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The Principle of Least Action in Geometry and Dynamics

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Preface

The motion of classical mechanical systems is determined by Hamilton's differential equations:

$$\begin{cases} \dot{x}(t) = \partial_y H(x(t), y(t)) \\ \dot{y}(t) = -\partial_x H(x(t), y(t)) \end{cases}$$

For instance, if we consider the motion of n particles in a potential field, the Hamiltonian function

$$H = \frac{1}{2} \sum_{i=1}^{n} y_i^2 \; - V(x_1, \ldots, x_n)$$

is the sum of kinetic and potential energy; this is just another formulation of Newton's Second Law.

A distinguished class of Hamiltonians on a cotangent bundle T^*X consists of those satisfying the Legendre condition. These Hamiltonians are obtained from Lagrangian systems on the configuration space X, with coordinates $(x, \dot{x}) =$ (*space, velocity*), by introducing the new coordinates $(x, y) =$ (*space, momentum*) on its phase space T^*X. Analytically, the Legendre condition corresponds to the convexity of H with respect to the fiber variable y. The Hamiltonian gives the energy value along a solution (which is preserved for time–independent systems) whereas the Lagrangian describes the action. Hamilton's equations are equivalent to the Euler–Lagrange equations for the Lagrangian:

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \partial_{\dot{x}} L(x(t), \dot{x}(t)) = \partial_x L(x(t), \dot{x}(t)).$$

These equations express the variational character of solutions of the Lagrangian system. A curve $x : [t_0, t_1] \to \mathbb{R}^n$ is a Euler–Lagrange trajectory if, and only if, the first variation of the action integral, with end points held fixed, vanishes:

$$\delta \int_{t_0}^{t_1} L(x(t), \dot{x}(t))\, \mathrm{d}t \,\Big|_{x(t_0)}^{x(t_1)} = 0.$$

In other words, solutions extremize the action with fixed end points on each finite time interval.

This is not quite what one usually remembers from school[1], namely that solutions should *minimize* the action. The crucial point here is that the minimizing property holds only for short times. For instance, when looking at geodesics on the round sphere, the movement along a great circle ceases to be the shortest connection as soon as one comes across the antipodal point.

However, under certain circumstances there may well be action minimizing trajectories. The investigation of these minimal objects is one of the central topics of the present work. In fact, they do not always exist as genuine solutions, but they do so as invariant measures. This is the outcome of a theory by Mather and Mañé which generalizes Aubry–Mather theory from one to more degrees of freedom. In particular, there exist action minimizing measures with any prescribed "asymptotic direction" (described by a homological rotation vector). Associating to each such rotation vector the action of a minimal measure, we obtain the *minimal action* functional

$$\alpha : H_1(X, \mathbb{R}) \to \mathbb{R}.$$

By construction, the minimal action does not describe the full dynamics but concentrates on a very special part of it. The fundamental question is how much information about the original system is contained in the minimal action?

The first two chapters of this book provide the necessary background on Aubry–Mather and Mather–Mañé theories. In the following chapters, we investigate the minimal action in four different settings:

1. convex billiards
2. fixed points and invariant tori
3. Hofer's geometry
4. symplectic geometry.

We will see that the minimal action plays an important role in all four situations, underlining the significance of that particular variational principle.

1. Convex billards. Can one hear the shape of a drum? This was Kac' pointed formulation of the inverse spectral problem: is a manifold uniquely determined by its Laplace spectrum? We do know now that this is not true in full generality; for the class of smooth convex domains in the plane, however, this problem is still open.

We ask a somewhat weaker question for the length spectrum (i.e., the set of lengths of closed geodesics) rather than the Laplace spectrum, which is closely related to the previous one: how much geometry of a convex domain is determined by its length spectrum? The crucial observation is that one can consider this geometric problem from a more dynamical viewpoint. Namely,

[1] depending on the school, of course...

following a geodesic inside a convex domain that gets reflected at the boundary, is equivalent to iterating the so–called billiard ball map. The latter is a monotone twist map for which the minimal action is defined.

The main results from Chapter 3 can be summarized as follows.

Theorem 1. *For planar convex domains, the minimal action is invariant under continuous deformations of the domain that preserve the length spectrum.*

In particular, every geometric quantity that can be written in terms of the minimal action is automatically a length spectrum invariant.

In fact, the minimal action is a complete invariant and puts all previously known ones (e.g., those constructed in [2, 19, 63, 87]) into a common framework.

2. Fixed points and invariant tori. We consider a symplectic diffeomorphism in a neighbourhood of an elliptic fixed point in $\mathbb{R}^2$. If the fixed point is of "general" type, the symplectic character of the map makes it possible (under certain restrictions) to find new symplectic coordinates in which the map takes a particularly simple form, the so–called Birkhoff normal form. The coefficients of this normal form, called Birkhoff invariants, are symplectically invariant.

The Birkhoff normal form describes an asymptotic approximation, in the sense that it coincides with the original map only up to a term that vanishes asymptotically when one approaches the fixed point. In general, it does not give any information about the dynamics away from the fixed point.

The main result in this context introduces the minimal action as a symplectically invariant function that contains the Birkhoff normal form, but also reflects part of the dynamics near the fixed point.

Theorem 2. *Associated to an area–preserving map near a general elliptic fixed point there is the minimal action α, which is symplectically invariant.*

It is a local *invariant, i.e., it contains information about the dynamics near the fixed point. Moreover, the Taylor coefficients of the convex conjugate α^* are the Birkhoff invariants.*

Area–preserving maps near a fixed point occur as Poincaré maps of closed characteristics of three–dimensional contact flows. A particular example is given by the geodesic flow on a two–dimensional Riemannian manifold. In this case, the minimal action is determined by the length spectrum of the surface, and we obtain the following result.

Theorem 3. *Associated to a general elliptic closed geodesic on a two–dimensional Riemannian manifold there is the germ of the minimal action, which is a length spectrum invariant under continuous deformations of the Riemannian metric.*

The minimal action carries information about the geodesic flow near the closed geodesic; in particular, it determines its C^0–integrability.

In higher dimensions, we consider a symplectic diffeomorphism ϕ in a neighbourhood of an invariant torus Λ. If we assume that the dynamics on Λ satisfy a certain non–resonance condition, one can transform ϕ into Birkhoff normal form again. If this normal form is positive definite the map ϕ determines the germ of the minimal action α, and we will show again that the minimal action contains the Birkhoff invariants as Taylor coefficients of α^*.

3. Hofer's geometry. Whereas the first three settings had many features in common, the viewpoint here is quite different. Instead of looking at a single Hamiltonian system, we investigate all Hamiltonian systems on a symplectic manifold (M, ω) at once, collected in the Hamiltonian diffeomorphism group $\mathrm{Ham}(M, \omega)$. It is one of the cornerstones of symplectic topology that this group carries a bi–invariant Finsler metric d, usually called Hofer metric, which is constructed as follows.

Think of $\mathrm{Ham}(M, \omega)$ as infinite–dimensional Lie group whose Lie algebra consists of all smooth, compactly supported functions $H : M \to \mathbb{R}$ with mean value zero. Introduce any norm $\| \cdot \|$ on those functions that is invariant under the adjoint action $H \mapsto H \circ \psi^{-1}$. Then the Hofer distance of a diffeomorphism ϕ from the identity is defined as the infimum of the lengths of all paths in $\mathrm{Ham}(M, \omega)$ that connect ϕ to the identity:

$$d(\mathrm{id}, \phi) = \inf \Big\{ \int_0^1 \|H_t\| \,\mathrm{d}t \mid \varphi_H^1 = \phi \Big\}.$$

The problem is to choose the norm $\| \cdot \|$. The Hamiltonian system is determined by the first derivatives of H, but $\|\mathrm{d}H\|_{C^0}$, for instance, is not invariant under the adjoint action. It turns out that the oscillation norm

$$\| \cdot \| = \mathrm{osc} := \max - \min$$

is the right choice although it seems to have nothing to do with the dynamics. Loosely speaking, the Hofer metric generates a C^{-1}–topology and measures how much energy is needed to generate a given map.

The resulting geometry is far from being understood completely. This is due to the fact that, despite its simple definition, the Hofer distance is very hard to compute. After all, one has to take *all* Hamiltonians into account that generate the same time–1–map. A fundamental question concerns the relation between the Hofer geometry and dynamical properties of a Hamiltonian diffeomorphism: does the dynamical behaviour influence the Hofer geometry and, vice versa, can one infer knowledge about the dynamics from Hofer's geometry? Only little is known in this direction.

In Chap. 5, we take up this question for Hamiltonians on the cotangent bundle $T^*\mathbb{T}^n$ satisfying a Legendre condition. This leads to convex Lagrangians on $T\mathbb{T}^n$ for which the minimal action α is defined. On the other hand, the Hamiltonians under consideration are unbounded and do not fit into the framework of Hofer's metric. Therefore, we have to restrict them to a compact part of $T^*\mathbb{T}^n$, e.g., to the unit ball cotangent bundle $B^*\mathbb{T}^n$, but in such a way that we stay in the range of Mather's theory.

Let α denote the minimal action associated to a convex Hamiltonian diffeomorphism on $B^*\mathbb{T}^n$. Our main result in this context shows that the oscillation of α^*, which is nothing but $\alpha(0)$, is a lower bound for the Hofer distance. This establishes a link between Hofer's geometry of convex Hamiltonian mappings and their dynamical behaviour.

Theorem 4. *If $\phi \in Ham(B^*\mathbb{T}^n)$ is generated by a convex Hamiltonian then*

$$d(id, \phi) \geq \operatorname{osc} \alpha^* = \alpha(0).$$

4. Symplectic geometry. Consider the cotangent bundle $T^*\mathbb{T}^n$ with its canonical symplectic form $\omega_0 = \mathrm{d}\lambda$. Here, λ is the Liouville 1–form which is $y\,\mathrm{d}x$ in local coordinates (x, y). Suppose $H : T^*\mathbb{T}^n \to \mathbb{R}$ is a convex Hamiltonian. Because H is time–independent the energy is preserved under the corresponding flow, i.e., all trajectories lie on (fiberwise) convex $(2n-1)$–dimensional hypersurfaces $\Sigma = \{H = \text{const.}\}$. Of particular importance in classical mechanics are so–called KAM–tori. i.e., invariant tori carrying quasiperiodic motion. These are graphs over the base manifold $\mathbb{T}^n$, with the additional property that the symplectic form ω_0 vanishes on them; submanifolds with the latter property are called Lagrangian submanifolds.

We want to study symplectic properties of Lagrangian submanifolds on convex hypersurfaces. To do so, we observe that a Lagrangian submanifold possesses a Liouville class a_Λ, induced by the pull-back of the Liouville form λ to Λ. The Liouville class is invariant under Hamiltonian diffeomorphisms, i.e., it belongs to the realm of symplectic geometry. On the other hand, being a graph is certainly *not* a symplectic property. Our starting question in this context is as follows: is it possible to move a Lagrangian submanifold Λ on some convex hypersurface Σ by a Hamiltonian diffeomorphism inside the domain U_Σ bounded by Σ?

In a first part, we will see that, under certain conditions on the dynamics on Λ, it is impossible to move Λ at all; we call this phenomenon *boundary rigidity.* In fact, the Liouville class a_Λ already determines Λ uniquely.

Theorem 5. *Let Λ be a Lagrangian submanifold with conservative dynamics that is contained in a convex hypersurface Σ, and let K be another Lagrangian submanifold inside U_Σ. Then*

$$a_\Lambda = a_K \iff \Lambda = K.$$

What can happen if boundary rigidity fails? Surprisingly, even if it is possible to push Λ partly inside the domain U_Σ, it cannot be done completely. Certain pieces of Λ have to stay put, and we call them *non–removable intersections.* In the case where Σ is some distinguished "critical" level set, these non–removable intersections always contain an invariant subset with specific dynamical behaviour; this subset is the so–called Aubry set from Mather–Mañé theory. This result reveals a hidden link between aspects of symplectic geometry and Mather–Mañé theory in modern dynamical systems.

Finally, we come back to the somewhat annoying fact that the property of being a Lagrangian section is not preserved under Hamiltonian diffeomorphisms. For this, we consider

Theorem 6. *Let U be a (fiberwise) convex subset U of $T^*\mathbb{T}^n$. Then every cohomology class that can be represented as the Liouville class of some Lagrangian submanifold in U, can actually be represented by a Lagrangian* section *contained in U.*

So, from this rather vague point of view at least, Lagrangian sections actually do belong to symplectic geometry.

Furthermore, the above result allows *symplectic* descriptions of seemingly non–symplectic objects: the stable norm from geometric measure theory, and also our favourite, the minimal action.

Theorem 7. *The stable norm of a Riemannian metric g on $\mathbb{T}^n$, and the minimal action of a convex Lagrangian $L : T\mathbb{T}^n \to \mathbb{R}$, both admit a symplectically invariant description.*

This closes the circle for our investigation of the Principle of Least Action in geometry and dynamics.

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Contents

1

Aubry–Mather theory

The Principle of Least Action states that, for sufficiently short times, trajectories of a Lagrangian system minimize the action amongst all paths in configuration space with the same end points. If the time interval becomes larger, however, the Euler–Lagrange equations describe just critical points of the action functional; they may well be saddle points.

In the eighties, Aubry [5] and Mather [64] discovered independently that monotone twist maps on an annulus possess orbits of any given rotation number which minimize the (discrete) action with fixed end points on *all* time intervals. Roughly speaking, the rotation number of a geodesic describes the direction in which the geodesic, lifted to the universal cover, travels. Those minimal orbits turned out to be of crucial importance for a deeper understanding of the complicated orbit structure of monotone twist mappings.

Later, Mather [69] developed a similar theory for Lagrangian systems in higher dimensions. There was, however, an old example by Hedlund [41] of a Riemannian metric on $\mathbb{T}^3$, having only three directions for which minimal geodesics existed. Therefore, Mather's generalization deals with minimal invariant measures instead of minimal orbits.

A different approach was suggested by Mañé [62] who introduced a certain critical energy value at which the dynamics of a Lagrangian systems change. It turned out that this approach essentially contains Mather's theory, but in a more both geometrical and dynamical setting.

We will deal with these generalizations of Aubry–Mather theory to higher dimensions in Chap. 2.

1.1 Monotone twist mappings

Let

$$\mathbb{S}^1 \times (a, b) \subset \mathbb{S}^1 \times \mathbb{R} = T^*\mathbb{S}^1$$

be a plane annulus with $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, where we allow the cases $a = -\infty$ or $b = +\infty$ (or both). Given a diffeomorphism ϕ of $\mathbb{S}^1 \times (a, b)$ we consider a lift $\widetilde{\phi}$

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of ϕ to the universal cover $\mathbb{R} \times (a, b)$ of $\mathbb{S}^1 \times (a, b)$ with coordinates x, y. Since ϕ is a diffeomorphism, so is $\widetilde{\phi}$, and we have $\widetilde{\phi}(x+1, y) = \widetilde{\phi}(x, y) + (1, 0)$. In this section, we will always work with (fixed) lifts for which we drop the tilde again and keep the notation ϕ.

In the case when a or b is finite we assume that ϕ extends continuously to $\mathbb{R} \times [a, b]$ by rotations by some fixed angles:

$$\phi(x, a) = (x + \omega_-, a) \quad \text{and} \quad \phi(x, b) = (x + \omega_+, b). \tag{1.1}$$

The numbers $\omega_\pm$ are unique after we have fixed the lift. For simplicity, we set $\omega_\pm = \pm\infty$ if $a = -\infty$ or $b = \infty$.

Definition 1.1.1. *A monotone* twist map *is a C^1–diffeomorphism*

$$\begin{aligned} \phi : \mathbb{R} \times (a, b) &\to \mathbb{R} \times (a, b) \\ (x_0, y_0) &\mapsto (x_1, y_1) \end{aligned}$$

satisfying $\phi(x_0 + 1, y_0) = \phi(x_0, y_0) + (1, 0)$ as well as the following conditions:

1. *ϕ preserves orientation and the boundaries of $\mathbb{R} \times (a, b)$, in the sense that $y_1(x_0, y_0) \to a, b$ as $y_0 \to a, b$;*
2. *if a or b is finite ϕ extends to the boundary by a rotation, i.e., it satisfies* (1.1)*;*
3. *ϕ satisfies a monotone* twist condition
$$\frac{\partial x_1}{\partial y_0} > 0; \tag{1.2}$$
4. *ϕ is exact symplectic; in other words, there is a C^2–function h, called a* generating function *for ϕ, such that*
$$y_1 \,\mathrm{d}x_1 - y_0 \,\mathrm{d}x_0 = \mathrm{d}h(x_0, x_1). \tag{1.3}$$

The interval $(\omega_-, \omega_+) \subset \mathbb{R}$, which can be infinite, is called the twist interval *of ϕ.*

Remark 1.1.2. The twist condition (1.2) states that images of verticals are graphs over the x–axis; see Fig. 1.1. This implies that ϕ can be described in the coordinates x_0, x_1 rather than x_0, y_0. In other words, for every choice of x–coordinates x_0 and x_1 (corresponding to the configuration space), there are unique choices y_0 and y_1 for the y–coordinates (corresponding to the velocities) such that the image of (x_0, y_0) under ϕ is (x_1, y_1).

Remark 1.1.3. A generating function h for a twist map ϕ is defined on the strip

$$\{(\xi, \eta) \in \mathbb{R}^2 \mid \omega_- < \eta - \xi < \omega_+\}$$

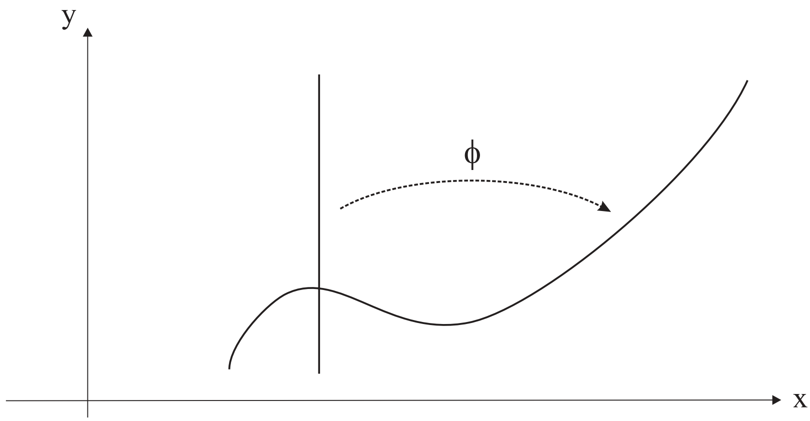


Fig. 1.1. The twist condition

and can be extended continuously to its closure. It is unique up to additive constants. Equation (1.3) is equivalent to the system

$$\begin{cases} \partial_1 h(x_0, x_1) = -y_0 \\ \partial_2 h(x_0, x_1) = y_1 \end{cases} \tag{1.4}$$

Here, the expression ∂_i denotes the partial derivative of a function with respect to the i–th variable. The equivalent of the twist condition (1.2) for a generating function is

$$\partial_1 \partial_2 h < 0. \tag{1.5}$$

Finally, a generating function satisfies the periodicity condition $h(\xi+1, \eta+1) = h(\xi, \eta)$.

Monotone twist maps are not as artificial as they might seem. They appear in a variety of situations, often unexpected and detected only by clever coordinate choices. In the following, we give a few examples. The reader my consult

Example 1.1.4. The simplest example is what is called an *integrable twist map* which, by definition, preserves the radial coordinate[1]. In this case, the property of being area–preserving implies that an integrable twist map is of the following form:

$$\phi(x_0, y_0) = (x_0 + f(y_0), y_0)$$

with $f' > 0$. Then the generating function (up to additive constants) is given by

[1] In the context of integrable Hamiltonian systems, this means that (x, y) are already the angle–action–variables.

$$h = h(x_1 - x_0),$$

with $h' = f^{-1}$; in other words, h is strictly convex.

Example 1.1.5. In some sense the "simplest" non–integrable monotone twist map is the so–called standard map

$$\phi : (x, y) \mapsto \big(x + y + \frac{k}{2\pi} \sin 2\pi x, y + \frac{k}{2\pi} \sin 2\pi x\big)$$

where $k \geq 0$ is a parameter. This map has been the subject of extensive analytical and numerical studies. Famous pictures illustrate the transition from integrability ($k = 0$) to "chaos" ($k \approx 10$).

Example 1.1.6. A particularly interesting class of monotone twist maps comes from planar convex billiards; we will deal with convex billiards in Chap. 3. The investigation of such systems goes back to Birkhoff [15] who introduced them as model case for nonlinear dynamical systems; for a modern survey see [101].

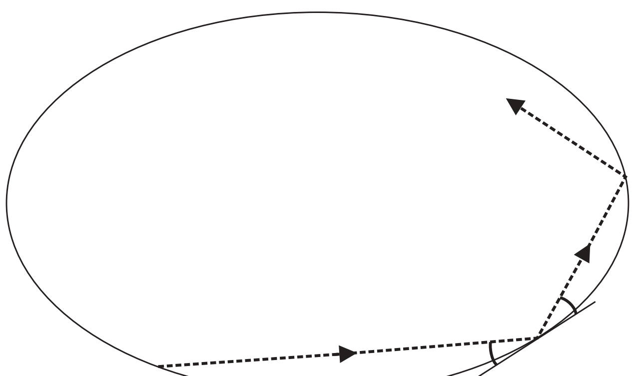


Fig. 1.2. The billard in a strictly convex domain

Given a strictly convex domain Ω in the Euclidean plane with smooth boundary $\partial\Omega$, we play the following game. Let a mass point move freely inside Ω, starting at some initial point on the boundary with some initial direction pointing into Ω. When the "billiard ball" hits the boundary, it gets reflected according to the rule "angle of incidence = angle of reflection"; see Fig. 1.2. The billiard map associates to a pair (point on the boundary, direction), respectively (s, ψ) = (arclength parameter divided by total length, angle with the tangent), the corresponding data when the points hits the boundary again. The lift of this map, which is then defined on $\mathbb{R} \times (0, \pi)$, is not a monotone twist map.

However, elementary geometry shows [101] that the map preserves the 2–form

$$\sin\psi \, \mathrm{d}\psi \wedge \mathrm{d}s = \mathrm{d}(-\cos\psi) \wedge \mathrm{d}s.$$

Hence the billiard map preserves the standard area form $\mathrm{d}x \wedge \mathrm{d}y$ in the new coordinates

$$(x, y) = (s, -\cos\psi) \in \mathbb{R} \times (-1, 1).$$

Moreover, if you increase the angle with the positive tangent to $\partial\Omega$ for the initial direction, the point where you hit $\partial\Omega$ again will move around $\partial\Omega$ in positive direction. This means that

$$\frac{\partial x_1}{\partial y_0} > 0,$$

so the billiard map in the new coordinates does satisfy the monotone twist condition.

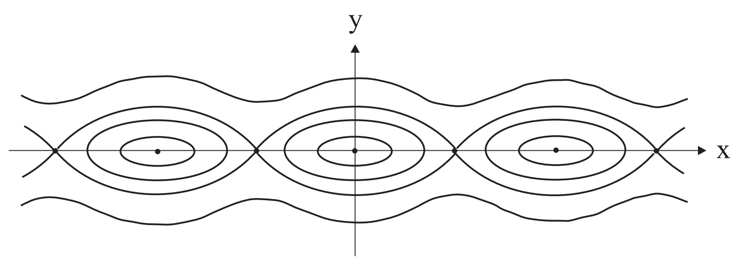


Fig. 1.3. The phase portrait of the mathematical pendulum

Example 1.1.7. Consider a particle moving in a periodic potential on the real line. According to Newton's Second Law, the motion of the particle is determined by the differential equation

$$\ddot{x}(t) = V'(x(t)).$$

This can be written as a Hamiltonian system

$$\begin{cases} \dot{x}(t) = \partial_y H(x(t), y(t)) \\ \dot{y}(t) = -\partial_x H(x(t), y(t)) \end{cases}$$

with the Hamiltonian $H(x, y) = y^2/2 - V(x)$. For small enough $t > 0$, we have

$$\begin{aligned} \frac{\partial x(t; x(0), y(0))}{\partial y(0)} &= \frac{\partial}{\partial y(0)} \int_0^t \dot{x}(\tau; x(0), y(0)) \, d\tau \\ &= \int_0^t \frac{\partial y(\tau; x(0), y(0))}{\partial y(0)} \, d\tau \\ &> 0. \end{aligned}$$

Therefore the time–t–map φ_H^t is a monotone twist map provided t is small. In fact, this holds true not only for Hamiltonians of the form "kinetic energy + potential energy", but for more general Hamiltonians which are fiberwise convex in the second variable (corresponding to the momentum).

A particular case is that of a mathematical pendulum where x is the angle to the vertical and $V'(x) = -\sin 2\pi x$. The phase portrait in $\mathbb{R} \times \mathbb{R}$, see Fig. 1.3, shows two types of invariant curves: closed circles around the stable equilibrium ("librational" circles), and curves homotopic to the real line above and below the separatrices ("rotational" curves).

Note that, by the monotone twist condition, an orbit $((x_i, y_i))_{i\in\mathbb{Z}}$ of a monotone twist map ϕ is completely determined by the sequence $(x_i)_{i\in\mathbb{Z}}$ via the relations

$$y_i = \partial_2 h(x_{i-1}, x_i) = -\partial_1 h(x_i, x_{i+1}).$$

Similarly, an arbitrary sequence $(\xi_i)_{i\in\mathbb{Z}}$ corresponds to an orbit of a monotone twist map ϕ if and only if

$$\partial_2 h(\xi_{i-1}, \xi_i) + \partial_1 h(\xi_i, \xi_{i+1}) = 0 \tag{1.6}$$

for all $i \in \mathbb{Z}$. Thus, on a formal level, orbits of a monotone twist mapping may be regarded as "critical points" of the discrete action "functional"

$$(\xi_i)_{i\in\mathbb{Z}} \mapsto \sum_{i\in\mathbb{Z}} h(\xi_i, \xi_{i+1})$$

on $\mathbb{R}^{\mathbb{Z}}$. This point of view leads to the following notion of minimal orbits.

1.2 Minimal orbits

Let $\phi : (x_0, y_0) \mapsto (x_1, y_1)$ be a monotone twist map with generating function $h(x_0, x_1)$. We have seen above that the ϕ–orbit of a point (x_0, y_0) is completely determined by the sequence (x_i) of the first coordinates. Moreover, an arbitrary sequence (ξ_i) corresponds to an orbit if, and only if, it satisfies the recursive relation (1.6). Loosely speaking, orbits are "critical points" of the action "functional"

$$(\xi_i)_{i\in\mathbb{Z}} \mapsto \sum_{i\in\mathbb{Z}} h(\xi_i, \xi_{i+1}).$$

In this section, we are interested in minima, i.e. in points which minimize the action.

This, of course, makes only sense if we restrict the action of a sequence $(\xi_i)_{i\in\mathbb{Z}}$ to finite parts. In analogy to the classical Principle of Least Action, we define minimal orbits in such a way that they minimize the action with the end points held fixed.

Definition 1.2.1. *Let h be a generating function of a monotone twist map ϕ. A sequence $(x_i)_{i\in\mathbb{Z}}$ with $\xi_i \in \mathbb{R}$ is called* minimal *if every finite segment minimizes the action with fixed end points, i.e., if*

$$\sum_{i=k}^{l-1} h(x_i, x_{i+1}) \leq \sum_{i=k}^{l-1} h(\xi_i, \xi_{i+1})$$

for all finite segments $(\xi_k, \ldots, \xi_l) \in \mathbb{R}^{l-k+1}$ with $\xi_k = x_k$ and $\xi_l = x_l$.

By (1.6), each minimal sequence $(x_i)_{i\in\mathbb{Z}}$ corresponds to a ϕ–orbit $((x_i, y_i))_{i\in\mathbb{Z}}$; these are called minimal orbits *of ϕ.*

Given an orbit (x_i, y_i) in $\mathbb{S}^1 \times (a, b)$, the twist map ϕ induces a circle mapping on the first coordinates x_i. This leads to the definition of the rotation number of an orbit of a monotone twist map.

Definition 1.2.2. *The* rotation number *of an orbit $((x_i, y_i))_{i\in\mathbb{Z}}$ of a monotone twist map is given by*

$$\omega := \lim_{|i|\to\infty} \frac{x_i}{i} = \lim_{|i|\to\infty} \frac{x_i - x_0}{i}$$

if this limit exists.

Example 1.2.3. The simplest orbits for which the rotation number always exists are periodic orbits, i.e., orbits $((x_i, y_i))_{i\in\mathbb{Z}}$ with

$$x_{i+q} = x_i + p$$

for all $i \in \mathbb{Z}$, where p, q are integers with $q > 0$. In order to have q as the minimal period one assumes that p and q are relatively prime. Then the rotation number is given by

$$\omega = \frac{p}{q}.$$

The questions arises whether there are orbits for a monotone twist map of any given rotation number in the twist interval. Actually, this is the core of Aubry–Mather theory, which yields an affirmative answer. The classical result in this context is a theorem by G.D. Birkhoff [15] who proved that monotone twist maps possess periodic orbits for each rational rotation number in their twist interval. Perhaps because monotone twist maps were not that popular in the mid-20th century, it took 60 years to generalize Birkhoff's result to all rotation numbers.

Theorem 1.2.4 (Birkhoff). *Let ϕ be a monotone twist map with twist interval (ω_-, ω_+), and $p/q \in (\omega_-, \omega_+)$ a rational number in lowest terms. Then ϕ possesses at least two periodic orbits with rotation number p/q.*

Proof. The proof is a nice illustration of the use of variational methods in the construction of specific orbits for monotone twist maps.

Consider the finite action functional

$$H(\xi_0, \ldots, \xi_q) := \sum_{i=0}^{q-1} h(\xi_i, \xi_{i+1})$$

on the set of all ordered $(q+1)$–tuples with

$$\xi_0 \le \xi_1 \le \ldots \le \xi_q = \xi_0 + p.$$

Since these tuples form a compact set, the continuous function H has a minimum, corresponding to a periodic orbit of the monotone twist map ϕ. What we need to show is that this minimum does not lie on the boundary, which consists of degenerate orbits of length less than q.

Suppose that there is a periodic orbit with

$$\xi_{j-1} < \xi_j = \xi_{j+1} < \xi_{j+2}$$

for some index j; the case of more than two equal values is treated analogously. Then the recursive relation (1.6) yields

$$\begin{aligned} \partial_2 h(\xi_{j-1}, \xi_j) + \partial_1 h(\xi_j, \xi_{j+1}) &= 0 \\ \partial_2 h(\xi_j, \xi_{j+1}) + \partial_1 h(\xi_{j+1}, \xi_{j+2}) &= 0 \end{aligned}$$

Since $\xi_j = \xi_{j+1}$, substracting the two equations gives

$$\partial_2 h(\xi_{j-1}, \xi_j) - \partial_2 h(\xi_j, \xi_j) + \partial_1 h(\xi_{j+1}, \xi_{j+1}) - \partial_1 h(\xi_{j+1}, \xi_{j+2}) = 0.$$

This can be written as

$$\partial_1 \partial_2 h(\eta_1, \xi_j)\,(\xi_{j-1} - \xi_j) + \partial_2 \partial_1 h(\xi_{j+1}, \eta_2)\,(\xi_{j+1} - \xi_{j+2}) = 0,$$

where η_1, η_2 are two intermediate values. But the left hand side is strictly negative, due to (1.6) and the assumptions, which is a contradiction. □

Birkhoff's theorem is sharp in the sense that, in general, one cannot expect more than two periodic orbits with a given rotation number. For example, in the elliptical billiard, there are precisely two 2–periodic orbits, corresponding to the two axes of symmetry.

1.3 The minimal action for monotone twist mappings

Of particular importance for the dynamics of a (projection of a) monotone twist map $\phi : \mathbb{S}^1 \times (a, b) \to \mathbb{S}^1 \times (a, b)$ are closed invariant curves. They fall into two classes: an invariant curve is either contractible or homotopically non-trivial. Lifted to the strip $\mathbb{R} \times (a, b)$, this means that we consider ϕ–invariant curves which are either closed or homotopic to $\mathbb{R}$.

Definition 1.3.1. *An* invariant circle *of a monotone twist map ϕ is an embedded, homotopically nontrivial, ϕ–invariant curve in $\mathbb{S}^1 \times (a, b)$, respectively, its lift to $\mathbb{R} \times (a, b)$.*

Example 1.3.2. Considering the phase space $\mathbb{R} \times \mathbb{R}$ of the mathematical pendulum (see Fig. 1.3), the librational circles around the stable equilibria are not invariant circles according to our definition. On the other hand, the rotational curves above and below the separatrices do represent invariant circles. Finally, the union of all the upper, respectively lower, separatrices also form (non–smooth) invariant circles.

It turns out that invariant circles of monotone twists maps cannot take any form. Indeed, another classical result by G.D. Birkhoff states that they must project injectively onto the base. More precisely, we have the following theorem.

Theorem 1.3.3 (Birkhoff). *Any invariant circle of a monotone twist map is the graph of a Lipschitz function.*

There are essentially two different proofs of this result. The original topological approach is indicated in [15, §44] and [16, §3]; precise, and even more general, proofs along this line can be found in [28, 42, 51, 66, 70]. The second approach [94] is different and more dynamical. We give a sketch of its main idea here and refer to [94] for details.

Proof ([94]). Assume, by contradiction, that there is an invariant circle Γ of a monotone twist map ϕ which is not a graph. Then we have a situation like that indicated in Fig. 1.4.

Let us apply ϕ once and see what happens to the area of the domain Ω_0. Since the preimage $\phi^{-1}(v_1)$ is a graph in view of the monotone twist condition, and since ϕ is area–preserving, the application of ϕ pushes more area into the fold, i.e., the area of Ω_1 is bigger than that of Ω_0.

Now iterate ϕ, and consider the domains Ω_n for $n \geq 1$. Each application lets the area of Ω_n grow:

$$|\Omega_n| > |\Omega_{n-1}| > \ldots > |\Omega_1| > |\Omega_0|.$$

On the other hand, everything takes place in a bounded domain because Γ is an invariant curve. Therefore, we conclude that $\sup_n |\Omega_n| < \infty$ which implies the areas of the additional pieces tend to zero:

$$\lim_{n \to \infty} |\Omega_n \setminus \Omega_{n-1}| = 0.$$

But it is easy to see that this means that Γ must have a point of self–intersection and, hence, is not embedded.

This contradiction proves the theorem. $\square$

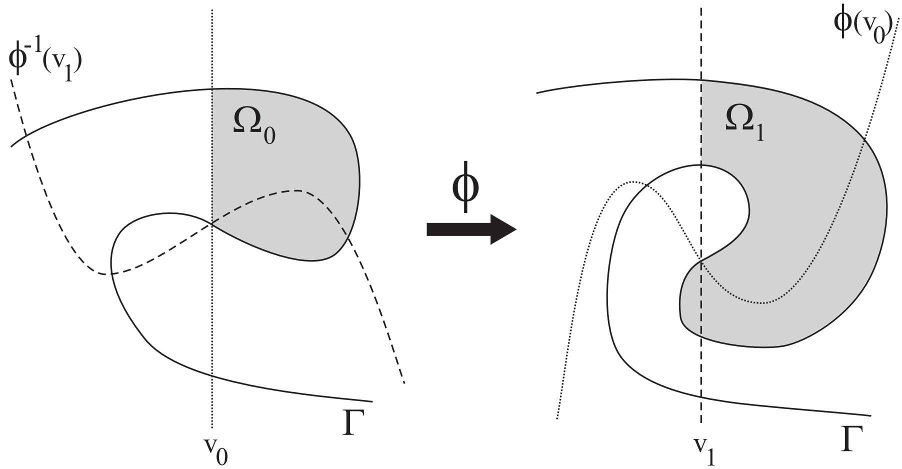


Fig. 1.4. Applying a monotone twist map in a non–graph situation

Let us return to the question whether there are orbits of *any* given rotation number for a monotone twist map. Theorem 1.2.4 asserts that there are always periodic orbits for a given rational rotation number in the twist interval. By taking limits of these orbits, one can construct also orbits of irrational rotation numbers. All of these orbits are minimal.

Minimal orbits resemble invariant circles in the sense that they, too, project injectively onto the base. In other words, minimal orbits lie on Lipschitz graphs. Moreover, if there happens to be an invariant circle, then every orbit on it is minimal.

The following theorem is the basic result in Aubry–Mather theory. The reader may consult [6, 34, 51, 72, 74] for more details.

Theorem 1.3.4. *A monotone twist map possesses minimal orbits for every rotation number in its twist interval; for rational rotation numbers there are always at least two periodic minimal orbits.*

Every minimal orbit lies on a Lipschitz graph over the x–axis. Moreover, if there exists an invariant circle then every orbit on that circle is minimal.

Remark 1.3.5. Theorem 1.3.4 remains true if one considers the more general setting of a monotone twist map on an invariant annulus $\{(x, y) \mid u_-(x) \leq y \leq u_+(x)\}$ between the graphs of two functions $u_\pm$; see [72].

From the existence of orbits of any given rotation number, we can build a function which will play a central role in our discussion. Namely, consider a monotone twist with generating function h. Then we associate to each ω in the twist interval the average h–action of some (and hence any) minimal orbit $((x_i, y_i))_{i\in\mathbb{Z}}$ having that rotation number ω.

Definition 1.3.6. *Let ϕ be a monotone twist map with generating function h and twist interval (ω_-, ω_+). Then the* minimal action *of ϕ is defined as the function $\alpha : (\omega_-, \omega_+) \to \mathbb{R}$ with*

$$\alpha(\omega) := \lim_{N\to\infty} \frac{1}{2N} \sum_{i=-N}^{N-1} h(x_i, x_{i+1}).$$

The minimal action can be seen as a "marked" Principle of Least Action: it gives the (average) action of action–minimizing orbits, together with the information to which topological type the corresponding minimal orbits belong. We wills see in Chap. 4 how this relates to the marked length spectrum of a Riemannian manifold.

Does the minimal action tell us anything about the dynamics of the underlying twist map? This question is central from the dynamical systems point of view. It turns out that, indeed, the minmal action does contain information about the dynamical behaviour of the twist map.

The following theorem lists useful analytical properties of the minimal action α.

Theorem 1.3.7. *Let ϕ be a monotone twist map, and α its minimal action. The the following holds true.*

1. *α is strictly convex; in particular, it is continuous.*
2. *α is differentiable at all irrational numbers.*
3. *If $\omega = p/q$ is rational, α is differentiable at p/q if and only if there is an ϕ–invariant circle of rotation number p/q consisting entirely of periodic minimal orbits.*
4. *If Γ_ω is an ϕ–invariant circle of rotation number ω then α is differentiable at ω with $\alpha'(\omega) = \int_{\Gamma_\omega} y\,dx$.*

Proof. Everything is well known and can be found in [72, 68], except perhaps for the precise value of $\alpha'(\omega)$ in the last part. This follows from the more general Thm. 2.1.24 and Rem. 2.1.7 in the next section. □

For later purposes, we need a certain continuity property of the minimal action as a functional. Namely, what happens with the minimal action if we perturb the monotone twist map? It turns out that, at least for perturbations of integrable twist maps, the minimal action behaves continuously. This is made precise in the next proposition.

