance. The math is a little more complex for a varying force, but the concepts are the same.

Question: Suppose a pencil is balanced on its tip and then falls over. The pencil is simultaneously changing its height and rotating, so the height change is different for different parts of the object. The bottom of the pencil doesn't lose any height at all. What do you do in this situation?
Answer: The general philosophy of energy is that an object's energy is found by adding up the energy of every little part of it. You could thus add up the changes in potential energy of all the little parts of the pencil to find the total change in potential energy. Luckily there's an easier way! The derivation of the equation for gravitational potential energy used Newton's second law, which deals with the acceleration of the object's center of mass (i.e. its balance point). If you just define Δy as the height change of the center of mass, everything works out. A huge Ferris wheel can be rotated without putting in or taking out any PE, because its center of mass is staying at the same height.

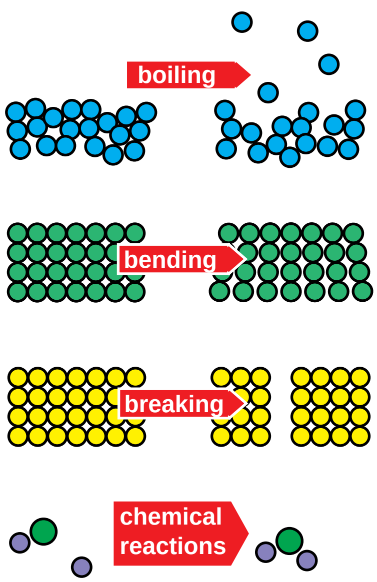
Self-Check

A ball thrown straight up will have the same speed on impact with the ground as a ball thrown straight down at the same speed. How can this be explained using potential energy?

Discussion Question

You throw a steel ball up in the air. How can you prove based on conservation of energy that it has the same speed when it falls back into your hand? What if you threw a feather up? Is energy not conserved in this case?

2.3 All Energy is Potential or Kinetic

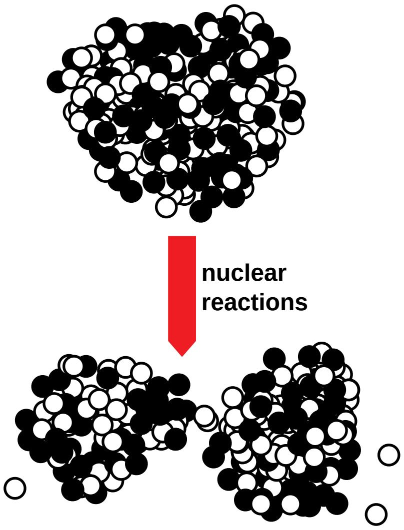


All these energy transformations turn out at the atomic level to be changes in potential energy resulting from changes in the distances between atoms.

In the same way that we found that a change in temperature is really only a change in kinetic energy at the atomic level, we now find that every other form of energy turns out to be a form of potential energy. Boiling, for instance, means knocking some of the atoms (or molecules) out of the liquid and into the space above, where they constitute a gas. There is a net attractive force between essentially any two atoms that are next to each other, which is why matter always prefers to be packed tightly in the solid or liquid state unless we supply enough potential energy to pull it apart into a gas. This explains why water stops getting hotter when it reaches the boiling point: the power being pumped into the water by your stove begins going into potential energy rather than kinetic energy.

As shown in the figure on the left, every stored form of energy that we encounter in everyday life turns out to be a form of potential energy at the atomic level. The forces between atoms are electrical and magnetic in nature, so these are actually electrical and magnetic potential energies.

Both balls start from the same height and end at the same height, so they have the same Δy. This implies that their losses in potential energy are the same, so they must both have gained the same amount of kinetic energy.



This figure looks similar to the previous ones, but the scale is a million times smaller. The little balls are the neutrons and protons that make up the tiny nucleus at the center of the uranium atom. When the nucleus splits (fissions), the potential energy change is partly electrical and partly a change in the potential energy derived from the force that holds atomic nuclei together (known as the strong nuclear force).

Even if we wish to include nuclear reactions in the picture, there still turn out to be only four fundamental types of energy:

kinetic energy (including heat)
gravitational potential energy
electrical and magnetic potential energy
nuclear potential energy

Astute students often ask me how light fits into this picture. This is a very good question, and in fact it could be argued that it is the basic question that led to Einstein's theory of relativity as well as the modern quantum picture of nature. Since these are topics for books 4, 5, and 6 of this series, we will have to be content with half an answer at this point. Essentially we may think of light energy as a form of kinetic energy, but one for which kinetic energy is not given by $\frac{1}{2}mv^2$ but rather by some other equation. (We know that $\frac{1}{2}mv^2$ would not make sense, because light has no mass, and furthermore, high-energy beams of light do not differ in speed from low-energy ones.)

Discussion Question

Referring back to the pictures at the beginning of the chapter, how do all these forms of energy fit into the shortened list of categories given above?

Summary

Selected Vocabulary

potential energy .... the energy having to do with the distance between to objects that interact via a noncontact force

Notation

PE .... potential energy

Alternative Notation to be Aware of

U or V .... symbols used for potential energy in the scientific literature and in most advanced textbooks

Summary

Historically, the energy concept was only invented to include a few phenomena, but it was later generalized more and more to apply to new situations, for example nuclear reactions. This generalizing process resulted in an undesirably long list of types of energy, each of which apparently behaved according to its own rules.

The first step in simplifying the picture came with the realization that heat was a form of random motion on the atomic level, i.e. heat was nothing more than the kinetic energy of atoms.

A second and even greater simplification was achieved with the realization that all the other apparently mysterious forms of energy actually had to do with changing the distances between atoms (or similar processes in nuclei). This type of energy, which relates to the distance between objects that interact via a force, is therefore of great importance. We call it potential energy.

Most of the important ideas about potential energy can be understood by studying the example of gravitational potential energy. The change in an object's gravitational potential energy is given by

$$\Delta PE_{grav} = -F_{grav}\Delta y \quad , \quad \text{[if } F_{grav} \text{ is constant, i.e. the motion is all near the Earth's surface]}$$

The most important thing to understand about potential energy is that there is no unambiguous way to define it in an absolute sense. The only thing that everyone can agree on is how much the potential energy has changed from one moment in time to some later moment in time.

Homework Problems

1. Can gravitational potential energy ever be negative? Note that the question refers to *PE*, not Δ*PE*, so that you must think about how the choice of a reference level comes into play. [Based on a problem by Serway and Faughn.]

2. A ball rolls up a ramp, turns around, and comes back down. At what position is its gravitational potential energy at a maximum? At what position is its kinetic energy at a maximum? Based on a problem by Serway and Faughn.]

3. (a) You release a magnet on a tabletop near a big piece of iron, and the magnet leaps across the table to the iron. Does the magnetic potential energy increase or decrease? Explain. (b) Suppose instead that you have two repelling magnets. You give them an initial push towards each other, so they decelerate while approaching each other. Does the magnetic potential energy increase or decrease? Explain.

4. Let E_b be the energy required to boil one kg of water. (a) Find an equation for the minimum height from which a bucket of water must be dropped if the energy released on impact is to vaporize it. Assume that all the heat goes into the water, not into the dirt it strikes, and ignore the relatively small amount of energy required to heat the water from room temperature to 100°C. [Numerical check, not for credit: Plugging in E_b=2.3 MJ/kg should give a result of 230 km.] (b) Show that the units of your answer in part a come out right based on the units given for E_b.

5 S. A grasshopper with a mass of 110 mg falls from rest from a height of 310 cm. On the way down, it dissipates 1.1 mJ of heat due to air resistance. At what speed, in m/s, does it hit the ground?

6. A person on a bicycle is to coast down a ramp of height *h* and then pass through a circular loop of radius *r*. What is the smallest value of *h* for which the cyclist will complete the loop without falling? (Ignore the kinetic energy of the spinning wheels.)

7★ S. A skateboarder starts at nearly rest at the top of a giant cylinder, and begins rolling down its side. (If she started exactly at rest and exactly at the top, she would never get going!) Show that her board loses contact with the pipe after she has dropped by a height equal to one third the radius of the pipe.

8★. (a) A circular hoop of mass *m* and radius *r* spins like a wheel while its center remains at rest. Its period (time required for one revolution) is *T*. Show that its kinetic energy equals $2\pi^2mr^2/T^2$. (b) If such a hoop rolls with its center moving at velocity *v*, its kinetic energy consists equals $(1/2)mv^2$, plus the amount of kinetic energy found in the first part of this problem. Show that a hoop rolls down an inclined plane with half the acceleration that a frictionless sliding block would have.

9 S. Students are often tempted to think of potential energy and kinetic energy as if they were always related to each other, like yin and yang. To show this is incorrect, give examples of physical situations in which (a) PE

S A solution is given in the back of the book. ★ A difficult problem.
✓ A computerized answer check is available. ∫ A problem that requires calculus.

is converted to another form of PE, and (b) KE is converted to another form of KE.

10 ✓. Lord Kelvin, a physicist, told the story of how he encountered James Joule when Joule was on his honeymoon. As he traveled, Joule would stop with his wife at various waterfalls, and measure the difference in temperature between the top of the waterfall and the still water at the bottom. (a) It would surprise most people to learn that the temperature increased. Why should there be any such effect, and why would Joule care? How would this relate to the energy concept, of which he was the principal inventor? (b) How much of a gain in temperature should there be between the top and bottom of a 50-meter waterfall? (c) What assumptions did you have to make in order to calculate your answer to part b? In reality, would the temperature change be more than or less than what you calculated? [Based on a problem by Arnold Arons.]

11 S. Make an order-of-magnitude estimate of the power represented by the loss of gravitational energy of the water going over Niagara Falls. If a hydroelectric plant was built at the bottom of the falls, and could convert 100% of this to electrical power, roughly how many households could be powered?

12. When you buy a helium-filled balloon, the seller has to inflate it from a large metal cylinder of the compressed gas. The helium inside the cylinder has energy, as can be demonstrated for example by releasing a little of it into the air: you hear a hissing sound, and that sound energy must have come from somewhere. The total amount of energy in the cylinder is very large, and if the valve is inadvertently damaged or broken off, the cylinder can behave like bomb or a rocket.

Suppose the company that puts the gas in the cylinders prepares cylinder A with half the normal amount of pure helium, and cylinder B with the normal amount. Cylinder B has twice as much energy, and yet the temperatures of both cylinders are the same. Explain, at the atomic level, what form of energy is involved, and why cylinder B has twice as much.

13. A microwave oven works by twisting molecules one way and then the other, counterclockwise and then clockwise about their own centers, millions of times a second. If you put an ice cube or a stick of butter in a microwave, you'll observe that the oven doesn't heat the solid very quickly, although eventually melting begins in one small spot. Once a melted spot forms, it grows rapidly, while the more distant solid parts remain solid. In other words, it appears based on this experiment that a microwave oven heats a liquid much more rapidly than a solid. Explain why this should happen, based on the atomic-level description of heat, solids, and liquids.

3 Work: The Transfer of Mechanical Energy

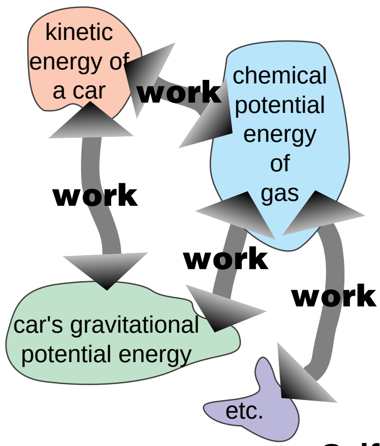
3.1 Work: The Transfer of Mechanical Energy

The concept of work

The mass contained in a closed system is a conserved quantity, but if the system is not closed, we also have ways of measuring the amount of mass that goes in or out. The water company does this with a meter that records your water use.

Likewise, there are many situations in which we would like to know how much energy comes in or out of a system that is not closed. Energy, however, is not a physical substance like water, so energy transfer cannot be measured with the same kind of meter. How can we tell, for instance, how much useful energy a tractor can “put out” on one tank of gas?

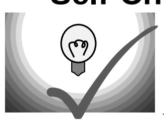
The law of conservation of energy guarantees that all the chemical energy in the gasoline will reappear in some form, but not necessarily in a



form that is useful for doing farm work. Tractors, like cars, are extremely inefficient, and typically 90% of the energy they consume is converted directly into heat, which is carried away by the exhaust and the air flowing over the radiator. We wish to distinguish the energy that comes out directly as heat from the energy that served to accelerate a trailer or to plow a field, and we define a technical meaning of the ordinary word "work" to express the distinction:

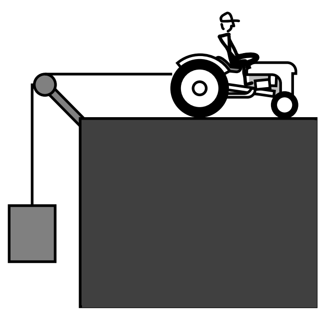
> **definition of work**
> Work is the amount of energy transferred into or out of a system, not counting the energy transferred by heat conduction.

Self-Check

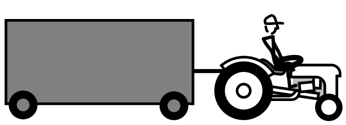


Based on this definition, is work a vector or a scalar? What are its units?

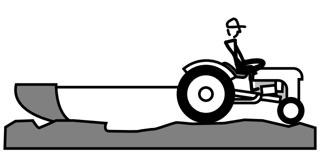
The conduction of heat is to be distinguished from heating by friction. When a hot potato heats up your hands by conduction, the energy transfer occurs without any force, but when friction heats your car's brake shoes, there is a force involved. The transfer of energy with and without a force are measured by completely different methods, so we wish to include heat transfer by frictional heating under the definition of work, but not heat transfer by conduction. The definition of work could thus be restated as the amount of energy transferred by forces.



(a) The tractor raises the weight over the pulley, increasing its gravitational potential energy.



(b) The tractor accelerates the trailer, increasing its kinetic energy.



(c) The tractor pulls a plow. Energy is expended in frictional heating of the plow and the dirt, and in breaking dirt clods and lifting dirt up to the sides of the furrow.

Calculating work as force multiplied by distance

The examples in the figures on the left show that there are many different ways in which energy can be transferred. Even so, all these examples have two things in common:

- A force is involved.
- The tractor travels some distance as it does the work.

In example (a), the increase in the height of the weight, Δy, is the same as the distance the tractor travels, call it d. For simplicity, we discuss the case where the tractor raises the weight at constant speed, so that there is no change in the kinetic energy of the weight, and we assume that there is negligible friction in the pulley, so that the force the tractor applies to the rope is the same as the rope's upward force on the weight. By Newton's first law, these forces are also of the same magnitude as the earth's gravitational force on the weight. The increase in the weight's potential energy is given by $F\Delta y$, so the work done by the tractor on the weight equals Fd, the product of the force and the distance moved:

$$W = Fd \quad .$$

In example (b), the tractor's force on the trailer accelerates it, increasing its kinetic energy. If frictional forces on the trailer are negligible, then the increase in the trailer's kinetic energy can be found using the same algebra that was used in the previous chapter to find the potential energy due to gravity. Just as in example (a), we have

$$W = Fd \quad .$$

Work is defined as the transfer of energy, so like energy it is a scalar with units of joules.

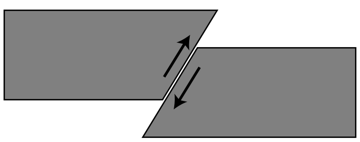
Does this equation always give the right answer? Well, sort of. In example (c), there are two quantities of work you might want to calculate, the work done by the tractor on the plow and the work done by the plow on the dirt. These two quantities can't both equal Fd, because the work done by the plow on the dirt is decreased by the heat lost in the plow itself. It turns out that the equation $W=Fd$ gives the work done by the tractor, not the work done by the plow. How are you supposed to know when the equation will work and when it won't? The somewhat complex answer is postponed until section 3.6. Until then, we will restrict ourselves to examples in which $W=Fd$ gives the right answer.

We have also been using examples in which the force is in the same direction as the motion, and the force is constant. (If the force was not constant, we would have to represent it with a function, not a symbol that stands for a number.) To summarize, we have:

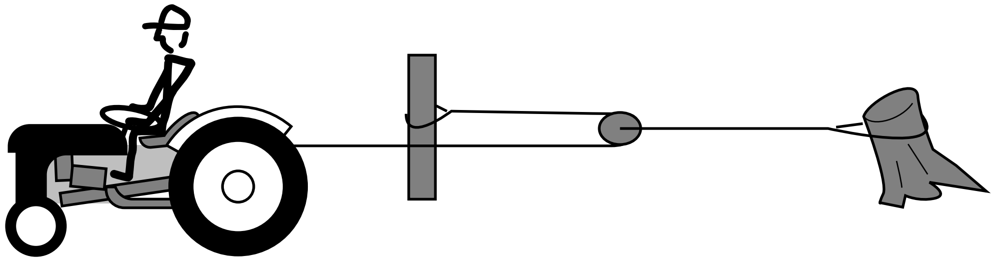
rule for calculating work (simplest version)
The work done by a force can be calculated as

$$W = Fd \quad ,$$

if the force is constant and in the same direction as the motion. Some ambiguities are encountered in cases such as kinetic friction.



Example: mechanical work done in an earthquake
Question: In 1998, geologists discovered evidence for a big prehistoric earthquake in Pasadena, between 10,000 and 15,000 years ago. They found that the two sides of the fault moved 6.7 m relative to one another, and estimated that the force between them was 1.3×10^{17} N. How much energy was released?
Solution: Multiplying the force by the distance gives 9×10^{17} J. For comparison, the Northridge earthquake of 1994, which killed 57 people and did 40 billion dollars of damage, released 22 times less energy.



Machines can increase force, but not work.

The figure above shows a pulley arrangement for doubling the force supplied by the tractor (book 1, section 5.6). The tension in the left-hand rope is equal throughout, assuming negligible friction, so there are two forces pulling the pulley to the left, each equal to the original force exerted by the tractor on the rope. This doubled force is transmitted through the right-hand rope to the stump.

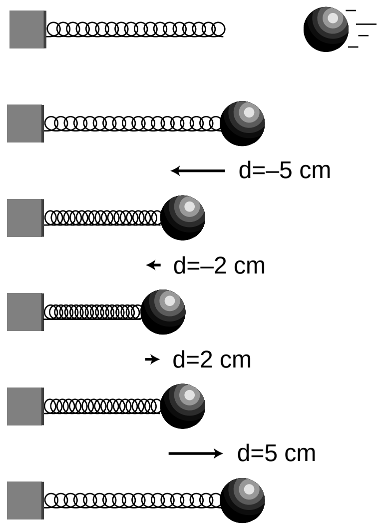
It might seem as though this arrangement would also double the work done by the tractor, but look again. As the tractor moves forward 2 meters, 1 meter of rope comes around the pulley, and the pulley moves 1 m to the left. Although the pulley exerts double the force on the stump, the pulley and stump only move half as far, so the work done on the stump is no greater that it would have been without the pulley.

The same is true for any mechanical arrangement that increases or decreases force, such as the gears on a ten-speed bike. You can't get out more work than you put in, because that would violate conservation of energy. If you shift gears so that your force on the pedals is amplified, the result is that you just have to spin the pedals more times.

No work is done without motion.

It strikes most students as nonsensical when they are told that if they stand still and hold a heavy bag of cement, they are doing no work on the bag. Even if it makes sense mathematically that $W=Fd$ gives zero when d is zero, it seems to violate common sense. You would certainly become tired standing there. The solution is simple. Physicists have taken over the common word "work" and given it a new technical meaning, which is the transfer of energy. The energy of the bag of cement is not changing, and that is what the physicist means by saying no work is done on the bag.

There is a transformation of energy, but it is taking place entirely within your own muscles, which are converting chemical energy into heat. Physiologically, a human muscle is not like a tree limb, which can support a weight indefinitely without the expenditure of energy. Each muscle cell's contraction is generated by zillions of little molecular machines, which take turns supporting the tension. When a particular molecule goes on or off duty, it moves, and since it moves while exerting a force, it is doing work. There is work, but it is work done by one molecule in a muscle cell on another.



Positive and negative work

When object A transfers energy to object B, we say that A does positive work on B. B is said to do negative work on A. In other words, a machine like a tractor is defined as doing positive work. This use of the plus and minus signs relates in a logical and consistent way to their use in indicating the directions of force and motion in one dimension. In the example shown on the left, suppose we choose a coordinate system with the x axis pointing to the right. Then the force the spring exerts on the ball is always a positive number. The ball's motion, however, changes directions. The symbol d is really just a shorter way of writing the familiar quantity Δx, whose positive and negative signs indicate direction.

While the ball is moving to the left, we use $d<0$ to represent its direction of motion, and the work done by the spring, Fd, comes out negative. This indicates that the spring is taking kinetic energy out of the ball, and accepting it in the form of its own potential energy.

As the ball is reaccelerated to the right, it has $d>0$, Fd is positive, and the spring does positive work on the ball. Potential energy is transferred out of the spring and deposited in the ball as kinetic energy.

In summary:

rule for calculating work (including cases of negative work)
The work done by a force can be calculated as

$$W = Fd \ ,$$

if the force is constant and along the same line as the motion. The quantity d is to be interpreted as a synonym for Δx, i..e. positive and negative signs are used to indicate the direction of motion. Some ambiguities are encountered in cases such as kinetic friction.

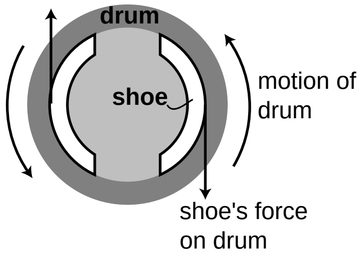
Self-Check

What about the work done by the ball on the spring?

There are many examples where the transfer of energy out of an object cancels out the transfer of energy in. When the tractor pulls the plow with a rope, the rope does negative work on the tractor and positive work on the plow. The total work done by the rope is zero, which makes sense, since it is not changing its energy.

It may seem that when your arms do negative work by lowering a bag of cement, the cement is not really transferring energy into your body. If your body was storing potential energy like a compressed spring, you would be able to raise and lower a weight all day, recycling the same energy. The bag of cement does transfer energy into your body, but your body accepts it as heat, not as potential energy. The tension in the muscles that control the speed of the motion also results in the conversion of chemical energy to heat, for the same physiological reasons discussed previously in the case where you just hold the bag still.

Whenever energy is transferred out of the spring, the same amount has to be transferred into the ball, and vice versa. As the spring compresses, the ball is doing positive work on the spring (giving up its KE and transferring energy into the spring as PE), and as it decompresses the ball is doing negative work (extracting energy).



Because the force is in the opposite direction compared to the motion, the brake shoe does negative work on the drum, i.e. accepts energy from it in the form of heat.

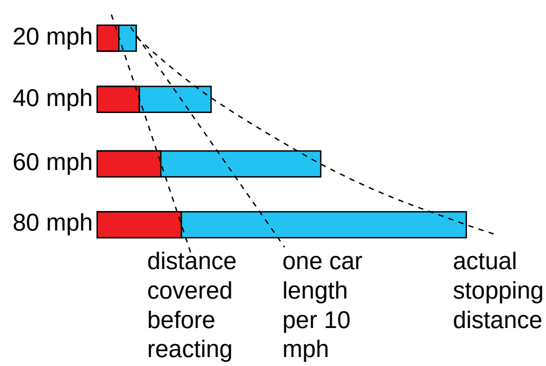
One of the advantages of electric cars over gasoline-powered cars is that it is just as easy to put energy back in a battery as it is to take energy out. When you step on the brakes in a gas car, the brake shoes do negative work on the rest of the car. The kinetic energy of the car is transmitted through the brakes and accepted by the brake shoes in the form of heat. The energy cannot be recovered. Electric cars, however, are designed to use regenerative braking. The brakes don't use friction at all. They are electrical, and when you step on the brake, the negative work done by the brakes means they accept the energy and put it in the battery for later use. This is one of the reasons why an electric car is far better for the environment than a gas car, even if the ultimate source of the electrical energy happens to be the burning of oil in the electric company's plant. The electric car recycles the same energy over and over, and only dissipates heat due to air friction and rolling resistance, not braking. (The electric company's power plant can also be fitted with expensive pollution-reduction equipment that would be prohibitively expensive or bulky for a passenger car.)

Discussion Questions

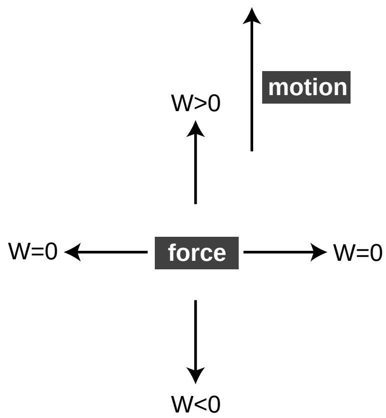
A. Besides the presence of a force, what other things differentiate the processes of frictional heating and heat conduction?

B. Criticize the following incorrect statement: "A force doesn't do any work unless it's causing the object to move."

C. To stop your car, you must first have time to react, and then it takes some time for the car to slow down. Both of these times contribute to the distance you will travel before you can stop. The figure shows how the average stopping distance increases with speed. Because the stopping distance increases more and more rapidly as you go faster, the rule of one car length per 10 m.p.h. of speed is not conservative enough at high speeds. In terms of work and kinetic energy, what is the reason for the more rapid increase at high speeds?



3.2 Work in Three Dimensions



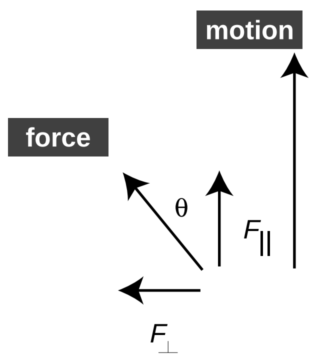
A force perpendicular to the motion does no work.

Suppose work is being done to change an object's kinetic energy. A force in the same direction as its motion will speed it up, and a force in the opposite direction will slow it down. As we have already seen, this is described as doing positive work or doing negative work on the object. All the examples discussed up until now have been of motion in one dimension, but in three dimensions the force can be at any angle θ with respect to the direction of motion.

What if the force is perpendicular to the direction of motion? We have already seen that a force perpendicular to the motion results in circular motion at constant speed. The kinetic energy does not change, and we conclude that no work is done when the force is perpendicular to the motion.

