Configurational Forces as Basic Concepts of Continuum Physics

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Springer

Applied Mathematical Sciences
Volume 137

Springer
New York
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Mathematics Subject Classification (1991): 73bxx, 73m25, 73a05

With seven illustrations.

Library of Congress Cataloging-in-Publication Data
Gurtin, Morton E.
Configurational forces as basic concepts of continuum physics /
Morton E. Gurtin.
p. cm. — (Applied mathematical sciences ; 137)
Includes bibliographical references.
ISBN 0-387-98667-7 (cloth : alk. paper)
1. Field theory (Physics) 2. Configuration space. I. Title.
II. Series: Applied mathematical sciences (Springer-Verlag New York
Inc.) ; v. 137.
QA1.A647 vol. 137
[QC173.7]
510 s—dc21
[530.14] 98-55407

ISBN 0-387-98667-7 Springer-Verlag New York Berlin Heidelberg SPIN 10698130

For my grandchildren Katie, Grant, and Liza

Contents

CHAPTER 1

Introduction[1]

a. Background

The notion of *force* is central to all of continuum mechanics. Classically, the response of a body to deformation is described by *standard* (Newtonian) forces consistent with balance laws for linear and angular momentum; these forces are well understood. That additional *configurational*[2] forces may be needed to describe phenomena associated with the material itself is clear from the beautiful work of Eshelby[3] on lattice defects and is at least intimated by Gibbs[4] in his discussion of multiphase equilibria.

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[1]I gratefully acknowledge many valuable discussions with P. Cermelli, E. Fried, A. I. Murdoch, P. Podio-Guidugli, A. Struthers, and P. Voorhees; much of the research discussed here was done with them. In particular, the insight afforded by the use of bulk and interfacial Eshelby tensors was pointed out to me by P. Podio-Guidugli, a comment that was central to my understanding of configurational forces. I would like to express my gratitude to the National Science Foundation, the Army Research Office, and the Department of Energy for their support of the research on which much of this book is based.

[2]I use the adjective *configurational* to differentiate these forces from classical Newtonian forces, which I refer to as *standard*. In the past I used the terms *accretive* and *deformational* rather than *configurational* and *standard*.

[3][1951, 1956, 1970, 1975]. Eshelby [1951] remarks that the idea of a force on a lattice defect goes back to "an interesting paper" of Burton [1892], a work that I am unable to comprehend. Cf. Peach and Koehler [1950], who discuss the configurational force on a dislocation loop, and Maugin [1993], whose monograph presents a comprehensive treatment of configurational forces (there called *material forces*) with a lengthy list of references.

Cf. also Nozieres [1989, p. 26], who uses the term *chemical* rather than *configurational* and writes: "Such a concept of 'chemical stresses,' although somewhat misleading, is often useful in assessing equilibrium shapes."

[4][1878, pp. 314–331].

Gibb's discussion is paraphrased by Cahn[5] as follows: "Solid surfaces can have their physical area changed in two ways, either by creating or destroying surface without changing surface structure and properties per unit area, or by an elastic strain . . . along the surface keeping the number of surface lattice sites constant . . . ." The creation of surface involves configurational forces, while stretching the surface involves standard forces.

The studies of Gibbs and Eshelby, and most related work, relegate configurational forces to a subsidiary status, because the statical theories are based on variational arguments and the generalizations to dynamics obtained by manipulation of the standard momentum balances. I take a different point of view. While I am not in favor of the capricious introduction of "fundamental physical laws," I do believe that *configurational forces should be viewed as basic objects consistent with their own force balance*. To help explain my reasons for this point of view, I sketch the typical treatment of a two-phase elastic solid within the formal framework of the calculus of variations.[6]

b. Variational definition of configurational forces

Consider a two-phase elastic body[7] B, neglecting thermal and compositional influences and interfacial energy. Suppose that the phases, α and β, occupy closed complementary subregions B_α and B_β of B, with the interface $\mathscr{S} = B_\alpha \cap B_\beta$ a smooth, oriented surface whose continuous *unit normal field* $\mathbf{m}$ points outward from B_α (Figure 1.1). Then, granted coherency, a *deformation* of B is a *continuous* function $\boldsymbol{y}$ that assigns to each material $\boldsymbol{X}$ in B a point $\boldsymbol{x} = \boldsymbol{y}(\boldsymbol{X})$ of space, has *deformation gradient*

$$\boldsymbol{F} = \nabla \boldsymbol{y}$$

smooth up to the interface from either side (but generally not across $\mathscr{S}$), has $\det \boldsymbol{F} > 0$, and for this discussion, is prescribed on ∂B.

Consider constitutive equations given the bulk free energy[8] Ψ at any point $\boldsymbol{X}$ in B when the deformation gradient $\boldsymbol{F}$ at $\boldsymbol{X}$ is known:

$$\Psi = \Psi_\alpha(\boldsymbol{F}, \boldsymbol{X}) \quad \text{in } B_\alpha, \qquad \Psi = \Psi_\beta(\boldsymbol{F}, \boldsymbol{X}) \quad \text{in } B_\beta, \tag{1–1}$$

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[5][1980].

[6]Cf. Eshelby [1970], Robin [1974], Larche and Cahn [1978], Grinfeld [1981], James [1981], Gurtin [1983].

[7]The body is identified with the region B of Euclidean space it occupies in a fixed reference configuration; to emphasize this, B is generally referred to as the *reference body*. Stresses and body forces are measured per unit area and volume in the *reference configuration*.

[8]I use the term *free energy* in a generic sense. The thermodynamic potential actually involved depends on which thermodynamic theory this purely mechanical theory is meant to approximate. The current theory is independent of such considerations.

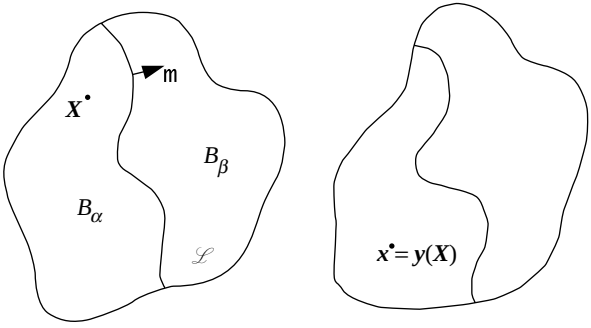

FIGURE 1.1. The regions B_α and B_β occupied by the phases α and β in the undeformed body; $\mathscr{S}$ is the interface and $\mathbf{m}$ is the unit normal to the interface.

with response functions $\Psi_\alpha(\boldsymbol{F}, \boldsymbol{X})$ and $\Psi_\beta(\boldsymbol{F}, \boldsymbol{X})$ defined for all $\boldsymbol{F}$ with $\det \boldsymbol{F} > 0$ and all $\boldsymbol{X}$ in B. (The notation $\Psi = \Psi_\alpha(\boldsymbol{F}, \boldsymbol{X})$, say, is shorthand for $\Psi(\boldsymbol{X}) = \Psi_\alpha(\boldsymbol{F}(\boldsymbol{X}), \boldsymbol{X})$.)

As is customary in variational treatments, the stress $\boldsymbol{S}$ is defined as the partial derivative of the energy with respect to $\boldsymbol{F}$,

$$\boldsymbol{S} = \partial_{\boldsymbol{F}} \Psi_\alpha(\boldsymbol{F}, \boldsymbol{X}) \quad \text{in } B_\alpha, \qquad \boldsymbol{S} = \partial_{\boldsymbol{F}} \Psi_\beta(\boldsymbol{F}, \boldsymbol{X}) \quad \text{in } B_\beta. \tag{1–2}$$

In conjunction with this, I define a *body force* $\boldsymbol{g}$ through

$$\boldsymbol{g} = -\partial_{\boldsymbol{X}} \Psi_\alpha(\boldsymbol{F}, \boldsymbol{X}) \quad \text{in } B_\alpha, \qquad \boldsymbol{g} = -\partial_{\boldsymbol{X}} \Psi_\beta(\boldsymbol{F}, \boldsymbol{X}) \quad \text{in } B_\beta. \tag{1–3}$$

The traditional definition of stable equilibrium requires that the deformation of the body and the position of the interface minimize the total energy

$$E(\mathscr{S}, \boldsymbol{y}) = \int_{B_\alpha} \Psi dv + \int_{B_\beta} \Psi dv \tag{1–4}$$

and hence result in a vanishing first variation, $\delta E(\mathscr{S}, \boldsymbol{y}) = 0$, a restriction that I will use to deduce appropriate field equations and interface conditions.

The variation $\delta E(\mathscr{S}, \boldsymbol{y})$ is defined as follows: assume that $\boldsymbol{y}(\boldsymbol{X})$ and $\mathscr{S}$ are values at $\varepsilon = 0$ of one-parameter families $\boldsymbol{y}_\varepsilon(\boldsymbol{X})$ and $\mathscr{S}_\varepsilon$, with ε a small parameter and $\boldsymbol{y}_\varepsilon(\boldsymbol{X}) = \boldsymbol{y}(\boldsymbol{X})$ on ∂B for all ε; then

$$\delta E(\mathscr{S}, \boldsymbol{y}) = \frac{d}{d\varepsilon} E(\mathscr{S}_\varepsilon, \boldsymbol{y}_\varepsilon)\big|_{\varepsilon=0},$$

where $E(\mathscr{S}_\varepsilon, \boldsymbol{y}_\varepsilon)$ is defined by (1–4) with $\Psi(\boldsymbol{X}) = \Psi_\alpha(\nabla \boldsymbol{y}_\varepsilon(\boldsymbol{X}), \boldsymbol{X})$ in $B_\alpha = B_\alpha(\varepsilon)$ and similarly in $B_\beta = B_\beta(\varepsilon)$.

To formally compute $\delta E(\mathscr{S}, \boldsymbol{y})$, define the variations $\delta \boldsymbol{y}(\boldsymbol{X})$ and $\delta \boldsymbol{F}(\boldsymbol{X})$ through

$$\delta \boldsymbol{y}(\boldsymbol{X}) = \frac{\partial}{\partial \varepsilon} \boldsymbol{y}_\varepsilon(\boldsymbol{X})\big|_{\varepsilon=0}, \qquad \delta F(\boldsymbol{X}) = \frac{\partial}{\partial \varepsilon} \nabla \boldsymbol{y}_\varepsilon(\boldsymbol{X})\big|_{\varepsilon=0},$$

so that

$$\delta \boldsymbol{y} = \boldsymbol{0} \quad \text{on } \partial B, \qquad \delta \boldsymbol{F} = \nabla(\delta \boldsymbol{y}). \tag{1–5}$$

Further, assume that $\mathscr{S}_\varepsilon$ admits a parametrization $\boldsymbol{X} = \hat{\boldsymbol{X}}_\varepsilon(\sigma)$, $\sigma = (\sigma_1, \sigma_2)$, and define the *normal* variation $\delta\mathscr{S}(\boldsymbol{X})$ of $\mathscr{S}$ to be the scalar field

$$\delta\mathscr{S}(\boldsymbol{X}) = \mathbf{m}(\boldsymbol{X}) \cdot \left(\frac{\partial}{\partial\varepsilon}\right) \hat{\boldsymbol{X}}_\varepsilon(\sigma)\big|_{\varepsilon=0}.$$

Finally, let $[\![f]\!]$ denote the jump in a field f across the interface (the limit from β minus that from α), and let $\langle\!\langle f \rangle\!\rangle$ designate the average of the interfacial limits of f. The divergence theorem, the compatibility condition[9]

$$[\![\delta\boldsymbol{y}]\!] = -(\delta\mathscr{S})[\![\boldsymbol{F}]\!]\mathbf{m},$$

the identity $[\![fg]\!] = \langle\!\langle \boldsymbol{f} \rangle\!\rangle [\![g]\!] + \langle\!\langle g \rangle\!\rangle [\![f]\!]$, and the conditions (1–5) then imply that

$$\begin{aligned}
-\delta E(\mathscr{S}, \boldsymbol{y}) &= -\int_{B_\alpha} \boldsymbol{S} \cdot \nabla(\delta\boldsymbol{y})\, dv - \int_{B_\beta} \boldsymbol{S} \cdot \nabla(\delta\boldsymbol{y})\, dv + \int_{\mathscr{S}} [\![\Psi]\!]\delta\mathscr{S}\, da \\
&= \int_{B_\alpha} \operatorname{Div}\boldsymbol{S} \cdot \delta\boldsymbol{y}\, dv + \int_{B_\alpha} \operatorname{Div}\boldsymbol{S} \cdot \delta\boldsymbol{y}\, dv \\
&\quad + \int_{\mathscr{S}} [\![(\boldsymbol{S}\mathbf{m}) \cdot (\delta\boldsymbol{y})]\!]\, da + \int_{\mathscr{S}} [\![\Psi]\!]\delta\mathscr{S}\, da \\
&= \int_{B_\alpha} \operatorname{Div}\boldsymbol{S} \cdot \delta\boldsymbol{y}\, dv + \int_{B_\beta} \operatorname{Div}\boldsymbol{S} \cdot \delta\boldsymbol{y}\, dv \\
&\quad + \int_{\mathscr{S}} \left\{ [\![\boldsymbol{S}]\!]\mathbf{m} \cdot \langle\!\langle \delta\boldsymbol{y} \rangle\!\rangle + ([\![\Psi]\!] - \langle\!\langle \boldsymbol{S}\mathbf{m} \rangle\!\rangle \cdot [\![\boldsymbol{F}]\!]\mathbf{m}) \,\delta\mathscr{S} \right\} da. \qquad (1\text{–}6)
\end{aligned}$$

Assume that $\delta E(\mathscr{S}, \boldsymbol{y}) = 0$ for all variations $\delta\boldsymbol{y}$ and $\delta\mathscr{S}$. Then because $\delta\boldsymbol{y}$ can be specified arbitrarily away from $\mathscr{S}$, while $\langle\!\langle \delta\boldsymbol{y} \rangle\!\rangle$ and $\delta\mathscr{S}$ can be specified arbitrarily on $\mathscr{S}$, (1–6) yields the standard equilibrium equation

$$\operatorname{Div}\boldsymbol{S} = \mathbf{0} \quad \text{in bulk} \qquad (1\text{–}7)$$

(that is, in B_α and in B_β), the standard force balance

$$[\![\boldsymbol{S}]\!]\mathbf{m} = \mathbf{0} \quad \text{on the interface,} \qquad (1\text{–}8)$$

and an additional condition

$$[\![\Psi]\!] = [\![\boldsymbol{F}\mathbf{m} \cdot \boldsymbol{S}\mathbf{m}]\!] \quad \text{on the interface,} \qquad (1\text{–}9)$$

often referred to as the *Maxwell relation.*

Since (1–9) cannot be derived from balance of forces alone, this leads to the question of whether the Maxwell relation represents an additional "force balance." In fact it does. To see this, consider the "stress tensor"

$$\boldsymbol{C} = \Psi\mathbf{1} - \boldsymbol{F}^\top\boldsymbol{S} \qquad (1\text{–}10)$$

introduced by Eshelby in his discussion of defects. In terms of the Eshelby tensor, the Maxwell relation has the simple form $\mathbf{m} \cdot [\![\boldsymbol{C}]\!]\mathbf{m} = 0$. Further, the continuity of $\boldsymbol{y}$

[9] Cf., e.g., Larché and Cahn [1978, eq. (6)]; if the parameter ε is viewed as "time," then this condition is the classical Hadamard condition for shocks (cf. Truesdell and Toupin [1960, eq. (189.1)]).

across the interface implies that $[\boldsymbol{F}]\boldsymbol{t} = \boldsymbol{0}$ for any vector $\boldsymbol{t}$ tangent to the interface, so that (1–8) yields $\boldsymbol{t} \cdot [\boldsymbol{C}]\mathtt{m} = 0$. Thus

$$[\boldsymbol{C}]\mathtt{m} = \boldsymbol{0} \quad \text{on the interface,} \tag{1–11}$$

implying continuity of the Eshelby traction across the interface.[10] Further, a computation based on (1–2), (1–3), and (1–7) yields the conclusion

$$\operatorname{Div} \boldsymbol{C} + \boldsymbol{g} = \boldsymbol{0} \quad \text{in bulk,} \tag{1–12}$$

so that Eshelby tensor $\boldsymbol{C}$ and the body force $\boldsymbol{g}$ satisfy a balance law; in fact, (1–11) and (1–12) together imply the integral balance

$$\int_{\partial P} \boldsymbol{C}\boldsymbol{n}\, da + \int_{P} \boldsymbol{g}\, dv = \boldsymbol{0} \tag{1–13}$$

for every subregion P of B,where $\boldsymbol{n}$ is the outward unit normal to ∂P. I will refer to $\boldsymbol{g}$ as the *internal configurational body force*, where, for now, the term *internal* can be thought of as arising from the fact that, by (1–3), $\boldsymbol{g}$ is a measure of material inhomogeneity.

I henceforth use the term *standard balance* for balances such as (1–7) and (1–8) involving the standard Piola stress[11] $\boldsymbol{S}$, as opposed to the term *configurational balance*, which I reserve for balances of the form (1–13) involving the Eshelby tensor $\boldsymbol{C}$ and the body force $\boldsymbol{g}$.

This analysis leads to the questions:

- Is there a formulation in which $\boldsymbol{C}$ and $\boldsymbol{g}$ are primitive quantities, consistent with a force balance of the type (1–13), and in which the Eshelby relation (1–10) follows as a natural consequence?
- Aside from a possible better understanding of the underlying physics, does the introduction of configurational forces lead to new results?

The chief purpose of this book is to answer these questions.

c. Interfacial energy. A further argument for a configurational force balance

The argument in support of a configurational force balance is even more compelling when the free energy of the interface is accounted for in the total energy (1–4) by a term of the form

$$\int_{\mathscr{S}} \psi\, da. \tag{1–14}$$

[10] Cf. Kaganova and Roitburd [1988].

[11] Called Piola-Kirchhoff stress in the terminology of Truesdell and Noll [1965] and Gurtin [1981].

Here ψ, assumed, for convenience, to be *constant*, represents the interfacial free energy per unit *referential* area. The variation of (1–14) is

$$-\int_{\mathscr{S}} \psi K \delta \mathscr{S}\, da, \tag{1–15}$$

with K twice the mean curvature of $\mathscr{S}$, and this term results in the following generalization of the interface condition (1–9):

$$\mathbf{m} \cdot [\boldsymbol{C}]\mathbf{m} + \psi K = 0. \tag{1–16}$$

Here $\boldsymbol{C}$ is the bulk Eshelby stress (1–10), and, granted the identification of surface tension with surface free energy, (1–16) resembles a classical identity for fluids equating the jump in pressure across an interface to the product of surface tension and twice the mean curvature. Here, however, *this identity takes place in the configurational system*.

Further, (1–16), the argument in the paragraph containing (1–11), and well-known differential-geometric identities yield the local balance

$$[\boldsymbol{C}]\mathbf{m} + \mathrm{Div}_{\mathscr{S}}\, \mathbf{C} = \mathbf{0}, \tag{1–17}$$

where $\mathrm{Div}_{\mathscr{S}}$ represents the surface divergence on $\mathscr{S}$, while $\mathbf{C}$ is the tensor

$$\mathbf{C} = \psi \mathbf{P},$$

with $\mathbf{P} = \mathbf{1} - \mathbf{m} \otimes \mathbf{m}$ the projection onto the interface; equivalently, relative to an orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ with $\boldsymbol{e}_3 = \mathbf{m}$,

$$\mathbf{C} = \begin{pmatrix} \psi & 0 & 0 \\ 0 & \psi & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The identity (1–17) represents a local balance law relating the configurational bulk stress $\boldsymbol{C}$ and the configurational surface stress $\mathbf{C}$; in fact, given any subregion P of B, if $\mathscr{G}$, assumed nonempty, represents the portion of $\mathscr{S}$ in P, and if $\mathbf{n}$, a vector field tangent to $\mathscr{S}$, denotes the outward unit normal to the boundary curve $\partial\mathscr{G}$, then (1–12) and (1–17) yield the integral balance

$$\int_{\partial P} \boldsymbol{C}\boldsymbol{n}\, da + \int_P \boldsymbol{g}\, dv + \int_{\partial\mathscr{G}} \psi \mathbf{n}\, ds = \mathbf{0}, \tag{1–18}$$

which relates the forces' exerted by the traction $\boldsymbol{C}\boldsymbol{n}$ on ∂P and the body force $\boldsymbol{g}$ on P to the tensile force $\psi\mathbf{n}$ exerted on P across $\partial\mathscr{G}$ by surface tension.

Here it is important to note that *the balances* (1–16)–(1–18) *concern configurational forces, not standard forces*; the introduction of a constant interfacial energy ψ, measured per unit area in the reference configuration, leaves the standard balance (1–8) unchanged.

To allow for surface tension in the standard force system necessitates *strain-dependent* surface energies.[12] To quote Herring[13] on crystalline materials: "The principal cause of surface tension is the fact that surface atoms are bound by fewer neighbors than internal atoms; surface tension is therefore mainly a measure of the change in the number of atoms in the surface layer." I interpret this as implying that surface tension in crystalline materials is primarily configurational. Compare this to fluids, where interfacial energy is a constant when measured in the deformed configuration and is hence dependent on $\boldsymbol{F}$ (through the surface Jacobian) when measured with respect to a fixed reference; for that reason, interfacial energy in fluids gives rise to surface tension in the standard force system.

d. Configurational forces as basic objects

It is difficult to imagine distinct force systems acting concurrently at each point of a body, which is perhaps why configurational forces have never been considered more than derived quantities. Unfortunately, the current entrenched, facile view of force in terms of "pushes" and "pulls" has led to a sense of security in which force is seen as a real quantity rather than as a *mathematical concept*. Such a feeling of "understanding," while a natural outgrowth of experience and an aid to pedagogy, is a major drawback to the acceptance of new ideas, whose very youth generally precludes a deep understanding of their physical nature.

In this book I will:

- present a framework in which configurational forces are treated as basic objects;
- give a discussion of configurational forces that provides at least an intuitive understanding of their physical nature.

In the words of Pierce:[14]

> [Force is] "the great conception which, developed in the early part of the seventeenth century from the rude idea of a cause, and constantly improved upon since, has shown us how to explain all the changes of motion which bodies experience, and how to think about physical phenomena; which has given birth to modern science; and which . . . has played a principal part in directing the course of modern thought . . . . It is, therefore worth some pains to comprehend it."
>
> Those who believe the notion of force is obvious should read the scientific literature of the period following Newton. Truesdell[15] notes that "D'Alembert spoke of Newtonian forces as 'obscure and metaphysical beings, capable of nothing but spreading darkness over a science clear by itself,' " while Jammer[16] paraphrases a

[12] Cf. Herring [1951], Gurtin and Struthers [1990], Gurtin [1995]; see also the sentence following (21–17).
[13] [1951b].
[14] [1934, p. 262].
[15] [1966].
[16] [1957, pp. 209, 215].

remark of Maupertuis, "we speak of forces only to conceal our ignorance," and one of Carnot, "an obscure metaphysical notion, that of force."[17]

What I believe to be a major roadblock to the acceptance of a configurational force balance lies in the fact that Gibbs's[18] masterpiece, so central to the subsequent development of materials science, is based on variational arguments; force is not primitive. But arguments appropriate to the statical setting within which Gibbs framed his theory seem inappropriate to dynamical situations involving dissipation.

