Symmetries and Conservation Laws of Evolution Equations

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Contents

Chapter 1

Introduction and some history

1.1 Some history

> There has been a revolution in nonlinear physics over the past twenty years. Two great discoveries, each of which, incidentally, was made with the aid of computer experiments, have radically changed the thinking of scientists about the nature of nonlinearity and introduced two new theoretical constructs into the field of dynamics. The first of these is the soliton and the second is the strange attractor [New85].

Developments in the theory of these strange attractors, so-called "chaos theory", gave a gradual clarification of the erratic and unpredictable properties in natural phenomena. This thesis considers the equally puzzling and almost opposite challenge: to explain the striking predictability and regularity of the "soliton solution", showing a remarkable survivability under conditions where one might normally expect such a feature to be destroyed.

We first give a sketch of how the subject began. The purpose is purely motivational, not to give a historical survey of the field. There are several places in the literature with eye-witness descriptions such as [vdB78], [Kru78], [EvH81], [New85], [Kon87], [AC91] and [Pal97]. The following narrative is based mainly on these sources.

The discovery of the physical soliton is attributed to John `Scott` Russell's observation in 1834: A boat was rapidly drawn along a narrow channel by a pair of horses. When the boat suddenly stopped before a bridge, the bow wave detached from the boat and rolled forward with great velocity assuming the form of large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel without change the original form or diminution of the speed, as observed by `Scott` Russell who followed on horseback.

Although `Scott` Russell spent a major part of his professional life carrying out experiments to determine the properties of the great wave, it is doubtful that he appreciated their true soliton properties, i.e., the ability of these waves to interact and come out of these interactions without change of form, as if they were particles.

Nevertheless, the dash on the horseback exerts a powerful appeal.

The discovery of mathematical soliton started with an investigation of the solutions of nonlinear partial differential equations, such as the work of `Boussinesq` and `Rayleigh`, independently, in the 1870's.

It was in 1895 that `Korteweg` and de `Vries` derived the equation for water waves in shallow channels, which confirmed the existence of solitary waves. The equation which now bears their names[1] is of the form [2]

$$u_t = u_3 + uu_1 \quad (\textbf{KdV equation}). \tag{1.1.1}$$

This was the first stage of discovery. The primary thrust was to establish the physical and mathematical existence and robustness of the wave. The discovery of its additional properties was to await the appearance of computers.

In 1955, `Fermi`, `Pasta` and `Ulam` (FPU) undertook a numerical study of the one-dimensional anharmonic lattice of equal masses coupled by nonlinear springs. The computations were carried out on the Maniac I computer. They predicted that any smooth initial state would eventually reach equilibrium due to the nonlinear coupling, according to the ergodic hypothesis. Much to their surprise, the energy recollected after some time in the degree of freedom where it was when the experiment was started. Thus the experiment failed to produce the expected result. Instead it produced a difficult challenge.

Fortunately, the curious results were not ignored altogether. In 1965, `Kruskal` and `Zabusky` approached the FPU problem from the continuum viewpoint. They amazingly rederived the KdV equation and found its stable pulse-like waves by numerical experimentation. A remarkable property of these solitary waves was that they preserved their shapes and speeds after two of them collide, interact and then spread apart again. They named such waves **soliton**s.

The discovery by `Kruskal` and `Zabusky` attracted the attention and stimulated the curiosity of many physicists and mathematicians throughout the world. They took up the intriguing challenge of the analytical understanding of the numerical results. The stability and particle-like behavior of the solutions could only be explained by the existence of many conservation laws; this started the search for the conservation laws for the KdV equation. A conservation law has the form $D_tU+D_xF=0$; U is called the **conserved density** and F is called **conserved flux**. The expressions for the conservation of momentum and energy were classically known:

$$D_tu - D_x(u_2 + \frac{u^2}{2}) = 0, \; D_t(\frac{u^2}{2}) - D_x(uu_2 - \frac{u_1^2}{2} + \frac{u^3}{3}) = 0.$$

`Whitham` found a third conserved density, which corresponds to `Boussinesq`'s famous moment of instability. `Zabusky` and `Kruskal` continued searching and found two

[1] According to R. `Pego`, in a letter to the Notices of the AMS, 1998, volume 45, number 3, this equation appears in a footnote of a paper by `Boussinesq`, *Essai sur la théorie des eaux courantes*, presented in 1872 to the French Academy and published in 1877.

[2] where $u_i = \frac{\partial^i u}{\partial x^i}$.

more densities of order 2 and 3 (the highest derivative in the expression). Since they had made an algebraic mistake, they did not find a conserved density of order 4. This caused a delay of more than a year before they went back on the right track.

`Kruskal` somewhat later asked `Miura` to search for a conserved density of order 5. `Miura` found one and then quickly filled in the missing order 4. After the order 6 and 7 were found, `Kruskal` and `Miura` were fairly certain that there was an infinite number. However, `Miura` was challenged to find the order 8 conserved density since there were rumors that order 7 was the limit. He did this during a two-week vacation in the summer of 1966. Later, it was proved that there was indeed a conserved density of each order [MGK68]. Moreover, in [SW97b] it is proven that there are no other conservation laws besides the known conservation laws of the KdV equation.

The existence of an infinite number of conservation laws was an important link in the chain of discovery. After the search for conserved densities of the KdV equation (1.1.1), Miura found that the **Modified Korteweg–de Vries equation**

$$v_t = v_3 + v^2 v_1 \quad (\mathtt{mKdV}) \tag{1.1.2}$$

also had an infinite number of conserved densities. He showed that

$$u_t - (u_3 + uu_1) = (2v + \sqrt{-6}D_x)(v_t - (v_3 + v^2 v_1)),$$

under the transformation $u = v^2 + \sqrt{-6}v_1$, which now bears his name. Therefore, if $v(x,t)$ is a solution of (1.1.2), $u(x,t)$ is a solution of (1.1.1). From this observation, the famous inverse scattering method was developed and the Lax pair was found [Lax68]. `Gardner` was the first to notice that the KdV equation could be written in a Hamiltonian framework. Later, `Zakharov` and `Faddeev` showed how this could be interpreted as a completely integrable Hamiltonian system in the same sense as finite dimensional integrable Hamiltonian systems [ZF71] where one finds for every degree of freedom a conserved quantity, the **action**.

The conserved geometric features of solitons are intimately bound up with notions of symmetry. The symmetry groups of differential equations were first studied by Sophus Lie. Roughly speaking, a symmetry group of a system consists of those transformations of the variables which leave the system invariant. In the classical framework of Lie, these groups consist of only geometric transformations on the space of independent and dependent variables of the system, the so-called geometric symmetries. There are four such linear independent symmetries for the KdV equation, namely arbitrary translations in x and t, Galilean boost and scaling.

In 1918, Emmy `Noether` proved the remarkable theorem giving a one-to-one correspondence between symmetry groups and conservation laws for the Euler–Lagrange equations [Noe18]. The question was raised how to explain the infinitely many conserved densities for the KdV equation. One started to search for the hidden symmetries, **generalized symmetries**, which are 'groups' whose infinitesimal generators depend not only on the independent and dependent variables of the system, but also the derivatives of the dependent variables.

In fact, generalized symmetries first appeared in [Noe18]. Somehow, they were neglected for many years and have since been rediscovered several times. The great advantage of searching for symmetries is that they can be found by explicit computation. Moreover, the entire procedure is rather mechanical and, indeed, several symbolic programs have been developed for this task [HZ95], [Her96].

In 1977, `Olver` provided a method for the construction of infinitely many symmetries of evolution equations, originally due to `Lenard` [GGKM74]. This is the **recursion operator** [Olv77], which maps a symmetry to a new symmetry. For the KdV equation, a recursion operator is

$$\mathfrak{R}_{\mathtt{KdV}} = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_1 D_x^{-1}.$$

Here D_x^{-1} stands for the left inverse of D_x, so the recursion operator is only defined on $\operatorname{Im} D_x$.

Almost at the same time, `Magri` studied the connections between conservation laws and symmetries from the geometric point of view [Mag78]. He observed that the object of the theory of conservation laws, the gradients of the conserved densities (covariants), was dual to that of the theory of the symmetries. This problem required the introduction of a "metric operator", called **symplectic operator** if it maps the symmetries to the cosymmetries, or called **Hamiltonian (cosymplectic) operator** in the reverse direction. He found that some systems admitted two distinct but compatible Hamiltonian structures (Hamiltonian pairs). He called them twofold Hamiltonian system, now called bi-Hamiltonian systems. The KdV equation is a bi-Hamiltonian system. It can be written

$$u_t = D_x(u_2 + \frac{1}{2}u^2) = (D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_1)u,$$

where these two operators are a Hamiltonian pair.

Actually, the two operators had made their appearance before. Lenard used them to rederive the KdV hierarchy, infinitely many equations sharing all t-independent conservation laws. Lax also used them to produce infinitely many conservation laws for the KdV equation [Lax76]. This scheme is now called the **Lenard scheme**.

There appeared naturally a special kind of operator, called the **Nijenhuis** or **hereditary operator**. The defining relation for this operator was originally found as a necessary condition for an almost complex structure to be complex, i.e., as an integrability condition. Its important property is to construct an abelian Lie algebra. . Precisely speaking, for any given vectorfield Q_0 leaving the Nijenhuis operator $\mathfrak{R}$ invariant, the $Q_j = \mathfrak{R}^j(Q_0),\ j = 0, 1, \cdots$, leave $\mathfrak{R}$ invariant again and commute in pairs. This property was independently given by `Magri` [Mag80] and `Fuchssteiner` [Fuc79], where it was called hereditary symmetry. In the paper [GD79], the authors also introduced Nijenhuis operators, called regular structures.

Interrelations between Hamiltonian pairs and Nijenhuis operators were discovered by Gel'fand & `Dorfman` [GD79] and `Fuchssteiner` & `Fokas` [FF80], [FF81].

For example, the recursion operator of the KdV equation

$$\mathfrak{R}_{\mathtt{KdV}} = (D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_1)D_x^{-1}$$

is a Nijenhuis operator and it produced the higher KdV equations,

$$u_t = \Re^j(u_1), j = 0, 1, \cdots.$$

This is the KdV hierarchy, which shares infinitely many commuting symmetries produced by the same recursion operator.

From this point in time on, there has been an explosion of research activity in algebraic and geometric aspects of nonlinear partial differential systems, both the applications to concrete physical systems and the development of the theory itself. See [Oev84], [Zak91], [Olv93], [Dor93] and [FG96].

There were other important developments in the field, such as inverse scattering transformation ("S-integrability"), the `Hirota` method, the Painlevé method and the theory of "C-integrability", linearization of the systems by change of variables. Since this thesis does not contribute to these aspects, we shall not mention them any further.

We list the properties of the KdV equation, on which we shall focus:

- It possesses infinitely many conservation laws,
- It possesses infinitely many commuting symmetries,
- It is a bi-Hamiltonian system.

These aspects pertain not only to the KdV equation, but are found for the whole KdV hierarchy. Such systems are given a special designation, "integrable", or, more accurately, "exactly solvable".

The above properties are not equivalent. It is known that **Burgers' equation**

$$u_t = u_2 + uu_1,$$

possesses infinitely many symmetries produced by a recursion operator, but it is not a Hamiltonian system and it has only one nontrivial conservation law. However, it can be transformed to the exactly solvable heat equation $v_t = v_2$ by the transformation $u = \frac{2v_1}{v}$.

What are the relations among these properties, cf. [Fok87]? What kind of systems are integrable? These questions lead to heated discussions on "What is Integrability", the title of the book [Zak91].

1.2 Motivation

Despite the rapid development, which was stimulated by the application of soliton theory, one still finds unanswered questions at a remarkably elementary level.

Basically two kinds of question motivated the present work.

- Why is it that after the initial gold rush it was so difficult to find any new integrable systems, i.e., systems not already in the hierarchy of some known integrable system.

- Conservation laws and symmetries come in hierarchies with periodic gaps, like for the KdV equation one finds only odd order symmetries. Where do these gaps come from?

As it turned out, these questions are strongly related.

1.3 Summary

Underlying much of the theory of generalized symmetries, conservation laws and Hamiltonian structures, there is an important construct known as the "complex of formal variational calculus", which presents all different objects for nonlinear evolution systems as a unified whole. In chapter 2 and 3 of the thesis, we build up such a complex from a given ring based on `Dorfman`'s work [Dor93]. However, our complex is more general since the ring we used can contain t-dependent functions. To this end we have set up the whole framework using Leibniz algebras in stead of Lie algebras. In this complex one finds all the important objects in the study of symmetries and conservation laws, such as cosymmetries, recursion operators, symplectic forms.

We prove in section 2.9 that the folklore conjecture, "if a system has one nontrivial symmetry, it has infinitely many", is true under certain technical conditions. The statement and the proof is purely Lie (or `Leibniz`) algebraic theory, but the conditions can be checked by symbolic methods, as formulated in chapters 7 and 8 and diophantine approximation theory. The results in this section are essential for the classification of integrable equations.

In chapter 4 we motivate the definition of the Nijenhuis operator and derive its main properties and we formulate the notions of symplectic and Hamiltonian operators in the abstract context which we used to set up the complex. We derive some of the classical properties and relations of these notions.

In chapter 5 we apply the abstract machinery to the complex of formal variational calculus and we give expressions for all kinds of invariants of the evolution equation in terms of Fréchet derivatives. This links the abstract approach to the more usual definitions.

In chapter 6 we formulate and prove several theorems regarding the form of recursion and Nijenhuis operators. These results are very useful in computations, since they allow one to split messy expressions in terms of known symmetries and cosymmetries. They also allow one to conclude that under rather weak conditions these operators are well defined, that is: they will, starting from a given root, produce an infinite hierarchy of symmetries. We give a list of examples where these results are applied.

In chapter 7 we introduce the symbolic method, which enables us to translate questions about the integrability of nonlinear differential equations into divisibility questions of polynomials.

In chapter 8 we use the symbolic method to classify scalar λ-homogeneous equations. For $\lambda > 0$ we give the complete list of 10 integrable equations. This proof of the classification theorem gives the answer to the questions in section 1.2. For $\lambda = 0$ we give the complete analysis. The result is that the integrable equations turn out to be Hamiltonian, with the exception of one family deriving from the Potential `Burgers` equation.

In chapter 9 we give a list of 39 integrable systems, together with their recursion-, symplectic- and cosymplectic operators, and the roots of symmetries and scalings. Either these were known in the literature or they can be found by our new methods. This information allows one to produce the symmetries and cosymmetries of each given equation.

In the appendices we collect some material, proofs and examples, which did not quite fit in the main text, but seemed to be interesting enough to include here.

1.4 Suggestions to the reader

The text goes from the abstract to the concrete. The theoretically inclined reader can just start at the beginning and read sequentially.

For readers familiar with the standard theory as presented in [Olv93], [Dor93], whose main interest is in applications, chapter 5 might be a good point to start reading, since it connects the abstract approach with the standard theory. From here one can go forward reading either the results on recursion and Nijenhuis operators in chapter 6 or the classification results in chapters 7 and 8, using the results in section 2.9. Or one can go backward and read chapter 4, followed by chapters 2 and 3.

For those who consider even this as too theoretical, the examples in 6.3 are computed in great detail and may be a good starting point. Chapter 9 might be read as a handbook on integrable evolution equations. If needed, one can then backtrack.

Chapter 2

Connection and curvature

In this chapter and the next we set up the foundations of the theory of connections, curvature and cohomology. Although the intention was not to do anything new, the final result is not quite standard. For that reason all the proofs are explicitly given, even if they are of a rather mechanical nature. This has the disadvantage of pages filled with formulae, but the advantage that it can be read easily, since the computations usually only take one step at a time. There are only a few exceptions to this rule.

If one wants to restrict to Lie algebras, one substitutes $\mathcal{A} = \mathcal{C} = \mathbb{C}$, $\bar{V} = V$ and $\mathfrak{g} = \mathfrak{h}$, for the connection π_0^1 one reads the adjoint representation and for ∇_0^0 any representation. In this case π_m^n and ∇_m^n are zero for $m > 0$. General sources on Lie algebra cohomology and related issues are: [Fuk86], [Kos50], [HS53], [God64], [Kna88], [Lod91], [Pal61], [Mac87], and on the application of Lie symmetries to differential equations [Dor93], [Olv93], [Oev84].

The goal is to built a complex, in which one finds all the important objects in the study of symmetries and conservation laws, such as cosymmetries, recursion operators, symplectic forms. In this complex one can define cohomology, which is used here as a language to separate the trivial from the nontrivial and to motivate the definition of the Nijenhuis tensor and functionals. We do not use any results from general cohomology theory, since in the application one is interested in the individual elements, and their properties are not the subject of the theory. This implies that all the real problems remain hidden in the cohomology spaces until in each concrete problem these (or at least some of them) are computed. It is at that point that one usually has to require more properties from the underlying rings. We compute some small examples in appendix C.

2.1 Introduction

This chapter gives the abstract material on which we will build the theoretical framework. In section 2.2 the basic definitions are given of rings, modules and derivations. In section 2.3 we give the basic definitions of Lie and Poisson algebras, which will serve as a motivation for the later developments. In section 2.4 we construct a Lie algebra from a given ring. This is done by first formally defining the space of Kähler differentials and then considering the dual space, just as one normally defines the tangent space using the cotangent space. Then we define connections on modules, generalizing the concept of Lie and Poisson algebra and their representations. This definition is then naturally lifted to chains and cochains in section 2.6. In section 2.7 we define the curvature of a connection, inspired on the definition of the `Riemannian` curvature of the Levi–Civita connection. We show that the curvature is again a connection. Zero curvature is equivalent to the Jacobi identity for Lie algebras. In section 2.8 we give some elementary examples, based on classical mechanics, to illustrate the concepts of connection and curvature. Finally, in section 2.9 we give an implicit function theorem to be used later in our classification of λ-homogeneous equations, cf. chapter 8.

2.2 Rings, modules and derivations

Abstract 2-1. *In this section we give the basic definitions for modules over not necessarily commutative rings and their derivations.*

Notation 2-2. *We denote by $\mathcal{C}$ a commutative ring with unity $1_{\mathcal{C}}$ ($\neq 0_{\mathcal{C}}$, the zero element), which will be the basic ring in all that follows. The field of invertible elements in $\mathcal{C}$ will be denoted by $\mathcal{C}^\star$. The reader may think of $\mathcal{C}$ as $\mathbb{R}, \mathbb{C}$ or $\mathbb{Q}$, but actually not much is needed of $\mathcal{C}$.*

Definition 2-3. *Let $\mathcal{R}$ be a ring with multiplication $\mu_{\mathcal{R}} : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$. We say that V is a* **left $\mathcal{R}$-module** *if it is an abelian group with group operation $(v_1, v_2) \mapsto v_1 + v_2$ and $\mathcal{R}$ acts on V, i.e., there is a map $\lambda_{\mathcal{R}} : \mathcal{R} \times V \to V$ such that for $l_1, l_2 \in \mathcal{R}$ and $v, v_1, v_2 \in V$,*

1. $\lambda_{\mathcal{R}}(l_1 + l_2, v) = \lambda_{\mathcal{R}}(l_1, v) + \lambda_{\mathcal{R}}(l_2, v)$.

2. $\lambda_{\mathcal{R}}(0_{\mathcal{R}}, v) = 0$.

3. $\lambda_{\mathcal{R}}(1_{\mathcal{R}}, v) = v$.

4. $\lambda_{\mathcal{R}}(l_1, v_1 + v_2) = \lambda_{\mathcal{R}}(l_1, v_1) + \lambda_{\mathcal{R}}(l_1, v_2)$.

5. $\lambda_{\mathcal{R}}(\mu_{\mathcal{R}}(l_1, l_2), v) = \lambda_{\mathcal{R}}(l_1, \lambda_{\mathcal{R}}(l_2, v))$.

We define a **right $\mathcal{R}$-module** *in the same way, with $\rho_{\mathcal{R}} : \mathcal{R} \times V \to V$ such that $\rho_{\mathcal{R}}(\mu_{\mathcal{R}}(l_1, l_2), v) = \rho_{\mathcal{R}}(l_2, \rho_{\mathcal{R}}(l_1, v))$. We write $\lambda_{\mathcal{R}}(l, v) = l \circ v$, $\rho_{\mathcal{R}}(l, v) = v \circ l$ and $\mu_{\mathcal{R}}(l_1, l_2) = (l_1 l_2)$. When V is both a left and a right $\mathcal{R}$-module, we call it an* **$\mathcal{R}$-module**. .

When $\mathcal{R}$ is commutative, $\mathcal{R}$ itself is an $\mathcal{R}$-module.

Notation 2-4. *We denote by $\mathcal{A}$ a (not necessarily commutative) ring with unity which is also a $\mathcal{C}$-module. Let in the sequel $\mathcal{R}$ be the* **center of** $\mathcal{A}$, *i.e.,*

$$\mathcal{R} = \{r \in \mathcal{A} | rs = sr, \forall s \in \mathcal{A}\}.$$

We imbed $\mathcal{C}$ in $\mathcal{A}$ by $\lambda_{\mathcal{C}}(l_1, 1_{\mathcal{A}})$ and we assume $\lambda_{\mathcal{C}}(\mathcal{C}) \subset \mathcal{R}$.

Example 2-5. *A typical example would be: Let $\mathcal{C}$ be the ring of real valued C^∞-functions in t and x and let $\mathcal{A}$ be generated by $u = u(t,x)$ and its x-derivatives with coefficients from $\mathcal{C}$. So we would have $sin(t+x)u_1^2 \in \mathcal{A}$, where u_1 stands for $\frac{\partial u}{\partial x}$.*

Definition 2-6. *We say that a $\mathcal{C}$-module V is a* **filtered module** *if there exist $\mathcal{C}$-submodules $V^{(i)}, i = 0, \cdots, \infty$ such that*

- $V = V^{(0)} \supset V^{(1)} \supset \cdots \supset V^{(i)} \supset \cdots,$
- $\bigcap_{i=0}^{\infty} V^{(i)} = 0.$

Definition 2-7. *Let V and W be filtered modules. We say that $\phi : V \times W \to W$ defines a* **filtered action** *of V on W if $\phi(V^{(i)}, W^{(j)}) \subset W^{(i+j)}$.*

Definition 2-8. *We call two $\mathcal{C}$-modules $\bar{V}$ and V a* **direct pair**, *if V is a direct summand of $\bar{V}$. We denote the* **retract** *by $\pi_V : \bar{V} \to V$. When V is an $\mathcal{A}$-module, we denote this by $(\bar{V}, V)_{\mathcal{A}}$.*

Remark 2-9. *If in the direct pair $(\bar{V}, V)_{\mathcal{A}}$, the module V is zero, some care should be taken with the definitions in the sequel, but we will not mention this again.*

Notation 2-10. *Let V, W be $\mathcal{C}$-modules and $Hom_{\mathcal{C}}(V, W)$ be the space of $\mathcal{C}$-linear transformations of V to W. We write $End_{\mathcal{C}}(V)$ for $Hom_{\mathcal{C}}(V, V)$. If $(\bar{V}, V)_{\mathcal{A}}$ is a direct pair, we denote by $End_{\mathcal{C}}^{V}(\bar{V})$ those $\mathcal{C}$-linear homomorphisms that leave V invariant.*

If V is an $\mathcal{A}$-module, then $End_{\mathcal{C}}(V)$ is automatically also a left $\mathcal{A}$-module, with $(rB)v = r(Bv)$ for $r \in \mathcal{A},\ v \in V,\ B \in End_{\mathcal{C}}(V)$.

Definition 2-11. *Let V be a $\mathcal{A}$-module. The map $\mathfrak{d} : \mathcal{A} \to V$ is a* **derivation** *if it satisfies the Leibniz rule*

$$\mathfrak{d}\mu_{\mathcal{A}}(f, g) = \lambda_{\mathcal{A}}(f, \mathfrak{d}(g)) + \rho_{\mathcal{A}}(g, \mathfrak{d}(f)), \quad f, g \in \mathcal{A}.$$

We say that $\mathfrak{d}$ is $\mathcal{C}$-linear if it is a map of $\mathcal{C}$-modules. The space $Der_{\mathcal{C}}(\mathcal{A}, V)$ of all $\mathcal{C}$-linear derivations $\mathcal{A} \to V$ is naturally a left $\mathcal{R}$-module, with the left action defined by

$$f\mathfrak{d} : g \mapsto f(\mathfrak{d}(g)) \in V, \quad f \in \mathcal{R}, g \in \mathcal{A}.$$

We write $Der_{\mathcal{C}}(\mathcal{A})$ for $Der_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$.

Remark 2-12. *We write $1 = 1_{\mathcal{A}} = 1_{\mathcal{C}}$. Since*

$$\mathfrak{d}(1) = \mathfrak{d}(\mu_{\mathcal{A}}(1,1)) = \lambda_{\mathcal{C}}(1, \mathfrak{d}(1)) + \rho_{\mathcal{C}}(1, \mathfrak{d}(1)) = 2\mathfrak{d}(1),$$

we have $\mathfrak{d}(1) = 0$. It follows that $\mathfrak{d}(a) = \mathfrak{d}(\lambda_{\mathcal{C}}(1, a)) = \rho_{\mathcal{C}}(a, \mathfrak{d}(1)) = \rho_{\mathcal{C}}(a, 0) = 0$ for $a \in \mathcal{C}$ due to the $\mathcal{C}$-linearity of $\mathfrak{d}$.

2.3 Representations of Lie algebras

Abstract 2-13. *In this section we introduce Lie algebras, Poisson algebras and their representations. This motivates the definition of connection in section 2.5.*

Definition 2-14. *We say that a $\mathcal{C}$-module $\mathfrak{g}$ is a* **Lie algebra** *if there exists a $\mathcal{C}$-bilinear operation $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called* **Lie bracket***, satisfying*

$$[X,Y] = -[Y,X] \text{ (antisymmetry)},$$
$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \text{ (Jacobi identity)},$$

where $X, Y, Z \in \mathfrak{g}$.

Definition 2-15. *If $\mathfrak{g} = \mathcal{A}$, then $\mathfrak{g}$ is called a* **Poisson algebra***, if it is a Lie algebra and*

$$[p, rq] = r[p,q] + [p,r]q, \forall p,q,r \in \mathcal{A}, \text{ (Leibniz rule)},$$

Example 2-16. *$End_{\mathcal{C}}(V)$ is a Lie algebra with Lie bracket $[A,B] = AB - BA$. This construction works for any associative algebra.*

Definition 2-17. *A* **representation of a Lie algebra** *is a $\mathcal{C}$-linear homomorphism of Lie algebras $\nabla_0^0 : \mathfrak{g} \to End_{\mathcal{C}}(V)$, i.e., $\nabla_0^0([X,Y]) = [\nabla_0^0(X), \nabla_0^0(Y)] = \nabla_0^0(X)\nabla_0^0(Y) - \nabla_0^0(Y)\nabla_0^0(X)$.*

Definition 2-18. *If, moreover, V is a left $\mathcal{A}$-module and there exists a representation γ_0^0 of $\mathfrak{g}$ in $Der_{\mathcal{C}}(\mathcal{A})$, such that*

$$\nabla_0^0(X)(r \circ v) = r \circ \nabla_0^0(X)v + \gamma_0^0(X)(r) \circ v,\ \ r \in \mathcal{A}, v \in V, X \in \mathfrak{g},$$

then ∇_0^0 is called an **$\mathcal{A}$-representation** *(cf. definition 2-28).*

For any $X_1, X_2 \in \mathfrak{g}$, we denote the map $X_2 \mapsto [X_1, X_2]$ by $ad(X_1)$. Then $\mathbf{ad} : \mathfrak{g} \to End_{\mathcal{C}}(\mathfrak{g})$ is a homomorphism of Lie algebras due to the Jacobi identity, called the **adjoint representation** of $\mathfrak{g}$.

Definition 2-19. *Let $(\mathfrak{g}, \mathfrak{h})_{\mathcal{A}}$ be a direct pair and $\mathfrak{h}$ be an ideal of Lie algebra $\mathfrak{g}$, i.e., $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$. We say that $\mathfrak{g}$ is an* **$\mathcal{A}$-Lie algebra** *if the adjoint representation* **ad** *is an $\mathcal{A}$-representation on $\mathfrak{h}$.*

Definition 2-20. *When there exists a representation $\nabla_0^0 : \mathfrak{g} \to End_{\mathcal{C}}(V)$, we call V a* **(left) $\mathfrak{g}$-module***.*

Definition 2-21. *We say that Lie algebra $\mathfrak{g}$ is a* **graded Lie algebra** *if there exist $\mathcal{C}$-submodules $\mathfrak{g}^{(i)}, i \in \mathbb{Z}$ such that*

- $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}^{(i)}$,
- $[\mathfrak{g}^{(k)}, \mathfrak{g}^{(l)}] \subset \mathfrak{g}^{(k+l)}$.

Definition 2-22. *We say that a filtered module $\mathfrak{f}$ is a* **filtered Lie algebra** *if ad is a filtered action, i.e., $[\mathfrak{f}^{(k)}, \mathfrak{f}^{(l)}] \subset \mathfrak{f}^{(k+l)}$, where the $\mathfrak{f}^{(k)}, k = 0, \cdots, \infty$ are submodules.*

If we have a graded Lie algebra $\mathfrak{g} = \bigoplus_{i \in \mathbb{N}} \mathfrak{g}^{(i)}$, we can view it as a filtered Lie algebra by putting $\mathfrak{f}^{(i)} = \bigoplus_{i \leq j} \mathfrak{g}^{(j)}$.

2.4 Lie algebra of a ring

Abstract 2-23. *In this section we show that we can construct a Lie algebra given a ring. If one thinks of the ring as the ring of functions on a manifold, the construction is analogous to the construction of the tangent space as the dual of the cotangent space. The construction is standard and closely follows [Eis95], except for the fact that we do not require $\mathcal{A}$ to be commutative.*

The reader who identifies $\mathcal{C}$ and $\mathcal{A}$ can skip this section altogether. We denote the product xy as (xy) if this improves the readability.

Definition 2-24. *The space of* **Kähler differential***s of $\mathcal{A}$ over $\mathcal{C}$, written $\Omega^1_{\mathcal{A}/\mathcal{C}}$, is the $\mathcal{A}$-module generated by the set $\{d(f)|f \in \mathcal{A}\}$ subject to the relations*

$$\begin{aligned} d(xy) &= x \circ d(y) + d(x) \circ y, \quad x, y \in \mathcal{A}, \text{(Leibniz rule)} && (2.4.1) \\ d(c_1x + c_2y) &= c_1d(x) + c_2d(y), \quad c_1, c_2 \in \mathcal{C}. && (2.4.2) \end{aligned}$$

Often one writes dx for $d(x)$.

The map $d : \mathcal{A} \to \Omega^1_{\mathcal{A}/\mathcal{C}}$, defined by $d : x \mapsto dx$ is a $\mathcal{C}$-linear derivation. The map d has, by its definition, the following universal property: given any $\mathcal{A}$-module V and $\mathcal{C}$-linear derivation $\mathfrak{d} : \mathcal{A} \to V$, there is an unique $\mathcal{A}$-linear homomorphism $X : \Omega^1_{\mathcal{A}/\mathcal{C}} \to V$ such that $\mathfrak{d} = Xd$.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "d"] \arrow[d, "\mathfrak{d}"] & \Omega^1_{\mathcal{A}/\mathcal{C}} \arrow[dl, "X"] \\ V & \end{tikzcd}$$

Indeed, X is defined by the formula $X(dy) = \mathfrak{d}y$. One sees that X is an $\mathcal{A}$-linear homomorphism (by construction) since the relations among the dy are also satisfied by the $\mathfrak{d}y$:

$$\begin{aligned} X(x \circ dy \circ z) &= X(x \circ dy) \circ z = x \circ X(dy) \circ z = x \circ \mathfrak{d}y \circ z \\ (Xd)(xy) &= \mathfrak{d}(xy) = x \circ \mathfrak{d}(y) + \mathfrak{d}(x) \circ y \\ X(d(xy)) &= X(x \circ dy) + X(dx \circ y) = x \circ X(dy) + X(dx) \circ y \\ &= x \circ \mathfrak{d}(y) + \mathfrak{d}(x) \circ y. \end{aligned}$$

We may consider the construction of $\Omega^1_{\mathcal{A}/\mathcal{C}}$ as a linearization of the construction of derivations. It is therefore the simpler object to work with.

If $\mathcal{A}$ is generated (as a ring) by elements x_ι, then $\Omega^1_{\mathcal{A}/\mathcal{C}}$ is generated as an $\mathcal{A}$-module by the elements dx_ι. For example, if $g = p(x_1, \cdots, x_r)$ is a polynomial in the x_ι with coefficients in $\mathcal{C}$, then using the Leibniz rule we show that $dg = \sum_{\iota=1}^r \frac{\partial p}{\partial x_\iota} dx_\iota$.

We now consider the case $V=\mathcal{A}$. Let $\mathfrak{d}_1,\mathfrak{d}_2\in Der_{\mathcal{C}}(\mathcal{A})$. Then the Lie bracket

$$[\mathfrak{d}_1,\mathfrak{d}_2]=\mathfrak{d}_1\mathfrak{d}_2-\mathfrak{d}_2\mathfrak{d}_1$$

is again an element of $Der_{\mathcal{C}}(\mathcal{A})$:

$$\begin{aligned}
[\mathfrak{d}_1,\mathfrak{d}_2](xy) &= \mathfrak{d}_1\mathfrak{d}_2(xy)-\mathfrak{d}_2\mathfrak{d}_1(xy)\\
&= \mathfrak{d}_1(x\circ\mathfrak{d}_2(y)+\mathfrak{d}_2(x)\circ y)-\mathfrak{d}_2(x\circ\mathfrak{d}_1(y)+\mathfrak{d}_1(x)\circ y)\\
&= \mathfrak{d}_1(x\circ\mathfrak{d}_2(y))+\mathfrak{d}_1(\mathfrak{d}_2(x)\circ y)-\mathfrak{d}_2(x\circ\mathfrak{d}_1(y))-\mathfrak{d}_2(\mathfrak{d}_1(x)y)\\
&= x\circ\mathfrak{d}_1(\mathfrak{d}_2(y))+\mathfrak{d}_1(x)\circ\mathfrak{d}_2(y)+\mathfrak{d}_1(\mathfrak{d}_2(x))\circ y+\mathfrak{d}_2(x)\circ\mathfrak{d}_1(y)\\
&\quad -\mathfrak{d}_2(x)\circ\mathfrak{d}_1(y)-x\circ\mathfrak{d}_2(\mathfrak{d}_1(y))-\mathfrak{d}_2(\mathfrak{d}_1(x))\circ y-\mathfrak{d}_1(x)\circ\mathfrak{d}_2(y)\\
&= x\circ[\mathfrak{d}_1,\mathfrak{d}_2](y)+[\mathfrak{d}_1,\mathfrak{d}_2](x)\circ y.
\end{aligned}$$

By the universal property of $d:\mathcal{A}\to\Omega^1_{\mathcal{A}/\mathcal{C}}$, the maps $\mathfrak{d}_1,\mathfrak{d}_2,[\mathfrak{d}_1,\mathfrak{d}_2]$ must be of the form Xd,Yd,Zd, respectively, for $\mathcal{A}$-module homomorphisms $X,Y,Z:\Omega^1_{\mathcal{A}/\mathcal{C}}\to\mathcal{A}$. Clearly

$$[\mathfrak{d}_1,\mathfrak{d}_2]=\mathfrak{d}_1\mathfrak{d}_2-\mathfrak{d}_2\mathfrak{d}_1=XdYd-YdXd=(XdY-YdX)d,$$

so one might guess that $Z=XdY-YdX$. But this does not work, since the right hand side is not a homomorphism of $\mathcal{A}$-modules. Indeed,

$$\begin{aligned}
&Z(x\circ dy)=XdY(x\circ dy)-YdX(x\circ dy)=\\
&= Xd(x\circ Y(dy))-Yd(x\circ X(dy))\\
&= X(x\circ dY(dy))+X(dx\circ Y(dy))-Y(dx\circ X(dy))-Y(x\circ dX(dy))\\
&= x\circ X(dY(dy))+(X(dx)Y(dy))-(Y(dx)X(dy))-x\circ Y(dX(dy))\\
&= x\circ Z(dy)+(X(dx)Y(dy))-(Y(dx)X(dy)).
\end{aligned}$$

So we have an obstruction of the form $(X(dx)Y(dy))-(Y(dx)X(dy))$. To compensate for this obstruction, let

$$\omega(X,Y)(x,y)=(X(dx)Y(dy))-(Y(dx)X(dy)).$$

We define

$$\begin{aligned}
&[X,Y](x\circ dy\circ z)=XdY(x\circ dy\circ z)-YdX(x\circ dy\circ z)\\
&+x\circ\omega(X,Y)(y,z)-\omega(X,Y)(x,y)\circ z.
\end{aligned}$$

Then it follows that

$$\begin{aligned}
&[X,Y](x\circ dy\circ z)\\
=\ & XdY(x\circ dy\circ z)-YdX(x\circ dy\circ z)+x\circ\omega(X,Y)(y,z)-\omega(X,Y)(x,y)\circ z\\
=\ & Xd(x(Y(dy)z))-Yd(x(X(dy)z))+x\circ\omega(X,Y)(y,z)-\omega(X,Y)(x,y)\circ z\\
=\ & (X(dx)Y(dy))\circ z+x\circ Xd(Y(dy))\circ z+x\circ(Y(dy)X(dz))\\
-\ & (Y(dx)X(dy))\circ z-x\circ YdX(dy)\circ z-x\circ(X(dy)Y(dz))\\
+\ & x\circ\omega(X,Y)(y,z)-\omega(X,Y)(x,y)\circ z\\
=\ & x\circ[X,Y](dy)\circ z.
\end{aligned}$$

We show that $[,]$ defines a Lie bracket on $\mathfrak{g} = Hom_{\mathcal{A}}(\Omega^1_{\mathcal{A}/\mathcal{C}}, \mathcal{A})$. Let $X, Y, Z \in Hom_{\mathcal{A}}(\Omega^1_{\mathcal{A}/\mathcal{C}}, \mathcal{A})$. Antisymmetry being obvious, we prove the Jacobi identity. It suffices to prove the identity on the generators dy.

$$\begin{aligned}
& \left([X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]]\right)(dy) \\
= \; & (Xd[Y,Z] - [Y,Z]dX + Zd[X,Y] \\
& -[X,Y]dZ + Yd[Z,X] - [Z,X]dY)(dy) \\
= \; & (Xd(YdZ - ZdY) - (YdZ - ZdY)dX + Zd(XdY - YdX) \\
& -(XdY - YdX)dZ + Yd(ZdX - XdZ) - (ZdX - XdZ)dY)(dy) \\
= \; & 0.
\end{aligned}$$

If $\mathcal{A}$ is commutative, $Der_{\mathcal{C}}(\mathcal{A})$ is an $\mathcal{A}$-module, and this induces an action of $\mathcal{A}$ on $\mathfrak{g}$. Indeed, $r \circ X$ is defined by the formula $r \circ X(dy) = r \circ \mathfrak{d}y$. The Lie bracket is **not** $\mathcal{A}$-linear, but defines an $\mathcal{A}$-Lie algebra (cf. definition 2-19). We have, with $\gamma_0^0(X) = Xd$ and $X, Y \in \mathfrak{g}$

$$\begin{aligned}
[X, r \circ Y](dy) &= Xd(r \circ Y(dy)) - r \circ YdX(dy) \\
&= (X(dr)Y(dy)) + r \circ XdY(dy) - r \circ YdX(dy) \\
&= Xd(r) \circ Y(dy) + r \circ [X,Y](dy) \\
&= r[X,Y](dy) + \gamma_0^0(X)(r)Y(dy).
\end{aligned}$$

Proposition 2-25. *Assume that $\varphi : \mathcal{A} \to \mathcal{A}'$ is a ring homomorphism. Then there exists an unique Lie algebra homomorphism*

$$\varphi^\star : Hom_{\mathcal{A}}(\Omega^1_{\mathcal{A}/\mathcal{C}}, \mathcal{A}) \to Hom_{\mathcal{A}'}(\Omega^1_{\mathcal{A}'/\mathcal{C}'}, \mathcal{A}').$$

Proof. First we construct the map $\varphi^\star$ with the help of the following diagram:

$$\begin{tikzcd}
\Omega^1_{\mathcal{A}/\mathcal{C}} \arrow[r, "\varphi_\star"] \arrow[d, shift left, "X"] & \Omega^1_{\mathcal{A}'/\mathcal{C}'} \arrow[d, shift left, "\varphi^\star X"] \\
\mathcal{A} \arrow[u, shift left, "d"] \arrow[ur, "\varphi'"] \arrow[r, "\varphi"'] & \mathcal{A}' \arrow[u, shift left, "d'"]
\end{tikzcd}$$

Given $\varphi : \mathcal{A} \to \mathcal{A}'$, we define $\varphi' : \mathcal{A} \to \Omega^1_{\mathcal{A}'/\mathcal{C}'}$ by $\varphi' = d'\varphi$. The map d has, by the definition, the following universal property: There exists a unique map $\varphi_\star : \Omega^1_{\mathcal{A}/\mathcal{C}} \to \Omega^1_{\mathcal{A}'/\mathcal{C}'}$ such that $\varphi_\star d = \varphi'$. This means that $\varphi_\star(x \circ dy \circ z) = \varphi(x) \circ d'\varphi(y) \circ \varphi(z)$. Therefore $\varphi^\star X$ can be defined on $\operatorname{Im}\varphi_\star$ by $(\varphi^\star X)\varphi_\star = \varphi X$. We have $\varphi^\star X \varphi_\star d = \varphi^\star X \varphi' = \varphi^\star X d'\varphi = \varphi X d$.

First we check that $\varphi^\star X$ is an $\mathcal{A}'$-linear homomorphism.

$$\begin{aligned}
& \varphi^\star X \varphi_\star (x \circ dy \circ z) = \\
& \quad = \; \varphi^\star X(\varphi(x) \circ d'\varphi(y) \circ \varphi(z))
\end{aligned}$$

$$
\begin{aligned}
&= \varphi X(x \circ dy \circ z)\\
&= \varphi(x \circ X dy \circ z)\\
&= \varphi(x) \circ \varphi X dy \circ \varphi(z)\\
&= \varphi(x) \circ \varphi^\star X d'\varphi(y) \circ \varphi(z).
\end{aligned}
$$

We check such $\varphi^\star$ is a Lie algebra homomorphism. For any $r \in \mathcal{A}$ and $X, Y \in Hom_{\mathcal{A}}(\Omega^1_{\mathcal{A}/\mathcal{C}}, \mathcal{A})$, we have

$$
\begin{aligned}
&(\varphi^\star[X,Y])d'\varphi(r) = \varphi([X,Y]dr) =\\
&= \varphi(XdY(dr)) - \varphi(YdX(dr))\\
&= (\varphi^\star X)d'\varphi(Y(dr)) - (\varphi^\star Y)d'\varphi(X(dr))\\
&= (\varphi^\star X)d'(\varphi^\star Y)d'\varphi(r) - (\varphi^\star Y)d'(\varphi^\star X)d'\varphi(r)\\
&= [\varphi^\star X, \varphi^\star Y]d'\varphi(r),
\end{aligned}
$$

and this proves the statement. □

In chapter 3, we will be assuming the existence of $\mathfrak{g}$ and the maps X, Y will be written there as $\iota^1_0(X), \iota^1_0(Y)$, where $X, Y \in \mathfrak{g}$, cf. definition 3-8.

The reader should realize that although the constructions given here seem to be of a very general nature, this does not imply that they always lead to useful results. This depends on the existence of derivations on the ring (cf. [Lod91], E.1.3.5). But in chapter 5 we will see that they apply in a natural way to the complex of formal variational calculus.

2.5 Connections

Abstract 2-26. *In this section we introduce connections and their elementary properties.*

We assume $(\bar{V}, V)_{\mathcal{A}}$, $(\bar{W}, W)_{\mathcal{A}}$ and $(\mathfrak{g}, \mathfrak{h})_{\mathcal{A}}$ to be a direct pairs.

Definition 2-27. *We define $\mathfrak{h}^0_{\mathcal{R}} = \mathfrak{h}$ and, for $m \geq 0$, $\mathfrak{h}^{m+1}_{\mathcal{R}} = \mathfrak{h}^m_{\mathcal{R}} \otimes_{\mathcal{R}} \mathfrak{h}^m_{\mathcal{R}}$ and $\mathfrak{g}^0_{\mathcal{C}} = \mathfrak{g}$, $\mathfrak{g}^{m+1}_{\mathcal{C}} = \mathfrak{g}^m_{\mathcal{C}} \otimes_{\mathcal{C}} \mathfrak{g}^m_{\mathcal{C}}$. We consider $\mathfrak{h}^m_{\mathcal{R}}$ as an $\mathcal{A}$-module by the action of $\mathcal{A}$ on the last component. Of course, when $\mathcal{R} = \mathcal{A}$, this is compatible with the usual action of $\mathcal{A}$ on the tensor product over $\mathcal{A}$.*

Definition 2-28. *Suppose there exists a $\gamma^0_m : \mathfrak{g}^m_{\mathcal{C}} \to End_{\mathcal{C}}(\mathcal{A})$. We say that $\nabla^\bullet_m : \mathfrak{g}^m_{\mathcal{C}} \to End^V_{\mathcal{C}}(\bar{V})$ is a* **connection** *of $\mathfrak{g}^m_{\mathcal{C}}$ on V with* **anchor** *γ^0_m if*

$$
\nabla^\bullet_m(X)(l \circ v) = l \circ \nabla^\bullet_m(X)v + \gamma^0_m(X)(l) \circ v, \quad l \in \mathcal{A}, v \in V,
$$

and we write $\nabla^\bullet_m \in \Gamma^\bullet_m(\mathfrak{g}, V, \mathcal{A})$. A connection $\nabla^\bullet_m$ is said to be **$\mathcal{A}$-linear** *if $\nabla^\bullet_m(rY) = r\nabla^\bullet_m(Y)$ for $Y \in \mathfrak{h}^m_{\mathcal{R}}$ and $r \in \mathcal{A}$.*

Remark 2-29.
- *The terminology* anchor *is introduced in [Mac87].*

- *Compare this with the definition of an $\mathcal{A}$-representation.*
- *If here $\nabla^\bullet_m(X)$ is only defined on V, it can easily be extended by using the projection of $\bar{V}$ on V to a connection on $\bar{V}$.*
- *The defining property of a connection only reflects its behavior on V.*
- *$\Gamma^\bullet_m(\mathfrak{g}, V, \mathcal{A})$ is a left $\mathcal{A}$-module since $(l \circ \nabla^\bullet_m)(X)v = l \circ \nabla^\bullet_m(X)v$.*

$\star$ **Remark 2-30.** *Similar definitions in the literature usually assume $\mathfrak{g}^m_{\mathcal{C}}$ to be a Lie algebra and γ^0_m a representation. This makes it possible to define a connection in terms of extensions of Lie algebras and clearly state the obstructions to the existence of connections. In the present approach one could do similar things if one already has a connection on $\mathfrak{h}^m_{\mathcal{C}}$ and wishes to define a 'compatible' connection on V, cf. [Hue90], [KT71].*

We will not always mention the anchor, but assume that for all connections in $\Gamma^\bullet_m$ there is one and the same γ^0_m. If one sees $\Gamma^\bullet_m(\mathfrak{g}, V, \mathcal{C})$, then the anchor is automatically zero, since $Der_{\mathcal{C}}(\mathcal{C}) = 0$ (cf. proposition 2-31).

Proposition 2-31. *If $\nabla^\bullet_m \in \Gamma^\bullet_m(\mathfrak{g}, V, \mathcal{A})$ then $\gamma^0_m(X) \in Der_{\mathcal{C}}(\mathcal{A})$.*

Proof. An immediate consequence of definition 2-28 is that

$$\nabla^\bullet_m(X)((l_1 l_2) \circ v) \;=\; (l_1 l_2) \circ \nabla^\bullet_m(X)v + \gamma^0_m(X)(l_1 l_2) \circ v$$

and

$$\begin{aligned} &\nabla^\bullet_m(X)((l_1 l_2) \circ v) = \nabla^\bullet_m(X)(l_1 \circ l_2 \circ v) \\ &= l_1 \circ \nabla^\bullet_m(X)(l_2 \circ v) + \gamma^0_m(X)(l_1) \circ l_2 \circ v \\ &= l_1 \circ l_2 \circ \nabla^\bullet_m(X)v + l_1 \circ \gamma^0_m(X)(l_2) \circ v + \gamma^0_m(X)(l_1) \circ l_2 \circ v \\ &= (l_1 l_2) \circ \nabla^\bullet_m(X)v + (l_1 \gamma^0_m(X)(l_2)) \circ v + (\gamma^0_m(X)(l_1) l_2) \circ v. \end{aligned}$$

It follows that

$$\gamma^0_m(X)(l_1 l_2) = (l_1 \gamma^0_m(X)(l_2)) + (\gamma^0_m(X)(l_1) l_2),$$

i.e., $\gamma^0_m(X) \in Der_{\mathcal{C}}(\mathcal{A})$. But we can also write this as

$$\gamma^0_m(X)(l_1 \circ l_2) = l_1 \circ \gamma^0_m(X)(l_2) + \gamma^0_m(X)(l_1) \circ l_2,$$

and conclude that $\gamma^0_m \in \Gamma^\bullet_m(\mathfrak{g}, \mathcal{A}, \mathcal{A})$, where $\mu_{\mathcal{A}}$ is the left action of $\mathcal{A}$ into itself. □

Notice that if $\tilde{\nabla}^\bullet_m \in \tilde{\Gamma}^\bullet_m(\mathfrak{g}, V, \mathcal{A})$ and $\bar{\nabla}^\bullet_m \in \bar{\Gamma}^\bullet_m(\mathfrak{g}, V, \mathcal{A})$ then $\nabla^\bullet_m = \tilde{\nabla}^\bullet_m + \bar{\nabla}^\bullet_m \in \Gamma^\bullet_m(\mathfrak{g}, V, \mathcal{A})$, with anchor $\gamma^0_m = \tilde{\gamma}^0_m + \bar{\gamma}^0_m$.

Proof.

$$\begin{aligned} &(\tilde{\nabla}^\bullet_m + \bar{\nabla}^\bullet_m)(X)(l \circ v) = \tilde{\nabla}^\bullet_m(X)(l \circ v) + \bar{\nabla}^\bullet_m(X)(l \circ v) \\ &= l \circ \tilde{\nabla}^\bullet_m(X)v + \tilde{\gamma}^0_m(X)(l) \circ v + l \circ \bar{\nabla}^\bullet_m(X)v + \bar{\gamma}^0_m(X)(l) \circ v \\ &= l \circ (\tilde{\nabla}^\bullet_m + \bar{\nabla}^\bullet_m)(X)v + (\tilde{\gamma}^0_m + \bar{\gamma}^0_m)(X)(l) \circ v. \end{aligned}$$

This proves the statement. □

Besides adding two connections, one can also multiply them.

Definition 2-32. *Consider $W \otimes_{\mathcal{R}} V$ as a left $\mathcal{A}$-module by defining*

$$r \circ (w \otimes_{\mathcal{R}} v) = w \otimes_{\mathcal{R}} r \circ v, \;\; r \in \mathcal{A}.$$

Assume that $\bar{\nabla}^{\bullet}_m \in \Gamma^{\bullet}_m(\mathfrak{g}, V, \mathcal{A})$ and $\tilde{\nabla}^{\bullet}_m \in \Gamma^{\bullet}_m(\mathfrak{g}, W, \mathcal{A})$. We define the product $\bar{\nabla}^{\bullet}_m \boxtimes \tilde{\nabla}^{\bullet}_m$ by

$$\bar{\nabla}^{\bullet}_m \boxtimes \tilde{\nabla}^{\bullet}_m(X)(w \otimes_{\mathcal{R}} v) = \bar{\nabla}^{\bullet}_m(X)w \otimes_{\mathcal{R}} v + w \otimes_{\mathcal{R}} \tilde{\nabla}^{\bullet}_m(X)v.$$

Proposition 2-33. *One has $\bar{\nabla}^{\bullet}_m \boxtimes \tilde{\nabla}^{\bullet}_m \in \Gamma^{\bullet}_m(\mathfrak{g}, W \otimes_{\mathcal{R}} V, \mathcal{A})$.*

Proof. Indeed,

$$\begin{aligned}
&\bar{\nabla}^{\bullet}_m \boxtimes \tilde{\nabla}^{\bullet}_m(X) r \circ (w \otimes_{\mathcal{R}} v) = \\
&\quad = \bar{\nabla}^{\bullet}_m \boxtimes \tilde{\nabla}^{\bullet}_m(X)(w \otimes_{\mathcal{R}} r \circ v) \\
&\quad = \bar{\nabla}^{\bullet}_m(X) w \otimes_{\mathcal{R}} r \circ v + w \otimes_{\mathcal{R}} \tilde{\nabla}^{\bullet}_m(X) r \circ v \\
&\quad = \bar{\nabla}^{\bullet}_m(X) w \otimes_{\mathcal{R}} r \circ v + w \otimes_{\mathcal{R}} r \circ \tilde{\nabla}^{\bullet}_m(X) v + w \otimes_{\mathcal{R}} \gamma^0_m(X)(r) \circ v \\
&\quad = r \circ (\bar{\nabla}^{\bullet}_m(X) w \otimes_{\mathcal{R}} v + w \otimes_{\mathcal{R}} \tilde{\nabla}^{\bullet}_m(X) v) + \gamma^0_m(X)(r) \circ w \otimes_{\mathcal{R}} v \\
&\quad = r \circ (\bar{\nabla}^{\bullet}_m \boxtimes \tilde{\nabla}^{\bullet}_m)(X)(w \otimes_{\mathcal{R}} v) + \gamma^0_m(X)(r) \circ w \otimes_{\mathcal{R}} v.
\end{aligned}$$

This show that we have indeed a connection. □

Remark 2-34. *The definition of connection is a special case of this product under the identification $V \backsimeq \mathcal{A} \otimes_{\mathcal{R}} V$. One has $\nabla^{\bullet}_m \circ \lambda_{\mathcal{A}} = \lambda_{\mathcal{A}} \circ \gamma^0_m \boxtimes \nabla^{\bullet}_m$, where $\lambda_{\mathcal{A}}(r \otimes v) = r \circ v$, as usual.*

Proposition 2-35. *Any connection $\nabla^{\bullet}_m \in \Gamma^{\bullet}_m(\mathfrak{g}, V, \mathcal{A})$ induces an* **adjoint connection** *$\hat{\nabla}^{\bullet}_m \in \Gamma^{\bullet}_m(\mathfrak{g}, End_{\mathcal{C}}(V), \mathcal{A})$ by*

$$\hat{\nabla}^{\bullet}_m(X)B = \nabla^{\bullet}_m(X)B - B\nabla^{\bullet}_m(X), \quad B \in End_{\mathcal{C}}(V).$$

Proof. Indeed, for $v \in V$,

$$\begin{aligned}
&\hat{\nabla}^{\bullet}_m(X) rBv = \nabla^{\bullet}_m(X) rBv - rB\nabla^{\bullet}_m(X)v \\
&\quad = r\nabla^{\bullet}_m(X)Bv + \gamma^0_m(X)(r)Bv - rB\nabla^{\bullet}_m(X)v \\
&\quad = r\hat{\nabla}^{\bullet}_m(X)Bv + \gamma^0_m(X)(r)Bv,
\end{aligned}$$

and we have shown that it is well defined. □

Remark 2-36. *$\nabla^{\bullet}_m$ itself is also a connection on $End_{\mathcal{C}}(V)$ since we have $(\nabla^{\bullet}_m B)v = \nabla^{\bullet}_m(Bv)$ for $v \in V$ and $B \in End_{\mathcal{C}}(V)$.*

2.6 Connections on chains and cochains

Abstract 2-37. *In this section we introduce chains and cochains and show how to induce connections to them.*

Definition 2-38. *$\mathcal{R}$-linear n-tensors of $\mathfrak{h}_{\mathcal{R}}^m$ are called n-**chain**s and their space is denoted by $\bigotimes_{\mathcal{R}}^n \mathfrak{h}_{\mathcal{R}}^m$.*

Remark 2-39. *Notice that $\bigotimes_{\mathcal{R}}^n \mathfrak{h}_{\mathcal{R}}^m = \bigotimes_{\mathcal{R}}^{2^m n} \mathfrak{h}$. This makes sense as long as $2^m n$ is an integer.*

Definition 2-40. *If a direct pair $(\bar{V}, V)_{\mathcal{A}} = (\mathfrak{g}_{\mathcal{C}}^m, \mathfrak{h}_{\mathcal{R}}^m)_{\mathcal{A}}$ in definition 2-28, we write the connection as $\pi_m^1 \in \Gamma_m^1(\mathfrak{g}, \mathfrak{h}_{\mathcal{R}}^m, \mathcal{A})$. Then we inductively define the connection π_m^n on n-chains by $\pi_m^{n+1} = \pi_m^1 \boxtimes \pi_m^n$, with $\pi_m^n \in \Gamma_m^n(\mathfrak{g}, \bigotimes_{\mathcal{R}}^n \mathfrak{h}_{\mathcal{R}}^m, \mathcal{A})$.*

Proposition 2-41. *$\pi_m^n \in \Gamma_m^n(\mathfrak{g}, \bigotimes_{\mathcal{R}}^n \mathfrak{h}_{\mathcal{R}}^m, \mathcal{A})$.*

Proof. This follows immediately from proposition 2-33. $\square$

Remark 2-42. *Whenever we have an expression like π_m^n (or ∇_m^n, see definition 2-51) the lower index m will indicate the basic $\mathfrak{g}_{\mathcal{C}}^m$. We call 2^m the* word size *in the expression, where we think of individual elements in $\mathfrak{g}$ as* bits. *The upper index then indicates the total number of words that the expression acts on. This total number will be of the form $\frac{p}{2^q} \in \mathbb{N}[\frac{1}{2}]$. For instance if π has an argument of the form $X_1 \otimes X_2$, $X_1, X_2 \in \mathfrak{g}$ and acts on $Z \in \mathfrak{h}$, we write $\pi_1^{\frac{1}{2}}(X_1 \otimes X_2)Z$. This means that the word size is 2^1. In any expression the number m should be the same everywhere; in an equality it may be different in the left and right hand side. Clearly, if the group size is halved ($m+1 \mapsto m$), the number of groups doubles ($n \mapsto 2n$).*

Definition 2-43. *We call $\pi_m^1 \in \Gamma_m^1(\mathfrak{g}, \mathfrak{h}_{\mathcal{R}}^m, \mathcal{A})$ an* **antisymmetric connection** *if $\pi_m^1(X_1)X_2 + \pi_m^1(X_2)X_1 = 0$ for all $X_1, X_2 \in \mathfrak{g}_{\mathcal{C}}^m$.*

Remark 2-44. *Notice that an $\mathcal{A}$-linear antisymmetric connection necessarily has $\gamma_0^0 = 0$, so the two properties are rather incompatible. This explains why Lie brackets and connections (the last are always supposed to be $\mathcal{A}$-linear in the literature) are never treated within one framework.*

Definition 2-45. *Suppose $\pi_m^1 \in \Gamma_m^1(\mathfrak{g}, \mathfrak{h}_{\mathcal{R}}^m, \mathcal{A})$. We say that $\mathfrak{a}$ an m-**ideal** in $\mathfrak{g}_{\mathcal{C}}^m$ is if $\pi_m^1(X)Y \in \mathfrak{a}$ for all $X \in \mathfrak{g}_{\mathcal{C}}^m, Y \in \mathfrak{a} \subset \mathfrak{g}_{\mathcal{C}}^m$.*

Definition 2-46. *Let $\mathfrak{a}$ be an m-ideal. We say that $\mathfrak{a}$ is an* **abelian ideal** *if*

- $\pi_m^1(Z_1)Z_2 = 0, \forall Z_1, Z_2 \in \mathfrak{a}$,
- $\pi_m^1(Z)X + \pi_m^1(X)Z = 0, \forall Z \in \mathfrak{a}, X \in \mathfrak{g}_{\mathcal{C}}^m$.