So far we have been reasoning about the case of a single force acting on an object, and changing only its kinetic energy. The result is more generally true, however. For instance, imagine a hockey puck sliding across the ice. The ice makes an upward normal force, but does not transfer energy to or from the puck

Forces at other angles



Suppose the force is at some other angle with respect to the motion, say θ=45°. Such a force could be broken down into two components, one along the direction of the motion and the other perpendicular to it. The force vector equals the vector sum of its two components, and the principle of vector addition of forces thus tells us that the work done by the total force cannot be any different than the sum of the works that would be done by the two forces by themselves. Since the component perpendicular to the motion does no work, the work done by the force must be

$$W = F_{||}\,|\mathbf{d}| \quad , \qquad \text{[work done by a constant force]}$$

where the vector $\mathbf{d}$ is simply a less cumbersome version of the notation $\Delta\mathbf{r}$. This result can be rewritten via trigonometry as

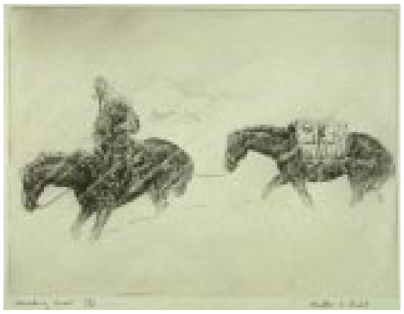
$$W = |\mathbf{F}|\,|\mathbf{d}| \cos\theta \quad . \qquad \text{[work done by a constant force]}$$

Even though this equation has vectors in it, it depends only on their magnitudes, and the magnitude of a vector is a scalar. Work is therefore still a scalar quantity, which only makes sense if it is defined as the transfer of energy. Ten gallons of gasoline have the ability to do a certain amount of mechanical work, and when you pull in to a full-service gas station you don't have to say "Fill 'er up with 10 gallons of south-going gas."

Students often wonder why this equation involves a cosine rather than a sine, or ask if it would ever be a sine. In vector addition, the treatment of sines and cosines seemed more equal and democratic, so why is the cosine so special now? The answer is that if we are going to describe, say, a velocity vector, we must give both the component *parallel* to the x axis and the component *perpendicular* to the x axis (i.e. the y component). In calculating work, however, the force component perpendicular to the motion is irrelevant — it changes the direction of motion without increasing or decreasing the energy of the object on which it acts. In this context, it is *only* the parallel force component that matters, so only the cosine occurs.

Self-Check

(a) Work is the transfer of energy. According to this definition, is the horse in the picture doing work on the pack? (b) If you calculate work by the method described in this section, is the horse doing work on the pack?



Breaking Trail, by Walter E. Bohl.

Example: pushing a broom

Question: If you exert a force of 21 N on a push broom, at an angle 35 degrees below horizontal, and walk for 5.0 m, how much work do you do? What is the physical significance of this quantity of work?

Solution: Using the second equation above, the work done equals

$$(21\ \text{N})(5.0\ \text{m})(\cos 35°) = 86\ \text{J} \quad .$$

The form of energy being transferred is heat in the floor and the broom's bristles. This comes from the chemical energy stored in your body. (The majority of the calories you burn are dissipated directly as heat inside your body rather than doing any work on the broom. The 86 J is only the amount of energy transferred through the broom's handle.)

(a) No. The pack is moving at constant velocity, so its kinetic energy is staying the same. It is only moving horizontally, so its gravitational potential energy is also staying the same. No energy transfer is occurring. (b) No. The horse's upward force on the pack forms a 90-degree angle with the direction of motion, so $\cos\theta=0$, and no work is done.

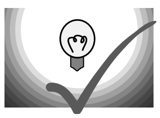
3.3 Varying Force

Up until now we have done no actual calculations of work in cases where the force is not constant. The question of how to treat such cases is mathematically analogous to the issue of how to generalize the equation (distance) = (velocity)(time) to cases where the velocity was not constant. There, we found that the correct generalization was to find the area under the graph of velocity versus time. The equivalent thing can be done with work:

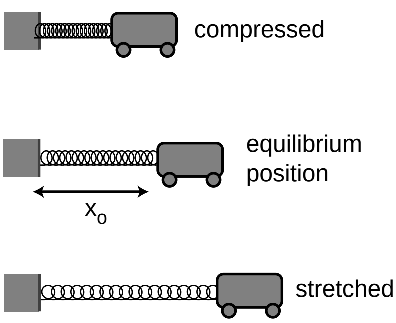
> **general rule for calculating work**
> The work done by a force F equals the area under the curve on a graph of $F_{\|}$ versus x. (Some ambiguities are encountered in cases such as kinetic friction.)

The examples in this section are ones in which the force is varying, but is always along the same line as the motion, so F is the same as $F_{\|}$.

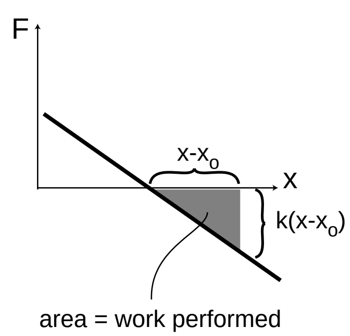
Self-Check



In which of the following examples would it be OK to calculate work using Fd, and in which ones would you have to use the area under the F-x graph?
(a) A fishing boat cruises with a net dragging behind it.
(b) A magnet leaps onto a refrigerator from a distance.
(c) Earth's gravity does work on an outward-bound space probe.



(a) The spring does work on the cart. (Unlike the ball in section 3.1, the cart is attached to the spring.)



(b)The area of the shaded triangle gives the work done by the spring as the cart moves from the equilibrium position to position x.

An important and straightforward example is the calculation of the work done by a spring that obeys Hooke's law,

$$F \approx -k(x-x_o) \quad .$$

The minus sign is because this is the force being exerted by the spring, not the force that would have to act on the spring to keep it at this position. That is, if the position of the cart is to the right of equilibrium, the spring pulls back to the left, and vice-versa.

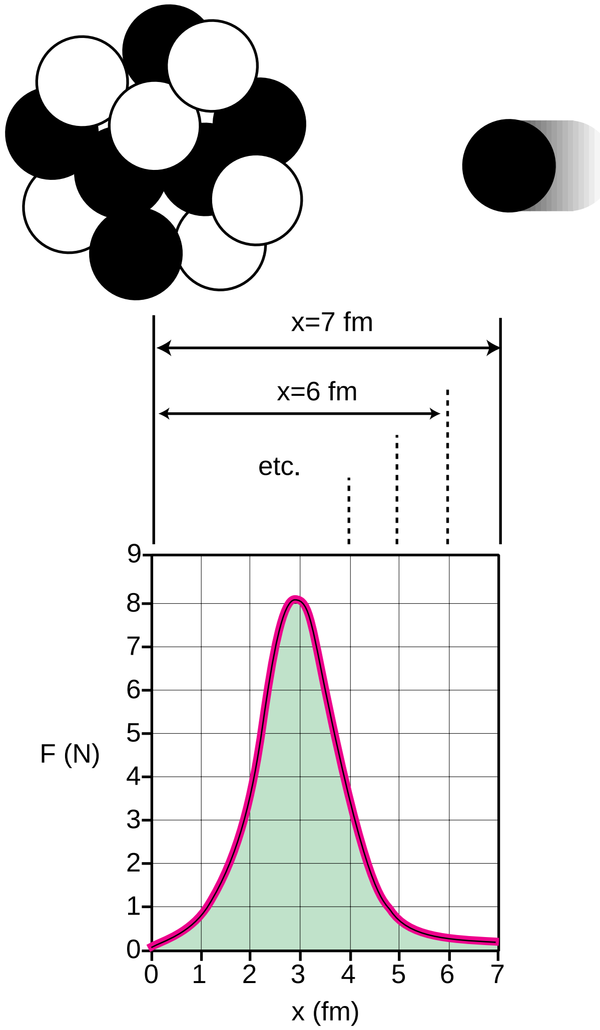
We calculate the work done when the spring is initially at equilibrium and then decelerates the car as the car moves to the right. The work done by the spring on the cart equals the minus area of the shaded triangle, because the triangle hangs below the x axis. The area of a triangle is half its base multiplied by its height, so

$$W = -\tfrac{1}{2}k(x-x_o)^2 \quad .$$

This is the amount of kinetic energy lost by the cart as the spring decelerates it.

It was straightforward to calculate the work done by the spring in this case because the graph of F versus x was a straight line, giving a triangular area. But if the curve had not been so geometrically simple, it might not have been possible to find a simple equation for the work done, or an equation might have been derivable only using calculus. Optional section 3.4 gives an important example of such an application of calculus.

(a) If the boat is cruising at constant speed, then the forces are all presumably constant, so Fd is correct. (b) The force is changing: weaker at first, and stronger as the magnet approaches the fridge. Fd would give the wrong answer. (c) Gravity is getting weaker and weaker and the probe moves away from the earth. Fd would give the wrong answer.



Example: energy production in the sun

The sun produces energy through nuclear reactions in which nuclei collide and stick together. The figure depicts one such reaction, in which a single proton (hydrogen nucleus) collides with a carbon nucleus, consisting of six protons and six neutrons. Neutrons and protons attract other neutrons and protons via the strong nuclear force, so as the proton approaches the carbon nucleus it is accelerated. In the language of energy, we say that it loses nuclear potential energy and gains kinetic energy. Together, the seven protons and six neutrons make a nitrogen nucleus. Within the newly put-together nucleus, the neutrons and protons are continually colliding, and the new proton's extra kinetic energy is rapidly shared out among all the neutrons and protons. Soon afterward, the nucleus calms down by releasing some energy in the form of a gamma ray, which helps to heat the sun.

The graph shows the force between the carbon nucleus and the proton as the proton is on its way in, with the distance in units of femtometers (1 fm=10^{-15} m). Amusingly, the force turns out to be a few newtons: on the same order of magnitude as the forces we encounter ordinarily on the human scale. Keep in mind, however, that a force this big exerted on a single subatomic particle such as a proton will produce a truly fantastic acceleration (on the order of 10^{27} m/s^2!).

Why does the force have a peak around x=3 fm, and become smaller once the proton has actually merged with the nucleus? At x=3 fm, the proton is at the edge of the crowd of protons and neutrons. It feels many attractive forces from the left, and none from the right. The forces add up to a large value. However if it later finds itself at the center of the nucleus, x=0, there are forces pulling it from all directions, and these force vectors cancel out.

We can now calculate the energy released in this reaction by using the area under the graph to determine the amount of mechanical work done by the carbon nucleus on the proton. (For simplicity, we assume that the proton came in "aimed" at the center of the nucleus, and we ignore the fact that it has to shove some neutrons and protons out of the way in order to get there.) The area under the curve is about 17 squares, and the work represented by each square is

$$(1\text{ N})(10^{-15}\text{ m}) = 10^{-15}\text{ J} \quad ,$$

so the total energy released is about

$$(10^{-15}\text{ J/square})(17\text{ squares}) = 1.7\times10^{-14}\text{ J} \quad .$$

This may not seem like much, but remember that this is only a reaction between the nuclei of two out of the zillions of atoms in the sun. For comparison, a typical *chemical* reaction between two atoms might transform on the order of 10^{-19} J of electrical potential energy into heat — 100,000 times less energy!

As a final note, you may wonder why reactions such as these only occur in the sun. The reason is that there is a repulsive electrical force between nuclei. When two nuclei are close together, the electrical forces are typically about a million times weaker than the nuclear forces, but the nuclear forces fall off much more quickly with distance than the electrical forces, so the electrical force is the dominant one at longer ranges. The sun is a very hot gas, so the random motion of its atoms is extremely rapid, and a collision between two atoms is sometimes violent enough to overcome this initial electrical repulsion.

3.4 ∫ Applications of Calculus

The student who has studied integral calculus will recognize that the graphical rule given in the previous section can be reexpressed as an integral,

$$W = \int_{x_1}^{x_2} F \, dx \quad .$$

We can then immediately find by the fundamental theorem of calculus that force is the derivative of work with respect to position,

$$F = \frac{\mathrm{d}W}{\mathrm{d}x} \quad .$$

For example, a crane raising a one-ton block on the moon would be transferring potential energy into the block at only one sixth the rate that would be required on Earth, and this corresponds to one sixth the force.

Although the work done by the spring could be calculated without calculus using the area of a triangle, there are many cases where the methods of calculus are needed in order to find an answer in closed form. The most important example is the work done by gravity when the change in height is not small enough to assume a constant force. Newton's law of gravity is

$$F = \frac{GMm}{r^2} \quad ,$$

which can be integrated to give

$$\begin{aligned} \mathrm{W} &= \int_{r_1}^{r_2} \frac{GMm}{r^2} dr \\ &= GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \quad . \end{aligned}$$

3.5 Work and Potential Energy

The techniques for calculating work can also be applied to the calculation of potential energy. If a certain force depends only on the distance between the two participating objects, then the energy released by changing the distance between them is defined as the potential energy, and the amount of potential energy lost equals minus the work done by the force,

$$\Delta PE = -W \quad .$$

The minus sign occurs because positive work indicates that the potential energy is being expended and converted to some other form.

Work and potential energy are not the same thing. We are simply using work as a way of calculating potential energy.

It is sometimes convenient to pick some arbitrary position as a reference position, and derive an equation for once and for all that gives the potential energy relative to this position

$$PE_x = -W_{ref\rightarrow x} \quad . \text{ [potential energy at a point x]}$$

To find the energy transferred into or out of potential energy, one then subtracts two different values of this equation.

These equations might almost make it look as though work and energy were the same thing, but they are not. First, potential energy measures the energy that a system *has* stored in it, while work measures how much energy is *transferred* in or out. Second, the techniques for calculating work can be used to find the amount of energy transferred in many situations where there is no potential energy involved, as when we calculate the amount of kinetic energy transformed into heat by a car's brake shoes.

Example: a toy gun

Question: A toy gun uses a spring with a spring constant of 10 N/m to shoot a ping-pong ball of mass 5 g. The spring is compressed to 10 cm shorter than its equilibrium length when the gun is loaded. At what speed is the ball released?

Solution: The equilibrium point is the natural choice for a reference point. Using the equation found previously for the work, we have

$$PE_x = \frac{1}{2}k\left(x - x_o\right)^2 \quad .$$

The spring loses contact with the ball at the equilibrium point, so the final potential energy is

$$PE_f = 0 \quad .$$

The initial potential energy is

$$PE_i = (1/2)(10\text{ N/m})(0.10\text{ m})^2 \quad .$$
$$= 0.05\text{ J.}$$

The loss in potential energy of 0.05 J means an increase in kinetic energy of the same amount. The velocity of the ball is found by solving the equation $KE = \frac{1}{2}mv^2$ for v,

$$v = \sqrt{\frac{2KE}{m}}$$
$$= \sqrt{\frac{(2)(0.05\ J)}{0.005\ kg}}$$
$$= 4\text{ m/s} \quad .$$

Example: gravitational potential energy

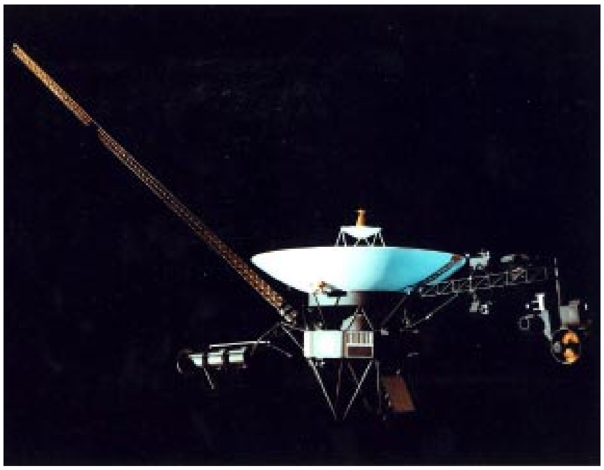
Question: We have already found the equation $\Delta PE = -F\Delta y$ for the gravitational potential energy when the change in height is not enough to cause a significant change in the gravitational force F. What if the change in height is enough so that this assumption is no longer valid? Use the equation $W = GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$ derived in the previous section to find the potential energy, using $r=\infty$ as a reference point.

Solution: The potential energy equals minus the work that would have to be done to bring the object from $r_1 = \infty$ to $r = r_2$, which is

$$PE = -\frac{GMm}{r} .$$

This is simpler than the equation for the work, which is an example of why it is advantageous to record an equation for potential energy relative to some reference point, rather than an equation for work.

Although the equations derived in the previous two examples may seem arcane and not particularly useful except for toy designers and rocket scientists, their usefulness is actually greater than it appears. The equation for the potential energy of a spring can be adapted to any other case in which an object is compressed, stretched, twisted, or bent. While you are not likely to use the equation for gravitational potential energy for anything practical, it is directly analogous to an equation that is extremely useful in chemistry, which is the equation for the potential energy of an electron at a distance r from the nucleus of its atom. As discussed in more detail later in the course, the electrical force between the electron and the nucleus is proportional to $1/r^2$, just like the gravitational force between two masses. Since the equation for the force is of the same form, so is the equation for the potential energy.



The twin Voyager space probes were perhaps the greatest scientific successes of the space program. Over a period of decades, they flew by all the planets of the outer solar system, probably accomplishing more of scientific interest than the entire space shuttle program at a tiny fraction of the cost. Both Voyager probes completed their final planetary flybys with speeds greater than the escape velocity at that distance from the sun, and so headed on out of the solar system on hyperbolic orbits, never to return. Radio contact has been lost, and they are now likely to travel interstellar space for billions of years without colliding with anything or being detected by any intelligent species.

Discussion Questions

A. What does the graph of $PE_x = \frac{1}{2}k(x - x_o)^2$ look like as a function of x? Discuss the physical significance of its features.

B. What does the graph of $PE = -\frac{GMm}{r}$ look like as a function of *r*? Discuss the physical significance of its features. How would the equation and graph change if some other reference point was chosen rather than $r=\infty$?

C. Starting at a distance *r* from a planet of mass *M*, how fast must an object be moving in order to have a hyperbolic orbit, i.e. one that never comes back to the planet? This velocity is called the escape velocity. Interpreting the result, does it matter in what direction the velocity is? Does it matter what mass the object has? Does the object escape because it is moving too fast for gravity to act on it?

D. Does a spring have an “escape velocity”?

E. Calculus-based question: If the form of energy being transferred is potential energy, then the equations $F = \mathrm{d}W / \mathrm{d}x$ and $W = \int F\,\mathrm{d}x$ become $F = \mathrm{d}PE / \mathrm{d}x$ and $PE = \int F\,\mathrm{d}x$. How would you then apply the following calculus concepts: zero derivative at minima and maxima, and the second derivative test for concavity up or down.

3.6* When Does Work Equal Force Times Distance?

In the first section of this chapter I gave an example of a case where the work done by a force did not equal *Fd*. The purpose of this section is to explain more fully how the quantity *Fd* can and cannot be used. To simplify things, I write *Fd* throughout this section, but more generally everything said here would be true for the area under the graph of $F_{||}$ versus *d*.

The following two theorems allow most of the ambiguity to be cleared up.

> **the work-kinetic energy theorem**
> The change in the kinetic energy associated with the motion of an object's center of mass is related to the total force acting on it and to the distance traveled by its center of mass according to the equation $\Delta KE_{cm} = F_{total} d_{cm}$.

This can be proven based on Newton’s second law and the equation $KE=\frac{1}{2}mv^2$. Note that despite the traditional name, it does not necessarily tell the amount of work done, since the forces acting on the object could be changing other types of energy besides the *KE* associated with its center of mass motion.

The second theorem does relate directly to work:

> When a contact force acts between two objects and the two surfaces do not slip past each other, the work done equals *Fd*, where *d* is the distance traveled by the point of contact.

This one has no generally accepted name, so we refer to it simply as the second theorem.

A great number of physical situations can be analyzed with these two theorems, and often it is advantageous to apply both of them to the same situation.

Example: an ice skater pushing off from a wall

- The work-kinetic energy theorem tells us how to calculate the skater's kinetic energy if we know the amount of force and the distance her center of mass travels while she is pushing off.
- The second theorem tells us that the wall does no work on the skater. This makes sense, since the wall does not have any source of energy.

Example: absorbing an impact without recoiling?

Question: Is it possible to absorb an impact without recoiling? For instance, would a brick wall "give" at all if hit by a ping-pong ball?

Answer: There will always be a recoil. In the example proposed, the wall will surely have some energy transferred to it in the form of heat and vibration. The second theorem tells us that we can only have nonzero work if the distance traveled by the point of contact is nonzero.

Example: dragging a refrigerator at constant velocity

- Newton's first law tells us that the total force on the refrigerator must be zero: your force is canceling the floor's kinetic frictional force. The work-kinetic energy theorem is therefore true but useless. It tells us that there is zero total force on the refrigerator, and that the refrigerator's kinetic energy doesn't change.
- The second theorem tells us that the work you do equals your hand's force on the refrigerator multiplied by the distance traveled. Since we know the floor has no source of energy, the only way for the floor and refrigerator to gain energy is from the work you do. We can thus calculate the total heat dissipated by friction in the refrigerator and the floor.

Note that there is no way to find how much of the heat is dissipated in the floor and how much in the refrigerator.

Example: accelerating a cart

If you push on a cart and accelerate it, there are two forces acting on the cart: your hand's force, and the static frictional force of the ground pushing on the wheels in the opposite direction.

- Applying the second theorem to your force tells us how to calculate the work you do.
- Applying the second theorem to the floor's force tells us that the floor does no work on the cart. There is no motion at the point of contact, because the atoms in the floor are not moving. (The atoms in the surface of the wheel are also momentarily at rest when they touch the floor.) This makes sense, since the floor does not have any source of energy.
- The work-kinetic energy theorem refers to the total force, and because the floor's backward force cancels part of your force, the total force is less than your force. This tells us that only part of your work goes into the kinetic energy associated with the forward motion of the cart's center of mass. The rest goes into rotation of the wheels.

3.7* The Dot Product

Up until now, we have not found any physically useful way to define the multiplication of two vectors. It would be possible, for instance, to multiply two vectors component by component to form a third vector, but there are no physical situations where such a multiplication would be useful.

The equation $W = |\mathbf{F}|\,|\mathbf{d}| \cos\theta$ is an example of a sort of multiplication of vectors that is useful. The result is a scalar, not a vector, and this is therefore often referred to as the *scalar product* of the vectors ***F*** and ***d***. There is a standard shorthand notation for this operation,

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos\theta \quad ,$$

[definition of the notation $\mathbf{A}\cdot\mathbf{B}$;

θ is the angle between vectors ***A*** and ***B***]

and because of this notation, a more common term for this operation is the *dot product.* In dot product notation, the equation for work is simply

$$W = \mathbf{F}\cdot\mathbf{d}$$

The dot product has the following geometric interpretation:

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}| \times (\text{component of } \mathbf{B} \text{ parallel to } \mathbf{A})$$

$$= |\mathbf{B}| \times (\text{component of } \mathbf{A} \text{ parallel to } \mathbf{B})$$

The dot product has some of the properties possessed by ordinary multiplication of numbers,

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$$

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$$

$$(c\mathbf{A})\cdot\mathbf{B} = c(\mathbf{A}\cdot\mathbf{B}) \quad ,$$

but it lacks one other: the ability to undo multiplication by dividing.

If you know the components of two vectors, you can easily calculate their dot product as follows:

$$\mathbf{A}\cdot\mathbf{B} = A_x B_x + A_y B_y + A_z B_z \quad .$$

This can be proven by first analyzing the special case where each vector has only an *x* component, and the similar cases for *y* and *z*. We can then apply the rule $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$ to generalize by writing each vector as the sum of its *x*, *y*, and *z* components.

Summary

Selected Vocabulary

work .... the amount of energy transferred into or out of a system, excluding energy transferred by heat conduction

Notation

W .... work

Summary

Work is a measure of the transfer of mechanical energy, i.e. the transfer of energy by a force rather than by heat conduction. When the force is constant, work can usually be calculated as

$$W = F_{||} |\mathbf{d}| \quad , \qquad \text{[only if the force is constant]}$$

where $\mathbf{d}$ is simply a less cumbersome notation for $\Delta\mathbf{r}$, the vector from the initial position to the final position. Thus,

- A force in the same direction as the motion does positive work, i.e. transfers energy into the object on which it acts.
- A force in the opposite direction compared to the motion does negative work, i.e. transfers energy out of the object on which it acts.
- When there is no motion, no mechanical work is done. The human body burns calories when it exerts a force without moving, but this is an internal energy transfer of energy within the body, and thus does not fall within the scientific definition of work.
- A force perpendicular to the motion does no work.

When the force is not constant, the above equation should be generalized as the area under the graph of $F_{||}$ versus d.

Machines such as pulleys, levers, and gears may increase or decrease a force, but they can never increase or decrease the amount of work done. That would violate conservation of energy unless the machine had some source of stored energy or some way to accept and store up energy.

There are some situations in which the equation $W = F_{||} |\mathbf{d}|$ is ambiguous or not true, and these issues are discussed rigorously in section 3.6. However, problems can usually be avoided by analyzing the types of energy being transferred before plunging into the math. In any case there is no substitute for a physical understanding of the processes involved.

The techniques developed for calculating work can also be applied to the calculation of potential energy. We fix some position as a reference position, and calculate the potential energy for some other position, x, as

$$PE_x = -W_{ref \rightarrow x} \quad .$$

The following two equations for potential energy have broader significance than might be suspected based on the limited situations in which they were derived:

$$PE = \frac{1}{2} k(x - x_o)^2 \quad .$$

[potential energy of a spring having spring constant k, when stretched or compressed from the equilibrium position x_o; analogous equations apply for the twisting, bending, compression, or stretching of any object.]

$$PE = -\frac{GMm}{r}$$

[gravitational potential energy of objects of masses M and m, separated by a distance r; an analogous equation applies to the electrical potential energy of an electron in an atom.]

Homework Problems

1. Two cars with different masses each have the same kinetic energy. (a) If both cars have the same brakes, capable of supplying the same force, how will the stopping distances compare? Explain. (b) Compare the times required for the cars to stop.

2. In each of the following situations, is the work being done positive, negative, or zero? (a) a bull paws the ground; (b) a fishing boat pulls a net through the water behind it; (c) the water resists the motion of the net through it; (d) you stand behind a pickup truck and lower a bale of hay from the truck's bed to the ground. Explain.

3. In the earth's atmosphere, the molecules are constantly moving around. Because temperature is a measure of kinetic energy per molecule, the average kinetic energy of each type of molecule is the same, e.g. the average KE of the O_2 molecules is the same as the average KE of the N_2 molecules. (a) If the mass of an O_2 molecule is eight times greater than that of a He atom, what is the ratio of their average speeds? Which way is the ratio, i.e. which is typically moving faster? (b) Use your result from part a to explain why any helium occurring naturally in the atmosphere has long since escaped into outer space, never to return. (Helium is obtained commercially by extracting it from rocks.)

