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Solitons and Geometry

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Lecture 1
Introduction. Plan of the lectures. Poisson structures.

The theory of Solitons ("solitary waves") deals with the propagation of non-linear waves in continuum media. Their famous discovery has been done in the period 1965-1968 (by M. Kruscal and N. Zabuski, 1965; G. Gardner, I. Green, M. Kruscal and R. Miura, 1967; P. Lax, 1968 —see the survey [8] or the book [22]).

The familiar KdV (Korteweg-de Vries) non-linear equation was found to be exactly solvable in some profound nontrivial sense by the so-called "Inverse Scattering Transform" (IST) at least for the class of rapidly decreasing initial data $\Phi(x)$:

$$\Phi_t = 6\Phi\Phi_x - \Phi_{xxx} \qquad \text{[KdV]}.$$

Some other famous systems are also solvable by analogous procedures. The following are examples of $1+1$-systems:

$$\begin{aligned} &\Phi_{\eta\zeta} = \sin\Phi && \text{[SG]} \\ &i\Phi_t = -\Phi_{xx} \pm |\Phi|^2\Phi && [\text{NS}_\pm] \\ &\cdots . \end{aligned}$$

The most interesting integrable $2+1$-systems are the following:

$$\begin{cases} w_x = u_y \\ u_t = 6uu_x + u_{xxx} + 3\alpha^2\omega_y \\ \alpha^2 = \pm 1 \end{cases} \qquad \text{[KP]},$$

$$\begin{cases} V_t = \Re(V_{zzz} + (aV)_z) \\ \overline{V} = V, a_{\overline{z}} = 3V_z, \partial_z = \partial_x - i\partial_y \end{cases} \qquad \text{[2-dimKdV]}.$$

There are different beautiful connections between Solitons and Geometry, which we will now shortly describe.

Solitons and 3-dimensional Geometry.

a) The sine-Gordon equation $\Phi_{\eta\zeta} = \sin\Phi$ appeared for the first time in a problem of 3-dimensional geometry: it describes locally the isometric imbeddings of the Lobatchevski 2-plane L^2 (i.e. the surface with constant negative Gaussian curvature) in the Euclidean 3-space $\mathbf{R}^3$. Here Φ is the angle between two asymptotic directions (η, ζ) on the surface along which the second

(curvature) form is zero. It has been used by Bianchi, Lie and Backlund for the construction of new imbeddings ("Backlund transformations", discovered by Bianchi).
b) The elliptic equation

$$\triangle\Phi = \sinh\Phi$$

appeared recently for the description of genus 1 surfaces ("topological tori") in $\mathbf{R}^3$ with constant mean curvature (H. Wente, 1986; R. Walter, 1987).

Starting from 1989 F. Hitchin, U. Pinkall, N. Ercolani, H. Knorrer, E. Trubowitz, A. Bobenko used in this field the technique of the "periodic IST" —see [5].

Solitons and algebraic geometry.
a) There is a famous connection of Soliton theory with algebraic geometry. It appeared in 1974-1975 . The solution of the periodic problems of Soliton theory led to beautiful analytical constructions involving Riemann surfaces and their Jacobian varieties, Θ-functions and later also Prym varieties and so on (S. Novikov, 1974; B. Dubrovin - S.Novikov, 1974; B. Dubrovin, 1975; A. Its - V. Matveev,1975; P. Lax, 1975; H. McKean - P. Van Moerbeke, 1975 —see [8]).

Many people worked in this area later (see [22], [8], [7] and [6]). Very important results were obtained in different areas, including classical problems in the theory of Θ-functions and construction of the harmonic analysis on Riemann surfaces in connection with the "string theory" —see [15], [16], [17] and [6].
b) Some very new and deep connection of the KdV theory with the topology of the moduli spaces of Riemann surfaces appeared recently in the works of M. Kontzevich (1992) in the development of the so-called "2-d quantum gravity". It is a byproduct of the theory of "matrix models" of D. Gross-A. Migdal, E. Bresin-V. Kasakov and M. Duglas - N. Shenker, in which Soliton theory appeared as a theory of the "renormgroup" in 1989-90.

Soliton theory and Riemannian geometry.
Let us recall that the systems of Soliton theory (like KdV) sometimes describe the propagation of non-linear waves. For the solution of some problems we are going to develop an asymptotic method which may be considered as a natural non-linear analogue of the famous WKB approximation in Quantum Mechanics. It leads to the structures of Riemannian Geometry; some nice

classes of infinite-dimensional Lie algebras appeared in this theory. This will be exactly the subject of the present lectures (see also [10]). *All beautiful constructions of Soliton theory are available for Hamiltonian systems only (nobody knows why).* Therefore we will start with an elementary introduction to Symplectic and Poisson Geometry (see also [21], [7], [10] and [11]).

Plan of the lectures.

1. Symplectic and Poisson structures on finite-dimensional manifolds. Dirac monopole in classical mechanics. Complete integrability and Algebraic Geometry.
2. Local Poisson Structures on loop spaces. First-order structures and finite dimensional Riemannian Geometry. Hydrodynamic-Type systems. Infinite-dimensional Lie Algebras. Riemann Invariants and classical problems of differential Geometry. Orthogonal coordinates in $\mathbf{R}^n$.
3. Nonlinear analogue of the WKB-method. Hydrodynamics of Soliton Lattices. Special analysis for the KdV equation. Dispersive analogue of the shock wave. Genus 1 solution for the hydrodynamics of Soliton Lattices.

Symplectic and Poisson structures

Let M be a finite-dimensional manifold with a system $(y^1, \ldots, y^m)$ of (local) coordinates.

Definition. Any non-degenerate closed 2-form

$$\Omega = \omega_{\alpha\beta} dy^\alpha \wedge dy^\beta$$

generates a *symplectic structure* on the manifold M. Non-degeneracy means exactly that the skew-symmetric matrix $(\omega_{\alpha\beta})$ is non-singular for all points $y \in M$, i.e.

$$\det(\omega_{\alpha\beta}(y)) \neq 0.$$

Remark. Since a skew-symmetric matrix in odd dimension is necessarily singular we have that if M has a symplectic structure then it has even dimension.

By definition a symplectic structure is just a special skew-symmetric scalar product of the tangent vectors: if $V = (V^\alpha)$ and $W = (W^\beta)$ are coordinates of tangent vectors we set:

$$(V, W) = \omega_{\alpha\beta} V^\alpha W^\beta = -(W, V).$$

Let $\omega^{\alpha\beta}$ denote the inverse matrix:

$$\omega^{\alpha\beta}\omega_{\beta\gamma} = \delta^\alpha_\beta.$$

This inverse matrix $(\omega^{\alpha\beta})$ determines everything important in the theory of Hamiltonian systems. Therefore we shall start with the following definition.

Definition. A skew-symmetric C^∞-tensor field $(\omega^{\alpha\beta})$ on the manifold M generates a *Poisson structure* if the Poisson bracket (defined below) turns the space $\mathrm{C}^\infty(M)$ into a Lie algebra: for any two functions $f, g \in \mathrm{C}^\infty(M)$ we define their Poisson bracket as a scalar product of the gradients:

$$\{f, g\} = \omega^{\alpha\beta} \frac{\partial f}{\partial y^\alpha} \frac{\partial g}{\partial y^\beta} = -\{g, f\}.$$

This operation obviously satisfies the following requirements:

$$\begin{aligned}
&\{f, g\} = -\{g, f\}, \\
&\{f + g, h\} = \{f, h\} + \{g, h\}, \\
&\{fg, h\} = f\{g, h\} + g\{f, h\},
\end{aligned}$$

so only the Jacobi identity is non-obvious.
Remark. Using the coordinate functions we have that

$$\{y^\alpha, y^\beta\} = \omega^{\alpha\beta}$$

$$\{\{y^\alpha, y^\beta\}, y^\gamma\} = \frac{\partial \omega^{\alpha\beta}}{\partial y^k}\frac{\partial y^\gamma}{\partial y^p}\omega^{kp} = \frac{\partial \omega^{\alpha\beta}}{\partial y^k}\omega^{k\gamma}$$

and it is easily checked that the Jacobi identity is equivalent to:

$$\frac{\partial \omega^{\alpha\beta}}{\partial y^k}\omega^{k\gamma} + \frac{\partial \omega^{\gamma\alpha}}{\partial y^k}\omega^{k\beta} + \frac{\partial \omega^{\beta\gamma}}{\partial y^k}\omega^{k\alpha} = 0 \quad \forall \alpha, \beta, \gamma.$$

In case $(\omega^{\alpha\beta})$ is non-singular and $(\omega_{\alpha\beta})$ denotes the inverse matrix this is also equivalent to:

$$\frac{\partial \omega_{\alpha\beta}}{\partial y^\gamma} + \frac{\partial \omega_{\gamma\alpha}}{\partial y^\beta} + \frac{\partial \omega_{\beta\gamma}}{\partial y^\alpha} = 0 \quad \forall \alpha, \beta, \gamma$$

i.e. to

$$d\left(\sum_{\alpha<\beta} \omega_{\alpha\beta} dy^\alpha \wedge dy^\beta\right) = 0$$

i.e. to closedness of the 2-form $\omega_{\alpha\beta}dy^\alpha \wedge dy^\beta$. (Recall however that the inverse matrix does not exist in some important cases.)
Definition. A function $f \in \mathrm{C}^\infty(M)$ is called a *Casimir* for the given Poisson bracket if it belongs to the kernel (or annihilator) of the Poisson bracket, i.e. if for any function $g \in \mathrm{C}^\infty(M)$ we have

$$\{f, g\} = 0.$$

Lecture 2.

Poisson Structures on Finite-dimensional Manifolds. Hamiltonian Systems. Completely Integrable Systems.

As in Lecture 1 we are dealing with a finite-dimensional manifold M with (local) coordinates $(y^1, \dots, y^m)$ and a Poisson tensor field $-\omega^{ij} = \omega^{ji}$ such that the corresponding Poisson bracket

$$\{f, g\} = \omega^{ij} \frac{\partial t}{\partial y^i} \frac{\partial g}{\partial y^j}$$

generates a Lie algebra structure on the space $\mathrm{C}^\infty(M)$.

Definition. Any smooth function $H(y)$ on M or a closed 1-form $H_\alpha dy^\alpha$ generates a *Hamiltonian system* by the formula

$$\dot{y}^\alpha = \omega^{\alpha\beta} H_\beta \qquad \left(H_\beta = \frac{\partial H}{\partial y^\beta} \right).$$

For any function $f \in \mathrm{C}^\infty(M)$ we define

$$\dot{f} = \{f, H\} = \omega^{\alpha\beta} H_\beta f_\alpha.$$

Definition. We will say a vector field V with coordinates (V^α) is a *Hamiltonian vector field generated by the Hamiltonian* $H \in \mathrm{C}^\infty(M)$ if $V^\alpha = w^{\beta\alpha} \partial H / \partial y^\beta$.

A well-known lemma states that the commutator of any pair of Hamiltonian vector fields is also Hamiltonian and it is generated by the Poisson bracket of the corresponding Hamiltonians.

We define a *Poisson algebra* as a commutative associative algebra C with an additional Lie algebra operation ("bracket")

$$C \times C \ni (f, g) \mapsto \{f, g\} \in C$$

such that

$$\{f \cdot g, h\} = f \cdot \{g, h\} + g \cdot \{f, h\}.$$

Definition. We call *integral* of a Hamiltonian system a function f such that $\dot{f} = 0$.

Lemma. The centralizer $Z(Q)$ of any set Q of elements of a Poisson algebra C is a Poisson algebra. In particular, for $C = \mathrm{C}^\infty(M)$ and $Q = \{H\}$ the centralizer of H is exactly the collection of all integrals of the Hamiltonian system generated by H, and therefore this collection is a Poisson algebra.

Examples.