This definition is motivated by the kind of expressions one obtains when one tries to imitate the construction of a central extension for modules with connection.

Definition 2-47. *Let* $\mathfrak{z} = \{X_1 \in \mathfrak{g}_{\mathcal{C}}^m : \pi_m^1(X_1)X_2 = 0, \quad \forall X_2 \in \mathfrak{g}_{\mathcal{C}}^m\}$. *We say that* $\mathfrak{z}$ *is the* **center of** $\mathfrak{g}_{\mathcal{C}}^m$. *For a given subset* $\mathfrak{k} \subset \mathfrak{g}_{\mathcal{C}}^m$ *we define the* **centralizer** $\mathfrak{g}_{\mathfrak{k}}^m$ *as* $\{X \in \mathfrak{g}_{\mathcal{C}}^m : \pi_m^1(Z)X = 0, \quad \forall Z \in \mathfrak{k}\}$.

Example 2-48. *Assume* $\mathcal{A}$ *to be commutative. Suppose we have*

$$\gamma_0^0 \in \Gamma_0^0(\mathfrak{b}, \mathcal{A}, \mathcal{A}).$$

Using γ_0^0 *we construct* $\pi_0^1 \in \Gamma_0^1(\mathfrak{b} \oplus \mathfrak{h}, \mathfrak{h}, \mathcal{A})$, *where* $\mathfrak{h}$ *is the Lie algebra of* $\mathcal{A}$.

Since $\gamma_0^0(X) \in Der_{\mathcal{C}}(\mathcal{A})$ *for any* $X \in \mathfrak{b}$, *this induces (cf. section 2.4) an element in the Lie algebra* $\mathfrak{h}$ *of* $\mathcal{A}$, *which we denote by* $\tilde{X}$, *such that* $\gamma_0^0(X) = \tilde{X}d$. *This element induces an action of* $\mathfrak{b}$ *on* $Y \in \mathfrak{h}$ *as follows:* $\tilde{\pi}_0^1(X)Y = [\tilde{X}, Y]$. *One has*

$$\tilde{\pi}_0^1(X)rY = [\tilde{X}, rY] = r[\tilde{X}, Y] + \gamma_0^0(X)(r)Y.$$

We can now define a connection $\pi_0^1 \in \Gamma_m^1(\mathfrak{b} \oplus \mathfrak{h}, \mathfrak{h}, \mathcal{A})$ *by*

$$\pi_0^1(X + Y)(W + Z) = [\tilde{X} + Y, \tilde{W} + Z],$$

with $X, W \in \mathfrak{b}, Y, Z \in \mathfrak{h}$. *In fact,*

$$\begin{aligned} \pi_0^1(X + Y)rZ &= [\tilde{X} + Y, rZ] = \\ &= r[\tilde{X} + Y, Z] + (\tilde{X} + Y)d(r)Z \\ &= r\pi_0^1(X + Y)Z + \hat{\gamma}_0^0(X + Y)(r)Z, \end{aligned}$$

where $\hat{\gamma}_0^0(X + Y) = (\tilde{X} + Y)d = \gamma_0^0(X) + Yd$. *This will be used in chapter 5.*

Definition 2-49. *Let* $C_m^0(\mathfrak{h}, V, -) = V$, *the space of* 0-**cochain***s. For* $n > 0$ *we define* $C_m^n(\mathfrak{h}, V, \mathcal{C}\mathcal{A}^k)$ *as the space of* $\mathcal{C}$*-linear maps of* $\bigotimes_{\mathcal{R}}^n \mathfrak{h}_{\mathcal{R}}^m$ *to* V *which are* $\mathcal{A}$*-linear in their last* k *variables, similarly called* n-**cochains** *(or* n-**forms***) of* $\mathfrak{h}_{\mathcal{R}}^m$. *The extremal cases are* $\mathcal{C}\mathcal{A}^0 = \mathcal{C}$ *and* $\mathcal{C}\mathcal{A}^n = \mathcal{A}$. *We use the notation* $\mathcal{S} = \mathcal{C}\mathcal{A}^k$ *if we do not want to specify* k. *Observe that if* $(\bar{V}, V)_{\mathcal{A}}$ *is a direct pair, so is* $(C_m^n(\mathfrak{h}, \bar{V}, \mathcal{C}\mathcal{A}^k), C_m^n(\mathfrak{h}, V, \mathcal{C}\mathcal{A}^k))_{\mathcal{A}}$.

Remark 2-50. *If a cochain is not* $\mathcal{A}$*-linear we may as well allow its arguments to be in* $\mathfrak{g}_{\mathcal{C}}^m$. *In order not to make the notation any heavier, we will not do this here.*

Definition 2-51. *If* $V = C_m^0(\mathfrak{h}, V, -)$ *in definition 2-28, we write the connection* $\nabla_m^0 \in \Gamma_m^0(\mathfrak{g}, V, \mathcal{A})$. *Given a* ∇_m^0 *and* $\pi_m^1 \in \Gamma_m^1(\mathfrak{g}, \mathfrak{h}_{\mathcal{R}}^m, \mathcal{A})$, *we define* ∇_m^n *by*

$$(\nabla_m^n(X)\omega_n)(Y) = \nabla_m^0(X)(\omega_n(Y)) - \omega_n(\pi_m^n(X)Y), n > 0, \tag{2.6.1}$$

with $X \in \mathfrak{g}_{\mathcal{C}}^m$, $Y \in \bigotimes_{\mathcal{R}}^n \mathfrak{h}_{\mathcal{R}}^m$ *and* $\omega_n \in C_m^n(\mathfrak{h}, V, \mathcal{S})$.

Remark 2-52.

- *Notice* $\nabla_m^0 = \gamma_m^0$ *if* $V = \mathcal{A}$.
- *When the space of* 0*-cochains is* $\mathfrak{h}_{\mathcal{R}}^m$, *we use* $\nabla_m^0 \in \Gamma_m^0(\mathfrak{g}, \mathfrak{h}_{\mathcal{R}}^m, \mathcal{A})$ *and* $\pi_m^1 \in \Gamma_m^1(\mathfrak{g}, \mathfrak{h}_{\mathcal{R}}^m, \mathcal{A})$ *to distinguish the actions on the cochains and chains.*

$\star$ **Remark 2-53.** *When ω_n defines a geometric structure, e.g., with $n = 2$ one might think of a Lie bracket, a symplectic form or a Riemannian metric, then one says that ∇_m^0 is a Lie-, symplectic- or Riemannian connection, respectively, if $\nabla_m^n(X)\omega_n = 0$ for all $X \in \mathfrak{g}_{\mathcal{C}}^m$.*

Proposition 2-54. *$\nabla_m^n \in \Gamma_m^n(\mathfrak{g}, C_m^n(\mathfrak{h}, V, \mathcal{S}), \mathcal{A})$, which stands for the space of connections: $\mathfrak{g}_{\mathcal{C}}^m \to End_{\mathcal{C}}^{C_m^n(\mathfrak{h},V,\mathcal{S})}(C_m^n(\mathfrak{h}, \bar{V}, \mathcal{S}))$ with anchor γ_m^0.*

Proof. Let $r \in \mathcal{A}$, $X \in \mathfrak{g}_{\mathcal{C}}^m$, $Y \in \bigotimes_{\mathcal{R}}^n \mathfrak{h}_{\mathcal{R}}^m$, $\omega_n \in C_m^n(\mathfrak{h}, V, \mathcal{S})$. Then

$$\begin{aligned}
\nabla_m^n(X)(r\omega_n)(Y) &= \nabla_m^0(X) r\omega_n(Y) - r\omega_n(\pi_m^n(X)Y) \\
&= r\nabla_m^0(X)\omega_n(Y) + \gamma_m^0(X)(r)\omega_n(Y) - r\omega_n(\pi_m^n(X)(Y)) \\
&= r\nabla_m^n(X)(\omega_n)(Y) + \gamma_m^0(X)(r)\omega_n(Y).
\end{aligned}$$

This proves the proposition. □

Proposition 2-55. *If $\omega_n \in C_m^n(\mathfrak{h}, V, \mathcal{CA})$, then $\nabla_m^n(X)\omega_n \in C_m^n(\mathfrak{h}, V, \mathcal{CA})$ for all $X \in \mathfrak{g}_{\mathcal{C}}^m$ and $n \in \mathbb{N}$.*

Proof. For any $Y \in \bigotimes_{\mathcal{R}}^n \mathfrak{h}_{\mathcal{R}}^m$ and $r \in \mathcal{A}$, we have

$$\begin{aligned}
& (\nabla_m^n(X)\omega_n)(rY) \\
= \ & \nabla_m^0(X)\omega_n(rY) - \omega_n(\pi_m^n(X)rY) \\
= \ & \nabla_m^0(X)(r\omega_n(Y)) - \omega_n(\pi_m^n(X)rY) \\
= \ & r\nabla_m^0(X)(\omega_n(Y)) + \gamma_m^0(X)(r)\omega_n(Y) - \omega_n(r\pi_m^n(X)Y + \gamma_m^0(X)(r)Y) \\
= \ & r\nabla_m^0(X)(\omega_n(Y)) - r\omega_n(\pi_m^n(X)Y) \\
= \ & r(\nabla_m^n(X)\omega_n)(Y).
\end{aligned}$$

This proves the proposition. □

2.7 Curvature

Abstract 2-56. *In this section we define the notion of curvature of a connection and use it to define representations.*

Definition 2-57. *Given $\nabla_m^\bullet \in \Gamma_m^\bullet(\mathfrak{g}, V, \mathcal{A})$ and $\pi_m^1 \in \Gamma_m^1(\mathfrak{g}, \mathfrak{h}_{\mathcal{R}}^m, \mathcal{A})$, define for $X_1, X_2 \in \mathfrak{g}_{\mathcal{C}}^m$, $\mathcal{C}(\nabla_m^\bullet)$, the* **curvature of** *$\nabla_m^\bullet$, by,*

$$\mathcal{C}(\nabla_m^\bullet)(X_1, X_2) = \hat{\nabla}_m^\bullet(X_1)\nabla_m^\bullet(X_2) - \nabla_m^\bullet(\pi_m^1(X_1)X_2), \tag{2.7.1}$$

with $\hat{\nabla}_m^\bullet$ the adjoint connection of $\nabla_m^\bullet$ as in proposition 2-35. When the curvature is zero, we say that we have a **flat connection**. *When $\mathcal{C}(\nabla_m^\bullet)(X, Y) = 0$ for all $X \in \mathfrak{g}_{\mathcal{C}}^m, Y \in \mathfrak{h}_{\mathcal{R}}^m$, we say that $\nabla_m^\bullet$ is an* **almost flat connection**.

When $\mathcal{C}(\nabla_m^\bullet) = 0$, one has $\nabla_m^\bullet(\pi_m^1(X_1)X_2) = [\nabla_m^\bullet(X_1), \nabla_m^\bullet(X_2)]$. This gives the ordinary Lie bracket of endomorphisms (cf. example 2-16), that is, one represents π_m^1 in $End_{\mathcal{C}}^V(\bar{V})$.

Example 2-58. *If the curvature of π^1_m equals zero, $\pi^1_m(X_1)X_2 + \pi^1_m(X_2)X_1$ is an element of $\mathfrak{z}$, the center of $\mathfrak{g}^m_{\mathcal{C}}$, for any $X_1, X_2 \in \mathfrak{g}^m_{\mathcal{C}}$, since we have*

$$\pi^1_m(\pi^1_m(X_1)X_2 + \pi^1_m(X_2)X_1) = \mathcal{C}(\pi^1_m)(X_1, X_2) + \mathcal{C}(\pi^1_m)(X_2, X_1) = 0.$$

Example 2-59. *We compute the curvature of the adjoint connection (as defined in 2-35) and show that it is the adjoint of the curvature. Let $B \in \mathrm{End}_{\mathcal{C}}(V)$. Then*

$$\begin{array}{rl} \multicolumn{2}{l}{\mathcal{C}(\hat{\nabla}^\bullet_m)(X,Y)B = \hat{\nabla}^\bullet_m(X)\hat{\nabla}^\bullet_m(Y)B - \hat{\nabla}^\bullet_m(Y)\hat{\nabla}^\bullet_m(X)B - \hat{\nabla}^\bullet_m(\pi^1_m(X)Y)B} \\ = & \hat{\nabla}^\bullet_m(X)(\nabla^\bullet_m(Y)B - B\nabla^\bullet_m(Y)) - \hat{\nabla}^\bullet_m(Y)(\nabla^\bullet_m(X)B - B\nabla^\bullet_m(X)) \\ - & \hat{\nabla}^\bullet_m(\pi^1_m(X)Y)B \\ = & \nabla^\bullet_m(X)\nabla^\bullet_m(Y)B - \nabla^\bullet_m(Y)B\nabla^\bullet_m(X) - \nabla^\bullet_m(X)B\nabla^\bullet_m(Y) + B\nabla^\bullet_m(Y)\nabla^\bullet_m(X) \\ - & \nabla^\bullet_m(Y)(\nabla^\bullet_m(X)B - B\nabla^\bullet_m(X)) + (\nabla^\bullet_m(X)B - B\nabla^\bullet_m(X))\nabla^\bullet_m(Y) \\ - & \nabla^\bullet_m(\pi^1_m(X)Y)B + B\nabla^\bullet_m(\pi^1_m(X)Y) \\ = & \mathcal{C}(\nabla^\bullet_m)(X,Y)B - B\mathcal{C}(\nabla^\bullet_m)(X,Y) \\ = & \widehat{\mathcal{C}(\nabla^\bullet_m)}(X,Y)B. \end{array}$$

Definition 2-60. *We say that $\nabla^\bullet_m \in \Gamma^\bullet_m(\mathfrak{g}, V, \mathcal{A})$ is an m-***representation** *of $\mathfrak{g}^m_{\mathcal{C}}$ if its curvature is almost flat, and $\bar{V}$ is called a* **left $\mathfrak{g}^m_{\mathcal{C}}$-module***.*

Definition 2-61. *If $\pi^1_m \in \Gamma^1_m(\mathfrak{g}, \mathfrak{h}^m_{\mathcal{R}}, \mathcal{A})$ is flat, we say that $\mathfrak{g}^m_{\mathcal{C}}$ is a* **Leibniz algebra***.*

Proposition 2-62. *Let π^1_0 be a flat antisymmetric connection $\mathfrak{g} \to End^{\mathfrak{h}}_{\mathcal{C}}(\mathfrak{g})$. Then we can define a Lie bracket on $\mathfrak{g}$ by*

$$[X_1, X_2] = \pi^1_0(X_1)X_2.$$

If moreover $\mathfrak{g} = \mathfrak{h} = \mathcal{A}$, then this defines a Poisson algebra on $\mathcal{A}$, with $\pi^1_0 = \gamma^0_0$.

Proof. Antisymmetry being clear from the definition, we check the Jacobi identity first.

$$\begin{array}{rl} \multicolumn{2}{l}{[X_1,[X_2,X_3]] + [X_2,[X_3,X_1]] + [X_3,[X_1,X_2]] =} \\ = & \pi^1_0(X_1)\pi^1_0(X_2)X_3 + \pi^1_0(X_2)\pi^1_0(X_3)X_1 + \pi^1_0(X_3)\pi^1_0(X_1)X_2 \\ = & -\pi^1_0(\pi^1_0(X_2)X_3)X_1 + \pi^1_0(X_2)\pi^1_0(X_3)X_1 - \pi^1_0(X_3)\pi^1_0(X_2)X_1 \\ = & \mathcal{C}(\pi^1_0)(X_2, X_3)X_1 \\ = & 0. \end{array}$$

In the case $\mathfrak{g} = \mathfrak{h} = \mathcal{A}$, the Leibniz rule follows from $\gamma^0_0(q)r = [q, r]$ and

$$[p, rq] = \gamma^0_0(p)rq = r\gamma^0_0(p)q + \gamma^0_0(p)(r)q = r[p,q] + [p,r]q,$$

and this proves the statement. $\square$

Corollary 2-63. *If $\mathfrak{g}_{\mathcal{C}}^m$ is a Leibniz algebra and π_m^1 is antisymmetric, then $\mathfrak{g}_{\mathcal{C}}^m$ is an $\mathcal{A}$-Lie algebra.* □

Proposition 2-64. *For $\bar{\nabla}_m^\bullet \in \Gamma_m^\bullet(\mathfrak{g}, V, \mathcal{A})$ and $\tilde{\nabla}_m^\bullet \in \Gamma_m^\bullet(\mathfrak{g}, W, \mathcal{A})$ one has*

$$\mathcal{C}(\bar{\nabla}_m^\bullet \boxtimes \tilde{\nabla}_m^\bullet) = \mathcal{C}(\bar{\nabla}_m^\bullet) \boxtimes \mathcal{C}(\tilde{\nabla}_m^\bullet).$$

Proof. Let $X_1, X_2 \in \mathfrak{g}_{\mathcal{C}}^m, v \in \bar{V}, w \in \bar{W}$. Then

$$\begin{aligned}
&\mathcal{C}(\bar{\nabla}_m^\bullet \boxtimes \tilde{\nabla}_m^\bullet)(X_1, X_2)(v \otimes w) = \\
&= \bar{\nabla}_m^\bullet \boxtimes \tilde{\nabla}_m^\bullet(X_1) \bar{\nabla}_m^\bullet \boxtimes \tilde{\nabla}_m^\bullet(X_2)(v \otimes w) - \bar{\nabla}_m^\bullet \boxtimes \tilde{\nabla}_m^\bullet(X_2) \bar{\nabla}_m^\bullet \boxtimes \tilde{\nabla}_m^\bullet(X_1)(v \otimes w) \\
&- \bar{\nabla}_m^\bullet \boxtimes \tilde{\nabla}_m^\bullet(\pi_m^1(X_1)X_2)(v \otimes w) \\
&= \bar{\nabla}_m^\bullet \boxtimes \tilde{\nabla}_m^\bullet(X_1)(\bar{\nabla}_m^\bullet(X_2)v \otimes w) + \bar{\nabla}_m^\bullet \boxtimes \tilde{\nabla}_m^\bullet(X_1)(v \otimes \tilde{\nabla}_m^\bullet(X_2)w) \\
&- \bar{\nabla}_m^\bullet \boxtimes \tilde{\nabla}_m^\bullet(X_2)(\bar{\nabla}_m^\bullet(X_1)v \otimes w) - \bar{\nabla}_m^\bullet \boxtimes \tilde{\nabla}_m^\bullet(X_2)(v \otimes \tilde{\nabla}_m^\bullet(X_1)w) \\
&- \bar{\nabla}_m^\bullet(\pi_m^1(X_1)X_2)v \otimes w - v \otimes \tilde{\nabla}_m^\bullet(\pi_m^1(X_1)X_2)w \\
&= \bar{\nabla}_m^\bullet(X_1)\bar{\nabla}_m^\bullet(X_2)v \otimes w + \bar{\nabla}_m^\bullet(X_1)v \otimes \tilde{\nabla}_m^\bullet(X_2)w - \bar{\nabla}_m^\bullet(X_2)\bar{\nabla}_m^\bullet(X_1)v \otimes w \\
&- \bar{\nabla}_m^\bullet(X_2)v \otimes \tilde{\nabla}_m^\bullet(X_1)w + \bar{\nabla}_m^\bullet(X_2)v \otimes \tilde{\nabla}_m^\bullet(X_1)w + v \otimes \tilde{\nabla}_m^\bullet(X_1)\tilde{\nabla}_m^\bullet(X_2)w \\
&- \bar{\nabla}_m^\bullet(X_1)v \otimes \tilde{\nabla}_m^\bullet(X_2)w - v \otimes \tilde{\nabla}_m^\bullet(X_2)\tilde{\nabla}_m^\bullet(X_1)w - \bar{\nabla}_m^\bullet(\pi_m^1(X_1)X_2)v \otimes w \\
&- v \otimes \tilde{\nabla}_m^\bullet(\pi_m^1(X_1)X_2)w \\
&= \bar{\nabla}_m^\bullet(X_1)\bar{\nabla}_m^\bullet(X_2)v \otimes w - \bar{\nabla}_m^\bullet(X_2)\bar{\nabla}_m^\bullet(X_1)v \otimes w - \bar{\nabla}_m^\bullet(\pi_m^1(X_1)X_2)v \otimes w \\
&+ v \otimes \tilde{\nabla}_m^\bullet(X_1)\tilde{\nabla}_m^\bullet(X_2)w - v \otimes \tilde{\nabla}_m^\bullet(X_2)\tilde{\nabla}_m^\bullet(X_1)w - v \otimes \tilde{\nabla}_m^\bullet(\pi_m^1(X_1)X_2)w \\
&= \mathcal{C}(\bar{\nabla}_m^\bullet)(X_1, X_2)v \otimes w + v \otimes \mathcal{C}(\tilde{\nabla}_m^\bullet)(X_1, X_2)w \\
&= \mathcal{C}(\bar{\nabla}_m^\bullet)(X_1, X_2) \boxtimes \mathcal{C}(\tilde{\nabla}_m^\bullet)(X_1, X_2)(v \otimes w).
\end{aligned}$$

This shows that the curvature behaves naturally with respect to products of connections. □

Corollary 2-65. *The curvature of a connection with anchor γ_m^0 is a connection with anchor $\mathcal{C}(\gamma_m^0)$.*

Remark 2-66. *Compare this with theorem 2.15 in [Hue90], relating this construction to the Picard group $Pic(\mathcal{A})$.*

Remark 2-67. *In the literature one usually takes π_m^1 to be the adjoint representation. Its curvature is zero by the Jacobi identity, and therefore its anchor γ_m^0 is also flat, i.e., $\mathcal{C}(\gamma_m^0) = 0$. This implies that the curvature is a* **tensor**. *This was the original motivation for the definition of the curvature in* ***Riemannian*** *geometry.*

Proof. [1] This follows immediately from proposition 2-64 with

$$\mathcal{C}(\nabla_m^\bullet) = \mathcal{C}(\nabla_m^\bullet \boxtimes \gamma_m^0) = \mathcal{C}(\nabla_m^\bullet) \boxtimes \mathcal{C}(\gamma_m^0).$$

This tells us that $\mathcal{C}(\gamma_m^0)$ is the anchor of $\mathcal{C}(\nabla_m^\bullet)$ if γ_m^0 is the anchor of $\nabla_m^\bullet$. □

[1] Of corollary 2-65.

We show that curvature is compatible with the definitions using the following diagram.

$$
\begin{tikzcd}
\pi^1_m \arrow[r, "Definition"] \arrow[d, "\mathcal{C}"] & \overbrace{\pi^n_m, \nabla^0_m} \arrow[r, "Definition"] \arrow[d, shift right=2, "\mathcal{C}"'] \arrow[d, shift left=2, "\mathcal{C}"] & \nabla^n_m \arrow[d, "\mathcal{C}"'] \\
\mathcal{C}(\pi^1_m) \arrow[r, "Definition"] & \underbrace{\mathcal{C}(\pi^n_m), \mathcal{C}(\nabla^0_m)} \arrow[r, "Definition"] & \mathcal{C}(\nabla^n_m)
\end{tikzcd}
$$

Due to proposition 2-64, it is obvious for chains. For cochains, we prove the following lemma.

Lemma 2-68. *The ∇^n_m is well defined and does not depend on the way in which it is obtained from the diagram.*

Proof. Let $X_1, X_2 \in \mathfrak{g}^m_{\mathcal{C}}, Y \in \bigotimes^n_{\mathcal{R}} \mathfrak{h}^m_{\mathcal{R}}$ and $\omega_n \in C^n_m(\mathfrak{h}, V, \mathcal{S})$. Then

$$
\begin{aligned}
&(\mathcal{C}(\nabla^n_m)(X_1, X_2)\omega_n)(Y) = \hat{\nabla}^n_m(X_1)\nabla^n_m(X_2)\omega_n(Y) - \nabla^n_m(\pi^1_m(X_1)X_2)\omega_n(Y) = \\
={}& \nabla^0_m(X_1)\nabla^n_m(X_2)\omega_n(Y) - \nabla^n_m(X_2)\omega_n(\pi^n_m(X_1)Y) - \nabla^0_m(X_2)\nabla^n_m(X_1)\omega_n(Y) \\
+{}& \nabla^n_m(X_1)\omega_n(\pi^n_m(X_2)Y) - \nabla^0_m(\pi^1_m(X_1)X_2)\omega_n(Y) + \omega_n(\pi^n_m(\pi^1_m(X_1)X_2)Y) \\
={}& \nabla^0_m(X_1)\nabla^0_m(X_2)\omega_n(Y) - \nabla^0_m(X_1)\omega_n(\pi^n_m(X_2)Y) - \nabla^0_m(X_2)\omega_n(\pi^n_m(X_1)Y) \\
+{}& \omega_n(\pi^n_m(X_2)\pi^n_m(X_1)Y) - \nabla^0_m(X_2)\nabla^0_m(X_1)\omega_n(Y) + \nabla^0_m(X_2)\omega_n(\pi^n_m(X_1)Y) \\
+{}& \nabla^0_m(X_1)\omega_n(\pi^n_m(X_2)Y) - \omega_n(\pi^n_m(X_1)\pi^n_m(X_2)Y) - \nabla^0_m(\pi^1_m(X_1)X_2)\omega_n(Y) \\
+{}& \omega_n(\pi^n_m(\pi^1_m(X_1)X_2)Y) \\
={}& \nabla^0_m(X_1)\nabla^0_m(X_2)\omega_n(Y) - \nabla^0_m(X_2)\nabla^0_m(X_1)\omega_n(Y) - \nabla^0_m(\pi^1_m(X_1)X_2)\omega_n(Y) \\
+{}& \omega_n(\pi^n_m(X_2)\pi^n_m(X_1)Y) - \omega_n(\pi^n_m(X_1)\pi^n_m(X_2)Y) + \omega_n(\pi^n_m(\pi^1_m(X_1)X_2)Y) \\
={}& \mathcal{C}(\nabla^0_m)(X_1, X_2)\omega_n(Y) - \omega_n(\mathcal{C}(\pi^n_m)(X_1, X_2)Y).
\end{aligned}
$$

This proves the lemma. ☐

Corollary 2-69. *If π^1_m is a representation, then π^n_m are representations for all n. If, moreover, ∇^0_m is a representation, so are the ∇^n_m.*

2.8 Some examples

Abstract 2-70. *In this section we give two examples, inspired on the classical mechanics of finite dimensional systems.*

Example 2-71. *Let $\mathcal{C} = \mathbb{R}$ and let $\mathcal{A}$ be the ring of C^∞-functions from $\mathbb{R}$ into itself. Then $(l_1 l_2)(x) = l_1(x) l_2(x)$. Let $V = \mathcal{A}$ and let $\mathfrak{g} = \mathfrak{h}$ be the operators of type $f(x)\frac{\partial}{\partial x}$, with $f \in \mathcal{A}$, and $l_1 \circ f(.)\frac{\partial}{\partial x} = l_1(.)f(.)\frac{\partial}{\partial x}$. Put, with $\alpha, \beta \in \mathcal{C}$,*

$$\pi^1_0(f_1 \frac{\partial}{\partial x}) f_2 \frac{\partial}{\partial x} = \alpha f_1 \frac{\partial f_2}{\partial x}\frac{\partial}{\partial x} - \beta \frac{\partial f_1}{\partial x} f_2 \frac{\partial}{\partial x}$$

and

$$\gamma_0^0(f\frac{\partial}{\partial x})h = \alpha f\frac{\partial h}{\partial x}.$$

Then $\gamma_0^0 \in \Gamma_0^0(\mathfrak{h}, \mathcal{A}, \mathcal{A})$ *and* $\pi_0^1 \in \Gamma_0^1(\mathfrak{h}, \mathfrak{h}, \mathcal{A})$.

Proof. We check

$$\begin{aligned}
\gamma_0^0(f\frac{\partial}{\partial x})(rs) &= \alpha f(\frac{\partial r}{\partial x}s + r\frac{\partial s}{\partial x}) \\
&= (\gamma_0^0(f\frac{\partial}{\partial x})(r)s) + (r\gamma_0^0(f\frac{\partial}{\partial x})(s))
\end{aligned}$$

and

$$\begin{aligned}
\pi_0^1(f\frac{\partial}{\partial x})r \circ h\frac{\partial}{\partial x} &= \alpha f(\frac{\partial r}{\partial x}h + r\frac{\partial h}{\partial x})\frac{\partial}{\partial x} - \beta\frac{\partial f}{\partial x}r \circ h\frac{\partial}{\partial x} \\
&= \alpha rf\frac{\partial h}{\partial x}\frac{\partial}{\partial x} + \alpha f\frac{\partial r}{\partial x}h\frac{\partial}{\partial x} - r\beta \circ \frac{\partial f}{\partial x}h\frac{\partial}{\partial x} \\
&= r \circ \pi_0^1(f\frac{\partial}{\partial x})h\frac{\partial}{\partial x} + \gamma_0^0(f\frac{\partial}{\partial x})(r) \circ h\frac{\partial}{\partial x}.
\end{aligned}$$

Clearly π_0^1 is $\mathcal{A}$-linear if $\beta = 0$. □

We now compute the curvature of π_0^1.

$$\begin{aligned}
&\mathcal{C}(\pi_0^1)(f_1\frac{\partial}{\partial x}, f_2\frac{\partial}{\partial x})f_3\frac{\partial}{\partial x} = \\
&= \pi_0^1(f_1\frac{\partial}{\partial x})\pi_0^1(f_2\frac{\partial}{\partial x})f_3\frac{\partial}{\partial x} - \pi_0^1(f_2\frac{\partial}{\partial x})\pi_0^1(f_1\frac{\partial}{\partial x})f_3\frac{\partial}{\partial x} \\
&- \pi_0^1(\pi_0^1(f_1\frac{\partial}{\partial x})f_2\frac{\partial}{\partial x})f_3\frac{\partial}{\partial x} \\
&= (\beta - \alpha)\left(\alpha\frac{\partial f_1}{\partial x}f_2\frac{\partial f_3}{\partial x} - \beta\frac{\partial \frac{\partial f_1}{\partial x}f_2}{\partial x}f_3\right)\frac{\partial}{\partial x} \\
&= (\beta - \alpha)\pi_0^1(\frac{\partial f_1}{\partial x}f_2\frac{\partial}{\partial x})f_3\frac{\partial}{\partial x}.
\end{aligned}$$

It follows that π_0^1 *is a 0-representation, iff* $\alpha = \beta$. *In this case it is also antisymmetric, so it defines a Lie bracket, which of course is the familiar Lie bracket of vectorfields on the real line.*

The curvature of γ_0^0 *is given by*

$$\begin{aligned}
&\mathcal{C}(\gamma_0^0)(f_1\frac{\partial}{\partial x}, f_2\frac{\partial}{\partial x}) = \\
&= \gamma_0^0(f_1\frac{\partial}{\partial x})\gamma_0^0(f_2\frac{\partial}{\partial x}) - \gamma_0^0(f_2\frac{\partial}{\partial x})\gamma_0^0(f_1\frac{\partial}{\partial x}) - \gamma_0^0(\pi_0^1(f_1\frac{\partial}{\partial x})f_2\frac{\partial}{\partial x}) \\
&= \alpha\gamma_0^0(f_1\frac{\partial}{\partial x})f_2\frac{\partial}{\partial x} - \alpha\gamma_0^0(f_2\frac{\partial}{\partial x})f_1\frac{\partial}{\partial x} - \alpha\gamma_0^0(f_1\frac{\partial f_2}{\partial x}\frac{\partial}{\partial x}) + \beta\gamma_0^0(\frac{\partial f_1}{\partial x}f_2\frac{\partial}{\partial x}) \\
&= \alpha(\beta - \alpha)\frac{\partial f_1}{\partial x}f_2\frac{\partial}{\partial x}.
\end{aligned}$$

Example 2-72. *Let $\mathcal{C} = \mathbb{R}$ and let $\mathcal{A}$ be the ring of C^∞-functions from $\mathbb{R}^2$ into $\mathbb{R}$ and $\mathcal{B}$ be the ring of C^∞-functions from $\mathbb{R}$ into itself. Let $V = \mathcal{A}$ and let $\mathfrak{g}$ be the operators of type $\alpha(t)\frac{\partial}{\partial t} + f(t,x)\frac{\partial}{\partial x}$, with $\alpha \in \mathcal{B}, f \in \mathcal{A}$ and $\mathfrak{h}$ those of type $f(t,x)\frac{\partial}{\partial x}$. We see that $\mathfrak{g}$ is an $\mathcal{B}$-module and $\mathfrak{h}$ is an $\mathcal{A}$-module. Put $\gamma_0^0(\alpha\frac{\partial}{\partial t}+f\frac{\partial}{\partial x})h = \alpha\frac{\partial h}{\partial t} + f\frac{\partial h}{\partial x}$. We check*

$$\begin{aligned}\gamma_0^0(\alpha\frac{\partial}{\partial t} + f\frac{\partial}{\partial x})rs = \alpha(\frac{\partial r}{\partial t}s + r\frac{\partial s}{\partial t}) + f(\frac{\partial r}{\partial x}s + r\frac{\partial s}{\partial x})\\ = \gamma_0^0(\alpha\frac{\partial}{\partial t} + f\frac{\partial}{\partial x})(r)s + r\gamma_0^0(\alpha\frac{\partial}{\partial t} + f\frac{\partial}{\partial x})(s).\end{aligned}$$

We now define $\pi_0^1 \in \Gamma_0^1(\mathfrak{g}, \mathfrak{h}, \mathcal{A})$ by

$$\pi_0^1(\alpha\frac{\partial}{\partial t} + f\frac{\partial}{\partial x})(\beta\frac{\partial}{\partial t} + h\frac{\partial}{\partial x}) = (\alpha\frac{\partial h}{\partial t} + f\frac{\partial h}{\partial x} - h\frac{\partial f}{\partial x})\frac{\partial}{\partial x}.$$

Notice that if we define $\frac{d}{dt} = \pi_0^1(\frac{\partial}{\partial t} + f\frac{\partial}{\partial x})$, this gives us exactly the notation as it is used in classical mechanics.

We check

$$\begin{aligned}\pi_0^1(\alpha\frac{\partial}{\partial t} + f\frac{\partial}{\partial x})rh\frac{\partial}{\partial x} = (\alpha\frac{\partial rh}{\partial t} + f\frac{\partial rh}{\partial x} - rh\frac{\partial f}{\partial x})\frac{\partial}{\partial x}\\ = (\alpha\frac{\partial r}{\partial t}h + \alpha r\frac{\partial h}{\partial t} + f\frac{\partial r}{\partial x}h + fr\frac{\partial h}{\partial x} - rh\frac{\partial f}{\partial x})\frac{\partial}{\partial x}\\ = r\pi_0^1(\alpha\frac{\partial}{\partial t} + f\frac{\partial}{\partial x})h\frac{\partial}{\partial x} + \gamma_0^0(\alpha\frac{\partial}{\partial t} + f\frac{\partial}{\partial x})(r)h\frac{\partial}{\partial x}.\end{aligned}$$

If we now compute its curvature, we see that

$$\mathcal{C}(\pi_0^1)(\alpha\frac{\partial}{\partial t} + f\frac{\partial}{\partial x}, g\frac{\partial}{\partial x}) = 0.$$

Therefore, π_0^1 is almost flat. But it is not flat and **not** *antisymmetric on $\mathfrak{g}$. It induces a Lie algebra structure on $\mathfrak{h}$.*

2.9 An implicit function theorem

Abstract 2-73. *We prove an implicit function theorem, stating that one only has to solve a problem up to a certain order to know that the solution exists and the computation will converge (in the filtration topology) to this solution. We use this theorem in our classification of scalar λ-homogeneous equations (cf. chapter 8), but it could also be used for the classification of other objects like cosymmetries or recursion operators.*

Let a Leibniz algebra $\mathfrak{g}_{\mathcal{C}}^m$ and a left $\mathfrak{g}_{\mathcal{C}}^m$-module $\bar{V}$ be filtered. We assume that π_m^1 and $\nabla_m^\bullet$ are filtered actions. For simplification, we write $\mathfrak{g}_{\mathcal{C}}^{m\,(i)}$ as $\mathfrak{g}_m^{(i)}$.

Definition 2-74. *We call* $\nabla^{\bullet}_m(S^0)$, $S^0 \in \mathfrak{g}_m^{(0)}$ **relatively** l**-prime** *with respect to* $\nabla^{\bullet}_m(K^0)$, $K^0 \in \mathfrak{g}_m^{(0)}$ *if* $\nabla^{\bullet}_m(S^0)v^j \in \operatorname{Im} \nabla^{\bullet}_m(K^0) \pmod{\bar{V}^{(j+1)}}$, *then it implies that* $v^j \in \operatorname{Im} \nabla^{\bullet}_m(K^0)|_{\bar{V}^{(j)}} \pmod{\bar{V}^{(j+1)}}$ *for all* $j \geq l$ *and* $v^j \in \bar{V}^{(j)}$.

Definition 2-75. *We call* $\nabla^{\bullet}_m(K^0)$, $K^0 \in \mathfrak{g}_m^{(0)}$, **nonlinear injective** *if for all* $v^l \in \bar{V}^{(l)}, l > 0$, $\nabla^{\bullet}_m(K^0)v^l \in \bar{V}^{(l+1)} \Rightarrow v^l \in \bar{V}^{(l+1)}$.

The following theorem states that under certain technical conditions the existence of a symmetry of an equation, i.e., an S such that $\pi^1_m(K)S = 0$ for given $K \in \mathfrak{g}_m^{(0)}$, is enough to show finite determinacy, i.e., from a finite order computation one can conclude the existence of a solution of the equation $\nabla^{\bullet}_m(K)Q = 0$. The technical conditions look strange, but are perfectly natural in at least one important class of equations, cf. remark 2-78.

Theorem 2-76. *Let* $K^i, S^i \in \mathfrak{g}_m^{(i)}$, $i = 0, 1$. *Put* $K = K^0 + K^1$ *and* $S = S^0 + S^1$. *Suppose that*

- $\pi^1_m(K)S = 0$,
- $\nabla^{\bullet}_m(K^0)$ *is nonlinear injective,*
- $\nabla^{\bullet}_m(S^0)$ *is relatively* $l+1$*-prime with respect to* $\nabla^{\bullet}_m(K^0)$ *(this implies* $S \neq K$*),*

and there exists some $\hat{Q} \in \bar{V}^{(0)}$ *such that*

- $\nabla^{\bullet}_m(K)\hat{Q} \in \bar{V}^{(l+1)}$ *and* $\nabla^{\bullet}_m(S)\hat{Q} \in \bar{V}^{(1)}$.

Then there exists a unique $Q = \hat{Q} + Q^{l+1}, Q^{l+1} \in \bar{V}^{(l+1)}$ *such that* $\nabla^{\bullet}_m(K)Q = \nabla^{\bullet}_m(S)Q = 0$.

Remark 2-77. *In the envisioned applications* Q *will be explicitly computable, reflecting the fact that e.g., the symmetries of the* `Korteweg`*-de* `Vries` *equation are all polynomial. If this is not the case, the convergence is in the filtration topology.*

$\star$ **Remark 2-78.** *If one thinks of the application of this theorem to the computation of symmetries of evolution equations, with* $m = 0$, $\bar{V} = \mathfrak{g}$, *and* $\pi^1_0 = ad$, *the adjoint action given by the Lie bracket, then this proves (at least up till the existence of* $\hat{Q}$*) the long held belief that one nontrivial symmetry* S *of the equation* K *is enough for integrability. With such a strong result one has to inspect the conditions. The strangest of these seems to be the relative prime condition. In chapter 7, however, we show that for scalar equations with linear part* $u_t = u_k$ *any symmetry* S *starting with* $u_s, s \neq 1, k$, *satisfies the conditions of the theorem with* $l = 1$.

$\star$ **Remark 2-79.** *Although the* $\nabla^{\bullet}_m(K)$*-invariant* Q *and the* $\pi^1_m(K)$*-invariant* S *are completely unrelated in theorem 2-76, we later on use the result in a context where* $\hat{Q}$ *is directly derived from* S*, and one can think of this as a way to generate the hierarchy in which* S *is contained, with* Q *as an arbitrary element of this hierarchy.*

Proposition 2-80. *$\nabla^\bullet_m(S)\hat{Q} \in \bar{V}^{(k+1)}$ if $\nabla^\bullet_m(S)\hat{Q} \in \bar{V}^{(k)}$ and $\nabla^\bullet_m(K)\hat{Q} \in \bar{V}^{(k+1)}$ under the conditions of theorem 2-76,*

Proof. We use the fact that $\nabla^\bullet_m$ is a flat connection.

$$\begin{aligned}
&\nabla^\bullet_m(K^0)\nabla^\bullet_m(S)\hat{Q} = \\
&\quad= \nabla^\bullet_m(\pi^1_m(K^0)S)\hat{Q} + \nabla^\bullet_m(S)\nabla^\bullet_m(K^0)\hat{Q} \\
&\quad\equiv -\nabla^\bullet_m(\pi^1_m(K^1)S)\hat{Q} - \nabla^\bullet_m(S)\nabla^\bullet_m(K^1)\hat{Q} \pmod{\bar{V}^{(k+1)}} \\
&\quad\equiv -\nabla^\bullet_m(K^1)\nabla^\bullet_m(S)\hat{Q} \pmod{\bar{V}^{(k+1)}} \\
&\quad\equiv 0 \pmod{\bar{V}^{(k+1)}}.
\end{aligned}$$

By the nonlinear injectiveness of $\nabla^\bullet_m(K^0)$ we have that $\nabla^\bullet_m(S)\hat{Q} \in \bar{V}^{(k+1)}$. □

Proof. [2] We prove by induction on p that $\nabla^\bullet_m(S)\hat{Q} \in \bar{V}^{(p)}$ for $p \leq l+1$. For $p = 1$ this is true by assumption. Suppose it is true for all $p \leq q < l+1$. Then the conditions of proposition 2-80 are satisfied. This implies that $\nabla^\bullet_m(S)\hat{Q} \in \bar{V}^{(q+1)}$. It follows that $\nabla^\bullet_m(S)\hat{Q} \in \bar{V}^{(l+1)}$.
Next we suppose that $\tilde{Q}$ satisfies the conditions $\nabla^\bullet_m(K)\tilde{Q} \in \bar{V}^{(p)}$ and $\nabla^\bullet_m(S)\tilde{Q} \in \bar{V}^{(p)}$, $p > l$. We know that for $p = l+1$ we can take $\tilde{Q} = \hat{Q}$. We find

$$\begin{aligned}
&\nabla^\bullet_m(K^0)\nabla^\bullet_m(S)\tilde{Q} - \nabla^\bullet_m(S^0)\nabla^\bullet_m(K)\tilde{Q} = \\
&\quad= (\nabla^\bullet_m(K) - \nabla^\bullet_m(K^1))\nabla^\bullet_m(S)\tilde{Q} - (\nabla^\bullet_m(S) - \nabla^\bullet_m(S^1))\nabla^\bullet_m(K)\tilde{Q} \\
&\quad= \nabla^\bullet_m(\pi^1_m(K)S)\tilde{Q} - \nabla^\bullet_m(K^1)\nabla^\bullet_m(S)\tilde{Q} + \nabla^\bullet_m(S^1)\nabla^\bullet_m(K)\tilde{Q} \\
&\quad= -\nabla^\bullet_m(K^1)\nabla^\bullet_m(S)\tilde{Q} + \nabla^\bullet_m(S^1)\nabla^\bullet_m(K)\tilde{Q} \in \bar{V}^{(p+1)}.
\end{aligned}$$

Since $\nabla^\bullet_m(S^0)$ is relatively $l+1$-prime with respect to $\nabla^\bullet_m(K^0)$, we see that $\nabla^\bullet_m(K)\tilde{Q} \in \operatorname{Im}\nabla^\bullet_m(K^0)$. So we can (uniquely up to $\bar{V}^{(p+1)}$ terms because $\nabla^\bullet_m(K^0)$ is nonlinear injective) define $Q^p \in \bar{V}^{(p)}$ by

$$\nabla^\bullet_m(K^0)Q^p = -\nabla^\bullet_m(K)\tilde{Q}.$$

We then automatically have $\nabla^\bullet_m(K)(\tilde{Q} + Q^p) \in \bar{V}^{(p+1)}$. That $\nabla^\bullet_m(S)(\tilde{Q} + Q^p) \in \bar{V}^{(p+1)}$ then follows from proposition 2-80. Therefore there exists a convergent (in the filtration topology) sequence with limit $Q = \hat{Q} + \sum_{p=l+1}^\infty Q^p$ such that $\nabla^\bullet_m(K)Q$ and $\nabla^\bullet_m(S)Q$ vanish. Uniqueness follows from the assumption that $\bigcap_{p=0}^\infty \bar{V}^{(p)} = 0$. This proves the statement. □

[2] Of theorem 2-76

Chapter 3

Construction of a complex

We first give the abstract definitions of precomplex, complex and cohomology. We then show that the cochain spaces form a precomplex and, when the curvature is almost flat, an m-complex. Once this is done, we start working towards the applications by introducing the reduction procedure, Fréchet and Lie derivatives Fréchet derivative and conjugate and adjoint operators.

Apparently the first to notice (in January 1989) that the construction of a complex can be lifted from the antisymmetric case to the general case was `Loday` (c.f. [Lod91], Chapter 10), who speaks of a *simple, but striking* result. In the present construction, which is based on an exercise in `Bourbaki` [Bou68], one does not even have to change the definition of the coboundary operator.

3.1 Introduction

In section 3.2 we define a precomplex and a complex, and then we introduce cohomology spaces. Our next goal is to show in section 3.3 that the cochains form a precomplex, and, when the connection is almost flat, a complex. In section 3.4 we derive explicit formulae for the coboundary operator, which are useful to answer symmetry and $\mathcal{A}$-linearity questions. The $\mathcal{A}$-linearity is treated in section 3.5, where it is shown that under certain technical conditions $\mathcal{A}$-linearity in the last two variables is preserved under the coboundary operator. The antisymmetric case is treated in section 3.6. We show that antisymmetric cochains are mapped onto themselves by the coboundary operator if the connection is $\mathcal{A}$-linear. In section 3.7 we collect all the previous results and show that we now have a number of different complexes. We now turn more into the applied direction, but we try to formulate everything in the abstract context. In section 3.8 we describe the reduction procedure which allows us to define functionals and D_x-commuting vectorfields later on in chapter 5 on the complex of formal variational calculus. Next we define in section 3.9 the Fréchet derivative and derive some of its properties. This connects the abstract coboundary operator approach with the usual variational calculus in terms of Fréchet derivatives. We are then in a position to define the Lie derivative in section 3.10. Finally we define conjugate and adjoint operators in section 3.11.

3.2 Complex and cohomology

Abstract 3-1. *We give the definitions of precomplex, m-complex and its cohomology.*

Definition 3-2. *Suppose one has*

- $\gamma_m^0 \in \Gamma_m^0(\mathfrak{g}, \mathcal{A}, \mathcal{A})$,
- $\pi_m^1 \in \Gamma_m^1(\mathfrak{g}, \mathfrak{h}_{\mathcal{R}}^m, \mathcal{A})$,

and a collection of

- $\mathcal{A}$*-modules* Ω_m^n,
- *maps* $\iota_m^n : \mathfrak{h}_{\mathcal{R}}^m \to Hom_{\mathcal{C}}(\Omega_m^n, \Omega_m^{n-1}), \quad n \geq 1$,
- *maps* $d_m^n \in Hom_{\mathcal{C}}(\Omega_m^n, \Omega_m^{n+1})$,
- *connections* $\nabla_m^n \in \Gamma_m^n(\mathfrak{g}, \Omega_m^n, \mathcal{A})$,

such that

1. $\iota_m^n(Y)\nabla_m^n(X) - \nabla_m^{n-1}(X)\iota_m^n(Y) = -\iota_m^n(\pi_m^1(X)Y), X \in \mathfrak{g}_{\mathcal{C}}^m, Y \in \mathfrak{h}_{\mathcal{R}}^m$,
2. $\nabla_m^n(Y) = \iota_m^{n+1}(Y)d_m^n + d_m^{n-1}\iota_m^n(Y), Y \in \mathfrak{h}_{\mathcal{R}}^m$.

Then we say that the Ω_m^n *form a* **precomplex** *over a direct pair* $(\mathfrak{g}_{\mathcal{C}}^m, \mathfrak{h}_{\mathcal{R}}^m)_{\mathcal{A}}$ *with ring* $\mathcal{A}$.

Definition 3-3. *A precomplex is called an* m**-complex** *if, moreover, for some* m,

1. *the* ∇_m^n *are m-representations,*
2. $\nabla_m^{n+1}(X)d_m^n = d_m^n\nabla_m^n(X)$, $X \in \mathfrak{g}_{\mathcal{C}}^m$,
3. $d_m^n d_m^{n-1} = 0$.

Once the maps $d_m^n \in Hom_{\mathcal{C}}(\Omega_m^n, \Omega_m^{n+1})$ are given for all n, we define the following spaces:

Definition 3-4. *let the space of* n**-cocycles** *(or* **closed** n**-forms***)* $Z_m^n(\Omega^\bullet)$ *be defined as* $\operatorname{Ker} d_m^n$ *and the space of* n**-coboundaries** *(or* **exact** n**-forms***)* $B_m^n(\Omega^\bullet)$ *as* $\operatorname{Im} d_m^{n-1}$.

If $d_m^n d_m^{n-1} = 0$, the cohomologies can be defined as usual:

Definition 3-5. *The* n^{th}**-cohomology module** *is* $H_m^n(\Omega^\bullet) = Z_m^n(\Omega^\bullet)/B_m^n(\Omega^\bullet)$.

Proposition 3-6. *Assume that* $H_m^1(\Omega^\bullet) = H_m^0(\Omega^\bullet) = \{0\}$. *Then,* $d_m^1\omega_1 = 0$ *and* $\nabla_m^1(X)\omega_1 = 0$ *for* $X \in \mathfrak{g}_{\mathcal{C}}^m$ *and* $\omega_1 \in \Omega^1$ *if and only if there exists a unique* $\omega_0 \in \Omega^0$ *such that* $d_m^0\omega_0 = \omega_1$ *and* $\nabla_m^0(X)\omega_0 = 0$.

Proof. Take $\omega_1 \in \Omega^1$ with $d^1_m\omega_1 = 0$ and $\nabla^1_m(X)\omega_1 = 0$ for some $X \in \mathfrak{g}^m_{\mathcal{C}}$. Since $H^1_m(\Omega^\bullet) = 0$, there exists a unique $\omega_0 \in \Omega^0$ such that $d^0_m\omega_0 = \omega_1$. From the definition of m-complex, we have

$$0 = \nabla^1_m(X)\omega_1 = \nabla^1_m(X)d^0_m\omega_0 = d^0_m\nabla^0_m(X)\omega_0. \tag{3.2.1}$$

So $\nabla^0_m(X)\omega_0 = 0$ due to $H^0_m(\Omega^\bullet) = 0$. Uniqueness follows from the fact that $H^0_m(\Omega^\bullet) = 0$.

In the other direction we define $\omega_1 = d^0_m\omega_0$. Then $d^1_m\omega_1 = d^1_m d^0_m\omega_0 = 0$ and $\nabla^1_m(X)\omega_1 = \nabla^1_m(X)d^0_m\omega_0 = d^0_m\nabla^0_m(X)\omega_0 = 0$. □

3.3 The coboundary operator

Abstract 3-7. *This section gives the definition of the coboundary operator d^n_m for the n-cochains $C^n_m(\mathfrak{h}, V, \mathcal{S})$, and shows that $d^n_m d^{n-1}_m = 0$ when the connections are representations.*

Definition 3-8. *Let $\iota^{n+1}_m\colon \mathfrak{h}^m_{\mathcal{R}} \to Hom_{\mathcal{C}}(C^{n+1}_m(\mathfrak{h}, V, \mathcal{S}), C^n_m(\mathfrak{h}, V, \mathcal{S}))$ be defined by*

$$(\iota^{n+1}_m(Y)\omega_{n+1})(Z) = \omega_{n+1}(Y \otimes_{\mathcal{R}} Z), \tag{3.3.1}$$

where $n \geq 0$, $Y \in \mathfrak{h}^m_{\mathcal{R}}$, $Z \in \bigotimes^n_{\mathcal{R}} \mathfrak{h}^m_{\mathcal{R}}$ and $\omega_{n+1} \in C^{n+1}_m(\mathfrak{h}, V, \mathcal{S})$.

Remark 3-9. $\bigcap_{Y\in\mathfrak{h}^m_{\mathcal{R}}} ker(\iota^n_m(Y)) = 0$.

Remark 3-10. *By pulling out the first argument we preserve $\mathcal{A}$-linearity of the last k arguments, with $\mathcal{S} = \mathcal{C}\mathcal{A}^k$.*

⋆ **Remark 3-11.** *In the literature one sometimes finds different definitions. The reason for this is that one views the cochains as a representation space of (a subgroup of) the permutation group of n elements acting on $\bigotimes^n_{\mathcal{R}} \mathfrak{h}^m_{\mathcal{R}}$. The most common example is the alternating representation in the case of Lie algebra cohomology theory. Another class is the alternating representation of $\mathbb{Z}/n$ giving rise to cyclic cohomology. With the present definition we include the first example, but exclude the second.*

Lemma 3-12. *One has, with $X \in \mathfrak{g}^m_{\mathcal{C}}, Y \in \mathfrak{h}^m_{\mathcal{R}}$,*

$$\iota^n_m(Y)\nabla^n_m(X) - \nabla^{n-1}_m(X)\iota^n_m(Y) = -\iota^n_m(\pi^1_m(X)Y). \tag{3.3.2}$$

Notice that the right hand side makes sense, since π^1_m is a connection and therefore $\pi^1_m(X)$ leaves $\mathfrak{h}^m_{\mathcal{R}}$ invariant.

Proof. [1] For any $\omega_n \in C^n_m(\mathfrak{h}, V, \mathcal{S})$, $X \in \mathfrak{g}^m_{\mathcal{C}}, Y \in \mathfrak{h}^m_{\mathcal{R}}$, and $Z \in \bigotimes^{n-1} \mathfrak{h}^m_{\mathcal{R}}$, we have

$$(\iota^n_m(Y)\nabla^n_m(X)\omega_n - \nabla^{n-1}_m(X)\iota^n_m(Y)\omega_n)(Z)$$

[1]Of lemma 3-12.

$$
\begin{aligned}
&= (\nabla^n_m(X)\omega_n)(Y\otimes Z) - \nabla^0_m(X)\iota^n_m(Y)\omega_n(Z) + \iota^n_m(Y)\omega_n(\pi^{n-1}_m(X)Z)\\
&= \nabla^0_m(X)(\omega_n(Y\otimes Z)) - \omega_n(\pi^n_m(X)(Y\otimes Z))\\
&- \nabla^0_m(X)(\iota^n_m(Y)\omega_n(Z)) + \omega_n(Y\otimes \pi^{n-1}_m(X)Z)\\
&= -\omega_n(\pi^1_m(X)Y\otimes Z) - \omega_n(Y\otimes \pi^{n-1}_m(X)Z) + \omega_n(Y\otimes \pi^{n-1}_m(X)Z)\\
&= -\iota^n_m(\pi^1_m(X)Y)\omega_n(Z), \qquad (3.3.3)
\end{aligned}
$$

and this proves the statement. □

Lemma 3-13. *There exist a unique* **coboundary operator** d^n_m *satisfying*

$$
\iota^{n+1}_m(Y)d^n_m + d^{n-1}_m\iota^n_m(Y) = \nabla^n_m(Y), Y \in \mathfrak{h}^m_{\mathcal{R}}, \qquad (3.3.4)
$$

where $m,n\in\mathbb{N}$, *and* $d^n_m \in Hom_{\mathcal{C}}(C^n_m(\mathfrak{h},V,\mathcal{CA}^k), C^{n+1}_m(\mathfrak{h},V,\mathcal{CA}^k))$, *with* $k\le n$.

Proof. We prove it by induction on n. For $n=0$, $Y\in\mathfrak{h}^m_{\mathcal{R}}$, the formula reads $d^0_m\omega_0(Y) = \iota^1_m(Y)d^0_m\omega_0 = \nabla^0_m(Y)\omega_0$.

If we have d^p_m uniquely defined for all $p\le n$, then d^{n+1}_m follows from (3.3.4). Its uniqueness follows from the fact that for any δ^{n+1}_m satisfying (3.3.4), we have $(d^{n+1}_m - \delta^{n+1}_m)\omega_{n+1} \in \text{Ker } \iota^n_m(Y)$ for all $Y\in\mathfrak{h}^m_{\mathcal{R}}$ (since $d^n_m=\delta^n_m$). This implies $d^{n+1}_m - \delta^{n+1}_m = 0$. □

Example 3-14. *We continue example 2-71. Take* $V=\mathfrak{h}$ *in definition 2-49 and let* $\nabla^0_0 = \pi^1_0$. *We see that*

$$
\iota^1_0(f\frac{\partial}{\partial x})d^0_0 g\frac{\partial}{\partial x} = d^0_0 g\frac{\partial}{\partial x}(f\frac{\partial}{\partial x}) = \pi^1_0(f\frac{\partial}{\partial x})g\frac{\partial}{\partial x} = \alpha f\frac{\partial g}{\partial x}\frac{\partial}{\partial x} - \beta\frac{\partial f}{\partial x}g\frac{\partial}{\partial x}.
$$

And we have

$$
\begin{aligned}
&\iota^1_0(g\frac{\partial}{\partial x})\iota^2_0(f\frac{\partial}{\partial x})d^1_0 d^0_0 h\frac{\partial}{\partial x} =\\
&= \iota^1_0(g\frac{\partial}{\partial x})(\nabla^1_0(f\frac{\partial}{\partial x}) - d^0_0\iota^1_0(f\frac{\partial}{\partial x}))d^0_0 h\frac{\partial}{\partial x}\\
&= \nabla^0_0(f\frac{\partial}{\partial x})\iota^1_0(g\frac{\partial}{\partial x})d^0_0 h\frac{\partial}{\partial x} - \iota^1_0(\pi^1_0(f\frac{\partial}{\partial x})g\frac{\partial}{\partial x})d^0_0 h\frac{\partial}{\partial x}\\
&- \iota^1_0(g\frac{\partial}{\partial x})d^0_0\iota^1_0(f\frac{\partial}{\partial x})d^0_0 h\frac{\partial}{\partial x}\\
&= \pi^1_0(f\frac{\partial}{\partial x})\pi^1_0(g\frac{\partial}{\partial x})(h\frac{\partial}{\partial x}) - \pi^1_0(\pi^1_0(f\frac{\partial}{\partial x})g\frac{\partial}{\partial x})(h\frac{\partial}{\partial x})\\
&- \pi^1_0(g\frac{\partial}{\partial x})\pi^1_0(f\frac{\partial}{\partial x})(h\frac{\partial}{\partial x})\\
&= \mathcal{C}(\pi^1_0)(f\frac{\partial}{\partial x}, g\frac{\partial}{\partial x})h\frac{\partial}{\partial x}.
\end{aligned}
$$

Proposition 3-15. *Let* $\theta^n_m(X) = \nabla^{n+1}_m(X)d^n_m - d^n_m\nabla^n_m(X)$. *Then*

$$
\iota^{n+1}_m(Y)\theta^n_m(X) + \theta^{n-1}_m(X)\iota^n_m(Y) = \mathcal{C}(\nabla^n_m)(X,Y)
$$

for $X\in\mathfrak{g}^m_{\mathcal{C}}, Y\in\mathfrak{h}^m_{\mathcal{R}}$.