4. Weiping lifts a rock with a weight of 1.0 N through a height of 1.0 m, and then lowers it back down to the starting point. Bubba pushes a table 1.0 m across the floor at constant speed, requiring a force of 1.0 N, and then pushes it back to where it started. Compare the total work done by Weiping and Bubba.

5. ✓ In one of his more flamboyant moments, Galileo wrote "Who does not know that a horse falling from a height of three or four cubits will break his bones, while a dog falling from the same height or a cat from a height of eight or ten cubits will suffer no injury? Equally harmless would be the fall of a grasshopper from a tower or the fall of an ant from the distance of the moon." Find the speed of an ant that falls to earth from the distance of the moon at the moment when it is about to enter the atmosphere. Assume it is released from a point that is not actually near the moon, so the moon's gravity is negligible.

6. [Problem 6 has been deleted.]

S A solution is given in the back of the book.
✓ A computerized answer check is available.
★ A difficult problem.
∫ A problem that requires calculus.

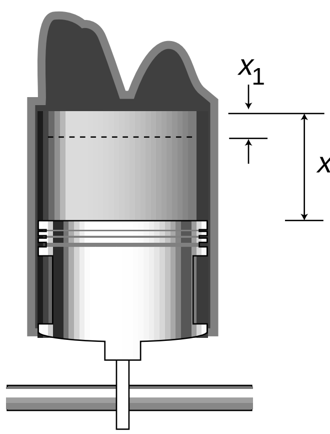
7. (a) The crew of an 18th century warship is raising the anchor. The anchor has a mass of 5000 kg. The water is 30 m deep. The chain to which the anchor is attached has a mass per unit length of 150 kg/m. Before they start raising the anchor, what is the total weight of the anchor plus the portion of the chain hanging out of the ship? (Assume that the buoyancy of the anchor and is negligible.)

(b) After they have raised the anchor by 1 m, what is the weight they are raising?

(c) Define y=0 when the anchor is resting on the bottom, and y=+30 m when it has been raised up to the ship. Draw a graph of the force the crew has to exert to raise the anchor and chain, as a function of y. (Assume that they are raising it slowly, so water resistance is negligible.) It will not be a constant! Now find the area under the graph, and determine the work done by the crew in raising the anchor, in joules.

(d✓) Convert your answer from (c) into units of kcal.

8. In the power stroke of a car's gasoline engine, the fuel-air mixture is ignited by the spark plug, explodes, and pushes the piston out. The exploding mixture's force on the piston head is greatest at the beginning of the explosion, and decreases as the mixture expands. It can be approximated by $F = a / x$, where x is the distance from the cylinder to the piston head, and a is a constant with units of N·m. (Actually $a/x^{1.4}$ would be more accurate, but the problem works out more nicely with a/x!)The piston begins its stroke at $x=x_1$, and ends at $x=x_2$. The 1965 Rambler had six cylinders, each with a=220 N·m, x_1=1.2 cm, and x_2=10.2 cm.



Problem 8: A cylinder from the 1965 Rambler's engine. The piston is shown in its pushed out position. The two bulges at the top are for the valves that let fresh air-gas mixture in.

Based on a figure from Motor Service's Automotive Encyclopedia, Toboldt and Purvis.

(a) Draw a neat, accurate graph of F vs x, on graph paper.

(b✓) From the area under the curve, derive the amount of work done in one stroke by one cylinder.

(c✓) Assume the engine is running at 4800 r.p.m., so that during one minute, each of the six cylinders performs 2400 power strokes. (Power strokes only happen every other revolution.) Find the engine's power, in units of horsepower (1 hp=746 W).

(d) The compression ratio of an engine is defined as x_2/x_1. Explain in words why the car's power would be exactly the same if x_1 and x_2 were, say, halved or tripled, maintaining the same compression ratio of 8.5. Explain why this would *not* quite be true with the more realistic force equation $F=a/x^{1.4}$.

9. ∫ The magnitude of the force between two magnets separated by a distance r can be approximated as kr^{-3} for large values of r. The constant k depends on the strengths of the magnets and the relative orientations of their north and south poles. Two magnets are released on a slippery surface at an initial distance r_i, and begin sliding towards each other. What will be the total kinetic energy of the two magnets when they reach a final distance r_f? (Ignore friction.)

10. ∫ A car starts from rest at $t=0$, and starts speeding up with constant acceleration. (a) Find the car's kinetic energy in terms of its mass, m, acceleration, a, and the time, t. (b) Your answer in the previous part also equals the amount of work, W, done from $t=0$ until time t. Take the derivative of the previous expression to find the power expended by the car at time t. (c) Suppose two cars with the same mass both start from rest at the same time, but one has twice as much acceleration as the other. At any moment, how many times more power is being dissipated by the more quickly accelerating car? (The answer is not 2.)

11.★ ∫ A space probe of mass m is dropped into a previously unexplored spherical cloud of gas and dust, and accelerates toward the center of the cloud under the influence of the cloud's gravity. Measurements of its velocity allow its potential energy, U, to be determined as a function of the distance r from the cloud's center. The mass in the cloud is distributed in a spherically symmetric way, so its density, $\rho(r)$, depends only on r and not on the angular coordinates. Show that by finding $U(r)$, one can infer $\rho(r)$ as follows:

$$\rho(r) = \frac{1}{4\pi G m r^2}\frac{d}{dr}\left(r^2\frac{dU}{dr}\right) \quad .$$

12. ∫ A rail gun is a device like a train on a track, with the train propelled by a powerful electrical pulse. Very high speeds have been demonstrated in test models, and rail guns have been proposed as an alternative to rockets for sending into outer space any object that would be strong enough to survive the extreme accelerations. Suppose that the rail gun capsule is launched straight up, and that the force of air friction acting on it is given by $F=be^{-cx}$, where x is the altitude, b and c are constants, and e is the base of natural logarithms. The exponential decay occurs because the atmosphere gets thinner with increasing altitude. (In reality, the force would probably drop off even faster than an exponential, because the capsule would be slowing down somewhat.) Find the amount of kinetic energy lost by the capsule due to air friction between when it is launched and when it is completely beyond the atmosphere. (Gravity is negligible, since the air friction force is much greater than the gravitational force.)

13 . A certain binary star system consists of two stars with masses m_1 and m_2, separated by a distance b. A comet, originally nearly at rest in deep space, drops into the system and at a certain point in time arrives at the midpoint between the two stars. For that moment in time, find its velocity, v, symbolically in terms of b, m_1, m_2, and fundamental constants. [Numerical check: For $m_1=1.5\text{x}10^{30}$ kg, $m_2=3.0\text{x}10^{30}$ kg, and $b=2.0\text{x}10^{11}$ m you should find $v=7.7\text{x}10^4$ m/s.]

14 ∫. An airplane flies in the positive direction along the x axis, through crosswinds that exert a force $\boldsymbol{F} = (a + bx)\hat{\boldsymbol{x}} + (c + dx)\hat{\boldsymbol{y}}$. Find the work done by the wind on the plane, and by the plane on the wind, in traveling from the origin to position x.

15. ∫ In 1935, Yukawa proposed an early theory of the force that held the neutrons and protons together in the nucleus. His equation for the potential energy of two such particles, at a center-to-center distance r, was $PE(r)=g\,r^{-1}\,e^{-r/a}$, where g parametrizes the strength of the interaction, e is the base of natural logarithms, and a is about 10^{-15} m. Find the force between two nucleons that would be consistent with this equation for the potential energy.

16. Prove that the dot product defined in section 3.7 is rotationally invariant in the sense of book 1, section 7.5.

17. Fill in the details of the proof of $\boldsymbol{A} \cdot \boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z$ in section 3.7.

18 S. Does it make sense to say that work is conserved?

19. (a) Suppose work is done in one-dimensional motion. What happens to the work if you reverse the direction of the positive coordinate axis? Base your answer directly on the definition of work. (b) Now answer the question based on the $W=Fd$ rule.

4 Conservation of Momentum

In many subfields of physics these days, it is possible to read an entire issue of a journal without ever encountering an equation involving force or a reference to Newton's laws of motion. In the last hundred and fifty years, an entirely different framework has been developed for physics, based on conservation laws.

The new approach is not just preferred because it is in fashion. It applies inside an atom or near a black hole, where Newton's laws do not. Even in everyday situations the new approach can be superior. We have already seen how perpetual motion machines could be designed that were too complex to be easily debunked by Newton's laws. The beauty of conservation laws is that they tell us something must remain the same, regardless of the complexity of the process.

So far we have discussed only two conservation laws, the laws of conservation of mass and energy. Is there any reason to believe that further conservation laws are needed in order to replace Newton's laws as a complete description of nature? Yes. Conservation of mass and energy do not relate in any way to the three dimensions of space, because both are scalars. Conservation of energy, for instance, does not prevent the planet earth from abruptly making a 90-degree turn and heading straight into the sun, because kinetic energy does not depend on direction. In this chapter, we develop a new conserved quantity, called momentum, which is a vector.

4.1 Momentum

A conserved quantity of motion

Your first encounter with conservation of momentum may have come as a small child unjustly confined to a shopping cart. You spot something interesting to play with, like the display case of imported wine down at the end of the aisle, and decide to push the cart over there. But being imprisoned by Dad in the cart was not the only injustice that day. There was a far greater conspiracy to thwart your young id, one that originated in the laws of nature. Pushing forward did nudge the cart forward, but it pushed you backward. If the wheels of the cart were well lubricated, it wouldn't matter how you jerked, yanked, or kicked off from the back of the cart. You could not cause any overall forward motion of the entire system consisting of the cart with you inside.

In the Newtonian framework, we describe this as arising from Newton's third law. The cart made a force on you that was equal and opposite to your force on it. In the framework of conservation laws, we cannot attribute your frustration to conservation of energy. It would have been perfectly possible for you to transform some of the internal chemical energy stored in your body to kinetic energy of the cart and your body.

The following characteristics of the situation suggest that there may be a new conservation law involved:

> **A closed system is involved**. All conservation laws deal with closed systems. You and the cart are a closed system, since the well-oiled wheels prevent the floor from making any forward force on you.
>
> **Something remains unchanged**. The overall velocity of the system started out being zero, and you cannot change it. This vague reference to "overall velocity" can be made more precise: it is the velocity of the system's center of mass that cannot be changed.
>
> **Something can be transferred back and forth without changing the total amount**: If we define forward as positive and backward as negative, then one part of the system can gain positive motion if another part acquires negative motion. If we don't want to worry about positive and negative signs, we can imagine that the whole cart was initially gliding forward on its well-oiled wheels. By kicking off from the back of the cart, you could increase your own velocity, but this inevitably causes the cart to slow down.

It thus appears that there is some numerical measure of an object's quantity of motion that is conserved when you add up all the objects within a system.

Momentum

Although velocity has been referred to, it is not the total velocity of a closed system that remains constant. If it was, then firing a gun would cause the gun to recoil at the same velocity as the bullet! The gun does recoil, but at a much lower velocity than the bullet. Newton's third law tells us

$$F_{\text{gun on bullet}} = -F_{\text{bullet on gun}} ,$$

and assuming a constant force for simplicity, Newton's second law allows us to change this to

$$m_{bullet}\frac{\Delta v_{bullet}}{\Delta t} = -m_{gun}\frac{\Delta v_{gun}}{\Delta t} \quad .$$

Thus if the gun has 100 times more mass than the bullet, it will recoil at a velocity that is 100 times smaller and in the opposite direction, represented by the opposite sign. The quantity mv is therefore apparently a useful measure of motion, and we give it a name, *momentum*, and a symbol, p. (As far as I know, the letter "p" was just chosen at random, since "m" was already being used for mass.) The situations discussed so far have been one-dimensional, but in three-dimensional situations it is treated as a vector.

definition of momentum for material objects
The momentum of a material object, i.e. a piece of matter, is defined as

$$\mathbf{p} = m\mathbf{v} \quad ,$$

the product of the object's mass and its velocity vector.

The units of momentum are kg·m/s, and there is unfortunately no abbreviation for this clumsy combination of units.

The reasoning leading up to the definition of momentum was all based on the search for a conservation law, and the only reason why we bother to define such a quantity is that experiments show it is conserved:

the law of conservation of momentum
In any closed system, the vector sum of all the momenta remains constant,

$$\mathbf{p}_{1i} + \mathbf{p}_{2i} + \ldots = \mathbf{p}_{1f} + \mathbf{p}_{2f} + \ldots \quad ,$$

where i labels the initial and f the final momenta. (A closed system is one on which no external forces act.)

This chapter first addresses the one-dimensional case, in which the direction of the momentum can be taken into account by using plus and minus signs. We then pass to three dimensions, necessitating the use of vector addition.

A subtle point about conservation laws is that they all refer to "closed systems," but "closed" means different things in different cases. When discussing conservation of mass, "closed" means a system that doesn't have matter moving in or out of it. With energy, we mean that there is no work or heat transfer occurring across the boundary of the system. For momentum conservation, "closed" means there are no external *forces* reaching into the system.

Example: a cannon

Question: A cannon of mass 1000 kg fires a 10-kg shell at a velocity of 200 m/s. At what speed does the cannon recoil?

Solution: The law of conservation of momentum tells us that

$$\boldsymbol{p}_{\text{cannon,i}} + \boldsymbol{p}_{\text{shell,i}} = \boldsymbol{p}_{\text{cannon,f}} + \boldsymbol{p}_{\text{shell,f}} \quad .$$

Choosing a coordinate system in which the cannon points in the positive direction, the given information is

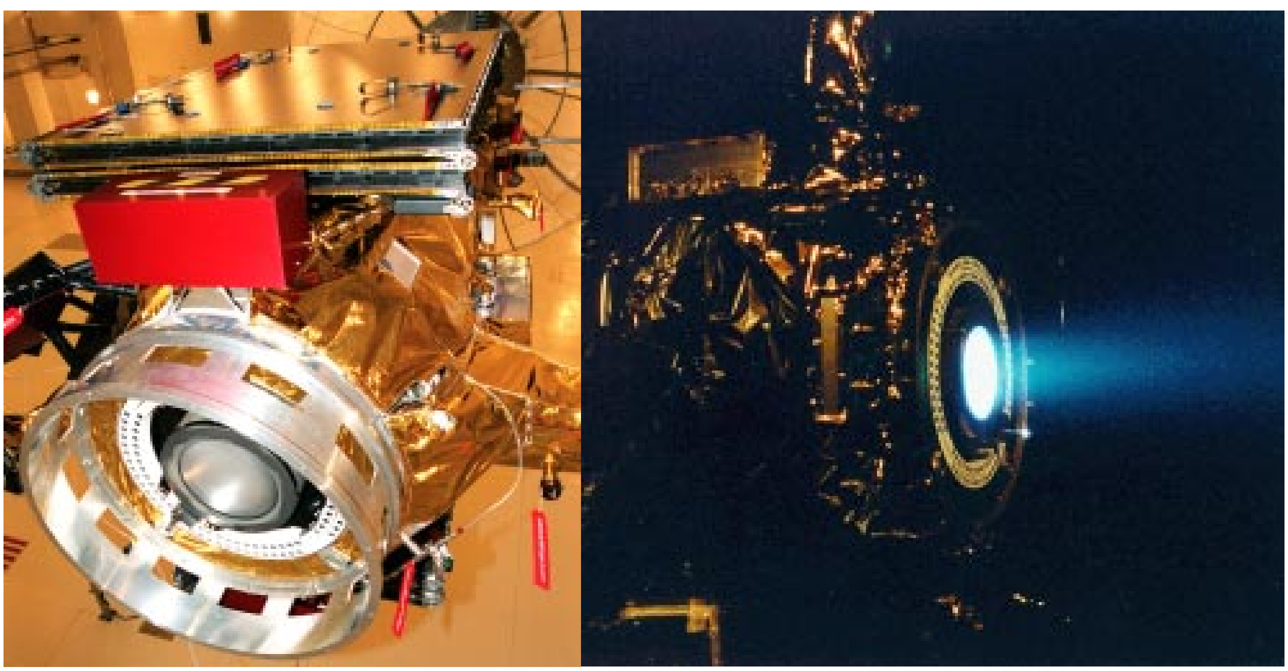
$$\boldsymbol{p}_{\text{cannon,i}} = 0$$
$$\boldsymbol{p}_{\text{shell,i}} = 0$$
$$\boldsymbol{p}_{\text{shell,f}} = 2000 \text{ kg} \cdot \text{m/s} \quad .$$

We must have $\boldsymbol{p}_{\text{cannon,f}} = -2000 \text{ kg} \cdot \text{m/s}$, so the recoil velocity of the cannon is 2 m/s.

Example: ion drive for propelling spacecraft

Question: The experimental solar-powered ion drive of the Deep Space 1 space probe expels its xenon gas exhaust at a speed of 30,000 m/s, ten times faster than the exhaust velocity for a typical chemical-fuel rocket engine. Roughly how many times greater is the maximum speed this spacecraft can reach, compared with a chemical-fueled probe with the same mass of fuel ("reaction mass") available for pushing out the back as exhaust?

Solution: Momentum equals mass multiplied by velocity. Both spacecraft are assumed to have the same amount of reaction mass, and the ion drive's exhaust has a velocity ten times greater, so the momentum of its exhaust is ten times greater. Before the engine starts firing, neither the probe nor the exhaust has any momentum, so the total momentum of the system is zero. By conservation of momentum, the total momentum must also be zero after all the exhaust has been expelled. If we define the positive direction as the direction the spacecraft is going, then the negative momentum of the exhaust is canceled by the positive momentum of the spacecraft. The ion drive allows a final



The ion drive engine of the NASA Deep Space 1 probe, shown under construction (left) and being tested in a vacuum chamber (right) prior to its October 1998 launch. Intended mainly as a test vehicle for new technologies, the craft nevertheless is scheduled to carry out a scientific program that includes a 1999 flyby of a near-earth asteroid and a rendezvous with a comet in 2004.

speed that is ten times greater. (This simplified analysis ignores the fact that the reaction mass expelled later in the burn is not moving backward as fast, because of the forward speed of the already-moving spacecraft.)

Generalization of the momentum concept

As with all the conservation laws, the law of conservation of momentum has evolved over time. In the 1800s it was found that a beam of light striking an object would give it some momentum, even though light has no mass, and would therefore have no momentum according to the above definition. Rather than discarding the principle of conservation of momentum, the physicists of the time decided to see if the definition of momentum could be extended to include momentum carried by light. The process is analogous to the process outlined in chapter 1 for identifying new forms of energy. The first step was the discovery that light could impart momentum to matter, and the second step was to show that the momentum possessed by light could be related in a definite way to observable properties of the light. They found that conservation of momentum could be successfully generalized by attributing to a beam of light a momentum vector in the direction of the light's motion and having a magnitude proportional to the amount of energy the light possessed. The momentum of light is negligible under ordinary circumstances, e.g. a flashlight left on for an hour would only absorb about 10^{-5} kg·m/s of momentum as it recoiled.

The reason for bringing this up is not so that you can plug numbers into a formulas in these exotic situations. The point is that the conservation

Modern Changes in the Momentum Concept

Einstein played a role in two major changes in the momentum concept in the 1900s.

First Einstein showed that the equation $m\mathbf{v}$ would not work for a system containing objects moving at very high speeds relative to one another. He came up with a new equation, to which $m\mathbf{v}$ is only the low-velocity approximation.

The second change, and a far stranger one, was the realization that at the atomic level, motion is inescapably random. The electron in a hydrogen atom doesn't really orbit the nucleus, it forms a vague cloud around it. It might seem that this would prove nonconservation of momentum, but in fact the random wanderings of the proton are exactly coordinated with those of the electron so that the total momentum stays exactly constant. In an atom of lead, there are 82 electrons plus the nucleus, all changing their momenta randomly from moment to moment, but all coordinating mysteriously with each other to keep the vector sum constant. In the 1930s, Einstein pointed out that the theories of the atom then being developed would require this kind of spooky coordination, and used this as an argument that there was something physically unreasonable in the new ideas. Experiments, however, have shown that the spooky effects do happen, and Einstein's objections are remembered today only as a historical curiousity.

Momentum is not always equal to mv. Halley's comet, shown here, has a very elongated elliptical orbit, like those of many other comets. About once per century, its orbit brings it close to the sun. The comet's head, or nucleus, is composed of dirty ice, so the energy deposited by the intense sunlight boils off water vapor. The bottom photo shows a view of the water boiling off of the nucleus from the European Giotto space probe, which passed within 596 km of the comet's head on March 13, 1986. The sunlight does not just carry energy, however. It also carries momentum. Once the steam boils off, the momentum of the sunlight impacting on it pushes it away from the sun, forming a tail as shown in in the top image, taken through a ground-based telescope. By analogy with matter, for which momentum equals mv, you would expect that massless light would have zero momentum, but the equation $p=mv$ is not the correct one for light, and light does have momentum. (Some comets also have a second tail, which is propelled by electrical forces rather than by the momentum of sunlight.)

laws have proven so sturdy exactly because they can easily be amended to fit new circumstances. Newton's laws are no longer at the center of the stage of physics because they did not have the same adaptability. More generally, the moral of this story is the provisional nature of scientific truth.

It should also be noted that conservation of momentum is not a consequence of Newton's laws, as is often asserted in textbooks. Newton's laws do not apply to light, and therefore could not possibly be used to prove anything about a concept as general as the conservation of momentum in its modern form.

Momentum compared to kinetic energy

Momentum and kinetic energy are both measures of the quantity of motion, and a sideshow in the Newton-Leibniz controversy over who invented calculus was an argument over whether mv (i.e. momentum) or mv^2 (i.e. kinetic energy without the 1/2 in front) was the "true" measure of motion. The modern student can certainly be excused for wondering why we need both quantities, when their complementary nature was not evident to the greatest minds of the 1700s. The following table highlights their differences.

Kinetic energy...	**Momentum...**
is a scalar.	is a vector.
is not changed by a force perpendicular to the motion, which changes only the direction of the velocity vector.	is changed by any force, since a change in either the magnitude or direction of the velocity vector will result in a change in the momentum vector.
is always positive, and cannot cancel out.	cancels with momentum in the opposite direction.
can be traded for forms of energy that do not involve motion. KE is not a conserved quantity by itself.	is always conserved in a closed system.
is quadrupled if the velocity is doubled.	is doubled if the velocity is doubled.

Here are some examples that show the different behaviors of the two quantities.

Example: a spinning top

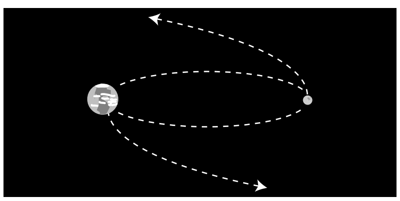
A spinning top has zero total momentum, because for every moving point, there is another point on the opposite side that cancels its momentum. It does, however, have kinetic energy.

Example: momentum and kinetic energy in firing a rifle

The rifle and bullet have zero momentum and zero kinetic energy to start with. When the trigger is pulled, the bullet gains some momentum in the forward direction, but this is canceled by the rifle's backward momentum, so the total momentum is still zero. The kinetic energies of the gun and bullet are both positive scalars, however, and do not cancel. The total kinetic energy is allowed to increase, because kinetic energy is being traded for other forms of energy. Initially there is chemical energy in the gunpowder. This chemical energy is converted into heat, sound, and kinetic energy. The gun's "backward'" kinetic energy does not refrigerate the shooter's shoulder!

Example: the wobbly earth

As the moon completes half a circle around the earth, its motion reverses direction. This does not involve any change in kinetic energy, and the earth's gravitational force does not do any work on the moon. The reversed velocity vector does, however, imply a reversed momentum vector, so conservation of momentum in the closed earth-moon system tells us that the earth must also change its momentum. In fact, the earth wobbles in a little "orbit" about a point below its surface on the line connecting it and the moon. The two bodies' momentum vectors always point in opposite directions and cancel each other out.



Example: the earth and moon get a divorce

Why can't the moon suddenly decide to fly off one way and the earth the other way? It is not forbidden by conservation of momentum, because the moon's newly acquired momentum in one direction could be canceled out by the change in the momentum of the earth, supposing the earth headed the opposite direction at the appropriate, slower speed. The catastrophe is forbidden by conservation of energy, because both their energies would have to increase greatly.

Example: momentum and kinetic energy of a glacier

A cubic-kilometer glacier would have a mass of about 10^{12} kg. If it moves at a speed of 10^{-5} m/s, then its momentum is 10^{7} kg·m/s. This is the kind of heroic-scale result we expect, perhaps the equivalent of the space shuttle taking off, or all the cars in LA driving in the same direction at freeway speed. Its kinetic energy, however, is only 50 J, the equivalent of the calories contained in a poppy seed or the energy in a drop of gasoline too small to be seen without a microscope. The surprisingly small kinetic energy is because kinetic energy is proportional to the square of the velocity, and the square of a small number is an even smaller number.

Discussion Questions

A. If a swarm of ants has a total momentum of zero, what can we conclude? What if their total kinetic energy is zero? [Based on a problem by Serway and Faughn.]

B. If all the air molecules in the room settled down in a thin film on the floor, would that violate conservation of momentum as well as conservation of energy?

C. A refrigerator has coils in back that get hot, and heat is molecular motion. These moving molecules have both energy and momentum. Why doesn't the refrigerator need to be tied to the wall to keep it from recoiling from the momentum it loses out the back?

4.2 Collisions in One Dimension


This Hubble Space Telescope photo shows a small galaxy (yellow blob in the lower right) that has collided with a larger galaxy (spiral near the center), producing a wave of star formation (blue track) due to the shock waves passing through the galaxies' clouds of gas. This is considered a collision in the physics sense, even though it is statistically certain that no star in either galaxy ever struck a star in the other. (This is because the stars are very small compared to the distances between them.)

Physicists employ the term "collision" in a broader sense than ordinary usage, applying it to any situation where objects interact for a certain period of time. A bat hitting a baseball, a radioactively emitted particle damaging DNA, and a gun and a bullet going their separate ways are all examples of collisions in this sense. Physical contact is not even required. A comet swinging past the sun on a hyperbolic orbit is considered to undergo a collision, even though it never touches the sun. All that matters is that the comet and the sun exerted gravitational forces on each other.

The reason for broadening the term "collision" in this way is that all of these situations can be attacked mathematically using the same conservation laws in similar ways. In the first example, conservation of momentum is all that is required.

Example: getting rear-ended

Question: Ms. Chang is rear-ended at a stop light by Mr. Nelson, and sues to make him pay her medical bills. He testifies that he was only going 35 miles per hour when he hit Ms. Chang. She thinks he was going much faster than that. The cars skidded together after the impact, and measurements of the length of the skid marks and the coefficient of friction show that their joint velocity immediately after the impact was 19 miles per hour. Mr. Nelson's Nissan weighs 3100 pounds, and Ms. Chang 's Cadillac weighs 5200 pounds. Is Mr. Nelson telling the truth?

