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SINGULARITIES OF TRANSITION PROCESSES IN DYNAMICAL SYSTEMS: QUALITATIVE THEORY OF CRITICAL DELAYS

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ABSTRACT. This monograph presents a systematic analysis of the singularities in the transition processes for dynamical systems. We study general dynamical systems, with dependence on a parameter, and construct relaxation times that depend on three variables: Initial conditions, parameters k of the system, and accuracy ε of the relaxation. We study the singularities of relaxation times as functions of (x_0, k) under fixed ε, and then classify the bifurcations (explosions) of limit sets. We study the relationship between singularities of relaxation times and bifurcations of limit sets. An analogue of the Smale order for general dynamical systems under perturbations is constructed. It is shown that the perturbations simplify the situation: the interrelations between the singularities of relaxation times and other peculiarities of dynamics for general dynamical system under small perturbations are the same as for the Morse-Smale systems.

CONTENTS

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Introduction

Are there "white spots" in topological dynamics? Undoubtedly, they exist: The transition processes in dynamical systems are still not very well known. As a consequence, it is difficult to interpret the experiments that reveal singularities of transition processes, and in particular, anomalously slow relaxation. "Anomalously slow" means here "unexpectedly slow"; but what can one expect from a dynamical system in a general case?

In this monograph, we study the transition processes in general dynamical systems. The approach based on the topological dynamics is quite general, but one pays for these generality by the loss of constructivity. Nevertheless, this stage of a general consideration is needed.

The limiting behaviour (as $t \to \infty$) of dynamical systems have been studied very intensively in the XX century [16, 37, 36, 68, 12, 56]. New types of limit sets ("strange attractors") were discovered [50, 1]. Fundamental results concerning the structure of limit sets were obtained, such as the Kolmogorov–Arnold–Moser theory [11, 55], the Pugh lemma [61], the qualitative [66, 47, 68] and quantitative [38, 79, 40] Kupka–Smale theorem, etc. The theory of limit behaviour "on the average", the ergodic theory [45], was considerably developed. Theoretical and applied achievements of the bifurcation theory have become obvious [3, 13, 60]. The fundamental textbook on dynamical systems [39] and the introductory review [42] are now available.

The achievements regarding transition processes have not been so impressive, and only relaxations in linear and linearized systems are well known. The applications of this elementary theory received the name the "relaxation spectroscopy". Development of this discipline with applications in chemistry and physics was distinguished by Nobel Prize (M. Eigen [24]).

A general theory of transition processes of essentially non-linear systems does not exist. We encountered this problem while studying transition processes in catalytic reactions. It was necessary to give an interpretation on anomalously long transition processes observed in experiments. To this point, a discussion arose and even some papers were published. The focus of the discussion was: do the slow relaxations arise from slow "strange processes" (diffusion, phase transitions, and so on), or could they have a purely kinetic (that is dynamic) nature?

Since a general theory of relaxation times and their singularities was not available at that time, we constructed it by ourselves from the very beginning [35, 34, 32, 33, 25, 30]. In the present paper the first, topological part of this theory is presented. It is quite elementary theory, though rather lengthy $\varepsilon - \delta$ reasonings may require some time and effort. Some examples of slow relaxation in chemical systems, their theoretical and numerical analysis, and also an elementary introduction into the theory can be found in the monograph [78].

Two simplest mechanisms of slow relaxations can be readily mentioned: The delay of motion near an unstable fixed point, and the delay of motion in a domain where a fixed point appears under a small change of parameters. Let us give some simple examples of motion in the segment $[-1,1]$.

The delay near an unstable fixed point exists in the system $\dot{x}=x^2-1$. There are two fixed points $x=\pm 1$ on the segment $[-1,1]$, the point $x=1$ is unstable and the point $x=-1$ is stable. The equation is integrable explicitly:

$$x=[(1+x_0)\mathrm{e}^{-t}-(1-x_0)\mathrm{e}^{t}]/[(1+x_0)\mathrm{e}^{-t}+(1-x_0)\mathrm{e}^{t}],$$

where $x_0=x(0)$ is initial condition at $t=0$. If $x_0\neq 1$ then, after some time, the motion will come into the ε-neighborhood of the point $x=-1$, for whatever $\varepsilon>0$. This process requires the time

$$\tau(\varepsilon,x_0)=-\frac{1}{2}\ln\frac{\varepsilon}{2-\varepsilon}-\frac{1}{2}\ln\frac{1-x_0}{1+x_0}.$$

It is assumed that $1>x_0>\varepsilon-1$. If ε is fixed then τ tends to $+\infty$ as $x_0\to 1$ like $-\frac{1}{2}\ln(1-x_0)$. The motion that begins near the point $x=1$ remains near this point for a long time ($\sim -\frac{1}{2}\ln(1-x_0)$), and then goes to the point $x=-1$. In order to show it more clear, let us compute the time τ' of residing in the segment $[-1+\varepsilon,\ 1-\varepsilon]$ of the motion, beginning near the point $x=1$, i.e. the time of its stay outside the ε-neighborhoods of fixed points $x=\pm 1$. Assuming $1-x_0<\varepsilon$, we obtain

$$\tau'(\varepsilon,x_0)=\tau(\varepsilon,x_0)-\tau(2-\varepsilon,x_0)=-\ln\frac{\varepsilon}{2-\varepsilon}.$$

One can see that if $1-x_0<\varepsilon$ then $\tau'(\varepsilon,x_0)$ does not depend on x_0. This is obvious: the time τ' is the time of travel from point $1-\varepsilon$ to point $-1+\varepsilon$.

Let us consider the system $\dot{x}=(k+x^2)(x^2-1)$ on $[-1,1]$ and try to obtain an example of delay of motion in a domain where a fixed point appears under small change of parameter. If $k>0$, there are again only two fixed points $x=\pm 1$, $x=-1$ is a stable point and $x=1$ is an unstable. If $k=0$ there appears the third point $x=0$. It is not stable, but "semistable" in the following sense: If the initial position is $x_0>0$ then the motion goes from x_0 to $x=0$. If $x_0<0$ then the motion goes from x_0 to $x=-1$. If $k<0$ then apart from $x=\pm 1$, there are two other fixed points $x=\pm\sqrt{|k|}$. The positive point is stable, and the negative point is unstable. Let us consider the case $k>0$. The time of motion from the point x_0 to the point x_1 can be found explicitly ($x_{0,1}\neq\pm 1$):

$$t=\frac{1}{2}\ln\frac{1-x_1}{1+x_1}-\frac{1}{2}\ln\frac{1-x_0}{1+x_0}-\frac{1}{\sqrt{k}}\left(\arctan\frac{x_1}{\sqrt{k}}-\arctan\frac{x_0}{\sqrt{k}}\right).$$

If $x_0>0$, $x_1<0$, $k>0$, $k\to 0$, then $t\to\infty$ like $\pi/\sqrt{k}$. These examples do not exhaust all the possibilities; they rather illustrate two common mechanisms of slow relaxations appearance.

Below we study parameter-dependent dynamical systems. The point of view of topological dynamics is adopted (see [16, 37, 36, 56, 65, 80]). In the first place this means that, as a rule, the properties associated with the smoothness, analyticity and so on will be of no importance. The phase space X and the parameter space K are compact metric spaces: for any points x_1,x_2 from X (k_1,k_2 from K) the distance $\rho(x_1, x_2)$ ($\rho_K(k_1, k_2)$) is defined with the following properties:

$$\rho(x_1, x_2) = \rho(x_2, x_1), \quad \rho(x_1, x_2) + \rho(x_2, x_3) \geq \rho(x_1, x_3),$$
$$\rho(x_1, x_2) = 0 \text{ if and only if } x_1 = x_2 \text{ (similarly for } \rho_K).$$

The sequence x_i converges to x^* ($x_i \to x^*$) if $\rho(x_i, x^*) \to 0$. The compactness means that from any sequence a convergent subsequence can be chosen.

The states of the system are represented by the points of the phase space X. The reader can think of X and K as closed, bounded subsets of finite-dimensional Euclidean spaces, for example polyhedrons, and ρ and ρ_K are the standard Euclidean distances.

Let us define the phase flow (the transformation "shift over the time t"). It is a function f of three arguments: $x \in X$ (of the initial condition), $k \in K$ (the parameter value) and $t \geq 0$, with values in X: $f(t, x, k) \in X$. This function is assumed continuous on $[0, \infty) \times X \times K$ and satisfying the following conditions:

- $f(0, x, k) = x$ (shift over zero time leaves any point in its place);
- $f(t, f(t', x, k), k) = f(t + t', x, k)$ (the result of sequentially executed shifts over t and t' is the shift over $t + t'$);
- if $x \neq x'$, then $f(t, x, k) \neq f(t, x', k)$ (for any t distinct initial points are shifted in time t into distinct points for.

For a given parameter value $k \in K$ and an initial state $x \in X$, the *ω-limit set* $\omega(x, k)$ is the set of all limit points of $f(t, x, k)$ as $t \to \infty$:

> y is in $\omega(x, k)$ if and only if there exists a sequence $t_i \geq 0$ such that
> $t_i \to \infty$ and $f(t_i, x, k) \to y$.

Examples of ω-limit points are stationary (fixed) points, points of limit cycles and so on.

The relaxation of a system can be understood as its motion to the ω-limit set corresponding to given initial state and value of parameter. *The relaxation time* can be defined as the time of this motion. However, there are several possibilities to make this definition precise.

Let $\varepsilon > 0$. For given value of parameter k we denote by $\tau_1(x, k, \varepsilon)$ the time during which the system will come from the initial state x into the ε-neighbourhood of $\omega(x, k)$ (for the first time). The (x, k)-motion can enter the ε-neighborhood of the ω-limit set, then this motion can leave it, then reenter it, and so on it can enter and leave the ε-neighbourhood of $\omega(x, k)$ several times. After all, the motion will enter this neighbourhood finally, but this may take more time than the first entry. Therefore, let us introduce for the (x, k)-motion the time of being outside the ε-neighborhood of $\omega(x, k)$ (τ_2) and the time of final entry into it (τ_3). Thus, we have a system of relaxation times that describes the relaxation of the (x, k)-motion to its ω-limit set $\omega(x, k)$:

$$\tau_1(x, k, \varepsilon) = \inf\{t > 0 : \rho^*(f(t, x, k),\ \omega(x, k)) < \varepsilon\};$$
$$\tau_2(x, k, \varepsilon) = \operatorname{meas}\{t > 0 : \rho^*(f(t, x, k),\ \omega(x, k)) \geq \varepsilon\};$$
$$\tau_3(x, k, \varepsilon) = \inf\{t > 0 : \rho^*(f(t', x, k),\ \omega(x, k)) < \varepsilon \text{ for } t' > t\}.$$

Here meas is the Lebesgue measure (on the real line it is length), ρ^* is the distance from the point to the set: $\rho^*(x, P) = \inf_{y \in P} \rho(x, y)$.

The ω-limit set depends on an initial state (even under the fixed value of k). The limit behavior of the system can be characterized also by the ***total limit set***

$$\omega(k) = \bigcup_{x\in X} \omega(x,k).$$

The set $\omega(k)$ is the union of all $\omega(x,k)$ under given k. Whatever initial state would be, the system after some time will be in the ε-neighborhood of $\omega(k)$. The relaxation can be also considered as a motion towards $\omega(k)$. Introduce the corresponding system of relaxation times:

$$\eta_1(x,k,\varepsilon) = \inf\{t>0 : \rho^*(f(t,x,k),\ \omega(k)) < \varepsilon\};$$
$$\eta_2(x,k,\varepsilon) = \operatorname{meas}\{t>0 : \rho^*(f(t,x,k),\ \omega(k)) \geq \varepsilon\};$$
$$\eta_3(x,k,\varepsilon) = \inf\{t>0 : \rho^*(f(t',x,k),\ \omega(k)) < \varepsilon \text{ for } t'>t\}.$$

Now we are able to define a ***slow transition process***. There is no distinguished scale of time, which could be compared with relaxation times. Moreover, by decrease of the relaxation accuracy ε the relaxation times can become of any large amount even in the simplest situations of motion to unique stable fixed point. For every initial state x and given k and ε all relaxation times are finite. But the set of relaxation time values for various x and k and given $\varepsilon>0$ can be unbounded. Just in this case we speak about the slow relaxations.

Let us consider the simplest example. Let us consider the differential equation $\dot{x} = x^2 - 1$ on the segment $[-1,1]$. The point $x=-1$ is stable, the point $x=1$ is unstable. For any fixed $\varepsilon>0$, $\varepsilon<\frac{1}{2}$ the relaxation times $\tau_{1,2,3}, \eta_3$ have the singularity: $\tau_{1,2,3}, \eta_3(x,k,\varepsilon) \to \infty$ as $x\to 1$, $x<1$. The times η_1, η_2 remain bounded in this case.

Let us say that the system has τ_i- (η_i)-***slow relaxations***, if for some $\varepsilon>0$ the function $\tau_i(x,k,\varepsilon)$ ($\eta_i(x,k,\varepsilon)$) is unbounded from above in $X\times K$, i.e. for any $t>0$ there are such $x\in X$, $k\in K$, that $\tau_i(x,k,\varepsilon)>t$ ($\eta_i(x,k,\varepsilon)>t$).

One of the possible reasons of slow relaxations is a sudden jump in dependence of the ω-limit set $\omega(x,k)$ of x,k (as well as a jump in dependence of $\omega(k)$ of k). These "explosions" (or bifurcations) of ω-limit sets are studied in Sec. 1. In the next Sec. 2 we give the theorems, providing necessary and sufficient conditions of slow relaxations. Let us mention two of them.

Theorem 2.9$'$. *A system has τ_1-slow relaxations if and only if there is a singularity on the dependence $\omega(x,k)$ of the following kind: There exist points $x^*\in X$, $k^*\in K$, sequences $x_i\to x^*$, $k_i\to k^*$, and number $\delta>0$, such that for any i, $y\in\omega(x^*,k^*)$, $z\in\omega(x_i,k_i)$ the distance satisfies $\rho(y,z)>\delta$.*

The singularity of $\omega(x,k)$ described in the statement of the theorem indicates that the ω-limit set $\omega(x,k)$ makes a jump: the distance from any point of $\omega(x_i,k_i)$ to any point of $\omega(x^*,k^*)$ is greater than δ.

By the next theorem, necessary and sufficient conditions of τ_3-slow relaxations are given. Since $\tau_3\geq\tau_1$, the conditions of τ_3-slow relaxations are weaker than the conditions of Theorem 2.9$'$, and τ_3-slow relaxations are "more often" than τ_1-slow relaxation (the relations between different kinds of slow relaxations with corresponding examples are given below in Subsec. 3.2). That is why the discontinuities of ω-limit sets in the following theorem are weaker.

Theorem 2.20. *τ_3-slow relaxations exist if and only if at least one of the following conditions is satisfied:*

(1) *There are points $x^* \in X$, $k^* \in K$, $y^* \in \omega(x^*, k^*)$, sequences $x_i \to x^*$, $k_i \to k^*$ and number $\delta > 0$ such that for any i and $z \in \omega(x_i, k_i)$ the inequality $\rho(y^*, z) > \delta$ is valid (The existence of one such y is su cient, compare it with Theorem 2.9′).*
(2) *There are $x \in X$, $k \in K$ such that $x \notin \omega(x, k)$, for any $t > 0$ can be found $y(t) \in X$, for which $f(t, y(t), k) = x$ ($y(t)$ is a shift of x over $-t$), and for some $z \in \omega(x, k)$ can be found such a sequence $t_i \to \infty$ that $y(t_i) \to z$. That is, the (x, k)-trajectory is a generalized loop: the intersection of its ω-limit set and α-limit set (i.e., the limit set for $t \to -\infty$) is non-empty, and x is not a limit point for the (x, k)-motion.*

An example of the point satisfying the condition 2 is provided by any point lying on the loop, that is the trajectory starting from the fixed point and returning to the same point.

Other theorems of Sec. 2 also establish connections between slow relaxations and peculiarities of the limit behaviour under different initial conditions and parameter values. In general, in topological and differential dynamics the main attention is paid to the limit behavior of dynamical systems [16, 37, 36, 68, 12, 56, 65, 80, 57, 41, 18, 39, 42]. In applications, however, it is often of importance how rapidly the motion approaches the limit regime. In chemistry, long-time delay of reactions far from equilibrium (induction periods) have been studied since Van't-Hoff [73] (the first Nobel Prize laureate in Chemistry). It is necessary to mention the classical monograph of N.N. Semjonov [30] (also the Nobel Prize laureate in Chemistry), where induction periods in combustion are studied. From the latest works let us note [69]. When minimizing functions by relaxation methods, the similar delays can cause some problems. The paper [29], for example, deals with their elimination. In the simplest cases, the slow relaxations are bound with delays near unstable fixed points. In the general case, there is a complicated system of interrelations between different types of slow relaxations and other dynamical peculiarities, as well as of different types of slow relaxations between themselves. These relations are the subject of Sects. 2, 3. The investigation is performed generally in the way of classic topological dynamics [16, 37, 36]. There are, however, some distinctions:

- From the very beginning not only one system is considered, but also practically more important case of parameter dependent systems;
- The motion in these systems is defined, generally speaking, only for positive times.

The last circumstance is bound with the fact that for applications (in particular, for chemical ones) the motion is defined only in a positively invariant set (in balance polyhedron, for example). Some results can be accepted for the case of general semidynamical systems [72, 14, 54, 70, 20], however, for the majority of applications, the considered degree of generality is more than sufficient.

For a separate semiflow f (without parameter) η_1-slow relaxations are impossible, but η_2-slow relaxations can appear in a separate system too (Example 2.4). Theorem 3.2 gives the necessary conditions for η_2-slow relaxations in systems without parameter.

Let us recall the definition of *non-wandering points*. A point $x^* \in X$ is the non-wandering point for the semiflow f, if for any neighbourhood U of x^* and for any $T > 0$ there is such $t > T$ that $f(t,U) \bigcap U \neq \varnothing$. Let us denote by ω_f the complete ω-limit set of one semiflow f (instead of $\omega(k)$).

Theorem 3.2. *Let a semiflow f possess η_2-slow relaxations. Then there exists a non-wandering point $x^* \in X$ which does not belong to $\overline{\omega_f}$.*

For of smooth systems it is possible to obtain results that have no analogy in topological dynamics. Thus, it is shown in Sec. 2 that "almost always" η_2-slow relaxations are absent in one separately taken C^1-smooth dynamical system (system, given by differential equations with C^1-smooth right parts). Let us explain what "almost always" means in this case. A set Q of C^1-smooth dynamical systems with common phase space is called nowhere-dense in C^1-topology, if for any system from Q an infinitesimal perturbation of right hand parts can be chosen (perturbation of right hand parts and its first derivatives should be smaller than an arbitrary given $\varepsilon > 0$) such that the perturbed system should not belong to Q and should exist $\varepsilon_1 > 0$ ($\varepsilon_1 < \varepsilon$) such that under ε_1-small variations of right parts (and of first derivatives) the perturbed system could not return in Q. The union of finite number of nowhere-dense sets is also nowhere-dense. It is not the case for countable union: for example, a point on a line forms nowhere-dense set, but the countable set of rational numbers is dense on the real line: a rational number is on any segment. However, both on line and in many other cases countable union of nowhere-dense sets (the sets of *first category*) can be considered as very "meagre". Its complement is so-called "residual set". In particular, for C^1-smooth dynamical systems on compact phase space the union of countable number of nowhere-dense sets has the following property: any system, belonging to this union, can be eliminated from it by infinitesimal perturbation. The above words "almost always" meant: except for union of countable number of nowhere-dense sets.

In two-dimensional case (two variables), "almost any" C^1-smooth dynamical system is rough, i.e. its phase portrait under small perturbations is only slightly deformed, qualitatively remaining the same. For rough two-dimensional systems ω-limit sets consist of fixed points and limit cycles, and the stability of these points and cycles can be verified by linear approximation. The correlation of six different kinds of slow relaxations between themselves for rough two-dimensional systems becomes considerably more simple.

Theorem 3.12. *Let M be C^∞-smooth compact manifold,* $\dim M = 2$, *F be a structural stable smooth dynamical system over M, $F|_X$ be an associated with M semiflow over connected compact positively invariant subset $X \subset M$. Then:*

(1) *For $F|_X$ the existence of τ_3-slow relaxations is equivalent to the existence of $\tau_{1,2}$- and η_3-slow relaxations;*
(2) *$F|_X$ does not possess τ_3-slow relaxations if and only if $\omega_F \bigcap X$ consists of one fixed point or of points of one limit cycle;*
(3) *$\eta_{1,2}$-slow relaxations are impossible for $F|_X$.*

For smooth rough two-dimensional systems it is easy to estimate the measure (area) of the region of durable delays $\mu_i(t) = \text{meas}\{x \in X : \tau_i(x,\varepsilon) > t\}$ under fixed sufficiently small ε and large t (the parameter k is absent because a separate system is studied). Asymptotical behaviour of $\mu_i(t)$ as $t \to \infty$ does not depend on

i and

$$\lim_{t\to\infty}\frac{\ln\mu_i(t)}{t} = -\min\{\varkappa_1,\dots,\varkappa_n\},$$

where n is a number of unstable limit motions (of fixed points and cycles) in X, and the numbers are determined as follows. We denote by $B_i,\dots,B_n$ the unstable limit motions lying in X.

(1) Let B_i be an unstable node or focus. Then $\varkappa_1$ is the trace of matrix of linear approximation in the point b_i.
(2) Let b_i be a saddle. Then $\varkappa_1$ is positive eigenvalue of the matrix of linear approximation in this point.
(3) Let b_i be an unstable limit cycle. Then $\varkappa_i$ is characteristic indicator of the cycle (see [15, p. 111]).

Thus, the area of the region of initial conditions, which result in durable delay of the motion, in the case of smooth rough two-dimensional systems behaves at large delay times as $\exp(-\varkappa t)$, where t is a time of delay, $\varkappa$ is the smallest number of $\varkappa_i,\dots,\varkappa_n$. If $\varkappa$ is close to zero (the system is close to bifurcation [12, 15]), then this area decreases slowly enough at large t. One can find here analogy with linear time of relaxation to a stable fixed point

$$\tau_l = -1/\max\mathrm{Re}\lambda$$

where λ runs through all the eigenvalues of the matrix of linear approximation of right parts in this point, $\max\mathrm{Re}\lambda$ is the largest (the smallest by value) real part of eigenvalue, $\tau_l\to\infty$ as $\mathrm{Re}\lambda\to 0$.

However, there are essential differences. In particular, τ_l comprises the eigenvalues (with negative real part) of linear approximation matrix in that (stable) point, to which the motion is going, and the asymptotical estimate μ_i comprises the eigenvalues (with positive real part) of the matrix in that (unstable) point or cycle, near which the motion is retarded.

In typical situations for two-dimensional parameter depending systems the singularity of τ_l entails existence of singularities of relaxation times τ_i (to this statement can be given an exact meaning and it can be proved as a theorem). The inverse is not true. As an example should be noted the delays of motions near unstable fixed points. Besides, for systems of higher dimensions the situation becomes more complicated, the rough systems cease to be "typical" (this was shown by S. Smale [67], the discussion see in [12]), and the limit behaviour even of rough systems does not come to tending of motion to fixed point or limit cycle. Therefore the area of reasonable application the linear relaxation time τ_l to analysis of transitional processes becomes in this case even more restricted.

Any real system exists under the permanent perturbing influence of the external world. It is hardly possible to construct a model taking into account all such perturbations. Besides that, the model describes the internal properties of the system only approximately. The discrepancy between the real system and the model arising from these two circumstances is different for different models. So, for the systems of celestial mechanics it can be done very small. Quite the contrary, for chemical kinetics, especially for kinetics of heterogeneous catalysis, this discrepancy can be if not too large but, however, not such small to be neglected. Strange as it may seem, the presence of such an unpredictable divergence of the model and reality can simplify the situation: The perturbations "conceal" some fine details of dynamics, therefore these details become irrelevant to analysis of real systems.

Sec. 4 is devoted to the problems of slow relaxations in presence of small perturbations. As a model of perturbed motion here are taken ε-*motions*: the function of time $\varphi(t)$ with values in X, defined at $t \geq 0$, is called ε-motion ($\varepsilon > 0$) under given value of $k \in K$, if for any $t \geq 0$, $\tau \in [0, T]$ the inequality $\rho(\varphi(t+\tau),\ f(\tau, \varphi(t), k)) < \varepsilon$ holds. In other words, if for an arbitrary point $\varphi(t)$ one considers its motion on the force of dynamical system, this motion will diverge $\varphi(t+\tau)$ from no more than at ε for $\tau \in [0, T]$. Here $[0, T]$ is a certain interval of time, its length T is not very important (it is important that it is fixed), because later we shall consider the case $\varepsilon \to 0$.

There are two traditional approaches to the consideration of perturbed motions. One of them is to investigate the motion in the presence of small constantly acting perturbations [22, 51, 28, 46, 52, 71, 53], the other is the study of fluctuations under the influence of small stochastic perturbations [59, 74, 75, 43, 44, 76]. The stated results join the first direction, but some ideas bound with the second one are also used. The ε-motions were studied earlier in differential dynamics, in general in connection with the theory of Anosov about ε-trajectories and its applications [41, 6, 77, 26, 27], see also [23].

When studying perturbed motions, we correspond to each point “a bundle” of ε-motions, $\{\varphi(t)\}$, $t \geq 0$ going out from this point ($\varphi(0) = x$) under given value of parameter k. The totality of all ω-limit points of these ε-motions (of limit points of all $\varphi(t)$ as $t \to \infty$) is denoted by $\omega^{\varepsilon}(x, k)$. Firstly, it is necessary to notice that $\omega^{\varepsilon}(x, k)$ does not always tend to $\omega(x, k)$ as $\varepsilon \to 0$: the set $\omega^0(x, k) = \bigcap_{\varepsilon > 0} \omega^{\varepsilon}(x, k)$ may not coincide with $\omega(x, k)$. In Sec. 4 there are studied relaxation times of ε-motions and corresponding slow relaxations. In contrast to the case of nonperturbed motion, all natural kinds of slow relaxations are not considered because they are too numerous (eighteen), and the principal attention is paid to two of them, which are analyzed in more details than in Sec. 2.

The structure of limit sets of one perturbed system is studied. The analogy of general perturbed systems and Morse-Smale systems as well as smooth rough two-dimensional systems is revealed. Let us quote in this connection the review by Professor A. M. Molchanov of the thesis [31] of A. N. Gorban [1] (1981):

> After classic works of Andronov, devoted to the rough systems on the plane, for a long time it seemed that division of plane into finite number of cells with source and drain is an example of structure of multidimensional systems too... The most interesting (in the opinion of opponent) is the fourth chapter “Slow relaxations of the perturbed systems”. Its principal result is approximately as follows. If a complicated dynamical system is made rough (by means of ε-motions), then some its important properties are similar to the properties of rough systems on the plane. This is quite positive result, showing in what sense the approach of Andronov can be generalized for arbitrary systems.

To study limit sets of perturbed system, two relations are introduced in [30] for general dynamical systems: the preorder $\succsim$ and the equivalence $\sim$:

- $x_1 \succsim x_2$ if for any $\varepsilon > 0$ there is such a ε-motion $\varphi(t)$ that $\varphi(0) = x_1$ and $\varphi(\tau) = x_2$ for some $\tau > 0$;

[1] This paper is the first complete publication of that thesis.