Proof. Using lemma 3-12 and lemma 3-13, we have

$$
\begin{aligned}
&\iota_m^{n+1}(Y)\theta_m^n(X) + \theta_m^{n-1}(X)\iota_m^n(Y) = \\
&= \iota_m^{n+1}(Y)(\nabla_m^{n+1}(X)d_m^n - d_m^n\nabla_m^n(X)) \\
&+ (\nabla_m^n(X)d_m^{n-1} - d_m^{n-1}\nabla_m^{n-1}(X))\iota_m^n(Y) \\
&= \nabla_m^n(X)\iota_m^{n+1}(Y)d_m^n - \iota_m^{n+1}(\pi_m^1(X)Y)d_m^n - \iota_m^{n+1}(Y)d_m^n\nabla_m^n(X) \\
&+ \nabla_m^n(X)d_m^{n-1}\iota_m^n(Y) - d_m^{n-1}\iota_m^n(Y)\nabla_m^n(X) - d_m^{n-1}\iota_m^n(\pi_m^1(X)Y) \\
&= \nabla_m^n(X)(\iota_m^{n+1}(Y)d_m^n + d_m^{n-1}\iota_m^n(Y)) \\
&- (\iota_m^{n+1}(Y)d_m^n + d_m^{n-1}\iota_m^n(Y))\nabla_m^n(X) \\
&- d_m^{n-1}\iota_m^n(\pi_m^1(X)Y) - \iota_m^{n+1}(\pi_m^1(X)Y)d_m^n \\
&= \nabla_m^n(X)\nabla_m^n(Y) - \nabla_m^n(Y)\nabla_m^n(X) - \nabla_m^n(\pi_m^1(X)Y) \\
&= \mathcal{C}(\nabla_m^n)(X,Y),
\end{aligned}
$$

and this proves the statement. □

Corollary 3-16. *If ∇_m^0 and π_m^1 are representations, then*

$$\nabla_m^{n+1}(X)d_m^n = d_m^n\nabla_m^n(X),\ X \in \mathfrak{g}_{\mathcal{C}}^m.$$

Proof. For $X \in \mathfrak{g}_{\mathcal{C}}^m, Y \in \mathfrak{h}_{\mathcal{R}}^m$, we have $\iota_m^1(Y)\theta_m^0(X) = \mathcal{C}(\nabla_m^0)(X,Y) = 0$. Therefore, $\theta_m^0(X) \in \mathrm{Ker}\ \iota_m^1(Y)$. This implies $\theta_m^0(X) = 0$. Since $\mathcal{C}(\nabla_m^n)(X,Y) = 0$ for all n, we obtain that $\theta_m^n(X) = 0$ by induction and prove the statement. □

Notice that, for $Y \in \mathfrak{h}_{\mathcal{R}}^m$,

$$
\begin{aligned}
&\theta_m^n(Y) = \nabla_m^{n+1}(Y)d_m^n - d_m^n\nabla_m^n(Y) = \\
&= (\iota_m^{n+2}(Y)d_m^{n+1} + d_m^n\iota_m^{n+1}(Y))d_m^n - d_m^n(\iota_m^{n+1}(Y)d_m^n + d_m^{n-1}\iota_m^n(Y)) \\
&= \iota_m^{n+2}(Y)d_m^{n+1}d_m^n - d_m^n d_m^{n-1}\iota_m^n(Y)
\end{aligned}
$$

and $\theta_m^0(Y) = \iota_m^2(Y)d_m^1 d_m^0$. We know that if ∇_m^0 and π_m^1 are representations, the ∇_m^n are representations for all n by corollary 2-69. This leads to

Corollary 3-17. *If ∇_m^0 and π_m^1 are representations, then $d_m^{n+1}d_m^n = 0$ and*

$$\nabla_m^{n+1}(X)d_m^n = d_m^n\nabla_m^n(X),\ X \in \mathfrak{g}_{\mathcal{C}}^m.$$

3.4 Explicit formulae of d_m^n

Abstract 3-18. *In this section we give explicit formulae for the coboundary operator. In the literature this is often the way the operator is defined. These formulae are handy to have in studying its (lack of) $\mathcal{A}$-linearity.*

We first give some notational conventions that we need later.

Notation 3-19. • *For $Y_j \in \mathfrak{h}^m_{\mathcal{R}}, j = 1, \cdots, n \in \mathbb{N}$, $1 \leq l < k \leq n$, let*

$$\begin{aligned}
\alpha^n &= Y_1 \otimes \cdots \otimes Y_n, \\
\alpha^n_l &= Y_1 \otimes \cdots \otimes Y_{l-1} \otimes Y_{l+1} \otimes \cdots \otimes Y_n, \\
\alpha^n_{lk} &= Y_1 \otimes \cdots \otimes Y_{l-1} \otimes Y_{l+1} \otimes \cdots \otimes Y_{k-1} \otimes Y_{k+1} \otimes \cdots \otimes Y_n, \\
\alpha^n_{[lk]} &= Y_1 \otimes \cdots Y_{l-1} \otimes Y_{l+1} \cdots Y_{k-1} \otimes \pi^1_m(Y_l) Y_k \otimes Y_{k+1} \cdots \otimes Y_n, \\
\iota_m(\alpha^{n+1}) &= \iota^1_m(Y_{n+1}) \cdots \iota^{n+1}_m(Y_1).
\end{aligned}$$

where $\otimes$ stands for $\otimes_{\mathcal{R}}$.

- *The span of all $Y_1 \wedge \ldots \wedge Y_n$, the* **antisymmetric** n**-chain***s, is denoted $\bigwedge^n_{\mathcal{R}} \mathfrak{h}^m_{\mathcal{R}}$, and the span of all* **symmetric** n**-chain***s by $\bigvee^n_{\mathcal{R}} \mathfrak{h}^m_{\mathcal{R}}$.*

Proposition 3-20. *We can express ∇^n_m in terms of ∇^0_m as follows:*

$$\iota_m(\alpha^{n+1}_1) \nabla^n_m(Y_1) = \nabla^0_m(Y_1) \iota_m(\alpha^{n+1}_1) - \sum_{l=2}^{n+1} \iota_m(\alpha^{n+1}_{[1,l]}). \tag{3.4.1}$$

Proof. We prove this by induction on n. For $n = 1$, it is implied by the recursion relation (3.3.2). Assuming its truth for $n - 1$, we compute

$$\begin{aligned}
&\iota_m(\alpha^{n+1}_1) \nabla^n_m(Y_1) = \\
&= \iota^1_m(Y_{n+1}) \cdots \iota^n_m(Y_2) \nabla^n_m(Y_1) \\
&= \iota^1_m(Y_{n+1}) \cdots \iota^{n-1}_m(Y_3) (\nabla^{n-1}_m(Y_1) \iota^n_m(Y_2) - \iota^n_m(\pi^1_m(Y_1) Y_2)) \\
&= \iota_m(\alpha^{n+1}_{12}) \nabla^{n-1}_m(Y_1) \iota^n_m(Y_2) - \iota_m(\alpha^{n+1}_{[12]}) \\
&= \nabla^0_m(Y_1) \iota_m(\alpha^{n+1}_1) - \sum_{l=3}^{n+1} \iota_m(\alpha^{n+1}_{[1,l]}) - \iota_m(\alpha^{n+1}_{[12]}) \\
&= \nabla^0_m(Y_1) \iota_m(\alpha^{n+1}_1) - \sum_{l=2}^{n+1} \iota_m(\alpha^{n+1}_{[1,l]}),
\end{aligned}$$

and this proves the statement. □

Proposition 3-21. *We can express d^n_m in terms of ∇^0_m as follows:*

$$\iota_m(\alpha^{n+1}) d^n_m = \sum_{l=1}^{n+1} (-1)^{l+1} \left(\nabla^0_m(Y_l) \iota_m(\alpha^{n+1}_l) - \sum_{k=l+1}^{n+1} \iota_m(\alpha^{n+1}_{[lk]}) \right).$$

Proof. For $n = 0$ the expression reduces to $\iota_m(\alpha^1) d^0_m = \iota^1_m(Y_1) d^0_m = \nabla^0_m(Y_1)$ which is the formula (3.3.4) when $n = 0$. Assuming the formula to be true for n, we prove it for $n + 1$.

$$\begin{aligned}
&\iota_m(\alpha^{n+2}) d^{n+1}_m = \\
&= \iota_m(\alpha^{n+2}_1) \iota^{n+2}_m(Y_1) d^{n+1}_m
\end{aligned}$$

$$\begin{aligned}
&= \iota_m(\alpha_1^{n+2})(\nabla_m^{n+1}(Y_1) - d_m^n \iota_m^{n+1}(Y_1)) \\
&= \nabla_m^0(Y_1)\iota_m(\alpha_1^{n+2}) - \sum_{l=2}^{n+2} \iota_m(\alpha_{[1,l]}^{n+2}) \\
&+ \sum_{l=2}^{n+2}(-1)^{l+1}\left(\nabla_m^0(Y_l)\iota_m(\alpha_l^{n+2}) - \sum_{k=l+1}^{n+2} \iota_m(\alpha_{[lk]}^{n+2})\right) \\
&= \sum_{l=1}^{n+2}(-1)^{l+1}\left(\nabla_m^0(Y_l)\iota_m(\alpha_l^{n+2}) - \sum_{k=l+1}^{n+2} \iota_m(\alpha_{[lk]}^{n+2})\right).
\end{aligned}$$

The statement follows by induction. □

We write out explicitly d_m^0, d_m^1 and d_m^2, which are used later.

$$\begin{aligned}
\iota_m(\alpha^1)d_m^0 &= \nabla_m^0(Y_1), \\
\iota_m(\alpha^2)d_m^1 &= \sum_{l=1}^{2}(-1)^{l+1}\nabla_m^0(Y_l)\iota_m(\alpha_l^2) - \iota_m(\alpha_{[1,2]}^2) \\
&= \nabla_m^0(Y_1)\iota_m^1(Y_2) - \nabla_m^0(Y_2)\iota_m^1(Y_1) - \iota_m^1(\pi_m^1(Y_1)Y_2), \qquad (3.4.2) \\
\iota_m(\alpha^3)d_m^2 &= \sum_{l=1}^{3}(-1)^{l+1}\left(\nabla_m^0(Y_l)\iota_m(\alpha_l^3) - \sum_{m=l+1}^{3}\iota_m(\alpha_{[lm]}^3)\right) \\
&= \nabla_m^0(Y_1)\iota_m(\alpha_1^3) - \iota_m(\alpha_{[1,2]}^3) - \iota_m(\alpha_{[1,3]}^3) \\
&- \nabla_m^0(Y_2)\iota_m(\alpha_2^3) - \iota_m(\alpha_{[2,3]}^3) + \nabla_m^0(Y_3)\iota_m(\alpha_3^3) \\
&= \nabla_m^0(Y_1)\iota_m^1(Y_3)\iota_m^2(Y_2) - \nabla_m^0(Y_2)\iota_m^1(Y_3)\iota_m^2(Y_1) \\
&+ \nabla_m^0(Y_3)\iota_m^1(Y_2)\iota_m^2(Y_1) - \iota_m^1(Y_3)\iota_m^2(\pi_m^1(Y_1)Y_2) \\
&- \iota_m^1(\pi_m^1(Y_1)Y_3)\iota_m^2(Y_2) + \iota_m^1(\pi_m^1(Y_2)Y_3)\iota_m^2(Y_1). \qquad (3.4.3)
\end{aligned}$$

Let d_m^1 act on $\omega_1 \in C_m^1(\mathfrak{h}, V, \mathcal{S})$. (3.4.2) becomes

$$(d_m^1\omega_1)(Y_1, Y_2) = \nabla_m^0(Y_1)\omega_1(Y_2) - \nabla_m^0(Y_2)\omega_1(Y_1) - \omega_1(\pi_m^1(Y_1)Y_2). \qquad (3.4.4)$$

If π_m^1 is antisymmetric, we see by interchanging Y_1 and Y_2 that d_m^1 maps $C_m^1(\mathfrak{h}, V, \mathcal{S})$ into $C_{m,\wedge}^2(\mathfrak{h}, V, \mathcal{S})$, where we indicate by the $\wedge$ that we consider antisymmetric cochains here. Similarly we use $\circ$ to denote cyclic cochains and $\vee$ for symmetric cochains. E.g., when $\omega_n(Y_1, \cdots, Y_n) = (-1)^{n+1}\omega_n(Y_n, Y_1, \cdots, Y_{n-1})$, with $Y_i \in \mathfrak{h}$, then $\omega_n \in C_{0,\circ}^n(\mathfrak{h}, V, \mathcal{S})$.

Example 3-22. *We can consider $\nabla_m^\bullet$ as an element in $C_m^1(\mathfrak{h}, End_\mathcal{C}(V), \mathcal{S})$. From remark 2-36, we know $\nabla_m^\bullet \in \Gamma_m^\bullet(\mathfrak{g}, End_\mathcal{C}(V), \mathcal{A})$. Take it as the connection on 0-cochains $End_\mathcal{C}(V)$. Then*

$$\begin{aligned}
&(d_m^1\nabla_m^\bullet)(Y_1, Y_2) \\
&= \nabla_m^\bullet(Y_1)\nabla_m^\bullet(Y_2) - \nabla_m^\bullet(Y_2)\nabla_m^\bullet(Y_1) - \nabla_m^\bullet(\pi_m^1(Y_1)Y_2)) = \mathcal{C}(\nabla_m^\bullet)(Y_1, Y_2).
\end{aligned}$$

In other words, $\mathcal{C}(\nabla_m^\bullet) \in B_m^1(\mathfrak{h}, End_\mathcal{C}(V), \mathcal{S})$. We see that $\nabla_m^\bullet$ is closed if $\nabla_m^\bullet$ is an m-representation. Moreover, $\nabla_m^\bullet$ is exact, since $d_m^0 id_V = \nabla_m^\bullet$.

Example 3-23. *Take $\hat{\nabla}^\bullet_m \in \Gamma^\bullet_m(\mathfrak{g}, End_{\mathcal{C}}(V), \mathcal{A})$ as defined in proposition 2-35 as the connection on 0-cochains $End_{\mathcal{C}}(V)$. One has, with $\hat{d}^n_m$ the coboundary induced by $\hat{\nabla}^n_m$ and π^1_m,*

$$\mathcal{C}(\nabla^\bullet_m) = \hat{d}^1_m \nabla^\bullet_m - [\nabla^\bullet_m, \nabla^\bullet_m],$$

where, as usual, $[\nabla^\bullet_m, \nabla^\bullet_m](Y_1, Y_2) = \nabla^\bullet_m(Y_1)\nabla^\bullet_m(Y_2) - \nabla^\bullet_m(Y_2)\nabla^\bullet_m(Y_1)$. This formula appears in physical literature (cf. [GSW88]) as expressing the gauge field strength $\mathcal{C}(\nabla^\bullet_m)$ in terms of the gauge field $\nabla^\bullet_m$.

Proof. Let $Y_1, Y_2 \in \mathfrak{h}^m_{\mathcal{R}}$. Then

$$\begin{aligned}
&\hat{d}^1_m \nabla^\bullet_m(Y_1, Y_2) = \\
= \; & \hat{\nabla}^\bullet_m(Y_1)\nabla^\bullet_m(Y_2) - \hat{\nabla}^\bullet_m(Y_2)\nabla^\bullet_m(Y_1) - \nabla^\bullet_m(\pi^1_m(Y_1)Y_2) \\
= \; & 2(\nabla^\bullet_m(Y_1)\nabla^\bullet_m(Y_2) - \nabla^\bullet_m(Y_2)\nabla^\bullet_m(Y_1)) - \nabla^\bullet_m(\pi^1_m(Y_1)Y_2) \\
= \; & \mathcal{C}(\nabla^\bullet_m)(Y_1, Y_2) + [\nabla^\bullet_m, \nabla^\bullet_m](Y_1, Y_2).
\end{aligned}$$

The formula is now proved. □

Example 3-24. *We continue example 3-14. Assume that $\alpha = \beta = 1$. Let the space of 0-cochains be $V = C^\infty(\mathbb{R}, \mathbb{R})$. If $\nabla^0_0(f\frac{\partial}{\partial x})h = f\frac{\partial h}{\partial x}$, $h \in V$, and $\omega_1(f\frac{\partial}{\partial x}) = \frac{\partial^2 f}{\partial x^2}$, $\omega_1 \in C^1_0(\mathfrak{h}, V, \mathbb{C})$, then*

$$\begin{aligned}
& d^1_0\omega_1(f\frac{\partial}{\partial x}, g\frac{\partial}{\partial x}) \\
= \; & \nabla^0_0(f\frac{\partial}{\partial x})\omega_1(g\frac{\partial}{\partial x}) - \nabla^0_0(g\frac{\partial}{\partial x})\omega_1(f\frac{\partial}{\partial x}) - \omega_1(\pi^1_0(f\frac{\partial}{\partial x})g\frac{\partial}{\partial x}) \\
= \; & f\frac{\partial^3 g}{\partial x^3} - g\frac{\partial^3 f}{\partial x^3} - \frac{\partial^2(f\frac{\partial g}{\partial x} - g\frac{\partial f}{\partial x})}{\partial x^2} \\
= \; & \frac{\partial g}{\partial x}\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial x}\frac{\partial^2 g}{\partial x^2}.
\end{aligned}$$

Similarly, let d^2_m act on $\omega_2 \in C^2_m(\mathfrak{h}, V, \mathcal{S})$. (3.4.3) becomes

$$\begin{aligned}
& (d^2_m\omega_2)(Y_1, Y_2, Y_3) \\
= \; & \nabla^0_m(Y_1)\omega_2(Y_2, Y_3) - \nabla^0_m(Y_2)\omega_2(Y_1, Y_3) + \nabla^0_m(Y_3)\omega_2(Y_1, Y_2) \\
- \; & \omega_2(\pi^1_m(Y_1)Y_2, Y_3) - \omega_2(Y_2, \pi^1_m(Y_1)Y_3) + \omega_2(Y_1, \pi^1_m(Y_2)Y_3).
\end{aligned} \tag{3.4.5}$$

Example 3-25. *Take $\hat{\nabla}^\bullet_m \in \Gamma^\bullet_m(\mathfrak{g}, End_{\mathcal{C}}(V), \mathcal{A})$ as defined in proposition 2-35 as the connection on 0-cochains $End_{\mathcal{C}}(V)$. One has*

$$\hat{d}^2_m\mathcal{C}(\nabla^\bullet_m)(Y_1, Y_2, Y_3) = -\nabla^\bullet_m(\mathcal{C}(\pi^1_m)(Y_1, Y_2)Y_3).$$

This is the **Bianchi identity**. *In particular, $\mathcal{C}(\nabla^\bullet_m)$ is closed when $\mathfrak{g}^m_{\mathcal{C}}$ is a Leibniz algebra. We prove it as follows.*

$$\hat{d}^2_m\mathcal{C}(\nabla^\bullet_m)(Y_1, Y_2, Y_3) =$$

$$
\begin{aligned}
&= \hat{\nabla}^{\bullet}_m(Y_1)\mathcal{C}(\nabla^{\bullet}_m)(Y_2,Y_3) - \hat{\nabla}^{\bullet}_m(Y_2)\mathcal{C}(\nabla^{\bullet}_m)(Y_1,Y_2) + \hat{\nabla}^{\bullet}_m(Y_3)\mathcal{C}(\nabla^{\bullet}_m)(Y_1,Y_2)\\
&- \mathcal{C}(\nabla^{\bullet}_m)(\pi^1_m(Y_1)Y_2,Y_3) - \mathcal{C}(\nabla^{\bullet}_m)(Y_2,\pi^1_m(Y_1)Y_3) + \mathcal{C}(\nabla^{\bullet}_m)(Y_1,\pi^1_m(Y_2)Y_3)\\
&= \nabla^{\bullet}_m(Y_1)\nabla^{\bullet}_m(Y_2)\nabla^{\bullet}_m(Y_3) - \nabla^{\bullet}_m(Y_1)\nabla^{\bullet}_m(Y_3)\nabla^{\bullet}_m(Y_2) - \nabla^{\bullet}_m(Y_1)\nabla^{\bullet}_m(\pi^1_m(Y_2)Y_3)\\
&- \nabla^{\bullet}_m(Y_2)\nabla^{\bullet}_m(Y_3)\nabla^{\bullet}_m(Y_1) + \nabla^{\bullet}_m(Y_3)\nabla^{\bullet}_m(Y_2)\nabla^{\bullet}_m(Y_1) + \nabla^{\bullet}_m(\pi^1_m(Y_2)Y_3)\nabla^{\bullet}_m(Y_1)\\
&- \nabla^{\bullet}_m(Y_2)\nabla^{\bullet}_m(Y_1)\nabla^{\bullet}_m(Y_3) + \nabla^{\bullet}_m(Y_2)\nabla^{\bullet}_m(Y_3)\nabla^{\bullet}_m(Y_1)) + \nabla^{\bullet}_m(Y_2)\nabla^{\bullet}_m(\pi^1_m(Y_1)Y_3)\\
&+ \nabla^{\bullet}_m(Y_1)\nabla^{\bullet}_m(Y_3)\nabla^{\bullet}_m(Y_2) - \nabla^{\bullet}_m(Y_3)\nabla^{\bullet}_m(Y_1))\nabla^{\bullet}_m(Y_2) - \nabla^{\bullet}_m(\pi^1_m(Y_1)Y_3)\nabla^{\bullet}_m(Y_2)\\
&+ \nabla^{\bullet}_m(Y_3)\nabla^{\bullet}_m(Y_1)\nabla^{\bullet}_m(Y_2) - \nabla^{\bullet}_m(Y_3)\nabla^{\bullet}_m(Y_2)\nabla^{\bullet}_m(Y_1)) - \nabla^{\bullet}_m(Y_3)\nabla^{\bullet}_m(\pi^1_m(Y_1)Y_2)\\
&- \nabla^{\bullet}_m(Y_1)\nabla^{\bullet}_m(Y_2)\nabla^{\bullet}_m(Y_3) + \nabla^{\bullet}_m(Y_2)\nabla^{\bullet}_m(Y_1)\nabla^{\bullet}_m(Y_3) + \nabla^{\bullet}_m(\pi^1_m(Y_1)Y_2)\nabla^{\bullet}_m(Y_3)\\
&- \nabla^{\bullet}_m(\pi^1_m(Y_1)Y_2)\nabla^{\bullet}_m(Y_3) + \nabla^{\bullet}_m(Y_3)\nabla^{\bullet}_m(\pi^1_m(Y_1)Y_2) + \nabla^{\bullet}_m(\pi^1_m(\pi^1_m(Y_1)Y_2)Y_3)\\
&- \nabla^{\bullet}_m(Y_2)\nabla^{\bullet}_m(\pi^1_m(Y_1)Y_3) + \nabla^{\bullet}_m(\pi^1_m(Y_1)Y_3)\nabla^{\bullet}_m(Y_2) + \nabla^{\bullet}_m(\pi^1_m(Y_2)\pi^1_m(Y_1)Y_3)\\
&+ \nabla^{\bullet}_m(Y_1)\nabla^{\bullet}_m(\pi^1_m(Y_2)Y_3) - \nabla^{\bullet}_m(\pi^1_m(Y_2)Y_3)\nabla^{\bullet}_m(Y_1) - \nabla^{\bullet}_m(\pi^1_m(Y_1)\pi^1_m(Y_2)Y_3)\\
&= -\nabla^{\bullet}_m(\mathcal{C}(\pi^1_m)(Y_1,Y_2)Y_3).
\end{aligned}
$$

Proposition 3-26. *Let π^1_m be antisymmetric. Then $d^2_m C^2_{m,\circ}(\mathfrak{h},V,\mathcal{C}) \subset C^3_{m,\circ}(\mathfrak{h},V,\mathcal{C})$.*

Proof. We show that $(d^2_m\omega_2)(Y_1,Y_2,Y_3)$ is invariant under a generator of the group of cyclic transformations.

$$
\begin{aligned}
&(d^2_m\omega_2)(Y_1,Y_2,Y_3) - (d^2_m\omega_2)(Y_3,Y_1,Y_2) =\\
&= \nabla^0_m(Y_1)\omega_2(Y_2,Y_3) - \nabla^0_m(Y_2)\omega_2(Y_1,Y_3) + \nabla^0_m(Y_3)\omega_2(Y_1,Y_2)\\
&+ \nabla^0_m(Y_1)\omega_2(Y_3,Y_2) - \nabla^0_m(Y_2)\omega_2(Y_3,Y_1) - \nabla^0_m(Y_3)\omega_2(Y_1,Y_2)\\
&+ \omega_2(Y_1,\pi^1_m(Y_2)Y_3) - \omega_2(Y_2,\pi^1_m(Y_1)Y_3) - \omega_2(\pi^1_m(Y_1)Y_2,Y_3)\\
&+ \omega_2(Y_1,\pi^1_m(Y_3)Y_2) + \omega_2(\pi^1_m(Y_3)Y_1,Y_2) - \omega_2(Y_3,\pi^1_m(Y_1)Y_2)\\
&= \nabla^0_m(Y_1)(\omega_2(Y_2,Y_3) + \omega_2(Y_3,Y_2)) - \nabla^0_m(Y_2)(\omega_2(Y_1,Y_3) + \omega_2(Y_3,Y_1))\\
&+ \omega_2(Y_1,\pi^1_m(Y_2)Y_3) + \pi^1_m(Y_3)Y_2) + \omega_2(\pi^1_m(Y_3)Y_1 + \pi^1_m(Y_1)Y_3,Y_2)\\
&= 0,
\end{aligned}
$$

since the 2-forms are antisymmetric. □

Proposition 3-27. *Let π^1_m be antisymmetric. Then $d^2_m C^2_{m,\wedge}(\mathfrak{h},V,\mathcal{C}) \subset C^3_{m,\wedge}(\mathfrak{h},V,\mathcal{C})$.*

Proof. First we show that $(d^2_m\omega_2)(Y_1,Y_2,Y_3)$ is invariant under the exchange of Y_1 and Y_2.

$$
\begin{aligned}
&(d^2_m\omega_2)(Y_1,Y_2,Y_3) + (d^2_m\omega_2)(Y_2,Y_1,Y_3) =\\
&= \nabla^0_m(Y_1)\omega_2(Y_2,Y_3) - \nabla^0_m(Y_2)\omega_2(Y_1,Y_3) + \nabla^0_m(Y_3)\omega_2(Y_1,Y_2)\\
&- \nabla^0_m(Y_1)\omega_2(Y_2,Y_3) + \nabla^0_m(Y_2)\omega_2(Y_1,Y_3) + \nabla^0_m(Y_3)\omega_2(Y_2,Y_1)\\
&- \omega_2(\pi^1_m(Y_1)Y_2,Y_3) - \omega_2(Y_2,\pi^1_m(Y_1)Y_3) + \omega_2(Y_1,\pi^1_m(Y_2)Y_3)\\
&- \omega_2(\pi^1_m(Y_2)Y_1,Y_3) + \omega_2(Y_2,\pi^1_m(Y_1)Y_3) - \omega_2(Y_1,\pi^1_m(Y_2)Y_3)\\
&= \nabla^0_m(Y_3)(\omega_2(Y_1,Y_2) + \omega_2(Y_2,Y_1)) - \omega_2(\pi^1_m(Y_1)Y_2 + \pi^1_m(Y_2)Y_1,Y_3)\\
&= 0,
\end{aligned}
$$

using the antisymmetry of both π^1_m and ω_2. Together with proposition 3-26 this proves the proposition for the generators of the symmetric group. □

We generalize this proposition from d^2_m to d^n_m in proposition 3-38.

$\star$ **Remark 3-28.** *In the formula for $d^2_m\omega_2$ one recognizes the* **Christoffel symbol***s of the first kind*

$$\left[\begin{array}{c} i\ \ j \\ k \end{array}\right] = \nabla^0_m(Y_i)\omega_2(Y_k, Y_j) - \nabla^0_m(Y_k)\omega_2(Y_i, Y_j) + \nabla^0_m(Y_j)\omega_2(Y_i, Y_k).$$

If $\omega_2 \in C^2_{m,\vee}(\mathfrak{h}, V, \mathcal{S})$, one finds that

$$\left[\begin{array}{c} i\ \ j \\ k \end{array}\right] = \left[\begin{array}{c} j\ \ i \\ k \end{array}\right].$$

Here we compute some simple cohomologies for the concrete cochain spaces $\Omega^n_m = C^n_m(\mathfrak{h}, V, \mathcal{S})$ and write the cohomology $H^n_m(\Omega^\bullet)$ as $H^n_m(\mathfrak{h}, V, \mathcal{S})$.

$\star$ **Remark 3-29.** *Let $\mathfrak{h}$ be a Lie algebra, π^1_0 the adjoint representation, ω_2 its Killing form and ∇^0_0 the trivial representation. Then ω_2 is invariant under ∇^2_0 due to its associativity, i.e., $\omega_2([Y_1, Y_2], Y_3) = \omega_2(Y_1, [Y_2, Y_3])$. We see that $d^2_0\omega_2(Y_1, Y_2, Y_3) = \omega_2(Y_1, [Y_2, Y_3])$. If $\mathfrak{h}$ is semisimple, this gives us a nontrivial element in $H^3_{0,\wedge}(\mathfrak{h}, \mathcal{C}, \mathcal{C})$, the usual (cf. [Car36]) counterexample to what otherwise might have been called* `Whitehead`'s third lemma. *One has to check that it is in the kernel of d^3_0, but in our case this is trivial since $d^3_0 d^2_0 = 0$.*

Example 3-30. $H^0_m(\mathfrak{h}, V, \mathcal{S}) = Z^0_m(\mathfrak{h}, V, \mathcal{S})$ *for any V, since $B^0_m(\mathfrak{h}, V, \mathcal{S}) = 0$ by convention. Since $C^0_m(\mathfrak{h}, V, -) = V$, one has $(d^0_m v)(Y) = \nabla^0_m(Y)v$. Therefore, $H^0_m(\mathfrak{h}, V, \mathcal{S}) = V^{\mathfrak{h}}$, the subspace of* **invariant***s in V under ∇^0_m.*

3.5 $\mathcal{A}$-linearity

Abstract 3-31. *In this section we show that under certain conditions d^n_m preserves $\mathcal{A}$-linearity in the last component(s).*

Proposition 3-32. *Assume that $\omega_n \in C^n_m(\mathfrak{h}, V, \mathcal{CA})$ and $\nabla^0_m|\mathfrak{h}^m_{\mathcal{R}}$ is $\mathcal{A}$-linear. Then $d^n_m\omega_n \in C^{n+1}_m(\mathfrak{h}, V, \mathcal{CA})$.*

Proof. Using proposition 3-21, for $r \in \mathcal{A}$ we have

$$\begin{aligned}
&(d^n_m\omega_n)(Y_1, \cdots, rY_{n+1}) = \\
&= \sum_{l=1}^{n}(-1)^{l+1}\nabla^0_m(Y_l)\omega_n(Y_1, \cdots, \hat{Y}_l, \cdots, rY_{n+1}) \\
&+ (-1)^n\nabla^0_m(rY_{n+1})\omega_n(Y_1, \cdots, Y_n) \\
&- \sum_{l=1}^{n}(-1)^{l+1}\sum_{k=l+1}^{n}\omega_n(Y_1, \cdots, \hat{Y}_l, \cdots, \pi^1_m(Y_l)Y_k, \cdots, rY_{n+1})
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{l=1}^{n}(-1)^{l+1}\omega_n(Y_1,\cdots,\hat{Y}_l,\cdots,\pi^1_m(Y_l)rY_{n+1})\\
&= r\sum_{l=1}^{n+1}(-1)^{l+1}\nabla^0_m(Y_l)\omega_n(Y_1,\cdots,\hat{Y}_l,\cdots,Y_{n+1})\\
&+ \sum_{l=1}^{n}(-1)^{l+1}\gamma^0_m(Y_l)(r)\omega_n(Y_1,\cdots,\hat{Y}_l,\cdots,Y_{n+1})\\
&- r\sum_{l=1}^{n}(-1)^{l+1}\sum_{k=l+1}^{n}\omega_n(Y_1,\cdots,\hat{Y}_l,\cdots,\pi^1_m(Y_l)Y_k,\cdots,Y_{n+1})\\
&- r\sum_{l=1}^{n}(-1)^{l+1}\omega_n(Y_1,\cdots,\hat{Y}_l,\cdots,\pi^1_m(Y_l)Y_{n+1})\\
&- \sum_{l=1}^{n}(-1)^{l+1}\gamma^0_m(Y_l)(r)\omega_n(Y_1,\cdots,\hat{Y}_l,\cdots,Y_{n+1})\\
&= r(d^n_m\omega_n)(Y_1,\cdots,Y_{n+1}).
\end{aligned}
$$

This proves the statement. □

Corollary 3-33. *If $d^n_m\omega_n$ is antisymmetric or symmetric, this implies that $d^n_m\omega_n \in C^{n+1}_{m,\wedge}(\mathfrak{h},V,\mathcal{A})$ or $C^{n+1}_{m,\vee}(\mathfrak{h},V,\mathcal{A})$, respectively.*

Notice that d^1_m maps $C^1_m(\mathfrak{h},V,\mathcal{A})$ into $C^2_{m,\wedge}(\mathfrak{h},V,\mathcal{CA})$. So, we have $d^1_m C^1_m(\mathfrak{h},V,\mathcal{A}) \subset C^2_{m,\wedge}(\mathfrak{h},V,\mathcal{A})$.

Proposition 3-34. *If $\omega_n \in C^n_m(\mathfrak{h},V,\mathcal{CA}^2)$, $\nabla^0_m|\mathfrak{h}^m_{\mathcal{R}}$ is $\mathcal{A}$-linear, and π^1_m is antisymmetric, then $d^n_m\omega_n \in C^{n+1}_m(\mathfrak{h},V,\mathcal{CA}^2)$.*

Proof. From proposition 3-21, for $r \in \mathcal{A}$ we have

$$
\begin{aligned}
&(d^n_m\omega_n)(Y_1,\cdots,rY_n,Y_{n+1}) =\\
&= \sum_{l=1}^{n-1}(-1)^{l+1}\nabla^0_m(Y_l)\omega_n(Y_1,\cdots,\hat{Y}_l,\cdots,rY_n,Y_{n+1})\\
&+ (-1)^n\nabla^0_m(Y_{n+1})\omega_n(Y_1,\cdots,Y_{n-1},rY_n)\\
&+ (-1)^{n+1}\nabla^0_m(rY_n)\omega_n(Y_1,\cdots,Y_{n-1},Y_{n+1})\\
&- \sum_{l=1}^{n-1}(-1)^{l+1}\sum_{k=l+1}^{n-1}\omega_n(Y_1,\cdots,\hat{Y}_l,\cdots,\pi^1_m(Y_l)Y_k,\cdots,rY_n,Y_{n+1})\\
&- \sum_{l=1}^{n-1}(-1)^{l+1}\omega_n(Y_1,\cdots,\hat{Y}_l,\cdots,rY_n,\pi^1_m(Y_l)Y_{n+1})\\
&- \sum_{l=1}^{n-1}(-1)^{l+1}\omega_n(Y_1,\cdots,\hat{Y}_l,\cdots,Y_{n-1},\pi^1_m(Y_l)rY_n,Y_{n+1})
\end{aligned}
$$

$$
\begin{aligned}
&- && (-1)^{n+1}\omega_n(Y_1,\cdots,Y_{n-1},\pi_m^1(rY_n)Y_{n+1})\\
&= && r\sum_{l=1}^{n+1}(-1)^{l+1}\nabla_m^0(Y_l)\omega_n(Y_1,\cdots,\hat{Y}_l,\cdots,Y_n,Y_{n+1})\\
&- && r\sum_{l=1}^{n}(-1)^{l+1}\sum_{k=l+1}^{n+1}\omega_n(Y_1,\cdots,\hat{Y}_l,\cdots,\pi_m^1(Y_l)Y_k,\cdots,Y_n,Y_{n+1})\\
&= && r(d_m^n\omega_n)(Y_1,\cdots,Y_n,Y_{n+1}).
\end{aligned}
$$

This together with proposition 3-32 proves the statement. □

For later reference we consider here the following special case.

Proposition 3-35. *If $\omega_2 \in C_m^2(\mathfrak{h},V,\mathcal{A})$, $\nabla_m^0|\mathfrak{h}_{\mathcal{R}}^m$ is $\mathcal{A}$-linear, and π_m^1 is antisymmetric, then*

$$
d_m^2\omega_2(rY_1,Y_2,Y_3) = rd_m^2\omega_2(Y_1,Y_2,Y_3) + \gamma_m^0(Y_3)(r)(\omega_2(Y_2,Y_1)+\omega_2(Y_1,Y_2)).
$$

Proof. For $r \in \mathcal{A}$, we compute

$$
\begin{aligned}
&d_m^2\omega_2(rY_1,Y_2,Y_3) = \\
&\quad = \quad \nabla_m^0(rY_1)\omega_2(Y_2,Y_3) - \nabla_m^0(Y_2)\omega_2(rY_1,Y_3) + \nabla_m^0(Y_3)\omega_2(rY_1,Y_2)\\
&\quad - \quad \omega_2(\pi_m^1(rY_1)Y_2,Y_3) - \omega_2(Y_2,\pi_m^1(rY_1)Y_3) + \omega_2(rY_1,\pi_m^1(Y_2)Y_3)\\
&\quad = \quad r\nabla_m^0(Y_1)\omega_2(Y_2,Y_3) - \nabla_m^0(Y_2)r\omega_2(Y_1,Y_3) + \nabla_m^0(Y_3)r\omega_2(Y_1,Y_2)\\
&\quad + \quad \omega_2(\pi_m^1(Y_2)rY_1,Y_3) + \omega_2(Y_2,\pi_m^1(Y_3)rY_1) + r\omega_2(Y_1,\pi_m^1(Y_2)Y_3)\\
&\quad = \quad r\nabla_m^0(Y_1)\omega_2(Y_2,Y_3) - r\nabla_m^0(Y_2)\omega_2(Y_1,Y_3) + r\nabla_m^0(Y_3)\omega_2(Y_1,Y_2)\\
&\quad + \quad r\omega_2(\pi_m^1(Y_2)Y_1,Y_3) + r\omega_2(Y_2,\pi_m^1(Y_3)Y_1) + r\omega_2(Y_1,\pi_m^1(Y_2)Y_3)\\
&\quad + \quad \gamma_m^0(Y_3)(r)\omega_2(Y_2,Y_1) + \gamma_m^0(Y_3)(r)\omega_2(Y_1,Y_2)\\
&\quad = \quad rd_m^2\omega_2(Y_1,Y_2,Y_3) + \gamma_m^0(Y_3)(r)(\omega_2(Y_2,Y_1)+\omega_2(Y_1,Y_2)).
\end{aligned}
$$

This clearly shows the obstruction to $\mathcal{A}$-linearity. □

Remark 3-36. *Assume that $\nabla_m^0|\mathfrak{h}_{\mathcal{R}}^m$ is $\mathcal{A}$-linear, and π_m^1 is antisymmetric. We start to lose $\mathcal{A}$-linearity when we apply d_m^2. This explains why the Levi–Civita connection is not a tensor (cf. appendix B), nor is the Christoffel symbol (take $\pi_m^1 = 0$ in the last case).*

3.6 The antisymmetric case

Abstract 3-37. *We show that $d_m^n C_{m,\wedge}^n(\mathfrak{h},V,\mathcal{A}) \subset C_{m,\wedge}^{n+1}(\mathfrak{h},V,\mathcal{A})$.*

Proposition 3-38. *If π_m^1 is antisymmetric and $\nabla_m^0|\mathfrak{h}_{\mathcal{R}}^m$ is $\mathcal{A}$-linear, then*

$$
d_m^n C_{m,\wedge}^n(\mathfrak{h},V,\mathcal{A}) \subset C_{m,\wedge}^{n+1}(\mathfrak{h},V,\mathcal{A}).
$$

Proof. We have to show that, for $\omega_n \in C^n_{m,\wedge}(\mathfrak{h}, V, \mathcal{A})$ and any $1 \leq q \leq n+1$,

$$(d^n_m \omega_n)(Y_1, \cdots, Y_q, Y_{q+1}, \cdots, Y_{n+1}) = -(d^n_m \omega_n)(Y_1, \cdots, Y_{q+1}, Y_q, \cdots, Y_{n+1}).$$

We check it directly by using proposition 3-21.

$$\begin{aligned}
&(d^n_m)\omega_n(Y_1, \cdots, Y_q, Y_{q+1}, \cdots, Y_{n+1}) = \\
= \ & \sum_{l=1}^{q-1}(-1)^{l+1}\nabla^0_m(Y_l)\omega_n(Y_1, \cdots, \hat{Y}_l, \cdots, Y_q, Y_{q+1}, \cdots, Y_{n+1}) \\
+ \ & (-1)^{q+1}\nabla^0_m(Y_q)\omega_n(Y_1, \cdots, \hat{Y}_q, Y_{q+1}, \cdots, Y_{n+1}) \\
+ \ & (-1)^{q}\nabla^0_m(Y_{q+1})\omega_n(Y_1, \cdots, Y_q, \hat{Y}_{q+1}, \cdots, Y_{n+1}) \\
+ \ & \sum_{l=q+2}^{n+1}(-1)^{l+1}\nabla^0_m(Y_l)\omega_n(Y_1, \cdots, Y_q, Y_{q+1}, \cdots, \hat{Y}_l, \cdots, Y_{n+1}) \\
- \ & \sum_{l=1}^{q-1}(-1)^{l+1}\sum_{k=l+1}^{q-1}\omega_n(Y_1, \cdots, \hat{Y}_l, \cdots, \pi^1_m(Y_l)Y_k, \cdots, Y_q, Y_{q+1}, \cdots, Y_{n+1}) \\
- \ & \sum_{l=1}^{q-1}(-1)^{l+1}\omega_n(Y_1, \cdots, \hat{Y}_l, \cdots, \pi^1_m(Y_l)Y_q, Y_{q+1}, \cdots, Y_{n+1}) \\
- \ & \sum_{l=1}^{q-1}(-1)^{l+1}\omega_n(Y_1, \cdots, \hat{Y}_l, \cdots, Y_q, \pi^1_m(Y_l)Y_{q+1}, \cdots, Y_{n+1}) \\
- \ & \sum_{l=1}^{q-1}(-1)^{l+1}\sum_{k=q+2}^{n+1}\omega_n(Y_1, \cdots, \hat{Y}_l, \cdots, Y_q, Y_{q+1}, \cdots, \pi^1_m(Y_l)Y_k, \cdots, Y_{n+1}) \\
- \ & (-1)^{q+1}\omega_n(Y_1, \cdots, \pi^1_m(Y_q)Y_{q+1}, \cdots, Y_{n+1}) \\
- \ & (-1)^{q+1}\sum_{k=q+2}^{n+1}\omega_n(Y_1, \cdots, \hat{Y}_q, \cdots, \pi^1_m(Y_q)Y_k, \cdots, Y_{n+1}) \\
- \ & (-1)^{q}\sum_{k=q+2}^{n+1}\omega_n(Y_1, \cdots, Y_q, \hat{Y}_{q+1}, \cdots, \pi^1_m(Y_{q+1})Y_k, \cdots, Y_{n+1}) \\
- \ & \sum_{l=q+2}^{n}(-1)^{l+1}\sum_{k=l+1}^{n+1}\omega_n(Y_1, \cdots, Y_q, Y_{q+1}, \cdots, \hat{Y}_l, \cdots, \pi^1_m(Y_l)Y_k, \cdots, Y_{n+1}) \\
= \ & -\sum_{l=1}^{q-1}(-1)^{l+1}\nabla^0_m(Y_l)\omega_n(Y_1, \cdots, \hat{Y}_l, \cdots, Y_{q+1}, Y_q, \cdots, Y_{n+1}) \\
- \ & (-1)^{q}\nabla^0_m(Y_q)\omega_n(Y_1, \cdots, Y_{q+1}, \hat{Y}_q, \cdots, Y_{n+1}) \\
- \ & (-1)^{q+1}\nabla^0_m(Y_{q+1})\omega_n(Y_1, \cdots, \hat{Y}_{q+1}, Y_q, \cdots, Y_{n+1}) \\
- \ & \sum_{l=q+2}^{n+1}(-1)^{l+1}\nabla^0_m(Y_l)\omega_n(Y_1, \cdots, Y_{q+1}, Y_q, \cdots, \hat{Y}_l, \cdots, Y_{n+1})
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{l=1}^{q-1}(-1)^{l+1}\sum_{k=l+1}^{q-1}\omega_n(Y_1,\cdots,\hat{Y}_l,\cdots,\pi^1_m(Y_l)Y_k,\cdots,Y_{q+1},Y_q,\cdots,Y_{n+1})\\
&+ \sum_{l=1}^{q-1}(-1)^{l+1}\omega_n(Y_1,\cdots,\hat{Y}_l,\cdots,Y_{q+1},\pi^1_m(Y_l)Y_q,\cdots,Y_{n+1})\\
&+ \sum_{l=1}^{q-1}(-1)^{l+1}\omega_n(Y_1,\cdots,\hat{Y}_l,\cdots,\pi^1_m(Y_l)Y_{q+1},Y_q,\cdots,Y_{n+1})\\
&+ \sum_{l=1}^{q-1}(-1)^{l+1}\sum_{k=q+2}^{n+1}\omega_n(Y_1,\cdots,\hat{Y}_l,\cdots,Y_{q+1},Y_q,\cdots,\pi^1_m(Y_l)Y_k,\cdots,Y_{n+1})\\
&+ (-1)^{q+2}\sum_{k=q+2}^{n+1}\omega_n(Y_1,\cdots,Y_{q+1},\hat{Y}_q,\cdots,\pi^1_m(Y_q)Y_k,\cdots,Y_{n+1})\\
&+ (-1)^{q+1}\omega_n(Y_1,\cdots,\pi^1_m(Y_{q+1})Y_q,\cdots,Y_{n+1})\\
&+ (-1)^{q+1}\sum_{k=q+2}^{n+1}\omega_n(Y_1,\cdots,\hat{Y}_{q+1},Y_q,\cdots,\pi^1_m(Y_{q+1})Y_k,\cdots,Y_{n+1})\\
&+ \sum_{l=q+2}^{n}(-1)^{l+1}\sum_{k=l+1}^{n+1}\omega_n(Y_1,\cdots,Y_{q+1},Y_q,\cdots,\hat{Y}_l,\cdots,\pi^1_m(Y_l)Y_k,\cdots,Y_{n+1})\\
&= -(d^n_m\omega_n)(Y_1,\cdots,Y_{q+1},Y_q,\cdots,Y_{n+1}).
\end{aligned}
$$

The $\mathcal{A}$-linearity follows from corollary 3-33, using the $\mathcal{A}$-linearity of $\nabla^0_m|\mathfrak{h}^m_{\mathcal{R}}$. □

One should notice that there are three different operations involved here:

- Using the antisymmetry of ω_n itself.
- Using the factor $(-1)^{l+1}$.
- Using the antisymmetry of π^1_m.

This prevents repeating the proof for the symmetric case, since the factor $(-1)^{l+1}$ does not cooperate.

3.7 The complexes

Abstract 3-39. *In this section we show that we have now constructed a number of (pre)complexes.*

Theorem 3-40. *1. $C^{\bullet}_{\bullet}(\mathfrak{h},V,\mathcal{C})$ is a precomplex.*

2. If $\nabla^0_{\bullet}|\mathfrak{h}^{\bullet}_{\mathcal{R}}$ is $\mathcal{A}$-linear, $C^{\bullet}_{\bullet}(\mathfrak{h},V,\mathcal{C}\mathcal{A})$ is a precomplex.

3. If π^1_m is antisymmetric and $\nabla^0_{\bullet}|\mathfrak{h}^{\bullet}_{\mathcal{R}}$ is $\mathcal{A}$-linear, $C^{\bullet}_{\bullet}(\mathfrak{h},V,\mathcal{C}\mathcal{A}^2)$ is a precomplex.

4. If π^1_m is antisymmetric and $\nabla^0_\bullet|\mathfrak{h}^\bullet_{\mathcal{R}}$ is $\mathcal{A}$-linear, $C^\bullet_{\bullet,\wedge}(\mathfrak{h}, V, \mathcal{A})$ is a precomplex.

Proof. We recollect the definition of a precomplex to be a collection of

- $\mathcal{A}$-modules Ω^n_m (the n-cochains), $n \in \mathbb{N}$, which we take to be $C^n_m(\mathfrak{h}, V, \mathcal{S})$, and
- Maps $\iota^n_m : \mathfrak{h}^m_{\mathcal{R}} \to Hom_{\mathcal{C}}(\Omega^n_m, \Omega^{n-1}_m)$, following definition 3-8,
- Maps $d^n_m \in Hom_{\mathcal{C}}(\Omega^n_m, \Omega^{n+1}_m)$, as defined in lemma 3-13, (where the $\mathcal{A}$-linearity is proved in proposition 3-32 and 3-34; the antisymmetric case follows from proposition 3-38) and
- The connections $\nabla^n_m : \mathfrak{g}^m_{\mathcal{C}} \to Hom_{\mathcal{C}}(\Omega^n_m, \Omega^n_m)$ as in definition 2-51

such that (with the convention that $d^{-1}_m = 0$ and $\iota^0_m = 0$)

1. $\iota^n_m(Y)\nabla^n_m(X) - \nabla^{n-1}_m(X)\iota^n_m(Y) = -\iota^n_m(\pi^1_m(X)Y), X \in \mathfrak{g}^m_{\mathcal{C}}, Y \in \mathfrak{h}^m_{\mathcal{R}}$, following from lemma 3-12,
2. $\nabla^n_m(Y) = \iota^{n+1}_m(Y)d^n_m + d^{n-1}_m\iota^n_m(Y), Y \in \mathfrak{h}^m_{\mathcal{R}}$, as in lemma 3-13.

This shows that $C^\bullet_\bullet(\mathfrak{h}, V, \mathcal{S})$ is indeed a precomplex. □

Theorem 3-41. *If ∇^0_m and π^1_m are representations,*

1. $C^\bullet_\bullet(\mathfrak{h}, V, \mathcal{C})$ is a complex.

2. If $\nabla^0_\bullet|\mathfrak{h}^\bullet_{\mathcal{R}}$ is $\mathcal{A}$-linear, $C^\bullet_\bullet(\mathfrak{h}, V, \mathcal{CA})$ is a complex.

3. If π^1_m is antisymmetric and $\nabla^0_\bullet|\mathfrak{h}^\bullet_{\mathcal{R}}$ is $\mathcal{A}$-linear, $C^\bullet_\bullet(\mathfrak{h}, V, \mathcal{CA}^2)$ is a complex.

4. If π^1_m is antisymmetric and $\nabla^0_\bullet|\mathfrak{h}^\bullet_{\mathcal{R}}$ is $\mathcal{A}$-linear, $C^\bullet_{\bullet,\wedge}(\mathfrak{h}, V, \mathcal{A})$ is a complex.

Proof. 1. ∇^n_m are m-representations proved in corollary 2-69,

2. $\nabla^{n+1}_m(X)d^n_m = d^n_m\nabla^n_m(X), X \in \mathfrak{g}^m_{\mathcal{C}}$ follows from corollary 3-16,

3. $d^n_m d^{n-1}_m = 0$ follows from corollary 3-17.

This, together with theorem 3-40, proves the theorem. □

Remark 3-42. *The concept of Lie algebra complex as defined in [Dor93] is a special case of a complex when $\mathfrak{h} = \mathfrak{g}$ is a Lie algebra.*

3.8 Reduction procedure of a complex

Abstract 3-43. *We show how a given m-complex $C^\bullet_m(\mathfrak{h}, V, \mathcal{S})$ over a direct pair $(\mathfrak{g}^m_{\mathcal{C}}, \mathfrak{h}^m_{\mathcal{R}})_{\mathcal{A}}$ reduces to a new m-complex $\Omega^n_m[\mathfrak{k}]$ over a direct pair $(\mathfrak{g}^m_{\mathfrak{k}}, \mathfrak{h}^m_{\mathfrak{k}})_{\mathcal{A}_{\mathfrak{k}}}$ with respect to a finitely generated linear subspace $\mathfrak{k} \subset \mathfrak{g}^m_{\mathcal{C}}$.*

Consider an m-complex $C^\bullet_m(\mathfrak{h}, V, \mathcal{S})$. Assume that π^1_m and ∇^0_m are both flat. Given a finitely generated linear subspace $\mathfrak{k} \subset \mathfrak{g}^m_{\mathcal{C}}$, let $\mathfrak{h}^m_{\mathfrak{k}}$(or $\mathfrak{g}^m_{\mathfrak{k}}$) be the centralizer of $\mathfrak{k}$ in $\mathfrak{h}^m_{\mathcal{R}}$(or $\mathfrak{g}^m_{\mathcal{C}}$) and $\mathcal{A}_{\mathfrak{k}}$ the $\mathfrak{k}$-invariant elements of $\mathcal{A}$:

$$\begin{aligned}
\mathfrak{h}^m_{\mathfrak{k}} &= \{Y \in \mathfrak{h}^m_{\mathcal{R}} : \pi^1_m(Z)Y = 0, \quad \forall Z \in \mathfrak{k}\}, \\
\mathfrak{g}^m_{\mathfrak{k}} &= \{X \in \mathfrak{g}^m_{\mathcal{C}} : \pi^1_m(Z)X = 0, \quad \forall Z \in \mathfrak{k}\}, \\
\mathcal{A}_{\mathfrak{k}} &= \{r \in \mathcal{A} : \gamma^0_m(Z)r = 0, \quad \forall Z \in \mathfrak{k}\}, \\
\mathcal{R}_{\mathfrak{k}} &= \mathcal{A}_{\mathfrak{k}} \bigcap \mathcal{R}.
\end{aligned}$$

Notice that $\mathfrak{h}^m_{\mathfrak{k}}$ is an $\mathcal{A}_{\mathfrak{k}}$-module since for $Y \in \mathfrak{h}^m_{\mathfrak{k}}, r \in \mathcal{A}_{\mathfrak{k}}$,

$$\pi^1_m(Z)rY = r\pi^1_m(Z)Y + \gamma^0_m(Z)(r)Y = 0.$$

We construct $\Omega^n_m[\mathfrak{k}]$, the cochain spaces, in the following way:

$$\Omega^n_m[\mathfrak{k}] = \Omega^n_m / \{\sum_k \nabla^n_m(Z_k)\omega_k, Z_k \in \mathfrak{k}, \omega_k \in \Omega^n_m\},$$

where Ω^n_m is the space of maps of $\bigotimes^n_{\mathcal{R}_{\mathfrak{k}}} \mathfrak{h}^m_{\mathfrak{k}}$ to V.

Proposition 3-44. $\pi^1_m : \mathfrak{g}^m_{\mathfrak{k}} \to End^{\mathfrak{h}^m_{\mathfrak{k}}}_{\mathcal{C}}(\mathfrak{g}^m_{\mathfrak{k}})$.

Proof. Take $X_1, X_2 \in \mathfrak{g}^m_{\mathfrak{k}}$, then

$$\pi^1_m(Z)\pi^1_m(X_1)X_2 = \pi^1_m(X_1)\pi^1_m(Z)X_2 + \pi^1_m(\pi^1_m(Z)X_1)X_2 = 0.$$

Therefore $\pi^1_m(X_1)X_2 \in \mathfrak{g}^m_{\mathfrak{k}}$. By the same argument we show that $\mathfrak{h}^m_{\mathfrak{k}}$ is an invariant subspace for all $\pi^1_m(X_1)$. ☐

We have to check that the maps $\iota^n_m(Y)$, $\nabla^n_m(X)$ and d^n_m are well defined in this new context. Let $X \in \mathfrak{g}^m_{\mathfrak{k}}$ and $Y \in \mathfrak{h}^m_{\mathfrak{k}}$.

- Define $\nabla^n_m(X)[\omega] = [\nabla^n_m(X)\omega]$. This is well defined since

$$\nabla^n_m(X) \sum_k \nabla^n_m(Z_k)\omega_k = \sum_k \nabla^n_m(Z_k)\nabla^n_m(X)\omega_k.$$

- Define $\iota^n_m(Y)[\omega] = [\iota^n_m(Y)\omega]$. This is well defined since

$$\iota^n_m(Y) \sum_k \nabla^n_m(Z_k)\omega_k = \sum_k \nabla^{n-1}_m(Z_k)\iota^n_m(Y)\omega_k$$

(since $\pi^1_m(Z_k)Y = 0$ for $Z_k \in \mathfrak{k}$).

- Define $d^n_m[\omega] = [d^n_m\omega]$. This is well defined since

$$d^n_m \sum_k \nabla^n_m(Z_k)\omega_k = \sum_k \nabla^{n+1}_m(Z_k) d^n_m \omega_k.$$

- The relations follow automatically since all the operands factor through the equivalence classes, i.e., if $\mathcal{E}$ is a relation, then we have

$$\mathcal{E}[\omega] = [\mathcal{E}\omega] = [0].$$

3.9 The Fréchet derivative and its properties

Abstract 3-45. *We give an abstract definition of Fréchet derivative on the ring $\mathcal{A}$ and, using a finiteness assumption, extend it to chains and cochains.*

Definition 3-46. *The* **Fréchet derivative** *of $\sigma \in \mathcal{A}$ in the direction $h \in \mathfrak{h}^m_{\mathfrak{k}}$ is defined as*

$$D_\sigma[h] = \gamma^0_m(h)\sigma.$$

This definition directly leads to the following property:

$$D_{\sigma_1\sigma_2}[\cdot] = D_{\sigma_1}[\cdot]\sigma_2 + \sigma_1 D_{\sigma_2}[\cdot],\ \ \sigma_1, \sigma_2 \in \mathcal{A}.$$

We could have defined the Fréchet derivative for $h \in \mathfrak{g}^m_C$. But this would not be in accordance with the usage in the literature. This explains the occurrence of $\frac{\partial}{\partial t}$ terms in the formulae for the Lie derivatives, cf. theorem 5-10.