1. As an example of non-degenerate Poisson structure we may choose local coordinates $(y^1, \ldots, y^m)$ such that

$$\omega^{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

2. As an example of a degenerate Poisson structure with constant rank we may choose local coordinates $(y^1, \ldots, y^m)$ such that

$$\omega^{ij} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

3. Let us consider a Poisson structure ω^{ij} whose coefficients are linear functions of some coordinates $(y^1, \ldots, y^m)$:

$$\omega^{ij} = C^{ij}_k y^k, \qquad C^{ij}_k = \text{const}.$$

Remark that

$$\{y^i, y^j\} = C^{ij}_k y^k;$$

therefore the collection of all linear functions is a Lie algebra which is finite-dimensional for the finite-dimensional manifold M; it is much smaller than the whole algebra $\mathrm{C}^\infty(M)$ in any case ("Lie-Poisson bracket").

The annihilator of this bracket is exactly the collection of "Casimirs", i.e. the center of the enveloping associative algebra $U(L)$ for L (this is a non-obvious theorem). This bracket has been invented by Sophus Lie about 100 years ago and later rediscovered by F. Beresin in 1960; it has been seriously used by Kirillov and Costant in representation theory —see [7] and [12].

4. (For this and the next example see the survey [21].) Let L be a semisimple Lie algebra with non-degenerate Killing form, $M = L^* = L$. For any diagonal quadratic Hamiltonian function $H(y) = \sum_i q_i (y^i)^2$ on the corresponding Hamiltonian system has the "Euler form":

$$\dot{Y} = [Y, \Omega]$$
$$Y \in L,\ \Omega \in L^*,\ \Omega = \frac{\partial H}{\partial Y},\ H \in S^2 L.$$

Suppose $L = so_n$. In this case the index (i) is exactly the pair

$$i = (\alpha, \beta),\ \alpha < \beta,\ \alpha, \beta = 1, \ldots, n,\ m = n(n-1)/2.$$

By [1] the generalized "rigid body" system corresponds to the case:

$$q_i = q_{(\alpha,\beta)} = q_\alpha + q_\beta, q_\alpha \geq 0.$$

More generally, let two collections of numbers

$$a_1, \ldots, a_n,\ b_1, \ldots, b_n$$

be given in such a way that

$$q_i = q_{(\alpha,\beta)} = \frac{a_\alpha - a_\beta}{b_\alpha - b_\beta}, \alpha < \beta.$$

The Euler system in this case admits the following so-called "λ-representation" (analogous to the one constructed in 1974 for the finite-gap solutions of KdV and the finite-gap potentials of the Schrödinger operator):

$$\partial_t(Y - \lambda U) = [Y - \lambda U, \Omega - \lambda V].$$

Here Y and Ω are skew-symmetric matrices and

$$U = \operatorname{diag}(a_1, \ldots, a_n), \quad V = \operatorname{diag}(b_1, \ldots, b_n)$$

(Manakov, 1976).

The collection of conservation laws might be obtained from the coefficients of the algebraic curve Γ:

$$\Gamma : \det(Y - \lambda U - \mu 1) = P(\lambda, \mu) = 0.$$

We shall return to this type of examples later. Remark that a Riemann surface already appeared here.

5. Consider now the Lie algebra L of the group E_3 (the isometry group of euclidean 3-space $\mathbf{R}^3$). The Lie algebra L is 6-dimensional: it has a set of generators $\{\underline{M}_1, \underline{M}_2, \underline{M}_3, \underline{p}_1, \underline{p}_2, \underline{p}_3\}$ satisfying the relations:

$$\begin{aligned}
&[\underline{M}_i, \underline{M}_j] = \varepsilon_{ijk}\underline{M}_k,\ \varepsilon_{ijk} = \pm 1,\\
&[\underline{M}_i, \underline{p}_j] = \varepsilon_{ijk}\underline{p}_k,\\
&[\underline{p}_i, \underline{p}_j] = 0.
\end{aligned}$$

We set $M = L = L^*$ and we denote by M_i and p_i the coordinates along $\underline{M}_i$ and $\underline{p}_i$ respectively. There are exactly two independent functions

$$f_1 = p^2 = \sum_i p_i^2, \qquad f_2 = ps = \sum_i M_i p_i$$

such that

$$\{f_\alpha, C^\infty(M)\} = 0 \quad \alpha = 1, 2.$$

We shall consider later the Hamiltonians

$$\begin{aligned} &\text{a) } 2H = \sum a_i M_i^2 + \sum b_{ij}(p_i M_j + M_i p_j) + \sum c_{ij} p_i p_j; \\ &\text{b) } 2H = \sum a_i M_i^2 + 2W(l^i p_i); \end{aligned}$$

(in case b we have $p^2 = 1$).

Definition. A Hamiltonian system is called *completely integrable in the sense of Liouville* if it admits a "large enough" family of independent integrals which are in involution (i.e. have trivial pairwise Poisson brackets), where "large enough" means exactly $(\dim M)/2$ for non-degenerate Poisson structures (symplectic manifolds) or $k + s$ if $\dim M = 2k + s$ and the rank of the Poisson tensor (ω^{ij}) is equal to 2k.

Let us fix for the sequel a completely integrable Hamiltonian system and a family $\{f_1, ..., f_{k+s}\}$ of integrals as in the definition. The gradients of these integrals are linearly independent at the generic point. Therefore the generic level surface:

$$\begin{cases} f_1 = c_1 \\ \dots \\ f_{k+s} = c_{k+s} \end{cases}$$

(where the c_i's are constants) is a k-dimensional non-singular manifold N^k in M.

Theorem. The manifold N^k is the factor of $\mathbf{R}^k$ by a lattice:

$$N^k = \mathbf{R}^k/\Gamma.$$

On N^k there are natural coordinates $(\varphi_1, \ldots, \varphi_k)$ such that $\dot{\varphi}_i = \text{const}$.

Corollary. If the level surface N^k is compact then it is diffeomorphic to the torus $T^k = (S^1)^k$.

Let us assume now that $s = 0$, i.e. that the tensor (ω^{ij}) is invertible. As usual we denote by Ω the (closed) 2-form with local expression $\sum_{i<j} \omega_{ij} dy^i \wedge$

dy^j where by definition $\omega^{il}\omega_{lj} = \delta^i_j$. Let us consider a compact level surface N^k. In a small neighborhood $U(N^k)$ in M we may introduce an important 1-form ω such that

$$d\omega = \Omega$$

$(\Omega = 0$ on $N^k)$. In the domain $U(N^k)$ there exist "action-angle" coordinates

$$(\varphi_1, \ldots, \varphi_k, s_1, \ldots, s_k)$$

such that

$$\{\varphi_i, \varphi_j\} = \{s_i, s_j\} = 0, \quad \{\varphi_i, s_j\} = \delta_{ij},$$
$$|s_i| \leq 1/(2\pi) \oint_{\gamma_i} \omega, \quad 0 \leq \varphi_i \leq 2\pi.$$

Here $\gamma_i \in H_1(N^k, \mathbf{Z})$ is the basic cycle on the torus N^k represented by the coordinates:

$$(\varphi_1, ..., \varphi_k), \; 0 \leq \varphi_i \leq 2\pi, \; \varphi_j = 0 \text{ for } j \neq i.$$

Lecture 3

Classical Analogue of the Dirac Monopole. Complete Integrability and Algebraic Geometry.

1. Let us consider again the important Example 5 of Lecture 2. Let $L^* = M$ where L is the Lie algebra of the group E_3; L has six generators $(\underline{M}_1, \underline{M}_2, \underline{M}_3, \underline{p}_1, \underline{p}_2, \underline{p}_3)$ and two "Casimirs"

$$ps = \sum M_i p_i = f_2, \qquad p^2 = \sum p_i^2 = f_1.$$

We recall that this means the following:

$$\{f_1, \mathrm{C}^\infty(M)\} = 0, \qquad \{f_2, \mathrm{C}^\infty(M)\} = 0$$

The ordinary top in the constant gravity field might be considered as a system on this phase space corresponding to the Hamiltonian:

$$2H = \sum_i a_i M_i^2 + g p_3, \quad p^2 = 1.$$

The original phase space is $T^*(\mathrm{SO}(3))$. There is at least one additional integral of motion apart from energy ("area integral") because the gravity force is axially symmetric in this case. Consider now the Poisson subalgebra A of $\mathrm{C}^\infty(T^*(\mathrm{SO}(3))$ which annihilates the area integral: $A = Z(f)$.
Lemma. The algebra A is isomorphic to $\mathrm{C}^\infty(M)$ factorized by the relation $(p^2 = 1)$. Here $M = L^*$ and L is the Lie algebra of the group E_3. The function $f \in A$ corresponds to the Casimir element

$$f_2 = ps = \sum M_i p_i$$

(the restriction $p^2 = 1$ has to be added). For any value $s = f_2$ this algebra is naturally isomorphic to $\mathrm{C}^\infty(T^*(S^2))$ as a commutative algebra.
Proof. The conclusion is obvious in case $\sum_i M_i p_i = s = 0$ and $p^2 = 1$. The sphere S^2 is exactly $p^2 = 1$ and M is the basis for the cotangent space.

Assume now that $s \neq 0$; we introduce the new variables

$$\sigma_i = M_i - \gamma p_i$$

such that

$$\sum \sigma_i p_i = 0, \gamma = s$$

and

$$\sum_i \sigma_i p_i = \sum_i M_i p_i - \gamma p^2 = s - \gamma = 0.$$

The lemma is proved. $\square$

Consider now the standard phase space $M = T^*(N)$ with the new symplectic structure:

$$\Omega = \sum_{i=1}^{k} dp_i \wedge dx^i + e \sum_{i<j} b_{ij} dx^i \wedge dx^j$$

Here $B = b_{ij} dx^i \wedge dx^j$ is a closed 2-form on the manifold N with local coordinates $(x^1, \dots, x^k)$, and $B = B(x)$ is the "magnetic field". The corresponding Poisson tensor has the form corrected by the magnetic field:

$$\omega^{ab} = \begin{pmatrix} -B(x)e & 1 \\ -1 & 0 \end{pmatrix},$$

which means that

$$\{p_i, p_j\} = eB_{ij}(x),$$
$$\{p_i, x^j\} = \delta_i^j,$$
$$\{x^i, x^j\} = 0.$$

After quantization the operators $\hat{p}_i$ will have the form

$$\hat{p}_j = \frac{\hbar}{i} \frac{\partial}{\partial x_j} + eA_j(x), \quad [p_i, p_j] = \hbar e B_{ij}(x),$$
$$d(\sum A_i(x) dx^i) = B, \quad b_{ij} = \partial_i A_j - \partial_j A_i.$$

The quantities $p_i - eA_i(x)$ will have trivial Poisson Bracket (often abbreviated by PB in the sequel):

$$\{p_i - eA_i, p_j - eA_j\} = 0,$$

and therefore they are the "canonical momenta" with the standard brackets. If $[B] \in H_r(N, \mathbf{R})$ is non-zero, the canonical momenta do not exist globally.