- $x_1 \sim x_2$ if $x_1 \succsim x_2$ and $x_2 \succsim x_1$.

For smooth dynamical systems with finite number of "basic attractors" similar relation of equivalence had been introduced with the help of action functionals in studies on stochastic perturbations of dynamical systems ([76] p. 222 and further). The concepts of ε-motions and related topics can be found in [23]. For the Morse-Smale systems this relation is the Smale order [68].

Let $\omega^0 = \bigcup_{x \in X} \omega^0(x)$ (k is omitted, because only one system is studied). Let us identify equivalent points in ω^0. The obtained factor-space is totally disconnected (each point possessing a fundamental system of neighborhoods open and closed simultaneously). Just this space $\omega^0/\sim$ with the order over it can be considered as a system of sources and drains analogous to the system of limit cycles and fixed points of smooth rough two-dimensional dynamical system. The sets $\omega^0(x)$ can change by jump only on the boundaries of the region of attraction of corresponding "drains" (Theorem 4.43). This totally disconnected factor-space $\omega^0/\sim$ is the generalization of the Smale diagrams [68] defined for the Morse-Smale systems onto the whole class of general dynamical systems. The interrelation of six principal kinds of slow relaxations in perturbed system is analogous to their interrelation in smooth rough two-dimensional system described in Theorem 3.12.

Let us enumerate the most important results of the investigations being stated.

(1) It is not always necessary to search for "foreign" reasons of slow relaxations, in the first place one should investigate if there are slow relaxations of dynamical origin in the system.
(2) One of possible reasons of slow relaxations is the existence of bifurcations (explosions) of ω-limit sets. Here, it is necessary to study the dependence $\omega(x,k)$ of limit set both on parameters and initial data. It is violation of the continuity with respect to $(x,k) \in X \times K$ that leads to slow relaxations.
(3) The complicated dynamics can be made "rough" by perturbations. The useful model of perturbations in topological dynamics provide the ε-motions. For $\varepsilon \to 0$ we obtain the rough structure of sources and drains similar to the Morse-Smale systems (with totally disconnected compact instead of finite set of attractors).
(4) The interrelations between the singularities of relaxation times and other peculiarities of dynamics for general dynamical system under small perturbations are the same as for the Morse-Smale systems, and, in particular, the same as for rough two-dimensional systems.
(5) There is a large quantity of different slow relaxations, unreducible to each other, therefore for interpretation of experiment it is important to understand which namely of relaxation times is large.
(6) Slow relaxations in real systems often are "bounded slow", the relaxation time is large (essentially greater than could be expected proceeding from the coefficients of equations and notions about the characteristic times), but nevertheless bounded. When studying such singularities, appears to be useful the following method, ascending to the works of A.A. Andronov: the considered system is included in appropriate family for which slow relaxations are to be studied in the sense accepted in the present work. This study together with the mention of degree of proximity of particular systems to the initial one can give an important information.

1. Bifurcations (Explosions) of ω-limit Sets

Let X be a compact metric space with the metrics ρ, and K be a compact metric space (the space of parameters) with the metrics ρ_K,

$$f : [0, \infty) \times X \times K \to X \tag{1.1}$$

be a continuous mapping for any $t \geq 0$, $k \in K$; let mapping $f(t, \cdot, k) : X \to X$ be homeomorphism of X into subset of X and under every $k \in K$ let these homeomorphisms form monoparametric semigroup:

$$f(0, \cdot, k) = \text{id}, \; f(t, f(t', x, k), k) = f(t + t', x, k) \tag{1.2}$$

for any $t, t' \geq 0, x \in X$.

Below we call the semigroup of mappings $f(t, \cdot, k)$ under fixed k a ***semiflow of homeomorphisms*** (or, for short, semiflow), and the mapping (1.1) a family of semiflows or simply a system (1.1). It is obvious that all results, concerning the system (1.1), are valid also in the case when X is a phase space of dynamical system, i.e. when every semiflow can be prolonged along t to the left onto the whole axis $(-\infty, \infty)$ up to flow (to monoparametric group of homeomorphisms of X onto X).

1.1. **Extension of Semiflows to the Left.** It is clear that for fixed x and k the mapping $f(\cdot, x, k)$: $t \to f(t, x, k)$ can be, generally speaking, defined also for certain negative t, preserving semigroup property (1.2). In fact, for fixed x and k consider the set of all non-negative t for which there is point $q_i \in X$ such that $f(t, q_i, k) = x$. Let us denote the upper bound of this set by $T(x, k)$:

$$T(x, k) = \sup\{t : \exists q_t \in X, \; f(t, q_t, k) = x\}. \tag{1.3}$$

For given t, x, k the point q_t, if it exists, has a single value, since the mapping $f(t, \cdot, k) : X \to X$ is homeomorphism. Introduce the notation $f(-t, x, k) = q_t$. If $f(-t, x, k)$ is determined, then for any τ within $0 \leq \tau \leq t$ the point $f(-\tau, x, k)$ is determined: $f(-\tau, x, k) = f(t - \tau, f(-t, x, k), k)$. Let $T(x, k) < \infty$, $\quad T(x, k) > t_n > 0 \quad (n = 1, 2, \dots)$, $\quad t_n \to T$. Let us choose from the sequence $f(-t_n, x, k)$ a subsequence converging to some $q^* \in X$ and denote it by $\{q_j\}$, and the corresponding times denote by $-t_j$ ($q_j = f(-t_j, x, k)$). Owing to the continuity of f we obtain: $f(t_j, q_j, k) \to f(T(x, k), q^*, k)$, therefore $f(T(x, k), q^*, k) = x$. Thus, $f(-T(x, k), x, k) = q^*$.

So, for fixed x, k the mapping f was determined in interval $[-T(x, k), \infty)$, if $T(x, k)$ is finite, and in $(-\infty, \infty)$ in the opposite case. Let us denote by S the set of all triplets (t, x, k), in which f is now determined. For enlarged mapping f the semigroup property in following form is valid:

Proposition 1.1 (Enlarged semigroup property)**.**

(A) *If* (τ, x, k) *and* $(t, f(\tau, x, k), k) \in S$*, then* $(t + \tau, x, k) \in S$ *and*

$$f(t, f(t, x, k), k) = f(t + \tau, x, k)\,. \tag{1.4}$$

(B) *Simmilarly, if* $(t + \tau, x, k)$ *and* $(\tau, x, k) \in S$*, then* $(t, f(\tau, x, k), k) \in S$ *and (1.4) holds.*

Thus, if the left part of the equality (1.4) makes sense, then its right part is also determined and the equation is valid. If there are determined both the right part and $f(\tau, x, k)$ in the left part, then the whole left part makes sense and (1.4) hold.

Proof. We consider several possible cases. Since the parameter k is fixed, for short notation, it is omitted in the formulas.
Case (1)

$$f(t, f(-\tau, x)) = f(t - \tau, x) \quad (t, \tau > 0); \quad \text{case (a) } t > \tau > 0.$$

Let the left part make sense, i.e., $f(-\tau, x)$ is determined. Then, taking into account that $t - \tau > 0$, we have

$$f(t, f(-\tau, x)) = f(t - \tau + \tau, f(-\tau, x)) = f(t - \tau, f(\tau, f(-\tau, x))) = f(t - \tau, x),$$

since $f(\tau, f(-\tau, x)) = x$ by definition. Therefore, the equality is true (the right part makes sense since $t > \tau$)- the part for the case 1a is proved. Similarly, if $f(-\tau, x)$ is determined, then the whole left part ($t > 0$) makes sense, and then according to the proved the equality is true. The other cases are considered in analogous way. □

Proposition 1.2. *The set* S *is closed in* $(-\infty, \infty) \times X \times K$ *and the mapping* $f : S \to X$ *is continuous.*

Proof. Denote by $\langle -T(x, k), \infty)$ the interval $[-T(x, k), \infty)$, if $T(x, k)$ is finite, and the whole axis $(-\infty, \infty)$ in opposite case. Let $t_n \to t^*,\ x_n \to x^*,\ k_n \to k^*$, and $t_n \in \langle -T(x_n, k_n), \infty)$. To prove the proposition, it should be made certain that $t^* \in \langle -T(x^*, k^*), \infty)$ and $f(t_n, x_n, k_n) \to f(t^*, x^*, k^*)$. If $t^* > 0$, this follows from the continuity of f in $[0, \infty) \times X \times K$. Let $t^* \le 0$. Then it can be supposed that $t_n < 0$. Let us re-denote by changing the signs t_n by $-t_n$ and t^* by $-t^*$. Let us choose from the sequence $f(-t_n, x_n, k_n)$ using the compactness of X a subsequence converging to some $q^* \in X$. let us denote it by q_j, and the sequences of corresponding t_n, x_n and k_n denote by t_j, x_j and k_j. The sequence $f(t_j, q_j, k_j)$ converges to $f(t^*, q^*, k^*)$ ($t_j > 0,\ t^* > 0$). But $f(t_j, q_j, k_j) = x_j \to x^*$. That is why $f(t^*, q^*, k^*) = x^*$ and $f(-t^*, x^*, k^*) = q^*$ is determined. Since q^* is an arbitrary limit point of $\{q_n\}$, and the point $f(-t^*, x^*, k^*)$, if it exists, is determined by given t^*, x^*, k^* and has a single value, the sequence q_n converges to q^*. The proposition is proved. □

Later on we shall call the mapping $f(\cdot, x, k) : \langle -T(x, k), \omega) \to X$ k-*motion* of the point x ((k, x)-motion), the image of (k, x)-motion – k-*trajectory* of the point x ((k, x)-trajectory), the image of the interval $\langle -T(x, k), 0)$ a negative, and the image of $0, \infty)$ a positive k-*semitrajectory* of the point x ((k, x)-semitrajectory). If $T(x, k) = \infty$, then let us call the k-motion of the point x the *whole* k-motion, and the corresponding k-trajectory the whole k-trajectory.

Let $(x_n, k_n) \to (x^*, k^*),\ t_n \to t^*,\ t_n, t^* > 0$ and for any n the (k_n, x_n)-motion be determined in the interval $[-t_n, \infty)$, i.e. $[-t_n, \infty) \subset \langle -T(x_n, k_n), \infty)$. Then (k^*, x^*)-motion is determined in $[-t^*, \infty]$. In particular, if all (k_n, x_n)-motions are determined in $[-\bar{t}, \infty)$ ($\bar{t} > 0$), then (k^*, x^*)-motion is determined in too. If $t_n \to \infty$ and (k_n, x_n)-motion is determined in $[-t_n, \infty)$, then (k^*, x^*)-motion is determined in $(-\infty, \infty)$ and is a whole motion. In particular, if all the (k_n, x_n)-motions are whole, then (k^*, x^*)-motion is whole too. All this is a direct consequence of the closure of the set S, i.e. of the domain of definition of extended mapping f. It should be noted that from $(x_n, k_n) \to (x^*, k^*)$ and $[-t^*, \infty) \subset \langle -T(x^*, k^*), \infty)$ does not follow that for any $\varepsilon > 0$ $[-t^* + \varepsilon, \infty) \subset \langle -T(x_n, k_n), \infty)$ for n large enough.

Let us note an important property of uniform convergence in compact intervals. Let $(x_n, k_n) \to (x^*, k^*)$ and all (k_n, x_n)-motions and correspondingly (k^*, x^*)-motion be determined in compact interval $[a, b]$. Then (k_n, x_n)-motions converge uniformly in $[a, b]$ to (k^*, x^*)-motion: $f(t, x_n, k_n) \rightrightarrows f(t, x^*, k^*)$. This is a direct consequence of continuity of the mapping $f : S \to X$

1.2. **Limit Sets.**

Definition 1.3. Point $p \in X$ is called ω- (α-)-*limit point* of the (k, x)-motion (correspondingly of the whole (k, x)-motion), if there is such sequence $t_n \to \infty$ ($t_n \to -\infty$) that $f(t_n, x, k) \to p$ as $n \to \infty$. The totality of all ω- (α-)-limit points of (k, x)-motion is called its ω- (α-)-*limit set* and is denoted by $\omega(x, k)$ ($\alpha(x, k)$).

Definition 1.4. *A set $W \subset X$ is called k-invariant set, if for any $x \in W$ the (k, x)-motion is whole and the whole (k, x)-trajectory belongs W. In similar way, let us call a set $V \subset X$ $(k, +)$-invariant ((k,positively)-invariant), if for any $x \in V$, $t > 0$, $f(t, x, k) \in V$.*

Proposition 1.5. *The sets $\omega(x, k)$ and $\alpha(x, k)$ are k-invariant.*

Proof. Let $p \in \omega(x, k)$, $t_n \to \infty$, $x_n = f(t_n, x, k) \to p$. Note that (k, x_n)-motion is determined at least in $[-t_n, \infty)$. Therefore, as it was noted above, (k, p)-motion is determined in $(-\infty, \infty)$, i.e. it is whole. Let us show that the whole (k, p)-trajectory consists of ω-limit points of (k, x)-motion. Let $f(\bar{t}, p, k)$ be an arbitrary point of (k, p)-trajectory. Since $t \to \infty$, from some $n$is determined a sequence $f(\bar{t} + t_n, x, k))$. It converges to $f(\bar{t}, p, k)$, since $f(\bar{t} + t_n, x, k) = f(\bar{t}, f(t_n, x, k), k)$ (according to Proposition 1.1), $f(t_n, x, k) \to p$ and $f : S \to X$ is continuous (Proposition 1.2).

Now, let $q \in \alpha(x, k)$, $t_n \to -\infty$ and $x_n = f(t_n, x, k) \to q$. Since (according to the definition of α-limit points) (k, x)-motion is whole, then all (k, x_n)-motions are whole too. Therefore, as it was noted, (k, q)-motion is whole. Let us show that every point $f(\bar{t}, q, k)$ of (k, q)-trajectory is α-limit for (k, x)-motion. Since (k, x)-motion is whole, then the semigroup property and continuity of f in S give

$$f(\bar{t} + t_n, x, k) = f(\bar{t}, f(t_n, x, k), k) \to f(\bar{t}, q, k),$$

and since $\bar{t} + t_n \to -\infty$, then $f(\bar{t}, q, k)$ is α-limit point of (k, x)-motion. Proposition 1.5 is proved. □

Further we need also the complete ω-limit set $\omega(k) : \omega(k) = \bigcup_{x \in X} \omega(x, k)$. The set $\omega(k)$ is k-invariant, since it is the union of k-invariant sets.

Proposition 1.6. *The sets $\omega(x, k)$, $\alpha(x, k)$ (the last in the case when (k, x)-motion is whole) are nonempty, closed and connected.*

The proof practically coincides with the proof of similar statements for usual dynamical systems ([56, p.356-362]). The set $\omega(k)$ might not be closed.

Example 1.7 (The set $\omega(k)$ might not be closed)**.** Let us consider the system given by the equations $\dot{x} = y(x - 1)$, $\dot{y} = -x(x - 1)$ in the circle $x^2 + y^2 \le 1$ on the plane.

The complete ω-limit set is $\omega = \{(1, 0)\} \bigcup \{(x, y) : x^2 + y^2 < 1\}$. It is unclosed. The closure of ω coincides with the whole circle ($x^2 + y^2 \le 1$), the boundary of ω consists of two trajectories: of the fixed point $(1, 0) \in \omega$ and of the loop $\{(x, y) : x^2 + y^2 = 1,\ x \neq 1\} \not\subseteq \omega$

Proposition 1.8. *The sets $\partial\omega(k)$, $\partial\omega(k) \setminus \omega(k)$ and $\partial\omega(k) \bigcap \omega(k)$ are $(k, +)$-invariant. Furthermore, if $\partial\omega(k) \setminus \omega(k) \neq \varnothing$, then $\partial\omega(k) \bigcap \omega(k) \neq \varnothing$ ($\partial\omega(k) = \overline{\omega(k)} \setminus \mathrm{int}\omega(k)$ is the boundary of the set $\omega(k)$).*

Let us note that for the propositions 1.6 and 1.8 to be true, the compactness of X is important, because for non-compact spaces analogous propositions are incorrect, generally speaking.

To study slow relaxations, we need also sets that consist of ω-limit sets $\omega(x,k)$ as of elements (the sets of ω-limit sets):

$$\begin{gathered}\Omega(x,k)=\{\omega(x',k):\omega(x',k)\subset\omega(x,k),\ x'\in X\};\\ \Omega(k)=\{\omega(x,k):x\in X\},\end{gathered}\tag{1.5}$$

where $\Omega(x,k)$ is the set of all ω-limit sets, lying in $\omega(x,k)$, $\Omega(k)$ is the set of ω-limit sets of all k-motions.

1.3. **Convergence in the Spaces of Sets.** Further we consider the connection between slow relaxations and violations of continuity of the dependencies $\omega(x,k)$, $\omega(k)$, $\Omega(x,k)$, $\Omega(k)$. Let us introduce convergences in spaces of sets and investigate the mappings continuous with respect to them. One notion of continuity, used by us, is well known (see [48, Sec. 18] and [49, Sec. 43] lower semicontinuity). Two other ones are some more "exotic". In order to reveal the resemblance and distinctions between these convergences, let us consider them simultaneously (all the statements, concerning lower semicontinuity, are variations of known ones, see [48, 49]).

Let us denote the set of all nonempty subsets of X by $B(X)$, and the set of all nonempty subsets of $B(X)$ by $B(B(X))$.

Let us introduce in $B(X)$ the following proximity measures: let $p,q\in B(X)$, then

$$d(p,q)=\sup_{x\in p}\inf_{y\in q}\rho(x,y);\tag{1.6}$$

$$r(p,q)=\inf_{x\in p,y\in q}\rho(x,y).\tag{1.7}$$

The "distance" $d(p,q)$ represents "a half" of the known Hausdorff metrics ([49, p.223]):

$$\operatorname{dist}(p,q)=\max\{d(p,q),\ d(q,p)\}.\tag{1.8}$$

It should be noted that, in general, $d(p,q)\neq d(q,p)$. Let us determine in $B(X)$ converges using the introduced proximity measures. Let q_n be a sequence of points of $B(X)$. We say that q_n d-converges to $p\in B(X)$, if $d(p,q_n)\to 0$. Analogously, q_n r-converges to $p\in B(X)$, if $r(p,q_n)\to 0$. Let us notice that d-convergence defines topology in $B(X)$ with a countable base in every point and the continuity with respect to this topology is equivalent to d-continuity (λ-topology [48, p.183]). As a basis of neighborhoods of the point $p\in B(X)$ in this topology can be taken, for example, the family of sets $\{q\in B(X):d(p,q)<1/n\ (n=1,2,\dots)\}$. The topology conditions can be easily verified, since the triangle inequality

$$d(p,s)\le d(p,q)+d(q,s)\tag{1.9}$$

is true (in regard to these conditions see, for example, [19, p.19-20]), r-convergence does not determine topology in $B(X)$. To prove this, let us use the following obvious property of convergence in topological spaces: if $p_i\equiv p$, $q_i\equiv q$ and $s_i\equiv s$ are constant sequences of the points of topological space and $p_i\to q$, $q_i\to s$, then $p_i\to s$. This property is not valid for r-convergence. To construct an example, it is enough to take two points $x,y\in X$ $(x\neq y)$ and to make $p=\{x\}$, $q=\{x,y\}$, $s=\{y\}$. Then $r(p,q)=r(q,s)=0$, $r(p,s)=\rho(x,y)>0$. Therefore

$p_i \to q,\ q_i \to s,\ p_i \not\to s$, and r-convergence does not determine topology for any metric space $X \neq \{x\}$.

Introduce also a proximity measure in $B(B(X))$ (that is the set of nonempty subsets of $B(X)$): let $P, Q \in B(B(X))$, then

$$D(P,Q) = \sup_{p\in P} \inf_{q\in Q} r(p,q). \tag{1.10}$$

Note that the formula (1.10) is similar to the formula (1.6), but in (1.10) appears $r(p,q)$ instead of $\rho(x,y)$. The expression (1.10) can be somewhat simplified by introducing the following denotations. Let $Q \in B(B(X))$. Let us define $SQ = \bigcup_{q\in Q} q, SQ \in B(X)$; then

$$D(P,Q) = \sup_{p\in P} r(p, SQ). \tag{1.11}$$

Let us introduce convergence in $B(B(X))$ (D-convergence):

$$Q_n \to P, \text{ if } D(P,Q_n) \to 0.$$

D-convergence, as well as r-convergence, does not determine topology. This can be illustrated in the way similar to that used for r-convergence. Let $x, y \in X$, $x \neq y$, $P = \{\{x\}\}$, $Q = \{\{x,y\}\}$, $R = \{\{y\}\}$, $P_i = P$, $Q_i = Q$. Then $D(Q,P) = D(R,Q) = 0$, $P_i \to Q$, $Q_i \to R$, $D(R,P) = \rho(x,y) > 0$, $P_i \not\to R$.

Later we need the following criteria of convergence of sequences in $B(X)$ and in $B(B(X))$.

Proposition 1.9 ([48]). *The sequence of sets $q_n \in B(X)$ d-converges to $p \in B(X)$ if and only if $\inf_{y\in q_n} \rho(x,y) \to 0$ as $n \to \infty$ for any $x \in p$.*

Proposition 1.10. *The sequence of sets $q_n \in B(X)$ r-converges to $p \in B(X)$ if and only if there are such $x_n \in p$ and $y_n \in q_n$ that $\rho(x_n, y_n) \to 0$ as $n \to \infty$.*

This follows immediately from the definition of r-proximity. Before treating the criterion of D-convergence, let us prove the following topological lemma.

Lemma 1.11. *Let p_n, q_n $(n = 1, 2, \dots)$ be subsets of compact metric space X and $r(p_n, q_n) > \varepsilon > 0$ for any n. Then there are such $\gamma > 0$ and an infinite set of indices J that $r(p_N, q_n) > \gamma$ for $n \in J$ and for some number N.*

Proof. Choose in X $\varepsilon/5$-network M; let to each $q \subset X$ correspond $q^M \subset M$:

$$q^M = \big\{m \in M \mid \inf_{x\in q} \rho(x,m) \leq \varepsilon/5\big\}. \tag{1.12}$$

For any two sets $p, q \subset X$ $r(p^M, q^M) + \frac{2}{5}\varepsilon \geq r(p,q)$. Therefore $r(p_n^M, q_n^M) > 3\varepsilon/5$. Since the number of different pairs p^M, q^M is finite (M is finite), there exists an infinite set J of indices n, for which the pairs p_n^M, q_n^M coincide: $p_n^M = p^M$, $q_n^M = q^M$ as $n \in J$. For any two indices $n, l \in J$ $r(p_n^M, q_l^M) = r(p^M, q^M) > 3\varepsilon/5$, therefore $r(p_n, q_l) > \varepsilon/5$, and this fact completes the proof of the lemma. It was proved more important statement really: there exists such infinite set J of indices that for any $n, l \in J$ $r(p_n, q_l) > \gamma$ (and not only for one N). □

Proposition 1.12. *The sequence of sets $Q_n \in B(B(X))$ D-converges to $p \in B(X))$ if and only if $\inf_{q\in Q} r(p,q) \to 0$ for any $p \in P$.*

Proof. In one direction this is obvious: if $Q_n \to P$, then according to definition $D(P, Q_n) \to 0$, i.e. the upper bound by $p \in P$ of the value $\inf_{q \in Q_n} r(p, q)$ tends to zero and all the more for any $p \in P$ $\inf_{q \in Q} r(p, q) \to 0$. Now, suppose that for any $p \in P$ $\inf_{q \in Q_n} r(p, q) \to 0$. If $D(P, Q_n) \not\to 0$, then one can consider that $D(P, Q_n) > \varepsilon > 0$. Therefore (because of (1.11)) there are such $p_n \in P$ for which $r(p_n, SQ_n) > \varepsilon\big(SQ_n = \bigcup_{q \in Q_n} q\big)$. Using Lemma 1.11, we conclude that for some N $r(p_N, SQ_n) > \gamma > 0$, i.e. $\inf_{q \in Q_n} \ r(p_N, q) \not\to 0$. The obtained contradiction proves the second part of Proposition 1.12. □

For the rest of this monograph, if not stated otherwise, the convergence in $B(X)$ implies d-convergence, and the convergence in $B(B(X))$ implies D-convergence, and as continuous are considered the functions with respect to these convergences.

1.4. **Bifurcations of ω-limit Sets.**

Definition 1.13. We say that the system (1.1) possesses:

(A) $\omega(x, k)$-*bifurcations*, if $\omega(x, k)$ is not continuous function in $X \times K$;
(B) $\omega(k)$-*bifurcations*, if $\omega(k)$ is not continuous function in K;
(C) $\Omega(x, k)$-*bifurcations*, if $\Omega(x, k)$ is not continuous function in $X \times K$;
(D) $\Omega(k)$-*bifurcations*, if $\Omega(k)$ is not continuous function in K.

The points of $X \times K$ or K, in which the functions $\omega(x, k)$, $\omega(k)$, $\Omega(x, k)$, $\Omega(k)$ are not d- or not D-continuous, we call *the points of bifurcation*. The considered discontinuities in the dependencies $\omega(x, k)$, $\omega(k)$, $\Omega(x, k)$, $\Omega(k)$ could be also called *"explosions"* of ω-limit sets (compare with the explosion of the set of non-wandering points in differential dynamics ([57], Sec. 6.3., p.185-192, which, however, is a violation of semidiscontinuity from above).

Proposition 1.14. (A) *If the system (1.1) possesses $\Omega(k)$-bifurcations, then it possesses $\Omega(x, k)$-, $\omega(x, k)$- and $\omega(x, k)$-bifurcations.*
(B) *If the system (1.1) possesses $\Omega(x, k)$-bifurcations, then it possesses $\omega(x, k)$-bifurcations.*
(C) *If the system (1.1) possesses $\omega(k)$-bifurcations, then it possesses $\omega(x, k)$-bifurcations.*

It is convenient to illustrate Proposition 1.14 by the scheme (the word "bifurcation" is omitted):

$$\begin{array}{ccccc} & \swarrow & \Omega(k) & \searrow & \\ \Omega(x,k) & & & & \omega(k) \\ & \searrow & \omega(x,k) & \swarrow & \end{array} \tag{1.13}$$

Proof. Let us begin from item C. Let the system (1.1) (family of semiflows) possess $\omega(k)$-bifurcations. This means that there are such $k^* \in K$ (point of bifurcation), $\varepsilon > 0$, $x^* \in \omega(k^*)$ and sequence $k_n \in K$, $k_n \to k^*$, for which $\inf_{y \in \omega(x_0, k_n)} \rho(x^*, y) > \varepsilon$ for any n (according to Proposition 1.9). The point x^* belongs to some $\omega(x_0, k^*)$ $(x_0 \in X)$. Note that $\omega(x_0, k_n) \subset \omega(k_n)$, consequently, $\inf_{y \in \omega(k_n)} \rho(x^*, y) > \varepsilon$, therefore the sequence $\omega(x_0, k_n)$ does not converge to $\omega(x_0, k^*)$: there exist $\omega(x, k)$-bifurcations, and the point of bifurcation is (x_0, k^*).