Solution: Since the cars skidded together, we can write down the equation for conservation of momentum using only two velocities, v for Mr. Nelson's velocity before the crash, and v' for their joint velocity afterward:

$$m_N v = m_N v' + m_C v' \quad .$$

Solving for the unknown, v, we find

$$v = \left(1 + \frac{m_C}{m_N}\right) v' \quad .$$

Although we are given the weights in pounds, a unit of force, the ratio of the masses is the same as the ratio of the weights, and we find v=51 miles per hour. He is lying.

The above example was simple because both cars had the same velocity afterward. In many one-dimensional collisions, however, the two objects do not stick. If we wish to predict the result of such a collision, conservation of momentum does not suffice, because both velocities after the collision are unknown, so we have one equation in two unknowns.

Conservation of energy can provide a second equation, but its application is not as straightforward, because kinetic energy is only the particular form of energy that has to do with motion. In many collisions, part of the kinetic energy that was present before the collision is used to create heat or sound, or to break the objects or permanently bend them. Cars, in fact, are carefully designed to crumple in a collision. Crumpling the car uses up energy, and that's good because the goal is to get rid of all that kinetic energy in a relatively safe and controlled way. At the opposite extreme, a superball is "super" because it emerges from a collision with almost all its original kinetic energy, having only stored it briefly as potential energy while it was being squashed by the impact.

Collisions of the superball type, in which almost no kinetic energy is converted to other forms of energy, can thus be analyzed more thoroughly, because they have $KE_f=KE_i$, as opposed to the less useful inequality $KE_f<KE_i$ for a case like a tennis ball bouncing on grass.

Algebraic Proof of the Result in the Example

The equation $A+B=C+D$ says that the change in one ball's velocity is equal and opposite to the change in the other's. We invent a symbol $x=C-A$ for the change in ball 1's velocity. The second equation can then be rewritten as

$$A^2+B^2=(A+x)^2+(B-x)^2 \quad .$$

Squaring out the quantities in parentheses gives

$$A^2+B^2$$
$$=A^2+2Ax+x^2+B^2-2Bx+x^2 \quad ,$$

which simplifies to

$$0=Ax-Bx+x^2 \quad .$$

The equation has the trivial solution $x=0$, i.e. neither ball's velocity is changed, but this is physically impossible because the balls cannot travel through each other like ghosts. Assuming $x\neq0$, we can divide by x and solve for $x=B-A$. This means that ball 1 has gained an amount of velocity exactly sufficient to make it equal to ball 2's initial velocity, and vice-versa. The balls must have swapped velocities.

Example: pool balls colliding head-on

Question: Two pool balls collide head-on, so that the collision is restricted to one dimension. Pool balls are constructed so as to lose as little kinetic energy as possible in a collision, so under the assumption that no kinetic energy is converted to any other form of energy, what can we predict about the results of such a collision?

Solution: Pool balls have identical masses, so we use the same symbol m for both. Conservation of energy and no loss of kinetic energy give us the two equations

$$mv_{1i}+mv_{2i}=mv_{1f}+mv_{2f}$$
$$\frac{1}{2}mv_{1i}^2+\frac{1}{2}mv_{2i}^2=\frac{1}{2}mv_{1f}^2+\frac{1}{2}mv_{2f}^2$$

The masses and the factors of 1/2 can be divided out, and we eliminate the cumbersome subscripts by replacing the symbols v_{1i},... with the symbols A, B, C, and D.

$$A+B=C+D$$
$$A^2+B^2=C^2+D^2 \quad .$$

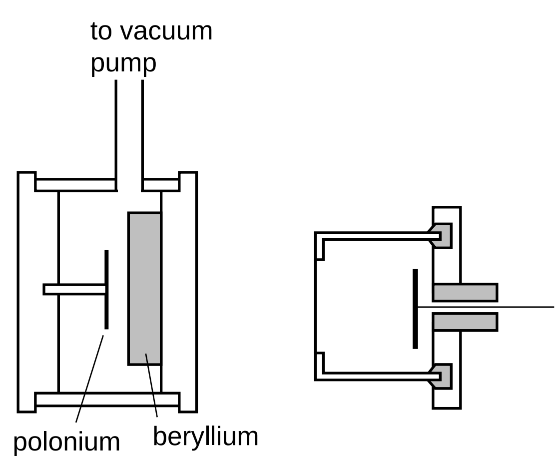
A little experimentation with numbers shows that given values of A and B, it is impossible to find C and D that satisfy these equations unless C and D equal A and B, or C and D are the same as A and B but swapped around. An algebraic proof is given in the box on the left. In the special case where ball 2 is initially at rest, this tells us that ball 1 is stopped dead by the collision, and ball 2 heads off at the velocity originally possessed by ball 1. This behavior will be familiar to players of pool.

Often, as in the example above, the details of the algebra are the least interesting part of the problem, and considerable physical insight can be gained simply by counting the number of unknowns and comparing to the number of equations. Suppose a beginner at pool notices a case where her cue ball hits an initially stationary ball and stops dead. "Wow, what a good trick," she thinks. "I bet I could never do that again in a million years." But she tries again, and finds that she can't help doing it even if she doesn't want to. Luckily she has just learned about collisions in her physics course. Once she has written down the equations for conservation of energy and no loss of kinetic energy, she really doesn't have to complete the algebra. She knows that she has two equations in two unknowns, so there must be a well-defined solution. Once she has seen the result of one such collision, she knows that the same thing must happen every time. The same thing would happen with colliding marbles or croquet balls. It doesn't matter if the masses or velocities are different, because that just multiplies both equations by some constant factor.

The discovery of the neutron

This was the type of reasoning employed by James Chadwick in his 1932 discovery of the neutron. At the time, the atom was imagined to be made out of two types of fundamental particles, protons and electrons. The protons were far more massive, and clustered together in the atom's core, or nucleus. Attractive electrical forces caused the electrons to orbit the nucleus in circles, in much the same way that gravitational forces kept the planets from cruising out of the solar system. Experiments showed that the helium nucleus, for instance, exerted exactly twice as much electrical force on an electron as a nucleus of hydrogen, the smallest atom, and this was explained by saying that helium had two protons to hydrogen's one. The trouble was that according to this model, helium would have two electrons and two protons, giving it precisely twice the mass of a hydrogen atom with one of each. In fact, helium has about four times the mass of hydrogen.

Chadwick suspected that the helium nucleus possessed two additional particles of a new type, which did not participate in electrical forces at all, i.e. were electrically neutral. If these particles had very nearly the same mass as protons, then the four-to-one mass ratio of helium and hydrogen could be explained. In 1930, a new type of radiation was discovered that seemed to fit this description. It was electrically neutral, and seemed to be coming from the nuclei of light elements that had been exposed to other types of radiation. At this time, however, reports of new types of particles were a dime a dozen, and most of them turned out to be either clusters made of previously known particles or else previously known particles with higher energies. Many physicists believed that the "new" particle that had attracted Chadwick's interest was really a previously known particle called a gamma ray, which was electrically neutral. Since gamma rays have no mass, Chadwick decided to try to determine the new particle's mass and see if it was nonzero and approximately equal to the mass of a proton.



Chadwick's subatomic pool table. A disk of the naturally occurring metal polonium provides a source of radiation capable of kicking neutrons out of the beryllium nuclei. The type of radiation emitted by the polonium is easily absorbed by a few mm of air, so the air has to be pumped out of the left-hand chamber. The neutrons, Chadwick's mystery particles, penetrate matter far more readily, and fly out through the wall and into the chamber on the right, which is filled with nitrogen or hydrogen gas. When a neutron collides with a nitrogen or hydrogen nucleus, it kicks it out of its atom at high speed, and this recoiling nucleus then rips apart thousands of other atoms of the gas. The result is an electrical pulse that can be detected in the wire on the right. Physicists had already calibrated this type of apparatus so that they could translate the strength of the electrical pulse into the velocity of the recoiling nucleus. The whole apparatus shown in the figure would fit in the palm of your hand, in dramatic contrast to today's giant particle accelerators.

Unfortunately a subatomic particle is not something you can just put on a scale and weigh. Chadwick came up with an ingenious solution. The masses of the nuclei of the various chemical elements were already known, and techniques had already been developed for measuring the speed of a rapidly moving nucleus. He therefore set out to bombard samples of selected elements with the mysterious new particles. When a direct, head-on collision occurred between a mystery particle and the nucleus of one of the target atoms, the nucleus would be knocked out of the atom, and he would measure its velocity.

Suppose, for instance, that we bombard a sample of hydrogen atoms with the mystery particles. Since the participants in the collision are fundamental particles, there is no way for kinetic energy to be converted into heat or any other form of energy, and Chadwick thus had two equations in three unknowns:

equation #1: conservation of momentum
equation #2: no loss of kinetic energy
unknown #1: mass of the mystery particle
unknown #2: initial velocity of the mystery particle
unknown #3: final velocity of the mystery particle

The number of unknowns is greater than the number of equations, so there is no unique solution. But by creating collisions with nuclei of another element, nitrogen, he gained two more equations at the expense of only one more unknown:

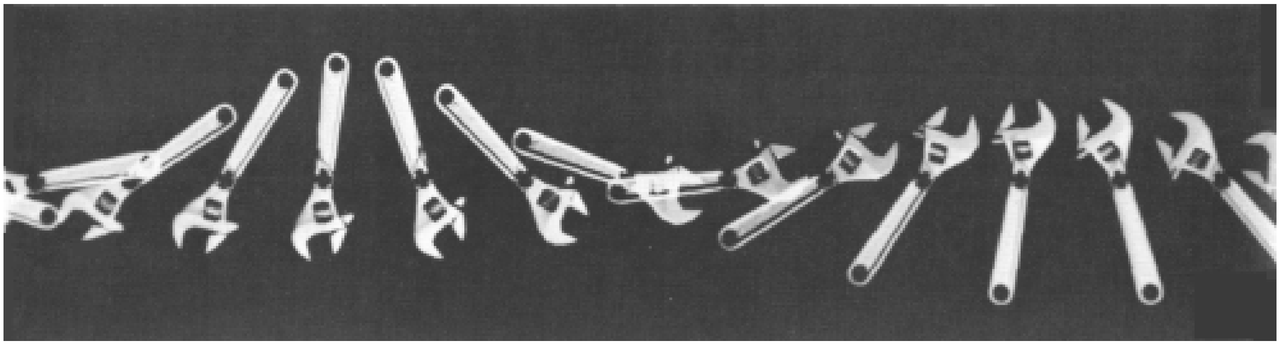
equation #3: conservation of momentum in the new collision
equation #4: no loss of kinetic energy in the new collision
unknown #4: final velocity of the mystery particle in the new collision

He was thus able to solve for all the unknowns, including the mass of the mystery particle, which was indeed within 1% of the mass of a proton. He named the new particle the neutron, since it is electrically neutral.

Discussion Question

Good pool players learn to make the cue ball spin, which can cause it not to stop dead in a head-on collision with a stationary ball. If this does not violate the laws of physics, what hidden assumption was there in the example above?

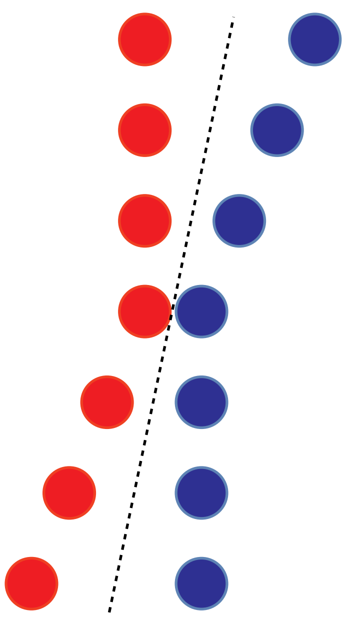
4.3* Relationship of Momentum to the Center of Mass



In this multiple-flash photograph, we see the wrench from above as it flies through the air, rotating as it goes. Its center of mass, marked with the black cross, travels along a straight line, unlike the other points on the wrench, which execute loops.

We have already discussed the idea of the center of mass in the first book of this series, but using the concept of momentum we can now find a mathematical method for defining the center of mass, explain why the motion of an object's center of mass usually exhibits simpler motion than any other point, and gain a very simple and powerful way of understanding collisions.

The first step is to realize that the center of mass concept can be applied to systems containing more than one object. Even something like a wrench, which we think of as one object, is really made of many atoms. The center of mass is particularly easy to visualize in the case shown on the left, where two identical hockey pucks collide. It is clear on grounds of symmetry that their center of mass must be at the midpoint between them. After all, we previously defined the center of mass as the balance point, and if the two hockey pucks were joined with a very lightweight rod whose own mass was negligible, they would obviously balance at the midpoint. It doesn't matter that the hockey pucks are two separate objects. It is still true that the motion of their center of mass is exceptionally simple, just like that of the wrench's center of mass.



Two hockey pucks collide. Their mutual center of mass traces the straight path shown by the dashed line.

The x coordinate of the hockey pucks' center of mass is thus given by $x_{cm}=(x_1+x_2)/2$, i.e. the arithmetic average of their x coordinates. Why is its motion so simple? It has to do with conservation of momentum. Since the hockey pucks are not being acted on by any net external force, they constitute a closed system, and their total momentum is conserved. Their total momentum is

$$\begin{aligned} mv_1+mv_2 &= m(v_1+v_2) \\ &= m\left(\frac{\Delta x_1}{\Delta t}+\frac{\Delta x_2}{\Delta t}\right) \\ &= \frac{m}{\Delta t}\Delta\left(x_1+x_2\right) \\ &= m\frac{2\Delta x_{cm}}{\Delta t} \\ &= m_{total}v_{cm,x} \end{aligned}$$

In other words, the total momentum of the system is the same as if all its mass was concentrated at the center of mass point. Since the total momentum is conserved, the *x* component of the center of mass's velocity vector cannot change. The same is also true for the other components, so the center of mass must move along a straight line at constant speed.

The above relationship between the total momentum and the motion of the center of mass applies to any system, even if it is not closed.

> **total momentum related to center of mass motion**
> The total momentum of any system is related to its total mass and the velocity of its center of mass by the equation
> $$\mathbf{p}_{total} = m_{total}\mathbf{v}_{cm} \quad .$$

What about a system containing objects with unequal masses, or containing more than two objects? The reasoning above can be generalized to a weighted average

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \ldots}{m_1 + m_2 + \ldots} \quad ,$$

with similar equations for the *y* and *z* coordinates.

Momentum in different frames of reference

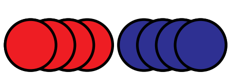
Absolute motion is supposed to be undetectable, i.e. the laws of physics are supposed to be equally valid in all inertial frames of reference. If we first calculate some momenta in one frame of reference and find that momentum is conserved, and then rework the whole problem in some other frame of reference that is moving with respect to the first, the numerical values of the momenta will all be different. Even so, momentum will still be conserved. All that matters is that we work a single problem in one consistent frame of reference.

One way of proving this is to apply the equation $\mathbf{p}_{total}=m_{total}\mathbf{v}_{cm}$. If the velocity of frame B relative to frame A is $\mathbf{v}_{BA}$, then the only effect of changing frames of reference is to change $\mathbf{v}_{cm}$ from its original value to $\mathbf{v}_{cm}+\mathbf{v}_{BA}$. This adds a constant onto the momentum vector, which has no effect on conservation of momentum.

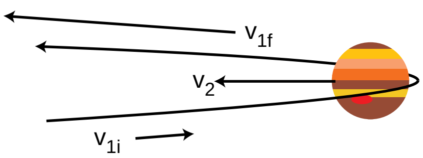
The center of mass frame of reference

A particularly useful frame of reference in many cases is the frame that moves along with the center of mass, called the center of mass (c.m.) frame. In this frame, the total momentum is zero. The following examples show how the center of mass frame can be a powerful tool for simplifying our understanding of collisions.

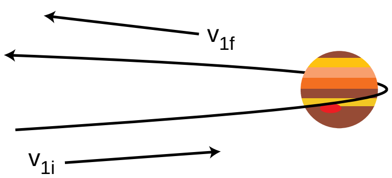
Example: a collision of pool balls viewed in the c.m. frame
If you move your head so that your eye is always above the point halfway in between the two pool balls, you are viewing things in the center of mass frame. In this frame, the balls come toward the center of mass at equal speeds. By symmetry, they must therefore recoil at equal speeds along the lines on which they entered. Since the balls have essentially swapped paths in the center of mass frame, the same must also be true in any other frame. This is the same result that required laborious algebra to prove previously without the concept of the center of mass frame.



Moving our head so that we are always looking down from right above the center of mass, we observe the collision of the two hockey pucks in the center of mass frame.



(a) The slingshot effect viewed in the sun's frame of reference. Jupiter is moving to the left, and the collision is head-on.



(b) The slingshot viewed in the frame of the center of mass of the Jupiter-spacecraft system.

Example: the slingshot effect
It is a counterintuitive fact that a spacecraft can pick up speed by swinging around a planet, if arrives in the opposite direction compared to the planet's motion. Although there is no physical contact, we treat the encounter as a one-dimensional collision, and analyze it in the center of mass frame. Since Jupiter is so much more massive than the spacecraft, the center of mass is essentially fixed at Jupiter's center, and Jupiter has zero velocity in the center of mass frame, as shown in figure (b). The c.m. frame is moving to the left compared to the sun-fixed frame used in (a), so the spacecraft's initial velocity is greater in this frame. Things are simpler in the center of mass frame, because it is more symmetric. In the sun-fixed frame, the incoming leg of the encounter is rapid, because the two bodies are rushing toward each other, while their separation on the outbound leg is more gradual, because Jupiter is trying to catch up. In the c.m. frame, Jupiter is sitting still, and there is perfect symmetry between the incoming and outgoing legs, so by symmetry we have $v_{1f}=-v_{1i}$. Going back to the sun-fixed frame, the spacecraft's final velocity is increased by the frames' motion relative to each other. In the sun-fixed frame, the spacecraft's velocity has increased greatly. The result can also be understood in terms of work and energy. In Jupiter's frame, Jupiter is not doing any work on the spacecraft as it rounds the back of the planet, because the motion is perpendicular to the force. But in the sun's frame, the spacecraft's velocity vector at the same moment has a large component to the left, so Jupiter is doing work on it.

Discussion Questions

A. Make up a numerical example of two unequal masses moving in one dimension at constant velocity, and verify the equation $p_{total}=m_{total}v_{cm}$ over a time interval of one second.

B. A more massive tennis racquet or baseball bat makes the ball fly off faster. Explain why this is true, using the center of mass frame. For simplicity, assume that the racquet or bat is simply sitting still before the collision, and that the hitter's hands do not make any force large enough to have a significant effect over the short duration of the impact.

4.4 Momentum Transfer

The rate of change of momentum

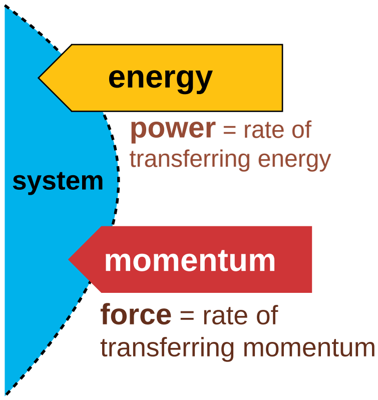
As with conservation of energy, we need a way to measure and calculate the transfer of momentum into or out of a system when the system is not closed. In the case of energy, the answer was rather complicated, and entirely different techniques had to be used for measuring the transfer of mechanical energy (work) and the transfer of heat by conduction. For momentum, the situation is far simpler.

In the simplest case, the system consists of a single object acted on by a constant external force. Since it is only the object's velocity that can change, not its mass, the momentum transferred is

$$\Delta \boldsymbol{p} = m\Delta \boldsymbol{v}$$

which with the help of $\boldsymbol{a}=\boldsymbol{F}/m$ and the constant-acceleration equation $\boldsymbol{a}=\Delta\boldsymbol{v}/\Delta t$ becomes

$$\begin{aligned}\Delta \boldsymbol{p} &= m\boldsymbol{a}\Delta t \\ &= \boldsymbol{F}\Delta t \quad .\end{aligned}$$



Thus the rate of transfer of momentum, i.e. the number of kg·m/s absorbed per second, is simply the external force,

$$\mathbf{F} = \Delta\mathbf{p}/\Delta t \ .$$

[relationship between the force on an object and the rate of change of its momentum; valid only if the force is constant]

This equation is really just a restatement of Newton's second law, and in fact Newton originally stated it this way. As shown in the diagram, the relationship between force and momentum is directly analogous to that between power and energy.

The situation is not materially altered for a system composed of many objects. There may be forces between the objects, but the internal forces cannot change the system's momentum — if they did, then removing the external forces would result in a closed system that was able to change its own momentum, violating conservation of momentum. The equation above becomes

$$\mathbf{F}_{total} = \Delta\mathbf{p}_{total}/\Delta t \ .$$

[relationship between the total external force on a system and the rate of change of its total momentum; valid only if the force is constant]

Example: walking into a lamppost

Question: Starting from rest, you begin walking, bringing your momentum up to 100 kg·m/s. You walk straight into a lamppost. Why is the momentum change of -100 kg·m/s so much more painful than the change of +100 kg·m/s when you started walking?

Solution: The situation is one-dimensional, so we can dispense with the vector notation. It probably takes you about 1 s to speed up initially, so the ground's force on you is $F=\Delta p/\Delta t\approx 100$ N. Your impact with the lamppost, however, is over in the blink of an eye, say 1/10 s or less. Dividing by this much smaller Δt gives a much larger force, perhaps thousands of newtons. (The negative sign simply indicates that the force is in the opposite direction.)

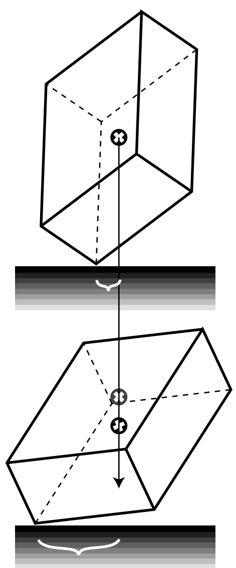
This is also the principle of airbags in cars. The time required for the airbag to decelerate your head is fairly long, the time required for your face to travel 20 or 30 cm. Without an airbag, your face would have been hitting the dashboard, and the time interval would have been the much shorter time taken by your skull to move a couple of centimeters while your face compressed. Note that either way, the same amount of mechanical work has to be done on your head: enough to eliminate all its kinetic energy.

Example: ion drive for spacecraft

Question: The ion drive of the Deep Space 1 spacecraft, pictured earlier in the chapter, produces a thrust of 90 mN (millinewtons). It carries about 80 kg of reaction mass, which it ejects at a speed of 30000 m/s. For how long can the engine continue supplying this amount of thrust before running out of reaction mass to shove out the back?

Solution: Solving the equation $\mathbf{F}=\Delta\mathbf{p}/\Delta t$ for the unknown Δt, and treating force and momentum as scalars since the problem is one-dimensional, we find

$$\begin{aligned} \Delta t &= \frac{\Delta p}{F} \\ &= \frac{m_{\text{exhaust}}\Delta v_{\text{exhaust}}}{F} \\ &= \frac{(80\text{ kg})(30000\text{ m/s})}{0.090\text{ N}} \\ &= 2.7\text{x}10^7\text{ s} \\ &= 300\text{ days} \end{aligned}$$



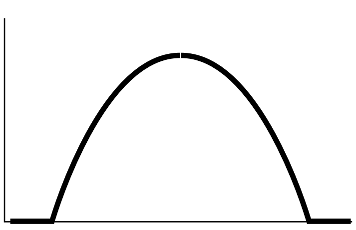
Example: a toppling box

If you place a box on a frictionless surface, it will fall over with a very complicated motion that is hard to predict in detail. We know, however, that its center of mass moves in the same direction as its momentum vector points. There are two forces, a normal force and a gravitational force, both of which are vertical. (The gravitational force is actually many gravitational forces acting on all the atoms in the box.) The total force must be vertical, so the momentum vector must be purely vertical too, and the center of mass travels vertically. This is true even if the box bounces and tumbles. [Based on an example by Kleppner and Kolenkow.]

The area under the force-time graph

Few real collisions involve a constant force. For example, when a tennis ball hits a racquet, the strings stretch and the ball flattens dramatically. They are both acting like springs that obey Hooke's law, which says that the force is proportional to the amount of stretching or flattening. The force is therefore small at first, ramps up to a maximum when the ball is about to reverse directions, and ramps back down again as the ball is on its way back out. The equation $F=\Delta p/\Delta t$, derived under the assumption of constant acceleration, does not apply here, and the force does not even have a single well-defined numerical value that could be plugged in to the equation.

As with similar-looking equations such as $v=\Delta p/\Delta t$, the equation $F=\Delta p/\Delta t$ is correctly generalized by saying that the force is the slope of the p-t graph.



Conversely, if we wish to find Δp from a graph such as the one shown on the left, one approach would be to divide the force by the mass of the ball, rescaling the F axis to create a graph of acceleration versus time. The area under the acceleration-versus-time graph gives the change in velocity, which can then be multiplied by the mass to find the change in momentum. An unnecessary complication was introduced, however, because we began by dividing by the mass and ended by multiplying by it. It would have made just as much sense to find the area under the original *F-t* graph, which would have given us the momentum change directly.

Discussion Question

Many collisions, like the collision of a bat with a baseball, appear to be instantaneous. Most people also would not imagine the bat and ball as bending or being compressed during the collision. Consider the following possibilities:

(1) The collision is instantaneous.

(2) The collision takes a finite amount of time, during which the ball and bat retain their shapes and remain in contact.

(3) The collision takes a finite amount of time, during which the ball and bat are bending or being compressed.

How can two of these be ruled out based on energy or momentum considerations?

4.5 Momentum in Three Dimensions

In this section we discuss how the concepts applied previously to one-dimensional situations can be used as well in three dimensions. Often vector addition is all that is needed to solve a problem:

Example: an explosion

Question: Astronomers observe the planet Mars as the Martians fight a nuclear war. The Martian bombs are so powerful that they rip the planet into three separate pieces of liquified rock, all having the same mass. If one fragment flies off with velocity components $v_{1x}=0$, $v_{1y}=1.0\text{x}10^4$ km/hr, and the second with $v_{2x}=1.0\text{x}10^4$ km/hr, $v_{2y}=0$, what is the magnitude of the third one's velocity?