Let us introduce new variables $(\Psi, \Theta, p_\Psi, p_\Theta)$ by the relations:

$$\begin{aligned}
p_1 &= p\cos\Theta\cos\Psi, \\
p_2 &= p\cos\Theta\sin\Psi, \\
p_3 &= p\sin\Theta, \\
\sigma_1 &= p_\Psi\tan\Theta\cos\Psi - p_\Theta\sin\Psi, \\
\sigma_2 &= p_\Psi\tan\Theta\sin\Psi + p_\Theta\cos\Psi, \\
\sigma_3 &= -p_\Psi.
\end{aligned}$$

(Recall that $\sigma_i = M_i - \frac{s}{p}p_i$ and that we have $p^2 = 1$ in case b.) Here Ψ is defined modulo 2π and $-\pi/2 \le \Theta \le \pi/2$. Therefore (Ψ, Θ) are local coordinates on the sphere S^2 and $(p_\Theta, p_\Psi, \Theta, \Psi)$ are local coordinates on the space $T^*(S^2)$, their brackets being:

$$\begin{aligned}
&\{\Theta, \Psi\} = \{\Psi, p_\Theta\} = \{\Theta, p_\Psi\} = 0, \\
&\{\Theta, p_\Theta\} = \{\Psi, p_\Psi\} = 1, \\
&\{p_\Theta, p_\Psi\} = s\cos\Theta.
\end{aligned}$$

As a conclusion we have the following result:
Theorem. The motion of the top in constant gravity field might be reduced (after factorization by the area integral) to the system isomorphic to the "Dirac monopole" describing a particle with non-trivial electric charge moving on the sphere S^2 with a Riemannian metric under the influence of magnetic forces and potential forces. The corresponding magnetic flux is proportional to the value of the area integral:

$$\int\int_{S^2} s\cos\Theta d\Theta \wedge d\Psi = 4\pi s.$$

Proof. The Poisson Bracket was described above in the variables $(p_\Theta, p_\Psi, \Theta, \Psi)$ on $T^*(S^2)$. It corresponds exactly to the symplectic form

$$\Omega = d\Theta \wedge dp_\Theta + d\Psi \wedge dp_\Psi + s\cos\Theta d\Theta \wedge d\Psi.$$

Now consider the Hamiltonian:

$$\begin{aligned}
2H &= \sum a_i M_i^2 + gp_3 = \sum a_i(\sigma_i - sp_i)^2 + gp_i \\
&= \sum a_i\sigma_i^2 - 2\sum a_i sp_i\sigma_i + \sum s^2 a_i p_i^2 + gp_3.
\end{aligned}$$

In the new variables $(\Theta, \Psi, p_\Theta, p_\Psi)$ it has the form:

$$2H = g^{ab}\xi_a\xi_b + \tilde{A}^a\xi_a + W(x^1, x^2)$$

where

$$x^1 = \Theta,\ x^2 = \Psi,\ \xi_1 = p_\Theta,\ \xi_2 = p_\Psi$$

and we have:

$$\begin{aligned}
&\sum a_i\sigma_i^2 = g^{ab}(x)\xi_a\xi_b > 0, \\
&\tilde{A}^a\xi_a = s(\sum a_ip_i\sigma_i), \\
&2W = s^2(\sum a_ip_i^2 + gp_3, \\
&\tilde{A}_a = g_{ab}\tilde{A}^b(x), \quad a, b = 1, 2.
\end{aligned}$$

The full "magnetic term" will be:

$$d\left(\tilde{A}_a dx^a + s\sin\Theta d\Psi\right) = d\left(\sum \tilde{A}_a dx^a\right) + s\cos\Theta d\Theta d\Psi.$$

The first term $\left(\sum_a \tilde{A}_a dx^a\right)$ represents a part of a vector-potential which is a globally defined 1-form on the sphere S^2 proportional to s. All vector-potentials have the local form:

$$A_a dx^a = \tilde{A}_a dx^a + s\sin x_1 dx_2.$$

It is well-defined only with the area $\Theta \neq \pm\pi/2$.

The magnetic flux is equal to $4\pi s$. □

The previous theorem was proved by the author and E. Schmeltzer in 1981 (see [21]); it was known before only with $s = 0$, in which case there is no magnetic term at all.

The action functional $S\{\gamma\}$ is multi-valued on the loop space $\Omega(S^2)$ in the case of a Dirac monopole, which means that δS is a well-defined closed 1-form on the loop space.

2. Consider now Arnold's generalization of the Euler system

$$\dot{Y} = [Y, \Omega]$$

in the general Manakov integrable case (see Example 4 in Lecture 2), for the Lie algebras so_n, $n \geq 3$.

This equation itself looks like a "Lax representation" or an isospectral deformation of the matrix Y. But this representation is actually very poor. As a consequence of this fact we have only the trivial conservation laws:

$$\det(Y - \mu 1) = 0.$$

The coefficients of the characteristic polynomial are exactly the Casimirs only. Even the energy integral is not presented here.

The λ-representation

$$\partial_t \tilde{Y}(\lambda) = \partial_t (Y - \lambda U) = [Y - \lambda U, \Omega - \lambda V]$$

gives us much more: in fact it contains as coefficients of the algebraic curve:

$$\Gamma = \{\det(\tilde{Y}(\lambda) - \mu 1) = P(\lambda, \mu) = 0\}$$

a whole collection of commuting integrals sufficient for the complete integrability of the system. It is difficult to prove this fact directly if one forgets of the origin of the integrals from the periodic analogue of the IST based on the λ-representations —it was done in papers of Fomenko and Mischenko (see the references quoted in [7]). It is much better to avoid the direct elementary analysis of the integrals and use algebraic geometry like in the periodic IST for KdV.

The eigenvector $\Psi(\lambda, \mu)$ such that

$$\tilde{Y}(\lambda)\Psi = \mu\Psi$$

is meromorphic on the algebraic curve $\Gamma \backslash \infty$ and has an exponential asymptotic for $\lambda \to \infty$. Its analytical properties are such that its components have a collection of first order poles (γ_j) on the curve Γ and may be completely determined using these poles. In many cases this collection consists of exactly g poles, g being the genus of Γ. As a consequence the phase space of the system admits the new coordinates

$$\{\Gamma, \gamma_1, \dots, \gamma_g\},$$

where Γ varies in some subspace of the space of moduli of algebraic curves. For the collection of poles we have:

$$(\gamma_1, \dots, \gamma_g) \sim (\gamma_{i_1}, \dots, \gamma_{i_g})$$

for every permutation i on $\{1, ..., g\}$, i.e. the collection of poles $(\gamma_1, \ldots, \gamma_g)$ varies in the g-th symmetric power $S^g(\Gamma)$ of Γ because the ordering of the poles is not important.

Everybody knows that the manifold $S^g\Gamma$ is birationally isomorphic to the complex algebraic torus (Jacobian variety) $J(\Gamma)$. There is a natural "Abel map"

$$A : S^g\Gamma \longrightarrow J(\Gamma) = C^g/\Gamma.$$

In almost all important cases the Abel map allows to introduce the "angle coordinates" for the original system and hence to re-write the motion using the Θ-functions.

The computation of the action variables is very interesting: it has been carried out by H. Flashka and D. McLaughlin in 1976 and by A.Veselov and the author in 1982 for the most important systems (see [7]).

Starting from the λ-representation we are arrived to consider phase spaces M whose points have the form (Γ, γ), where $\gamma \in S^g(\Gamma)$ and Γ belongs to a subspace of the space of moduli of algebraic curves. In the most important classical cases this subspace contains only hyperelliptic curves Γ written in one of the following forms:

$$\text{(a)} \;\; \Gamma = \left\{ \mu^2 = P_{2g+1}(\lambda) = \prod_{j=0}^{2g} (\lambda - \lambda_j) \right\},$$

$$\text{(b)} \;\; \Gamma = \left\{ \mu^2 = P_{2g+2}(\lambda) = \prod_{j=0}^{2g+1} (\lambda - \lambda_j) \right\}.$$

(This is false for the case of generalized top considered above. The algebraic curves in this case are more complicated.)

We are going to introduce now the "algebro-geometric Poisson brackets". Consider a phase space M as above, with Γ varying in a subspace F. Suppose that $\dim F \geq g$, $\dim F = g + s$, and consider the natural projection

$$M \ni (\Gamma, (\gamma_1, ..., \gamma_g)) \overset{p}{\longmapsto} \Gamma \in F.$$

1) Our Poisson brackets should contain the annihilator $A \subset \mathrm{C}^\infty(M)$, such that

$$A \subset p^* \mathrm{C}^\infty(F)$$

i.e. the annihilator depends on Γ only.

For any $\Gamma \in F$ consider a differential form $Qd\lambda$ on Γ or on a branched covering $\hat{\Gamma} \to \Gamma$ depending on $\gamma \in \Gamma$ such that for any direction τ tangent to the common level of all the annihilator functions, the derivative $\bigtriangledown_\tau Q$ is a single-valued algebraic differential form on the curve Γ. (For the definition of $\bigtriangledown_\tau Q$ we have to fix a holomorphic flat connection.)
2) The Poisson bracket in the most important cases can be written in the following form (the annihilator has been already fixed):

$$\{Q(\gamma_j), \gamma_k\} = \delta_{jk},$$

$$\{\gamma_j, \gamma_k\} = \{Q(\gamma_j), Q(\gamma_k)\} = 0.$$

3) Let $\bigtriangledown_\tau Q$ be the first kind form on Γ for all τ tangent to the level of the annihilator A. The linear structure of the angle variables coincides with the natural linear structure on the Jacobian variety $J(\Gamma)$.

If the forms $\bigtriangledown_\tau Q$ have poles we shall obtain a non-abelian complex torus. This fact happens exactly for the geodesic flows on the ellipsoida in $\mathbf{R}^3$ written using the natural Hamiltonian formalism with respect to which the time is a natural parameter.

If one wants to get the linear coordinates on $J(\Gamma)$ as angle variables he has to change time variable and Hamiltonian formalism (see [18] and [19]). But people usually do not ask what is happening in the natural time: it might be important only in case one really needed the physical action variables.

The computation of the action variables requires the knowledge of the "real structure" on the complex phase space M and therefore on the Jacobian varieties $J(\Gamma)$, which leads to a subgroup N of $H_1(\Gamma, \mathbf{Z})$ with fixed basis $a_1, \ldots, a_g \in N$ such that:

$$a_i \circ a_j = 0.$$

The cycles a_j generate the "real subgroup" $H_1(T^n, \mathbf{Z}) \subset H_1(J(\Gamma), \mathbf{Z}) = H_1(\Gamma)$. The action variables are equal to the 1-dimensional integrals on these curves:

$$J_j = \frac{1}{2\pi} \oint_{a_j} Qd\lambda.$$

This formula holds for all known non-trivial integrable cases but the real structures are sometimes non-trivial.

In the integrable case of Arnold's systems (which we are considering) we have some additional problem. Our system

$$\partial_t(Y - \lambda U) = [Y - \lambda U, \Omega - \lambda V]$$

is in fact determined on the space of all $n \times n$ matrices. There is an involution

$$\sigma : Y \mapsto -Y^T, \Omega \mapsto -\Omega^T$$

commuting with our system. Therefore we are going to restrict our system to the invariant subspace of the skew-symmetric matrices, i.e. we assume that $\sigma Y = Y$ and $\sigma\Omega = \Omega$. For this subspace we shall have the algebraic curves Γ with an involution

$$\sigma : \Gamma \to \Gamma, \quad \sigma^2 = 1.$$

The divisor of poles $D = \gamma_1 + \ldots + \gamma_g$ should belong to some "Prym" subvariety of the Jacobian variety (the so-called "Prym Θ-functions" appear in this problem). But we want to select the real solutions only. This leads to the anti-holomorphic involution κ:

$$\kappa : M \to M, \ \kappa\sigma = \sigma\kappa, \ \kappa^2 = \sigma^2 = 1.$$

The same situation appeared also in the (2+1)-soliton theory associated with some very interesting 2-dimensional variants of the KdV system based on the IST method for the 2-dimensional Schrödinger operator and some analogue of the Lax-type representation (see [23]).