Prove the statement in item B. Let the system (1.1) possess $\Omega(x, k)$-bifurcations. Then, (according to Proposition 1.12) there are such $(x^*, k^*) \in X \times K$ (the point of bifurcation), $\omega(x_0, k^*) \subset \omega(x^*, k^*)$ and sequence $(x_n, k_n) \to (x^*, k^*)$ that

$$r(\omega(x_0, k^*), S\,\Omega(x_n, k_n)) > \varepsilon > 0 \quad \text{for any } n.$$

But the above statement implies $r(\omega(s_0,k^*),\omega(x_n,k_n)) > \varepsilon > 0$ and, consequently,

$$\inf_{y\in\omega(x_n,k_n)} \rho(\xi,y) > \varepsilon \quad \text{for any } \xi \in \omega(x^0,k^*).$$

Since $\xi \in \omega(x^*,k^*)$, the existence of $\omega(x,k)$-bifurcations follows and (x^*,k^*) is the point of bifurcation.

Statement in item A. Let the system (1.1) possess $\Omega(k)$-bifurcations. Then there are $k^* \in K$ (the point of bifurcation), $\varepsilon > 0$ and sequence of points $k_n, k_n \to k^*$, for which $D(\Omega(k^*),\Omega(k_n)) > \varepsilon$ for any n, that is for any n there is such $x_n \in X$ that $r(\omega(x_n,k^*),\omega(k_n)) > \varepsilon$ (according to (1.11)). By Lemma 1.11 there are such $\gamma > 0$ and a natural N that for infinite set J of indices $r(\omega(x_N,k^*),\omega(k_n)) > \gamma$ for $n \in J$. Furthermore, $r(\omega(x_N,k^*),\omega(x_N,k_n)) > \gamma$ $(n \in J)$, consequently, there are $\Omega(x,k)$-bifurcations:

$$\begin{aligned} &(x_N,k_n) \to (x_N,k^*) \quad \text{as } n \to \infty,\ n \in J; \\ D(\Omega(x_N,k^*),\Omega(x_N,k_n)) &= \sup_{\omega(x,k^*)\subset\Omega(x_N,k^*)} r(\omega(x,k^*),\omega(x_N,k_n)) \\ &\geq r(\omega(x_N,k^*),\omega(x_N,k_n)) > \gamma. \end{aligned}$$

Therefore, the point of bifurcation is (x_N,k^*).

We need to show only that if there are $\Omega(k)$-bifurcations, then $\omega(k)$-bifurcations exist. Let us prove this. Let the system (1.1) possess $\Omega(k)$-bifurcations. Then, as it was shown just above, there are such $k^* \in K$, $x^* \in X$, $\gamma > 0$ $(x^* = x_N)$ and a sequence of points $k_n \in K$ that $k_n \to k^*$ and $r(\omega(x^*,k^*),\omega(k_n)) > \gamma$. Furthermore, for any $\xi \in \omega(x^*,k^*)$, $\inf_{y\in\omega(k_n)} \rho(\xi,y) > \gamma$; therefore $d(\omega(k^*),\omega(k_n)) > \gamma$ and there are $\omega(k)$-bifurcations (k^* is the point of bifurcation). Proposition 1.14 is proved. ☐

Proposition 1.15. *The system* (1.1) *possesses* $\Omega(x,k)$*-bifurcations if and only if* $\omega(x,k)$ *is not* r*-continuous function in* $X \times K$.

Proof. Let the system (1.1) possess $\Omega(x,k)$-bifurcations, then there are $(x^*,k^*) \in X \times K$, the sequence $(x_n,k_n) \in X \times K, (x_n,k_n) \to (x^*,k^*)$ for which for any n,

$$D(\Omega(x^*,k^*),\Omega(x_n,k_n)) > \varepsilon > 0.$$

The last means that for any n there is $x_n^* \in X$ for which $\omega(x_n^*,k^*) \subset \omega(x^*,k^*)$, and $r(\omega(x_n^*,k^*),\omega(x_n,k_n)) > \varepsilon$. ¿From Lemma 1.11 follows the existence of such $\gamma > 0$ and natural N that for infinite set J of indices $r(\omega(x_N^*,k^*),\omega(x_n,k_n)) > \gamma$ as $n \in J$. Let x_0^* be an arbitrary point of $\omega(x_N^*,k^*)$. As it was noted already, (k^*,x_0^*)-trajectory lies in $\omega(x_N^* < k^*)$ and because of the closure of the last $\omega(x_0^*,k^*) \subset \omega(x_N^*,k^*)$. Therefore, $r(\omega(x_n,k_n)) > \gamma$ as $n \in J$. As $x_0^* \in \omega(x^*,k^*)$, there is such sequence $t_i \to \infty, t_i > 0$, that $f(t_i,x^*,k^*) \to x_0^*$ as $i \to \infty$. Using the continuity of f, choose for every i such $n(i) \in J$ that $\rho(f(t_i,x^*,k^*),f(t_i,x_{n(i)},k_{n(i)})) < 1/i$. Denote $f(t_i,x_{n(i)},k_{n(i)}) = x'_i,\ k_{n(i)} = k'_i$. Note that $\omega(x'_i,k'_i) = \omega(x_{n(i)},k_{n(i)})$. Therefore, for any i $r(\omega(x_0^*,k^*),\omega(x'_i,k'_i)) > \gamma$. Since $(x'_i,k'_i) \to (x_0^*,k^*)$, we conclude that $\omega(x,k)$ is not r-continuous function in $X \times K$.

Let us emphasize that the point of $\Omega(x,k)$-bifurcations can be not the point of r-discontinuity.

Now, suppose that $\omega(x,k)$ is not r-continuous in $X \times K$. Then there exist $(x^*,k^*) \in X \times K$, sequence of points $(x_n,k_n) \in X \times K$, $(x_n,k_n) \to (x^*,k^*)$, and $\varepsilon > 0$, for which $r(\omega(x^*,k^*),\omega(x_n,k_n)) > \varepsilon$ for any n. But, according to (1.11), from this it follows that $D(\Omega(x^*,k^*),\Omega(x_n,k_n)) > \varepsilon$ for any n. Therefore, (x^*,k^*) is the point of $\Omega(x,k)$-bifurcation. Proposition 1.15 is proved. □

The $\omega(k)$- and $\omega(x,k)$-bifurcations can be called bifurcations with appearance of new ω-limit points, and $\Omega(k)$- and $\Omega(x,k)$-bifurcations with appearance of ω-limit sets. In the first case there is such sequence of points k_n (or (x_n,k_n)), converging to the point of bifurcation k^* (or (x^*,k^*)) that there is such point $x_0 \in \omega(k^*)$ (or $x_0 \in \omega(x^*,k^*)$) which is removed away from all $\omega(k_n)(\omega(x_n,k_n))$ other than some $\varepsilon > 0$. It could be called the "new" ω-limit point. In the second case, as it was shown, the existence of bifurcations is equivalent to existence of a sequence of the points k_n (or $(x_n,k_n) \in X \times K$), converging to the point of bifurcation k^* (or (x^*,k^*)), together with existence of some set $\omega(x_0,k^*) \subset \omega(k^*)$ $(\omega(x_0,k^*) \subset \omega(x^*,k^*))$, being r-removed from all $\omega(k_n)$ $(\omega(x_n,k_n))$ other than $\gamma > 0$: $\rho(x,y) > \gamma$ for any $x \in \omega(x_0,k^*)$ and $y \in \omega(k_n)$. It is natural to call the set $\omega(x_0,k^*)$ the "new" ω-limit set. A question arises: are there bifurcations with appearance of new ω-limit points, but without appearance of new ω-limit sets? The following example gives positive answer to this question.

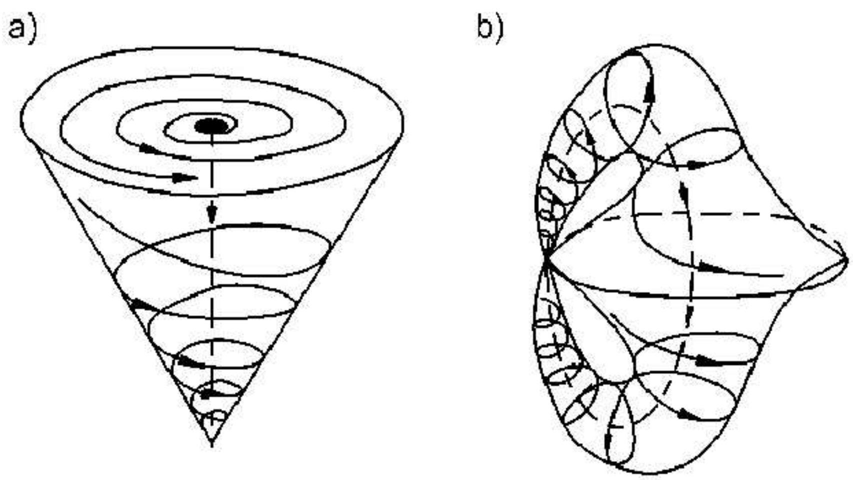

FIGURE 1. $\omega(x,k)$-, but not $\Omega(x,k)$-bifurcations: *a* - phase portrait of the system (1.14); *b* - the same portrait after gluing all fixed points.

Example 1.16. ($\omega(x,k)$-, but not $\Omega(x,k)$-bifurcations). Consider at first the system, given in the cone $x^2+y^2 \le z^2$, $0 \le z \le 1$ by differential equations (in cylindrical coordinates)

$$\begin{aligned} \dot r &= r(2z-r-1)^2 - 2r(1-r)(1-z);\\ \dot\varphi &= r\cos\varphi + 1;\\ \dot z &= -z(1-z)^2. \end{aligned} \tag{1.14}$$

The solutions of (1.14) under initial conditions $0 \le z(0) \le 1$, $0 \le r(0) \le z(0)$ and arbitrary φ tend to their unique ω-limit point as $t \to \infty$ (this point is the equilibrium $z = r = 0$). If $0 < r(0) < 1$, then the solution tends to the circumference $z = r = 1$

as $t \to \infty$. If $z(0) = 1$, $r(0) = 0$, then the ω-limit point is unique: $z = 1$, $r = 0$. If $z(0) = r(0) = 1$, then the ω-limit point is also unique: $z = r = 1$, $\varphi = \pi$ (Fig. 1). Thus,

$$\omega(r_0, \varphi_0, z_0) = \begin{cases} (z = r = 0), & \text{if } z_0 < 1; \\ \{(r, \varphi, z) : r = z = 1\}, & \text{if } z_0 = 1,\ r_0 \neq 0, 1; \\ (z = r = 1),\ \varphi = \pi, & \text{if } z_0 = r_0 = 1; \\ (r = 0, z = 1), & \text{if } z_0 = 1,\ r_0 = 0\,. \end{cases}$$

Consider the sequence of points of the cone $(r_n, \varphi_n, z_n) \to (r^*, \varphi^*, 1)$, $r^* \neq 0, 1$ and $z_n < 1$ for all n. For any point of the sequence the ω-limit set includes one point, and for $(r^*, \varphi, 1)$ the set includes the circumference. If all the positions of equilibrium were identified, then there would be $\omega(x, k)$-, but not $\omega(x, k)$-bifurcations.

The correctness of the identification procedure should be guaranteed. Let the studied semiflow f have fixed points $x_i, \dots, x_n$. Define a new semiflow $\tilde{f}$ as follows:

$$\tilde{X} = X \setminus \{x_i, \dots, x_n\} \cup \{x^*\}$$

is a space obtained from X when the points $x_i, \dots, x_n$ are deleted and a new point x^* is added. Let us give metrics over $\tilde{X}$ as follows: Let $x, y \in \tilde{X}$, $x \neq x^*$,

$$\tilde{\rho}(x, y) = \begin{cases} \min\{\rho(x, y), \min_{1 \le j \le n} \rho(x, x_j) + \min_{1 \le j \le n} \rho(y, x_j)\}, & \text{if } y \neq x^*; \\ \min_{1 \le j \le n} \rho(x, x_j), & \text{if } y = x^*. \end{cases}$$

Let $\tilde{f}(t, x) = f(t, x)$ if $x \in X \bigcap \tilde{X}$, $\tilde{f}(t, x^*) = x^*$.

Lemma 1.17. *The mapping $\tilde{f}$ determines a semiflow in $\tilde{X}$.*

Proof. Injectivity and semigroup property are obvious from the corresponding properties of f. If $x \in X \bigcap \tilde{X}$, $t \ge 0$ then the continuity of $\tilde{f}$ in the point (t, x) follows from the fact that $\tilde{f}$ coincides with f in some neighbourhood of this point. The continuity of $\tilde{f}$ in the point (t, x^*) follows from the continuity of f and the fact that any sequence converging in $\tilde{X}$ to x^* can be divided into finite number of sequences, each of them being either (a) a sequence of points $X \bigcap \tilde{X}$, converging to one of x_j or (b) a constant sequence, all elements of which are x^* and some more, maybe, a finite set. Mapping $\tilde{f}$ is a homeomorphism, since it is continuous and injective, and $\tilde{X}$ is compact. □

Proposition 1.18. *Let each trajectory lying in $\omega(k)$ be recurrent for any k. Then the existence of $\omega(x, k)$- ($\omega(k)$-)-bifurcations is equivalent to the existence of $\Omega(x, k)$- ($\Omega(k)$-)-bifurcations. More precisely:*
(A) if $(x_n, k_n) \to (x^*, k^*)$ and $\omega(x_n, k_n) \not\to \omega(x^*, k^*)$, then $\Omega(x_n, k_n) \not\to \Omega(x^*, k^*)$.
(B) if $k_n \to k^*$ and $\omega(k_n) \not\to \omega(k^*)$, then $\Omega(k_n) \not\to \Omega(k^*)$.
(Let us recall that the convergence in $B(X)$ implies d-convergence, and the convergence in $B(B(X))$ implies D-convergence, and continuity is considered as continuity with respect to these convergence, if there are no other mentions.)

Proof. (A) Let $(x_n, k_n) \to (x^*, k^*)$, $\omega(x_n, k_n) \not\to \omega(x^*, k^*)$. Then, according to Proposition 1.9, there exists $\tilde{x} \in (x^*, k^*)$ such that $\inf_{y \in \omega(x_n, k_n)} \rho(\tilde{x}, y) \not\to 0$. Therefore, from $\{(x_n, k_n)\}$ we can choose a subsequence (denoted as $\{(x_m, k_m)\}$) for which there exists such $\varepsilon > 0$ that $\inf_{y \in \omega(x_m, k_m)} \rho(\tilde{x}, y) > \varepsilon$ for any $m = 1, 2 < \dots$. Denote by L the set of all limit points of sequences of the kind $\{y_m\}$, $y_m \in$

$\omega(x_m, k_m)$. The set L is closed and k^*-invariant. Note that $\rho^*(\tilde{x}, L) \geq \varepsilon$. Therefore, $\omega(\tilde{x}, k^*) \bigcap L = \varnothing$ as $\omega(\tilde{x}, k^*)$ is a minimal set (Birkhoff's theorem, see [56, p.404]). From this follows the existence of such $\delta > 0$ that $r(\omega(\tilde{x}, k^*), L) > \delta$ and from some M $r(\omega(\tilde{x}, k^*),\ (x_m, k_m)) > \delta/2$ (when $m > M$). Therefore, (Proposition 1.12) $\Omega(x_m, k_m) \not\to \Omega(x^*, k^*)$.
(B) The proof practically coincides with that for the part A (it should be substituted $\omega(k)$ for $\omega(x, k)$). □

Corollary 1.19. *For every pair* $(x, k) \in X \times K$ *let the* ω*-limit set be minimal:* $\Omega(x, k) = \{\omega(x, k)\}$*. Then statements A, B of Proposition 1.18 hold.*

Proof. According to one of Birkhoff's theorems [56, p.402], each trajectory lying in minimal set is recurrent. Therefore, Proposition 1.18 is applicable. □

2. Slow Relaxations

2.1. **Relaxation Times.** The principal object of our consideration is *the relaxation time*. The system of the relaxation times is defined in Introduction.

Proposition 2.1. *For any* $x \in X$, $k \in K$ *and* $\varepsilon > 0$ *the numbers* $\tau_i(x, k, \varepsilon)$ *and* $\eta_i(x, k, \varepsilon)$ $(i = 1, 2, 3)$ *are defined, and the inequalities* $\tau_i \geq \eta_i$, $\tau_1 \leq \tau_2 \leq \tau_3$, $\eta_1 \leq \eta_2 \leq \eta_3$ *hold.*

Proof. If τ_i, η_i are defined, then the validity of inequalities is obvious ($\omega(x, k) \subset \omega(k)$, the time of the first entry in the ε-neighbourhood of the set of limit points is included into the time of being outside of this neighbourhood, and the last is not larger than the time of final entry in it). The numbers τ_i, η_i are definite (bounded): there are $t_n \in [0, \infty)$, $t_n \to \infty$ and $y \in \omega(x, k)$, for which $f(t_n, x, k) \to y$ and from some n, $\rho(f(t_n, x, k), y) < \varepsilon$; therefore ,the sets $\{t > 0 : \rho^*(f(t, x, k), \omega(x, k)) < \varepsilon\}$ and $\{t > 0 : \rho^*(f(t, x, k), \omega(k)) < \varepsilon\}$ are nonempty. Since X is compact, there is such $t(\varepsilon) > 0$ that for $t > t(\varepsilon)$ $\rho^*(f(t, x, k), \omega(x, k)) < \varepsilon$. In fact, let us suppose the contrary: there are such $t_n > 0$ that $t_n \to \infty$ and $\rho^*(f(t_n, x, k), \omega(x, k)) > \varepsilon$. Let us choose from the sequence $f(t_n, x, k)$ a convergent subsequence and denote its limit x^*; x^* satisfies the definition of ω-limit point of (k, x)-motion, but it lies outside of $\omega(x, k)$. The obtained contradiction proves the required, consequently, τ_3 and η_3 are defined. According to the proved, the sets

$$\{t > 0 : \rho^*(f(t, x, k), \omega(x, k)) \geq \varepsilon\},$$
$$\{t > 0 : \rho^*(f(t, x, k), \omega(k)) \geq \varepsilon\}$$

are bounded. They are measurable because of the continuity with respect to t of the functions $\rho^*(f(t, x, k), \omega(x, k))$ and $\rho^*(f(t, x, k), \omega(k))$. The proposition is proved. □

Note that the existence (finiteness) of $\tau_{2,3}$ and $\eta_{2,3}$ is associated with the compactness of X.

Definition 2.2. We say that the system (1.1) possesses τ_i- (η_i-)*-slow relaxations*, if for some $\varepsilon > 0$ the function $\tau_i(x, k, \varepsilon)$ (respectively $\eta_i(x, k, \varepsilon)$) is not bounded above in $X \times K$.

Proposition 2.3. For any semiflow (k is fixed) the function $\eta_1(x, \varepsilon)$ is bounded in X for every $\varepsilon > 0$.

Proof. Suppose the contrary. Then there is such sequence of points $x_n \in X$ that for some $\varepsilon > 0$ $\eta_1(x_n, \varepsilon) \to \infty$. Using the compactness of X and, if it is needed, choosing a subsequence, assume that $x_n \to x^*$. Let us show that for any $t > 0$ $\rho^*(f(t, x^*), \omega(k)) > \varepsilon/2$. Because of the property of uniform continuity on bounded segments there is such $\delta = \delta(\tau) > 0$ that $\rho(f(t, x^*), f(t, x)) < \varepsilon/2$ if $0 \leq t \leq \tau$ and $\rho(x, x^*) < \delta$. Since $\eta_1(x_n, \varepsilon) \to \infty$ and $x_n \to x^*$, there is such N that $\rho(x_N, x^*) < \delta$ and $\eta_1(x_N, \varepsilon) > \tau$, i.e. $\rho^*(f(t, x_N), \omega(k)) \geq \varepsilon$ under $0 \leq t \leq \tau$. From this we obtain the required: for $0 \leq t \leq \tau$ $\rho^*(f(t, x^*), \omega(k)) > \varepsilon/2$ or $\rho^*(f(t, x^*), \omega(k)) > \varepsilon/2$ for any $t > 0$, since τ was chosen arbitrarily. This contradicts to the finiteness of $\eta_1(x^*, \varepsilon/2)$ (Proposition 2.1). Proposition 2.3 is proved. For $\eta_{2,3}$ and $\tau_{1,2,3}$ does not exist proposition analogous to Proposition 2.3, and slow relaxations are possible for one semiflow too. □

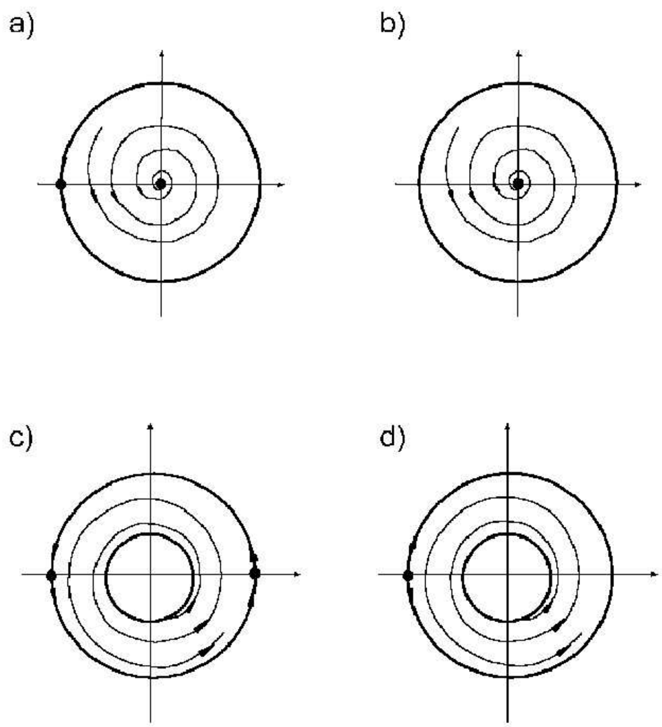

FIGURE 2. Phase portraits of the systems: *a* - (2.1); *b* - (2.2); *c* - (2.3); *d* - (2.3)

Example 2.4 (η_2-slow relaxations for one semiflow)**.** Consider a system on the plane in the circle $x^2 + y^2 \leq 1$ given by the equations in the polar coordinates

$$\begin{aligned} \dot{r} &= -r(1-r)(r \ \cos\varphi + 1); \\ \dot{\varphi} &= r \ \cos\varphi + 1. \end{aligned} \tag{2.1}$$

The complete ω-limit set consists of two fixed points $r = 0$ and $r = 1$, $\varphi = \pi$ (Fig. 2,a), $\eta_2((r, \varphi), 1/2) \to \infty$ as $r \to 1$, $r < 1$.

The following series of simple examples is given to demonstrate the existence of slow relaxations of some kinds without some other kinds.

Example 2.5 (η_3- but not η_2-slow relaxations)**.** Let us modify the previous example, substituting unstable limit cycle for the boundary loop:

$$\begin{aligned} \dot{r} &= -r(1-r); \\ \dot{\varphi} &= 1. \end{aligned} \tag{2.2}$$

Now the complete ω-limit set includes the whole boundary circumference and the point $r = 0$ (Fig. 2,b), the time of the system being outside of its ε-neighborhood is bounded for any $\varepsilon > 0$. Nevertheless, $\eta_3((r,\varphi), 1/2) \to \infty$ as $r \to 1,\ r \neq 1$

Example 2.6 (τ_1, but not $\eta_{2,3}$-slow relaxations)**.** Let us analyze in the ring $\frac{1}{2} \leq x^2 + y^2 \leq 1$ a system given by differential equations in polar coordinates

$$\begin{aligned} \dot{r} &= (1-r)(r\cos\varphi + 1)(1 - r\cos\varphi); \\ \dot{\varphi} &= (r\cos\varphi + 1)(1 - r\cos\varphi). \end{aligned}$$

In this case the complete ω-limit set is the whole boundary circumference $r = 1$ (Fig. 2,c). Under $r = 1,\ \varphi \to \pi,\ \varphi > \pi\ \tau_1(r, \varphi, 1/2) \to \infty$ since for these points $\omega(r,\varphi) = \{(r = 1,\ \varphi = 0)\}$.

Example 2.7 (τ_3, but not $\tau_{1,2}$ and not η_3-slow relaxations)**.** Let us modify the preceding example of the system in the ring, leaving only one equilibrium point on the boundary circumference $r = 1$:

$$\begin{aligned} \dot{r} &= (1-r)(r\cos\varphi + 1); \\ \dot{\varphi} &= r\cos\varphi + 1. \end{aligned} \tag{2.3}$$

In this case under $r = 1,\ \varphi \to \pi,\ \varphi \to \pi\ \tau_3((r,\varphi), 1/2) \to \infty$ and $\tau_{1,2}$ remain bounded for any fixed $\varepsilon > 0$, because for these points $\omega(r,\varphi) = \{(r = 1,\ \varphi = \pi)\}$ (Fig. 2,d). $\eta_{2,3}$ are bounded, since the complete ω-limit set is the circumference $r = 1$.

Example 2.8 (τ_2, but not τ_1 and not η_2-slow relaxations)**.** We could not find a simple example on the plane without using Lemma 1.17. Consider at first a semiflow in the circle $x^2 + y^2 \leq 2$ given by the equations

$$\begin{aligned} \dot{r} &= -r(1-r)^2[(r\cos\varphi + 1)^2 + r^2 \sin\varphi]; \\ \dot{\varphi} &= (r\cos\varphi + 1)^2 + r^2\sin^2\varphi. \end{aligned}$$

the ω-limit sets of this system are as follows (Fig. 3,a):

$$\omega(r_0, \varphi_0) = \begin{cases} \text{circumference } r = 1, & \text{if } r_0 > 1; \\ \text{point } (r = 1,\ \varphi = \pi), & \text{if } r_0 = 1; \\ \text{point } (r = 0), & \text{if } r_0 < 1. \end{cases}$$

Let us identify the fixed points $(r = 1, \varphi = \pi)$ and $(r = 0)$ (Fig. 3,b). We obtain that under $r \to 1,\ r < 1\ \tau_2(r, \varphi, 1/2) \to \infty$, although τ_1 remains bounded as well as η_2. However, η_3 is unbounded.

The majority of the above examples is represented by non-rough systems, and there are serious reasons for this non-roughness. In rough systems on a plane $\tau_{1,2,3}$- and η_3-slow relaxations can occur only simultaneously (see Subsection 3.3).

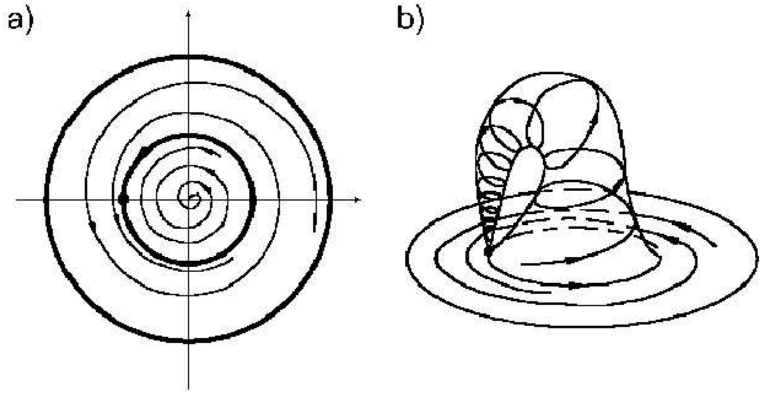

FIGURE 3. Phase portrait of the system (2.4):
a - without gluing the fixed points; *b* - after gluing.