Those reluctant to accept a separate balance for configurational forces should note that a balance law for moments was not part of Newtonian mechanics. As remarked by Truesdell and Toupin,[19] "It should be, but unfortunately it is not, unnecessary to comment that the laws of Newton are ... [not] sufficiently general to serve as a foundation for continuum mechanics," Indeed, a balance law for moments—first stated explicitly by Euler [1776] almost a century after the appearance of Newton's *Principia* [1687]—need join balance of forces as a basic axiom.

A framework that considers as fundamental both configurational and classical forces requires a concept that unifies disparate notions of force. Here the unifying concept is "the rate at which work is performed" or, more simply, "the working." Roughly speaking, to each *independent* kinematical descriptor I assign an associated system of forces, and to each density of force, whether it be a surface traction or a body force, I associate a *work-conjugate* generalized velocity, the rate of change of the kinematical descriptor, such that

$$\text{density of working} = \{\text{force density}\} \cdot \{\text{generalized velocity}\}.$$

The paradigm I use requires an answer to the question: What makes a kinematical quantity independent? The answer is the need for an independent observer to measure its generalized velocity. Such observers are essential to the development of the theory, because invariance of the thermodynamics to changes in observer yields the underlying mechanical balance laws. In variational treatments, independent kinematical quantities may be independently varied, and each such variation yields a corresponding Euler-Lagrange balance. In dynamics with general forms of dissipation there is no encompassing variational principle; the use of independent observers provides a dynamical theory with a rational basis for determining mechanical balance laws.

There is a large literature that uses the *principle of virtual work* to derive balance laws for force. I prefer to not consider such variational forms of balance as basic, but rather as consequences of more classically formulated balances.[20] My reasons are the following:

- The principle of virtual work, which is variational in nature, is physically well-grounded, as the test functions are virtual velocities, but the variational form

[17] Cf. the remarks of Maugin [1993, p. 4].

[18] [1878, pp. 55–371].

[19] [1960, §196].

[20] But one should bear in mind that the *weaker* variational balances are powerful tools of analysis.

of other balance laws such as that for energy seem devoid of meaning, chiefly because the associated test functions have no readily identifiable physical interpretation. I prefer a consistent presentation in which all of the relevant balances have classical forms.

- The principle of virtual work requires an a priori notion of stress, while classically formulated balances may be based on the more fundamental notion of a traction, with stress derived via Cauchy's theorem.[21]

e. The nature of configurational forces

Configurational forces are related to the integrity of a body's material structure and perform work in the transfer of material and the evolution of material structures such as defects and phase interfaces. With this in mind, I introduce three nonclassical kinematical notions used to capture physics related to the transfer of material:

- control volumes $P(t)$ that *migrate through the reference body* B;
- material observers that view the reference configuration and measure, e.g., velocities associated with migrating control volumes; these observers are used independently of the classical spatial observers that view motions of B;
- time-dependent changes in reference configuration.

The net working of both standard and configurational forces plays a central role in the underlying thermodynamics; since much of the theory is mechanical, a thermodynamics based on work and energy is introduced, with energy represented by a *free energy* density Ψ.[22] A standard precept of continuum *mechanics* is that when writing basic laws for a control volume P, all that is external to P may be accounted for by the action of forces on P. Consistent with this, I base the theory on a nonclassical version of the *second law* requiring that, for each migrating control volume $P = P(t)$,

$$(d/dt)\{\text{free energy of } P(t)\} \leq \{\text{rate at which work is performed on } P(t)\};$$

in so doing I account for the working of *both* configurational forces and standard forces, but only implicitly for a flow of free energy across $\partial P(t)$ as it migrates.[23] This form of the second law is central to the theory:

- the *Eshelby relation* (1–10) is derived as a consequence of the requirement that the second law be independent of the choice of velocity field describing the migration of ∂P;

---

[21] But because this derivation is well known, I here assume the existence of stress.

[22] Also discussed is a more general formulation based on balance of energy and growth of entropy.

[23] Gurtin [1995, §3c].

- invariance of the working under changes in spatial observer results in the *standard* force balance;
- invariance under changes in *material* observer yields an additional balance for *configurational* forces.[24]

An important feature of the theory as presented here is that all basic equations and thermodynamic inequalities are derived without recourse to constitutive equations, a feature not present in variational treatments and one that renders the theory applicable to the dynamics of a general class of dissipative materials.

f. Configurational stress and residual stress. Internal configurational forces

Configurational stress is often confused with *residual* (standard) *stress*, which is the stress in the reference configuration when the body is undeformed. In the absence of deformation $\boldsymbol{F} = \mathbf{1}$ and the Eshelby relation (1–10) yields $\boldsymbol{C} = \Psi\mathbf{1} - \boldsymbol{S}$; in particular, $\boldsymbol{C}$ need not vanish when $\boldsymbol{S}$ vanishes, because then $\boldsymbol{C} = \Psi\mathbf{1}$.

A major difference between the standard and configurational force systems is the presence of *internal configurational forces* such as the body force $\boldsymbol{g}$. These forces are related to the material structure of the body B; to each configuration of B there correspond a distribution of material and internal configurational forces that act to hold the material in place in that configuration. Such forces characterize the resistance of the material to structural changes and are basic when discussing temporal changes associated with phenomena such as the breaking of atomic bonds during fracture.

To better understand the role of internal forces, note the difference between the body's reference configuration and the deformed (actual) configurations assumed by the body during a motion. In the latter the body is *free* to move about in a manner dictated by the standard (Newtonian) forces acting on it, forces that result from the interaction of separate parts of the body and from the interaction of the body with its environment. There are no internal forces. But the body is *not free* to move about in the reference, and a basic presumption of the theory is that there are internal configurational forces that pin, in place, the material points of the body, thereby maintaining its internal structure.[25]

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[24]This derivation of the standard balance is due to Noll [1963] (cf. Green and Rivlin [1964]), that of the configurational balance is due to Gurtin and Struthers [1990].

Pedagogically, I prefer to postulate force balances as consequences of invariance, chiefly because of the nonintuitive nature of configurational forces and because of the opposition I have encountered to the introduction of a configurational force balance.

[25]Internal configurational forces will be discussed in more detail in §5a.

g. Configurational forces and indeterminacy

Indeterminate forces arise as a response to kinematic constraints and are essentially irrelevant to the underlying thermodynamics because they are not generally found in local forms of the second law. For that reason such forces are not specified constitutively. Classical indeterminate forces are those associated with the pressure in an incompressible fluid and the stress in a rigid body.[26]

Indeterminacy arises in the configurational system whenever there is no change in material structure. For example, consider the equilibrium of a hyperelastic body B that is free of defects. Within this classical framework, configurational forces are indeterminant, in fact, superfluous; granted appropriate boundary data, if the problem has a solution, then the stress $\boldsymbol{S}$ and the free energy Ψ are known, and the configurational stress $\boldsymbol{C}$ and internal body force $\boldsymbol{g}$ can be computed using (1–3) and (1–10).

More illuminating, assume that ∂B is free of applied standard and configurational tractions.[27] Then, neglecting *surface stresses* within ∂B, $\boldsymbol{Sn} = \boldsymbol{0}$, with $\boldsymbol{n}$ the outward unit normal to ∂B. Hence, by the Eshelby relation, there is a configurational traction $\boldsymbol{Cn} = -\Psi\boldsymbol{n}$ exerted at the free surface by the bulk material. If configurational forces are to be balanced, there must be an internal configurational surface force $\boldsymbol{g}^{\partial B}$ distributed over ∂B that opposes this traction. The force $\boldsymbol{g}^{\partial B}$ is indeterminate, because ∂B is fixed; $\boldsymbol{g}^{\partial B}$ is, in fact, trivially equal to $\Psi\boldsymbol{n}$. On the other hand, were I to allow material to be (freely) added and removed at the boundary, then ∂B would not be a material surface. In this case (the normal part of) $\boldsymbol{g}^{\partial B}$ would *not* be indeterminate; in fact, its constitution would help to characterize temporal changes of ∂B.

Similarly, the internal configurational force associated with an interface in a composite material is indeterminate, since such interfaces do not migrate, but the analogous force associated with a moving phase interface or grain boundary would have a constitutive specification. As a general rule,

> *the bulk material and all material structures such as free surfaces and interfaces have associated internal configurational forces, with such forces indeterminate when and only when the associated structures are fixed in the material.*

Another example is furnished by a propagating crack: The tip migrates and hence has an associated internal configurational force that characterizes its kinetics; the crack faces behind the tip also have associated internal configurational forces, but these are indeterminate because the faces are fixed in the material.

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[26] Cf. Truesdell and Noll [1965, §30] and Gurtin and Podio-Guidugli [1973] for general discussions of the classical theory of constraints.

[27] An example of null configurational tractions is furnished by an environment composed of a fluid with vanishing enthalpy (cf. §6d).

h. Scope of the book

The book begins with a discussion of configurational forces within a classical context; this allows an acquaintance with their physical nature and provides the derivation of several important relations.

As a first departure from a classical context, I consider migrating material structures such as phase interfaces; here, so as to not introduce too much new material at once, I neglect configurational stresses, such as surface tension, that act within the interface, and focus, instead, on the internal configurational forces that characterize the exchange of material at the interface. In subsequence sections I consider more general theories that include surface stress; here the underlying mathematical structure is differential geometry, and to keep the book reasonably self-contained, I discuss in some detail the main geometric concepts and results on which the theory is based.

Configurational forces are also relevant in purely thermal situations, a central example being solidification as described by the Stefan problem and its generalizations to include surface distributions of energy and entropy. I discuss such theories in detail. A major and somewhat surprising consequence of the treatment of the Stefan problem within the framework of configurational forces is that the classical free-boundary condition equating the temperature to the melting temperature is *not* a constitutive assumption but instead a consequence of the configurational force balance applied across the interface, at least in those situations for which the energy and entropy of the interface are negligible.

The book closes with a discussion of fracture, concentrating on the configurational forces most influential in the motion of the crack tip. Discussed at length are the propagation of a running crack, crack initiation with and without kinking, and crack curving. In particular, a criterion for determining the direction of a running crack is proposed; in contrast to previous criteria based on minimizing the energy release rate, the criterion proposed here chooses directions that maximize dissipation.

Most of the presentation is based on finite deformations, as the underlying concepts are most transparent within a framework that distinguishes between reference and deformed configurations. However, because many applications of configurational forces presume infinitesimal deformations, I also discuss the theory within that context.

i. On operational definitions and mathematics

Many of the concepts concerning configurational forces are nonstandard. For that reason I have tried to give simple interpretations of these concepts, fully realizing that such explanations are strongly prejudiced by my background. What is important is the mathematical framework, and that is what the reader should take most seriously, supplying his or her own metaphysical “footnotes” whenever mine

seem inappropriate. In this regard note that the early explanation of gravitational forces in terms of transmission through an all-pervasive ether is no longer tenable to most scientists; but even so, the mathematical (nonrelativistic) description of these forces remains as set down by Newton more than three centuries ago.

j. General notation. Tensor analysis

j1. On direct notation

I generally use notation and terminology standard in continuum mechanics.[28] In particular, I use direct (coordinate-free) notation, and for two reasons:

- Direct notation makes the statement of physical laws transparent and, in so doing, helps to underline their beauty.
- The physical sense of, say, stress seems most clearly conveyed when considered as a linear transformation $\boldsymbol{T}$ that assigns to the normal $\boldsymbol{n}$ of a surface $\mathscr{S}$ the force $\boldsymbol{Tn}$ transmitted across $\mathscr{S}$.

j2. Vectors and tensors. Fields

Scalars are denoted by lightface letters, *vectors* (and points) by lowercase boldface letters (although $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{Z}$ denote vectors). A *dot*, as in $\boldsymbol{u}\cdot\boldsymbol{v}$, designates the *inner product*, irrespective of the space in question. *Tensors* are linear transformations of vectors into vectors and are denoted by uppercase boldface letters. The *unit tensor* $\mathbf{1}$ is defined by $\mathbf{1}\boldsymbol{u}=\boldsymbol{u}$ for every vector $\boldsymbol{u}$; the *tensor product* $\boldsymbol{a}\otimes\boldsymbol{b}$ of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is the tensor defined by

$$(\boldsymbol{a}\otimes\boldsymbol{b})\boldsymbol{u}=(\boldsymbol{b}\cdot\boldsymbol{u})\boldsymbol{a} \qquad \text{for all vectors } \boldsymbol{u};$$

$\boldsymbol{A}^\top$, $\operatorname{tr}\boldsymbol{A}$, $\boldsymbol{A}^{-1}$, and $\det\boldsymbol{A}$, respectively, denote the *transpose*, *trace*, *inverse*, and *determinant* of a tensor $\boldsymbol{A}$; the inner product of tensors $\boldsymbol{A}$ and $\boldsymbol{C}$ is defined by $\boldsymbol{A}\cdot\boldsymbol{C}=\operatorname{tr}(\boldsymbol{A}^\top\boldsymbol{C})$. In Cartesian components with summation over repeated indices implied, $(\boldsymbol{Aa})_i=A_{ij}a_j$, $(\boldsymbol{a}\otimes\boldsymbol{b})_{ij}=a_ib_j$, $(\boldsymbol{A}^\top)_{ij}=A_{ji}$, $\operatorname{tr}\boldsymbol{A}=A_{ii}$, $\boldsymbol{A}\cdot\boldsymbol{C}=A_{ij}C_{ij}$. The transpose is defined by the requirement that

$$\boldsymbol{u}\cdot\boldsymbol{Av}=(\boldsymbol{A}^\top\boldsymbol{u})\cdot\boldsymbol{v} \qquad \text{for all vectors } \boldsymbol{u} \text{ and } \boldsymbol{v}.$$

An identity bearing formal similarity to this definition concerns the inner product of tensors and has the form

$$\boldsymbol{U}\cdot(\boldsymbol{AV})=(\boldsymbol{A}^\top\boldsymbol{U})\cdot\boldsymbol{V} \qquad \text{for all tensors } \boldsymbol{U} \text{ and } \boldsymbol{V};$$

this identity will be used repeatedly.

The term *field* signifies a function of position $\boldsymbol{X}$ (in this subsection) or, more generally, a function of position $\boldsymbol{X}$ and time t. The symbols ∇ and Div denote the

[28] Cf., e.g., Truesdell and Noll [1965], Gurtin [1981].

gradient and *divergence*. It is most convenient to define these operations abstractly, as such definitions extend naturally to surfaces. For φ a smooth[29] scalar field, the gradient $\nabla\varphi$, a vector field, is defined by the chain-rule: for any vector function $\boldsymbol{z}(\alpha)$ of a scalar variable α,

$$\frac{d}{d\alpha}\,\varphi(\boldsymbol{z}(\alpha)) = [\nabla\varphi(\boldsymbol{z}(\alpha))]\cdot\dot{\boldsymbol{z}}(\alpha), \tag{1–19}$$

or, more succinctly,

$$\varphi(\boldsymbol{z})^{\cdot} = \nabla\varphi(\boldsymbol{z})\cdot\dot{\boldsymbol{z}}.$$

(Here and for the remainder of this subsection the superposed dot denotes ordinary differentiation with respect to a scalar variable, but in the body of the text a superposed dot denotes differentiation with respect to time holding material points fixed.)

A sketch of the proof that, given any $\boldsymbol{X}$, (1–19) defines a unique vector $\nabla\varphi(\boldsymbol{X})$ proceeds as follows. One shows that, for $\boldsymbol{z}(\alpha) = \boldsymbol{X} + \alpha\boldsymbol{a}$, $\varphi(\boldsymbol{z})^{\cdot}$ at $\alpha = 0$ is a linear function of $\boldsymbol{a}$; one then uses the fact that any such scalar-valued linear function can be written as the inner product of a unique fixed vector, written $\nabla\varphi(\boldsymbol{X})$, with $\boldsymbol{a}$. Similar arguments apply to the gradients of vector and tensor fields, but there only linearity need be shown.

For $\boldsymbol{u}$ a vector field, $\nabla\boldsymbol{u}$ is the tensor field defined by

$$\boldsymbol{u}(\boldsymbol{z})^{\cdot} = \nabla\boldsymbol{u}(\boldsymbol{z})\dot{\boldsymbol{z}}$$

for all vector functions $\boldsymbol{z}(\alpha)$, and $\operatorname{Div}\boldsymbol{u}$ is the scalar field

$$\operatorname{Div}\boldsymbol{u} = \operatorname{tr}\nabla\boldsymbol{u}.$$

The divergence of a tensor field $\boldsymbol{T}$ is the vector field $\operatorname{Div}\boldsymbol{T}$ defined by the requirement that

$$\boldsymbol{a}\cdot\operatorname{Div}\boldsymbol{T} = \operatorname{Div}(\boldsymbol{T}^{\top}\boldsymbol{a})$$

for all *constant* vectors $\boldsymbol{a}$. The Cartesian components of these fields are

$$\begin{aligned} (\nabla\varphi)_i &= \partial\varphi/\partial X_i, & (\nabla\boldsymbol{u})_{ij} &= \partial u_i/\partial X_j,\\ \operatorname{Div}\boldsymbol{u} &= \partial u_i/\partial X_i, & (\operatorname{Div}\boldsymbol{T})_i &= \partial T_{ij}/\partial X_j. \end{aligned}$$

Classical identities, which will generally be used without mention, are

$$\operatorname{Div}(\varphi\boldsymbol{u}) = \varphi\operatorname{Div}\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\varphi, \tag{1–20a}$$

$$\operatorname{Div}(\boldsymbol{T}^{\top}\boldsymbol{u}) = \boldsymbol{u}\cdot\operatorname{Div}\boldsymbol{T} + \boldsymbol{T}\cdot\nabla\boldsymbol{u}, \tag{1–20b}$$

$$\operatorname{Div}(\boldsymbol{u}\otimes\boldsymbol{v}) = (\operatorname{Div}\boldsymbol{v})\boldsymbol{u} + (\nabla\boldsymbol{u})\boldsymbol{v}, \tag{1–20c}$$

$$\operatorname{Div}(\nabla\boldsymbol{u}^{\top}) = \nabla\operatorname{Div}\boldsymbol{u}. \tag{1–20d}$$

[29] Assumptions of smoothness and regularity are generally left as tacit, although precise assumptions are specified for defects such as interfaces and crack tips, where they are crucial.

The verification of (1–20b) is an excellent example of direct tensor analysis. Assume that $\boldsymbol{T}$ is constant. The definition of $\nabla \boldsymbol{u}$ then yields

$$[\boldsymbol{T}^\top \boldsymbol{u}(z)]^{\cdot} = \boldsymbol{T}^\top [\boldsymbol{u}(z)^{\cdot}] = \boldsymbol{T}^\top [\nabla \boldsymbol{u}(z)\dot{z}] = [\boldsymbol{T}^\top \nabla \boldsymbol{u}(z)]\dot{z};$$

thus, by definition, $\nabla(\boldsymbol{T}^\top \boldsymbol{u}) = \boldsymbol{T}^\top \nabla \boldsymbol{u}$. Dropping the assumption that $\boldsymbol{T}$ be constant, by the product rule for differentiation, which holds for all "products" involving vectors and tensors, $\operatorname{Div}(\boldsymbol{T}^\top \boldsymbol{u})$ is equal to the divergence of $\boldsymbol{T}^\top \boldsymbol{u}$ holding $\boldsymbol{T}$ fixed plus the divergence of $\boldsymbol{T}^\top \boldsymbol{u}$ holding $\boldsymbol{u}$ fixed. For $\boldsymbol{T}$ fixed, $\nabla(\boldsymbol{T}^\top \boldsymbol{u}) = \boldsymbol{T}^\top \nabla \boldsymbol{u}$; therefore $\operatorname{Div}(\boldsymbol{T}^\top \boldsymbol{u}) = \operatorname{tr} \nabla(\boldsymbol{T}^\top \boldsymbol{u}) = \operatorname{tr}(\boldsymbol{T}^\top \nabla \boldsymbol{u}) = \boldsymbol{T} \cdot \nabla \boldsymbol{u}$. On the other hand, the definition of $\operatorname{Div} \boldsymbol{T}$ implies that, for $\boldsymbol{u}$ fixed, $\operatorname{Div}(\boldsymbol{T}^\top \boldsymbol{u}) = \boldsymbol{u} \cdot \operatorname{Div} \boldsymbol{T}$.

Various consequences of the divergence theorem, for $\boldsymbol{u}$ and $\boldsymbol{T}$ smooth fields on a sufficiently regular region P, take the form

$$\int_{\partial P} \boldsymbol{u} \cdot \boldsymbol{n}\, da = \int_P \operatorname{Div} \boldsymbol{u}\, dv, \tag{1–21a}$$

$$\int_{\partial P} \boldsymbol{T}\boldsymbol{n}\, da = \int_P \operatorname{Div} \boldsymbol{T}\, dv, \tag{1–21b}$$

$$\int_{\partial P} \boldsymbol{T}\boldsymbol{n} \cdot \boldsymbol{u}\, da = \int_P (\boldsymbol{u} \cdot \operatorname{Div} \boldsymbol{T} + \boldsymbol{T} \cdot \nabla \boldsymbol{u})\, dv. \tag{1–21c}$$

The identities (1–21bc) are consequences of the standard identity (1–21a). For example, take the inner product of the left side of (1–21b) with an arbitrary constant vector $\boldsymbol{a}$ and apply (1–21a) with $\boldsymbol{u} = \boldsymbol{T}^\top \boldsymbol{a}$.

j3. Third-order tensors (3-tensors). The operation $\boldsymbol{T}$:$\boldsymbol{\Lambda}$

The tensors under consideration are generally of second order, and it would burden the text to repeatedly use the term second-order tensor. Since third-order tensors are occasionally needed, I adopt the convention that the term *tensor* by itself signify a tensor of second order (i.e., a linear transformation of vectors into vectors), and that third-order tensors always be referred to as 3-*tensors*.

Precisely a **3-tensor** $\boldsymbol{\Lambda}$ is a linear transformation of vectors into (second-order) tensors: for any fixed vector $\boldsymbol{a}$, $\boldsymbol{\Lambda}\boldsymbol{a}$ is a linear transformation that assigns to each vector $\boldsymbol{b}$ a vector $(\boldsymbol{\Lambda}\boldsymbol{a})\boldsymbol{b}$. In components, $(\boldsymbol{\Lambda}\boldsymbol{a})_{ij} = \Lambda_{ijk} a_k$. (This definition is most convenient; third-order tensors could also be defined as trilinear forms or as linear transformations of second-order tensors into vectors.)