In the 1+1-dimensional case we have the simplest picture for the famous KdV equation (see Lecture 1). There is a standard "Lax representation" for it:

$$\begin{aligned}
&\frac{\partial L}{\partial t} = [L, A], \ \Psi_t = -\Psi_{xxx} + 6\Psi\Psi_x, \\
&L = -\partial_x^2 + \Psi(x,t), \ A = -4\partial_x^3 + \frac{3}{2}(\Psi\partial_x + \partial_x\Psi).
\end{aligned}$$

There is also a "KdV hierarchy":

$$\begin{aligned}
&\vec{t} = (t_0, t_1, t_2, \ldots), \ t_0 = x, \ t_1 = t, \\
&\Psi(x, t, t_2, t_3, \ldots) = \Psi(\vec{t}\,)
\end{aligned}$$

such that for each "time" t_n our function satisfies the equation KdV_n defined as follows:

$$\begin{aligned}
&\mathrm{KdV}_0 : \ \Psi_x = \Psi_{t_0}, \\
&\mathrm{KdV}_1 = \mathrm{KdV} : \ \Psi_{t_1} = \Psi_t = 6\Psi\Psi_x - \Psi_{xxx}, \\
&\qquad \ldots \\
&\mathrm{KdV}_n : \ \Psi_{t_n} = (\mathrm{const})\Psi_x^{(2n+1)} + \ldots
\end{aligned}$$

and each equation KdV_n admits the "Lax representation":

$$\frac{\partial L}{\partial t} = [L, A_n], \quad L = -\partial_x^2 + \Psi,$$
$$A_n = (\mathrm{const})\partial_x^{(2n+1)} + a_{1n}\partial_x^{(2n-1)} + \sum_{j\geq 2} a_{jn}\partial_x^{(2n-j)}.$$

The coefficients a_{jn} are polynomials in the variables $(\Psi, \Psi_x, \ldots, \Psi^{(2n-1)})$. All these systems commute with each other.

There is a second "zero-curvature" representation found in 1974 for the study of periodic problems —see [20]. Consider two linear systems for the 2×2 matrices:

$$\begin{cases} \dfrac{\partial \varphi(\lambda, x, t, \ldots)}{\partial t_n} = \Lambda_n \varphi \\ \dfrac{\partial \varphi}{\partial x} = Q\varphi = \Lambda_0 \varphi \end{cases}$$

$$Q = \begin{pmatrix} 0 & 1 \\ \Psi - \lambda & 1 \end{pmatrix} = \Lambda_0, \qquad \Lambda_n = \begin{pmatrix} a_n & b_n \\ c_n & -a_n \end{pmatrix},$$

$$a_1 = -\Psi', \ b_1 = 2\Psi + 4\lambda, \ c_1 = 2\Psi^2 - \Psi'' - 4\lambda^2 + 2\lambda\Psi.$$

For Λ_n we have that:

$$b_n = (\mathrm{const})(\lambda^n + \frac{\Psi}{2}\lambda^{n-1} + \ldots)$$

is a polynomial in λ of degree n whose coefficients are polynomials in $(\Psi, \Psi_x, \ldots)$, such that

$$b = (\mathrm{const}) \prod (\lambda - \gamma_j), \quad \sum \gamma_j = -\Psi/2.$$

The compatibility condition $(\varphi_x)_{t_n} = (\varphi_{t_n})_x$ leads to the matrix equation ("zero-curvature representation"):

$$\left[\frac{\partial}{\partial x} - Q, \frac{\partial}{\partial t_n} - \Lambda_n\right] = 0$$

or

$$\frac{\partial \Lambda_n}{\partial x} - \frac{\partial Q}{\partial t_n} = [\Lambda_n, Q].$$

For the stationary problem we have

$$\frac{\partial \Lambda_n}{\partial x} = [\Lambda_n, Q], \quad \frac{\partial Q}{\partial t} = 0.$$

Therefore we have the so-called "λ-representation" for the stationary system which is polynomial in λ.

The algebraic curve

$$\Gamma = \{\det(\Lambda_n(\lambda) - \mu 1) = 0\}$$

has the form

$$\mu^2 = R_{2n+1}(\lambda) = (\mathrm{const})\lambda^{2n+1} + \cdots .$$

The stationary generic real solutions are the *finite-gap potentials.* They are periodic or quasi-periodic in the generic case.

The hyperelliptic complex curve Γ determines completely the spectrum of the Schrödinger operator $L = -\partial_x^2 + \Psi$ with periodic potential Ψ acting on the space $L_2(R)$. This means exactly that the family of periodic finite-gap Schrödinger operators with given spectrum may be identified with the real part of the Jacobian variety of the algebraic curve Γ.

The general stationary system for the KdV hierarchy is obtained as a linear combination of the "higher KdV's":

$$\Lambda = \Lambda_n + c_n\Lambda_{n-1} + \ldots + c_n\Lambda_0, \qquad \frac{\partial \Lambda}{\partial x} = [\Lambda, Q].$$

The last equation describes the set of all n-gap potentials (their Bloch-Floquet eigenfunction is meromorphic on the Riemann surphace with genus $g = n$). The Bloch-Floquet eigenfunction is by definition a solution of the equation

$$L\varphi = \lambda\varphi \ , \ L = -\partial_x^2 + \Psi$$

such that

$$\varphi(x + T, \lambda) = \exp(ip(\lambda)T)\varphi(x, \lambda)$$

if

$$\Psi(x + T) = \Psi(x).$$

For the quasi-periodic potentials $\Psi(x)$ the Bloch-Floquet eigenfunction is such that the function χ

$$\chi(x, \lambda) = \frac{\partial}{\partial x}(\log \varphi)$$

is quasi-periodic with the same periods as the potential $\Psi(x)$. For real bounded smooth potentials all branching points are real and coincide exactly with the endpoints of the spectrum of the operator L acting on the space $L_2(R)$. People use the expression "finite-gap potentials" in this case, though the expression "algebro-geometric potentials" would probably be more suitable. The Θ-functional formula can be written in the following very simple form (found by A. Its and V. Matveev in 1975):

$$\Psi(x, t_1, t_2, \ldots) = \text{const} - 2\partial_x^2 \log \Theta(\vec{U}_0 x + \sum \vec{U}_j t_j + \vec{V}_0).$$

Here the Θ-function and the vectors

$$\vec{U}_j = (U_{j_1}, \ldots, U_{j_n}), \quad j \geq 0$$

are completely determined by the algebraic curve Γ. The initial phase $\vec{U}_0$ corresponds to the divisor of the poles $\gamma_1, \ldots, \gamma_n$.

The Bloch-Floquet eigenfunction of a periodic real smooth potential has the following analytical properties:
1) It is meromorphic away from ∞ on the hyperelliptic algebraic curve

$$\Gamma = \left\{\mu^2 = R_{2n+1}(\lambda) = \prod(\lambda - \lambda_j)\right\}$$

with the real branching points λ_j

$$\lambda_0 < \lambda_1 < \ldots < \lambda_{2n}$$

which are exactly the endpoints of the spectrum in $L_2(R)$.
2) It has exactly g poles, $g = n$,

$$\gamma_1, \ldots, \gamma_n \quad \in \Gamma$$

whose projections on the λ-plane $\mathbf{C}$ are inside the finite gaps:

$$\lambda_{2j-1} \leq p(\gamma_j) \leq \lambda_{2j}, \quad j = 1, 2, \ldots, n.$$

3) It has exponential asymptotics for $\lambda \to \infty$ of the form

$$\varphi \sim \exp\left(\sum t_j k^{2j+1}\right)(1 + \mathrm{O}(k^{-1})), \quad k^2 = \lambda,\ t_0 = x,\ t_1 = t.$$

Lecture 4

Poisson Structures on Loop Spaces.
Systems of Hydrodynamic Type and Differential Geometry.

We are going to consider now "local" infinite-dimensional systems. The phase space $M = \mathrm{Map}(S^1 \to Y)$ consists of all C^∞ maps of the circle into some C^∞ manifold Y with local coordinates $(u^1, \ldots, u^N)$, $N = \dim Y$, $u(x) = (u^p(x))$, $p = 1, \ldots, N$, $x \in S^1$.

The coordinate x here is periodic by definition. It is convenient to forget now about the boundary conditions for the formal investigations. All our constructions will be local in the variable x. In fact we shall work only with "local" functionals of the following type:

$$I = \int_{S^1} j(u(x), u'(x), \ldots, u^{(m)}(x), x)dx,$$

whose density depends on a finite number of derivatives of u at the same point x. Its variation

$$\delta I = \int \frac{\delta I}{\delta u^p(x)} \delta u^p(x) dx$$

determines the "Euler-Lagrange" expression of the "variational derivative"

$$\delta I / \delta u^p(x) \; ,$$

which has the form

$$\frac{\delta I}{\delta u^p(x)} = \frac{\partial j}{\partial u^p} - \left(\frac{\partial j}{\partial u^p_x} \right)_x + \ldots + (-1)^m \left(\frac{\partial j}{\partial u^{p(m)}_{x \ldots x}} \right)^{(m)}_{x \ldots x} .$$

The pair (p, x) is the right ∞-dimensional analogue of the "index" p: instead of summation $\sum$ we shall have $\sum_p \int (\ldots) dx$ in the formulas.

Definition. A *local Poisson Bracket* (PB) of *order* κ on the space $M = \mathrm{Map}(S^1 \to Y)$ is given by the formal expression

$$\{u^p(x), u^q(y)\} = \omega^{pq}(x, y) = \sum_{k=0}^{m} b^{pq}_k(u, u', \ldots, u^{(l_k)}, x) \delta^{(k)}(x - y).$$

Here κ is given by $\max_{0\le k\le m}(k+l_k)$ and all the derivatives $u', u^{(2)}, \ldots$ should be computed at the same point x of S^1. For any pair of functionals (I_1, I_2) their PB is equal to the expression

$$\{I_1, I_2\} = \int_x \int_y \frac{\delta I_1}{\delta u^p(x)} \omega^{pq}(x,y) \frac{\delta I^q}{\delta u^q(y)} dx dy.$$

Using locality we integrate in the variable y and obtain that:

$$\{I_1, I_2\} = \int \frac{\delta I_1}{\delta u^p(x)} A^{pq} \frac{\delta I_2}{\delta u^q(x)} dx$$

where

$$A^{pq} = \sum_{k=0}^{m} b_k^{pq}(u(x), \ldots, u^{(l_k)}(x), x)\partial_x^k.$$

These expressions determine correctly a local PB if and only if the Jacobi identity holds. It is difficult to find an acceptable general criterion for the Jacobi identity. Many people worked on this (for example I. Gelfand and L. Dickey in 1978, I. Gelfand and I. Dorfman in 1979 and 1981, A. Astashov and A. Vinogradov in 1981 —see the references quoted in [10]).