Proposition 1.3.8. *Let h_0 be a generating function for an integrable twist map such that*

$$h_0(s) = c(s-\gamma)^k + \mathcal{O}((s-\gamma)^{k+1})$$

as $s \to \gamma$ with $c > 0$ and $k \geq 2$. Let h be a generating function for another (not necessarily integrable) twist map such that

$$h(\xi, \eta) = h_0(\eta - \xi) + \mathcal{O}((\eta - \xi - \gamma)^{k+m})$$

as $\eta - \xi \to \gamma$ *with* $2m \in \mathbb{N} \setminus \{0\}$.

Then the corresponding minimal actions α_0 *and* α *satisfy*

$$\alpha_0(\omega) = h_0(\omega),$$

as well as

$$\alpha(\omega) = \alpha_0(\omega) + \mathcal{O}((\omega - \gamma)^{k+m})$$

as $\omega \to \gamma$.

Proof. Let us first convince ourselves that $\alpha_0 = h_0$. This follows from the fact that all orbits of rotation number ω lie on the invariant circle $\mathbb{S}^1 \times \{(h_0')^{-1}(\omega)\}$ and have the same average action $h_0(\omega)$. Hence the minimal action $\alpha_0(\omega)$ is indeed $h_0(\omega)$.

For the continuity of the minimal action with respect to the generating function, we will use a monotonicity argument which is standard in the calculus of variations; compare also [8]. Let us consider the minimal action $\alpha = \lim_{N\to\infty} 1/2N \sum_{i=-N}^{N-1} h(x_i, x_{i+1})$, where (x_i) is h–minimal, i.e.,

$$h(x_i, x_{i+1}) \leq h(\xi_i, \xi_{i+1})$$

for all finite sequences (ξ_i) with the same end points. Note that the action of an arbitrary segment (not necessarily part of an orbit) is monotone in the generating function: if $h_1 \leq h_2$ then

$$\sum_i h_1(\xi_i, \xi_{i+1}) \leq \sum_i h_2(\xi_i, \xi_{i+1}).$$

Moreover, the minimality of a sequence (x_i) is defined by a minimization process over all sequences (ξ_i), a set which does not depend on the generating function h. Hence, not just the action, but also the minimal action is monotone in the generating function.

The monotonicity of the minimal action implies the second assertion. □

Later, we will apply this proposition when $\gamma = \omega_-$ is the lower boundary point of the twist interval. Note that in this case we may have $k = 3$, for instance, which would be forbidden if γ were a point in the twist interval because then h_0 would not fulfill the generating function condition $\partial_1\partial_2 h_0 = -h_0'' < 0$.

Finally, since α is a convex function by Thm. 1.3.7, it possesses a *convex conjugate* (or *Fenchel transform*) α^* defined by

$$\alpha^*(I) := \sup_\omega (\omega I - \alpha(\omega)). \tag{1.7}$$

Actually, α is strictly convex, so the supremum is a maximum, and α^* is a convex, real-valued C^1–function with

$$(\alpha^*)'(\alpha'(\omega)) = \omega$$

whenever $\alpha'(\omega)$ exists [90, Thm. 11.13]. Flat parts of α^* correspond to points of non–differentiability of α.[2]

[2] See [90] for any question about smooth or non–smooth convex analysis.

2

Mather–Mañé theory

It was well known that the theory of Aubry and Mather concerning action–minimizing *orbits* is valid only in two dimensions. For, there is a classical example by Hedlund [41] of a Riemannian metric on $\mathbb{T}^3$ such that minimal geodesics exist only in three directions. Hedlund's construction modifies the flat metric on $\mathbb{T}^3$ in such a way that there are three directions, corresponding to three disjoint "highway tunnels", along which the metric is very small, so that the particle can travel along these highways and gather almost no action. Hedlund shows that any minimal geodesic changes between the tunnels only finitely often. Therefore, the asymptotic directions of minimal geodesics are confined to the three tunnel directions.

Hedlund's example showed that any generalization of Aubry–Mather theory to higher dimensions could not deal with minimal orbits. Instead, Mather [69] developed a corresponding theory of action–minimizing invariant *measures* for positive definite Lagrangian systems. Later, Mañé [62] gave another approach using a so-called critical value. This value singled out the energy value at which certain dynamically relevant orbits appear. Essentially, these are two sides of one coin.

In this section, we will give an introduction to the relevant notions and results. For further details we refer to [21, 29, 72].

2.1 Mather's minimal action

The setting for Mather's generalization of the theory of minimal orbits to higher dimensions are convex Lagrangian (or Hamiltonian) systems on the tangent (or cotangent) bundle of a compact manifold; we will restrict ourselves to the case of the n–dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

K.F. Siburg: LNM 1844, pp. 15–35, 2004.

2.1.1 The minimal action for convex Lagrangians

For the convenience of the reader, we present a quick review of the classical Lagrangian calculus of variations; for details we refer to [29, 33]. We denote by x, p the canonical coordinates on the tangent bundle $T\mathbb{T}^n = \mathbb{T}^n \times \mathbb{R}^n$. Any C^2–function $L : \mathbb{S}^1 \times T\mathbb{T}^n \to \mathbb{R}$, the so-called Lagrangian, gives rise to the Euler–Lagrange flow φ_L on $T\mathbb{T}^n$, defined as follows.

The *action* of a C^1–curve $\gamma : [a, b] \to \mathbb{T}^n$ is defined as the integral

$$A(\gamma) := \int_a^b L(t, \gamma(t), \dot{\gamma}(t))dt.$$

Curves that extremize the action among all curves with the same end points are characterized by the *Euler–Lagrange equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial p}(t, \gamma(t), \dot{\gamma}(t)) = \frac{\partial L}{\partial x}(t, \gamma(t), \dot{\gamma}(t)) \tag{2.1}$$

for all $t \in [a, b]$. Equation (2.1) is equivalent to

$$\frac{\partial^2 L}{\partial p^2}(t, \gamma(t), \dot{\gamma}(t))\, \ddot{\gamma}(t) = \frac{\partial L}{\partial x}(t, \gamma(t), \dot{\gamma}(t)) - \frac{\partial^2 L}{\partial x \partial p}(t, \gamma(t), \dot{\gamma}(t))\, \dot{\gamma}(t). \tag{2.2}$$

If the Lagrangian satisfies the so-called *Legendre condition*

$$\det \frac{\partial^2 L}{\partial p^2} \neq 0,$$

then one can solve (2.2) for $\ddot{\gamma}$ and, therefore, define a time–dependent vector field $X_L(t, x, p) = ((x, p), (p, X_L^\pi(t, x, p))$ on $T\mathbb{T}^n$ such that the solutions of $\ddot{\gamma}(t) = X_L^\pi(t, \gamma(t), \dot{\gamma}(t))$ are precisely the curves satisfying the Euler–Lagrange equation (2.1). The vector field X_L is called the Euler–Lagrange vector field, and its flow is called the *Euler–Lagrange flow* φ_L. It turns out that φ_L is C^1, even if L is only C^2.

Definition 2.1.1. *A* convex Lagrangian *is a C^2–function*

$$L : \mathbb{S}^1 \times T\mathbb{T}^n \to \mathbb{R}$$

such that the following conditions hold.

1. *Restricted to every fiber $\{t\} \times T_x\mathbb{T}^n$, L is strictly convex; this means that L has fiberwise positive definite Hessian:*

$$\frac{\partial^2 L}{\partial p^2} > 0.$$

2. *L has fiberwise superlinear growth (with respect to some, and hence any, Riemannian metric on $\mathbb{T}^n$); this means that*

$$\lim_{|p|\to\infty} \frac{L(x,p)}{|p|} = \infty$$

uniformly in x.

3. *The Euler–Lagrange flow φ_L is complete, i.e., its solutions exist for all times.*

Example 2.1.2. A prime example of a flow generated by a convex Lagrangian is the geodesic flow on $\mathbb{T}^n$ with respect to some Riemannian metric, where one considers the free motion of a particle on $\mathbb{T}^n$. The Lagrangian is then given by

$$L(x,p) = \frac{1}{2}\,|p|_x^2.$$

If one adds a potential V on $\mathbb{T}^n$, the Lagrangian changes to

$$L(x,p) = \frac{1}{2}\,|p|_x^2 - V(x).$$

Remark 2.1.3. A Lagrangian is by no means uniquely defined by the Euler–Lagrange flow. Indeed, if L generates the flow φ_L, then also the new Lagrangian

$$L(x,p) - \nu_x(p),$$

where ν is any closed 1–form on $\mathbb{T}^n$, generates the same φ_L.

This can be seen as follows. The actions of a curve γ with respect to L and $L-\nu$ differ by the term $\int_\gamma \nu$. Since ν is closed, Stokes' Theorem implies that this term does not depend on the curve γ (in the same homotopy class). Therefore the actions differ only by some additive constant, and so the extremal curves are the same.

Note that for convex L, the new Lagrangian L_ν is also convex.

Let L be a convex Lagrangian. In the following, we will not deal with orbits of the Euler–Lagrange flow φ_L, but rather with invariant probability measures. To do so, we denote by $\mathcal{M}_L$ the set of φ_L–invariant probability measures on $T\mathbb{T}^n$. For $\mu \in \mathcal{M}_L$ we call

$$A(\mu) = \int L\,d\mu \qquad \in \mathbb{R}\cup\{+\infty\}$$

its *action*. To each $\mu \in \mathcal{M}_L$, one associates the linear functional

$$H^1(\mathbb{T}^n,\mathbb{R}) \to \mathbb{R} \quad , \quad [\nu] \mapsto \int \nu\,d\mu$$

where we view a 1-form ν as a function on $T\mathbb{T}^n$ that is linear on the fibers. By duality, there is a unique class $\rho(\mu) \in H_1(\mathbb{T}^n,\mathbb{R})$ such that

$$\int \nu \, d\mu = \langle [\nu], \rho(\mu) \rangle \tag{2.3}$$

for all $[\nu] \in H^1(\mathbb{T}^n, \mathbb{R})$.

Definition 2.1.4. *Let $\mu \in \mathcal{M}_L$ be an invariant measure for a convex Lagrangian L. Then the class $\rho(\mu) \in H_1(\mathbb{T}^n, \mathbb{R})$, defined by* (2.3)*, is called the* rotation vector *of μ.*

Remark 2.1.5. The rotation vector of an invariant measure is related to Schwartzman's asymptotic cycles [91]; see [21].

In analogy to Aubry–Mather theory in two dimensions, we want to minimize the action of all invariant measures having the same rotation vector. Although the tangent bundle of $\mathbb{T}^n$ is not compact, this can be dealt with by taking its one point compactification, adding a point at infinity; see [69]. Then $\mathcal{M}_L$ becomes compact with respect to the vague (weak*) topology [14], and we actually can minimize the action over the set of invariant probability measures having a given rotation vector.

Definition 2.1.6. *Let L be a convex Lagrangian. Then the function*

$$\begin{aligned} \alpha \, : \, H_1(\mathbb{T}^n, \mathbb{R}) &\to \mathbb{R} \\ h &\mapsto \min\{A(\mu) \mid \mu \in \mathcal{M}_L, \rho(\mu) = h\} \end{aligned}$$

is called the minimal action *of L.*

Any invariant measure $\mu \in \mathcal{M}_L$ realizing this minimum, i.e. with $A(\mu) = \alpha(\rho(\mu))$, is called a minimal measure. *For a fixed rotation vector $h \in H_1(\mathbb{T}^n, \mathbb{R})$, the set of all minimal measures with $\rho(\mu) = h$ is denoted by $\mathcal{M}_h$.*

Remark 2.1.7. In the case of one degree of freedom ($n = 1$), the theory of Mather–Mañé reproduces the discrete Aubry–Mather theory from Chap. 1. To see this, one uses the result by Moser [78] that every monotone twist map on the cylinder is the time–1–map of a convex Lagrangian; see also [93]. Then it is shown in [67] that the minimal action $\alpha(\rho(\mu))$ in the continuous setting considered here is, perhaps after adding a constant, the same as the minimal action $\alpha(\omega)$ in the discrete framework of Aubry–Mather theory where $\rho(\mu) = \omega$. Hence we need not distinguish between the two.

Remark 2.1.8. The relation between minimizing measures and globally minimizing orbits is quite delicate, and we refer to [21] for details. We mentioned Hedlund's example [41] showing that minimal orbits for an arbitrary rotation vector need not always exist. At least, every trajectory that lies in the union of all supports of minimal measures in $\mathcal{M}_h$ minimizes the action among all curves in the universal cover $\mathbb{R}^n$ with the same end points [69, Prop. 3]. The dynamics on the set of minimizing trajectories is not limited to any particular behaviour—it can be as complicated as that of any vector field on the base manifold [60].

Let us consider the minimal action α. Recall from Thm. 1.3.7 that, in the two-dimensional discrete setting of Aubry–Mather theory, the minimal action is a strictly convex function. We want to prove a similar result for the higher dimensional case.

Proposition 2.1.9. *The minimal action* $\alpha : H_1(\mathbb{T}^n, \mathbb{R}) \to \mathbb{R}$ *is a convex, superlinear function.*

Proof. Let $h_1, h_2 \in H_1(\mathbb{T}^n, \mathbb{R})$ and $\lambda \in [0, 1]$. Choose minimal measures $\mu_1, \mu_2 \in \mathcal{M}_L$ such that $\rho(\mu_i) = h_i$. Then the convex combination

$$\mu := \lambda\mu_1 + (1 - \lambda)\mu_2$$

lies in $\mathcal{M}_L$ and has rotation vector $\rho(\mu) = \lambda h_1 + (1-\lambda)h_2$. Since both μ_1 and μ_2 are minimal, we conclude that

$$\alpha(\lambda h_1 + (1 - \lambda)h_2) \leq A(\mu) = \lambda\alpha(h_1) + (1 - \lambda)\alpha(h_2),$$

which proves the convexity of α.

As for the superlinearity, we refer to [69] or [29][Thm. 4.4.5]. □

Remark 2.1.10. In contrast to the two–dimensional case, the function α need not be strictly convex.

As a convex function, α possesses a convex conjugate

$$\alpha^* \,:\, H^1(\mathbb{T}^n, \mathbb{R}) \to \mathbb{R} \tag{2.4}$$

$$c \mapsto \sup_{h \in H_1} \left(\langle c, h\rangle - \alpha(h)\right) \tag{2.5}$$

Since α is superlinear, the supremum is a maximum and attained at $h_c \in H_1(\mathbb{T}^n, \mathbb{R})$ if, and only if,

$$\alpha(h) \geq \alpha(h_c) + \langle c, h - h_c\rangle$$

for all h, in other words, if c is a subgradient of α at h_c; compare, for instance, [90, 29]. We arrive at the following equivalent formulations for the minimality of a measure μ:

- there exists a homology class $h \in H_1(\mathbb{T}^n, \mathbb{R})$, namely the rotation vector $\rho(\mu)$, such that μ minimizes the action $\int L \, d\mu$ amongst all measures in $\mathcal{M}_L$ with rotation vector h;
- there exists a cohomology class $c = [\nu] \in H^1(\mathbb{T}^n, \mathbb{R})$, namely any subgradient of α at $\rho(\mu)$, such that μ minimizes $\int L - \nu \, d\mu$ amongst all measures in $\mathcal{M}_L$.

Note that $L - \nu$ is again a convex Lagrangrian and generates the same flow as L because ν is closed. Therefore, $\mathcal{M}_{L-\nu} = \mathcal{M}_L$; see Rem. 2.1.3.

Let us continue with the idea to prove results, analogous to those in Aubry–Mather theory, in the more general setting of Mather's theory of minimal measures. Recall that Thm. 1.3.4 stated that minimal orbits of monotone twist maps always lie on Lipschitz graphs. Thus, one is lead to the conjecture that the supports of minimal measures (corresponding to minimal orbits) should lie on Lipschitz graphs over $\mathbb{T}^n$ (seen as the zero section in $T\mathbb{T}^n$).

In fact, this conjecture is true. The following is Mather's so-called Lipschitz Graph Theorem from [69]; see also [21].

Theorem 2.1.11. *For every $h \in H_1(\mathbb{T}^n, \mathbb{R})$, the union of the supports of all minimal measures in $\mathcal{M}_h$ lies on a Lipschitz graph over $\mathbb{T}^n$. Moreover, the Lipschitz constant depends only on the Lagrangian L and not on the rotation vector h.*

Important dynamical objects for twist maps are invariant circles; in higher dimensions, the corresponding objects are invariant tori. We know from Thm. 1.3.4 that orbits on invariant circles are automatically minimal. What is the corresponding result in higher dimensions? We point out that invariant tori of convex Lagrangian systems are only shown to be graphs under certain assumptions on their dynamics; see [10] for a generalization of Birkhoff's Theorem 1.3.3 to higher dimensions.

In order to deal with invariant tori, it is convenient to reformulate everything in the Hamiltonian, rather than in the Lagrangian, framework. Given a convex Lagrangian $L : \mathbb{S}^1 \times T\mathbb{T}^n \to \mathbb{R}$, the so-called *Legendre transformation*

$$\begin{aligned} \ell \,:\, \mathbb{S}^1 \times T\mathbb{T}^n &\to \mathbb{S}^1 \times T^*\mathbb{T}^n \\ (t,x,p) &\mapsto (t,x,y := \partial_p L) \end{aligned} \tag{2.6}$$

is a diffeomorphism between the tangent and the cotangent bundle. It yields the *convex Hamiltonian* $H : \mathbb{S}^1 \times T^*\mathbb{T}^n \to \mathbb{R}$ defined by

$$H(t,x,y) := \langle y, p \rangle - L(t,x,p)|_{p=(\partial_p L)^{-1}(y)}.$$

The Hamiltonian H gives rise to the *Hamiltonian flow* φ_H on the cotangent bundle via the Hamiltonian equations, written in local coordinates as

$$\begin{cases} \dot{x}(t) = \partial_y H(t, x(t), y(t)) \\ \dot{y}(t) = -\partial_x H(t, x(t), y(t)) \end{cases} \tag{2.7}$$

Then the Legendre transformation provides a conjugation between the Hamiltonian flow φ_H on $T^*\mathbb{T}^n$ and the Euler–Lagrange flow φ_L on $T\mathbb{T}^n$. We refer to [33, 21] for more details.

Given a Hamiltonian flow φ_H on $T^*\mathbb{T}^n$, we denote its time–t–map by

$$\varphi_H^t : T^*\mathbb{T}^n \to T^*\mathbb{T}^n.$$

This yields a one–to–one correspondence between φ_L–invariant probability measures and φ_H– or φ_H^1–invariant ones. For simplicity, we do not introduce three different notations but write μ for any of those. Likewise, we define the minimal action associated to a convex Hamiltonian H to be that associated to L and write α in either case. We say that a φ_H– or φ_H^1–invariant probability measure is minimal if its φ_L–invariant counterpart is.

2.1.2 A bit of symplectic geometry

The Hamiltonian viewpoint is the viewpoint of symplectic geometry. Let us recall a few notions; see [73] for a comprehensive introduction to symplectic geometry.

Definition 2.1.12. *A* symplectic form ω *on a manifold* M *is a closed nondegenerate 2–form. A* symplectic manifold (M, ω) *is a manifold* M*, equipped with a symplectic form* ω*.*

Example 2.1.13. The $2n$–dimensional Euclidean space $\mathbb{R}^{2n}$, together with the so–called canonical symplectic form

$$\omega_0 := \mathrm{d}y \wedge \mathrm{d}x = \sum_{i=1}^{n} \mathrm{d}y_i \wedge \mathrm{d}x_i,$$

is called the *standard symplectic space.* Note that the dimension of a symplectic manifold must always be even in view of the nondegeneracy condition on the 2-form ω.

Example 2.1.14. An important example of a symplectic manifold is the cotangent bundle T^*X of an n–dimensional manifold X. It carries a canonical symplectic form $\omega = \mathrm{d}\lambda$ that is not just closed but even exact. Here, the 1-form λ is the so–called *Liouville form* which, in local coordinates, is given by

$$\lambda := y\,\mathrm{d}x = \sum_{i=1}^{n} y_i\,\mathrm{d}x_i.$$

This local definition admits a global interpretation as follows. Let

$$\theta : T^*X \to X$$

be the canonical projection, and $\xi \in T_{(x,y)}T^*X$. Then

$$\lambda_{(x,y)}(\xi) = (\theta^* y_x)(\xi).$$

Of particular interest in symplectic geometry are submanifolds $\Lambda \subset M$ of s symplectic manifold (M, ω) on which the symplectic form vanishes:

$$\omega|_{T\Lambda} = 0.$$

Such submanifolds are called *isotropic.* It follows from the nondegeneracy of ω that $\dim \Lambda \leq 1/2 \dim M$ for isotropic submanifolds Λ.

Definition 2.1.15. *A* Lagrangian submanifold Λ *of a symplectic manifold* (M, ω) *is an isotropic manifold of maximal dimension; in other words, we have*

$$\dim \Lambda = \frac{1}{2} \dim M \quad \textit{and} \quad \omega|_{T\Lambda} = 0.$$

Example 2.1.16. In the standard symplectic space, the submanifold $\{(x,y) \in \mathbb{R}^{2n} \mid y = 0\}$ is a Lagrangian submanifold, whereas $\{(x,y) \in \mathbb{R}^{2n} \mid x = 0\}$ is not.

Example 2.1.17. Let ν be a 1–form on some manifold X. Then the graph

$$\operatorname{gr} \nu := \{(x, \nu_x) \mid x \in X\}$$

is a Lagrangian submanifold of $(T^*X, \mathrm{d}\lambda)$ if, and only if, the 1–form ν is closed. Such a Lagrangian manifold, which projects injectively onto the base, is called a *Lagrangian graph* or *Lagrangian section.*

In our case where $M = T^*\mathbb{T}^n$, any Lagrangian submanifold that is diffeomorphic to $\mathbb{T}^n$ is called a *Lagrangian torus.* For instance, if $n = 1$, any circle on the cylinder is a Lagrangian torus (or circle, rather).

We want to define Hamiltonian flows on symplectic manifolds. To do so, let $H : \mathbb{S}^1 \times M \to \mathbb{R}$ be a time–periodic Hamiltonian on some symplectic manifold (M, ω), and denote by $H_t : M \to \mathbb{R}$ the function for fixed t.

Definition 2.1.18. *The Hamiltonian vector field X_H on M associated to a Hamiltonian H is defined by*

$$i_{X_H}\omega = -\mathrm{d}H_t,$$

where $i_{X_H}\omega := \omega(X_H, \cdot)$ is the usual contraction of a form by a vector field.

Example 2.1.19. If $(M, \omega) = (\mathbb{R}^{2n}, \omega_0)$ is the standard symplectic space then the Hamiltonian vector field is given by

$$X_H(x,y) = J\nabla H_t(x,y),$$

where J is the $2n \times 2n$–matrix

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In other words, we arrive at our familiar system (2.7).

The invariance group of a symplectic manifold consists of all diffeomorphisms that leave the symplectic form invariant.

Definition 2.1.20. *A map $\phi : M \to M$ of a symplectic manifold (M, ω) is called symplectic if it preserves the symplectic form ω:*

$$\phi^*\omega = \omega.$$

Example 2.1.21. Certainly, the identity is symplectic. More generally, every time–t–map φ_H^t of a Hamiltonian flow is symplectic since, by Cartan's formula for the Lie derivative, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_H^t)^*\omega = L_{X_H}\omega = \mathrm{d}i_{X_H}\omega + i_{X_H}\mathrm{d}\omega = \mathrm{d}(-\mathrm{d}H_t) = 0.$$

Example 2.1.22. On a cotangent bundle T^*X with coordinates (x, y) and canonical symplectic form (see Ex. 2.1.14), we have the symplectic shift mapping

$$(x, y) \mapsto (x, y - \nu)$$

where ν is some closed 1–form on X.

2.1.3 Invariant tori and the minimal action

Let us return to our original setting. We know from Thm. 1.3.4 that invariant circles of monotone twist maps carry minimal orbits. In higher dimensions, a similar statement is true. Namely, let $\phi = \varphi_H^1$ be generated by a convex Hamiltonian on $\mathbb{S}^1 \times T^*\mathbb{T}^n$, and suppose that ϕ possesses an invariant Lagrangian torus Λ which is a graph. This situation occurs, for instance, in KAM–theory where one considers small perturbations of convex, completely integrable Hamiltonian systems.

Definition 2.1.23. *Consider a cotangent bundle $\theta : T^*X \to X$ with its canonical symplectic form $\omega = \mathrm{d}\lambda$. We denote by $\mathcal{L}$ the class of all Lagrangian submanifolds of T^*X which are Lagrangian isotopic to the zero section $\mathcal{O}$.*

Given $\Lambda \in \mathcal{L}$, the natural projection $\theta|_\Lambda : \Lambda \to X$ induces an isomorphism between the cohomology groups $H^1(X, \mathbb{R})$ and $H^1(\Lambda, \mathbb{R})$. The preimage $a_\Lambda \in H^1(X, \mathbb{R})$ of $[\lambda|_\Lambda] \in H^1(\Lambda, \mathbb{R})$ under this isomorphism is called the Liouville class *of Λ.*

The next theorem, firstly, says that Λ consists of supports of minimal measures and, secondly, shows that the Liouville class of Λ is a subgradient of the minimal action. Recall that a vector $v \in \mathbb{R}^n$ is a *subgradient* of a function $f : \mathbb{R}^n \to \mathbb{R}$ at $x \in \mathbb{R}^n$ if

$$f(y) \geq f(x) + \langle v, y - x \rangle$$

for all $y \in \mathbb{R}^n$. If we have a strict inequality for all $y \neq x$, we say that v is a subgradient with only one point of tangency. For instance, if f is differentiable at x then, of course, its gradient $\nabla f(x)$ is its unique subgradient at x. See [90] for more details.

Theorem 2.1.24. *Let $\phi = \varphi_H^1$ be generated by a convex Hamiltonian H on $\mathbb{S}^1 \times T^*\mathbb{T}^n$. Suppose that ϕ possesses an invariant Lagrangian torus Λ in $(T^*\mathbb{T}^n, d\lambda)$ such that Λ is homologous to the zero section and $\phi|_\Lambda$ is conjugated to a translation on $\mathbb{T}^n$ by some fixed vector ρ.*

Then every ϕ–invariant probability measure with support in Λ is minimal, and $a_\Lambda \in H^1(\mathbb{T}^n, \mathbb{R})$ is a subgradient of the minimal action α of H at ρ with only one point of tangency. Vice versa, every minimal measure of rotation vector ρ has support in Λ.

We point out that an observation by Herman [43, Prop. 3.2] shows that the condition on Λ being Lagrangian can be dropped if the vector $(1, \rho)$ is rationally independent, e.g. for invariant KAM–tori; in this case the minimal measure supported on Λ is unique.

Proof. We proceed in three steps and reduce each to the previous one. First of all, by a higher-dimensional version of Birkhoff's Theorem [10] the tori $\varphi_H^t(\Lambda)$ are graphs for all $t \in [0, 1]$; for our assumption on $\varphi|_\Lambda$ says, in particular, that $\varphi|_\Lambda$ preserves a measure which is positive on open sets (cf. [10, Prop. 1.2.(ii)]). Note that, as a Lagrangian graph, $\varphi_H^t(\Lambda)$ is the graph of a closed 1-form ν_t; by invariance, $\nu_0 = \nu_1$.

Case 1: Our starting point is the simplest possible, where $\Lambda = \mathcal{O}$ is the zero section and remains invariant under the flow, i.e.

$$\nu_t = 0$$

for all t. Then

$$0 = y(t) = \partial_p L(t, x(t), \dot{x}(t))$$

and

$$0 = \frac{d}{dt}\partial_p L(t, x(t), \dot{x}(t)) = \partial_x L(t, x(t), \dot{x}(t))$$

for all orbits starting (and hence lying) on $\mathcal{O}$. Note that $\ell^{-1}\mathcal{O}$, the preimage of $\mathcal{O}$ under the Legendre transformation, will depend on t unless $\dot{x}(t) = \rho$ for all t.

In any case, we have $\nabla L_t|_{\ell^{-1}\mathcal{O}} = 0$ which, by convexity of L, implies that

$$L_t(x, p) = \min L_t \iff (x, p) \in \ell^{-1}\mathcal{O}.$$

Consequently, an invariant measure $\mu \in \mathcal{M}_L$ is *globally* minimizing if, and only if, its support lies in $\ell^{-1}\mathcal{O}$. Since all orbits in $\mathcal{O}$ have rotation vector ρ we see that for any such μ we have

$$\int L \, d\mu = \int_0^1 \min L_t \, dt = \alpha(\rho).$$

In addition, $\alpha(h) > \alpha(\rho)$ if $h \neq \rho$, so $0 = a_\mathcal{O}$ is a subgradient of α at ρ with only one point of tangency.

Case 2: Next we consider the case when Λ is still the zero section but does not stay invariant under the flow; more precisely, we assume that

$$\nu_t = dS_t$$

for some function S_t on $\mathbb{T}^n$ with $\nu_0 = \nu_1$. (S_t is a generating function in the simplest case where the Lagrangian $\varphi_H^t(\mathcal{O})$ is a graph.)

We define the new Hamiltonian

$$K(t, x, y) = \partial_t S_t(x) + H(t, x, y + dS_t(x)).$$

It is convex, and if we write $\varphi_H^t(x,y) = (x(t), y(t))$ the transformation law of Hamiltonian vector fields yields

$$\varphi_K^t(x,y) = \varphi_{\partial_t S_t}^t \circ \varphi_H^t(x,y) = (x(t), y(t) - dS_t(x(t))).$$

Therefore, $\varphi_K^1 = \varphi_H^1 = \varphi$ and $\varphi_K^t(x,0) = (x(t), 0)$.

Now we are in the first case with H replaced by K. This changes the (minimal) action only by an additive constant [67], and the same conclusions hold as before.

Case 3: In the general case, we apply the symplectic shift

$$(x,y) \mapsto (x, y - \nu_0).$$

This maps Λ onto the zero section $\mathcal{O}$ and the 1–form λ onto $\widetilde{\lambda} = \lambda - \lambda|_\Lambda$; the new flow $\varphi_{\widetilde{H}}^t$ maps $\mathcal{O}$ onto the graph of $\widetilde{\nu}_t = \nu_t - \nu_0$. A generating function for $\varphi_{\widetilde{H}}^t$ is given by

$$S_t(x) = \int_0^t -\widetilde{H}(s, \varphi_{\widetilde{H}}^s \circ (\varphi_{\widetilde{H}}^t)^{-1}(x, \nu_t(x)))\, ds.$$

Thus we are back in Case 2, but this time with with a different 1-form $\widetilde{\lambda}$ instead of λ. It is shown in [67] that the actions behave like

$$\widetilde{A}(\mu) = A(\mu) + \langle [\widetilde{\lambda} - \lambda], \rho(\mu) \rangle = A(\mu) - \langle a_\Lambda, \rho(\mu) \rangle.$$

From Case 2 we know that every μ with support in Λ minimizes $\widetilde{A}(\mu)$ among all measures with $\rho(\mu) = \rho$. But under this constraint the correction term $\langle a_\Lambda, \rho \rangle$ is a mere constant, so μ minimizes $A(\mu)$, too. Moreover, 0 is a subgradient of $\widetilde{\alpha}(h) = \alpha(h) - \langle a_\Lambda, h \rangle$ at ρ with only one point of tangency. That means that

$$\alpha(h) \geq \alpha(\rho) + \langle a_\Lambda, h - \rho \rangle$$

with equality only for $h = \rho$, so a_Λ is a subgradient of α at ρ with only one point of tangency.

This finishes the proof of the theorem. $\square$

Corollary 2.1.25. *If, under the assumptions of Theorem 2.1.24, the invariant torus Λ is invariant under the flow of H (and not just its time–1–map) then*

$$\alpha^*(a_\Lambda) = \int_0^1 H(t, \cdot)|_\Lambda \, dt.$$

Proof. Theorem 2.1.24 implies that $\alpha^*(a_\Lambda) = \langle a_\Lambda, \rho \rangle - \alpha(\rho) = \int \lambda|_\Lambda - L\, d\mu$ for every invariant measure μ supported in Λ. Since Λ is invariant under the Hamiltonian flow, the function $H(t, \cdot)$ is constant on Λ with $H = \lambda|_\Lambda - L$. $\square$

Remark 2.1.26. If H is autonomous, Corollary 2.1.25 determines the energy level of invariant tori with given $\alpha^*(a_\Lambda)$; compare [25].

2.2 Mañé's critical value

Another approach to a generalization of Aubry–Mather theory to higher dimensions was suggested by Mañé [62]. Its main idea is to single out a certain energy level at which a significant change of the dynamical behaviour takes place. This produces a "critical" energy value for each convex Lagrangian. It turns out that this value is the minimum of the actions of *all* invariant measures in $\mathcal{M}_L$, and that one can recover Mather's minimal action from it (and vice versa).

2.2.1 The critical value for convex Lagrangians

Let $L : T\mathbb{T}^n \to \mathbb{R}$ be a time-independent convex Lagrangian on the tangent bundle of the n–torus. Let $\ell : T\mathbb{T}^n \to T^*\mathbb{T}^n$ be the Legendre transformation, and $H : T^*\mathbb{T}^n \to \mathbb{R}$ the Hamiltonian corresponding to L. The push–forward of the Euler–Lagrange flow φ_L on $T\mathbb{T}^n$ by the Legendre transformation is the Hamiltonian flow φ_H on $T^*\mathbb{T}^n$ with respect to the canonical symplectic structure on $T^*\mathbb{T}^n$; see Sect. 2.1.2. The *energy* of L is the function $E : T\mathbb{T}^n \to \mathbb{R}$ defined by

$$E(x,p) := \langle \frac{\partial L}{\partial p}(x,p), p\rangle - L(x,p) = H(\ell(x,p)).$$

It is a first integral of the Euler–Lagrange flow φ_L.

Recall that a curve $\gamma : [a,b] \to \mathbb{T}^n$ is called *absolutely continuous* if for every $\epsilon > 0$ there exists $\delta > 0$ so that for each finite collection of pairwise disjoint open intervals (s_i, t_i) in $[a,b]$ of total length less than δ one has $\sum_i \operatorname{dist}(\gamma(t_i), \gamma(s_i)) < \epsilon$; here dist is any Riemannian distance on $\mathbb{T}^n$. As before, the action of an absolutely continuous curve $\gamma : [a,b] \to \mathbb{T}^n$ is defined by

$$A_L(\gamma) := \int_a^b L(\gamma(t), \dot\gamma(t))\, dt.$$

We keep the subscript L in order to distinguish between the actions for different Lagrangians.

Given two points $x_1, x_2 \in \mathbb{T}^n$ and some $T > 0$, denote by $\mathcal{C}_T(x_1, x_2)$ the set of absolutely continuous curves $\gamma : [0,T] \to \mathbb{T}^n$ with $\gamma(0) = x_1$ and $\gamma(T) = x_2$. For each $k \in \mathbb{R}$, we define

$$\Phi_k(x_1, x_2; T) := \inf\{A_{L+k}(\gamma) \mid \gamma \in \mathcal{C}_T(x_1, x_2)\}$$

as the infimum of the $(L+k)$–actions over all curves connecting x_1 and x_2 in time T.

Definition 2.2.1. *Let $L : T\mathbb{T}^n \to \mathbb{R}$ be a convex Lagrangian. Then the* action potential $\Phi_k : \mathbb{T}^n \times \mathbb{T}^n \to \mathbb{R} \cup \{-\infty\}$ *of L is defined as*

$$\begin{aligned} \Phi_k \ : \ \mathbb{T}^n \times \mathbb{T}^n &\to \mathbb{R} \cup \{-\infty\} \\ (x_1, x_2) &\mapsto \inf_{T>0} \Phi_k(x_1, x_2; T) \end{aligned}$$

The critical value *of L is given by*

$$c(L) := \inf\{k \in \mathbb{R} \mid \Phi_k(x,x) > -\infty \textit{ for some } x \in \mathbb{T}^n\}.$$

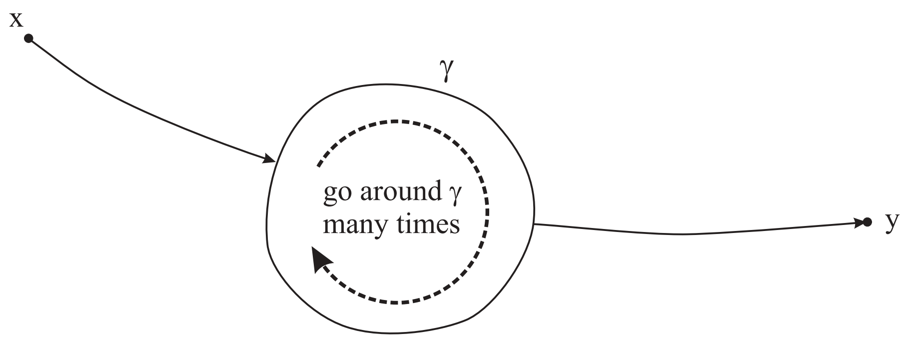


Fig. 2.1. If $A_{L+k}(\gamma) < 0$ then $\Phi_k(x,y) = -\infty$

Remark 2.2.2. Observe that if there is some closed curve γ with $A_{L+k}(\gamma) < 0$ then immediately $\Phi_k(x,y) = -\infty$ for all x, y. This follows by considering the curve going from x to some point on γ, going around as many times as one wishes (gathering as much negative action as one wants), and finally going to y; see Fig. 2.1. Thus, we could replace the word "some" in the definition of $c(L)$ by "all". Since the action potential is monotone in k, we then have

$$c(L) = \sup\{k \in \mathbb{R} \mid \text{there is a closed curve } \gamma \text{ with } A_{L+k}(\gamma) < 0\}, \qquad (2.8)$$

which gives another description of the critical value.

Remark 2.2.3. We will explain why $c(L)$ is a real number. Think of some Lagrangian, and pick a point $x \in \mathbb{T}^n$. Consider the infimum of the actions of all closed curves through x. Since the time interval is free, you will get $-\infty$ for the infimum as soon as you have just one closed curve with negative action.

Now let L be a fixed Lagrangian. We want to see what happens if we shift L by some constant k. If $k < -\min_x L(x,0)$ then $L(x,0)+k$ is negative at some point x, we can choose the constant curve at x, and end up with $\Phi_k(x,x) = -\infty$. On the other hand, the fact that L is convex implies that L is bounded from below. Therefore, if $k > -\min L$ then $L+k$ is positive, and we must have $\Phi_k(x_1,x_2;T) > 0$ for all x_1, x_2, T. This shows that

$$c(L) < \infty$$

is a real number.

The critical value can in fact be characterized in a variety of ways [60, 20, 22, 23]. Each of these characterizations gives new insight into the geometry or the dynamics of the given Lagrangian system. In the following, we will explain the relation between the critical value and the minimal action defined in Sect. 2.1.

Let $\mathcal{M}_L$ be the set of invariant probability measures on $T\mathbb{T}^n$. The next result, due to Mañé [60], states that the critical value of a convex Lagrangian is equal to the minmal action of *all* measures in $\mathcal{M}_L$, regardless of their rotation vector.