We restrict the space using the following assumption: $\mathfrak{h}^m_{\mathfrak{k}}$ is a finitely generated free $\mathcal{A}$-module, which can be realized by the reduction procedure as described in section 3.8. This may seem to be in contradiction with the fact that $\mathfrak{h}^m_{\mathfrak{k}}$ is a left $\mathcal{A}_{\mathfrak{k}}$-module, not a left $\mathcal{A}$-module. We assume that there is a new action of the ring, which makes $\mathfrak{h}^m_{\mathfrak{k}}$ into a left $\mathcal{A}$-module. These allows us to write $h = \sum_\alpha h^\alpha e_\alpha$ for any $h \in \mathfrak{h}^m_{\mathfrak{k}}$, where $h^\alpha \in \mathcal{A}$ and $e_\alpha \in \mathfrak{h}^m_{\mathfrak{k}}$, and the set of α is finite. Therefore, we can define the Fréchet derivatives of chains and cochains as follows.

Definition 3-47. *The Fréchet derivative of $h_0 = \sum_\alpha h^\alpha_0 e_\alpha \in \mathfrak{h}^m_{\mathfrak{k}}$ in the direction $h \in \mathfrak{h}^m_{\mathfrak{k}}$ is*

$$D_{h_0}[h] = \sum_\alpha D_{h^\alpha_0}[h] e_\alpha.$$

Definition 3-48. *Let $\omega_n \in C^n_0(\mathfrak{h}_{\mathfrak{k}}, \Omega^0_m[\mathfrak{k}], \mathcal{S})$ and $\Omega^0_m[\mathfrak{k}] \subset \mathcal{A}$. Then we define the Fréchet derivative of ω_n in the direction $h \in \mathfrak{h}^m_{\mathfrak{k}}$ by*

$$D_{\omega_n}[h](h_1, \cdots, h_n) = D_{\omega_n(h_1,\cdots,h_n)}[h] - \sum_{k=1}^n \omega_n(h_1, \cdots, D_{h_k}[h], \cdots, h_n),$$

where $h_i \in \mathfrak{h}^m_{\mathfrak{k}}$ and $i = 1, \cdots, n$.

Definition 3-49. *If now ϕ is a map from $\mathfrak{h}^m_{\mathfrak{k}}$ into itself, we can define the Fréchet derivative of of ϕ, D_ϕ in the direction $h \in \mathfrak{h}^m_{\mathfrak{k}}$ by*

$$D_\phi[h](g) = D_{\phi g}[h] - \phi(D_g[h]),\ \forall g \in \mathfrak{h}^m_{\mathfrak{k}}.$$

In this way we again have the Leibniz rule by construction.

3.10 The Lie derivative

Abstract 3-50. *We define the Lie derivative, combining π^n_m and ∇^n_m in one notation.*

We now define the **Lie derivative** $L_X, X \in \mathfrak{g}^m_{\mathcal{C}}$ on an m-complex as follows. For $\alpha^n \in \bigotimes^n \mathfrak{h}^m_{\mathcal{R}}$ (Notation 3-19) we let $L_X\alpha^n = \pi^n_m(X)(\alpha^n)$ and for $\omega_n \in \Omega^n_m$ we let $L_X\omega_n = \nabla^n_m(X)\omega_n$. Clearly we have

- $L_{L_XY} = L_XL_Y - L_YL_X,\ Y \in \mathfrak{h}^m_{\mathcal{R}}$,
- $L_X(r\cdot) = rL_X\cdot + \gamma^0_m(X)(r)\cdot\ ,\ r \in \mathcal{A}$.

and the chain rule

$$L_X(\omega_n(\alpha^n)) = (L_X\omega_n)(\alpha^n) + \omega_n(L_X\alpha^n).$$

Definition 3-51. *For $Y \in \mathfrak{g}^m_{\mathcal{C}}$ and α (either α^n or ω_n), if there exists a constant $\lambda_\alpha \in \mathcal{C}$ such that*

$$L_Y\alpha = \lambda_\alpha\alpha,$$

*we say, when $\lambda_\alpha \neq 0$, that $Y \in \mathfrak{g}^m_{\mathcal{C}}$ is a λ_α-***scaling symmetry** *of α and that α is* **homogeneous** *with respect to Y and that λ_α is the* **grading** *of α.*

If $\lambda_\alpha = 0$, we call α an **invariant** *of Y. If, moreover, $Y \in \mathfrak{h}^m_{\mathcal{R}}$, we say Y is a* **symmetry** *of α.*

Proposition 3-52. *If there exists $Y \in \mathfrak{h}^m_{\mathcal{R}}$ such that $L_YZ = \lambda_ZY$ and $L_Y\alpha = \lambda_\alpha\alpha$, then $L_Y(L_Z\alpha) = (\lambda_Z + \lambda_\alpha)L_Z\alpha$ for $Z \in \mathfrak{h}^m_{\mathcal{R}}$.*

Proof. The statement follows directly from $L_Y(L_Z\alpha) = L_Z(L_Y\alpha) + L_{L_YZ}(\alpha)$. □

Proposition 3-53. *Assume that $\omega_n \in Z^n_m(\mathfrak{h}, V, \mathcal{C})$ is homogeneous with respect $Y \in \mathfrak{h}^m_{\mathcal{R}}$ with invertible grading $\lambda_{\omega_n} \in \mathcal{C}$. Then $\omega_n \in B^n_m(\mathfrak{h}, V, \mathcal{C})$.*

Proof. One has,

$$\begin{aligned}\omega_n &= L_{\lambda^{-1}_{\omega_n}Y}\omega_n = (\iota^{n+1}_m(\lambda^{-1}_{\omega_n}Y)d^n_m + d^{n-1}_m\iota^n_m(\lambda^{-1}_{\omega_n}Y))\omega_n \\ &= d^{n-1}_m\iota^n_m(\lambda^{-1}_{\omega_n}Y)\omega_n\end{aligned}$$

and this shows that indeed $\omega_n \in B^n_m(\mathfrak{h}, V, \mathcal{C})$. □

Corollary 3-54. *If for certain $Y \in \mathfrak{h}^m_{\mathcal{R}}$ one has that*

$$\Omega^\bullet_m = \Omega^\bullet_{m,0} \oplus \bigoplus_{\lambda_{\omega_n} \in \mathcal{C}^\star} \Omega^\bullet_{m,\lambda_{\omega_n}},$$

where $\omega_n \in \Omega^n_{m,\lambda_{\omega_n}}$ if $L_Y\omega_n = \lambda_{\omega_n}\omega_n$, then the cohomology is contained in the space of invariants of Y.

3.11 Conjugate and adjoint operators

Abstract 3-55. *We define the notions of conjugate and adjoint, carefully avoiding any (unnatural) identification of* $\mathfrak{h}^m_{\mathcal{R}}$ *with* Ω^1_m.

Definition 3-56. *For arbitrary* $Y \in \mathfrak{h}^m_{\mathcal{R}}$ *and any* 1*-form* $\omega_1 \in \Omega^1_m$ *define the* **pairing** *of* Y *and* ω_1 *by the formula:*

$$(\omega_1, Y) = \iota^1_m(Y)\omega_1 \in \Omega^0_m.$$

Once the pairing is given, the **conjugate operator** can be defined as follows:

Definition 3-57. *Given an operator* $S : \mathfrak{h}^m_{\mathcal{R}} \to \mathfrak{h}^m_{\mathcal{R}}$ *(or* $S : \Omega^1_m \to \Omega^1_m$*), we call the operator* $S^\star : \Omega^1_m \to \Omega^1_m$ *(or* $S^\star : \mathfrak{h}^m_{\mathcal{R}} \to \mathfrak{h}^m_{\mathcal{R}}$*) the* **conjugate operator** *to* S *if* $(\omega_1, S(Y)) = (S^\star\omega_1, Y)$ *(or* $(S(\omega_1), Y) = (\omega_1, S^\star(Y))$*) for all* $Y \in \mathfrak{h}^m_{\mathcal{R}},\ \omega_1 \in \Omega^1_m$.

Here we also give the definition of the adjoint operator.

Definition 3-58. *Given an operator* $S : \mathfrak{h}^m_{\mathcal{R}} \to \Omega^1_m$ *(or* $S : \Omega^1_m \to \mathfrak{h}^m_{\mathcal{R}}$*), we call the operator* $S^\dagger : \mathfrak{h}^m_{\mathcal{R}} \to \Omega^1_m$ *(or* $S^\dagger : \Omega^1_m \to \mathfrak{h}^m_{\mathcal{R}}$*) the* **adjoint operator** *to* S *if* $(S(Y_1), Y_2) = (S^\dagger(Y_2), Y_1)$ *(or* $(\omega_1, S(\omega_2)) = (\omega_2, S^\dagger(\omega_1))$*) for all* $Y_1, Y_2 \in \mathfrak{h}^m_{\mathcal{R}},\ \omega_1, \omega_2 \in \Omega^1_m$.

Definition 3-59. $S : \mathfrak{h}^m_{\mathcal{R}} \to \Omega^1_m$ *(or* $S : \Omega^1_m \to \mathfrak{h}^m_{\mathcal{R}}$*) is called a* **symmetric operator** *if* $S^\dagger = S$*, and a* **antisymmetric operator** *if* $S^\dagger = -S$.

Chapter 4

Geometric structures

We show how the deformation of a representation leads to the definition of a Nijenhuis operator and derive very useful properties of such an operator, which will be used to generate the symmetries and cosymmetries of evolution equations. Furthermore, we give the Hamiltonian formalism, which is well known in the classical Lie algebra context and can be found in any modern text on the foundations of classical mechanics, e.g., [AM78].

4.1 Introduction

In section 4.2 we generalize `Dorfman`'s approach in [Dor93] to define the Nijenhuis operator on an arbitrary Leibniz algebra cochain complex. Although somewhat more complicated than for Lie algebras, we see that the main ideas survive without a scratch. In section 4.3 we prove the usual properties of Nijenhuis operators in a formal way, i.e., assuming that $\mathfrak{R}^k X$ always exists. We return to the existence question in chapter 6. In section 4.4 we prove the corresponding properties for the conjugate Nijenhuis operator. Finally, we show in section 4.5 how one obtains the classical symplectic and cosymplectic structures for an arbitrary complex. Only the most elementary results are given here. Most of the theory in the literature is involved in choosing a ring $\mathcal{A}$ of (germs of) functions on a manifold and derive the consequences, depending on the topology of the underlying manifold.

4.2 Deformations of Leibniz algebra and Nijenhuis operators

Abstract 4-1. *We set up the deformation equations and derive the definition of the Nijenhuis tensor from them.*

Consider the m-complex $C^n_m(\mathfrak{h}, \mathfrak{h}^m_{\mathcal{R}}, \mathcal{S})$ with $\nabla^0_m = \pi^1_m$ and $\mathfrak{g}^m_{\mathcal{C}} = \mathfrak{h}^m_{\mathcal{R}}$. We see that π^1_m induces a 2-form ω_2 by $\omega_2(Y_1, Y_2) = \pi^1_m(Y_1)Y_2$.

It will be shown that the assumption of an additional representation leads to the definition of a Nijenhuis operator. The sections 4.2-4.4 are based on [Dor93].

For the connection $\tilde{\pi}^1_m \in \tilde{\Gamma}^1_m(\mathfrak{h}, \mathfrak{h}^m_{\mathcal{R}}, \mathcal{A})$, consider a λ-parametrized family operations

$$\pi^1_{m,\lambda} = \pi^1_m + \lambda\tilde{\pi}^1_m, \ \ \lambda \in \mathcal{C}. \tag{4.2.1}$$

If the connection $\pi^1_{m,\lambda}$ endows $\mathfrak{h}^m_{\mathcal{R}}$ with Leibniz algebra structure, we say that $\tilde{\pi}^1_m$ generates a **deformation** of the Leibniz algebra $\mathfrak{h}^m_{\mathcal{R}}$.

Evidently, this requirement, $\mathcal{C}(\pi^1_{m,\lambda}) = 0$, is equivalent to the following conditions

$$\begin{aligned} 0 &= \pi^1_m(X)\tilde{\pi}^1_m(Y) - \pi^1_m(Y)\tilde{\pi}^1_m(X) - \pi^1_m(\tilde{\pi}^1_m(X)Y) \\ &+ \tilde{\pi}^1_m(X)\pi^1_m(Y) - \tilde{\pi}^1_m(Y)\pi^1_m(X) - \tilde{\pi}^1_m(\pi^1_m(X)Y), \end{aligned} \tag{4.2.2}$$

$$0 = \tilde{\pi}^1_m(X)\tilde{\pi}^1_m(Y) - \tilde{\pi}^1_m(Y)\tilde{\pi}^1_m(X) - \tilde{\pi}^1_m(\tilde{\pi}^1_m(X)Y). \tag{4.2.3}$$

Thus, $\tilde{\pi}^1_m$ must itself be a representation, satisfying condition (4.2.3). We can present (4.2.2) in the short form according to (3.4.5), viewing, as before, $\tilde{\pi}^1_m$ as an element $\omega_2 \in C^2_m(\mathfrak{h}, \mathfrak{h}^m_{\mathcal{R}}, \mathcal{C})$,

$$d^2_m\omega_2(X, Y, Z) = \pi^1_m(Z)\tilde{\pi}^1_m(X)Y + \pi^1_m(\tilde{\pi}^1_m(X)Y)Z. \tag{4.2.4}$$

We call a deformation is a **trivial deformation** if there exists $\mathfrak{R} \in C^1_m(\mathfrak{h}, \mathfrak{h}^m_{\mathcal{R}}, \mathcal{C})$ such that for $\mathcal{T}_\lambda = id + \lambda\mathfrak{R}$ there holds

$$\mathcal{T}_\lambda\pi^1_{m,\lambda}(X)Y = \pi^1_m(\mathcal{T}_\lambda X)\mathcal{T}_\lambda Y.$$

Since we have

$$\mathcal{T}_\lambda\pi^1_{m,\lambda}(X)Y = \pi^1_m(X)Y + \lambda(\tilde{\pi}^1_m(X)Y + \mathfrak{R}\pi^1_m(X)Y) + \lambda^2\mathfrak{R}\tilde{\pi}^1_m(X)Y$$

and

$$\pi^1_m(\mathcal{T}_\lambda X)\mathcal{T}_\lambda Y = \pi^1_m(X)Y + \lambda(\pi^1_m(\mathfrak{R}X)Y + \pi^1_m(X)\mathfrak{R}Y) + \lambda^2\pi^1_m(\mathfrak{R}X)\mathfrak{R}Y,$$

the triviality of the deformation is equivalent to the conditions

$$\tilde{\pi}^1_m(X)Y = \pi^1_m(\mathfrak{R}X)Y + \pi^1_m(X)\mathfrak{R}Y - \mathfrak{R}\pi^1_m(X)Y, \tag{4.2.5}$$

$$\mathfrak{R}\tilde{\pi}^1_m(X)Y = \pi^1_m(\mathfrak{R}X)\mathfrak{R}Y. \tag{4.2.6}$$

Similarly, (4.2.5) can be represented as

$$\tilde{\pi}^1_m(X)Y = (d^1_0\mathfrak{R})(X, Y) + \pi^1_m(\mathfrak{R}X)Y + \pi^1_m(Y)\mathfrak{R}X,$$

and this is the solution of (4.2.4).

Definition 4-2. *We define the* **Nijenhuis tensor** *[Nij51] to be given by*

$$N_{\mathfrak{R}}(X, Y) = \pi^1_m(\mathfrak{R}X)\mathfrak{R}Y - \mathfrak{R}\pi^1_m(\mathfrak{R}X)Y - \mathfrak{R}\pi^1_m(X)\mathfrak{R}Y + \mathfrak{R}^2\pi^1_m(X)Y. \tag{4.2.7}$$

Alternatively, and more in the spirit of the setup using connections, we may put $\Pi_{\mathfrak{R}}(X) = \pi^1_m(\mathfrak{R}X)\mathfrak{R} - \mathfrak{R}\pi^1_m(\mathfrak{R}X) - \mathfrak{R}\pi^1_m(X)\mathfrak{R} + \mathfrak{R}^2\pi^1_m(X)$.

Proposition 4-3. *If $\mathfrak{R}$ is $\mathcal{A}$-linear, then $N_{\mathfrak{R}} \in C^2_m(\mathfrak{h}, \mathfrak{h}^m_{\mathcal{R}}, \mathcal{C}\mathcal{A})$ or, in other words, the anchor of $\Pi_{\mathfrak{R}}$ is 0.*

Proof. We check directly the $\mathcal{A}$-linearity in the second argument. For $r \in \mathcal{A}$,

$$\begin{aligned}
&N_{\mathfrak{R}}(X, rY) = \\
&= \pi^1_m(\mathfrak{R}X)r\mathfrak{R}Y - \mathfrak{R}\pi^1_m(\mathfrak{R}X)rY - \mathfrak{R}\pi^1_m(X)r\mathfrak{R}Y + \mathfrak{R}^2\pi^1_m(X)rY \\
&= r\pi^1_m(\mathfrak{R}X)\mathfrak{R}Y + \gamma^0_0(\mathfrak{R}X)(r)\mathfrak{R}Y - r\mathfrak{R}\pi^1_m(\mathfrak{R}X)Y - \mathfrak{R}\gamma^0_0(\mathfrak{R}X)(r)Y \\
&- \mathfrak{R}r\pi^1_m(X)\mathfrak{R}Y - \mathfrak{R}\gamma^0_0(X)(r)\mathfrak{R}Y + \mathfrak{R}^2 r\pi^1_m(X)Y + \mathfrak{R}^2\gamma^0_0(X)(r)Y \\
&= rN_{\mathfrak{R}}(X, Y),
\end{aligned}$$

and this proves the statement. $\square$

Definition 4-4. *An $\mathcal{A}_{\mathfrak{k}}$-linear operator $\mathfrak{R} : \mathfrak{h}^m_{\mathcal{R}} \to \mathfrak{h}^m_{\mathcal{R}}$ is called a* **Nijenhuis operator** *if for all $X, Y \in \mathfrak{h}^m_{\mathcal{R}} \bigcap dom(\mathfrak{R})$,*

$$N_{\mathfrak{R}}(X, Y) = 0. \tag{4.2.8}$$

Combining (4.2.5) and (4.2.6), we have $N_{\mathfrak{R}}(X, Y) = 0$. This implies that any trivial deformation produces a Nijenhuis operator. Notably, the converse is also valid, as the following theorem shows.

Theorem 4-5. *Let $\mathfrak{R} : \mathfrak{h}^m_{\mathcal{R}} \to \mathfrak{h}^m_{\mathcal{R}}$ be a Nijenhuis operator. Then a deformation of π^1_m can be obtained by putting*

$$\tilde{\pi}^1_m(X) = \pi^1_m(\mathfrak{R}X) + \pi^1_m(X)\mathfrak{R} - \mathfrak{R}\pi^1_m(X).$$

One has $\mathcal{C}(\tilde{\pi}^1_m) = 0$, so one can call $\mathfrak{h}^m_{\mathcal{R}}$ a **Leibniz bialgebra**. *The induced 2-form $\tilde{\omega}_2$ (induced by $\tilde{\omega}_2(X, Y) = \tilde{\pi}^1_m(X)Y$) is trivial when π^1_m is antisymmetric.*

Proof. This can be proved by directly checking (4.2.2), (4.2.3), (4.2.5) and (4.2.6). The curvature computation is straightforward. $\square$

Remark 4-6. *Notice that m does not play an essential role. We will drop it when we study the properties of Nijenhuis operators by saying $\mathfrak{R} : \mathfrak{h} \to \mathfrak{h}$.*

Definition 4-7. *Given $\mathcal{C}$-linear maps $\mathfrak{R}, \mathfrak{R}'$ and Φ between two Leibniz algebras $\mathfrak{h}$ and $\mathfrak{h}'$ according to the following diagram:*

$$\begin{tikzcd}
\mathfrak{h} \arrow[r, "\mathfrak{R}"] \arrow[d, "\Phi"'] & \mathfrak{h} \arrow[d, "\Phi"] \\
\mathfrak{h}' \arrow[r, "\mathfrak{R}'"] & \mathfrak{h}'
\end{tikzcd}$$

When then diagram commutes $\mathfrak{R}$ and $\mathfrak{R}'$ are called **Φ-intertwined**.

Proposition 4-8. *Assume that $\mathfrak{R}$ and $\mathfrak{R}'$ are Φ-intertwined and Φ is a Lie algebra homomorphism. If $\mathfrak{R}$ is a Nijenhuis operator, then $\mathfrak{R}'$ is a Nijenhuis operator on $Im(\Phi) \subset \mathfrak{h}'$.*

Proof. This can be proved by using Φ acting both sides of (4.2.8). □

We constructed a Lie algebra from a certain ring in section 2.4, where we also proved that any ring homomorphism φ leads to Lie algebra homomorphism $\varphi^\star$. We will show the relation between $\mathfrak{R}, \mathfrak{R}'$ and φ from the following example.

⋆ **Example 4-9.** *Consider the Modified Korteweg–de Vries equation*

$$u_t = u_3 + u^2 u_1.$$

Its ***Miura*** *transformation $w = u^2 + \sqrt{-6}u_1$ transforms it into the* ***Korteweg****–de* ***Vries*** *equation*

$$w_t = w_3 + ww_1.$$

As we know $\mathfrak{R} = D_x^2 + \frac{2}{3}w + \frac{w_1}{3}D_x^{-1}$ is a Nijenhuis recursion operator for KdV. We will compute the corresponding Nijenhuis operator for mKdV. Notice that $\varphi X d = (\varphi^\star X)d'\varphi$ (Proposition 2-25). Therefore, for any $f[w]\partial_w$ there is a vector-field $h[u]\partial_u = \varphi^\star(f[w]\partial_w)$ satisfying with $f[u^2+\sqrt{-6}u_1] = 2uh[u] + \sqrt{-6}D_x(h[u]) = (2u+\sqrt{-6}D_x)h[u]$. This leads to

$$\begin{aligned} & (2u+\sqrt{-6}D_x)\mathfrak{R}'(h) = (\mathfrak{R}(f))[u^2+\sqrt{-6}u_1] \\ = \ & (D_x^2 + \frac{2}{3}w + \frac{w_1}{3}D_x^{-1})(2u+\sqrt{-6}D_x)h[u]. \end{aligned}$$

So $\mathfrak{R}' = D_x^2 + \frac{2}{3}u^2 + \frac{2u_1}{3}D_x^{-1} \cdot u$.

Remark 4-10. *In general, $\mathfrak{R}' = D_\varphi^{-1}\varphi\mathfrak{R}\varphi^{-1}D_\varphi = D_\varphi^{-1}(\varphi\mathfrak{R})D_\varphi$, where $\varphi : \mathcal{A} \to \mathcal{A}'$, $\mathfrak{h} = Hom_{\mathcal{A}}(\Omega^1_{\mathcal{A}/\mathcal{C}}, \mathcal{A})$, $\mathfrak{h}' = Hom_{\mathcal{A}'}(\Omega^1_{\mathcal{A}'/\mathcal{C}'}, \mathcal{A}')$ and $\mathfrak{R} : \mathfrak{h} \to \mathfrak{h}$ (cf. proposition 2.1 in [Fok87]).*

4.3 Properties of Nijenhuis operators

Abstract 4-11. *We derive the recursion formulae for a Nijenhuis operator. These will be used to compute hierarchies of symmetries.*

In the following, we encounter expressions like $\mathfrak{R}^k X$. We tacitly assume that $\mathfrak{R}^k X$ exists. In Chapter 6, we address the existence question.

Proposition 4-12. *Let $\mathfrak{R} : \mathfrak{h} \to \mathfrak{h}$ be a Nijenhuis operator. For arbitrary elements $X, Y \in \mathfrak{h}$ and arbitrary $j, k \in \mathbb{N}$,*

$$\pi^1(\mathfrak{R}^j X)\mathfrak{R}^k Y - \mathfrak{R}^k \pi^1(\mathfrak{R}^j X)Y - \mathfrak{R}^j \pi^1(X)\mathfrak{R}^k Y + \mathfrak{R}^{j+k}\pi^1(X)Y = 0. \quad (4.3.1)$$

Proof. Fix $j = 1$ and prove (4.3.1) for arbitrary $k > 0$. For $k = 1$ the formula is evidently valid. With the help of (4.2.8) we get

$$\begin{aligned}
&\pi^1(\mathfrak{R}X)\mathfrak{R}^{k+1}Y - \mathfrak{R}^{k+1}\pi^1(\mathfrak{R}X)Y - \mathfrak{R}\pi^1(X)\mathfrak{R}^{k+1}Y + \mathfrak{R}^{k+2}\pi^1(X)Y = \\
&= \mathfrak{R}\pi^1(\mathfrak{R}X)\mathfrak{R}^kY - \mathfrak{R}^{k+1}\pi^1(\mathfrak{R}X)Y - \mathfrak{R}^2\pi^1(X)\mathfrak{R}^kY + \mathfrak{R}^{k+2}\pi^1(X)Y \\
&= \mathfrak{R}(\pi^1(\mathfrak{R}X)\mathfrak{R}^kY - \mathfrak{R}^k\pi^1(\mathfrak{R}X)Y - \mathfrak{R}\pi^1(X)\mathfrak{R}^kY + \mathfrak{R}^{k+1}\pi^1(X)Y).
\end{aligned}$$

By induction it follows that, for any $k \in \mathbb{N}$,

$$\pi^1(\mathfrak{R}X)\mathfrak{R}^kY - \mathfrak{R}^k\pi^1(\mathfrak{R}X)Y - \mathfrak{R}\pi^1(X)\mathfrak{R}^kY + \mathfrak{R}^{k+1}\pi^1(X)Y = 0. \tag{4.3.2}$$

Now applying this formula to the element $\mathfrak{R}^jX$ instead of the element X, we obtain

$$\begin{aligned}
&\pi^1(\mathfrak{R}^{j+1}X)\mathfrak{R}^kY - \mathfrak{R}^k\pi^1(\mathfrak{R}^{j+1}X)Y - \mathfrak{R}^{j+1}\pi^1(X)\mathfrak{R}^kY + \mathfrak{R}^{k+j+1}\pi^1(X)Y = \\
&= \mathfrak{R}\pi^1(\mathfrak{R}^jX)\mathfrak{R}^kY - \mathfrak{R}^{k+1}\pi^1(\mathfrak{R}^jX)Y - \mathfrak{R}^{j+1}\pi^1(X)\mathfrak{R}^kY + \mathfrak{R}^{k+j+1}\pi^1(X)Y \\
&= \mathfrak{R}(\pi^1(\mathfrak{R}^jX)\mathfrak{R}^kY - \mathfrak{R}^k\pi^1(\mathfrak{R}^jX)Y - \mathfrak{R}^j\pi^1(X)\mathfrak{R}^kY + \mathfrak{R}^{k+j}\pi^1(X)Y).
\end{aligned}$$

So, the induction can be made with respect to j starting from the formula (4.3.2). Thus we prove the validity of (4.3.1) for $j, k \in \mathbb{N}$. □

Now we note that the formula (4.3.1) gets a natural interpretation in terms of the Lie derivative. Namely, if we consider its left-hand side as the result of the action of some operator on the element $Y \in \mathfrak{h}$, then we get for a Nijenhuis operator $\mathfrak{R}$

$$L_{\mathfrak{R}^jX}(\mathfrak{R}^k) = \mathfrak{R}^jL_X(\mathfrak{R}^k), \ \forall\ X \in \mathfrak{h}. \tag{4.3.3}$$

This leads to the following statement, which turns out to be fundamental for the construction of evolution equations with infinitely many commuting symmetries.

Theorem 4-13. *Suppose $\mathfrak{R} : \mathfrak{h} \to \mathfrak{h}$ is a Nijenhuis operator. Let $X \in \mathfrak{h}$ is a symmetry of $\mathfrak{R}$ and $Y \in \mathfrak{h}$ be a scaling symmetry for both the operator $\mathfrak{R}$ and X with the grading $\lambda_{\mathfrak{R}}$ and λ_X respectively, i.e., $L_YX = \lambda_XX$, $L_Y\mathfrak{R} = \lambda_{\mathfrak{R}}\mathfrak{R}$. Defining $X_k = \mathfrak{R}^kX$ and $Y_j = \mathfrak{R}^jY$, then $L_{X_k}\mathfrak{R}^j = 0$ for all $j, k \geq 0$. Furthermore,*

$$L_XX_k = \mathfrak{R}^kL_XX, \ L_{X_j}X_k = \mathfrak{R}^kL_{X_j}X, \ L_{Y_j}X_k = \mathfrak{R}^kL_{Y_j}X + k\lambda_{\mathfrak{R}}X_{k+j}.$$

Proof. As X is a symmetry, we have $L_X\mathfrak{R} = 0$ and by the chain rule for the Lie derivative $L_X\mathfrak{R}^k = 0$. By (4.3.3), all the elements of $\mathfrak{R}^jX$ are symmetries of $\mathfrak{R}^k$, i.e., $L_{X_j}\mathfrak{R}^k = 0$. This implies that

$$L_XX_k = (L_X\mathfrak{R}^k)X + \mathfrak{R}^kL_XX = \mathfrak{R}^kL_XX.$$

By by putting $X = Y$ in the formula (4.3.1), we get $L_{X_j}X_k = \mathfrak{R}^kL_{X_j}X$.

Similarly, to prove the second identity,

$$L_{Y_j}X_k = \mathfrak{R}^kL_{Y_j}X + \mathfrak{R}^jL_YX_k - \mathfrak{R}^{k+j}L_YX = \mathfrak{R}^kL_{Y_j}X + k\lambda_{\mathfrak{R}}X_{k+j}.$$

This concludes the proof. □

Corollary 4-14. *If, moreover, π^1 is antisymmetric, then*

1. $L_{X_j}X_k = 0,\ L_{Y_j}X_k = (k\lambda_{\mathfrak{R}} + \lambda_X)X_{k+j},\ L_{Y_j}Y_k = (k-j)\lambda_{\mathfrak{R}}Y_{k+j}$.
2. *If $Z \in \mathfrak{h}$ is another symmetry of $\mathfrak{R}$ and $L_X Z = 0$, then $L_{\mathfrak{R}^k X}(\mathfrak{R}^j Z) = 0$ for $j, k \in \mathbb{N}$.*

Proof. The first two identities of the first part follow from the antisymmetry of π^1. For the third identity,

$$L_{Y_j}(Y_k) = \mathfrak{R}^k L_{Y_j}Y + \mathfrak{R}^j L_Y(\mathfrak{R}^k Y) = (k-j)\lambda_{\mathfrak{R}}Y_{k+j}.$$

The second part follows from (4.3.1). □

4.4 Conjugate of Nijenhuis operators

Abstract 4-15. *We derive recursion formulae for the conjugate Nijenhuis operator. These are to be used to compute hierarchies of cosymmetries.*

Consider an arbitrary m-complex $C^n_m(\mathfrak{h}, V, \mathcal{S})$ with the representations π^1_m and ∇^0_m and define the conjugate operator $\mathfrak{R}^\star$ as in section 3.11.

Proposition 4-16. *For $\mathfrak{R} : \mathfrak{h}^m_{\mathcal{R}} \to \mathfrak{h}^m_{\mathcal{R}}$, and $X, Y \in \mathfrak{h}^m_{\mathcal{R}}$, $\omega_1 \in C^1_m(\mathfrak{h}, V, \mathcal{S})$ the following is valid:*

$$\begin{aligned} & d^1_m\omega_1(\mathfrak{R}X, \mathfrak{R}Y) - d^1_m(\mathfrak{R}^\star\omega_1)(X, \mathfrak{R}Y) \\ - \quad & d^1_m(\mathfrak{R}^\star\omega_1)(\mathfrak{R}X, Y) + d^1_m(\mathfrak{R}^{\star 2}\omega_1)(X, Y) \\ = \quad & -(\omega_1, N_{\mathfrak{R}}(X, Y)). \end{aligned} \tag{4.4.1}$$

Proof. Using the formula (3.4.4), we compute $d^1_m\omega_1$, $d^1_m(\mathfrak{R}^\star\omega_1)$ and $d^1_m(\mathfrak{R}^{\star 2}\omega_1)$.
We obtain

$$\begin{aligned} d^1_m\omega_1(\mathfrak{R}X, \mathfrak{R}Y) \quad & = \nabla^0_m(\mathfrak{R}X)\omega_1(\mathfrak{R}Y) - \nabla^0_m(\mathfrak{R}Y)\omega_1(\mathfrak{R}X) \\ & -\omega_1(\pi^1_m(\mathfrak{R}X)\mathfrak{R}Y), \\ d^1_m(\mathfrak{R}^\star\omega_1)(\mathfrak{R}X, Y) \quad & = \nabla^0_m(\mathfrak{R}X)\mathfrak{R}^\star\omega_1(Y) - \nabla^0_m(Y)\mathfrak{R}^\star\omega_1(\mathfrak{R}X) \\ & -\mathfrak{R}^\star\omega_1(\pi^1_m(\mathfrak{R}X)Y), \\ d^1_m(\mathfrak{R}^{2\star}\omega_1)(X, Y) \quad & = \nabla^0_m(X)\mathfrak{R}^{2\star}\omega_1(Y) - \nabla^0_m(Y)\mathfrak{R}^{2\star}\omega_1(X) \\ & -\mathfrak{R}^{2\star}\omega_1(\pi^1_m(X)Y). \end{aligned}$$

Substituting the expressions obtained into the left-hand side of (4.4.1) and using the definition of $\mathfrak{R}^\star$, we get its right-hand side. □

Suppose that $\mathfrak{R}$ is a Nijenhuis operator. In this case the right side of (4.4.1) vanishes and we get

$$\begin{aligned} & d^1_m(\mathfrak{R}^{2\star}\omega_1)(X, Y) = \\ = \quad & d^1_m(\mathfrak{R}^\star\omega_1)(X, \mathfrak{R}Y) + d^1_m(\mathfrak{R}^\star\omega_1)(\mathfrak{R}X, Y) - d^1_m\omega_1(\mathfrak{R}X, \mathfrak{R}Y). \end{aligned}$$

It can easily be deduced that if for arbitrary $\omega_{1,0} \in C^1_m(\mathfrak{h}, V, \mathcal{S})$ we construct the sequence of 1-forms $\omega_{1,k} = \mathfrak{R}^{k\star}\omega_{1,0} \in C^1_m(\mathfrak{h}, V, \mathcal{S})$, then

$$\begin{aligned} & d^1_m\omega_{1,k+2}(X,Y) = \\ & = \; d^1_m\omega_{1,k+1}(X, \mathfrak{R}Y) + d^1_m\omega_{1,k+1}(\mathfrak{R}X, Y) - d^1_m\omega_{1,k}(\mathfrak{R}X, \mathfrak{R}Y). \end{aligned} \qquad (4.4.2)$$

The following theorem is a direct consequence of the formula (4.4.2). It explains how the infinite series of conservation laws of an evolution equation arises, cf. chapter 6.

Theorem 4-17. *Let $\mathfrak{R} : \mathfrak{h}^m_{\mathcal{R}} \to \mathfrak{h}^m_{\mathcal{R}}$, be a Nijenhuis operator, $\omega_{1,0} \in Z^1_m(\mathfrak{h}, V, \mathcal{S})$ be a 1-form such that $d^1_m\mathfrak{R}^\star\omega_{1,0} = 0$. Then all $\omega_{1,k} = \mathfrak{R}^{k\star}\omega_{1,0} \in Z^1_m(\mathfrak{h}, V, \mathcal{S})$, $k \geq 0$.*

4.5 Symplectic, cosymplectic and Poisson structures

Abstract 4-18. *In this section we derive the symplectic formalism starting from an exact 2-form.*

Let $\Omega^\bullet_m$ be a complex. We fix $\omega_2 \in Z^2_m(\Omega^\bullet_m)$, i.e., $d^2_m\omega_2 = 0$. Remark that ω_2 is not required to be antisymmetric. We call ω_2 a **symplectic form** if it is nondegenerate.

Definition 4-19. *If for given $\alpha_1 \in \Omega^1_m$, we have $Y \in \mathfrak{h}^m_{\mathcal{R}}$ such that $\iota^2_m(Y)\omega_2 = \alpha_1$, we write $Y = \mathfrak{H}(\alpha_1)$. So $\mathfrak{H}$ maps $\Omega^1_m \to \mathfrak{h}^m_{\mathcal{R}}$ and is called the* **cosymplectic operator**. *In the other direction, we let $\mathfrak{I}(Y) = \iota^2_m(Y)\omega_2$ define a map $\mathfrak{I} : \mathfrak{h}^m_{\mathcal{R}} \to \Omega^1_m$, the* **symplectic operator**.

In practice, things are sometimes done the other way around: one starts with an operator $\mathfrak{I}$ and defines a form by

$$\omega_2(Y_1, Y_2) = \mathfrak{I}(Y_1)(Y_2).$$

The verification that $\omega_2 \in Z^2_m(\Omega^\bullet_m)$ determines the properties that $\mathfrak{I}$ needs to have in order to be called a symplectic operator.

Notation 4-20. *We define the following spaces:*

$$W_{\mathfrak{I}} = \{\alpha_1 \in W \subset \Omega^1_m \mid \exists Y \in \mathfrak{h}^m_{\mathcal{R}} : \mathfrak{I}(Y) = \alpha_1\},$$

$$\tilde{\Omega}^0_m = \{H_0 \in \Omega^0_m \mid \exists Y \in \mathfrak{h}^m_{\mathcal{R}} : \iota^2_m(Y)\omega_2 = d^0_m H_0\}.$$

Definition 4-21. *Assuming ι^2_m is $\mathcal{A}$-linear, we can define a flat connection ρ^1_m on $\Omega^1_{m,\mathfrak{I}}$ as follows:*

$$\rho^1_m(\alpha_1)\beta_1 = \mathfrak{I}(\pi^1_m(\mathfrak{H}(\alpha_1))\mathfrak{H}(\beta_1)), \alpha_1, \beta_1 \in \Omega^1_{m,\mathfrak{I}}.$$

Proof.

$$\begin{aligned}
&\rho^1_m(\alpha_1) r\beta_1 = \\
&\quad = \ \mathfrak{I}(\pi^1_m(\mathfrak{H}(\alpha_1))\mathfrak{H}(r\beta_1)) \\
&\quad = \ \mathfrak{I}(\pi^1_m(\mathfrak{H}(\alpha_1)) r\mathfrak{H}(\beta_1)) \\
&\quad = \ \mathfrak{I}(r\pi^1_m(\mathfrak{H}(\alpha_1))\mathfrak{H}(\beta_1) + \gamma^0_m(\mathfrak{H}(\alpha_1))(r)\mathfrak{H}(\beta_1)) \\
&\quad = \ r\mathfrak{I}(\pi^1_m(\mathfrak{H}(\alpha_1))\mathfrak{H}(\beta_1) + \gamma^0_m(\mathfrak{H}(\alpha_1))(r)\mathfrak{I}(\mathfrak{H}(\beta_1))
\end{aligned}$$

This shows that we have indeed a connection, with anchor $\gamma^0_m \circ \mathfrak{H}$. We now show that it is flat.

$$\begin{aligned}
&\mathcal{C}(\rho^1_m)(\alpha_1, \alpha_2)\beta_1 = \\
&\quad = \ \rho^1_m(\alpha_1)\rho^1_m(\alpha_2)\beta_1 - \rho^1_m(\alpha_2)\rho^1_m(\alpha_1)\beta_1 - \rho^1_m(\rho^1_m(\alpha_1)\alpha_2)\beta_1 \\
&\quad = \ \rho^1_m(\alpha_1)\mathfrak{I}(\pi^1_m(\mathfrak{H}(\alpha_2))\mathfrak{H}(\beta_1)) - \rho^1_m(\alpha_2)\mathfrak{I}(\pi^1_m(\mathfrak{H}(\alpha_2))\mathfrak{H}(\beta_1)) \\
&\quad - \ \rho^1_m(\mathfrak{I}(\pi^1_m(\mathfrak{H}(\alpha_1))\mathfrak{H}(\alpha_2))\beta_1 \\
&\quad = \ \mathfrak{I}(\pi^1_m(\mathfrak{H}(\alpha_1))\mathfrak{H}(\mathfrak{I}(\pi^1_m(\mathfrak{H}(\alpha_2))\mathfrak{H}(\beta_1)))) \\
&\quad - \ \mathfrak{I}(\pi^1_m(\mathfrak{H}(\alpha_2))\mathfrak{H}(\mathfrak{I}(\pi^1_m(\mathfrak{H}(\alpha_1))\mathfrak{H}(\beta_1)))) \\
&\quad - \ \mathfrak{I}(\pi^1_m(\mathfrak{H}(\mathfrak{I}(\pi^1_m(\mathfrak{H}(\alpha_1))\mathfrak{H}(\alpha_2)))\mathfrak{H}(\beta_1)) \\
&\quad = \ \mathfrak{I}(\pi^1_m(\mathfrak{H}(\alpha_1))\pi^1_m(\mathfrak{H}(\alpha_2))\mathfrak{H}(\beta_1)) - \mathfrak{I}(\pi^1_m(\mathfrak{H}(\alpha_2))\pi^1_m(\mathfrak{H}(\alpha_1))\mathfrak{H}(\beta_1)) \\
&\quad - \ \mathfrak{I}(\pi^1_m(\pi^1_m(\mathfrak{H}(\alpha_1))\mathfrak{H}(\alpha_2)\mathfrak{H}(\beta_1)) \\
&\quad = \ \mathfrak{I}(\mathcal{C}(\pi^1_m(\mathfrak{H}(\alpha_1), \mathfrak{H}(\alpha_2))\mathfrak{H}(\beta_1)) \\
&\quad = \ 0.
\end{aligned}$$

This concludes the proof. □

Proposition 4-22. *If $\alpha_1, \beta_1 \in Z^1_{m,\mathfrak{I}}(\Omega^\bullet_m)$, then $\rho^1_m(\alpha_1)\beta_1 \in B^1_{m,\mathfrak{I}}(\Omega^\bullet_m)$.*

Proof. First of all, we find that

$$\begin{aligned}
0 \ &= \ d^1_m \alpha_1 \\
&= \ d^1_m \iota^2_m(\mathfrak{H}(\alpha_1))\omega_2 \\
&= \ \nabla^2_m(\mathfrak{H}(\alpha_1))\omega_2 - \iota^3_m(\mathfrak{H}(\alpha_1)) d^2_m \omega_2 \\
&= \ \nabla^2_m(\mathfrak{H}(\alpha_1))\omega_2.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\rho^1_m(\alpha_1)\beta_1 = \iota^2_m(\pi^1_m(\mathfrak{H}(\alpha_1))\mathfrak{H}(\beta_1))\omega_2 \\
&\quad = \ \nabla^1_m(\mathfrak{H}(\alpha_1))\iota^2_m(\mathfrak{H}(\beta_1))\omega_2 - \iota^2_m(\mathfrak{H}(\beta_1))\nabla^2_m(\mathfrak{H}(\alpha_1))\omega_2 \\
&\quad = \ \nabla^1_m(\mathfrak{H}(\alpha_1))\beta_1 \\
&\quad = \ \iota^2_m(\mathfrak{H}(\alpha_1)) d^1_m \beta_1 + d^0_m \iota^1_m(\mathfrak{H}(\alpha_1))\beta_1 \\
&\quad = \ d^0_m \iota^1_m(\mathfrak{H}(\alpha_1))\beta_1 \in B^1_m(\Omega^\bullet_m).
\end{aligned}$$

It belongs to $\Omega^1_{m,\mathfrak{I}}$ by construction, and therefore sits in $B^1_{m,\mathfrak{I}}(\Omega^\bullet_m)$. □

Let

$$\mathfrak{Ham}_{\omega_2}(\mathfrak{h}^m_{\mathcal{R}}) = \left\{Y \in \mathfrak{h}^m_{\mathcal{R}} | \iota^2_m(Y)\omega_2 \in B^1_m(\Omega^\bullet_m)\right\},$$
$$\mathfrak{Sym}_{\omega_2}(\mathfrak{h}^m_{\mathcal{R}}) = \left\{Y \in \mathfrak{h}^m_{\mathcal{R}} | \iota^2_m(Y)\omega_2 \in Z^1_m(\Omega^\bullet_m)\right\}.$$

We write $Y = \mathfrak{X}_{H_0}$ if $\iota^2_m(Y)\omega_2 = d^0_m H_0$ and we call $\mathfrak{X}_{H_0}$ a **Hamiltonian vectorfield**, in analogy with the situation in classical mechanics. An element in $\mathfrak{Sym}_{\omega_2}(\mathfrak{h}^m_{\mathcal{R}})$ is called a **symplectic vectorfield**.

Remark 4-23. *Notice that* $\mathfrak{Ham}_{\omega_2}(\mathfrak{h}^m_{\mathcal{R}}) \subset \mathfrak{Sym}_{\omega_2}(\mathfrak{h}^m_{\mathcal{R}})$ *and that symplectic vectorfields are Hamiltonian vectorfields if* $H^1_m(\Omega^\bullet_m) = 0$.

Proposition 4-24. $\pi^1_m(\mathfrak{Sym}_{\omega_2}(\mathfrak{h}^m_{\mathcal{R}}))\mathfrak{Sym}_{\omega_2}(\mathfrak{h}^m_{\mathcal{R}}) \subset \mathfrak{Ham}_{\omega_2}(\mathfrak{h}^m_{\mathcal{R}})$.

Proof. Let $Y_1 \in \mathfrak{Sym}_{\omega_2}(\mathfrak{g}^m_{\mathcal{C}}), Y_2 \in \mathfrak{Sym}_{\omega_2}(\mathfrak{g}^m_{\mathcal{C}})$. Then

$$\begin{aligned}
&\iota^2_m(\pi^1_m(Y_1)Y_2)\omega_2 = \\
&= \nabla^1_m(Y_1)\iota^2_m(Y_2)\omega_2 - \iota^2_m(Y_2)\nabla^2_m(Y_1)\omega_2 \\
&= \iota^2_m(Y_1)d^1_m\iota^2_m(Y_2)\omega_2 + d^0_m\iota^1_m(Y_1)\iota^2_m(Y_2)\omega_2 \\
&= d^0_m\iota^1_m(Y_1)\iota^2_m(Y_2)\omega_2 \in B^1_m(\Omega^\bullet_m).
\end{aligned}$$

This proves the statement in the proposition. □

Corollary 4-25. $\mathfrak{Ham}_{\omega_2}(\mathfrak{h}^m_{\mathcal{R}})$ *is an ideal of* $\mathfrak{Sym}_{\omega_2}(\mathfrak{h}^m_{\mathcal{R}})$.

Chapter 5

Complex of formal variational calculus

In this chapter, we construct and investigate a special complex, the complex of formal variational calculus based on the ring $\mathcal{A}$. The coboundary operator d_0^0 in the complex leads naturally to the definition of the Euler operator. Furthermore, we define the pairing between $\Omega_0^1[\mathfrak{k}]$ and $\mathfrak{h}_{\mathfrak{k}}^0$. Its nondegeneracy allows us to write the Lie derivatives in an explicit form.

5.1 Introduction

We will now restrict our ring to be a ring of differentiable functions in two independent variables, time t and space x, and symbols $u_{i,j}^{\alpha}$, with α in some finite index set, representing the dependent variables and their t, x derivatives. At this point we have to make a choice: do we want to work with polynomials, formal power series, analytic functions, smooth functions? And do we want the underlying space to be, e.g., $\mathbb{R}^2$, or maybe $\mathbb{R} \times S^1$? This choice is not very important for our analysis, since everything is set up for abstract rings anyway, but it determines the cohomology spaces. Let us give a fairly trivial example here. Suppose H_0^0 is the space of solutions of the equation

$$\frac{\partial f}{\partial x} = 0.$$

We can solve this and the solutions are of the form $f = f(t)$. But, as we will see later on, we are interested in these solutions modulo the image of the total differential operator D_x. So if we can write $f(t) = D_x x f(t)$, the cohomology is trivial. But this depends on our choice of space (and possibly on other choices too): if we choose $\mathbb{R} \times S^1$ to be the underlying space, x is not a function, since it is not periodic and therefore the cohomology is nontrivial. If the underlying space is trivial, the local cohomology can also be nontrivial depending on the choice of ring. In the notation to be introduced later, $\int \frac{1}{u} \in \operatorname{Ker} d_0^1$, but not in $\operatorname{Im} d_0^0$ if we restrict ourselves to rational functions in the dependent variables, cf. [Dor93], pp. 62–73.

We make the blanket assumption that the zeroth and first cohomology in the space of functionals is zero. This assumption is implicit in some of the proofs, but made explicit by some remarks. Besides the cohomology, also the existence of connections may depend on the choice of ring, cf. [KT71].

5.2 Definition of the complex

Abstract 5-1. *We construct a special complex that is most important in building hierarchies of for instance symmetries and conservation laws of nonlinear evolution equations. The construction is universal and the complexes considered differ only by the choice of the basic ring.*

Consider the ring $\mathcal{B}$ of smooth functions depending on t and x only. Then let $\mathcal{S}$ be the $\mathcal{B}$-algebra generated by the symbols $w^\alpha_{i,j}$ subject to the relations $w^\alpha_{i,j}w^\beta_{k,l} = w^\beta_{k,l}w^\alpha_{i,j}$, with $i,j,k,l \in \mathbb{N}$ and $\alpha,\beta \in \Im$, with $|\Im| < \infty$. The index α taken from some set of indices $\Im$ enumerates dependent variables or unknown functions of partial differential equations and the index i,j indicates the number of x,t-derivatives. The precise nature of $\mathcal{S}$ is not specified at this point, but is subject to the considerations sketched above.

Let $\bar{D}_x w^\alpha_{i,j} = w^\alpha_{i+1,j}$ and $\bar{D}_t w^\alpha_{i,j} = w^\alpha_{i,j+1}$ be derivations on $\mathcal{S}$. We assume that one is given relations in $\mathcal{S}$ of the form $w^\alpha_{i,1} = D^i_x K^\alpha(t,x;w_{0,0},\cdots,w_{n,0})$, where $D_x = \frac{\partial}{\partial x} + \sum_{i,\alpha} w^\alpha_{i+1,0}\frac{\partial}{\partial w^\alpha_{i,0}}$. This covers the theory of evolution equations of the form

$$u^\alpha_t = K^\alpha(t,x;u,\cdots,u_n), \tag{5.2.1}$$

with $u^\alpha_i = w^\alpha_{i,0}$. Using these relations we can eliminate $w^\alpha_{i,j}$ with $j > 0$. This is what makes evolution equations relatively simple to handle. Other types of equations may have mixed derivatives in their normal form after elimination.

Whenever one has $\frac{\partial u^\alpha_i}{\partial t}$, this is to be replaced by $\bar{D}_t w^\alpha_{i,0}$ and then eliminated using the relations. The quotient of $\mathcal{S}$ over these relations is denoted by $\mathcal{A}$. The dynamics of the evolution equation are now built into the ring $\mathcal{A}$.

We construct an $\mathcal{A}$-Lie algebra $\mathfrak{h} = Hom_{\mathcal{A}}(\Omega^1_{\mathcal{A}/\mathcal{B}},\mathcal{A})$ over $\mathcal{A}$ according to section 2.4, denoting the d by d_v, and a $\mathcal{B}$-Lie algebra $\mathfrak{b} = Hom_{\mathcal{B}}(\Omega^1_{\mathcal{B}/\mathcal{C}},\mathcal{B})$, denoting the d by d_h. Observe that the construction of Kähler differentials behaves well under taking the quotient in the ring $\mathcal{S}$ by simply eliminating $d_v w^\alpha_{i,1} = d_v D^i_x K^\alpha$.

For any $F \in \mathcal{A}$, $d_v F = \sum_{i,\alpha}\frac{\partial F}{\partial u^\alpha_i} d_v u^\alpha_i$. By definition $Y d_v = \partial$ and with conventions like $\partial_{u^\alpha_i} d_v u^\alpha_j = \delta^i_j$, any vectorfield $Y \in \mathfrak{h}$ can be written as

$$Y d_v = (\sum_{i,\alpha} h^{i,\alpha}\partial_{u^\alpha_i}) d_v = \sum_{i,\alpha} h^{i,\alpha}\frac{\partial}{\partial u^\alpha_i}, \tag{5.2.2}$$

where $h^{i,\alpha} \in \mathcal{A}$. Likewise any vectorfield in $Z \in \mathfrak{b}$ can be written as

$$\bar{\gamma}^0_0(Z) = Z d_h = (p\partial_t + q\partial_x) d_h = p\frac{\partial}{\partial t} + q\frac{\partial}{\partial x}, \tag{5.2.3}$$

where $p, q \in \mathcal{B}$ and $\bar{\gamma}_0^0 \in \bar{\Gamma}_0^0(\mathfrak{b}, \mathcal{B}, \mathcal{B})$. We trivially extend the action of $\bar{\gamma}_0^0$ to $\mathcal{A}$.

Let $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{h}$. Any vectorfield $X \in \mathfrak{g}$ is written as

$$X = b + h = p\partial_t + q\partial_x + \sum_{i,\alpha} h^{i,\alpha}\partial_{u_i^\alpha}, \tag{5.2.4}$$

where $b \in \mathfrak{b}$ and $h \in \mathfrak{h}$, i.e., $p, q \in \mathcal{B}$ and $h^{i,\alpha} \in \mathcal{A}$.

From the Lie algebras $\mathfrak{h}$ and $\mathfrak{b}$ we construct the connection π_0^1 of $\mathfrak{g}$ on $\mathfrak{h}$ using $\bar{\gamma}_0^0$ as is done in example 2-48, i.e.,

$$\pi_0^1(X_1)X_2 = [p_1\partial_t + q_1\partial_x + \sum_{i,\alpha} h_1^{i,\alpha}, p_2\partial_t + q_2\partial_x + \sum_{i,\alpha} h_2^{i,\alpha}].$$

Moreover, the anchor γ_0^0 of π_0^1 is defined by

$$\gamma_0^0(Z + Y) = \bar{\gamma}_0^0(Z) + Yd_v, \; Z \in \mathfrak{b}, \; Y \in \mathfrak{h}.$$

Notice that $\mathfrak{h}$ is a direct summand of $\mathfrak{g}$ and an $\mathcal{A}$-module. So $(\mathfrak{g}, \mathfrak{h})_{\mathcal{A}}$ is a direct pair and $\mathfrak{g}$ is an $\mathcal{A}$-Lie algebra. Using formula (2.6.1), taking $V = \mathcal{A}$, we obtain a 0-complex $C_0^n(\mathfrak{h}, \mathcal{A}, \mathcal{S})$ from theorem 3-41 with $\nabla_0^0 = \gamma_0^0$.

Consider the one-dimensional space $\mathfrak{k} \subset \mathfrak{g}$ spanned by X^0, where

$$\begin{aligned} \gamma_0^0(X^0) &= X^0 d = \\ &= \partial_x d_h + \sum_{i,\alpha} u_{i+1}^\alpha \partial_{u_i^\alpha} d_v = \frac{\partial}{\partial x} + \sum_{i,\alpha} u_{i+1}^\alpha \frac{\partial}{\partial u_i^\alpha} = D_x. \end{aligned} \tag{5.2.5}$$

We perform the reduction procedure with respect to $\mathfrak{k}$ as given in section 3.8. In this way there arises a complex over $(\mathfrak{g}_{\mathfrak{k}}^0, \mathfrak{h}_{\mathfrak{k}}^0)_{\mathcal{A}_{\mathfrak{k}}}$. It is called the **complex of formal variational calculus** based on the ring $\mathcal{A}$.

We now describe the objects involved in detail. Any element $X \in \mathfrak{g}$ given by (5.2.4) commutes with X^0 given by (5.2.5) iff $D_x p = D_x q = 0$ and $h^{i,\alpha} = D_x(h^{i-1,\alpha})$. It is evident that any $X \in \mathfrak{g}$ commuting with X^0 can be recovered from its action on the variables dx, dt and du^α, $X(du^\alpha) = h^\alpha$, and we have

$$h^{i,\alpha} = X(du_i^\alpha) = XdX^0du_{i-1}^\alpha = X^0dXdu_{i-1}^\alpha = D_x(h^{i-1,\alpha}) = D_x^i(h^\alpha).$$

The equation (5.2.1), simply written as $u_t = K$, corresponds to $X = \partial_t + \sum_{i,\alpha} D_x^i(K^\alpha)\partial_{u_i^\alpha} \in \mathfrak{g}_{\mathfrak{k}}^0$.

The vectorfields in the form $h\partial_u = \sum_{i,\alpha} D_x^i(h^\alpha)\partial_{u_i^\alpha}$, conventionally written as h, consist of $\mathfrak{h}_{\mathfrak{k}}^0$ with Lie bracket

$$[h, g]^\beta = \sum_{i,\alpha} \left(\frac{\partial g^\beta}{\partial u_i^\alpha} D_x^i(h^\alpha) - \frac{\partial h^\beta}{\partial u_i^\alpha} D_x^i(g^\alpha) \right). \tag{5.2.6}$$

$\Omega_0^0[\mathfrak{k}]$ consists of elements in the quotient space $\mathcal{A}/D_x(\mathcal{A})$ since

$$L_{X^0} f = \gamma_0^0(X^0)f = D_x(f), \; \forall f \in \mathcal{A} = \Omega_0^0.$$

Such an element will be called a **functional** and denoted by $\int f$. The integral symbol is both standard and appropriate, since one has $\int D_x f = 0$. Remark that we circumvent the usual problems, forcing one to make assumptions on the underlying space and the test function space in order to make this formula true in the case one does real integration.

One has $\Omega_0^n[\mathfrak{k}] = \Omega_0^n / L_{X^0}\Omega_0^n$. By the definition of the Lie derivative, for $\omega_n \in \Omega_0^n$,

$$
\begin{aligned}
& (L_{X^0}\omega_n)(h_1, \cdots, h_n) \\
= \quad & \gamma_0^0(X^0)\omega_n(h_1, \cdots, h_n) - \sum_{i=1}^{n} \omega_n(h_1, \cdots, \pi_0^1(X^0)h_i, \cdots, h_n) \\
= \quad & D_x\omega_n(h_1, \cdots, h_n).
\end{aligned}
$$

Therefore, $\Omega_0^n[\mathfrak{k}] = C_0^n(\mathfrak{h}_{\mathfrak{k}}^0, \mathcal{A}/D_x(\mathcal{A}), \mathcal{A}_{\mathfrak{k}})$. When dealing with $\omega_n \in \Omega_0^n[\mathfrak{k}]$, $n > 0$, we can freely throw away the ImD_x part of $\omega_n(h_1, \cdots, h_n)$ as is done for $n = 0$. The analysis later relies on this fact.