In some cases the criterion is trivial. For example the Jacobi identity holds whenever the "Poisson tensor" $\omega^{pq}(x,y)$ does not depend on the "point" u of the space M, which means that

$$b_k^{pq} = b_k^{pq}(x) \qquad \partial b_k^{pq}/\partial u^p = 0.$$

The important brackets are:

1. 0-order brackets ("ultralocal" PB), i.e. brackets enjoying the property:

$$A^{pq} = \omega^{pq}(u)\ ,\ u \in Y.$$

Brackets of this type are generated by the finite-dimensional PB's on the manifold Y. In the classical case we have:

$$\omega^{pq}(u) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad Y = \mathbf{R}^{2k}.$$

2. First-order homogeneous brackets ("Hydrodynamic Type" PB, or HTPB), enjoying:

$$A^{pq} = g^{pq}(u)\partial_x + b_k^{pq}(u)u_x^k.$$

The simplest case is $Y = \mathbf{R}$. We have here the so-called "Gardner-Zakharov-Faddeev" (or, briefly, GZF) bracket:

$$\{u(x), u(y)\} = \delta'(x - y) \ , \quad A = \partial_x.$$

The functional $I_{-1} = \int u dx$ belongs to the annihilator of the GZF bracket. The generalized GZF bracket is by definition the following one:

$$\begin{aligned} &\{u^p(x), u^q(y)\} = g_0^{pq}\delta'(x - y), \\ &g_0^{qp} = g_0^{pq} = (\text{const}), \quad A^{pq} = g_0^{pq}\partial_x. \end{aligned}$$

All the Hamiltonian systems generated by the Poisson brackets in exam have the form

$$\frac{\partial u^p}{\partial t} = A^{pq}\frac{\delta H}{\delta u^q(x)}$$

for any Hamiltonian H. For example, all the systems of the KdV hierarchy have the following "Gardner form":

$$u_t = \partial_x\left(\frac{\delta H_n}{\delta u(x)}\right), \qquad H_n = I_n.$$

For $n = 0, 1$ we have:

$$\begin{aligned} n = 0: &\quad u_t = \partial_x\left(\frac{\delta I_0}{\delta u(x)}\right), \qquad I_0 = (\text{const})\int u^2 dx, \\ n = 1: &\quad u_t = \partial_x\left(\frac{\delta I_1}{\delta u(x)}\right), \qquad I_1 = (\text{const})\int\left(\frac{u_x^2}{2} + u^3\right)dx. \end{aligned}$$

Let us give the general description of the functionals H_n: consider the formal solution of the Riccati equation:

$$\chi(x, k) = k + \sum_{m\geq 1}\chi_m/(2k)^m,$$

$$\chi' + \chi^2 = u - \lambda, \quad k^2 = \lambda.$$

For $m = 2k + 1$ we have

$$\int\chi_1 dx = (\text{const})\int u dx = I_{-1},$$

$$\int \chi_3 dx = (\text{const}) \int u^2 dx = I_0,$$
$$\int \chi_5 dx = (\text{const}) \int (u_x^2/2 + u^3) dx = I_1$$
$$\dots$$
$$\int \chi_{2k+3} dx = (\text{const}) I_k.$$

3. There is a very interesting "Lenard-Magri bracket" defined by:

$$\{u(x), u(y)\} = c\delta^{(3)}(x-y) + 2u(x)\delta' + u'\delta,$$
$$A = c\partial_x^3 + u\partial_x + \partial_x u.$$

This PB has no local annihilator. It gives us also the Hamiltonian form for the KdV hierarchy but with a "shift" of the Hamiltonians ($c = 1$):

$$\begin{aligned} &\text{KdV}_0: \quad u_t = u_x = A(\delta I_{-1}/\delta u(x)),\\ &\text{KdV}_1: \quad u_t = 6uu_x - u_{xxx} = A(\delta I_0/\delta u(x)),\\ &\dots\\ &\text{KdV}_n: \quad u_t = (\text{const})u^{(2n+1)} + \dots = A(\delta I_{n-1}/\delta u(x)). \end{aligned}$$

Now, going back to the beginning of the lecture, we generalize the situation by considering $M = \text{Map}(T^n \to Y)$, the space of smooth functions $u(x)$ from the n-torus to an N-dimensional manifold Y. We have $x = (x^1, \dots, x^n)$ and locally $u = (u^1, \dots, u^N)$.

Definition. We call Hydrodinamic (or Riemann) Type (or, briefly, HT) equation an equation having the following form in local coordinates:

$$u_t^p = V_q^{p,\alpha}(u(x))u_\alpha^q.$$

Here the coefficients $V_q^{p,\alpha}$ define the "velocity" tensor fields $\hat{V}^\alpha$, $\alpha = 1, \dots, n$, $\hat{V} = V_q^p(u)$, on the manifold Y.

The main problems which we will discuss are the following:

1. How can we construct the local Hamiltonian formalism for HT Systems?
2. How can we find HT systems which are completely integrable? Are the natural HT systems (which already appeared in the description of the "slow modulation of parameters" in the so-called Whitham method —i.e. in the "non-linear analogue of WKB-asymptotics" for KdV, SG, NS) completely integrable?

Problem 1 was solved in 1983 in a joint work of B. Dubrovin and the author; problem 2 was solved by S. Tsarev in 1985 on the base of Hamiltonian formalism —see [10]. We will explain here some ideas used in the solution of these problems. The applications to soliton theory will be discussed in Lecture 5.

Definition. We call Riemann Invariants (or, briefly, RI) coordinates $(u^1, \dots, u^N)$ on the target manifold Y with the property that all the velocity tensors $\hat{V}^\alpha$ are diagonal:

$$\hat{V}^\alpha = V_q^{p,\alpha}(u) = V^{p,\alpha}(u)\delta_q^p \qquad \alpha = 1, \dots, n.$$

According to a classical result (of Riemann) we have that for $n = 1$ and $N = 2$ RI always exist. For $N \geq 3$ they do not exist in general (even for $n = 1$). In case $N = 2$ and $n = 1$ the HT equation is linear in the inverse functions

$$x(u^1, u^2)\ ,\ t(u^1, u^2).$$

The map $(x, t) \mapsto (u^1, u^2)$ is the "hodograf transformation"; it is quite natural to ask whether there exists any analogue of the "hodograf" for $n = 1$ and $N \geq 3$. Such an analogue has been found by Tsarev for Hamiltonian systems.

Definition. The functional $H = \int h(u)dx$ is called a functional of *Hydrodynamic Type* (or, briefly, HT) if its density does not depend on the derivatives u_x, u_{xx}, etc.

Definition. We call *Hydrodynamic Type Poisson Bracket* (or, briefly, HTPB) a bracket having the form

$$\{u^p(x), u^q(y)\} = g^{pq,\alpha}(u(x))\delta_\alpha(x-y) + b_k^{pq,\alpha}(u(x))u_\alpha^k\delta(x-y),$$

$$A^{pq} = g^{pq,\alpha}\partial_\alpha^k + b_k^{pq,\alpha}u_\alpha^k.$$

Here $\partial_\alpha = \partial/\partial x^\alpha$, $u_\alpha^k = \partial u^k/\partial x^\alpha$, $\delta_\alpha = \partial\delta/\partial x^\alpha$. (We will use the symbol ∂_α with the meaning of $\partial/\partial x^\alpha$ only with Greek subscripts α, while in the sequel we will use ∂_k, with Latin subscripts k, to denote the local derivative $\partial/\partial u^k$ on Y.)

The HT Hamiltonians generate the HT equations through the HTPB's:

$$H = \int h(u)d^n x, \quad u_t^p = V_q^{p,\alpha}(u)u_\alpha^q,$$
$$V_q^{p,\alpha}(u) = g^{ps}\,\nabla_s^{(\alpha)}\,\nabla_q^{(\alpha)}h(u),$$

(but $\bigtriangledown_q^{(\alpha)} h(u) = \partial_q h(u)$ because $h(u)$ is a scalar). For the covector fields we define:

$$\bigtriangledown_p^{(\alpha)} X_q = \partial_p X_q - \Gamma_{pq}^{s,\alpha} X_s, \quad b_k^{pq,\alpha} = \Gamma_{sk}^{q,\alpha} g^{ps,\alpha}.$$

Theorem 1 ([9]). Let $\det(g^{pq,\alpha}) \neq 0$ for $\alpha = 1, \ldots, n$. The collection of coefficients of the Hydrodynamic Type PB transforms in the following way: the coefficients $g^{pq,\alpha}$ as components of a tensor of rank 2; the coefficients $\Gamma_{sk}^{q,\alpha}$ as Kristoffel symbols. The tensors $(g^{pq,\alpha})$ are symmetric. The covariant derivative $\bigtriangledown_k^{(\alpha)} g^{pq,\alpha}$ is zero. Torsion and Curvature of the connection $(\Gamma_k^{pq,\alpha})$ are zero.

Corollary. If $n = 1$ there exists a system of "flat" coordinates $(u^1, \ldots, u^N)$ such that

$$g^{pq} = g_0^{pq} = \text{const}, \qquad b_k^{pq} = 0.$$

The signature of the metric $g^{pq}(u)$ is a complete local invariant of the HTPB if $\det(g^{pq})(u) \neq 0$. Therefore the covariant derivatives commute with each other in any coordinates:

$$\bigtriangledown_p \bigtriangledown_q = \bigtriangledown_q \bigtriangledown_p \,.$$

For the velocity tensor of a Hamiltonian HT system we have:

$$V_{pq} = g_{ps} V_q^s = V_{qp} = \bigtriangledown_p \bigtriangledown_q h(u) = \bigtriangledown_q \bigtriangledown_p h(u) \qquad (g_{ps} g^{sq} = \delta_p^q)$$

(it is a symmetric tensor).

Theorem 2. Let $n > 1$ and $\det(g^{pq,1}) \neq 0$. In the coordinates $(u^1, \ldots, u^N)$ which are flat for the metric $(g^{pq,1})$ we have:

$$g^{pq,\alpha} = C_k^{pq,\alpha} u^k + \text{const}, \quad C_k^{pq,\alpha} = \text{const},$$

$$b_k^{pq,\alpha} = \text{const}, \ \alpha \geq 2, \quad C_k^{pq,\alpha} = b_k^{pq,\alpha} + b_k^{qp,\alpha}, \quad (b_k^{pq,1} = 0).$$

(This result was proved by Dubrovin and the author in 1984 for $N \geq 3$ and by Mokhov (later) for $N = 2$; some mistakes of the original proof for $N \geq 3$ were fixed by Mokhov in 1988 —see [10] and the references quoted therein.)

Remark. The case of a HTPB when the tensor (g^{pq}) is singular and has constant rank has also been investigated. The kernels of the metric (g^{pq}) generate an integrable foliation on the manifold Y with interesting geometric structures which have never been completely classified. A very interesting problem is also to investigate the generic singularities of a HTPB when the rank of (g^{pq}) is not a constant.

Examples.
1. Flat coordinates $g^{pq} = g_0^{pq}$, $\det(g^{pq}) \neq 0$. Our bracket is exactly the generalized GZF bracket. The standard GZF bracket δ' has HT but the KdV Hamiltonian does not have HT. The second LM bracket does not have HT, but the LM Hamiltonian for KdV has HT.
2. For the Lie algebra $L^{(n)}$ of the vector fields in T^n (or $\mathbf{R}^n$) we have

$$[V, W]^p = (\partial_k V^p) W^k - (\partial_k W^p) V^k.$$

Here $N = n$. The corresponding Lie-Poisson bracket has the following form:

$$\begin{aligned} \{p_\alpha(x), p_\beta(y)\} = &\quad p_\alpha(x)\delta_\beta(x-y) - p_\beta(y)\delta_\alpha(y-x) \\ = &\quad p_\alpha(x)\delta_\beta(x-y) + p_\beta(y)\delta_\alpha(x-y). \end{aligned}$$

The variables p_α describe the adjoint space L_n^*. They are such that the expression:

$$\int p_\alpha(x) V^\alpha(x) d^n x$$

is invariant under changes of coordinates $x = x(x')$. Therefore the objects (p_α) transform like the components of the density of a covector ("density of momentum" in HT theory).

In the hydrodynamic of a compressible liquid (without viscosity) we have two more fields:

$$\begin{aligned} &\rho \text{ (density of mass)}, \qquad M = \int \rho(x) d^n x, \\ &s \text{ (density of entropy)}, \qquad S = \int s(x) d^n x, \end{aligned}$$

with PB:

$$\begin{aligned} &\{p_\alpha(x), \rho(y)\} = \rho(x)\partial_\alpha\delta(x-y), \\ &\{p_\alpha(x), s(y)\} = s(x)\partial_\alpha\delta(x-y), \\ &\{\rho, \rho\} = \{s, s\} = \{\rho, s\} = 0. \end{aligned}$$

The Hamiltonian has the standard form:

$$H = \text{"energy"} = \int \left[\frac{p^2}{2\rho} + \varepsilon_0(\rho, s)\right] d^n x, \qquad p^2 = \sum_{\alpha=1}^{n} p_\alpha^2.$$

By definition we have (as "velocities"):

$$V_\alpha = p_\alpha/\rho.$$

The functionals $M = \int \rho dx$, $S = \int s dx$, $\vec{P} = \int \vec{p} dx$ belong to annihilator .