2.2. **Slow Relaxations and Bifurcations of ω-limit Sets.** In the simplest situations the connection between slow relaxations and bifurcations of ω-limit sets is obvious. We should mention the case when the motion tending to its ω-limit set is retarded near unstable equilibrium position. In general case the situation becomes more complicated at least because there are several relaxation times (and consequently several corresponding kinds of slow relaxations). Except that, as it will be shown below, bifurcations are not a single possible reason of slow relaxation appearance. Nevertheless, for the time of the first entering (both for the proper time τ_1 and for the non-proper one η_1) the connection between bifurcations and slow relaxations is manifest.

Theorem 2.9. *The system (1.1) possesses τ_1-slow relaxations if and only if it possesses $\Omega(x,k)$-bifurcations.*

Proof. Let the system possess $\Omega(x,k)$-bifurcations, (x^*,k^*) be the point of bifurcation. This means that there are such $x' \in X, \varepsilon > 0$ and sequence of points $(x_n,k_n) \in X \times K$, for which $\omega(x',k^*) \subset \omega(x^*,k^*)$, $(x_n,k_n) \to (x^*,k^*)$, and $r(\omega(x',k^*),\omega(x_n,k_n)) > \varepsilon$ for any n. Let $x_0 \in \omega(x',k^*)$. Then $\omega(x_0,k^*) \subset \omega(x',k^*)$ and $r(\omega(x_0,k^*),\omega(x_n,k_n)) > \varepsilon$ for any n. Since $x_0 \in \omega(x^*,k^*)$, there is such sequence $t_i > 0,\ t \to \infty$, for which $f(t_i,x^*,\ k^*) \to x_0$. As for every i $f(t_i,x_n,k_n) \to f(t_i,x^*,k^*)$, then there is such sequence $n(i)$ that $f(t_i,x_{n(i)},k_{n(i)}) \to x_0$ as $i \to \infty$. Denote $k_{n(i)}$ as k_i' and $f(t_i,x_{n(i)},k_{n(i)})$ as y_i. It is obvious that $\omega(y,k_i') = \omega(x_{n(i)},k_{n(i)})$. Therefore $r(\omega(x',k^*),\omega(y_i,k_i')) > \varepsilon$.

Let us show that for any $\tau > 0$ there is i such that $\tau_1(y_i,k_i',\varepsilon/2) > \tau$. To do that, use the uniform continuity of f on compact segments and choose $\delta > 0$ such that $\rho(f(t,x_0,k^*),f(t,y_i,k_i')) < \varepsilon/2$ if $0 \le t \le \tau, \rho(x_0,y_i) + \rho_K(k^*,k_i') < \delta$. The last inequality is true for some i_0 (when $i > i_0$), since $y_i \to x_0$ and $k_i' \to k^*$. For any $t \in (-\infty,\infty)$, $f(t,x_0,k^*) \in \omega(x',k^*)$, consequently, $\rho^*(f(t,y_i,k_i'),\omega(y_i,k_i')) > \varepsilon/2$ for $i > i_0$, $0 \le t \le \tau$; therefore, for these i $\tau_1(y_i,k_i',\varepsilon/2) > \tau$. The existence of τ_1-slow relaxations is proved. □

Now, let us suppose that there are τ_1-slow relaxations: There can be found a sequence $(x_n,k_n) \in X \times K$ such that for some $\varepsilon > 0$, $\tau_1(x_n,k_n,\varepsilon) \to \infty$. Using the compactness of $X \times K$, choose from this sequence a convergent one, preserving the denotations: $(x_n, k_n) \to (x^*, k^*)$. For any $y \in \omega(x^*, k^*)$ there is $n = n(y)$ such that when $n > n(y)$ $\rho^*(y, \omega(x_n, k_n)) > \varepsilon/2$. In deed as $y \in \omega(x^*, k^*)$, there is $t > 0$ such that $\rho(f(t, x^*, k^*), y) < \varepsilon/4$. Since $(x_n, k_n) \to (x^*, k^*)$, $\tau_1(x_n, k_n, \varepsilon) \to \infty$, there is n (we denote it by $n(y)$) such that for $n > n(y)$ $\rho^*(\bar{t}, x_n, k_n)) < \varepsilon/4$, $\tau_1(x_n, k_n, \varepsilon) > t$. Therefore, since $\rho^*(f(\bar{t}, x_n, k_n), \omega(x_n, k_n)) > \varepsilon$, it follows that $\rho^*(f(\bar{t}, x^*, k^*), \omega(x_n, k_n)) > 3\varepsilon/4$, and, consequently, $\rho^*(y, \omega(x_n, k_n)) > \varepsilon/2$. Let $y_i, \dots, y_m$ be $\varepsilon/4$-network in $\omega(x^*, k^*)$. Let $N = \max\ n(y_i)$. Then for $n > N$ and for any i $(1 \le i \le m)$, $\rho^*(y_i, \omega(x_n, k_n)) > \varepsilon/2$. Consequently, for any $y \in \omega(x^*, k^*)$ for $n > N$ $\rho^*(y, \omega(x_n, k_n) > \varepsilon/4$, i.e. for $n > N$ $r(\omega(x^*, k^*), \omega(x_n, k_n)) > \varepsilon/4$. The existence of $\Omega(x, k)$-bifurcations is proved (according to Proposition 1.12). Using Theorem 2.9 and Proposition 1.15 we obtain the following theorem.

Theorem 2.9′ *The system (1.1) possesses τ_1-slow relaxations if and only if $\omega(x, k)$ is not r-continuous function in $X \times K$.*

Theorem 2.10. *The system (1.1) possesses η_1-slow relaxations if and only if it possesses $\Omega(k)$-bifurcations.*

Proof. Let the system possess $\Omega(k)$-bifurcations. Then (according to Proposition 1.12) there is such sequence of parameters $k_n \to k^*$ that for some $\omega(x^*, k^*) \in \Omega(k^*)$ and $\varepsilon > 0$ for any n $r(\omega(x^*, k^*), \omega(k_n)) > \varepsilon$. Let $x_0 \in \omega(x^*, k^*)$. Then for any n and $t \in (-\infty, \infty)$ $\rho^*(f(t, x_0, k^*),\ \omega(k_n)) > \varepsilon$ because $f(t, x_0, k^*) \in \omega(x^*, k^*)$. Let us prove that $\eta_1(x_0, k_n, \varepsilon/2) \to \infty$ as $n \to \infty$. To do this, use the uniform continuity of f on compact segments and for any $\tau > 0$ find such $\delta = \delta(\tau) > 0$ that $\rho(f(t, x_0, k^*), f(t, x_0, k_n)) < \varepsilon/2$ if $0 \le t \le \tau$ and $\rho_K(k^*, k_n) < \delta$. Since $k_n \to k^*$, there is such $N = N(\tau)$ that for $n > N$ $\rho_K(k_n, k) < \delta$. Therefore, for $n > N$, $0 \le t \le \tau \rho^*(f(t, x_0, k_n), \omega(k_n)) > \varepsilon/2$. The existence of η_1-slow relaxations is proved. □

Now, suppose that there exist η_1-slow relaxations: there are such $\varepsilon > 0$ and sequence $(x_n, k_n) \in X \times K$ that $\eta_1(x_n, k_n, \varepsilon) \to \infty$. Use the compactness of $X \times K$ and turn to converging subsequence (retaining the same denotations): $(x_n, k_n) \to (x^*, k^*)$. Using the way similar to the proof of Theorem 2.9, let us show that for any $y \in \omega(x^*, k^*)$ there is such $n = n(y)$ that if $n > n(y)$, then $\rho^*(y, \omega(k_n)) > \varepsilon/2$. Really, there is such $\tilde{t} > 0$ that $\rho(f(\tilde{t}, x^*, k^*), y) < \varepsilon/4$. As $\eta_1(x_n, k_n, \varepsilon) \to \infty$ and $(x_n, k_n) \to (x^*, k^*)$, there is such $n = n(y)$ that for $n > n(y)$ $\rho(f(\tilde{t}, x^*, k^*), f(\tilde{t}, x_n, k_n)) < \varepsilon/4$ and $\eta_1(x_n, k_n, \varepsilon) > \tilde{t}$. Thereafter we obtain

$$
\begin{aligned}
&\rho^*(y, \omega(k_n)) \\
&\ge \rho^*(f(t, x_n, k_n), \omega(k_n)) - \rho(y, f(\tilde{t}, x^*, k^*)) - \rho(f(\tilde{t}, x^*, k^*), f(\tilde{t}, x_n, k_n)) > \varepsilon/2.
\end{aligned}
$$

Further the reasonings about $\varepsilon/4$-network of the set $\omega(x^*, k^*)$ (as in the proof of Theorem 2.9) lead to the inequality $r(\omega(x^*, k^*), \omega(k_n)) > \varepsilon/4$ for n large enough. On account of Proposition 1.12 the existence of $\Omega(k)$-bifurcations is proved, therefore is proved Theorem 2.10.

Theorem 2.11. *If the system (1.1) possesses $\omega(x, k)$-bifurcations then it possesses τ_2-slow relaxations.*

Proof. Let the system (1.1) possess $\omega(x,k)$-bifurcations: there is a sequence $(x_n,k_n) \in X \times K$ and $\varepsilon > 0$ such that $(x_n,k_n) \to (x^*,k^*)$ and

$$\rho^*(x',\omega(x_n,k_n)) > \varepsilon \text{ for any } n \text{ and some } x' \in \omega(x^*,k^*).$$

Let $t > 0$. Define the following auxiliary function:

$$\Theta(x^*,x',t,\varepsilon) = \operatorname{meas}\{t' \geq 0 : t' \leq t,\ \rho(f(t',x^*,k^*),x') < \varepsilon/4\}, \tag{2.4}$$

$\Theta(x^*,x',t,\varepsilon)$ is "the time of residence" of (k^*,x^*)-motion in $\varepsilon/4$-neighbourhood of x over the time segment $[0,t]$. Let us prove that $\Theta(x^*,x',t,\varepsilon) \to \infty$ as $t \to \infty$. We need the following corollary of continuity of f and compactness of X □

Lemma 2.12. *Let $x_0 \in X$, $k \in K$, $\delta > \varepsilon > 0$. Then there is such $t_0 > 0$ that for any $x \in X$ the inequalities $\rho(x,x_0) < \varepsilon$ and $0 \leq t' < t_0$ lead to $\rho(x_0,f(t',x,k)) < \delta$.*

Proof. Let us suppose the contrary: there are such sequences x_n and t_n that $\rho(x_0,x_n) < \varepsilon$, $t'_n \to 0$, and $\rho(x_0,f(t'_n,x_n,k)) \geq \delta$. Due to the compactness of X one can choose from the sequence x_n a convergent one. Let it converge to $\bar{x}$. The function $\rho(x_0,f(t,x,k))$ is continuous. Therefore, $\rho(x_0,f(t'_n,x_n,k)) \to \rho(x_0,f(0,x,k)) = \rho(x_0,\bar{x})$. Since $\rho(x_0,x_n) < \varepsilon$, then $\rho(x_0,\bar{x}) \leq \varepsilon$. This contradicts to the initial supposition $(\rho(x_0,f(t'_n,x_n,k)) \geq \delta \geq \varepsilon)$.

Let us return to the proof of Theorem 2.11. Since $x' \in \omega(x^*,k^*)$, then there is such monotonic sequence $t_j \to \infty$ that for any j $\rho(f(t_j,x^*,k^*),x') < \varepsilon/8$. According to Lemma 2.12 there is $t_0 > 0$ for which $\rho(f(t_j+\tau,x^*,k^*),x') < \varepsilon/4$ as $0 \leq \tau \leq t_0$. Suppose (turning to subsequence, if it is necessary) that $t_{j+1} - t_j > t_0$. $\Theta(x^*,x',t,\varepsilon) > jt_0$ if $t > t_j + t_0$. For any $j = 1,2,\dots$ there is such $N(j)$ that $\rho(f(t,x_n,k_n),f(t,x^*,k^*)) < \varepsilon/4$ under the conditions $n > N(j)$, $0 \leq t \leq t_j + t_0$. If $n > N(j)$, then $\rho(f(t,x_n,k_n),x') < \varepsilon/2$ for $t_j \leq t \leq t_j + t_0$ $(i \leq j)$. Consequently, $\tau_2(x_n,k_n,\varepsilon/2) > jt_0$ if $n > N(j)$. The existence of τ_2 slow relaxations is proved. □

Theorem 2.13. *If the system (1.1) possesses $\omega(k)$-bifurcations, then it possesses η_2-slow relaxations too.*

Proof. Let the system (1.1) possess $\omega(k)$-bifurcations: there are such sequence $k_n \in K$ and such $\varepsilon > 0$ that $k_n > k^*$ and $\rho^*(x',\omega(k_n)) > \varepsilon$ for some $x' \in \omega(k^*)$ and any n. The point x' belongs to the ω-limit set of some motion: $x' \in \omega(x^*,k^*)$. Let $\tau > 0$ and t^* be such that $\Theta(x^*,x',t^*,\varepsilon) > \tau$ (the existence of such t^* is shown when proving Theorem 2.11). Due to the uniform continuity of f on compact intervals there is such N that $\rho(f(x^*,k^*),f(t,x^*,k_n)) < \varepsilon/4$ for $0 \leq t \leq t^*$, $n > N$. But from this fact it follows that $\eta_2(x^*,k_n,\varepsilon/2) \geq \Theta(x^*,x',t^*,\varepsilon) > \tau$ $(n > N)$. Because of the arbitrary choice of τ Theorem 2.13 is proved. □

The two following theorems provide supplementary sufficient conditions of τ_2 - and η_2 -slow relaxations.

Theorem 2.14. *If for the system (1.1) there are such $x \in X, k \in K$ that (k,x)-motion is whole and $\alpha(x,k) \not\subset \omega(x,k)$, then the system (1.1) possesses τ_2-slow relaxations.*

Proof. Let there be such x and k that (k,x)-motion is whole and $\alpha(x,k) \not\subset \omega(x,k)$. Let us denote by x^* an arbitrary α-, but not ω-limit point of (k,x)-motion. Since $\omega(x,k)$ is closed, $\rho^*(x^*,\omega(x,k)) > \varepsilon > 0$. Define an auxiliary function

$$\varphi(x,x^*,t,\varepsilon) = \operatorname{meas}\{t' : -t \leq t' \leq 0,\ \rho(f(t',x,k),x^*) < \varepsilon/2\}.$$

Let us prove that $\varphi(x, x^*, \varepsilon) \to \infty$ as $t \to \infty$. According to Lemma 2.12 there is such $t_0 > 0$ that $\rho(f(t, y, k), x^*) < \varepsilon/2$ if $0 \le t \le t_0$ and $\rho(x^*, y) < \varepsilon/4$. Since x^* is α-limit point of (k, x)-motion, there is such sequence $t_j < 0$, $t_{j+1} - t_j < -t_0$, for which $\rho(f(t_j, x, k), x^*) < \varepsilon/4$. Therefore, by the way used in the proof of Theorem 2.11 we obtain: $\varphi(x, x^*, t_j, \varepsilon) > jt_0$. This proves Theorem 2.14, because $\tau_2(f(-t, x, k), k, \varepsilon/2) \ge \varphi(x, x^*, t, \varepsilon)$. □

Theorem 2.15. *If for the system (1.1) exist such $x \in X, k \in K$ that (k, x)-motion is whole and $\alpha(x, k) \not\subset \overline{\omega(k)}$, then the system (1.1) possesses η_2-slow relaxations.*

Proof. Let (k, x)-motion be whole and

$$\alpha(x, k) \not\subset \overline{\omega(k)},\ x^* \in \alpha(x, k) \setminus \overline{\omega(k)},\ \rho^*(x^*, \overline{\omega(k)}) = \varepsilon > 0.$$

As in the proof of the previous theorem, let us define the function $\varphi(x, x^*, t, \varepsilon)$. Since $\varphi(x, x^*, t, \varepsilon) \to \infty$ as $t \to \infty$ (proved above) and $\eta_2(f(-t, x, k), k, \varepsilon/2) \ge \varphi(x, x^*, t, \varepsilon)$, the theorem is proved. □

Note that the conditions of the theorems 2.14, 2.15 do not imply bifurcations.

Example 2.16 (τ_2-, η_2-slow relaxations without bifurcations). *Examine the system given by the set of equations (2.1) in the circle $x^2 + y^2 \le 1$ (see Fig. 2,a, Example 2.4). Identify the fixed points $r = 0$ and $r = 1$, $\varphi = \pi$ (Fig. 4). The complete ω-limit set of the system obtained consists of one fixed point. For initial data $r_0 \to 1, r_0 < 1$ (φ_0 is arbitrary) the relaxation time $\eta_2(r_0, \varphi_0, 1/2) \to \infty$ (hence, $\tau_2(r_0, \varphi_0, 1/2) \to \infty$).*

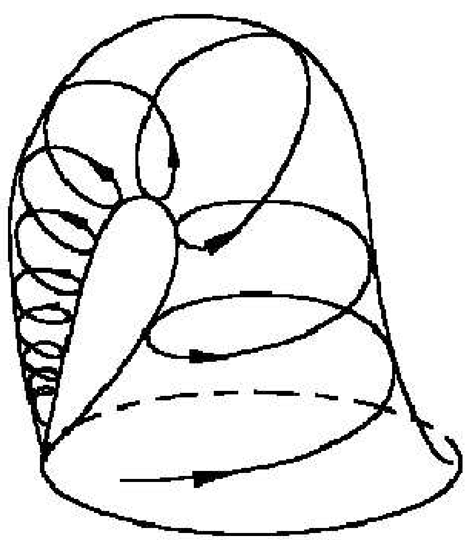

FIGURE 4. Phase portrait of the system (2.1) after gluing the fixed points.

Before analyzing τ_3, η_3-slow relaxations, let us define *Poisson's stability* according to [56], p.363: (k, x)-motion is it Poisson's positively stable (P^+*-stable*), if $x \in \omega(x, k)$.

Note that any P^+-stable motion is whole.

Lemma 2.17. *If for the system (1.1) exist such $x \in X,\ k \in K$ that (k,x)-motion is whole but not P^+-stable, then the system (1.1) possesses τ_3-slow relaxations.*

Proof. Let $\rho^*(x,\omega(x,k)) = \varepsilon > 0$ and (k,x)-motion be whole. Then $\tau_3(f(-t,x,k),k,\varepsilon) \geq t$, since $f(t,f(-t,x,k),k) = x$ and $\rho^*(x,\omega(f(-t,x,k),k)) = \varepsilon$ (because $\omega(f(-t,x,k),k) = \omega(x,k)$). Therefore, τ_3-slow relaxations exist. □

Lemma 2.18. *If for the system (1.1) exist such $x \in X,\ k \in K$ that (k,x)-motion is whole and $x \notin \omega(k)$, then this system possesses η_3 -slow relaxations.*

Proof. Let $\rho^*(x,\omega(k)) = \varepsilon > 0$ and (k,x)-motion be whole. Then

$$\eta_3(f(-t,x,k),k,\varepsilon)) \geq t,$$

$$\text{since } f(t,f(-t,x,k),k) = x \text{ and } \rho^*(x,\omega(k)) = \rho^*(x,\overline{\omega(k)}) = \varepsilon.$$

Consequently, η_3-slow relaxations exist.

Lemma 2.19. *Let for the system (1.1) be such $x_0 \in X,\ k \in K$ that (k_0,x_0)-motion is whole. If $\omega(x,k)$ is d-continuous function in $X \times K$ (there are no $\omega(x,k)$-bifurcations), then:*

1. $\omega(x^*,k_0) \subset \omega(x_0,k_0)$ *for any* $x^* \in \alpha(x_0,k_0)$*, that is* $\omega(\alpha(x_0,k_0),k_0) \subset \omega(x_0,k_0)$*;*
2. *In particular,* $\omega(x_0,k_0) \bigcap \alpha(x_0,k_0) \neq \varnothing$*.*

Proof. Let $x^* \in \alpha(x_0,k_0)$. Then there are such $t_n > 0$ that $t_n \to \infty$ and $x_n = f(-t_n,x_0,k_0) \to x^*$. Note that $\omega(x_n,k_0) = \omega(x_0,k_0)$. If $\omega(x^*,k_0) \not\subset \omega(x_0,k_0)$, then, taking into account closure of $\omega(x_0,k_0)$, we obtain $d(\omega(x^*,k_0),\omega(x_0,k_0)) > 0$. In this case $x_n \to x^*$, but $\omega(x_n,k_0) - / \to \omega(x^*,k_0)$, i.e. there is $\omega(x,k)$-bifurcation. But according to the assumption there are no $\omega(x,k)$-bifurcations. The obtained contradiction proves the first statement of the lemma. The second statement follows from the facts that $\alpha(x_0,k_0)$ is closed, k_0-invariant and nonempty. Really, let $x^* \in \alpha(x_0,k_0)$. Then $\overline{f((-\infty,\infty),x^*,k_o)} \subset \alpha(x_0,k_0)$ and, in particular, $\omega(x^*,k_0) \subset \alpha(x_0,k_0)$. But it has been proved that $\omega(x^*,k_0) \subset \omega(x_0,k_0)$. Therefore, $\omega(x_0,k_0) \bigcap \alpha(x_0,k_0) \supset \omega(x^*,k_0) \neq \varnothing$. □

Theorem 2.20. *The system (1.1) possesses τ_3-slow relaxations if and only if at least one of the following conditions is satisfied:*

1. *There are $\omega(x,k)$-bifurcations;*
2. *There are such $x \in X,\ k \in K$ that (k,x)-motion is whole but not P^+-stable.*

Proof. If there exist $\omega(x,k)$-bifurcations, then the existence of τ_3-slow relaxations follows from Theorem 2.11 and the inequality $\tau_2(x,k,\varepsilon) \leq \tau_3(x,k,\varepsilon)$. If the condition 2 is satisfied, then the existence of τ_3-slow relaxations follows from Lemma 2.17. To finish the proof, it must be ascertained that if the system (1.1) possesses τ_3-slow relaxations and does not possess $\omega(x,k)$-bifurcations, then there exist such $x \in X,\ k \in K$ that (k,x)-motion is whole and not P^+-stable. Let there be τ_3-slow relaxations and $\omega(x,k)$-bifurcations be absent. There can be chosen such convergent (because of the compactness of $X \times K$) sequence $(x_n,k_n) \to (x^*,k^*)$ that $\tau_3(x_n,k_n,\varepsilon) \to \infty$ for some $\varepsilon > 0$. Consider the sequence $y_n = f(\tau_3(x_n,k_n,\varepsilon),x_n,k_n)$. Note that $\rho^*(y_n,\omega(x_n,k_n)) = \varepsilon$. This follows from the definition of relaxation time and continuity of the function $\rho^*(f(t,x,k),s)$ of t at any $(x,k) \in X \times K,\ s \subset X$. Let us choose from the sequence y_n a convergent one (preserving the denotations y_n, x_n, k_n). Let us denote its limit: $y_n \to x_0$. It is clear that (k^*, x_0)-motion is whole. This follows from the results of Subection 1.1 and the fact that (k_n, y_n)-motion is defined in the time interval $[-\tau_3(x_n, k_n, \varepsilon), \infty)$, and $\tau_3(x_n, k_n, \varepsilon) \to \infty$ as $n \to \infty$. Let us prove that (k^*, x_0)-motion is not P^+-stable, i.e. $x_0 \notin \omega(x_0, k^*)$. Suppose the contrary: $x_0 \in \omega(x_0, k^*)$. Since $y_n \to x_0$, then there is such N that $\rho(x_0, y_n) < \varepsilon/2$ for any $n \geq N$. For the same $n \geq N$ $\rho^*(x_0, \omega(y_n, k_n)) > \varepsilon/2$, since $\rho^*(y_n, \omega(y_n, k_n)) = \varepsilon$. But from this fact and from the assumption $x_0 \in \omega(x_0, k^*)$ it follows that for $n \geq N$ $d(\omega(x_0, k^*), \omega(y_n, k_n)) > \varepsilon/2$, and that means that there are $\omega(x, k)$-bifurcations. So far as it was supposed d-continuity of $\omega(x, k)$, it was proved that (k^*, x_0)-motion is not P^+-stable, and this completes the proof of the theorem. □

Using Lemma 2.19, Theorem 2.20 can be formulated as follows.

Theorem 2.20′. *The system (1.1) possesses τ_3-slow relaxations if and only if at least one of the following conditions is satisfied:*

1. *There are $\omega(x, k)$-bifurcations;*
2. *There are such $x \in X$, $k \in K$ that (k, x)-motion is whole but not P^+-stable and possesses the following property: $\omega(\alpha(x, k), k) \subset \omega(x, k)$.*

As an example of motion satisfying the condition 2 can be considered a trajectory going from a fixed point to the same point (for example, the loop of a separatrice), or a homoclinical trajectory of a periodical motion.

Theorem 2.21. *The system (1.1) possesses η_3-slow relaxations if and only if at least one of the following conditions is satisfied:*

1. *There are $\omega(k)$-bifurcations;*
2. *There are such $x \in X$, $k \in K$ that (k, x)-motion is whole and $x \notin \overline{\omega(k)}$.*

Proof. If there are $\omega(k)$-bifurcations, then, according to Theorem 2.13, there are η_2- and all the more η_3-slow relaxations. If the condition 2 holds, then the existence of η_3-slow relaxations follows from Lemma 2.18. To complete the proof, it must be established that if the system (1.1) possesses η_3-slow relaxations and does not possess $\omega(k)$-bifurcations then the condition 2 of the theorem holds: there are such $x \in X$, $k \in K$ (k, x)-motion is whole and $x \notin \overline{\omega(k)}$. Let there be η_3 -slow relaxation and $\omega(k)$-bifurcations be absent. Then we can choose such convergent (because of the compactness of $X \times K$) sequence $(x_n, k_n) \to (x^*, k^*)$ that $\eta_3(x_n, k_n, \varepsilon) \to \infty$ for some $\varepsilon > 0$. Consider the sequence $y_n = f(\eta_3(x_n, k_n, \varepsilon), x_n, k_n)$. Note that $\rho^*(y_n, \omega(k_n)) = \varepsilon$. Choose from the sequence y_n a convergent one (preserving the denotations y_n, x_n, k_n). Let us denote its limit by $x_0 : y_n \to x_0$. From the results of Subection 1.1 and the fact that (k_n, y_n)-motion is defined at least on the segment $[-\eta_3(x_n, k_n, \varepsilon), \infty)$ and $\eta_3(x_n, k_n, \varepsilon) \to \infty$ we obtain that (k^*, x_0)-motion is whole. Let us prove that $x_0 \notin \overline{\omega(k)}$. Really, $y_n \to x_0$, hence there is such N that for any $n \geq N$ the inequality $\rho(x_0, y_n) < \varepsilon/2$ is true. But $\rho^*(y_n, \overline{\omega(k_n)}) = \varepsilon$, consequently for $n > N$ $\rho^*(x_0, \overline{\omega(k_n)}) > \varepsilon/2$. If x_0 belonged to $\overline{\omega(k^*)}$, then for $n > N$ the inequality $d(\overline{\omega(k^*)}, \omega(k)n)) > \varepsilon/2$ would be true and there would exist $\omega(k)$-bifurcations. But according to the assumption they do not exist. Therefore is proved that $x_0 \notin \overline{\omega(k^*)}$. □

Formulate now some corollaries for $\omega(k)$ from the proved theorems.