An example of a 3-tensor is furnished by the values of the gradient $\nabla \boldsymbol{T}$ of a (second-order) tensor field $\boldsymbol{T}$, where $\nabla \boldsymbol{T}$ is defined by the chain rule:

$$\boldsymbol{T}(z)^{\cdot} = [\nabla \boldsymbol{T}(z)]\dot{z} \tag{1–22}$$

for any vector function $\boldsymbol{z}(\alpha)$. The following three identities, in which $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant vectors and $\boldsymbol{F} = \nabla \boldsymbol{y}$, are useful:

$$[\nabla(\boldsymbol{T}\boldsymbol{a})]\boldsymbol{b} = [(\nabla \boldsymbol{T})\boldsymbol{b}]\boldsymbol{a}, \tag{1–23a}$$

$$[(\nabla \boldsymbol{F})\boldsymbol{b}]\boldsymbol{a} = [(\nabla \boldsymbol{F})\boldsymbol{a}]\boldsymbol{b}, \tag{1–23b}$$

$$(\nabla \boldsymbol{F})\boldsymbol{a} = \nabla(\boldsymbol{F}\boldsymbol{a}). \tag{1–23c}$$

In these identities the placement of parentheses and brackets is crucial.

To verify (1–23a), let $\bar{\boldsymbol{a}}(\boldsymbol{X}) = \boldsymbol{T}(\boldsymbol{X})\boldsymbol{a}$ for all $\boldsymbol{X}$. Fix a point $\boldsymbol{X}$ and a vector $\boldsymbol{b}$, and let β denote a scalar variable. Then the left side of (1–23a), at $\boldsymbol{X}$, is given by $[\nabla\bar{\boldsymbol{a}}(\boldsymbol{X})]\boldsymbol{b}$, and this, in turn, is equal to

$$\frac{\partial}{\partial\beta}\,\bar{\boldsymbol{a}}(\boldsymbol{X}+\beta\boldsymbol{b})\big|_{\beta=0} = \left[\frac{\partial}{\partial\beta}\,\boldsymbol{T}(\boldsymbol{X}+\beta\boldsymbol{b})\big|_{\beta=0}\right]\boldsymbol{a} = [(\nabla\boldsymbol{T}(\boldsymbol{X}))\boldsymbol{b}]\boldsymbol{a},$$

which is the right side of (1–23a). Consider (1–23b). Fix a point $\boldsymbol{X}$ and let α and β denote scalar variables. Then

$$\frac{\partial^2}{\partial\beta\partial\alpha}\,\boldsymbol{y}(\boldsymbol{X}+\alpha\boldsymbol{a}+\beta\boldsymbol{b})\big|_{\alpha=\beta=0} = \left[\frac{\partial}{\partial\beta}\,\boldsymbol{F}(\boldsymbol{X}+\beta\boldsymbol{b})\big|_{\beta=0}\right]\boldsymbol{a} = [(\nabla\boldsymbol{F}(\boldsymbol{X}))\boldsymbol{b}]\boldsymbol{a}.$$

But (assuming that $\boldsymbol{y}$ is smooth) the order of the α and β differentiations is irrelevant, and this yields (1–23b). The result (1–23c) is the consequence of (1–23a) and (1–23b), because these relations imply the identity $[(\nabla\boldsymbol{F})\boldsymbol{a}]\boldsymbol{b} = [\nabla(\boldsymbol{F}\,\boldsymbol{a})]\boldsymbol{b}$ for all vectors $\boldsymbol{b}$. (In components, $(\nabla\boldsymbol{F})_{ijk} = \partial F_{ij}/\partial X_k$, and the symmetry (1–23b) may be established as follows: $(\nabla\boldsymbol{F})_{ijk} = \partial^2 y_i/\partial X_j\partial X_k = \partial^2 y_i/\partial X_k\partial X_j = (\nabla\boldsymbol{F})_{ikj}$.)

Let $\boldsymbol{T}$ be a tensor and $\boldsymbol{\Lambda}$ a 3-tensor; then $\boldsymbol{\Lambda T}$, a 3-tensor, and $\boldsymbol{T}{:}\boldsymbol{\Lambda}$, a vector, are defined by

$$(\boldsymbol{\Lambda T})\boldsymbol{a} = \boldsymbol{\Lambda}(\boldsymbol{T}\,\boldsymbol{a}), \tag{1–24a}$$

$$(\boldsymbol{T}{:}\boldsymbol{\Lambda})\cdot\boldsymbol{a} = \boldsymbol{T}\cdot(\boldsymbol{\Lambda a}) \tag{1–24b}$$

for all vectors $\boldsymbol{a}$. In components, $(\boldsymbol{\Lambda T})_{ijk} = \Lambda_{ijm}T_{mk}$, $(\boldsymbol{T}{:}\boldsymbol{\Lambda})_k = T_{ij}\Lambda_{ijk}$. The following identities, for $\boldsymbol{T}$ a tensor field and $\boldsymbol{F} = \nabla\boldsymbol{y}$, are useful:

$$(\boldsymbol{T}{:}\nabla\boldsymbol{F})\cdot\boldsymbol{a} = \boldsymbol{T}\cdot\nabla(\boldsymbol{F}\,\boldsymbol{a}) \tag{1–25}$$

for all constant vectors $\boldsymbol{a}$, and

$$\operatorname{Div}(\boldsymbol{F}^{\top}\boldsymbol{T}) = \boldsymbol{F}^{\top}\operatorname{Div}\boldsymbol{T} + \boldsymbol{T}{:}\nabla\boldsymbol{F}. \tag{1–26}$$

Equation (1–25) is a consequence of (1–23c). To verify (1–26), choose a constant vector $\boldsymbol{a}$. Then, by (1–20b) (with $\boldsymbol{u} = \boldsymbol{F}\,\boldsymbol{a}$) and (1–25),

$$\boldsymbol{a}\cdot\operatorname{Div}(\boldsymbol{F}^{\top}\boldsymbol{T}) = \operatorname{Div}(\boldsymbol{T}^{\top}\boldsymbol{F}\,\boldsymbol{a}) = (\boldsymbol{F}\,\boldsymbol{a})\cdot\operatorname{Div}\boldsymbol{T} + \boldsymbol{T}\cdot\nabla(\boldsymbol{F}\,\boldsymbol{a}) = \boldsymbol{a}\cdot\boldsymbol{F}^{\top}\operatorname{Div}\boldsymbol{T} + (\boldsymbol{T}{:}\nabla\boldsymbol{F})\cdot\boldsymbol{a},$$

which implies (1–26), because $\boldsymbol{a}$ is arbitrary. Note that $(\boldsymbol{T}{:}\nabla\boldsymbol{F})_k = T_{ij}(\partial F_{ij}/\partial X_k)$, so that $\boldsymbol{T}{:}\nabla\boldsymbol{F}$ is the gradient of $\boldsymbol{T}\cdot\boldsymbol{F}$ holding $\boldsymbol{T}$ fixed.

Finally, for $\boldsymbol{G}$ and $\boldsymbol{T}$ tensors and $\boldsymbol{\Lambda}$ a 3-tensor,

$$\boldsymbol{G}(\boldsymbol{T}{:}\boldsymbol{\Lambda}) = \boldsymbol{T}{:}(\boldsymbol{\Lambda}\boldsymbol{G}^{\top}). \tag{1–27}$$

j4. Functions of tensors

The derivative of a scalar function $\Phi(\boldsymbol{T})$ of a tensor $\boldsymbol{T}$ is written $\partial_{\boldsymbol{T}}\Phi(\boldsymbol{T})$ and is defined by the chain rule: For any tensor function $\boldsymbol{T}(\alpha)$ of a scalar variable α,

$$\frac{d}{d\alpha}\,\Phi(\boldsymbol{T}(\alpha)) = [\partial_{\boldsymbol{T}}\Phi(\boldsymbol{T}(\alpha))]\cdot\dot{\boldsymbol{T}}(\alpha),$$

or, more succintly,

$$\Phi(\boldsymbol{T})^{\cdot} = \partial_{\boldsymbol{T}}\Phi(\boldsymbol{T}) \cdot \dot{\boldsymbol{T}}. \tag{1–28}$$

In components, $(\partial_{\boldsymbol{T}}\Phi)_{ij} = \partial\Phi/\partial T_{ij}$. A consequence of this definition is that, for $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{X})$,

$$\nabla\Phi(\boldsymbol{T}) = \partial_{\boldsymbol{T}}\Phi(\boldsymbol{T}){:}\nabla\boldsymbol{T} \tag{1–29}$$

(where the gradient on the left is the gradient of $\Phi(\boldsymbol{T}(\boldsymbol{X}))$ with respect to $\boldsymbol{X}$).

For functions $\Phi(\boldsymbol{a}, \boldsymbol{b}, \ldots)$ of scalar, tensor, and vector variables, $\partial_{\boldsymbol{a}}\Phi(\boldsymbol{a}, \boldsymbol{b}, \ldots)$, say, will denote the partial derivative with respect to the variable $\boldsymbol{a}$.

Part A

Configurational Forces within a Classical Context

Much is to be gained by a discussion of configurational forces within a context that neglects evolving material structures such as defects and phase interfaces, even though within that context such forces are extraneous to the solution of actual boundary-value problems.

CHAPTER 2

Kinematics

a. Reference body. Material points. Motions

I write $\mathscr{E}$ for three-dimensional Euclidean space and restrict attention to a given open time interval. To avoid cumbersome statements I use the phrase "all t" to mean "all t in that interval," and so on.

I consider a body identified with the region B of Euclidean space $\mathscr{E}$ it occupies in a fixed configuration; I refer to B as the **reference body** and to points $\boldsymbol{X} \in B$ as **material points**.

A smooth mapping $\boldsymbol{y}$ that assigns to each t and each $\boldsymbol{X} \in B$ a point $\boldsymbol{x} = \boldsymbol{y}(\boldsymbol{X}, t)$ in $\mathscr{E}$ represents a **motion** (of B) if $\boldsymbol{y}(\boldsymbol{X}, t)$ is one-to-one as a function of $\boldsymbol{X}$ and if the **deformation gradient**

$$\boldsymbol{F} = \nabla \boldsymbol{y} \tag{2–1}$$

satisfies $\det \boldsymbol{F} > 0$; $\boldsymbol{x} = \boldsymbol{y}(\boldsymbol{X}, t)$ is then the **place** occupied by $\boldsymbol{X}$ at time t,

$$\bar{B}(t) = \boldsymbol{y}(B, t) \tag{2–2}$$

is the **deforming body** at t, and[1]

$$\dot{\boldsymbol{y}}(\boldsymbol{X}, t) = \frac{\partial}{\partial t} \boldsymbol{y}(\boldsymbol{X}, t) \tag{2–3}$$

is the **motion velocity**.

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[1]It is convenient to denote by an overbar a quantity that has been transported, via the motion, to the deformed configuration. In particular, this is done with sets, so that $\bar{B}(t)$ is the deformed body and not the closure of B. The following notation is used throughout: $(\,)^{\cdot}$ (a dot) denotes the derivative with respect to t holding $\boldsymbol{X}$ fixed; ∇ and Div are the gradient and divergence with respect to $\boldsymbol{X}$ holding t fixed; when the place $\boldsymbol{x}$ and time t are used as variables, $(\,)'$ (a prime) denotes the derivative with respect to t holding $\boldsymbol{x}$ fixed.

Since $\boldsymbol{x} = \boldsymbol{y}(\boldsymbol{X}, t)$ is invertible at each fixed t, the material point $\boldsymbol{X}$ may be considered as a function,

$$\boldsymbol{X} = \boldsymbol{Y}(\boldsymbol{x}, t), \tag{2–4}$$

of the place $\boldsymbol{x}$ and time t. I will refer to the mapping (2–4) as the **inverse motion**. Because $\boldsymbol{y}(\boldsymbol{Y}(\boldsymbol{x}, t), t) = \boldsymbol{x}$, it follows that

$$\dot{\boldsymbol{y}} = -\boldsymbol{F}\,\boldsymbol{Y}' \tag{2–5}$$

with

$$\boldsymbol{Y}'(\boldsymbol{x}, t) = \frac{\partial}{\partial t}\,\boldsymbol{Y}(\boldsymbol{x}, t) \tag{2–6}$$

the **inverse-motion velocity**.

b. Material and spatial vectors. The sets $\mathscr{E}_{\text{space}}$ and $\mathscr{E}_{\text{matter}}$

$\bar{B}(t)$ is the set actually observed during the motion of a body; the reference body B serves only to be label material points; any other configuration could equally well have been used as reference. That is why it is useful to differentiate between $\mathscr{E}_{\text{space}}$, the copy of $\mathscr{E}$ that represents the ambient space for $\bar{B}(t)$, and $\mathscr{E}_{\text{matter}}$, the copy that represents the ambient space for B. In accord with this, I use the following terminology:

material vector: vector associated with $\mathscr{E}_{\text{matter}}$;
spatial vector: vector associated with $\mathscr{E}_{\text{space}}$.

The motion velocity $\dot{\boldsymbol{y}}(\boldsymbol{X}, t)$ is then a spatial vector, while the deformation gradient $\boldsymbol{F}(\boldsymbol{X}, t)$ is a linear transformation of material vectors into spatial vectors.

For convenience I use a single symbol $\boldsymbol{o}$ for an arbitrary but fixed choice of "origin" for $\mathscr{E}_{\text{matter}}$ and $\mathscr{E}_{\text{space}}$, leaving it to the context to decide which space is intended.

The presumption that $\bar{B}(t)$ and B do not belong to the same space seems natural. $\bar{B}(t)$ represents the body during an actual motion, a motion that could, in principle, be seen or felt by any of us. On the other hand, the set B, while essential to the mathematical structure of continuum mechanics, is virtual; the body need never occupy B, although it might. Here it is useful to consider, within the framework of particle mechanics, a system consisting of, say, a red particle, and a blue particle. B is the counterpart of the set of particle labels, which could be {1, 2}, or {red,blue}, or the initial partial positions $\{\boldsymbol{x}_1(0), \boldsymbol{x}_2(0)\}$, and, with respect to these choices, $\mathscr{E}_{\text{matter}}$ is the analog of the integers, or the set of primary colors, or three-dimensional Euclidean space.

c. Material and spatial observers[2]

I consider two *independent* classes of observers: **spatial observers** that describe $\mathscr{E}_{\text{space}}$ and **material observers** that describe $\mathscr{E}_{\text{matter}}$. For *each class* I restrict attention to changes in observer for which the observers, in motion relative to each other, are *coincident* at some arbitrarily chosen time. The phrase *invariant under a change in observer* then signifies invariance at the time of coincidence.

For a **change in spatial observer** the relative velocity at time of coincidence has the form

$$\text{velocity} = \boldsymbol{w} + \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{o}) \qquad (\boldsymbol{w}, \boldsymbol{\omega} = \text{spatial vectors}) \tag{2–7}$$

and the motion velocity $\dot{\boldsymbol{y}}$ transforms according to

$$\dot{\boldsymbol{y}} \to \dot{\boldsymbol{y}} + \boldsymbol{w} + \boldsymbol{\omega} \times (\boldsymbol{y} - \boldsymbol{o}). \tag{2–8}$$

The discussion of material observers is delicate. I view the foregoing description of $\mathscr{E}_{\text{matter}}$ in which the reference body and its material points are independent of time as a description obtained by a **rest observer**. I consider **changes in material observer** from this rest observer to a Galilean observer who views the rest observer in motion with

$$\text{velocity} = \boldsymbol{a} \qquad (\boldsymbol{a} = \text{material vector}). \tag{2–9}$$

Under such a change in observer the points observed as stationary by the moving observer *do not represent material points*; *material points as viewed by the moving observer are seen to migrate with velocity* $\boldsymbol{a}$. Indeed, the Galilean observer views the points

$$\tilde{\boldsymbol{X}} = \boldsymbol{X} - (t - t_0)\boldsymbol{a} \qquad (t_0 = \text{time of coincidence}) \tag{2–10}$$

as stationary; but the $\tilde{\boldsymbol{X}}$s do not represent material points, which continue to be labeled by $\boldsymbol{X}$s. Thus *material time derivatives measured by the moving observer remain derivatives holding material points* $\boldsymbol{X}$ *fixed.*

I could consider the more general case of a moving (non-Galilean) observer with

$$\text{velocity} = \boldsymbol{a} + \boldsymbol{\gamma} \times (\boldsymbol{X} - \boldsymbol{o}) \qquad (\boldsymbol{a}, \boldsymbol{\gamma} = \text{material vectors}) \tag{2–11}$$

at the time of coincidence, but the additional generality would add nothing essential to the discussion (cf. the paragraph containing (5–11)).

d. Consistency requirement. Objective fields

Because spatial observers view spatial vectors and are oblivious to material vectors, and because the reverse is true for material observers, the following general rule seems appropriate.

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[2]Cf. the detailed discussion of Gurtin and Struthers [1990, §4].

Consistency requirement for vector fields: Those spatial vector fields that represent physical quantities should be invariant under changes in material observer; material vector fields that represent physical quantities should be invariant under changes in spatial observer.

For example, the motion velocity $\dot{\boldsymbol{y}}$ represents the time derivative of the motion holding material points $\boldsymbol{X}$ fixed; because the transformation to $\tilde{\boldsymbol{X}}$ does not affect this computation,

$$\dot{\boldsymbol{y}} \text{ is invariant under a change in material observer.} \tag{2–12}$$

Many of the fields of interest are objective in the sense that their transformation at any given time t obeys the standard rules for the transformation of scalars, vectors, and tensors under the observer change at t.[3] Here the stipulation that we restrict attention to the time of observer coincidence rules out the necessity of considering orientational changes and, consequently, allows for a simple definition of objectivity: A field Φ is **objective** if Φ is invariant (i.e., $\Phi \to \Phi$) under both spatial and material changes in observer.

[3]Cf., e.g., Truesdell and Noll [1965, §17].

CHAPTER 3

Standard Forces. Working

I begin with a discussion of the standard forces that form the basis for classical continuum mechanics. I consider inertia as represented through an internal body force.

a. Forces

Motions are accompanied by forces. Classically, forces in continuum mechanics are described by body forces distributed over the volume and tractions distributed over oriented surfaces. Such body forces and tractions may be measured per unit volume and area in the reference body or per unit volume and area in the deformed body; even so, the resulting forces are always *spatial* vectors. Here it is most convenient to measure forces in the reference body, so that, in particular, *stresses* are *Piola stresses*.[1]

Specifically, I restrict attention to a standard force system described by the fields:

$\boldsymbol{S}$ **stress**

$\boldsymbol{b}$ **external body force**

with $\boldsymbol{b}$ presumed to include inertia. The traction exerted across an oriented surface $\mathscr{S}$ is represented by the action $\boldsymbol{Sn}$ of the stress $\boldsymbol{S}$ on the unit normal $\boldsymbol{n}$ to $\mathscr{S}$, and both $\boldsymbol{Sn}$ and $\boldsymbol{b}$ perform work over spatial velocities; thus $\boldsymbol{S}(\boldsymbol{X}, t)$ is a linear transformation of material vectors into spatial vectors, while $\boldsymbol{b}(\boldsymbol{X}, t)$ is a spatial

[1] Referred to as *first Piola-Kirchhoff stresses* by Truesdell and Noll [1965, §43A] and as *Piola-Kirchoff stresses* by Gurtin [1981, §27].

vector. I assume that

$$\boldsymbol{S} \text{ and } \boldsymbol{b} \text{ are objective.} \tag{3–1}$$

There is, I believe, a basic misconception that inertial body forces are not objective.[2] Consider an inertial observer, an inertial body force $\boldsymbol{b} = -\rho\ddot{\boldsymbol{y}}$ (ρ = reference density), and the noninertial observer change defined by the transformation $\tilde{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{z}(t)$. Then relative to the new observer the motion is given by $\tilde{\boldsymbol{y}}(\boldsymbol{X}, t) = \boldsymbol{y}(\boldsymbol{X}, t) + \boldsymbol{z}(t)$ and $\tilde{\boldsymbol{b}}$ is defined by $\tilde{\boldsymbol{b}} = -\rho(\ddot{\tilde{\boldsymbol{y}}} - \ddot{\boldsymbol{z}})$, so that $\tilde{\boldsymbol{b}} = \boldsymbol{b}$; thus, trivially, $\boldsymbol{b}$ is invariant, although it does not preserve its form, because $\tilde{\boldsymbol{b}}$ is not $-\rho$ times the acceleration $\ddot{\tilde{\boldsymbol{y}}}$ measured by the noninertial observer.

b. Working. Standard force and moment balances as consequences of invariance under changes in spatial observer[3]

Let P be a (referential) **control volume** (i.e., a bounded subregion of B with smooth boundary ∂P) and let $\boldsymbol{n}$ denote the outward unit normal to ∂P. I define the **working** on P through the classical relation

$$W(P) = \int_{\partial P} \boldsymbol{S}\boldsymbol{n} \cdot \dot{\boldsymbol{y}}\, da + \int_P \boldsymbol{b} \cdot \dot{\boldsymbol{y}}\, dv \tag{3–2}$$

and require that $W(P)$ *be invariant under changes in spatial observer*. Then, by (2–8) and (3–1),

$$\begin{aligned}\int_{\partial P} \boldsymbol{S}\boldsymbol{n} \cdot \dot{\boldsymbol{y}}\, da + \int_P \boldsymbol{b} \cdot \dot{\boldsymbol{y}}\, dv = \int_{\partial P} \boldsymbol{S}\boldsymbol{n} \cdot [\dot{\boldsymbol{y}} + \boldsymbol{w} + \boldsymbol{\omega} \times (\boldsymbol{y} - \boldsymbol{o})]\, da \\ + \int_P \boldsymbol{b} \cdot [\dot{\boldsymbol{y}} + \boldsymbol{w} + \boldsymbol{\omega} \times (\boldsymbol{y} - \boldsymbol{o})]\, dv;\end{aligned} \tag{3–3}$$

hence

$$\boldsymbol{0} = \left\{\int_{\partial P} \boldsymbol{S}\boldsymbol{n}\, da + \int_P \boldsymbol{b}\, dv\right\} \cdot \boldsymbol{w} + \left\{\int_{\partial P} (\boldsymbol{y} - \boldsymbol{o}) \times \boldsymbol{S}\boldsymbol{n}\, da + \int_P (\boldsymbol{y} - \boldsymbol{o}) \times \boldsymbol{b}\, dv\right\} \cdot \boldsymbol{\omega} \tag{3–4}$$

for all P and all vectors $\boldsymbol{w}$ and $\boldsymbol{\omega}$. Invariance of the working therefore yields the **standard force and moment balances**

$$\int_{\partial P} \boldsymbol{S}\boldsymbol{n}\, da + \int_P \boldsymbol{b}\, dv = \boldsymbol{0}, \tag{3–5a}$$

$$\int_{\partial P} (\boldsymbol{y} - \boldsymbol{o}) \times \boldsymbol{S}\boldsymbol{n}\, da + \int_P (\boldsymbol{y} - \boldsymbol{o}) \times \boldsymbol{b}\, dv = \boldsymbol{0} \tag{3–5b}$$

for all P; or equivalently,[4]

$$\operatorname{Div} \boldsymbol{S} + \boldsymbol{b} = \boldsymbol{0}, \tag{3–6a}$$

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[2] Cf. the discussion of Noll [1995].
[3] Cf. Noll [1963].
[4] Cf., e.g., Gurtin [1981, §27].

$$\boldsymbol{S}\boldsymbol{F}^{\top} = \boldsymbol{F}\,\boldsymbol{S}^{\top}. \tag{3–6b}$$

The assertion (3–5a) $\Leftrightarrow$ (3–6a) is a direct consequence of the divergence theorem. To show that, granted (3–6a), (3–5b) $\Leftrightarrow$ (3–6b), consider the tensor

$$\boldsymbol{M}(P) = \int_{\partial P} (\boldsymbol{y} - \boldsymbol{o}) \otimes \boldsymbol{S}\boldsymbol{n}\, da + \int_{P} (\boldsymbol{y} - \boldsymbol{o}) \otimes \boldsymbol{b}\, dv.$$

Then (3–5b) is equivalent to the assertion that $\boldsymbol{M}(P)$ be symmetric: $\boldsymbol{M}(P) = \boldsymbol{M}(P)^{\top}$. Since

$$\int_{\partial P} (\boldsymbol{y} - \boldsymbol{o}) \otimes \boldsymbol{S}\boldsymbol{n}\, da = \int_{P} (\boldsymbol{y} - \boldsymbol{o}) \otimes \operatorname{Div} \boldsymbol{S}\, dv + \int_{P} \boldsymbol{F}\,\boldsymbol{S}^{\top} dv,$$

(3–6a) yields the conclusion

$$\boldsymbol{M}(P) = \int_{P} \boldsymbol{F}\,\boldsymbol{S}^{\top}\, dv,$$

and $\boldsymbol{M}(P) = \boldsymbol{M}(P)^{\top}$ for all P if and only if (3–6b) is satisfied.