Proposition 2.2.4.

$$c(L) = -\min\{A_L(\mu) \mid \mu \in \mathcal{M}_L\}.$$

Proof. First of all, one can show that

$$\min\{A_L(\mu) \mid \mu \in \mathcal{M}_L\} = \min\{A_L(\mu_\gamma) \mid \gamma \text{ abs. cont. curve}\}$$

where μ_γ is the measure equally distributed along some absolutely continuous curve γ; see [21]. So we will prove that

$$-c(L) = \min\{A_L(\mu_\gamma) \mid \gamma \text{ abs. cont. curve}\}.$$

For any curve γ, we have $A_{L+c(L)}(\mu_\gamma) \geq 0$ by definition of $c(L)$. Therefore,

$$-c(L) \leq \min\{A_L(\mu_\gamma) \mid \gamma \text{ abs. cont. curve}\}.$$

To prove the reversed inequality, we observe that, whenever $k < c(L)$, there exists a curve γ with $A_{L+k}(\mu_\gamma) < 0$, which implies

$$-k \geq \min\{A_L(\mu_\gamma) \mid \gamma \text{ abs. cont. curve}\}.$$

Now let k tend to $c(L)$. $\square$

Remark 2.2.5. The fact that there is an invariant measure μ with $A_L(\mu) = \inf\{A_L(\mu) \mid \mu \in \mathcal{M}_L\}$ follows from the compactness of $\mathcal{M}_L$ as in Sect. 2.1.1.

Recall that

$$\alpha^*(c) = \max_{h \in H_1} \left(\langle c, h\rangle - \alpha(h)\right)$$

is the convex conjugate of the minimal action $\alpha : H_1(\mathbb{T}^n, \mathbb{R}) \to \mathbb{R}$ defined in Def. 2.1.6. Therefore,

$$\alpha^*(0) = -\min_h \min\{A_L(\mu) \mid \rho(\mu) = h\} = -\min\{A_L(\mu) \mid \mu \in \mathcal{M}_L\},$$

which yields the following description of the (convex conjugate of the) minimal action in terms of the critical value.

Corollary 2.2.6. *For every closed 1–form ν on $\mathbb{T}^n$, we have*

$$c(L-\nu) = \alpha^*([\nu]).$$

Thus, at least in the case of the torus, the theories of the minimal action and the critical value are equivalent.

It turns out the critical value $c(L)$ of a convex Lagrangian can be, in fact, recovered also from the Hamiltonian H, as the following result [22] shows. Namely, we have

$$c(L) = \inf_{u \in C^\infty(\mathbb{T}^n,\mathbb{R})} \max_{x \in \mathbb{T}^n} H(x, \mathrm{d}u(x)). \tag{2.9}$$

In other words, the critical value is a minimax value of H over all exact Lagrangian graphs. We will give a purely symplectic description of the critical value in Sect. 6.3.2.

Let $L : T\mathbb{T}^n \to \mathbb{R}$ be a convex Lagrangian, and $\mathcal{M}_L$ denote the set of invariant probability measures on $T\mathbb{T}^n$.

Definition 2.2.7. *A measure $\mu_0 \in \mathcal{M}_L$ is called* globally minimizing *if it minimizes the action amongst all invariant measures, i.e., if*

$$A_L(\mu_0) = \min\{A_L(\mu) \mid \mu \in \mathcal{M}_L\}.$$

The Mather set in $T\mathbb{T}^n$ *is defined as the closure of the union of the supports of globally minimizing measures:*

$$\tilde{\mathcal{M}} := \overline{\cup\{\mathit{supp}(\mu) \mid \mu \text{ globally minimizing}\}}.$$

Note that a globally minimizing measure must have zero rotation vector. Therefore, in view of Thm. 2.1.11, the set $\tilde{\mathcal{M}}$ is a Lipschitz graph with respect to the canonical projection

$$\tau : T\mathbb{T}^n \to \mathbb{T}^n.$$

We call the set

$$\mathcal{M} := \tau(\tilde{\mathcal{M}})$$

the *projected Mather set*. It is known [25] that $\tilde{\mathcal{M}}$ is contained in the energy level $E^{-1}(c(L))$. Finally, we define the *Mather set in* $T^*\mathbb{T}^n$ as the image of $\tilde{\mathcal{M}}$ under the Legendre transform:

$$\tilde{\mathcal{M}}^* := \ell(\tilde{\mathcal{M}}).$$

2.2.2 Weak KAM solutions

In the study of the dynamics of a Lagrangian system on $T\mathbb{T}^n$, a particular role is played by invariant tori. Since we consider time–independent convex Lagrangians, we know that the energy $E(x,p)$ on $T\mathbb{T}^n$ (or the Hamiltonian

$H(x,y)$ on $T^*\mathbb{T}^n$) is an integral of the corresponding flow. If an invariant torus is a graph of some closed 1–form ν, then ν must satisfy the equation $H(x,\nu_x) = k = \text{const}$. Actually, we can restrict ourselves to exact 1–form by considering $L - \nu$ instead of L; compare Rem. 2.1.3. Finding a smooth exact invariant torus $\operatorname{gr} \mathrm{d}u$, where $f : \mathbb{T}^n \to \mathbb{R}$ is some smooth function, is then equivalent to finding a smooth solution of the (time–independent) *Hamilton–Jacobi equation*

$$H(x, \mathrm{d}u(x)) = k \tag{2.10}$$

Now, smooth solutions of the Hamilton–Jacobi equation will, in general, not always exist. There is a general theory, developed by Fathi [29], that deals with Lipschitz continuous solutions of (2.10) and gives another approach to the critical value. Recall from (2.9) that the critical value is characterized as the infimum of energy values k such that the sublevel set $\{H < k\}$ contains a smooth solution of (2.10). Moreover, the only energy level that might support a $C^{1+\mathrm{Lip}}$–solution of (2.10) is the level $H^{-1}(c(L))$ where $c(L)$ is the critical value.

Example 2.2.8. It might be instructive to consider the simplest nontrivial case. Let $L : T\mathbb{S}^1 \to \mathbb{R}$ be the Lagrangian of the mathematical pendulum:

$$L(x,p) = \frac{1}{2}|p|^2 - \cos(2\pi x).$$

The corresponding Hamiltonian is given by

$$H(x,y) = \frac{1}{2}|y|^2 - \cos(2\pi x)$$

whose level sets are shown in Fig. 2.2. Since we are in one degree of freedom, the level sets of H consist of solutions of the Hamiltonian flow, and we end up with the phase portrait of the pendulum (Fig. 1.3).

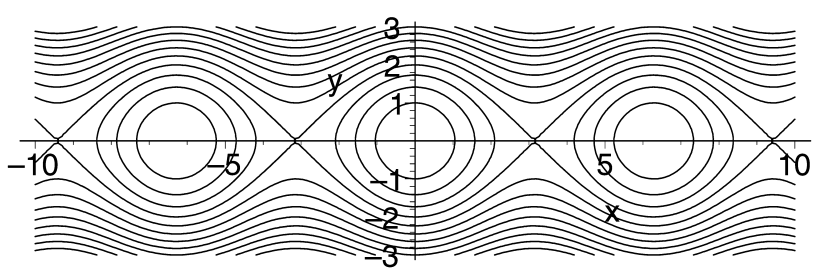


Fig. 2.2. The level sets of $H(x,y) = y^2/2 - \cos x$

Note that the energy of the separatrices is 1. For energies below 1, each level set is a closed C^∞–curve around the point $(0,0)$; thus, no level set contains even a graph. For energies above 1, each level set consists of two C^∞–curves that go around the cylinder, one above and one below the separatrices;

but none of these curves is exact, i.e., the graph of an exact 1–form. The energy level $H^{-1}(1)$, however, is the union of two graphs of differentials of $C^{1+\mathrm{Lip}}$–functions $u_\pm$ on $\mathbb{S}^1$, namely the upper and lower separatrix.

Therefore, the critical value is 1. In fact, $c(L) = \max V$ for any Lagrangian $L(x,p) = 1/2\,|p|_x^2 - V(x)$ on $T\mathbb{T}^n$.

Given a continuous function $u : \mathbb{T}^n \to \mathbb{R}$, we write

$$u \prec L + c$$

whenever $u(x) - u(y) \leq \Phi_c(y,x)$ for all $x, y \in \mathbb{T}^n$. Here, Φ_c is the action potential for the critical value

$$c := c(L).$$

Remark 2.2.9. Fathi showed that a function u satisfies $u \prec L + c$ if, and only if, it is Lipschitz continuous and fulfills the inequality

$$H(x, \mathrm{d}u(x)) \leq c$$

for almost every $x \in \mathbb{T}^n$; see [29]. In other words, for Lipschitz continuous functions u, the condition $u \prec L + c$ is equivalent to u being a subsolution of the Hamilton–Jacobi equation (2.10).

Note that, by Rademacher's theorem, Lipschitz functions are differentiable almost everywhere.

Definition 2.2.10. *We say that a continuous function $u_+ : \mathbb{T}^n \to \mathbb{R}$ is a* positive weak KAM solution *if u_+ satisfies the following two conditions:*

1. *$u_+ \prec L + c$*
2. *for all $x \in \mathbb{T}^n$ there exists an absolutely continuous curve $\gamma_+^x : [0,\infty) \to \mathbb{T}^n$ such that $\gamma_+^x(0) = x$ and*

$$u_+(\gamma_+^x(t)) - u_+(x) = \int_0^t (L+c)(\gamma_+^x(s), \dot\gamma_+^x(s))\, ds$$

for all $t \geq 0$.

Similarly, we say that a continuous function $u_- : \mathbb{T}^n \to \mathbb{R}$ is a negative weak KAM solution *if u_- satisfies the following two conditions:*

1. *$u_- \prec L + c$*
2. *for all $x \in \mathbb{T}^n$, there exists an absolutely continuous curve $\gamma_-^x : (-\infty, 0] \to \mathbb{T}^n$ such that $\gamma_-^x(0) = x$ and*

$$u_-(x) - u_-(\gamma_-^x(-t)) = \int_{-t}^0 (L+c)(\gamma_-^x(s), \dot\gamma_-^x(s))\, ds$$

for all $t \geq 0$.

We denote by $\mathcal{S}_\pm$ the set of all positive (respectively, negative) weak KAM solutions. A pair of functions (u_-, u_+) is said to be conjugate *if $u_\pm \in \mathcal{S}_\pm$ and $u_- = u_+$ on the projected Mather set $\mathcal{M}$.*

Fathi's Weak KAM–Theorem asserts that positive and negative weak KAM solutions always exist [29]. Moreover, at any point x_0 of differentiability of a weak KAM solution u, Conditions 1 and 2 of the above definition imply that u satisfies

$$H(x_0, \mathrm{d}u(x_0)) = c.$$

In fact, the points x_0 of differentiablity of u_+ (respectively, u_-) are precisely those for which the curve $\gamma_+^{x_0}$ (resp. $\gamma_-^{x_0}$) is unique. The following result [29, Thm. 5.1.2] states that any function u with $u \prec L+c$ can be squeezed between a (unique) pair of conjugate functions.

Theorem 2.2.11. *If $u : \mathbb{T}^n \to \mathbb{R}$ is a function such that $u \prec L+c$, then there exists a unique pair of conjugate functions (u_-, u_+) such that*

$$u_+ \leq u \leq u_-.$$

Finally, we will need the notion of Peierls barrier which goes back to Mather [71].

Definition 2.2.12. *The* Peierls barrier *$h : \mathbb{T}^n \times \mathbb{T}^n \to \mathbb{R}$ is defined by*

$$h(x, y) := \liminf_{T \to \infty} \Phi_c(x, y; T).$$

The function h is Lipschitz continuous and, by definition, satisfies

$$h(x, y) \geq \Phi_c(x, y).$$

Moreover, the Peierls barrier can be written as

$$h(x, y) = \max_{(u_-, u_+)} (u_-(y) - u_+(x)) \tag{2.11}$$

where the maximum is taken over all pairs (u_-, u_+) of conjugate functions; see [29].

2.2.3 The Aubry set

By definition, two conjugate functions $u_\pm$ coincide on the projected Mather set $\mathcal{M}$. In general, however, there is a bigger set with this property. To define this set we set

$$\mathcal{I}_{(u_-, u_+)} := \{x \in \mathbb{T}^n \mid u_-(x) = u_+(x)\}.$$

Definition 2.2.13. *The* projected Aubry set $\mathcal{A}$ *is the set of points in* $\mathbb{T}^n$ *at which all pairs of conjugate functions coincide:*

$$\mathcal{A} := \bigcap_{(u_-,u_+)} \mathcal{I}_{(u_-,u_+)}$$

where the intersection is taken over all pairs of conjugate functions.

It follows from the definition of conjugacy that $\mathcal{M} \subset \mathcal{A}$.

In order to define the Aubry set in $T^*\mathbb{T}^n$, we note that the functions u_- and u_+ are differentiable at every point $x \in \mathcal{I}_{(u_-,u_+)}$ with the same derivative. Moreover, the map

$$\mathcal{I}_{(u_-,u_+)} \ni x \mapsto du_-(x) = du_+(x) \in T^*\mathbb{T}^n$$

is Lipschitz continuous [29, Thm. 5.2.2]. That map defines a set

$$\tilde{\mathcal{I}}_{(u_-,u_+)} \subset T^*\mathbb{T}^n$$

that projects injectively onto $\mathcal{I}_{(u_-,u_+)}$ and contains the Mather set. The *Aubry set* in $T^*\mathbb{T}^n$ is defined as

$$\tilde{\mathcal{A}}^* := \bigcap_{(u_-,u_+)} \tilde{\mathcal{I}}_{(u_-,u_+)}$$

where, again, the intersection is taken over all pairs (u_-, u_+) of conjugate functions. As the notation suggests, one can prove that

$$\mathcal{A} = \theta(\tilde{\mathcal{A}}^*)$$

where $\theta : T^*\mathbb{T}^n \to \mathbb{T}^n$ is the canonical projection. As usual, we denote the preimage of $\tilde{\mathcal{A}}^*$ under the Legendre transform by $\tilde{\mathcal{A}}$ and call it the Aubry set in $T\mathbb{T}^n$.

The sets $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{A}}$ are compact and invariant under the Euler–Lagrange flow ϕ_t. It turns out that the Aubry set consists of a distinguished kind of orbits. To make this precise, we give the following definition due to Mañé. Recall that $c = c(L)$ denotes the critical value of the Lagrangian L.

Definition 2.2.14. *We say that an absolutely continuous curve* $\gamma : [a, b] \to \mathbb{T}^n$ *is* semistatic *if*

$$A_{L+c}(\gamma) = \Phi_c(\gamma(a), \gamma(b)).$$

An absolutely continuous curve on an infinite interval is called semistatic if it is semistatic on every finite interval.

Semistatic curves are solutions of the Euler–Lagrange equation because of their minimizing properties. It is not hard to check that semistatic curves have energy precisely c.

Definition 2.2.15. *An absolutely continuous curve $\gamma : [a,b] \to \mathbb{T}^n$ is called* static *if it is semistatic and satisfies*

$$\Phi_c(\gamma(a), \gamma(b)) + \Phi_c(\gamma(b), \gamma(a)) = 0.$$

An absolutely continuous curve on an infinite interval is called static if it is static on every finite interval.

In fact, one does not even need to require that a static curve be semistatic. If one defines static curves by the condition that $A_{L+c}(\gamma) = -\Phi_c(\gamma(b), \gamma(a))$ then it follows that a static curve is semistatic [21].

The following proposition gives a useful characterization of the Aubry set. Its proof is well known to experts; nevertheless, we include it for the sake of completeness.

Proposition 2.2.16. *The Aubry set $\tilde{\mathcal{A}}$ consists precisely of those orbits whose projections to $\mathbb{T}^n$ are static curves.*

Proof. Take $(x,v) \in \tilde{\mathcal{A}}$. We want to show that the curve

$$\gamma(t) := \tau(\phi_t(x,v))$$

is a static curve. By the definition of $\tilde{\mathcal{A}}$ and [29, Thm. 5.2.2] we have for any pair (u_-, u_+) of conjugate functions that

$$u_+(\gamma(t)) - u_-(\gamma(s)) = u_+(\gamma(t)) - u_+(\gamma(s)) = A_{L+c}(\gamma|_{[s,t]})$$

for all $s \leq t$. Using (2.11) we can choose a pair (u_-, u_+) of conjugate functions for which the Peierls barrier h satisfies

$$h(\gamma(t), \gamma(s)) = u_-(\gamma(s)) - u_+(\gamma(t)).$$

Therefore, we can estimate

$$A_{L+c}(\gamma|_{[s,t]}) + \Phi_c(\gamma(t), \gamma(s)) \leq u_+(\gamma(t)) - u_-(\gamma(s)) + h(\gamma(t), \gamma(s)) = 0. \quad (2.12)$$

It is easy to show that Φ_c satisfies the triangle inequality

$$\Phi_c(x,y) \leq \Phi_c(x,z) + \Phi_c(z,y)$$

as well as $\Phi_c(x,x) = 0$ for all $x \in \mathbb{T}^n$. Hence we have

$$\begin{aligned} 0 &= \Phi_c(\gamma(s), \gamma(s)) \\ &\leq \Phi_c(\gamma(s), \gamma(t)) + \Phi_c(\gamma(t), \gamma(s)) \\ &\leq A_{L+c}(\gamma|_{[s,t]}) + \Phi_c(\gamma(t), \gamma(s)) \\ &\leq 0 \end{aligned}$$

in view of (2.12). This implies that γ is a static curve.

Suppose now that $\gamma : \mathbb{R} \to \mathbb{T}^n$ is a static curve. Then γ is a semistatic curve with energy c, and given $s < t$ and $\epsilon > 0$, there exists a curve $\bar{\gamma}$ connecting $\gamma(t)$ to $\gamma(s)$ such that

$$A_{L+c}\left(\gamma|_{[s,t]}\right) + A_{L+c}(\bar{\gamma}) \leq \epsilon.$$

Looking at the loop formed by $\gamma|_{[s,t]}$ and $\bar{\gamma}$, we conclude that $h(\gamma(t), \gamma(t)) \leq 0$. But $h(x, x) \geq \Phi_c(x, x) = 0$ for all $x \in \mathbb{T}^n$, and hence $h(\gamma(t), \gamma(t)) = 0$. It follows from (2.11) that $\gamma(t) \in \mathcal{A}$, and thus $(\gamma(t), \dot{\gamma}(t)) \in \tilde{\mathcal{A}}$, as we wanted to prove. $\square$

We refer the interested reader to [21, 29] for more details on Mather–Mañé theory.

3

The minimal action and convex billiards

In the classical spectral problem one would like to calculate or estimate the spectrum of a given Riemannian manifold. The inverse spectral problem asks the opposite question: how much information about the underlying manifold is encoded in its spectrum? In other words, to what extent is the geometry determined by the spectrum? Or, formulated by Kac [49] in his famous title: can one hear the shape of a drum?

The term "spectrum" can have different meanings here. Either it stands for the eigenvalue spectrum of the Laplacian, or it means the length spectrum of the geodesic flow. These two interpretations are not completely independent, and there are subtle relations via the Poisson relation; see, for instance, [18, 38]. It is known by now that the answer to Kac' question is negative. There are whole families of isospectral, non–isometric manifolds; see, e.g., the survey in [35]. An in–depth survey on positive results concerning the inverse spectral problem has recently been given by Zelditch in [105].

In the following, we want to consider the inverse spectral problem for strictly convex domains Ω in $\mathbb{R}^2$. The Laplace spectrum with Dirichlet boundary conditions describes the sound you hear when you beat the "drum" Ω. The notion of length spectrum needs some explanation because Ω has a boundary. By definition, geodesics in a bounded domain Ω are geodesics (in our case, straight lines) that get reflected at the boundary according to the law "angle of reflection = angle of incidence". Such geodesics are often called *broken geodesics*. Then the length spectrum consists of the lengths of all closed (broken) geodesics, together with their multiples.

One way to think of broken geodesics is to image a room Ω lined with mirrors on its side $\partial\Omega$; broken geodesics are light rays in this mirrored chamber. Another way of looking at broken geodesics is to think of sound travelling inside the room Ω. Finally, and this is the point of view that we will adopt, broken geodesics are the trajectories of a billiard "ball" (which is just a point rather) being played inside the billiard table Ω and going around without friction.

K.F. Siburg: LNM 1844, pp. 37–57, 2004.

The main question is whether we can recognize the domain Ω, respectively its boundary curve, from the knowledge of its length spectrum. One way to attack this problem is to construct length spectrum invariants (LS–invariants, for short) and to relate them to geometry. Difficulties may arise in a twofold way – to prove that a certain geometric quantity is an LS–invariant, or to give a geometric meaning to some known LS–invariant.

The crucial observation is that, for planar convex domains, one can find coordinates such that the billiard ball map is a monotone twist map. Moreover, the length of a closed geodesic is, up to sign, the action of the corresponding orbit; hence length maximizing geodesics correspond to minimal orbits. This observation allows us to apply techniques from Aubry–Mather theory. We will see that the minimal action is invariant under continuous deformations of the domain that preserve the length spectrum. In addition, many geometric quantities—such as the lengths and Lazutkin parameters of convex caustics—can be read off from the minimal action. Note that, once we know that the minimal action is an LS–invariant, the proof that some geometric quantity obtained from it is also an invariant becomes trivial. Finally, we show that the asymptotics of the minimal action is determined by the Dirichlet spectrum of Ω.

3.1 Convex billiards

Let Ω be a strictly convex domain in $\mathbb{R}^2$ with C^3–boundary $\partial\Omega$. As for the regularity of $\partial\Omega$, we just point out the surprising observation by Halpern [40] that a C^2–curve $\partial\Omega$ may produce a geodesic flow which is not defined for all times. Our C^3–condition guarantees the completeness of the flow. In this chapter, we will always assume that the length $l(\partial\Omega)$ of the boundary curve is normalized to 1.

A broken geodesic in Ω is completely determined by its reflection points, together with the angles of reflection. The map

$$\begin{aligned} \phi : \mathbb{S}^1 \times (0,\pi) &\to \mathbb{S}^1 \times (0,\pi) \\ (s_0,\psi_0) &\mapsto (s_1,\psi_1) \end{aligned}$$

that associates to a pair (s,ψ)=(arclength on $\partial\Omega$, angle with the positive tangent) the corresponding data at the next reflection, is called the *billiard map* associated to Ω. Let us denote by

$$h(s,s') = -|P(s) - P(s')|$$

the negative Euclidean distance between two points on $\partial\Omega$. Elementary geometry (see Fig. 3.1) shows that

$$\begin{cases} \frac{\partial h}{\partial s}(s_0,s_1) = \cos\psi_0 \\ \frac{\partial h}{\partial s'}(s_0,s_1) = -\cos\psi_1 \end{cases} \tag{3.1}$$

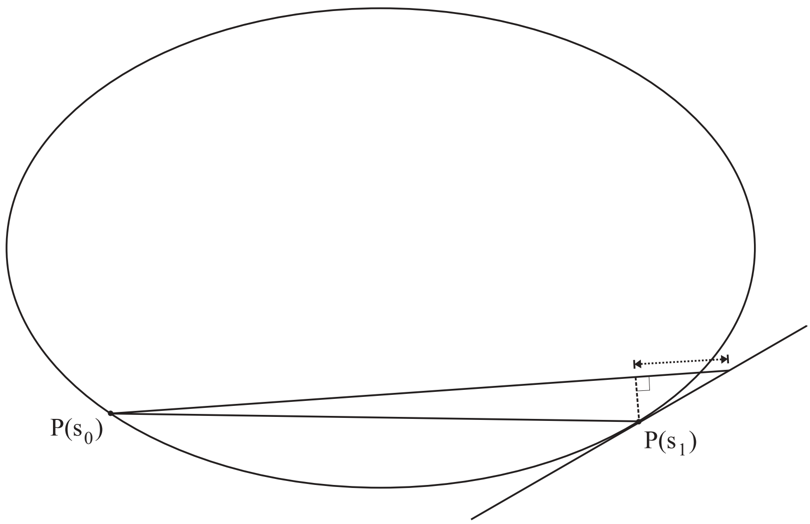


Fig. 3.1. Proof of (3.1)

as well as $\partial s_1/\partial\psi_0 > 0$.

Thus, if we lift everything to the universal cover and introduce new coordinates $(x, y) = (s, -\cos\psi) \in \mathbb{R} \times (-1, 1)$, we have

$$y_1\,\mathrm{d}x_1 - y_0\,\mathrm{d}x_0 = \mathrm{d}h(x_0, x_1)$$

as well as

$$\frac{\partial x_1}{\partial y_0} = \frac{1}{\sin\psi_0}\,\frac{\partial s_1}{\partial\psi_0} > 0.$$

This proves the following proposition.

Proposition 3.1.1. *In the coordinates (x, y), the billiard map*

$$\phi : \mathbb{R} \times (-1, 1) \to \mathbb{R} \times (-1, 1)$$

is a twist map, with the negative Euclidean distance being a generating function.

Moreover, we may extend the billiard map to the closed strip $\mathbb{S}^1 \times [-1, 1]$ by fixing the boundaries pointwise.

A periodic orbit of the billiard map ϕ corresponds to a closed (broken) geodesic inside the domain Ω. In order to distinguish topologically different closed geodesics, we associate to each periodic orbit its rotation number as defined in Def. 1.2.2. Let us give a more geometric definition here.

Definition 3.1.2. *The* rotation number *of a periodic billiard trajectory (respectively, a closed broken geodesic) is the rational number*

$$\frac{m}{n} = \frac{\textit{winding number}}{\textit{number of reflections}} \in \left(0, \frac{1}{2}\right],$$

where the winding number $m \geq 1$ is defined as follows. Fix the positive orientation of $\partial\Omega$ and pick any reflection point of the closed geodesic on $\partial\Omega$; then follow the trajectory and measure how many times it goes around $\partial\Omega$ in the positive direction until it comes back to the starting point.

Note that we restrict ourselves to rotation numbers less than or equal to $1/2$, since a closed geodesic with rotation number ω can be seen as one with rotation number $1-\omega$, traversed in the backward direction; see Fig. 3.2.

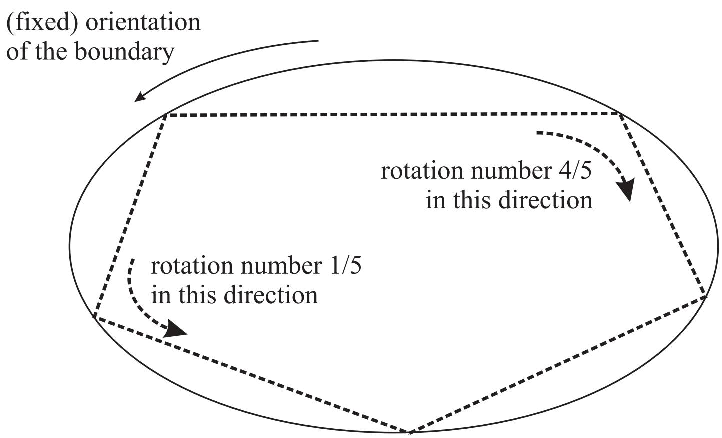


Fig. 3.2. Closed geodesics of rotation number 1/5 and 4/5

It was G.D. Birkhoff who introduced convex billiards as a conceptually simple, yet mathematically complicated, dynamical system. Applied to convex billiards, Birkhoff's theorem (Thm. 1.2.4) shows that for every $m/n \in (0, 1/2]$ in lowest terms, there are at least two closed geodesics of rotation number m/n. In fact, one of them is an inscribed n–gon with winding number m, that maximizes the perimeter amongst all such n–gons; the other one corresponds to a saddle point of the length functional.

Example 3.1.3. Consider 2–periodic billiard trajectories in a strictly convex domain Ω. Geometrically, they correspond exactly to diameters of Ω, i.e., segments that meet the boundary at a right angle at both ends. Therefore, we have $n = 2$ and $m = 1$. Already the ellipse is an example with precisely two 2–periodic billard trajectories, showing that the lower bound in Birkhoff's theorem is sharp.

Definition 3.1.4. *The* marked length spectrum *of a strictly convex domain Ω is the map*

$$\mathcal{ML}(\Omega) : \mathbb{Q} \cap \big(0, \frac{1}{2}\big] \to \mathbb{R}$$

that associates to any m/n in lowest terms the maximal length of closed geodesics having rotation number m/n.

The length spectrum *of Ω is defined as the set*

$$\mathcal{L}(\Omega) := \mathbb{N}\,\{\textit{lengths of closed geodesics in } \Omega\} \cup \mathbb{N}\, l(\partial\Omega).$$

Note that, due to Birkhoff's theorem, the marked length spectrum is a well defined map. Moreover, $l(\partial\Omega) = 1$ by our standing assumption that the boundary length is normalized to 1.

The length spectrum contains information about *all* closed geodesics, albeit in an "unformatted" form. In contrast, the marked length spectrum does give the labelling by the rotation number but only for the closed geodesics of maximal length.

Have you ever visited the great basilica in Rome and stood in its huge dome (42m in diameter)? If you are inside the domed roof and try to communicate with a friend on the other side of the dome. Rather than shouting into the air, get close to the circular wall and whisper along the wall – you will be heard clearly on the other side.

This is the effect of what is usually called a "whispering gallery". The sound waves get reflected and travel along the wall, always staying close to it. In the context of billiards, such a whispering gallery is called a caustic.

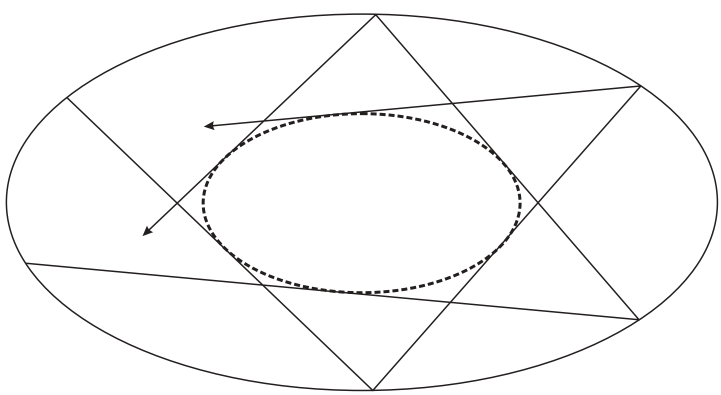


Fig. 3.3. A convex caustic

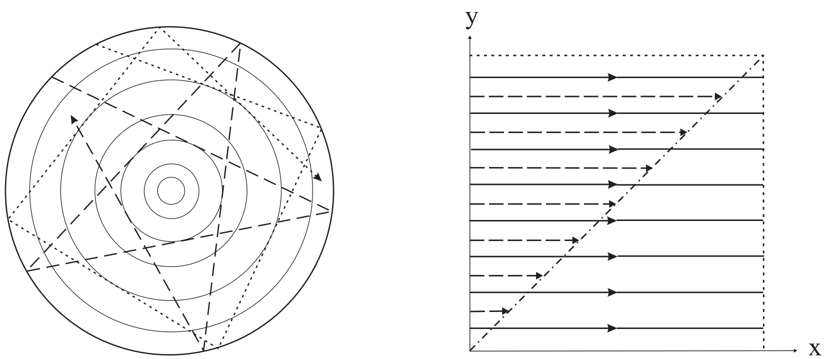
Definition 3.1.5. *Let Ω be a strictly convex bounded domain in $\mathbb{R}^2$. A* convex caustic *is a closed C^1–curve in the interior of Ω, bounding itself a strictly convex domain, with the property that each trajectory that is tangent to it stays tangent after each reflection; see Fig. 3.3.*

More generally, a caustic is defined a a continuous curve inside Ω with the above property; it need not be differentiable, nor bound a convex domain. For our purposes, however, it is sufficient to restrict ourselves to convex caustics.

Example 3.1.6. The simplest example is given by the disk Ω of perimeter 1. In the original coordinates (s, ψ), the billiard map is given by

$$(s_1, \psi_1) = (s_0 + \frac{\psi_0}{\pi}, \psi_0).$$

Since it preserves the angle, it is an integrable twist map in the coordinates $(x, y) = (s, -\cos\psi)$. Its phase space is foliated by invariant circles; the phase portrait is shown in Fig. 3.4. Moreover, the disk Ω is foliated by concentric circles, each of which is a convex caustic for the circular billiard; see also Fig. 3.4.



(a) Caustics are concentric circles

(b) Phase portrait

Fig. 3.4. The billiard inside a disk

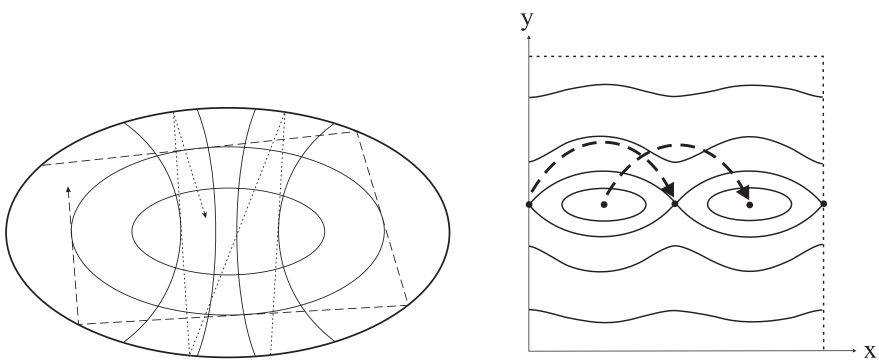
Example 3.1.7. The next simple example is the billiard inside an ellipse. It is known that this system possesses an integral, i.e., there is a (non–constant) quantity which is preserved along a trajectory [24]. This integral has the following geometric interpretation.

Consider the billiard inside an ellipse with foci F_1, F_2. Then each trajectory either

1. always intersects the open segment between the two foci, or
2. always passes through the two foci alternately, or
3. never intersects the closed segment between the foci.

In fact, each trajectory, which does not pass through a focal point, is always tangent to precisely one confocal conic section, either a confocal ellipse (in which case the trajectory never intersects the segment between the foci) or the two branches of a confocal hyperbola (where the trajectory always intersects the segment between the foci); see Fig. 3.5. The eccentricity of the corresponding conic section, for example, can be taken as an integral for the elliptical billiard. For proofs and further remarks, the reader may consult [24, 101].

Thus, the confocal ellipses inside an elliptical billiard are convex caustics in accordance with Def. 3.1.5, so the elliptical billiard is foliated by convex caustics (up to the segment between the foci). The branches of the confocal hyperbolae can then be seen as caustics in the more general sense mentioned above.



(a) Caustics are confocal ellipses and hyperbolae

(b) Phase portrait

Fig. 3.5. The billiard inside an ellipse

The phase portrait of an elliptical billiard is also shown in Fig. 3.5. Although it looks like the phase portrait of the pendulum (Fig. 1.3), the dynamics are quite different. The points $(0,0)$ and $(1/2,0)$ and its translates do not represent equilibrium points anymore, but belong to the two–periodic orbits corresponding one of the half–axes of the ellipse, and similarly for the other half–axis. Their rotation number is $1/2$, which implies that the islands are not fixed, but "wander": they are mapped onto each other.

Bounding the islands we see separatrices, corresponding to the orbits through the foci. The invariant curves above and below the seperatrices represent the orbits not intersecting the segment between the foci (i.e., being tangent to confocal hyperbola).

As an aside, we mention here that a famous conjecture, usually attributed to Birkhoff, states that the elliptical is, in fact, the only convex billiard with an integral.

Let us return to the general case of a convex billiard Ω. Suppose for a moment that the billiard possesses a convex caustic $\mathfrak{c}$. Then one can associate the following two parameters to $\mathfrak{c}$:

1. its rotation number $\omega \in (0, 1/2)$, defined as the rotation number of the circle homeomorphism on $\mathfrak{c}$ induced by the geodesic flow via the points of tangency;
2. its length $l(\mathfrak{c})$.

It turns out that there is a third parameter associated to a convex caustic, the so–called Lazutkin parameter.

Definition 3.1.8. *Let Ω be a convex billiard with a convex caustic $\mathfrak{c}$. Then the* Lazutkin parameter *of $\mathfrak{c}$ is defined as*

$$Q(\mathfrak{c}) = |A - P| + |P - B| - |\widehat{AB}|,$$

where P is any point on $\partial\Omega$ and $A, B \in \mathfrak{c}$ are the points of tangency of $\mathfrak{c}$ seen from P; see Fig. 3.6. Moreover, $|\widehat{AB}|$ denotes the length of the caustic's part from A to B, where we have oriented the caustic according to the geodesics touching it.

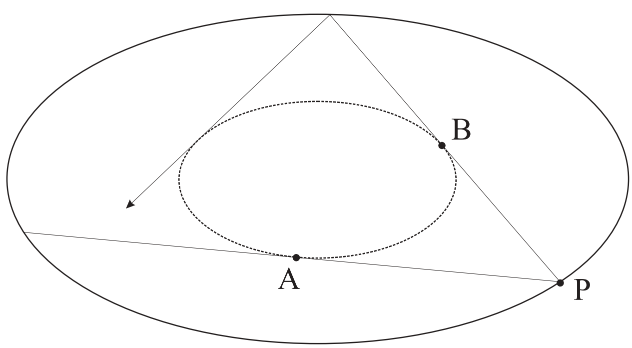


Fig. 3.6. The Lazutkin parameter of a convex caustic

In fact, if $\mathfrak{c}$ is not a caustic but just any closed convex curve inside Ω, the Lazutkin parameter can be defined in the same manner but may depend on the point $P \in \partial\Omega$. It is independent of P if, and only if, $\mathfrak{c}$ is a caustic [55, 1]. Therefore, the Lazutkin parameter of a caustic is well defined.

What is the relation between (convex) caustics of a convex billiard Ω and invariant circles for the corresponding billiard map ϕ? Certainly, to a convex caustic in Ω corresponds an invariant circle for the billiard map, i.e. a simply closed, homotopically nontrivial curve Γ in $\mathbb{S}^1 \times (-1, 1)$ with $\phi(\Gamma) = \Gamma$. The converse, however, is not entirely true. By a theorem of Birkhoff (see [94] and the references therein), invariant circles of twist maps are graphs and therefore do give rise to caustics; but these caustics need neither be convex nor differentiable.

Finally, one may ask whether convex caustics exist for some arbitrarily given convex billiard. Lazutkin proved [55] that the billiard map of a convex billiard with sufficiently smooth boundary possesses a Cantor set of invariant circles near the boundary, a result which is based on Moser's twist theorem on invariant curves [75] (hence the condition on being sufficiently smooth). Fortunately, invariant circles near the boundary always correspond to caustics which are convex C^1–curves, so every convex billiard with C^∞–boundary possesses (uncountably many) convex caustics according to Def.3.1.5.