Corollary 2.22. *Let any trajectory from $\omega(k)$ be recurrent for any $k \in K$ and there be not such $(x,k) \in X \times K$ that (k,x)-motion is whole, not P^+-stable and $\omega(\alpha(x,k),k) \subset \omega(x,k)$ (or weaker, $\omega(x,k) \bigcap \alpha(x,k) \neq \varnothing$). Then the existence of τ_3-slow relaxations is equivalent to the existence of $\tau_{1,2}$-slow relaxations.*

Obviously, this follows from Theorem 2.20 and Proposition 1.18.

Corollary 2.23. *Let the set $\omega(x,k)$ be minimal $(\Omega(x,k) = \{\omega(x,k)\})$ for any $(x,k) \in X \times K$ and there be not such $(x,k) \in X \times K$ that (k,x)-motion is whole, not P^+-stable and $\omega(\alpha(x,k),k) \subset \omega(x,k)$ (or weaker, $\alpha(x,k) \bigcap \omega(x,k) \neq \varnothing$). Then the existence of τ_3-slow relaxations is equivalent to the existence of $\tau_{1,2}$ -slow relaxations.*

This follows from Theorem 2.20 and Corollary 1.19 of Proposition 1.18.

3. Slow Relaxations of One Semiflow

In this section we study one semiflow f. Here and further we denote by ω_f and Ω_f the complete ω-limit sets of one semiflow f (instead of $\omega(k)$ and $\Omega(k)$).

3.1. η_2-slow Relaxations.

As it was shown (Proposition 2.3), η_1-slow relaxations of one semiflow are impossible. Also was given an example of η_2-slow relaxations in one system (Example 2.4). It is be proved below that a set of smooth systems possessing η_2-slow relaxations on a compact variety is a set of first category in C^1-topology. As for general dynamical systems, for them is true the following theorem.

Let us recall the definition of *non-wandering points*.

Definition 3.1. A point $x^* \in X$ is the non-wandering point for the semiflow f, if for any neighbourhood U of x^* and for any $T > 0$ there is such $t > T$ that $f(t,U) \bigcap U \neq \varnothing$.

Theorem 3.2. *Let a semiflow f possess η_2-slow relaxations. Then there exists a non-wandering point $x^* \in X$ which does not belong to $\overline{\omega_f}$.*

Proof. Let for some $\varepsilon > 0$ the function $\eta_2(x,\varepsilon)$ be unbounded in X. Consider a sequence $x_n \in X$ for which $\eta_2(x_n,\varepsilon) \to \infty$. Let V be a closed subset of the set $\{x \in X : \rho^*\ (x,\omega_f) \geq \varepsilon\}$. Define an auxiliary function: the residence time of x-motion in the intersection of the closed δ-neighbourhood of the point $y \in V$ with V:

$$\psi(x,y,\delta,V) = \operatorname{meas}\{t > 0 : \rho(f(t,x),y) \leq \delta,\ f(t,x) \in V\}. \tag{3.1}$$

From the inequality $\psi(x,y,\delta,V) \leq \eta_2(x,\varepsilon)$ and the fact that finite $\eta_2(x,\varepsilon)$ exists for each $x \in X$ (see Proposition 2.1) it follows that the function ψ is defined for any $x, y, \delta > 0$ and V with indicated properties (V is closed, $r(V,f) \geq \varepsilon$).

Let us fix some $\delta > 0$. Suppose that $V_0 = \{x \in X : \rho^*(x,\omega_f) \geq \varepsilon\}$. Let $\bar{U}_\delta(y_j)$ be a closed sphere of radius δ centered in $y_j \in V_0$). Consider a finite covering of V_0 with closed spheres centered in some points from V_0: $V_0 \subset \bigcup_{j=1}^k \bar{U}_\delta(y_i)$. The inequality

$$\sum_{j=1}^{k} \psi(x,y_i,\delta,V_0) \geq \eta_2(x,\varepsilon) \tag{3.2}$$

is true (it is obvious: being in V_0, x-motion is always in some of $\bar{U}_\delta(y_i)$). From (3.2) it follows that $\sum_{j=1}^k \psi(x,y_i,\delta,V_0) \to \infty$ as $n \to \infty$. Therefore there is j_0 $(1 \leq j_0 \leq$

k) for which there is such subsequence $\{x_{m(i)}\} \subset \{x_n\}$ that $\psi(x_{m(i)}, y_{j_0}, \delta, V_0) \to \infty$. Let $y_0^* = y_{j_0}$.

Note that if $\rho(x, y_0^*) < \delta$ then for any $I > 0$ there is $t > T$ for which

$$f(t, \bar{U}_{2\delta}(x)) \bigcap \bar{U}_{2\delta}(x) \neq \varnothing.$$

Let us denote $V_1 = \bar{U}_\delta(y_0^*) \bigcap V_0$. Consider the finite covering of V_1 with closed spheres of radius $\delta/2$ centered in some points from V_1: $V_1 = \bigcup_{j=1}^{k_1} \bar{U}_{\delta/2}(y_j^1)$; $y_j^1 \in V_1$. The following inequality is true:

$$\sum_{j=1}^{k_1} \psi(x, y_j^1, \delta/2, V_1) \geq \psi(x, y_0^*, \delta, V_0). \tag{3.3}$$

Therefore exists j_0' $(1 \geq j_0' \geq k_1)$ for which there is such sequence $\{x_{l(i)}\} \subset \{x_{m(i)}\} \subset \{x_n\}$ that $\psi(x_{l(i)}, y^1_{j_0'}, \delta/2, V_1) \to \infty$ as $i \to \infty$. We denote $y_1^* = y^1_{j_0'}$.

Note that if $\rho(x, y_1^*) \leq \delta/2$ then for any $T > 0$ there is such $t > T$ that

$$f(t, \bar{U}_\delta(x)) \bigcap \bar{U}_\delta(x) \neq \varnothing.$$

Let us denote $V_2 = \bar{U}_{\delta/2}(y_1^*) \bigcap V_1$ and repeat the construction, substituting $\delta/2$ for $\delta, \delta/4$ for $\delta/2$, $V_{1,2}$ for $V_{0,1}$.

Repeating this constructing, we obtain the fundamental sequence $y_0^*, y_1^*, \dots$. We denote its limit x^*, $\rho^*(x^*, \omega_f) \geq \varepsilon$ because $x^* \in V_0$. The point x^* is non-wandering: for any its neighbourhood U and for any $T > 0$ there is such $t > T$ that $f(t, U) \bigcap U \neq \varnothing$. Theorem 3.2 is proved. □

The inverse is not true in general case.

Example 3.3 (The existence of non-wandering point $x^* \notin \overline{\omega_f}$ without η_2-slow relaxations)**.** Consider a cylinder in R^3 : $x^2 + y^2 \leq 1$, $-1 \leq z \leq 1$. Define in it a motion by the equations $\dot{x} = \dot{y} = 0$, $\dot{z} = (x, y, z)$, where is a smooth function, ≥ 0, and it is equal to zero only at (all) points of the sets $(z = -1, x \leq 0)$ and $(z = 1, x \geq 0)$. Since the sets are closed, such function exists (even infinitely smooth). Identify the opposite bases of the cylinder, preliminary turning them at angle π. In the obtained dynamical system the closures of trajectories, consisting of more than one point, form up Zeifert foliation (Fig. 5) (see, for example, [12], p.158).

Trajectory of the point $(0, 0, 0)$ is a loop, tending at $t \to \pm\infty$ to one point which is the identified centers of cylinder bases. The trajectories of all other nonfixed points are also loops, but before to close they make two turns near the trajectory $(0, 0, 0)$ The nearer is the initial point of motion to $(0, 0, 0)$, the larger is the time interval between it and the point of following entering of this motion in small neighborhood of $(0, 0, 0)$ (see Fig. 5).

3.2. **Slow Relaxations and Stability.** Let us recall the definition of Lyapunov stability of closed invariant set given by Lyapunov (see [80], p.31-32), more general approach is given in [7].

Definition 3.4. A closed invariant set $W \subset X$ is *Lyapunov stable* if and only if for any $\varepsilon > 0$ there is such $\delta = \delta(\varepsilon) > 0$ that if $\rho^*(x, W) < \delta$ then the inequality $\rho^*(f(t, x), W) < \varepsilon$ is true for all $t \geq 0$.

The following lemma follows directly from the definition.

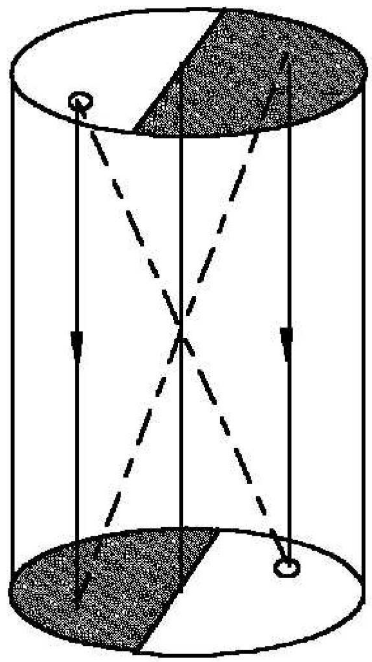

FIGURE 5. Phase space of the system (Example 3.3). All the points of the axis are non-wandering; ◯ is the place of delay near fixed points.

Lemma 3.5. *A closed invariant set W is Lyapunov stable if and only if it has a fundamental system of positively invariant closed neighborhoods: for any ε there are such $\delta > 0$ and closed positively invariant set $V \subset X$ that*

$$\{x \in X : \rho^*(x, W) < \delta\} \subset V \subset \{x \in X : \rho^*(x, W) < \varepsilon\}. \tag{3.4}$$

To get the set V, one can take for example the closure of following (it is obviously positively invariant) set: $\{f(t, x) : \rho^*(x, W) \le \delta = \delta(\varepsilon/2),\ t \in [0, \infty)\}$, i.e. of the complete image (for all $t \le 0$) of δ-neighbourhood of W, where $\delta(\varepsilon)$ is that spoken about in Definition 3.4.

The following lemma can be deduced from the description of Lyapunov stable sets ([80], Sec. 11, p.40-49).

Lemma 3.6. *Let a closed invariant set $W \subset X$ be not Lyapunov stable. Then for any $\lambda > 0$ there is such $y_0 \in X$ that the y_0-motion is whole, $\rho^*(y_0, W) < \lambda$, $d(\alpha(y_0), W) < \lambda$ (i.e. the α-limit set of the y_0-motion lies in λ-neighbourhood of W), and $y_0 \notin W$.*

Definition 3.7. An f-invariant set $W \subset X$ for the semiflow f is called isolated, if there is such $\lambda > 0$ that for any $y \in \omega_f$ from the condition $\rho^*(y, W) < \lambda$ it follows that $y \in W$, that is, any ω-limit point $y \in \omega_f$ from the λ-neighbourhood of W belongs to W.

Theorem 3.8. *If for semiflow f there exists closed isolated and not Lyapunov stable invariant set $W \subset X$ then this semiflow possesses η_3-slow relaxations.*

Proof. Let W be a closed invariant isolated Lyapunov unstable set. Let $\lambda > 0$ be the value from the definition of isolation. Then Lemma 3.6 guarantees the existence of such $y_0 \in X$ that y_0-motion is whole, $\rho^*(y_0, W) < \lambda$ and $y_0 \notin W$. It gives (due to closure of W) $\rho^*(y_0, W) = d > 0$. Let $\delta = \min\{d/2, (\lambda - d)/2\}$. Then δ-neighbourhood of the point y_0 lies outside of the set W, but in its λ-neighbourhood, and the last is free from the points of the set $\omega_f \setminus W$ (isolation of W). Thus, δ-neighbourhood of the point y_0 is free from the points of the set $\omega_f \subseteq W \bigcup (\omega_f \setminus W)$, consequently $y_0 \notin \overline{\omega_f}$. Since y_0-motion is whole, Theorem 2.21 guarantees the presence of η_3-slow relaxations. Theorem 3.8 is proved. □

Lemma 3.9. *Let X be connected and $\overline{\omega_f}$ be disconnected, then $\overline{\omega_f}$ is not Lyapunov stable.*

Proof. Since $\overline{\omega_f}$ is disconnected, there are such nonempty closed W_1, W_2 that $\overline{\omega_f} = W_1 \bigcup W_2$ and $W_1 \bigcap W_2 = \varnothing$. Since any x-trajectory is connected and $\overline{\omega_f}$ is invariant, then and W_1 and W_2 are invariant too. The sets $\omega(x)$ are connected (see Proposition 1.6); therefore, for any $x \in X$ $\omega(x) \subset W_1$ or $\omega(x) \subset W_2$. Let us prove that at least one of the sets W_i $(i = 1, 2)$ is not stable. Suppose the contrary: W_1 and W_2 are stable. Define for each of them ***attraction domain***:

$$At(W_i) = \{x \in X : \omega(x) \subset W_i\}. \tag{3.5}$$

It is obvious that $W_i \subset At(W_i)$ owing to closure and invariance of W_i. The sets $At(W_i)$ are open due to the stability of W_i. Really, there are non-intersecting closed positively invariant neighborhoods V_i of the sets W_i, since the last do not intersect and are closed and stable (see Lemma 3.5). Let $x \in At(W_i)$. Then there is such $t \geq$ that $f(t, x) \in \operatorname{int} V_i$. But because of the continuity of f there is such neighbourhood of x in X that for each its point x' $f(t, x) \in \operatorname{int} V_i$. Now positive invariance and closure of V_i ensure $\omega(x') \subset W_i$, i.e. $x' \in At(W_i)$. Consequently, x lies in $At(W_i)$ together with its neighbourhood and the sets $At(W_i)$ are open in X. Since $At(W_i) \bigcup At(W_2) = X$, $At(W_1) \bigcap At(W_2) = \varnothing$, the obtained result contradicts to the connectivity of X. Therefore at least one of the sets W is not Lyapunov stable. Prove that from this follows unstability of $\overline{\omega_f}$. Note that if a closed positively invariant set V is union of two non-intersecting closed sets, $V = V_1 \bigcup V_2$, $V_1 \bigcap V_2 = \varnothing$, then V_1 and V_2 are also positively invariant because of the connectivity of positive semitrajectories. If $\overline{\omega_f}$ is stable, then it possess fundamental system of closed positively invariant neighborhoods $V_1 \supset V_2 \supset \dots V_n \supset \dots$. Since $\overline{\omega_f} = W_1 \bigcup W_2$, $W_1 \bigcap W_2 = \varnothing$ and W_i are nonempty and closed, then from some N $V_n = V'_n \bigcup V''_n = \varnothing$ for $n \geq N$, and the families of the sets $V'_n \supset V'_{n+1} \supset \dots$, $V''_n \supset V''_{n+1} \dots$ form fundamental systems of neighborhoods of W_1 and W_2 correspondingly. So long as V'_n, V''_n are closed positively invariant neighborhoods, from this follows stability of both W_1 and W_2, but it was already proved that it is impossible. This contradiction shows that $\overline{\omega_f}$ is not Lyapunov stable and completes the proof of the lemma. □

Theorem 3.10. *Let X be connected and $\overline{\omega_f}$ be disconnected. Then the semiflow f possesses η_3- and $\tau_{1,2,3}$-slow relaxations.*

Proof. The first part (the existence of η_3-slow relaxations) follows from Lemma 3.9 and Theorem 3.8 (in the last as a closed invariant set one should take $\overline{\omega_f}$). Let us prove the existence of τ_1-slow relaxations. Let ω_f be disconnected: $\overline{\omega_f} = W_1 \bigcup W_2$, $W_1 \bigcap W_2 = \varnothing$, W_i $(i = 1, 2)$ are closed and, consequently, invariant due to the connectivity of trajectories. Consider the sets $At(W_i)$ (3.5). Note that at least one of these sets $At(W_i)$ does not include any neighbourhood of W_i. Really, suppose the contrary: $At(W_i)$ $(i = 1, 2)$ includes a ε-neighbourhood of W_i. Let $x \in At(W_i)$, $\tau = \tau_1(x, \varepsilon/3)$ be the time of the first entering of the x-motion into $\varepsilon/3$-neighbourhood of the set $\omega(x) \subset W_i$. The point x possesses such neighbourhood $U \subset X$ that for any $y \in U$ $\rho(f(t,x), f(t,y)) < \varepsilon/3$ as $y \in U$, $0 \leq t \leq \tau$. Therefore $d(f(\tau, U), W_i) \leq 2\varepsilon/3$, $U \subset At(W_i)$. Thus, x lies in $At(W_i)$ together with its neighbourhood: the sets $At(W_i)$ are open. This contradicts to the connectivity of X, since $X = At(W_1) \bigcup At(W_2)$ and $At(W_1) \bigcap At(W_2) = \varnothing$. To be certain, let $At(W_1)$ contain none neighbourhood of W_1. Then (owing to the compactness of X and the closure of W_1) there is a sequence $x_i \in At(W_2)$, $x_i \to y \in W_1$, $\omega(x_i) \subset W_2$, $\omega(y) \subset W_1$. Note that $r((x_i), \omega(y)) \geq r(W_1, W_2) > 0$, therefore there are $\Omega(x)$-bifurcations (y is the bifurcation point) and, consequently, (Theorem 2.9) there are τ_1-slow relaxations. This yields the existence of $\tau_{2,3}$-slow relaxations $(\tau_1 \leq \tau_2 \leq \tau_3)$. □

3.3. **Slow Relaxations in Smooth Systems.** In this subsection we consider the application of the approach to the semiflows associated with smooth dynamical systems developed above. Let M be a smooth (of class C^∞) finite-dimensional manifold, $F:\ (-\infty, \infty) \times M \to M$ be a smooth dynamical system over M, generated by vector field of class C^1, X be a compact set positively invariant with respect to the system F (in particular, $X = M$ if M is compact). The restriction of F to the set we call the semiflow over X, associated with F, and denote it by $F|_X$.

We shall often use the following condition: the semiflow $F|_X$ has not non-wandering points at the boundary of X (∂X); if X is positively invariant submanifold of M with smooth boundary, $\operatorname{int} X \neq \varnothing$, then this follows, for example, from the requirement of transversality of the vector field corresponding to the system F and the boundary of X. All the below results are valid, in particular, in the case when X is the whole manifold M and M is compact (the boundary is empty).

Theorem 3.11. *The complement of the set of smooth dynamical systems on compact manifold M possessing the following attribute 1, is the set of first category (in C^1-topology in the space of smooth vector fields).*
Attribute 1. *Every semiflow $F|_X$ associated with a system F on any compact positively invariant set $X \subset M$ without non-wandering points on ∂X has not η_2-slow relaxations.*

This theorem is a direct consequence of the Pugh closing lemma, the density theorem [61, 57], and Theorem 3.2 of the present work.

Note that if X is positively invariant submanifold in M with smooth boundary, $\operatorname{int} X \neq \varnothing$, then by infinitesimal (in C^1-topology) perturbation of F preserving positive invariance of X one can obtain that semiflow over X, associated with the perturbed system, would not have non-wandering points on ∂X. This can be easily proved by standard in differential topology reasonings about transversality. In the present case the transversality of vector field of "velocities" F to the boundary of X is meant.

The structural stable systems over compact two-dimensional manifolds are studied much better than in general case [8, 58]. They possess a number of characteristics which do not remain in higher dimensions. In particular, for them the set of non-wandering points consists of a finite number of limit cycles and fixed points, and the "loops" (trajectories whose α- and ω-limit sets intersect, but do not contain points of the trajectory itself) are absent. Slow relaxations in these systems also are different from the relaxations in the case of higher dimensions.

Theorem 3.12. *Let M be C^∞-smooth compact manifold, $\dim M = 2$, F be a structural stable smooth dynamical system over M, $F|_X$ be an associated with M semiflow over connected compact positively invariant subset $X \subset M$. Then:*

1. *For $F|_X$ the existence of τ_3-slow relaxations is equivalent to the existence of $\tau_{1,2}$- and η_3-slow relaxations;*
2. *$F|_X$ does not possess τ_3-slow relaxations if and only if $\omega_F \bigcap X$ consists of one fixed point or of points of one limit cycle;*
3. *$\eta_{1,2}$-slow relaxations are impossible for $F|_X$.*

Proof. To prove the part 3, it is sufficient to refer to Theorem 3.2 and Proposition 2.3. Let us prove the first and the second parts. Note that $\omega_{F|_X} = \omega_F \bigcap X$. Let $\omega_F \bigcap X$ consist of one fixed point or of points of one limit cycle. Then $\omega(x) = X \bigcap \omega_F$ for any $x \in X$. Also there are not such $x \in X$ that x-motion would be whole but not P^+-stable and $\alpha(x) \bigcap \omega(x) \neq \varnothing$ (owing to the structural stability). Therefore (Theorem 2.20) τ_3-slow relaxations are impossible. Suppose now that $\omega_F \bigcap X$ includes at least two limit cycles or a cycle and a fixed point or two fixed points. Then $\omega_{F|_X}$ is disconnected, and using Theorem 3.10 we obtain that $F|_X$ possesses η_3-slow relaxations. Consequently, exist τ_3-slow relaxations. From Corollary 2.22, i.e. the fact that every trajectory from ω_F is a fixed point or a limit cycle and also from the fact that rough two-dimensional systems have no loops we conclude that τ_1-slow relaxations do exist. Thus, if $\omega_F \bigcap X$ is connected, then $F|_X$ has not even τ_3-slow relaxations, and if $\omega_F \bigcap X$ is disconnected, then there are η_3 and $\tau_{1,2,3}$-slow relaxations. Theorem 3.12 is proved. □

In general case (for structural stable systems with dim $M > 2$) the statement 1 of Theorem 3.12 is not always true. Really, let us consider topologically transitive U-flow F over the manifold M [5]; $\omega_F = M$, therefore $\eta_3(x, \varepsilon) = 0$ for any $X \in M$, $\varepsilon > 0$. The set of limit cycles is dense in M. Let us choose two different cycles P_1 and P_2, whose stable (P_1) and unstable (P_2) manifolds intersect (such cycles exist, see for example [68, 18]). For the point x of their intersection $\omega(x) = P_1$, $\alpha(x) = P_2$, therefore x-motion is whole and not Poisson's positively stable, and (Lemma 2.17) τ_3-slow relaxations exist. And what is more, there exist τ_1-slow relaxations too. These appears because the motion beginning at point near P_2 of x-trajectory delays near P_2 before to enter small neighbourhood of P_1. It is easy to prove the existence of $\Omega(x)$-bifurcations too. Really, consider a sequence $t_1 \to \infty$, from the corresponding sequence $F(t_i, x)$ choose convergent subsequence: $F(t_j, x) \to y \in P_2$, $\omega(y) = P_2$, $\omega(F(t_j, x)) = P_1$, i.e. there are both τ_1-slow relaxations and $\Omega(x)$-bifurcations. For A-flows a weaker version of the statement 1 of Theorem 3.12 is valid (A-flow is called a flow satisfying S.Smale A-axiom [68], in regard to A-flows see also [18], p.106-143).

Theorem 3.13. *Let F be an A-flow over compact manifold M. Then for any compact connected positively invariant $X \subset M$ which does not possess non-wandering points of $F|_X$ on the boundary the existence of τ_3-slow relaxations involves the existence of $\tau_{1,2}$-slow relaxations for $F|_X$.*

Proof. Note that $\omega_{F|_X} = \omega_f \bigcap \operatorname{int} X$. If $\omega_F \bigcap \operatorname{int} X$ is disconnected, then, according to Theorem 3.10, $F|_X$ possesses η_3- and $\tau_{1,2,3}$-slow relaxations. Let $\omega_F \bigcap \operatorname{int} X$ be connected. The case when it consists of one fixed point or of points of one limit cycle is trivial: there are no any slow relaxations. Let $\omega_F \bigcap \operatorname{int} X$ consist of one non-trivial (being neither point nor cycle) basic set (in regard to these basic sets see [68, 18]): $\omega_F \bigcap \operatorname{int} X = \Omega_0$. Since there are no non-wandering points over ∂X, then every cycle which has point in X lies entirely in $\operatorname{int} X$. And due to positive invariance of X, unstable manifold of such cycle lies in X. Let P_1 be some limit cycle from X. Its unstable manifold intersects with stable manifold of some other cycle $P_2 \subset X$ [68]. This follows from the existence of hyperbolic structure on Ω_0 (see also [18], p.110). Therefore there is such $x \in X$ that $\omega(x) = P_2,\ \alpha(x) = P_1$. From this follows the existence of τ_1- (and $\tau_{2,3}$-)-slow relaxations. The theorem is proved. □

Remark. We have used only very weak consequence of the hyperbolicity of the set of non-wandering points: the existence in any non-trivial (being neither point nor limit cycle) isolated connected invariant set of two closed trajectories, stable manifold of one of which intersects with unstable manifold of another one. It seems very likely that the systems for which the statement of Theorem 3.13 is true are typical, i.e. the complement of their set in the space of flows is a set of first category (in C^1- topology).

4. Slow Relaxation of Perturbed Systems

4.1. **Limit Sets of ε-motions.** As models of perturbed motions let us take ε-*motions*. These motions are mappings $f^\varepsilon : [0, \infty) \to X$, which during some fixed time T depart from the real motions at most at ε.

Definition 4.1. Let $x \in X,\ k \in K,\ \varepsilon > 0,\ T > 0$. The mapping $f^\varepsilon : [0, \infty) \to X$ is called (k, ε, T)-*motion* of the point x for the system (1.1) if $f^\varepsilon(0) = x$ and for any $t \geq 0$, $\tau \in [0, T]$

$$\rho(f^\varepsilon(t + \tau),\ f(\tau, f^\varepsilon(t), k)) < \varepsilon. \tag{4.1}$$

We call (k, ε, T)-motion of the point x (k, x, ε, T)-motion and use the denotation $f^\varepsilon(t|x, k, T)$. It is obvious that if $y = f^\varepsilon(\tau|x, k, T)$ then the function $f^*(t) = f^\varepsilon(t + \tau|x, k, T)$ is (k, y, ε)-motion.

The condition (4.1) is fundamental in study of motion with constantly functioning perturbations. Different restrictions on the value of perturbations of the right parts of differential equations (uniform restriction, restriction at the average etc., see [9], p.184 and further) are used as a rule in order to obtain estimations analogous to (4.1). On the base of these estimations the further study is performed.

Let us introduce two auxiliary functions:

$$\varepsilon(\delta, t_0) = \sup\{\rho(f(t, x, k), f(t, x', k')) : 0 \leq t \leq t_0,\ \rho(x, x') < \delta,\ \rho_K(k, k') < \delta\}; \tag{4.2}$$

$$\delta(\varepsilon, t_0) = \sup\{\delta \geq 0 : \varepsilon(\delta, t_0) \leq \varepsilon\}. \tag{4.3}$$

Due to the compactness of X and K the following statement is true.

Proposition 4.2. *A) For any $\delta > 0$ and $t_0 > 0$ the function $\varepsilon(\delta, t_0)$ is defined; $\varepsilon(\delta, t_0) \to 0$ as $\delta \to 0$ uniformly over any compact segment $t_0 \in [t_1, t_2]$.*
B) For any $\varepsilon > 0$ and $t_0 > 0$ the function $\delta(\varepsilon, t_0) > 0$ is defined (is finite).