Solution: We work the problem in the center of mass frame, in which the planet initially had zero momentum. After the explosion, the vector sum of the momenta must still be zero. Vector addition can be done by adding components, so

$$mv_{1x} + mv_{2x} + mv_{3x} = 0 \text{ , and}$$

$$mv_{1y} + mv_{2y} + mv_{3y} = 0 \text{ ,}$$

where we have used the same symbol m for all the terms, because the fragments all have the same mass. The masses can be eliminated by dividing each equation by m, and we find

$$v_{3x} = -1.0\text{x}10^4 \text{ km/hr}$$

$$v_{3y} = -1.0\text{x}10^4 \text{ km/hr}$$

which gives a magnitude of

$$|\mathbf{v}_3| = \sqrt{v_{3x}^2 + v_{3y}^2}$$

$$= 1.4\text{x}10^4 \text{ km/hr}$$

The center of mass

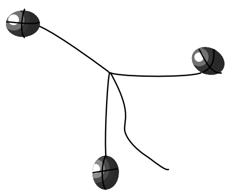
In three dimensions, we have the vector equations

$$\boldsymbol{F}_{total} = \Delta \boldsymbol{p}_{total}/\Delta t$$

and

$$\boldsymbol{p}_{total} = m_{total}\, \boldsymbol{v}_{cm} \quad .$$

The following is an example of their use.



Example: the bola

The bola, similar to the North American lasso, is used by South American gauchos to catch small animals by tangling up their legs in the three leather thongs. The motion of the whirling bola through the air is extremely complicated, and would be a challenge to analyze mathematically. The motion of its center of mass, however, is much simpler. The only forces on it are gravitational, so

$$\boldsymbol{F}_{total} = m_{total}\boldsymbol{g} \quad .$$

Using the equation $\boldsymbol{F}_{total} = \Delta \boldsymbol{p}_{total}/\Delta t$, we find

$$\Delta \boldsymbol{p}_{total}/\Delta t = m_{total}\boldsymbol{g} \quad ,$$

and since the mass is constant, the equation $\boldsymbol{p}_{total} = m_{total}\, \boldsymbol{v}_{cm}$ allows us to change this to

$$m_{total}\Delta \boldsymbol{v}_{cm}/\Delta t = m_{total}\boldsymbol{g} \quad .$$

The mass cancels, and $\Delta \boldsymbol{v}_{cm}/\Delta t$ is simply the acceleration of the center of mass, so

$$\boldsymbol{a}_{cm} = \boldsymbol{g} \quad .$$

In other words, the motion of the system is the same as if all its mass was concentrated at and moving with the center of mass. The bola has a constant downward acceleration equal to $\boldsymbol{g}$, and flies along the same parabola as any other projectile thrown with the same initial center of mass velocity. Throwing a bola with the correct rotation is presumably a difficult skill, but making it hit its target is no harder than it is with a ball or a single rock.

[Based on an example by Kleppner & Kolenkow.]

Counting equations and unknowns

Counting equations and unknowns is just as useful as in one dimension, but every object's momentum vector has three components, so an unknown momentum vector counts as three unknowns. Conservation of momentum is a single vector equation, but it says that all three components of the total momentum vector stay constant, so we count it as three equations. Of course if the motion happens to be confined to two dimensions, then we need only count vectors as having two components.

Example: a two-car crash with sticking

Suppose two cars collide, stick together, and skid off together. If we know the cars' initial momentum vectors, we can count equations and unknowns as follows:

unknown #1: x component of cars' final, total momentum
unknown #2: y component of cars' final, total momentum
equation #1: conservation of the total p_x
equation #2: conservation of the total p_y

Since the number of equations equals the number of unknowns, there must be one unique solution for their total momentum vector after the crash. In other words, the speed and direction at which their common center of mass moves off together is unaffected by factors such as whether the cars collide center-to-center or catch each other a little off-center.

Example: shooting pool

Two pool balls collide, and as before we assume there is no decrease in the total kinetic energy, i.e. no energy converted from KE into other forms. As in the previous example, we assume we are given the initial velocities and want to find the final velocities. The equations and unknowns are:

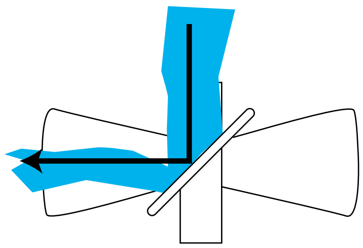
unknown #1: x component of ball #1's final momentum
unknown #2: y component of ball #1's final momentum
unknown #3: x component of ball #2's final momentum
unknown #4: y component of ball #2's final momentum
equation #1: conservation of the total p_x
equation #2: conservation of the total p_y
equation #3: no decrease in total KE

Note that we do not count the balls' final kinetic energies as unknowns, because knowing the momentum vector, one can always find the velocity and thus the kinetic energy. The number of equations is less than the number of unknowns, so no unique result is guaranteed. This is what makes pool an interesting game. By aiming the cue ball to one side of the target ball you can have some control over the balls' speeds and directions of motion after the collision.

It is not possible, however, to choose any combination of final speeds and directions. For instance, a certain shot may give the correct direction of motion for the target ball, making it go into a pocket, but may also have the undesired side-effect of making the cue ball go in a pocket.

Calculations with the momentum vector

The following example illustrates how a force is required to change the direction of the momentum vector, just as one would be required to change its magnitude.



Example: a turbine

Question: In a hydroelectric plant, water flowing over a dam drives a turbine, which runs a generator to make electric power. The figure shows a simplified physical model of the water hitting the turbine, in which it is assumed that the stream of water comes in at a 45° angle with respect to the turbine blade, and bounces off at a 90° angle at nearly the same speed. The water flows at a rate R, in units of kg/s, and the speed of the water is v. What are the magnitude and direction of the water's force on the turbine?

Solution: In a time interval Δt, the mass of water that strikes the blade is $R\Delta t$, and the magnitude of its initial momentum is $mv=vR\Delta t$. The water's final momentum vector is of the same magnitude, but in the perpendicular direction. By Newton's third law, the water's force on the blade is equal and opposite to the blade's force on the water. Since the force is constant, we can use the equation

$$\mathbf{F}_{\text{blade on water}} = \Delta \mathbf{p}_{\text{water}}/\Delta t \quad .$$

Choosing the x axis to be to the right and the y axis to be up, this can be broken down into components as

$$\begin{aligned} F_{\text{blade on water,x}} &= \Delta p_{\text{water,x}}/\Delta t \\ &= (-vR\Delta t-0)/\Delta t \\ &= -vR \end{aligned}$$

and

$$\begin{aligned} F_{\text{blade on water,y}} &= \Delta p_{\text{water,y}}/\Delta t \\ &= (0-(-vR\Delta t))/\Delta t \\ &= vR \quad . \end{aligned}$$

The water's force on the blade thus has

$$\begin{aligned} F_{\text{water on blade,x}} &= vR \\ F_{\text{water on blade,y}} &= -vR \quad . \end{aligned}$$

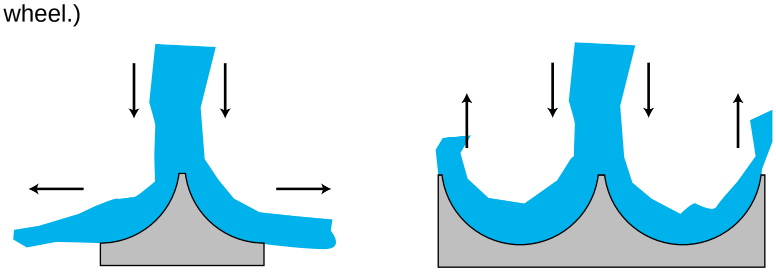
In situations like this, it is always a good idea to check that the result makes sense physically. The x component of the water's force on the blade is positive, which is correct since we know the blade will be pushed to the right. The y component is negative, which also makes sense because the water must push the blade down. The magnitude of the water's force on the blade is

$$|\mathbf{F}_{\text{water on blade}}| = \sqrt{2}\ vR$$

and its direction is at a 45-degree angle down and to the right.

Discussion Questions

The figures show a jet of water striking two different objects. How does the total downward force compare in the two cases? How could this fact be used to create a better waterwheel? (Such a waterwheel is known as a Pelton wheel.)



4.6∫ Applications of Calculus

By now you will have learned to recognized the circumlocutions I use in the sections without calculus in order to introduce calculus-like concepts without using the notation, terminology, or techniques of calculus. It will therefore come as no surprise to you that the rate of change of momentum can be represented with a derivative,

$$\mathbf{F}_{\text{total}} = \frac{\mathrm{d}\mathbf{p}_{\text{total}}}{\mathrm{d}t} \quad .$$

And of course the business about the area under the *F-t* curve is really an integral, $\Delta p_{\text{total}} = \int F_{\text{total}} \mathrm{d}t$, which can be made into an integral of a vector in the more general three-dimensional case:

$$\Delta \mathbf{p}_{\text{total}} = \int \mathbf{F}_{\text{total}} \mathrm{d}t \quad .$$

In the case of a material object that is neither losing nor picking up mass, these are just trivially rearranged versions of familiar equations, e.g. $F = m\frac{\mathrm{d}v}{\mathrm{d}t}$ rewritten as $F = \frac{\mathrm{d}(mv)}{\mathrm{d}t}$. The following is a less trivial example, where $F=ma$ alone would not have been very easy to work with.

Example: rain falling into a moving cart

Question: If 1 kg/s of rain falls vertically into a 10-kg cart that is rolling without friction at an initial speed of 1.0 m/s, what is the effect on the speed of the cart when the rain first starts falling?

Solution: The rain and the cart make horizontal forces on each other, but there is no external horizontal force on the rain-plus-cart system, so the horizontal motion obeys

$$F = \frac{\mathrm{d}(mv)}{\mathrm{d}t} = 0$$

We use the product rule to find

$$0 = \frac{\mathrm{d}m}{\mathrm{d}t} v + m\frac{\mathrm{d}v}{\mathrm{d}t} \quad .$$

We are trying to find how v changes, so we solve for dv/dt,

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}t} &= -\frac{v}{m}\frac{\mathrm{d}m}{\mathrm{d}t} \\ &= -\left(\frac{1\ \text{m/s}}{10\ \text{kg}}\right)(1\ \text{kg/s}) \\ &= -0.1\ \text{m/s}^2 \quad .\end{aligned}$$

(This is only at the moment when the rain starts to fall.)

Finally we note that there are cases where $F=ma$ is not just less convenient than $F=\mathrm{d}p/\mathrm{d}t$ but in fact $F=ma$ is wrong and $F=\mathrm{d}p/\mathrm{d}t$ is right. A good example is the formation of a comet's tail by sunlight. We cannot use $F=ma$ to describe this process, since we are dealing with a collision of light with matter, whereas Newton's laws only apply to matter. The equation $F=\mathrm{d}p/\mathrm{d}t$, on the other hand, allows us to find the force experienced by an atom of gas in the comet's tail if we know the rate at which the momentum vectors of light rays are being turned around by reflection from the molecule.

Summary

Selected Vocabulary

momentum .... a measure of motion, equal to $m\mathbf{v}$ for material objects
collision .... an interaction between moving objects that lasts for a certain time
center of mass .... the balance point or average position of the mass in a system

Notation

$\mathbf{p}$ .... the momentum vector
cm .... center of mass, as in x_{cm}, a_{cm}, etc.

Standard Terminology and Notation Not Used in This Book

impulse .... the amount of momentum transferred, $\Delta\mathbf{p}$
$\mathbf{I}$, $\mathbf{J}$ .... impulse
elastic collision .... one in which no KE is converted into other forms of energy
inelastic collision .... one in which some KE is converted to other forms of energy

Summary

If two objects interact via a force, Newton's third law guarantees that any change in one's velocity vector will be accompanied by a change in the other's which is in the opposite direction. Intuitively, this means that if the two objects are not acted on by any external force, they cannot cooperate to change their overall state of motion. This can be made quantitative by saying that the quantity $m_1\mathbf{v}_1+m_2\mathbf{v}_2$ must remain constant as long as the only forces are the internal ones between the two objects. This is a conservation law, called the conservation of momentum, and like the conservation of energy, it has evolved over time to include more and more phenomena unknown at the time the concept was invented. The momentum of a material object is

$$\mathbf{p} = m\mathbf{v} \quad ,$$

but this is more like a standard for comparison of momenta rather than a definition. For instance, light has momentum, but has no mass, and the above equation is not the right equation for light. The law of conservation of momentum says that the total momentum of any closed system, i.e. the vector sum of the momentum vectors of all the things in the system, is a constant.

An important application of the momentum concept is to collisions, i.e. interactions between moving objects that last for a certain amount of time while the objects are in contact or near each other. Conservation of momentum tells us that certain outcomes of a collision are impossible, and in some cases may even be sufficient to predict the motion after the collision. In other cases, conservation of momentum does not provide enough equations to find all the unknowns. In some collisions, such as the collision of a superball with the floor, very little kinetic energy is converted into other forms of energy, and this provides one more equation, which may suffice to predict the outcome.

The total momentum of a system can be related to its total mass and the velocity of its center of mass by the equation

$$\mathbf{p}_{total} = m_{total}\mathbf{v}_{cm} \quad .$$

The center of mass, introduced on an intuitive basis in book 1 as the "balance point" of an object, can be generalized to any system containing any number of objects, and is defined mathematically as the weighted average of the positions of all the parts of all the objects,

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + \ldots}{m_1 + m_2 + \ldots}$$

with similar equations for the y and z coordinates.

The frame of reference moving with the center of mass of a closed system is always a valid inertial frame, and many problems can be greatly simplified by working them in the inertial frame. For example, any collision between two objects appears in the c.m. frame as a head-on one-dimensional collision.

When a system is not closed, the rate at which momentum is transferred in or out is simply the total force being exerted externally on the system. If the force is constant,

$$\mathbf{F}_{total} = \Delta\mathbf{p}_{total}/\Delta t \quad .$$

When the force is not constant, the force equals the slope of the tangent line on a graph of p versus t, and the change in momentum equals the area under the F-t graph.

Homework Problems

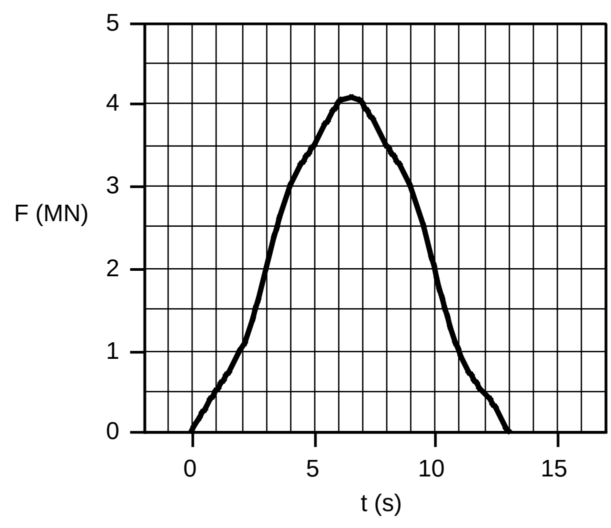
1. Derive a formula expressing the kinetic energy of an object in terms of its momentum and mass.

2. Two people in a rowboat wish to move around without causing the boat to move. What should be true about their total momentum? Explain.

3✓. A learjet traveling due east at 300 mi/hr collides with a jumbo jet which was heading southwest at 150 mi/hr. The jumbo jet's mass is five times greater than that of the learjet. When they collide, the learjet sticks into the fuselage of the jumbo jet, and they fall to earth together. Their engines stop functioning immediately after the collision. On a map, what will be the direction from the location of the collision to the place where the wreckage hits the ground? (Give an angle.)

4. A bullet leaves the barrel of a gun with a kinetic energy of 90 J. The gun barrel is 50 cm long. The gun has a mass of 4 kg, the bullet 10 g.
(a✓) Find the bullet's final velocity.
(b✓) Find the bullet's final momentum.
(c) Find the momentum of the recoiling gun.
(d✓) Find the kinetic energy of the recoiling gun, and explain why the recoiling gun does not kill the shooter.

5✓. The graph below shows the force, in meganewtons, exerted by a rocket engine on the rocket as a function of time. If the rocket's mass if 4000 kg, at what speed is the rocket moving when the engine stops firing? Assume it goes straight up, and neglect the force of gravity, which is much less than a meganewton.



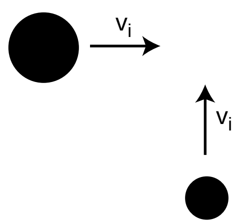
6. Cosmic rays are particles from outer space, mostly protons and atomic nuclei, that are continually bombarding the earth. Most of them, although they are moving extremely fast, have no discernible effect even if they hit your body, because their masses are so small. Their energies vary, however,

S A solution is given in the back of the book.
✓ A computerized answer check is available.
★ A difficult problem.
∫ A problem that requires calculus.

and a very small minority of them have extremely large energies. In some cases the energy is as much as several Joules, which is comparable to the KE of a well thrown rock! If you are in a plane at a high altitude and are so incredibly unlucky as to be hit by one of these rare ultra-high-energy cosmic rays, what would you notice, the momentum imparted to your body, the energy dissipated in your body as heat, or both? Base your conclusions on numerical estimates, not just random speculation. (At these high speeds, one should really take into account the deviations from Newtonian physics described by Einstein's special theory of relativity. Don't worry about that, though.)

7. Show that for a body made up of many *equal* masses, the equation for the center of mass becomes a simple average of all the positions of the masses.

8 S. The figure shows a view from above of a collision about to happen between two air hockey pucks sliding without friction. They have the same speed, v_i, before the collision, but the big puck is 2.3 times more massive than the small one. Their sides have sticky stuff on them, so when they collide, they will stick together. At what angle will they emerge from the collision? In addition to giving a numerical answer, please indicate by drawing on the figure how your angle is defined.



Problem 8.

9 ∫. A flexible rope of mass m and length L slides without friction over the edge of a table. Let x be the length of the rope that is hanging over the edge at a given moment in time.

(a) Show that x satisfies the equation of motion $\frac{d^2x}{dt^2} = \frac{g}{L}x$. [Hint: Use $F=dp/dt$, which allows you to ignore internal forces in the rope.]

(b) Give a physical explanation for the fact that a larger value of x on the right-hand side of the equation leads to a greater value of the acceleration on the left side.

(c) When we take the second derivative of the function $x(t)$ we are supposed to get essentially the same function back again, except for a constant out in front. The function e^x has the property that it is unchanged by differentiation, so it is reasonable to look for solutions to this problem that are of the form $x=be^{ct}$, where b and c are constants. Show that this does indeed provide a solution for two specific values of c (and for any value of b).

(d) Show that the sum of any two solutions to the equation of motion is also a solution.

(e) Find the solution for the case where the rope starts at rest at t=0 with some nonzero value of x.

10. A very massive object with velocity v collides head-on with an object at rest whose mass is very small. No kinetic energy is converted into other forms. Prove that the low-mass object recoils with velocity $2v$.

11S. When the contents of a refrigerator cool down, the changed molecular speeds imply changes in both momentum and energy. Why, then, does a fridge transfer *power* through its radiator coils, but not *force*?

12. A 10-kg bowling ball moving at 2.0 m/s hits a 1.0-kg bowling pin, which is initially at rest. The other pins are all gone already, and the collision is head-on, so that the motion is one-dimensional. Assume that negligible amounts of heat and sound are produced. Find the velocity of the pin immediately after the collision.

13 ∫★. A rocket ejects exhaust with an exhaust velocity u. The rate at which the exhaust mass is used (mass per unit time) is b. We assume that the rocket accelerates in a straight line starting from rest, and that no external forces act on it. Let the rocket's initial mass (fuel plus the body and payload) be m_i, and m_f be its final mass, after all the fuel is used up. (a) Find the rocket's final velocity, v, in terms of u, m_i, and m_f. (b) A typical exhaust velocity for chemical rocket engines is 4000 m/s. Estimate the initial mass of a rocket that could accelerate a one-ton payload to 10% of the speed of light, and show that this design won't work. (For the sake of the estimate, ignore the mass of the fuel tanks.)

14 S. A firework shoots up into the air, and just before it explodes it has a certain momentum and kinetic energy. What can you say about the momenta and kinetic energies of the pieces immediately after the explosion? [Based on a problem from PSSC Physics.]

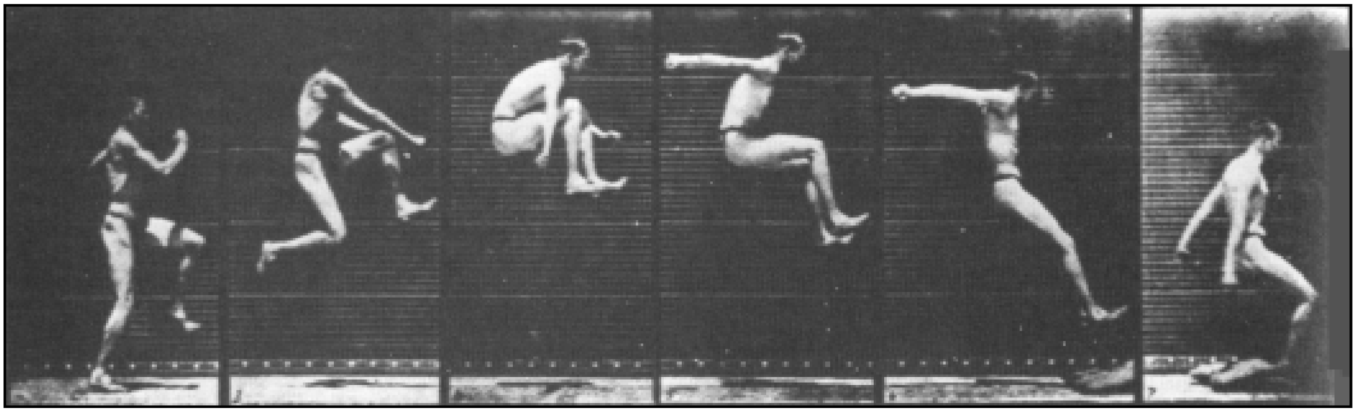
15 ★S. Suppose a system consisting of pointlike particles has a total kinetic energy K_{cm} measured in the center-of-mass frame of reference. Since they are pointlike, they cannot have any energy due to internal motion. (a) Prove that in a different frame of reference, moving with velocity $\boldsymbol{u}$ relative to the center-of-mass frame, the total kinetic energy equals $K_{cm}+M|\boldsymbol{u}|^2/2$, where M is the total mass. [Hint: You can save yourself a lot of writing if you express the total kinetic energy using the dot product.] (b) Use this to prove that if energy is conserved in one frame of reference, then it is conserved in every frame of reference. The total energy equals the total kinetic energy plus the sum of the potential energies due to the particles' interactions with each other, which we assume depends only on the distance between particles. [For a simpler numerical example, see problem 13 in ch. 1.]

A tornado touches down in Spring Hill, Kansas, May 20, 1957.

5 Conservation of Angular Momentum

"Sure, and maybe the sun won't come up tomorrow." Of course, the sun only appears to go up and down because the earth spins, so the cliche should really refer to the unlikelihood of the earth's stopping its rotation abruptly during the night. Why can't it stop? It wouldn't violate conservation of momentum, because the earth's rotation doesn't add anything to its momentum. While California spins in one direction, some equally massive part of India goes the opposite way, canceling its momentum. A halt to Earth's rotation would entail a drop in kinetic energy, but that energy could simply by converted into some other form, such as heat.

Other examples along these lines are not hard to find. A hydrogen atom spins at the same rate for billions of years. A high-diver who is rotating when he comes off the board does not need to make any physical effort to continue rotating, and indeed would be unable to stop rotating before he hit the water.



These observations have the hallmarks of a conservation law:

A closed system is involved. Nothing is making an effort to twist the earth, the hydrogen atom, or the high-diver. They are isolated from rotation-changing influences, i.e. they are closed systems.

Something remains unchanged. There appears to be a numerical quantity for measuring rotational motion such that the total amount of that quantity remains constant in a closed system.

Something can be transferred back and forth without changing the total amount: In the photo of the old-fashioned high jump above, the jumper wants to get his feet out in front of him so he can keep from doing a "face plant" when he lands. Bringing his feet forward would involve a certain quantity of counterclockwise rotation, but he didn't start out with any rotation when he left the ground. Suppose we consider counterclockwise as positive and clockwise as negative. The only way his legs can acquire some positive rotation is if some other part of his body picks up an equal amount of negative rotation. This is why he swings his arms up behind him, clockwise.

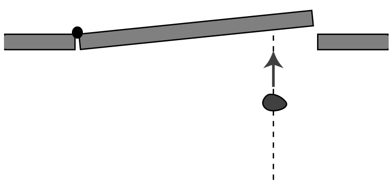
What numerical measure of rotational motion is conserved? Car engines and old-fashioned LP records have speeds of rotation measured in rotations per minute (r.p.m.), but the number of rotations per minute (or per second) is not a conserved quantity. A twirling figure skater, for instance, can pull her arms in to increase her r.p.m.'s. The first section of this chapter deals with the numerical definition of the quantity of rotation that results in a valid conservation law.

5.1 Conservation of Angular Momentum

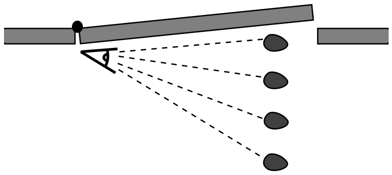
When most people think of rotation, they think of a solid object like a wheel rotating in a circle around a fixed point. Examples of this type of rotation, called rigid rotation or rigid-body rotation, include a spinning top, a seated child's swinging leg, and a helicopter's spinning propeller. Rotation, however, is a much more general phenomenon, and includes noncircular examples such as a comet in an elliptical orbit around the sun, or a cyclone, in which the core completes a circle more quickly than the outer parts.

If there is a numerical measure of rotational motion that is a conserved quantity, then it must include nonrigid cases like these, since nonrigid rotation can be traded back and forth with rigid rotation. For instance, there is a trick for finding out if an egg is raw or hardboiled. If you spin a hardboiled egg and then stop it briefly with your finger, it stops dead. But if you do the same with a raw egg, it springs back into rotation because the soft interior was still swirling around within the momentarily motionless shell. The pattern of flow of the liquid part is presumably very complex and nonuniform due to the asymmetric shape of the egg and the different consistencies of the yolk and the white, but there is apparently some way to describe the liquid's total amount of rotation with a single number, of which some percentage is given back to the shell when you release it.

The best strategy is to devise a way of defining the amount of rotation of a single small part of a system. The amount of rotation of a system such as a cyclone will then be defined as the total of all the contributions from its many small parts.