Given a control volume P, (3–6a) and the divergence theorem imply that

$$\int_{\partial P} \boldsymbol{S}\boldsymbol{n} \cdot \dot{\boldsymbol{y}}\, da + \int_{P} \boldsymbol{b} \cdot \dot{\boldsymbol{y}}\, dv = \int_{P} \boldsymbol{S} \cdot \dot{\boldsymbol{F}}\, dv, \tag{3–7}$$

and hence that, trivially,

$$W(P) = \int_{P} \boldsymbol{S} \cdot \dot{\boldsymbol{F}}\, dv. \tag{3–8}$$

This expression represents a **power balance** for P; $W(P)$ as defined in (3–2) represents the working of all forces external to P, and (3–8) relates this **external working** to the **internal working** $\int_P \boldsymbol{S} \cdot \dot{\boldsymbol{F}}\, dv$. The integrand $\boldsymbol{S} \cdot \dot{\boldsymbol{F}}$ is usually referred to as the **stress power**; $\boldsymbol{S} \cdot \dot{\boldsymbol{F}}$ represents internal working resulting from temporally varying strains.

A rigid motion has $\boldsymbol{F}$ orthogonal, so that $\boldsymbol{F}\boldsymbol{F}^{\top} = \boldsymbol{1}$, which, when differentiated, implies that $\dot{\boldsymbol{F}}\boldsymbol{F}^{\top}$ is skew. By (3–6b), $\boldsymbol{S}\boldsymbol{F}^{\top}$ is symmetric. Thus $\boldsymbol{S} \cdot \dot{\boldsymbol{F}} = \boldsymbol{S}\boldsymbol{F}^{\top} \cdot \dot{\boldsymbol{F}}\boldsymbol{F}^{\top} = 0$ and the stress power vanishes when the motion is rigid, a result that justifies the use of the term *strains* in the previous paragraph.

The tensor field

$$\boldsymbol{T} = (\det \boldsymbol{F})^{-1} \boldsymbol{S}\boldsymbol{F}^{\top}, \tag{3–9}$$

usually referred to as the **Cauchy stress**,[5] represents the stress measured per unit area in the deformed configuration. Similarly, $\bar{\boldsymbol{b}} = (\det \boldsymbol{F})^{-1} \boldsymbol{b}$ represents the body force measured per unit volume in the deformed configuration. Precisely, if $\mathscr{S}$ with (unit) normal $\boldsymbol{n}$ is an oriented surface in B then, considering $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{x}, t)$ and $\bar{\boldsymbol{b}} = \bar{\boldsymbol{b}}(\boldsymbol{X}, t)$ functions of $\boldsymbol{x} = \boldsymbol{y}(\boldsymbol{X}, t)$ and t,

$$\int_{\mathscr{S}} \boldsymbol{T}\boldsymbol{n}\, da = \int_{\bar{\mathscr{S}}} \boldsymbol{T}\bar{\boldsymbol{n}}\, d\bar{a}, \qquad \int_{P} \boldsymbol{b}\, dv = \int_{\bar{P}} \bar{\boldsymbol{b}}\, d\bar{v} \tag{3–10}$$

[5] Cf., e.g., Gurtin [1981, §14, §27].

(using the notation discussed in the paragraph following (2–3), so that $\bar{\mathscr{S}}$ with normal $\bar{\boldsymbol{n}}$ is the image of $\mathscr{S}$ under $\boldsymbol{y}$, $d\bar{a}$ is the element of area on $\bar{\mathscr{S}}$, and so on). Then the balance (3–5a) takes the form

$$\int_{\partial\bar{P}} \boldsymbol{T}\bar{\boldsymbol{n}}\, d\bar{a} + \int_{\bar{P}} \bar{\boldsymbol{b}}\, d\bar{v} = \boldsymbol{0}$$

and, letting div and grad, respectively, denote the spatial divergence and spatial gradient (with respect to $\boldsymbol{x}$), this yields the local balance

$$\operatorname{div} \boldsymbol{T} + \bar{\boldsymbol{b}} = \boldsymbol{0}. \tag{3–11}$$

Similarly, the moment balance (3–5b) has an analogous counterpart involving $\boldsymbol{T}$ and $\bar{\boldsymbol{b}}$ whose local form yields the symmetry of $\boldsymbol{T}$, a result that also follows from (3–6b). Finally, the working (3–2) has the equivalent forms

$$W(P) = \int_{\partial\bar{P}} \boldsymbol{T}\bar{\boldsymbol{n}} \cdot \dot{\boldsymbol{y}}\, d\bar{a} + \int_{\bar{P}} \bar{\boldsymbol{b}} \cdot \dot{\boldsymbol{y}}\, d\bar{v} = \int_{\bar{P}} \boldsymbol{T} \cdot \operatorname{grad} \dot{\boldsymbol{y}}\, d\bar{v},$$

so that $\boldsymbol{T} \cdot \operatorname{grad} \dot{\boldsymbol{y}}$ is the stress power measured per unit deformed volume, and

$$\boldsymbol{S} \cdot \dot{\boldsymbol{F}} = (\det \boldsymbol{F})\, \boldsymbol{T} \cdot \operatorname{grad} \dot{\boldsymbol{y}} = (\det \boldsymbol{F})\, \boldsymbol{T} \cdot \boldsymbol{D}, \qquad \boldsymbol{D} = \frac{1}{2}(\operatorname{grad} \dot{\boldsymbol{y}} + \operatorname{grad} \dot{\boldsymbol{y}}^{\top}). \tag{3–12}$$

CHAPTER 4

Migrating Control Volumes. Stationary and Time-Dependent Changes in Reference Configuration

To characterize the manner in which configurational forces perform work, a means of capturing the kinematics associated with the transfer of material is needed. I accomplish this with the aid of three notions, none of which is a standard. The first, that of material observers, has been examined in Chapter 2. The other two notions are:

1. control volumes $P(t)$ that migrate through B and thereby result in the transfer of material to $P(t)$ across $\partial P(t)$;
2. time-dependent changes in reference configuration.

In continuum mechanics one often uses the term *part* for a fixed subregion P of B; and the phrase *evolution of P with time* refers to the motion of the deformed part $\bar{P}(t) = \boldsymbol{y}(P, t)$. Parts should not be confused with control volumes $P(t)$, which are not fixed subregions of the reference body B but rather *migrate* through B. The phrase *transfer of material to ∂P* is meant in a general sense that allows for the "transfer of material *from* ∂P," and similarly for the phrase *addition of material to* ∂P.

a. Migrating control volumes $P = P(t)$. Velocity fields for $\partial P(t)$ and $\partial \bar{P}(t)$

Let $P = P(t)$ be a (smoothly) **migrating control volume** with U the (scalar) **normal velocity** of ∂P in the direction of the outward unit normal $\boldsymbol{n}$. To describe the working associated with the evolution of P, I introduce a field $\boldsymbol{q}$ interpreted as the velocity with which an external agency adds material to ∂P. Compatibility then requires that the normal component of $\boldsymbol{q}$ be U:

$$\boldsymbol{q} \cdot \boldsymbol{n} = U; \tag{4–1}$$

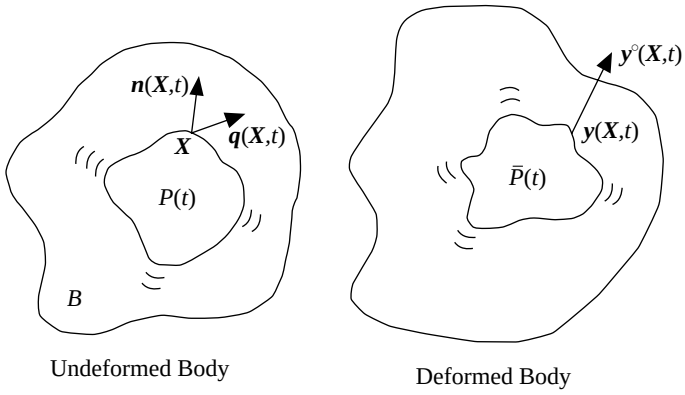

FIGURE 4.1. The time-dependent control volume $P(t)$, which deforms to $\bar{P}(t)$, with $\boldsymbol{q}(\boldsymbol{X}, t)$ a velocity field for $\partial P(t)$ and $\boldsymbol{y}^\circ$ the corresponding motion velocity following $\partial P(t)$.

$\boldsymbol{q}$ is otherwise arbitrary (Figure 4.1).

This discussion should motivate the following definition: An assignment, at each t, of a material vector $\boldsymbol{q}(\boldsymbol{X}, t)$ to each $\boldsymbol{X} \in \partial P(t)$ is a **velocity field for** ∂P if $\boldsymbol{q}$ is a smooth field that satisfies $\boldsymbol{q} \cdot \boldsymbol{n} = U$.

One might ask: Why not use, as velocity field, the vectorial normal velocity $U\boldsymbol{n}$, which is intrinsic? I have many reasons for not doing this:

1. If material is viewed as being transferred to ∂P via an external agency, then it would seem unreasonable to restrict the corresponding velocity to normality.
2. Changes in material observer do not preserve normality of the velocity field.
3. In the study of basic issues a powerful tool is the requirement that a theory be invariant under changes irrelevant to the physics; here invariance under changes in velocity field yields important and unexpected consequences.
4. An important example of a migrating control volume is a ball $P(t)$ of fixed radius centered at a point $\boldsymbol{Z}(t)$ that is migrating through B; in this case the spatially constant field $\boldsymbol{q}(t) = \dot{\boldsymbol{Z}}(t)$ represents a velocity field for $\partial P(t)$.
5. Granted smoothness, $\partial P(t)$ may be parametrized locally in time by a function of the form $\boldsymbol{X} = \hat{\boldsymbol{X}}(\zeta, t)$, $\zeta = (\zeta_1, \zeta_2)$; the field $\boldsymbol{q}(\boldsymbol{X}, t) = \partial \hat{\boldsymbol{X}}(\zeta, t)/\partial t$ then represents a velocity field for $\partial P(t)$.

Let $P = P(t)$ be a migrating control volume. A velocity field $\boldsymbol{q}$ for ∂P may be viewed as a velocity field for particles evolving on a migrating surface ∂P, with the trajectory $\boldsymbol{Z}(\tau)$ of the particle that passes through $\boldsymbol{X} \in \partial P(t)$ at time t the unique solution of

$$\dot{\boldsymbol{Z}}(\tau) = \boldsymbol{q}(\boldsymbol{Z}(\tau), \tau), \qquad \boldsymbol{Z}(t) = \boldsymbol{X}. \tag{4–2}$$

Given a field $\Phi(\boldsymbol{X}, t)$, the **time derivative of** Φ **following** ∂P, as described by $\boldsymbol{q}$, is the time derivative along such trajectories:

$$\mathring{\Phi}(\boldsymbol{X}, t) = \frac{d}{d\tau}\, \Phi(\boldsymbol{Z}(\tau), \tau)\big|_{\tau = t}. \tag{4–3}$$

Let $\boldsymbol{y}$ be a motion. Then $\mathring{\boldsymbol{y}}$, which I refer to as the **motion velocity following** ∂P, satisfies

$$\mathring{\boldsymbol{y}} = \dot{\boldsymbol{y}} + \boldsymbol{F}\boldsymbol{q}. \tag{4–4}$$

Further, writing

$$\bar{P}(t) = \boldsymbol{y}(P(t), t) \tag{4–5}$$

for the deformed control volume, the path $\boldsymbol{Z}(\tau)$ is mapped into a path $\boldsymbol{y}(\boldsymbol{Z}(\tau), \tau)$ that lies on $\partial\bar{P}(\tau)$ at each τ. Because $\mathring{\boldsymbol{y}}(\boldsymbol{X}, t)$ is the derivative of $\boldsymbol{y}(\boldsymbol{Z}(\tau), \tau)$ at $\tau = t$, $\mathring{\boldsymbol{y}}$ represents a velocity field for $\partial\bar{P}$. (Alternatively, each parametrization $\boldsymbol{X} = \hat{\boldsymbol{X}}(\sigma, t)$ of $\partial P(t)$ induces a corresponding parametrization $\boldsymbol{x} = \hat{\boldsymbol{x}}(\sigma, t) = \boldsymbol{y}(\hat{\boldsymbol{X}}(\sigma, t), t)$ for $\partial\bar{P}(t)$; if $\boldsymbol{q} = \partial\hat{\boldsymbol{X}}/\partial t$, then $\mathring{\boldsymbol{y}} = \partial\hat{\boldsymbol{x}}/\partial t$.) Note that $\mathring{\boldsymbol{y}}$ accounts for the evolution of $\partial\bar{P}$ through two terms; the motion velocity $\dot{\boldsymbol{y}}$ and the velocity $\boldsymbol{F}\boldsymbol{q}$ at which deformed material is being transferred to $\partial\bar{P}$.

The fields $\boldsymbol{q}$ and $\mathring{\boldsymbol{y}}$ transform according to

$$\boldsymbol{q} \to \boldsymbol{q} + \boldsymbol{a}, \qquad \mathring{\boldsymbol{y}} \to \mathring{\boldsymbol{y}} \tag{4–6}$$

under the change in material observer defined by (2–9), and according to

$$\mathring{\boldsymbol{y}} \to \mathring{\boldsymbol{y}} + \boldsymbol{w} + \boldsymbol{\omega} \times (\boldsymbol{y} - \boldsymbol{o}), \qquad \boldsymbol{q} \to \boldsymbol{q} \tag{4–7}$$

under the change in spatial observer defined by (2–7) (cf. the consistency requirement as stated in Section 2d).

b. Change in reference configuration

b1. Stationary change in reference configuration

Let $\boldsymbol{\kappa}$ be a smooth mapping

$$\overset{*}{\boldsymbol{X}} = \boldsymbol{\kappa}(\boldsymbol{X}) \tag{4–8}$$

of the reference body B onto a region

$$\overset{*}{B} = \boldsymbol{\kappa}(B)$$

of $\mathscr{E}_{\text{matter}}$, and let

$$\boldsymbol{K} = \nabla\boldsymbol{\kappa}, \qquad \mathscr{J} = \det \boldsymbol{K}. \tag{4–9}$$

Then $\boldsymbol{\kappa}$ is a **stationary change in reference** if $\boldsymbol{\kappa}$ is one-to-one with $\mathscr{J} > 0$.

Each field $\Phi(\boldsymbol{X}, t)$ associated with the body and hence defined over B will be presumed to have a unique representation, relative to $\boldsymbol{\kappa}$, as a field $\overset{*}{\Phi}(\overset{*}{\boldsymbol{X}}, t)$ over $\overset{*}{B}$. In particular, given a motion $\boldsymbol{y}(\boldsymbol{X}, t)$, $\overset{*}{\boldsymbol{F}}(\overset{*}{\boldsymbol{X}}, t)$ denotes the deformation gradient relative to the new reference; that is, the gradient of

$$\overset{*}{\boldsymbol{y}}(\overset{*}{\boldsymbol{X}}, t) = \boldsymbol{y}(\boldsymbol{\kappa}^{-1}(\overset{*}{\boldsymbol{X}}), t) \tag{4–10}$$

with respect to $\overset{*}{\boldsymbol{X}}$. Hence

$$\overset{*}{\boldsymbol{F}} = \boldsymbol{F}\boldsymbol{K}^{-1}. \tag{4–11}$$

Let $P = P(t)$ be a migrating control volume, $\boldsymbol{q}$ be a velocity field for ∂P, and $\boldsymbol{Z}(\tau)$ be defined through (4–2). Then, under $\boldsymbol{\kappa}$, P transforms to a control volume

$$\overset{*}{P}(t) = \boldsymbol{\kappa}(P(t)), \tag{4–12}$$

while $\boldsymbol{q}$ transforms to the velocity field $\overset{*}{\boldsymbol{q}}$ for $\partial\overset{*}{P}$ given by

$$\overset{*}{\boldsymbol{q}}(\overset{*}{\boldsymbol{X}}, t) \qquad \frac{d}{d\tau}\,\boldsymbol{\kappa}(\boldsymbol{Z}(\tau))\big|_{\tau=t} \tag{4–13}$$

for $\overset{*}{\boldsymbol{X}} = \boldsymbol{\kappa}(\boldsymbol{X})$; thus

$$\overset{*}{\boldsymbol{q}} = \boldsymbol{K}\boldsymbol{q}. \tag{4–14}$$

b2. Time-dependent change in reference configuration

A **time-dependent change in reference** is a smooth mapping

$$\overset{*}{\boldsymbol{X}} = \boldsymbol{\kappa}(\boldsymbol{X}, t) \tag{4–15}$$

with $\boldsymbol{\kappa}$ a stationary change in reference at each fixed time. The composition of the material at each point $\overset{*}{\boldsymbol{X}}$ in the range of $\boldsymbol{\kappa}$ will generally change with time; that is why such points will be referred to as **reference labels** rather than material points.

More useful than $\boldsymbol{\kappa}$ itself is its fixed-time inverse

$$\boldsymbol{X} = \hat{\boldsymbol{X}}(\overset{*}{\boldsymbol{X}}, t). \tag{4–16}$$

This mapping describes the trajectories of fixed labels $\overset{*}{\boldsymbol{X}}$ through the reference body B, while

$$\boldsymbol{q}(\boldsymbol{X}, t) = \frac{\partial}{\partial t}\,\hat{\boldsymbol{X}}(\overset{*}{\boldsymbol{X}}, t)\big|_{\overset{*}{\boldsymbol{X}}=\boldsymbol{\kappa}(\boldsymbol{X},t)} \tag{4–17}$$

represents the velocity field along such trajectories. Given a *fixed* region $\overset{*}{P}$ in the space of reference labels $\overset{*}{\boldsymbol{X}}$, consider the migrating control volume

$$P(t) = \hat{\boldsymbol{X}}(\overset{*}{P}, t) \tag{4–18}$$

and the corresponding deformed control volume $\bar{P}(t) = \boldsymbol{y}(P(t), t)$. Then $\boldsymbol{q}(\boldsymbol{X}, t)$ (restricted to $\partial P(t)$ at each t) represents a velocity field for $\partial P(t)$ with

$$\mathring{\boldsymbol{y}} = \dot{\boldsymbol{y}} + \boldsymbol{F}\boldsymbol{q} \tag{4–19}$$

the associated velocity field for $\partial\bar{P}(t)$.

An important example of a time-dependent change in reference is the motion $\boldsymbol{x} = \boldsymbol{y}(\boldsymbol{X}, t)$; here the inverse motion $\boldsymbol{Y}$ plays the role of $\hat{\boldsymbol{X}}$, $\boldsymbol{x}$ corresponds to $\overset{*}{\boldsymbol{X}}$,

and the fixed region $\overset{*}{P}$ is the deformed control volume $\bar{P}$, which is necessarily *stationary*. A consequence of (2–5), (4–17), and (4–19) is that, for this example,

$$\boldsymbol{q}(\boldsymbol{X}, t) = \boldsymbol{Y}'(\boldsymbol{x}, t)\big|_{\boldsymbol{x}=\boldsymbol{y}(\boldsymbol{X},t)}, \qquad \mathring{\boldsymbol{y}} = \boldsymbol{0}. \tag{4–20}$$

CHAPTER 5

Configurational Forces

Configurational forces are related to the integrity of a body's material structure; they act within the reference configuration and perform work in the transfer of material and in the evolution of structural defects.

Classically, body forces represent forces exerted on a body B by bodies *exterior* to B; in contrast, configurational force systems also require body forces that are *internal* to B.

a. Configurational forces

The configurational force system consists of three fields:

$\boldsymbol{C}$	**stress**
$\boldsymbol{g}$	**internal body force**
$\boldsymbol{e}$	**external body force**

As with standard forces, configurational tractions and body forces are measured in the reference body, with $\boldsymbol{Cn}$ the traction across any oriented referential surface $\mathscr{G}$ with unit normal $\boldsymbol{n}$. Thus and because $\boldsymbol{C}$, $\boldsymbol{g}$, and $\boldsymbol{e}$ perform work over velocities associated with the *reference body*, $\boldsymbol{C}(\boldsymbol{X}, t)$ is a linear transformation of material vectors into material vectors, while $\boldsymbol{g}(\boldsymbol{X}, t)$ and $\boldsymbol{e}(\boldsymbol{X}, t)$ are material vectors. I assume that, as for the standard force system (cf. (3–1))

$$\boldsymbol{C}, \boldsymbol{g}, \text{ and } \boldsymbol{e} \text{ are objective.} \tag{5–1}$$

A chief difference between the standard and configurational force systems is the presence of internal configurational forces. Such forces, being intimately connected with the material structure of the body, are best discussed within a framework more general than that discussed thus far. It is convenient, but somewhat misleading, to identify a body $\mathscr{B}$ with a region of space it might occupy, because that may change

from time to time; a more encompassing view of a body is as a set $\mathscr{B}$ of material points X together with a collection of possible configurations, where a configuration of $\mathscr{B}$ is a one-to-one mapping that assigns to each material point X a point $\boldsymbol{X} = \boldsymbol{\mu}(X)$ of Euclidean space.[1] To each configuration $\boldsymbol{\mu}$ there is a corresponding distribution of material and an associated system of configurational forces. I view $\boldsymbol{g}(\boldsymbol{X}, t)$ as the force needed to hold in place the material at $\boldsymbol{X}$ when the configuration is $\boldsymbol{\mu}$, or more simply to hold $\boldsymbol{X}$ in place in the configuration $\boldsymbol{\mu}$. Such forces are then "internal" to the *complete system*—material points plus region of space—that constitutes a body.

b. Working revisited

In discussing the working of the standard and configurational force systems, the reader should bear in mind the basic premise that *configurational forces perform work over positional changes in the reference body, while standard forces perform work over positional changes in space*.