Proof. A) Let $\delta > 0,\ t_0 > 0$. Finiteness of $\varepsilon(\delta, t_0)$ follows immediately from the compactness of x. Let $\delta_i > 0,\ \delta_i \to 0$. Let us prove that $\varepsilon(\delta_i, t_0) \to 0$. Suppose the contrary. In this case one can choose in $\{\delta_i\}$ such subsequence that corresponding $\varepsilon(\delta_i, t_0)$ are separated from zero by common constant: $\varepsilon(\delta_i, t_0) > \alpha > 0$. Let us turn to this subsequence, preserving the same denotations. For every i there are such $t_i, x_i, x_i', k_i, k_i'$ that $0 \le t_i \le t_0,\ \rho(x_i, x_i') < \delta_i,\ \rho_K(k_i, k_i') < \delta_i$ and $\rho(f(t_i, x_i, k_i), f(t_i, x_i', k_i')) > \alpha > 0$. The product $[0, t_0] \times X \times X \times K \times K$ is compact. Therefore from the sequence $(t_i, x_i, x_i', k_i, k_i')$ one can choose a convergent subsequence. Let us turn to it preserving the denotations: $(t_i, x_i, x_i', k_i, k_i') \to (\tilde{t}, x_0, x_0', k_0, k_0')$. It is obvious that $\rho(x_0, x_0') = \rho_K(k_0, k_0') = \rho_K(k_0, k_0') = 0$, therefore $x_0 = x_0',\ k_0 = k_0'$. Consequently, $f(\tilde{t}, x_0, k_0) = f(t, x_0', k_0')$. On the other hand, $\rho(f(t_i, x_i, k_i), f(t_i, x_i', k_i')) > \alpha > 0$, therefore $\rho(f(\tilde{t}, x_0, k_0), f(\tilde{t}, x_0', k_0')) \ge \alpha > 0$ and $f(\tilde{t}, x_0, k_0) \ne f(\tilde{t}, x_0', k_0')$. The obtained contradiction proves that $\varepsilon(\delta_i, t_0) \to 0$.

The uniformity of tending to 0 follows from the fact that for any $t_1, t_2 > 0,\ t_1 < t_2$ the inequality $\varepsilon(\delta, t_1) \le \varepsilon(\delta, t_2)$ is true, $\varepsilon(\delta, t)$ is a monotone function.

The statement of the point B) follows from the point A). □

The following estimations of divergence of the trajectories hold. Let $f^\varepsilon(t|x, k, T)$ be (k, x, ε, T)-motion. Then

$$\rho(f^\varepsilon(t|x, k, T), f(t, x, k)) \le \chi(\varepsilon, t, T), \tag{4.4}$$

where $\chi(\varepsilon, t, T) = \sum_{i=0}^{[t/T]} \varkappa_i,\ \varkappa_0 = \varepsilon,\ \varkappa_i = \varepsilon(\varkappa_{i-1}, T) + \varepsilon$. Here and further trivial verifications that follow directly from the triangle inequality are omitted. Let $f^{\varepsilon_1}(t|x_1, k_1, T),\ f^{\varepsilon_2}(t|x_2, k_2, T)$ be corresponding $(k_1, x_1, \varepsilon_1, T)$- and $(k_2, x_2, \varepsilon_2, T)$-motions. Then

$$\begin{aligned} &\rho(f^{\varepsilon_1}(t|x_1, k_1, T), f^{\varepsilon_2}(t|x_2, k_2, T)) \\ &\le \varepsilon(\max\{\rho(x_1, x_2, , \rho_K(k_1, k_2)\}, T) + \chi(\varepsilon_1, t, T) + \chi(\varepsilon_2, t, T). \end{aligned} \tag{4.5}$$

From Proposition 4.2 it follows that $\chi(\varepsilon, t, T) \to 0$ as $\varepsilon \to 0$ uniformly over any compact segment $t \in [t_1, t_2]$.

Let $T_2 > T_1 > 0$, $\varepsilon > 0$. Then any (k, x, ε, T_2)-motion is (k, x, ε, T_1)-motion, and any (k, x, ε, T_1)-motion is $(k, x, \chi(\varepsilon, T_2, T_1), T_2)$-motion. Since we are interested in perturbed motions behavior at $\varepsilon \to 0$, and $\chi(\varepsilon, T_2, T_1) \to 0$ as $\varepsilon \to 0$, then the choice of T is unimportant. Therefore, let us fix some $T > 0$ and omit references to it in formulas ((k, x, ε)-motion instead of (k, x, ε, T)-motion and $f^\varepsilon(t|x, k)$ instead of $f^\varepsilon(t|x, k, T)$). □

The following propositions allows us "to glue together" ε-motions.

Proposition 4.3. *Let* $\varepsilon_1,\ \varepsilon_2 > 0,\ f^{\varepsilon_1}(t|x, k)$ *be* ε_1*-motion,* $\tau > 0$*, and* $f^{\varepsilon_2}(t|f^{\varepsilon_1}(\tau|x, k), k)$ *be* ε_2*-motion. Then the mapping*

$$f^*(t) = \begin{cases} f^{\varepsilon_1}(t|x, k), & \text{if } 0 \le t \le \tau; \\ f^{\varepsilon_2}(t - \tau|f^{\varepsilon_1}(\tau|x, k), k), & \text{if } t \ge \tau, \end{cases}$$

is $(k, x, 2\varepsilon_1 + \varepsilon_2)$*-motion.*

Proposition 4.4. *Let* $\delta, \varepsilon_1, \varepsilon_2 > 0,\ f^{\varepsilon_1}(t|x, k)$ *be* ε_1*-motion,* $\tau > 0,\ f^{\varepsilon_2}(t|y, k')$ *be* ε_2*-motion,* $\rho_K(k, k') < \delta,\ \rho(y, f^{\varepsilon_1}(\tau|x, k)) < \delta$*. Then the mapping*

$$f^*(t) = \begin{cases} f^{\varepsilon_1}(t|x, k), & \text{if } 0 \le t < \tau; \\ f^{\varepsilon_2}(t - \tau|y, k), & \text{if } t \ge \tau, \end{cases}$$

is $(k, x, 2\varepsilon_1 + \varepsilon_2 + \varepsilon(\delta, T))$*-motion.*

Proposition 4.5. *Let* $\delta_j, \varepsilon_j > 0$*, the numbers* ε_j, δ_j *are bounded above,* $x_j \in X$*,* $k_j \in K$*,* $k^* \in K$*,* $\tau_0 > T$*,* $j = 0, 1, 2, \dots$*,* $i = 1, 2, \dots$*,* $f^{\varepsilon_j}(t|x_j, k_j)$ *be the* ε_j*-motions,* $\rho(f^{\varepsilon_j}(\tau_j|x_j, k_j), x_{j+1}) < \delta_j$*,* $\rho_K(k_j, k^*) < \delta_j/2$*. Then the mapping*

$$f^*(t) = \begin{cases} f^{\varepsilon_0}(t|x_0, k_0), & \text{if } 0 \le t < \tau_0; \\ f^{\varepsilon_j}\left(t - \sum_{j=0}^{i-1} \tau_j | x_j, k_j\right), & \text{if } \sum_{j=0}^{i-1} \tau_j \le t < \sum_{j=0}^{i} \tau_j, \end{cases}$$

is (k^*, x_0, β)*-motion, where*

$$\beta = \sup_{0 \le j < \infty} \{\varepsilon_{j+1} + \varepsilon(\varepsilon_j + \delta_j + \delta_{j+1} + \varepsilon(\delta_j, T), T)\}.$$

The proof of the propositions 4.3-4.5 follows directly from the definitions.

Proposition 4.6. *Let* $x_i \in X$*,* $k_i \in K$*,* $k_i \to k^*$*,* $\varepsilon_i > 0$*,* $\varepsilon_i \to 0$*,* $f^{\varepsilon_i}(t|x_i, k_i)$ *be* $(k_i, x_i, \varepsilon_i)$*-motions,* $t_i > 0$*,* $t_i > t_0$*,* $f^{\varepsilon_i}(t_i|x_i, k_i) \to x^*$*. Then* (k^*, x^*)*-motion is defined over the segment* $[-t_0, \infty)$ *and* $f^{\varepsilon_i}(t_0 + t|x_i, k_i)$ *tends to* $f(t, x^*, k^*)$ *uniformly over any compact segment from* $[-t_0, \infty)$*.*

Proof. Let us choose from the sequence $\{x_i\}$ a convergent subsequence (preserving the denotations): $x_i \to x_0$. Note that $f^{\varepsilon_i}(t|x_i, k_i) \to f(t, x_0, k^*)$ uniformly over any compact segment $t \in [t_1,\ t_2] \subset [0, \infty)$; this follows from the estimations (4.5) and Proposition 4.2. Particularly, $f(t_0, x_0, k^*) = x^*$. Using the injectivity of f, we obtain that x_0 is a unique limit point of the sequence $\{x_i\}$, therefore $f^{\varepsilon_i}(t_0 + t|y_i, k_i)$ tends to $f(t, x^*, k^*) = f(t_0 + t, x_0, k^*)$ uniformly over any compact segment $t \in [t_1, t_2] \subset [-t_0, \infty)$. □

Proposition 4.7. *Let* $x_i \in X$*,* $k_i \in K$*,* $k_i \to K^*$*,* $\varepsilon_i > 0$*,* $f^{\varepsilon_i}(t|x_i, k_i)$ *be* $(k_i, x_i, \varepsilon_i)$*-motions,* $t_i > 0$*,* $t_i \to \infty$*,* $f^{\varepsilon_i}(t_i|x_i, k_i) \to x^*$*. Then* (k^*, x^*)*-motion is whole and the sequence* $f^{\varepsilon_i}(t + t_i|x_i, k_i)$ *defined for* $t > t_0$ *for any* t_0*, from some* $i(t_0)$ *(for* $i \ge i(t_0)$*) tends to* $f(t, x^*, k^*)$ *uniformly over any compact segment.*

Proof. Let $t_0 \in (-\infty, \infty)$. From some i_0 $t_i > -t_0$. Let us consider the sequence of $(k_i, f^{\varepsilon_i}(t_i + t_0|x_i, k_i), \varepsilon_i)$-motions: $f^{\varepsilon_i}(t|f^{\varepsilon_i}(t_i + t_0|x_i, k_i), k_i) := f^{\varepsilon_i}(t + t_i + t_0|x_i, k_i)$. Applying to the sequence the precedent proposition, we obtain the required statement (due to the arbitrariness of t_0). □

Definition 4.8. Let $x \in X$, $k \in K$, $\varepsilon > 0$, $f^\varepsilon(t|x, k)$ be (k, x, ε)-motion. Let us call $y \in X$ ω*-limit point* of this ε-motion, if there is such a sequence $t_i \to \infty$ that $f^\varepsilon(t_i|x, k) \to y$. We denote the set of all ω-limit points of $f^\varepsilon(t|x, k)$ by $\omega(f^\varepsilon(t|x, k))$, the set of all ω-limit points of all (k, x, ε)-motions under fixed k, x, ε by $\omega^\varepsilon(x, k)$, and

$$\omega^0(x, k) := \bigcap_{\varepsilon > 0} \omega^\varepsilon(x, k).$$

Proposition 4.9. *For any* $\varepsilon > 0$*,* $\gamma > 0$*,* $x \in X$*,* $k \in K$

$$\overline{\omega^\varepsilon(x, k)} \subset \omega^{\varepsilon + \gamma}(x, k).$$

Proof. Let $y \in \overline{\omega^\varepsilon(x, k)}$. For any $\delta > 0$ there are (k, x, ε)-motion $f^\varepsilon(t|x, k)$ and subsequence $t_i \to \infty$, for which $\rho(f^\varepsilon(t_i|x, k), y) < \delta$. Let $\delta = \frac{1}{2}\delta(\gamma, T)$. As $(k, x, \varepsilon + \gamma)$-motion let us choose

$$f^*(t) = \begin{cases} f^*(t|x, k), & \text{if } t \ne t_i; \\ y, & \text{if } t = t_i (i = 1, 2, \dots, t_{i+1} - t_i > T). \end{cases}$$

y is the ω-limit point of $f^*(t)$, therefore, $y \in \omega^{\varepsilon+\gamma}(x,k)$. □

Proposition 4.10. *For any $x \in X$, $k \in K$ the set $\omega_0(x,k)$ is closed and k-invariant.*

Proof. From Proposition 4.9 it follows

$$\omega^0(x,k) = \bigcap_{\varepsilon>0} \overline{\omega^\varepsilon(x,k)}. \tag{4.6}$$

Therefore, $\omega^0(x,k)$ is closed. Let us prove that it is k-invariant. Let $y \in \omega^0(x,k)$. Then there are such sequences $\varepsilon_j > 0$, $\varepsilon_j \to 0$, $t_i^j \to \infty$ as $i \to \infty$ $(j = 1,2,\dots)$ and such family of (k,x,ε_j)-motions $f^{\varepsilon_j}(t|x,k)$ that $f^{\varepsilon_j}(t_i^j|x,k) \to y$ as $i \to \infty$ for any $j = 1,2,\dots$. From Proposition 4.7 it follows that (k,y)-motion is whole. Let $z = f(t_0,y,k)$. Let us show that $z \in \omega^0(x,k)$. Let $\gamma > 0$. Construct (k,x,γ)-motion which has z as its ω-limit point.

Let $t_0 > 0$. Find such $\delta_0 > 0$, $\varepsilon_0 > 0$ that $\chi(\varepsilon_0, t_0+T, T) + \varepsilon(\delta_0, t_0+T) < \gamma/2$ (this is possible according to Proposition 4.2). Let us take $\varepsilon_j < \varepsilon_0$ and choose from the sequence t_i^j $(i = 1,2,\dots)$ such monotone subsequence t_l $(l = 1,2,\dots)$ for which $t_{l+1} - t_l > t_0 + T$ and $\rho(f^{\varepsilon_j}(t_l|x,k), y) < \delta_0$. Let

$$f^*(t) = \begin{cases} f^{\varepsilon_j}(t|x,k), & \text{if } t \notin [t_l, t_l+t_0] \text{ for any } l = 1,2,\dots; \\ f(t-t_l,y,k), & \text{if } t \in [t_l, t_l+t_0] \ (l = 1,2,\dots). \end{cases}$$

z is the ω-limit point of this (k,x,γ)-motion. □

If $t_0 < 0$, then at first it is necessary to estimate the divergence of the trajectories for "backward motion". Let $\delta > 0$. Let us denote

$$\tilde\varepsilon(\delta,t_0,k) = \sup\big\{\rho(x,x') : \inf_{0\le t\le -t_0}\{\rho(f(t,x,k), f(t,x',k)\} \le \delta\big\}. \tag{4.7}$$

Lemma 4.11. *For any $\delta > 0$, $t_0 < 0$ and $k \in K$ $\tilde\varepsilon(\delta,t_0,k)$ is defined (finite). $\tilde\varepsilon(\delta,t_0,k) \to 0$ as $\delta \to 0$ uniformly by $k \in K$ and by t_0 from any compact segment $[t_1,t_2] \subset (-\infty,0]$.*

The proof can be easily obtained from the injectivity of $f(t,\cdot,k)$ and the compactness of X, K (similarly to Proposition 4.2).

Let us return to the proof of Proposition 4.10. Let $t_0 < 0$. Find $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that $\tilde\varepsilon(\chi(\varepsilon_0, T-t_0, T), t_0, K) + \tilde\varepsilon(\delta_0, t_0-T, k) < \gamma/2$. According to Proposition 4.2 and Lemma 4.11 this is possible. Let us take $\varepsilon_j < \varepsilon_0$ and choose from the sequence t_i^j $(i = 1,2,\dots)$ such monotone subsequence t_l $(l = 1,2,\dots)$ that $t_l > -t_0$, $\rho(f^{\varepsilon_i}(t_l|x,k), y) < \delta_0$ and $t_{l+1} - t_l > T - t_0$. Suppose

$$f^*(t) = \begin{cases} f^{\varepsilon_j}(t|x,k), & \text{if } t \notin [t_l+t_0, t_l] \text{ for any } l = 1,2,\dots; \\ f(t-t_l,y,k), & \text{if } t \in [t_l+t_0, t_l] \ (l = 1,2,\dots). \end{cases}$$

where $f^*(t)$ is (k,x,γ)-motion, and z is the ω-limit point of this motion.

Thus, $z \in \omega^\gamma(x,k)$ for any $\gamma > 0$. The proposition is proved.

Proposition 4.12. *Let $x \in \omega^0(x,k)$. Then for any $\varepsilon > 0$ there exists periodical (k,x,ε)-motion (This is a version of C^0-closing lemma).*

Proof. Let $x \in \omega^0(x,k)$, $\varepsilon > 0$, $\delta = \frac{1}{2}\delta(\frac{\varepsilon}{2}, T)$. There is (x,k,δ)-motion, and x is its ω-limit point: $x \in (f^\delta(t), x, k)$. There is such $t_0 > T$ that $\rho(f^\delta(t_0|x,k), x) < \delta$. Suppose

$$f^*(t) = \begin{cases} x, & \text{if } t = nt_0,\ n = 0,1,2,\dots; \\ f^\delta(t - nt_0|x,k), & \text{if } nt < t < (n+1)t_0. \end{cases}$$

Here $f^*(t)$ is a periodical (k,x,ε)-motion with the period t_0.

Thus, if $x \in \omega^0(x,k)$, then (k,x)-motion possesses the property of chain recurrence [23]. The inverse statement is also true: if for any $\varepsilon > 0$ there is a periodical (k,x,ε)-motion, then $x \in \omega^0(x,k)$ (this is obvious). □

Proposition 4.13. *Let $x_i \in X$, $k_i \in K$, $k_i \to k^*$, $\varepsilon_i > 0$, $\varepsilon_i \to 0$, $f^{\varepsilon_i}(t|x_i,k_i)$ be (k_i,x_i,ε_i)-motion, $y_i \in \omega(f^{\varepsilon_i}(t|x_i,k_i))$, $y_i \to y^*$. Then $y^* \in \omega^0(y,k)$. If in addition $x_i \to x^*$ then $y^* \in \omega^0(x,k)$.*

Proof. Let $\varepsilon > 0$ and $\delta = \frac{1}{2}\delta(\frac{\varepsilon}{2}, T)$. It is possible to find such i that $\varepsilon_i < \delta/2$, $\rho_K(k_i,k^*) < \delta$, and $\rho(f^{\varepsilon_i}(t_j|x_i,k_i), y^*) < \delta$ for some monotone sequence $t_j \to \infty$, $t_{j+1} - t_j > T$. Suppose

$$f^*(t) = \begin{cases} y^*, & \text{if } t = t_j - t_1\ (j = 1,2,\dots); \\ f^{\varepsilon_i}(t + t_1|x_i,k_i), & \text{otherwise,} \end{cases}$$

where $f^*(t)$ is (k^*,y^*,ε)-motion, $y^* \in \omega(f^*)$. Since $\varepsilon > 0$ was chosen arbitrarily, $y^* \in \omega^0(y^*,k^*)$. Suppose now that $x_i \to x^*$ and let us show that $y^* \in \omega^0(x^*,k^*)$. Let $\varepsilon > 0$, $\delta = \frac{1}{2}\delta(\frac{\varepsilon}{2}, T)$. Find such i for which $\varepsilon_\tau < \delta/2$, $\rho(x_i,x^*) < \delta$, $\rho_K(k_i,k^*) < \delta$ and there is such monotone subsequence $t_j \to \infty$ that $t_1 > T$, $t_{j+1} - t_j > T$; $\rho(f^{\varepsilon_i}(t_j|x_i,k_i), y^*) < \delta$. Suppose

$$f^*(t) = \begin{cases} x^*, & \text{if } t = 0; \\ y^*, & \text{if } t = t_j (j = 1,2,\dots); \\ f^{\varepsilon_i}(t|x_i,k_i), & \text{otherwise,} \end{cases}$$

where $f^*(t)$ is (k^*,x^*,ε)-motion and $y^* \in \omega(f^*)$. Consequently, $y \in \omega^0(x^*,k^*)$. □

Corollary 4.14. *If $x \in X$, $k \in K$, $y^* \in \omega^0(x,k)$ then $y^* \in \omega^0(y^*,k)$.*

Corollary 4.15. *Function $\omega^0(x,k)$ is upper semicontinuous in $X \times K$.*

Corollary 4.16. *For any $k \in K$*

$$\omega^0(k) := \bigcup_{x\in X} \omega^0(x,k) = \bigcup_{x\in X} \bigcap_{\varepsilon>0} \omega^\varepsilon(x,k) = \bigcap_{\varepsilon>0} \bigcup_{x\in X} \omega^\varepsilon(x,k). \tag{4.8}$$

Proof. The inclusion $\bigcup_{x\in X} \bigcap_{\varepsilon>0} \omega^\varepsilon(x,k) \subset \bigcap_{\varepsilon>0} \bigcup_{x\in X} \omega^\varepsilon(x,k)$ is obvious. To prove the equality, let us take arbitrary element y of the right part of this inclusion. For any natural n there is such $x_n \in X$ that $y \in \omega^{1/n}(x_n,k)$. Using Proposition 4.13, we obtain $y \in \omega^0(y,k) \subset \bigcup_{x\in X} \bigcap_{\varepsilon>0} \omega^\varepsilon(x,k)$, and this proves the corollary. □

Corollary 4.17. *For any $k \in K$ the set $\omega^0(k)$ is closed and k-invariant, and the function $\omega^0(k)$ is upper semicontinuous in K.*

Proof. The k-invariance of $\omega^0(k)$ follows from the k-invariance of $\omega^0(x,k)$ for any $x \in X$, $k \in K$ (Proposition 4.10), closure and semicontinuity follow from Proposition 4.13. □

Note that the statements analogous to the Corollaries 4.15. and 4.17 are incorrect for the true limit sets $\omega(x,k)$ and $\omega(k)$.

Proposition 4.18. *Let $k \in K$, $Q \subset \omega^0(k)$ and Q be connected. Then $Q \subset \omega^0(y,k)$ for any $y \in Q$.*

Proof. Let $y_1, y_2 \in Q$, $\varepsilon > 0$. Construct a periodical ε-motion which passes through the points y_1, y_2. Suppose $\delta = \frac{1}{2}\delta(\frac{\varepsilon}{2}, T)$. With Q being connected, there is such finite set $\{x_1, \dots, x_n\} \subset Q$ that $x_1 = y_1,\ x_n = y_2$ and $\rho(x_i, x_{i+1}) < \frac{1}{2}\delta$ $(i = 1, \dots, n-1)$ and for every $i = 1, \dots, n$ there is a periodical $(k, x_i, \delta/2)$-motion $f^{\delta/2}(t|x_i, k)$ (see Proposition 4.12 and Corollary 4.14). Let us choose for every $i = 1, \dots, n$ such $T_i > T$ that $f^{\delta/2}(T_i|x_i, k) = x_i$. Construct a periodical (k, y_1, ε)-motion passing through the points $x_1, \dots, x_n$ with the period $T_0 = 2\sum_{t=1}^{n} T_i - T_1 - T_n$: let $0 \le t \le T_0$, suppose

$$f^*(t) = \begin{cases} f^{\delta/2}(tx_1, k), & \text{if } 0 \le t < T; \\ f^{\delta/2}\Big(t - \sum_{i=1}^{j-1} T_i | x_i, k\Big), & \text{if } \sum_{i=1}^{j-1} T_i \le t < \sum_{i=1}^{j} T_i \\ & (j = 2, \dots, n); \\ f^{\delta/2}\Big(t - \sum_{i=1}^{n-1} T_i + \sum_{i=j+1}^{n} T_i | x_j, k\Big), & \text{if } \sum_{i=1}^{n-1} T_i + \sum_{i=j+1}^{n} T_i \\ & \le t < \sum_{i=1}^{n-1} T_i + \sum_{i=j}^{n} T_i. \end{cases}$$

If $mT_0 \le t < (m+1)T_0$, then $f^*(t) = f^*(t - mT_0)$. $f^*(t)$ is periodical (k, y_1, ε)-motion passing through y_2. Consequently (due to the arbitrary choice of $\varepsilon > 0$), $y_2 \in \omega^0(y, k)$ and (due to the arbitrary choice of $y_2 \in Q$) $Q \subset \omega^0(y_1, k)$. The proposition is proved. □

Definition 4.19. Let us say that (1.1) possesses $\omega^0(x,k)$-($\omega^0(k)$-)-*bifurcations*, if the function $\omega^0(x,k)$ ($\omega^0(k)$) is not lower semicontinuous (i.e. d-continuous) in $X \times K$. The point in which the lower semicontinuity gets broken is called the point of (corresponding) bifurcation.

Proposition 4.20. *If the system (1.1) possesses ω^0-bifurcations, then it possesses $\omega^0(x,k)$-bifurcations.*

Proof. Assume that there exist $\omega^0(k)$-bifurcations. Then there are $k^* \in K$ (the point of bifurcation), $x^* \in \omega^0(k^*)$, $\varepsilon > 0$, and sequence $k_i \to k$, such that $\rho^*(x^*, \omega^0(k_i)) > \varepsilon$ for any $i = 1, 2, \dots$.

Note that $\omega^0(x^*, k_i) \subset \omega^0(k_i)$, consequently, $\rho^*(x^*, \omega^0(x^*, k_i)) > \varepsilon$ for any i. However $x^* \in \omega^0(x^*, k^*)$ (Corollary 4.14). Therefore, $d(\omega^0(x^*, k^*), \omega^0(x^*, k_i)) > \varepsilon$, (x^*, k^*) is the point of $\omega^0(x,k)$-bifurcation. □

Proposition 4.21. *The sets of all points of discontinuity of the functions $\omega^0(x,k)$ and $\omega^0(k)$ are subsets of first category in $X \times K$ and K correspondingly. For each $k \in K$ the set of such $x \in X$ that (x,k) is the point of $\omega^0(x,k)$-bifurcation is $(k,+)$-invariant.*

Proof. The statement that the sets of points of $\omega^0(x,k)$- and $\omega^0(k)$-bifurcations are the sets of first category follows from the upper semicontinuity of the functions $\omega^0(x,k)$ and $\omega^0(k)$ and from known theorems about semicontinuous functions ([49, p.78-81]). Let us prove $(k,+)$-invariance. Note that for any $t > 0$, $\omega^0(f(t,x,k),k) = \omega^0(x,k)$. If $(x_i, k_i) \to (x,k)$, of $\omega^0(x,k)$-bifurcation for any $t > 0$. □

Let (x_0, k_0) be the point of $\omega^0(x,k)$-bifurcation, Γ be a set of such $\gamma > 0$ for which there exist $x^* \in \omega^0(x_0, k_0)$ and such sequence $(x_i, k_i) \to (x_0, k_0)$ that $\rho^*(x^*, \omega^0(x_i, k_i)) \geq \gamma$ for all $i = 1, 2, \dots$. Let us call the number $\tilde{\gamma} = \sup \Gamma$ *the value of discontinuity* of $\omega^0(x,k)$ in the point (x_0, k_0).

Proposition 4.22. *Let $\gamma > 0$. The set of those $(x,k) \in X \times K$, in which the function $\omega^0(x,k)$ is not continuous and the value of discontinuity $\tilde{\gamma} \geq \gamma$, is nowhere dense in $X \times K$.*

The proof can be easily obtained from the upper semicontinuity of the functions $\omega^0(x,k)$ and from known results about semicontinuous functions [49, p.78-81].