An overhead view of a piece of putty being thrown at a door. Even though the putty is neither spinning nor traveling along a curve, we must define it has having some kind of "rotation" because it is able to make the door rotate.



As seen by someone standing at the axis, the putty changes its angular position. We therefore define it as having angular momentum.

The quest for a conserved quantity of rotation even requires us to broaden the rotation concept to include cases where the motion doesn't repeat or even curve around. If you throw a piece of putty at a door, the door will recoil and start rotating. The putty was traveling straight, not in a circle, but if there is to be a general conservation law that can cover this situation, it appears that we must describe the putty as having had some "rotation," which it then gave up to the door. The best way of thinking about it is to attribute rotation to any moving object or part of an object that changes its angle in relation to the axis of rotation. In the putty-and-door example, the hinge of the door is the natural point to think of as an axis, and the putty changes its angle as seen by someone standing at the hinge. For this reason, the conserved quantity we are investigating is called *angular momentum*. The symbol for angular momentum can't be "a" or "m," since those are used for acceleration and mass, so the symbol L is arbitrarily chosen instead.

Imagine a 1-kg blob of putty, thrown at the door at a speed of 1 m/s, which hits the door at a distance of 1 m from the hinge. We define this blob to have 1 unit of angular momentum. When it hits the door, it will give up most of its own angular momentum to the door, which will recoil and start rotating.

Experiments show, not surprisingly, that a 2-kg blob thrown in the same way makes the door rotate twice as fast, so the angular momentum of the putty blob must be proportional to mass,

$$L \propto m \quad .$$

Similarly, experiments show that doubling the velocity of the blob will have a doubling effect on the result, so its angular momentum must be proportional to its velocity as well,

$$L \propto mv \quad .$$

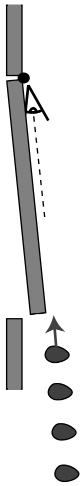
You have undoubtedly had the experience of approaching a closed door with one of those bar-shaped handles on it and pushing on the wrong side, the side close to the hinges. You feel like an idiot, because you have so little leverage that you can hardly budge the door. The same would be true with the putty blob. Experiments would show that the amount of rotation the blob can give to the door is proportional to the distance, r, from the axis of rotation, so angular momentum must be proportional to r as well,

$$L \propto mvr \quad .$$

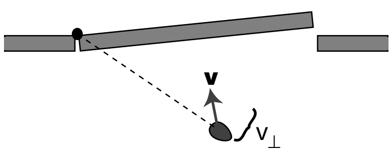
We are almost done, but there is one missing ingredient. We know on grounds of symmetry that a putty ball thrown directly inward toward the hinge will have no angular momentum to give to the door. After all, there would not even be any way to decide whether the ball's rotation was clockwise or counterclockwise in this situation. It is therefore only the component of the blob's velocity vector perpendicular to the door that should be counted in its angular momentum,

$$L = m v_{\perp} r \quad .$$

More generally, $v_{\perp}$ should be thought of as the component of the object's velocity vector that is perpendicular to the line joining the object to the axis of rotation.



A putty blob thrown directly at the axis has no angular motion, and therefore no angular momentum. It will not cause the door to rotate.



Only the component of the velocity vector perpendicular to the line connecting the object to the axis should be counted into the definition of angular momentum.

We find that this equation agrees with the definition of the original putty blob as having one unit of angular momentum, and we can now see that the units of angular momentum are (kg·m/s) · m, i.e. kg·m^2/s. This gives us a way of calculating the angular momentum of any material object or any system consisting of material objects:

angular momentum of a material object

The angular momentum of a moving particle is

$$L = m v_{\perp} r \quad ,$$

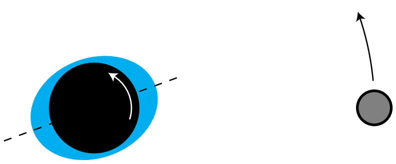
where m is its mass, $v_{\perp}$ is the component of its velocity perpendicular to the line joining it to the axis of rotation, and r is its distance from the axis of rotation. Positive and negative signs of angular momentum are used to describe opposite directions of rotation.

The angular momentum of a finite-sized object or a system of many objects is found by dividing it up into many small parts, applying the equation to each part, and adding to find the total amount of angular momentum.

Note that r is not necessarily the radius of a circle. (As implied by the qualifiers, matter isn't the only thing that can have angular momentum. Light can also have angular momentum, and the above equation would not apply to light.)

Conservation of angular momentum has been verified over and over again by experiment, and is now believed to be one of the three most fundamental principles of physics, along with conservation of energy and momentum.

A figure skater pulls in her arms so that she can execute a spin more rapidly.



A view of the earth-moon system from above the north pole. All distances have been highly distorted for legibility. The earth's rotation is counterclockwise from this point of view (arrow). The moon's gravity creates a bulge on the side near it, because its gravitational pull is stronger there, and an "anti-bulge" on the far side, since its gravity there is weaker. For simplicity, let's focus on the tidal bulge closer to the moon. Its frictional force is trying to slow down the earth's rotation, so its force on the earth's solid crust is toward the bottom of the figure. By Newton's third law, the crust must thus make a force on the bulge which is toward the top of the figure. This causes the bulge to be pulled forward at a slight angle, and the bulge's gravity therefore pulls the moon forward, accelerating its orbital motion about the earth and flinging it outward.

Example: a figure skater pulls her arms in

When a figure skater is twirling, there is very little friction between her and the ice, so she is essentially a closed system, and her angular momentum is conserved. If she pulls her arms in, she is decreasing r for all the atoms in her arms. It would violate conservation of angular momentum if she then continued rotating at the same speed, i.e. taking the same amount of time for each revolution, her arms' contributions to her angular momentum would have decreased, and no other part of her would have increased its angular momentum. This is impossible because it would violate conservation of angular momentum. If her total angular momentum is to remain constant, the decrease in r for her arms must be compensated for by an overall increase in her rate of rotation. That is, by pulling her arms in, she substantially reduces the time for each rotation.

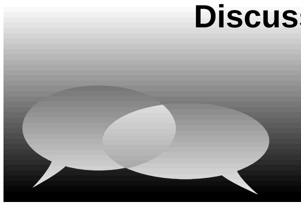
Example: Earth's slowing rotation and the receding moon

As noted in chapter 1, the earth's rotation is actually slowing down very gradually, with the kinetic energy being dissipated as heat by friction between the land and the tidal bulges raised in the seas by the earth's gravity. Does this mean that angular momentum is not really perfectly conserved? No, it just means that the earth is not quite a closed system by itself. If we consider the earth and moon as a system, then the angular momentum lost by the earth must be gained by the moon somehow. In fact very precise measurements of the distance between the earth and the moon have been carried out by bouncing laser beams off of a mirror left there by astronauts, and these measurements show that the moon is receding from the earth at a rate of 2 millimeters per year! The moon's greater value of r means that it has a greater angular momentum, and the increase turns out to be exactly the amount lost by the earth. In the days of the dinosaurs, the days were significantly shorter, and the moon was closer and appeared bigger in the sky.

But what force is causing the moon to speed up, drawing it out into a larger orbit? It is the gravitational forces of the earth's tidal bulges. The effect is described qualitatively in the caption of the figure. The result would obviously be extremely difficult to calculate directly, and this is one of those situations where a conservation law allows us to make precise quantitative statements about the outcome of a process when the calculation of the process itself would be prohibitively complex.

Restriction to rotation in a plane

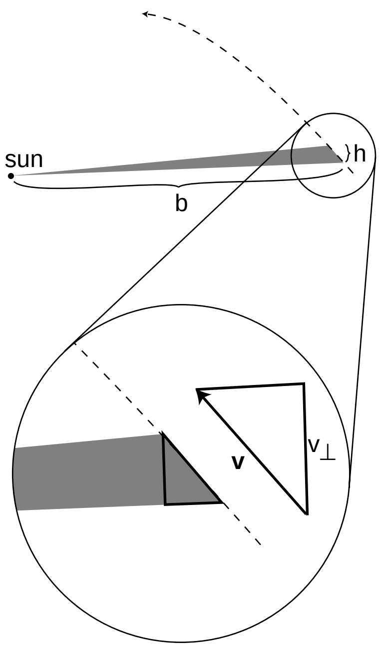
Is angular momentum a vector or a scalar? It does have a direction in space, but it's a direction of rotation, not a straight-line direction like the directions of vectors such as velocity or force. It turns out that there is a way of defining angular momentum as a vector, but in this book the examples will be confined to a single plane of rotation, i.e. effectively two-dimensional situations. In this special case, we can choose to visualize the plane of rotation from one side or the other, and to define clockwise and counterclockwise rotation as having opposite signs of angular momentum.



Discussion Question

Conservation of plain old momentum, ***p***, can be thought of as the greatly expanded and modified descendant of Galileo's original principle of inertia, that no force is required to keep an object in motion. The principle of inertia is counterintuitive, and there are many situations in which it appears superficially that a force *is* needed to maintain motion, as maintained by Aristotle. Think of a situation in which conservation of angular momentum, *L*, also seems to be violated, making it seem incorrectly that something external must act on a closed system to keep its angular momentum from "running down."

5.2 Angular Momentum in Planetary Motion



We now discuss the application of conservation of angular momentum to planetary motion, both because of its intrinsic importance and because it is a good way to develop a visual intuition for angular momentum.

Kepler's law of equal areas states that the area swept out by a planet in a certain length of time is always the same. Angular momentum had not been invented in Kepler's time, and he did not even know the most basic physical facts about the forces at work. He thought of this law as an entirely empirical and unexpectedly simple way of summarizing his data, a rule that succeeded in describing and predicting how the planets sped up and slowed down in their elliptical paths. It is now fairly simple, however, to show that the equal area law amounts to a statement that the planet's angular momentum stays constant.

There is no simple geometrical rule for the area of a pie wedge cut out of an ellipse, but if we consider a very short time interval, as shown in the figure, the shaded shape swept out by the planet is very nearly a triangle. We do know how to compute the area of a triangle. It is one half the product of the base and the height:

$$\text{area} = \frac{1}{2} bh \quad .$$

We wish to relate this to angular momentum, which contains the variables r and $v_\perp$. If we consider the sun to be the axis of rotation, then the variable r is identical to the base of the triangle, $r=b$. Referring to the magnified portion of the figure, $v_\perp$ can be related to h, because the two right triangles are similar:

$$\frac{h}{\text{distance traveled}} = \frac{v_\perp}{|\boldsymbol{v}|}$$

The area can thus be rewritten as

$$\text{area} = \frac{1}{2} r \frac{v_\perp (\text{distance traveled})}{|\boldsymbol{v}|} \quad .$$

The distance traveled equals $|\boldsymbol{v}|\Delta t$, so this simplifies to

$$\text{area} = \frac{1}{2} r\, v_\perp \Delta t \quad .$$

We have found the following relationship between angular momentum and the rate at which area is swept out:

$$L = 2m \frac{\text{area}}{\Delta t} .$$

The factor of 2 in front is simply a matter of convention, since any conserved quantity would be an equally valid conserved quantity if you multiplied it by a constant. The factor of m was not relevant to Kepler, who did not know the planets' masses, and who was only describing the motion of one planet at a time.

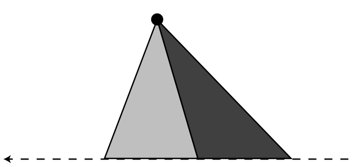
We thus find that Kepler's equal-area law is equivalent to a statement that the planet's angular momentum remains constant. But wait, why should it remain constant? — the planet is not a closed system, since it is being acted on by the sun's gravitational force. There are two valid answers. The first is that it is actually the total angular momentum of the sun plus the planet that is conserved. The sun, however, is millions of times more massive than the typical planet, so it accelerates very little in response to the planet's gravitational force. It is thus a good approximation to say that the sun doesn't move at all, so that no angular momentum is transferred between it and the planet.

The second answer is that to change the planet's angular momentum requires not just a force but a force applied in a certain way. In section 5.4 we discuss the transfer of angular momentum by a force, but the basic idea here is that a force directly in toward the axis does not change the angular momentum.

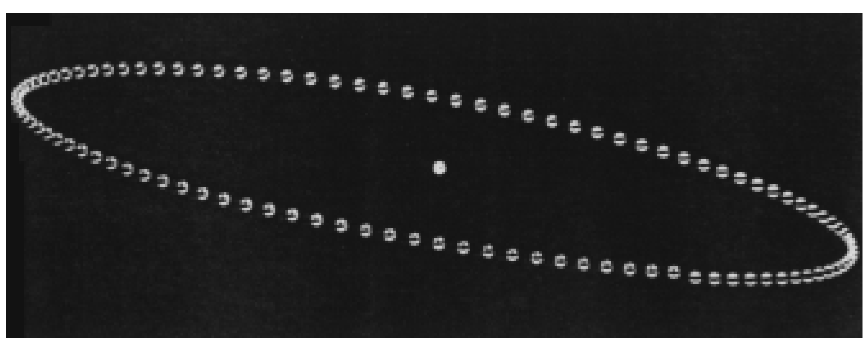
Discussion Questions

A. Suppose an object is simply traveling in a straight line at constant speed. If we pick some point not on the line and call it the axis of rotation, is area swept out by the object at a constant rate? Would it matter if we chose a different axis?

B. The figure below is a strobe photo of a pendulum bob, taken from underneath the pendulum looking straight up. The black string can't be seen in the photograph. The bob was given a slight sideways push when it was released, so it did not swing in a plane. The bright spot marks the center, i.e. the position the bob would have if it hung straight down at us. Does the bob's angular momentum appear to remain constant if we consider the center to be the axis of rotation? What if we choose a different axis?



Discussion question A.



Discussion question B.

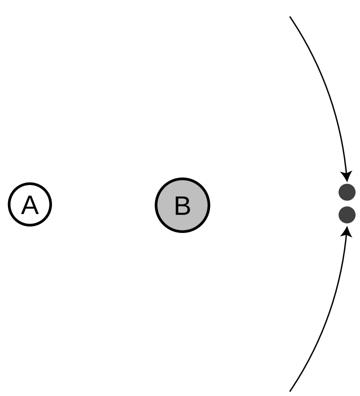
5.3 Two Theorems About Angular Momentum

With plain old momentum, ***p***, we had the freedom to work in any inertial frame of reference we liked. The same object could have different values of momentum in two different frames, if the frames were not at rest with respect to each other. Conservation of momentum, however, would be true in either frame. As long as we employed a single frame consistently throughout a calculation, everything would work.

The same is true for angular momentum, and in addition there is an ambiguity that arises from the definition of an axis of rotation. For a wheel, the natural choice of an axis of rotation is obviously the axle, but what about an egg rotating on its side? The egg has an asymmetric shape, and thus no clearly defined geometric center. A similar issue arises for a cyclone, which does not even have a sharply defined shape, or for a complicated machine with many gears. The following theorem, the first of two presented in this section without proof, explains how to deal with this issue. Although I have put descriptive titles above both theorems, they have no generally accepted names.

choice of axis theorem

It is entirely arbitrary what point one defines as the axis for purposes of calculating angular momentum. If a closed system's angular momentum is conserved when calculated with one choice of axis, then it will also be conserved for any other choice of axis. Likewise, any inertial frame of reference may be used.



Example: colliding asteroids described with different axes

Observers on planets A and B both see the two asteroids colliding. The asteroids are of equal mass and their impact speeds are the same. Astronomers on each planet decide to define their own planet as the axis of rotation. Planet A is twice as far from the collision as planet B. The asteroids collide and stick. For simplicity, assume planets A and B are both at rest.

With planet A as the axis, the two asteroids have the same amount of angular momentum, but one has positive angular momentum and the other has negative. Before the collision, the total angular momentum is therefore zero. After the collision, the two asteroids will have stopped moving, and again the total angular momentum is zero. The total angular momentum both before and after the collision is zero, so angular momentum is conserved if you choose planet A as the axis.

The only difference with planet B as axis is that *r* is smaller by a factor of two, so all the angular momenta are halved. Even though the angular momenta are different than the ones calculated by planet A, angular momentum is still conserved.

The earth spins on its own axis once a day, but simultaneously travels in its circular one-year orbit around the sun, so any given part of it traces out a complicated loopy path. It would seem difficult to calculate the earth's angular momentum, but it turns out that there is an intuitively appealing shortcut: we can simply add up the angular momentum due to its spin plus that arising from its center of mass's circular motion around the sun. This is a special case of the following general theorem:

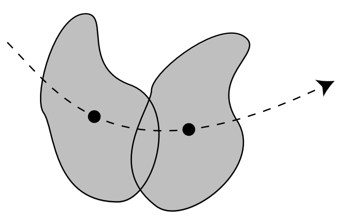
Everyone has a strong tendency to think of the diver as rotating about his own center of mass. However, he is flying in an arc, and he also has angular momentum because of this motion.

spin theorem

An object's angular momentem with respect to some outside axis A can be found by adding up two parts:
(1) The first part is the object's angular momentum found by using its own center of mass as the axis, i.e. the angular momentum the object has because it is spinning.
(2) The other part equals the angular momentum that the object would have with respect to the axis A if it had all its mass concentrated at and moving with its center of mass.

Example: a system with its center of mass at rest

In the special case of an object whose center of mass is at rest, the spin theorem implies that the object's angular momentum is the same regardless of what axis we choose. (This is an even stronger statement than the choice of axis theorem, which only guarantees that angular momentum is conserved for any given choice of axis, without specifying that it is the same for all such choices.)



This rigid object has angular momentum both because it is spinning about its center of mass and because it is moving through space.

Example: angular momentum of a rigid object

Question: A motorcycle wheel has almost all its mass concentrated at the outside. If the wheel has mass m and radius r, and the time required for one revolution is T, what is the spin part of its angular momentum?

Solution: This is an example of the commonly encountered special case of rigid motion, as opposed to the rotation of a system like a hurricane in which the different parts take different amounts of time to go around. We don't really have to go through a laborious process of adding up contributions from all the many parts of a wheel, because they are all at about the same distance from the axis, and are all moving around the axis at about the same speed. The velocity is all perpendicular to the spokes,

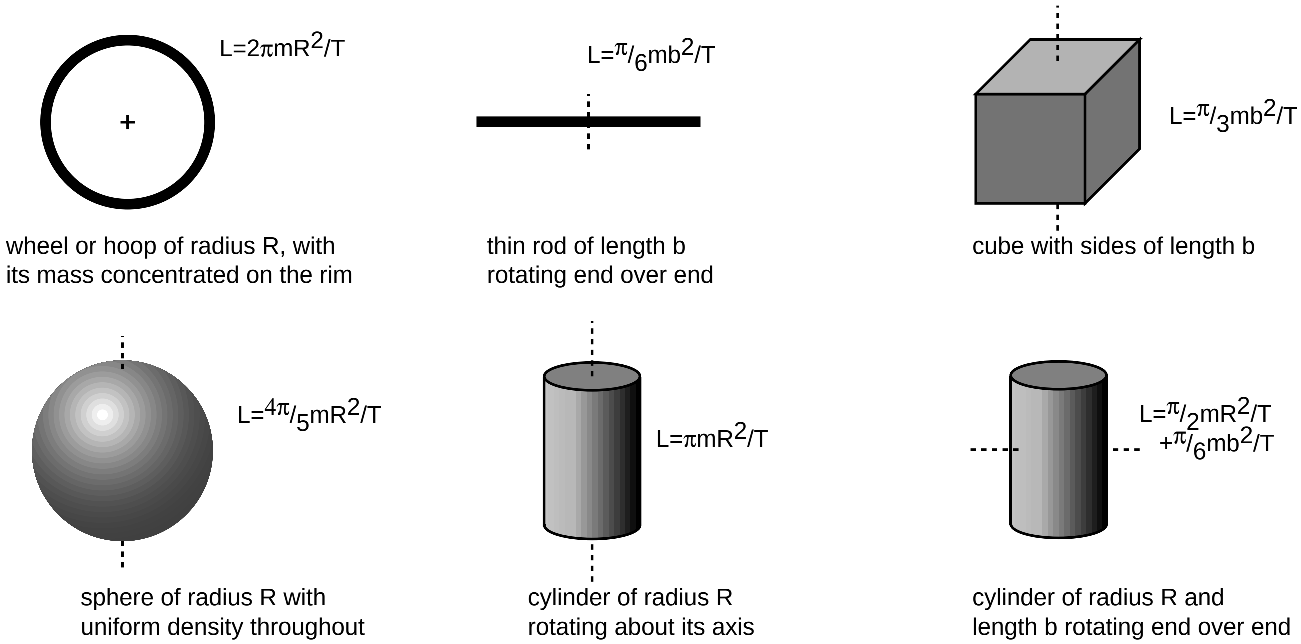
$$\begin{aligned} v_\perp &= v \\ &= \text{(circumference)}/T \\ &= 2\pi r/T \end{aligned}$$

and the angular momentum of the wheel about its center is

$$\begin{aligned} L &= mv_\perp r \\ &= m(2\pi r/T)r \\ &= 2\pi mr^2/T \ . \end{aligned}$$

Note that although the factors of 2π in this expression is peculiar to a wheel with its mass concentrated on the rim, the proportionality to m/T would have been the same for any other rigidly rotating object. Although an object with a noncircular shape does not have a radius, it is also true in general that angular momentum is proportional to the square of the object's size for fixed values of m and T. For instance doubling an object's size doubles both the $v_\perp$ and r factors in the contribution of each of its parts to the total angular momentum, resulting in an overall factor of four increase.

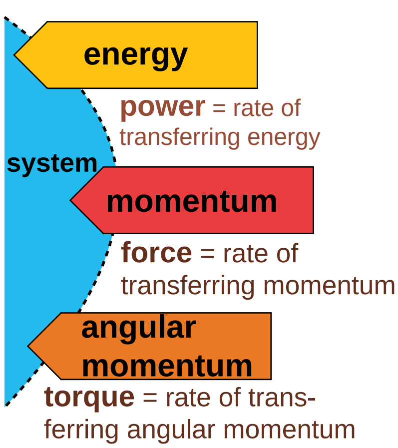
The figure shows some examples of angular momenta of various shapes rotating about their centers of mass. The equations for their angular momenta were derived using calculus, using methods discussed in supplement 2-7. Do not memorize these equations!



Discussion Question

In the example of the colliding asteroids, suppose planet A was moving toward the top of the page, at the same speed as the bottom asteroid. How would planet A's astronomers describe the angular momenta of the asteroids? Would angular momentum still be conserved?

5.4 Torque: the Rate of Transfer of Angular Momentum



Force can be interpreted as the rate of transfer of momentum. The equivalent in the case of angular momentum is called *torque* (rhymes with "fork"). Where force tells us how hard we are pushing or pulling on something, torque indicates how hard we are twisting on it. Torque is represented by the Greek letter tau, τ, and the rate of change of an object's angular momentum equals the total torque acting on it,

$$\tau_{total} = \Delta L/\Delta t \quad .$$

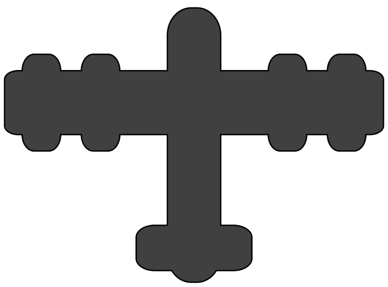
(If the angular momentum does not change at a constant rate, the total torque equals the slope of the tangent line on a graph of L versus t.)

As with force and momentum, it often happens that angular momentum recedes into the background and we focus our interest on the torques. The torque-focused point of view is exemplified by the fact that many scientifically untrained but mechanically apt people know all about torque, but none of them have heard of angular momentum. Car enthusiasts eagerly compare engines' torques, and there is a tool called a torque wrench which allows one to apply a desired amount of torque to a screw and avoid overtightening it.

Torque distinguished from force

Of course a force is necessary in order to create a torque — you can't twist a screw without pushing on the wrench — but force and torque are two different things. One distinction between them is direction. We use positive and negative signs to represent forces in the two possible directions along a line. The direction of a torque, however, is clockwise or counter-clockwise, not a linear direction.

The other difference between torque and force is a matter of leverage. A given force applied at a door's knob will change the door's angular momentum twice as rapidly as the same force applied halfway between the knob and the hinge. The same amount of force produces different amounts of torque in these two cases.



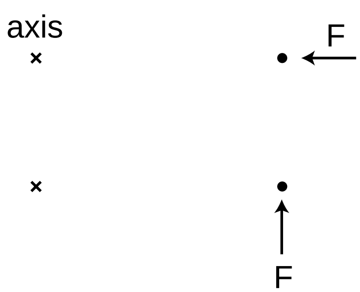
The plane's four engines produce zero total torque but not zero total force.

It is possible to have a zero total torque with a nonzero total force. An airplane with four jet engines would be designed so that their forces are balanced on the left and right. Their forces are all in the same direction, but the clockwise torques of two of the engines are canceled by the counter-clockwise torques of the other two, giving zero total torque.

Conversely we can have zero total force and nonzero total torque. A merry-go-round's engine needs to supply a nonzero torque on it to bring it up to speed, but there is zero total force on it. If there was not zero total force on it, its center of mass would accelerate!

Relationship between force and torque

How do we calculate the amount of torque produced by a given force? Since it depends on leverage, we should expect it to depend on the distance between the axis and the point of application of the force. We'll derive an equation relating torque to force for a particular very simple situation, and state without proof that the equation actually applies to all situations.



The simple physical situation we use to derive an equation for torque. A force that points directly in at or out away from the axis produces neither clockwise nor counterclockwise angular momentum. A force in the perpendicular direction does transfer angular momentum.

Consider a pointlike object which is initially at rest at a distance r from the axis we have chosen for defining angular momentum. We first observe that a force directly inward or outward, along the line connecting the axis to the object, does not impart any angular momentum to the object.

A force perpendicular to the line connecting the axis and the object does, however, make the object pick up angular momentum. Newton's second law gives

$$a = F/m \quad ,$$

and assuming for simplicity that the force is constant, the constant acceleration equation $a=\Delta v/\Delta t$ allows us to find the velocity the object acquires after a time Δt,

$$\Delta v = F\Delta t/m \quad .$$

We are trying to relate force to a change in angular momentum, so we multiply both sides of the equation by mr to give

$$m\Delta vr = F\Delta tr$$

$$\Delta L = F\Delta tr \quad .$$

Dividing by Δt gives the torque:

$$\Delta L/\Delta t = Fr$$

$$\tau = Fr \quad .$$

If a force acts at an angle other than 0 or 90° with respect to the line joining the object and the axis, it would be only the component of the force perpendicular to the line that would produce a torque,

$$\tau = F_{\perp} r \quad .$$

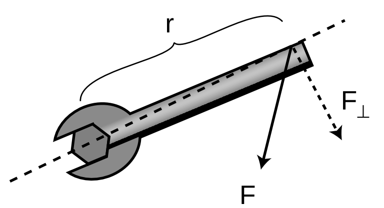
Although this result was proved under a simplified set of circumstances, it is more generally valid.

relationship between force and torque

The rate at which a force transfers angular momentum to an object, i.e. the torque produced by the force, is given by

$$|\tau| = r\,|F_{\perp}| \quad ,$$

where r is the distance from the axis to the point of application of the force, and $F_{\perp}$ is the component of the force that is perpendicular to the line joining the axis to the point of application.