3.2 Length spectrum invariants

In this section, we start to investigate which geometric data of a strictly convex domain $\Omega \subset \mathbb{R}^2$ are determined by its length spectrum. Recall from Def. 3.1.4 that the length spectrum is the set $\mathcal{L}(\Omega)$ consisting of all multiples of lengths of closed (broken) geodesics in Ω. The marked length, on the other hand, is the map associating to every rational m/n the maximal length of a closed geodesic of rotation number m/n. In order to study the length spectrum one looks for length spectrum invariants, i.e., quantities that depend only on the length spectrum; see Def. 3.2.1 below. In particular, we are interested in invariants that carry some geometric information about the domain Ω.

In general, it is not clear to what extent the marked length spectrum is determined by the length spectrum, or vice versa. For this reason, we make the following definition.

Definition 3.2.1. *A quantity (number, function, etc.) is called a* (marked) length spectrum invariant *if for any two strictly convex domains $\Omega_0, \Omega_1 \subset \mathbb{R}^2$ with the same (marked) length spectrum this quantity is the same.*

We will write *LS–invariant* for length spectrum invariant, and *MLS–invariant* for marked length spectrum invariant.

As mentioned before, the notions of LS–invariant and MLS–invariant may differ. For continuous deformations of smooth (C^∞) domains, however, one has the following result.

Proposition 3.2.2. *Suppose $\Omega_s, s \in [0,1]$, is a continuous family of strictly convex domains with C^∞–boundaries such that $\mathcal{L}(\Omega_s) = \mathcal{L}(\Omega_0)$ for all s. Then*

$$\mathcal{ML}(\Omega_s) = \mathcal{ML}(\Omega_0)$$

for all s.

Proof. Pick a rational rotation number m/n and a closed geodesic in Ω_s of rotation number m/n having maximal length. We claim that the corresponding value $\mathcal{ML}(\Omega_s)(m/n)$ is independent of s.

The chosen geodesic corresponds to a periodic orbit of the corresponding twist map, respectively, to an $(n+1)$–tuple with $\xi_0 < \xi_1 < \ldots < \xi_n = \xi_0 + m$ which is a minimum of the finite action functional

$$H_s(\xi_0, \ldots, \xi_{n+1}) = \sum_{i=0}^{n} h_s(\xi_i, \xi_{i+1}),$$

where the generating function h_s is nothing but the negative length function in Ω_s. By Sard's Theorem, the critical values of H_s, i.e., the values of $\mathcal{L}(\Omega_s)$, form a set of Lebesgue measure zero. In general, the maxima of a family of smooth functions depend continuously on the parameter. In our case, they lie in a set of Lebesgue measure zero, so they must stay constant.

This means that the maximal length of closed geodesics having a fixed rotation number m/n does not depend on s. Hence the functions $\mathcal{ML}(\Omega_s)$, $s \in [0,1]$, are all the same. □

Corollary 3.2.3. *For continuous deformations of smooth strictly convex domains Ω_s, MLS–invariants are also LS–invariants. In other words, if a certain quantity stays invariant under deformations with $\mathcal{ML}(\Omega_s) = \mathcal{ML}(\Omega_0)$, it is also invariant under deformations with $\mathcal{L}(\Omega_s) = \mathcal{L}(\Omega_0)$.*

We have seen in the previous section that the billiard map associated to a convex domain Ω is a monotone twist map on $\mathbb{S}^1 \times (-1,1)$ generated by h, the negative distance between points on $\partial\Omega$. Therefore, closed geodesics of maximal length correspond to minimal orbits, and the marked length spectrum is essentially nothing but the minimal action; more precisely,

$$\alpha\Big(\frac{m}{n}\Big) = -\frac{1}{n}\,\mathcal{ML}(\Omega)\Big(\frac{m}{n}\Big) \tag{3.2}$$

for every $m/n \in (0, 1/2]$ in lowest terms.

This simple observation turns out quite fruitful because it implies the following principle.

Main Principle. Every quantity that can be calculated from the minimal action is, by tautology, a marked length spectrum invariant.

Remark 3.2.4. One of the main advantages of this principle is that the actual proof that a certain quantity is a MLS–invariant becomes trivial, once it is clear how the quantity can e calculated from the minimal action; we will see applications of this remark in the following sections.

Vice versa, we see from the identity (3.2) that every MLS–invariant must be hidden in the minimal action—the only question is how. Let us formulate the above principle as a theorem.

Theorem 3.2.5. *The minimal action $\alpha : [0,1] \to \mathbb{R}$ for the billiard map of a strictly convex C^3–domain $\Omega \subset \mathbb{R}^2$ is a complete MLS–invariant, i.e., $\mathcal{ML}(\Omega_0) = \mathcal{ML}(\Omega_1)$ if and only if $\alpha_0 = \alpha_1$.*

Moreover, α is a strictly convex function on $[0,1]$, symmetric with respect to the point 1/2, and three times differentiable at the boundary points with $\alpha'(0) = -l(\partial\Omega) = -1$.

Proof. The first assertion follows from (3.2) and the continuity of α. The strict convexity is contained in Proposition 1.3.7; the symmetry property is obvious.

For the last part, we make use of a special choice of coordinates near the boundary $\{\psi = 0\}$ which is due to Lazutkin [55, 56]. Namely, let us introduce

$$\xi = C \int_0^s \rho^{-2/3}(\tau)\, d\tau \quad , \quad \eta = 4C\rho^{1/3}(s) \sin\frac{\psi}{2}$$

where $\rho \in C^1(\mathbb{S}^1, \mathbb{R})$ is the radius of curvature and $C = (\int_0^1 \rho^{-2/3})^{-1}$. Then $\{\psi = 0\}$ corresponds to $\{\eta = 0\}$, and the invariant symplectic form $\sin\psi\, d\psi \wedge ds$ takes the form

$$\frac{1}{4C^3}\,\eta\, d\eta \wedge d\xi = d\big(\frac{1}{8C^3}\,\eta^2 d\xi\big).$$

Moreover, one calculates that

$$\phi(\xi,\eta) = (\xi + \eta, \eta) + \mathcal{O}(\eta^2)$$

as $\eta \to 0$. Hence ϕ is a perturbation of the integrable twist map $(\xi,\eta) \mapsto (\xi+\eta,\eta)$, and we obtain for the generating function[1]

$$h(\xi_0,\xi_1) = \frac{1}{24C^3}\,(\xi_1 - \xi_0)^3 + \mathcal{O}((\xi_1 - \xi_0)^4)$$

as $\xi_1 - \xi_0 \to 0$. Prop. 1.3.8 implies that the minimal action—in the coordinates ξ, η—can be written as

$$\alpha_{\xi,\eta}(\omega) = \frac{1}{24C^3}\,\omega^3 + \mathcal{O}(\omega^4)$$

as $\omega \to 0$.

Transforming back to the coordinates x, y means adding the linear term $-\omega$ to the action [67] because the cohomology class $[y\,dx - (8C^3)^{-1}\eta^2\, d\xi]$ is -1; the latter is due to the fact that the boundary $\{\psi = 0\}$ corresponds to $\{y = -1\}$ respectively $\{\eta = 0\}$. Summarizing, we have for the minimal action

$$\alpha(\omega) = -\omega + \frac{1}{24C^3}\,\omega^3 + \mathcal{O}(\omega^4) \tag{3.3}$$

as $\omega \to 0$. By symmetry, an analogous formula holds near $\omega = 1$. □

[1] Note that the exponent in the first term is 3, due to the nonstandard form of the symplectic form.

Recall from (1.7) that the convex conjugate of α is defined as $\alpha^*(I) = \max_\omega [\omega I - \alpha(\omega)]$ with $(\alpha^*)'(\alpha'(\omega)) = \omega$. Hence, in view of Theorem 3.2.5, the domain of definition of α^* is the interval $[-1, 1]$. By way of illustration, let us calculate the minimal action and its convex conjugate in the simplest example.

Example 3.2.6. Take Ω to be the disk of perimeter 1 in $\mathbb{R}^2$. The billiard map is integrable and given by $(s_1, \psi_1) = (s_0 + \psi_0/\pi, \psi_0)$ with generating function

$$h(s, s') = -\frac{1}{\pi} \sin \pi(s - s').$$

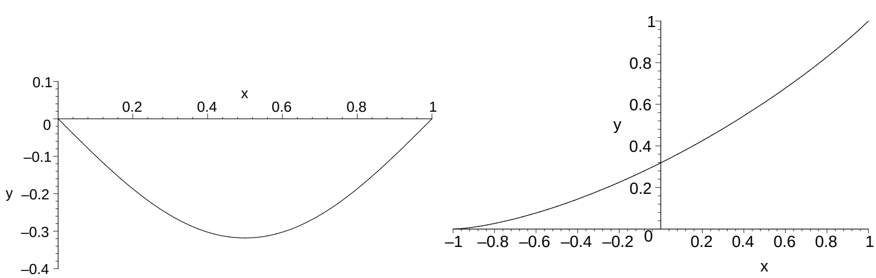
The whole phase space is foliated by invariant circles $\{y = \text{const.}\}$, and the minimal action is just

$$\alpha(\omega) = -\frac{1}{\pi} \sin \pi\omega.$$

In view of the identity $\sin \arccos(-x) = \sqrt{1 - x^2}$, its convex conjugate is

$$\alpha^*(I) = \frac{1}{\pi} \left(\arccos(-I) \cdot I + \sqrt{1 - I^2} \right) \in [0, 1]$$

for $I \in [-1, 1]$. The graphs of α and its convex conjugate α^* are depicted in Fig. 3.7.



(a) The minimal action α (b) The convex conjugate α^*

Fig. 3.7. The minimal action for the disk and its convex conjugate

The asymptotics for $\omega \to 0$ and $I \to -1$, respectively, are as follows:

$$\alpha(\omega) = -\omega + \frac{\pi^2}{6}\omega^3 - \frac{\pi^4}{120}\omega^5 + \mathcal{O}(\omega^7)$$

$$\alpha^*(I) = \frac{\sqrt{2}}{\pi} \left(\frac{2}{3}(I+1)^{3/2} + \frac{1}{30}(I+1)^{5/2} + \frac{3}{560}(I+1)^{7/2} \right) + \mathcal{O}((I+1)^{9/2})$$

α can be extended to an odd smooth function on $\mathbb{R}$. α^* has a singularity of order $3/2$ at $I = -1$; the function $(\alpha^*)^{2/3}$ is smooth.

If we ask whether the minimal action can be recovered from the length spectrum instead of the marked length spectrum, we have the following result for continuous deformations.

Corollary 3.2.7. *For continuous deformations of smooth strictly convex domains, α is an LS–invariant function on* $[0,1]$.

Proof. Immediate from Corollary 3.2.3. □

Finally, we point out that the minimal action is actually invariant under arbitrary symplectic (i.e., area–preserving) coordinate changes in the phase space. For, by (3.11), the actions of periodic orbits are symplectically invariant, and they determine α.

3.2.1 Classical invariants

What kind of *geometric* data can be recovered from the (marked) length spectrum? It is well known, for instance, that the perimeter and the diameter of the domain Ω are invaraints of the spectrum, a fact which follows readily from the Main Principle.

Proposition 3.2.8. *The boundary length $l(\partial\Omega)$, the diameter $\operatorname{diam}\Omega$, and the curvature integral $\int \rho^{-2/3}$ are MLS–invariants.*

Proof. This is an immediate consequence of our Main Principle, together with (3.3),since $l(\partial\Omega) = -\alpha'(0)$, $\operatorname{diam}\Omega = -\alpha(1/2)$, and $(\int \rho^{-2/3})^3 = 4\alpha'''(0)$. □

Remark 3.2.9. In view of Cor. 3.2.7, the above quantities are LS–invariants for continuous deformations of smooth strictly convex domains.

Are there other geometric data hidden in the minimal action? The following theorem shows that the parameters connected to a convex caustic can be read off from the minimal action.

Theorem 3.2.10. *Let Ω be a strictly convex C^3–domain in $\mathbb{R}^2$, and suppose that $\mathfrak{c}_\omega$ is a convex caustic of rotation number ω. Then the length of $\mathfrak{c}_\omega$ and its Lazutkin parameter are given by*

$$l(\mathfrak{c}_\omega) = -\alpha'(\omega)$$

and

$$Q(\mathfrak{c}_\omega) = \alpha^*(\alpha'(\omega)).$$

Proof. Call $T(s)$ and $N(s)$ the unit tangent vector and unit inward normal, respectively, at a point $P(s) \in \partial\Omega$, and set $U(s) = \cos\psi(s)T(s) + \sin\psi(s)N(s)$; here, $\psi(s) \in (0, \pi/2)$ is the unique angle such that the ray from $P(s)$ having direction $U(s)$ touches $\mathfrak{c}_\omega$. Then there is a function $\tau(s)$ such that

$$A(s) = P(s) + \tau(s)U(s) \in \mathfrak{c}_\omega$$

and

$$\dot{A}(s) = T(s) + \dot{\tau}(s)U(s) + \tau(s)\dot{U}(s) \parallel U(s).$$

Since $\dot{A}(s) \neq 0$, we can write

$$|\dot{A}(s)| = \langle \dot{A}(s), U(s) \rangle = \cos\psi(s) + \dot{\tau}(s)$$

so that

$$l(\mathfrak{c}_\omega) = \int_{\mathbb{S}^1} |\dot{A}|\, ds = \int_{\mathbb{S}^1} \cos\psi\, ds = -\int_{\Gamma_\omega} y\, dx = -\alpha'(\omega)$$

by Proposition 1.3.7.

By definition of the Lazutkin parameter (see Fig. 3.6), we have

$$(N+1)Q(\mathfrak{c}_\omega) = |A_1 - P_1| + \sum_{i=1}^{N} |P_i - P_{i+1}| + |P_{N+1} - B_{N+1}| - \sum_{i=1}^{N+1} |\widehat{A_i B_i}|$$

for $N \geq 1$. Hence

$$\begin{aligned} Q(\mathfrak{c}_\omega) &= \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} |P_i - P_{i+1}| - \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} |\widehat{A_i B_i}| \\ &= -\alpha(\omega) - \omega L(\mathfrak{c}_\omega) \\ &= \omega\alpha'(\omega) - \alpha(\omega) \\ &= \alpha^*(\alpha'(\omega)) \end{aligned}$$

□

Corollary 3.2.11. *The length and Lazutkin parameter of a convex caustic are MLS–invariants, respectively, LS–invariants under continuous deformations.*

Remark 3.2.12. The result that lengths and Lazutkin parameters of convex caustics are spectrally determined is due to Amiran [2]. The above proof, however, is new.

Remark 3.2.13. In view of Theorem 3.2.10, one might call $-\alpha'$ and α^* the generalized "length" and "Lazutkin parameter", even if there is no convex caustic of the corresponding rotation number.

Given a convex caustic $\mathfrak{c}$, one can reconstruct $\partial\Omega$ by wrapping a string of length $l(\mathfrak{c}) + Q(\mathfrak{c})$ around $\mathfrak{c}$, pulling it tight, and going along $\mathfrak{c}$. These "string length parameters" of convex caustics are, of course, also MLS–invariants of the domain Ω.

The following result shows how analytical properties of α translate into geometric properties of Ω.

Proposition 3.2.14. *Let $\Omega \subset \mathbb{R}^2$ be a strictly convex C^3–domain with associated minimal action $\alpha : [0,1] \to \mathbb{R}$. Then the following statements are equivalent:*

1. *α is differentiable at the point $1/2$;*
2. *$\partial\Omega$ is a curve of constant width.*

Proof. The differentiability of α at $1/2$ is, in view of Thm. 1.3.7, equivalent to the existence of an invariant circle consisting entirely of minimal 2–periodic orbits, i.e., diameters of the same length. □

Another well known fact is the spectral rigidity of the disk, i.e., that the disk is completely determined by its spectrum. In our context, we have the following rigidity result.

Theorem 3.2.15. *Let $\Omega \subset \mathbb{R}^2$ be a strictly convex C^3–domain with associated minimal action $\alpha : [0,1] \to \mathbb{R}$. Then the following statements are equivalent:*

1. *α is differentiable;*
2. *α is analytic;*
3. *Ω is a disk.*

Proof. Since 3. $\Rightarrow$ 2. $\Rightarrow$ 1. in view of Example 3.2.6, it suffices to show 1. $\Rightarrow$ 3.

According to Thm. 1.3.7, the differentiability of α at the point $1/2$ is equivalent to the existence of an invariant circle consisting of minimal periodic orbits. Taking limits of these curves, we obtain invariant circles for all rotation numbers, consisting entirely of minimal orbits.

We claim that they foliate the phase space. Indeed, if there was a gap, its boundary curves would necessarily have the same rotation number (otherwise, there would be rotation numbers without invariant circles). But this is impossible, due to the graph property of the set of minimal orbits in Theorem 1.3.4.

Now the assertion follows from Bialy's result [9] that the only billiard whose phase space is foliated by invariant circles is a circular one. □

Remark 3.2.16. We conjecture that the statement "α is differentiable near the point $1/2 \Rightarrow \Omega$ is a disk" is true, which would mean that the differentiability of a minimal action, stemming from a strictly convex billard, near $1/2$ already implies its global differentiability.

Corollary 3.2.17. *Suppose Ω is a strictly convex C^3–domain with the same marked length spectrum as the disk of the same perimeter. Then Ω is a disk.*

Remark 3.2.18. In view of Corollary 3.2.3, we have the following rigidity result for deformations: any continuous deformation of the disk (inside the class of smooth convex domains) that preserves the length spectrum must be trivial.

Remark 3.2.19. Corollary 3.2.17 has an analogue in differential geometry: a Riemannian 2–torus, having the same marked length spectrum as a flat torus, is flat [7, Thm. 6.1].

3.2.2 The Marvizi–Melrose invariants

So far, we have considered the most general case of domains with C^3–boundaries where we do not know, for instance, whether there are any convex caustics at all. If we assume smooth enough boundaries, however, then Moser's twist theorem [75] can be applied and guarantees the existence of a Cantor family of convex caustics that accumulate at the boundary $\partial\Omega$; this was first proven by Lazutkin [55]. From now on, we assume that Ω is smooth (C^∞). Then there is the following KAM–theorem by Kovachev and Popov [52, Thm. 2] which is based on the work of Pöschel [89].

Theorem 3.2.20. *Let Ω be a smooth strictly convex domain of unit boundary length. Then there are symplectic coordinates (θ, I) near $\{y = -1\} \leftrightarrow \{I = -1\}$ such that the billiard map $\phi : (\theta_0, I_0) \mapsto (\theta_1, I_1)$ is generated by*

$$S(\theta_0, I_1) = \theta_0 I_1 + K(I_1)^{3/2} + R(\theta_0, I_1),$$

i.e.,

$$\begin{cases} I_0 = \partial_1 S = I_1 + \partial_1 R \\ \theta_1 = \partial_2 S = \theta_0 + \frac{3}{2} K(I_1)^{1/2} K'(I_1) + \partial_2 R \end{cases} \tag{3.4}$$

Here, $K \in C^\infty(\mathbb{R}, \mathbb{R})$ with $K(-1) = 0, K'(-1) > 0$, and $R \in C^\infty(\mathbb{R}^2, \mathbb{R})$ is 1–periodic in the first variable. Moreover, there exists a Cantor set $\mathcal{C}^* \subset [-1, -1 + \epsilon^*)$ with $-1 \in \mathcal{C}^*$, where $\epsilon^* > 0$ is some small number, such that $R \equiv 0$ on $\mathbb{R} \times \mathcal{C}^*$.

We see that the perturbation term R vanishes on $\mathbb{R} \times \mathcal{C}^*$ with all its derivatives. Each curve $\mathbb{R} \times \{I\}, I \in \mathcal{C}^*$, gives rise to an invariant circle for the billiard map on which it is conjugated to the rigid rotation

$$(\theta, I) \mapsto (\theta + \omega, I) \tag{3.5}$$

with $\omega = 3/2 K(I)^{1/2} K'(I)$. Since these invariant circles lie near the boundary, they correspond to a Cantor set of convex caustics near and accumulating at $\partial\Omega$.

In view of (3.5), $\theta_0 I_1 + \alpha^*(I_1)$ generates $\phi : (\theta_0, I_0) \mapsto (\theta_1, I_1)$ on the Cantor set $\mathbb{R} \times \mathcal{C}^*$ of invariant circles accumulating at $\mathbb{R} \times \{-1\}$. We also know from Theorem 3.2.20 that ϕ is generated by $S = \theta_0 I_1 + K(I_1)^{3/2} + R(\theta_0, I_1)$, where K is smooth and R vanishes with all its derivatives on $\mathbb{R} \times \mathcal{C}^*$. Since $\alpha^*(-1) = 0 = K(-1)$, we must have that

$$\alpha^*(I) = K(I)^{3/2} \tag{3.6}$$

for all $I \in \mathcal{C}^*$. Thus, α^* yields a generating function for ϕ, restricted to a Cantor set of invariant KAM–circles.

The smooth function K can be written as

$$K(I) = K'(-1) \cdot (I + 1) + \mathcal{O}((I + 1)^2) \tag{3.7}$$

with $K'(-1) > 0$. Combining (3.6) and (3.7) we obtain that

$$\alpha(\omega) = a_1\omega + a_3\omega^3 + \mathcal{O}(\omega^5)$$

is smooth on the Cantor set

$$\mathcal{C} = \left\{ \frac{3}{2} K(I)^{1/2} K'(I) \mid I \in \mathcal{C}^* \right\} \tag{3.8}$$

containing 0.

Let us return to the original question which geometric data can be recovered from the (marked) length spectrum. Concerning the lengths and Lazutkin parameters of convex caustics—whose existence is now guaranteed by Thm. 3.2.20— Cor. 3.2.11 immediately implies the following assertion.

Theorem 3.2.21. *The function* $\omega \mapsto (l(\mathfrak{c}_\omega), Q(\mathfrak{c}_\omega))$*, defined on a Cantor set in* $[0,1]$ *containing* 0 *and* 1*, is an MLS–invariant, respectively an LS–invariant under continuous deformations.*

Remark 3.2.22. Again, we point out that Thm. 3.2.21 is not new: it is the main result in [88]. Popov's proof, however, relies on the existence of invariant KAM–circles, and a good part of it is hard analysis. Our approach is simpler and shows that the invariance property of the caustic parameters is definitely not a phenomenon inside KAM–theory.

In the smooth case, Theorem 3.2.10 also implies a functional dependence of $l(\mathfrak{c})$ and $Q(\mathfrak{c})$.

Theorem 3.2.23. *There is a formal power series expansion*

$$l = 1 + \sum_{k\geq 1} b_k Q^{2k/3}$$

as $Q \to 0$*, whose coefficients are MLS–invariants, respectively LS–invariants under continuous deformations.*

Proof. $l(\mathfrak{c}_\omega) = -\alpha'(\omega)$ and $Q(\mathfrak{c}_\omega)^{2/3} = \alpha^*(\alpha'(\omega))^{2/3} = K(\alpha'(\omega))$ are smooth functions on $\mathcal{C}$ with $K'(-1) > 0$. The claim follows from the implicit function theorem. $\square$

Remark 3.2.24. Cor. 3.2.23 was first stated explicitly in [2, (3.1)]. It follows, however, already from formulae obtained by Lazutkin, namely (1.11) and (1.12) in [55], where he expresses the length and his parameter in terms of the rotation number.

Let us go on and see if there are other spectral invariants hidden in the minimal action. In 1982, Marvizi and Melrose [63] defined a sequence of MLS–invariants by investigating an integrable approximation of the billiard map near the boundary. The *Marvizi–Melrose invariants* are given by the asymptotics of a so–called interpolating Hamiltonian. By definition, an interpolating Hamiltonian is a smooth function ζ on $\mathbb{S}^1 \times [-1,1]$, whose time–$\zeta^{1/2}$–map is the billiard map, up to a diffeomorphism that fixes the boundary to infinite order. The integral invariants are then defined as the Taylor coefficients of

$$J(r) := \frac{1}{\zeta'(\zeta^{-1}(r))} \tag{3.9}$$

at $r = 0$ for any interpolating Hamiltonian ζ in action–angle–variables.

We claim that, in the context of minimal action, we can find a quite simple interpolating Hamiltonian. Indeed, from (3.4) and (3.6), we see that

$$\zeta(I) := \left(\frac{3}{2}\right)^{2/3} K(I) = \left(\frac{3}{2}\,\alpha^*(I)\right)^{2/3}$$

is an interpolating Hamiltonian, at least on a Cantor set containing the boundary. Since the Taylor coefficients of (3.9) at the boundary point $r = 0$ only depend on the behaviour of $\zeta(I)$ on any sequence accumulating at 0, they only depend on the Taylor coefficients of α^* at the corresponding boundary point -1. Summarizing, we obtain the following result.

Theorem 3.2.25. *The integral invariants of Marvizi and Melrose are algebraically equivalent to the Taylor coefficients of α^* at -1.*

Marvizi and Melrose [63] go further and prove an asymptotic formula for the lengths of closed geodesics which is then shown to be spectrally determined. In order to integrate this formula into our context, we consider a closed geodesic g_{mn} of rotation number m/n. Its length is given by

$$l(g_{mn}) = -\sum_{g_{mn}} h. \tag{3.10}$$

We want to rewrite this in (θ, I)–coordinates and relate it to the generating function $S(\theta_0, I_1)$ from Theorem 3.2.20. We have the following transformations:

$$\Phi : (\theta, I) \mapsto (x, y) \quad \text{with} \quad \Phi^*(y\,dx) - I\,d\theta = dH$$
$$\Psi = \Phi^{-1} \circ \phi \circ \Phi$$
$$\theta_1\,dI_1 + I_0\,d\theta_0 = dS \quad \text{with} \quad S = \theta_0 I_1 + K(I_1)^{3/2} + R(\theta_0, I_1)$$

A straightforward calculation shows that the generating function transforms according to the formula

$$\Psi^*(I\,d\theta) - I\,d\theta = d(h \circ \Phi + H - H \circ \Psi). \tag{3.11}$$

On the other hand, we can write the left hand side as

$$\begin{aligned}
I_1\,d\theta_1 - I_0\,d\theta_0 &= (-\theta_1\,dI_1 - I_0\,d\theta_0) + (\theta_1\,dI_1 + I_1\,d\theta_1)\\
&= -dS + \Big(\frac{\partial S}{\partial I_1}\,dI_1 + I_1\,d(\frac{\partial S}{\partial I_1})\Big)\\
&= d\Big(I_1\,\frac{\partial S}{\partial I_1} - S\Big)\\
&= dS^*
\end{aligned}$$

From this we conclude that

$$-h\circ\Phi = H - H\circ\Psi - S^* + \text{const.}$$

Summed over a closed orbit, the term $H - H\circ\Psi$ adds to zero, so (3.10) yields

$$\frac{1}{n}\,l(g_{mn}) = \frac{1}{n}\sum_{g_{mn}} -S^* + \frac{\text{const.}}{n}.$$

Since $S^* = (K(I)^{3/2})^* + I\,\partial R/\partial I - R$ has the same Taylor series at the boundary point as $(K(I)^{3/2})^*$, which coincides on $\mathbb{S}^1\times\mathcal{C}$ with $(\alpha^*)^* = \alpha$, we see that $l(g_{mn})/n$ has the same Taylor series for $n\to\infty$ as $-\alpha$. Loosely speaking, all closed geodesics become minimal as they approach $\partial\Omega$. More precisely,

$$l(g_{mn}) = m + \sum_{k\geq 1} c_{mk} n^{-2k}$$

where

$$c_{mk} = -\frac{m^{2k+1}}{(2k+1)!}\,\alpha^{(2k+1)}(0). \tag{3.12}$$

This formula implies the following

Theorem 3.2.26. *If Ω is smooth, the asymptotics of the length spectrum as $n\to\infty$ are equivalent to that of the minimal action as $\omega\to 0$.*

3.2.3 The Gutkin–Katok width

Another application of Theorem 3.2.5 concerns regions in Ω which are free of convex caustics. Gutkin and Katok gave estimates for their area in terms of the geometry of Ω. In particular, they proved [39, Prop. 1.3] that a convex caustic $\mathfrak{c}_\omega$ with rotation number ω cannot lie too far from the boundary:

$$\max_{P\in\mathfrak{c}_\omega} d(P,\partial\Omega) < \sqrt{\operatorname{diam}\Omega\cdot Q(\mathfrak{c}_\omega)}.$$

See Fig. 3.8 for an illustration.

We will see that the Gutkin–Katok width $\sqrt{\operatorname{diam}\Omega\cdot Q(\mathfrak{c}_\omega)}$ has an MLS–invariant interpretation. Indeed, in view of Prop. 3.2.8 and Theorem 3.2.10, this number can also be written as

$$\sqrt{\operatorname{diam}\Omega\cdot Q(\mathfrak{c}_\omega)} = \sqrt{-\alpha(1/2)\cdot\alpha^*(\alpha'(\omega))}.$$

According to our Main Principle, this proves the following result.

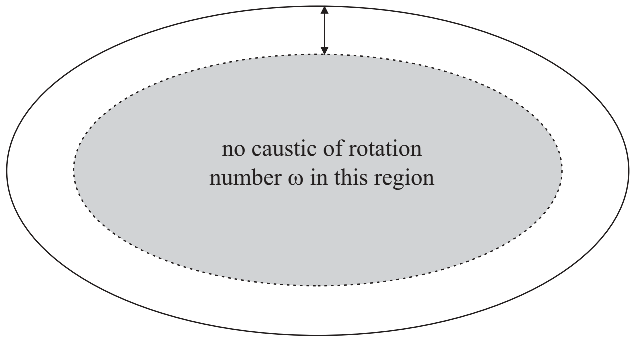


Fig. 3.8. The Gutkin–Katok width

Theorem 3.2.27. *Suppose Ω_0, Ω_1 are two strictly convex C^3–domains with the same marked length spectrum. Then, for a fixed rotation number ω, every convex caustic $\mathfrak{c}_\omega$ in Ω_0, respectively Ω_1, is contained in a strip around $\partial\Omega_0$, respectively $\partial\Omega_1$, of one and the same width.*

3.3 Laplace spectrum invariants

As already mentioned in the introduction, there is a relation between the length spectrum $\mathcal{L}(\Omega)$ of a strictly convex domain and the spectrum of the Laplacian with Dirichlet boundary conditions:

$$\begin{cases} \triangle u = \lambda^2 u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.13}$$

Indeed, due to a Poisson relation for compact Riemannian manifolds with geometrically convex boundary [38], the expression

$$\sigma(t) = \sum_{\lambda^2 \in \operatorname{spec} \triangle} \cos \lambda t$$

is well defined as a distribution, which is smooth away from $\mathcal{L}(\Omega)$ [3]. More precisely, if $T > 0$ is in the singular support of σ then T lies in the length spectrum of Ω.

Conversely, whether some given $T \in \mathcal{L}(\Omega)$ belongs to the singular support of σ, depends on possible cancellations of singularities stemming from different closed geodesics of the same length. It is known [38] that T lies in the singular support of *sigma* if there is exactly one closed geodesic of length T, whose Poincaré map does not have an eigenvalue 1. Marvizi and Melrose [63] showed that a much weaker non–coincidence condition on Ω suffices to conclude that almost all *maximal* lengths of geodesics having rotation number $1/n$ lie in the singular support of σ. Popov [88] generalized this to geodesics of rotation number m/n with $m > 1$, provided (m, n) is "near" the Cantor set described in (3.8). In particular, that non–coincidence condition is satisfied by all curvature functions in a C^1–neighbourhood of the constants.

Thus, the values $\alpha(1/n)$, respectively $\alpha(m/n)$, with sufficiently large n are spectral invariants of the domain. This is also true for the coefficients c_{1k} in (3.12), and hence for the Taylor coefficients of $-\alpha$ at 0. Therefore we can state the following result which is an analogue of [63, Thm. 7.4].

Theorem 3.3.1. *Suppose $\Omega \subset \mathbb{R}^2$ is a smooth strictly convex domain with unit boundary length, such that* 1 *is not a limit point of lengths of closed geodesics having fixed rotation number m/n with $m > 1$. Then the Taylor series of the minimal action at* 0 *is completely determined by the Dirichlet spectrum* (3.13).

Applying Popov's more general result, one can show that also the values of α on the Cantor set $\mathcal{C}$ (and hence all the caustic parameters $L(\mathfrak{c})$ and $Q(\mathfrak{c})$) are spectral invariants under the non–coincidence condition in [88, (6.1)].

For an exhaustive treatise on the inverse (Laplace) spectral problem problem we refer to the survey by Zelditch [105].

4

The minimal action near fixed points and invariant tori

When investigating a dynamical system it is often a first important step to transform it into a form as simple as possible. If the system belongs to some restricted class these normal forms should be invariant under the corresponding invariance transformations.

In classical mechanics, the class under consideration consists of Hamiltonian systems, respectively symplectic transformations. We consider a symplectic map near a fixed point (in two dimensions), respectively, near an invariant torus (in arbitrary dimensions). Under certain nondegeneracy conditions on the linearization of the map , G.D. Birkhoff constructed a normal form that is invariant under symplectic coordinate changes. This Birkhoff normal form describes an integrable approximation of the original map. Its asymptotics at the fixed point define a set of symplectic invariants; for obvious reasons, we call these Birkhoff invariants asymptotic.

The goal of this chapter is the construction of a new *local* symplectic invariant that, in particular, includes the asymptotic Birkhoff invariants. To do so, we will associate to the germ of a symplectic map at a fixed point/invariant torus the germ of its minimal action. This is symplectically invariant and, in contrast to the Birkhoff normal form, reflects part of the dynamical behaviour in a neighbourhood of the fixed point/invariant torus. We show that the classical Birkhoff invariants are encoded in the minimal action as the Taylor coefficients of its convex conjugate. Moreover, in the integrable case, the minimal action determines the map completely, a fact which is also not true for the Birkhoff normal form (unless the map is analytic).

Symplectic mappings near a fixed point appear as Poincaré section maps of a closed trajectory. We explain this for the geodesic flow on a compact surface. It turns out that the minimal action depends only on the length spectrum of the Riemannian manifold. Therefore, the minimal action is a new local length spectrum invariant for compact two–dimensional manifolds.

In the final section we investigate the role of the minimal action near an invariant torus of a symplectic map.

K.F. Siburg: LNM 1844, pp. 59–80, 2004.

4.1 The minimal action near plane elliptic fixed points

We consider an area–preserving diffeomorphism ϕ of the plane in the vicinity of a fixed point. We can shift the fixed point into the origin, so we assume that $\phi(0) = 0$. Moreover, we are only interested in the local behaviour of ϕ near 0. Therefore, we call two diffeomorphisms equivalent if they coincide on some open neighbourhood of the origin. In other words, we consider the equivalence class of ϕ, i.e., we consider the germ of a symplectic diffeomorphism at the fixed point $0 \in (\mathbb{R}^2, \Omega)$ where Ω is some area form on $\mathbb{R}^2$. In the following, whenever we pick a representative $\phi : U \to \mathbb{R}^2$, we assume that U is a simply connected neighbourhood of 0; this is no loss of generality. Then, by Poincaré's Lemma, the symplectic form Ω is exact, i.e., there is a 1–form λ with $\Omega = d\lambda$.

Suppose for a moment that there is a point $p \in U \setminus \{0\}$ whose iterates $p_i = \phi^i(p)$ exist for all $i \in \mathbb{Z}$. We want to define the average action and the rotation number of the orbit $(p_i)_{i\in\mathbb{Z}}$. Since ϕ is symplectic, the 1–form $\phi^*\lambda - \lambda$ on U is closed, hence exact:

$$\phi^*\lambda - \lambda = dS.$$

Definition 4.1.1. *Let $\phi : U \to \mathbb{R}^2$ be a symplectic diffeomorphism, and λ be a (local) primitive of the area form Ω.*

Any function $S : U \to \mathbb{R}$ satisfying $\phi^*\lambda - \lambda = dS$ is called a generating function *for ϕ. Any two generating functions differ by some additive constant, and we normalize S by setting $S(0) = 0$; this makes the generating function unique.*

If $(p_i)_{i\in\mathbb{Z}}$ is an orbit of ϕ then the average action *of (p_i) is defined as*

$$A((p_i)) := \lim_{N\to\infty} \frac{1}{2N} \sum_{i=-N}^{N-1} S(p_i)$$

if this limits exists.

Remark 4.1.2. As an aside, we remark that the normalization of the generating function is in accordance with setting $H(t, 0) = 0$ when H is a Hamiltonian whose flow generates ϕ and leaves 0 fixed; for, then the generating function $\int \lambda - H\,dt$ vanishes at 0.

Lemma 4.1.3. *The average action does not depend on the choice of the 1–form λ. Moreover, it is invariant under local symplectic coordinate changes Φ fixing the origin.*

Proof. We claim that the definition of the average action does not depend on the choice of the 1–form λ. Indeed, taking another 1–form λ' with $d\lambda' = d\lambda = \Omega$, the closed 1–form $\lambda' - \lambda$ is exact: $\lambda' - \lambda = dF$. The new (normalized) generating function is given by $S' = S + \phi^*F - F$. But the average of ϕ^*F over an orbit is the same as that of F, so the summation of $\phi^*F - F$ over the orbit vanishes. This proves our first claim.

Moreover, we claim that the average action is invariant under local symplectic coordinate changes Φ fixing the origin. Such a transformation Φ is generated by some function F, due to Poincaré's Lemma. Then a short calculation shows that the generating function for $\Phi \circ \phi \circ \Phi^{-1}$ is given by $\Phi^* S + F - (\Phi \circ \phi \circ \Phi^{-1})^* F$. Now the claim follows by the same argument as above. $\square$

Next, we want to define the rotation number of an orbit $(p_i)_{i\in\mathbb{Z}}$ in $U \setminus \{0\}$. Roughly speaking, this is its average winding number around the origin. More precisely, we introduce polar coordinates on $\mathbb{R}^2 \setminus \{0\} = \mathbb{S}^1 \times (0,\infty)$, and lift $\phi : U \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ to a map $\widetilde{\phi}$ which is then defined on some strip in the universal cover $\mathbb{R} \times (0,\infty)$ of $\mathbb{S}^1 \times (0,\infty)$. Since ϕ is an orientation–preserving diffeomorphism, $\widetilde{\phi}$ is a diffeomorphism of degree 1. Given an orbit $(\widetilde{p_i})$ of $\widetilde{\phi}$ projecting onto (p_i) and a natural number N, we choose a curve $\widetilde{\Gamma} : [-N,N] \to \mathbb{R} \times (0,\infty)$ with $\widetilde{\Gamma}(i) = \widetilde{p_i}$. Call $\Gamma : [-N,N] \to \mathbb{R}^2 \setminus \{0\}$ the projection of $\widetilde{\Gamma}$, and close it up to a closed curve Γ_N by adding a "short" piece (whose lift upstairs lies inside one fundamental domain); see Fig. 4.1.