Proposition 4.23. *If there is such $\gamma > 0$ that for any $\varepsilon > 0$ there are $(x,k) \in X \times K$ for which $d(\omega^\varepsilon(x,k), \omega^0(x,k)) > \gamma$ then the system (1.1) possesses $\omega^0(x,k)$-bifurcations with the discontinuity $\tilde{\gamma} \geq \gamma$.*

Proof. Let the statement of the proposition be true for some $\gamma > 0$. Then there are sequences $\varepsilon_i > 0$, $\varepsilon_i \to 0$ and $(x_i, k_i) \in X \times K$, for which $d(\omega^{\varepsilon_i}(x_i, k_i), \omega^0(x_i, k_i)) > \gamma$. For every $i = 1, 2, \dots$ choose such point $y_i \in \omega^{\varepsilon_i}(x_i, k_i)$ that $\rho^*(y_i, \omega^0(x_i, k_i)) > \gamma$. Using the compactness of X and K, choose subsequence (preserving the denotations) in such a way that the new subsequences y_i and (x_i, k_i) would be convergent: $y_i \to y_0$, $(x_i, k_i) \to (x_0, k_0)$. According to Proposition 4.13 $y_0 \in \omega^0(x_0, k_0)$. For any $\varkappa > 0$ $\rho^*(y_0, \omega^0(x_i, k_i)) > \gamma - \varkappa$ from some $i = i(\varkappa)$. Therefore (x_0, k_0) is the point of $\omega^0(x,k)$-bifurcation with the discontinuity $\tilde{\gamma} \geq \gamma$. $\square$

4.2. **Slow Relaxations of ε-motions.** Let $\varepsilon > 0$, $f^\varepsilon(t|x,k)$ be (k,x,ε)-motion, $\gamma > 0$. Let us define the following relaxation times:

(a) $\tau_1^\varepsilon(t|x,k), \gamma) = \inf\{t \geq 0 : \rho^*(f^\varepsilon(t|x,k), \omega^\varepsilon(x,k)) < \gamma\}$;
(b) $\tau_2^\varepsilon(f^\varepsilon(t|x,k), \gamma) = \overline{\text{meas}}\{t \geq 0 : \rho^*(f^\varepsilon(t|x,k), \omega^\varepsilon(x,k)) \geq \gamma\}$;
(c) $\tau_3^\varepsilon(f^\varepsilon(t|x,k), \gamma) = \inf\{t \geq 0 : \rho^*(f^\varepsilon(t'|x,k), \omega^\varepsilon(x,k)) < \gamma \text{ for } t' > t\}$;
(d) $\eta_1^\varepsilon(f^\varepsilon(t|x,k), \gamma) = \inf\{t \geq 0 : \rho^*(t|x,k), \omega^\varepsilon(k)) < \gamma\}$;
(e) $\eta_2^\varepsilon(f^\varepsilon(t|x,k), \gamma) = \overline{\text{meas}}\{t \geq 0 : \rho^*(f^\varepsilon(t|x,k), \omega^\varepsilon(k)) \geq \gamma\}$;
(f) $\eta_3^\varepsilon(f^\varepsilon(t|x,k), = \inf\{t \geq 0 : \rho^*(f^\varepsilon(t'|x,k), \omega^\varepsilon(k)) < \gamma \text{ for } t' > t\}$.

Here $\overline{\text{meas}}\{\cdot\}$ is the external measure, $\omega^\varepsilon(k) = \bigcup_{x \in X} \omega^\varepsilon(x,k)$.

There are another three important relaxation times. They are bound up with the relaxation of a ε-motion to its ω-limit set. We do not consider them in this work.

Proposition 4.24. *For any $x \in X$, $k \in K$, $\varepsilon > 0$, $\gamma > 0$ and (k,x,ε)-motion $f^\varepsilon(t|x,k)$ the relaxation times (a)-(f) are defined (finite) and the inequalities $\tau_1^\varepsilon \leq \tau_2^\varepsilon \leq \tau_3^\varepsilon$, $\eta_1^\varepsilon \leq \eta_2^\varepsilon \leq \eta_3^\varepsilon$, $\tau_i^\varepsilon \geq \eta_i^\varepsilon$ ($i = 1, 2, 3$) are valid.*

Proof. The validity of the inequalities is obvious due to the corresponding inclusions relations between the sets or their complements from the right parts of (a)–(f). For the same reason it is sufficient to prove definiteness (finiteness) of $\tau_3^\varepsilon(f^\varepsilon(t|x,k), \gamma)$. Suppose the contrary: the set from the right part of (c) is empty for some $x \in X$, $k \in K$, $\gamma > 0$ and (k,x,ε)-motion $f^\varepsilon(t|x,k)$. Then there is such sequence $t_i \to \infty$ that $\rho^*(f^\varepsilon(t_i|x,k), \omega^\varepsilon(x,k)) \geq \gamma$. Due to the compactness of X, from the sequence $f^\varepsilon(t_i|x,k)$ can be chosen a convergent one. Denote its limit as y. Then y satisfies the definition of ω-limit point of (k,x,ε)-motion but does not lie in $\omega^\varepsilon(x,k)$. The obtained contradiction proves the existence (finiteness) of $\tau_3(f^\varepsilon(t|x,k), \gamma)$. $\square$

In connection with the introduced relaxation times (a)-(f) it is possible to study many different kinds of slow relaxations: infiniteness of the relaxation time for given ε, infiniteness for any ε small enough e.c. We confine ourselves to one variant only. The most attention will be paid to the times τ_1^ε and τ_3^ε.

Definition 4.25. We say that the system (1.1) possesses τ_i^0- (η_i^0-)*-slow relaxations*, if there are such $\gamma > 0$, sequences of numbers $\varepsilon_j > 0$, $\varepsilon_j \to 0$, of points $(x_j, k_j) \in X \times K$, and of $(k_j, x_j, \varepsilon_j)$-motions $f^{\varepsilon_j}(t|x_j, k_j)$ that $\tau_i^{\varepsilon_j}(f^{\varepsilon_j}(t|x_j, k_j), \gamma) \to \infty$ ($\eta_i^{\varepsilon_j}(f^{\varepsilon_j}(t|x_j, k_j), \gamma) \to \infty$) as $j \to \infty$.

Theorem 4.26. *The system (1.1) possesses τ_3^0-slow relaxations if and only if it possesses $\omega^0(x, k)$-bifurcations.*

Proof. Suppose that the system (1.1) possesses τ_0^3-slow relaxations: There are such $\gamma > 0$, sequences of numbers $\varepsilon_j > 0$, $\varepsilon_j \to 0$, of points $(x_j, k_j) \in X \times K$ and of $(k_j, x_j, \varepsilon_j)$-motions $f^{\varepsilon_j}(t|x_j, k_j)$ that

$$\tau_3^{\varepsilon_j}(f^{\varepsilon_j}(t|x, k), \gamma) \to \infty \tag{4.9}$$

as $j \to \infty$. Using the compactness of $X \times K$, choose from the sequence (x_j, k_j) a convergent one (preserving the denotations): $(x_j, k_j) \to (x^*, k^*)$. According to the definition of the relaxation time τ_3^ε there is such sequence $t_j \to \infty$ that

$$\rho^*(f^{\varepsilon_j}(t_j|x_j, k_j), \omega^{\varepsilon_j}(x_j, k_j)) \geq \gamma. \tag{4.10}$$

□

Choose again from (x_j, k_j) a sequence (preserving the denotations) in such a manner, that the sequence $y_j = f^{\varepsilon_j}(t_j|x_j, k_j)$ would be convergent: $y_j \to y^* \in X$. According to Proposition 4.7 (k^*, y^*)-motion is whole and $f^{\varepsilon_j}(t_j + t|x_j, k_j) \to f(t, y^*, k^*)$ uniformly over any compact segment $t \in [t_1, t_2]$. Two cases are possible: $\omega^0(y^*, k^*) \bigcap \alpha(y^*, k^*) \neq \varnothing$ or $\omega^0(y^*, k^*) \bigcap \alpha(y^*, k^*) = \varnothing$. We shall show that in the first case there are $\omega^0(x, k)$-bifurcations with the discontinuity $\tilde{\gamma} \geq \gamma/2$ ((y^*, k^*) is the point of bifurcation), in the second case there are $\omega^0(x, k)$-bifurcations too ((p, k^*) is the point of bifurcation, where p is any element from $\alpha(y^*, k^*)$), but the value of discontinuity can be less than $\gamma/2$. We need four lemmas.

Lemma 4.27. *Let $x \in X$, $k \in K$, $\varepsilon > 0$, $f^\varepsilon(t|x, k)$ be (k, x, k)-motion, $t > 0, y = f^\varepsilon(t|x, k)$. Then $\omega^0(y, k) \subset \omega^{2\varepsilon+\sigma}(x, k)$ for any $\sigma > 0$.*

The proof is an obvious consequence of the definitions and Proposition 4.3.

Lemma 4.28. *Let $x \in X$, $k \in K$, $t_0 > 0$, $y = f(t_0, x, k)$, $,\delta > 0$, $\varepsilon = \varepsilon(\chi(\delta, t_0, T), T) + \delta$. Then $\omega^\delta(x, k) \subset \omega^\varepsilon(y, k)$.*

Proof. Let $f^\delta(t|x, k)$ be (k, x, δ)-motion. Then

$$f^*(t) = \begin{cases} y, & \text{if } t = 0; \\ f^\delta(t + t_0|x, k), & \text{if } t > 0, \end{cases} \tag{4.11}$$

is (k, y, ε)-motion, $\omega(f^*) \subset \omega^\varepsilon(y, k)$, and $\omega(f^*) = \omega(f^\delta(t|x, k))$. □

Since $\varepsilon(\chi(\delta, t_0, T), T) \to 0$, for $\delta \to 0$ we obtain the following result.

Corollary 4.29. *Let $x \in X$, $k \in K$, $t_0 > 0$, $y = f(t_0, x, k)$. Then $\omega^0(x, k) = \omega^0(y, k)$.*

Lemma 4.30. *Let (k,y)-motion be whole and $\omega^0(y,k)\bigcap\alpha(y,k)\neq\varnothing$. Then $y\in\omega^0(y,k)$.*

Proof. Let $\varepsilon>0$, $p\in\omega^0(y,k)\bigcap\alpha(y,k)$. Construct a periodical (k,y,ε)-motion. Suppose that $\delta=\frac{1}{2}\delta(\frac{\varepsilon}{2},T)$. There is such $t_1>T$ that for some (k,y,δ)-motion $f^\delta(t|y,k)\rho(f^\delta(t_1|y,k),p)<\delta$. There is also such $t_2<0$ that $\rho(f(t_2,y,k),p)<\delta$. Then it is possible to construct a periodical (k,y,ε)-motion, due to the arbitrariness of $\varepsilon>0$ and $y\in\omega^0(y,k)$.

Lemma 4.31. *Let $y\in X$, $k\in K$, (k,y)-motion be whole. Then for any $p\in\alpha(y,k)$ $\omega^0(p,k)\supset\alpha(y,k)$.*

Proof. Let $p\in\alpha(y,k)$, $\varepsilon>0$. Construct a periodical (k,p,ε)-motion. Suppose that $\delta=\frac{1}{2}\delta(\varepsilon,T)$. There are two such $t_1,t_2<0$ that $t_1-t_2>T$ and $\rho(f(t_{1,2},y,k),p)<\delta$. Suppose

$$f^*(t)=\begin{cases}p, & \text{if } t=0 \text{ or } t=t_1-t_2;\\ f(t+t_2|y,k), & \text{if } 0<t<t_1-t_2,\end{cases}$$

where $f^*(t+n(t_1-t_2))=f^*(t)$. Periodical (k,p,ε)-motion is constructed. Since $\varepsilon>0$ is arbitrary, $p\in\omega^0(p,k)$. Using Proposition 4.18 and the connectivity of $\alpha(y,k)$, we obtain the required: $\alpha(y,k)\subset\omega^0(p,k)$. □

Let us return to the proof of Theorem 4.26. Note that according to Proposition 4.23 if there are not $\omega^0(x,k)$-bifurcations with the discontinuity $\tilde\gamma\geq\gamma/2$, then from some $\varepsilon_0>0$ (for $0<\varepsilon\leq\varepsilon_0$)$d(\omega^\varepsilon(x,k),\omega^0(x,k))\leq\gamma/2$ for any $x\in X$, $k\in K$. Suppose that the system has τ_3^0-slow relaxations and does not possess $\omega^0(x,k)$-bifurcations with the discontinuity $\tilde\gamma\geq\gamma/2$. Then from (4.10) it follows that for $0<\varepsilon\leq\varepsilon_0$

$$p^*(f^{\varepsilon_j}(t_j|x_j,k_j),\omega^\varepsilon(x_j,k_j))\geq\gamma/2. \tag{4.12}$$

According to Lemma 4.27 $\omega^0(y_j,k_j)\subset\omega^{3\varepsilon_j}(x_i,k_j)$. Let $0<\varkappa<\gamma/2$. From some j_0 (for $j>j_0$) $3\varepsilon_j<\varepsilon_0$ and $\rho(f^{\varepsilon_j}(t_j|x_j,k_j),y^*)<\gamma/2-\varkappa$. For $j>j_0$ from (4.12) we obtain

$$\rho^*(y^*,\omega^0(y_j,k_j))>\varkappa. \tag{4.13}$$

If $\omega^0(y^*,k^*)\bigcap\alpha(y^*,k^*)\neq\varnothing$, then from (4.13) and Lemma 4.30 follows the existence of $\omega^0(x,k)$-bifurcations with the discontinuity $\tilde\gamma\geq\gamma/2$. The obtained contradiction (if $\omega^0(y^*,k^*)\bigcap\alpha(y^*,k^*)\neq\varnothing$ and there are not $\omega^0(x,k)$-bifurcations with the discontinuity $\tilde\gamma\geq\gamma/2$, then they are) proves in this case the existence of $\omega^0(x,k)$-bifurcations with the discontinuity $\tilde\gamma\geq\gamma/2$. If $\omega^0(y^*,k^*)\bigcap\alpha(y^*,k^*)=\varnothing$, then there also exist $\omega^0(x,k)$-bifurcations. Really, let $p\in\alpha(y^*,k^*)$. Consider such a sequence $t_i\to-\infty$ that $f(t_i,y^*,k^*)\to p$. According to Corollary 4.29 $\omega^0(f(t_i,y^*,k^*),k^*)=\omega^0(y^*,k^*)$, consequently, according to Lemma 4.31, $d(\omega^0(p,k^*),\omega^0(f(t_i,y^*,k^*))\geq d(\alpha(y^*,k^*),\omega^0(y^*,k^*))>0$ and there are $\omega^0(x,k)$-bifurcations. The theorem is proved. □

Note that inverse to Theorem 4.26 is not true: for unconnected X from the existence of $\omega^0(x,k)$-bifurcations does not follow the existence of τ_3^0-slow relaxations.

Example 4.32. ($\omega^0(x,k)$-bifurcations without τ_3^0 -slow relaxations). Let X be a subset of plane, consisting of points with coordinates $(\frac{1}{n},0)$ and vertical segment $J=\{(x,y)|x=0,\ y\in[-1,1]\}$. Let us consider on X a trivial dynamical system $f(t,x)\equiv x$. In this case $\omega_f^0((\frac{1}{n},0))=\{(\frac{1}{n},0)\}$, $\omega_f^0((0,y))=J$. There are $\omega^0(x,k)$

bifurcations: $(\frac{1}{n},0)\to(0,0)$ as $n\to\infty$, $\omega_f^0((\frac{1}{n},0))=\left\{(\frac{1}{n},0)\right\}$, $\omega_f^0((0,0))=J$. But there are not τ_0^3 -slow relaxations: $\tau_3^\varepsilon(f^\varepsilon(t|x),\gamma)=0$ for any (x,ε)-motion $f^\varepsilon(t|x)$ and $\gamma>0$. This is associated with the fact that for any (x,ε)-motion and arbitrary $t_0\ge 0$ the function

$$f^*(t)=\begin{cases} f^\varepsilon(t|x), & \text{if } 0\le t\le t_0;\\ f^\varepsilon(t_0|x), & \text{if } t\ge t_0\end{cases}$$

is (x,ε)-motion too, consequently, each (x,ε)-trajectory consists of the points of $\omega_f^\varepsilon(x)$.

For connected X the existence of $\omega^0(x,k)$-bifurcations is equivalent to the existence of τ_3^0-slow relaxations.

Theorem 4.33. *Let X be connected. In this case the system (1.1) possesses τ_3^0 -slow relaxations if and only if it possesses $\omega^0(x,k)$-bifurcations.*

One part of Theorem 4.33 (only if) follows from the Theorem 4.26. Let us put off the proof of the other part of Theorem 4.33 till Subsection 4.4, and the remained part of the present subsection devote to the study of the set of singularities of the relaxation time τ_2 for perturbed motions.

Theorem 4.34. *Let $\gamma>0$, $\varepsilon_i>0$, $\varepsilon_i\to 0$, $(x_i,k_i)\in X\times K$, $f^{\varepsilon_i}(t|x_i,k_i)$ be (k_i,x_i,ε_i)-motions, $\tau_2^{\varepsilon_i}(f^{\varepsilon_i}(t|x_i,k_i),\gamma)\to\infty$. Then any limit point of the sequence $\{(x_i,k_i)\}$ is a point of $\omega^0(x,k)$-bifurcation with the discontinuity $\tilde\gamma\ge\gamma$.*

Proof. Let (x_0,k_0) be limit point of the sequence $\{(x_i,k_i)\}$. Turning to subsequence and preserving the denotation, let us write down $(x_i,k_i)\to(x_0,k_0)$. Let $X=\bigcup_{j=1}^n V_j$ be a finite open covering of X. Note that

$$\begin{aligned}&\tau_2^{\varepsilon_i}(f^{\varepsilon_i}(t|x_i,k_i)\gamma)\\ &\le\sum_{j=1}^n\overline{\operatorname{meas}}\{t\ge 0: f^{\varepsilon_j}(t|x_i,k_i)\in V_j,\rho^*(f^{\varepsilon_i}(t|x_i,k_i),\omega^{\varepsilon_i}(x_i,k_i))\ge\gamma\}.\end{aligned}$$

Using this remark, consider a sequence of reducing coverings. Let us find (similarly to the proof of Theorem 3.2) such $y_0\in X$ and subsequence in $\{(x_i,k_i)\}$ (preserving for it the previous denotation) that for any neighbourhood V of the point y_0

$$\overline{\operatorname{meas}}\{t\ge 0: f^{\varepsilon_i}(t|x_i,k_i)\in V,\ \rho(f^{\varepsilon_i}(t|x_i,k_i),\omega^{\varepsilon_i}(x_i,k_i))\ge\gamma\}\to\infty.$$

Let us show that $y_0\in\omega^0(x_0,k_0)$. Let $\varepsilon>0$. Construct such a (k_0,x_0,ε)-motion that y_0 is its ω-limit point. Suppose $\delta=\frac{1}{2}\delta(\frac{\varepsilon}{3},T)$. From some i_0 (for $i>i_0$) the following inequalities are true: $\rho(x_i,x_0)<\delta$, $\rho_K(k_i,k_0)<\delta$, $\varepsilon_i<\delta$, and

$$\overline{\operatorname{meas}}\{t\ge 0:\rho(f^{\varepsilon_i}(t|x_i,k_i),y_0)<\delta,\ \rho^*(f^{\varepsilon_i}(t|x_i,k_i),\ \omega^{\varepsilon_i}(x_i,k_i))\ge\gamma\}>T.$$

On account of the last of these inequalities for every $i>i_0$ there are such $t_1,t_2>0$ that $t_2-t_1>T$ and $\rho(f^{\varepsilon_i}(t_{1,2}|x_i,k_i),y_0)<\delta$. Let $i>i_0$. Suppose

$$f^*(t)=\begin{cases} x_0, & \text{if } t=0;\\ f^{\varepsilon_i}(t|x_i,k_i), & \text{if } 0<t<t_2,\ t\ne t_1;\\ y_0, & \text{if } t=t_1.\end{cases}$$

If $t\ge t_1$, then $f^*(t+n(t_2-t_1))=f^*(t)$, $n=0,1,\dots$. By virtue of the construction f^* is (k_0,x_0,ε)-motion, $y_0\in\omega(f^*)$. Consequently (due to the arbitrary choice of $\varepsilon>0$), $y_0\in\omega^0(x_0,k_0)$. Our choice of the point y_0 guarantees that $y_0\notin\omega^{\varepsilon_i}(x_i,k_i)$

from some i. Furthermore, for any $\varkappa > 0$ exists such $i = i(\varkappa)$ that for $i > i(\varkappa)$ $\rho^*(y_0, \omega^{\varepsilon_i}(x_i, k_i) > \gamma - \varkappa$. Consequently, (x_0, k_0) is the point of $\omega^0(x,k)$-bifurcation with the discontinuity $\tilde{\gamma} \geq \gamma$. □

Corollary 4.35. *Let $\gamma > 0$. The set of all points $(x,k) \in X \times K$, for which there are such sequences of numbers $\varepsilon_i > 0$, $\varepsilon \to 0$, of points $(x_i, k_i) \to (x,k)$, and of $(k_i, x_i, \varepsilon_i)$-motions $f^{\varepsilon_i}(t|x_i, k_i)$ that $\tau_2^{\varepsilon_i}(f^{\varepsilon_i}(t|x_i, k_i), \gamma) \to \infty$, is nowhere dense in $X \times K$. The union of all $\gamma > 0$ these sets (for all $\gamma > 0$) is a set of first category in $X \times K$.*

4.3. **Smale Order and Smale Diagram for General Dynamical Systems.** Everywhere in this subsection one semiflow of homeomorphisms f on X is studied. We study here the equivalence and preorder relations generated by semiflow.

Definition 4.36. Let $x_1, x_2 \in X$. Say that points x_1 and x_2 are f-*equivalent* (denotation $x_1 \sim x_2$), if for any $\varepsilon > 0$ there are such (x_1, ε)- and (x_2, ε)-motions $f^\varepsilon(t|x_1)$ and $f^\varepsilon(t|x_2)$ that for some $t_1, t_2 > 0$

$$f^\varepsilon(t_1|x_1) = x_2, \; f^\varepsilon(t_2|x_2) = x_1.$$

Proposition 4.37. *The relation $\sim$ is a closed f-invariant equivalence relation: the set of pairs (x_1, x_2), for which $x_1 \sim x_2$ is closed in $X \times K$; if $x_1 \sim x_2$ and $x_1 \neq x_2$, then x_1- and x_2-motions are whole and for any $t \in (-\infty, \infty)$ $f(t, x_1) \sim f(t, x_2)$. If $x_1 \neq x_2$, then $x_1 \sim x_2$ if and only if $\omega_f^0(x_1) = \omega_f^0(x_2)$, $x_1 \in \omega_f^0(x_1)$, $x_2 \in \omega_f^0(x_2)$.*

Compare with [76, Ch.6, Sec. 1], where analogous propositions are proved for equivalence relation defined by action functional.

Proof. Symmetry and reflexivity of the relation $\sim$ are obvious. Let us prove its transitivity. Let $x_1 \sim x_2$, $x_2 \sim x_3$, $\varepsilon > 0$. Construct ε-motions which go from x_1 to x_3, and from x_3 to x_1, gluing together δ-motions, going from x_1 to x_2, from x_2 to x_3 and from x_3 to x_2, from x_2 to x_1. Suppose that $\delta = \varepsilon/4$. Then, according to Proposition 4.3, as a result of the gluing we obtain ε-motions with required properties. Therefore $x_1 \sim x_3$.

Let us consider the closure of the relation $\sim$. Let $\varepsilon > 0$, $x_i, y_i \in X$, $x_i \to x$, $y_i \to y$, $x_i \sim y_i$. Suppose $\delta = \frac{1}{2}\delta(\frac{\varepsilon}{3}, T)$. There is such i that $\rho(x_i, x) < \delta$ and $\rho(y_i, y) < \delta$. Since $x_i \sim y_i$, there are binding them δ-motions $f^\delta(t|y_i)$ and $f^\delta(t|x_i):$ $f^\delta(t_{1i}|x_i) = y_i$, $f^\delta(t_{2i}|y_i) = x_i$, for which $t_{1i}, t_{2i} > 0$. Suppose that

$$f_1^*(t) = \begin{cases} x, & \text{if } t = 0; \\ y, & \text{if } t = t_{1i}; \\ f^\delta(t|x_i), & \text{if } t \neq 0, t_{1i}, \end{cases}$$

$$f^*(t) = \begin{cases} y, & \text{if } t = 0; \\ x, & \text{if } t = t_{2i}; \\ f^\delta(t|y_i), & \text{if } t \neq 0, t_{2i}, \end{cases}$$

here f_1^* and f_2^* are correspondingly (x, ε)- and (y, ε)-motions, $f_1^*(t_{1i}) = y$, $f^*(t_{2i}) = x$. Since was chosen arbitrarily, it is proved that $x \sim y$. Let $x_{1,2} \in X$, $x_1 \sim x_2$, $x_1 \neq x_2$. Show that $\omega_f^0(x_1) = \omega_f^0(x_2)$ and $x_{1,2} \in \omega_f^0(x_1)$. Let $\varepsilon > 0$, $y \in \omega_f^0(x_2)$. Prove that $y \in \omega_f^\varepsilon(x_1)$. Really, let $f^{\varepsilon/3}(t|x_1)$ be $(x_1, \varepsilon/3)$-motion, $f^{\varepsilon/3}(t_0|x_1) =$

x_2, $f^{\varepsilon/3}(t|x_2)$ be $(x_2, \varepsilon/3)$-motion, $y \in \omega(f^{\varepsilon/3}(f^{\varepsilon/3}(t|x_2))$. Suppose

$$f^*(t) = \begin{cases} f^{\varepsilon/3}(t|x_1), & \text{if } 0 \le t \le t_0; \\ f^{\varepsilon/3}(t_0|x_2), & \text{if } t > t_0, \end{cases}$$

here f^* is (x_1, ε)-motion (in accordance with Proposition 4.3), $y \in \omega(f^*)$. Consequently, $y \in \omega^\varepsilon(x_1)$ and, due to arbitrary choice of $\varepsilon > 0$, $y \in \omega^0(x_1)$. Similarly $\omega^0(x_1) \subset \omega^0(x_2)$; therefore, $\omega^0(x_1) = \omega^0(x_2)$. It can be shown that $x_1 \in \omega^0_f(x_1), x_2 \in \omega^0_f(x_2)$. According to Proposition 4.10, the sets $\omega^0_f(x_{1,2})$ are invariant and $x_{1,2}$-motions are whole.