Now consider the space $\Omega_0^1[\mathfrak{k}]$. Since any $\bar{\omega}_1 \in \Omega_0^1$ can be written as $\bar{\omega}_1 = \sum_{i,\alpha} \bar{\omega}_1^{i,\alpha} du_i^\alpha$, $\bar{\omega}_1^{i,\alpha} \in \mathcal{A}$, one has

$$
\int \bar{\omega}_1(h) = \int \sum_{i,\alpha} \bar{\omega}_1^{i,\alpha} D_x^i(h^\alpha) = \int \sum_{\alpha} (\sum_{i} (-D_x)^i \bar{\omega}_1^{i,\alpha}) h^\alpha = \int \omega_1(h),
$$

where $\omega_1 = \sum_{i,\alpha} (-D_x)^i \bar{\omega}_1^{i,\alpha} du^\alpha = \sum_\alpha \omega_1^\alpha du^\alpha$, with $\omega_1^\alpha = \sum_i (-D_x)^i \bar{\omega}_1^{i,\alpha}$. This leads to the understanding that any element $\omega_1 \in \Omega_0^1[\mathfrak{k}]$ is completely defined by the collection $\{\omega_1^\alpha \in \mathcal{A}\}$.

Remark 5-2. *Notice that $\mathcal{A}/D_x(\mathcal{A})$ does not inherit the ring structure from $\mathcal{A}$, since $D_x(\mathcal{A})$ is not a multiplicative ideal in $\mathcal{A}$. Therefore we cannot construct the Lie algebra from $\mathcal{A}/D_x(\mathcal{A})$ as in section 2.4. This is the reason that we perform the reduction procedure after constructing the complex. One way out of this difficulty might be to take a direct summand to $\operatorname{Im} D_x$ which is multiplicatively closed. This is, e.g., the case when the direct summand is the kernel of a derivation. This is described in [SR94], [SW97a], where D_x is imbedded in a* **Heisenberg** *algebra. This leads to rules like:*

$$
\int u^2 \int u_1^2 = \int u^2 \int \frac{1}{2}(u_1^2 - uu_2) = \frac{1}{2} \int u^2(u_1^2 - uu_2) = 2 \int u^2 u_1^2.
$$

5.3 The pairing and Euler operator

Abstract 5-3. *In this section, we define the pairing between $\Omega_0^1[\mathfrak{k}]$ and $\mathfrak{h}_{\mathfrak{k}}^0$. Its non-degeneracy allows us to write operators, such as L_X, in an explicit form. We also define the Euler operator which follows naturally from d_0^0 in the complex of formal variational calculus.*

First we define the pairing between $\omega_1 \in \Omega_0^1[\mathfrak{k}]$ and $h \in \mathfrak{h}_{\mathfrak{k}}^0$. According to definition 3-56, it is given by

$$(\omega_1, h) = \int \omega_1 \cdot h \in \mathcal{A}/D_x(\mathcal{A}), \tag{5.3.1}$$

where $\omega_1 \cdot h = \sum_\alpha \omega_1^\alpha h^\alpha \in \mathcal{A}$.

Proposition 5-4. *The pairing between $\Omega_0^1[\mathfrak{k}]$ and $\mathfrak{h}_{\mathfrak{k}}^0$ given by (5.3.1) is nondegenerate.*

Proof. This is equivalent to the statement: If there is a given $f \in \mathcal{A}$ such that $\int fg = 0$, i.e., $fg \in ImD_x$ for arbitrary $g \in \mathcal{A}$, then $f = 0$.

Suppose that $g = 1$, then $f \in ImD_x$, which means that f depends linearly on the highest-order derivative $u_{N_\alpha}^\alpha$ of the variables u^α involved in f. Now take $g = u_{N_\alpha}^\alpha$, then $fg \notin ImD_x$. This contradiction shows that f must be equal to zero. ☐

For any $f \in \mathcal{A}$, we calculate d_0^0 of a functional as follows

$$\iota_0^1(h) d_0^0 \int f = \int \sum_{i,\alpha} \frac{\partial f}{\partial u_i^\alpha} D_x^i(h^\alpha) = \sum_\alpha \int h^\alpha \sum_i (-D_x)^i (\frac{\partial f}{\partial u_i^\alpha}).$$

Introduce the operator $\mathbf{E} : \mathcal{A}/D_x(\mathcal{A}) \to \Omega_0^1[\mathfrak{k}]$ called the **Euler operator** by

$$\mathbf{E} = \sum_\alpha \mathbf{E}_\alpha du^\alpha, \quad \mathbf{E}_\alpha(\int f) = \sum_i (-D_x)^i \frac{\partial f}{\partial u_i^\alpha}. \tag{5.3.2}$$

Now we can express d_0^0 in terms of the Euler operator, that is,

$$\iota_0^1(h) d_0^0 \int f = (d_0^0 \int f, h) = \sum_\alpha \int h^\alpha \mathbf{E}_\alpha(f) = (\mathbf{E}(\int f), h).$$

Notice that $\mathbf{E}(\int f) = d_0^0 \int f$ due to the nondegeneracy of the pairing. So we have $\mathbf{E}(0) = 0$ which can, of course, also be checked by direct calculation. This implies that the Euler operator does not depend on the choice of a representative f in the equivalence class $\int f$. Therefore, the operator is well defined.

Remark 5-5. *If we assume that $H_0^0(\mathfrak{h}, \mathcal{A}/D_x(\mathcal{A}), \mathcal{A}_{\mathfrak{k}}) = 0$, then $\mathbf{E}(\int f) = 0$ implies $\int f = 0$, i.e., $f \in ImD_x$ (cf. [Olv93] p. 248). This is useful to know when one studies conservation laws (cf. proposition 5-12 and theorem 6-8).*

Since the index set for α is finite, this implies $dim_{\mathcal{A}} \mathfrak{h}_{\mathfrak{k}}^0 < \infty$, we can define the Fréchet derivatives on both chains and cochains. For $h, g \in \mathfrak{h}_{\mathfrak{k}}^0$, we have

$$D_h[g]^\beta = \sum_{i,\alpha} \frac{\partial h^\beta}{\partial u_i^\alpha} D_x^i(g^\alpha).$$

Therefore, the Lie bracket (5.2.6) can simply be written as

$$[h, g] = D_g[h] - D_h[g].$$

Notice that $D_h : \mathfrak{h}^0_{\mathfrak{k}} \to \mathfrak{h}^0_{\mathfrak{k}}$ ($h \in \mathfrak{h}^0_{\mathfrak{k}}$) and $D_{\omega_1} : \mathfrak{h}^0_{\mathfrak{k}} \to \Omega^1_0[\mathfrak{k}]$ ($\omega_1 \in \Omega^1_0[\mathfrak{k}]$). We have the conjugate operator $D^\star_h : \Omega^1_0[\mathfrak{k}] \to \Omega^1_0[\mathfrak{k}]$ and the adjoint operator $D^\dagger_{\omega_1} : \mathfrak{h}^0_{\mathfrak{k}} \to \Omega^1_0[\mathfrak{k}]$ satisfying, for all $g \in \mathfrak{h}^0_{\mathfrak{k}}$,

$$\begin{aligned}(\omega_1, D_h[g]) &= (D^\star_h[\omega_1], g);\\ (D_{\omega_1}[g], h) &= (D^\dagger_{\omega_1}[h], g).\end{aligned}$$

Definition 5-6. *For $\omega_1 \in \Omega^1_0[\mathfrak{k}]$, if D_{ω_1} is a symmetric operator, i.e., $D_{\omega_1} = D^\dagger_{\omega_1}$, we call ω_1* **self-adjoint***.*

It is easy to obtain the following two important formula for the pairing from the Leibniz rule:

$$D_{\omega_1\cdot h}[g] = D_{\omega_1}[g]\cdot h + \omega_1 \cdot D_h[g], \tag{5.3.3}$$
$$\mathbf{E}((\omega_1, h)) = D^\dagger_{\omega_1}[h] + D^\star_h[\omega_1]. \tag{5.3.4}$$

Proposition 5-7. *For any $\omega_1 \in \Omega^1_0[\mathfrak{k}]$, $d^1_0\omega_1 = 0$ is equivalent to $D_{\omega_1} = D^\dagger_{\omega_1}$.*

Proof. We compute $d^1_0\omega_1$ for $\omega_1 \in \Omega^1_0[\mathfrak{k}]$ by formula (3.4.4). We have

$$\begin{aligned}&d^1_0\omega_1(h_1, h_2) =\\ &= \int (D_{\omega_1\cdot h_2}[h_1] - D_{\omega_1\cdot h_1}[h_2] - \omega_1\cdot(D_{h_2}[h_1] - D_{h_1}[h_2]))\\ &= \int (D_{\omega_1}[h_1]\cdot h_2 - D_{\omega_1}[h_2]\cdot h_1) = \int (D_{\omega_1} - D^\dagger_{\omega_1})[h_1]\cdot h_2.\end{aligned}$$

The result now follows from the nondegeneracy of (5.3.1). □

Remark 5-8. *There arises a question: if $D_{\omega_1} = D^\dagger_{\omega_1}$, can we find $\int f \in \mathcal{A}/D_x\mathcal{A}$ satisfying $\mathbf{E}(\int f) = \omega_1$. Generally, the answer depends on the choice of ring. It is yes when $\mathcal{A}$ are smooth functions or polynomials. This is due to the vanishing of the first cohomology space: $H^1_0(\mathfrak{h}, \mathcal{A}/D_x(\mathcal{A}), \mathcal{A}_{\mathfrak{k}}) = \operatorname{Ker} d^1_0/\operatorname{Im} d^0_0 = 0$. Such an $\int f$ is called the* **density** *of ω_1. For details and the procedure of finding the solution see [Dor93], pp. 62–73.*

5.4 Lie derivatives expressed in Fréchet derivatives

Abstract 5-9. *This section is devoted to explicit forms for Lie derivatives L_X of some basic objects, where $X = \partial_t + \sum_{i,\alpha} D^i_x(K^\alpha)\partial_{u^\alpha_i}$ with $K^\alpha \in \mathcal{A}$. They are very useful for studying evolution equations since, as we know, they can be treated as vectorfields in such a form. Moreover, symmetries can be considered as the elements in the Lie algebra $\mathfrak{h}^0_{\mathfrak{k}}$.*

Theorem 5-10. *The Lie derivatives L_X are given by the following formulae:*

$$\begin{array}{lll}
\int f \in \mathcal{A}/D_x(\mathcal{A}) & L_X \int f & = \int(\frac{\partial f}{\partial t} + D_f[K]), \\
h \in \mathfrak{h}^0_{\mathfrak{k}} & L_X h & = \frac{\partial h}{\partial t} + D_h[K] - D_K[h], \\
\omega_1 \in \Omega^1_0[\mathfrak{k}] & L_X \omega_1 & = \frac{\partial \omega_1}{\partial t} + D_{\omega_1}[K] + D^\star_K[\omega_1], \\
\mathfrak{H} : \Omega^1_0[\mathfrak{k}] \to \mathfrak{h}^0_{\mathfrak{k}} & L_X \mathfrak{H} & = \frac{\partial \mathfrak{H}}{\partial t} + D_{\mathfrak{H}}[K] - D_K \mathfrak{H} - \mathfrak{H} D^\star_K, \\
\mathfrak{I} : \mathfrak{h}^0_{\mathfrak{k}} \to \Omega^1_0[\mathfrak{k}] & L_X \mathfrak{I} & = \frac{\partial \mathfrak{I}}{\partial t} + D_{\mathfrak{I}}[K] + \mathfrak{I} D_K + D^\star_K \mathfrak{I}, \\
\mathfrak{R} : \mathfrak{h}^0_{\mathfrak{k}} \to \mathfrak{h}^0_{\mathfrak{k}} & L_X \mathfrak{R} & = \frac{\partial \mathfrak{R}}{\partial t} + D_{\mathfrak{R}}[K] - D_K \mathfrak{R} + \mathfrak{R} D_K, \\
\mathfrak{T} : \Omega^1_0[\mathfrak{k}] \to \Omega^1_0[\mathfrak{k}] & L_X \mathfrak{T} & = \frac{\partial \mathfrak{T}}{\partial t} + D_{\mathfrak{T}}[K] + D^\star_K \mathfrak{T} - \mathfrak{T} D^\star_K, \\
\omega_n \in \Omega^n_0[\mathfrak{k}] & L_X \omega_n & = \frac{\partial \omega_n}{\partial t} + D_{\omega_n}[K] + D^\star_K[\omega_n],
\end{array}$$

where $\star$ means conjugation, and by definition

$$D^\star_K[\omega_n](h_1, \cdots, h_n) = \sum_{i=1}^{n} \omega_n(h_1, \cdots, D_K[h_i], \cdots, h_n).$$

Moreover, for the operators, the formulae are only valid on the domain of the left hand sides of the identities.

Proof. The first two formulae follow directly from the definition of Lie derivative. We now prove the third one. For $\omega_1 \in \Omega^1_0[\mathfrak{k}]$ and any $g \in \mathfrak{h}^0_{\mathfrak{k}}$, we have

$$\begin{aligned}
&(L_X\omega_1)(g) = L_X\omega_1(g) - \omega_1(L_X g) = \\
&= \gamma^0_0(X)\omega_1(g) - \omega_1(\pi^1_0(X)g) \\
&= \int \left(\frac{\partial \omega_1}{\partial t} \cdot g + \omega_1 \cdot \frac{\partial g}{\partial t} + \omega_1 \cdot D_g[K] + D_{\omega_1}[K] \cdot g \right) \\
&\quad - \int \omega_1 \cdot (\frac{\partial g}{\partial t} + D_g[K] - D_K[g]) \\
&= \left(\frac{\partial \omega_1}{\partial t} + D_{\omega_1}[K] + D^\star_K[\omega_1], g \right).
\end{aligned}$$

By the nondegeneracy of the pairing, we obtain the formula. From the chain rule of the Lie derivative and the first three formulae, it follows, for any $\omega_1 \in \Omega^1_0[\mathfrak{k}]$,

$$\begin{aligned}
&(L_X\mathfrak{H})(\omega_1) = L_X\mathfrak{H}(\omega_1) - \mathfrak{H}(L_X\omega_1) = \\
&= \frac{\partial \mathfrak{H}(\omega_1)}{\partial t} + D_{\mathfrak{H}(\omega_1)}[K] - D_K[\mathfrak{H}(\omega_1)] - \mathfrak{H}(\frac{\partial \omega_1}{\partial t} + D_{\omega_1}[K] + D^\star_K[\omega_1]) \\
&= (\frac{\partial \mathfrak{H}}{\partial t} + D_{\mathfrak{H}}[K] - D_K\mathfrak{H} - \mathfrak{H}D^\star_K)(\omega_1).
\end{aligned}$$

Therefore, we prove the formula for $L_X\mathfrak{H}$. Similarly, we compute

$$\begin{aligned}
&(L_X\mathfrak{I})(g) = L_X\mathfrak{I}(g) - \mathfrak{I}(L_X g) = \\
&= \frac{\partial \mathfrak{I}(g)}{\partial t} + D_{\mathfrak{I}(g)}[K] + D^\star_K[\mathfrak{I}(g)] - \mathfrak{I}(\frac{\partial g}{\partial t} + D_g[K] - D_K[g]) \\
&= (\frac{\partial \mathfrak{I}}{\partial t} + D_{\mathfrak{I}}[K] + \mathfrak{I}D_K + D^\star_K\mathfrak{I})(g).
\end{aligned}$$

And

$$\begin{aligned}
(L_X\mathfrak{R})(g) &= L_X\mathfrak{R}(g) - \mathfrak{R}(L_X g) = \\
&= \frac{\partial \mathfrak{R}(g)}{\partial t} + D_{\mathfrak{R}(g)}[K] - D_K[\mathfrak{R}(g)] - \mathfrak{R}(\frac{\partial g}{\partial t} + D_g[K] - D_K[g]) \\
&= (\frac{\partial \mathfrak{R}}{\partial t} + D_{\mathfrak{R}}[K] - D_K\mathfrak{R} + \mathfrak{R}D_K)(g),
\end{aligned}$$

$$\begin{aligned}
(L_X\mathfrak{T})(\omega_1) &= L_X\mathfrak{T}(\omega_1) - \mathfrak{T}(L_X\omega_1) = \\
&= \frac{\partial \mathfrak{T}(\omega_1)}{\partial t} + D_{\mathfrak{T}(\omega_1)}[K] + D_K^\star[\mathfrak{T}(\omega_1)] - \mathfrak{T}(\frac{\partial \omega_1}{\partial t} + D_{\omega_1}[K] + D_K^\star[\omega_1]) \\
&= (\frac{\partial \mathfrak{T}}{\partial t} + D_{\mathfrak{T}}[K] + D_K^\star\mathfrak{T} - \mathfrak{T}D_K^\star)(\omega_1).
\end{aligned}$$

Finally, for $h_i \in \mathfrak{h}_{\mathfrak{k}}^0$ $(i = 1, \cdots, n)$ and $\omega_n \in \Omega_0^n[\mathfrak{k}]$, it leads to

$$\begin{aligned}
&(L_X\omega_n)(h_1, \cdots, h_n) = \\
&= L_X\omega_n(h_1, \cdots, h_n) - \sum_{i=1}^n \omega_n(h_1, \cdots, L_X h_i, \cdots, h_n) \\
&= \frac{\partial \omega_n(h_1, \cdots, h_n)}{\partial t} + D_{\omega_n(h_1,\cdots,h_n)}[K] \\
&\quad -\sum_{i=1}^n \omega_n(h_1, \cdots, \frac{\partial h_i}{\partial t} + D_{h_i}[K] - D_K[h_i], \cdots, h_n) \\
&= (\frac{\partial \omega_n}{\partial t} + D_{\omega_n}[K] + D_K^\star[\omega_n])(h_1, \cdots, h_n).
\end{aligned}$$

This concludes the proof for all cases. □

Notation 5-11. *Let* $X = \partial_t + \sum_{i,\alpha} D_x^i(K^\alpha)\partial_{u_i^\alpha}$. *When the following objects are invariant under* X*, i.e., in the kernel of* L_X*, then we call*

$\int f \in \mathcal{A}/D_x(\mathcal{A})$	*a conserved density*
$h \in \mathfrak{h}_{\mathfrak{k}}^0$	*a symmetry*
$\omega_1 \in \Omega_0^1[\mathfrak{k}]$	*a* **cosymmetry**
$\mathfrak{H} : \Omega_0^1[\mathfrak{k}] \to \mathfrak{h}_{\mathfrak{k}}^0$	*a cosymplectic operator*
$\mathfrak{I} : \mathfrak{h}_{\mathfrak{k}}^0 \to \Omega_0^1[\mathfrak{k}]$	*a symplectic operator*
$\mathfrak{R} : \mathfrak{h}_{\mathfrak{k}}^0 \to \mathfrak{h}_{\mathfrak{k}}^0$	*a* **recursion operator**
$\mathfrak{T} : \Omega_0^1[\mathfrak{k}] \to \Omega_0^1[\mathfrak{k}]$	*a* **conjugate recursion operator**

of the equation $u_t^\alpha = K^\alpha$ $(u_t = K)$*, where* $\mathfrak{H}$ *is a cosymplectic operator and* $\mathfrak{I}$ *is a symplectic operator as defined in chapter 4.*

Proposition 5-12. *Consider the equation* $u_t = K$*. If* $\int f \in \mathcal{A}/D_x(\mathcal{A})$ *is a conserved density of the equation, then* $\mathbf{E}(\int f)$ *is its cosymmetry. Moreover, if* $\omega_1 \in \Omega_0^1[\mathfrak{k}]$ *is a cosymmetry of the equation and self-adjoint, then the density of* ω_1 *is its conserved density.*

Proof. We know from remark 5-5 and remark 5-8 that the cohomology spaces $H_0^0(\mathfrak{h}, \mathcal{A}/D_x(\mathcal{A}), \mathcal{A}_{\mathfrak{k}})$ and $H_0^1(\mathfrak{h}, \mathcal{A}/D_x(\mathcal{A}), \mathcal{A}_{\mathfrak{k}})$ vanish.

One sees that $\int f \in \mathcal{A}/D_x(\mathcal{A})$ is a conserved density of the equation, i.e.,

$$\int (\frac{\partial f}{\partial t} + D_f[K]) = \int (\frac{\partial f}{\partial t} + (\mathbf{E}(\int f), K)) = 0.$$

So $\mathbf{E}(\int \frac{\partial f}{\partial t} + (\mathbf{E}(\int f), K)) = 0$. Using formula (5.3.4), we can rewrite this as

$$\frac{\partial \mathbf{E}(\int f)}{\partial t} + D^{\dagger}_{\mathbf{E}(\int f)}[K] + D^{\star}_K[\mathbf{E}(\int f)] = 0.$$

It is easy to check $D_{\mathbf{E}(\int f)} = D^{\dagger}_{\mathbf{E}(\int f)}$. Therefore $\mathbf{E}(\int f)$ is a cosymmetry.

Since $D_{\omega_1} = D^{\dagger}_{\omega_1}$, there exists unique $\int f \in \mathcal{A}/D_x(\mathcal{A})$ such that $\omega_1 = \mathbf{E}(\int f)$. The second statement can be proved by reversing the proof for first statement. □

Remark 5-13. *This proposition is a concrete version of proposition 3-6.*

Definition 5-14. *If $\omega_1 \in \Omega_0^1[\mathfrak{k}]$ is a cosymmetry of an evolution equation and self-adjoint, we call ω_1 a* **covariant**

Definition 5-15. *An object α will be called* **time-independent** *if $L_{\partial_t}\alpha = 0$.*

Chapter 6

On Nijenhuis recursion operators

Integrable evolution equations in one space variable, like the KdV equation, are often characterized by the possession of a recursion operator, which is an operator invariant under the flow of the equation, carrying symmetries of the equation into its (new) symmetries.

Example 6-1. *The operator $\mathfrak{R} = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_1 D_x^{-1}$ is a Nijenhuis recursion operator (cf. section 4.2) for the* Korteweg *–de Vries equation $u_t = u_3 + uu_1$. It is remarkable that any $\mathfrak{R}^l u_1$ is a local function ([Olv93], p. 312), i.e., $\mathfrak{R}^l u_1 \in Im D_x$ for $l \geq 0$.*

The fact that the image under repeated application of the recursion operator is again a local function is often not proved, with the KdV equation as an exception to this rule; some proofs can be found in [Dor93], relying on the bi-Hamiltonian character of the equations. But usually one finds in the literature a few explicit calculations, followed by the remark that this goes on. It is the goal of this chapter to prove that this is indeed the case for at least a large class of examples (no exception being known to us, cf. however [Li91]). We give a general theorem to that effect, valid for systems of evolution equations in one evolution system spatial variable and apply this theorem to a number of characteristic examples.

We do not use the property that the operator is a recursion operator of any given equation, only that it is a Nijenhuis operator. Of course, the fact that these operators are recursion operators makes them interesting in the study of integrable systems. We show that Nijenhuis operators of the form that one finds in the study of evolution equations are well defined, i.e., they map into their own domain under some weak conditions and can therefore produce hierarchies of symmetries which are all local. Moreover, the nonlocal part of the operator contains the candidates of roots, starting points for a hierarchy of symmetries.

Apart from the theoretical interest, this splitting (which reminds one of the factorization of the operator in symplectic and cosymplectic operators, if they exist) is useful in the actual computation of a recursion operator for a given system. For this can be done iteratively treating D_x as a symbol, which is fine as long as its power is nonnegative, but fails for D_x^{-1}. It is here that one can proceed to split off a symmetry, and move the remaining part after the D_x^{-1}. This turns out to be an effective algorithm to compute recursion operators.

6.1 Construction of recursion operators

Abstract 6-2. *We show that if the recursion operator is of a specific form, its non-local part, namely containing the D_x^{-1} term(s), can be written as the outer products of symmetries and cosymmetries.*

In this chapter we make the blanket assumptions that

- The operators and vectorfields are t-independent (cf. definition 5-15).
- There exists an universal scaling symmetry (cf. definition 3-51). E.g. in the KdV equation we have a scaling symmetry $xu_1+2u \in \mathfrak{h}^0_{\mathfrak{k}}$, such that $\lambda_{u_t\partial_u} = 3$ and $\lambda_{\mathfrak{R}[u]} = 2$.

For any operator $\mathfrak{R} = \sum_{i=0}^n \mathfrak{R}^{(i)} D_x^i + \sum_{j\in\Gamma} h^{(j)} \otimes D_x^{-1}\xi^{(j)}$, where $\mathfrak{R}^{(i)} \in \mathfrak{h}^0_{\mathfrak{k}} \otimes \Omega^1_0[\mathfrak{k}]$, $h^{(j)} \in \mathfrak{h}^0_{\mathfrak{k}}$, $\xi^{(j)} \in \Omega^1_0[\mathfrak{k}]$ and Γ is the set of gradings, i.e., $\lambda_{h^{(j)}} = j$. Remark that this does not hold for the $\xi^{(j)}$, but we have $j + \lambda_{\xi^{(j)}} = 1 + \lambda_{\mathfrak{R}}$. We denote $\mathfrak{R} \sim \sum_{j\in\Gamma} h^{(j)} \otimes D_x^{-1}\xi^{(j)}$. In order to be able to give some estimates later, we introduce the **gap length**

$$\gamma(\mathfrak{R}) = max_{j\in\Gamma} j - min_{j\in\Gamma} j.$$

We denote by $\mathcal{G}^{\perp}_{\mathfrak{R}}$ the space of all $g \in \mathfrak{h}^0_{\mathfrak{k}}$ such that $(\xi^{(j)}, g) = 0$, where the $\xi^{(j)}$ are given in the recursion operator.

Remark 6-3. *In fact, we have in $\mathfrak{R}$ a tensor product $\otimes_{\mathcal{A}_{\mathfrak{k}}}$, but since $\mathfrak{R}$ is t-independent, we might as well consider it to be $\otimes_{\mathcal{C}}$. This is important since the functions of t can influence the gradings. We come back to this issue in remark 6-5 and example 6-15.*

Lemma 6-4. *Let $\mathfrak{R} = \sum_{i=0}^n \mathfrak{R}^{(i)} D_x^i + \sum_{j\in\Gamma} h^{(j)} \otimes D_x^{-1}\xi^{(j)}$ be a recursion operator of the equation $u_t = K$, with $K\partial_u \in \mathfrak{h}^0_{\mathfrak{k}}$ and $|\lambda_{K\partial_u}| > \gamma(\mathfrak{R})$. Then the $h^{(j)}$ are symmetries of the equation and the $\xi^{(j)}$ are cosymmetries for any $j \in \Gamma$.*

Proof. Since $\mathfrak{R}$ is a recursion operator of $u_t = K$, it satisfies $L_X\mathfrak{R} = 0$, where $X = \partial_t + \sum_{i,\alpha} D_x^i K^\alpha \partial_{u_i^\alpha}$ (notation 5-11). Notice that

$$L_X\mathfrak{R} \sim \sum_{j\in\Gamma} \left(h^{(j)} \otimes D_x^{-1} L_X\xi^{(j)} + L_X h^{(j)} \otimes D_x^{-1}\xi^{(j)}\right).$$

We have either $\lambda_{K\partial_u} > \gamma(\mathfrak{R})$, in which case $\lambda_{L_X h^{(j)}} = \lambda_{K\partial_u} + j > \gamma(\mathfrak{R}) + j \geq max_{j\in\Gamma} j$, or $\lambda_{K\partial_u} < -\gamma(\mathfrak{R})$ and then $\lambda_{L_X h^{(j)}} = \lambda_{K\partial_u} + j < -\gamma(\mathfrak{R}) + j \leq min_{j\in\Gamma} j$. Therefore $L_X h^{(j)} = 0$ and $L_X\xi^{(j)} = 0$. The proof is finished according to notation 5-11. ∎

Remark 6-5. *In the t-dependent case, one can have complications of the following form. Assume $L_X h^{(j)} = \mu_j h^{(j)}$, with $\mu_j \in \mathcal{A}_\mathfrak{k}$. And assume $L_X \xi^{(j)} = -\mu_j \xi^{(j)}$. Under these assumptions one can also solve the equation. To analyze this completely, one has to compute the matrices of L_X on the $h^{(j)}$ and the $\xi^{(j)}$ with coefficients in $\mathcal{A}_\mathfrak{k}$, and do the linear algebra. One then expects a result of the type: There exist $\mu_j \in \mathcal{A}_\mathfrak{k}$ and $\kappa_j \in \mathcal{A}_\mathfrak{k}$, with $L_X \kappa_j = \mu_j \kappa_j$, such that $\kappa_j^{-1} h^{(j)}$ are symmetries of the equation and the $\kappa_j \xi^{(j)}$ are cosymmetries for any $j \in \Gamma$. The case $\mu_j = 0$ and $\kappa_j = 1$ corresponds with the t-independent case.*

$\star$ **Remark 6-6.** *A similar result holds for symplectic and cosymplectic operators once we assume they are in the same form as the operator $\mathfrak{R}$, i.e., their D_x^{-1} part (if it exists) can be written as the product of cosymmetries and symmetries, respectively. This observation is of great help in computing the splitting of a recursion operator, see section 6.3.7,* The new Nijenhuis operator (3D)*, for an example.*

We mention that this result also appeared in the paper [Bil93]. The author gave the condition that $h^{(j)}$ and $\xi^{(j)}$ are independent differential functions, which seems not enough for the proof.

6.2 Hierarchies of symmetries

Abstract 6-7. *It is shown that under certain conditions (which hold for all examples known to us) Nijenhuis operators are well defined, i.e., they give rise to hierarchies of infinitely many commuting symmetries of the operator. Moreover, the nonlocal part of a Nijenhuis operator contains the candidates of roots and coroots.*

We make a distinction between $\mathfrak{R}$ being an invariant of $X = \partial_t + K^\alpha \partial_{u_i^\alpha} \in \mathfrak{g}_\mathfrak{k}^0$ and of $Y = K^\alpha \partial_{u_i^\alpha} \in \mathfrak{h}_\mathfrak{k}^0$. In the first case, we say $\mathfrak{R}$ is a recursion operator of $u_t = K$ (cf. notation 5-11), but in the second that Y is a symmetry of $\mathfrak{R}$ (cf. definition 3-51). When $\mathfrak{R}$ is t-independent, the $\frac{\partial}{\partial t}$ in X does not play a role. These two act in the same way.

The operator $\mathfrak{R}$ is $\mathcal{A}_\mathfrak{k}$-linear. If Y is a symmetry of $\mathfrak{R}$, then fY, with $f \in \mathcal{A}_\mathfrak{k}$, is also a symmetry of $\mathfrak{R}$. However, when Y is a symmetry of $u_t = K$, in general fY will not be a symmetry unless $f \in \mathcal{C}$.

We finally remark that $u_1 \partial_u$ is a trivial symmetry for any operator and that any t-independent operator is a recursion operator of the equation $u_t = u_1$.

Theorem 6-8. *Let $\mathfrak{R} = \sum_{i=0}^n \mathfrak{R}^{(i)} D_x^i + \sum_{j\in\Gamma} h^{(j)} \otimes D_x^{-1} \xi^{(j)}$ be a Nijenhuis operator and $\xi^{(r)}$ be self-adjoint for $r \in \Gamma$. Then $Q_l = \mathfrak{R}^l Q_0 \in \mathcal{G}_\mathfrak{R}^\perp$ for any $l \geq 0$, where $Q_0 \in \mathfrak{h}_\mathfrak{k}^0$ is any symmetry of the Nijenhuis operator with $|\lambda_{Q_0}| > \gamma(\mathfrak{R})$ and $\lambda_\mathfrak{R} \lambda_{Q_0} \geq 0$. Moreover, the Q_l commute.*

Proof. Since $\mathfrak{R}$ is a Nijenhuis operator, for any $l \geq 0$ and any symmetry $Q_0 \in \mathfrak{h}_\mathfrak{k}^0$ (i.e., $L_{Q_0} \mathfrak{R} = 0$ by definition 3-51), it follows from (4.3.3) that $Q_l \in \mathfrak{h}_\mathfrak{k}^0$ satisfies

$L_{Q_l}\mathfrak{R} = 0$. We have

$$L_{Q_l}\mathfrak{R} \sim \sum_{j\in\Gamma} \left(h^{(j)} \otimes D_x^{-1} L_{Q_l}\xi^{(j)} + L_{Q_l}h^{(j)} \otimes D_x^{-1}\xi^{(j)}\right).$$

Due to the assumption that $\lambda_{\mathfrak{R}}\lambda_{Q_0} \geq 0$, $|\lambda_{Q_l}| = |l\lambda_{\mathfrak{R}} + \lambda_{Q_0}| \geq |\lambda_{Q_0}| > \gamma(\mathfrak{R})$. Therefore $L_{Q_l}h^{(j)} = 0$ and $L_{Q_l}\xi^{(j)} = 0$ by the same reason as in the proof of lemma 6-4. We have,

$$d_0^0 \int \xi^{(j)} \cdot Q_l = d_0^0 \iota_0^1(Q_l)\xi^j = L_{Q_l}\xi^{(j)} - \iota_0^2(Q_l) d_0^1 \xi^j = 0.$$

This implies $\xi^{(j)} \cdot Q_l \in ImD_x$ (remark 5-5) and we prove $Q_l \in \mathcal{G}_{\mathfrak{R}}^{\perp}$ by induction. The commuting of Q_l follows from corollary 4-14. □

Remark 6-9. *The operator* $\mathfrak{R}$ *can often be written as*

$$\sum_{i=0}^{n} \mathfrak{R}^{(i)} D_x^i + \sum_{j\in\Gamma} h^{(j)} \otimes D_x^{-1} d_0^0 T^{(j)},$$

where the $T^{(j)}$ *are the densities of* $\xi^{(j)}$ *(remark 5-8), i.e.,* $\mathbf{E}(T^{(j)}) = \xi^{(j)}$.

Our assumptions for lemma 6-4 and theorem 6-8 are not sharp. For a given operator which may be t-dependent, the proof may still go through (cf. example 6-15 and remark 6-5).

Recursion operators of nonevolution equations appear to have a similar form. Compare with [vBGKS97], where the same splitting of the D_x^{-1} *and* D_y^{-1} *terms is found. It would be interesting to see whether one can indeed obtain similar results.*

Definition 6-10. *Let* $\mathfrak{R}$ *be a Nijenhuis operator. If the* $\mathfrak{R}^l Q_0 \neq 0$ *exist for all* $l \geq 0$ *and are commuting symmetries of* $\mathfrak{R}$*, we call* $Q_0 \in \mathfrak{h}_{\mathfrak{k}}^0$ *but* $\notin Im\mathfrak{R}$ *a* **root of** $\mathfrak{R}$.

Corollary 6-11. *If, moreover,* $\mathfrak{R}$ *is a recursion operator of an equation and* Q_0 *is a symmetry of the equation, all* Q_l *for* $l \geq 0$ *consist of a hierarchy of commuting symmetries of the equation.*

Definition 6-12. *Let* $\mathfrak{R}$ *be a recursion operator of a given equation. If the* $\mathfrak{R}^l Q_0 \neq 0$ *exist for all* $l \geq 0$ *and are commuting symmetries of the equation, we call* $Q_0 \in \mathfrak{h}_{\mathfrak{k}}^0$ *but* $\notin Im\mathfrak{R}$ *a* **root of symmetries** *for the equation.*

Similarly, we define $Q^0 \in \Omega_0^1[\mathfrak{k}]$ to be the **coroot of a Nijenhuis operator** $\mathfrak{R}$, from which we produce a hierarchy consisting of all the self-adjoint elements $\mathfrak{R}^{\star l} Q^0$ for all $l \geq 0$ and **coroot of covariants** for the equation, when the operator $\mathfrak{R}$ is its recursion operator, from which we produce a hierarchy of covariants (furthermore, conserved densities) for the equation (cf. section 4.4).

Theorem 6-13. *Let* $\mathfrak{R} = \sum_{i=0}^n \mathfrak{R}^{(i)} D_x^i + \sum_{j\in\Gamma} h^{(j)} \otimes D_x^{-1}\xi^{(j)}$ *be a Nijenhuis operator. And assume that* $\mathfrak{R}h^{(j)}$ *exist and that* $\xi^{(j)}$ *are self-adjoint for* $j \in \Gamma$*. Then* $L_{h^{(j)}}\mathfrak{R} \mid_{\Delta} = 0$ *and* $\iota_0^2(H) d_0^1 \mathfrak{R}^{\star}\xi^{(j)} = 0$, $H \in \Delta$, *where*

$$\Delta = \{H \in dom\mathfrak{R} \mid |\lambda_{\mathfrak{R}H}| > \gamma(\mathfrak{R})\}.$$

Proof. We know that $\mathfrak{R}$ is a Nijenhuis operator, that is $L_{\mathfrak{R}H}\mathfrak{R} = \mathfrak{R}L_H\mathfrak{R}$ for $H \in dom(\mathfrak{R})$. We have

$$\begin{aligned}
\mathfrak{R}L_H\mathfrak{R} &\sim \sum_{r\in\Gamma} \mathfrak{R}L_H h^{(r)} \otimes D_x^{-1}\xi^{(r)} + \mathfrak{R}h^{(r)} \otimes D_x^{-1}L_H\xi^{(r)} \\
&\quad + \sum_{j\in\Gamma} h^{(j)} \otimes D_x^{-1}(L_H\mathfrak{R})^\star\xi^{(j)}, \\
L_{\mathfrak{R}H}\mathfrak{R} &\sim \sum_{j\in\Gamma} L_{\mathfrak{R}H}h^{(j)} \otimes D_x^{-1}\xi^{(j)} + \sum_{j\in\Gamma} h^{(j)} \otimes D_x^{-1}L_{\mathfrak{R}H}\xi^{(j)}.
\end{aligned}$$

Since $H \in dom(\mathfrak{R})$, $d_0^0 \int \xi^{(j)} \cdot H = 0$. We know that

$$d_0^0 \int \xi^{(j)} \cdot H = d_0^0 \iota_0^1(H)\xi^{(j)} = L_H\xi^{(j)} - \iota_0^2(H)d_0^1\xi^{(j)}.$$

Therefore, $L_H\xi^{(j)} = 0$.

Due to the same analysis as in the proof of lemma 6-4, for any $H \in \Delta$, we draw the following conclusions:

$$\mathfrak{R}L_H h^{(j)} = L_{\mathfrak{R}H}h^{(j)}, \tag{6.2.1}$$

$$(L_H\mathfrak{R})^\star\xi^{(j)} = L_{\mathfrak{R}H}\xi^{(j)}. \tag{6.2.2}$$

Formula (6.2.1) implies that $-\mathfrak{R}L_{h^{(j)}}H = -\mathfrak{R}L_{h^{(j)}}H - (L_{h^{(j)}}\mathfrak{R})H$. Hence $L_{h^{(j)}}\mathfrak{R}\mid_\Delta = 0$.

Notice that $(L_H\mathfrak{R})^\star = L_H\mathfrak{R}^\star$. Therefore, formula (6.2.2) implies that

$$L_H(\mathfrak{R}^\star\xi^{(j)}) - \mathfrak{R}^\star L_H\xi^{(j)} = L_{\mathfrak{R}H}\xi^{(j)},$$

i.e., $L_H(\mathfrak{R}^\star\xi^{(j)}) = L_{\mathfrak{R}H}\xi^{(j)}$ since $L_H\xi^{(j)} = 0$. This leads to

$$\begin{aligned}
\iota_0^2(H)d_0^1\mathfrak{R}^\star\xi^{(j)} = L_H(\mathfrak{R}^\star\xi^{(j)}) - d_0^0 \int \mathfrak{R}^\star\xi^{(j)} \cdot H = \\
&= L_{\mathfrak{R}H}\xi^{(j)} - d_0^0 \int \xi^{(j)} \cdot \mathfrak{R}H = \iota_0^2(\mathfrak{R}H)d_0^1\xi^{(j)} \\
&= 0.
\end{aligned}$$

The statement is proved now. □

Remark 6-14. *This theorem theoretically gives us the candidates of roots and coroots. One notices that the restriction on the space Δ is due to the technical problem in the proof. In practice, they are indeed roots and coroots. If this were not the case, one would have a formula, e.g., derived by computer algebra, such that it would vanish for all $H \in \Delta$, but not for all $H \in dom(\mathfrak{R})$. This is hard to imagine.*

Example 6-15. *Consider the* **Cylindrical Korteweg–de Vries equation** *(cf. section 9.9)*

$$u_t = u_3 + uu_1 - \frac{u}{2t}$$

and let a Nijenhuis recursion operator be given by

$$\mathfrak{R} = t(D_x^2 + \frac{2}{3}u + \frac{1}{3}u_1 D_x^{-1}) + \frac{1}{3}x + \frac{1}{6}D_x^{-1} \sim (\frac{t}{3}u_1 + \frac{1}{6})D_x^{-1}.$$

There exists a scaling symmetry $-3t\partial_t + (xu_1 + 2u)\partial_u$ *such that*

$$\lambda_{u_t\partial_u} = 3,\ \lambda_{\mathfrak{R}} = -1,\ \lambda_{\frac{t}{3}u_1+\frac{1}{6}} = -2.$$

However, $h^{(-2)} = \frac{t}{3}u_1 + \frac{1}{6}$ *is* **not** *a symmetry and* $\xi^{(-2)} = 1$ *is* **not** *a cosymmetry of the equation. The lemma 6-4 fails since* $L_X\xi^{(-2)} = -\frac{1}{2t}\xi^{(-2)}$ *and* $-\frac{1}{2t} \in \mathcal{A}_{\mathfrak{k}}$ *which commutes with* D_x^{-1}*. We now compute* $L_X\kappa_{-2} = -\frac{1}{2t}\kappa_{-2}$*, to find* $\frac{\partial\kappa_{-2}}{\partial t} = -\frac{1}{2t}\kappa_{-2}$*, as in remark 6-5. This implies* $\kappa_{-2} = \frac{1}{\sqrt{t}}$*.*

We rewrite the nonlocal part of $\mathfrak{R}$ *as* $\mathfrak{R} \sim \sqrt{t}(\frac{t}{3}u_1 + \frac{1}{6})D_x^{-1}\frac{1}{\sqrt{t}}$*, and we have that* $\lambda_{\sqrt{t}(\frac{u_1}{3}+\frac{1}{6t})} = -\frac{7}{2}$ *and* $\lambda_{\frac{1}{\sqrt{t}}} = \frac{5}{2}$*. By remark 6-5 or by direct computation, we know that* $h^{(-\frac{7}{2})}$ *is a symmetry of the equation and* $\xi^{(-\frac{7}{2})}$ *is a cosymmetry.*

Notice that $L_{h^{(-\frac{7}{2})}}\mathfrak{R} = 0$ *and the proof of theorem 6-8 can be applied. Therefore, it produces a hierarchy of symmetries of the operator* $\mathfrak{R}$*. So we conclude that* $\sqrt{t}(\frac{u_1}{3} + \frac{1}{6t})$ *is a root of symmetries for the equation and* $\frac{1}{\sqrt{t}}$ *is a coroot.*

The trivial symmetry u_1 *of* $\mathfrak{R}$ *cannot produce a hierarchy of symmetries, consistent with the fact* $\lambda_{u_1}\lambda_{\mathfrak{R}} = -1$ *and it does not satisfy the conditions of theorem 6-8.*

6.3 Examples

Abstract 6-16. *A number of examples is given, exhibiting the structure of the Nijenhuis operator and proving the existence of the hierarchies.*

In the following examples we do not check whether the recursion operator is in fact a Nijenhuis recursion operator except for Burgers' equation. Proofs in the literature are usually in the forms 'after a long and boring calculation it follows that...'. We show they satisfy the other conditions of the theorem.

6.3.1 Burgers' equation

Consider Burgers' equation (cf. section 9.1)

$$u_t = f = u_2 + uu_1.$$

We explicitly check all the conditions we need in order to prove that a hierarchy of the symmetries for the equation and the operator exists.

First we check that $\mathfrak{R} = D_x + \frac{1}{2}u + \frac{1}{2}u_1 D_x^{-1}$ is a recursion operators of the equation, i.e., that $L_X\mathfrak{R}$ is equal to zero, where $X = \partial_t + \sum_i D_x^i f \partial_{u_i}$. According the theorem 5-10, we have

$$\begin{aligned}
& L_X\mathfrak{R} = \frac{\partial \mathfrak{R}}{\partial t} + D_{\mathfrak{R}}[f] - D_f\mathfrak{R} + \mathfrak{R}D_f \\
= \;& \frac{1}{2}f + \frac{1}{2}D_x(f)D_x^{-1} - (D_x^2 + uD_x + u_1)(D_x + \frac{1}{2}u + \frac{1}{2}u_1D_x^{-1}) \\
& +(D_x + \frac{1}{2}u + \frac{1}{2}u_1D_x^{-1})(D_x^2 + uD_x + u_1) \\
= \;& \frac{u_2 + uu_1}{2} + \frac{u_3 + uu_2 + u_1^2}{2}D_x^{-1} \\
& -(D_x^3 + \frac{3u}{2}D_x^2 + \frac{5u_1 + u^2}{2}D_x + \frac{3u_2 + 3uu_1}{2} + \frac{u_3 + uu_2 + u_1^2}{2}D_x^{-1}) \\
& +(D_x^3 + \frac{3u}{2}D_x^2 + \frac{5u_1 + u^2}{2}D_x + u_2 + uu_1) \\
= \;& 0.
\end{aligned}$$

Furthermore, there exists a vectorfield $(xu_1 + u)\partial_u$ as a scaling symmetry such that $\lambda_{u_2+uu_1} = 2,\ \lambda_{\mathfrak{R}} = 1,\ \lambda_{u_1} = 1$. It is easy to see that for $\mathfrak{R}$, the conditions of lemma 6-4 are satisfied since $\gamma(\mathfrak{R}) = 0$. So, u_1 is a symmetry of the equation.

Now we check $\mathfrak{R}$ is a Nijenhuis operator. For all $H \in dom\mathfrak{R}$, i.e., $H = D_xP$, we compute

$$\begin{aligned}
L_{\mathfrak{R}H}\mathfrak{R} &= D_{\mathfrak{R}}[D_x(H) + \frac{1}{2}D_x(uP)] - (D_xD_H + \frac{1}{2}D_x\cdot(uD_P + P))\mathfrak{R} \\
&\quad +\mathfrak{R}(D_xD_H + \frac{1}{2}D_x\cdot(uD_P + P)) \\
&= \frac{1}{2}(D_x(H) + \frac{1}{2}D_x(uP)) + \frac{1}{2}(D_x^2(H) + \frac{1}{2}D_x^2(uP))D_x^{-1} \\
&\quad -(\mathfrak{R}D_H + \frac{1}{2}(PD_x + H))\mathfrak{R} + \mathfrak{R}(\mathfrak{R}D_H + \frac{1}{2}(PD_x + H)) \\
&= \mathfrak{R}^2D_H - \mathfrak{R}D_H\mathfrak{R} + \frac{1}{2}D_x(H) + \frac{1}{4}uH + \frac{1}{4}u_1P \\
&\quad +\frac{1}{2}(D_x^2(H) + \frac{1}{2}(uD_x(H) + 2u_1H + u_2P))D_x^{-1} \\
&\quad -\frac{1}{4}u_1P - \frac{1}{4}(u_2P + u_1H)D_x^{-1} + \frac{1}{2}HD_x + \frac{1}{2}D_x(H) \\
&= \mathfrak{R}^2D_H - \mathfrak{R}D_H\mathfrak{R} + \frac{1}{2}HD_x + D_x(H) + \frac{1}{4}uH \\
&\quad +\frac{1}{2}(D_x^2(H) + \frac{1}{2}(uD_x(H) + u_1H))D_x^{-1}, \\
\mathfrak{R}L_H\mathfrak{R} &= \mathfrak{R}(\frac{1}{2}H + \frac{1}{2}D_x(H)D_x^{-1} - D_H\mathfrak{R} + \mathfrak{R}D_H) \\
&= \mathfrak{R}^2D_H - \mathfrak{R}D_H\mathfrak{R} + \frac{1}{2}HD_x + D_x(H) + \frac{1}{4}uH \\
&\quad +\frac{1}{2}(D_x^2(H) + \frac{1}{2}(uD_x(H) + u_1H))D_x^{-1}.
\end{aligned}$$

Therefore, $L_{\mathfrak{R}H}\mathfrak{R} = \mathfrak{R}L_H\mathfrak{R}$, for all $H \in dom\mathfrak{R}$.

Notice that $\lambda_{\mathfrak{R}}\lambda_{u_1} > 0$ and $L_{u_1}\mathfrak{R} = 0$. From theorem 6-8 and its corollary, a hierarchy of symmetries of the equation are $\mathfrak{R}^l u_1$ for $l \geq 0$. This confirms our remark 6-14 that u_1 is the root of $\mathfrak{R}$. There is no coroot for it since $\mathfrak{R}^\star(1) = 0$. This reflects the fact there is only one conservation laws for Burgers' equation.

6.3.2 Krichever – Novikov equation

The **Krichever–Novikov equation** (cf. section 9.12) is given by

$$u_t = u_3 - \frac{3}{2}u_1^{-1}u_2^2$$

and it has a Nijenhuis recursion operator of the form

$$\mathfrak{R} = D_x^2 - 2u_1^{-1}u_2 D_x + (u_1^{-1}u_3 - u_1^{-2}u_2^2) + u_1 D_x^{-1}\xi^{(1)},$$

where $\xi^{(1)} = 3u_1^{-4}u_2^3 - 4u_1^{-3}u_2u_3 + u_1^{-2}u_4 = \mathbf{E}(\frac{1}{2}u_1^{-2}u_2^2)$. First we have $\gamma(\mathfrak{R}) = 0$ and $\lambda_{\mathfrak{R}} = 2$, $\lambda_{u_1} = 1$, $\lambda_{u_t\partial_u} = 3$ with respect to a scaling symmetry $xu_1 \in \mathfrak{h}^0_{\mathfrak{k}}$. Therefore, u_1 is a symmetry and $\xi^{(1)}$ is a cosymmetry of the equation by lemma 6-4. Moreover, we compute $L_{u_1}\mathfrak{R} = 0$ and $\mathfrak{R}^\star\xi^{(1)} = \mathbf{E}(-\frac{1}{2}u_1^{-2}u_3^2 + \frac{3}{8}u_1^{-4}u_2^4)$. We conclude that u_1 is a root of a hierarchy and $\xi^{(1)}$ is a coroot generating a hierarchy of covariants for the equation.

6.3.3 Diffusion system

We consider the **Diffusion system** (cf. section 9.23)

$$\left\{ \begin{array}{l} u_t = u_2 + v^2 \\ v_t = v_2, \end{array} \right.$$

with Nijenhuis recursion operator given by

$$\mathfrak{R}[u, v] = \begin{pmatrix} D_x & vD_x^{-1} \\ 0 & D_x \end{pmatrix} \sim \begin{pmatrix} v \\ 0 \end{pmatrix} \otimes D_x^{-1} \begin{pmatrix} 0, & 1 \end{pmatrix}.$$

We have $\Gamma = \{1\}$, $\lambda_{\mathfrak{R}} = 1$ with respect to the scaling symmetry $\begin{pmatrix} xu_1 \\ xv_1 + v \end{pmatrix}$ and $L_{h^{(1)}}\mathfrak{R} = 0$, $\xi^{(1)}$ is obviously self-adjoint. Thereby, $h^{(1)}$ fulfills the conditions of lemma 6-4 and theorem 6-8. It is a root of $\mathfrak{R}$ and also produces a hierarchy of symmetries of the equation.

Another candidate is the trivial symmetry $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$ and it satisfies the conditions of theorem 6-8, so it is a root of a hierarchy of symmetries, which includes the equation itself, for both the equation and $\mathfrak{R}$.

6.3.4 Boussinesq system

We consider the **Boussinesq system** (cf. section 9.29)

$$\begin{cases} u_t = v_1 \\ v_t = \frac{1}{3}u_3 + \frac{8}{3}uu_1, \end{cases}$$

with Nijenhuis recursion operator given by

$$\mathfrak{R}(u,v) =$$
$$\begin{pmatrix} 3v + 2v_1 D_x^{-1} & D_x^2 + 2u + u_1 D_x^{-1} \\ \frac{1}{3}D_x^4 + \frac{10}{3}uD_x^2 + 5u_1 D_x + 3u_2 + \frac{16}{3}u^2 + 2v_t D_x^{-1} & 3v + v_1 D_x^{-1} \end{pmatrix}$$
$$\sim \begin{pmatrix} 2v_1 \\ \frac{2}{3}u_3 + \frac{16}{3}uu_1 \end{pmatrix} \otimes D_x^{-1} \begin{pmatrix} 1, & 0 \end{pmatrix} + \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \otimes D_x^{-1} \begin{pmatrix} 0, & 1 \end{pmatrix}.$$

There exists a scaling symmetry $\begin{pmatrix} xu_1 + 2u \\ xv_1 + 3v \end{pmatrix}$ such that $\Gamma = \{1, 2\}$ and $\lambda_{\mathfrak{R}} = 3$. Since the equation is t-independent, it is also a symmetry of $\mathfrak{R}$. Notice that $\gamma(\mathfrak{R}) = 1$ and both $\xi^{(1)}$ and $\xi^{(2)}$ are self-adjoint. So $h^{(2)}$ obeys the estimates in theorem 6-8 and is therefore a root since $\lambda_{\mathfrak{R}} > 2$.

For $h^{(1)}$ we explicitly compute

$$\mathfrak{R}h^{(1)} = \begin{pmatrix} v_3 + 4u_1 v + 4uv_1 \\ \frac{1}{3}u_5 + 4uu_3 + 8u_1u_2 + \frac{32}{3}u^2u_1 + 4vv_1 \end{pmatrix}$$

and $\lambda_{\mathfrak{R}h^{(1)}} = 4$. So $\mathfrak{R}h^{(1)}$ satisfies the conditions of theorem 6-8. Therefore the $h^{(j)}$, $j = 1, 2$ are roots of hierarchies of symmetries for the equation (of $\mathfrak{R}$).

6.3.5 Derivative Schrödinger system

Consider the **Derivative Schrödinger system** (cf. section 9.27)

$$\begin{cases} u_t = -v_2 - (u^2 + v^2)u_1 \\ v_t = u_2 - (u^2 + v^2)v_1, \end{cases}$$

with Nijenhuis recursion operator given by

$$\mathfrak{R}(u,v) =$$
$$\begin{pmatrix} vD_x^{-1} \cdot v_1 - u_1 D_x^{-1} \cdot u - \frac{u^2+v^2}{2} & -D_x - u_1 D_x^{-1} \cdot v - vD_x^{-1} \cdot u_1 \\ D_x - v_1 D_x^{-1} \cdot u - uD_x^{-1} \cdot v_1 & uD_x^{-1} \cdot u_1 - v_1 D_x^{-1} \cdot v - \frac{u^2+v^2}{2} \end{pmatrix}$$
$$\sim \begin{pmatrix} vD_x^{-1} \cdot v_1 - u_1 D_x^{-1} \cdot u & -u_1 D_x^{-1} \cdot v - vD_x^{-1} \cdot u_1 \\ -v_1 D_x^{-1} \cdot u - uD_x^{-1} \cdot v_1 & uD_x^{-1} \cdot u_1 - v_1 D_x^{-1} \cdot v \end{pmatrix}$$
$$= \begin{pmatrix} v \\ -u \end{pmatrix} \otimes D_x^{-1} \begin{pmatrix} v_1, & -u_1 \end{pmatrix} + \begin{pmatrix} -u_1 \\ -v_1 \end{pmatrix} \otimes D_x^{-1} \begin{pmatrix} u, & v \end{pmatrix}.$$

We find that $\lambda_{\mathfrak{R}} = 1$ and $\Gamma = \{0, 1\}$ with respect to the scaling symmetry $\begin{pmatrix} xu_1 + \frac{u}{2} \\ xv_1 + \frac{v}{2} \end{pmatrix}$, implying $\gamma(\mathfrak{R}) = 1$. And we have that both $\xi^{(0)}$ and $\xi^{(1)}$ are self-adjoint. However, the condition for $h^{(0)}$ and $h^{(1)}$ in theorem 6-8 is not satisfied. First we notice that $h^{(1)} = -\mathfrak{R}h^{(0)}$. So all we have to check explicitly are the conditions for $h^{(1)}$. First all, $L_{h^{(1)}}\mathfrak{R} = 0$ trivially. Secondly, $\mathfrak{R}h^{(1)} = \begin{pmatrix} -u_t \\ -v_t \end{pmatrix} \neq 0$ and $\lambda_{\mathfrak{R}h^{(1)}} = 2$. We now see that $\mathfrak{R}h^{(1)}$ satisfies the conditions. So $h^{(0)} = \begin{pmatrix} v \\ -u \end{pmatrix}$ is a root of a hierarchy.

6.3.6 Sine–Gordon equation in the laboratory coordinates

Consider the **Sine–Gordon equation** (cf. section 9.24) in the form of

$$\left\{ \begin{array}{l} u_t = v \\ v_t = u_2 - \alpha \sin(u), \alpha \in \mathbb{R}. \end{array} \right.$$

with Nijenhuis recursion operator given by

$$\begin{aligned} \mathfrak{R}(u, v) &= \begin{pmatrix} \mathfrak{R}_{11} & \mathfrak{R}_{12} \\ \mathfrak{R}_{21} & \mathfrak{R}_{22} \end{pmatrix} \\ &\sim \begin{pmatrix} u_1 + v \\ u_2 + v_1 - \alpha \sin(u) \end{pmatrix} \otimes D_x^{-1} \begin{pmatrix} -(u_2 + v_1 - \alpha \sin(u)), & u_1 + v \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} \mathfrak{R}_{11} &= 4D_x^2 - 2\alpha \cos(u) + (u_1 + v)^2 - (u_1 + v)D_x^{-1}(u_2 + v_1 - \alpha \sin(u)), \\ \mathfrak{R}_{12} &= 4D_x + (u_1 + v)D_x^{-1}(u_1 + v), \\ \mathfrak{R}_{21} &= 4D_x^3 + (u_1 + v)^2 D_x - 4\alpha \cos(u) D_x + 2u_1 \alpha \sin(u) + (u_2 + v_1)(u_1 + v) \\ &- (u_2 + v_1 - \alpha \sin(u))D_x^{-1}(u_2 + v_1 - \alpha \sin(u)), \\ \mathfrak{R}_{22} &= 4D_x^2 + (u_1 + v)^2 - 2\alpha \cos(u) + (u_2 + v_1 - \alpha \sin(u))D_x^{-1}(u_1 + v). \end{aligned}$$

This is one of the 'new' operators which appeared in [FOW87], p. 53, when $\alpha = 1$. The system is not homogeneous in 2-dimensional space, namely u and v. Let us consider the extended system

$$\left\{ \begin{array}{l} u_t = v \\ v_t = u_2 - \alpha \sin(u) \\ \alpha_t = 0. \end{array} \right.$$

Then

$$\tilde{\mathfrak{R}}(u, v, \alpha) = \begin{pmatrix} \mathfrak{R}_{11} & \mathfrak{R}_{12} & 0 \\ \mathfrak{R}_{21} & \mathfrak{R}_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim h^{\tilde{(1)}} \otimes D_x^{-1} \xi^{\tilde{(1)}}$$

and $\lambda_{\mathfrak{R}} = 2$ with respect to the scaling symmetry $\begin{pmatrix} xu_1 \\ xv_1 + v \\ x\alpha_x + 2\alpha \end{pmatrix}$. We see that $\gamma(\tilde{\mathfrak{R}}) = 0$, $\xi^{\tilde{(1)}}$ is self-adjoint and $L_{h^{\tilde{(1)}}}\tilde{\mathfrak{R}} = 0$. So $h^{\tilde{(1)}}$ is a root of a hierarchy of symmetries of the Nijenhuis operator $\tilde{\mathfrak{R}}$. This leads to the same result if we take α as a constant, i.e., $h = \begin{pmatrix} u_1 + v \\ u_2 + v_1 - \alpha\sin(u) \end{pmatrix}$ is a root of $\mathfrak{R}$.