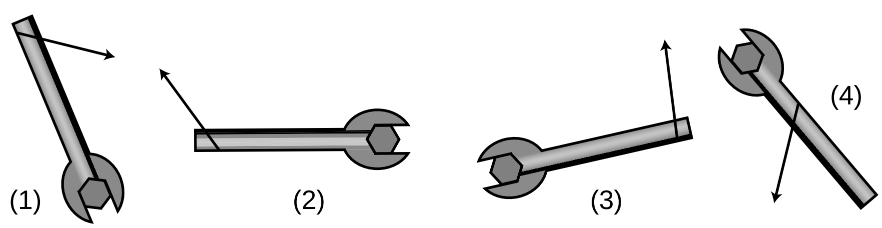
The geometric relationships refered to in the relationship between force and torque.

The equation is stated with absolute value signs because the positive and negative signs of force and torque indicate different things, so there is no useful relationship between them. The sign of the torque must be found by physical inspection of the case at hand.

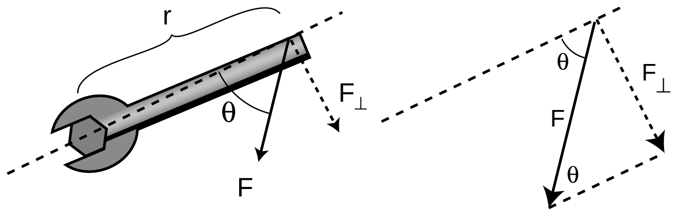
From the equation, we see that the units of torque can be written as newtons multiplied by meters. Metric torque wrenches are calibrated in N·m, but American ones use foot-pounds, which is also a unit of distance multiplied by a unit of force. We know from our study of mechanical work that newtons multiplied by meters equal joules, but torque is a completely different quantity from work, and nobody writes torques with units of joules, even though it would be technically correct.

Self-Check

Compare the magnitudes and signs of the four torques shown in the figure. [Answer on next page.]



Example: how torque depends on the direction of the force
Question: How can the torque applied to the wrench in the figure be expressed in terms of r, $|\mathbf{F}|$, and the angle θ?
Solution: The force vector and its $F_\perp$ component form the hypotenuse and one leg of a right triangle,

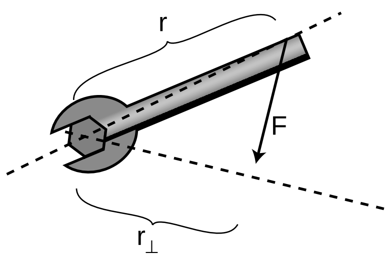


and the interior angle opposite to $F_\perp$ equals θ. The absolute value of $F_\perp$ can thus be expressed as

$$F_\perp = |\mathbf{F}| \sin\theta ,$$

leading to

$$|\tau| = r|\mathbf{F}|\sin\theta .$$



Sometimes torque can be more neatly visualized in terms of the quantity $r_\perp$ shown in the figure on the left, which gives us a third way of expressing the relationship between torque and force:

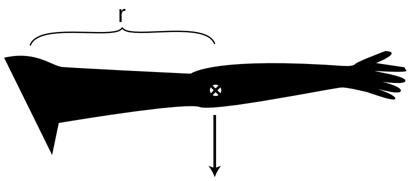
$$|\tau| = r_\perp |F| .$$

Of course you would not want to go and memorize all three equations for torque. Starting from any one of them you could easily derive the other two using trigonometry. Familiarizing yourself with them can however clue you in to easier avenues of attack on certain problems.

The torque due to gravity

Up until now we've been thinking in terms of a force that acts at a single point on an object, such as the force of your hand on the wrench. This is of course an approximation, and for an extremely realistic calculation of your hand's torque on the wrench you might need to add up the torques exerted by each square millimeter where your skin touches the wrench. This is seldom necessary. But in the case of a gravitational force, there is never any single point at which the force is applied. Our planet is exerting a separate tug on every brick in the Leaning Tower of Pisa, and the total gravitational torque on the tower is the sum of the torques contributed by all the little forces. Luckily there is a trick that allows us to avoid such a massive calculation. It turns out that for purposes of computing the total gravitational torque on an object, you can get the right answer by just pretending that the whole gravitational force acts at the object's center of mass.

[Answer to self-check on previous page.] 1, 2, and 4 all have the same sign, because they are trying to twist the wrench clockwise. The sign of 3 is opposite to the signs of 1, 2, and 4. The magnitude of 3 is the greatest, since it has a large r and the force is nearly all perpendicular to the wrench. Torques 1 and 2 are the same because they have the same values of r and $F_\perp$. Torque 4 is the smallest, due to its small r.



Example: gravitational torque on an outstretched arm

Question: Your arm has a mass of 3.0 kg, and its center of mass is 30 cm from your shoulder. What is the gravitational torque on your arm when it is stretched out horizontally to one side, taking the shoulder to be the axis?

Solution: The total gravitational force acting on your arm is

$$|\boldsymbol{F}| = (3.0\ \text{kg})(9.8\ \text{m/s}^2) = 29\ \text{N} \quad .$$

For the purpose of calculating the gravitational torque, we can treat the force as if it acted at the arm's center of mass. The force is straight down, which is perpendicular to the line connecting the shoulder to the center of mass, so

$$F_\perp = |\boldsymbol{F}| = 29\ \text{N} \quad .$$

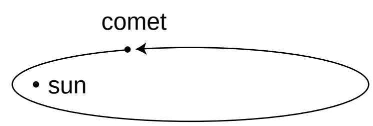
Continuing to pretend that the force acts at the center of the arm, r equals 30 cm = 0.30 m, so the torque is

$$\tau = rF_\perp = 9\ \text{N·m} \quad .$$

Discussion Questions

A. This series of discussion questions deals with past students' incorrect reasoning about the following problem.

Suppose a comet is at the point in its orbit shown in the figure. The only force on the comet is the sun's gravitational force.



Throughout the question, define all torques and angular momenta using the sun as the axis.

(1) Is the sun producing a nonzero torque on the comet? Explain.

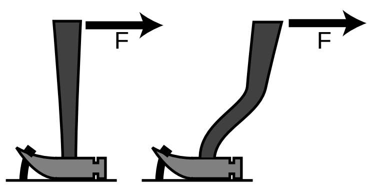
(2) Is the comet's angular momentum increasing, decreasing, or staying the same? Explain.

Explain what is wrong with the following answers. In some cases, the answer is correct, but the reasoning leading up to it is wrong.

(a) Incorrect answer to part (1): "Yes, because the sun is exerting a force on the comet, and the comet is a certain distance from the sun."

(b) Incorrect answer to part (1): "No, because the torques cancel out."

(c) Incorrect answer to part (2): "Increasing, because the comet is speeding up."



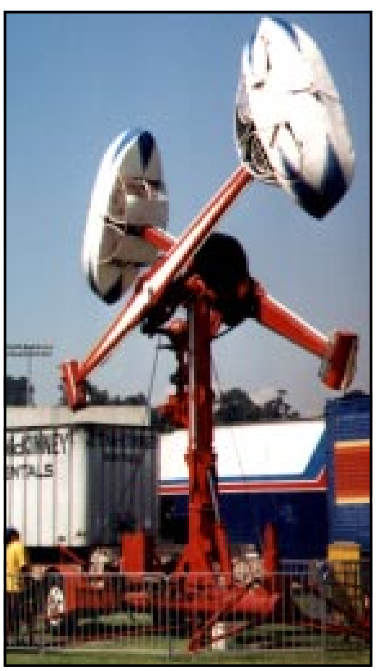
Discussion question B.

B. Which claw hammer would make it easier to get the nail out of the wood if the same force was applied in the same direction?

C. You whirl a rock over your head on the end of a string, and gradually pull in the string, eventually cutting the radius in half. What happens to the rock's angular momentum? What changes occur in its speed, the time required for one revolution, and its acceleration? Why might the string break?

D. A helicopter has, in addition to the huge fan blades on top, a smaller propeller mounted on the tail that rotates in a vertical plane. Why?

E. The photo shows an amusement park ride whose two cars rotate in opposite directions. Why is this a good design?



Discussion question E.

5.5 Statics

Equilibrium

There are many cases where a system is not closed but maintains constant angular momentum. When a merry-go-round is running at constant angular momentum, the engine's torque is being canceled by the torque due to friction.

The windmills are not closed systems, but angular momentum is being transferred out of them at the same rate it is transferred in, resulting in constant angular momentum. To get an idea of the huge scale of the modern windmill farm, note the sizes of the trucks and trailers.

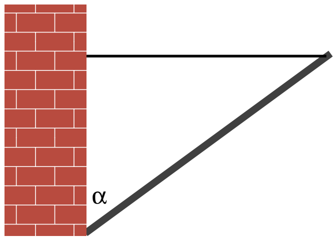
When an object has constant momentum and constant angular momentum, we say that it is in *equilibrium*. This is a scientific redefinition of the common English word, since in ordinary speech nobody would describe a car spinning out on an icy road as being in equilibrium.

Very commonly, however, we are interested in cases where an object is not only in equilibrium but also at rest, and this corresponds more closely to the usual meaning of the word. Trees and bridges have been designed by evolution and engineers to stay at rest, and to do so they must have not just zero total force acting on them but zero total torque. It is not enough that they don't fall down, they also must not tip over. *Statics* is the branch of physics concerned with problems such as these.

Solving statics problems is now simply a matter of applying and combining some things you already know:

- You know the behaviors of the various types of forces, for example that a frictional force is always parallel to the surface of contact.
- You know about vector addition of forces. It is the vector sum of the forces that must equal zero to produce equilibrium.
- You know about torque. The total torque acting on an object must be zero if it is to be in equilibrium.
- You know that the choice of axis is arbitrary, so you can make a choice of axis that makes the problem easy to solve.

In general, this type of problem could involve four equations in four unknowns: three equations that say the force components add up to zero, and one equation that says the total torque is zero. Most cases you'll encounter will not be this complicated. In the example below, only the equation for zero total torque is required in order to get an answer.



Example: a flagpole

Question: A 10-kg flagpole is being held up by a lightweight horizontal cable, and is propped against the foot of a wall as shown in the figure. If the cable is only capable of supporting a tension of 70 N, how great can the angle α be without breaking the cable?

Solution: All three objects in the figure are supposed to be in equilibrium: the pole, the cable, and the wall. Whichever of the three objects we pick to investigate, all the forces and torques on it have to cancel out. It is not particularly helpful to analyze the forces and torques on the wall, since it has forces on it from the ground that are not given and that we don't want to find. We could study the forces and torques on the cable, but that doesn't let us use the given information about the pole. The object we need to analyze is the pole.

The pole has three forces on it, each of which may also result in a torque: (1) the gravitational force, (2) the cable's force, and (3) the wall's force.

We are free to define an axis of rotation at any point we wish, and it is helpful to define it to lie at the bottom end of the pole, since by that definition the wall's force on the pole is applied at r=0 and thus makes no torque on the pole. This is good, because we don't know what the wall's force on the pole is, and we are not trying to find it.

With this choice of axis, there are two nonzero torques on the pole, a counterclockwise torque from the cable and a clockwise torque from gravity. Choosing to represent counterclockwise torques as positive numbers, and using the equation $|\tau| = r|\mathbf{F}|\sin\theta$, we have

$$r_{cable}|\mathbf{F}_{cable}|\sin\theta_{cable} - r_{grav}|\mathbf{F}_{grav}|\sin\theta_{grav} = 0 \quad .$$

A little geometry gives $\theta_{cable}=90°-\alpha$ and $\theta_{grav}=\alpha$, so

$$r_{cable}|\mathbf{F}_{cable}|\sin(90°-\alpha) - r_{grav}|\mathbf{F}_{grav}|\sin\alpha = 0 \quad .$$

The gravitational force can be considered as acting at the pole's center of mass, i.e. at its geometrical center, so r_{cable} is twice r_{grav}, and we can simplify the equation to read

$$2|\mathbf{F}_{cable}|\sin(90°-\alpha) - |\mathbf{F}_{grav}|\sin\alpha = 0 \quad .$$

These are all quantities we were given, except for α, which is the angle we want to find. To solve for α we need to use the trig identity sin(90°–x) = cos x,

$$2|\mathbf{F}_{cable}|\cos\alpha - |\mathbf{F}_{grav}|\sin\alpha = 0 \quad ,$$

which allows us to find

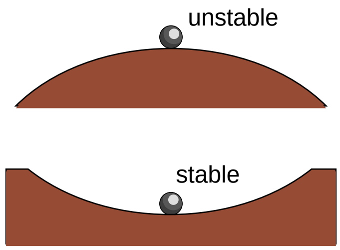
$$\tan\alpha = 2\frac{|F_{cable}|}{|F_{grav}|} \quad ,$$

$$\alpha = \tan^{-1}\left(2\frac{|F_{cable}|}{|F_{grav}|}\right)$$

$$= \tan^{-1}\left(2\times\frac{70\text{ N}}{98\text{ N}}\right)$$

$$= 55° \quad .$$

Stable and unstable equilibria



A pencil balanced upright on its tip could theoretically be in equilibrium, but even if it was initially perfectly balanced, it would topple in response to the first air current or vibration from a passing truck. The pencil can be put in equilibrium, but not in stable equilibrium. The things around us that we really do see staying still are all in stable equilibrium.

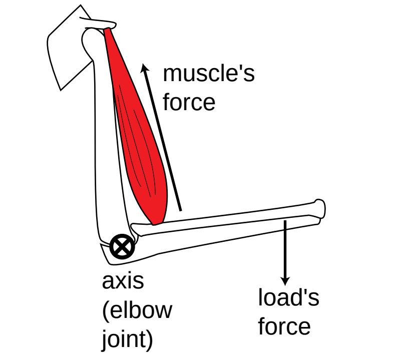
Why is one equilibrium stable and another unstable? Try pushing your own nose to the left or the right. If you push it a millimeter to the left, it responds with a gentle force to the right. If you push it a centimeter to the left, its force on your finger becomes much stronger. The defining characteristic of a stable equilibrium is that the farther the object is moved away from equilibrium, the stronger the force is that tries to bring it back.

The opposite is true for an unstable equilibrium. In the top figure, the ball resting on the round hill theoretically has zero total force on it when it is exactly at the top. But in reality the total force will not be exactly zero, and the ball will begin to move off to one side. Once it has moved, the net force on the ball is greater than it was, and it accelerates more rapidly. In an unstable equilibrium, the farther the object gets from equilibrium, the stronger the force that pushes it farther from equilibrium.

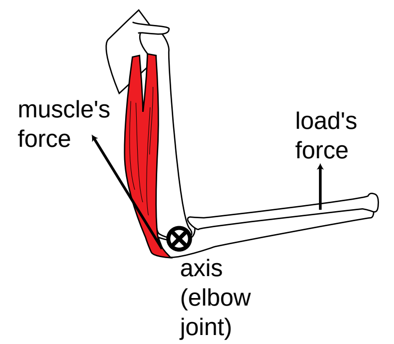
Note that we are using the term “stable” in a weaker sense than in ordinary speech. A domino standing upright is stable in the sense we are using, since it will not spontaneously fall over in response to a sneeze from across the room or the vibration from a passing truck. We would only call it unstable in the technical sense if it could be toppled by *any* force, no matter how small. In everyday usage, of course, it would be considered unstable, since the force required to topple it is so small.

Pooh’s equilibrium is unstable.
(c) 1926 E.H. Shepard

5.6 Simple Machines: The Lever



(a) The biceps muscle flexes the arm.



(b) The triceps muscle extends the arm.

Although we have discussed some simple machines such as the pulley, without the concept of torque we were not yet ready to address the lever, which is the machine nature used in designing living things, almost to the exclusion of all others. (We can speculate what life on our planet might have been like if living things had evolved wheels, gears, pulleys, and screws.) The figures show two examples of levers within your arm. Different muscles are used to flex and extend the arm, because muscles work only by contraction.

Analyzing example (a) physically, there are two forces that do work. When we lift a load with our biceps muscle, the muscle does positive work, because it brings the bone in the forearm in the direction it is moving. The load's force on the arm does negative work, because the arm moves in the direction opposite to the load's force. This makes sense, because we expect our arm to do positive work on the load, so the load must do an equal amount of negative work on the arm. (If the biceps was lowering a load, the signs of the works would be reversed. Any muscle is capable of doing either positive or negative work.)

There is also a third force on the forearm: the force of the upper arm's bone exerted on the forearm at the elbow joint (not shown with an arrow in the figure). This force does no work, because the elbow joint is not moving.

Because the elbow joint is motionless, it is natural to define our torques using the joint as the axis. The situation now becomes quite simple, because the upper arm bone's force exerted at the elbow neither does work nor creates a torque. We can ignore it completely. In any lever there is such a point, called the fulcrum.

If we restrict ourselves to the case in which the forearm rotates with constant angular momentum, then we know that the total torque on the forearm is zero,

$$\tau_{muscle} + \tau_{load} = 0 \quad .$$

If we choose to represent counterclockwise torques as positive, then the muscle's torque is positive, and the load's is negative. In terms of their absolute values,

$$|\tau_{muscle}| = |\tau_{load}| \quad .$$

Assuming for simplicity that both forces act at angles of 90° with respect to the lines connecting the axis to the points at which they act, the absolute values of the torques are

$$r_{muscle}F_{muscle} = r_{load}F_{arm} \quad ,$$

where r_{muscle}, the distance from the elbow joint to the biceps' point of insertion on the forearm, is only a few cm, while r_{load} might be 30 cm or so. The force exerted by the muscle must therefore be about ten times the force exerted by the load. We thus see that this lever is a force reducer. In general, a lever may be used either to increase or to reduce a force.

Why did our arms evolve so as to reduce force? In general, your body is built for compactness and maximum speed of motion rather than maximum force. This is the main anatomical difference between us and the

Neanderthals (their brains covered the same range of sizes as those of modern humans), and it seems to have worked for us.

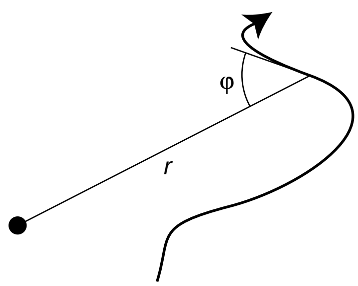
As with all machines, the lever is incapable of changing the amount of mechanical work we can do. A lever that increases force will always reduce motion, and vice versa, leaving the amount of work unchanged.

It is worth noting how simple and yet how powerful this analysis was. It was simple because we were well prepared with the concepts of torque and mechanical work. In anatomy textbooks, whose readers are assumed not to know physics, there is usually a long and complicated discussion of the different types of levers. For example, the biceps lever, (a), would be classified as a class III lever, since it has the fulcrum and load on the ends and the muscle's force acting in the middle. The triceps, (b), is called a class I lever, because the load and muscle's force are on the ends and the fulcrum is in the middle. How tiresome! With a firm grasp of the concept of torque, we realize that all such examples can be analyzed in much the same way. Physics is at its best when it lets us understand many apparently complicated phenomena in terms of a few simple yet powerful concepts.

5.7* Proof of Kepler's Elliptical Orbit Law

Kepler determined purely empirically that the planets' orbits were ellipses, without understanding the underlying reason in terms of physical law. Newton's proof of this fact based on his laws of motion and law of gravity was considered his crowning achievement both by him and by his contemporaries, because it showed that the same physical laws could be used to analyze both the heavens and the earth. Newton's proof was very lengthy, but by applying the more recent concepts of conservation of energy and angular momentum we can carry out the proof quite simply and succinctly, and without calculus.

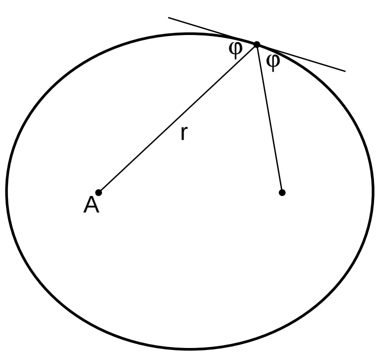
The basic idea of the proof is that we want to describe the shape of the planet's orbit with an equation, and then show that this equation is exactly the one that represents an ellipse. Newton's original proof had to be very complicated because it was based directly on his laws of motion, which include time as a variable. To make any statement about the shape of the orbit, he had to eliminate time from his equations, leaving only space variables. But conservation laws tell us that certain things don't change over time, so they have already had time eliminated from them.



There are many ways of representing a curve by an equation, of which the most familiar is $y=ax+b$ for a line in two dimensions. It would be perfectly possible to describe a planet's orbit using an x-y equation like this, but remember that we are applying conservation of angular momentum, and the space variables that occur in the equation for angular momentum are the distance from the axis, r, and the angle between the velocity vector and the $\mathbf{r}$ vector, which we will call φ. The planet will have φ=90° when it is moving perpendicular to the $\mathbf{r}$ vector, i.e. at the moments when it is at its smallest or greatest distances from the sun. When φ is less than 90° the planet is approaching the sun, and when it is greater than 90° it is receding from it. Describing a curve with an r-φ equation is like telling a driver in a parking lot a certain rule for what direction to steer based on the distance from a certain streetlight in the middle of the lot.

The proof is broken into the three parts for easier digestion. The first part is a simple and intuitively reasonable geometrical fact about ellipses, whose proof we relegate to the caption of a figure; you will not be missing much if you merely absorb the result without reading the proof.

(1) If we use one of the two foci of an ellipse as an axis for defining the variables r and φ, then the angle between the tangent line and the line drawn to the other focus is the same as φ, i.e. the two angles labeled φ in the figure are in fact equal.



Proof that the two angles labeled φ are in fact equal: The definition of an ellipse is that the sum of the distances from the two foci stays constant. If we move a small distance ℓ along the ellipse, then one distance shrinks by an amount $\ell\cos\varphi_1$, while the other grows by $\ell\cos\varphi_2$. These are equal, so $\varphi_1=\varphi_2$.

The other two parts form the meat of our proof. We state the results first and then prove them.

(2) A planet, moving under the influence of the sun's gravity with less then the energy required to escape, obeys an equation of the form

$$\sin\varphi = \frac{1}{\sqrt{-pr^2+qr}} \quad ,$$

where p and q are constants that depend on the planet's energy and angular momentum and p is greater than zero.

(3) A curve is an ellipse if and only if its r-φ equation is of the form

$$\sin\varphi = \frac{1}{\sqrt{-pr^2+qr}} \quad ,$$

where p and q are constants that depend on the size and shape of the ellipse and p is greater than zero.

Proof of part (2)

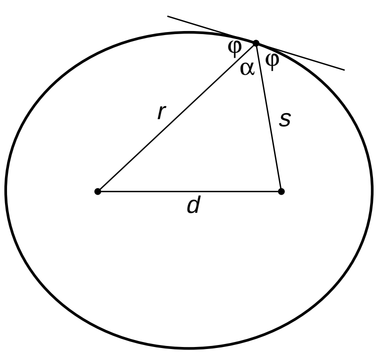
The component of the planet's velocity vector that is perpendicular to the $\boldsymbol{r}$ vector is $v_\perp = v \sin\varphi$, so conservation of angular momentum tells us that $L = mrv\sin\varphi$ is a constant. Since the planet's mass is a constant, this is the same as the condition

$$rv\sin\varphi = \text{constant} \quad .$$

Conservation of energy gives

$$\tfrac{1}{2}mv^2 - G\frac{Mm}{r} = \text{constant} \quad .$$

We solve the first equation for v and plug into the second equation to eliminate v. Straightforward algebra then leads to the equation claimed above, with the constant p being positive because of our assumption that the planet's energy is insufficient to escape from the sun, i.e. its total energy is negative.



Proof of part (3)

We define the quantities α, d, and s as shown in the figure. The law of cosines gives

$$d^2 = r^2 + s^2 - 2rs\cos\alpha \quad .$$

Using $\alpha=180°-2\varphi$ and the trigonometric identities $\cos(180°-x) = -\cos x$ and $\cos 2x = 1-2\sin^2 x$, we can rewrite this as

$$d^2 = r^2 + s^2 + 2rs\left(1-\sin^2\varphi\right) \quad .$$

Straightforward algebra transforms this into

$$\sin\varphi = \sqrt{\frac{d^2+\left(r+s\right)^2}{2rs}} \quad .$$

Since $r+s$ is constant, the top of the fraction is constant, and the denominator can be rewritten as $2rs=2r(\text{constant}-r)$, which is equivalent to the desired form.

Summary

Selected Vocabulary

angular momentum .... a measure of rotational motion; a conserved quantity for a closed system

axis .... An arbitrarily chosen point used in the definition of angular momentum. Any object whose direction changes relative to the axis is considered to have angular momentum. No matter what axis is chosen, the angular momentum of a closed system is conserved.

torque .... the rate of change of angular momentum; a numerical measure of a force's ability to twist on an object

equilibrium .... a state in which an object's momentum and angular momentum are constant

stable equilibrium .... one in which a force always acts to bring the object back to a certain point

unstable equilibrium .... one in which any deviation of the object from its equilibrium position results in a force pushing it even farther away

Notation

L .... angular momentum

τ .... torque

T .... the time required for a rigidly rotating body to complete one rotation

Standard Terminology and Notation Not Used in This Book

period .... a name for the variable T defined above

moment of inertia, I .... the proportionality constant in the equation $L = 2\pi I / T$

Summary

Angular momentum is a measure of rotational motion which is conserved for a closed system. This book only discusses angular momentum for rotation of material objects in two dimensions. Not all rotation is rigid like that of a wheel or a spinning top. An example of nonrigid rotation is a cyclone, in which the inner parts take less time to complete a revolution than the outer parts. In order to define a measure of rotational motion general enough to include nonrigid rotation, we define the angular momentum of a system by dividing it up into small parts, and adding up all the angular momenta of the small parts, which we think of as tiny particles. We arbitrarily choose some point in space, the *axis*, and we say that anything that changes its direction relative to that point possesses angular momentum. The angular momentum of a single particle is

$$L = mv_\perp r \quad ,$$

where $v_\perp$ is the component of its velocity perpendicular to the line joining it to the axis, and r is its distance from the axis. Positive and negative signs of angular momentum are used to indicate clockwise and counter-clockwise rotation.