Other interesting examples can be found in [21], Section 2, e.g. the superfluid liquid.

For $n = 1$ we have:

$$\{p(x), p(y)\} = [p(x) + p(y)]\delta'(x - y) = 2p(x)\delta'(x - y) + p'(x)\delta(x - y).$$

After the change of the function $p = u^2$ we are coming to the standard form of the GZF bracket:

$$\{u(x), u(y)\} = (\text{const})\delta'(x - y).$$

The "Lenard-Magri bracket":

$$\{p(x), p(y)\} = c\delta^{(3)}(x - y) + 2p(x)\delta'(x - y) + p'(x)\delta(x - y)$$

is therefore a central extension of the Lie-Poisson Bracket associated with the Lie algebra of vector fields on the circle S^1 (we are considering $n = 1$). It corresponds to the central extension of this algebra by the third order "Gelfand-Fucks cocycle" (see below). Therefore the LM bracket corresponds to the "Virasoro algebra" (Virasoro-Gelfand-Fucks-Bott: in fact the Virasoro algebra is a subalgebra of this one generated by the trigonometric polynomials):

$$[V, W] = V'W - W'V + \chi(V, W)t, \qquad [t, V] = 0,$$

$$\chi(V, W) = \oint_{S^1} V^{(3)}(x)W(x)dx, \qquad V = V(x)\partial_x, \quad W = W(x)\partial_x.$$

The general first-order translationally invariant Lie algebra structure on the space of C$^\infty$-vector functions on the circle S^1 with values in the space $\mathbf{R}^N = Y$ may be described in the following way: consider a basis $e^1, \ldots, e^N$ of $\mathbf{R}^N$ and a bilinear multiplication determined by structural constants

$$e^p \circ e^q = b_k^{pq} e^k.$$

We have an algebra B (which is $\mathbf{R}^N$ as a set).

Lemma. The operation [,] on the space $M = \mathrm{Map}(S^1 \to B) = L_B$ defined as

$$[P,Q](x) = P' \circ Q - Q' \circ P$$

for $P = f_p(x)e^p$ and $Q = g_p(x)e^p \in B$, satisfies the Jacobi identity if and only if for any three elements a, b, c of B we have

$$a(bc) = b(ac)\ ,\ [L_b, L_a] = 0, \qquad R_{[bc-cb]} = [R_b, R_c].$$

By definition, the left and right multiplication operators are:

$$L_a(b) = R_b(a) = a \circ b.$$

Remarks.
1. The (equivalent) facts of the previous Lemma are true in particular if B is a commutative associative algebra. (Remark that if B is commutative then it is also associative, but not conversely. Moreover if B has a unit then it is automatically commutative and associative.)
2. The algebra B which corresponds to the HTPB of a compressible liquid ($n = 1$) is 3-dimensional (generated by the fields p, ρ, s), non-commutative but associative. There are 3 basic elements:

e (corresponding to p), $\quad f$ (corresponding to ρ), $\quad g$ (corresponding to s)

and e is a left unit; we have

$$e \circ a = a,\ a \in B, \quad f \circ B = g \circ B = 0.$$

3. E. Zelmanov proved in 1987 that the finite-dimensional algebras B are never simple if $N = \mathrm{dim} B > 1$ for characteristic 0 (see [10]). They may be simple for $N = \infty$ or for fields of positive characteristic. There are very interesting examples constructed recently by Osborn which we shall not discuss (see [24]). There is also an “extension theory” for the algebras B (see [3]).

We get a particularly interesting class of examples from classical commutative associative “Frobenius algebras” B with unit . In our language the Frobenius property exactly means that:

$$\det(g^{pq}(u_0)) \neq 0$$

for the generic point

$$u_0 = (u_0^1, \ldots, u_0^N), \quad g^{pq} = c_k^{pq} u^k, \quad c_k^{pq} = b_k^{pq} + b_k^{qp} = 2b_k^{pq}.$$

The pseudo-Riemannian metric $(g^{pq}(u))$ has zero curvature by Theorem 1 above. Therefore there exists a system of flat coordinates. How to find it? In the Frobenius case it is easy. Fix some point (u_0) such that $\det(g^{pq}(u_0)) \neq 0$; consider the inverse matrix

$$g^0_{pq} g_0^{qs} = \delta_p^s$$

and the corresponding Kristoffel symbols at u_0:

$$F^p_{qk} = \Gamma^p_{qk}(u_0) = g^0_{qs} b_k^{ps} = F^p_{kq}$$

(they are symmetric because the torsion is equal to zero by Theorem 1 again). The quadratic trasformation

$$u^p = F^p_{qk} v^q v^k$$

introduces a flat coordinate system $(v^1, \ldots, v^N)$ (cfr. [3]). For $N = 1$ we have $u = v^2$ (as above). Such a transformation is unknown for general B-algebras.

A very interesting problem is to classify the central extensions of the Lie algebras L_B:

$$O \to R \to \tilde{L}_B \to L_B \to 0.$$

Especially interesting are local translationally invariant extensions of order $\tau = 0, 1, 2, 3$ written in the form

$$\chi(P, Q) = \oint_{S^1} f_p^{(\tau)} G_q \gamma^{pq} dx, \quad \gamma^{pq} = \text{const} = (-1)^{\tau+1} \gamma^{qp}.$$

For $\tau = 3$ we have the “Gelfand-Fucks” type cocycles. In the Frobenius case the group of “central extesions”

$$H^2(L_B, \mathbf{R})$$

contains many non-trivial non-degenerate cocycles of order $\tau = 3$:

$$\gamma^{pq} = G^{pq}(u_0).$$

(No proof has been published, but Balinsky claimed in his Ph.D. thesis that the group $H^2(L_B, \mathbf{R})$ contains in this case these cohomology classes only.)

For $\tau = 1$ the cocycles γ^{pq} are such that the "perturbed" metric tensor

$$\tilde{g}^{pq}(u) = g^{pq}(u) + \varepsilon\gamma^{pq} = c_k^{pq} u^k + \varepsilon\gamma^{pq}$$

determines a new HTPB with the same (b_k^{pq}).

For the PB of a compressible liquid we have $(n = 1, N = 3)$ and

$$\det(g^{pq}(u)) \equiv 0$$

but there is a cocycle γ^{pq} of order $\tau = 1$ such that

$$\det(g^{pq}(u) + \varepsilon\gamma^{pq}) \neq 0$$

at the generic point u_0.

Non-degenerate cocycles of order $\tau = 2$ might be very interesting.

We have different types of important coordinate structures associated with HTPB's:

- flat coordinates;
- Riemann Invariants;
- Lie algebra structures (linear coordinates);
- Physical Coordinates (to be defined).

We are going to introduce now the notion of "Physical Coordinates" or "Liouville" coordinates, which are important in soliton theory.

Definition. Let a HTPB be given by the tensor $(g^{pq}(u))$ and the quantities $(b_k^{pq}(u))$, written in the following *Liouville form*: there exists a tensor field $\gamma^{pq}(u)$ such that

$$g^{pq}(u) = \gamma^{pq}(u) + \gamma^{qp}(u), \qquad b_k^{pq}(u) = \frac{\partial\gamma^{pq}(u)}{\partial u^k};$$

coordinates (u) enjoying this property will be called *Liouville* or *Physical* ones.

Remarks.

1. Obviously for any Lie algebra the natural linear coordinates are Liouville ones because we may define:

$$\gamma^{pq} = b_k^{pq} u^k.$$

2. For the Liouville coordinates (u) any affine transformation

$$v^p = A_q^p u^q + v_0^p$$

defines new Liouville coordinates. Therefore the Liouville coordinates determine an "affine structure" on the target manifold Y.

Definition. The Liouville structure is called *strong* if the restriction of the tensor $\gamma^{pq}(u)$ to some subset of coordinates $(u^{p_1}, \ldots, u^{p_k})$ determines a well-defined Liouville structure on the subspace $\tilde{Y} \subset Y$ corresponding to such a set of coordinates.

This property should hold for any coordinates (v) obtained from (u) by affine transformation.

Remark. For $N = 3$ it is easy to check that the HTPB of a compressible liquid (see above) is written in the strongly Liouville form; the Physical Coordinates (p, ρ, s) in this case satisfy to the additional requirement.

The present author and B. Dubrovin made a mistake in the proof of Theorem 1 of Section 6 in [10]: the HTPB in the Physical coordinates is only Liouville, not strong. The paper of the present author in collaboration with the student Maltsev correcting this mistake will appear in Uspechi Math. Nauk (1993) issue 1 (January-February).

Probably very few linear HTPB's (i.e. Lie algebras) have tensors which are strongly Liouville. It is the author's opinion that it should be possible to classify all of them.

The role of Physical Coordinates will be clarified in Lecture 5.

We are coming now to the "integration theory" for Hamiltonian HT systems developed by S. Tsarev in his Ph.D. thesis in 1985. It was already known for many years that some HT systems which appear in the KdV theory through the so-called "Whitham method" (i.e. the "non-linear WKB method") admit Riemann Invariants (G. Whitham in 1973 for $n = 1$ and $N = 3$; H. Flashka, G. Forest and D. McLaughlin in 1980 for $N = 2m + 1$ and $n = 1$).

The differential-geometric Hamiltonian formalism (above) for these HT systems was developed by B. Dubrovin and the author in 1983; later the author formulated the following conjecture: *in the class of Hamiltonian HT systems existence of Riemann Invariants automatically implies complete integrability.* This conjecture has been proved by S. Tsarev in his remarkable paper [25].

We are going to explain now some ideas of Tsarev's work. Consider any HT system which is written in its Riemanian Invariants $(r_1, \ldots, r_N)$ as:

$$\frac{\partial r_k}{\partial t} = V^k(r(x))\frac{\partial r_k}{\partial x} \qquad \text{(no summation!)}.$$

Suppose we are dealing with the Hamiltonian system corresponding to the local Hamiltonian formalism developed above, and that in the generic point we have:

$$V^k(r) \neq V^l(r),\ k \neq l,\ \det(g^{kl}(r)) \neq 0.$$

It is easy to check the corresponding property of the metric tensor:

$$g^{kl}(r) = g^k(r)\delta^{kl}$$

(i.e. the tensor is diagonal). Therefore the Riemann Invariants are orthogonal coordinates for the pseudo-Riemannian metric (g^{kl}) on the target manifold Y, which generates the Hamiltonian formalism. The main result is:
Lemma. For any HT system for which the coordinates $(r_1, \ldots, r_N)$ are Riemann Invariants, i.e.

$$\frac{\partial r_k}{\partial t} = W^k(r(x))\frac{\partial r_k}{\partial x} \qquad \text{(no summation!)}$$

the following indentity is true:

$$\frac{\partial_i W_k}{W_i - W_k} = \Gamma^k_{ki}, \quad i \neq k.$$

All these systems commute (in fact all of them are HT Hamiltonian systems generated by the same HTPB).

The last equation might be considered as an equation for finding all the HT flows commuting with each other, provided the symbols Γ^k_{ki} are known. In particular we have for any diagonal metric:

$$\Gamma^k_{ki} = \partial_i \log |g_k(r)|^{1/2}, \quad g_k = 1/g^k(r).$$

As a consequence we have

$$\partial_i \left(\frac{\partial_j W_k}{W_j - W_k} \right) = \partial_j \left(\frac{\partial_i W_k}{W_i - W_k} \right), \quad i \neq j \neq k.$$

Definition. Any diagonal HT system (written in its RI) is called *semi-Hamiltonian* if the last equation is true for the velocities $W_k(r)$. (We actually have that all such systems being not Hamiltonian in the HTPB above are in fact Hamiltonian systems corresponding to some non-local Hamiltonian formalism generalizing the HTPB above, for which the metric is diagonal but some components of the curvature tensor may be non-zero. This fact was recently observed by Ferapontov, in 1990.)