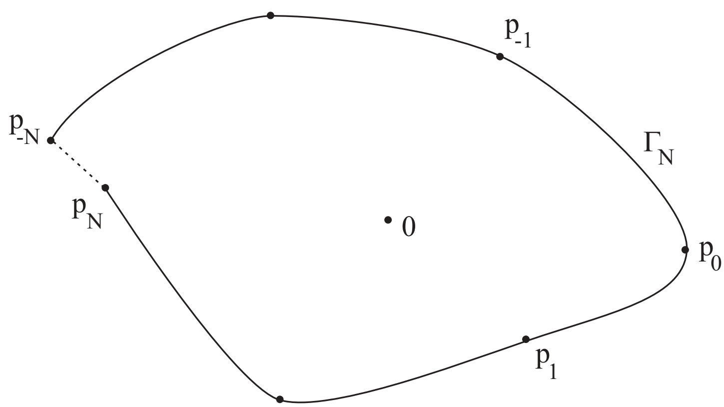


Fig. 4.1. The rotation number of an orbit $(p_i)_{i\in\mathbb{Z}}$

Then we define the rotation number of $(\widetilde{p_i})$ to be

$$\rho((\widetilde{p_i}),\widetilde{\phi}) := \lim_{N\to\infty} \frac{1}{2N} [\Gamma_N] \quad \in H_1(\mathbb{R}^2 \setminus \{0\},\mathbb{R})$$

if this limit exists. Clearly, if we fix the lift $\widetilde{\phi}$, the class ρ does not depend on the particular choice of $(\widetilde{p_i})$ and Γ_N. Moreover, choosing a different lift $\overline{\phi}$ means adding the class $[\overline{\phi} - \widetilde{\phi}]$ of the deck transformation $\overline{\phi} - \widetilde{\phi}$. Finally, we have a canonical identification $H_1(\mathbb{R}^2 \setminus \{0\},\mathbb{R}) = \mathbb{R}$ if we take as generator of $H_1(\mathbb{R}^2 \setminus \{0\},\mathbb{R})$ the class represented by the positively oriented unit circle.

Definition 4.1.4. *The* rotation number *of an orbit* $(p_i)_{i\in\mathbb{Z}}$ *contained in* $U \setminus \{0\}$ *is defined as the real number*

$$\rho((p_i)) := \rho((\widetilde{p_i}), \widetilde{\phi}),$$

where $\widetilde{p_i}$ *and* $\widetilde{\phi}$ *are defined as above. It is well defined up to integer shifts, and invariant under conjugation by homeomorphisms.*

Analogous to Aubry–Mather theory, we introduce the following variational principle for the symplectic map $\phi : U \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$. Having fixed some lift $\widetilde{\phi}$ of ϕ, we denote by $\alpha(\omega)$ the infimum of average actions of orbits in $U \setminus \{0\}$ with rotation number ω where, as usual, the infimum over the empty set is ∞.

Definition 4.1.5. *Consider the germ of a symplectic diffeomorphism at the fixed point* 0*. Then the* minimal action *is defined as the function*

$$\begin{aligned} \alpha : \mathbb{R} &\to (-\infty, \infty] \\ \omega &\mapsto \inf\{A((p_i)) \mid \rho((p_i)) = \omega\} \end{aligned} \tag{4.1}$$

taking the infimum of average actions over all orbits of a given rotation number.

The minimal action is well defined up to additive integers in the rotation number ω. What are the invariance properties of the minimal action?

Proposition 4.1.6. *The minimal action is invariant under (local) symplectic diffeomrophisms.*

Proof. In view of Lemma 4.1.3, the average action is invariant under symplectic coordinate changes, whereas the rotation number, as a real number up to integer shifts, is even invariant under homeomorphisms. □

Thus, we have constructed a symplectic invariant function, associated to a symplectic germ at a fixed point. The only problem is that this invariant may be trivial: the minimal action could well be infinite, i.e.,

$$\alpha \equiv +\infty,$$

because it is not clear whether there are any orbits in $U \setminus \{0\}$ at all. Therefore, we have to find situations where the minimal action is a nontrivial invariant, i.e., a real–valued function.

At this point we will make four additional assumptions on the symplectic map. First of all, we assume smoothness.

I. ϕ is a symplectic C^∞–diffeomorphism defined on a simply connected open neighbourhood U of $0 \in (\mathbb{R}^2, \Omega)$ with $\phi(0) = 0$.

Secondly, we make some restrictions on the linearization of ϕ at the fixed point. As a symplectic mapping, the eigenvalues λ_1 and λ_2 of $D\phi(0)$ are inverse to each other, i.e.,

$$\lambda_2 = \frac{1}{\lambda_1}.$$

We will assume that both eigenvalues have modulus 1. More precisely, we make the following definition.

Definition 4.1.7. *A fixed point* 0 *of a symplectic map is called* elliptic *if* $D\phi(0)$ *has eigenvalues* $\lambda, \overline{\lambda} \in \mathbb{S}^1 \setminus \{\pm 1\}$.

Our second assumption for our setting is then the ellipticity of the fixed point.

II. 0 is an elliptic fixed point of ϕ.

If 0 is an elliptic fixed point of ϕ then its linearization $D\phi(0)$ is a rotation. In order to remove the ambiguity in the rotation number, we write

$$\lambda = e^{2\pi i a}$$

with $0 \leq a < 1$ and fix the lift $\widetilde{\phi}$ in such a way that $\widetilde{\phi}(\theta, r) \to (\theta + a, r)$ as $r \to 0$. This means that we associate to the fixed point the rotation number a (and not some integer shift $a + k$ of it).

In the following, we want to construct a certain normal form of ϕ in the neighbourhood of an elliptic fixed point. This idea goes back to Birkhoff who proved the existence of a normal form under some additional assumptions on the eigenvalue λ. We will assume that λ satisfies the following condition.

III. λ satisfies the non–resonance condition $\lambda^k \neq 1$ for $1 \leq k \leq 4$.

Then, under the assumptions I.–III., there is an analytic symplectic change of coordinates fixing the fixed point 0 and transforming ϕ into a certain normal form in the standard symplectic space $(\mathbb{R}^2, \Omega_0 = dx \wedge dy)$. This is the ocntent of the next theorem, a proof of which can be found in [76].

Theorem 4.1.8 (Birkhoff normal form). *Suppose a map* ϕ *satisfies the conditions I.–III. Then there is an analytic symplectic change of coordinates transforming* ϕ *into the form*

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \begin{pmatrix} \cos 2\pi\Theta & -\sin 2\pi\Theta \\ \sin 2\pi\Theta & \cos 2\pi\Theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \mathcal{O}((x^2+y^2)^2) \\ \Theta &= a + b(x^2+y^2) \end{aligned} \tag{4.2}$$

as $x^2 + y^2 \to 0$*. The numbers* a, b *are called the* Birkhoff invariants *of* ϕ*; they are symplectically invariant. The leading term in* (4.2) *is called the* Birkhoff normal form *of the map* ϕ*.*

The symplectic invariance of the Birkhoff invariants means that the process of transforming ϕ into normal form yields the same result, regardless of the symplectic coordinates you chose at the beginning to represent ϕ.

Our last assumption on the map ϕ is a nonlinearity condition on the Birkhoff normal form.

IV. $b \neq 0$, respectively, $b > 0$.

Note that it is no loss of generality to assume that the second Birkhoff invariant is positive, i.e, $b > 0$. Indeed, if $b < 0$ we consider ϕ^{-1} instead of ϕ; this map has $-b > 0$ as second Birkhoff invariant.

Definition 4.1.9. *Suppose a map ϕ meets all four conditions I.–IV. Then the fixed point 0 is called a* general elliptic fixed point *of ϕ.*

We point out that the notion a general elliptic fixed point is intrinsic, i.e., the above conditions are invariant under smooth symplectic coordinate transformations.

In the following, we will construct the minimal action associated to a general elliptic fixed point. There is, however, a sligth detail to consider here. Since the rotation number of the fixed point is assumed to be a, the map ϕ will cease to have orbits of rotation numbers less than a. Therefore, the minimal action α will only be defined on some closed half–interval $[a, a+\epsilon)$, where the action of the fixed point is assumed to be zero:

$$\alpha(a) = 0.$$

Since we are interested in the local behaviour of ϕ, we pass to considering the germ of ϕ at the fixed point 0. Consequently, we need to introduce the following notion.

Definition 4.1.10. *The* half–sided germ *of a function at a point $x \in \mathbb{R}$ is the equivalence class of functions defined on intervals $[x, z)$, where two such functions are equivalent if they agree on some (maybe smaller) interval $[x, y)$.*

Finally, in accordance with Def. 1.3.1, the term *invariant circle* always means an invariant circle that goes around the fixed point.

The following is the main result in this section. Recall that $\lambda = e^{2\pi i a}$ with $0 \leq a < 1$.

Theorem 4.1.11. *Given the germ of a symplectic diffeomorphism ϕ at a general elliptic fixed point, the half–sided germ of the minimal action α at the point a is a nontrivial symplectic invariant. In addition, one has the following:*

1. *The Birkhoff invariants are the Taylor coefficients of the convex conjugate α^* at 0.*
2. *ϕ possesses an invariant circle of rotation number p/q, consisting of periodic orbits, if and only if α is differentiable at p/q.*

3. If ϕ has an invariant circle of rotation number ω, its enclosed area is given by $\alpha'(\omega)$.

Proof. Let $\alpha : \mathbb{R} \to (-\infty, \infty]$ be the minimal action for ϕ. We may assume that ϕ is already given in the form (4.2); since α is symplectically invariant this does not change anything. To prove that α is nontrivial (i.e. not identically ∞) we introduce symplectic polar coordinates $(\theta, r) \in \mathbb{S}^1 \times (0, \infty)$ on $\mathbb{R}^2 \setminus \{0\}$ by

$$\begin{cases} x = \sqrt{2r}\cos 2\pi\theta \\ y = \sqrt{2r}\sin 2\pi\theta \end{cases}$$

It is a straightforward calculation to show that

$$\frac{1}{2}(x\,dy - y\,dx) = 2\pi r\,d\theta$$

so that the map

$$\begin{gathered} (\mathbb{R}^2 \setminus \{0\}, \mathrm{d}x \wedge \mathrm{d}y) \to (\mathbb{S}^1 \times (0,\infty), 2\pi \mathrm{d}r \wedge \mathrm{d}\theta) \\ (x,y) \mapsto (\theta, r) \end{gathered}$$

is exact symplectic with respect to the 1–forms $1/2\,(x\,\mathrm{d}y - y\,\mathrm{d}x)$ and $2\pi\mathrm{d}r\,\mathrm{d}\theta$, respectively. Hence the average action of corresponding orbits stays the same if we pass to (θ, r)–coordinates. The map ϕ has the form

$$\phi : (\theta_0, r_0) \mapsto (\theta_1, r_1) = (\theta_0 + a + 2br_0, r_0) + \mathcal{O}(r_0^{3/2}) \tag{4.3}$$

as $r_0 \to 0$. For small enough $r_0 > 0, \phi$ satisfies the monotone twist condition $\partial\theta_1/\partial r_0 = 2b + \mathcal{O}(r_0^{1/2}) > 0$.

Since ϕ is smooth, KAM–theory applies and yields the existence of invariant circles accumulating at the fixed point, respectively the boundary circle $\mathbb{S}^1 \times \{0\}$; see [56, 77]. On each of these circles Γ_ω the map ϕ is conjugated to the rotation by some Diophantine number ω near a; since the twist constant b is positive we have $\omega > a$.

Therefore, perhaps after restriction to a smaller domain, ϕ is defined on an invariant annulus in $\mathbb{S}^1 \times (0, \infty)$ with lower boundary $S^1 \times \{0\}$. This annulus itself is divided into a sequence of invariant annuli A_k, approaching $\mathbb{S}^1 \times \{0\}$ as $k \to \infty$ and being bounded by KAM–circles $\Gamma_{\omega_k^\pm}$ with rotation numbers $\omega_k^+ > \omega_k^- > a$. According to (4.3), the map ϕ on each $A_k \cup A_{k+1}$ is a smooth monotone twist map whose generating function with $r_1\,d\theta_1 - r_0\,d\theta_0 = dh$ is given by

$$h(\xi, \eta) = \frac{1}{4b}\,(\eta - \xi - a)^2 + \mathcal{O}((\eta - \xi - a)^{5/2}) \tag{4.4}$$

as $\eta - \xi \to a$. The function

$$h_0(s) = \frac{1}{4b}\,(s - a)^2$$

describes the integrable twist map $\phi_0(\theta_0, r_0) = (\theta_0 + a + 2br_0, r_0)$ approximating ϕ. Notice that h is normalized according to our convention; namely, $h(\xi, \eta) \to 0$ as $\eta - \xi \to a$ which means that the (hypothetical) value of h at the fixed point is 0.

Now we apply Aubry–Mather theory for ϕ on each "double" annulus $A_k \cup A_{k+1}$. In view of Theorem 1.3.4 and Remark 1.3.5, there are minimal orbits for every rotation number $\omega \in (\omega_k^-, \omega_{k+1}^+)$. This allows us to define the minimal action α in the sense of Section 1.2, which is a strictly convex function on the interval $(\omega_k^-, \omega_{k+1}^+)$.

We claim that this α is the minimal action as defined in (4.1). First of all, the notions of average action and rotation number agree. Therefore, the only thing to check is that the set of orbits over which we minimize is the same in both settings. This follows from the fact that *all* orbits of rotation numbers $\omega \in (\omega_k^-, \omega_{k+1}^+)$ lie in the annulus $A_k \cup A_{k+1}$. Indeed, suppose that a monotone twist map possesses two invariant circles $\Gamma_{\omega^\pm}$ of rotation numbers $\omega^- < \omega^+$. Then, if an orbit lies outside the annulus formed by Γ_{ω^-} and Γ_{ω^+}, its rotation number must lie outside (ω^-, ω^+); this is a simple consequence of the twist property.

Thus, the minimal action α is a real valued, strictly convex function on each interval $(\omega_k^-, \omega_{k+1}^+)$. Note that the annuli $A_k \cup A_{k+1}$ overlap so each rotation number $\omega_k^\pm$ is an interior point at some stage, and the different pieces of α really fit together. Moreover, as $k \to \infty$, the rotation numbers $\omega_k^\pm$ tend to a and the average actions to zero, so that the minimal action extends to a strictly convex function $\alpha : [a, a+\delta) \to \mathbb{R}$ with $\alpha(a) = 0$.

This proves the first part of the theorem. The assertion that the minimal action determines the existence of periodic invariant circles as well as the enclosed areas of invariant circles follows immediately from Proposition 1.3.7. It remains to prove that the minimal action encodes the Birkhoff invariants. For this, we consider the convex conjugate $\alpha^*(I) = \max_\omega[\omega I - \alpha(\omega)]$ which is a strictly convex C^1–function defined on some interval $[0, \delta^*)$. Applying Theorem 1.3.8, we conclude from (4.4) that

$$\alpha(\omega) = h_0(\omega) + \mathcal{O}((\omega - a)^{5/2}) = \frac{1}{4b}(\omega - a)^2 + \mathcal{O}((\omega - a)^{5/2}) \qquad (4.5)$$

as $\omega \to a$ which implies an analogous formula for $\alpha^*(I)$ as $I \to 0$ [90, Ex. 8.8]:

$$\alpha^*(I) = h_0^*(I) + \mathcal{O}(I^{5/2}) = aI + bI^2 + \mathcal{O}(I^{5/2}). \qquad (4.6)$$

Hence the Taylor coefficients of α^* at 0 are indeed the Birkhoff invariants a and b, and the theorem is completely proven. □

Remark 4.1.12. Theorem 4.1.11 shows that the minimal action is a *local* invariant in the sense that it contains information not just about the asymptotic behaviour of ϕ at the fixed point, but also about the dynamics away from it.

Remark 4.1.13. The assumption that ϕ is smooth is not really necessary; in fact, Theorem 4.1.11 is true for C^5–diffeomorphisms [77]. For the sake of simplicity, however, we restrict ourselves to the smooth case.

Remark 4.1.14. If the Birkhoff normal form approximates the given map ϕ up to order $(x^2 + y^2)^k$ with $k \geq 2$, then the Taylor coefficients of α^* exist up to order k and Theorem 1.3.8 implies that they are precisely the k Birkhoff invariants of ϕ.

Remark 4.1.15. The fact that the Birkhoff invariants are encoded in the actions of periodic orbits (via the labelled length spectrum) was first formulated by Colin de Verdiere [19]. The minimal action, respectively its convex conjugate, can be viewed as an extension of the labelled length spectrum from the rational numbers to the reals.

Remark 4.1.16. The minimal action α may be seen as a "partial integral" for the map ϕ. This goes as follows. Consider the set $\mathcal{M} \subset U \setminus \{0\}$ of minimal orbits. Then the function $p \mapsto \alpha(\rho(\phi^i(p)))$ from $\mathcal{M}$ to $\mathbb{R}$ is constant along orbits but certainly not constant everywhere.

In general, the "partial integral" mentioned in Rem. 4.1.16 is neither defined in a whole neighbourhood of 0, nor is it differentiable. In the special situation when ϕ possesses a genuine integral, however, the minimal action turns out to be an integral. In this context, we recall the definition of integrability for an area–preserving map.

Definition 4.1.17. *Suppose that ϕ is a smooth area–preserving map defined near the elliptic fixed point 0. Then ϕ is called* integrable *if, perhaps after restricting ϕ to some smaller neighbourhood U of 0, there is a smooth fibration of $U \setminus \{0\}$ by invariant circles. More generally, ϕ is called* C^0–integrable *if there is a C^0–fibration by invariant circles.*

Theorem 4.1.18. *Given the germ of a symplectic diffeomorphism ϕ at a general elliptic fixed point, let α denote the associated minimal action. Then the following holds true:*

1. *If ϕ is integrable, α^* is an integrable Hamiltonian generating ϕ.*
2. *If α is differentiable then ϕ is C^0–integrable.*

Proof. In order to prove the first assertion, we pass to angle–action coordinates $(\theta_0, I_0) \in \mathbb{S}^1 \times (0, \epsilon)$ in which we have $\phi : (\theta_0, I_0) \mapsto (\theta_1, I_1) = (\theta_0 + H'(I_0), I_0)$ with a smooth strictly convex Hamiltonian H. Repeating the calculation following (3.11) we compute that $I_1\, d\theta_1 - I_0\, d\theta_0 = dS^*$ with $S(\theta_0, I_1) = \theta_0 I_1 + H(I_1)$, which means that $S^* = H^*$ is a generating function for the integrable twist map ϕ. Hence $H = \alpha^*$ is an autonomous integrable Hamiltonian generating ϕ.

We show the second assertion. According to Thm. 1.3.7, the minimal action is differentiable at irrational numbers, and it is differentiable at rationals if and only if there is an invariant circle of (periodic) minimal orbits of the corresponding rotation number. Therefore, if α is differentiable we obtain invariant circles for all rotation numbers by taking limits of rational ones, so ϕ is C^0–integrable. $\square$

Remark 4.1.19. We see that, in the integrable case, the dynamics of ϕ are completely determined by the symplectic invariant α. This is not true for the Birkhoff normal form unless ϕ is analytic; see [47].

Note that, as a strictly convex function, α is differentiable if and only if it is C^1 [90, Thm. 11.13].

Finally, we just mention that there are higher order Birkhoff normal forms near an elliptic fixed point if the eigenvalue $\lambda = e^{2\pi i a}$ at the fixed point satisfies non–resonance conditions of higher order. For instance, if λ is not a root of unity the Birkhoff normal form is a formal power series. In general, the coordinate transformation bringing ϕ to that normal form will be a divergent power series. We refer to [98] for proofs and more details. Everything in this section can also be formulated in this more general context, but we forgo such extensions.

4.2 Contact flows in three dimensions

Let M be a smooth compact three–dimensional manifold[1]. We want to define something like the odd–dimensional analogue of a symplectic form.

Definition 4.2.1. *A smooth 1–form η on M is called a* contact form *if*

$$\eta \wedge \mathrm{d}\eta \neq 0$$

pointwise, i.e., if $\eta \wedge \mathrm{d}\eta$ is a volume form on M.

This definition means that the kernel of a contact form η defines a maximally non–integrable hyperplane field in TM; see, e.g., [73] for more details and further references on contact geometry. Any contact form defines a unique vector field, and hence a unique flow on M, as follows.

Definition 4.2.2. *Let η be a contact form on M. Then the* Reeb vector field *X on M is defined by the equations*

$$i_X \mathrm{d}\eta = 0 \quad \text{and} \quad i_X \eta = 1. \tag{4.7}$$

The corresponding flow on M is called the Reeb flow. *Periodic trajectories of the Reeb flow are also called* closed characteristics.

[1] Most of what is said in this section can be formulated for odd–dimensional manifolds in higher dimensions. We are only interested in the 3–dimensional case, however.

Example 4.2.3. The basic example is the contact form η_0 on $\mathbb{R}^3$ given by

$$\eta_0 := \mathrm{d}z - y\mathrm{d}x$$

where x, y, z are coordinates on $\mathbb{R}^3$. Note that $y\mathrm{d}x$ is the standard Liouville form on $\mathbb{R}^2$ whose differential is the standard symplectic form $\mathrm{d}y \wedge \mathrm{d}x$ from Ex. 2.1.13.

Example 4.2.4. Here comes a less trivial and very important example. Let (N, g) be a 2–dimensional Riemannian manifold, and $M := T_1^*N$ its unit cotangent bundle. Let λ be the canonical Liouville form; see Ex. 2.1.14. Then the restriction

$$\eta := \lambda|_{T_1^*N}$$

is a contact form on M. The corresponding Reeb vector field is the Hamiltonian vector field on T_1^*N, conjugate to the geodesic vector field on T_1N given by the convex Lagrangian $1/2\, g_x(v, v)$.

Let η be a contact form on a 3–dimensional manifold M. Assume γ is a periodic trajectory of (prime) period T of the Reeb flow. In the following we want to reduce the 3–dimensional Reeb flow near the closed characteristic γ to a 2–dimensional mapping. For this, we consider a transverse local section W at some point $p \in \gamma$. This is a 2–dimensional manifold, and we equip it with the symplectic form $\omega := i^*\mathrm{d}\eta$ where $i : W \hookrightarrow M$ is the inclusion. For each point in W, we follow its trajectory until the first time it returns to the local section W again. Then we map the original point to that first return point. By continuity, this map is defined on some small neighbourhood of p in W. This is the so–called *Poincaré map*; see Fig. 4.2. The map associating to each point in W the time it returns to W is called the *first return time*.

Let us identify the small neighbourhood of p in W with a small neighbourhood U of 0 in $\mathbb{R}^2$. Denote by

$$\phi : U \to U$$

the Poincaré map, and by

$$S : U \to \mathbb{R}$$

the first return time. It follows that $\phi(0) = 0$ and $S(0) = T$.

It is well known that the Poincaré map ϕ is symplectic—this is just a reformulation of the fact that time and energy are conjugate variables in Hamiltonian mechanics. In fact, ϕ is even exact symplectic as the following observation shows; compare, for instance, [32, Prop. 2.1].

Lemma 4.2.5. *The Poincaré map ϕ defined above is exact symplectic with the first return time S being a generating function S:*

$$\phi^*\eta - \eta = \mathrm{d}S.$$

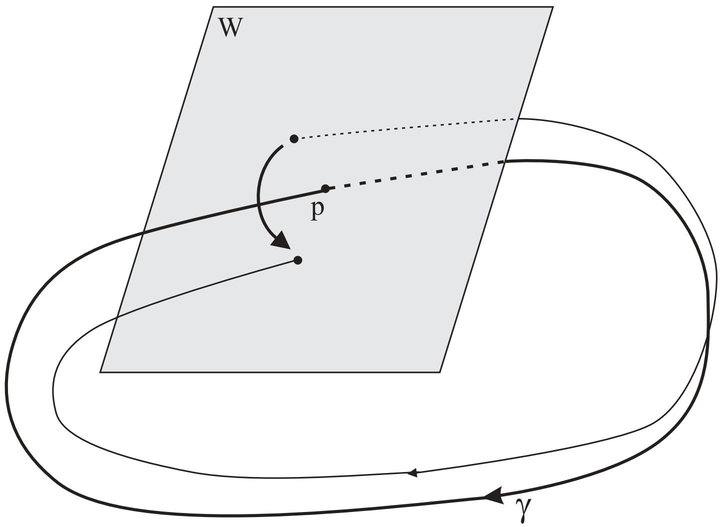


Fig. 4.2. The Poincaré map of a closed characteristic γ

In other words, the time is a generating function for the Poincaré return map.

Proof. Let X be the Reeb flow, and denote its flow by ψ^t. Consider the family of mappings

$$f_t(z) := \psi^{tS(z)}(z).$$

Then $f_1 = \phi$ and $\frac{\mathrm{d}}{\mathrm{d}t} f_t(z) = S(z)X(f_t(z))$. Therefore

$$\phi^*\eta - \eta = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f_t^* \eta \, dt = \int_0^1 f_t^* (i_{\frac{\mathrm{d}}{\mathrm{d}t} f_t} \mathrm{d}\eta + \mathrm{d} i_{\frac{\mathrm{d}}{\mathrm{d}t} f_t} \eta) \, dt = \mathrm{d}S$$

in view of (4.7). $\square$

The general question will be how much information about the geometry of the contact manifold M is encoded in the closed characteristics. This is the analogue of the question formulated and dealt with in Ch. 3 for convex domains in the plane, where we asked how much information is hidden in the length spectrum. In the setting of Reeb flows, however, the notion of length has no meaning yet. Therefore we replace the length spectrum by the period spectrum.

Definition 4.2.6. *Let M be a 3–dimensional compact manifold. The* period spectrum $\mathcal{P}(\eta)$ *of a contact form η on M is defined as the set of periods of all closed characteristics, together with all their (non–negative) multiples.*

In the following, let us consider a continuous deformation $\eta_s, s \in [0,1]$, of contact forms on M such that each (M, η_s) has the same period spectrum:

$$\mathcal{P}(\eta_s) = \mathcal{P}(\eta_0)$$

for each s. How can the dynamics of the corresponding Reeb flows change during the deformation?

4.2.1 Spectral invariants

Françoise and Guillemin [32] conjectured that, in any odd dimension, such a deformation preserving the period spectrum must be trivial, if it also fixes the set of (symplectic conjugacy classes of) linearized Poincaré maps at the closed characteristics. They proved that, if γ_s is a nondegenerate elliptic closed characteristic without resonances, the Birkhoff invariants of γ_s stay fixed during the deformation. This was generalized by Popov [87] who showed that the Birkhoff invariants as well as the Liouville classes of invariant tori stay fixed, even allowing resonances and dropping the condition that the deformation preserves the linearized Poincaré maps.

What we will do is to show that, for the three–dimensional case, there is a stronger invariant than just the Birkhoff normal form, namely the minimal action. In order to apply the results from Sect. 4.1, we reduce the three–dimensional Reeb flow to its two–dimensional Poincaré map near a closed characteristic. For the latter, we assume that this reduction leads to a map with a general elliptic fixed point (see Def. 4.1.9).

Definition 4.2.7. *A closed trajectory γ of a three–dimensional Reeb flow is called a* general elliptic closed characteristic *if the corresponding Poincaré map ϕ has 0 as a general elliptic fixed point.*

Note that this definition is independent of the choices of the point on γ and the transverse section because two Poincaré maps are symplectically conjugated and the conditions I.–IV. stated at the beginning of Sect. 4.1 are invariant under such conjugations.

Then, applying the theory from Sect. 4.1, we can associate to a general elliptic closed characteristic γ the half–sided germ of the minimal action α. To be really consistent with our notation from the previous section where we assumed that generating functions satisfy $S(0) = 0$, we replace the first return time $S(\cdot)$ by $S(\cdot) - T$, where T is the (prime) period of γ; for simplicity, we use the same letter S for the shifted first return time.

Now let $\eta_s, s \in [0,1]$, be a continuous family of contact forms on M, all having the same period spectrum, such that there is a continuous family of general elliptic closed characteristics γ_s. The next result states that the corresponding minimal actions α_s do not depend on s.

Theorem 4.2.8. *Suppose $\eta_s, s \in [0,1]$, is a continuous deformation of contact forms such that*
$$\mathcal{P}(\eta_s) = \mathcal{P}(\eta_0),$$
together with a continuous family of general elliptic closed characteristics γ_s. Then, as germs, we have
$$\alpha_s = \alpha_0$$
for all $s \in [0,1]$.

Proof. Associated to each closed characteristic γ_s, we have the half–sided germ of the minimal action α_s for the corresponding Poincaré return map. Being continuous, each α_s is uniquely defined by its values on $\mathbb{Q}$. We will show below that, for a fixed rational rotation number p/q, the values $\alpha_s(p/q)$ vary continuously with s. Postponing the proof, we claim that these values must be constant. Indeed, the period spectrum $\mathcal{P}(\eta_s)$, which is independent of s by assumption, has Lebesgue measure 0 in $\mathbb{R}$. This follows from Sard's Theorem since closed characteristics correspond to critical points of a smooth function. We had seen a similar argument in the proof of Prop. 3.2.2; for a detailed proof see, for instance, [87, Prop. 3.2]. Therefore the values $\alpha_s(p/q)$ vary continuously in a set of measure zero, so they must stay fixed.

It remains to prove that $\alpha_s(p/q)$ is continuous in s. For this, we recall from Theorem 1.3.4 that for rational rotation numbers there is always a *periodic* minimal orbit. Besides being periodic, these so–called Birkhoff orbits have the additional property that they are ordered as if they were orbits of a rigid rotation, and they can be found by minimizing the (discrete) action on the compact space of ordered periodic sequences [51, Thm. 9.3.7]. As minima, the corresponding minimal values $\alpha_s(p/q)$ are indeed continuous in s. □

In fact, one can even eliminate the assumption that we are given a family of general elliptic closed characteristics. Its existence follows already from the preservation of the period spectrum, as the next lemma shows; compare [87, Lemma 3.5] for a similar argument.

In the general theory of dynamical systems, the eigenvalues of a linearized Poincaré map $D\phi(0)$ of some closed trajectory γ are called the *Floquet multipliers* of γ. .

Lemma 4.2.9. *Suppose $\eta_s, s \in [0,1]$, is a continuous deformation of contact forms preserving the period spectrum, such that η_0 admits a general elliptic closed characteristic γ_0.*

Then there is a continuous family of general elliptic closed characteristics γ_s for each $\eta_s, s \in [0,1]$. Moreover, their periods and Floquet multipliers do not depend on s.

Proof. First of all, the condition that γ_0 is general guarantees that 1 is not a Floquet multiplier of γ_0. This implies that one can continue the fixed point of the Poincaré map, corresponding to γ_0, uniquely as a fixed point for small

$s > 0$, corresponding to a periodic trajectory γ_s. This is a standard technique, using the Implicit Function Theorem; see [98], for instance. Moreover, because everything changes continuously with s, the new closed characteristics γ_s are general elliptic provided s is small enough, say, for $s \in [0, \delta)$. In addition, since the period spectrum has Lebesgue measure 0, the periods of γ_s are all the same.

To each γ_s we associate the germ of the minimal action α_s. Thm. 4.2.8 implies

$$\alpha_s = \alpha_0.$$

It follows that the Birkhoff invariants of the Poincaré map, which are the Taylor coefficients of α_s^* in view of Thm. 4.1.11, do not change along the deformation. In particular, the Floquet multipliers stay fixed during the deformation. This proves the assertion for $s \in [0, \delta)$ where $\delta \in (0, 1]$ is assumed to be maximal.

We want to show that $\delta = 1$. Assume, on the contrary, that $\delta < 1$. Then, taking limits of the closed characteristics γ_s as $s \to \delta$, we find a closed characteristic for $s = \delta$. Moreover, the Poincaré maps of γ_s converge in the C^∞–topology to the Poincaré map of γ_δ. Our assumption that the period spectrum remains unchanged implies that γ_δ satisfies the conditions I.–III. from Sect. 4.1. On the other hand, applying Thm. 4.1.11 again, we know that the Birkhoff invariants of γ_δ are the same as those of γ_0. Thus, γ_δ is again a general elliptic closed characteristic. This proves that the set of parameters s, for which there is a continuous family of general elliptic closed characteristics, beginning with γ_0, is open and closed in $[0, 1]$. Hence $\delta = 1$, and the proof of the lemma is finished. □

Now we can translate our results for fixed points of symplectic mappings into the language of contact geometry. Again, we point out that the minimal action is a period spectrum invariant under continuous deformations of the contact form.

Theorem 4.2.10. *Suppose $\eta_s, s \in [0, 1]$, is a continuous family of contact forms on a three–dimensional manifold with*

$$\mathcal{P}(\eta_s) = \mathcal{P}(\eta_0),$$

such that η_0 admits a general elliptic closed characteristic γ_0.

Then there is a continuous family of general elliptic closed characteristics γ_s for $s \in [0, 1]$ whose half–sided germs of minimal actions α_s do not depend on s:

$$\alpha_s = \alpha_0.$$

In particular, this implies the following:

1. *The Birkhoff invariants of γ_0 and γ_1 are the same.*
2. *The Poincaré map ϕ_1 possesses an invariant circle of rotation number p/q, consisting entirely of periodic orbits, if and only if ϕ_0 does.*

3. *If ϕ_0 and ϕ_1 each have an invariant circle of rotation number ω, their enclosed areas agree.*
4. *If ϕ_0 is integrable then ϕ_1 is C^0–integrable.*

Proof. Under the above assumptions, Lemma 4.2.9 implies that we have a family of minimal actions α_s which, by Thm. 4.2.8, are all equal. Thus, the half–sided germ of the minimal action

$$\alpha_s = \alpha_0 =: \alpha$$

is a period spectrum invariant.

We prove the four implications. Since the Birkhoff invariants are the Taylor coefficients of α^* (Thm. 4.1.11), they are invariant too. Moreover, ϕ_1 possesses a periodic invariant circle of rotation number p/q if and only if α is differentiable at p/q (Thm. 1.3.7); since α is invariant, the same holds true for ϕ_0. A similar argument proves the third statement because the area enclosed by an invariant circle is given by $\alpha'(\omega)$ (Thm. 1.3.7). Finally, if ϕ_0 is integrable then α^* is an integrable Hamiltonian (Thm. 4.1.18); in particular, α is smooth, which implies the C^0–integrability of ϕ_1. □

4.2.2 Length spectrum invariants of surfaces

We consider a compact surface, i.e., a smooth compact two–dimensional Riemannian manifold (M, g). As usual, the *length spectrum*

$$\mathcal{L}(M, g)$$

is the set of lengths of all closed geodesics on M, together with all their multiples. The question is how much information about the dynamics of the geodesic flow is encoded in $\mathcal{L}(M, g)$.

According to Ex. 4.2.4, the Hamiltonian vector field on the unit cotangent bundle T_1^*M, conjugate to the geodesic vector field on the unit tangent bundle T_1M, is the Reeb vector field of the contact form

$$\eta := \lambda|_{T_1^*M}$$

where λ is the canonical Liouville form on the cotangent bundle T^*M; see Ex. 2.1.14. Therefore, we may repeat the results from Sect. 4.2.1 for the case of geodesic flows.

For this, we assume we have a continuous deformation $g_s, s \in [0, 1]$, of Riemannian metrics on M starting at $g_0 = g$ such that the length spectrum is preserved:

$$\mathcal{L}(M, g_s) = \mathcal{L}(M, g_0)$$

for all $s \in [0, 1]$. In addition, we suppose that g_0 possesses a general elliptic closed geodesic γ_0. Then, according to Lemma 4.2.9, there is a family of general elliptic closed geodesics γ_s for g_s. Let us call α_s the corresponding half–sided germs of minimal actions. Thm. 4.2.10 implies that

$$\alpha_s = \alpha_0$$

for all $s \in [0,1]$. We may formulate a more pointed version of this as the following principle.

Invariance Principle. Every quantity that can be calculated from the minimal action α is, by tautology, a length spectrum invariant under continuous deformations of the Riemannian metric.

More precisely, we have the following result which is Thm. 4.2.10 applied to the framework of Riemannian geometry.

Theorem 4.2.11. *Suppose $g_s, s \in [0,1]$, is a continuous family of Riemannian metrics on a compact two–dimensional manifold that preserves the length spectrum, such that g_0 admits a general elliptic closed geodesic γ_0.*

Then there is a continuous family of general elliptic closed geodesics γ_s whose half–sided germs of minimal actions do not depend on s. In particular, this implies the following:

1. *The Birkhoff invariants of γ_0 and γ_1 are the same.*
2. *The Poincaré map ϕ_1 possesses an invariant circle of rotation number p/q, consisting of periodic orbits, if and only if ϕ_0 does.*
3. *If ϕ_0 and ϕ_1 each have an invariant circle of rotation number ω, their enclosed areas agree.*
4. *If the geodesic flow of g_0 is integrable near γ_0 then the geodesic flow of g_1 is C^0–integrable near γ_1.*

Notice that an invariant circle of the Poincaré map gives rise to a two–dimensional invariant torus of the geodesic flow around the closed geodesic; see Fig. 4.3.

Remark 4.2.12. Concerning the last part of C^0–integrability, there is the following unpublished result by G. Forni and S. Zelditch, announced in [103]; see also [105]. Suppose you are given an analytic, rotationally symmetric metric g on S^2 with certain additional nondegeneracy conditions. In this case, the geodesic flow is integrable by Clairaut's Theorem. Forni and Zelditch showed that, if h is another metric with the same Laplace spectrum as g, the geodesic flow of h is C^0–integrable.

This remark leads to the question whether the Laplace spectrum characterizes a manifold up to isometries. For instance, Kac' question "Can one hear the shape of a drum?" asked if there are non–isometric domains in the plane with the same Laplace spectrum. It is well known that the answer is yes [35]. The Laplace spectrum is related to the length spectrum via trace formulae

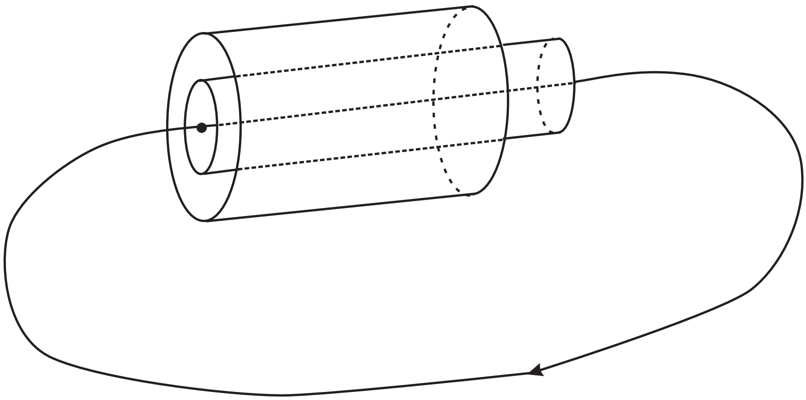


Fig. 4.3. Invariant tori of an integrable geodesic flow around a closed geodesic

and Poisson relations; we refer to [18, 38, 103] for details and more references. Zelditch [104] showed that a special class of real analytic surfaces of revolution is completely determined by the Laplace spectrum. The Birkhoff normal form is still an essential ingredient for the proof but does not suffice to obtain the full result.

4.3 The minimal action near positive definite invariant tori

In this section, we formulate appropriate versions of the results of Sect. 4.1 for invariant tori of symplectic mappings in higher dimensions, rather than for fixed points of area–preserving maps in the plane. The basic ideas are very similar to those in Sect. 4.1.