Consider the standard and configurational forces associated with a *migrating control volume* $P = P(t)$, with $\boldsymbol{q}$ a velocity field for ∂P and $\mathring{\boldsymbol{y}}$ the corresponding motion velocity following ∂P.

1. Working of the stresses

I view the traction $\boldsymbol{Cn}$ as a force that performs work in conjunction with the migration of ∂P and therefore I choose $\boldsymbol{q}$ as an appropriate work-conjugate velocity for $\boldsymbol{Cn}$. Classically, control volumes do not migrate and the standard traction $\boldsymbol{Sn}$ on ∂P is work-conjugate to the motion velocity $\dot{\boldsymbol{y}}$, but ∂P when migrating has no intrinsic material description, because material is continually being added and removed, and it would seem appropriate to use as work-conjugate velocity for $\boldsymbol{Sn}$ the motion velocity $\mathring{\boldsymbol{y}}$ following ∂P, as $\mathring{\boldsymbol{y}}$ represents a velocity field for the deformed boundary $\partial \bar{P}$ consistent with the choice of $\boldsymbol{q}$ as velocity field for ∂P. I therefore write the working of the standard and configurational stresses in the form

$$\int_{\partial P(t)} \boldsymbol{Cn} \cdot \boldsymbol{q}\, da + \int_{\partial P(t)} \boldsymbol{Sn} \cdot \mathring{\boldsymbol{y}}\, da. \tag{5–2}$$

Note that when $\boldsymbol{q} = \boldsymbol{0}$ the control volume P is stationary; in this case (4–4) yields $\mathring{\boldsymbol{y}} = \dot{\boldsymbol{y}}$ and (5–2) reduces to the classical relation

$$\int_{\partial P} \boldsymbol{Sn} \cdot \dot{\boldsymbol{y}}\, da. \tag{5–3}$$

2. Working of the internal configurational body force

Because $\boldsymbol{g}$ represents forces that hold in place the material (points) in the reference configuration and the material is there immobile, $\boldsymbol{g}$ performs no work.

[1] Cf. Noll [1958].

3. Working of the external body forces

The configurational force $\boldsymbol{e}$ *performs no work*, because the material points $\boldsymbol{X}$ are fixed in the reference body, but the standard force $\boldsymbol{b}$ views the points $\boldsymbol{X}$ as evolving in *space* via the mapping $\boldsymbol{x} = \boldsymbol{y}(\boldsymbol{X}, t)$ and hence performs work with $\dot{\boldsymbol{y}}$ its conjugate velocity. The *net working* of the external forces $\boldsymbol{e}$ and $\boldsymbol{b}$ therefore has the classical form

$$\int_{P(t)} \boldsymbol{b} \cdot \dot{\boldsymbol{y}}\, dv. \tag{5–4}$$

Based on these remarks, I take the following expression as the appropriate generalization of the working (3–2):

$$W(P(t)) = \int_{\partial P(t)} \boldsymbol{C}\boldsymbol{n} \cdot \boldsymbol{q}\, da + \int_{\partial P(t)} \boldsymbol{S}\boldsymbol{n} \cdot \mathring{\boldsymbol{y}}\, da + \int_{P(t)} \boldsymbol{b} \cdot \dot{\boldsymbol{y}}\, dv. \tag{5–5}$$

c. Configurational force balance as a consequence of invariance under changes in material observer[2]

By (2–8), (3–1), (4–7), and (5–1), invariance of (5–5) under changes in *spatial* observer yields the standard force and moment balance (3–6a), but nothing else.

Consider a change in *material* observer from an observer at rest to one who views the material in motion with velocity $\boldsymbol{a}$. Then $\boldsymbol{g}$ and $\boldsymbol{e}$, as observed by the moving observer, perform work, since material points as viewed by the new observer migrate with velocity $\boldsymbol{a}$. The working $W(P(t))$ as recorded by the observer in motion therefore has the form

$$W(P(t)) = \int_{\partial P(t)} \boldsymbol{C}\boldsymbol{n}\cdot(\boldsymbol{q}+\boldsymbol{a})\, da + \int_{P(t)} (\boldsymbol{g}+\boldsymbol{e})\cdot\boldsymbol{a}\, dv + \int_{\partial P(t)} \boldsymbol{S}\boldsymbol{n}\cdot\mathring{\boldsymbol{y}}\, da + \int_{P(t)} \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\, dv \tag{5–6}$$

(cf. (2–12), (3–1), (4–6), and (5–1)). The requirement that the *working* $W(P(t))$ *be invariant under changes in material observer* therefore leads to the conclusion

$$0 = \left\{ \int_{\partial P} \boldsymbol{C}\boldsymbol{n}\, da + \int_P (\boldsymbol{g} + \boldsymbol{e})\, dv \right\} \cdot \boldsymbol{a} \tag{5–7}$$

for all P and all vectors $\boldsymbol{a}$. A consequence of (5–7) is the **configurational force balance**

$$\int_{\partial P} \boldsymbol{C}\boldsymbol{n}\, da + \int_P (\boldsymbol{g} + \boldsymbol{e})\, dv = \boldsymbol{0}, \tag{5–8}$$

in which the individual terms

$$\int_{\partial P} \boldsymbol{C}\boldsymbol{n}\, da, \qquad \int_P \boldsymbol{g}\, dv, \qquad \int_P \boldsymbol{e}\, dv \tag{5–9}$$

represent the (net configurational) contact force, internal force, and external force on a control volume P. Finally, the requirement that (5–8) hold for all P yields

[2]Cf. Gurtin and Struthers [1990].

the local force balance

$$\operatorname{Div} \boldsymbol{C} + \boldsymbol{g} + \boldsymbol{e} = \boldsymbol{0}. \tag{5–10}$$

To account for non-Galilean material observers as defined in (2–11), the working (5–6) should have $\boldsymbol{a}$ replaced by $\boldsymbol{a} + \boldsymbol{\gamma} \times (\boldsymbol{X} - \boldsymbol{o})$ and should be augmented by the term

$$\int_P \{\boldsymbol{m} \cdot \boldsymbol{\gamma} + (\boldsymbol{g} + \boldsymbol{e}) \cdot [\boldsymbol{\gamma} \times (\boldsymbol{X} - \boldsymbol{o})]\}\, dv,$$

with $\boldsymbol{m}$ a material vector field that represents configurational body moments. (And perhaps there should also be a term that represents configurational couple stresses.) The configurational moment balance

$$\int_{\partial P} (\boldsymbol{X} - \boldsymbol{o}) \times \boldsymbol{C}\boldsymbol{n}\, da + \int_P (\boldsymbol{X} - \boldsymbol{o}) \times (\boldsymbol{g} + \boldsymbol{e})\, dv + \int_P \boldsymbol{m}\, dv = \boldsymbol{0} \tag{5–11}$$

then follows from invariance of the working under changes in material observer. The local form of (5–11), namely,

$$\boldsymbol{C} - \boldsymbol{C}^\top = -\boldsymbol{m}\times, \tag{5–12}$$

establishes the need for the body moment $\boldsymbol{m}$, as $\boldsymbol{C}$ need not be symmetric (cf. (6–9)). Certain defect structures generate configurational, but not standard, moments;[3] even so, for the situations discussed here there is sufficient indeterminacy in the configurational system to render the configurational moment balance superfluous.

d. Invariance under changes in velocity field for $\partial P(t)$. Configurational stress relation

Let $P = P(t)$ be a migrating control volume. I now require that the working be independent of the manner in which the external agency transfers material to ∂P; precisely, I require that $W(P)$, *given by* (5–5), *be independent of the choice of velocity field* $\boldsymbol{q}$ *corresponding to the prescribed motion* of ∂P as described by its normal velocity U. This requirement[4] has a major consequence, which I now derive.

By (4–4), (5–5) may be rewritten in the form

$$W(P) = \int_{\partial P} \boldsymbol{S}\boldsymbol{n} \cdot \dot{\boldsymbol{y}}\, da + \int_P \boldsymbol{b} \cdot \dot{\boldsymbol{y}}\, dv + \int_{\partial P} (\boldsymbol{F}^\top \boldsymbol{S}\boldsymbol{n} + \boldsymbol{C}\boldsymbol{n}) \cdot \boldsymbol{q}\, da. \tag{5–13}$$

Because of (4–1), changes in velocity field affect the tangential component of $\boldsymbol{q}$ but leave the normal component unaltered. The invariance of $W(P)$ under such changes is therefore equivalent to the requirement that

$$\int_{\partial P} (\boldsymbol{F}^\top \boldsymbol{S}\boldsymbol{n} + \boldsymbol{C}\boldsymbol{n}) \cdot \boldsymbol{t}\, da = 0$$

---

[3] For example, a phase transition gives rise to configurational moments distributed over the interface whenever the interfacial energy is anisotropic (cf., e.g., Gurtin and Struthers [1990, §4]; Gurtin [1993, eqs. (6–2)–(6–4)]; see also the paragraph following (18–12)).

[4] Referred to by Gurtin and Struthers [1990] and Gurtin [1995] as *invariance* under reparametrization, because it represents invariance under the choice of time-dependent parametrization for $\partial P(t)$.

for all tangential vector fields $\boldsymbol{t}$ on ∂P. Thus, letting $\boldsymbol{A} = \boldsymbol{F}^\top \boldsymbol{S} + \boldsymbol{C}$, $\boldsymbol{t} \cdot \boldsymbol{A}\boldsymbol{n} = 0$ on ∂P for all tangential vector fields $\boldsymbol{t}$. Thus, since ∂P and hence $\boldsymbol{n}$ are arbitrary, $\boldsymbol{A}\boldsymbol{n}$ must be parallel to $\boldsymbol{n}$ for all $\boldsymbol{n}$. Thus every vector must be an eigenvector of $\boldsymbol{A}$; hence there is a scalar field π such that

$$\boldsymbol{C} + \boldsymbol{F}^\top \boldsymbol{S} = \pi \boldsymbol{1}. \tag{5–14}$$

By (4–1), the working has the intrinsic form

$$W(P) = \int_{\partial P} \boldsymbol{S}\boldsymbol{n} \cdot \dot{\boldsymbol{y}}\, da + \int_P \boldsymbol{b} \cdot \dot{\boldsymbol{y}}\, dv + \int_{\partial P} \pi U\, da. \tag{5–15}$$

The scalar field π therefore represents a **bulk tension** that works to increase the volume of P through the addition of material at its boundary. Referring to the final term in (5–15) as the *net configurational working*, (5–15) may be written more suggestively as

$$\{\text{working}\} = \{\text{standard working}\} + \{\text{net configurational working}\}. \tag{5–16}$$

Note that πU is not due solely to the working of the configurational stress $\boldsymbol{C}$; the standard stress contributes also through the term $(\boldsymbol{S}\boldsymbol{n} \cdot \boldsymbol{F}\boldsymbol{n})U$, which accounts for the addition of deformed material to P.

Finally, (5–14) in the form

$$\boldsymbol{C} = \pi \boldsymbol{1} - \boldsymbol{F}^\top \boldsymbol{S} \tag{5–17}$$

will be referred to as the **stress relation**; this relation represents an expression for the configurational stress reminiscent of—but more general than—the Eshelby relation, because it is based only on notions of force and work and is therefore valid whether or not thermal or compositional effects are taken into account.

e. Invariance under time-dependent changes in reference. External and internal force relations

Consider a time-dependent change in reference configuration as discussed in (4–15)–(4–18). Let $\overset{*}{P}$ be a fixed region in the space of reference labels with $P(t) = \hat{\boldsymbol{X}}(\overset{*}{P}, t)$ the corresponding migrating control volume, and let $\boldsymbol{q}(\boldsymbol{X}, t)$ and $\mathring{\boldsymbol{y}}(\boldsymbol{X}, t)$ denote the velocity fields for $\partial P(t)$ and $\partial \bar{P}(t)$ defined in (4–17) and (4–19). Then, continuing to measure forces in the space of material points $\boldsymbol{X}$, the working of the tractions $\boldsymbol{C}\boldsymbol{n}$ and $\boldsymbol{S}\boldsymbol{n}$ is again given by (5–2). Further, the working of the *external* forces $\boldsymbol{e}$ and $\boldsymbol{b}$ may be described in terms of the trajectories of the reference labels $\overset{*}{\boldsymbol{X}}$, in which the fields $\boldsymbol{q}(\boldsymbol{X}, t)$ and $\mathring{\boldsymbol{y}}(\boldsymbol{X}, t)$, which are defined for all $\boldsymbol{X}$ and t, represent appropriate work-conjugate velocities. Finally, the *internal* force $\boldsymbol{g}$ does not perform work (cf. the passage following (5–3)).

The fact that the material composition at a fixed $\overset{*}{\boldsymbol{X}}$ may change with time is irrelevant to the working of $\boldsymbol{e}$ and $\boldsymbol{b}$, just as the material composition of $\partial P(t)$ does not affect the manner in which the tractions perform work. In this regard, consider a particle of mass $m(t)$ following a trajectory $\boldsymbol{y}(t)$ and acted on by an external force $\boldsymbol{f}(t)$. Here

the particle is losing mass, but that does not affect the working $\boldsymbol{f}_{\text{in}} \cdot \dot{\boldsymbol{y}}$; and while the inertial force $\boldsymbol{f}_{\text{in}} = -(m\boldsymbol{y})^{\cdot}$ depends on the rate at which mass is lost, the *form* $\boldsymbol{f}_{\text{in}} \cdot \dot{\boldsymbol{y}}$ of its working in no way reflects the loss. Both of these forces are external, and their working is simply force times the time rate of change of the associated positional parameter, irrespective of whether the material composition is varying or fixed.

Thus, in place of (5–5), the working takes the form

$$W(P(t)) = \int_{\partial P(t)} \boldsymbol{C}\boldsymbol{n} \cdot \boldsymbol{q}\, da + \int_{\partial P(t} \boldsymbol{S}\boldsymbol{n} \cdot \mathring{\boldsymbol{y}}\, da + \int_{P(t)} \boldsymbol{e} \cdot \boldsymbol{q}\, da + \int_{P(t)} \boldsymbol{y} \cdot \mathring{\boldsymbol{y}}\, \mathbf{da}. \quad (5\text{–}18)$$

Because the working itself should be the same for both descriptions,

$$\int_{P} \boldsymbol{e} \cdot \boldsymbol{q}\, da + \int_{P} \boldsymbol{b} \cdot \mathring{\boldsymbol{y}}\, da = \int_{P} \boldsymbol{b} \cdot \dot{\boldsymbol{y}}\, dv, \quad (5\text{–}19)$$

a relation that must be satisfied for all time-dependent changes in reference and all migrating control volumes $P(t)$ that are images of fixed regions $\overset{*}{P}$ in the space of reference labels.

Given a time t_0 and a vector $\boldsymbol{q}_0$, the time-dependent change in reference defined by $\hat{\boldsymbol{X}}(\overset{*}{\boldsymbol{X}}, t) = \overset{*}{\boldsymbol{X}} + (t - t_0)\boldsymbol{q}_0$ satisfies $\boldsymbol{q}(\boldsymbol{X}, t_0) = \boldsymbol{q}_0$ for all $\boldsymbol{X} \in B$; in addition, for P_0 a subregion of B and t sufficiently close to t_0, the migrating control volume $P(t) = \hat{\boldsymbol{X}}(P_0, t)$ satisfies $P(t_0) = P_0$. We now apply (5–19) to $P(t)$ at $t = t_0$; since t_0, $\boldsymbol{q}_0$, and P_0 are arbitrary, while $\mathring{\boldsymbol{y}} = \dot{\boldsymbol{y}} + \boldsymbol{F}\boldsymbol{q}$, this yields the conclusion $(\boldsymbol{e} + \boldsymbol{F}^{\top}\boldsymbol{b}) \cdot \boldsymbol{q}_0 = 0$ for all vectors $\boldsymbol{q}_0$. The standard force $\boldsymbol{b}$ therefore determines the configurational force $\boldsymbol{e}$ through the **external-force relation**

$$\boldsymbol{e} = -\boldsymbol{F}^{\top}\boldsymbol{b}. \quad (5\text{–}20)$$

Further, (5–20) and the stress relation (5–17) yield, by virtue of (1–26) and the force balances (3–6a) and (5–10), the **internal force relation**

$$\boldsymbol{g} = -\nabla\pi + \boldsymbol{S}{:}\nabla\boldsymbol{F}, \quad (5\text{–}21)$$

where $\boldsymbol{S}{:}\nabla\boldsymbol{F}$ is the material vector discussed in the paragraph containing (1–26). (In components $(\boldsymbol{S}{:}\nabla\boldsymbol{F})_k = S_{ij}(\partial F_{ij}/\partial X_k)$.)

Note that when $\boldsymbol{b}$ is conservative with potential φ, so that $\boldsymbol{b} = -\operatorname{grad}\varphi$ with "grad" the spatial gradient (with respect to the place $\boldsymbol{x}$ in the deformed configuration), then, by the chain rule, $\boldsymbol{b} = -\boldsymbol{F}^{-\top}\nabla\varphi$, and hence $\boldsymbol{e} = \nabla\varphi$.

f. Standard and configurational forms of the working. Power balance

In view of the remarks leading to (5–3), when the *undeformed control volume P is stationary*, the working is given by the classical expression

$$W(P) = \int_{\partial P} \boldsymbol{S}\boldsymbol{n} \cdot \dot{\boldsymbol{y}}\, da + \int_{P} \boldsymbol{b}\dot{\boldsymbol{y}}\, dv \quad (5\text{–}22)$$

involving only *standard forces*. On the other hand, when the *deformed control volume $\bar{P}$* corresponding to $P = P(t)$ *is stationary*, then (5–18) and the paragraph

containing (4–20) imply that this working may be written in a form

$$W(P) = \int_{\partial P} \boldsymbol{C}\boldsymbol{n} \cdot \boldsymbol{Y}' \, da + \int_{P} \boldsymbol{e} \cdot \boldsymbol{Y}' \, dv \tag{5–23}$$

involving only *configurational forces*.

Given a migrating control volume $P = P(t)$, (5–15) and (3–7) (which also hold when P migrates) yield a **power balance**

$$W(P) = \int_{P} \boldsymbol{S} \cdot \dot{\boldsymbol{F}} \, dv + \int_{\partial P} \pi U \, da \tag{5–24}$$

relating the **external working** $W(P)$ to the **internal working** as represented by the right side of (5–24).

CHAPTER 6

Thermodynamics. Relation Between Bulk Tension and Energy. Eshelby Identity

I now place the theory within a thermodynamical context. I consider a mechanical version of the second law, which I show to be a special case of a formulation that allows for thermal variations associated with the transfer of heat. The treatment is nonclassical; it accounts for configurational working and heating associated with the transfer of material to a migrating control volume.

a. Mechanical version of the second law

In the absence of thermal and compositional effects, classical continuum mechanics may be based on a "second law" that uses *stationary* control volumes P and has the form

$$\frac{d}{dt}\left\{\int_P \Psi\, dv\right\} \le \int_{\partial P} \boldsymbol{Sn}\cdot\dot{\boldsymbol{y}}\, da + \int_P \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\, dv \tag{6–1}$$

with $\Psi(\boldsymbol{X}, t)$ the **free energy**.[1] For a *migrating* control volume $P = P(t)$, with U the normal velocity of ∂P, the standard generalization of (6–1) would include the transport term

$$\{\text{inflow of free energy}\} = \int_{\partial P} \Psi U\, da \tag{6–2}$$

on the right side, but would not account for configurational forces.

I base the theory on what I believe to be a more fundamental version of the second law; specifically I write the second law for a migrating control volume

[1]This form of the second law follows from the laws of balance of energy and growth of entropy under isothermal conditions. Cf. the remark following (6–14).

$P = P(t)$ in a form

$$\frac{d}{dt}\{\text{free energy of } P(t)\} \leq \{\text{rate at which work is performed on } P(t)\} \tag{6–3}$$

that accounts for the working of both configurational forces and standard forces, but not explicitly for the flow (6–2) of free energy across $\partial P(t)$ as it migrates.[2] (As we shall see, this inflow of free energy will be accounted for implicitly in the working of the configurational forces.) Precisely, the **second law** is assumed to have the form

$$\frac{d}{dt}\left\{\int_{P(t)} \Psi\, dv\right\} \leq W(P(t)) \tag{6–4}$$

with the working $W(P(t))$ given by (5–5):

$$\frac{d}{dt}\left\{\int_{P(t)} \Psi\, dv\right\} \leq \int_{\partial P(t)} \boldsymbol{Cn}\cdot\boldsymbol{q}\, da + \int_{\partial P(t)} \boldsymbol{Sn}\cdot\mathring{\boldsymbol{y}}\, da + \int_{P(t)} \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\, dv. \tag{6–5}$$

Here $\boldsymbol{q}$ is a velocity field for ∂P, with $\mathring{\boldsymbol{y}}$ the corresponding motion velocity following ∂P, and

$$\frac{d}{dt}\left\{\int_{P(t)} \Psi\, dv\right\} = \frac{d}{dt}\left\{\int_{P(t)} \Psi(\boldsymbol{X}, t)\, dv(\boldsymbol{X})\right\}.$$

b. Eshelby relation as a consequence of the second law

By a standard transport theorem,

$$\frac{d}{dt}\left\{\int_{P(t)} \Psi\, dv\right\} = \int_{P(t)} \dot{\Psi}\, dv + \int_{\partial P(t)} \Psi U\, da. \tag{6–6}$$

Thus, appealing to (5–15),

$$\int_{P(t)} \dot{\Psi}\, dv \leq \int_{\partial P(t)} \boldsymbol{Sn}\cdot\dot{\boldsymbol{y}}\, da + \int_{P(t)} \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\, dv + \int_{\partial P(t)} (\pi - \Psi) U\, da. \tag{6–7}$$

Given a fixed time τ, it is possible to find a second control volume $\tilde{P}(t)$ with $\tilde{P}(\tau) = P(\tau)$, but with $\tilde{U}(\boldsymbol{X}, \tau)$, the normal velocity of $\partial\tilde{P}(\tau)$, an *arbitrary* scalar field on $\partial\tilde{P}(\tau)$; satisfaction of (6–7) for all such $\tilde{P}$ (and hence $\tilde{U}$) implies that

$$\pi = \Psi. \tag{6–8}$$

Bulk tension therefore coincides with bulk free-energy (a result analogous to the coincidence of surface tension and surface free-energy); thus, in the notation of

[2] Cf. Gurtin [1995, eq. (3–12)]

(5–16) and (6–2),

$$\{\text{net configurational working}\} = \{\text{inflow of free energy}\},$$

at least in this purely mechanical context, establishing consistency of the second law (6–4) with the more standard inequality (6–1) modified by (6–2):

$$\frac{d}{dt}\left\{\int_P \Psi\, dv\right\} \le \int_{\partial P} \Psi U\, da + \int_{\partial P} \boldsymbol{S}\boldsymbol{n}\cdot\dot{\boldsymbol{y}}\, da + \int_P \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\, dv.$$

What is more important, (5–17), (5–21), and (6–8) yield the **Eshelby relation**

$$\boldsymbol{C} = \Psi\mathbf{1} - \boldsymbol{F}^\top\boldsymbol{S} \tag{6–9}$$

and the **internal-force relation**

$$\boldsymbol{g} = -\nabla\Psi + \boldsymbol{S}{:}\nabla\boldsymbol{F}, \tag{6–10}$$

with $\boldsymbol{S}{:}\nabla\boldsymbol{F}$ defined in (1–24). The internal force $\boldsymbol{g}$ is therefore affected by material variations in the free energy and deformation gradient via the terms $\nabla\Psi$ and $\nabla\boldsymbol{F}$.