Theorem 3. A. The solutions of the system

$$\frac{\partial_i W_k}{W_i - W_k} = \Gamma^k_{ki}$$

give the commuting family of HT systems which is enough for Liouville integrability (of course this is a local differential-geometric statement). The statement is globally true if all the initial functions $r_k(x, 0)$ are convex (these flows generate the tangent space to the common level of the commuting integrals).

B. For any commuting pair of such systems

$$\frac{\partial r_k}{\partial t} = V_k(r) \frac{\partial r_k}{\partial x}, \qquad \frac{\partial r_k}{\partial \tau} = W_k(r) \frac{\partial r_k}{\partial \tau}$$

the solution $(r_k(x,t))$ of the equation

$$W_k(r) = V_k(r)t + x, \quad k = 1, \ldots, N, \ (r_k = r_k(x,t))$$

satisfies the first system.

(For the proof see [10].)

There are some very special classes of systems (called "weakly non-linear") found by S. Tsarev and M. Pavlov for which the construction of the above theorem may be realized directly. But our systems do not belong to these classes.

Using the Hamiltonian formalism we can construct families of solutions only if we know many HT integrals; we shall use this technique, but the natural integrals which we actually know do not give a complete family. The differential-geometric procedure of Tsarev is therefore not sufficiently effective. In the study of concrete important really non-trivial integrable HT systems we shall use the additional algebraic-geometric information about this HT system (see Lecture 5). For KdV all this problems have been quite recently solved by several people, among which Krichever, Potemin, Tian, Dubrovin and Krylov. We will present here only special results about very special important solutions (see Lecture 5), based on the results of Krichever and Potemin of 1989.

The latest results of other authors may be found in the transactions of the Conference dedicated to this subject held in Lyon in July 1991, which are going to be published soon.

Lecture 5

Non-linear WKB (Hydrodynamics of Weakly Deformed Solution Lattices).

Consider an evolutional system of non-linear differential equations

$$\Psi_t = K(\Psi, \Psi_x, \Psi_{xx}, \dots)$$

where K is a polynomial with constant coefficients in the variables $(\Psi, \Psi_x, \dots)$. Here Ψ may be a vector function. The simplest "hydrodynamic" approximation we get is for the class of very slowly varying functions, i.e. functions such that each derivative is much smaller then the previous one:

$$\Psi^{(n)} << \Psi^{(n-1)} << \Psi^{(n-2)} << \dots << \Psi.$$

It is convenient to introduce an algebraic "filtration" with the properties

$$f(\Psi) \geq 0, \quad f(\Psi^{(k)}) \geq k, \quad f(AB) \geq f(A) + f(B).$$

For the equation we get the decomposition

$$\Psi_t = K_0(\Psi) + K_1(\Psi)\Psi_x + \dots$$

where the remaining terms have filtration at least 2. Suppose all $\Psi = \text{const}$ are exact solutions. In this case $K_0(\Psi) = 0$.

We are coming to the HT system as the first hydrodynamic type approximation to the original system for the class of "very smooth" functions $\Psi(X, T)$, $X = \varepsilon x$, $T = \varepsilon t$ (slow modulations of the constant solutions):

$$\Psi_T = K_1(\Psi)\Psi_X.$$

This remark is trivial (but very useful). A much more profound "non-linear WKB method" has been developed by Whitham in 1965.

Suppose that we already know a family of exact periodic (or quasi-periodic) solutions

$$\Psi(x, t) = F(\vec{U}x + \vec{V}t + \vec{U}_0, u^1, \dots, u^N)$$

depending on the N parameters $u^1, \dots, u^N$. Here the vectors U and V (not U_0) should be already known as given functions of the parameters:

$$U_k(u^1, \dots, u^N), \quad V_k(u^1, \dots, u^N), \qquad k = 1, \dots, m.$$

The initial phase $\vec{U}_0$ may be an arbitrary m-vector. The function F should be periodic with period 2π in each of the first m variables:

$$F = F(\eta_1, \dots, \eta_m, u^1, \dots u^N) \qquad \eta_j(\eta_1, \dots, \eta_m) \in T^m.$$

For any (u = const) we get an exact solution of the original system ("m-phase solution"). For $m = 0$ we get constants (provided Ψ = const satisfies the original equation). The most popular case is $m = 1$ which is sufficiently non-trivial. Many systems have families of periodic solutions as "traveling waves":

$$x = U\eta + U_0, \quad \Psi(x - ct), \quad \Psi(\eta + T) = \Psi(\eta).$$

For integrable systems like KdV, SG and NS there exist very large families of m-phase quasi-periodic solutions ("finite-gap" or "algebro-geometric" solutions).

M. Ablowitz and D. Benney discussed in 1970 the "m-phase" analogue of G. Whitham's method (who developed it for $m = 1$ only) but there was no concrete base for serious development because the family of m-gap solutions was found for the first time in 1974.

Serious analogues of Whitham's theory for the cases $m > 1$ started in the late seventies only; they are due to H. Flashka, G. Forest, D. McLaughlin, Dobrokhotov and V. Maslov.

Suppose now that the parameters $(u^1, \dots, u^N)$ are not constants but only "slow functions":

$$u^p = u^p(X, T), \quad X = \varepsilon x, \quad T = \varepsilon t.$$

Consider a vector-function

$$(S_1, \dots, S_m) = S(X, T)$$

such that

$$\partial S_j / \partial X = U_j(u), \quad \partial S_j / \partial T = V_j(u).$$

Let us consider the following question: *is it possible to find the smooth functions* (u^p) *in such a way that the expression:*

$$\Psi = F(S(X,T), u^1(X,T), \dots, u^N(X,T))$$

satisfies the original evolutional equation asymptotically? This fact means by definition that

$$\Psi_t = K(\Psi, \Psi_x, \dots) + \mathrm{o}(1) \quad \text{for} \quad \varepsilon \to 0.$$

This question has been investigated well enough for the case $m = 1$ only, but the formal procedure is valid for $m > 1$ as well.

For non-degenerate Lagrangian systems the question was first faced by by G. Whitham in 1965, J. Luke in 1966, V. Maslov in 1969, Hayes in 1973 and others (see [8] and the references quoted therein). The main result is that there exists a collection of "velocity coefficients" $V^p_q(u)$ such that the following HT system gives the necessary condition for the expression $\Psi = F(S, u(X,T))$ to be an asymptotic solution of the original equation $\Psi_t = K$:

$$u^p_T = V^p_q(u(X,T))u^q_T.$$

The velocity coefficients were constructed for KdV also by Whitham for $m = 1$ and by Forest, Flashka and McLaughlin for $m > 1$. These systems admit Riemann Invariants for KdV, NS, SG.

Are these HT systems Hamiltonian? We can consider this problem in a more general form. Assume the original system is Hamiltonian, corresponding to some PB which is local and translationally invariant:

$$\{\Psi^l(x), \Psi^i(y)\}_0 = \sum_{k=0}^{m} B^{li}_k(\Psi, \Psi_x, \dots)\delta^{(k)}(x-y),$$

$$I_1 = H = \int i_1(\Psi, \Psi_x, \dots)dx.$$

Suppose that there exists a collection of N local integrals $H = I_1, \dots, I_n$:

$$I_k = \int J_k(\Psi, \Psi_x, \dots)dx$$

and a family of exact quasi-periodic solutions (on the tori $T^m(u)$)

$$\Psi = F(Ux + Vt + U_0; u^1, \dots, u^N)$$

such that:

1. the integrals are in involution, i.e. $\{I_k, I_l\}_0 = 0$;
2. the family of exact solutions $\Psi = F$ satisfies all the conditions above. This family is invariant under the flows generated by the integrals I_k. The restriction of these integrals to the family transforms it in the family of finite-dimensional completely integrable systems. This family should be non-degenerate in the sense that the components of the vector U generate the ring of functions in the variables u with trascendency degree equal to m. The number N should be equal to $2m + a$ where a is the number of integrals I_k who belong to the local annihilator of the original field-theoretical PB;
3. the averaged densities coincide exactly with the parameters u above:

$$J_k(\Psi, \Psi_x, \ldots) = u^k$$

(any function of the variables $(\Psi, \Psi_x, \ldots)$ might be averaged on the tori, as $\Psi, \Psi_x, \ldots$ vary on the tori).

Consider now the PB's of the densities:

$$\begin{aligned}&\{j_k(\Psi(x), \ldots), j_l(\Psi(y), \ldots)\}\\ &\qquad = A_0^{kl}(\Psi(x), \ldots)\delta(x-y) + A_0^{kl}(\varphi(x), \ldots)\delta'(x-y) + \cdots .\end{aligned}$$

As a consequence of the involutivity condition

$$\left\{\int j_k dx, \int j_l dy\right\} = 0$$

we have

$$A_0^{kl}(\varphi(x), \ldots) = \frac{\partial}{\partial x} Q^{kl}(\varphi(x), \varphi'(x), \ldots)$$

for any smooth function $\varphi(x) \in \mathrm{C}^\infty(S^1)$.

Let us define now

$$\gamma^{kl}(u) = \overline{Q}^{kl} + (\mathrm{const}), \qquad g^{kl}(u) = \overline{A}_1^{kl} = \gamma^{kl} + \gamma^{lk}.$$

Theorem 4. A. The coordinates $u^1, \ldots, u^N$ defined above are the Physical ones, which means that our formulas determine a 0-curvature metric in the Liouville form.

B. The functional
$$H = \int u^1 dx \ , \ u^1 = \overline{j}_1 = \overline{h}$$
determines exactly the Whitham equations ("the Hydrodynamic of Soliton Lattices" or "the system for slow modulations of the parameters") whenever they were constructed before using the HTPB defined by the metric (including the averaged KdV for all $m \geq 1$).

Remark. For non-degenerate Lagrangian systems and $N = 2m$ it was proved by Whitham and Hayes that the averaged HT system is also Lagrangian. It leads to the new "Clebsh-type" coordinates
$$(U_1, \dots, U_m, J_1, \dots, J_m) = (W^1, \dots, W^{2m})$$
with respect to which the HT system has the form:
$$\frac{\partial}{\partial_T} J_k = \partial_X \left(\frac{\delta H}{\delta U_k(X)} \right), \qquad \partial_T U_k = \partial_X \left(\frac{\delta H}{\delta J_k(X)} \right).$$
In this case we are coming to flat coordinates. The HTPB has the form
$$\{W^p(x), W^q(y)\} = g_0^{pq}(W(x))\delta'(x-y), \qquad g_0^{pq} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
The signature of this metric is (m,m). Our flat coordinates (W) are such that the second group of variables
$$J_k = W^{m+k}, \quad k = 1, \dots, m$$
exactly coincides with the group of "action variables" (defined above) of the finite-dimensional Hamiltonian subsystem of the original system whose solutions are quasi-periodic and generate the "non-linear WKB method" discussed above.

For KdV we have $N = 2m+1$, $m \geq 1$ and and the corresponding metric has signature $(m, m+1)$. More exactly, the HT systems corresponding to the KdV equation might be described using two different Hamiltonian structures (GZF and LM) —see Lecture 4. After averaging we get two different HTPB's corresponding to these structures; the restriction of the metric to the variables (U, J) is still the same but they do not give complete systems of coordinates; the action variables are different for GZF and LM. For GZF the additional flat coordinate of the corresponding HTPB is the averaged density of the annihilator

$$W^{2m+1} = \overline{u}.$$

An analogous result holds for LM, but the annihilator is non-local. Strictly speaking the requirements of the above theorem are not satisfied in this case, but we are able to prove the Jacobi property of the averaged HTPB using the method of Flashka, Forest and McLaughlin for the integrable systems. The metric in this coordinates has the following form:

$$g_0^{pq} = \begin{pmatrix} 0 & 1_m & 0 \\ 1_m & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore we know the flat coordinates in some important cases.