Let (M,ω) be a $2n$–dimensional symplectic manifold, and $\phi : M \to M$ be a smooth symplectic diffeomorphism; see Sect. 2.1.2 for definitions. Suppose that $\Lambda \subset M$ is an n–dimensional submanifold which is invariant under ϕ:

$$\phi(\Lambda) = \Lambda.$$

Suppose further that $\phi|_\Lambda$ is smoothly conjugate to the translation on the n–dimensional torus $\mathbb{T}^n$ by some vector $\rho \in \mathbb{R}^n$ satisfying the Diophantine condition

$$|k_0 + k \cdot \rho| \geq C\,|k|^{-\tau} \tag{4.8}$$

for all $k_0 \in \mathbb{Z}$ and $k \in \mathbb{Z}^n \setminus \{0\}$, where C and τ are some positive constants.

Under these assumptions, it follows from a remark by Herman [43] that Λ is actually a Lagrangian submanifold of M. Therefore, by Weinstein's Lagrangian neighbourhood theorem [73], there exists a neighbourhood of Λ in M

which is symplectically diffeomorphic to a neighbourhood of the zero section in the cotangent bundle $T^*\Lambda$ with its standard symplectic form. Note that Λ is diffeomorphic to $\mathbb{T}^n$ and invariant under ϕ. Therefore, it follows that we can find symplectic coordinates (x, y) in a neighbourhood of the zero section $\mathbb{T}^n \times \{0\}$ in $T^*\mathbb{T}^n = \mathbb{T}^n \times \mathbb{R}^n$ such that

$$\phi(x, y) = (x + \rho + A(x)y, B(x)y) + \mathcal{O}(|y|^2)$$

where $A(x), B(x)$ are $n\times n$–matrices depending on the point $x \in \mathbb{T}^n$. Since the standard symplectic form $\mathrm{d}y \wedge \mathrm{d}x$ is preserved under the symplectic map ϕ, $B(x)$ is the identity matrix and $A(x)$ is symmetric. Finally, the Diophantine condition (4.8) guarantees that, by averaging $A(x)$ over $\mathbb{T}^n$, one can choose symplectic coordinates in which the matrix $A(x)$ becomes independent of x; see [69] for details, or Thm. 4.3.6 for a refined version of this result.

Summarizing, under the above assumptions, there are symplectic coordinates (x, y) near the zero section in $T^*\mathbb{T}^n$ such that the map ϕ takes the form

$$\phi(x, y) = (x + \rho + Ay, y) + \mathcal{O}(|y|^2) \tag{4.9}$$

where A is some $n \times n$–matrix.

In the end, we want to interpolate ϕ by a Hamiltonian, respectively Lagrangian, flow and apply Mather's theory on minimizing measures. In order to do so, we need some positive definiteness condition.

Definition 4.3.1. *Let ϕ be a symplectic diffeomorphism with an invariant torus Λ, such that there are symplectic coordinates near Λ in which* (4.9) *holds. Then Λ is called* positive definite *if the matrix A in* (4.9) *is positive definite.*

Remark 4.3.2. It is actually sufficient to assume that A is just definite. For, if A is negative definite, one considers ϕ^{-1} instead of ϕ.

Remark 4.3.3. The positive definiteness of Λ is a well defined notion. Indeed, if we choose different coordinates (x', y') leading to a representation

$$\phi(x', y') = (x' + \rho + A'y', y') + \mathcal{O}(|y'|^2)$$

with a different symmetric matrix A' then one can calculate that $A' = B^\top AB$ where B is some invertible matrix. Hence A' is positive definite if, and only if, A is.

The following elementary lemma shows that the locally defined symplectic diffeomorphism ϕ can be embedded in a global Hamiltonian flow on $T^*\mathbb{T}^n$ generated by a convex Hamiltonian. Therefore, it fits into the setting of Mather–Mañé theory.

Lemma 4.3.4 ([69]). *Let Λ be a positive definite invariant torus of a symplectic diffeomorphism ϕ defined in a neighbourhood of Λ. Then ϕ can be written as the restriction of the time–1–map of a Hamiltonian flow on $T^*\mathbb{T}^n$ generated by a convex Hamiltonian on $\mathbb{S}^1 \times T^*\mathbb{T}^n$.*

Applying Mather–Mañé theory to the extended convex Hamiltonian, respectively Lagrangian, yields the corresponding minimal action defined on $H_1(\mathbb{T}^n,\mathbb{R})$. A priori, this function depends on the extension of ϕ and the choice of interpolating Lagrangian.

Concerning the dependence on the Lagrangian, Mather showed [67, 69] that there is a one–to–one correspondence between invariant measures of L and invariant measures of its time–1–map, and that the choice of a different interpolating Lagrangian affects the minimal action only by an affine function of the rotation vector. Let us identify $H_1(\mathbb{T}^n,\mathbb{R}) = \mathbb{R}^n$ in such a way that the homological rotation vector of the unique invariant measure supported on $\mathbb{T}^n \times \{0\}$ is mapped onto ρ. Then, up to irrelevant additive constants, the minimal action does not depend on the interpolating Lagrangian anymore.

The dependence on the extension is more crucial. It follows from the a priori bound on the Lipschitz constant in Thm. 2.1.11, however, that minimal invariant measures of rotation vectors near ρ (the rotation vector of the invariant torus Λ) lie also near Λ; see [69]. This localization result implies that the germ of the minimal action at ρ does *not* depend on the extension of ϕ.

Summarizing, we have the following result.

Proposition 4.3.5 ([69, 8]). *Associated to the germ of a symplectic diffeomorphism at a positive definite invariant torus with rotation vector ρ, there is the germ of the corresponding minimal action at ρ, which is a symplectic invariant.*

In the following, we want to show that the germ of the minimal action contains the Birkhoff invariants of the invariant torus, just as we did in Thm. 4.1.11 for the setting of a fixed point. The Birkhoff normal form of a symplectic diffeomorphism near an invariant torus is described in the following classical result; compare [26, App. 2] or [55, Prop. 9.13].

Theorem 4.3.6 (Birkhoff normal form). *Let Λ be a positive definite invariant torus of a symplectic diffeomorphism ϕ such that $\phi|_\Lambda$ is conjugate to the translation by a vector ρ satisfying the Diophantine condition* (4.8).

Then, for each $N \geq 2$, there is a symplectic change of coordinates transforming ϕ into the form

$$\phi(x,y) = (x + \nabla P_N(y), y) + \mathcal{O}(|y|^N) \tag{4.10}$$

as $y \to 0$ where P_N is a polynomial of degree N. The coefficients of P_N are called the Birkhoff invariants *of ϕ; they are symplectically invariant.*

Note that, according to (4.9), one has

$$P_N(y) = \rho \cdot y + \frac{1}{2} Ay \cdot y + \mathcal{O}(|y|^3)$$

if one ignores irrelevant additive constants.

Analogously to Sect. 4.1, we will show that the minimal action near an invariant torus comprises its Birkhoff invariants. This result was also proven by Bernard [8], with a slightly different approach.

Theorem 4.3.7. *Let Λ be a positive definite invariant torus of a symplectic diffeomorphism ϕ such that $\phi|_\Lambda$ is conjugate to the translation on $\mathbb{T}^n$ by a vector ρ satisfying the Diophantine condition* (4.8)*.*

Then the germ of the corresponding minimal action α at ρ is a symplectic invariant. Moreover, the Birkhoff invariants of ϕ are the Taylor coefficients of α^* at 0.

Proof. In view of Thm. 4.3.6, we can assume that ϕ is already given in the form

$$\phi(x,y) = (x + \nabla P_N(y), y) + \mathcal{O}(|y|^N) \tag{4.11}$$

for any given $N \geq 2$. Let us denote by

$$\phi_0(x,y) := (x + \nabla P_N(y), y)$$

the integrable part, i.e., the Birkhoff normal form, of ϕ. Then ϕ_0 can be written as the time–1–map of the flow of the integrable Hamiltonian

$$H_0(x,y) := \rho \cdot y + \frac{1}{2}\, Ay \cdot y + \mathcal{O}(|y|^3) = P_N(y).$$

Since A is assumed to be positive definite, the Hamiltonian $H_0 : T^*\mathbb{T}^n \to \mathbb{R}$ is convex and possesses a corresponding convex Lagrangian $L_0 : T\mathbb{T}^n \to \mathbb{R}$ which is the convex conjugate P_N^* of the polynomial P_N. A quick calculation shows that

$$L_0(x,p) = \frac{1}{2}\, A^{-1}(p-\rho)\cdot(p-\rho). \tag{4.12}$$

Up to now, everything was defined only in a neighbourhood of the invariant torus Λ. Lemma 4.3.4, however, allows us to extend the map and the convex Hamiltonian to the whole cotangent bundle $T^*\mathbb{T}^n$. Moreover, Prop. 4.3.5 guarantees that the germ of the minimal action corresponding to the extended convex Hamiltonian, respectively Lagrangian, is a symplectic invariant which is independent of the extension. Therefore, thinking in terms of germs now, the minimal action α_0 corresponding to the integrable Lagrangian L_0 is given by

$$\alpha_0 = L_0 = P_N^* \tag{4.13}$$

where we identified $H_1(\mathbb{T}^n, \mathbb{R}) = \mathbb{R}^n$ as above.

Next, we want to show a continuity property of the minimal action with respect to the Lagrangian, similar to Prop. 1.3.8 for the two–dimensional case. For this, we make use of the following characterization of the convex conjugate of the minimal action by Mañé's critical value, given in Cor. 2.2.6. Namely, we have

$$\alpha^*([\nu]) = c(L - \nu)$$

for every closed 1–form on $\mathbb{T}^n$, where the critical value $c(L)$ is defined in Def. 2.2.1. The main point is that the critical value $c(L)$ is monotone in the Lagrangian, i.e., if two Lagrangians satisfy the pointwise inequality $L_1 \leq L_2$ then $c(L_1) \geq c(L_2)$; this can be seen most easily by looking at (2.8). Reformulated for the minimal action, we conclude that

$$L_1 \leq L_2 \implies \alpha^*_{L_1} \geq \alpha^*_{L_2} \iff \alpha_{L_1} \leq \alpha_{L_2}.$$

Having proven the monotonicity of the minimal action with respect to the Lagrangian, it follows from (4.11) and (4.13) that

$$\alpha(h) = \alpha_0(h) + \mathcal{O}(|h|^N) = P^*_N(h) + \mathcal{O}(|h|^N).$$

Since the coefficients of P_N are the Birkhoff invariants of ϕ, the theorem is proven. □

5

The minimal action and Hofer's geometry

The classical dynamical way of investigating Hamiltonian systems is to look at one single system at a time. The development of symplectic topology changed this and added a more geometric point of view. Namely, for a given symplectic manifold (M, ω) one has the infinite–dimensional Lie group $\mathrm{Ham}(M, \omega)$ of all Hamiltonian diffeomorphisms. It was an astonishing discovery by Hofer in 1990 [44], later extended by Lalonde and McDuff [53], that this group comes equipped with an *intrinsic* geometry given by a bi–invariant Finsler metric. This opened the field of studying the geometry, and even the topology, of $\mathrm{Ham}(M, \omega)$ both of which are sometimes a bit mysterious.

On the other hand, definite progress has been made: geodesics in Hofer's geometry are completely understood [12], the existence of minimal geodesics has been established in some situations [45, 54, 92]—there are even purely symplectic bounds for the first eigenvalue of the Laplacian [83]. The methods that enter into the proofs are manifold: they range from dynamical systems over symplectic geometry to algebraic geometry. For recent developments in the study of the geometry of Hamiltonian diffeomorphism groups we refer to [46, 54, 82, 84, 85].

The motivation for the subsequent work was to find connections between the two branches in Hamiltonian mechanics mentioned above, the classical dynamical and the modern geometric one. In a sense, the geometric viewpoint seems simpler—the full dynamical system is being described by *one* Hamiltonian function and corresponds to *one* path in the Hamiltonian diffeomorphism group. The central problem is to deduce dynamical properties of the system from geometric properties of the path.

A distinguished class of Hamiltonian systems is rooted in classical mechanics. These are mechanical systems where the Hamiltonian is composed of two parts: kinetic energy plus potential energy. An essential feature of such systems is that they satisfy the Legendre condition which allows to switch between Hamiltonian and Lagrangian mechanics.

The main result of this chapter describes a relation between Hofer's geometry and Mather's theory of minimal action. We will see that minimizing

K.F. Siburg: LNM 1844, pp. 81–95, 2004.

measures play an important role, for the dynamics as well as for the geometry of a Hamiltonian system.

5.1 Hofer's geometry of Ham(M, ω)

Consider a Hamiltonian flow on a symplectic manifold. In classical mechanics, for instance, the flow of a Hamiltonian of the form $H(x, y) = 1/2\,|y|^2 - V(x)$, where V is some smooth potential, describes the motion of a particle in the potential V; compare Ex. 1.1.7. The value of the Hamiltonian, the energy, is constant along the trajectories; in the case indicated above, it is the sum of kinetic and potential energy.

Instead of following a trajectory continuously, we might also consider a "stroboscopic" picture of it, by looking at it only at discrete times, e.g., at all integer times. From the dynamical systems point of view, this means that we consider the time–1–map rather than the flow. In the case of a Hamiltonian flow, the time–1–map is a Hamiltonian diffeomorphism. Given a Hamiltonian diffeomorphism, one might ask the following "economical" question. Is it possible to generate the given map by a Hamiltonian flow with less energy?

This idea will eventually lead to a bi–invariant metric on the group of Hamiltonian diffeomorphisms, the so–called Hofer metric, which is one of the cornerstones of modern symplectic topology. The resulting geometry is called Hofer's geometry. Its geometric features, and its connection to dynamics, seem somewhat strange and are being studied quite intensively.

Let us, first of all, define Hofer's metric and deduce some simple properties. For a detailed exposition we refer to [46, 85]. Let (M, ω) be a symplectic manifold, i.e., a $2n$–dimensional manifold M with a closed, nondegenerate 2–form ω; compare Sect. 2.1.2. A Hamiltonian diffeomorphism is a diffeomorphism $\phi : M \to M$ which can be written as the time–1–map of a Hamiltonian flow, i.e.,

$$\phi = \varphi_H^1$$

for some time–periodic Hamiltonian $H : \mathbb{S}^1 \times M \to \mathbb{R}$. Let us denote by H_t the function $H(t, \cdot)$ on M. In the following, we will normalize the Hamiltonians, so that the ambiguity of adding constants is removed.

Definition 5.1.1. *Let (M, ω) be a symplectic manifold without boundary. If M is open, the set of* admissible Hamiltonians *is defined as*

$$\mathcal{H} := \{H : \mathbb{S}^1 \times M \to \mathbb{R} \mid \textit{supp } H_t \textit{ is compact for every } t\},$$

where supp denotes the support of a function. If M is compact, we set

$$\mathcal{H} := \{H : \mathbb{S}^1 \times M \to \mathbb{R} \mid \int_M H_t \,\mathrm{d}\omega^n = 0 \textit{ for every } t\}.$$

In any case, the group of Hamiltonian diffeomorphisms *of a symplectic manifold (M, ω) is given by*

$$Ham(M,\omega) := \{\phi : M \to M \mid \phi = \varphi_H^1 \textit{ for some } H \in \mathcal{H}\}.$$

A Hamiltonian $H : \mathbb{S}^1 \times M \to \mathbb{R}$ defines a path $t \mapsto \varphi_H^t$ in the diffeomorphism group $\mathrm{Ham}(M,\omega)$. We measure the length of this path by

$$\ell(H) := \int_{\mathbb{S}^1} \operatorname{osc} H_t \, \mathrm{d}t$$

where $\operatorname{osc} := \max - \min$ denotes the oscillation of a function on M. For $H \in \mathcal{H}$, it is clear that $\ell(H) = 0$ if, and only if, $H = 0$. Like in the case of Finsler geometry we measure the distance from the identity in $\mathrm{Ham}(M,\omega)$ by taking the infimum of lengths of all connecting paths.

Definition 5.1.2. *The* distance from the identity*, or* energy*, of an element* $\phi \in Ham(M,\omega)$ *is defined as*

$$d(id,\phi) := \inf\{\ell(H) \mid H \in \mathcal{H} \textit{ such that } \phi = \varphi_H^1\}.$$

Remark 5.1.3. Note that $\ell(H)$ measures the C^0–data of the Hamiltonian (corresponding to the energy) and not the C^1–data (which define the flow). Therefore, $d(\mathrm{id},\phi)$ indeed describes the minimal amount of energy necessary to generate a given map ϕ. On the other hand, it seems that d measures the "wrong" kind of data, at least from the dynamical systems viewpoint. Eliashberg and Polterovich [27]proved, however, that this is the only way of defining a bi–invariant metric.

Let us extend the distance to a bi–invariant function $d : \mathrm{Ham}(M,\omega) \times \mathrm{Ham}(M,\omega) \to [0,\infty)$ by setting

$$d(\phi,\psi) := d(\mathrm{id}, \psi \circ \phi^{-1}).$$

It follows quite easily from the definition of d and the transformation law of Hamiltonian vector fields that d defines a bi–invariant pseudo–metric on $\mathrm{Ham}(M,\omega)$, i.e., a function with

$$d(\phi \circ \chi, \psi \circ \chi) = d(\chi \circ \phi, \chi \circ \psi) = d(\phi,\psi)$$

that satisfies all the axioms of a metric, except that it might be degenerate. It is not clear at all whether $d(\mathrm{id},\phi) = 0$ should imply $\phi = \mathrm{id}$; compare Rem. 5.1.3. It was Hofer [44] who discovered that the pseudo–distance d is actually a genuine metric.

Theorem 5.1.4 (Hofer). *The pseudo–metric d is nondegenerate and, therefore, a bi–invariant metric on the Hamiltonian diffeomorphism group $Ham(M,\omega)$.*

This metric is called *Hofer's metric.* It is intrinsically defined, i.e., via the lengths of paths inside the group itself. Consequently, it defines a Finsler geometry on $\mathrm{Ham}(M,\omega)$. Of particular importance are globally length–minimizing paths.

Definition 5.1.5. *Let (M, ω) be a symplectic manifold and $H \in \mathcal{H}$ an admissible Hamiltonian. Then H is said to generate a* minimal geodesic *if*

$$d(id, \varphi_H^1) = \ell(H).$$

Hofer's geometry allows also the notions of geodesics, conjugate points, etc. We refer to [12, 54, 84, 85, 92] for further definitions and results.

We want to pursue the following idea here. A Hamiltonian dynamical system corresponds to one single path in the Hamiltonian diffeomorphism group, and vice versa. Therefore, this path contains all information about the Hamiltonian dynamical system (like periodic orbits, heteroclinic connections, etc.). The group $\mathrm{Ham}(M, \omega)$, on the other hand, is equipped with a geometry coming from Hofer's metric, and the path has certain properties with respect to this geometry (like being a geodesic, etc.). What kind of relations exist between the (finite–dimensional) dynamics of the Hamiltonian flow and the (infinite–dimensional) Hofer geometry of the corresponding path? In particular, what kind of information about Hofer's geometry can be retrieved from the classical dynamics? Take a look at Fig. 5.1 to see the two viewpoints of a Hamiltonian system, the dynamical and the geometric one.

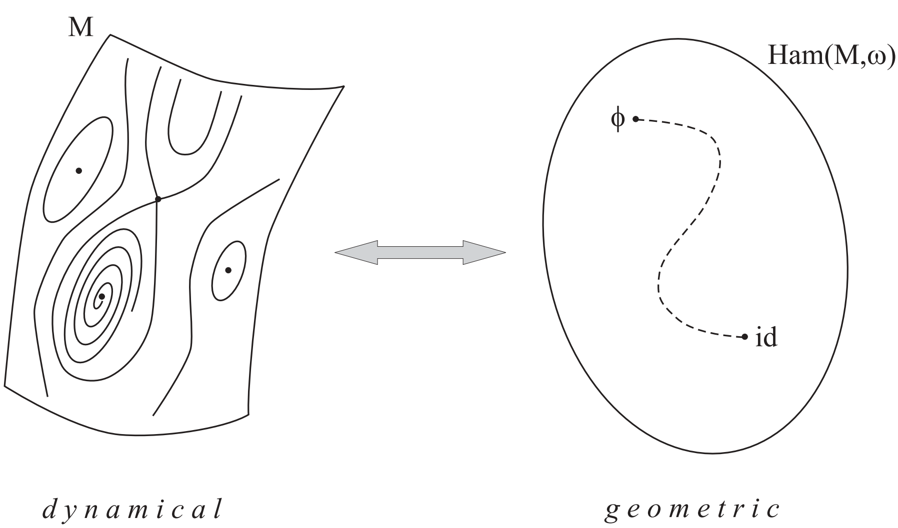


Fig. 5.1. Two viewpoints of a Hamiltonian system

In the following, we go back to the phase space of classical mechanics and consider the cotangent bundle $T^*\mathbb{T}^n$ with its canonical symplectic form $\omega_0 = \mathrm{d}\lambda$. In order to include Hamiltonians satisfying the Legendre condition (see Sect. 2.1.1) into the framework of Hofer's geometry, we have to restrict them to a compact part of $T^*\mathbb{T}^n$; otherwise, they would violate the compact support condition. Denote by

$$B^*\mathbb{T}^n := \mathbb{T}^n \times \{|y| \leq 1\}$$

the unit ball cotangent bundle of the torus. We also restrict the class of admissible Hamiltonians and consider the set $\mathcal{H}_0$ consisting of all smooth Hamiltonians $H : \mathbb{S}^1 \times B^*\mathbb{T}^n \to \mathbb{R}$ that satisfy the following two conditions:

1. $H : \mathbb{S}^1 \times B^*\mathbb{T}^n \to \mathbb{R}$ vanishes on the boundary of $B^*\mathbb{T}^n$, i.e., $H(\cdot, \cdot, y) = 0$ whenever $|y| = 1$;
2. $H : \mathbb{S}^1 \times B^*\mathbb{T}^n \to \mathbb{R}$ admits a smooth extension

$$\overline{H} : \mathbb{S}^1 \times T^*\mathbb{T}^n \to \mathbb{R}$$

   which is only a function of t and $|y|^2$ outside $\mathbb{S}^1 \times B^*\mathbb{T}^n$.

As before, we consider the group

$$\mathrm{Ham}(B^*\mathbb{T}^n) := \{\phi : B^*\mathbb{T}^n \to B^*\mathbb{T}^n \mid \phi = \varphi_H^1 \text{ for some } H \in \mathcal{H}_0\}$$

of Hamiltonian diffeomorphisms generated by Hamiltonians in $\mathcal{H}_0$.

In order to prove estimates for $d(\mathrm{id}, \phi)$ when ϕ is generated by a convex Hamiltonian $H \in \mathcal{H}_0$, we need the following notion. As usual, φ_H denotes the flow corresponding to a Hamiltonian function H.

Definition 5.1.6. *Given any $H \in \mathcal{H}_0$, the set*

$$\sigma_c(H) := \left\{ \int_\Gamma \lambda - H\mathrm{d}t \mid \Gamma \textit{ contractible 1–periodic orbit of } \varphi_H \right\}$$

is called the contractible action spectrum *of H.*

Note that $\int \lambda - H\mathrm{d}t$ corresponds to the action integral $\int L\,\mathrm{d}t$. Thus, $\sigma_c(H)$ collects the actions of all 1–periodic trajectories of the Hamiltonian flow that belong to contractible orbits. Hofer proved [45] that this set plays a crucial role in the geometry of the Hamiltonian diffeomorphism group $\mathrm{Ham}(\mathbb{R}^{2n}, \omega_0)$. We will see here that the smallest action of a contractible orbit yields a lower bound for the energy of a map which is generated by a convex Hamiltonian. The following theorem is a slight generalization of a result by Bialy and Polterovich [12, Prop. 4.3.A].

Theorem 5.1.7. *Suppose $\phi \in Ham(B^*\mathbb{T}^n)$ is generated by a convex Hamiltonian $H \in \mathcal{H}_0$. Then*

$$d(id, \phi) \geq \inf \sigma_c(H) > 0.$$

Proof. The first, main step is to show that

$$d(\mathrm{id}, \phi) \geq \inf \sigma_c(H). \tag{5.1}$$

The second step will be to prove that

$$\inf \sigma_c(H) > 0. \tag{5.2}$$

For the first inequality we could essentially use a proof of Hofer and Zehnder [46] if ϕ was defined on all of $T^*\mathbb{T}^n$ and had compact support. So we will build an appropriate "nice" extension

$$\overline{\phi} : T^*\mathbb{T}^n \to T^*\mathbb{T}^n$$

of $\phi : B^*\mathbb{T}^n \to B^*\mathbb{T}^n$, and find a procedure how to extend any given $K \in \mathcal{H}_0$ with $\varphi_K^1 = \phi$ to a Hamiltonian on $\mathbb{S}^1 \times T^*\mathbb{T}^n$ whose time–1–map coincides with $\overline{\phi}$.

Let us fix, once and for all, some $\epsilon > 0$. We define the map $\overline{\phi}$ in the following way. By definition of $\mathcal{H}_0$, there is an extension

$$\overline{H} : \mathbb{S}^1 \times T^*\mathbb{T}^n \to \mathbb{R}$$

of $H : \mathbb{S}^1 \times B^*\mathbb{T}^n \to \mathbb{R}$ which is a function of t and $|y|^2$ for $|y| \geq 1$. Since H is convex and satisfies the boundary condition $H = 0$ on $\{|y| = 1\}$, we have $H \leq 0$. This implies that the derivative $\overline{H}_t'(1)$, seen as a function of $|y|^2$, is positive. Therefore, we can pick an extension $\overline{H}$ with the following properties:

1. $\overline{H}_t' > 0$ for $1 \leq |y| < 2$
2. $0 \leq \overline{H} \leq \epsilon$ for $|y| \geq 1$
3. $\overline{H} = \epsilon$ for $|y| \geq 2$.

Note that for $|y(0)| \geq 1$ the time–t–map $\varphi_{\overline{H}}^t(x(0), y(0)) = (x(t), y(t))$ is given by

$$\begin{cases} x(t) &= x(0) + y(0) \int_0^t \overline{H}_s'(|y(0)|^2)\, \mathrm{d}s \\ y(t) &= y(0) \end{cases} \tag{5.3}$$

In other words, outside $B^*\mathbb{T}^n$, the flow of $\overline{H}$ at time t is just the geodesic flow at time $\int_0^t \overline{H}_s'\, \mathrm{d}s$. Let us call

$$\overline{\phi} := \varphi_{\overline{H}}^1.$$

Given any (not necessarily convex) $K \in \mathcal{H}_0$ with $\varphi_K^1 = \phi$, we claim that we can extend K to a smooth function $\overline{K} : \mathbb{S}^1 \times T^*\mathbb{T}^n \to \mathbb{R}$ such that

1. $\overline{K}$ is a function of t and $|y|^2$ for $|y| \geq 1$
2. $0 \leq \overline{K} \leq 3\epsilon$ for $|y| \geq 1$
3. $\overline{K} = \epsilon$ for $|y| \geq 2$
4. $\varphi_{\overline{K}}^1 = \overline{\varphi}$.

Of course, only the last point has to be checked. For that we pick any extension $\widetilde{K}$ satisfying the first three conditions with $0 \leq \widetilde{K} \leq \epsilon$ for $|y| \geq 1$, and define

$$\overline{K}(t, x, y) := \begin{cases} \widetilde{K}_t(|y|^2) + \int_{\mathbb{S}^1} (\overline{H}_t(|y|^2) - \widetilde{K}_t(|y|^2))\, \mathrm{d}t & \text{if } |y| \geq 1 \\ \widetilde{K}(t, x, y) & \text{if } |y| \leq 1 \end{cases}$$

By (5.3), $\overline{K}$ fulfills all four requirements provided it is smooth, which can be seen as follows. The smooth diffeomorphisms $\varphi^1_{\widetilde{K}}$ and $\varphi^1_{\overline{H}}$ coincide on $\{|y| \leq 1\}$, hence all their derivatives at points on $\{|y| = 1\}$ are the same. Therefore, $\int_{\mathbb{S}^1}(\overline{H}_t(|y|^2) - \widetilde{K}_t(|y|^2))\,\mathrm{d}t$ is a smooth function on $\{|y| \geq 1\}$ that vanishes with all its derivatives on $\{|y| = 1\}$. Thus, $\overline{K}$ is the extension we were looking for.

We are going to prove (5.1). The flow of the extended Hamiltonian $\overline{H} : \mathbb{S}^1 \times T^*\mathbb{T}^n \to \mathbb{R}$ has no (non–constant) contractible 1–periodic orbits outside $B^*\mathbb{T}^n$. We eliminate all non–contractible 1-periodic orbits of $\varphi_{\overline{H}}$ by choosing an appropriate covering of $\mathbb{T}^n$. Moreover, we can embed the compact part of $T^*\mathbb{T}^n$ with $|y| \leq 2$ symplectically into $\mathbb{R}^{2n}$ and view $\overline{\phi} = \varphi^1_{\overline{H}}$ as being generated by the compactly supported Hamiltonian $\overline{H} - \epsilon : \mathbb{S}^1 \times \mathbb{R}^{2n} \to (-\infty, 0]$. For the contractible action spectrum we have

$$\sigma_c(\overline{H} - \epsilon) = (\sigma_c(H) + \epsilon) \cup \{0\}.$$

Let us call d_0 the usual Hofer metric on the group $\mathrm{Ham}(\mathbb{R}^{2n}, \omega_0)$ of Hamiltonian diffeomorphisms generated by compactly supported Hamiltonians on $\mathbb{S}^1 \times \mathbb{R}^{2n}$. In this, setting, Hofer and Zehnder proved that the inequality

$$d_0(\mathrm{id}, \overline{\phi}) \geq \inf \sigma_c(H) \, + \epsilon \tag{5.4}$$

holds; see [46, Ch. 5] for details. Now we pick any $K \in \mathcal{H}_0$ such that $\varphi^1_K = \phi$ and

$$d(\mathrm{id}, \phi) \geq \int_{\mathbb{S}^1} \operatorname{osc} K_t \,\mathrm{d}t - \epsilon.$$

As described above, we can extend K to $\overline{K}$ with $\varphi^1_{\overline{K}} = \overline{\phi}$ and

$$\int_{\mathbb{S}^1} \operatorname{osc} K_t \,\mathrm{d}t \geq \int_{\mathbb{S}^1} \operatorname{osc} \overline{K}_t \,\mathrm{d}t - 6\epsilon,$$

so that we can estimate

$$d(\mathrm{id}, \phi) \geq \int_{\mathbb{S}^1} \operatorname{osc} \overline{K}_t \,\mathrm{d}t - 7\epsilon. \tag{5.5}$$

We point out here that non–contractible periodic orbits of φ_K and φ_H are homotopic with fixed end points because $\varphi^1_K = \varphi^1_H$. Thus, the covering needed to eliminate those orbits does not depend on the particular choice of Hamiltonian generating ϕ.

From (5.4) we conclude that

$$\int_{\mathbb{S}^1} \operatorname{osc} \overline{K}_t \,\mathrm{d}t \geq d_0(\mathrm{id}, \overline{\phi}) \geq \inf \sigma_c(H) + \epsilon$$

which, together with (5.5), finally implies that

$$d(\mathrm{id}, \phi) \geq \inf \sigma_c(H) - 6\epsilon.$$

Since $\epsilon > 0$ was arbitrarily small, this finishes the proof of (5.1).

As the final step, we have to show that the contractible action spectrum $\sigma_c(H)$ consists of positive numbers. This is already done in [12] and repeated here for the convenience of the reader. Pick any contractible 1–periodic orbit $\Gamma = (x(t), y(t))_{0 \leq t \leq 1}$, and let $\Lambda = \mathrm{gr}\ \nu$ be a Lagrangian section in $B^*\mathbb{T}^n$ (ν is a closed 1–form on $\mathbb{T}^n$). It follows that

$$\begin{aligned}
\int_\Gamma \lambda - H\mathrm{d}t &= \int_\Gamma \lambda - \pi^*\nu - H\mathrm{d}t \\
&= \int_{\mathbb{S}^1} \big[(y-\nu)\,\partial_y H - H\big]\,\mathrm{d}t \\
&\geq \int_{\mathbb{S}^1} -H(t, x(t), \nu(x(t)))\,\mathrm{d}t \qquad\qquad \text{by the convexity of } H \\
&\geq \int_{\mathbb{S}^1} \min(-H(t,\cdot)|_\Lambda)\,\mathrm{d}t.
\end{aligned}$$

The convexity of H, in conjunction with the zero boundary condition, implies that $H \leq 0$. Therefore, we have

$$\int_{\mathbb{S}^1} \min(-H(t,\cdot)|_\Lambda)\,\mathrm{d}t > 0$$

and, hence, $\inf \sigma_c(H) > 0$.

This proves (5.2) and completes the proof of Thm. 5.1.7. □

Remark 5.1.8. Thm. 5.1.7 holds also true if we replace the flat norm $|\cdot|$ by $|\cdot|_g$ where g is any Riemannian metric on $\mathbb{T}^n$ without contractible closed geodesics.

Remark 5.1.9. Iturriaga and Sánchez–Morgado generalized Thm. 5.1.7 to general cotangent bundles; see [48].

The action spectrum of an arbitrary convex Hamiltonian H defies computation. If we restrict ourselves to integrable Hamiltonians $H = H(t, y)$, however, the only contractible periodic solutions are constant. Therefore, $\sigma_c(H)$ consists of all values $-\int_{\mathbb{S}^1} H_t(y_0)\,\mathrm{d}t$ where y_0 is a critical point of H_t for all t. Thm. 5.1.7 then states that

$$d(\mathrm{id}, \varphi_H^1) \geq \min_{y_0} \Big(- \int_{\mathbb{S}^1} H_t(y_0)\,\mathrm{d}t \Big)$$

where y_0 is as above. Note that, in this inequality, the left hand side involves all Hamiltonians generating the map φ_H^1, not just the convex ones.

In fact, we can give a characterization of those integrable convex Hamiltonians that generate a minimal geodesic.

Theorem 5.1.10. *Let $H = H(t, y)$ be an integrable convex Hamiltonian in $\mathcal{H}_0$. Then H generates a minimal geodesic if, and only if, all H_t attain their minimum at one and the same point y_{min}.*

Proof. Suppose $H(t, y)$ is convex and has a fixed minimal point $y_{\min}$. Then

$$K(y) := \int_{\mathbb{S}^1} H(t, y)\,\mathrm{d}t \in \mathcal{H}_0$$

is convex, generates the same time–1–map as H, and has the same energy:

$$\ell(K) = \int_{\mathbb{S}^1} \operatorname{osc} K_t\,\mathrm{d}t = -K(y_{\min}) = \int_{\mathbb{S}^1} \operatorname{osc} H_t\,\mathrm{d}t = \ell(H).$$

But then $\sigma_c(K) = \{-K(y_{\min})\}$ implies, by Theorem 5.1.7, that K generates a minimal geodesic. Since $\ell(K) = \ell(H)$, H generates a minimal geodesic, too.

Conversely, the existence of a fixed minimal point is necessary for a Hamiltonian $H \in \mathcal{H}_0$ in order to generate a minimal geodesic. This is proven in [54, Prop. 2.1]. In fact, if H does not have a fixed minimal point then there exists a strictly shorter connection between the identity and φ_H^1. □

Remark 5.1.11. In the case of $\mathrm{Ham}(\mathbb{R}^{2n}, \omega_0)$, a Hamiltonian with (isolated) fixed minimal and maximal points generates a minimal geodesic as long as it does not generate non–constant closed orbits; see [92]. The proof involves perturbations of the Hamiltonian near the fixed minimal and maximal points. Since this is only local, a generalized version of Thm. 5.1.10 should hold for non–integrable convex Hamiltonians.

In particular, Thm. 5.1.10 states that every time–independent, convex, integrable Hamiltonian describes a minimal geodesic. This implies the following result.

Corollary 5.1.12. *The diameter of $(Ham(B^*\mathbb{T}^n), d)$ is infinite.*

5.2 Estimates via the minimal action

Consider a convex Hamiltonian $H : \mathbb{S}^1 \times B^*\mathbb{T}^n \to \mathbb{R}$ in $\mathcal{H}_0$, and extend it by some function of t and $|y|^2$ to a convex Hamiltonian $\overline{H} : \mathbb{S}^1 \times T^*\mathbb{T}^n \to \mathbb{R}$. Associated to this extension we have the minimal action

$$\alpha := \alpha_{\overline{H}} : H_1(\mathbb{T}^n, \mathbb{R}) \to \mathbb{R}.$$

Our aim in this section is to relate the function α to the Hofer distance $d(\mathrm{id}, \phi)$ of the time–1–map $\phi := \varphi_H^1$ on $B^*\mathbb{T}^n$.

The idea why this should be possible at all is prompted by the following observation. Suppose for a moment that $H = H_0(y)$ is integrable and convex. Then Thm. 5.1.10 states that H_0 generates a minimal geodesic:

$$d(\mathrm{id}, \phi) = \operatorname{osc} H_0.$$

On the other hand, we know from Cor. 2.1.25 that $\alpha^* = \overline{H_0}$ if we identify $H^1(\mathbb{T}^n, \mathbb{R}^n) = \mathbb{R}^n$. Hence we obtain

$$d(\mathrm{id}, \phi) = \operatorname*{osc}_{B^*\mathbb{T}^n} H_0 = \operatorname*{osc}_{B} \alpha^* \tag{5.6}$$

where the oscillation of α^* is taken over the unit ball $B \subset \mathbb{R}^n$. Thus, if we view $\alpha^*|_B$ as an integrable Hamiltonian on $B^*\mathbb{T}^n$, it generates a minimal geodesic in $\mathrm{Ham}(B^*\mathbb{T}^n)$!

Since $\alpha^* = 0$ on the boundary of the unit ball B, we have $\operatorname{osc} \alpha^* = -\min \alpha^*$, which is nothing but $\alpha(0)$ by (2.4), so that we may rewrite (5.6) as

$$d(\mathrm{id}, \phi) = \alpha(0). \tag{5.7}$$

Unfortunately, already the simplest non–integrable example will show that (5.6), respectively (5.7), does not hold in general.

Example 5.2.1. Consider the Lagrangian

$$L(x, p) = \frac{p^2}{2} - V(x)$$

of a particle in a periodic potential in one degree of freedom, e.g., the motion of a mathematical pendulum. We assume that V attains its minimum at $x = 0$ with $V(0) = 0$. Of course, the corresponding Hamiltonian

$$H(x, y) = \frac{y^2}{2} + V(x)$$

does not belong to $\mathcal{H}_0$ because it does not satisfy the boundary conditions. Therefore we fix a cut off function $\beta : [0, 1] \to [0, 1]$ with $\beta(s) = 1$ if $s \le 1/2$, and consider the modified Hamiltonian

$$H(x, y) := \frac{1}{2}(y^2 - 1) + \beta(y^2)V(x).$$

If $\max V$ is sufficiently small, H is a convex function and does belong to $\mathcal{H}_0$. Moreover, we assume that the outer separatrices, defined by $H = -1/2 + \max V$, lie in the region $\{y^2 \le 1/2\}$ where β plays no role anymore. Then these separatrices are the graphs

$$y(x) = \pm\sqrt{2(\max V - V(x))},$$

and the region between them is filled by fixed points, periodic orbits, and maybe also further separatrices; see Fig. 5.2.