In fact $\mathfrak{R}(h) = 2\mathfrak{R}(Q^0)$, where $Q^0 = \begin{pmatrix} u_1, & v_1 \end{pmatrix}$ which is considered as a start point in [FOW87] (so this is not in contradiction, since the same hierarchy will be generated by h and Q^0).

6.3.7 The new Nijenhuis operator (3D)

Consider the following Nijenhuis operator ([FOW87] p. 54):

$$
\begin{aligned}
\mathfrak{R}(u,\phi,\psi) &= \begin{pmatrix} 4u^2 & 0 & 1 \\ D_x & -2u^2 & 0 \\ 4u\psi & D_x - 4u\phi & -2u^2 \end{pmatrix} \\
&+ 4\begin{pmatrix} \phi \\ \psi \\ u_1 - 6u^2\phi \end{pmatrix} \otimes D_x^{-1}\begin{pmatrix} \psi + 6u^3, & -\phi, & u \end{pmatrix}.
\end{aligned}
$$

We see that $\lambda_{\mathfrak{R}} = \frac{2}{3}$ and $\mathfrak{R} \sim 4h^{(\frac{1}{3})} \otimes D_x^{-1}\xi^{(\frac{1}{3})}$ under the scaling symmetry

$$
\begin{pmatrix} xu_1 + \frac{u}{3} \\ x\phi_1 + \frac{2\phi}{3} \\ x\psi_1 + \psi \end{pmatrix}.
$$

Furthermore, we have $L_{h^{(\frac{1}{3})}}\mathfrak{R} = 0$ and $\xi^{(\frac{1}{3})}$ is self-adjoint. So the conditions of theorem 6-8 are satisfied for $h^{(\frac{1}{3})}$. Therefore from $h^{(\frac{1}{3})}$ a hierarchy of symmetries of $\mathfrak{R}$ is generated.

In fact $\mathfrak{R}(h^{(\frac{1}{3})}) = \begin{pmatrix} 4u_1, & 4\phi_1, & 4\psi_1 \end{pmatrix}$, which is considered as a starting point in [FOW87]. $h^{(\frac{1}{3})}$ must be a root, since the only (using a scaling argument and the fact that it should be in the domain of $\mathfrak{R}$) vectorfield that could generate $h^{(\frac{1}{3})}$ is $\begin{pmatrix} 0, & u, & \phi \end{pmatrix}$, and this gives not $h^{(\frac{1}{3})}$, but $h^{(\frac{1}{3})} - \begin{pmatrix} 0, & \psi + 2u^3, & 0 \end{pmatrix}$ and their difference is not in $\operatorname{Ker}\mathfrak{R}$.

Assume $\mathfrak{H}_2 = \mathfrak{R}\mathfrak{H}_1$ where $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are both cosymplectic operators. We know that $\mathfrak{H}_2$ must have the following term as its nonlocal part

$$
\begin{pmatrix} \phi \\ \psi \\ u_1 - 6u^2\phi \end{pmatrix} \otimes D_x^{-1}\begin{pmatrix} \phi \\ \psi \\ u_1 - 6u^2\phi \end{pmatrix}.
$$

This means $\left(\ \psi + 6u^3, \quad -\phi, \quad u\ \right) \mathfrak{H}_1 \equiv \left(\ \phi, \quad \psi, \quad u_1 - 6u^2\phi\ \right)$. Therefore

$$\mathfrak{H}_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 6u^2 \\ 0 & -6u^2 & -D_x \end{pmatrix}.$$

One can easily check that $\mathfrak{H}_1$ and $\mathfrak{H}_2$ form an cosymplectic pair (cf. [Olv93] pp. 444–454)

6.3.8 Landau – Lifshitz system

Consider the **Landau–Lifshitz system** (cf. section 9.31)

$$\begin{cases} u_t = -\sin(u)v_2 - 2\cos(u)u_1v_1 + (J_1 - J_2)\sin(u)\cos(v)\sin(v) \\ v_t = \frac{u_2}{\sin(u)} - \cos(u)v_1^2 + \cos(u)(J_1\cos^2(v) + J_2\sin^2(v) - J_3), \end{cases}$$

with Nijenhuis recursion operator given by

$$\begin{aligned} \mathfrak{R}(u,v) &= \\ &= \begin{pmatrix} \mathfrak{R}_{11} & \mathfrak{R}_{12} \\ \mathfrak{R}_{21} & \mathfrak{R}_{22} \end{pmatrix} \\ &\sim \begin{pmatrix} u_t \\ v_t \end{pmatrix} \otimes D_x^{-1} \left(\ \sin(u)v_1, \quad -\sin(u)u_1\ \right) - \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \otimes D_x^{-1} \left(\ S_1, \quad S_2,\ \right), \end{aligned}$$

where

$$\begin{aligned} \mathfrak{R}_{11} &= -D_x^2 - 2\sin^2(u)v_1^2 - u_1^2 + v_1^2 - (J_1 - J_2)\sin^2(u)\sin^2(v) \\ &\quad +(J_1 - J_3)\sin^2(u) + J_3 - J_2 + u_tD_x^{-1}\cdot(\sin(u)v_1) - u_1D_x^{-1}\cdot S_1, \\ \mathfrak{R}_{12} &= 2\cos(u)\sin(u)v_1D_x + \cos(u)\sin(u)v_2 - 3\sin^2(u)u_1v_1 + 2u_1v_1 \\ &\quad +u_tD_x^{-1}\cdot(-\sin(u)u_1) - u_1D_x^{-1}\cdot S_2, \\ \mathfrak{R}_{21} &= -2\cos(u)v_1D_x - \cos(u)v_2 + u_1v_1 \\ &\quad +v_tD_x^{-1}\cdot(\sin(u)v_1) - v_1D_x^{-1}\cdot S_1, \\ \mathfrak{R}_{22} &= -D_x^2 - 2\cos(u)u_1D_x - \cos(u)u_2 - (J_1 - J_2)\sin(u)\sin^2(v) \\ &\quad -2\sin^2(u)v_1^2 + v_1^2 + (J_1 - J_3)\sin^2(u) + J_3 - J_2 \\ &\quad +v_tD_x^{-1}\cdot(-\sin(u)u_1) - v_1D_x^{-1}\cdot S_2, \\ S_1 &= (J_1 - J_2)\cos(u)\sin(u)\sin^2(v) - (J_1 - J_3)\cos(u)\sin(u) \\ &\quad +\cos(u)\sin(u)v_1^2 - u_2, \\ S_2 &= (J_1 - J_2)\cos(v)\sin^2(u)\sin(v) - 2\cos(u)\sin(u)u_1v_1 - \sin^2(u)v_2. \end{aligned}$$

As we did in section 6.3.6, for the extended system, we have that $\lambda_{J_i} = 2$ for $i = 1, 2, 3$, $\lambda_{\mathfrak{R}} = 2$ and $\Gamma = \{1, 2\}$ with

$$h^{(1)} = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \ h^{(2)} = \begin{pmatrix} u_t \\ v_t \end{pmatrix}.$$

So $\gamma(\mathfrak{R}) = 1$. Notice that $\xi^{(1)}$ and $\xi^{(2)}$ are both self-adjoint. It follows that $h^{(2)}$ is a root of a hierarchy. For $h^{(1)}$, we have

$$\begin{aligned}
& \xi^{(1)} \cdot h^{(1)} = \begin{pmatrix} S_1, & S_2 \end{pmatrix} \cdot \begin{pmatrix} u_1 & v_1 \end{pmatrix} \\
= \;& (J_1 - J_2)\cos(u)\sin(u)\sin^2(v)u_1 + (J_1 - J_2)\cos(v)\sin^2(u)\sin(v)v_1 \\
& -(J_1 - J_3)\cos(u)\sin(u)u_1 - u_1u_2 - \cos(u)\sin(u)u_1v_1^2 - \sin^2(u)v_1v_2 \\
= \;& \frac{1}{2}D_x((J_1 - J_2)\sin^2(u)\sin^2(v) - (J_1 - J_3)sin^2(u) - u_1^2 - \sin^2(u)v_1^2)
\end{aligned}$$

and

$$\xi^{(2)} \cdot h^{(1)} = \begin{pmatrix} \sin(u)v_1 & -\sin(u)u_1 \end{pmatrix} \cdot \begin{pmatrix} u_1, & v_1 \end{pmatrix} = 0.$$

So $h^{(1)} \cdot \xi^{(j)} \in ImD_x$ for $j = 1, 2$. This implies that $\mathfrak{R}h^{(1)} \neq 0$ exists. Notice that $\lambda_{\mathfrak{R}h^{(1)}} = 3$. By theorem 6-8, we conclude that $h^{(1)}$ is a root of a hierarchy.

Chapter 7

The symbolic method

In this chapter, we introduce the symbolic notation and derive expressions for the (co)symmetries. The Lie derivatives of $u_k\partial_u$ acting on polynomials of degree $m+1$ are given, from which we define the functions $G_k^{(m)}$. The mutual divisibility of these functions play a role in proving the (non-)existence of (co)symmetries. It is interesting to note that the result that any nontrivial symmetry satisfies the conditions of our abstract theorem 2-76, relies (at the moment) on results using diophantine approximation theory.

The symbolic method was introduced by Gel'fand–Dikii [GD75] and used in [TQ81] to show (as an example) that the symmetries of the Sawada–Kotera equation have to be of order 1 or 5 $\pmod 6$. The basic idea of the symbolic method is very old, probably dating from the time when the position of index and power were not as fixed as they are today. In fact, the symbolic calculus of classical invariant theory relies on it. The idea is simply to replace u_i, where i is an index, in our case counting the number of derivatives, by $u\xi^i$, where ξ is now a symbol. We see that the basic operation of differentiation, i.e., replacing u_i by u_{i+1}, is now replaced by multiplication with ξ, as is the case in `Fourier` transformation theory. If one has multiple u's, as in u_iu_j, one replaces this by $\frac{1}{2}\left(\xi_1^i\xi_2^j+\xi_1^j\xi_2^i\right)u^2$. We have averaged over the permutation group Σ_2 to retain complete equality among the symbols, reflecting the fact that $u_iu_j = u_ju_i$. Differentiation now becomes multiplication with $\xi_1+\xi_2$.

7.1 Symbolic Notation

Abstract 7-1. *We introduce the Gel'fand–Dikii transformation and give the expressions for the (co)symmetries using the symbolic method. We also define the functions $G_k^{(m)}$ from the Lie derivatives of $u_k\partial_u$ acting on polynomials of degree $m+1$. As applications, we compute the (co)symmetries for the linear evolution equation.*

Notation 7-2. *1. Let $\mathcal{A}_n^k$ be the set of polynomials f of degree k in $n+1$ variables* *and* $\tilde{\mathcal{A}}_n^k$ *be the set of its symmetrized elements* $\tilde{f} \stackrel{\text{def}}{=} < f >$. *Here*

$$< f > (x_1, \cdots, x_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} f(\sigma(x_1), \cdots, \sigma(x_n)),$$

where Σ_n *is the permutation group on* n *elements.*

2. *For brevity,* $[u]$ *is used to denote the set of arguments* $u, u_1, u_2, \cdots$. *We denote by* $\mathcal{U}_n^k (n \geq 0, k \geq 0)$ *the set of polynomials of* $[u]$ *of degree* $k+1$ *and index* n, *that is*

$$\mathcal{U}_n^k = \{f | f = \sum_{\|\alpha\|=n, |\alpha|=k+1} C_{\alpha_0 \cdots \alpha_m} u^{\alpha_0} u_1^{\alpha_1} \cdots u_m^{\alpha_m}\},$$

   where $|\alpha| \stackrel{\text{def}}{=} \sum_{i=0}^m \alpha_i$ *and* $\|\alpha\| \stackrel{\text{def}}{=} \sum_{i=0}^m i\alpha_i$. *The ring of all polynomials of* $[u]$ *is denoted by* $\mathcal{U}$ *and* $\mathcal{U} = \bigoplus_{n \geq 0, k \geq 0} \mathcal{U}_n^k$.

3. $\tilde{\mathcal{A}}_n$, $\tilde{\mathcal{A}}^k$, $\mathcal{U}_n$ *and* $\mathcal{U}^k$ *make sense, e.g.,* $\mathcal{U}^k$ *is the set of polynomials of* $[u]$ *of degree* $k+1$.

Remark 7-3.
- *Notice that we consider* $k \geq 0$ *which excludes the constant case, i.e.,* $1 \notin \mathcal{U}$. *This case need to be treated separately.*
- *We construct the complex of formal variational calculus based on the ring* $\mathcal{U}$ *according to Chapter 5, denoted by* $(\mathfrak{g}, \mathfrak{h})_C$ *for simplification.* $\Omega_0^n[\mathfrak{k}]$ *are written as* Ω^n *for the same reason.*

With each polynomial in $\mathcal{U}_n^k$ we associate a form in $\mathcal{A}_k^n$ by the following rule:

$$u^{\alpha_0} u_1^{\alpha_1} \cdots u_m^{\alpha_m} \longmapsto u^{k+1} \xi_1^0 \cdots \xi_{\alpha_0}^0 \xi_{\alpha_0+1}^1 \cdots \xi_{\alpha_0+\alpha_1}^1 \cdots \xi_{k-\alpha_m+2}^m \cdots \xi_{k+1}^m.$$

Definition 7-4. *The* **Gel'fand–Dikii transformation** *[GD75] maps* $f \in \mathcal{U}_n^k$ *to* $\hat{f} = u^{k+1}\tilde{f}$, *where* $\tilde{f} \in \tilde{\mathcal{A}}_k^n$. *For a monomial it is defined as*

$$u^{\alpha_0} u_1^{\alpha_1} \cdots u_m^{\alpha_m} \longmapsto u^{k+1} < \xi_1^0 \cdots \xi_{\alpha_0}^0 \xi_{\alpha_0+1}^1 \cdots \xi_{\alpha_0+\alpha_1}^1 \cdots \xi_{k-\alpha_m+2}^m \cdots \xi_{k+1}^m > .$$

For any $f \in \mathcal{U}^k$, two important properties of Gel'fand–Dikii transformation are

$$\begin{array}{l} \widehat{D_x f}(\xi_1, \cdots, \xi_{k+1}) = u^{k+1} \tilde{f}(\xi_1, \cdots, \xi_{k+1}) \sum_{i=1}^{k+1} \xi_i, \\ \widehat{\frac{\partial f}{\partial u_i}}(\xi_1, \cdots, \xi_{k+1}) = \frac{1}{i!} \frac{\partial^{i+1} \hat{f}}{\partial u \partial \xi_{k+1}^i}(\xi_1, \cdots, \xi_k, 0). \end{array} \tag{7.1.1}$$

Consider that $\mathfrak{h}_n^k = \{D_x^i(h)\partial_{u_i},\ h \in \mathcal{U}_n^k\}$ (simply written as h) and $\mathfrak{h} = \bigoplus_{n,k \in \mathbb{N}} \mathfrak{h}_n^k$. We shall show that this is a graded Lie algebra with respect to both n and k.

Proposition 7-5. *If* $f \in \mathcal{U}_r^m$ *and* $h \in \mathfrak{h}_s^n$, *then* $D_f[h] \in \mathcal{U}_{r+s}^{m+n}$ *and*

$$\begin{aligned} &\widehat{D_f[h]} \\ = \ &(m+1)u^{m+n+1} < \tilde{f}(\xi_1, \cdots, \xi_m, \sum_{i=1}^{n+1} \xi_{m+i}) \tilde{h}(\xi_{m+1}, \cdots, \xi_{m+n+1}) > . \end{aligned}$$

Proof. Using (7.1.1), we compute, with $\hat{h} = u^{n+1}\tilde{h}(\zeta_1, \cdots, \zeta_{n+1})$,

$$\begin{aligned}
\widehat{D_f[h]} &=< \sum_j \widehat{\frac{\partial f}{\partial u_j}} \widehat{D_x^j h} > \\
&= \sum_j u^{m+n+1} < \frac{m+1}{j!} \frac{\partial^j \tilde{f}}{\partial \xi_{m+1}^j}(\xi_1, \cdots, \xi_m, 0)(\zeta_1 + \cdots + \zeta_{n+1})^j \tilde{h}(\zeta_1, \cdots, \zeta_{n+1}) > \\
&= (m+1)u^{m+n+1} < \tilde{f}(\xi_1, \cdots, \xi_m, \zeta_1 + \cdots + \zeta_{n+1})\tilde{h}(\zeta_1, \cdots, \zeta_{n+1}) >
\end{aligned}$$

and the conclusion follows. □

Therefore, the Lie bracket $[h, g] = D_g[h] - D_h[g] \in \mathfrak{h}_{r+s}^{m+n}$ if $h \in \mathfrak{h}_r^m$ and $g \in \mathfrak{h}_s^n$ since we define Fréchet derivative through the ring. This implies that $\mathfrak{h}$ is indeed a graded Lie algebra.

In the same way, we have $\Omega^1 = \bigoplus_{n,k\in\mathbb{N}}(\Omega^1)_n^k$, where $(\Omega^1)_n^k$ is defined by $\omega \in \mathcal{U}_n^k$. Notice that Ω^1 is a $\mathfrak{h}$-module under the Lie derivative action. We now check whether it is a graded $\mathfrak{h}$-module.

Proposition 7-6. *Let ξ_0 define by $\sum_{j=0}^{n+m+1} \xi_j = 0$. If $h \in \mathfrak{h}_r^m$ and $\omega \in (\Omega^1)_s^n$, then $D_h^\star[\omega] \in (\Omega^1)_{r+s}^{m+n}$ and*

$$\widehat{D_h^\star[\omega]} = (m+1)u^{m+n+1} < \tilde{h}(\xi_1, \cdots, \xi_m, \xi_0)\tilde{\omega}(\xi_{m+1}, \cdots, \xi_{m+n+1}) > .$$

Proof. Using (7.1.1), we compute

$$\begin{aligned}
< \widehat{D_h^\star[\omega]} > &=< \sum_j (-1)^j D_x^j (\widehat{\frac{\partial h}{\partial u_j}\omega}) > \\
&= \sum_j u^{m+n+1}(-\sum_{i=1}^m \xi_i - \sum_{s=1}^{n+1} \zeta_s)^j < \frac{m+1}{j!} \frac{\partial^j \tilde{h}}{\partial \xi_{m+1}^j}(\xi_1, \cdots, \xi_m, 0)\tilde{\omega}(\zeta_1, \cdots, \zeta_{n+1}) > \\
&= (m+1)u^{m+n+1} < \tilde{h}(\xi_1, \cdots, \xi_m, -(\sum_{i=1}^m \xi_i + \sum_{s=1}^{n+1} \zeta_s))\tilde{\omega}(\zeta_1, \cdots, \zeta_{n+1}) > \\
&= (m+1)u^{m+n+1} < \tilde{h}(\xi_1, \cdots, \xi_m, \xi_0)\tilde{\omega}(\xi_{m+1}, \cdots, \xi_{m+n+1}) >
\end{aligned}$$

and this proves the proposition. □

So $L_h\omega = D_\omega[h] + D_h^\star[\omega] \in (\Omega^1)_{r+s}^{m+n}$ when $h \in \mathfrak{h}_r^m$ and $\omega \in (\Omega^1)_s^n$. It follows that Ω^1 is a graded $\mathfrak{h}$-module. Surely, we can say $\mathfrak{h}$ is a graded $\mathfrak{h}$-module under the adjoint representation **ad**. If we do not specify $\mathfrak{h}$ or Ω^1, we use the notation $\mathcal{V}$.

Proposition 7-7. *Let $Q \in \mathcal{V}$, $K \in \mathfrak{h}$ and $Q = \sum Q_r^i$, $K = \sum K_s^j$, where $Q_r^i \in \mathcal{V}_r^i$ and $K_r^i \in \mathfrak{h}_r^i$. Then Q is a (co)symmetry of the equation $u_t = K$ iff*

$$\sum_{\substack{i+j=p \\ r+s=q}} \widehat{L_{K_s^j} Q_r^i} = 0, \ \ p, q \geq 0.$$

Proof. We know that Q is a (co)symmetry of the equation $u_t = K$ iff $L_K Q = D_Q[K] - D_K[Q] = 0$ ($L_K Q = D_Q[K] + D_K^{\star}[Q] = 0$) since Q is t-independent. By proposition 7-5 and 7-6, this can be proved directly. □

Definition 7-8. *Let ξ_0 be defined by $\sum_{j=0}^{m+1} \xi_j = 0$. We now have that*

$$< L_{u_k} Q^m >= G_k^{(m)} \hat{Q}^m,$$

where we define $G_k^{(m)}$ by

$$G_k^{(m)} = \sum_{i=1}^{m+1} \xi_i^k + \alpha^{k+1} \xi_0^k,$$

with $\alpha = -1$ if Q^m is a symmetry and $\alpha = 1$ if Q^m is a cosymmetry.

We give the following results as applications of proposition 7-7.

Notation 7-9. $(\mathcal{C})\mathcal{S}_f = \{$*(co)symmetries of $u_t = f$ in $\mathfrak{h}$ (Ω^1)* $\}$.

Proposition 7-10. *Consider the linear evolution equation*

$$u_t = f = \sum_{j=1}^{p} \lambda_j u_j,$$

where the λ_j are constants and $\lambda_p \neq 0$. Then

- $\mathcal{S}_f = \mathfrak{h}$ *iff* $p = 1$,
- $\mathcal{S}_f = \mathfrak{h}^0$ *iff* $p > 1$.

We see that for $p > 1$, L_f is nonlinear injective (for symmetries).

Proof. Notice $\sum_{j=1}^{p} \lambda_j u_j \in \mathfrak{h}^0$ and $\sum_{j=1}^{p} \lambda_j \hat{u}_j = u \sum_{j=1}^{p} \lambda_j \xi_1^j$. Let $Q \in \mathfrak{h}$ and $Q = \sum Q^i$, where $Q^i \in \mathfrak{h}^i$. By proposition 7-7, Q is a symmetry of this equation iff $\widehat{L_f Q^i} = 0$, for any $i \geq 0$. So

$$(i+1) < \tilde{Q}^i(\xi_1, \cdots, \xi_{i+1}) \sum_{j=1}^{p} \lambda_j \xi_{i+1}^j >= \sum_{j=1}^{p} \lambda_j (\xi_1 + \cdots + \xi_{i+1})^j \tilde{Q}^i(\xi_1, \cdots, \xi_{i+1}).$$

This implies

$$\sum_{j=1}^{p} \lambda_j (\xi_1^j + \cdots + \xi_{i+1}^j) = \sum_{j=1}^{p} \lambda_j (\xi_1 + \cdots + \xi_{i+1})^j.$$

Under the assumption, this holds iff either $p = 1$ or $p \neq 1$ and $i = 0$. □

Proposition 7-11. *Consider the linear evolution equation $u_t = f = \sum_{j=1}^{p} \lambda_j u_j$, where λ_j are constants and $\lambda_p \neq 0$. Then*

- $\mathcal{CS}_f = \Omega^1$ *iff* $p = 1$,

- $\mathcal{CS}_f = (\Omega^1)^0$ *iff* $p > 1$ *and all* j *are odd,*

- $\mathcal{CS}_f = 0$ *iff* $p > 1$ *and at least one of* j *is even.*

Remark 7-12. *We see that for* $p > 1$ *and all* j *odd (meaning that* $\lambda_j = 0$ *for* j *even),* L_f *is nonlinear injective (for cosymmetries). If one of the* j *is even, there are no solutions for* $L_f\omega = 0$ *in* Ω^1.

Proof. Notice that $\sum_{j=1}^p \lambda_j u_j \in \mathfrak{h}^0$. Let $\omega \in \Omega^1$ and $\omega = \sum \omega^i$, where $\omega^i \in (\Omega^1)^i$. By proposition 7-7, ω is a cosymmetry of this equation iff $\widehat{D_{\omega^i}[f]} + \widehat{D_f^\star[\omega^i]} = 0$, for any $i \geq 0$. So

$$(i+1) < \tilde{\omega^i}(\xi_1, \cdots, \xi_{i+1}) \sum_{j=1}^p \lambda_j \xi_{i+1}^j > + \sum_{j=1}^p \lambda_j \xi_0^j \tilde{\omega^i}(\xi_1, \cdots, \xi_{i+1}) = 0,$$

where $\sum_{j=0}^{i+1} \xi_j = 0$. This implies

$$\sum_{j=1}^p \lambda_j \sum_{k=0}^{i+1} \xi_k^j = 0.$$

Under the assumption, it holds iff either $p = 1$ or $p \neq 1$ and $i = 0$ when all j are odd. □

7.2 Divisibility of the $G_k^{(m)}$

Abstract 7-13. *In this section, we give the results about the mutual divisibility of the polynomials* $G_k^{(m)}$ *proved by F.* `Beukers`.

We notice that for cosymmetries or if k is odd for symmetries, $G_k^{(l)}$ is invariant under the natural action of the permutation group Σ_{l+2} on the coordinates $\xi_0, \cdots, \xi_{l+1}$.

Proposition 7-14. $G_k^{(m)}|_{\alpha=-1} = t_k^{(m)} g_k^{(m)}$, *where* $(g_k^{(m)}, g_l^{(m)}) = 1$ *for* $k < l$, *and* $t_k^{(m)}$ *is one of the following cases.*

- $m = 1$:
  - $k = 0 \pmod 2$: $\xi_1\xi_2$
  - $k = 3 \pmod 6$: $\xi_1\xi_2(\xi_1 + \xi_2)$
  - $k = 5 \pmod 6$: $\xi_1\xi_2(\xi_1 + \xi_2)(\xi_1^2 + \xi_1\xi_2 + \xi_2^2)$
  - $k = 1 \pmod 6$: $\xi_1\xi_2(\xi_1 + \xi_2)(\xi_1^2 + \xi_1\xi_2 + \xi_2^2)^2$.
- $m = 2$:
  - $k = 0 \pmod 2$: 1

  - $k = 1 \pmod 2$: $(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_2 + \xi_3)$.

- $m > 2$: 1.

.

Proof. This proposition is proved by F. `Beukers` using diophantine approximation theory ([Beu97] for $m = 1$ and Appendix A for $m > 1$). Despite the innocent look of the polynomials involved, we have not been able to find a simpler way of proving this case. It is conjectured that the g_k^m are $\mathbb{Q}[\xi]$-irreducible. $\square$

Chapter 8

Classification of the scalar polynomial evolution equations

We determine the existence of (infinitely many) symmetries for equations of the form

$$u_t = u_k + f(u, \cdots, u_{k-1}),$$

when their right hand sides are homogeneous with respect to the scaling symmetry $xu_1 + \lambda u \in \mathfrak{h}^0_{\mathfrak{k}}$ (so-called **λ-homogeneous equations**) with $\lambda \geq 0$.

Algorithms are given to determine whether a system has a symmetry in $\mathfrak{h}$. If it has one nontrivial symmetry, we prove it has infinitely many and these can be found using recursion operators or master symmetries. The method of proof uses the symbolic method and results from diophantine approximation theory. We list the 10 integrable hierarchies for $\lambda > 0$. The methods can be applied to the $\lambda \leq 0$ case, as demonstrated in section 8.5.

In principle they can also be used for systems of evolution equations, evolution system but so far this has only been demonstrated for one class of examples, cf. [BSW98].

8.1 Introduction

The existence of symmetries and conservation laws of scalar evolution equations is well understood, in the sense that today it would be difficult to find anything new. That this understanding is complete remains to be proven. The main questions in this respect are the following.

- Can we decide, given an equation, whether there exists a nontrivial symmetry (i.e., not u_1 or the equation itself) (the recognition problem)?
- And if so, can we answer the question whether this leads to infinitely many symmetries (the symmetry–integrability problem)?

- Given a class of equations with arbitrary parameters, possibly functions of given type, can we completely classify this class with respect to the existence of symmetries (the classification problem)?

In the literature there exist lists of integrable systems, i.e., systems which came through certain integrability tests (cf. [MSS91]). The main problem with these lists is that there is no recipe to check whether a given system is equivalent to a system in the list, and there is no proof that the lists are complete. This leads to interesting discussions. In this chapter we will not solve this problem, but we answer the questions with yes for a class of λ-homogeneous equations and mention that our methods can be extended to cover much in the lists. We should stress at the beginning that this classification only allows linear transformations (in the scalar case that means multiplication of u with a constant only), and so there might be relations among the hierarchies (by graded Lie–Bäcklund transformations), as there exist between `KdV` and Potential `KdV`.

This chapter was motivated by the observation that after quickly finding a number of hierarchies (`mKdV`, `Sawada–Kotera`, `Kaup–Kupershmidt`) soon after `KdV`, nothing more was found for polynomial scalar evolution equations which are linear in the highest order derivative.

We show that integrability of an equation of the form

$$u_t = u_k + f(u, \cdots, u_{k-1})$$

(with f a formal power series starting with at least quadratic terms) is determined by

- the existence of one nontrivial symmetry,
- the existence of approximate (co)symmetries.

Remark 8-1. *We have derived the formalism not only for symmetries but also for cosymmetries. We apply the formalism in this chapter only to the classification of the symmetries (symmetry–integrability). The classification of the cosymmetries is more complicated, since one also encounters equations (usually without a nontrivial symmetry) with only a finite number of cosymmetries. Some results for* **KdV–like equation***s ($\lambda = 2$) are given in [SW98].*

To this end we have formulated theorem 2-76 in the context of a filtered complex. We proved in an abstract setting the remark made in [Fok80]

> Another interesting fact regarding the symmetry structure of evolution equations is that in all known cases the existence of one generalized symmetry implies the existence of infinitely many. (However, this has not been proved in general.)

under fairly relaxed conditions. The result also confirms the remark made in [GKZ91]

> It turns out from practice that if the first integrability conditions [...] are fulfilled, then often all the others are fulfilled as well.

We should also remark however that the conjecture

> the existence of one symmetry implies the existence of (infinitely many) others

has been disproved. Using an example given in [Bak91], we show this in [BSW98] using the same techniques as in the present chapter. This example, however, does not contradict the spirit of our theorem, since it depends on the nonexistence of certain quadratic terms, the existence of which is one of the conditions in theorem 2-76.

Some of the strange conditions in theorem 2-76 have been inspired by the symbolic method (cf. chapter 7). With this method one can readily translate solvability questions into divisibility questions and we can use generating functions to handle infinitely many orders at once. While this does not mean that the questions are much easier to answer, we do now have the whole machinery which has been developed in number theory available (cf. section 7.2), and this makes a crucial difference.

In this chapter, we show that a nontrivial symmetry of a λ-homogeneous equation is part of a hierarchy starting at order $3, 5$ or 7 in the odd case, and at order 2 in the even case. When $\lambda > 0$, we can further reduce 7^{th}-order equations to 5^{th}-order. This result explains why despite systematic searches using computer algebra nothing new was ever found beyond 5^{th}-order (cf. [GKZ91]). Moreover, it enables us to completely analyze λ-homogeneous equations for positive λ. For $\lambda = 1$, this describes the family of **Burgers–like equation**s, for $\lambda = 2$ the family of KdV–like equations.

Finally we apply the method to the $\lambda \leq 0$ case.

8.2 Symmetries of λ-homogeneous equations

Abstract 8-2. *In this section we show that it suffices to compute the linear and quadratic or cubic terms of a symmetry to guarantee its existence, if one has a λ-homogeneous equation with a nontrivial symmetry. Moreover, if the order of the symmetry is > 7, we show that there exists a nontrivial symmetry of order ≤ 7. We can then replace the equation by its symmetry. This makes it possible to solve the complete classification problem.*

We consider n^{th}-order equations of the form

$$u_t = \sum_{i \geq 0} K^i_{n-\lambda i}, \quad (K^i_{n-\lambda i} \in \mathcal{U}^i_{n-\lambda i}), \tag{8.2.1}$$

where $n \geq 2$, $K^0_n = u_n$ and $\lambda \in \mathbb{Q}$ strictly positive.

In the notation of section 2.9 one has

$$K^0 = K^0_n, \quad (K^0_n \in \mathfrak{h}^0_n),$$

and

$$K^1 = \sum_{i>0} K^i_{n-\lambda i}, \quad (K^i_{n-\lambda i} \in \mathfrak{h}^i_{n-\lambda i}).$$

If $S \in \mathfrak{h}$ is an order m symmetry of (8.2.1), by proposition 7-7 the following formula holds for all $r \in \mathbb{N}$:

$$\sum_{i+j=r} \widehat{[S^j_{m-\lambda j}, K^i_{n-\lambda i}]} = 0. \tag{8.2.2}$$

The lowest upper index of S has to be zero, otherwise this equation cannot be solved, i.e., $S^0_m \neq 0$. Clearly we have $[S^0_m, K^0_n] = 0$. The next equation to be solved is

$$\widehat{[S^0_m, K^1_{n-\lambda}]} + \widehat{[S^1_{m-\lambda}, K^0_n]} = 0. \tag{8.2.3}$$

Remark 8-3. *If $K^i = 0$ for $i = 1, \cdots, j-1$, from (8.2.3) we have*

$$\tilde{S}^j_{m-j\lambda} = \frac{\tilde{K}^j_{n-j\lambda} G^{(j)}_m}{G^{(j)}_n}. \tag{8.2.4}$$

This equation can not be solved when $j \geq 3$, or when $j = 2$ and n is even since $G^{(j)}_m$ and $G^{(j)}_n$ have no common factors, and the degree of $K^j_{n-j\lambda}$ is $n - j\lambda < n$, which is the degree of $G^{(j)}_n$. This implies that there are no symmetries for such equations. When $j = 2$ and n is odd, it can only have odd order symmetries. In this case one can remark that if the equation can be solved for any m, it can also be solved for $m = 3$.

Note that if λ is not an integer, this leads automatically to $\tilde{K}^i_{n-i\lambda} = 0$ if $i\lambda \notin \mathbb{N}$. This restricts the number of relevant λ to a finite set.

We rewrite (8.2.4) as

$$\tilde{K}^1_{n-\lambda} = \frac{\tilde{S}^1_{m-\lambda}}{G^{(1)}_m} G^{(1)}_n. \tag{8.2.5}$$

It follows from the results of section 7.2 that this can be written in the form

$$\tilde{K}^1_{n-\lambda} = \frac{f(\xi_1, \xi_2)}{\xi_1 \xi_2 (\xi_1 + \xi_2)} G^{(1)}_n, \tag{8.2.6}$$

where f is Σ_2-symmetric and $\lim_{\xi_1+\xi_2 \to 0} f(\xi_1, \xi_2)$ exists. By the same reason,

$$\tilde{S}^1_{m-\lambda} = \frac{f(\xi_1, \xi_2)}{\xi_1 \xi_2 (\xi_1 + \xi_2)} G^{(1)}_m. \tag{8.2.7}$$

We now go to the next order

$$\sum_{i+j=2} \widehat{[S^j_{m-\lambda j}, K^i_{n-\lambda i}]} = 0, \tag{8.2.8}$$

or

$$\tilde{S}^2_{m-2\lambda} = \frac{\tilde{K}^2_{n-2\lambda} G^{(2)}_m + \widetilde{[S^1_{m-\lambda}, K^1_{n-\lambda}]}}{G^{(2)}_n}. \tag{8.2.9}$$

Definition 8-4. *We say that $f \in \mathcal{I}_1$ iff $\xi_1+\xi_2|f(\xi_1,\xi_2)$ and that $f \in \mathcal{I}_2$ iff $\xi_1|f(\xi_1,\xi_2)$.*

Proposition 8-5. *Suppose m and n are odd. Then $(\xi_1+\xi_2)(\xi_2+\xi_3)(\xi_1+\xi_3)$ divides $[\widetilde{S^1_{m-\lambda}, K^1_{n-\lambda}}]$ iff $f \in \mathcal{I}_1 \bigcup \mathcal{I}_2$.*

Proof. We compute

$$
\begin{aligned}
&[\widetilde{S^1_{m-\lambda}, K^1_{n-\lambda}}] = \\
&= 2 < \frac{f(\xi_1,\xi_2+\xi_3)f(\xi_2,\xi_3)}{\xi_1(\xi_2+\xi_3)^2(\xi_1+\xi_2+\xi_3)} G_m^{(1)}(\xi_1,\xi_2+\xi_3)G_n^{(1)}(\xi_2,\xi_3) > \\
&- 2 < \frac{f(\xi_1,\xi_2+\xi_3)f(\xi_2,\xi_3)}{\xi_1(\xi_2+\xi_3)^2(\xi_1+\xi_2+\xi_3)} G_n^{(1)}(\xi_1,\xi_2+\xi_3)G_m^{(1)}(\xi_2,\xi_3) > \\
&= \frac{2}{3}\frac{1}{\xi_1+\xi_2+\xi_3}\left[\frac{f(\xi_1,\xi_2+\xi_3)f(\xi_2,\xi_3)}{\xi_1(\xi_2+\xi_3)^2}\right. \\
&\quad (G_m^{(1)}(\xi_1,\xi_2+\xi_3)G_n^{(1)}(\xi_2,\xi_3) - G_n^{(1)}(\xi_1,\xi_2+\xi_3)G_m^{(1)}(\xi_2,\xi_3)) \\
&+ \frac{f(\xi_2,\xi_1+\xi_3)f(\xi_1,\xi_3)}{\xi_2(\xi_1+\xi_3)^2} \\
&\quad (G_m^{(1)}(\xi_2,\xi_1+\xi_3)G_n^{(1)}(\xi_1,\xi_3) - G_n^{(1)}(\xi_2,\xi_1+\xi_3)G_m^{(1)}(\xi_1,\xi_3)) \\
&+ \frac{f(\xi_3,\xi_1+\xi_2)f(\xi_1,\xi_2)}{\xi_3(\xi_1+\xi_2)^2} \\
&\quad \left.(G_m^{(1)}(\xi_3,\xi_1+\xi_2)G_n^{(1)}(\xi_1,\xi_2) - G_n^{(1)}(\xi_3,\xi_1+\xi_2)G_m^{(1)}(\xi_1,\xi_2))\right].
\end{aligned}
$$

We now prove that $\lim_{\xi_1+\xi_2\to 0}$ of this expression is zero. First we have

$$
\begin{aligned}
&\lim_{\xi_1+\xi_2\to 0}\left(G_m^{(1)}(\xi_1,\xi_2+\xi_3)G_n^{(1)}(\xi_2,\xi_3) - G_n^{(1)}(\xi_1,\xi_2+\xi_3)G_m^{(1)}(\xi_2,\xi_3)\right) \\
&= G_m^{(1)}(-\xi_2,\xi_2+\xi_3)G_n^{(1)}(\xi_2,\xi_3) - G_n^{(1)}(-\xi_2,\xi_2+\xi_3)G_m^{(1)}(\xi_2,\xi_3) \\
&= -G_m^{(1)}(\xi_2,\xi_3)G_n^{(1)}(\xi_2,\xi_3) + G_n^{(1)}(\xi_2,\xi_3)G_m^{(1)}(\xi_2,\xi_3) = 0.
\end{aligned}
$$

So the only interesting situation is the one with $\frac{1}{(\xi_1+\xi_2)^2}$. If we let

$$F_{\xi_2,\xi_3}(x) = G_m^{(1)}(\xi_3,x)G_n^{(1)}(x-\xi_2,\xi_2) - G_n^{(1)}(\xi_3,x)G_m^{(1)}(x-\xi_2,\xi_2),$$

then we see that $F_{\xi_2,\xi_3}(0) = 0$ and

$$
\begin{aligned}
&\frac{d}{dx}F_{\xi_2,\xi_3}(0) \\
&= G_m^{(1)}(\xi_3,0)\frac{d}{dx}G_n^{(1)}(x-\xi_2,\xi_2)|_{x=0} + G_n^{(1)}(-\xi_2,\xi_2)\frac{d}{dx}G_m^{(1)}(\xi_3,x)|_{x=0} \\
&- G_n^{(1)}(\xi_3,0)\frac{d}{dx}G_m^{(1)}(x-\xi_2,\xi_2)|_{x=0} - G_m^{(1)}(-\xi_2,\xi_2)\frac{d}{dx}G_n^{(1)}(\xi_3,x)|_{x=0} = 0.
\end{aligned}
$$

Moreover

$$\frac{d^2}{dx^2}F_{\xi_2,\xi_3}(0)$$

$$
\begin{aligned}
&= 2\frac{d}{dx}G_m^{(1)}(\xi_3,x)|_{x=0}\frac{d}{dx}G_n^{(1)}(x-\xi_2,\xi_2)|_{x=0}\\
&- 2\frac{d}{dx}G_m^{(1)}(x-\xi_2,\xi_2)|_{x=0}\frac{d}{dx}G_n^{(1)}(\xi_3,x)|_{x=0}\\
&= 2nm\left((-1)^n\xi_3^{m-1}\xi_2^{n-1}-(-1)^m\xi_3^{n-1}\xi_2^{m-1}\right)\neq 0.
\end{aligned}
$$

This implies that

$$
\lim_{\xi_1+\xi_2\to 0}\frac{1}{(\xi_1+\xi_2)^2}(G_m^{(1)}(\xi_3,\xi_1+\xi_2)G_n^{(1)}(\xi_1,\xi_2)-G_n^{(1)}(\xi_3,\xi_1+\xi_2)G_m^{(1)}(\xi_1,\xi_2))\neq 0
$$

and therefore $(\xi_1+\xi_2)\not|\widetilde{[S^1_{m-\lambda},K^1_{n-\lambda}]}$ unless $\xi_1+\xi_2|f(\xi_3,\xi_1+\xi_2)f(\xi_1,\xi_2)$, or, equivalently, $\xi_1+\xi_2|f(\xi_1,\xi_2)$ or $\xi_1|f(\xi_1,\xi_2)$.

By the symmetric property this statement follows. □

Corollary 8-6. *Assume m and n are odd. Then $(\xi_1+\xi_2)(\xi_2+\xi_3)(\xi_1+\xi_3)$ divides*

$$
\tilde{K}^2_{n-2\lambda}G_m^{(2)}+\widetilde{[S^1_{m-\lambda},K^1_{n-\lambda}]} \tag{8.2.10}
$$

iff $\xi_1+\xi_2|\tilde{K}^1_{n-\lambda}(\xi_1,\xi_2)$ or $\xi_1|\tilde{K}^1_{n-\lambda}(\xi_1,\xi_2)$.

Theorem 8-7. *If $S=\sum_{i\geq 0}S^i_{m-\lambda i}$ is an order m symmetry of equation (8.2.1) and $Q^1_{q-\lambda}$ exists, with $q\neq m,n$, $q\geq\lambda$ and q is odd if n is odd, such that $[K^0_n,Q^1_{q-\lambda}]+[K^1_{n-\lambda},Q^0_q]=0$, then there exists a unique $Q=\sum_{i\geq 0}Q^i_{q-\lambda i}$ such that Q is a symmetry commuting with S.*

Proof. For even n or m, this follows from theorem 2-76, since S^0_m is relatively 2-prime w.r.t. K^0_n.

We conclude from the existence of S that $(\xi_1+\xi_2)(\xi_2+\xi_3)(\xi_1+\xi_3)$ divides

$$
\tilde{K}^2_{n-2\lambda}G_m^{(2)}+\widetilde{[S^1_{m-\lambda},K^1_{n-\lambda}]}
$$

for odd n and m. In other words, $(\xi_1+\xi_2)|\tilde{K}^1_{n-\lambda}(\xi_1,\xi_2)$ or $\xi_1|\tilde{K}^1_{n-\lambda}(\xi_1,\xi_2)$.

We know from the proof of theorem 2-76 that

$$
g_n^{(2)}(\widetilde{[S^1,Q^1]}+\widetilde{[S^2,Q^0]})=g_m^{(2)}(\widetilde{[K^1,Q^1]}+\widetilde{[K^2,Q^0]}).
$$

Since $(g_m^{(2)},g_n^{(2)})=1$, and (by exactly the same argument as for S)

$$
(\xi_1+\xi_2)(\xi_2+\xi_3)(\xi_1+\xi_3)|(\widetilde{[K^1,Q^1]}+\widetilde{[K^2,Q^0]}),
$$

we may conclude that $G_n^{(2)}$ divides $\widetilde{[K^1,Q^1]}+\widetilde{[K^2,Q^0]}$, or

$$
\tilde{Q}^2_{q-2\lambda}=\frac{\widetilde{[Q^1,K^1]}+\widetilde{[Q^0,K^2]}}{G_n^{(2)}}
$$

is well defined. Since the $G_n^{(k)}$ are relative prime for $k>2$, this means that K^0_m is relatively 2-prime and we can apply theorem 2-76 to draw the conclusion that there indeed exists a symmetry Q commuting with S. □

Notation 8-8. *Let* $c_2 = \xi_1^2 + \xi_1\xi_2 + \xi_2^2$.

Now we can make a very interesting observation. Consider a given λ-homogeneous equation with odd n

$$u_t = \sum_{i\geq 0} K^i_{n-\lambda i}, \quad (K^i_{n-\lambda i} \in \mathcal{U}^i_{n-\lambda i}).$$

Then we pose the problem of finding all its symmetries. Suppose we have found a nontrivial symmetry with quadratic terms given by equation (8.2.5):

$$\tilde{Q}^1_{q-\lambda} = \frac{\tilde{K}^1_{n-\lambda} c_2^{s-s'} g^1_q}{g^1_n},$$

where $s' = \frac{n+3}{2} \pmod 3$ and $s = \frac{q+3}{2} \pmod 3$. This formula implies $\lambda \leq 3 + 2\min(s, s')$. Then $\tilde{Q}^1_{2s+3-\lambda}$, defined by

$$\tilde{Q}^1_{2s+3-\lambda} = \frac{\tilde{K}^1_{n-\lambda} c_2^{s-s'} g^1_{2s+3}}{g^1_n},$$

gives rise to a symmetry $Q = Q^0_{2s+3} + Q^1_{2s+3-\lambda} + \cdots$ of the original equation, using theorem 8-7 (Of course, this argument generates a whole hierarchy). This implies that Q and K have the same symmetries, so instead of considering K we may consider the equation given by Q, which is of order $2\hat{s}+3$ for $\hat{s} = 0, 1, 2$. It follows that we only need to find the symmetries of λ-homogeneous equations (with $\lambda \leq 7$) of order ≤ 7 in order to get the complete classification of symmetries of λ-homogeneous scalar polynomial equations starting with linear terms.

A similar observation can be made for even $n > 2$: Suppose we have found a nontrivial symmetry with quadratic terms given by equation (8.2.5):

$$\tilde{Q}^1_{q-\lambda} = \frac{\tilde{K}^1_{n-\lambda}}{g^1_n} \frac{G^{(1)}_q}{\xi_1\xi_2}.$$

This immediately implies $\lambda \leq 2$. Then $\tilde{Q}^1_{2-\lambda}$, defined by

$$\tilde{Q}^1_{2-\lambda} = 2\frac{\tilde{K}^1_{n-\lambda}}{g^1_n},$$

gives rise to a symmetry $Q = Q^0_2 + Q^1_{2-\lambda} + \cdots$ of the original equation, using theorem 8-7 (this argument generates a whole hierarchy with symmetries of every order). So Q and K have the same symmetries. Instead of considering K, if its order is > 2, we may consider the equation given by Q, which is of order 2. It follows that we only need to find the symmetries of equations of order 2, in order to get the complete classification of symmetries of λ-homogeneous scalar polynomial equations (with $\lambda \leq 2$) starting with an even linear term. We have proved the following

Theorem 8-9. *A nontrivial symmetry of a λ-homogeneous equation is part of a hierarchy starting at order $3, 5$ or 7 in the odd case, and at order 2 in the even case.*

8.3 Reduction of 7^{th}-order λ-homogeneous equations

Abstract 8-10. *We conclude from a rather extensive computer algebra computation that 7^{th}-order equations must have 5^{th}-order symmetries if they possess symmetries of order* $1 \pmod 6$.

Suppose one can show that for a given 7^{th}-order equation to have a symmetry implies that the quadratic terms of the equation in symbolic form divides through $\xi_1^2+\xi_1\xi_2+\xi_2^2$, then $\frac{\tilde{K}^1}{\xi_1^2+\xi_1\xi_2+\xi_2^2}$ is the quadratic part of a 5^{th}-order symmetry. Therefore we can in that case replace the 7^{th}-order equation by a 5^{th}-order symmetry.

To this end we have analyzed all order $6m+1$ symmetries ($m = 2, 3, \cdots$) of all 7^{th}-order λ-homogeneous equations for $\lambda = 1, \cdots, 7$.

We have done this using generating functions of the form

$$G_\infty^{(1)}(\tau) = \sum_{m=0}^{\infty} S^1_{m-\lambda} \tau^{6m+1}.$$

Using the relation (8.2.4) we obtain

$$\begin{aligned} G_\infty^{(1)}(\tau) &= \sum_{m=0}^{\infty} \frac{\tilde{K}^1_{n-\lambda} G_m^{(1)}}{G_7^{(1)}} \tau^{6m+1} = \frac{\tilde{K}^1_{n-\lambda}}{G_7^{(1)}} \sum_{m=0}^{\infty} G_m^{(1)} \tau^{6m+1} \\ &= \frac{\tilde{K}^1_{n-\lambda}}{7} \frac{\tau^6\left({\xi_1}^6+3{\xi_1}^5\xi_2+{\xi_1}^4{\xi_2}^2-3{\xi_1}^3{\xi_2}^3+{\xi_1}^2{\xi_2}^4+3\xi_1{\xi_2}^5+{\xi_2}^6\right)-7}{\left((\xi_1+\xi_2)^6\tau^6-1\right)\left({\xi_1}^6\tau^6-1\right)\left({\xi_2}^6\tau^6-1\right)}. \end{aligned}$$

As predicted by the theory in the preceding section, this does not directly lead to any conditions. So we have to go to the next term as in (8.2.9).

Using Maple ([CGG+91]) and Form ([Ver91]) we compute for each λ the Lie bracket of $G_\infty^{(1)}$ with the quadratic part of the equation $\tilde{K}^1_{7-\lambda}$ plus the product of $G_\infty^{(2)}$ and the cubic terms of the equation $\tilde{K}^2_{7-2\lambda}$. We check under what conditions $G_7^{(2)}$ divides the result. We then find for each λ that the quadratic terms can always be divided by $\xi_1^2+\xi_1\xi_2+\xi_2^2$ under these conditions.

This reduces the problem of 7^{th}-order equations to 5^{th}-order equations.

8.4 The list of integrable systems for $\lambda > 0$

Abstract 8-11. *We give the complete list of symmetry–integrable systems for λ-homogeneous equations with $\lambda > 0$. From the list we have removed the equations belonging to a hierarchy starting at a lower order. The infinitely many symmetries for these equations are generated by the recursion operators or master symmetries. References are given to the pertinent sections in chapter 9, where these equations are treated in more detail.*

8.4.1 Symmetries

We only have to look for 7^{th}-order symmetries, since any symmetry of order 3 or 5 (mod 6) automatically gives rise to a symmetry of order $1 \pmod 6$.

We assume that the equations have nonzero quadratic terms, since otherwise the analysis reduces to 3^{rd}-order equations.

As we have seen, the first order calculation does not lead to any obstructions, so one has to go to second order at least. Since a symmetry needs to be found, one can not stop at a certain order, even if the equation is totally determined. The problem has to be completely and explicitly solved. Although straightforward in principle, the calculation is quite large. Again using Maple and Form we have produced a complete list of λ-homogeneous ($\lambda > 0$) 5^{th}-order equations (with quadratic terms not equal to zero) with 7^{th}-order symmetry. From this list we have removed the equations belonging to a hierarchy starting at a lower order. A similar list appeared in [Bil93], but that list is not complete, e.g., the $\lambda = \frac{1}{2}$ case is missing.

8.4.1.1 $\lambda = 1$

Kupershmidt equation (cf. section 9.17)

$$f_1 = u_5 + 5u_1u_3 + 5u_2^2 - 5u^2u_3 - 20\,uu_1u_2 - 5u_1^3 + 5u^4u_1.$$

Potential Sawada–Kotera equation (cf. section 9.19)

$$f_2 = u_5 + u_1u_3 + \frac{1}{15}u_1^3.$$

Potential Kaup–Kupershmidt equation (cf. section 9.21)

$$f_3 = u_5 + 10u_1u_3 + \frac{15}{2}u_2^2 + \frac{20}{3}u_1^3.$$

8.4.1.2 $\lambda = 2$

Kaup–Kupershmidt equation (cf. section 9.20)

$$f_4 = u_5 + 10uu_3 + 25u_1u_2 + 20u^2u_1.$$

Sawada–Kotera equation (cf. section 9.18)

$$f_5 = u_5 + 5uu_3 + 5u_1u_2 + 5u^2u_1.$$

8.4.2 Symmetries

Using the same methods as before we find the following list of systems with a symmetry.

8.4.2.1 $\lambda = \frac{1}{2}$

Ibragimov–Shabat equation [IS81] (cf. section 9.10)

$$f_6 = u_3 + 3u^2u_2 + 9uu_1^2 + 3u^4u_1.$$

8.4.2.2 $\lambda = 1$

Potential Korteweg–de Vries equation (cf. section 9.6)

$$f_7 = u_3 + u_1^2.$$

Modified Korteweg–de Vries equation (cf. section 9.7)

$$f_8 = u_3 + u^2u_1.$$

8.4.2.3 $\lambda = 2$

KdV equation (cf. section 9.5)

$$f_9 = u_3 + uu_1.$$

8.4.3 Symmetries

The only 2^{nd}-order equation with a symmetry and $\lambda > 0$ is

8.4.3.1 $\lambda = 1$

Burgers' equation (cf. section 9.1)

$$f_{10} = u_2 + uu_1.$$

8.5 The integrable systems for $\lambda \leq 0$

Abstract 8-12. *We use the same method to solve the classification problem when $\lambda \leq 0$. We give the results for $\lambda = 0$ and from which we derive the correspondent results for $\lambda = -1$. The results given here describe research in progress. We remark that due to the infinite dimensional character of the search spaces, the computational details are much more involved than was the case for $\lambda > 0$. For $\lambda = 0$ we obtain ordinary differential equations as our obstruction equations, for $\lambda < 0$ we obtain partial differential equations.*

For $\lambda \leq 0$ the space of monomials of a fixed degree and λ-grading is no longer finite dimensional. But if we restrict the number of derivatives, as is natural from the definition of a system of given order, then the space is finitely generated. This can be seen as follows. We consider, for given n, monomials of the type

$$u^{k_0}u_1^{k_1}\cdots u_{n-1}^{k_{n-1}}.$$

Since the grading of u_n is $\lambda + n$, the monomial will have the same grading as u_n iff

$$\lambda + n = \sum_{j=0}^{n-1} (\lambda + j) k_j.$$

Since $\lambda \in \frac{1}{2}\mathbb{Z}$, this is a diophantine equation of the type considered by `Gordan`. This implies there is a Hilbert basis consisting of monomials of λ-grading 0, $< r_1, \cdots, r_{d_0} >$, and of λ-grading $\lambda + n$, $< m_1, \cdots, m_{d_{\lambda+n}} >$. This Hilbert basis can be computed for given $\lambda \in \frac{1}{2}\mathbb{Z}$ using the software described in [Pas95]. We write an arbitrary λ-homogeneous equation as

$$u_t = u_n + \sum_{j=1}^{d_{\lambda+n}} f_j(r_1, \cdots, r_{d_0}) m_j.$$

Observe that there may be relations of the type:

$$g_i(r_1, \cdots, r_{d_0}) m_i = g_j(r_1, \cdots, r_{d_0}) m_j.$$

Following the algorithm described in section 2.9, the f_j are formal power series. We can consider them as C^∞-functions with the same arguments $r_1, \cdots, r_{d_0}$ according to [Poè76]. One can now compute the Lie bracket of two arbitrary equations and derive the PDEs (or ODEs when $\lambda = 0$) which have to be satisfied to let the Lie bracket vanish. The system of PDEs can be analyzed using the algorithm in [BLOP95] and [BLOP97] as implemented in the Maple package Diffalg. This leads to a system of PDEs which has to be solved explicitly in order to obtain integrable equations with their symmetries. Some solutions may fall outside the category of formal power series, but since they solve the relation defining the symmetry, this does not seem to matter. That is to say, it does not matter for the existence of the symmetry. What may go wrong is the use of the symmetry in theorem 2-76. This needs careful consideration of the topologies involved. But anyway the procedure gives us good candidates for symmetry–integrable systems and these can be analyzed using completely different methods if necessary.

8.5.1 The case of $\lambda = 0$

8.5.1.1 Symmetries of 2^{nd}- and 3^{th}-order equations

We consider the family

$$u_t = u_3 + 3f(u)u_1u_2 + g(u)u_1^3, \tag{8.5.1}$$

where f and g are arbitrary formal power series, and look for 5^{th}-order symmetries.

To find a symmetry to a given order or a given class of equations is not too difficult in principle. One writes down the Lie bracket equation, looks for some grading or filtering to organize the computation with, and then proceeds to calculate, solve the equations and find the obstructions to the solution of the equations. The result, given in [Maw98], is a list of obstruction equations that in this particular case is very short, namely

$$\frac{\partial g}{\partial u} = \frac{\partial^2 f}{\partial u^2} + 2fg - 2f^3.$$

We can manually simplify this by putting

$$g = \frac{\partial f}{\partial u} + f^2 + h$$

and we obtain the obstruction equation

$$\frac{\partial h}{\partial u} = 2fh.$$

Notice that the evolution equation can only be polynomial if $h = 0$. The equation (8.5.1) has a recursion operator

$$\mathfrak{R} = (D_x + fu_1 + 2u_1 D_x^{-1} hu_1)(D_x + fu_1). \tag{8.5.2}$$

Thanks to the existence of a recursion operator, convergence of the formal power series is inherited by the symmetries.

The recursion operator can be split (if $h \neq 0$) as $\mathfrak{R} = \mathfrak{H}\mathfrak{I}$, where

$$\mathfrak{H} = (hD_x + hfu_1)^{-1}$$

and

$$\mathfrak{I} = h(D_x + fu_1)(D_x + fu_1 + 2u_1 D_x^{-1} hu_1)(D_x + fu_1)$$

are the cosymplectic and the symplectic operator, respectively. The Hamiltonian function is given by

$$\frac{1}{2}h\left(u_2^2 - (\frac{2}{3}\frac{\partial f}{\partial u} + \frac{1}{3}f^2 + \frac{1}{2}h)u_1^4\right).$$

We see that we have an example of a family of Hamiltonian systems which is not Hamiltonian in one exceptional point $h = 0$. This exceptional case derives from the 2^{nd}-order equation (cf. section 9.2)

$$u_t = u_2 + f(u)u_1^2,$$

which is known to be integrable [Fok80] with a recursion operator $D_x + f(u)u_1$.