This derivation of the Eshelby and internal force-relations (and that of the external force relation (5–20)) were accomplished without using constitutive equations or a variational principle; the derivations were based on a version of the second law appropriate to control volumes whose boundaries migrate with time.[3] This observation is not simply of pedagogical interest; it establishes these relations as appropriate to theories, such as plasticity and viscoelasticity, for which memory effects render variational derivations inappropriate.

The Eshelby relation (6–9), the external-force relation (5–20), and the configurational balance (5–10) may be considered as *defining* relations for $\boldsymbol{C}$, $\boldsymbol{e}$, and $\boldsymbol{g}$ in terms of the classical fields $\boldsymbol{y}$, Ψ, $\boldsymbol{S}$, and $\boldsymbol{b}$; *configurational forces are therefore superfluous within the framework of classical continuum mechanics*.

The Eshelby and external-force relations have somewhat similar structures, i.e., $\boldsymbol{C} = \Psi\mathbf{1} - \boldsymbol{F}^\top\boldsymbol{S}$ and $\boldsymbol{e} = -\boldsymbol{F}^\top\boldsymbol{b}$, structures that differ markedly from that of the internal force $\boldsymbol{g}$ as specified in (6–10).

Finally, restricting attention to stationary P yields, by virtue of (3–7) and (6–1), the local dissipation inequality

$$\dot{\Psi} \le \boldsymbol{S}\cdot\dot{\boldsymbol{F}}. \tag{6–11}$$

c. Thermomechanical theory

I now consider a more general thermodynamics, one that allows for the flow of heat. I write the first two laws for a migrating control volume $P = P(t)$ as

$$\frac{d}{dt}\{\text{internal energy}\} = \{\text{heating}\} + \{\text{working}\},$$

[3] Gurtin [1995]. Eshelby's [1951] derivation is variational and presumes elasticity.

$$\frac{d}{dt}\{\text{internal entropy}\} \geq \{\text{entropy flux induced by heating}\}$$

in which, paralleling (6–3), the right sides include an accounting of the work and *heat* required to transfer material to P but make no explicit mention of flows of internal energy and internal entropy across ∂P.

I consider the standard and configurational force systems supplemented by the classical thermodynamical fields, namely, the internal energy ε, the entropy η, the temperature T, the heat flux $\boldsymbol{h}$, and the external heat supply r, and I *define the free energy* Ψ through

$$\Psi = \varepsilon - T\eta. \tag{6–12}$$

In addition, I allow for a scalar field Q, the **configurational heating**; for $P = P(t)$ a migrating control volume with U the normal velocity of ∂P,

$$\int_{\partial P} QU\, da \qquad \text{and} \qquad \int_{\partial P} (Q/T)U\, da,$$

respectively, represent flows of heat and entropy into P associated with the transfer of material across ∂P.

The basic thermodynamical laws, for each migrating control volume $P(t)$, are **balance of energy and growth of entropy**:

$$\frac{d}{dt}\left\{\int_{P(t)} \varepsilon\, dv\right\} = -\int_{\partial P(t)} \boldsymbol{h}\cdot\boldsymbol{n}\, da + \int_{P(t)} r\, dv + \int_{\partial P(t)} QU\, da + W(P(t)), \tag{6–13a}$$

$$\frac{d}{dt}\left\{\int_{P(t)} \eta\, dv\right\} \geq -\int_{\partial P(t)} (\boldsymbol{h}/T)\cdot\boldsymbol{n}\, da + \int_{P(t)} (r/T)\, dv + \int_{\partial P(t)} (Q/T)U\, da \tag{6–13b}$$

with $W(P(t))$ given by (5–15). For P *stationary* (6–13), have the classical form

$$\frac{d}{dt}\left\{\int_P \varepsilon\, dv\right\} = -\int_{\partial P} \boldsymbol{h}\cdot\boldsymbol{n}\, da + \int_P r\, dv + \int_{\partial P} \boldsymbol{S}\boldsymbol{n}\cdot\dot{\boldsymbol{y}}\, da + \int_P \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\, dv, \tag{6–14a}$$

$$\frac{d}{dt}\left\{\int_P \eta\, dv\right\} \geq -\int_{\partial P} (\boldsymbol{h}/T)\cdot\boldsymbol{n}\, da + \int_P (r/T)\, dv, \tag{6–14b}$$

demonstrating consistency with classical ideas.

Remark. If $T \equiv$ constant, then (6–13) combine to form (6–4), while (6–14) reduces to (6–1).

Because the working $W(P)$ is as discussed in Chapter 5, invariance under changes in material and spatial observers yields the standard force and moment balances discussed in Chapter 3 and the configurational force balance discussed in Chapter 5. Further, arguing as before, (5–15) and the identity (6–6) applied to ε in (6–13a) and η in (6–13b) yield

$$\varepsilon = \pi + Q, \qquad \eta = Q/T, \tag{6–15}$$

relations that, when multiplied by U, express balance of energy and entropy across ∂P associated with the transfer of material to P. Further, the second of (6–15) has a measure-theoretic interpretation: when material is added to a control volume across its boundary, the entropy transferred, $\eta\, da$, induces a transfer of heat of amount $Q\, da = T(\eta\, da)$; more simply, $dQ = T\, d\eta$, which is classical. Important corollaries of these relations are that bulk tension and bulk free-energy coincide,

$$\pi = \Psi, \tag{6–16}$$

and that *the configurational stress* $\boldsymbol{C}$ *and internal force* $\boldsymbol{g}$ *be again given by the Eshelby and internal force relations* (6–9) *and* (6–10).

By (3–6a), (6–14a) localize to

$$\dot{\varepsilon} = -\operatorname{Div} \boldsymbol{h} + r + \boldsymbol{S} \cdot \dot{\boldsymbol{F}}, \tag{6–17a}$$

$$\dot{\eta} \geq -\operatorname{Div}(\boldsymbol{h}/T) + r/T, \tag{6–17b}$$

and yield the local free-energy inequality

$$\dot{\Psi} - \boldsymbol{S} \cdot \dot{\boldsymbol{F}} + \eta \dot{T} + T^{-1} \boldsymbol{h} \cdot \nabla T \leq 0. \tag{6–18}$$

d. Fluids. Current configuration as reference

Most of the previous discussion was linked to solids, but the mathematical theory itself is independent of the specific constitutive theory. Further, while a fixed reference configuration may be used to describe a fluid—and often is in the study of shock waves—constitutive equations for a fluid are independent of the specific choice of reference. That is why fluids are generally described using the current (deformed) configuration as reference. For this choice of reference,

$$\boldsymbol{F} = \mathbf{1} \tag{6–19}$$

and the Piola-Kirchhoff stress $\boldsymbol{S}(\boldsymbol{X}, t)$ reduces to the Cauchy stress $\boldsymbol{T}(\boldsymbol{x}, t)$ (cf. (3–9)). Thus, letting $\bar{\Psi}(\boldsymbol{x}, t)$ denote the free energy per unit volume, measured relative to the current configuration, the Eshelby relation (6–9) takes the form

$$\bar{\boldsymbol{C}} = \bar{\Psi}\mathbf{1} - \boldsymbol{T}, \tag{6–20}$$

with $\bar{\boldsymbol{C}}$ the configurational stress taking the current configuration as reference. This result is independent of the constitution of the material and, in particular, of whether the material is solid or fluid.

For an ideal fluid or an elastic fluid the stress $\boldsymbol{T}$ is a pressure,

$$\boldsymbol{T} = -p\mathbf{1}, \tag{6–21}$$

and the configurational stress is a uniform tension

$$\bar{\boldsymbol{C}} = (\bar{\Psi} + p)\mathbf{1}; \tag{6–22}$$

the term $\bar{\Psi} + p$ represents the *enthalpy* per unit current volume. On the other hand, by (6–20), configurational shearing stresses would generally accompany the flow of a viscous fluid.

CHAPTER 7

Inertia and Kinetic Energy. Alternative Versions of the Second Law

a. Inertia and kinetic energy

If the external body force $\boldsymbol{b}$ is inertial, then, granted an inertial observer,

$$\boldsymbol{b} = -\rho\ddot{\boldsymbol{y}}, \tag{7–1}$$

with $\rho(\boldsymbol{X}) \geq 0$, assumed smooth, the **mass density** in the reference configuration. ($\rho = 0$ characterizes quasi-static situations.) Then, by (5–20),

$$\boldsymbol{e} = \rho F^{\top}\ddot{\boldsymbol{y}}, \tag{7–2}$$

and the equations of motion (3–6a) and (5–10) take the form

$$\operatorname{Div}\boldsymbol{S} = \rho\ddot{\boldsymbol{y}}, \tag{7–3a}$$

$$\operatorname{Div}\boldsymbol{C} + \boldsymbol{g} = -\rho\boldsymbol{F}^{\top}\ddot{\boldsymbol{y}}. \tag{7–3b}$$

Let

$$\boldsymbol{p} = \rho\dot{\boldsymbol{y}}, \qquad k = \frac{1}{2}\,\rho|\dot{\boldsymbol{y}}|^2 \tag{7–4}$$

denote the densities of **momentum** and **kinetic energy**. Then, by (2–5), (5–20), and (7–1),

$$\boldsymbol{b}\cdot\dot{\boldsymbol{y}} = \boldsymbol{e}\cdot\boldsymbol{Y}' = -\dot{k}. \tag{7–5}$$

Thus, because $\boldsymbol{b} = -\dot{\boldsymbol{p}}$, (6–6) implies that, for $P = P(t)$ a migrating control volume,

$$\frac{d}{dt}\left\{\int_P \boldsymbol{p}\,dv\right\} - \int_{\partial P}\boldsymbol{p}U\,da = -\int_P \boldsymbol{b}\,dv, \tag{7–6a}$$

$$\frac{d}{dt}\left\{\int_P k\,dv\right\} - \int_{\partial P} kU\,da = -\int_P \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\,dv = -\int_P \boldsymbol{e}\cdot\boldsymbol{Y}'\,dv. \tag{7–6b}$$

These identities assert the equivalence of:

- the production of momentum with the negative inertial body force;
- the production of kinetic energy with the negative working of the inertial body force.[1]

These equivalences will form a basis for characterizing inertial forces when discussing phase transitions and fracture.

b. Alternative forms of the second law

By (5–22), (5–23), (6–4), and (7–6a),

$$\frac{d}{dt}\left\{\int_P (\Psi + k)\, dv\right\} \leq \int_{\partial P} \boldsymbol{S}\boldsymbol{n} \cdot \dot{\boldsymbol{y}}\, da \qquad \text{for } P \text{ stationary,} \quad (7\text{–}7a)$$

$$\frac{d}{dt}\left\{\int_P (\Psi + k)\, dv\right\} \leq \int_{\partial P} (\boldsymbol{C} + k\mathbf{1})\boldsymbol{n} \cdot \boldsymbol{Y}'\, da \qquad \text{for } \bar{P} \text{ stationary,} \quad (7\text{–}7b)$$

inequalities that represent standard and configurational versions of the second law. The term

$$\boldsymbol{C} + k\mathbf{1} = (\Psi + k)\mathbf{1} - \boldsymbol{F}^\top \boldsymbol{S} \qquad (7\text{–}8)$$

represents a dynamical Eshelby tensor, because it is based on the total energy density $\Psi + k$ (cf. (6–9)).

c. Pseudomomentum

The external body force (7–2) may be written in the form

$$\boldsymbol{e} = -\dot{\mathfrak{p}} + \nabla(-k) + \frac{1}{2}\dot{\boldsymbol{y}}^2 \nabla\rho, \qquad (7\text{–}9)$$

with

$$\mathfrak{p} = -F^\top \boldsymbol{p} = -\rho \boldsymbol{F}^\top \dot{\boldsymbol{y}} \qquad (7\text{–}10)$$

a field generally referred to as the **pseudomomentum**. Trivially, (7–3a) may be written as a momentum balance $\mathrm{Div}\, \boldsymbol{S} = \dot{\boldsymbol{p}}$. Similarly, (7–9) yields, as an alternative

[1] Cf. Podio-Guidugli [1997].

to (7–3b), the **configurational momentum balance**[2]

$$\mathrm{Div}(\boldsymbol{C} - k\mathbf{1}) + \boldsymbol{g} + \frac{1}{2}\dot{\boldsymbol{y}}^2\nabla\rho = \dot{\mathfrak{p}}. \tag{7–11}$$

Note that, by (6–9), the term

$$\mathfrak{C} = \boldsymbol{C} - k\mathbf{1} = \Lambda\mathbf{1} - \boldsymbol{F}^\top\boldsymbol{S} \tag{7–12}$$

representing stress in (7–11) has the form of an Eshelby stress with the free energy Ψ replaced by the Lagrangian (cf. (7–8))

$$\Lambda = \Psi - \boldsymbol{k}. \tag{7–13}$$

Next, by (6–6) and (7–9), for $P = P(t)$ a migrating control volume,

$$\frac{d}{dt}\left\{\int_P \mathfrak{p}\,dv\right\} - \int_P \mathfrak{p}U\,da = -\int_P \boldsymbol{e}\,dv - \int_{\partial P} k\boldsymbol{n}\,da + \int_P \frac{1}{2}\dot{\boldsymbol{y}}^2\nabla\rho\,dv, \tag{7–14}$$

showing that, in contrast to the identity (7–6a) for the momentum $\boldsymbol{p}$, the production of pseudomomentum requires, for its balance, not only the negative internal configurational force on P, but also an inertial pressure k on ∂P and an inertial body force $\int_P \frac{1}{2}\dot{\boldsymbol{y}}^2\nabla\rho\,dv$ resulting from density variations within P.

Finally, the second law (6–5) is equivalent to the following inequality for migrating control volumes $P = P(t)$:[3]

$$\frac{d}{dt}\left\{\int_P (\Lambda + \boldsymbol{p}\cdot\dot{\boldsymbol{y}})\,dv\right\} \le \int_{\partial P} \mathfrak{C}\boldsymbol{n}\cdot\boldsymbol{q}\,da + \int_{\partial P} \boldsymbol{S}\boldsymbol{n}\cdot\mathring{\boldsymbol{y}}\,da + \int_{\partial P} U\mathfrak{p}\cdot\boldsymbol{q}\,da + \int_{\partial P} U\boldsymbol{p}\cdot\mathring{\boldsymbol{y}}\,da.$$

d. Lyapunov relations

Assume that B is bounded and the relevant fields are smooth up to ∂B, and consider the following two types of boundary conditions:

(i) **fixed boundary**:

$$\dot{\boldsymbol{y}} = \mathbf{0} \text{ on } \partial B \text{ for all time.} \tag{7–15}$$

(ii) **constant dead loads**: There is a constant tensor $\boldsymbol{S}_0$ such that

$$\boldsymbol{S}\boldsymbol{n} = \boldsymbol{S}_0\boldsymbol{n} \text{ on } \partial B \text{ for all time.} \tag{7–16}$$

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[2]Eshelby [1971], for an elastic body as a consequence of $\mathrm{Div}\,\boldsymbol{S} = \rho\ddot{\boldsymbol{y}}$. Within this framework, Maugin [1993, 1995] gives a detailed discussion of pseudomomentum, a notion with a large physical literature (cf., e.g., Nelson [1979], Peierls [1991]). Cermelli and Fried [1997] give an alternative treatment of inertia that is independent of constitution and results in a balance equivalent to (7–11). To these authors $\frac{1}{2}\dot{\boldsymbol{y}}^2\nabla\rho$ represents an *internal* force, a view with which I disagree; to me $\frac{1}{2}\dot{\boldsymbol{y}}^2\nabla\rho$ represents a portion of an *external* force (inertia) arising from variations in mass.

[3]Cermelli and Fried [1997].

Then, by (7–7a), *for a fixed boundary*,

$$\frac{d}{dt}\left\{\int_B (\Psi + k)\, dv\right\} \leq 0, \tag{7–17}$$

and *the total energy of the body cannot increase with time*.

On the other hand, *for a boundary under constant dead loads*,

$$\int_{\partial B} \boldsymbol{S}\boldsymbol{n} \cdot \dot{\boldsymbol{y}}\, da = \int_{\partial B} \boldsymbol{S}_0 \boldsymbol{n} \cdot \dot{\boldsymbol{y}}\, da = \int_B \boldsymbol{S}_0 \cdot \dot{\boldsymbol{F}}\, da = \frac{d}{dt}\left\{\int_B \boldsymbol{S}_0 \cdot \boldsymbol{F}\, da\right\} \tag{7–18}$$

represents the stress power of the dead loads, and

$$\frac{d}{dt}\left\{\int_B (\Psi - \boldsymbol{S}_0 \cdot \boldsymbol{F} + k)\, dv\right\} \leq 0; \tag{7–19}$$

thus *the energy of the body minus the stress power of the dead loads cannot increase with time*.

The relations (7–17) and (7–19) represent Lyapunov relations for the body; as derived here they are independent of specific constitutive equations.

CHAPTER 8

Change in Reference Configuration

Configurational forces are material and, consequently, they depend strongly on the choice of reference. With this in mind, I now give a detailed discussion of the manner in which the basic fields transform under changes in reference configuration (cf. Subsection 4b1).

a. Transformation laws for free energy and standard force

The following rules for changing integration-variable from $\boldsymbol{X}$ to $\overset{*}{\boldsymbol{X}} = \boldsymbol{\kappa}(\boldsymbol{X})$ will be useful:

$$\int_{\overset{*}{P}} \ldots d\overset{*}{v} = \int_{P} \ldots \mathscr{J}\, dv, \tag{8–1a}$$

$$\int_{\partial\overset{*}{P}} \ldots \overset{*}{\boldsymbol{n}}\, d\overset{*}{a} = \int_{\partial P} \ldots \mathscr{J}\boldsymbol{K}^{-\top}\boldsymbol{n}\, da, \tag{8–1b}$$

where $\overset{*}{\boldsymbol{n}}$ and $d\overset{*}{a}$ are the outward unit normal and element of area on $\partial\overset{*}{P}$, $d\overset{*}{v}$ is the element of volume on $\overset{*}{P}$, $\boldsymbol{K} = \nabla\boldsymbol{\kappa}$, $\mathscr{J} = \det\boldsymbol{K}$, and $\boldsymbol{K}^{-\top} = (\boldsymbol{K}^{-1})^{\top}$.

The transformation laws for the free energy Ψ, the stress $\boldsymbol{S}$, and the body force $\boldsymbol{b}$ are determined by the requirement that the net traction, body force, and free energy associated with each control volume be invariant:

$$\int_{\partial P} \boldsymbol{S}\boldsymbol{n}\, da = \int_{\partial\overset{*}{P}} \overset{*}{\boldsymbol{S}}\overset{*}{\boldsymbol{n}}\, d\overset{*}{a}, \qquad \int_{P} \boldsymbol{b}\, dv = \int_{\overset{*}{P}} \overset{*}{\boldsymbol{b}}\, d\overset{*}{v}, \qquad \int_{P} \Psi\, dv = \int_{\overset{*}{P}} \overset{*}{\Psi}\, d\overset{*}{v}. \tag{8–2}$$

Thus, by (8–1a),

$$\overset{*}{\boldsymbol{S}} = \mathscr{J}^{-1}\boldsymbol{S}\boldsymbol{K}^{\top}, \qquad \overset{*}{\boldsymbol{b}} = \mathscr{J}^{-1}\boldsymbol{b}, \qquad \overset{*}{\Psi} = \mathscr{J}^{-1}\Psi. \tag{8–3}$$

b. Transformation laws for configurational force

Consider a reference change $\overset{*}{\boldsymbol{X}} = \boldsymbol{\kappa}(\boldsymbol{X})$. The theory presented thus far must hold for any choice of reference. This observation has two important consequences: configurational forces in the new reference must be balanced,

$$\int_{\partial \overset{*}{P}} \overset{*}{\boldsymbol{C}}\overset{*}{\boldsymbol{n}}\, d\overset{*}{a} + \int_{\overset{*}{P}} (\overset{*}{\boldsymbol{g}} + \overset{*}{\boldsymbol{e}})\, d\overset{*}{v} = 0; \tag{8–4}$$

and the configurational stress $\overset{*}{\boldsymbol{C}}$ and external body $\overset{*}{\boldsymbol{e}}$ must be given by the Eshelby and external force relations (cf. (5–20), (6–9))

$$\overset{*}{\boldsymbol{C}} = \overset{*}{\Psi}\mathbf{1} - (\overset{*}{\boldsymbol{F}})^{\top}\overset{*}{\boldsymbol{S}}, \qquad \overset{*}{\boldsymbol{e}} = -(\overset{*}{\boldsymbol{F}})^{\top}\overset{*}{\boldsymbol{b}}. \tag{8–5}$$

Equations (4–11), (8–3), and (8–5) yield the transformation laws

$$\overset{*}{\boldsymbol{C}} = \mathscr{J}^{-1}\boldsymbol{K}^{-\top}\boldsymbol{C}\boldsymbol{K}^{\top}, \qquad \overset{*}{\boldsymbol{e}} = \mathscr{J}^{-1}\boldsymbol{K}^{-\top}\boldsymbol{e}. \tag{8–6}$$

Further, because (8–4) must hold for all $\overset{*}{P}$,

$$\operatorname{Div}^{*}\overset{*}{\boldsymbol{C}} + \overset{*}{\boldsymbol{g}} + \overset{*}{\boldsymbol{e}} = \mathbf{0}, \tag{8–7}$$

where Div^{*} denotes the divergence with respect to $\overset{*}{\boldsymbol{X}}$ in $\overset{*}{B}$. By (8–1b) and (8–6),

$$\int_{\partial P} \boldsymbol{C}\boldsymbol{n}\, da = \int_{\partial \overset{*}{P}} \boldsymbol{K}^{\top}\overset{*}{\boldsymbol{C}}\overset{*}{\boldsymbol{n}}\, d\overset{*}{a}, \tag{8–8}$$

so that, applying the divergence theorem to both sides of (8–8),

$$\operatorname{Div}\boldsymbol{C} = \mathscr{J}\operatorname{Div}^{*}(\boldsymbol{K}^{\top}\overset{*}{\boldsymbol{C}}). \tag{8–9}$$

Next, a straightforward calculation yields

$$\operatorname{Div}^{*}(\boldsymbol{K}^{\top}\overset{*}{\boldsymbol{C}}) = \boldsymbol{K}^{\top}\operatorname{Div}^{*}\overset{*}{\boldsymbol{C}} + (\overset{*}{\boldsymbol{C}}\boldsymbol{K}^{-\top}){:}\nabla\boldsymbol{K},$$

where, for $\boldsymbol{H} = \overset{*}{\boldsymbol{C}}\boldsymbol{K}^{-\top}$, the vector field $\boldsymbol{H}{:}\nabla\boldsymbol{K}$ is defined as in (5–21). Thus (5–10), (8–6), (8–7), and (8–9) imply that

$$\overset{*}{\boldsymbol{g}} + \overset{*}{\boldsymbol{e}} = \mathscr{J}^{-1}\boldsymbol{K}^{-\top}\{\boldsymbol{g} + \boldsymbol{e} + (\boldsymbol{K}^{-\top}\boldsymbol{C}){:}\nabla\boldsymbol{K}\}, \tag{8–10}$$

and, appealing to the second part of (8–6),

$$\overset{*}{\boldsymbol{g}} = \mathscr{J}^{-1}\boldsymbol{K}^{-\top}\{\boldsymbol{g} + (\boldsymbol{K}^{-\top}\boldsymbol{C}){:}\nabla\boldsymbol{K}\}. \tag{8–11}$$

The results (8–6) and (8–11) represent a complete set of transformation laws for the configurational force system. Note that (8–11) involves the stress $\boldsymbol{C}$ through

$(\boldsymbol{K}^{-\top}\boldsymbol{C})$:$\nabla\boldsymbol{K}$, a term that is absent when the change in reference is homogeneous $(\nabla\boldsymbol{K} \equiv \boldsymbol{0})$.