What is the value $\alpha(0)$ of the minimal action in this example? Setting $p = 0$ and $x = x_{\max}$ such that $V(x_{\max}) = \max V$, we see that a minimal

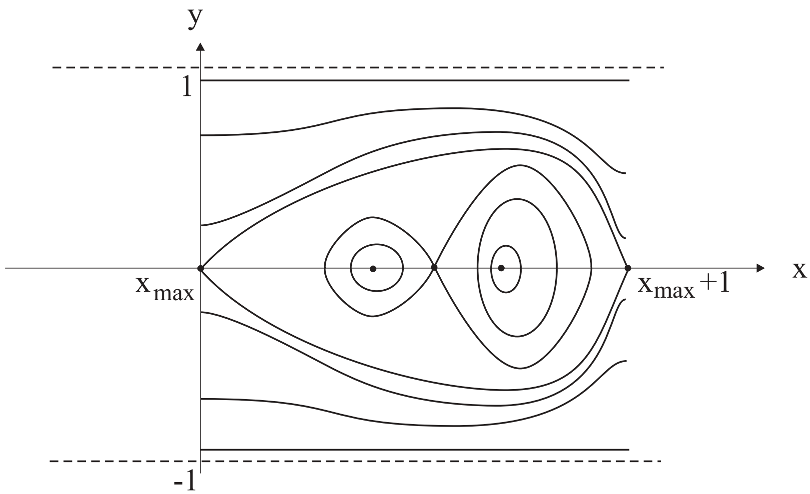


Fig. 5.2. The level sets of $H(x, y) = \frac{1}{2}(y^2 - 1) + \beta(y^2)V(x)$

measure of zero rotation vector is concentrated on the fixed point $(x_{\max}, 0)$, and hence we have

$$\alpha(0) = \frac{1}{2} - \max V. \tag{5.8}$$

This means that α does not "see" the entire region between the outer separatrices.

Observe now that a C^2-small potential V will admit only non–constant periodic orbits of period greater than one. Therefore, by a criterion due to Hofer [45, 92], H generates a minimal geodesic, so that

$$d(\mathrm{id}, \phi) = \frac{1}{2} > \frac{1}{2} - \max V = \alpha(0), \tag{5.9}$$

and (5.7) does indeed not hold.

We saw that the distance from the identity of the time–1–map of a convex Hamiltonian can be smaller than $\alpha(0)$. But what happens if we do not restrict the time to being at most 1 but let it tend to infinity? In fact, from the geometric point of view, the time $t = 1$ is not distinguished at all. If we consider a path in $\mathrm{Ham}(M, \omega)$, a more relevant piece of information would be the average distance from the identity over all times. This idea leads to the following notion.

Definition 5.2.2. *The* asymptotic distance from the identity *of an element $\phi \in Ham(M, \omega)$ is defined as*

$$d_\infty(id, \phi) := \lim_{N \to \infty} \frac{1}{N} d(id, \phi^N).$$

Remark 5.2.3. It follows from the triangle inequality that $d_\infty \leq d$. Hence the limit in Def. 5.2.2 always exists. Moreover, lower estimates for the asymptotic distance d_∞ are stronger than the same estimates for the distance d.

The asymptotic distance was introduced by Bialy and Polterovich [13]. It measures the deviation of a path from being a minimal geodesic. Note that, in general, minimal geodesics will eventually lose the property of being shortest connections.

In the following, we will give an estimate of the asymptotic distance of a convex Hamiltonian H by $\alpha(0)$ where α is the minimal action for some convex extension of H. To do so, we must first show that the value $\alpha(0)$ does not depend on the particular extension.

Let $H : \mathbb{S}^1 \times B^*\mathbb{T}^n \to \mathbb{R}$ be a convex Hamiltonian in $\mathcal{H}_0$. Consider any convex extension $\overline{H} : \mathbb{S}^1 \times T^*\mathbb{T}^n \to \mathbb{R}$ of H that is a function t and $|y|^2$ outside $B^*\mathbb{T}^n$, and let $\alpha_{\overline{H}} : H_1(\mathbb{T}^n, \mathbb{R}) \to \mathbb{R}$ be the corresponding minimal action.

Lemma 5.2.4. *The value $\alpha_{\overline{H}}(0)$ is independent of the particular extension $\overline{H}$.*

Proof. We identify $H^1(\mathbb{T}^n, \mathbb{R}) = \mathbb{R}^n$ in such a way that $[\lambda|_{\mathbb{T}^n \times \{y\}}] = y$. Note that, for $|y| \geq 1$, the torus $\mathbb{T}^n \times \{y\}$ is invariant under each $\varphi^t_{\overline{H}}$ which is a fixed rotation there. Cor. 2.1.25 implies that

$$\alpha^*_{\overline{H}}(y) = \int_0^1 \overline{H}\big(t, \frac{1}{2}|y|^2\big)\, dt$$

for $|y| \geq 1$; in particular, $\alpha^*_{\overline{H}}(y) = 0$ whenever $|y| = 1$. Since $\alpha^*_{\overline{H}}$ is convex it must attain its negative minimum in the unit ball where $\overline{H} = H$. Therefore, $-\min \alpha^*_{\overline{H}} = \alpha_{\overline{H}}(0)$ does not depend on the choice of extension of H. $\square$

In view of Lemma 5.2.4 we may ignore the particular choice of extension, and drop the index $\overline{H}$. We state the main result of this section.

Theorem 5.2.5. *Suppose $\phi \in Ham(B^*\mathbb{T}^n)$ is generated by a convex Hamiltonian. Then*

$$d_\infty(id, \phi) \geq \operatorname{osc} \alpha^* = \alpha(0).$$

Proof. Fix any convex $H \in \mathcal{H}_0$ such that $\varphi^1_H = \phi$. Recall from Thm. 5.1.7 that

$$d(\mathrm{id}, \phi) \geq \inf \sigma_c(H)$$

where $\sigma_c(H)$ is the contractible action spectrum of H. Each contractible 1–periodic orbit is the support of an invariant probability measure with rotation vector zero. Hence

$$\inf \sigma_c(H) \geq \alpha(0)$$

with $\alpha(0) = -\min \alpha^* = \operatorname{osc} \alpha^*$.

This proves that $d(\mathrm{id}, \phi) \geq \alpha(0)$. The theorem follows by observing that $\alpha(Nh, \phi^N) = N\alpha(h, \phi)$. $\square$

Remark 5.2.6. Bialy and Polterovich proved in [12, Thm. 1.4.A] that the Hofer distance is bounded from below by the quantity

$$C_H = \sup_{\Lambda} \int_{\mathbb{S}^1} \min(-H_t|_{\Lambda})\,\mathrm{d}t \; > 0$$

where Λ runs over all Lagrangian sections in $B^*\mathbb{T}^n$. In general, the dynamical meaning of C_H is not quite clear. For autonomous Hamiltonians, however, this number agrees with our lower bound since

$$C_H = -\inf_{c\in H^1} \inf_{[\nu]=c} \max H(x,\nu(x)) = -\inf_{c\in H^1} \alpha^*(c) = \alpha(0),$$

where the second last equality follows from Cor. 2.2.6 and (2.9).

Remark 5.2.7. The first nontrivial class of examples where d_∞ could actually be calculated is given by compactly supported autonomous Hamiltonians on a surface of infinite area; see [86].

Remark 5.2.8. Iturriaga and Sánchez–Morgado [48] gave a generalized version of Thm. 5.2.5 and proved the estimate $d(\mathrm{id},\phi) \geq \alpha(0)$ for convex Hamiltonians on general cotangent bundles.

Let us continue Ex. 5.2.1 of the motion of a particle in a 1–dimensional periodic potential.

Example 5.2.9 (cont.). Consider the Lagrangian

$$L(x,p) = \frac{p^2}{2} - V(x)$$

of a particle in a periodic potential in one degree of freedom. Let us cut off the corresponding Hamiltonian so that the new Hamiltonian H belongs to $\mathcal{H}_0$, and denote by ϕ its time–1–map. We had seen in (5.8) that $\alpha(0) = 1/2 - \max V$ so that Thm. 5.2.5 yields

$$d_\infty(\mathrm{id},\phi) \geq \frac{1}{2} - \max V.$$

We claim that we can estimate

$$d_\infty(\mathrm{id},\phi) \leq \frac{1}{2} - \frac{1}{2}\max V.$$

In order to prove the claim, we make use of a curve shortening procedure for autonomous Hamiltonians due to Bialy and Polterovich [13, Thm. 3.3.A]. They showed that

$$d_\infty(\mathrm{id},\phi) \leq \inf_{K\in\mathcal{H}_0} \frac{\mathrm{osc}(H + H\circ\varphi_K^1)}{2}.$$

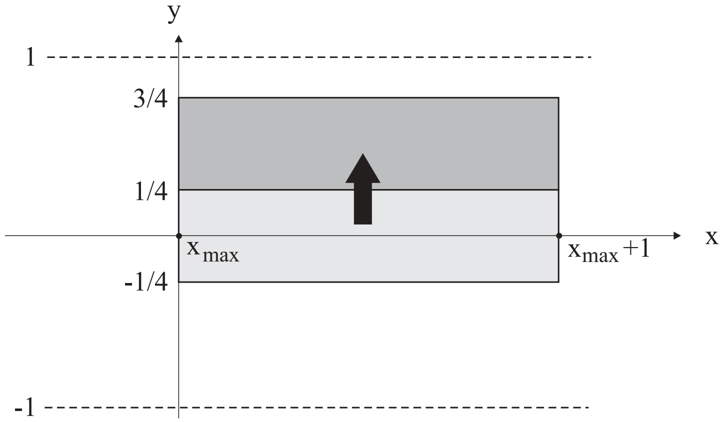


Fig. 5.3. The map φ_K^1 shifts the lower rectangle onto the upper one

We will construct a Hamiltonian $K \in \mathcal{H}_0$ such that

$$\operatorname{osc}(H + H \circ \varphi_K^1) \leq 1 - \max V$$

in the following way. Lift everything to the universal cover $\mathbb{R} \times [-1, 1]$ of $B^*\mathbb{T}^1$. Fix some point $x_{\max}$ where the potential V attains its maximum. Suppose that the outer separatrices, where $H = -1/2 + \max V$, lie in the region $\{|y| \leq 1/4\}$. Neglecting small pertubations near $\{x = x_{\max}\}$ and $\{x = x_{\max} + 1\}$, we can pick a $K \in \mathcal{H}_0$ such that $K(x, y) = x/2$ for $(x, y) \in [x_{\max}, x_{\max} + 1] \times [-3/4, 3/4]$. Then φ_K^1 shifts the rectangle $[x_{\max}, x_{\max} + 1] \times [-1/4, 1/4]$ onto $[x_{\max}, x_{\max} + 1] \times [1/4, 3/4]$, so the set $\{-1/2 \leq H \leq -1/2 + \max V\}$ will be mapped into the region where $H \geq -1/2 + \max V$; see Fig. 5.3. Hence at each point we have

$$0 \geq H + H \circ \varphi_K^1 \geq -\frac{1}{2} + \Big(-\frac{1}{2} + \max V \Big) = -1 + \max V,$$

and our claim is proven.

This example prompts the following question.

Open problem. *Suppose $\phi \in Ham(B^*\mathbb{T}^n)$ is generated by a convex Hamiltonian. Is it true that*

$$d_\infty(id, \phi) = \operatorname{osc} \alpha^* = \alpha(0)?$$

If the answer were yes, this would mean that α^*, seen as a non–smooth Hamiltonian, generated an asymptotically shortest connection. Thus, one would be lead to the investigation of non–smooth Hamiltonians and symplectic homeomorphisms in the context of Hofer geometry.

Finally, Thm. 5.2.5 can be applied to obtain converse KAM–results, in the sense that the location of invariant KAM–tori can be restricted to certain domains in phase space. See [59, 58] for classical estimates involving minimal orbits, and [12, 13] for results using Hofer's metric d in the autonomous case. For instance, Thm. 5.2.5 and Cor. 2.1.25 immediately imply the following result.

Corollary 5.2.10. *Suppose $\phi \in Ham(B^*\mathbb{T}^n)$ is generated by a convex Hamiltonian $H \in \mathcal{H}_0$ whose flow possesses an invariant KAM–torus Λ. Then*

$$\int_{\mathbb{S}^1} H_t|_\Lambda \,\mathrm{d}t \geq -d_\infty(id, \phi).$$

6

The minimal action and symplectic geometry

A hypersurface in a cotangent bundle is called convex if it bounds a fiberwise strictly convex domain. In this chapter, we will deal with Lagrangian submanifolds that lie in a convex hypersurface. A particularly important class of examples is given by invariant tori in classical mechanics where the hypersurface is the level set of a convex Hamiltonian H. However, we will consider this situation from the symplectic point of view, which is different from the dynamical one. For instance, the property of being a Lagrangian section is *not* a symplectic property.

In the first section, we establish a *boundary rigidity* phenomenon which, roughly speaking, can be formulated as follows. Certain Lagrangian submanifolds Λ in a convex hypersurface Σ cannot be mapped by a Hamiltonian diffeomorphism into the domain bounded by Σ. In fact, under certain assumptions on the dynamics on Λ, it is not possible to move Λ at all, so Λ is indeed "boundary rigid".

Furthermore, even when boundary rigidity fails, we often find another phenomenon called *non–removable intersection*. In this case, the Lagrangian submanifold Λ can partly be moved into the domain Σ but certain pieces of Λ stay put. At this point the link between symplectic geometry and Mather–Mañé theory appears. Namely, if Σ is the Mañé critical level set of H, these "non–removable intersections" always contain the Aubry set of Σ.

Finally, we discuss Lagrangian submanifolds lying in the open domain U bounded by a convex hypersurface. This leads to the notion of the shape of U, a symplectic invariant that describes all Liouville classes that can be represented by Lagrangian submanifolds in U. We will see that each class in the shape of a convex domain can actually be represented by a Lagrangian *section.* This allows us to give symplectic descriptions of, firstly, the stable norm in Riemannian geometry and, secondly, the convex conjugate of Mather's minimal action.

This chapter is based on joint work with Gabriel P. Paternain and Leonid Polterovich [81].

K.F. Siburg: LNM 1844, pp. 97–119, 2004.

6.1 Boundary rigidity in convex hypersurfaces

Let us recall some notation from Sect. 2.1.2. We denote by $\theta : T^*X \to X$ the cotangent bundle of a closed manifold X. It comes equipped with the canonical symplectic form $\omega = \mathrm{d}\lambda$ where λ is the Liouville form. We write $\mathcal{O}$ for the zero section, and denote by $\mathcal{L}$ the class of all Lagrangian submanifolds of T^*X which are Lagrangian isotopic to $\mathcal{O}$. The Liouville class of $\Lambda \in \mathcal{L}$ is the class $a_\Lambda \in H^1(X, \mathbb{R})$ defined as the preimage of $[\lambda_\Lambda]$ under the canonical isomorphism $H^1(X, \mathbb{R}) \to H^1(\Lambda, \mathbb{R})$; see Def. 2.1.23.

Definition 6.1.1. *A Lagrangian submanifold $\Lambda \in \mathcal{L}$ is* exact *if $a_\Lambda = 0$. The class of all exact Lagrangian submanifolds in $\mathcal{L}$ is denoted by $\mathcal{L}_0$.*

Finally, we define what we mean by a convex hypersurface in a cotangent bundle. Convexity will be a fundamental concept in this chapter. We refer to [81] for more general results concerning the non–convex case.

Definition 6.1.2. *A smooth, closed, fiberwise strictly convex hypersurface $\Sigma \subset T^*X$ is called a* convex hypersurface.

Fiberwise strict convexity means that Σ intersects each fiber T_x^*X along a hypersurface whose second fundamental form is positive definite.

In this section, we will establish a phenomenon called boundary rigidity which, roughly speaking, can be formulated as follows. Certain Lagrangian submanifolds lying in a convex hypersurface cannot be deformed into the domain bounded by that hypersurface. Boundary rigidity may seem unrelated to Mather–Mañé theory. However, as Thm. 6.2.11 will show, boundary rigidity can be seen as a particular case of non–removable intersection, to be discussed in Sect. 6.2. Moreover, the latter does have relations to Mather–Mañé theory, as we will see in Sect. 6.2.2. Thus, there is indeed some indirect connection to minimal action here.

6.1.1 Graph selectors for Lagrangian submanifolds

A particular example of an exact Lagrangian submanifold in T^*X is given by the graph of the differential of a smooth function $f : X \to \mathbb{R}$. A general exact Lagrangian submanifold Λ, of course, need not be a graph. However, we will see that, even in this case, it is possible to extract a "graph part" inside Λ.

The following theorem was outlined by Sikorav (in a talk held at Chaperon's seminar) and proven by Chaperon (in the framework of generating functions) and Oh (via Floer homology).

Theorem 6.1.3 (Sikorav, Chaperon [17], Oh [80]). *Let $\Lambda \subset T^*X$ be an exact Lagrangian submanifold in $\mathcal{L}_0$. Then there exists a Lipschitz continuous function $\Phi : X \to \mathbb{R}$, which is smooth on an open set $X_0 \subset X$ of full measure, such that*

$$(x, \mathrm{d}\Phi(x)) \in \Lambda \tag{6.1}$$

for every $x \in X_0$. Moreover, if $\mathrm{d}\Phi(x) = 0$ for all $x \in X_0$ then Λ coincides with the zero section $\mathcal{O}$.

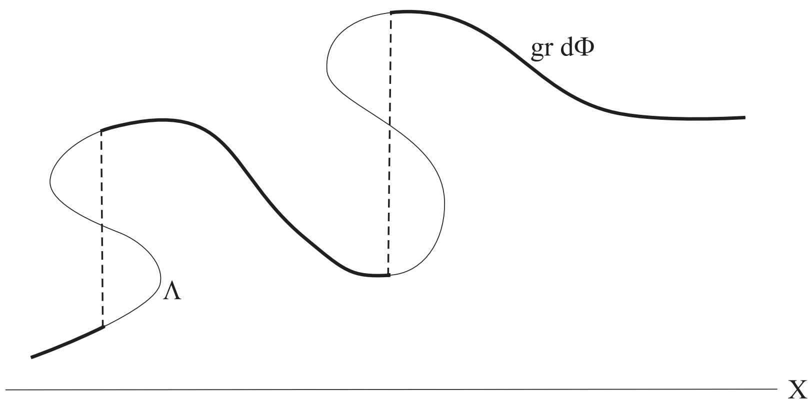


Fig. 6.1. A graph selector of an exact Lagrangian submanifold Λ

Definition 6.1.4. *Let $\Lambda \subset T^*X$ be an exact Lagrangian submanifold in $\mathcal{L}_0$. Any function $\Phi : X \to \mathbb{R}$ satisfying* (6.1) *is called a* graph selector *of Λ; compare Fig. 6.1.*

We will prove Thm. 6.1.3 by using generating functions quadratic at infinity, a powerful tool of symplectic topology in cotangent bundles. Although this proof of Thm 6.1.3 is well known to experts, it was probably published in [81]; we repeat it here for the convenience of the reader.

Let X be a closed manifold, and E a finite–dimensional real vector space. Denote by $\mathcal{O}_E$ the zero section of T^*E and set

$$V := T^*X \times \mathcal{O}_E \subset T^*X \times T^*E = T^*(X \times E).$$

Definition 6.1.5. *A smooth function $S : X \times E \to \mathbb{R}$ is called a* generating function quadratic at infinity *if*

$$S(x, \xi) = Q_x(\xi)$$

outside a compact subset of $X \times E$, where Q_x is a smooth family of nondegenerate quadratic forms on E, and $gr\,\mathrm{d}S$ is transversal to V in $T^*(X \times E)$.

In particular, $W := \mathrm{gr}\,\mathrm{d}S \cap V$ is a smooth closed submanifold of V of the same dimension as X. Let $\chi : V \to T^*X$ be the natural projection. One can show that the restriction of χ to W is a Lagrangian immersion; see [4, Sect. 19]. If $\chi|_W$ is an embedding then

$$\Lambda := \chi(W)$$

is a Lagrangian submanifold of T^*X. In this case we say that Λ possesses a generating function S quadratic at infinity, which means that

$$\Lambda = \{(x, d_x S(x,\xi)) \mid x \in X, \xi \in E, d_\xi S(x,\xi) = 0\}. \tag{6.2}$$

Proof (Thm. 6.1.3). Let $\Lambda \in \mathcal{L}_0$ be given. Then Λ possesses a generating function $S : X \times E \to \mathbb{R}$ quadratic at infinity [99]. The graph selector $\Phi : X \to \mathbb{R}$ will be defined by the following minimax procedure.

Fix a scalar product on E. Let $B_x : E \to E$ be a self–adjoint operator so that $Q_x(\xi) = (B_x\xi, \xi)$. Denote by E_x^- the subspace of E generated by all eigenvectors of B_x with negative eigenvalues. Set

$$E_x^a := \{\xi \in E \mid S_x(\xi) \le a\}$$

where $a \in \mathbb{R}$ and $S_x(\cdot) := S(x, \cdot)$. Pick an $N > 0$ such that $S(x,\xi) = Q_x(\xi)$ whenever $|Q_x(\xi)| \ge N$. All quadratic forms Q_x have the same index which we denote by m. The homology group $H_m(E_x^N, E_x^{-N}; \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2$, and its generator A_x is represented by the m-dimensional disc in E_x^- whose boundary lies in $\{Q_x(\xi) = -N\}$. For $a \in [-N, N]$, consider the natural morphism

$$I_{a,x} : H_m(E_x^a, E_x^{-N}; \mathbb{Z}_2) \to H_m(E_x^N, E_x^{-N}; \mathbb{Z}_2).$$

Now define the function $\Phi : X \to \mathbb{R}$ by

$$\Phi(x) := \inf\{a \mid A_x \in \mathrm{Image}(I_{a,x})\}.$$

We claim that Φ has all the properties stated in Thm. 6.1.3.

It follows from the definition that each value $\Phi(x)$ is a critical value of S_x. Consider the subset $X_0 \subset X$ consisting of all those x for which S_x is a Morse function whose critical points have pairwise distinct critical values. In any neighbourhood U of a point in X_0 there exists a smooth function $\varphi : U \to E$ such that $\varphi(x)$ is a critical point of S_x and $\Phi(x) = S(x, \varphi(x))$. Differentiating with respect to x and taking into account that $\mathrm{d}_\xi S(x, \varphi(x)) = 0$ we get that $\mathrm{d}\Phi(x) = \mathrm{d}_x S(x, \varphi(x))$. Thus, in view of (6.2), we have

$$(x, \mathrm{d}\Phi(x)) \in \Lambda$$

for all $x \in X_0$.

Claim. X_0 is an open subset of X of full measure.

Proof. Let $\theta : T^*X \to X$ be the natural projection. Then S_x is a Morse function if, and only if, x is a regular value of $\theta|_\Lambda$; see, e.g., [4, Sect. 21.2].

Denote the set of these $x \in X$ by X_1. It is an open subset of X and, by Sard's Theorem, has full measure.

Let $U \subset X_1$ be a sufficiently small open subset. The critical points of S_x depend smoothly on $x \in U$. Denote them by $\varphi_1(x), \ldots, \varphi_d(x)$, and put

$$a_{ij}(x) := S(x, \varphi_i(x)) - S(x, \varphi_j(x))$$

for $i \neq j$. Note that

$$\mathrm{d}a_{ij}(x) = \mathrm{d}_x S(x, \varphi_i(x)) - \mathrm{d}_x S(x, \varphi_j(x)) \neq 0$$

since the map $(x, \xi) \mapsto (x, \mathrm{d}_x S(x, \xi))$ is an embedding of $W = \operatorname{gr} \mathrm{d}S \cap V$ into T^*X. Therefore the sets $\{x \in U \mid a_{ij}(x) = 0\}$ are smooth hypersurfaces. It follows from the definition of X_0 that

$$X_0 \cap U = U \setminus \cup_{i \neq j} \{x \in U \mid a_{ij}(x) = 0\},$$

so $X_0 \cap U$ is indeed an open subset of full measure in $X \cap U$. $\square$

Claim. If $\mathrm{d}\Phi(x) = 0$ for all $x \in X_0$ then Λ coincides with the zero section of T^*X.

Proof. Identify X with the zero section of T^*X. Since X_0 has full measure, its closure equals X. Hence Λ contains X since $\mathrm{d}\Phi(x) = 0$ for $x \in X_0$, and thus $\Lambda = X$. $\square$

Claim. Φ is a Lipschitz function on X.

Proof. Since X is compact it suffices to prove this locally. Let $U \subset X$ be a sufficiently small open subset. There exists a smooth family of linear automorphisms $F_x : E \to E$ with $x \in U$, and a quadratic form Q on E, such that $Q_x \circ F_x = Q$ for all $x \in U$. It is easy to see that the function $S'(x, \xi) := S(x, F_x \xi)$ is again a generating function of Λ over U quadratic a infinity, and the functions Φ' and Φ coincide on U. Let us now work with S' instead of S, because the functions S'_x with $x \in U$ equal the same quadratic form Q outside a compact subset of E.

There exists a positive constant C such that for all $x, y \in U$ and $\xi \in E$ we have

$$|S'(x, \xi) - S'(y, \xi)| \leq C|x - y|. \tag{6.3}$$

Fix any $\epsilon > 0$ and $x \in U$, and set

$$a(y) := \Phi(x) + \epsilon + C|x - y|$$

where $y \in U$. It follows from inequality (6.3) that $E_x^{a(x)} \subset E_y^{a(y)}$ for all $y \in U$. By definition, the pair $(E_x^{a(x)}, E^{-N})$ contains a relative cycle representing the class A_x. Therefore, the same holds for the pair $(E_y^{a(y)}, E^{-N})$. This implies that $\Phi(y) \leq a(y)$, so that

$$\Phi(y) - \Phi(x) \leq C|x-y| + \epsilon.$$

Since $\epsilon > 0$ was arbitrary we have

$$\Phi(y) - \Phi(x) \leq C|x-y|.$$

Interchanging x and y shows that Φ is Lipschitz continuous, as we wanted to prove. $\square$

Summarizing, the function Φ satisfies all requirements and is indeed a graph selector of Λ. This finishes the proof of Thm. 6.1.3. $\square$

6.1.2 Boundary rigidity

Let Σ be a hypersurface in a cotangent bundle T^*X. Denote by σ the *characteristic foliation* of Σ, i.e., the 1–dimensional foliation tangent to the kernel of $\omega|_{T\Sigma}$. Note that σ is orientable and tangent to each Lagrangian submanifold contained in Σ.

Given a convex hypersurface Σ, we denote by U_Σ the closed(!) domain in T^*X bounded by Σ.

Definition 6.1.6. *An orientable 1–dimensional foliation on a closed manifold is called* conservative *if it admits a non–vanishing tangent vector field whose flow preserves a measure which is absolutely continuous with respect to some (and hence any) Riemannian measure on that manifold.*

Recall that $a_\Lambda \in H^1(X,\mathbb{R})$ is the Liouville class of $\Lambda \in \mathcal{L}$.

Theorem 6.1.7. *Let $\Lambda \in \mathcal{L}$ be a Lagrangian submanifold that is contained in some convex hypersurface Σ such that the restriction $\sigma|_\Lambda$ of the characteristic foliation is conservative. Let $K \in \mathcal{L}$ be*[1] *any Lagrangian submanifold lying inside U_Σ. Then*

$$a_K = a_\Lambda \iff K = \Lambda.$$

In particular, Λ cannot be deformed inside U_Σ by an exact Lagrangian isotopy, i.e., by a Lagrangian isotopy that preserves the Liouville class. This is the reason for the name "boundary rigidity".

Proof. First of all, by the multi–dimensional Birkhoff theorem [11], Λ is a Lagrangian section, i.e., $\Lambda = \operatorname{gr}\nu$ for some closed 1–form ν. By applying the symplectic shift $(x,p) \mapsto (x, p-\nu_x)$ we may assume that $\Lambda = \mathcal{O}$ is the zero section. Note that the transformed hypersurface, again denoted by Σ, remains convex.

Suppose now there is another Lagrangian submanifold $K \subset U_\Sigma$, obtained from Λ by an exact Lagrangian deformation. Thm. 6.1.3 implies that K admits

[1] We denote Lagrangian submanifolds by Greek letters, so this is a capital κ and not a capital k...

a graph selector $\Phi : X \to \mathbb{R}$, i.e., a function such that $(x, d\Phi(x)) \in K$ for all $x \in X_0$ where $X_0 \subset X$ is a set of full measure.

Pick a convex Hamiltonian function $H : T^*X \to \mathbb{R}$ such that Σ is a regular level set of H. Since Λ is the zero section, the vector $\partial_p H(x,0)$ gives the outer normal direction to the hypersurface $\Sigma \cap T_x^*X \subset T_x^*X$. Because Σ is convex and K is contained in U_Σ we have

$$\mathrm{d}\Phi(x) \cdot \partial_p H(x,0) < 0 \tag{6.4}$$

for all $x \in X_0$ with $\mathrm{d}\Phi(x) \neq 0$; see Fig 6.2.

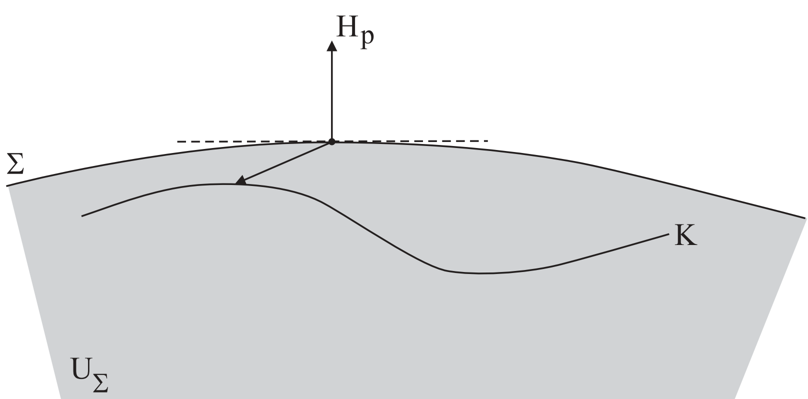


Fig. 6.2. An illustration for the inequality (6.4)

Let V be a non–singular vector field on Λ, tangent to the characteristic foliation, whose flow φ^t preserves a measure μ which is absolutely continuous with respect to some Riemannian measure. Then the Hamiltonian differential equations for H show that V is collinear to the vector field $\partial_p H(x,0)$ on Λ. In view of (6.4), we may assume that

$$\mathrm{d}\Phi(x) \cdot V(x) < 0 \tag{6.5}$$

for all $x \in X_0$ with $\mathrm{d}\Phi(x) \neq 0$. On the other hand, we claim that

$$\int_{X_0} \mathrm{d}\Phi(x) \cdot V(x) \mathrm{d}\mu(x) = 0. \tag{6.6}$$

Then the theorem is an immediate consequence of (6.6). Indeed, combining (6.6) and (6.5) we see that $\mathrm{d}\Phi$ must vanish on X_0, and hence

$$K = \mathcal{O} = \Lambda$$

in view of Thm. 6.1.3.

It remains to prove (6.6). Since the function Φ is Lipschitz continuous, the function $t \mapsto \Phi(\varphi^t(x)) - \Phi(x)$ is also Lipschitz continuous on $[0,1]$ for every $x \in X$. By Rademacher's theorem, it is differentiable almost everywhere with

$$\Phi(\varphi^1(x)) - \Phi(x) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\Phi(\varphi^t(x))\mathrm{d}t.$$

Since the flow φ^t preserves the measure μ we have

$$0 = \int_X [\Phi(\varphi^1(x)) - \Phi(x)]\,\mathrm{d}\mu(x) = \int_X \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\Phi(\varphi^t(x))\mathrm{d}t\,\mathrm{d}\mu(x).$$

Since X_0 has full measure with respect to μ and since φ^t preserves μ, we have

$$\begin{aligned} 0 &= \int_0^1 \int_X \frac{\mathrm{d}}{\mathrm{d}t}\Phi(\varphi^t(x))\mathrm{d}\mu(x)\,\mathrm{d}t \\ &= \int_0^1 \int_{\varphi^{-t}(X_0)} \mathrm{d}\Phi(\varphi^t(x)) \cdot V(\varphi^t(x))\mathrm{d}\mu(x)\,\mathrm{d}t \\ &= \int_0^1 \int_{X_0} \mathrm{d}\Phi(x) \cdot V(x)\mathrm{d}\mu(x)\,\mathrm{d}t \\ &= \int_{X_0} \mathrm{d}\Phi(x) \cdot V(x)\mathrm{d}\mu(x). \end{aligned}$$

This proves (6.6) and finishes the proof of the theorem. □

As the following example shows, the assumption about the dynamics of the characteristic foliation cannot be omitted.

Example 6.1.8. Consider $\Sigma = \{H = 1\} \subset T^*\mathbb{T}^2$ where

$$H(x_1, x_2, y_1, y_2) := (y_1 - \sin x_1)^2 + (y_2 - \cos x_1)^2. \tag{6.7}$$

Then Σ contains the zero section $\Lambda = \mathcal{O}$. However, the restriction $\sigma|_{\mathcal{O}}$ of the characteristic foliation is a Reeb foliation with exactly two limit cycles and, therefore, not conservative. We claim that $\mathcal{O}$ is not boundary rigid either. Indeed, the exact Lagrangian torus $K = \operatorname{gr} \mathrm{d}f$ with

$$f(x_1, x_2) := -\cos x_1$$

does lie in U_Σ.

It is worth mentioning that K intersects Σ precisely at the two limit cycles of the characteristic foliation. As we will see in Section 6.2.2, this is no coincidence.

6.2 Non–removable intersections

6.2.1 Mather–Mañé theory for minimizing hypersurfaces

In this section, we will see that many of the concepts presented in Sect. 2.2 do not really depend on the Lagrangian (or the Hamiltonian), but can rather be formulated in the more general framework of convex hypersurfaces. We concentrate on the torus $\mathbb{T}^n$ here, but all results and proofs in this section are, word by word, valid for general closed manifolds; see [81].

As usual, we let $\theta : T^*\mathbb{T}^n \to \mathbb{T}^n$ be the cotangent bundle of the torus $\mathbb{T}^n$, equipped with the canonical symplectic form $\omega = \mathrm{d}\lambda$, where λ is the Liouville form. Let $\Sigma \subset T^*\mathbb{T}^n$ be a convex hypersurface, and σ its characteristic foliation. Recall that σ is orientable; we choose the orientation defined by the Hamiltonian vector field of any convex Hamiltonian having Σ as a regular level set. Denote by U_Σ the closed domain in $T^*\mathbb{T}^n$ bounded by Σ.

Definition 6.2.1. *A convex hypersurface Σ is* minimizing *if the interior of U_Σ does not contain a Lagrangian submanifold from $\mathcal{L}_0$, but any open neighbourhood of U_Σ does.*

Remark 6.2.2. It will turn out that, in defining minimizing hypersurfaces, one can restrict to Lagrangian sections, rather than Lagrangian submanifolds. Indeed, Thm. 6.3.4 ensures that we obtain precisely the same concept.

Remark 6.2.3. Suppose Σ is a minimizing hypersurface, and H is a convex Hamiltonian having Σ as a regular level set $H^{-1}(k)$. Then, in view of (2.9), $k = c(L)$ is the Mañé critical value of the Lagrangian L corresponding to H.

Proposition 6.2.4. *If a convex hypersurface Σ contains a Lagrangian submanifold $\Lambda \in \mathcal{L}_0$ then Σ is minimizing.*

Proof. Any open neighbourhood of U_Σ contains the Lagrangian submanifold $\Lambda \in \mathcal{L}_0$. On the other hand, any other Lagrangian submanifold $K \in \mathcal{L}_0$ must intersect Λ by Gromov's theorem [36], so it cannot lie completely in the interior of U_Σ. □

In the following, we are going to replace the concept of minimizing measure for a convex Lagrangian L by a notion that depends only on the foliation σ of an energy surface, and not on the particular choice of L. The appropriate notion is that of a foliation cycle, introduced by Sullivan [100]. We briefly review these ideas.

Let M be a closed n–dimensional manifold and let Ω_p be the real vector space of smooth p–forms on M. This vector space has a natural topology which makes it a locally convex linear space. A continuous linear functional $f : \Omega_p \to \mathbb{R}$ is called a p–current. With a natural topology, the space Ω_p^* of p–currents becomes a locally convex linear space. Given a p–current f, we define its boundary ∂f as the $(p-1)$–current such that $\partial f(\omega) = f(\mathrm{d}\omega)$ for all $\omega \in \Omega_{p-1}$. Currents with zero boundary are called cycles.

Given a foliation of M, Sullivan considers a distinguished subset of Ω_1^* that he calls *foliation currents*. This subset is defined as follows. Let V be a vector field tangent to the foliation. For each $x \in M$, let $\delta_x : \Omega_1 \to \mathbb{R}$ be the Dirac 1–current defined by $\delta_x(\omega) := \omega_x(V(x))$. By definition, foliation currents are the elements of the closed convex cone in Ω_1^* generated by all the Dirac currents.

Definition 6.2.5. *A* foliation cycle *is a foliation current $f \in \Omega_1^*$ whose boundary ∂f is zero.*

Suppose now that V is a non–vanishing vector field on M. Then V defines a map $\mu \mapsto f_{V,\mu}$ from measures to 1–currents given by

$$f_{V,\mu}(\omega) := \int_M \omega(V)\,\mathrm{d}\mu.$$

Sullivan [100, Prop. II.24] shows that this map yields continuous bijections between the following objects:

1. nonnegative measures on M and foliation currents;
2. measures on M, invariant under the flow of V, and foliation cycles.

In our setting, the manifold M will be a minimizing hypersurface $\Sigma \subset T^*\mathbb{T}^n$. Pick a convex Hamiltonian H such that

$$\Sigma = H^{-1}(k)$$

is a regular level set, and let L be the corresponding Lagrangian. In view of Remark 6.2.3, we have

$$k = c := c(L).$$

The following simple observation allows us to translate the notion of globally minimizing measure into the languange of foliation cycles of the characteristic foliation. Namely, if (x, v) is a point in the critical energy level $E^{-1}(c) \subset T\mathbb{T}^n$ then

$$L(x, v) + c = \lambda(\mathrm{d}\ell(V(x, v))) \tag{6.8}$$

where V is the Euler–Lagrange vector field of L, and ℓ the Legendre transform. By Prop. 2.2.4, an invariant measure μ is globally minimizing if $\int_{T\mathbb{T}^n} L + c\,d\mu = 0$. We also know from [25] that globally minimizing measures have their support contained in the energy level $E^{-1}(c)$. Hence, the correct translation of the notion of globally minimizing measures into the language of foliation cycles is the following.

Definition 6.2.6. *Let Σ be a minimizing hypersurface in $T^*\mathbb{T}^n$, and σ its characteristic foliation. A foliation cycle f of σ is called* minimizing *if, and only if, $f(\lambda) = 0$.*

Minimizing foliation cycles are precisely those which can be represented by measures $\ell_* \mu$ on $T^*\mathbb{T}^n$, where μ is some minimizing measure for some Hamiltonian H with $\Sigma = H^{-1}(k)$. Observe also that, if we have two Hamiltonians H_1, H_2 with the same regular level set Σ, and two minimizing measures μ_1, μ_2 of H_1, H_2 representing the same foliation cycle f, then the supports μ_1 and μ_2 will coincide. Hence it makes sense to talk about the support of a foliation cycle f of σ.