By the results of Whitham of 1973 and Flashka, Forest and McLaughlin of 1979-80, the RI exist for the averaged KdV and coincide with the "branching points" of the Riemann surface Γ defined by

$$\mu^2 = R_{2m+1}(\lambda) = \prod_{j=0}^{2m} (\lambda - \lambda_j),$$

$$r_j = \lambda_j,\ j = 0, \ldots, 2m,\ N = 2m+1,\ \frac{\partial r_j}{\partial t} = V_j(r) \frac{\partial r_j}{\partial x}.$$

Consider the "quasi-momentum" and the "quasi-energy" for the Schrödinger operator with periodic potential:

$$\begin{aligned}
&\Psi(x+T) = \Psi(x), \quad L = -\partial_x^2 + \Psi, \\
&L\varphi = \lambda\varphi,\ \varphi(x+T) = \exp(ip(\lambda)T)\varphi(x), \\
&p(\lambda) = -i\overline{(\log\varphi)_x},\ q_n(\lambda) = -i\overline{(log\varphi)_t},\ n \geq 1.
\end{aligned}$$

The differential forms $d_\lambda p$ and $d_\lambda q_n$ are algebraic second kind forms with trivial a-periods:

$$\begin{aligned}
&\oint_{a_j} dp(\lambda) = 0,\ j = 1, \ldots, m, \qquad \oint_{a_j} dq_n(\lambda) = 0, \\
&p(a_j) = [\lambda_{2j-1}, \lambda_{2j}] \subset \mathbf{R}, \quad p : \Gamma \to \mathbf{C}, \quad \mu^2 = R_{2m+1}(\lambda).
\end{aligned}$$

The poles of the forms dp and dq_n are displaced at the point $\lambda = \infty$ and have the asymptotics:

$$dp = (dk + reg), \quad k^2 = \lambda,$$

$$dq_n = (dk^n + reg), \quad n \geq 1.$$

The canonical cycles are such that:

$$a_k \circ b_l = \delta_{kl}, \quad b_k \circ b_l = 0, \quad a_k \circ a_l = 0.$$

By definition we have:

$$U_j = \oint_{b_j} dp(\lambda), \quad V_j^{(n)} = \oint dq_n(\lambda), \quad p = q_0.$$

The quantity

$$p(\lambda) = \lambda^{1/2} + \sum_{s \geq 0} \frac{I_{s-1}}{(2\lambda^{1/2})^{2s+1}}$$

is in fact the integral from the solution of the Riccati equation (see above):

$$p(\lambda) = \int_0^T dx = \lim_{T \to \infty} \frac{1}{T} \int_0^T \chi(x) dx = \vec{\chi}.$$

For the velocities of the HT systems generated by the averaged integrals I_s through the HTPB we have the following formulas:

$$r_j = \lambda_j \ , \ j = 0, 1, \ldots, 2n$$

$$V_j^{(s)}(r) = \left(\frac{dq_s(\lambda)}{dp(\lambda)} \right)_{\lambda = \lambda_j}, \quad s \geq 0.$$

Here we have $(dq_0 = dp)$; the most important property is that:

$$V_j(\alpha r) = \alpha V_j(r).$$

Through Tsarev's procedure the averaged integrals I_s generate exact solutions ("averaged finite-gap" in our terminology). Krichever observed in 1987 that these exact solutions are in fact self-similar (see [10]).

Consider now the special case $m = 1$ which is non-trivial and physically important. We have:

$$\begin{aligned}
-V_0 &= \left(\sum_{j=0}^{2} r_j\right)/3 - \frac{2}{3}(r_1 - r_0)\frac{K}{K-E},\\
-V_1 &= \left(\sum_{j=0}^{2} r_j\right)/3 - \frac{2}{3}(r_1 - r_0)\frac{(1-s^2)k}{E-(1-s^2)K},\\
-V_2 &= \left(\sum_{j=0}^{2} r_j\right)/3 - \frac{2}{3}(r_2 - r_0)\frac{(1-s^2)K}{E},\\
s^2 &= \frac{r_1 - r_0}{r_2 - r_0},\ 0 \le s^2 \le 1,\\
V_2 &\ge V_1 \ge V_0,\quad r_2 \ge r_1 \ge r_0.
\end{aligned}$$

(K and E are the elliptic integrals —see [4]).

The special case $r_1 = r_0$ corresponds to a constant solution of KdV. The case $r_2 = r_1$ corresponds to a soliton which is rapidly descreasing . The generic "knoidal wave" has the form for KdV

$$u(x,t) = 2\mathcal{P}(x - ct) + \text{const}$$

($\mathcal{P}$ is a Weierstrass elliptic function corresponding to the algebraic curve Γ of genus $g = m = 1$). The generic solution generated by the averaged Kruskal integrals I_s has velocities written in the form:

$$\sum_{s\ge 0} a_s I_s \mapsto \left\{W_j(r) = \sum_{s\ge 0} a_s W_j^{(s)}(r)\right\}, \qquad W_j^{(s)}(r) = \left(\frac{dq_s(\lambda)}{dp(\lambda)}\right)_{\lambda=\lambda_j}.$$

Following Tsarev's procedure we have to solve the equation

$$\begin{aligned}
&W_j(r) = V_j(r)T + X,\\
&r_k = r(x,t),\ k = 0,\dots,N-1,\ N = 2m+1.
\end{aligned}$$

For the basic Kruskal Integrals

$$c_s = 1\ ,\ c_i = 0\ ,\ i \ne s$$

we obtain special self-similar solutions:

$$r_k(X,T) = T^{\gamma} R_k(XT^{-1-\gamma}), \quad \gamma = \frac{1}{s-2} \quad (s \neq 2).$$

For $s=4$ and $\gamma = 1/2$ we obtain a solution which G. Potemin identified (in his Ph.D. thesis) with the very important "dispersive analogue of the shock wave" defined by A. Gurevitch and L. Pitaevski in 1973 —see the reference quoted in [10].

In this case we have:

$$\begin{aligned} &W_k = \frac{1}{35}[(3V_k - a)f_k + f], \qquad f = 5a^3 - 12ab + c, \\ &a = \sum r_j, \ b = \sum_{i \le j-1} r_i r_j, \ c = r_0 r_1 r_2, \quad f_i = \frac{\partial f}{\partial r_i}, \ i = 0,1,2. \end{aligned}$$

This solution is defined on the interval

$$\triangle : -\sqrt{2} \le z \le \frac{\sqrt{10}}{27}, \quad z = XT^{-3/2}, \ \gamma = 1/2.$$

Outside the interval $\triangle$ this solution should be continued as the multi-valued C^1-function $R(z)$ such that:

$$\begin{aligned} &z + R/T = R^3 \ \ (z \notin \triangle), \quad R = r/\sqrt{T}, \\ &R\left(-\sqrt{2}\right) = R_2\left(-\sqrt{2}\right), \quad R_1\left(-\sqrt{2}\right) = R_0\left(-\sqrt{2}\right), \\ &R\left(\frac{\sqrt{10}}{27}\right) = R_0\left(\frac{\sqrt{10}}{27}\right), \quad R_2\left(\frac{\sqrt{10}}{27}\right) = R_1\left(\frac{\sqrt{10}}{27}\right). \end{aligned}$$

Outside the interval $\triangle$ we use the trivial Hopf equation ("variation of constants")

$$x + 6rt = r^3, \qquad r_t = 6rr_x.$$

Inside $\triangle$ we use the non-linear WKB method with $m = 1$.

The dispersive analogue of the shock wave appears in the following way. Let us start from the "very smooth" initial data for KdV and approximate the equation by the trivial one:

$$\Psi_t = 6\Psi\Psi_x, \quad \Psi_x << \Psi, \quad \Psi_{xx} << \Psi_x, \quad \cdots.$$

After some development we find that the first derivative Ψ_x tends to ∞ as $t \to t_0$ and $x \to x_0$; hence the trivial approximation does not work for "the shock wave situation", which may be locally approximated by a cubic curve. Some "Oscillation Zone" (briefly, OZ) for the original KdV equation appears in this case, and we shall see no discontinuities at all for any time.

Inside this OZ the non-linear WKB method works for large t; outside it we continue to use the trivial Hopf equation for any time. On the boundary of the OZ we have

$$\begin{aligned}
&\mathrm{OZ} = [x_-(t), x_+(t)],\\
&r(x_-) = r_1(x_-) = r_0(x_-) < r_2(x_-),\\
&r(x_+) = r_2(x_+) = r_1(x_+) > r_0(x_+).
\end{aligned}$$

For $x < x_-$ and $x > x_+$ we use the trivial Hopf equation for $r(x,t)$. It wuold be important to find some equation for the boundary of OZ

$$\frac{dx_-}{dt} =? \ , \ \frac{dx_+}{dt} =?$$

The multi-valued function $r(x,t)$ (which is single-valued for $x < x(t)$ and for $x > x_+(t)$) should be at least C^1 everywhere and satisfy the following asymptotic conditions for $x \to x_\pm(t)$:

$$r = r(x,t), \text{ if } x > x_+, \ x < x_-; \quad r = (r_0, r_1, r_2), \text{ if } x \in [x_-, x_+];$$

$$\begin{gathered}
r_0 = r_1 \text{ at } x = x_-, \quad r_1 = r_2 \text{at } x = x_+;\\
r(x,t) \text{ of class } \mathrm{C}^2 \text{ near } x_-(t), \ x \in \triangle;\\
0 < x - x_- = a_-(r - r_-)^2 + \mathrm{o}(r - r_-)^2;\\
0 > x - x_+ = a_+ f(1 - s^2) + \mathrm{o}(r - r_-)^2;\\
f(u) = u \log\left(\frac{16}{u} + 1/2\right), \quad u = 1 - s^2.
\end{gathered}$$

These conditions are discussed in [2], where also the following equations for $x_\pm(t)$ were found:

$$\dot{x}_+ = V_1^+ = V_2^+, \quad \dot{r}_+ = -|r_2^+ - r_1^+|/12a_+;$$

$$\dot{x}_- = V_0^- = V_1^-, \quad \dot{r}_- = -1/2a_-;$$

$$r_+ = r_2^+ = r_1^+,\ r_- = r_0^- = r_1^-.$$

The self-similar solution of Gurevitch and Pitaevski (found in 1973) has no singularities inside $\triangle$ (as Potemin proved in 1988-90); therefore it satisfies our requirements. The C^1-property was first missed and then proved in 1990; by definition the interval $\triangle$ is constant in the coordinate z

$$z = xt^{-1/2}.$$

For the function $r(x,t)$ we have

$$r(x,t) \to x^{1/3},\ x \to \pm\infty$$

because $r = \Psi$ outside $\triangle$. This fact means that the length of $\triangle$ is small enough (and we may use the cubic form of the curve which in fact is a local object). The author's numerical investigation (carried out in 1987) confirmed that the Gurevitch-Pitaevski dispersive shock wave is a stable solution of our problem, and it is asymptotically attractive for an open set of admissible multi-valued functions $r(x,t)$.

This numerical result has been proved and extended to a very large open set by Tian and others in 1991 on the basis of the exact analytic solution of the problem using the methods described above.

The influence of the small viscosity on the situation above has been investigated in 1987 by the author in collaboration with V. Avilov and Krichever (and also by L. Pitaevski and A. Gurevitch —see in [10]).

The foundation of the averaging (non-linear WKB) method for the KdVsystem with small dispersion term

$$\Psi_t = 6\Psi\Psi_x + \delta\Psi_{xxx}\ ,\ \delta \to 0$$

has been investigated by P. Lax and C. Levermore starting from 1983, and by S. Venakides starting from 1987. Some important asymptotic results were obtained also by R. Its, R. Biklaev and V. Novokshenov, using "isomonodromic" deformations, in 1983-1989 (see the references in [10]).

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