By (8–8), the net configurational traction on control volumes is generally not invariant under changes in reference. This is to be expected; the ambient space of B and that of $\overset{*}{B}$ need not be identified, and hence the *configurational forces*

$$\int_{\partial P} \boldsymbol{C}\boldsymbol{n}\, da, \qquad \int_{\partial \overset{*}{P}} \overset{*}{\boldsymbol{C}}\overset{*}{\boldsymbol{n}}\, d\overset{*}{a}$$

need bear no relation to one another. In fact, it is work rather than force that is basic, and working is invariant under stationary changes in reference; that is

$$\int_{\partial P} \boldsymbol{q}\cdot\boldsymbol{C}\boldsymbol{n}\, da = \int_{\partial \overset{*}{P}} \overset{*}{\boldsymbol{q}}\cdot\overset{*}{\boldsymbol{C}}\overset{*}{\boldsymbol{n}}\, d\overset{*}{a}, \tag{8–12}$$

a relation that follows from (4–14), (8–1b), and (8–6). Similarly, net forces on control volumes by external and internal configurational body forces are generally not invariant under changes in reference.

The preceding remarks vividly illustrate the nature of configurational forces. Changes in reference configuration generally change the spatial arrangement of the material in the reference and hence change the forces needed to hold this material in place. Thus configurational forces are not generally invariant under changes in reference. On the other hand, standard forces act on material in the deformed configuration and are consequently invariant under such changes.

CHAPTER 9

Elastic and Thermoelastic Materials

In this chapter I will develop the constitutive theories for elastic materials with and without thermal influences. I do this for two reasons:

- to demonstrate, within a very simple context, the procedure I use to develop constitutive theories. In future chapters I will apply this procedure in more complicated situations involving moving interfaces.
- to derive explicit expressions for the internal configurational force, expressions that help to better understand its physical nature.

My treatment of constitutive equations uses the Coleman-Noll procedure,[1] a procedure based on the premise that the second law be satisfied in all conceivable processes, irrespective of the difficulties involved in producing such processes in the laboratory. The rational application of this procedure requires external forces and supplies that may be assigned arbitrarily to ensure satisfaction of the underlying balances in all processes. This may seem artificial, but it is no more artificial than theories based on virtual work, a paradigm that requires arbitrary variations, which are not guaranteed to be consistent with the resulting evolution equations, granted a constitutive description. The Coleman-Noll procedure makes explicit the external fields needed to support the "virtual processes" used, and in so doing ensures that these external fields, whether virtual or not, enter the theory in a thermodynamically consistent manner.

[1] Cf. Coleman and Noll [1963], who discuss single-phase thermoelastic materials.

a. Mechanical theory

a1. Basic equations

The basic equations of the mechanical theory consist of the standard force and moment balances

$$\operatorname{Div} \boldsymbol{S} + \boldsymbol{b} = \boldsymbol{0}, \tag{9–1a}$$

$$\boldsymbol{S}\boldsymbol{F}^{\top} = \boldsymbol{F}\boldsymbol{S}^{\top}, \tag{9–1b}$$

supplemented by a free-energy inequality

$$\dot{\Psi} \leq \boldsymbol{S} \cdot \dot{\boldsymbol{F}} \tag{9–2}$$

that represents a local form of the second law. These are augmented by the configurational balance

$$\operatorname{Div} \boldsymbol{C} + \boldsymbol{g} + \boldsymbol{e} = \boldsymbol{0} \tag{9–3}$$

with $\boldsymbol{C}$, $\boldsymbol{g}$, and $\boldsymbol{e}$ given by the Eshelby relation

$$\boldsymbol{C} = \Psi \boldsymbol{1} - \boldsymbol{F}^{\top}\boldsymbol{S} \tag{9–4}$$

and the internal and external force relations

$$\boldsymbol{g} = -\nabla\Psi + \boldsymbol{S}{:}\nabla\boldsymbol{F}, \tag{9–5a}$$

$$\boldsymbol{e} = -\boldsymbol{F}^{\top}\boldsymbol{b}. \tag{9–5b}$$

By (9–4) and (9–5a), the configurational balance (9–3) is automatically satisfied whenever the standard balance (9–1a) is satisfied; the configurational fields are thus superfluous. This is to be expected, as the theory discussed thus far does not account for migrating material structures such as defects, phase interfaces, and grain boundaries.

a2. Constitutive theory

A *homogeneous* elastic body described relative to a homogeneous reference configuration is defined by constitutive equations giving the free energy Ψ and the stress $\boldsymbol{S}$ when the deformation gradient $\boldsymbol{F}$ is known:

$$\Psi = \hat{\Psi}(\boldsymbol{F}), \tag{9–6a}$$

$$\boldsymbol{S} = \hat{\boldsymbol{S}}(\boldsymbol{F}), \tag{9–6b}$$

where $\Psi = \hat{\Psi}(\boldsymbol{F})$ signifies $\Psi(\boldsymbol{X}, t) = \hat{\Psi}(\boldsymbol{F}(\boldsymbol{X}, t))$, and so forth. The **response functions** $\hat{\Psi}$ and $\hat{\boldsymbol{S}}$ determine the particular body under consideration and are defined on the set of all tensors $\boldsymbol{F}$ with $\det \boldsymbol{F} > 0$. I assume that $\hat{\boldsymbol{S}}$ is restricted by the requirement

$$\hat{\boldsymbol{S}}(\boldsymbol{F})\boldsymbol{F}^{\top} = \boldsymbol{F}\hat{\boldsymbol{S}}(\boldsymbol{F})^{\top}, \tag{9–7}$$

which ensures satisfaction of the local moment balance (9–1b).

Consider an arbitrary *constitutive process*; that is, a motion $\boldsymbol{y}(\boldsymbol{X}, t)$ together with fields $\Psi(\boldsymbol{X}, t)$ and $\boldsymbol{S}(\boldsymbol{X}, t)$ determined by that motion through the constitutive equations (9–6a). The standard force balance (9–1a) then gives the external body force $\boldsymbol{b}$ needed to support the process, granted the possibility of considering $\boldsymbol{b}$ as virtual (and hence arbitrary); this balance in no way restricts the class of processes possible for the material. On the other hand, unless the constitutive equations are suitably restricted, not all constitutive processes will be compatible with the second law in the form of the free-energy inequality (9–2). A basic hypothesis of the theory is that *all constitutive processes be consistent with* (9–2).

Consistency of the constitutive equations with the free-energy inequality has strong consequences. Granted (9–6), (9–2) is equivalent to

$$\left\{\partial_{\boldsymbol{F}}\hat{\Psi}(\boldsymbol{F}) - \hat{\boldsymbol{S}}(\boldsymbol{F})\right\} \cdot \dot{\boldsymbol{F}} \leq 0, \tag{9–8}$$

an inequality that must hold for all motions of the body. It is possible to find a motion in which $\boldsymbol{F}$ and $\dot{\boldsymbol{F}}$ have arbitrarily prescribed values at some point and time. (Choose, arbitrarily, tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ with $\det \boldsymbol{A} > 0$, choose a function $\delta(t)$ with $\delta(0) = 0$, $\dot{\delta}(0) = 1$, and $|\delta(t)|$ small enough that $\boldsymbol{F}(t) = \boldsymbol{A} + \delta(t)\boldsymbol{B}$ has strictly positive determinant for all t; then $\boldsymbol{y}(\boldsymbol{X}, t) = \boldsymbol{F}(t)\boldsymbol{X}$ is a motion with $\boldsymbol{F}(0) = \boldsymbol{A}$ and $\dot{\boldsymbol{F}}(0) = \boldsymbol{B}$.) Thus, because (9–8) is *linear* in $\dot{\boldsymbol{F}}$, this inequality can be satisfied for all motions only if the coefficient of $\dot{\boldsymbol{F}}$ vanishes; (9–6a) must therefore have the form

$$\Psi = \hat{\Psi}(\boldsymbol{F}), \qquad \boldsymbol{S} = \hat{\boldsymbol{S}}(\boldsymbol{F}) = \partial_{\boldsymbol{F}}\hat{\Psi}(\boldsymbol{F}). \tag{9–9}$$

Materials defined by (9–9), in which the stress is the derivative of the free energy with respect to the deformation gradient, are generally referred to as **hyperelastic**; the basic equations of hyperelasticity are the balance law (9–1a) and the restricted constitutive relations (9–9). For such materials (9–9) renders the free-energy inequality (9–2) an identity

$$\dot{\Psi} = \boldsymbol{S} \cdot \dot{\boldsymbol{F}}, \tag{9–10}$$

there being no dissipation.

The relations (1–29), (9–5a), and (9–9) yield vanishing internal forces:

$$\boldsymbol{g} = \boldsymbol{0}. \tag{9–11}$$

This is a direct consequence of homogeneity; for an *inhomogeneous* body or a homogeneous body described relative to an inhomogeneous reference, the constitutive equations, derived using the same procedure, have the form

$$\Psi = \hat{\Psi}(\boldsymbol{F}, \boldsymbol{X}), \qquad \boldsymbol{S} = \hat{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{X}) = \partial_{\boldsymbol{F}}\hat{\Psi}(\boldsymbol{F}, \boldsymbol{X}),$$

and yield

$$\boldsymbol{g} = -\partial_{\boldsymbol{X}}\hat{\Psi}, \tag{9–12}$$

the derivative of $\hat{\Psi}(\boldsymbol{F}, \boldsymbol{X})$ with respect to $\boldsymbol{X}$ holding $\boldsymbol{F}$ fixed.

As noted previously, $\boldsymbol{g}$ is interpreted as representing forces that pin the material in place at $\boldsymbol{X}$ in the reference configuration. The formulas (9–11) and (9–12) reinforce this view; no such forces are required when the points $\boldsymbol{X}$ label material arranged homogeneously, but material described relative to an inhomogeneous reference requires the internal force $\boldsymbol{g} = -\partial_{\boldsymbol{X}}\hat{\Psi}$ to hold the material in place.

Invariance under changes in spatial observer places additional restrictions on the constitutive equations (9–9); these are well known and will not be discussed here.[2]

An interesting relation for the configurational stress in a homogeneous elastic body under a change in reference $\overset{*}{\boldsymbol{X}} = \boldsymbol{\kappa}(\boldsymbol{X})$ was derived by Epstein and Maugin.[3] Let $\boldsymbol{K} = \nabla\boldsymbol{\kappa}$ and $\mathscr{J} = \det \boldsymbol{K}$. By (8–3), the response function $\hat{\Psi}_{\boldsymbol{K}}(\overset{*}{\boldsymbol{F}})$ for the free energy in the new reference is given by

$$\hat{\Psi}_{\boldsymbol{K}}(\overset{*}{\boldsymbol{F}}) = \mathscr{J}^{-1}\hat{\Psi}(\boldsymbol{F}), \qquad \boldsymbol{F} = \overset{*}{\boldsymbol{F}}\boldsymbol{K}$$

and generates the configurational stress $\overset{*}{\boldsymbol{C}}$ in the new reference, as defined in (8–5), through

$$\overset{*}{\boldsymbol{C}} = -\partial_{\boldsymbol{K}}\hat{\Psi}_{\boldsymbol{K}}(\overset{*}{\boldsymbol{F}})\boldsymbol{K}^{\top}.$$

b. Thermomechanical theory[4]

b1. Basic equations

The basic equations of the thermomechanical theory consist of the standard force and moment balances (9–1), the configurational relations (9–3)–(9–5), and the thermodynamical laws

$$\dot{\varepsilon} = -\operatorname{Div}\boldsymbol{h} + r + \boldsymbol{S}\cdot\dot{\boldsymbol{F}} \tag{9–13a}$$

$$\dot{\eta} \geq -\operatorname{Div}(\boldsymbol{h}/T) + r/T, \tag{9–13b}$$

expressing balance of energy and growth of entropy. Together these yield the free-energy inequality

$$\dot{\Psi} - \boldsymbol{S}\cdot\dot{\boldsymbol{F}} + \eta\dot{T} + T^{-1}\boldsymbol{h}\cdot\nabla T \leq 0, \tag{9–14}$$

with free energy defined by

$$\Psi = \varepsilon - T\eta. \tag{9–15}$$

Granted, the balance laws (9–1a) and (9–13a) for standard forces and energy, the inequalities (9–13b) and (9–14) are equivalent.

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[2] Cf., e.g., Truesdell and Noll [1965, §84]; Gurtin [1981, §§25, 28].

[3] [1990] (cf. Maugin [1993, eq. (6–11)]).

[4] Cf. Coleman and Noll [1963], Coleman and Mizel [1964].

b2. Constitutive theory

Temperature variations in a homogeneous elastic body described relative to a homogeneous reference are accounted for by constitutive equations giving the free energy, stress, entropy, and heat flux in terms of the deformation gradient, the temperature, and the temperature gradient:

$$\Psi = \hat{\Psi}(\boldsymbol{F}, T, \nabla T), \tag{9–16a}$$

$$\boldsymbol{S} = \hat{\boldsymbol{S}}(\boldsymbol{F}, T, \nabla T), \tag{9–16b}$$

$$\eta = \hat{\eta}(\boldsymbol{F}, T, \nabla T), \tag{9–16c}$$

$$\boldsymbol{h} = \hat{\boldsymbol{h}}(\boldsymbol{F}, T, \nabla T). \tag{9–16d}$$

Writing

$$\boldsymbol{p} = \nabla T, \tag{9–17}$$

the common domain of the response functions is the set of all $(\boldsymbol{F}, \mathrm{T}, \boldsymbol{p})$, with $\boldsymbol{F}$ a tensor satisfying $\det \boldsymbol{F} > 0$, $T > 0$, and $\boldsymbol{p}$ a vector.

Consider an arbitrary *constitutive process*; that is, a motion $\boldsymbol{y}(\boldsymbol{X}, t)$ and a temperature field $T(\boldsymbol{X}, t)$ together with fields $\Psi(\boldsymbol{X}, t)$, $\boldsymbol{S}(\boldsymbol{X}, t)$, $\eta(\boldsymbol{X}, t)$, and $\boldsymbol{h}(\boldsymbol{X}, t)$ determined by the constitutive relations (9–16). The force and energy balances (9–1a) and (9–13a) then give the external body force $\boldsymbol{b}$ and heat supply r needed to support the process. To ensure that the second law is satisfied in all such processes, I require that *all constitutive processes be consistent with the free-energy inequality* (9–14). Equivalently,

$$\left\{\partial_{\boldsymbol{F}} \hat{\Psi}(\boldsymbol{F}, T, \boldsymbol{p}) - \hat{\boldsymbol{S}}(\boldsymbol{F}, T, \boldsymbol{p})\right\} \cdot \dot{\boldsymbol{F}} + \left\{\partial_T \hat{\Psi}(\boldsymbol{F}, T, \boldsymbol{p}) + \hat{\eta}(\boldsymbol{F}, T, \boldsymbol{p})\right\} \cdot \dot{\boldsymbol{T}}$$
$$+ \partial_{\boldsymbol{p}} \hat{\Psi}(\boldsymbol{F}, T, \boldsymbol{p}) \cdot \dot{\boldsymbol{p}} + T^{-1} \hat{\boldsymbol{h}}(\boldsymbol{F}, T, \boldsymbol{p}) \cdot \boldsymbol{p} \leq 0. \tag{9–18}$$

It is always possible to find a motion and a temperature field in which $\boldsymbol{F}$, T, $\boldsymbol{p} = \nabla T$, $\dot{\boldsymbol{F}}$, $\dot{T}$, and $\dot{\boldsymbol{p}}$ have arbitrarily prescribed values at some point and time (consistent with the constraints $\det \boldsymbol{F} > 0$, $T > 0$). Granted this, arguing as in Subsection a2 leads to the conclusions:

$$\hat{S}(\boldsymbol{F}, T, \boldsymbol{p}) = \partial_{\boldsymbol{F}} \hat{\Psi}(\boldsymbol{F}, T, \boldsymbol{p}),$$
$$\hat{\eta}(\boldsymbol{F}, T, \boldsymbol{p}) = -\partial_T \hat{\Psi}(\boldsymbol{F}, T, \boldsymbol{p}),$$
$$\partial_{\boldsymbol{p}} \hat{\Psi}(\boldsymbol{F}, T, \boldsymbol{p}) = \boldsymbol{0}.$$

The general constitutive equations (9–16) must therefore be consistent with the following restrictions:

(i) the free energy Ψ, the stress $\boldsymbol{S}$, and the entropy η must be independent of $\boldsymbol{p} = \nabla T$ and related through

$$\Psi = \hat{\Psi}(\boldsymbol{F}, T), \qquad \boldsymbol{S} = \partial_{\boldsymbol{F}} \hat{\Psi}(\boldsymbol{F}, T), \qquad \eta = -\partial_T \hat{\Psi}(\boldsymbol{F}, T); \tag{9–19}$$

(ii) the heat flux must obey the following inequality for all values of its arguments:

$$\hat{\boldsymbol{h}}(\boldsymbol{F}, T, \boldsymbol{p}) \cdot \boldsymbol{p} \leq 0. \tag{9–20}$$

If $\hat{\boldsymbol{h}}(\boldsymbol{F}, T, \boldsymbol{p})$ is linear in $\boldsymbol{p}$, the most general form of the constitutive equation for $\hat{\boldsymbol{h}}$ consistent with (9–20) is

$$\boldsymbol{h} = -\boldsymbol{K}(\boldsymbol{F}, T)\nabla T, \tag{9–21}$$

with **conductivity tensor** $\boldsymbol{K}(\boldsymbol{F}, T)$ positive semidefinite.

The relations (9–19) and (9–21) are the most general constitutive equations of the form (9–16) that are consistent with the free-energy inequality (9–14) and have $\boldsymbol{h}$ linear in ∇T. The basic equations of the thermoelasticity consist of the balance laws (9–1a) and (9–13a) for force and energy in conjunction with the restricted constitutive equations.

An interesting feature of the Coleman-Noll procedure is that the local dissipation inequality generally suggests which fields should be given constitutive descriptions, a use of the second law that seems to lead—in all classical continuum theories—to the correct set of constitutive variables. This contrasts the standard formalism of studying balance laws to see where a lack of field equations may be compensated for by the introduction of constitutive relations.

By (9–5a) and (9–19),

$$\boldsymbol{g} = \eta\nabla T, \tag{9–22}$$

and the internal configurational force vanishes if and only if the temperature is materially uniform. (For an inhomogeneous body, $\boldsymbol{g} = \eta\nabla T - \partial_{\boldsymbol{X}}\hat{\Psi}$.)

The most general smooth constitutive equation of the form (9–16d) consistent with (9–20) is

$$\boldsymbol{h} = -\boldsymbol{K}(\boldsymbol{F}, T, \nabla T)\nabla T$$

with $\boldsymbol{p}\cdot\boldsymbol{K}(\boldsymbol{F}, T, \boldsymbol{p})\,\boldsymbol{p} \geq 0$ for all values of $\boldsymbol{F}$, T, and $\boldsymbol{p}$. To verify this result, consider the inequality

$$\boldsymbol{h}(\boldsymbol{q}, \boldsymbol{p})\cdot\boldsymbol{p} \leq 0, \tag{9–23}$$

with $\boldsymbol{q} \in \mathbf{R}^m$, $\boldsymbol{p} \in \mathbf{R}^n$, and $\boldsymbol{h}$ a smooth function from $\mathbf{R}^m \times \mathbf{R}^n$ into $\mathbf{R}^n$ that satisfies (9–23) for all $\boldsymbol{q} \in \mathbf{R}^m$ and $\boldsymbol{p} \in \mathbf{R}^n$. Because the variable $\boldsymbol{q}$ appears as a parameter, it may, without loss in generality, be suppressed. Then, for $\lambda > 0$, $\boldsymbol{h}(\lambda\boldsymbol{p})\cdot\lambda\boldsymbol{p} \leq 0$; hence $\boldsymbol{h}(\lambda\boldsymbol{p})\cdot\boldsymbol{p} \leq 0$. Let $\lambda \to 0$. Then $\boldsymbol{h}(\boldsymbol{0})\cdot\boldsymbol{p} \leq 0$ for all $\boldsymbol{p}$, so that $\boldsymbol{h}(\boldsymbol{0}) = \boldsymbol{0}$. Thus

$$\boldsymbol{h}(\boldsymbol{p}) = \left\{\int_0^1 \nabla\boldsymbol{h}(s\boldsymbol{p})\,ds\right\}\boldsymbol{p} \tag{9–24}$$

for all $\boldsymbol{p}$. Let $-\boldsymbol{K}(\boldsymbol{p})$ denote the quantity $\{\ldots\}$. Then $\boldsymbol{h}(\boldsymbol{p}) = -\boldsymbol{K}(\boldsymbol{p})\boldsymbol{p}$ for all $\boldsymbol{p}$. The general solution $\boldsymbol{h}$ of (9–23) is therefore

$$\boldsymbol{h}(\boldsymbol{q}, \boldsymbol{p}) = -\boldsymbol{K}(\boldsymbol{q}, \boldsymbol{p})\boldsymbol{p} \tag{9–25}$$

with $\boldsymbol{K}(\boldsymbol{q}, \boldsymbol{p})$, for each $(\boldsymbol{q}, \boldsymbol{p})$, a linear transformation from $\mathbf{R}^n$ into $\mathbf{R}^n$ consistent with the inequality

$$\boldsymbol{p}\cdot\boldsymbol{K}(\boldsymbol{q}, \boldsymbol{p})\boldsymbol{p} \geq 0. \tag{9–26}$$

Because of the dependence of $\boldsymbol{K}(\boldsymbol{q}, \boldsymbol{p})$ on $\boldsymbol{p}$, the inequality (9–26) is weaker than positive definiteness for $\boldsymbol{K}(\boldsymbol{q}, \boldsymbol{p})$. However, when $\boldsymbol{h}$ is *quasilinear*, that is, when

$\boldsymbol{h}(\boldsymbol{q}, \boldsymbol{p})$ is linear in $\boldsymbol{p}$ for each $\boldsymbol{q}$, then

$$\boldsymbol{h}(\boldsymbol{q}, \boldsymbol{p}) = -\boldsymbol{K}(\boldsymbol{q})\boldsymbol{p} \tag{9–27}$$

for all $(\boldsymbol{q}, \boldsymbol{p})$, with $\boldsymbol{K}(\boldsymbol{q})$ positive semidefinite.