Now, the *Mather set* of a minimizing hypersurface Σ is defined as the closure of the union of the supports of all minimizing foliation cycles. It coincides with the Mather set $\tilde{\mathcal{M}}^*$ in $T^*\mathbb{T}^n$ of any convex Hamiltonian H having Σ as regular level set.

In order to go further and define the Aubry set of Σ, we first have to explain what a weak KAM solution should be in our setting. Given a point $(x,p) \in \Sigma$, let $\Gamma^{\pm}(x,p)$ be the oriented positive (respectively, negative) half of the leaf $\Gamma_{(x,p)}$ of σ through (x,p).

Definition 6.2.7. *Let Σ be a minimizing hypersurface in $T^*\mathbb{T}^n$. A function $u_+ : \mathbb{T}^n \to \mathbb{R}$ is called a* positive weak KAM solution *of Σ if the following two conditions hold:*

1. *u_+ is Lipschitz, and $(x, \mathrm{d}u_+(x)) \in U_\Sigma$ for almost every $x \in \mathbb{T}^n$;*
2. *for every $x \in \mathbb{T}^n$, there exists $(x,p) \in \Sigma$ such that, if (y,p') is a point in $\Gamma^+_{(x,p)}$, then*
$$u_+(y) - u_+(x) = \int_{\Gamma^+_{(x,p)}(y,p')} \lambda$$
where $\Gamma^+_{(x,p)}(y,p')$ is the oriented part of the leaf between (x,p) and (y,p').

Similarly, a function $u_- : \mathbb{T}^n \to \mathbb{R}$ is called a negative weak KAM solution *of Σ if the following two conditions hold:*

1. *u_- is Lipschitz, and $(x, \mathrm{d}u_-(x)) \in U_\Sigma$ for almost every $x \in \mathbb{T}^n$;*
2. *for every $x \in \mathbb{T}^n$, there exists $(x,p) \in \Sigma$ such that, if (y,p') is a point in $\Gamma^-_{(x,p)}$, then*
$$u_-(x) - u_-(y) = \int_{\Gamma^-_{(x,p)}(y,p')} \lambda$$
where $\Gamma^-_{(x,p)}(y,p')$ is the oriented part of the leaf between (y,p') and (x,p).

Again, (6.8) shows that the sets
$$\mathcal{S}_\pm = \mathcal{S}_\pm(\Sigma)$$
of positive (respectively, negative) weak KAM solutions depend only on Σ and not on the particular choice of H. A pair of functions (u_-, u_+) is called *conjugate* if $u_\pm \in \mathcal{S}_\pm$ and $u_- = u_+$ on the projected Mather set. Setting
$$\mathcal{I}_{(u_-,u_+)} := \{x \in \mathbb{T}^n \mid u_-(x) = u_+(x)\}$$

for a pair of conjugate functions, we see as before that the functions $u_\pm$ are differentiable on $\mathcal{I}_{(u_-,u_+)}$ with the same derivative. Therefore, the map $x \mapsto \mathrm{d}u_-(x) = \mathrm{d}u_+(x)$ defines a set $\tilde{\mathcal{I}}_{(u_-,u_+)}$ in $T^*\mathbb{T}^n$ that contains the Mather set of Σ. The *Aubry set* of Σ in $T^*\mathbb{T}^n$ is then given by

$$\tilde{\mathcal{A}}^* = \tilde{\mathcal{A}}^*(\Sigma) := \cap_{(u_-,u_+)}\tilde{\mathcal{I}}_{(u_-,u_+)}$$

where the intersection is taken over all pairs (u_-, u_+) of conjugate functions.

Having defined the Aubry set, one would now like to study the dynamics on it and single out a certain dynamically relevant set inside the Aubry set. For this, we need the following general definition.

Definition 6.2.8. *Let φ^t be a continuous flow on a compact metric space (X, d). Given $\epsilon > 0$ and $T > 0$, a* strong (ϵ, T)–chain *from x to y in X is a finite sequence $(x_i, t_i)_{1\le i\le n}$ in $X \times \mathbb{R}$ such that $x_1 = x, x_n = y$, and $t_i > T$ for all i, as well as*

$$\sum_{i=1}^{n-1} d(\varphi^{t_i}(x_i), x_{i+1}) < \epsilon.$$

A point $x \in X$ is said to be strongly chain recurrent *if for all $\epsilon > 0$ and $T > 0$, there exists a strong (ϵ, T)–chain that begins and ends in x. The set of strongly chain recurrent points in X is denoted by $\mathcal{R}$.*

The set $\mathcal{R}$ contains the nonwandering set[2], but it is easy to give examples showing that it could be strictly larger. The notion of strong chain recurrence strengthens the usual notion of chain recurrence where one requires only $d(\varphi^{t_i}(x_i), x_{i+1}) < \epsilon$ for every single i.

Given a smooth orientable 1–dimensional foliation σ on a closed manifold, the strong chain recurrent set of σ is the strong chain recurrent set of the flow of any non–vanishing vector field V tangent to σ. In the case where σ is the characteristic foliation of a hypersurface $\Sigma \subset T^*\mathbb{T}^n$, we denote by

$$\mathcal{R}^*(\sigma) \subset \Sigma$$

the strong chain recurrent set in $T^*\mathbb{T}^n$, and by $\mathcal{R}(\sigma) \subset T\mathbb{T}^n$ its preimage under the Legendre transform.

Theorem 6.2.9. *Let Σ be a minimizing hypersurface in $T^*\mathbb{T}^n$, and $\Lambda \subset \Sigma$ be an exact Lagrangian submanifold (not necessarily in $\mathcal{L}$). Then*

$$\mathcal{R}^*(\sigma|_\Lambda) \subset \tilde{\mathcal{A}}^*(\Sigma).$$

In particular, $\mathcal{R}^*(\sigma|_\Lambda)$ is a Lipschitz graph over $\mathbb{T}^n$.

[2] A point $x \in X$ is nonwandering if, and only if, for every neighbourhood U of x there exists a $T > 1$ such that $\phi_T(U) \cap U \neq \emptyset$; this implies that there are also arbitrarily large T with that property.

Proof. Choose a convex Hamiltonian H having Σ as a regular level set, and let L be the corresponding Lagrangian. For the proof, we will work in the tangent bundle $T\mathbb{T}^n$.

Endow $T\mathbb{T}^n$ and $\mathbb{T}^n$ with auxiliar Riemannian distances $d_{T\mathbb{T}^n}$ and $d_{\mathbb{T}^n}$ in such a way that the natural projection $\tau : T\mathbb{T}^n \to \mathbb{T}^n$ does not increase the distances. Consider $(x, v) \in \mathcal{R}$ and consider the curve γ with

$$\gamma(t) := \tau(\varphi_L^t(x, v)).$$

In view of Prop. 2.2.16, it suffices to show that γ is static. This will imply that $\mathcal{R} \subset \tilde{\mathcal{A}}$.

Take $s \le t$ and set

$$\xi := \varphi_L^s(x, v) \quad , \quad \eta := \varphi_L^t(x, v).$$

We claim that, for any given $\epsilon > 0$, there exists a strong $(\epsilon, 1)$–chain from η to ξ. To see this, let us start with a strong (δ, T)–chain from $x_1 := \eta$ to $x_{n+1} := \eta$ where $T > 1$ is large compared to $t - s$, and replace x_{n+1} by $\varphi_L^{t_n-(t-s)}(x_n)$. If $\delta > 0$ is chosen sufficiently small, the point $\varphi_L^{t_n-(t-s)}(x_n)$ lies in an ϵ–neighbourhood of ξ, and we obtain a strong $(\epsilon, 1)$–chain from η to ξ. Let us call this chain $(\eta_i, t_i)_{1\le i\le n+1}$ with $\eta_1 = \eta$, $\eta_{n+1} = \xi, t_i > 1$, and

$$\sum_{i=1}^{n} d_{T\mathbb{T}^n}(\varphi_L^{t_i}(\eta_i), \eta_{i+1}) < \epsilon.$$

Set $p_i := \tau(\eta_i)$ and $q_i := \tau(\varphi_L^{t_i}(\eta_i))$. Using (6.8) and the fact that Λ is exact, we have

$$\Phi_c(p_i, q_i) \le A_{L+c}(\tau \circ \varphi_L^t(\eta_i)|_{[0,t_i]}) = g(\phi_{t_i}(\eta_i)) - g(\eta_i), \tag{6.9}$$

where $g : \ell^{-1}(\Lambda) \to \mathbb{R}$ is a smooth function such that $\mathrm{d}(g \circ \ell^{-1}|_\Lambda) = \lambda|_\Lambda$.

Recall that the action potential Φ_c satisfies the triangle inequality

$$\Phi_c(x, y) \le \Phi_c(x, z) + \Phi_c(z, y)$$

and $\Phi_c(x, x) = 0$. Hence we can estimate

$$\Phi_c(p_1, p_{n+1}) \le \Phi_c(p_1, q_1) + \Phi_c(q_1, p_2) + \ldots + \Phi_c(p_n, q_n) + \Phi_c(q_n, p_{n+1}).$$

Given $p, q \in \mathbb{T}^n$ let $\gamma : [0, d_{\mathbb{T}^n}(p, q)] \to \mathbb{T}^n$ be a unit speed minimizing geodesic from p to q. Then we have

$$\Phi_c(p, q) \le \int_0^{d_{\mathbb{T}^n}(p,q)} (L + c)(t, \gamma(t), \dot\gamma(t))\, \mathrm{d}t \le \kappa_1 d_{\mathbb{T}^n}(p, q)$$

where $\kappa_1 := \max\{|(L + c)(x, v)| \mid (x, v) \in T\mathbb{T}^n \text{ and } |v| = 1\}$. Thus

$$\sum_i \Phi_c(q_i, p_{i+1}) \le \kappa_1 \sum_i d_{\mathbb{T}^n}(q_i, p_{i+1}) \le \kappa_1 \epsilon. \tag{6.10}$$

Combining (6.10) and(6.9), we obtain

$$\Phi_c(p_1, p_{n+1}) \le \sum_i \Phi_c(p_i, q_i) + \Phi_c(q_i, p_{i+1}) \le \kappa_1\epsilon + \kappa_2\epsilon + g(\xi) - g(\eta),$$

where κ_2 is a Lipschitz constant for g. On the other hand,

$$\Phi_c(p_{n+1}, p_1) \le A_{L+c}(\gamma|_{[s,t]}) = g(\eta) - g(\xi).$$

Therefore, we obtain

$$0 = \Phi_c(p_1, p_1) \le \Phi_c(p_1, p_{n+1}) + \Phi_c(p_{n+1}, p_1) \le (\kappa_1 + \kappa_2)\epsilon.$$

Since $\epsilon > 0$ was arbitrary, we conclude that

$$\Phi_c(p_1, p_{n+1}) + \Phi_c(p_{n+1}, p_1) = 0.$$

Using the triangle inequality for Φ_c as in the proof of Prop. 2.2.16, we finally see that γ is a static curve. □

6.2.2 The Aubry set and non–removable intersections

Let $\Sigma \subset T^*\mathbb{T}^n$ be a convex hypersurface bounding the closed domain U_Σ. We want to study the following question. Suppose Σ contains a Lagrangian submanifold $\Lambda \in \mathcal{L}$. Is it possible to deform Λ into the interior of U_Σ? In other words, if you think of Σ being an energy surface $H^{-1}(k)$ of a convex Hamiltonian, can one push Λ into the region where $H < k$? Compare Fig. 6.3 for an illustration.

Of course, there is no problem in deforming Λ into the interior of U_Σ by a Lagrangian isotopy: just apply symplectic shifts $(x, y) \mapsto (x, y - \nu_x)$ with some closed 1–form ν. So the whole point is to ask for *exact* Lagrangian isotopies which preserve the Liouville class. In fact, one can forgo the isotopy and ask whether there exists another Lagrangian K in the interior of U_Σ with $a_K = a_\Lambda$.

It turns out that there are situations where any such Lagrangian K cannot lie completely in the interior of U_Σ. In fact, we will see that it cannot even be disjoint from Λ. In this case we say that a *non–removable intersection* phenomenon occurs for Λ. It is clear now that boundary rigidity from Sect. 6.1.2 can be seen as a particular case of non–removable interesctions.

The following theorem establishes a non–removable intersection phenomenon in the context of Mather–Mañé theory.

Theorem 6.2.10. *Let Σ be a minimizing hypersurface such that U_Σ contains a Lagrangian submanifold $\Lambda \in \mathcal{L}_0$. Then*

$$\tilde{\mathcal{A}}^* \subset \Lambda \cap \Sigma.$$

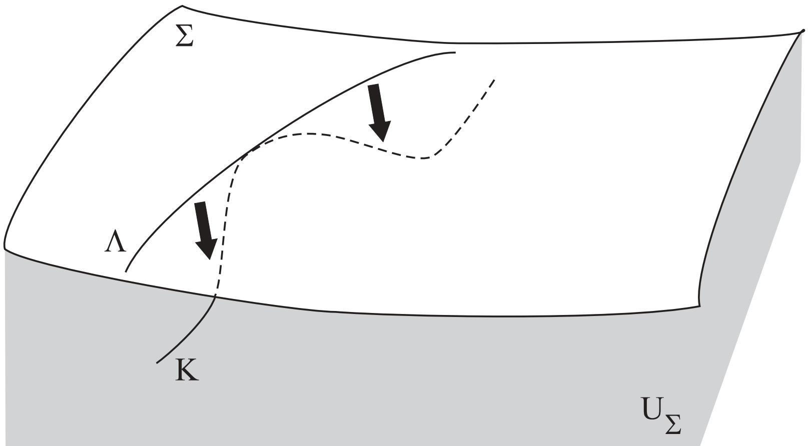


Fig. 6.3. A deformation of Λ into the domain U_Σ

Proof. Let $u : \mathbb{T}^n \to \mathbb{R}$ be a graph selector of Λ, whose existence is guaranteed by Thm. 6.1.3. The function u is Lipschitz continuous and satisfies

$$(x, \mathrm{d}u(x)) \in \Lambda \tag{6.11}$$

for every point $x \in \mathbb{T}^n$ where U is differentiable; these points form a set of full measure. Choose a Hamiltonian H such that Σ is a regular level set of H. Rem. 6.2.3 shows that

$$\Sigma = H^{-1}(c)$$

where $c = c(L)$ is the critical value of the Lagrangian L corresponding to H. By Rem. 2.2.9 and Thm. 2.2.11, there exists a pair of conjugate functions (u_-, u_+) with

$$u_+ \leq u \leq u_-.$$

At any point $x \in \mathcal{I}_{(u_-,u_+)}$, the three functions are differentiable with the same derivative. Hence $\mathrm{d}u(x)$ exists for each $x \in \mathcal{I}_{(u_-,u_+)}$ and satisfies $(x, \mathrm{d}u(x)) \in \Sigma$.

We claim that we have

$$(x, \mathrm{d}u(x)) \in \Lambda$$

for every $x \in \mathcal{I}_{(u_-,u_+)}$. This is the main step of the proof since, a priori, we only know that this is true for almost every x.

In order to prove our claim we let

$$C_x(\Lambda) := \mathrm{conv}(\Lambda \cap T^*_{x\mathbb{T}^n})$$

denote the convex hull of $\Lambda \cap T^*_x\mathbb{T}^n$. The set $\Lambda \cap T^*_x\mathbb{T}^n$ is compact, so $C_x(\Lambda)$ is compact, too, as well as

$$C(\Lambda) := \cup_{x\in\mathbb{T}^n} C_x(\Lambda).$$

This implies that for any point x of differentiability of u we have

$$(x, \mathrm{d}u(x)) \in C(\Lambda);$$

see [90]. But since $\Sigma \cap T_x^*\mathbb{T}^n$ is strictly convex, and (6.11) holds with $\Lambda \subset U_\Sigma$, the point $(x, \mathrm{d}u(x))$ is an extreme point of $C_x(\Lambda)$. But any extreme point in the convex hull $\mathrm{conv}(\Lambda \cap T_x^*\mathbb{T}^n)$ belongs to $\Lambda \cap T_x^*\mathbb{T}^n$ itself, and therefore we have $(x, \mathrm{d}u(x)) \in \Lambda$. This proves our claim.

Now, by definition of the Aubry set, $\tilde{\mathcal{A}}$ is contained in $\tilde{\mathcal{I}}_{(u_-,u_+)}$ for any pair of conjugate functions. This finishes the proof of the theorem. □

As mentioned before, Thm. 6.2.10 can be applied in order to establish boundary rigidity results. The following theorem is a generalization of Thm. 6.1.7, because the assumption on the dynamics on Σ are weaker. Note, however, that the proof of Thm. 6.1.7 did not need Mather–Mañé theory.

Theorem 6.2.11. *Let $\Lambda \in \mathcal{L}$ be a Lagrangian submanifold contained in some convex hypersurface Σ such that the restriction $\sigma|_\Lambda$ of the characteristic foliation is strongly chain recurrent. Let $K \in \mathcal{L}$ be any Lagrangian submanifold lying inside U_Σ. Then*

$$a_K = a_\Lambda \iff K = \Lambda.$$

Proof. Since the multi–dimensional Birkhoff theorem is valid if $\sigma|_\Lambda$ is chain recurrent [11, Prop. 1.2], we may, as in the proof of Thm. 6.1.7, apply a symplectic shift and assume that $\Lambda = \mathcal{O} \subset T^*\mathbb{T}^n$. By Prop. 6.2.4 the shifted hypersurface obtained from Σ is still minimizing since it contains $\mathcal{O}$. But then Thm. 6.2.9 implies that $\mathcal{O} \subset \tilde{\mathcal{A}}^*$. Since the natural projection $\theta|_{\tilde{\mathcal{A}}^*} : \tilde{\mathcal{A}}^* \to \mathcal{A}$ is a homeomorphism [29] we must have

$$\tilde{\mathcal{A}}^* = \mathcal{O}.$$

Thm. 6.1.3 states that K possesses a graph selector; choose one. As in the proof of Thm. 6.2.10, it will be differentiable at every point in $\theta(\tilde{\mathcal{A}}^*) = \mathbb{T}^n$ with zero derivative. But this means that K coincides with the zero section, and so

$$K = \mathcal{O} = \Lambda$$

as we wanted to prove. □

Example 6.2.12 (cont.). Let us come back to Ex. 6.1.8. Recall that we consider the zero section $\mathcal{O}$ of $T^*\mathbb{T}^2$ lying inside the convex hypersurface

$$\Sigma = \{(y_1 - \sin x_1)^2 + (y_2 - \cos x_1)^2 = 1\}.$$

The restriction $\sigma|_\mathcal{O}$ of the characteristic foliation is a Reeb foliation; see Fig. 6.4. Denote by Z the union of the two limit cycles. Note that Z is the strong chain recurrent set of $\sigma|_\mathcal{O}$, and so, by Thm. 6.2.9, we have

$$Z \subset \tilde{\mathcal{A}}^*. \tag{6.12}$$

Since Σ contains the zero section it is minimizing in view of Prop. 6.2.4. Applying Thm. 6.2.10, we see that

$$Z \subset K \cap \Sigma$$

for every Lagrangian submanifold $K \in \mathcal{L}_0$ contained in U_Σ. This explains the remark at the end of Ex. 6.1.8.

In fact, we can show that the Aubry and Mather sets of Σ coincide with Z:

$$\tilde{\mathcal{M}}^* = Z = \tilde{\mathcal{A}}^*.$$

Indeed, we noticed in Ex. 6.1.8 that the graph of $\mathrm{d}f$ with $f(x_1, x_2) = -\cos x_1$ intersects Σ precisely along Z. Hence, by Thm. 6.2.10, we obtain $\tilde{\mathcal{A}}^* \subset Z$. Together with (6.12) this yields $Z = \tilde{\mathcal{A}}^*$. Furthermore, each of the two limit cycles in Z is a foliation cycle; it vanishes on the Liouville form since $\lambda|_{\mathcal{O}} = 0$. Hence we also see that $\tilde{\mathcal{M}}^* = Z$.

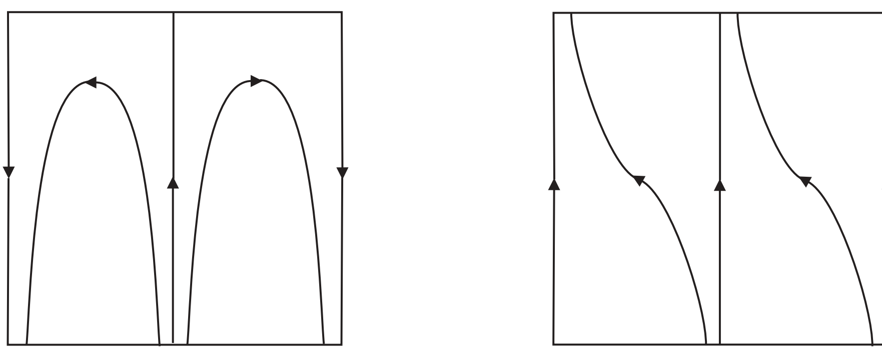


Fig. 6.4. The dynamics on the zero section in Ex. 6.2.12 (left) and Ex. 6.2.13 (right)

Example 6.2.13. Let us investigate the zero setion in $T^*\mathbb{T}^2$ with different dynamics. For this, we pick a diffeomorphism $f : S^1 \to S^1$ with exactly two fixed points such that the fixed points are neither attractors nor repellors. Let V be the unit norm vector field on $\mathbb{T}^2$ obtained by suspending f. Write

$$V(x_1, x_2) =: (a_1(x_1, x_2), a_2(x_1, x_2))$$

and let H be the convex Hamiltonian

$$H(x_1, x_2, y_1, y_2) := (y_1 - a_1(x_1, x_2))^2 + (y_2 - a_2(x_1, x_2))^2.$$

Consider the convex hypersurface $\Sigma := \{H = 1\} \subset T^*\mathbb{T}^2$. Since Σ contains the zero section $\mathcal{O}$ it is minimizing in view of Prop. 6.2.4. If we identify $\mathcal{O}$

with $\mathbb{T}^2$ then V is tangent to the characteristic foliation $\sigma|_{\mathcal{O}}$. Note that $\sigma|_{\mathcal{O}}$ is strongly chain recurrent, hence $\mathcal{O}$ is boundary rigid by Thm. 6.2.11.

In this example, we will find that

$$\tilde{\mathcal{M}}^* = Z \neq \mathcal{O} = \tilde{\mathcal{A}}^*.$$

Indeed, Theorems 6.2.9 and 6.2.10 yield $\tilde{\mathcal{A}}^* \subset \mathcal{O}$ and $\mathcal{O} \subset \tilde{\mathcal{A}}^*$, respectively, so $\tilde{\mathcal{A}}^* = \mathcal{O}$. On the other hand, the same argument as in Ex. 6.2.12 shows that $\tilde{\mathcal{M}}^* = Z$.

6.3 Symplectic shapes and the minimal action

This section deals with certain symplectic properties of domains in a cotangent bundle $(T^*X, \omega = \mathrm{d}\lambda)$ of some closed manifold X. Namely, given some domain $U \subset T^*X$, we ask which cohomology classes in $H^1(X, \mathbb{R})$ can be represented as Liouville classes of Lagrangian submanifolds lying in U. We refer to Def. 2.1.23 for the definition of the Liouville class of a Lagrangian submanifold in $\mathcal{L}$.

Definition 6.3.1. *The* shape *of a subset $U \subset T^*X$ is defined as*

$$sh(U) := \{a_\Lambda \in H^1(X, \mathbb{R}) \mid \Lambda \in \mathcal{L} \text{ with } \Lambda \subset U\}.$$

The notion of shape allows an elegant formulation of Gromov's theorem on Lagrangian intersections proven in [36]: shapes of disjoint subsets in T^*X are disjoint. As a consequence, if Σ is a hypersurface in T^*X bounding the domain U_Σ, then every Lagrangian submanifold $\Lambda \in \mathcal{L}$ with $a_\Lambda \in \partial \mathrm{sh}(U_\Sigma)$ must intersect Σ.

The shape of U is an exact symplectic invariant of U; in particular, it is preserved by Hamiltonian diffeomorphisms of T^*X. From the dynamical point of view, a very important class of Lagrangian submanifolds are Lagrangian sections, i.e., graphs of closed 1–forms. This leads to the following definition.

Definition 6.3.2. *The* sectional shape *of a subset $U \subset T^*X$ is defined as*

$$sh_0(U) := \{a_\Lambda \in H^1(X, \mathbb{R}) \mid \Lambda \in \mathcal{L} \text{ is a section with } \Lambda \subset U\}.$$

It is clear that

$$\mathrm{sh}_0(U) \subset \mathrm{sh}(U).$$

In contrast to the shape, however, the sectional shape is *not* preserved under Hamiltonian diffeomorphisms and does, therefore, not belong to the realm of symplectic geometry.

The question arises whether there are natural situations in which the sectional shape and the shape coincide. We will see that this is the case for the class of fiberwise convex domains. For simplicity, we call a subset $U \subset T^*X$ *convex* if it is fiberwise convex.

6.3.1 Lagrangian sections in convex domains

Suppose $U \subset T^*X$ be an open convex domain. We want to prove that every class $a \in \mathrm{sh}(U)$ can be represented by a Lagrangian *section* of the cotangent bundle. Indeed, this an immediate consequence of the following theorem[3]. Let us denote the fiberwise convex hull of a set $S \subset T^*X$ by $\mathrm{conv}(S)$.

Theorem 6.3.3. *Given a Lagrangian submanifold $\Lambda \in \mathcal{L}$, the fiberwise convex hull conv(W) of any neighbourhood W of Λ contains a Lagrangian section $\Lambda_0 \in \mathcal{L}$ with $a_{\Lambda_0} = a_\Lambda$.*

Proof. We may assume that Λ is an exact Lagrangian submanifold, by applying the symplectic shift $(x, y) \mapsto (x, y - \nu_x)$ where ν is the closed 1–form on X representing the Liouville class a_Λ.

Let $\Phi : X \to \mathbb{R}$ be a graph selector of Λ as described in Thm. 6.1.3; namely, Φ is Lipschitz continuous, smooth on an open subset $X_0 \subset X$ of full measure, and satisfies

$$\mathrm{gr}\, \mathrm{d}\Phi|_{X_0} \subset \Lambda. \tag{6.13}$$

The proof of Thm. 6.3.3 is divided into two steps.

Smoothing: We are going to regularize the Lipschitz continuous function Φ by a convolution argument, similar to the proof of Prop. 7 in [22]. For this, we embed X into some Euclidean space $\mathbb{R}^N$. Denote by V_r the r–neighbourhood of X in $\mathbb{R}^N$, where $r > 0$ is chosen small enough so that the orthogonal projection

$$\pi : V_r \to X$$

is well defined. We extend $\Phi : X \to \mathbb{R}$ to a function $\bar{\Phi} : V_r \to \mathbb{R}$ by setting

$$\bar{\Phi} := \Phi \circ \pi.$$

For each $s \in (0, r/2)$ we pick a smooth cut–off function $u : [0, \infty) \to [0, \infty)$ with support in $[0, s]$ such that u is constant near 0 and satisfies

$$\int_{\mathbb{R}^N} u(|z|)dz = 1.$$

Define the function $\bar{\Psi} : V_s \to \mathbb{R}$ to be the convolution

$$\bar{\Psi}(z) := (\bar{\Phi} * u)(z) = \int_{R^N} \bar{\Phi}(y)u(|z - y|)\mathrm{d}y.$$

Since $\bar{\Phi}$ is Lipschitz continuous, it is differentiable almost everywhere and weakly differentiable. Therefore, $\bar{\Psi}$ is a smooth function on V_s with

[3] A slightly more general version of it was proven independently in [30, App.].

$$\begin{aligned}\mathrm{d}\bar{\Psi}(z) &= \int_{\mathbb{R}^N} \bar{\Phi}(y)\mathrm{d}_z u(|z-y|)\mathrm{d}y \\ &= -\int_{\mathbb{R}^N} \bar{\Phi}(y)\mathrm{d}_y u(|z-y|)\mathrm{d}y \\ &= \int_{\mathbb{R}^N} \mathrm{d}\bar{\Phi}(y) u(|z-y|)\mathrm{d}y.\end{aligned}$$

Denote by

$$\Psi := \bar{\Psi}|_X$$

the restriction of $\bar{\Psi}$ to X, and let $B_s(x) \subset V_s \subset \mathbb{R}^N$ be the open ball of radius s centered at $x \in X$. Because X_0 has full measure in X, we conclude that

$$\mathrm{d}\Psi(x) = \int_{\pi^{-1}(X_0)\cap B_s(x)} \mathrm{d}\bar{\Phi}(y)|_{T_x X} u(|x-y|)\mathrm{d}y. \tag{6.14}$$

Note that, for this formula to make sense, we identify each $T_y\mathbb{R}^N$ (where $y \in \mathbb{R}^N$) with $\mathbb{R}^N$, and each T_xX (where $x \in X$) with a linear subspace of $\mathbb{R}^N$.

Analising formula (6.14)*:* For each $x \in X$, we write

$$P_x : T_x\mathbb{R}^N \cong \mathbb{R}^N \to T_xX$$

for the orthogonal projection. Write $|\cdot|$ for the Euclidean norm on $\mathbb{R}^N$ and $|\cdot|^*$ for the dual norm on $(\mathbb{R}^N)^*$. Introduce a distance function on T^*X by setting

$$\mathrm{dist}((x,\xi),(y,\eta)) := |x-y| + |\xi \circ P_x - \eta \circ P_y|^*. \tag{6.15}$$

For $x \in X$, we define the set

$$\mathcal{G}_s(x) := \{(x, \mathrm{d}\bar{\Phi}(y)|_{T_xX}) \mid y \in \pi^{-1}(X_0) \cap B_s(x)\} \subset T^*X.$$

For a subset $Z \subset T^*X$, we denote by $W_\epsilon(Z)$ the ϵ–neighbourhood of Z with respect to the distance defined in (6.15).

Claim. For every $\epsilon > 0$ there is an $s > 0$ such that

$$\mathcal{G}_s(x) \subset W_{\epsilon/2}(\mathrm{gr}\,\mathrm{d}\Phi|_{X_0})$$

for each $x \in X$.

Proof. Pick any point

$$\eta_1 = (x, \mathrm{d}\bar{\Phi}(y)|_{T_xX}) \in \mathcal{G}_s(x)$$

with $x \in X$ and $y \in \pi^{-1}(X_0) \cap B_s(x)$. We will show that the distance between η_1 and

$$\eta_2 := (\pi(y), \mathrm{d}\Phi(\pi(y))) \in \mathrm{gr}\,\mathrm{d}\Phi|_{X_0}$$

becomes as small as we wish, uniformly in x and y, when $s \to 0$.

Indeed, denote by $c > 0$ the Lipschitz constant of Φ with respect to the induced distance on $X \subset \mathbb{R}^N$. Let Q_y be the differential of the projection π at y, where we consider Q_y as an endomorphism of $\mathbb{R}^N$. Finally, write $\|\cdot\|$ for the operator norm on $\mathrm{End}(\mathbb{R}^N)$. Now we can estimate

$$\begin{aligned} \operatorname{dist}(\eta_1, \eta_2) &= |x - \pi(y)| + |\mathrm{d}\bar{\Phi}(y)|_{T_x X} \circ P_x - \mathrm{d}\Phi(\pi(y)) \circ P_{\pi(y)}|^* \\ &= |x - \pi(y)| + |\mathrm{d}\Phi(\pi(y)) \circ Q_y \circ P_x - \mathrm{d}\Phi(\pi(y)) \circ P_{\pi(y)}|^* \\ &\leq |x - y| + |y - \pi(y)| + c\|Q_y \circ P_x - P_{\pi(y)}\|. \end{aligned}$$

Note that $|x - y| + |y - \pi(y)| \leq 2s \to 0$ as $s \to 0$. Therefore, it remains to handle the term $\|Q_y \circ P_x - P_{\pi(y)}\|$. Using that $\|P_x\| = \|P_{\pi(y)}\| = 1$ we obtain

$$\begin{aligned} \|Q_y \circ P_x - P_{\pi(y)}\| &= \|Q_y \circ P_x - P_{\pi(y)} \circ P_x + P_{\pi(y)} \circ P_x - P_{\pi(y)} \circ P_{\pi(y)}\| \\ &\leq \|Q_y - P_{\pi(y)}\| + \|P_x - P_{\pi(y)}\| \to 0 \end{aligned}$$

as $s \to 0$, and the convergence is uniform in $x \in X$ and $y \in B_s(x)$.

This finishes the proof of our claim. □

Now the proof of Thm. 6.3.3 follows immediately. Namely, given any $\epsilon > 0$, we choose s as given in our claim. Then (6.14) and (6.13) imply that

$$(x, \mathrm{d}\Psi(x)) \in \operatorname{conv}(W_{\epsilon/2}(\mathcal{G}_s(x))) \subset \operatorname{conv}(W_\epsilon(\operatorname{gr} \mathrm{d}\Phi|_{X_0})) \subset \operatorname{conv}(W_\epsilon(\Lambda))$$

for each $x \in X$. Therefore, the Lagrangian section $\Lambda_0 := \operatorname{gr} \mathrm{d}\Psi$ satisfies

$$\Lambda_0 \subset \operatorname{conv}(W_\epsilon(\Lambda)).$$

Since $\epsilon > 0$ was arbitrary the proof of Thm. 6.3.3 is completed. □

6.3.2 Symplectic descriptions of the stable norm and the minimal action

In this final section, we focus on Lagrangian submanifolds contained in some convex subset of a cotangent bundle. Recall Def. 6.3.1 and Def. 6.3.2 of the shape and sectional shape of a subset $U \subset T^*\mathbb{T}^n$, respectively. We mentioned the fact that the shape is preserved under Hamiltonian diffeomorphisms, whereas the sectional shape is not.

The following theorem is the main result of this section. It states that for open convex sets $U \subset T^*\mathbb{T}^n$ both notions coincide.

Theorem 6.3.4. *Let $U \subset T^*\mathbb{T}^n$ be open and convex. Then every class $a \in sh(U)$ can be represented by a Lagrangian* section *of the cotangent bundle. In other words,*

$$sh_0(U) = sh(U).$$

Proof. Let $a \in \mathrm{sh}(U)$ be represented by a Lagrangian $\Lambda \in \mathcal{L}$ contained in U. Since U is open and convex, it contains the fiberwise convex hull $\mathrm{conv}(W)$ of some small neighbourhood W of Λ. Now, Thm. 6.3.3 guarantees that there is a Lagrangian section $\Lambda_0 \subset W$ with $a_{\Lambda_0} = a_\Lambda$. □

By taking convex combinations of Lagrangian sections, the following is a direct consequence of Thm.6.3.4.

Corollary 6.3.5. *The shape of an open convex subset of $T^*\mathbb{T}^n$ is an open convex subset of $H^1(\mathbb{T}^n, \mathbb{R})$.*

Note that the shape of an open subset is always open; this follows immediately from Weinstein's Lagrangian neighbourhood theorem. Therefore, the main statement here is about convexity.

Example 6.3.6. Take a Riemannian metric g on $\mathbb{T}^n$ and consider the corresponding open unit co–ball bundle

$$B_g^*\mathbb{T}^n := \{(x,p) \in T^*\mathbb{T}^n \mid |p|_g < 1\}.$$

In geometric measure theory, one defines a particular norm on $H^1(\mathbb{T}^n, \mathbb{R})$, called the *stable norm.* Let us illustrate the stable co–norm here, i.e., the corresponding dual norm $\|\cdot\|$ on $H_1(\mathbb{T}^n, \mathbb{R})$. If we write $\ell(h)$ for the minimal length of a closed geodesic representing an integer homology class $h \in H_1(\mathbb{T}^n, \mathbb{Z})$ then

$$\|h\| := \lim_{N\to\infty} \frac{\ell(Nh)}{N}.$$

Let us denote by $B_{\mathrm{st}}^*\mathbb{T}^n \subset H^1(\mathbb{T}^n, \mathbb{R})$ the open unit ball of the stable norm. Gromov proved [37] that

$$B_{\mathrm{st}}^*\mathbb{T}^n = \mathrm{sh}_0(B_g^*\mathbb{T}^n).$$

In view of Thm. 6.3.4, we now have the following result.

Theorem 6.3.7. *Let g be a Riemannian metric on $\mathbb{T}^n$ and $B_g^*\mathbb{T}^n$ the corresponding unit ball bundle. Then the unit ball of the stable norm coincides with the shape of $B_g^*\mathbb{T}^n$:*

$$B_{st}^*\mathbb{T}^n = sh(B_g^*\mathbb{T}^n).$$

Thus, for the Riemannian case, Theorem 6.3.4 leads to a geometric description of the symplectic shape of a Riemannian unit co–ball bundle and, vice versa, to a symplectic characterization of the unit stable norm ball.

We come back to our favourite setting and consider a convex Lagrangian $L : T\mathbb{T}^n \to \mathbb{R}$. Recall from Ch. 2 that, associated to L, there is the minimal action $\alpha : H_1(\mathbb{T}^n, \mathbb{R}) \to \mathbb{R}$ and its convex conjugate $\alpha^* : H^1(\mathbb{T}^n, \mathbb{R}) \to \mathbb{R}$. The following result translates Mather's variational construction of the minimal action into the language of symplectic geometry.

Theorem 6.3.8. *Let $L : T\mathbb{T}^n \to \mathbb{R}$ be a convex Lagrangian, and $H : T^*\mathbb{T}^n \to \mathbb{R}$ the corresponding convex Hamiltonian. Then the convex conjugate $\alpha^* : H^1(\mathbb{T}^n, \mathbb{R}) \to \mathbb{R}$ of the minimal action of L can be written as*

$$\alpha^*(c) = \inf\{k \in \mathbb{R} \mid c \in sh(\{H < k\})\}.$$

Proof. Recall from (2.9) that the critical value $c(L)$ of L allows the representation

$$c(L) = \inf_u \max_x H(x, \mathrm{d}u(x)).$$

It describes $c(L)$ as the least value k such that the sublevel set $\{H < k\}$ contains an exact Lagrangian section. Moreover, Cor. 2.2.6 showed that the convex conjugate $\alpha^* : H^1(\mathbb{T}^n, \mathbb{R}) \to \mathbb{R}$ of the minimal action can be calculated via the critical value as

$$\alpha^*([\nu]) = c(L - \nu).$$

Therefore, we have

$$\alpha^*(c) = \inf\{k \in \mathbb{R} \mid c \in \mathrm{sh}_0(\{H < k\})\}.$$

Since H is convex, each sublevel set $\{H < k\}$ is a (fiberwise) convex subset of $T^*\mathbb{T}^n$. Therefore, Thm. 6.3.4 implies that

$$\mathrm{sh}_0(\{H < k\}) = \mathrm{sh}(\{H < k\}).$$

This finishes the proof of Thm. 6.3.8. $\square$

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