8.5.1.2 Symmetries of 5^{th}-order equations

We consider the following family

$$\begin{aligned} u_t &= u_5 + A_{21}u_1u_4 + A_{20}u_2u_3 + A_{31}u_1^2u_3 + A_{30}u_1u_2^2 \\ &\quad + A_{40}u_1^3u_2 + A_{50}u_1^5, \end{aligned} \tag{8.5.3}$$

where the parameters are arbitrary functions with respect to u. We look for the condition that the equation possesses a 7^{th}-order symmetry by the same way that we did for 3^{rd}-order equation before. The results are the following:

Case I

$$
\begin{aligned}
\frac{\partial A_{30}}{\partial u} &= \frac{3}{2}\frac{\partial^2 A_{20}}{\partial u^2} + \frac{1}{5}A_{30}A_{20} - \frac{3}{100}A_{20}^3,\\
A_{21} &= \frac{1}{2}A_{20},\\
A_{31} &= -\frac{1}{2}\frac{\partial A_{20}}{\partial u} + A_{30} - \frac{1}{20}A_{20}^2,\\
A_{40} &= -\frac{9}{20}A_{20}\frac{\partial A_{20}}{\partial u} + \frac{1}{2}A_{30}A_{20} - \frac{13}{200}A_{20}^3 + \frac{\partial^2 A_{20}}{\partial u^2},\\
A_{50} &= \frac{1}{10}\frac{\partial^3 A_{20}}{\partial u^3} + \frac{1}{25}A_{20}\frac{\partial^2 A_{20}}{\partial u^2} + \frac{3}{200}(\frac{\partial A_{20}}{\partial u})^2 - \frac{2}{25}A_{30}\frac{\partial A_{20}}{\partial u}\\
&\quad -\frac{3}{250}A_{20}^2\frac{\partial A_{20}}{\partial u} + \frac{3}{50}A_{30}^2 + \frac{1}{500}A_{30}A_{20}^2 - \frac{31}{20000}A_{20}^4.
\end{aligned}
$$

Notice that the quadratic terms are equal to $A_{20}(\frac{1}{2}u_1u_4+u_2u_3)$. Its correspondent symbolic expression is

$$
\frac{A_{20}}{4}(\xi_1\xi_2^4+2\xi_1^2\xi_2^3+2\xi_1^3\xi_2^2+\xi_1^4\xi_2) = \frac{A_{20}}{4}\xi_1\xi_2(\xi_1+\xi_2)(\xi_1^2+\xi_1\xi_2+\xi_2^2).
$$

Therefore, this equation has a 3^{rd}-order symmetry since $\xi_1^2+\xi_1\xi_2+\xi_2^2$ divides the quadratic terms. As it turns out, it is the image under the recursion operator (8.5.2) of the 3^{rd}-order family (8.5.1) we just found with the identifications $A_{20}=10f(u)$, $A_{30}=15g(u)$.

Case II

$$
\begin{aligned}
\frac{\partial A_{21}}{\partial u} &= \frac{1}{5}A_{21}^2 - \frac{1}{10}A_{21}A_{20} + \frac{1}{2}\frac{\partial A_{20}}{\partial u},\\
A_{31} &= -\frac{1}{5}A_{20}^2 + \frac{4}{5}A_{20}A_{21} - \frac{2}{5}A_{21}^2 + \frac{\partial A_{20}}{\partial u},\\
A_{30} &= -\frac{1}{5}A_{20}^2 + \frac{11}{10}A_{20}A_{21} - \frac{4}{5}A_{21}^2 + \frac{3}{2}\frac{\partial A_{20}}{\partial u},\\
A_{40} &= \frac{4}{5}A_{20}A_{21}^2 - \frac{18}{25}A_{21}^3 + \frac{3}{5}A_{21}\frac{\partial A_{20}}{\partial u} + \frac{\partial^2 A_{20}}{\partial u^2} - \frac{1}{5}A_{21}A_{20}^2,\\
A_{50} &= \frac{1}{10}\frac{\partial^3 A_{20}}{\partial u^3} + \frac{1}{10}A_{21}\frac{\partial^2 A_{20}}{\partial u^2} - \frac{1}{100}A_{20}\frac{\partial^2 A_{20}}{\partial u^2} + \frac{3}{100}(\frac{\partial A_{20}}{\partial u})^2\\
&\quad -\frac{1}{25}A_{21}^2\frac{\partial A_{20}}{\partial u} + \frac{7}{100}A_{20}A_{21}\frac{\partial A_{20}}{\partial u} - \frac{19}{1000}A_{20}^2\frac{\partial A_{20}}{\partial u}\\
&\quad +\frac{1}{625}A_{20}^4 - \frac{8}{125}A_{21}^4 + \frac{7}{125}A_{21}^3A_{20} - \frac{9}{1000}A_{21}A_{20}^3.
\end{aligned}
$$

Let us first consider the special case $A_{21}=\frac{1}{2}A_{20}$. This turns out to be the only case the equation can be polynomial. This leads to

$$
A_{31} = \frac{1}{10}A_{20}^2 + \frac{\partial A_{20}}{\partial u},
$$

$$
\begin{aligned}
A_{30} &= \frac{3}{20}A_{20}^2 + \frac{3}{2}\frac{\partial A_{20}}{\partial u},\\
A_{40} &= \frac{1}{100}A_{20}^3 + \frac{3}{10}A_{20}\frac{\partial A_{20}}{\partial u} + \frac{\partial^2 A_{20}}{\partial u^2},\\
A_{50} &= \frac{1}{10}\frac{\partial^3 A_{20}}{\partial u^3} + \frac{A_{20}}{25}\frac{\partial^2 A_{20}}{\partial u^2} + \frac{3}{100}(\frac{\partial A_{20}}{\partial u})^2 + \frac{3A_{20}^2}{500}\frac{\partial A_{20}}{\partial u} + \frac{A_{20}^4}{10000}.
\end{aligned}
$$

Notice that such a solution also satisfies the conditions of case I. Indeed, this subcase $A_{21} = \frac{1}{2}A_{20}$ can be identified with Case I, $h(u) = 0$, i.e., it derives from the 2^{nd}-order equation $u_t = u_2 + \frac{1}{10}A_{20}u_1^2$.

Otherwise, let $Z = A_{21} - \frac{1}{2}A_{20}$. Then [1]

$$
\begin{aligned}
\frac{\partial Z}{\partial u} &= \frac{1}{5}A_{21}Z,\\
A_{31} &= -\frac{4}{5}Z^2 - \frac{2}{5}A_{21}Z + \frac{2}{5}A_{21}^2 + 2\frac{\partial A_{21}}{\partial u},\\
A_{30} &= \frac{3}{5}A_{21}^2 - \frac{6}{5}A_{21}Z - \frac{4}{5}Z^2 + 3\frac{\partial A_{21}}{\partial u},\\
A_{40} &= \frac{2}{25}A_{21}^3 - \frac{8}{25}A_{21}^2 Z - \frac{4}{5}A_{21}Z^2 + \frac{6}{5}A_{21}\frac{\partial A_{21}}{\partial u}\\
&\quad +2\frac{\partial^2 A_{21}}{\partial u^2} - \frac{2}{5}Z\frac{\partial A_{21}}{\partial u},\\
A_{50} &= \frac{1}{625}A_{21}^4 - \frac{2}{125}A_{21}^3 Z - \frac{8}{125}A_{21}^2 Z^2 + \frac{16}{625}Z^4\\
&\quad -\frac{2}{25}A_{21}Z\frac{\partial A_{21}}{\partial u} - \frac{4}{25}\frac{\partial A_{21}}{\partial u}Z^2 + \frac{3}{25}(\frac{\partial A_{21}}{\partial u})^2\\
&\quad +\frac{6}{125}A_{21}^2\frac{\partial A_{21}}{\partial u} + \frac{4}{25}A_{21}\frac{\partial^2 A_{21}}{\partial u^2} + \frac{1}{5}\frac{\partial^3 A_{21}}{\partial u^3}.
\end{aligned}
$$

The evolution equation (8.5.3) in this case has a recursion operator

$$
\begin{aligned}
\mathfrak{R} &= (D_x + \frac{2}{5}Zu_1 + \frac{1}{5}A_{21}u_1 - \frac{8}{25}u_1 D_x^{-1}Z^2 u_1)\cdot\\
&\quad (D_x + \frac{2}{5}Zu_1 + \frac{1}{5}A_{21}u_1)\cdot\\
&\quad (D_x + \frac{1}{5}A_{21}u_1)\cdot\\
&\quad (D_x - \frac{2}{5}Zu_1 + \frac{1}{5}A_{21}u_1)\cdot\\
&\quad (D_x - \frac{2}{5}Zu_1 + \frac{1}{5}A_{21}u_1 - \frac{8}{25}u_1 D_x^{-1}Z^2 u_1)\cdot\\
&\quad (D_x + \frac{1}{5}A_{21}u_1).
\end{aligned}
\tag{8.5.4}
$$

The recursion operator can be split (if $Z \neq 0$) as $\mathfrak{R} = \mathfrak{H}\mathfrak{I}$, where

$$
\mathfrak{H} = (\frac{8}{25}Z^2 D_x + \frac{8}{125}Z^2 A_{21}u_1)^{-1}
$$

[1] The case $A_{21} = 0$ corresponds with the Potential Kupershmidt equation.

and $\mathfrak{I} = \mathfrak{H}^{-1}\mathfrak{R}$ are the cosymplectic and the symplectic operator, respectively. The Hamiltonian function is given by

$$
\begin{aligned}
&-\frac{4}{234375} Z^2 \cdot \big(9375 u_3^2 + \{6250Z - 5625A_{21}\}u_2^3 \\
&+\{3750A_{21}Z + 7500Z^2 - 1125A_{21}^2 - 16875\frac{\partial A_{21}}{\partial u}\}u_1^2u_2^2 \\
&+\{80Z^4 + 525(\frac{\partial A_{21}}{\partial u})^2 - 40ZA_{21}^3 + 750\frac{\partial^3 A_{21}}{\partial u^3} + 3A_{21}^4 - 600Z^2\frac{\partial A_{21}}{\partial u} \\
&-180Z^2A_{21}^2 - 300ZA_{21}\frac{\partial A_{21}}{\partial u} + 120A_{21}^2\frac{\partial A_{21}}{\partial u} + 450A_{21}\frac{\partial^2 A_{21}}{\partial u^2}\}u_1^6\Big)
\end{aligned}
$$

Example 8-13. *If Z is a formal power series, then all coefficients can be determined as formal power series. Let us take $A_{21} = 5$. Then $Z = \alpha e^u$, where α is constant. This leads to, if $\alpha = 1$,*

$$
\begin{aligned}
u_t &= u_5 + 5u_1u_4 + 2(5 - e^u)u_2u_3 + (10 - 2e^u - \frac{4}{5}e^{2u})u_1^2u_3 \\
&+ (15 - 6e^u - \frac{4}{5}e^{2u})u_1u_2^2 + (10 - 8e^u - 4e^{2u})u_1^3u_2 \\
&+ (1 - 2e^u - \frac{8}{5}e^{2u} + \frac{16}{625}e^{4u})u_1^5.
\end{aligned}
$$

8.5.2 Some consequences for $\lambda = -1$

In this subsection, we derive the symmetry–integrable equations for $\lambda = -1$ from the results we found for $\lambda = 0$. The method is to put $u = v_1$ and derive the equation for v, cf. propositions 2-25 and 4-8.

8.5.2.1 Symmetries of 3^{th}-order equations

Putting $u = v_1$ and derive the equation for v from the equation (8.5.1). This leads to

$$
v_t = D_x^{-1}(v_4 + 3f(v_1)v_2v_3 + g(v_1)v_2^3) = v_3 + \frac{3}{2}f(v_1)v_2^2 + D_x^{-1}((g(v_1) - \frac{3}{2}\frac{\partial f}{\partial v_1})v_2^3).
$$

We can make this equation local by requiring $g = \frac{3}{2}\frac{\partial f}{\partial v_1}$. In that case we recover the condition

$$
\frac{\partial^2 f}{\partial {v_1}^2} = 6f\frac{\partial f}{\partial v_1} - 4f^3
$$

for the family

$$
v_t = v_3 + \frac{3}{2}f(v_1)v_2^2,
$$

which gives us an integrable 3^{rd}-order family for $\lambda = -1$, that is, the existence of a 5^{th}-order symmetry for the class of equations

$$
u_t = u_3 + f(u_1)u_2^2,
$$

iff

$$9\frac{\partial^2 f}{\partial {u_1}^2} - 36f\frac{\partial f}{\partial u_1} + 16f^3 = 0.$$

Special cases are (numbers in [] refer to [MSS91]) $f_0 = -\frac{3}{2u_1}$, ([4.1.16]) the Krichever–Novikov equation (cf. section 9.12), $f_\infty = -\frac{3}{4u_1}$ ([4.1.19]) (cf. section 9.10) and $f_i = -\frac{3u_1}{2(1+u_1^2)}$ ([4.1.14]).

This equation can easily be reduced to a first order equation (cf. [Kam43], 6.43, or [Ibr96a], p. 204 for a Lie symmetry approach)

$$(\left(\frac{1}{f}\right)' + \frac{4}{3})^2 + \frac{9}{16}(\left(\frac{1}{f}\right)' + \frac{2}{3})c_1^2 = 0.$$

The general solution ($c_1 \neq 0$) is given by

$$f_{c_1}(u_1) = \frac{3c_1}{2}\frac{(c_1u_1 + c_2)}{1 - (c_1u_1 + c_2)^2}$$

and we see that it reduces to ([4.1.14]) upon scaling if we allow for imaginary scaling factors. If not, then we should also allow

$$f_1(u_1) = \frac{3}{2}\frac{u_1}{1 - u_1^2}.$$

One notices that the number of solutions f of the condition is very low, because the equation has internal symmetry and that this is not the general family.

It will be interesting to see whether the fact that the differential obstructions can be solved by Lie symmetry methods holds in general.

8.5.2.2 Symmetries of 5^{th}-order equations

Similarly, we put $u = v_1$ and derive the 5^{th}-order equation for v. This leads to

$$v_t = v_5 + \tilde{A}_{21}v_2v_4 + \tilde{A}_{20}v_3^2 + \tilde{A}_{30}v_2^2v_3 + \tilde{A}_{40}v_2^4,$$

where the parameters are formal power series functions with respect to v_1 with the following relations

$$\begin{aligned}
\tilde{A}_{21} &= A_{21},\\
\tilde{A}_{20} &= \frac{1}{2}(A_{20} - A_{21}),\\
\tilde{A}_{30} &= A_{31} - \frac{\partial A_{21}}{\partial v_1},\\
\tilde{A}_{40} &= \frac{1}{4}(A_{40} - \frac{\partial A_{31}}{\partial v_1} + \frac{\partial^2 A_{21}}{\partial {v_1}^2}),\\
A_{30} &= \frac{1}{2}(\frac{\partial A_{20}}{\partial u} - 5\frac{\partial A_{21}}{\partial u} + 4A_{31}),\\
A_{50} &= \frac{1}{4}(\frac{\partial A_{40}}{\partial u} - \frac{\partial^2 A_{31}}{\partial u^2} + \frac{\partial^3 A_{21}}{\partial u^3}).
\end{aligned}$$

Using the Maple package *Diffalg*, we get the three cases corresponding to those when $\lambda = 0$ as follows:

Case I

$$\begin{aligned}
\tilde{A}_{21} &= 2\tilde{A}_{20},\\
\frac{\partial^2 \tilde{A}_{20}}{\partial {v_1}^2} &= -\frac{16}{25}\tilde{A}_{20}^3 + \frac{12}{5}\tilde{A}_{20}\frac{\partial \tilde{A}_{20}}{\partial v_1},\\
\tilde{A}_{30} &= 3\frac{\partial \tilde{A}_{20}}{\partial v_1} + \frac{4}{5}\tilde{A}_{20}^2,\\
\tilde{A}_{40} &= \frac{19}{10}\tilde{A}_{20}\frac{\partial \tilde{A}_{20}}{\partial v_1} - \frac{2}{5}\tilde{A}_{20}^3.
\end{aligned}$$

Case IIa

$$\tilde{A}_{20} = 0,\ \frac{\partial \tilde{A}_{21}}{\partial v_1} = \frac{1}{5}\tilde{A}_{21}^2,\ \tilde{A}_{30} = \frac{1}{5}\tilde{A}_{21}^2,\ \tilde{A}_{40} = 0.$$

Case IIb

$$\begin{aligned}
\tilde{A}_{21}\tilde{A}_{20} &= \frac{5}{4}\frac{\partial \tilde{A}_{20}}{\partial v_1} + \tilde{A}_{20}^2,\\
\frac{\partial^2 \tilde{A}_{20}}{\partial {v_1}^2}\tilde{A}_{20} &= -\frac{1}{100}(16\tilde{A}_{20}^4 - 80\tilde{A}_{20}^2\frac{\partial \tilde{A}_{20}}{\partial v_1} - 125(\frac{\partial \tilde{A}_{20}}{\partial v_1})^2),\\
\tilde{A}_{30}\tilde{A}_{20}^2 &= -\frac{1}{80}(-16\tilde{A}_{20}^4 - 280\tilde{A}_{20}^2\frac{\partial \tilde{A}_{20}}{\partial v_1} - 25(\frac{\partial \tilde{A}_{20}}{\partial v_1})^2),\\
\tilde{A}_{40}\tilde{A}_{20} &= -\frac{1}{200}(16\tilde{A}_{20}^4 - 120\tilde{A}_{20}^2\frac{\partial \tilde{A}_{20}}{\partial v_1} - 225(\frac{\partial \tilde{A}_{20}}{\partial v_1})^2).
\end{aligned}$$

Comparing with the list in [MSS91], [4.2.10] satisfies Case I, [4.2.11] satisfies Case IIa, [4.2.12] and [4.2.13] satisfy Case IIb if we take all the parameters in the equations zero.

8.6 Concluding remarks, open problems

It seems, based on this chapter and [BSW98], that the symbolic method used together with diophantine approximation theory and/or p-adic analysis gives a powerful method to do automated (co)symmetry classification of evolution equations. The goal would be to produce the final list of all integrable scalar evolution equations with a recognition algorithm. Once the classification is done for all gradings, one can systematically try to find all such relations, since they might exist whenever the hierarchies are alike. This one can measure by writing down the Hilbert function $H_f(\tau) = \sum_{k=0}^{\infty} \dim Sym_k(f)\tau^k$, where $Sym_k(f)$ is spanned by the f-symmetries of grading k. E.g. for `KdV` this will be, if we restrict ourselves to polynomial symmetries with constant coefficients,

$$H_{\mathtt{KdV}}(\tau) = \frac{\tau^3}{1-\tau^2}.$$

The main difficulty in applying the symbolic method in actual computations is the rather quick expression swell. In the programs developed for the present chapter we have countered this by using Form, whenever Maple gave us *object too large* errors. However, one can improve on this by doing the Lie bracket calculations in the classical domain and the division using the symbolic method. This method even allows one to compute recursion operators by a long division procedure.

Another interesting possibility is to use the method in the noncommutative case, cf. [OS98].

Chapter 9

Examples of integrable equations

A general reference for the present chapter is [Ibr96b], [Ibr96a], where lists of integrable equations with their properties are given and the basic theoretical results regarding the objects listed in the tables are presented (but not proved).

The following is **not** a classification list and does not claim to be complete in any sense.

For every equation we give a table containing (if it exists)

- The equation itself,
- its Hamiltonian function corresponding to the cosymplectic operator,
- its cosymplectic operator,
- its symplectic operator,
- its recursion operator (possibly resulting from the cosymplectic and symplectic operators) , or its master symmetry,
- roots of its symmetries and
- its scaling symmetry.

The source of these results is indicated where possible. Some of the material is new, as far as could be verified, such as in sections 9.10, 9.13, 9.32, 9.37 and other isolated results (especially the decomposition of the recursion operator in symplectic and cosymplectic operators). The new results mainly rely on the theory in chapter 6.

We hope that this material can serve as a source of motivation for future research.

In the sequel we assume $\mathcal{C} = \mathbb{R}$ or $\mathbb{C}$.

9.1 Burgers' equation

Reference: [Olv93] p. 315, [Oev84] p. 38, section 6.3.1 ;

Equation	$u_t = u_2 + uu_1$	
Hamiltonian	None	[Fuc79]
Cosymplectic	None	[FF81]
Symplectic	None	[FF81]
Recursion	$\mathfrak{R}_1 = D_x + \frac{1}{2}u + \frac{1}{2}u_1 D_x^{-1}$	[Olv77]
	$\mathfrak{R}_2 = tD_x + \frac{1}{2}(tu + x) + \frac{1}{2}(tu_1 + 1)D_x^{-1}$	[Olv93]
Root	$u_1,\ tu_1 + 1$	$\star$
Scaling	$-2t\partial_t + (xu_1 + u)\partial_u$	

$\star$ Since $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are both recursion operators of the equation, we obtain a double infinity of the symmetries, by applying $\mathfrak{R}_1$ or $\mathfrak{R}_2$ successively to u_1 and tu_1+1. Note that since $\mathfrak{R}_1\mathfrak{R}_2 = \mathfrak{R}_2\mathfrak{R}_1 + \frac{1}{2}$ and $\mathfrak{R}_1(tu_1+1) = \mathfrak{R}_2(u_1)$, if we are only interested in independent symmetries, it does not matter in which order $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are applied.

Notice that there is a difference between the root of an operator and the root of symmetries for an equation (cf. definition 6-10 and 6-12). For $\mathfrak{R}_1$, we can take $tu_1 + 1$ as a root of symmetries for Burgers' equation since $\mathfrak{R}_1 = D_x + \frac{1}{2}D_x(uD_x^{-1}\cdot)$, but it is not a root of $\mathfrak{R}_1$ since it is not its symmetry.

9.2 Potential Burgers' equation

Reference: [Olv93] pp. 311, 317;

Equation	$u_t = u_2 + u_1^2$	
Hamiltonian	None	[Fuc79]
Cosymplectic	None	[FF81]
Symplectic	None	[FF81]
Recursion	$\mathfrak{R}_1 = D_x + u_1$	
	$\mathfrak{R}_2 = tD_x + tu_1 + \frac{1}{2}x$	[Olv93]
Root	1	
Scaling	xu_1	[Olv93]

The same arguments hold here as Burgers' equation since $\mathfrak{R}_1\mathfrak{R}_2 = \mathfrak{R}_2\mathfrak{R}_1 + \frac{1}{2}$.

In the paper [Fok80], the author found the 2^{nd}-order equations of the form $u_t = u_2 + f(u, u_1)$, which possess a 3^{rd}-order symmetry and obtained the following equations:

$$u_t = u_2 + \frac{f''(u)}{f'(u)}u_1^2 + \alpha f(u)u_1, \tag{9.2.1}$$

where α is constant and $f(u)$ is an arbitrary function, with the Nijenhuis recursion operator $D_x + \frac{f''(u)}{f'(u)}u_1 + \frac{1}{2}\alpha f(u) + \frac{1}{2}\alpha u_1 D_x^{-1} f'(u)$.

$$u_t = u_2 + \frac{\gamma - f'(u)}{f(u)}u_1^2 + \alpha f(u), \tag{9.2.2}$$

where $f(u)$ is an arbitrary function and α, γ are constant, with the Nijenhuis recursion operator $D_x + \frac{\gamma - f'(u)}{f(u)}u_1$.

Notice that the (potential) Burgers' equation is a particular case of (9.2.1). If we take $\alpha = 1$ and $\frac{\gamma - f'(u)}{f(u)} = 1$, i.e., $f(u) = \beta \exp(-u) + \gamma$, it leads to the nontrivial equation:

$$u_t = u_2 + u_1^2 + \beta \exp(-u) + \gamma.$$

9.3 Diffusion equation

Reference: [Oev84] p. 39;

Equation	$u_t = u^2 u_2$
Hamiltonian	None
Cosymplectic	None
Symplectic	None
Recursion	$uD_x + u^2 u_2 D_x^{-1} \frac{1}{u^2}$
Root	$u^2 u_2$
Scaling	$\alpha x u_1 + \beta u, \alpha, \beta \in \mathcal{C}$.

9.4 Nonlinear diffusion equation

Reference: [Olv93] Ex. 5.10;

Equation	$u_t = D_x(\frac{u_1}{u^2})$
Hamiltonian	None
Cosymplectic	None
Symplectic	None
Recursion	$D_x^2 \frac{1}{u} D_x^{-1} = \frac{1}{u} D_x - \frac{2u_1}{u^2} - u_t D_x^{-1}$
Root	u_t
Scaling	$\alpha x u_1 + \beta u, \alpha, \beta \in \mathcal{C}$.

9.5 Korteweg–de Vries equation

Reference: [Olv93] p. 312, [Oev84] pp. 18, 67, 78, 84, 97, [Dor93] pp. 85, 151, 158, 162, [Oev90] pp. 27, 60;

Equation	$u_t = u_3 + u u_1$	[KdV95]
Hamiltonian	$\frac{u^2}{2}$	
Cosymplectic	$D_x^3 + \frac{1}{3}(uD_x + D_x u)$	
Symplectic	D_x^{-1}	
Recursion	$D_x^2 + \frac{2}{3}u + \frac{1}{3}u_1 D_x^{-1}$	[Olv77]
Root	u_1	
Scaling	$x u_1 + 2u$	

9.6 Potential Korteweg–de Vries equation

Reference: [Dor93] p. 125;

Equation	$u_t = u_3 + 3u_1^2$
Hamiltonian	$\frac{1}{2}uu_4 + 2uu_1u_2$
Cosymplectic	D_x^{-1}
Symplectic	$D_x^3 + 2(u_1D_x + D_xu_1)$
Recursion	$D_x^2 + 4u_1 - 2D_x^{-1}u_2$
Root	1
Scaling	$xu_1 + u$

9.7 Modified Korteweg–de Vries equation

Reference: [Olv93] Ex. 5.11, [Oev84] p. 97, [Oev90] pp. 29, 60;

Equation	$u_t = u_3 + u^2u_1$
Hamiltonian	$\frac{u^2}{2}$
Cosymplectic	$D_x^3 + \frac{2}{3}D_xuD_x^{-1}uD_x$
Symplectic	D_x^{-1}
Recursion	$D_x^2 + \frac{2}{3}u^2 + \frac{2}{3}u_1D_x^{-1}u$ [Olv77]
Root	u_1
Scaling	$xu_1 + u$

9.8 Potential modified Korteweg–de Vries equation

Reference: [Olv93] Ex.5.11;

Equation	$u_t = u_3 + \frac{1}{3}u_1^3$
Hamiltonian	$\frac{1}{2}uu_4 + \frac{1}{4}uu_1^2u_2$
Cosymplectic	D_x^{-1}
Symplectic	$D_x^3 + \frac{2}{3}D_xu_1D_x^{-1}u_1D_x$
Recursion	$D_x^2 + \frac{2}{3}u_1^2 - \frac{2}{3}u_1D_x^{-1}u_2$
Root	u_1
Scaling	$\alpha xu_1 + \beta u, \alpha, \beta \in \mathcal{C}$.

In the paper [Fok80], the author found the 3rd-order equations, not involving 2nd-order derivatives, i.e., of the form $u_t = u_3 + f(u, u_1)$, which possess a 5th-order symmetry and obtained the following equations:

$$u_t = u_3 + \alpha u_1^3 + \beta u_1^2 + \gamma u_1, \tag{9.8.1}$$

where α, β, γ are constant, with the Nijenhuis recursion operator $D_x^2 + 2\alpha u_1^2 + \frac{4}{3}\beta u_1 - \frac{2}{3}(3\alpha u_1 + \beta)D_x^{-1}u_2 + \gamma$.

$$u_t = u_3 + \alpha u_1^3 + f(u)u_1, \tag{9.8.2}$$

where $f(u)$ satisfies $f''' + 8\alpha f' = 0$, with the Nijenhuis recursion operator $D_x^2 + 2\alpha u_1^2 + \frac{2}{3}f(u) - \frac{1}{3}u_1D_x^{-1}(6\alpha u_2 - f')$.

pKdV ($\alpha = 0$) and pmKdV ($\beta = 0$) are particular cases of (9.8.1). pmKdV ($f(u) = 0$), KdV ($\alpha = 0$ and $f(u) = u$) and mKdV ($\alpha = 0$ and $f(u) = u^2$) are particular cases of (9.8.2) including Calogero–Degasperis–Fokas equation [CD81]:

$$u_t = u_3 - \frac{1}{8}u_1^3 + (a\exp(u) + b\exp(-u) + c)u_1.$$

9.9 Cylindrical Korteweg–de Vries equation

Reference: [OF84], [ZC86], [Cho87a], example 6-15 in this thesis;

Equation	$u_t = u_3 + uu_1 - \frac{u}{2t}$
Hamiltonian	None
Cosymplectic	$D_x^3 + \frac{1}{3}(uD_x + D_x u) + \frac{1}{6t}(xD_x + D_x x)$
Symplectic	tD_x^{-1}
Recursion	$t(D_x^2 + \frac{2}{3}u + \frac{1}{3}u_1 D_x^{-1}) + \frac{1}{3}x + \frac{1}{6}D_x^{-1}$
Root	$\sqrt{t}(\frac{u_1}{3} + \frac{1}{6t})$
Scaling	$-3t\partial_t + (2u + xu_1)\partial_u$

For the generalized Korteweg–de Vries equation [Cho87a] (the author also studied generalized mKdV in the same way [Cho87b]):

$$u_t + u_3 + 6uu_1 + 6f(t)u - x(f_t' + 12f^2) = 0,$$

where f is an arbitrary function of t. It possesses the following Nijenhuis recursion operator:

$$\mathfrak{R} = \frac{1}{g(t)}(D_x^2 + 4(u - xf(t)) + 2(u_1 - f(t))D_x^{-1})$$

with the root $\frac{1}{\sqrt{g}}(u_1 - f)$, where $g(t) = \exp(-\int 12f\,dt)$.

If we take $f(t) = \frac{1}{12t}$ and then do transformation $\tilde{u} = \frac{1}{6}u$ and $\tilde{t} = -t$, we get the Cylindrical Korteweg–de Vries equation.

9.10 Ibragimov–Shabat equation

Reference: [IS81], [Cal87];

Equation	$u_t = u_3 + 3u^2u_2 + 9uu_1^2 + 3u^4u_1$
Hamiltonian	None
Cosymplectic	None
Symplectic	None
Root	u_1
Scaling	$xu_1 + \frac{1}{2}u$
Master Symmetries	$xu_t + \frac{3}{2}u_2 + 5u_1u^2 + \frac{1}{2}u^5$

No recursion operator seems to be known for this equation.

We should mention that this equation possesses infinitely many symmetries [IS81], but only one local conservation law u^2 [Kap82]. The transformation $u = \sqrt{\frac{w_1}{2w}}$ [Cal87] transforms it into $w_t = w_3 - \frac{3}{4}\frac{w_2^2}{w_1}$ and the master symmetry becomes

$xw_t + \frac{1}{2}w_2$, which is the master symmetry for this new equation. Notice that the new equation has a recursion operator

$$\mathfrak{R} = D_x^2 - \frac{w_2}{w_1}D_x + \frac{w_3}{2w_1} - \frac{w_2^2}{4w_1^2} - D_x^{-1}\left(\frac{w_4}{2w_1} - \frac{w_2 w_3}{w_1^2} + \frac{w_2^3}{2w_1^3}\right),$$

where $\frac{w_4}{2w_1} - \frac{w_2 w_3}{w_1^2} + \frac{w_2^3}{2w_1^3} = E(\frac{w_2^2}{4w_1})$.

9.11 Harry Dym equation

Reference: [Olv93] Ex.5.15, [Oev84] p. 107, [Dor93] p. 85;

Equation	$u_t = u^3 u_3$	
Hamiltonian	$-\frac{1}{u}$	
Cosymplectic	$u^3 D_x^3 u^3$	
Symplectic	$\frac{1}{u^2} D_x^{-1} \frac{1}{u^2}$	
Recursion	$u^3 D_x^3 u D_x^{-1} \frac{1}{u^2}$	[LLS+83]
	$= u^2 D_x^2 - u u_1 D_x + u u_2 + u^3 u_3 D_x^{-1} \frac{1}{u^2}$	
Root	$u^3 u_3$	
Scaling	$\alpha x u_1 + \beta u, \alpha, \beta \in \mathcal{C}$.	

Sometimes the equation is transformed into $u_t = D_x^3(\frac{1}{\sqrt{u}})$ like in [Dor93].

9.12 Krichever–Novikov equation

Reference: [Dor93] p. 121;

Equation	$u_t = u_3 - \frac{3}{2}\frac{u_2^2}{u_1}$
Hamiltonian	$\frac{u_2^2}{2u_1^2}$
Cosymplectic	$2(\frac{1}{u_1^2}D_x + D_x \frac{1}{u_1^2})^{-1}$
Symplectic	$\frac{1}{2}(\frac{1}{u_1^2}D_x^3 + D_x^3 \frac{1}{u_1^2}) + (\frac{u_3}{u_1^3} - \frac{3u_2^2}{u_1^4})D_x + D_x(\frac{u_3}{u_1^3} - \frac{3u_2^2}{u_1^4})$
Recursion	$\begin{cases} D_x^2 - \frac{2u_2}{u_1}D_x + (\frac{u_3}{u_1} - \frac{u_2^2}{u_1^2}) - u_1 D_x^{-1}\xi, \\ \xi = \frac{3u_2^3}{u_1^4} - \frac{4u_2 u_3}{u_1^3} + \frac{u_4}{u_1^2} = \mathbf{E}(\frac{u_2^2}{2u_1^2}) \end{cases}$
Root	u_1
Scaling	$\alpha x u_1 + \beta u, \alpha, \beta \in \mathcal{C}$.

9.13 Cavalcante–Tenenblat equation

Reference: [CT88];

Equation	$u_t = D_x^2(u_1^{-\frac{1}{2}}) + u_1^{\frac{3}{2}}$
Hamiltonian	$-2\sqrt{u_1}$
Cosymplectic	$D_x - u_1 D_x^{-1} u_1$
Symplectic	$u_1^{-\frac{1}{2}} D_x u_1^{-\frac{1}{2}} - \frac{1}{4} u_1^{-\frac{3}{2}} u_2 D_x^{-1} u_1^{-\frac{3}{2}} u_2$
Recursion	$\frac{1}{u_1} D_x^2 - \frac{3u_2}{2u_1^2} D_x - \frac{u_3}{2u_1^2} + \frac{3u_2^2}{4u_1^3} - u_1 + \frac{u_t}{2} D_x^{-1} u_1^{-\frac{3}{2}} u_2$
Root	u_t
Scaling	xu_1

9.14 Sine–Gordon equation

Reference: [Olv93] Ex.5.12, [Dor93] pp. 133, 163;

Equation	$u_{xt} = \sin u$	Oevel[16]
Hamiltonian	$-\cos u$	$\star$
Cosymplectic	D_x^{-1}	
Symplectic	$D_x^3 + D_x u_1 D_x^{-1} u_1 D_x$	
Recursion	$D_x^2 + u_1^2 - u_1 D_x^{-1} u_2$	[Olv77]
Root	u_1	
Scaling	$\alpha x u_1 + \beta u, \alpha, \beta \in \mathcal{C}$.	

$\star$ Actually, the equation is treated as an evolution equation $u_t = D_x^{-1} \sin u$.

9.15 Liouville equation

Reference: [Dor93] pp. 134, 164;

Equation	$u_{xt} = \exp(u)$	see `Pogrebkov`, 1987
Hamiltonian	$\exp(u)$	$\star$
Cosymplectic	D_x^{-1}	
Symplectic	$D_x^3 - D_x u_1 D_x^{-1} u_1 D_x$	
Recursion	$D_x^2 - u_1^2 + u_1 D_x^{-1} u_2$	
Root	u_1	
Scaling	$\alpha x u_1 + \beta u, \alpha, \beta \in \mathcal{C}$.	

$\star$ As we mentioned for the Sine–Gordon equation, this equation is also treated as an evolution equation $u_t = D_x^{-1} \exp(u)$.

The **Sinh–Gordon equation** $u_{xt} = \sinh u$ has exactly the same geometric structure. [AC91].

9.16 Klein–Gordon equations

Reference: [AC91] p. 366, [Kon87] p. 41, [FG80];

Equation	$u_{xt} = \alpha \exp(-2u) + \beta \exp(u)$
Hamiltonian	$-\frac{\alpha}{2} \exp(-2u) + \beta \exp(u)$
Cosymplectic	D_x^{-1}
Symplectic	$\mathfrak{I}$
Recursion	$\mathfrak{R}$
Root	$u_1, u_5 + 5u_2u_3 - 5u_1^2u_3 - 5u_1u_2^2 + u_1^5$
Scaling	xu_1

$$
\begin{aligned}
\mathfrak{I} &= D_x^7 + 3(u_2D_x^5 + D_x^5u_2) - 3(u_1^2D_x^5 + D_x^5u_1^2) - 8(u_4D_x^3 + D_x^3u_4) \\
&+10(u_1u_3D_x^3 + D_x^3u_1u_3) + \frac{29}{2}(u_2^2D_x^3 + D_x^3u_2^2) - 3(u_1^2u_2D_x^3 + D_x^3u_1^2u_2) \\
&+\frac{9}{2}(u_1^4D_x^3 + D_x^3u_1^4) + 5(u_6D_x + D_xu_6) - 6(u_1u_5D_x + D_xu_1u_5) \\
&-25(u_2u_4D_x + D_xu_2u_4) + 3(u_1^2u_4D_x + D_xu_1^2u_4) - 21(u_3^2D_x + D_xu_3^2) \\
&+8(u_1u_2u_3D_x + D_xu_1u_2u_3) - 8(u_1^3u_3D_x + D_xu_1^3u_3) \\
&+6(u_2^3D_x + D_xu_2^3) - 44(u_1^2u_2^2D_x + D_xu_1^2u_2^2) - 2(u_1^6D_x + D_xu_1^6) \\
&+2u_2D_x^{-1}(u_6 + 5u_2u_4 + 5u_3^2 - 5u_1^2u_4 - 20u_1u_2u_3 - 5u_2^3 + 5u_1^4u_2) \\
&+2(u_6 + 5u_2u_4 + 5u_3^2 - 5u_1^2u_4 - 20u_1u_2u_3 - 5u_2^3 + 5u_1^4u_2)D_x^{-1}u_2 \\
\mathfrak{R} &= D_x^6 + 6(u_2 - u_1^2)D_x^4 + 9(u_3 - 2u_1u_2)D_x^3 \\
&+(5u_4 - 22u_1u_3 - 13u_2^2 - 6u_1^2u_2 + 9u_1^4)D_x^2 \\
&+(u_5 - 8u_1u_4 - 15u_2u_3 - 3u_1^2u_3 - 6u_1u_2^2 + 18u_1^3u_2)D_x \\
&-4u_1u_5 + 20u_1^3u_3 - 20u_1u_2u_3 + 20u_1^2u_2^2 - 4u_1^6 \\
&+2u_1D_x^{-1}(u_6 + 5u_2u_4 + 5u_3^2 - 5u_1^2u_4 - 20u_1u_2u_3 - 5u_2^3 + 5u_1^4u_2) \\
&+2(u_5 + 5u_2u_3 - 5u_1^2u_3 - 5u_1u_2^2 + u_1^5)D_x^{-1}u_2
\end{aligned}
$$

It shares its recursion operator [Bil93] with the Potential Kupershmidt equation, i.e., $u_t = u_5 + 5u_2u_3 - 5u_1^2u_3 - 5u_1u_2^2 + u_1^5$ (equation (4.2.7) in [MSS91]).

Klein–Gordon equations $u_{xt} = f(u)$ possess a nontrivial symmetry if and only if $f(u) = \alpha \exp(-\lambda u) + \beta \exp(\lambda u)$ or $f(u) = \alpha \exp(-2\lambda u) + \beta \exp(\lambda u)$ [ZS79].

9.17 Kupershmidt equation

Reference: [MSS91] Equation (4.2.6), [FG80], [Bil93];

Equation	$u_t = u_5 + 5u_1u_3 + 5u_2^2 - 5u^2u_3 - 20uu_1u_2 - 5u_1^3 + 5u^4u_1$
Hamiltonian	$\frac{u_2^2}{2} - \frac{5u_1^3}{6} + \frac{5u^2u_1^2}{2} + \frac{u^6}{6}$
Cosymplectic	D_x
Symplectic	$\mathfrak{I}$
Recursion	$\mathfrak{R}$
Root	u_1, u_t
Scaling	$xu_1 + u$

$$
\mathfrak{I} = D_x^5 + 3(u_1D_x^3 + D_x^3u_1) - 3(u^2D_x^3 + D_x^3u^2) - 3(u_1u^2D_x + D_xu_1u^2)
$$

$$
\begin{aligned}
& +\frac{5}{2}(u_1^2 D_x + D_x u_1^2) - 2(u_3 D_x + D_x u_3) + \frac{9}{2}(u^4 D_x + D_x u^4) \\
& -2(u u_2 D_x + D_x u u_2) - 2(u_4 - 5u^2 u_2 - 5u u_1^2 + 5u_1 u_2 + u^5) D_x^{-1} u \\
& -2u D_x^{-1}(u_4 - 5u^2 u_2 - 5u u_1^2 + 5u_1 u_2 + u^5) \\
\mathfrak{R} \;=\; & D_x^6 + 6u_1 D_x^4 - 6u^2 D_x^4 - 30u u_1 D_x^3 + 15u_2 D_x^3 + 9u^4 D_x^2 - 6u^2 u_1 D_x^2 \\
& -40u u_2 D_x^2 - 31u_1^2 D_x^2 + 14u_3 D_x^2 - 9u^2 u_2 D_x + 54u^3 u_1 D_x - 18u u_1^2 D_x \\
& -30u u_3 D_x - 63u_1 u_2 D_x + 6u_4 D_x - 4u^6 + 38u^3 u_2 + 74u^2 u_1^2 \\
& -3u^2 u_3 - 12u u_4 - 38u u_1 u_2 + u_5 - 6u_1^3 - 23u_1 u_3 - 15u_2^2 \\
& -2u_t D_x^{-1} u - 2u_1 D_x^{-1}\left(u_4 - 5u^2 u_2 - 5u u_1^2 + 5u_1 u_2 + u^5\right)
\end{aligned}
$$

9.18 Sawada–Kotera equation

Reference: [SK74], [CDG76], [FO82], [Oev84] p. 105, [FOW87], [Oev90] p. 30, [MSS91] Equation (4.2.2), [Bil93];

Equation	$u_t = u_5 + 5uu_3 + 5u_1u_2 + 5u^2u_1$
Hamiltonian	$\frac{u^3}{6} - \frac{u_1^2}{2}$
Cosymplectic	$D_x\left(D_x + 2(D_x^{-1}u + uD_x^{-1})\right)D_x$
Symplectic	$\left(D_x + D_x^{-1}u\right)D_x\left(D_x + uD_x^{-1}\right)$
Recursion	$\mathfrak{R}$
Root	u_1, u_t
Scaling	$xu_1 + 2u$

$$
\begin{aligned}
\mathfrak{R} \;=\; & D_x^6 + 6uD_x^4 + 9u_1 D_x^3 + 9u^2 D_x^2 + 11u_2 D_x^2 + 10u_3 D_x + 21u u_1 D_x \\
& +4u^3 + 16u u_2 + 6u_1^2 + 5u_4 + u_1 D_x^{-1}(2u_2 + u^2) + u_t D_x^{-1}
\end{aligned}
$$

9.19 Potential Sawada–Kotera equation

Reference: [MSS91] Equation (4.2.4), [Bil93];

Equation	$u_t = u_5 + 5u_1u_3 + \frac{5}{3}u_1^3$
Hamiltonian	$\frac{u_2^2}{2} - \frac{u_1^3}{6}$
Cosymplectic	$D_x + 2(u_1D_x^{-1} + D_x^{-1}u_1)$
Symplectic	$\left(D_x + u_1D_x^{-1}\right)D_x^3\left(D_x + D_x^{-1}u_1\right)$
Recursion	$\mathfrak{R}$
Root	$u_1, 1$
Scaling	$xu_1 + u$

$$
\begin{aligned}
\mathfrak{R} \;=\; & D_x^6 + 6u_1 D_x^4 + 3u_2 D_x^3 + 8u_3 D_x^2 + 9u_1^2 D_x^2 + 2u_4 D_x + 3u_2 u_1 D_x \\
& +3u_5 + 13u_3 u_1 + 3u_2^2 + 4u_1^3 - 2u_1 D_x^{-1}\left(u_4 + u_2 u_1\right) \\
& -2D_x^{-1}\left(u_6 + 3u_4 u_1 + 6u_3 u_2 + 2u_2 u_1^2\right)
\end{aligned}
$$

9.20 Kaup–Kupershmidt equation

Reference: [Kau80], [FO82], [FOW87], [MSS91] Equation (4.2.3), [Bil93];

Equation	$u_t = u_5 + 5uu_3 + \frac{25}{2}u_1u_2 + 5u^2u_1$
Hamiltonian	$\frac{2u^3}{3} - \frac{u_1^2}{2}$
Cosymplectic	$D_x\left(D_x + \frac{1}{2}(uD_x^{-1} + D_x^{-1}u)\right)D_x$
Symplectic	$D_x^3 + \frac{3}{2}(uD_x + D_xu) + D_x^2uD_x^{-1} + D_x^{-1}uD_x^2$
	$+2(u^2D_x^{-1} + D_x^{-1}u^2)$
Recursion	$\mathfrak{R}$
Root	u_1, u_t
Scaling	$xu_1 + 2u$

$$\begin{aligned}\mathfrak{R} \;=\; & D_x^6 + 6uD_x^4 + 18u_1D_x^3 + 9u^2D_x^2 + \frac{49}{2}u_2D_x^2 + 30uu_1D_x + \frac{35}{2}u_3D_x \\ & +4u^3 + \frac{41}{2}uu_2 + \frac{69}{4}u_1^2 + \frac{13}{2}u_4 + \frac{1}{2}u_1D_x^{-1}(u_2 + 2u^2) + u_tD_x^{-1}\end{aligned}$$

9.21 Potential Kaup–Kupershmidt equation

Reference: [MSS91] Equation (4.2.5), [Bil93];

Equation	$u_t = u_5 + 10u_1u_3 + \frac{15}{2}u_2^2 + \frac{20}{3}u_1^3$
Hamiltonian	$\frac{u_2^2}{2} - \frac{4u_1^3}{3}$
Cosymplectic	$D_x + u_1D_x^{-1} + D_x^{-1}u_1$
Symplectic	$D_x^5 + 5(u_1D_x^3 + D_x^3u_1) - 3(u_3D_x + D_xu_3)$
	$+8(u_1^2D_x + D_xu_1^2)$
Recursion	$\mathfrak{R}$
Root	$u_1, 1$
Scaling	$xu_1 + u$

$$\begin{aligned}\mathfrak{R} \;=\; & D_x^6 + 12u_1D_x^4 + 24u_2D_x^3 + 25u_3D_x^2 + 36u_1^2D_x^2 + 10u_4D_x + 48u_1u_2D_x \\ & +3u_5 + 21u_2^2 + 34u_1u_3 + 32u_1^3 - 2u_1D_x^{-1}\left(u_4 + 8u_1u_2\right) \\ & -D_x^{-1}\left(u_6 + 12u_1u_4 + 24u_2u_3 + 32u_1^2u_2\right)\end{aligned}$$

9.22 Dispersiveless Long Wave system

Reference: [AC91], [Gök96];

Equation	$\left\{\begin{array}{l} u_t = u_1 v + u v_1 \\ v_t = u_1 + v v_1 \end{array}\right.$
Hamiltonian	$\frac{u^2+uv^2}{2}$
Cosymplectic	$\begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix}$
Symplectic	$\begin{pmatrix} 2D_x^{-1} & vD_x^{-1} \\ D_x^{-1}v & uD_x^{-1} + D_x^{-1}u \end{pmatrix}$
Recursion	$\begin{pmatrix} v & 2u + u_1 D_x^{-1} \\ 2 & v + v_1 D_x^{-1} \end{pmatrix}$
Root	$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 \\ xv_1 \end{pmatrix} + \alpha \begin{pmatrix} 2u \\ v \end{pmatrix}, \alpha \in \mathcal{C}$

9.23 Diffusion system

Reference: [Oev84] p. 41, section 6.3.3 ;

Equation	$\left\{\begin{array}{l} u_t = u_2 + v^2 \\ v_t = v_2 \end{array}\right.$
Hamiltonian	None
Cosymplectic	None
Symplectic	None
Recursion	$\begin{pmatrix} D_x & vD_x^{-1} \\ 0 & D_x \end{pmatrix}$
Root	$\begin{pmatrix} v \\ 0 \end{pmatrix}, \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 + 2u \\ xv_1 + 2v \end{pmatrix} + \alpha \begin{pmatrix} 2u \\ v \end{pmatrix}, \alpha \in \mathcal{C}$

9.24 Sine–Gordon equation in the laboratory coordinates

Reference: [CLL87] , section 6.3.6 ;

Equation	$\begin{cases} u_t = v \\ v_t = u_2 - \sin(u) \end{cases}$
Hamiltonian	$\frac{1}{2}(u_1^2 + v^2) - \cos(u)$
Cosymplectic	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
Symplectic	$\begin{pmatrix} -\mathfrak{R}_{21} & -\mathfrak{R}_{22} \\ \mathfrak{R}_{11} & \mathfrak{R}_{12} \end{pmatrix}$
Recursion	$\mathfrak{R} = \begin{pmatrix} \mathfrak{R}_{11} & \mathfrak{R}_{12} \\ \mathfrak{R}_{21} & \mathfrak{R}_{22} \end{pmatrix}$
Root	$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$
Scaling	None

$$
\begin{aligned}
\mathfrak{R}_{11} &= 4D_x^2 - 2\cos(u) + (u_1+v)^2 - (u_1+v)D_x^{-1}(u_2+v_1-\sin(u)), \\
\mathfrak{R}_{12} &= 4D_x + (u_1+v)D_x^{-1}(u_1+v), \\
\mathfrak{R}_{21} &= 4D_x^3 + (u_1+v)^2 D_x - 4\cos(u)D_x + 2u_1\sin(u) + (u_2+v_1)(u_1+v) \\
&- (u_2+v_1-\sin(u))D_x^{-1}(u_2+v_1-\sin(u)), \\
\mathfrak{R}_{22} &= 4D_x^2 + (u_1+v)^2 - 2\cos(u) + (u_2+v_1-\sin(u))D_x^{-1}(u_1+v).
\end{aligned}
$$

9.25 AKNS equation

Reference: [Oev84] p. 100;

Equation	$\begin{cases} u_t = -u_2 + 2u^2 v \\ v_t = v_2 - 2v^2 u \end{cases}$
Hamiltonian	$\frac{1}{2}(uv_1 - vu_1)$
Cosymplectic	$\begin{pmatrix} 2uD_x^{-1}u & D_x - 2uD_x^{-1}v \\ D_x - 2vD_x^{-1}u & 2vD_x^{-1}v \end{pmatrix}$
Symplectic	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
Recursion	$\begin{pmatrix} -D_x + 2uD_x^{-1}v & 2uD_x^{-1}u \\ -2vD_x^{-1}v & D_x - 2vD_x^{-1}u \end{pmatrix}$
Root	$\begin{pmatrix} -u \\ v \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 + u \\ xv_1 + v \end{pmatrix}$

9.26 Nonlinear Schrödinger equation

Reference: [Oev84] p. 102, [Dor93] p. 135, [Oev90] pp. 31, 61;

Equation	$\begin{cases} u_t = v_2 \mp v(u^2+v^2) \\ v_t = -u_2 \pm u(u^2+v^2) \end{cases}$
Hamiltonian	$\frac{1}{2}(uv_1 - vu_1)$
Cosymplectic	$\begin{pmatrix} D_x \mp 2vD_x^{-1}v & \pm 2vD_x^{-1}u \\ \pm 2uD_x^{-1}v & D_x \mp 2uD_x^{-1}u \end{pmatrix}$
Symplectic	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
Recursion	$\begin{pmatrix} \mp 2vD_x^{-1}u & D_x \mp 2vD_x^{-1}v \\ -D_x \pm 2uD_x^{-1}u & \pm 2uD_x^{-1}v \end{pmatrix}$
Root	$\begin{pmatrix} -v \\ u \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 + u \\ xv_1 + v \end{pmatrix}$

The system can be written as $iq_t = q_2 \mp q^2 q^\star$, where $i^2 = -1$ [AC91].

9.27 Derivative Schrödinger system

Reference: [Oev84] p. 103, section 6.3.5 ;

Equation	$\begin{cases} u_t = -v_2 - (u^2+v^2)u_1 \\ v_t = u_2 - (u^2+v^2)v_1 \end{cases}$
Hamiltonian	$\frac{1}{2}(uv_1 - vu_1)$
Cosymplectic	$\begin{pmatrix} -D_x & \frac{u^2+v^2}{2} \\ -\frac{u^2+v^2}{2} & -D_x \end{pmatrix} - \begin{pmatrix} v \\ -u \end{pmatrix} D_x^{-1} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$
	$- \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} D_x^{-1} \begin{pmatrix} v \\ -u \end{pmatrix}$
Symplectic	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
Recursion	$\begin{pmatrix} -\frac{u^2+v^2}{2} & -D_x \\ D_x & -\frac{u^2+v^2}{2} \end{pmatrix} + \begin{pmatrix} v \\ -u \end{pmatrix} D_x^{-1} \begin{pmatrix} v_1 \\ -u_1 \end{pmatrix}^\dagger$
	$- \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} D_x^{-1} \begin{pmatrix} u \\ v \end{pmatrix}^\dagger$
Root	$\begin{pmatrix} v \\ -u \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 + \frac{u}{2} \\ xv_1 + \frac{v}{2} \end{pmatrix}$

9.28 Modified derivative Schrödinger system

Reference: [WHV95], [Gök96];

Equation	$\begin{cases} u_t = D_x(u^3 + uv^2 + \beta u - v_1) \\ v_t = D_x(vu^2 + v^3 + u_1) \end{cases}$
Hamiltonian	$\frac{1}{2}(u^2 + v^2)$
Cosymplectic	$\begin{pmatrix} \beta D_x + 2uD_xu & -D_x^2 + 2vD_xu \\ D_x^2 + 2uD_xv & 2vD_xv \end{pmatrix}$ $-2\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} D_x^{-1} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$
Symplectic	$\begin{pmatrix} D_x^{-1} & 0 \\ 0 & D_x^{-1} \end{pmatrix}$
Recursion	$\begin{pmatrix} \beta + 2u^2 & -D_x + 2uv \\ D_x + 2uv & 2v^2 \end{pmatrix}$ $+2\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} D_x^{-1} \begin{pmatrix} u \\ v \end{pmatrix}^{\dagger}$
Root	$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 + \frac{u}{2} \\ xv_1 + \frac{v}{2} \\ x\beta_x + \beta \end{pmatrix}$ (cf. section 6.3.6).