Now, let us show that if $x_2 \in \omega^0_f(x_1)$ and $x_1 \in \omega^0_f(x_2)$ then $x_1 \sim x_2$. Let $x_2 \in \omega^0_f(x_1)$, $\varepsilon > 0$. Construct a ε-motion going from x_1 to x_2. Suppose that $\delta = \delta = \frac{1}{2}\delta(\frac{\varepsilon}{2}, T)$. There is such a (x_1, δ)-motion that x_2 is its ω-limit point: $f^\delta(t_1|x_1) \to x_2$, $t_1 \to \infty$. There is such $t_0 > 0$ that $\rho(f^\delta(t_0|x_1), x_2) < \delta$. Suppose that

$$f^*(t) = \begin{cases} f^\delta(t|x_1), & \text{if } t \ne t_0; \\ x_2, & \text{if } t = t_0, \end{cases}$$

where $f^*(t)$ is (x_1, ε)-motion and $f^*(t_0) = x_2$. Similarly, if $x_1 \in \omega^0_f(x_2)$, then for any $\varepsilon > 0$ exists (x_2, ε)-motion which goes from x_2 to x_1. Thus, if $x_1 \ne x_2$, then $x_1 \sim x_2$ if and only if $x_1 \in \omega^0(x_2)$ and $x_2 \in \omega^0_f(x_1)$. In this case $\omega^0_f(x_1) = \omega^0_f(x_2)$. The invariance of the relation $\sim$ follows now from the invariance of the sets $\omega^0_f(x)$ and the fact that $\omega^0_f(x) = \omega^0_f(f(t,x))$ if $f(t,x)$ is defined. The proposition is proved. □

Let us remind, that topological space is called ***totally disconnected*** if there exist a base of topology, consisting of sets which are simultaneously open and closed. Simple examples of such spaces are discrete space and Cantor discontinuum.

Proposition 4.38. *Factor space $\omega^0_f/\sim$ is compact and totally disconnected.*

Proof. This proposition follows directly from Propositions 4.18, 4.37 and Corollary 4.17. □

Definition 4.39 (Preorder, generated by semiflow)**.** Let $x_1, x_2 \in X$. Let say $x_1 \succsim x_2$ if for any $\varepsilon > 0$ exists such a (x_1, ε)-motion $f^\varepsilon(t|x_1)$ that $f^\varepsilon(t_0|x_1) = x_2$ for some $t_0 \ge 0$.

Proposition 4.40. *The relation $\succsim$ is a closed preorder relation on X.*

Proof. Transitivity of $\succsim$ easily follows from Proposition 4.3 about gluing of ε-motions. The reflexivity is obvious. The closure can be proved similarly to the proof of the closure of $\sim$ (Proposition 4.37, practically literal coincidence). □

Proposition 4.41. *Let $x \in X$. Then*

$$\omega^0_f(x) = \{y \in \omega^0_f : x \succsim y\}.$$

Proof. Let $y \in \omega^0_f(x)$. Let us show that $x \succsim y$. Let $\varepsilon > 0$. Construct a ε-motion going from x to y. Suppose $\delta = \frac{1}{2}\delta(\frac{\varepsilon}{3}, T)$. There is such a (x, δ)-motion $f^\delta(t|x)$ that y is its ω-limit point: $f^\delta(t_j|x) \to$ for some sequence $t_j \to \infty$. There is such $t_0 > 0$ that $\rho(f^\delta(t_0|x), y) < \delta$. Suppose that

$$f^*(t) = \begin{cases} f^\delta(t|x), & \text{if } t \ne t_0; \\ y, & \text{if } t = t_0, \end{cases}$$

here $f^*(t)$ is (x,ε)-motion and $f^*(t_0) = y$. Consequently, $x \succsim y$. Now suppose that $y \in \omega_f^0$, $x \succsim y$. Let us show that $y \in \omega_f^0(x)$. Let $\varepsilon > 0$. Construct such a (x,ε)-motion that y is its ω-limit point. To do this, use Proposition 4.12 and Corollary 4.14 and construct a periodical $(y,\varepsilon/3)$-motion $f^{\varepsilon/3}$: $f^{\varepsilon/3}(nt_0|y) = y$, $n = 0, 1, \dots$.. Glue it together with $(x,\varepsilon/3)$-motion going from x to $y(f^{\varepsilon/3}(t_1|x) = y)$:

$$f^*(t) = \begin{cases} f^{\varepsilon/3}(t|x), & \text{if } 0 \le t \le t_1; \\ f^{\varepsilon/3}(t - t_1|y), & \text{if } t \ge t_1, \end{cases}$$

where $f^*(t)$ is (x,ε)-motion, $y \in \omega(f^*)$, consequently ($\varepsilon > 0$ is arbitrary), $y \in \omega_f^0(x)$. The proposition is proved. □

We say that the set $Y \subset \omega_f^0$ is *saturated downwards*, if for any $y \in Y$,

$$\{x \in \omega_f^0 | y \succsim x\} \subset Y.$$

It is obvious that every saturated downwards subset in ω_f^0 is saturated also for the equivalence relation $\sim$.

Proposition 4.42. *Let $Y \subset \omega_f^0$ be open (in ω_f^0) saturated downwards set. Then the set $At^0(Y) = \{x \in X : \omega_f^0(x) \subset Y\}$ is open in X.*

Proof. Suppose the contrary. Let $x \in At^0(Y)$, $x_i \to x$ and for every $i = 1, 2, \dots$ there is $y_i \in \omega_f^0(x_i) \setminus Y$. On account of the compactness of $\omega_f^0 \setminus Y$ there is a subsequence in $\{y_i\}$, which converges to $y^* \in \omega_f^0 \setminus Y$. Let us turn to corresponding subsequences in $\{x_i\}$, $\{y_i\}$, preserving the denotations: $y_i \to y^*$. Let us show that $y \in \omega_f^0(x)$. Let $\varepsilon > 0$. Construct a ε-motion going from x to y. Suppose that $\delta = \frac{1}{2}\delta(\frac{\varepsilon}{3}, T)$. From some i_0 $\rho(x_i, x) < \delta$ and $\rho(y_i, y^*) < \delta$. Let $i > i_0$. There is (x_i, δ)-motion going from x_i to y_i : $f^\delta(t_0|x_i) = y_i$ (according to Proposition 4.41). Suppose that

$$f^*(t) = \begin{cases} f^*, & \text{if } t = 0; \\ y^*, & \text{if } t = t_0; \\ f^\delta(t|x_i), & \text{if } t \ne 0, t_0, \end{cases}$$

where f^* is $(x,)$-motion going from x to y^*. Since $\varepsilon > 0$ is arbitrary, from this it follows that $x \succsim y^*$ and, according to Proposition 4.41, $y^* \in \omega_f^0(x)$. The obtained contradiction ($y^* \in \omega_f^0(x) \setminus Y$, but $\omega_f^0(x) subset Y$) proves the proposition. □

Theorem 4.43. *Let $x \in X$ be a point of $\omega_f^0(x)$-bifurcation. Then there is such open in ω_f^0 saturated downwards set W that $x \in \partial At^0(W)$.*

Proof. Let $x \in X$ be a point of ω_f^0-bifurcation: there are such sequence $x_i \to x$ and such $y^* \in \omega_f^0(x)$ that $\rho^*(y^*, \omega_f^0(x_i)) > \alpha > 0$ for all $i = 1, 2, \dots$. Let us consider the set $\omega = \bigcup_{i=1}^{\infty} \omega_f^0(x_i)$. The set ω is saturated downwards (according to Proposition 4.41). We have to prove that it possesses open (in ω_f^0) saturated downwards neighbourhood which does not contain y^*. Beforehand let us prove the following lemma. □

Lemma 4.44. *Let $y_1, y_2 \in \omega_f^0$, $y_1 \notin \omega_f^0(y_2)$. Then there exists such an open saturated downwards set Y that $y_2 \in Y$, $y_1 \notin Y$, and $Y \subset \omega_f^\varepsilon(y_2)$ for some $\varepsilon > 0$.*

Proof. $\omega_f^0(y_2) = \bigcap_{\varepsilon>0} \omega^\varepsilon(y_2)$ (according to Proposition 4.9). There are such $\varepsilon_0 > 0$, $\tau > 0$ that if $0 < \varepsilon \le \varepsilon_0$ then $\rho^*(y_1, \omega^\varepsilon(y_2)) > \lambda$. This follows from the compactness of X and the so-called Shura-Bura lemma [2, p.171-172]: let a subset V of compact space be intersection of some family of closed sets. Then for any neighbourhood of V exists a finite collection of sets from that family, intersection of which contains in given neighbourhood. Note now that if $\rho^*(z, \omega_f^0(y_2)) < \delta = \frac{1}{2}\delta(\frac{\varepsilon}{3}, T)$ and $z \in \omega_f^0$, then $z \in \omega_f^\varepsilon(y_2)$. Really, in this case there are such $p \in \omega_f^0(y_2)$, (y_2, δ)-motion $f^\delta(t|y_2)$, and monotone sequence $t_i \to \infty$ that $\rho(z,p) < \delta$, $t_{j+1} - t_j > T$ and $\rho(f^\delta(t_i|y_2), p) < \delta$. Suppose

$$f^*(t) = \begin{cases} f^\delta(t|y_2), & \text{if } t \neq t_j; \\ z, & \text{if } t = t_j, \end{cases}$$

here $f^*(t)$ is (y_2, ε)-motion and $z \in \omega(f^*) \subset \omega^\varepsilon(y_2)$. Strengthen somewhat this statement. Let $z \in \omega_f^0$ and for some $n > 0$ exist such chain $\{z_1, z_2, \dots, z_n\} \in \omega_f^0$ that $y_2 = z_1$, $z = z_n$ and for any $i = 1, 2, \dots, n-1$ either $z_i \succsim z_{i+1}$ or $\rho(z_i, z_{i+1}) < \delta = \frac{1}{2}\delta(\frac{\varepsilon}{7}, T)$. Then $z \in \omega^\varepsilon(y_2)$ and such a (y_2, ε)-motion that z is its ω-limit point is constructed as follows. If $z_i \succsim z_{i+1}$, then find (z_i, δ)-motion going from z_i to z_{i+1}, and for every $i = 1, \dots, n$ find a periodical (z_i, δ)-motion. If $z_1 \succsim z_2$, then suppose that f_1^* is (z_1, δ)-motion going from z_1 to z_2, $f_1^*(t_1) = z_2$; and if $\rho(z_1, z_2) < \delta$, $z_1 \not\succsim z_2$, then suppose that f_1^* is a periodical (z_2, δ)-motion and $t_1 > 0$ is such a number that $t_1 > T$ and $f_1^*(t_1) = z_2$. Let $f_1^*, \dots, f_k^*$, $t_1, \dots, t_k$ be already determined. Determine f_{k+1}^*. Four cases are possible:

1. (1) f^* is periodical (z_i, δ)-motion, $i < n$, $z_i \succsim z_{i+1}$, then f_{k+1}^* is (z_i, δ)-motion going from z_i to z_{i+1} $f_{k+1}^*(t_{k+1}) = z_{i+1}$;
2. (2) f_k^* is periodical (z_i, δ)-motion, $i < n$, $\rho(z_i, z_{i+1}) < \delta$, then f_{k+1}^* is periodical (z_{i+1}, δ)-motion, $f_{k+1}^*(t_{k+1}) = z_{i+1}$, $t_{k+1} > T$;
3. (3) f_k^* is (z_i, δ)-motion going from z_i to z_{i+1}, then f_{k+1}^* is periodical (z_{i+1}, δ)-motion, $f_{k+1}^*(t_{k+1}) = z_{i+1}$, $t_{k+1} > T$;
4. (4) f_k^* is periodical (z_n, δ)-motion, then the constructing is finished.

After constructing the whole chain of δ-motions f_k^* and time moments t_k, let us denote the number of its elements by q and assume that

$$f^*(t) = \begin{cases} z_1, & \text{if } t = 0; \\ f_1^*(t), & \text{if } 0 < t \le t_1; \\ f_k^*\Big(t - \sum_{j=1}^{k-1} t_j\Big), & \text{if } \sum_{j=1}^{k-1} t_j < t \le \sum_{j=1}^{k} t_j (k < q); \\ f_q^*\Big(t - \sum_{j=1}^{q-1} t_j\Big), & \text{if } t > \sum_{j=1}^{q-1} t_j. \end{cases}$$

Here $f^*(t)$ is (y_2, ε)-motion, and $z_n = z$ is its ω-limit point. The set of those $z \in \omega_f^0$ for which exist such chains $z_1, \dots, z_n$ $(n = 1, 2, \dots)$ is an openly-closed (in ω_f^0) subset of ω_f^0, saturated downwards. Supposing $0 < \varepsilon \le \varepsilon_0$, we obtain the needed result. Even more strong statement was proved: Y can be chosen openly-closed (in ω_f^0), not only open.

Let us return to the proof of Theorem 4.43. Since $\omega = \bigcup_{i=1}^\infty \omega_f^0(x_i)$ and each $z \in \omega_f^0(x_i)$ has an open (in ω_f^0) saturated downwards neighbourhood W_z which does not contain y^*, then the union of these neighborhoods is an open (in ω_f^0) saturated downwards set which includes ω but does not contain y^*. Denote this set by W:

$W = \bigcup_{z\in\omega} W_z$. Since $x_i \in At^0(W)$, $x \notin At^0(W)$ and $x_i \to x$, then $x \in \partial At^0(W)$. The theorem is proved. □

The following proposition will be used in Subsection 4.4 when studying slow relaxations of one perturbed system.

Proposition 4.45. *Let X be connected, ω_f^0 be disconnected. Then there is such $x \in X$ that x-motion is whole and $x \notin \omega_f^0$. There is also such partition of ω_f^0 into openly-closed (in ω_f^0) subsets:*

$$\omega_f^0 = W_1 \cup W_2, \quad W \cap W_2 = \varnothing, \quad \alpha_f(x) \subset W_1 \quad \text{but} \quad \omega_f^0(x) \subset W_2.$$

Proof. Repeating the proof of Lemma 3.9 (the repetition is practically literal, ω_f^0 should be substituted instead of $\overline{\omega_f}$), we obtain that ω_f^0 is not Lyapunov stable. Then, according to Lemma 3.6, there is such $x \in X$ that x-motion is whole and $x \notin \omega_f^0$. Note now that the set $\alpha_f(x)$ lies in equivalence class by the relation $\sim$, and the set ω_f^0 is saturated by the relation $\sim$ (Proposition 4.37, Lemma 4.31). $\alpha_f(x) \bigcap \omega_f^0(x) = \varnothing$, otherwise, according to Proposition 4.37 and Lemma 4.30, $x \in \omega_f^0$. Since $\omega_f^0/\sim$ is totally disconnected space (Proposition 4.38), there exists partition of it into openly-closed subsets, one of which contains image of $\alpha_f(x)$ and the other contains image of $\omega_f^0(x)$ (under natural projection $\omega_f^0 \to \omega_f^0/\sim$). Prototypes of these openly-closed sets form the needed partition of ω_f^0. The proposition is proved. □

4.4. **Slow Relaxations in One Perturbed System.** In this subsection, as in the preceding one, we investigate one semiflow of homeomorphisms f over a compact space X.

Theorem 4.46. *η_1^0- and η_2^0-slow relaxations are impossible for one semiflow.*

Proof. It is enough to show that η_2^0-slow relaxations are impossible. Suppose the contrary: there are such $\gamma > 0$ and such sequences of numbers $\varepsilon_i > 0$ $\varepsilon_i \to 0$, of points $x_i \in X$ and of (x_i, ε_i)-motions $f^{\varepsilon_i}(t|x_i)$ that $\eta_2^{\varepsilon_i}(f^{\varepsilon_i}(t|x_i), \gamma) \to \infty$. Similarly to the proofs of the theorems 4.34 and 3.2, find a subsequence in $\{f^{\varepsilon_i}(t|x_i)\}$ and such $y^* \in X$ that $\rho^*(y^*, \omega_f^0) \geq \gamma$ and, whatever be the neighbourhood V of the point y^* in X, $\overline{\text{meas}}\{t \geq 0 : f^{\varepsilon_i}(t|x_i) \in V\} \to \infty$ ($i \to \infty, f^{\varepsilon_i}(t|x_i)$ belongs to the chosen subsequence). As in the proof of Theorem 4.34, we have $y^* \in \omega_f^0(y^*) \subset \omega_f^0$. But, according to the constructing, $\rho^*(y^*, \omega_f^0) \geq \gamma > 0$. The obtained contradiction proves the absence of η_2^0-slow relaxations. □

Theorem 4.47. *Let X be connected. Then, if ω_f^0 is connected then the semiflow f has not $\tau_{1,2,3}^0$- and η_3^0-slow relaxations. If ω_f^0 is disconnected, then f possesses $\tau_{1,2,3}^0$- and η_3^0-slow relaxations.*

Proof. Let X and ω_f^0 be connected. Then, according to the propositions 4.37 and 4.38, $\omega_f^0(x) = \omega_f^0$ for any $x \in X$. Consequently, $\omega^0(x)$-bifurcations are absent. Therefore (Theorem 4.26) τ_3-slow relaxations are absent. Consequently, there are not other τ_i^0- and η_i^0-slow relaxations due to the inequalities $\tau_i^\varepsilon \leq \tau_3^\varepsilon$ and $\eta_i^\varepsilon \leq \tau_3^\varepsilon$ ($i = 1, 2, 3$) 1,2,3) (see Proposition 4.24). The first part of the theorem is proved.

Suppose now that X is connected and ω_f^0 is disconnected. Let us use Proposition 4.45. Find such $x \in X$ that x-motion is whole, $x \notin \omega_f^0$, and such partition of ω_f^0 into openly-closed subsets $\omega_f^0 = W_1 \bigcup W_2$, $W_1 \bigcap W_2 = \varnothing$ that $\alpha_f(x) \subset W_1$, $\omega_f^0(x) \subset W_2$. Suppose $\gamma = \frac{1}{3} r(W_1, W_2)$. There is such t_0 that for $t < t_0$ $\rho^*(f(t,x), W_2) > 2\gamma$. Let $p \in \alpha_f(x)$, $t_j < t_0$, $t_j \to -\infty$, $f(t_j, x) \to p$. For each $j = 1, 2, \dots$ exists such $\delta_j > 0$ that for $\varepsilon < \delta_j$ $d(\omega_f^\varepsilon(f(t_j, x)), \omega_f^0(f(t_i, x))) < \gamma$ (this follows from the Shura-Bura lemma and Proposition 4.10). Since $\omega_f^0(f(t_j, x)) = \omega_f^0(x)$ (Corollary 4.29), for $\varepsilon < \delta_j$ $d(\omega_f^\varepsilon(f(t_j, x)), W_2) < \gamma$. Therefore $\rho^*(f(t,x), \omega_f^\varepsilon(f(t_j, x))) > \gamma$ if $t \in [t_j, t_0]$, $\varepsilon > \delta_j$. Suppose $x_i = f(t_j,\ x)$, $\varepsilon_j > 0$, $\varepsilon_j < \delta_j$, $\varepsilon_j \to 0$, $f^{\varepsilon_j}(t|x_j) = f(t, x_j)$. Then $\tau_1^{\varepsilon_j}(f^{\varepsilon_j}(t|x_j), \gamma) \geq t_0 - t_i \to \infty$. The existence of τ_1- (and consequently of $\tau_{2,3}$-)-slow relaxations is proved. To prove the existence of η_3-slow relaxations we need the following lemma. □

Lemma 4.48. *For any* $\varepsilon > 0$ *and* $\varkappa > 0$, $\overline{\omega_f^\varepsilon} \subset \omega_f^{\varepsilon+\varkappa}$.

Proof. Let $y \in \overline{\omega_f^\varepsilon}$: there are such sequences of points $x_i \in X$, $y_i \in \omega_f^\varepsilon$, of (x_i, ε)-motions $f^\varepsilon(t|x_i)$, of numbers $t_j^i > 0$, $t_j^i \to \infty$ as $j \to \infty$ that $y_i \to y$, $f^\varepsilon(t_j^i|x_i) \to y_i$ as $j \to \infty$. Suppose that $\delta = \frac{1}{2}\delta(\frac{\varkappa}{3}, T)$. There is such y_i that $\rho(y_i, y) < \delta$. For this y_i there is such monotone sequence $t_j \to \infty$ that $t_j - t_{j-1} > T$ and $\rho(y_i, f^\varepsilon(t_j|x_i)) < \delta$. Suppose

$$f^*(t) = \begin{cases} f^\varepsilon(t|x_i), & \text{if } t \neq t_j; \\ y, & \text{if } t = t_j \ (j = 1, 2, \dots), \end{cases}$$

where $f^*(t)$ is $(x_i, \varkappa + \varepsilon)$-motion and $y \in \omega(f^*)$. Consequently, $y \in \omega_f^{\varepsilon+\varkappa}$. The lemma is proved. □

Corollary 4.49. $\omega_f^0 = \bigcap_{\varepsilon > 0} \overline{\omega_f^\varepsilon}$.

Let us return to the proof of Theorem 4.47 and show the existence of η_3^0-slow relaxations if X is connected and ω_f^0 is not. Suppose that $\gamma = \frac{1}{5} r(W_1, W_2)$. Find such $\varepsilon_0 > 0$ that for $\varepsilon < \varepsilon_0$ $d(\omega_f^\varepsilon, \omega_f^0) > \gamma$ (it exists according to Corollary 4.49 and the Shura-Bura lemma). There is t_1 for which $d(f(t_1, x), \omega_f^0) > 2\gamma$. Let $t_j < t_1$, $t_j \to -\infty$, $x_j = f(x_j, x)$. As (x_j, ε)-motions let choose true motions $f(t, x_j)$. Suppose that $\varepsilon_j \to 0$, $0 < \varepsilon_j < \varepsilon_0$. Then $\eta^{\varepsilon_j}(f(t, x_j), \gamma) > t_1 - t_j \to \infty$ and, consequently, η_3^0-slow relaxations exist. The theorem is proved.

In conclusion of this subsection let us give *the proof of Theorem 4.33.* We consider again the family of parameter depending semiflows.

Proof of Lemma 4.49. Let X be connected and $\omega^0(x,k)$-bifurcations exist. Even if for one $k \in K$ $\omega^0(k)$ is disconnected, then, according to Theorem 4.47, τ_3^0-slow relaxations exist. Let $\omega^0(k)$ be connected for any $k \in K$. Then $\omega^0(x,k) = \omega^0(k)$ for any $x \in X$, $k \in K$. Therefore from the existence of $\omega^0(x,k)$-bifurcations follows in this case the existence of $\omega^0(k)$-bifurcations. Thus, Theorem 4.33 follows from the following lemma which is of interest by itself. □

Lemma 4.50. *If the system (1.1) possesses* $\omega^0(k)$*-bifurcations, then it possesses* τ_3^0*- and* η_3^0*-slow relaxations.*

Proof. Let k^* be a point of $\omega^0(k)$-bifurcation: and there are such $\alpha > 0$, $y^* \in \omega^0(k^*)$ that $\rho^*(y^*, \omega^0(k_i)) > \alpha > 0$ for any $i = 1, 2, \dots$. According to Corollary 4.49 and the Shura-Bura lemma, for every i exists $\delta_i > 0$ for which $\rho^*(y^*, \omega^{\delta_i}(k_i)) > 2\alpha/3$. Suppose that $0 < \varepsilon_i \leq \delta_i$, $\varepsilon_i \to 0$. As the ε-motions appearing in the definition of slow relaxations take the real (k_i, y_i)-motions, where $y_i = f(-t_i, y^*, k^*)$, and t_i are determined as follows:

$$t_i = \sup\{t > 0 : \rho(f(t', x, k), f(t'x, k')) < \alpha/3\}$$

under the conditions $t' \in [0, t],\ x \in X,\ \rho_K(k, k') < \rho_K(k^*, k_i)\}$.

Note that $\rho^*(f(t_i, y_i, k_i), \omega^{\varepsilon_i}(k_i)) \geq \alpha/3$, consequently, $\eta_3^{\varepsilon_i}(f(t, y_i, k_i), \alpha/4) > t_i$ and $t_i \to \infty$ as $i \to \infty$. The last follows from the compactness of X and K (see the proof of Proposition 4.2). Thus, η_3^0-slow relaxations exist and then τ_3^0-slow relaxations exist too. Lemma 4.50 and Theorem 4.33 are proved. □

Summary

In Sections 1-4 the basic notions of the theory of transition processes and slow relaxations are stated. Two directions of further development of the theory are possible: introduction of new relaxation times and performing the same studies for them or widening the circle of solved problems and supplementing the obtained existence theorems with analytical results.

Among interesting but unsufficiently explored relaxation times let us mention the approximation time

$$\tau(x, k, \varepsilon) = \inf\{t \geq 0 : d(\omega(x, k), f([0, t], x, k)) < \varepsilon\}$$

and the averaging time

$$\tau_v(x, k, \varepsilon, \varphi) = \inf\Big\{t \geq 0 : \Big|\frac{1}{t'}\int_0^{t'} \varphi(f(\tau, x, k))d\tau - \langle\varphi\rangle_{x,k}\Big| < \varepsilon \text{ for } t' > t\Big\},$$

where $\varepsilon > 0$, φ is a continuous function over the phase space X,

$$\langle\varphi\rangle_{x,k} = \lim_{t\to\infty} \frac{1}{t}\int_0^t \varphi(f(t, x, k))d\tau$$

(if the limit exists).

The approximation time is the time necessary for the motion to visit the ε-neighbourhood of each its ω-limit point. The averaging time depends on continuous function φ and shows the time necessary for establishing the average value of φ with accuracy ε along the trajectory.

As the most important problem of analytical research, one should consider the problem of studying the asymptotical behaviour under $T \to \infty$ of "domains of delay", that is, the sets of those pairs (x, k) (the initial condition, parameter) for which $\tau_i(x, k, \varepsilon) > T$ (or $\eta_i(x, k, \varepsilon) > T$). Such estimations for particular two-dimensional system are given in the work [10].

"Structurally stable systems are not dense". It would not be exaggeration to say that the so titled work by Smale [67] opened a new era in the understanding of dynamics. Structurally stable (rough) systems are those whose phase portraits do not change qualitatively under small perturbations (accurate definitions with detailed motivation see in [12]). Smale constructed such structurally unstable system that any other system close enough to it is also structurally unstable. This result broke the hopes to classify if not all then "almost all" dynamical systems. Such hopes were associated with the successes of classification of two-dimensional dynamical systems [3, 4] among which structurally stable ones are dense.

There are quite a number of attempts to correct the catastrophic situation with structural stability: to invent such natural notion of stability, for which almost all systems would be stable. The weakened definition of structural stability is proposed in the works [21, 62, 63]: the system is stable if almost all trajectories change little under small perturbations. This stability is already typical, almost all systems are stable in this sense.

The other way to get rid of the "Smale nightmare" (the existence of domains of structurally unstable systems) is to consider the ε-motions, subsequently considering (or not) the limit $\varepsilon \to 0$. The picture obtained (even in limit $\varepsilon \to 0$) is more stable than the phase portrait (the accurate formulation see above in Section 4). It seems to be obvious that one should first study those (more rough) details of dynamics, which do not disappear under small perturbations.

The approach based on consideration of limit sets of ε-motions, in the form stated here was proposed in the paper [30]. It is necessary to note the conceptual proximity of this approach to the method of quasi-averages in statistical physics [17]. By analogy, the stated approach could be called the method of "quasi-limit" sets.

Unfortunately, elaborated analytical or numerical methods of studying (constructing or, wider, localizing) limit sets of ε-motions for dynamical systems of general type are currently absent. However, the author does not give up the hope for the possibility of elaboration of such methods.

Is the subject of this work in the "mainstream" of Dynamics? I don't know, but let us imagine an experimental situation: we observe a dynamic of a system. It might be a physical or a chemical system, or just a computational model, the precise nature of the system is not important. How long should we monitor the system in order to study the limit behaviour? When does the transition process turn into the limit dynamics? This work tries to state these problems mathematically and to answer them, at least partially.

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