The *choice of axis theorem* states that any axis may be used for defining angular momentum. If a system's angular momentum is constant for one choice of axis, then it is also constant for any other choice of axis.

The *spin theorem* states that an object's angular momentum with respect to some outside axis A can be found by adding up two parts:

(1) The first part is the object's angular momentum found by using its own center of mass as the axis, i.e. the angular momentum the object has because it is spinning.

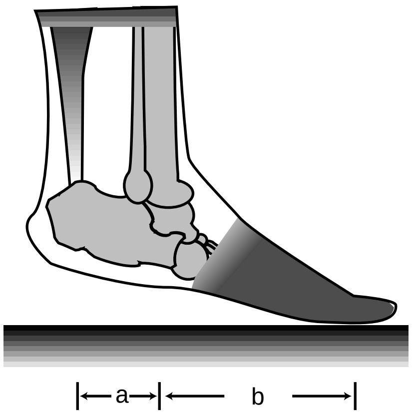
(2) The other part equals the angular momentum that the object would have with respect to the axis A if it had all its mass concentrated at and moving with its center of mass.

Torque is the rate of change of angular momentum. The torque a force can produce is a measure of its ability to twist on an object. The relationship between force and torque is

$$|\tau| = r|F_\perp| \quad ,$$

where r is the distance from the axis to the point where the force is applied, and $F_\perp$ is the component of the force perpendicular to the line connecting the axis to the point of application. Statics problems can be solved by setting the total force and total torque on an object equal to zero and solving for the unknowns.

Homework Problems



Problem 5.

In about five billion years, our own sun will become a white dwarf like the small dot at the center of this cloud of cast-off gas.

1✓. You are trying to loosen a stuck bolt on your RV using a big wrench that is 50 cm long. If you hang from the wrench, and your mass is 55 kg, what is the maximum torque you can exert on the bolt?

2✓. A physical therapist wants her patient to rehabilitate his injured elbow by laying his arm flat on a table, and then lifting a 2.1 kg mass by bending his elbow. In this situation, the weight is 33 cm from his elbow. He calls her back, complaining that it hurts him to grasp the weight. He asks if he can strap a bigger weight onto his arm, only 17 cm from his elbow. How much mass should she tell him to use so that he will be exerting the same torque? (He is raising his forearm itself, as well as the weight.)

3. An object thrown straight up in the air is momentarily at rest when it reaches the top of its motion. Does that mean that it is in equilibrium at that point? Explain.

4. An object is observed to have constant angular momentum. Can you conclude that no torques are acting on it? Explain. [Based on a problem by Serway and Faughn.]

5. A person of weight W stands on the ball of one foot. Find the tension in the calf muscle and the force exerted by the shinbones on the bones of the foot, in terms of W, a, and b. For simplicity, assume that all the forces are at 90-degree angles to the foot, i.e. neglect the angle between the foot and the floor.

6. Two objects have the same momentum vector. Can you conclude that their angular momenta are the same? Explain. [Based on a problem by Serway and Faughn.]

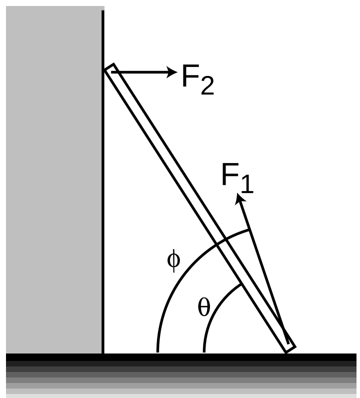
7. The sun turns on its axis once every 26.0 days. Its mass is 2.0×10^{30} kg and its radius is 7.0×10^{8} m. Assume it is a rigid sphere of uniform density.

(a✓) What is the sun's angular momentum?

In a few billion years, astrophysicists predict that the sun will use up all its sources of nuclear energy, and will collapse into a ball of exotic, dense matter known as a white dwarf. Assume that its radius becomes 5.8×10^{6} m (similar to the size of the Earth.) Assume it does not lose any mass between now and then. (Don't be fooled by the photo, which makes it look like nearly all of the star was thrown off by the explosion. The visually prominent gas cloud is actually thinner than the best laboratory vacuum every produced on earth. Certainly a little bit of mass is actually lost, but it is not at all unreasonable to make an approximation of zero loss of mass as we are doing.)

(b) What will its angular momentum be?
(c✓) How long will it take to turn once on its axis?

S A solution is given in the back of the book.	★ A difficult problem.
✓ A computerized answer check is available.	∫ A problem that requires calculus.



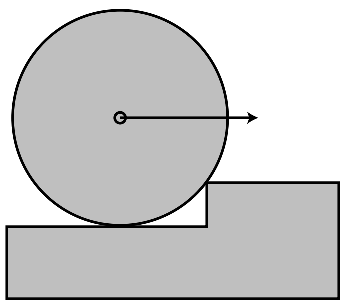
Problems 8 and 9.

8. A uniform ladder of mass *m* and length *L* leans against a smooth wall, making an angle θ with respect to the ground. The dirt exerts a normal force and a frictional force on the ladder, producing a force vector with magnitude F_1 at an angle ϕ with respect to the ground. Since the wall is smooth, it exerts only a normal force on the ladder; let its magnitude be F_2.

(a) Explain why ϕ must be greater than θ. No math is needed.

(b) Choose any numerical values you like for *m* and *L*, and show that the ladder can be in equilibrium (zero torque and zero total force vector) for θ=45.00° and ϕ=63.43°.

9 ★. Continuing the previous problem, find an equation for ϕ in terms of θ, and show that *m* and *L* do not enter into the equation. Do not assume any numerical values for any of the variables. You will need the trig identity $\sin(a-b) = \sin a \cos b - \sin b \cos a$. (As a numerical check on your result, you may wish to check that the angles given in part b of the previous problem satisfy your equation.)


Problem 10.

10. (a) Find the minimum horizontal force which, applied at the axle, will pull a wheel over a step. Invent algebra symbols for whatever quantities you find to be relevant, and give your answer in symbolic form. [Hints: There are three forces on the wheel at first, but only two when it lifts off. Normal forces are always perpendicular to the surface of contact. Note that the corner of the step cannot be perfectly sharp, so the surface of contact for this force really coincides with the surface of the wheel.]

(b) Under what circumstances does your result become infinite? Give a physical interpretation.

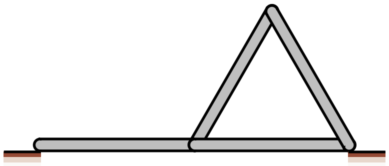
11 ★. A yo-yo of total mass *m* consists of two solid cylinders of radius *R*, connected by a small spindle of negligible mass and radius *r*. The top of the string is held motionless while the string unrolls from the spindle. Show that the acceleration of the yo-yo is $g/(1+R^2/2r^2)$. [Hint: The acceleration and the tension in the string are unknown. Use $\tau=\Delta L/\Delta t$ and $F=ma$ to determine these two unknowns.]

12. A ball is connected by a string to a vertical post. The ball is set in horizontal motion so that it starts winding the string around the post. Assume that the motion is confined to a horizontal plane, i.e. ignore gravity. Michelle and Astrid are trying to predict the final velocity of the ball when it reaches the post. Michelle says that according to conservation of angular momentum, the ball has to speed up as it approaches the post. Astrid says that according to conservation of energy, the ball has to keep a constant speed. Who is right? [Hint: How is this different from the case where you whirl a rock in a circle on a string and gradually pull in the string?]

13. In the 1950's, serious articles began appearing in magazines like *Life* predicting that world domination would be achieved by the nation that could put nuclear bombs in orbiting space stations, from which they could be dropped at will. In fact it can be quite difficult to get an orbiting object to come down. Let the object have energy $E=KE+PE$ and angular momentum L. Assume that the energy is negative, i.e. the object is moving at less than escape velocity. Show that it can never reach a radius less than

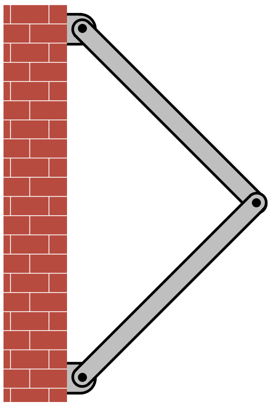
$$r_{min} = \frac{GMm}{2E}\left(-1+\sqrt{1+\frac{2EL^2}{G^2M^2m^3}}\right) \quad .$$

[Note that both factors are negative, giving a positive result.]



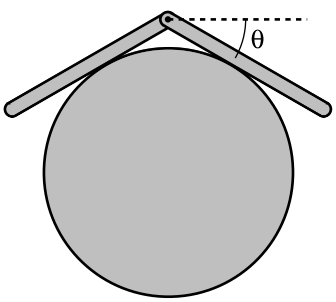
Problem 14.

14. The figure shows a bridge made out of four identical trusses, each of weight W. How much force must be supplied by each pier to hold up the bridge?



Problem 15.

15★. Two bars slanted at 45 degrees are attached to each other and to a wall as shown in the figure. Each bar has weight W. The goal is to show that the middle joint must be able to handle a strain $W/2$, and the top and bottom joints $\sqrt{5}W/4$. [The problem could be set up with three equations of equilibrium for the top bar and three for the bottom bar, giving a total of six equations in six unknowns (two unknowns for the components of each of the three forces). A less tedious approach is as follows. First prove that the horizontal forces at all three joints have the same magnitude, X, and figure out which are to the right and which are to the left. Next, choose the top joint as an axis, and use the fact that the total torque equals zero to prove $X=W/2$. Finally, find the vertical forces.]

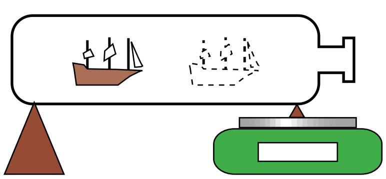


Problem 16.

16★. Two bars of length L are connected with a hinge and placed on a frictionless cylinder of radius r. (a) Show that the angle θ shown in the figure is related to the unitless ratio r/L by the equation

$$\frac{r}{L} = \frac{\cos^2\theta}{2\tan\theta} \quad .$$

(b) Discuss the physical behavior of this equation for very large and very small values of r/L.



Problem 17.

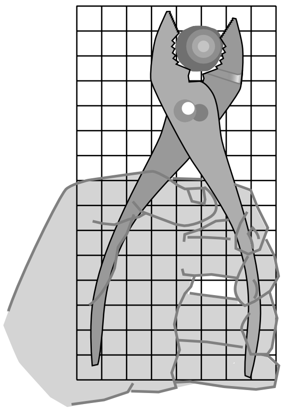
17. You wish to determine the mass of a ship in a bottle without taking it out. Show that this can be done with the setup shown in the figure, with a scale supporting the bottle at one end, provided that it is possible to take readings with the ship slid to two different locations.

18 ∫. Two atoms will interact via electrical forces between their protons and electrons. One fairly good approximation to the potential energy is the Lennard-Jones potential,

$$PE(r) = k\left[\left(\frac{a}{r}\right)^{12} - 2\left(\frac{a}{r}\right)^{6}\right] \quad ,$$

where r is the center-to-center distance between the atoms.

Show that (a) there is an equilibrium point at $r=a$, (b) the equilibrium is stable, and (c) the energy required to bring the atoms from their equilibrium separation to infinity is k. [Hints: The first two parts can be done more easily by setting $a=1$, since the value of a only changes the distance scale. One way to do part b is by graphing.]

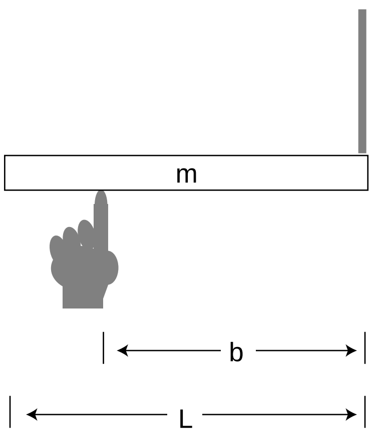

Problem 20.

19. Suppose that we lived in a universe in which Newton's law of gravity gave forces proportional to r^{-7} rather than r^{-2}. Which, if any, of Kepler's laws would still be true? Which would be completely false? Which would be different, but in a way that could be calculated with straightforward algebra?

20 S. The figure shows scale drawing of a pair of pliers being used to crack a nut, with an appropriately reduced centimeter grid. Warning: do not attempt this at home; it is bad manners. If the force required to crack the nut is 300 N, estimate the force required of the person's hand.

21. Show that a sphere of radius R that is rolling without slipping has angular momentum and momentum in the ratio $L/p=(2/5)R$.

22. Suppose a bowling ball is initially thrown so that it has no angular momentum at all, i.e. it is initially just sliding down the lane. Eventually kinetic friction will bring its angular velocity up to the point where it is rolling without slipping. Show that the final velocity of the ball equals 5/7 of its initial velocity. [Hint: You'll need the result of problem 21.]



Problem 23.

23. The rod in the figure is supported by the finger and the string.
(a) Find the tension, T, in the string, and the force, F, from the finger, in terms of m, b, L, and g. (b) Comment on the cases $b=L$ and $b=L/2$. (c) Are any values of b unphysical?

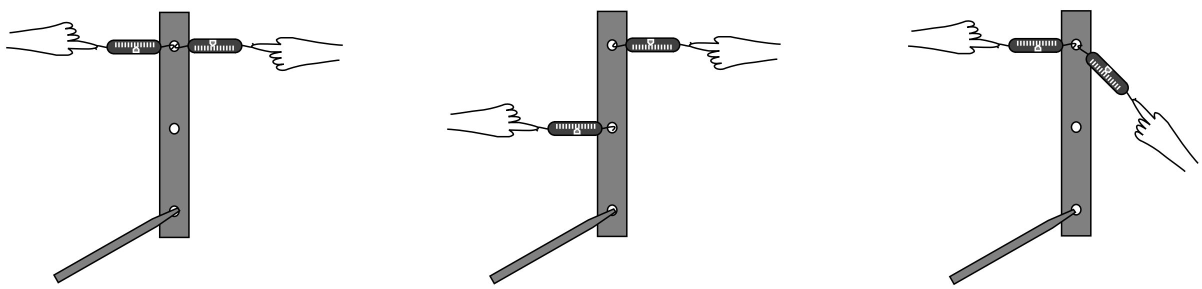
Exercises

Exercise 5A: Torque

Equipment:

rulers with holes in them

spring scales (two per group)



While one person holds the pencil which forms an axle for the ruler, the other members of the group pull on the scales and take readings. In each case, determine whether the total torque on the ruler appears to equal zero to roughly within the aqccuracy of the measurement.

Solutions to Selected Problems

Chapter 1

7. A force is an interaction between two objects, so while the bullet is in the air, there is no force. There is only a force while the bullet is in contact with the book. There is energy the whole time, and the total amount doesn't change. The bullet has some kinetic energy, and transfers some of it to the book as heat, sound, and the energy required to tear a hole through the book.

8. (a) The energy stored in the gasoline is being changed into heat via frictional heating, and also probably into sound and into energy of water waves. Note that the kinetic energy of the propeller and the boat are not changing, so they are not involved in the energy transformation. (b) The crusing speed would be greater by a factor of the cube root of 2, or about a 26% increase.

9. We don't have actual masses and velocities to plug in to the equation, but that's OK. We just have to reason in terms of ratios and proportionalities. Kinetic energy is proportional to mass and to the square of velocity, so B's kinetic energy equals

$$(13.4\text{ J})(3.77)/(2.34)^2 = 9.23\text{ J}$$

11. Room temperature is about 20°C. The fraction of the power that actually goes into heating the water is

$$\frac{(250\text{ g}) / (0.24\text{ J/g¡C}) \times (100¡\text{C} - 20¡\text{C}) / 126\text{ s}}{1.25\times 10^3\text{ J/s}} = 0.53$$

So roughly half of the energy is wasted. The wasted energy might be in several forms: heating of the cup, heating of the oven itself, or leakage of microwaves from the oven.

Chapter 2

5.

$$\begin{aligned}
E_{\text{total,i}} &= E_{\text{total,f}} \\
PE_i + heat_i &= PE_f + KE_f + heat_f \\
\tfrac{1}{2}mv^2 &= PE_i - PE_f + heat_i - heat_f \\
&= -\Delta PE - \Delta heat \\
v &= \sqrt{2\left(\frac{-\Delta PE - \Delta heat}{m}\right)} \\
&= 6.4\text{ m/s}
\end{aligned}$$

7. Let θ be the angle she by which she has progressed around the pipe. Conservation of energy gives

$$\begin{aligned}
E_{\text{total,i}} &= E_{\text{total,f}} \\
PE_i &= PE_f + KE_f
\end{aligned}$$

Let's make PE=0 at the top, so

$$0 = mgr(\cos\theta - 1) + \tfrac{1}{2}mv^2 \quad .$$

While she is still in contact with the pipe, the radial component of her acceleration is

$$a_r = v^2/r \quad ,$$

and making use of the previous equation we find

$$a_r = 2g(1-\cos\theta) \quad .$$

There are two forces on her, a normal force from the pipe and a downward gravitation force from the earth. At the moment when she loses contact with the pipe, the normal force is zero, so the radial component, $mg\cos\theta$, of the gravitational force must equal ma_r,

$$mg\cos\theta = 2mg(1-\cos\theta) \quad ,$$

which gives

$$\cos\theta = 2/3 \quad .$$

The amount by which she has dropped is $r(1-\cos\theta)$, which equals $r/3$ at this moment.

9. (a) Example: As one child goes up on one side of a see-saw, another child on the other side comes down. (b) Example: A pool ball hits another pool ball, and transfers some KE.

11. Suppose the river is 1 m deep, 100 m wide, and flows at a speed of 10 m/s, and that the falls are 100 m tall. In 1 second, the volume of water flowing over the falls is 10^3 m^3, with a mass of 10^6 kg. The potential energy released in one second is (10^6 kg)(g)(100 m)=10^9 J, so the power is 10^9 W. A typical household might have 10 hundred-watt applicances turned on at any given time, so it consumes about 10^3 watts on the average. The plant could supply a about million households with electricity.

Chapter 3

18. No. Work describes how energy was transferred by some process. It isn't a measurable property of a system.

Chapter 4

8. Let m be the mass of the little puck and M=2.3m be the mass of the little one. All we need to do is find the direction of the total momentum vector before the collision, because the total momentum vector is the same after the collision. Given the two components of the momentum vector $p_x=mv$ and $p_y=Mv$, the direction of the vector is $\tan^{-1}(p_y/p_x)$=23° counterclockwise from the big puck's original direction of motion.

11. Momentum is a vector. The total momentum of the molecules is always zero, since the momenta in different directions cancal out on the average. Cooling changes individual molecular momenta, but not the total.

15) (a) Particle i had velocity $\mathbf{v}_i$ in the center-of-mass frame, and has velocity $\mathbf{v}_i+\mathbf{u}$ in the new frame. The total kinetic energy is

$$\frac{1}{2}m_1\left(\mathbf{v}_1+\mathbf{u}\right)^2+\ldots$$

where "..." indicates that the sum continues for all the particles. Rewriting this in terms of the vector dot product, we have

$$\frac{1}{2}m_1\left(\mathbf{v}_1+\mathbf{u}\right)\cdot\left(\mathbf{v}_1+\mathbf{u}\right)+\ldots$$

$$=\frac{1}{2}m_1\left(\mathbf{v}_1\cdot\mathbf{v}_1+2\mathbf{u}\cdot\mathbf{v}_1+\mathbf{u}\cdot\mathbf{u}\right)+\ldots$$

When we add up all the terms like the first one, we get K_{cm}. Adding up all the terms like the third one, we get $M|\mathbf{u}|^2/2$. The terms like the second term cancel out:

$$m_1\mathbf{u}\cdot\mathbf{v}_1+\ldots$$

$$=\mathbf{u}\cdot\left(m_1\mathbf{v}_1+\ldots\right)\quad,$$

where the sum in brackets equals the total momentum in the center-of-mass frame, which is zero by definition. (b) Changing frames of reference doesn't change the distances between the particles, so the potential energies are all unaffected by the change of frames of reference. Suppose that in a given frame of reference,frame 1, energy is conserved in some process: the initial and final energies add up to be the same. First let's transform to the center-of-mass frame. The potential energies are unaffected by the transformation, and the total kinetic energy is simply reduced by the quantity $M|\mathbf{u}_1|^2/2$, where $\mathbf{u}_1$ is the velocity of frame 1 relative to the center of mass. Subtracting the same constant from the initial and final energies still leaves them equal. Now we transform to frame 2. Again, the effect is simply to change the initial and final energies by adding the same constant.

Chapter 5

20. The pliers are not moving, so their angular momentum remains constant at zero, and the total torque on them must be zero. Not only that, but each half of the pliers must have zero total torque on it. This tells us that the magnitude of the torque at one end must be the same as that at the other end. The distance from the axis to the nut is about 2.5 cm, and the distance from the axis to the centers of the palm and fingers are about 8 cm. The angles are close enough to 90° that we can pretend they're 90 degrees, considering the rough nature of the other assumptions and measurements. The result is (300 N)(2.5 cm)=(F)(8 cm), or F=90 N.

Glossary

Angular momentum. A measure of rotational motion; a conserved quantity for a closed system.

Axis. An arbitrarily chosen point used in the definition of angular momentum. Any object whose direction changes relative to the axis is considered to have angular momentum. No matter what axis is chosen, the angular momentum of a closed system is conserved.

Center of mass. The balance point or average position of the mass in a system.

Collision. An interaction between moving objects that lasts for a certain time.

Energy. A numerical scale used to measure the heat, motion, or other properties th.at would require fuel or physical effort to put into an object; a scalar quantity with units of joules (J).

Equilibrium. A state in which an object's momentum and angular momentum are constant.

Heat. The energy that an object has because of its temperature. Heat is different from temperature (q.v.) because an object with twice as much mass requires twice as much heat to increase its temperature by the same amount. There is a further distinction in the terminology, not emphasized in this book, between heat and thermal energy. See the entry under thermal energy for a discussion of this distinction.

Kinetic energy. The energy an object posesses because of its motion. Cf. potential energy.

Momentum. A measure of motion, equal to *mv* for material objects.

Potential energy. The energy having to do with the distance between to objects that interact via a noncontact force. Cf. Kinetic energy.

Power. The rate of transferring energy; a scalar quantity with units of watts (W).

Stable equilibrium. One in which a force always acts to bring the object back to a certain point.

Temperature. What a thermometer measures. Objects left in contact with each other tend to reach the same temperature. Roughly speaking, temperature measures the average kinetic energy per molecule. For the distinction between temperature and heat, see the glossary entry for heat.

Thermal energy.Careful writers make a distinction between heat and thermal energy, but the distinction is often ignored in casual speech, even among physicists. Properly, thermal energy is used to mean the total amount of energy posessed by an object, while heat indicates the amount of thermal energy transferred in or out. The term heat is used in this book to include both meanings.

Torque. The rate of change of angular momentum; a numerical measure of a force's ability to twist on an object.

Unstable equilibrium. One in which any deviation of the object from its equilibrium position results in a force pushing it even farther away.

Work. The amount of energy transferred into or out of a system, excluding energy transferred by heat conduction.

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Useful Data

Metric Prefixes

M-	mega-	10^{6}
k-	kilo-	10^{3}
m-	milli-	10^{-3}
μ- (Greek mu)	micro-	10^{-6}
n-	nano-	10^{-9}

(Centi-, 10^{-2}, is used only in the centimeter.)

Notation and Units

quantity	unit	symbol
distance	meter, m	x, Δx
time	second, s	t, Δt
mass	kilogram, kg	m
area	m^2 (square meters)	A
volume	m^3 (cubic meters)	V
density	kg/m^3	ρ
force	newton, 1 N=1 $kg{\cdot}m/s^2$	$\boldsymbol{F}$
velocity	m/s	$\boldsymbol{v}$
acceleration	m/s^2	$\boldsymbol{a}$
energy	joule, J	E
momentum	kg·m/s	$\boldsymbol{p}$
angular momentum	$kg{\cdot}m^2/s$	L

symbol	meaning
∝	is proportional to
≈	is approximately equal to
~	on the order of

The Greek Alphabet

α	Α	alpha	ν	Ν	nu
β	Β	beta	ξ	Ξ	xi
γ	Γ	gamma	ο	Ο	omicron
δ	Δ	delta	π	Π	pi
ε	Ε	epsilon	ρ	Ρ	rho
ζ	Ζ	zeta	σ	Σ	sigma
η	Η	eta	τ	Τ	tau
θ	Θ	theta	υ	Υ	upsilon
ι	Ι	iota	φ	Φ	phi
κ	Κ	kappa	χ	Χ	chi
λ	Λ	lambda	ψ	Ψ	psi
μ	Μ	mu	ω	Ω	omega

Conversions

Conversions between SI and other units:

1 inch	=	2.54 cm (exactly)
1 mile	=	1.61 km
1 pound	=	4.45 N
(1 kg)·g	=	2.2 lb
1 gallon	=	$3.78\text{x}10^{3}$ cm^3
1 horsepower	=	746 W
1 kcal*	=	$4.18\text{x}10^{3}$ J

*When speaking of food energy, the word "Calorie" is used to mean 1 kcal, i.e. 1000 calories. In writing, the capital C may be used to indicate 1 Calorie=1000 calories.

Conversions between U.S. units:

1 foot	=	12 inches
1 yard	=	3 feet
1 mile	=	5280 ft

Earth, Moon, and Sun

body	mass (kg)	radius (km)	radius of orbit (km)
earth	$5.97\text{x}10^{24}$	$6.4\text{x}10^{3}$	$1.49\text{x}10^{8}$
moon	$7.35\text{x}10^{22}$	$1.7\text{x}10^{3}$	$3.84\text{x}10^{5}$
sun	$1.99\text{x}10^{30}$	$7.0\text{x}10^{5}$	

The radii and radii of orbits are average values. The moon orbits the earth and the earth orbits the sun.

Subatomic Particles

particle	mass (kg)	radius (m)
electron	$9.109\text{x}10^{-31}$	? – less than about 10^{-17}
proton	$1.673\text{x}10^{-27}$	about $1.1\text{x}10^{-15}$
neutron	$1.675\text{x}10^{-27}$	about $1.1\text{x}10^{-15}$

The radii of protons and neutrons can only be given approximately, since they have fuzzy surfaces. For comparison, a typical atom is about 10^{-9} m in radius.

Fundamental Constants

speed of light	$c=3.00\text{x}10^{8}$ m/s
gravitational constant	$G=6.67\text{x}10^{-11}$ $N{\cdot}m^2{\cdot}kg^{-2}$