9.29 Boussinesq system

Reference: [Olv93] p. 459, section 6.3.4 ;

Equation	$\begin{cases} u_t = v_1 \\ v_t = \frac{1}{3}u_3 + \frac{8}{3}uu_1 \end{cases}$
Hamiltonian	$\frac{1}{2}v$
Cosymplectic	$\begin{pmatrix} D_x^3 + 2uD_x + u_1 & 3vD_x + 2v_1 \\ 3vD_x + v_1 & \mathfrak{H}_{22} \end{pmatrix}$
Symplectic	$\begin{pmatrix} 0 & D_x^{-1} \\ D_x^{-1} & 0 \end{pmatrix}$
Recursion	$\begin{pmatrix} 3v + 2v_1D_x^{-1} & D_x^2 + 2u + u_1D_x^{-1} \\ \mathfrak{R}_{21} & 3v + v_1D_x^{-1} \end{pmatrix}$
Root	$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_t \\ v_t \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 + 2u \\ xv_1 + 3v \end{pmatrix}$

$$\begin{aligned} \mathfrak{H}_{22} &= \frac{1}{3}D_x^5 + \frac{5}{3}(uD_x^3 + D_x^3u) - (u_2D_x + D_xu_2) + \frac{16}{3}uD_xu \\ \mathfrak{R}_{21} &= \frac{1}{3}D_x^4 + \frac{10}{3}uD_x^2 + 5u_1D_x + 3u_2 + \frac{16}{3}u^2 + 2v_tD_x^{-1} \end{aligned}$$

9.30 Modified Boussinesq system

Reference: [FG81];

Equation	$\left\{\begin{array}{l} u_t = 3v_2 + 6uv_1 + 6u_1v \\ v_t = -u_2 - 6vv_1 + 2uu_1 \end{array}\right.$
Hamiltonian	$\frac{1}{2}(uv_1 - u_1v - 2v^3 + 2vu^2)$
Cosymplectic	$\begin{pmatrix} 3D_x & 0 \\ 0 & D_x \end{pmatrix}$
Symplectic	$\begin{pmatrix} \frac{1}{3}D_x^{-1}\mathfrak{R}_{11} & \frac{1}{3}D_x^{-1}\mathfrak{R}_{12} \\ D_x^{-1}\mathfrak{R}_{21} & D_x^{-1}\mathfrak{R}_{22} \end{pmatrix}$
Recursion	$\begin{pmatrix} \mathfrak{R}_{11} & \mathfrak{R}_{12} \\ \mathfrak{R}_{21} & \mathfrak{R}_{22} \end{pmatrix}$
Root	$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_t \\ v_t \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 + u \\ xv_1 + v \end{pmatrix}$

$$\begin{aligned}
\mathfrak{R}_{11} &= 6vD_x^2 + 9v_1D_x + 3v_2 - 12u_1v_1 - 2u_tD_x^{-1}u - 6u_1D_x^{-1}(2uv + v_1), \\
\mathfrak{R}_{12} &= 3D_x^3 + 6uD_x^2 + 9u_1D_x - 3u^2D_x - 9v^2D_x + 3u_2 - 6u^3 - 36vv_1 \\
&- 18uv^2 - 6u_tD_x^{-1}v + 6u_1D_x^{-1}(u_1 - u^2 + 3v^2), \\
\mathfrak{R}_{21} &= -D_x^3 + 2uD_x^2 + u^2D_x + 3u_1D_x + 3v^2D_x + u_2 - 6uv^2 - 2u^3 + 4uu_1 \\
&- 2v_tD_x^{-1}u - 6v_1D_x^{-1}(v_1 + 2uv), \\
\mathfrak{R}_{22} &= -6vD_x^2 - 9v_1D_x - 12uv^2 + 12u_1v - 3v_2 + 36v^3 \\
&- 6v_tD_x^{-1}v + 6v_1D_x^{-1}(u_1 - u^2 + 3v^2).
\end{aligned}$$

9.31 Landau–Lifshitz system

Reference: [vBK91] , section 6.3.8 ;

Equation	$\left\{\begin{array}{l} u_t = -\sin(u)v_2 - 2\cos(u)u_1v_1 + (J_1 - J_2)\sin(u)\cos(v)\sin(v) \\ v_t = \frac{u_2}{\sin(u)} - \cos(u)v_1^2 + \cos(u)(J_1\cos^2(v) + J_2\sin^2(v) - J_3) \end{array}\right.$
Hamiltonian	$\frac{1}{2}(\sin^2(u)(J_1\cos^2(v) + J_2\sin^2(v) - J_3) + J_3 - u_1^2 - \sin^2(u)v_1^2)$
Cosymplectic	$\frac{1}{\sin(u)}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
Symplectic	$\sin(u)\begin{pmatrix} \mathfrak{R}_{21} & \mathfrak{R}_{22} \\ -\mathfrak{R}_{11} & -\mathfrak{R}_{12} \end{pmatrix}$
Recursion	$\begin{pmatrix} \mathfrak{R}_{11} & \mathfrak{R}_{12} \\ \mathfrak{R}_{21} & \mathfrak{R}_{22} \end{pmatrix}$
Root	$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_t \\ v_t \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 \\ xv_1 \\ xJ_x + 2J \end{pmatrix}$, where $J = (J_1, J_2, J_3)$.

where

$$\mathfrak{R}_{11} = -D_x^2 - 2\sin^2(u)v_1^2 - u_1^2 + v_1^2 - (J_1 - J_2)\sin^2(u)\sin^2(v)$$

$$
\begin{array}{rcl}
 & & +(J_1-J_3)\sin^2(u)+J_3-J_2+u_tD_x^{-1}\cdot(\sin(u)v_1)-u_1D_x^{-1}\cdot S_1 \\
\mathfrak{R}_{12} & = & 2\cos(u)\sin(u)v_1D_x+\cos(u)\sin(u)v_2-3\sin^2(u)u_1v_1+2u_1v_1 \\
 & & +u_tD_x^{-1}\cdot(-\sin(u)u_1)-u_1D_x^{-1}\cdot S_2 \\
\mathfrak{R}_{21} & = & -2\cos(u)v_1D_x-\cos(u)v_2+u_1v_1 \\
 & & +v_tD_x^{-1}\cdot(\sin(u)v_1)-v_1D_x^{-1}\cdot S_1 \\
\mathfrak{R}_{22} & = & -D_x^2-2\cos(u)u_1D_x-\cos(u)u_2-(J_1-J_2)\sin(u)\sin^2(v) \\
 & & -2\sin^2(u)v_1^2+v_1^2+(J_1-J_3)\sin^2(u)+J_3-J_2 \\
 & & +v_tD_x^{-1}\cdot(-\sin(u)u_1)-v_1D_x^{-1}\cdot S_2 \\
S_1 & = & (J_1-J_2)\cos(u)\sin(u)\sin^2(v)-(J_1-J_3)\cos(u)\sin(u) \\
 & & +\cos(u)\sin(u)v_1^2-u_2, \\
S_2 & = & (J_1-J_2)\cos(v)\sin^2(u)\sin(v)-2\cos(u)\sin(u)u_1v_1-\sin^2(u)v_2.
\end{array}
$$

9.32 Wadati–Konno–Ichikawa system

Reference: [WKI79], [BPT83], [Kon87] p. 88;

System	$\begin{cases} u_t = D_x^2(\frac{u}{\sqrt{1+uv}}) \\ v_t = -D_x^2(\frac{v}{\sqrt{1+uv}}) \end{cases}$
Hamiltonian	$2\sqrt{1+uv}$
Cosymplectic	$\begin{pmatrix} 0 & D_x^2 \\ -D_x^2 & 0 \end{pmatrix}$
Symplectic	$\begin{pmatrix} 0 & \frac{2}{1+uv} \\ -\frac{2}{1+uv} & 0 \end{pmatrix}$
	$-\begin{pmatrix} \frac{v}{\sqrt{1+uv}} \\ \frac{u}{\sqrt{1+uv}} \end{pmatrix}^{\dagger} D_x^{-1} \begin{pmatrix} \frac{v_1}{(1+uv)^{\frac{3}{2}}} \\ -\frac{u_1}{(1+uv)^{\frac{3}{2}}} \end{pmatrix}^{\dagger}$
	$-\begin{pmatrix} \frac{v_1}{(1+uv)^{\frac{3}{2}}} \\ -\frac{u_1}{(1+uv)^{\frac{3}{2}}} \end{pmatrix}^{\dagger} D_x^{-1} \begin{pmatrix} \frac{v}{\sqrt{1+uv}} \\ \frac{u}{\sqrt{1+uv}} \end{pmatrix}^{\dagger}$
Root	$\begin{pmatrix} u_t \\ v_t \end{pmatrix}, \begin{pmatrix} D_x^2(\frac{u_1}{(1+uv)^{\frac{3}{2}}}) \\ D_x^2(\frac{v_1}{(1+uv)^{\frac{3}{2}}}) \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 \\ xv_1 \end{pmatrix}$

9.33 Hirota–Satsuma system

Reference: [HS81], [Fuc82], [Kon87] p. 207, [Oev90] pp. 32, 61, [Oev84] pp. 31, 84;

Equation	$\begin{cases} u_t = \frac{1}{2}u_3 + 3uu_1 - 6vv_1 \\ v_t = -v_3 - 3uv_1 \end{cases}$
Hamiltonian	$\frac{1}{2}u^2 - v^2$
Cosymplectic	$\begin{pmatrix} \frac{1}{2}D_x^3 + uD_x + D_xu & vD_x + D_xv \\ vD_x + D_xv & \frac{1}{2}D_x^3 + uD_x + D_xu \end{pmatrix}$
Symplectic	$\begin{pmatrix} \frac{1}{2}D_x + uD_x^{-1} + D_x^{-1}u & -2D_x^{-1}v \\ -2vD_x^{-1} & -2D_x \end{pmatrix}$
Recursion	$\mathfrak{R}$
Root	$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_t \\ v_t \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 + 2u \\ xv_1 + 2v \end{pmatrix}$

$$\begin{aligned}
\mathfrak{R}(u,v) &= \begin{pmatrix} \frac{1}{2}D_x^3 + D_x \cdot u + uD_x & D_x \cdot v + vD_x \\ D_x \cdot v + vD_x & \frac{1}{2}D_x^3 + D_x \cdot u + uD_x \end{pmatrix} \cdot \\
&\quad \begin{pmatrix} \frac{1}{2}D_x + D_x^{-1} \cdot u + uD_x^{-1} & -2D_x^{-1} \cdot v \\ -2vD_x^{-1} & -2D_x \end{pmatrix} \\
&\sim \begin{pmatrix} \frac{1}{2}u_3 + 3uu_1 - 6vv_1 \\ -v_3 - 3uv_1 \end{pmatrix} \otimes D_x^{-1} \begin{pmatrix} 1, & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \otimes D_x^{-1} \begin{pmatrix} u, & -2v \end{pmatrix}
\end{aligned}$$

9.34 The Symmetrically–coupled Korteweg–de Vries system

Reference: [Fuc82];

Equation	$\begin{cases} u_t = u_3 + v_3 + 6uu_1 + 4uv_1 + 2u_1v \\ v_t = u_3 + v_3 + 6vv_1 + 4vu_1 + 2v_1u \end{cases}$
Hamiltonian	$\frac{1}{2}(u+v)^2$
Cosymplectic	$\begin{pmatrix} D_x^3 + 2(uD_x + D_xu) & 0 \\ 0 & D_x^3 + 2(vD_x + D_xv) \end{pmatrix}$
Symplectic	$\begin{pmatrix} D_x^{-1} & D_x^{-1} \\ D_x^{-1} & D_x^{-1} \end{pmatrix}$
Recursion	$\begin{pmatrix} D_x^2 + 4u + 2u_1D_x^{-1} & D_x^2 + 4u + 2u_1D_x^{-1} \\ D_x^2 + 4v + 2v_1D_x^{-1} & D_x^2 + 4v + 2v_1D_x^{-1} \end{pmatrix}$
Root	$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 + 2u \\ xv_1 + 2v \end{pmatrix}$

9.35 The Complexly–coupled Korteweg–de Vries system

Reference: [Fuc82];

Equation	$\begin{cases} u_t = u_3 + 6uu_1 + 6vv_1 \\ v_t = v_3 + 6uv_1 + 6vu_1 \end{cases}$
Hamiltonian	$\frac{1}{2}(u^2 + v^2)$
Cosymplectic	$\begin{pmatrix} D_x^3 + 2(uD_x + D_x u) & 2D_x v + 2vD_x \\ 2D_x v + 2vD_x & D_x^3 + 2(uD_x + D_x u) \end{pmatrix}$
Symplectic	$\begin{pmatrix} D_x^{-1} & 0 \\ 0 & D_x^{-1} \end{pmatrix}$
Recursion	$\begin{pmatrix} D_x^2 + 4u + 2u_1 D_x^{-1} & 4v + 2v_1 D_x^{-1} \\ 4v + 2v_1 D_x^{-1} & D_x^2 + 4u + 2u_1 D_x^{-1} \end{pmatrix}$
Root	$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} v_1 \\ u_1 \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 + 2u \\ xv_1 + 2v \end{pmatrix}$

9.36 Coupled nonlinear wave system (Ito system)

Reference: [Ito82], [AF87], [Dor93] p. 94;

System	$\begin{cases} u_t = u_3 + 6uu_1 + 2vv_1 \\ v_t = 2uv_1 + 2u_1 v \end{cases}$
Hamiltonian	$\frac{u^2+v^2}{2}$
Cosymplectic	$\begin{pmatrix} D_x^3 + 4uD_x + 2u_1 & 2vD_x \\ 2vD_x + 2v_1 & 0 \end{pmatrix}$
Symplectic	$\begin{pmatrix} D_x^{-1} & 0 \\ 0 & D_x^{-1} \end{pmatrix}$
Recursion	$\begin{pmatrix} D_x^2 + 4u + 2u_1 D_x^{-1} & 2v \\ 2v + 2v_1 D_x^{-1} & 0 \end{pmatrix}$
Root	$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 + 2u \\ xv_1 + 2v \end{pmatrix}$

9.37 Drinfel'd–Sokolov system

Reference: [Gök96], [Gök98];

System	$\begin{cases} u_t = 3vv_1 \\ v_t = 2v_3 + u_1 v + 2uv_1 \end{cases}$
Hamiltonian	$\frac{v^2}{2}$
Cosymplectic	$\begin{pmatrix} 2D_x^3 + 2uD_x + u_1 & 2vD_x + v_1 \\ 2vD_x + v_1 & 2D_x^3 + 2uD_x + u_1 \end{pmatrix}$
Symplectic	$\begin{pmatrix} D_x^3 + \frac{5}{2}(uD_x + D_x u) & -15vD_x + \frac{15}{2}v_1 \\ -15vD_x - \frac{45}{2}v_1 & -27D_x^3 - \frac{27}{2}(uD_x + D_x u) \end{pmatrix}$
	$-(\begin{pmatrix} 1 \\ 0 \end{pmatrix}^\dagger D_x^{-1} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}^\dagger + \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}^\dagger D_x^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}^\dagger)$
	$-18 \begin{pmatrix} 0 \\ v \end{pmatrix}^\dagger D_x^{-1} \begin{pmatrix} 0 \\ v \end{pmatrix}^\dagger,$
Root	$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}; \begin{pmatrix} u_t \\ v_t \end{pmatrix}; \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 + 2u \\ xv_1 + 2v \end{pmatrix}$

$$\begin{aligned}
\eta_1 &= -u_2 - 2u^2 + 3v^2, \\
\eta_2 &= 9v_2 + 6uv, \\
h_1 &= -2u_5 - 10uu_3 - 25u_1u_2 + 30vv_3 + 45v_1v_2 - 10u^2u_1 + 15v^2u_1 + 30uvv_1, \\
h_2 &= 18v_5 + 10vu_3 + 35u_2v_1 + 45u_1v_2 + 30uv_3 + 10uu_1v + 10u^2v_1 + 15v^2v_1.
\end{aligned}$$

9.38 Benney system

Reference: [Ben73]; [AF87];

System	$\begin{cases} u_t = vv_1 + 2D_x(uw) \\ v_t = 2u_1 + D_x(vw) \\ w_t = 2v_1 + 2ww_1 \end{cases}$
Hamiltonian	$uw + \frac{v^2}{2}$
Cosymplectic	$\begin{pmatrix} uD_x + D_x u & vD_x & wD_x \\ D_x v & 0 & 2D_x \\ D_x w & 2D_x & 0 \end{pmatrix}$
Symplectic	$\begin{pmatrix} 0 & 0 & D_x^{-1} \\ 0 & D_x^{-1} & 0 \\ D_x^{-1} & 0 & 0 \end{pmatrix}$
Recursion	$\begin{pmatrix} w & v & 2u + u_1 D_x^{-1} \\ 2 & 0 & v + v_1 D_x^{-1} \\ 0 & 2 & w + w_1 D_x^{-1} \end{pmatrix}$
Root	$\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 \\ xv_1 \\ xw_1 \end{pmatrix} + \alpha \begin{pmatrix} 3u \\ 2v \\ w \end{pmatrix}; \alpha \in \mathcal{C}$

9.39 Dispersive water wave system

Reference: [AF87];

System	$\begin{cases} u_t = D_x(uw) \\ v_t = -v_2 + 2D_x(vw) + uu_1 \\ w_t = w_2 - 2v_1 + 2ww_1 \end{cases}$
Hamiltonian	$vw + \frac{u^2}{2}$
Cosymplectic	$\begin{pmatrix} 0 & D_x u & 0 \\ uD_x & vD_x + D_x v & -D_x^2 + wD_x \\ 0 & D_x^2 + D_x w & -2D_x \end{pmatrix}$
Symplectic	$\begin{pmatrix} D_x^{-1} & 0 & 0 \\ 0 & 0 & D_x^{-1} \\ 0 & D_x^{-1} & 0 \end{pmatrix}$
Recursion	$\begin{pmatrix} 0 & 0 & u + u_1 D_x^{-1} \\ u & -D_x + w & 2v + v_1 D_x^{-1} \\ 0 & -2 & D_x + w + w_1 D_x^{-1} \end{pmatrix}$
Root	$\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 + \frac{3}{2}u \\ xv_1 + 2v \\ xw_1 + w \end{pmatrix}$

If $u = 0$, this system reduces to the `Broer–Kaup` system studied in [Gök96].

Appendix A

Some irreducibility results, by F. Beukers

The results in this appendix are obtained by F. Beukers, Mathematical Department, University of Utrecht and are published here with his kind permission. They are used in the proof of proposition 7-14.

Theorem A-1. *Consider the polynomial $g_k = x^k + y^k + z^k + (-x-y-z)^k$. Then g_k is absolutely irreducible if k is even. When k is odd it factors as $(x+y)(x+z)(y+z)h_k$ where h_k is absolutely irreducible.*

Proof. Consider the projective curve C defined by $g_k = 0$. Suppose $g_k = A.B$ where A, B are two polynomials of positive degree. Geometrically the curve C now consists of two components C_1, C_2 given by $A = 0, B = 0$ respectively. The curves C_1 and C_2 intersect in at least one point, which implies that the curve C has a singularity.

Let us now determine the singularities of C, i.e., the projective points (x, y, z) where all partial derivatives of g_k vanish. Hence

$$\begin{aligned} kx^{k-1} - k(-x-y-z)^{k-1} &= 0 \\ ky^{k-1} - k(-x-y-z)^{k-1} &= 0 \\ kz^{k-1} - k(-x-y-z)^{k-1} &= 0. \end{aligned}$$

We see that $x^{k-1} = y^{k-1} = z^{k-1} = w^{k-1}$ where $w = -x-y-z$. By taking $z = 1$, say, we can assume that x, y, w are $k-1$-st roots of unity such that $x+y+w+1 = 0$. Note that four complex numbers of the same absolute value can only add up to zero if they form the sides of a parallellogram with equal sides. Hence one of the x, y, w is -1 and the others are opposite. Suppose without loss of generality that $w = -1$ and $x = -y$. If k is even we see that $1 = z^{k-1} = -(-1) = -w^{k-1}$, contradicting $z^{k-1} = w^{k-1}$. Hence C is non-singular if k is even. In particular C is irreducible in this case.

Now suppose that k is odd. Then we have $3k-6$ singular points, namely $(\zeta, -\zeta, 1)$, $(\zeta, -1, 1)$, $(-1, \zeta, 1)$ where $\zeta^{k-1} = 1$. Note that we have a priori $3k-3$ singular points, but some of them coincide. Consider such a singular point, say

$(\zeta, -\zeta, 1)$ We study the singular point locally by introducing the coordinates $x = \zeta + u, y = -\zeta + v$. Up to 3^{rd}-order terms we find the local equation $((\zeta(u+v) - (u-v))(u+v) + \cdots$. Since the quadratic part factors in two distinct factors the singularity is simple, i.e., there are two distinct tangent lines through the point. Consider now the curves $(x+y)(x+z)(y+z) = 0$ and $h_k = 0$. These curves intersect in $3(k-3)$ points. Moreover, the first curve has 3 singularities. This accounts for the $3k-6$ singular points we found. Hence $h_k = 0$ cannot have any singular points and in particular it is irreducible. $\square$

Theorem A-2. *For any positive integer k the polynomial $G_k = x^k + y^k + z^k + u^k + (-x-y-z-u)^k$ is irreducible over $\mathbb{C}$.*

Proof. Suppose $G_k = A.B$ with A, B polynomials of positive degree. Then the projective hypersurface S given by $G_k = 0$ consists of two components S_1, S_2 given by $A = 0, B = 0$ respectively. These components intersect in an infinite number of points, which should all be singularities of S. Thus it suffices to show that S has finitely many singular points. We compute these singular points by setting the partial derivatives of G_k equal to zero,

$$\begin{aligned}
kx^{k-1} - k(-x-y-z-u)^{k-1} &= 0\\
ky^{k-1} - k(-x-y-z-u)^{k-1} &= 0\\
kz^{k-1} - k(-x-y-z-u)^{k-1} &= 0\\
ku^{k-1} - k(-x-y-z-u)^{k-1} &= 0.
\end{aligned}$$

From these equations follows in particular that $x^{k-1} = y^{k-1} = z^{k-1} = u^{k-1}$. Hence the coordinates differ by a $k-1$-st root of unity. In particular we get finitely many singular points. $\square$

Appendix B

Levi–Civita connections

Abstract B-1. *We define the notions of torsion and Levi–Civita connection, assuming the existence of a metric tensor g_2 and derive the classical* ***Bianchi*** *identities. This is to illustrate the use of a complex that is not formed by antisymmetric cochains in* ***Riemannian*** *geometry.*

We assume $X, Y.Z \in \mathfrak{h}^m_{\mathcal{R}}$ in this appendix.

B.1 Torsion

Definition B-2. *Let $id_{\mathfrak{h}^m_{\mathcal{R}}} \in C^1_m(\mathfrak{h}, \mathfrak{h}^m_{\mathcal{R}}, \mathcal{A})$ be defined by $id_{\mathfrak{h}^m_{\mathcal{R}}}(Y) = Y$. The* **torsion** *of a connection $\nabla^0_m \in \Gamma^0_m(\mathfrak{h}, \mathfrak{h}^m_{\mathcal{R}}, \mathcal{A})$ is $d^1_m id_{\mathfrak{h}^m_{\mathcal{R}}}$, i.e., $T : \mathfrak{h}^{m+1}_{\mathcal{R}} \to \mathfrak{h}^m_{\mathcal{R}}$ given by*

$$T(X, Y) = \nabla^0_m(X)Y - \nabla^0_m(Y)X - \pi^1_m(X)Y.$$

A connection ∇^0_m is called π^1_m**-symmetric** *if its torsion is zero.*

Proposition B-3. *The statements*

- *π^1_m is antisymmetric.*
- *π^1_m is $2\pi^1_m$-symmetric.*

are equivalent.

Proof. Take $\nabla^0_m = \pi^1_m$ and $\pi^1_m = 2\pi^1_m$ in definition B-2. Then

$$0 = \pi^1_m(X)Y - \pi^1_m(Y)X - 2\pi^1_m(X)Y = -\pi^1_m(X)Y - \pi^1_m(Y)X$$

and we are done. □

Proposition B-4. *Suppose $\nabla^0_m \in \Gamma^0_m(\mathfrak{h}, \mathfrak{h}^m_{\mathcal{R}}, \mathcal{A})$ is π^1_m-symmetric. Then*

$$\mathcal{C}(\nabla^0_m)(X, Y)Z + \mathcal{C}(\nabla^0_m)(Z, X)Y + \mathcal{C}(\nabla^0_m)(Y, Z)X = \mathcal{C}(\pi^1_m)(X, Y)Z.$$

Proof. We compute

$$\begin{aligned}
&\mathcal{C}(\nabla^0_m)(X,Y)Z + \mathcal{C}(\nabla^0_m)(Z,X)Y + \mathcal{C}(\nabla^0_m)(Y,Z)X = \\
&= \nabla^0_m(X)\nabla^0_m(Y)Z - \nabla^0_m(Y)\nabla^0_m(X)Z - \nabla^0_m(\pi^1_m(X)Y)Z \\
&+ \nabla^0_m(Z)\nabla^0_m(X)Y - \nabla^0_m(X)\nabla^0_m(Z)Y - \nabla^0_m(\pi^1_m(Z)X)Y \\
&+ \nabla^0_m(Y)\nabla^0_m(Z)X - \nabla^0_m(Z)\nabla^0_m(Y)X - \nabla^0_m(\pi^1_m(Y)Z)X \\
&= \nabla^0_m(X)\pi^1_m(Y)Z - \nabla^0_m(\pi^1_m(Y)Z)X + \nabla^0_m(Y)\pi^1_m(Z)X \\
&- \nabla^0_m(\pi^1_m(Z)X)Y + \nabla^0_m(Z)\pi^1_m(X)Y - \nabla^0_m(\pi^1_m(X)Y)Z \\
&= \pi^1_m(X)\pi^1_m(Y)Z - \pi^1_m(Y)\pi^1_m(X)Z - \pi^1_m(\pi^1_m(X)Y)Z \\
&= \mathcal{C}(\pi^1_m)(X,Y)Z.
\end{aligned}$$

This proves the proposition. □

B.2 The Levi–Civita connection

Assume that $g_2 \in C^2_m(\mathfrak{h}, V, \mathcal{A})$ such that, for $\nabla^0_m \in \Gamma^0_m(\mathfrak{h}, V, \mathcal{A})$,

$$\nabla^0_m(Z)g_2(X,Y) = g_2(\bar{\pi}^1_m(Z)X, Y) + g_2(X, \bar{\pi}^1_m(Z)Y),$$

where $\bar{\pi}^1_m \in C^1_m(\mathfrak{h}, End_{\mathcal{C}}(\mathfrak{h}^m_{\mathcal{R}}), \mathcal{S})$.

Proposition B-5. *Then $\bar{\pi}^1_m \in \Gamma^1_m(\mathfrak{h}, \mathfrak{h}^m_{\mathcal{R}}, \mathcal{A})$, assuming g_2 to be nondegenerate, i.e., $g_2(Y,X) = 0$ for all $Y \in \mathfrak{h}^m_{\mathcal{R}}$ implies $X = 0$.*

Proof. We compute, for $r \in \mathcal{A}$,

$$\begin{aligned}
&\nabla^0_m(Z)(r \circ g_2(X,Y)) = r \circ \nabla^0_m(Z)g_2(X,Y) + \gamma^0_m(Z)(r) \circ g_2(X,Y) = \\
&= r \circ g_2(\bar{\pi}^1_m(Z)X, Y) + r \circ g_2(X, \bar{\pi}^1_m(Z)Y) + g_2(\gamma^0_m(Z)(r) \circ X, Y)
\end{aligned}$$

and

$$\begin{aligned}
&\nabla^0_m(Z)g_2(r \circ X, Y) = g_2(\bar{\pi}^1_m(Z)(r \circ X), Y) + g_2(r \circ X, \bar{\pi}^1_m(Z)Y) = \\
&= g_2(\bar{\pi}^1_m(Z)(r \circ X), Y) + r \circ g_2(X, \bar{\pi}^1_m(Z)Y).
\end{aligned}$$

Since $g_2 \in C^2_m(\mathfrak{h}, V, \mathcal{A})$, we have $\nabla^0_m(Z)rg_2(X,Y) = \nabla^0_m(Z)g_2(rX,Y)$. It leads to $\bar{\pi}^1_m(Z)(r \circ X) = r \circ \bar{\pi}^1_m(Z)X + \gamma^0_m(Z)(r) \circ X$ from the nondegeneracy of g_2. □

Proposition B-6. *Let $\bar{\pi}^1_m$ be π^1_m-symmetric (cf. def. B-2). If $\pi^1_m \in \Gamma^1_m(\mathfrak{h}, \mathfrak{h}^m_{\mathcal{R}}, \mathcal{A})$ is an antisymmetric connection, then $\bar{\pi}^1_m$ is an $\mathcal{A}$-linear connection. Moreover,*

$$d^2_m g_2(X,Y,Z) = 2g_{2,\vee}(Y, \bar{\pi}^1_m(Z)X),$$

where $g_{2,\vee}(X,Y) = \frac{1}{2}(g_2(X,Y) + g_2(Y,X))$ is the symmetric part of g_2.

Proof. We know that $\bar{\pi}^1_m$ is π^1_m-symmetric. This implies that

$$\bar{\pi}^1_m(X)Y - \bar{\pi}^1_m(Y)X = \pi^1_m(X)Y.$$

The $\mathcal{A}$-linearity of $\bar{\pi}^1_m$ follows from, for $r \in \mathcal{A}$,

$$\begin{aligned}
&\bar{\pi}^1_m(r \circ X)Y = \pi^1_m(r \circ X)Y + \bar{\pi}^1_m(Y)(r \circ X) = \\
&= \ -\pi^1_m(Y)(r \circ X) + r \circ \bar{\pi}^1_m(Y)X + \gamma^0_m(Y)(r) \circ X \\
&= \ -r \circ \pi^1_m(Y)X + r \circ \bar{\pi}^1_m(Y)X \\
&= \ r \circ \bar{\pi}^1_m(X)Y.
\end{aligned}$$

Using formula 3.4.5 we obtain

$$\begin{aligned}
&d^2_m g_2(X, Y, Z) = \\
&= \ \nabla^0_m(X)g_2(Y, Z) - \nabla^0_m(Y)g_2(X, Z) + \nabla^0_m(Z)g_2(X, Y) \\
&- \ g_2(\pi^1_m(X)Y, Z) - g_2(Y, \pi^1_m(X)Z) + g_2(X, \pi^1_m(Y)Z) \\
&= \ g_2(\bar{\pi}^1_m(X)Y, Z) + g_2(Y, \bar{\pi}^1_m(X)Z) - g_2(\bar{\pi}^1_m(Y)X, Z) \\
&- \ g_2(X, \bar{\pi}^1_m(Y)Z) + g_2(\bar{\pi}^1_m(Z)X, Y) + g_2(X, \bar{\pi}^1_m(Z)Y) \\
&- \ g_2(\bar{\pi}^1_m(X)Y, Z) + g_2(\bar{\pi}^1_m(Y)X, Z) + g_2(Y, \bar{\pi}^1_m(Z)X) \\
&- \ g_2(Y, \bar{\pi}^1_m(X)Z) + g_2(X, \bar{\pi}^1_m(Y)Z) - g_2(X, \bar{\pi}^1_m(Z)Y) \\
&= \ g_2(Y, \bar{\pi}^1_m(Z)X) + g_2(\bar{\pi}^1_m(Z)X, Y) \\
&= \ 2g_{2,\vee}(Y, \bar{\pi}^1_m(Z)X).
\end{aligned}$$

Thus we prove the proposition. □

One can turn the above construction around:

Proposition B-7. *Let $g_2 \in C^2_{m,\vee}(\mathfrak{h}, V, \mathcal{A})$ (i.e., $g_2 = g_{2,\vee}$) be nondegenerate. Define the $\bar{\pi}^1_m$ by $2g_{2,\vee}(Y, \bar{\pi}^1_m(Z)X) = d^2_m g_2(X, Y, Z)$. Then we have*

$$\begin{aligned}
&\bar{\pi}^1_m(X)Y - \bar{\pi}^1_m(Y)X = \pi^1_m(X)Y, \\
&\nabla^0_m(Z)g_2(X, Y) = g_2(\bar{\pi}^1_m(Z)X, Y) + g_2(X, \bar{\pi}^1_m(Z)Y),
\end{aligned}$$

where $\pi^1_m \in \Gamma^1_m(\mathfrak{h}, \mathfrak{h}^m_{\mathcal{R}}, \mathcal{A})$ is an antisymmetric connection.

Proof. We have

$$\begin{aligned}
&2g_{2,\vee}(Y, \bar{\pi}^1_m(X)Z - \bar{\pi}^1_m(Z)X) - \pi^1_m(X)Z = \\
&= \ \nabla^0_m(Z)g_2(Y, X) - \nabla^0_m(Y)g_2(Z, X) + \nabla^0_m(X)g_2(Z, Y) \\
&- \ g_2(\pi^1_m(Z)Y, X) - g_2(Y, \pi^1_m(Z)X) + g_2(Z, \pi^1_m(Y)X) \\
&- \ \nabla^0_m(X)g_2(Y, Z) + \nabla^0_m(Y)g_2(X, Z) - \nabla^0_m(Z)g_2(X, Y) \\
&+ \ g_2(\pi^1_m(X)Y, Z) + g_2(Y, \pi^1_m(X)Z) - g_2(X, \pi^1_m(Y)Z) \\
&- \ 2g_2(Y, \pi^1_m(X)Z) \\
&= \ 0.
\end{aligned}$$

Therefore, $\bar{\pi}^1_m(X)Z - \bar{\pi}^1_m(Z)X = \pi^1_m(X)Z$ due to the nondegeneracy of g_2.

Notice that $2\nabla^0_m(Z)g_2(X,Y) = d^2_m g_2(X,Y,Z) + d^2_m g_2(Y,X,Z)$ when g_2 is symmetric. The second property follows immediately. □

By proposition B-5, $\bar{\pi}^1_m$ is a connection. This leads to the following definition.

Definition B-8. *Let $g_2 \in C^2_{m,\vee}(\mathfrak{h}, V, \mathcal{A})$ be nondegenerate and π^1_m be an antisymmetric connection. Then*

$$2g_2(Y, \bar{\pi}^1_m(Z)X) = d^2_m g_2(X,Y,Z)$$

defines a **Levi–Civita connection** *([Nic96] p. 120) with the following properties:*

- $\bar{\pi}^1_m(X)Y - \bar{\pi}^1_m(Y)X = \pi^1_m(X)Y,$
- $\nabla^0_m(Z)g_2(X,Y) = g_2(\bar{\pi}^1_m(Z)X, Y) + g_2(X, \bar{\pi}^1_m(Z)Y).$

Theorem B-9. *The* **Riemann curvature tensor** $\mathcal{C}(\bar{\pi}^1_m)$ *satisfies the following identities*

1. $g_2(\mathcal{C}(\bar{\pi}^1_m)(X,Y)U,V) + g_2(\mathcal{C}(\bar{\pi}^1_m)(Y,X)U,V) = 0,$
2. $g_2(\mathcal{C}(\bar{\pi}^1_m)(X,Y)U,V) + g_2(\mathcal{C}(\bar{\pi}^1_m)(X,Y)V,U) = \mathcal{C}(\nabla^0_m)(X,Y)g_2(U,V),$
3. *If* $\mathcal{C}(\nabla^0_m) = \mathcal{C}(\pi^1_m) = 0,$ $g_2(\mathcal{C}(\bar{\pi}^1_m)(X,Y)U,V) = g_2(\mathcal{C}(\bar{\pi}^1_m)(U,V)X,Y).$

Proof. Ad 1: Clearly, $\mathcal{C}(\bar{\pi}^1_m)(X,Y) + \mathcal{C}(\bar{\pi}^1_m)(Y,X) = 0.$

Ad 2: It is sufficient to show that $g_2(\mathcal{C}(\bar{\pi}^1_m)(X,Y)U,U) = 0$. Since we have $\nabla^0_m(Z)g_2(X,X) = 2g_2(\bar{\pi}^1_m(Z)X,X)$, it follows that

$$\begin{aligned}
&2g_2(\mathcal{C}(\bar{\pi}^1_m)(X,Y)U,U) = \\
&= \; 2g_2([\bar{\pi}^1_m(X), \bar{\pi}^1_m(Y)]U - \bar{\pi}^1_m(\pi^1_m(X)Y)U, U) \\
&= \; 2g_2(U, \bar{\pi}^1_m(X)\bar{\pi}^1_m(Y)U - \bar{\pi}^1_m(Y)\bar{\pi}^1_m(X)U) - \nabla^0_m(\pi^1_m(X)Y)g_2(U,U) \\
&= \; 2\nabla^0_m(X)g_2(U, \bar{\pi}^1_m(Y)U) - 2g_2(\bar{\pi}^1_m(X)U, \bar{\pi}^1_m(Y)U) \\
&- \; 2\nabla^0_m(Y)g_2(U, \bar{\pi}^1_m(X)U) + 2g_2(\bar{\pi}^1_m(Y)U, \bar{\pi}^1_m(X)U) - \nabla^0_m(\pi^1_m(X)Y)g_2(U,U) \\
&= \; \nabla^0_m(X)\nabla^0_0(Y)g_2(U,U) - \nabla^0_m(Y)\nabla^0_m(X)g_2(U,U) - \nabla^0_m(\pi^1_m(X)Y)g_2(U,U) \\
&= \; \mathcal{C}(\nabla^0_m)(X,Y)g_2(U,U).
\end{aligned}$$

Ad 3: We start by listing the 24 permutations of

$$R_{1234} = g_2(\mathcal{C}(\bar{\pi}^1_m)(X_1,X_2)X_3,X_4)$$

and we group these as orbits of cyclic permutations of the first three arguments. By adding these we obtain 8 relations using proposition B-4. Due to the antisymmetry in the first two and the last two arguments this reduces to 4 equations. We eliminate the antisymmetries and write R_{ij} for R_{ijkl} with $i < j$ and $k < l$. There are 6 of these R_{ij} and the obey the following 4 equations.

$$\begin{aligned} R_{12} - R_{13} + R_{23} &= 0 \\ R_{14} + R_{12} - R_{24} &= 0 \\ R_{34} - R_{13} + R_{14} &= 0 \\ R_{23} - R_{24} + R_{34} &= 0. \end{aligned}$$

We now introduce $\Delta_1 = R_{12} - R_{34}, \Delta_2 = R_{13} - R_{24}, \Delta_3 = R_{14} - R_{23}$ and $\Delta_0 = R_{24} - R_{23} - R_{34}$. The equations then reduce to

$$\begin{aligned} \Delta_1 - \Delta_2 &= \Delta_0 \\ \Delta_1 + \Delta_3 &= \Delta_0 \\ -\Delta_2 + \Delta_3 &= \Delta_0 \\ \Delta_0 &= 0, \end{aligned}$$

and it follows immediately that $\Delta_i = 0$, $i = 0, \cdots, 3$. This proves the statement. It follows that the R_{ijkl} are generated by the two expressions $R_\pm = R_{1234} \pm R_{1324}$. $\square$

Remark B-10. *In the two dimensional situation where one has at every point only two independent elements X_1 and X_2 in $\mathfrak{h}$, this reduces to $R_+ = 2R_{1221}$ and $R_- = 0$. In this case R_+ is called* **the** *curvature of g_2.*

Appendix C

Examples of cohomology computations

Abstract C-1. *In this appendix we compute the cohomologies for some concrete examples, which are not very important by themselves. They illustrate how complicated things can be even in the simplest cases. We also determine the class of Hamiltonian and symplectic vectorfields in some cases.*

In the following sections some examples are given illustrating the abstract theory developed in chapters 2, 3 and 4.

C.1 Hopf fibration

The following example is based on an example in [Mac87], appendix A, describing the `Hopf` bundle.

We let $\mathcal{C} = \mathcal{A} = \mathbb{R}$, $\mathfrak{g} = \mathfrak{h} = \mathbb{C}$, $V = \mathbb{C}^2$, $\gamma_0^0 = \pi_0^1 = 0$ and

$$\nabla_0^0(x) = \begin{pmatrix} 0 & x \\ \bar{x} & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathcal{C}(\nabla_0^0)(x, y) &= \begin{pmatrix} 0 & x \\ \bar{x} & 0 \end{pmatrix}\begin{pmatrix} 0 & y \\ \bar{y} & 0 \end{pmatrix} - \begin{pmatrix} 0 & y \\ \bar{y} & 0 \end{pmatrix}\begin{pmatrix} 0 & x \\ \bar{x} & 0 \end{pmatrix} \\ &= (x\bar{y} - \bar{x}y)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

Clearly, the curvature of ∇_1^0, with $\nabla_1^0(x \wedge y) = \mathcal{C}(\nabla_0^0)(x, y)$, is zero. We now compute some cohomology. First we start with $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \in C_0^0(\mathbb{C}, \mathbb{C}^2, \mathbb{R})$. Then

$$\begin{aligned} d_0^0\alpha(x) &= \nabla_0^0(X)\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 & x \\ \bar{x} & 0 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \alpha_2 x \\ \alpha_1 \bar{x} \end{pmatrix} \\ &= \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_1 \end{pmatrix}\begin{pmatrix} x \\ \bar{x} \end{pmatrix}. \end{aligned}$$

Next, let $\beta \in C_0^1(\mathbb{C}, \mathbb{C}^2, \mathbb{R})$ be defined by

$$\beta(x) = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} \cdot \begin{pmatrix} x \\ \bar{x} \end{pmatrix} = \begin{pmatrix} 0 & \beta_{12} \\ \beta_{21} & 0 \end{pmatrix} \begin{pmatrix} x \\ \bar{x} \end{pmatrix} + d_0^0 \begin{pmatrix} \beta_{22} \\ \beta_{11} \end{pmatrix} (x).$$

Then

$$\begin{aligned} d_0^1\beta(x, y) &= \nabla_0^0(x)\beta(y) - \nabla_0^0(y)\beta(x) \\ &= (x\bar{y} - \bar{x}y) \begin{pmatrix} \beta_{22} \\ -\beta_{11} \end{pmatrix}. \end{aligned}$$

We see that we cannot define cohomology here, since the curvature was not zero. We now repeat this with arguments in $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$. Then, with $\alpha \in C_1^0(\mathbb{C}, \mathbb{C}^2, \mathbb{R})$ we find

$$\begin{aligned} d_1^0\alpha(x \otimes y) &= \nabla_1^0(x \otimes y)\alpha = (x\bar{y} - \bar{x}y) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \\ &= \begin{pmatrix} x & \bar{x} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y \\ \bar{y} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}. \end{aligned}$$

We draw the conclusion that $H_1^0(\mathbb{C}, \mathbb{C}^2, \mathbb{R}) = 0$. Next, let $\beta \in C_1^1(\mathbb{C}, \mathbb{C}^2, \mathbb{R})$ be defined by $\beta = \begin{pmatrix} \beta^1 \\ \beta^2 \end{pmatrix}$,

$$\beta^i(x) = \begin{pmatrix} x & \bar{x} \end{pmatrix} \begin{pmatrix} \beta^i_{11} & \beta^i_{12} \\ \beta^i_{21} & \beta^i_{22} \end{pmatrix} \begin{pmatrix} y \\ \bar{y} \end{pmatrix}, i = 1, 2.$$

Then

$$\begin{aligned} d_1^1\beta(x \otimes y, u \otimes v) &= \nabla_1^0(x \otimes y)\beta(u \otimes v) - \nabla_1^0(u \otimes v)\beta(x \otimes y) \\ &= (x\bar{y} - \bar{x}y) \begin{pmatrix} \beta^1 \\ -\beta^2 \end{pmatrix} (u \otimes v) - (u\bar{v} - \bar{u}v) \begin{pmatrix} \beta^1 \\ -\beta^2 \end{pmatrix} (x \otimes y) \\ &= (x\bar{y} - \bar{x}y) \begin{pmatrix} \beta^1_{11}uv + \beta^1_{12}u\bar{v} + \beta^1_{21}\bar{u}v + \beta^1_{22}\bar{u}\bar{v} \\ -\beta^2_{11}uv - \beta^2_{12}u\bar{v} - \beta^2_{21}\bar{u}v - \beta^2_{22}\bar{u}\bar{v} \end{pmatrix} \\ &- (u\bar{v} - \bar{u}v) \begin{pmatrix} \beta^1_{11}xy + \beta^1_{12}x\bar{y} + \beta^1_{21}\bar{x}y + \beta^1_{22}\bar{x}\bar{y} \\ -\beta^2_{11}xy - \beta^2_{12}x\bar{y} - \beta^2_{21}\bar{x}y - \beta^2_{22}\bar{x}\bar{y} \end{pmatrix} \\ &= (x\bar{y}\bar{u}v - \bar{x}yu\bar{v}) \begin{pmatrix} (\beta^1_{21} + \beta^1_{12}) \\ -(\beta^2_{12} + \beta^2_{21}) \end{pmatrix} \\ &+ ((x\bar{y} - \bar{x}y)uv - xy(u\bar{v} - \bar{u}v)) \begin{pmatrix} \beta^1_{11} \\ -\beta^2_{11} \end{pmatrix} \\ &- ((x\bar{y} - \bar{x}y)\bar{u}\bar{v} - \bar{x}\bar{y}(u\bar{v} - \bar{u}v)) \begin{pmatrix} \beta^1_{22} \\ -\beta^2_{22} \end{pmatrix}. \end{aligned}$$

We draw the conclusion that $H_1^1(\mathbb{C}, \mathbb{C}^2, \mathbb{R}) = 0$.

C.2 A very small example

We consider the Lie algebra $\mathfrak{b}_+$ spanned by the elements h and m, with commutation relation $\pi_0^1(h)m = 2m$. Let $V = \mathbb{R}^2$ and $\mathcal{A} = \mathbb{R}$, then a representation is defined by

$$\nabla_0^0(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \nabla_0^0(m) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We denote an element $X \in \mathfrak{b}_+$ by $X = x_1 h + x_2 m$.

Remark C-2. *In general, one can consider the x_i as elements in $\mathfrak{g}^\star$, that is linear $\mathcal{C}$-valued functionals. This notation enables us to write for instance linear functionals, i.e., elements in $C_0^1(\mathfrak{h}, V, \mathcal{C})$ as endomorphisms of $\mathfrak{h}^\star$.*

We now compute the $d_0^0 y(X) = \nabla_0^0(X)y$ of an element $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in V$:

$$d_0^0 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(X) = \begin{pmatrix} x_1 & x_2 \\ 0 & -x_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ -y_2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Let $\alpha_1 \in C_0^1(\mathfrak{b}_+, V, \mathcal{C})$ be defined by

$$\alpha_1(h) = \begin{pmatrix} \alpha_1^{11} \\ \alpha_1^{12} \end{pmatrix}, \quad \alpha_1(m) = \begin{pmatrix} \alpha_1^{21} \\ \alpha_1^{22} \end{pmatrix}.$$

Then, for $Y = y_1 h + y_2 m$,

$$\begin{aligned} d_0^1\alpha_1(X,Y) &= \nabla_0^0(X)\alpha_1(Y) - \nabla_0^0(Y)\alpha_1(X) - \alpha_1(\pi_0^1(X)Y) \\ &= \begin{pmatrix} x_1 & x_2 \\ 0 & -x_1 \end{pmatrix} \begin{pmatrix} \alpha_1^{11} & \alpha_1^{21} \\ \alpha_1^{12} & \alpha_1^{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &- \begin{pmatrix} y_1 & y_2 \\ 0 & -y_1 \end{pmatrix} \begin{pmatrix} \alpha_1^{11} & \alpha_1^{21} \\ \alpha_1^{12} & \alpha_1^{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - 2 \begin{pmatrix} \alpha_1^{21} \\ \alpha_1^{22} \end{pmatrix} (x_1 y_2 - x_2 y_1) \\ &= - \begin{pmatrix} \alpha_1^{12} + \alpha_1^{21} \\ 3\alpha_1^{22} \end{pmatrix} (x_1 y_2 - x_2 y_1). \end{aligned}$$

We see that $Z_0^1(\mathfrak{b}_+, V, \mathcal{C}) = B_0^1(\mathfrak{b}_+, V, \mathcal{C})$, i.e., $H_0^1(\mathfrak{b}_+, V, \mathcal{C}) = 0$.

⋆ **Remark C-3.** *This does not follow from* **Whitehead***'s first lemma, since $\mathfrak{b}_+$ contains the nontrivial commutative ideal spanned by m, and therefore is not semisimple.*

Since an arbitrary element of $C_{0,\wedge}^2(\mathfrak{b}_+, V, \mathcal{C})$ can be written as

$$\gamma(X,Y) = (x_1 y_2 - x_2 y_1) \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix},$$

and $d_0^2\gamma = 0$ (since $\bigwedge^3 b_+ = 0$), we see that $B_{0,\wedge}^2(\mathfrak{b}_+, V, \mathcal{C}) = Z_{0,\wedge}^2(\mathfrak{b}_+, V, \mathcal{C})$, or $H_{0,\wedge}^2(\mathfrak{b}_+, V, \mathcal{C}) = 0$. Since (by a fairly long, but straightforward computation) we obtain $Z^2_0(\mathfrak{b}_+, V, \mathcal{C}) = Z^2_{0,\wedge}(\mathfrak{b}_+, V, \mathcal{C})$, the result follows for all $\omega_2 \in Z^2_0(\mathfrak{b}_+, V, \mathcal{C})$ and we have $H^2_0(\mathfrak{b}_+, V, \mathcal{C}) = 0$.

It follows from an elementary calculation that the Hamiltonian vectorfields consist of the ideal generated by m. The symplectic structure is $\omega_{2,<m>}(X,Y) = (x_1y_2 - x_2y_1)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The Hamiltonian of x_2m is $\begin{pmatrix} -x_2 \\ 0 \end{pmatrix}$.

We now compute the symplectic vectorfield. For $X \in \mathfrak{b}_+$, we have

$$(\nabla^2_0(X)\omega_2)(Y,Z) = (y_1z_2 - y_2z_1)\begin{pmatrix} -\omega^1_2x_1 + \omega^2_2x_2 \\ -3\omega^2_2x_1 \end{pmatrix},$$

where $\omega_2(Y,Z) = (y_1z_2 - y_2z_1)\begin{pmatrix} \omega^1_2 \\ \omega^2_2 \end{pmatrix}$. The condition of symplectic vectorfield $\nabla^2_0(X)\omega_2 = 0$ reduces to $x_1 = 0$ and $\omega^2_2 = 0$. It follows that $\mathfrak{Sym}_{\omega_2}(\mathfrak{b}_+)$ is also spanned by m, so that we have $\mathfrak{Ham}_{\omega_2}(\mathfrak{b}_+) = \mathfrak{Sym}_{\omega_2}(\mathfrak{b}_+)$. The general theory leads to exactly the same conclusion, but we have included the computations here as an example. symplectic form

C.3 Same Lie algebra, another representation

We now let $V = \mathfrak{b}_+ \wedge \mathfrak{b}_+$. The representation is the ad_2 representation, i.e., $\nabla^0_0(X)h \wedge m = \pi^1_m(X)h \wedge m + h \wedge \pi^1_m(X)m$. Take $\beta_0 = \beta^1_0 h \wedge m$. Then, with $X = x_1h + x_2m$,

$$d^0_0\beta_0(X) = \beta^1_0x_2\pi^1_m(m)h \wedge m + \beta^1_0x_1h \wedge \pi^1_m(h)m = 2\beta^1_0x_1h \wedge m.$$

Next we let $\alpha_1(X) = (\alpha^1_1x_1 + \alpha^2_1x_2)h \wedge m$. Then

$$\begin{aligned} d^1_0\alpha_1(X,Y) &= \nabla^0_0(X)\alpha_1(Y) - \nabla^0_0(Y)\alpha_1(X) - \alpha_1(\pi^1_m(X)Y) = \\ &= (\gamma_1y_1 + \gamma_2y_2)2x_1h \wedge m - (\gamma_1x_1 + \gamma_2x_2)2y_1h \wedge m \\ &\quad -2\gamma_2(x_1y_2 - x_2y_1)h \wedge m \\ &= 0. \end{aligned}$$

It follows that $H^1_0(\mathfrak{b}_+, \mathfrak{b}_+ \wedge \mathfrak{b}_+, \mathcal{C})$ is spanned by α_1 such that $\alpha_1(X) = x_2h \wedge m$.

An arbitrary element of $C^2_{m,\wedge}(\mathfrak{b}_+, \mathfrak{b}_+ \wedge \mathfrak{b}_+, \mathcal{C})$ is of the form $\gamma(X,Y) = \gamma_0(x_1y_2 - x_2y_1)h \wedge m$. Again $d^2_0\gamma = 0$ since $\bigwedge^3 \mathfrak{b}_+ = 0$. Therefore $H^2_{0,\wedge}$ is spanned by $\gamma(X,Y) = (x_1y_2 - x_2y_1)h \wedge m$. The higher cohomology spaces are all zero.

C.4 Another small example

In this section we let $\mathfrak{g} = \mathfrak{gl}(\mathbb{R}^2)$ and $V = \mathbb{R}$. As the representation we take $\nabla^0_0(X) = tr(X)$. Since $\nabla^0_0(\pi^1_0(X)Y) = 0$, this is indeed a representation.

Let $\beta_0 \in V = C^0_0(\mathfrak{h}, V, \mathcal{C})$. Then $d^0_0\beta_0(X) = \nabla^0_0(X)\beta_0 = \beta_0tr(X)$. Next, let $\alpha \in C^1_0(\mathfrak{h}, V, \mathcal{C})$, i.e.,

$$\alpha_1(X) = \alpha^{11}_1x_{11} + \alpha^{12}_1x_{12} + \alpha^{21}_1x_{21} + \alpha^{22}_1x_{22},$$

where $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$. Then

$$\begin{array}{rl} d_0^1\alpha_1(X,Y) = \nabla_0^0(X)\alpha_1(Y) - \nabla_0^0(Y)\alpha_1(X) - \alpha_1(\pi_0^1(X)Y) = & \\ = & (x_{11}+x_{22})(\alpha_1^{11}y_{11}+\alpha_1^{12}y_{12}+\alpha_1^{21}y_{21}+\alpha_1^{22}y_{22}) \\ - & (\alpha_1^{11}x_{11}+\alpha_1^{12}x_{12}+\alpha_1^{21}x_{21}+\alpha_1^{22}x_{22})(y_{11}+y_{22}) \\ - & \alpha_1^{11}(x_{12}y_{21}-x_{21}y_{12}) - \alpha_1^{12}(x_{11}y_{12}+x_{12}y_{22}-x_{12}y_{11}-x_{22}y_{12}) \\ - & \alpha_1^{21}(x_{21}y_{11}+x_{22}y_{21}-x_{11}y_{21}-x_{21}y_{22}) - \alpha_1^{22}(x_{21}y_{12}-x_{12}y_{21}) \\ = & (\alpha_1^{11}-\alpha_1^{22})(x_{22}y_{11}-x_{11}y_{22}-x_{12}y_{21}+x_{21}y_{12}) \\ + & 2\alpha_1^{12}(x_{22}y_{12}-x_{12}y_{22}) + 2\alpha_1^{21}(x_{11}y_{21}-x_{21}y_{11}). \end{array}$$

We see that $d_0^1\alpha_1 = 0$ is equivalent to $\alpha_1 = d_0^0\alpha_1^{11}$. In other words, $Z_0^1(\mathfrak{g}, V, \mathcal{C}) = B_0^1(\mathfrak{g}, V, \mathcal{C})$, i.e., $H_0^1(\mathfrak{h}, V, \mathcal{C}) = 0$.

$\star$ **Remark C-4.** *This is again not following from* ***Whitehead****'s first lemma, since $\mathfrak{gl}(2,\mathbb{R})$ is not semisimple, as it is containing the commutative ideal generated by the identity. But this ideal is equal to the center $\mathfrak{z}$, and has zero intersection with $[\mathfrak{g},\mathfrak{g}]$. Thus one can write $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{sl}(2,\mathbb{R})$. Therefore one has $H_0^1(\mathfrak{h}, V, \mathcal{C}) = H_0^1(\mathfrak{z}, V, \mathcal{C})$.*

An again rather extensive computation shows that one also has $H_0^2(\mathfrak{h}, V, \mathcal{C}) = 0$, so that we can write an arbitrary element $\omega_2 \in Z_0^2(\mathfrak{g}, V, \mathcal{C})$ as $d_0^1\alpha_1$. We can now ask for which $X \in \mathfrak{g}$ we can write

$$\iota_0^2(X)d_0^1\alpha_1 = d_0^0\beta_0.$$

Since the representation space is one dimensional, there is basically only one Hamiltonian possible, namely 1 (or multiples of 1). Thus one can only hope to find for a given X a symplectic structure $d_0^1\alpha_1$ which produces X by

$$\iota_0^2(X)d_0^1\alpha_1 = d_0^0\beta_0.$$

We write $\iota_0^2(X)d_0^1\alpha_1 = d_0^0\beta_0$ in matrix form

$$\iota_0^2(X)d_0^1\alpha_1 = \begin{pmatrix} -2\alpha_1^{21}x_{21} + (\alpha_1^{11}-\alpha_1^{22})x_{22} \\ (\alpha_1^{11}-\alpha_1^{22})x_{21} + 2\alpha_1^{12}x_{22} \\ 2\alpha_1^{21}x_{11} + (\alpha_1^{22}-\alpha_1^{11})x_{12} \\ (\alpha_1^{22}-\alpha_1^{11})x_{11} - 2\alpha_1^{12}x_{12} \end{pmatrix}^\top = \beta_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}^\top.$$

If we let $\beta_0 = -4\alpha_1^{12}\alpha_1^{21} - (\alpha_1^{11}-\alpha_1^{22})^2$, and assume $\beta_0 \neq 0$, then we find

$$X = \begin{pmatrix} -(\alpha_1^{11}-\alpha_1^{22}) & -2\alpha_1^{21} \\ -2\alpha_1^{12} & \alpha_1^{11}-\alpha_1^{22} \end{pmatrix}.$$

This implies that if $tr(X) = 0$ and $det(X) \neq 0$, then X is Hamiltonian. The occurrence of the *det* is not so strange as it may seem, since $det(X) = tr(\wedge^2 X)$.

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Dutch Summary – Nederlandse Samenvatting

Het onderwerp van dit proefschrift is: symmetrieën en behoudswetten van evolutievergelijkingen. Het beantwoord elementaire vragen als:

- Waarom is het zo moeilijk om nieuwe integreerbare systemen te vinden d.w.z. systemen die niet al in de hierarchie van een bekend integreerbaar systeem aanwezig zijn.
- Behoudenswetten en symmetrieën komen in hierarchieën met periodieke gaten. bijvoorbeeld voor de KdV vergelijkingen vindt men alleen symmetrieën van oneven orde. Waar komen deze gaten vandaan?

Een belanglijke constructie, namelijk het "complex van de formele variationele calculus", waarin alle verschillende objecten voor niet-lineaire evolution systemen worden samengebracht, ligt ten grondslag aan de theorie van gegeneraliseerde symmetrieën, behoudswetten en Hamiltoniaanse structuren. In hoofdstuk 2 and 3, bouwen we zo'n complex, geinspireerd door het werk van Dorfman [Dor93]. Ons complex is echter algemener aangezien de ring die we gebruiken tijdsafhankelijke functies kan bevatten, en het complex bestaat uit vormen die niet noodzakelijk antisymmetrisch zijn. Verder gebruiken we Leibniz algebra's in plaats van Lie algebra's. In dit complex vindt men al datgeen wat van belang is voor de studie van symmetrieën en behoudswetten van niet-lineaire evolutievergelijkingen, zoals cosymmetrieën, recursie operatoren, symplectische vormen.

We bewijzen in paragraaf 2.9 dat het vermoeden, "als een systeem één niet-triviale symmetrie heeft, dan heeft het er oneindig vele", waar is onder zekere technische condities. De stelling en het bewijs is pure Lie (of Leibniz) algebra, maar de condities kunnen geverifieerd worden door symbolische methoden, zoals geformuleerd in de hoofdstukken 7 en 8 en gebruikmakend van diophantische benaderingstheorie.

In principe kan de methode ook gebruikt worden voor systemen van evolutie vergelijkingen, maar tot dusver zijn de condities van de stelling alleen geverifieerd voor één klasse van voorbeelden met behulp van p-adische analyse [BSW98].

In hoofdstuk 4 motiveren we de definitie van de Nijenhuis operator en leiden we de voornaamste eigenschappen hiervan af en we formuleren de begrippen van symplectische en Hamiltoniaanse operatoren in de abstracte context waarin we het complex opgezet hebben. We leiden enige van de klassiek bekende eigenschappen en relaties van deze begrippen af.

In hoofdstuk 5 passen we de abstracte concepten toe op het complex van variationele calculus en we geven uitdrukkingen voor diverse soorten van invarianten van de evolutie vergelijking in termen van Fréchet afgeleiden. Dit verbindt de abstracte benadering met de meer gebruikelijke definities.

In hoofdstuk 6 formuleren en bewijzen we verschillende stellingen over de vorm van recursie en Nijenhuis operatoren. Deze resultaten zijn zeer nuttig in berekeningen, aangezien zij aangeven hoe men gecompliceerde uitdrukkingen in termen van bekende symmetrieën en cosymmetrieën kan weergeven. Zij maken ook mogelijk om te concluderen dat onder tamelijk zwakke condities deze operatoren goed gedefieerd zijn, dat wil zeggen: ze produceren, beginnend met een gegeven wortel, een oneindige hierarchie van symmetrieën. We geven een lijst van voorbeelden waar deze resultaten worden toegepast.

In hoofdstuk 7 introduceren we de symbolische methode, die ons in staat stelt vragen over de oplosbaarheid van niet-lineaire differentiaal vergelijkingen te vertalen in vragen over de deelbaarheid van polynomen.

In hoofdstuk 8 gebruiken we de symbolische methode om λ-homogenene scalaire vergelijkingen te klassificeren. For $\lambda > 0$ geven we de volledige lijst van 10 integreerbare vergelijkingen. Dit bewijs van de klassifikatie stelling geeft het antwoord op de vragen die in het begin gesteld werden.

De volledige analyse voor $\lambda = 0$ wordt gegeven in paragraaf 8.5.1. Het is interessant om te zien dat het enige niet-Hamiltoniaanse systeem afgeleid is van de *Potential Burgers* vergelijking.

In hoofdstuk 9 geven we een lijst of 39 integreerbare vergelijkingen, met hun recursie-, symplectic- en cosymplectic operators, voorzover bekend, en de wortels van de symmetrieën en schalingen. Deze zijn of reeds bekend in de literatuur of ze kunnen gevonden worden met onze nieuwe methoden. Met deze informatie kan men de symmetrieën en cosymmetrieën van iedere gegeven vergelijking berekenen.

In de appendices is materiaal verzameld, bestaande uit bewijzen en voorbeelden, dat niet zo goed paste in de tekst, maar toch interessant genoeg leek om hier toe te voegen.