More generally, the relation (9–27) holds to first order in $\boldsymbol{p}$:

$$\boldsymbol{h}(\boldsymbol{q}, \boldsymbol{p}) = -\boldsymbol{K}(\boldsymbol{q})\boldsymbol{p} + \boldsymbol{o}(|\boldsymbol{p}|) \qquad \text{as } \boldsymbol{p} \to \boldsymbol{0} \tag{9–28}$$

with $\boldsymbol{K}(\boldsymbol{q})$ positive semidefinite; and, for $\boldsymbol{q}$ and $\boldsymbol{p}$ both small,

$$\boldsymbol{h}(\boldsymbol{q}, \boldsymbol{p}) = -\boldsymbol{K}\boldsymbol{p} + \boldsymbol{o}(|\boldsymbol{q}| + |\boldsymbol{p}|) \qquad \text{as } (\boldsymbol{q}, \boldsymbol{p}) \to 0 \tag{9–29}$$

with $\boldsymbol{K}$ constant and positive semidefinite.

Part B

The Use of Configurational Forces to Characterize Coherent Phase Interfaces

Configurational forces are central to the study of evolving material structures such as defects and phase interfaces. In this part, I will discuss the dynamics of phase interfaces modelled as smoothly evolving surfaces.[1] I base the discussion on fundamental laws which—when restricted to control volumes that do not intersect the interface—reduce to those introduced earlier; that is why the local results established thus far[2] will be valid in **bulk** (i.e., away from the interface). Here I concentrate on deriving corresponding results for the interface.

[1] This part follows the presentation of Gurtin and Podio-Guidugli [1996a], although that work does not use observer-invariance to characterize balance laws. Cf. the earlier work of Gurtin [1995], but the treatment of inertia there is somewhat lacking.

[2] E.g., the Eshelby relation (6–9), the standard and configurational balances (3–6a) and (5–10), and the dissipation inequality (6–11).

CHAPTER 10

Interface Kinematics

I consider a two-phase body whose phases α and β occupy closed complementary subregions $B_\alpha(t)$ and $B_\beta(t)$ of the reference body B, with the **interface** $\mathscr{S}(t) = B_\alpha(t) \cap B_\beta(t)$ a smoothly evolving surface whose unit normal field $\mathbf{m}(\boldsymbol{X}, t)$ points outward from $B_\alpha(t)$ (cf. Figure 1.1).

For $\Phi(\boldsymbol{X}, t)$ a field that is continuous away from the interface and up to the interface from either side, $\Phi^\pm$ denote the interfacial limits of Φ,

$$\Phi^\pm(\boldsymbol{X}, t) = \Phi(\boldsymbol{X} \pm 0\mathbf{m}(\boldsymbol{X}, t), t) \qquad \text{for } \boldsymbol{X} \in \mathscr{S}(t),$$

while $[\Phi]$ and $\langle\Phi\rangle$ designate the **jump** in Φ across the interface and the **average** of the interfacial limits of Φ:

$$[\Phi] = \Phi^+ - \Phi^-, \qquad \langle\Phi\rangle = \frac{1}{2}(\Phi^+ + \Phi^-), \tag{10–1}$$

so that the jump is phase β minus phase α.

Motions y are defined as before, except that $\boldsymbol{y}$ is no longer presumed to be smooth; precisely, $\boldsymbol{y}$ is continuous across the interface,[1] smooth away from the interface, and smooth up to the interface from either side. These assumptions yield the **compatibility conditions**

$$[\dot{\boldsymbol{y}}] = -V[\boldsymbol{F}]\mathbf{m}, \tag{10–2a}$$

$$[\boldsymbol{F}]\mathbf{P} = \mathbf{0}, \tag{10–2b}$$

where V is the (scalar) **normal velocity** of the interface in the direction $\mathbf{m}$, while

$$\mathbf{P} = \mathbf{1} - \mathbf{m} \otimes \mathbf{m} \tag{10–3}$$

[1] The interface is therefore coherent. Cf. Cermelli and Gurtin [1994a,b] for a discussion of incoherent interfaces.

is the **projection** onto the interface. The result (10–2b) asserts that $\boldsymbol{F}$, restricted to its action on tangent vectors, is continuous across the interface.

An assignment, at each t, of material vector $\boldsymbol{v}(\boldsymbol{X}, t)$ to each $\boldsymbol{X} \in \mathscr{S}(t)$ is a **velocity field for** $\mathscr{S}$ if $\boldsymbol{v}$ is a smooth field that satisfies

$$\boldsymbol{v} \cdot \mathbf{m} = V$$

(cf. Chapter 4). As with migrating control volumes, the velocity field $\boldsymbol{v}$ for $\mathscr{S}$ may be viewed as a velocity field for evolving particles constrained to $\mathscr{S}$, with the path $\boldsymbol{Z}(\tau)$ traversed by the particle that passes through $\boldsymbol{X} \in \mathscr{S}(t)$ at time t the unique solution of (4–2). The **motion velocity following** $\mathscr{S}$ is then the time derivative following such particles:

$$\overset{\square}{\boldsymbol{y}}(\boldsymbol{X}, t) = \frac{d}{d\tau}\boldsymbol{y}(\boldsymbol{Z}(\tau), \tau)|_{\tau=t}. \tag{10–4}$$

By the chain rule and (10–1),

$$\overset{\square}{\boldsymbol{y}} = \dot{\boldsymbol{y}}^{\pm} + \boldsymbol{F}^{\pm}\boldsymbol{v} = \langle \dot{\boldsymbol{y}} \rangle + \langle \boldsymbol{F} \rangle \boldsymbol{v}. \tag{10–5}$$

Under $\boldsymbol{y}$, $\mathscr{S}(t)$ deforms to a surface

$$\bar{\mathscr{S}}(t) = \boldsymbol{y}(\mathscr{S}(t), t),$$

and $\overset{\square}{\boldsymbol{y}}$ represents a velocity field for $\bar{\mathscr{S}}$.

The fields $\boldsymbol{v}$ and $\overset{\square}{\boldsymbol{y}}$ transform according to

$$\boldsymbol{v} \to \boldsymbol{v} + \boldsymbol{a}, \qquad \overset{\square}{\boldsymbol{y}} \to \overset{\square}{\boldsymbol{y}} \tag{10–6}$$

under the change in material observer defined by (2–9), and according to

$$\overset{\square}{\boldsymbol{y}} \to \overset{\square}{\boldsymbol{y}} + \boldsymbol{w} + \boldsymbol{\omega} \times (\boldsymbol{X} - \boldsymbol{o}), \qquad \boldsymbol{v} \to \boldsymbol{v} \tag{10–7}$$

under the change in spatial observer defined by (2–7) (cf. (4–6), (4–7)).

Basic to what follows are four integral identities. Let Φ be a scalar field, $\boldsymbol{T}$ a tensor field, and $\boldsymbol{w}$ a vector field, with Φ, $\boldsymbol{T}$, and $\boldsymbol{w}$ smooth away from the interface and up to the interface from either side. Then, for $P = P(t)$ a migrating control volume,

$$\frac{d}{dt}\left\{\int_P \Phi\, dv\right\} = \int_P \dot{\Phi}\, dv - \int_{\mathscr{S}\cap P} [\Phi] V\, da + \int_{\partial P} \Phi U\, da, \tag{10–8a}$$

$$\int_{\partial P} \Phi \boldsymbol{n}\, da = \int_P \nabla\Phi\, dv + \int_{\mathscr{S}\cap P} [\Phi]\mathbf{m}\, da, \tag{10–8b}$$

$$\int_{\partial P} \boldsymbol{T}\boldsymbol{n}\, da = \int_P \operatorname{Div}\boldsymbol{T}\, dv + \int_{\mathscr{S}\cap P} [\boldsymbol{T}]\mathbf{m}\, da, \tag{10–8c}$$

$$\int_{\partial P} \boldsymbol{T}\boldsymbol{n}\cdot\boldsymbol{w}\, da = \int_P (\boldsymbol{w}\cdot\operatorname{Div}\boldsymbol{T} + \boldsymbol{T}\cdot\nabla\boldsymbol{w})\, dv + \int_{\mathscr{S}\cap P} [\boldsymbol{T}\mathbf{m}\cdot\boldsymbol{w}]\, da. \tag{10–8d}$$

(Integrals such as $\int_P \dot{\Phi}\, dv$, $\int_P \nabla\Phi\, dv$, and $\int_P \operatorname{Div}\boldsymbol{T}\, dv$ are treated as ordinary integrals with piecewise continuous integrands; the jump discontinuities in Φ and $\boldsymbol{T}$ are accounted for by the terms involving $[\Phi]$ and $[\boldsymbol{T}]$.)

The transport identity (10–8a) is a direct consequence of (6–6). Indeed, let P_α and P_β, respectively, denote the portions of P in B_α and B_β. Then $\mathbf{m}$ and V represent the outward normal and normal velocity of ∂P_α on $\mathscr{S}$, while $-\mathbf{m}$ and $-V$ represent the analogous quantities for ∂P_β on $\mathscr{S}$. Thus applying (6–6) to each of P_α and P_β and then adding the resulting equations yields (10–8a). The proofs of (10–8a,b,c,d) are similar. For example, to verify (10–8b) apply the divergence theorem to each of $\int_{\partial P_\alpha} \Phi \boldsymbol{n}\, da$ and $\int_{\partial P_\beta} \Phi \boldsymbol{n}\, da$ and then add the resulting equations.

Also important is the following result in which Φ is continuous away from the interface and up to the interface from either side, while φ is continuous on the interface. Let

$$F(P) = \int_P \Phi\, dv + \int_{\mathscr{S}\cap P} \varphi\, da \tag{10–9}$$

for all control volumes P; then

$$F(P) = 0 \quad \text{for all } P \qquad \Rightarrow \qquad \varphi = 0, \tag{10–10a}$$

$$F(P) \geq 0 \quad \text{for all } P \qquad \Rightarrow \qquad \varphi \geq 0. \tag{10–10b}$$

The verification of (10–10) follows upon **shrinking P to the interface**. Precisely, let $\mathscr{G}$ be an arbitrary subsurface of $\mathscr{S}$ and choose a family P_δ ($\delta > 0$) of control volumes such that $\mathscr{S} \cap P_\delta = \mathscr{G}$ for all δ but $\mathrm{vol}(P_\delta) \to 0$. Then $F(P_\delta) \to \int_{\mathscr{G}} \varphi\, da$ and, because $\mathscr{G}$ is arbitrary, the assertions (10–10) follow.

CHAPTER 11

Interface Forces. Second Law

To simplify the presentation, I do not allow for interfacial energy, nor for forces, such as surface tension, that act within the interface. I do, however, consider counterparts, for the interface, of the body forces $\boldsymbol{b}$, $\boldsymbol{e}$, and $\boldsymbol{g}$.

a. Interface forces

To the standard and configurational force systems introduced earlier, with stresses and body forces now presumed smooth away from the interface and up to the interface from either side, I add three fields *defined on the interface* for all time:

$\boldsymbol{b}^{\mathscr{S}}$	**external standard force**
$\boldsymbol{g}^{\mathscr{S}}$	**internal configurational force**
$\boldsymbol{e}^{\mathscr{S}}$	**external configurational force**

Here $\boldsymbol{b}^{\mathscr{S}}(\boldsymbol{X}, t)$ is a spatial vector, $\boldsymbol{g}^{\mathscr{S}}(\boldsymbol{X}, t)$ and $\boldsymbol{e}^{\mathscr{S}}(\boldsymbol{X}, t)$ are material vectors, and

$$\boldsymbol{b}^{\mathscr{S}}, \boldsymbol{g}^{\mathscr{S}}, \text{ and } \boldsymbol{e}^{\mathscr{S}} \text{ are objective.} \tag{11–1}$$

The interface forces $\boldsymbol{b}^{\mathscr{S}}$, $\boldsymbol{g}^{\mathscr{S}}$, and $\boldsymbol{e}^{\mathscr{S}}$ have physical interpretations identical to the body forces $\boldsymbol{b}$, $\boldsymbol{g}$, and $\boldsymbol{e}$, except that $\boldsymbol{b}^{\mathscr{S}}$, $\boldsymbol{g}^{\mathscr{S}}$, and $\boldsymbol{e}^{\mathscr{S}}$ are concentrated at the interface. I associate $\boldsymbol{g}^{\mathscr{S}}$ with the rearrangement of material at the interface during its evolution.

b. Working

Assume that the material is viewed by a rest observer. The working $W(P)$ on a migrating control volume $P = P(t)$ then has two contributions: a contribution

$$\int_{\partial P} (\boldsymbol{C}\boldsymbol{n} \cdot \boldsymbol{q} + \boldsymbol{S}\boldsymbol{n} \cdot \mathring{\boldsymbol{y}})\, da + \int_P \boldsymbol{b} \cdot \dot{\boldsymbol{y}}\, dv \tag{11–2}$$

associated with the bulk material, where $\boldsymbol{q}$ is a velocity field for ∂P with $\mathring{\boldsymbol{y}}$ the corresponding motion velocity following ∂P (cf. (5–5)); and a contribution, which I will now derive, that accounts for the interface $\mathscr{S} = \mathscr{S}(t)$.

Assume that $\mathscr{S}$ is *stationary*. In view of the remark following (11–1), because $\boldsymbol{g}$ and $\boldsymbol{e}$ perform no work, neither should $\boldsymbol{g}^{\mathscr{S}}$ and $\boldsymbol{e}^{\mathscr{S}}$, and because $\boldsymbol{b}$ is work-conjugate to $\dot{\boldsymbol{y}}$, so also should $\boldsymbol{b}^{\mathscr{S}}$. Therefore

$$\int_{\mathscr{S} \cap P} \boldsymbol{b}^{\mathscr{S}} \cdot \dot{\boldsymbol{y}}\, da \tag{11–3}$$

would seem the appropriate expression for the working associated with a stationary interface.

If $\mathscr{S}$ *is not stationary*, then (11–3) must be modified. Let $\boldsymbol{v}$ be a velocity field for $\mathscr{S}$ and $\overset{\square}{\boldsymbol{y}}$ be the corresponding motion velocity following $\mathscr{S}$. I view $\boldsymbol{e}^{\mathscr{S}}$ as a force that performs work in conjunction with the migration of $\mathscr{S}$ and therefore choose $\boldsymbol{v}$ as an appropriate work-conjugate velocity for $\boldsymbol{e}^{\mathscr{S}}$. Further, if $\mathscr{S}$ is not stationary, then $\bar{\mathscr{S}}$ has no intrinsic material description, and it would seem appropriate to use as work-conjugate velocity for $\boldsymbol{b}^{\mathscr{S}}$ the motion velocity $\overset{\square}{\boldsymbol{y}}$ following $\mathscr{S}$, as $\overset{\square}{\boldsymbol{y}}$ represents a velocity field for $\bar{\mathscr{S}}$ consistent with the choice of $\boldsymbol{v}$ as velocity field for $\mathscr{S}$. Finally, the field $\boldsymbol{g}^{\mathscr{S}}$ represents *internal* forces that pin, in place, those reference points $\boldsymbol{X}$ that mark the current location of the interface. If the material observer is at rest, the points $\boldsymbol{X}$ are viewed stationary and $\boldsymbol{g}^{\mathscr{S}}$ performs no work. I therefore take

$$\int_{\mathscr{S} \cap P} (\boldsymbol{e}^{\mathscr{S}} \cdot \boldsymbol{v} + \boldsymbol{b}^{\mathscr{S}} \cdot \overset{\square}{\boldsymbol{y}})\, da$$

as the working associated with a migrating interface. Note the similarity between $\boldsymbol{e}^{\mathscr{S}} \cdot \boldsymbol{v} + \boldsymbol{b}^{\mathscr{S}} \cdot \overset{\square}{\boldsymbol{y}}$ and the integrand $\boldsymbol{C}\boldsymbol{n} \cdot \boldsymbol{q} + \boldsymbol{S}\boldsymbol{n} \cdot \mathring{\boldsymbol{y}}$ in (11–2): $\overset{\square}{\boldsymbol{y}}$ rather than $\dot{\boldsymbol{y}}$ is the work-conjugate velocity for $\boldsymbol{b}^{\mathscr{S}}$, just as $\mathring{\boldsymbol{y}}$ rather than $\dot{\boldsymbol{y}}$ is the velocity for $\boldsymbol{S}\boldsymbol{n}$, and $\boldsymbol{v}$ is the work-conjugate velocity for $\boldsymbol{e}^{\mathscr{S}}$, just as $\boldsymbol{q}$ is for $\boldsymbol{C}\boldsymbol{n}$.

Summarizing, I write the **working** $W(P)$ on a migrating control volume $P = P(t)$ in the form

$$\begin{aligned} W(P) = &\int_{\partial P} \left(\boldsymbol{C}\boldsymbol{n} \cdot \boldsymbol{q} + \boldsymbol{S}\boldsymbol{n} \cdot \mathring{\boldsymbol{y}}\right)\, da + \int_P \boldsymbol{b} \cdot \dot{\boldsymbol{y}}\, dv \\ &+ \int_{\mathscr{S} \cap P} \left(\boldsymbol{e}^{\mathscr{S}} \cdot \boldsymbol{v} + \boldsymbol{b}^{\mathscr{S}} \cdot \overset{\square}{\boldsymbol{y}}\right)\, da. \end{aligned} \tag{11–4}$$

c. Standard and configurational force balances at the interface

Consider the change in material and spatial observers defined in the paragraphs containing (2–7) and (2–9). Then $\boldsymbol{g}^{\mathscr{S}}$, as observed by the moving material observer, performs work, because material points as viewed by that observer migrate with velocity $\boldsymbol{a}$. Thus and by the transformation laws for $\boldsymbol{q}$, $\mathring{\boldsymbol{y}}$, $\dot{\boldsymbol{y}}$, $\boldsymbol{v}$, and $\overset{\square}{\boldsymbol{y}}$ specified in (2–8), (2–12), (4–6), (4–7), (10–6), and (10–7), the working $W(P)$ as recorded by the new observers has the form

$$\begin{aligned} W(P) = &\int_{\partial P} \boldsymbol{C}\boldsymbol{n}\cdot(\boldsymbol{q}+\boldsymbol{a})\,da + \int_P (\boldsymbol{g}+\boldsymbol{e})\cdot\boldsymbol{a}\,dv + \int_{\mathscr{S}\cap P}\left(\boldsymbol{g}^{\mathscr{S}}\cdot\boldsymbol{a}+\boldsymbol{e}^{\mathscr{S}}\cdot(\boldsymbol{v}+\boldsymbol{a})\right)\,da \\ &+\int_{\partial P}\boldsymbol{S}\boldsymbol{n}\cdot\left(\mathring{\boldsymbol{y}}+\boldsymbol{w}+\boldsymbol{\omega}\times(\boldsymbol{y}-\boldsymbol{o})\right)\,da = \int_P \boldsymbol{b}\cdot(\dot{\boldsymbol{y}}+\boldsymbol{w}+\boldsymbol{\omega}\times(\boldsymbol{y}-\boldsymbol{o}))\,dv \\ &+\int_{\mathscr{S}\cap P}\boldsymbol{b}^{\mathscr{S}}\cdot\left(\overset{\square}{\boldsymbol{y}}+\boldsymbol{w}+\boldsymbol{\omega}\times(\boldsymbol{y}-\boldsymbol{o})\right)\,da. \end{aligned} \tag{11–5}$$

The requirement that the *working* be invariant under changes in material and spatial observer requires the equivalence of (11–4) and (11–5) for all vectors $\boldsymbol{a}$, $\boldsymbol{w}$, and $\boldsymbol{\omega}$, and hence yields the **standard force** and **moment balances**

$$\int_{\partial P}\boldsymbol{S}\boldsymbol{n}\,da + \int_P \boldsymbol{b}\,dv + \int_{\mathscr{S}\cap P}\boldsymbol{b}^{\mathscr{S}}\,da = \boldsymbol{0}, \tag{11–6a}$$

$$\int_{\partial P}(\boldsymbol{y}-\boldsymbol{o})\times\boldsymbol{S}\boldsymbol{n}\,da + \int_P(\boldsymbol{y}-\boldsymbol{o})\times\boldsymbol{b}\,dv + \int_{\mathscr{S}\cap P}(\boldsymbol{y}-\boldsymbol{o})\times\boldsymbol{b}^{\mathscr{S}}\,da = \boldsymbol{0} \tag{11–6b}$$

and the **configurational force balance**

$$\int_{\partial P}\boldsymbol{C}\boldsymbol{n}\,da + \int_P(\boldsymbol{g}+\boldsymbol{e})\,dv + \int_{\mathscr{S}\cap P}(\boldsymbol{g}^{\mathscr{S}}+\boldsymbol{e}^{\mathscr{S}})\,da = \boldsymbol{0}. \tag{11–7}$$

Since $\operatorname{Div}\boldsymbol{S} = -\boldsymbol{b}$ and $\operatorname{Div}\boldsymbol{C} = -\boldsymbol{g}-\boldsymbol{e}$ in bulk, (11–6a), (11–7), and the identity (10–8c) imply that

$$\int_{\mathscr{S}\cap P}\left([\boldsymbol{S}]\mathtt{m}+\boldsymbol{b}^{\mathscr{S}}\right)\,da = \boldsymbol{0}, \qquad \int_{\mathscr{S}\cap P}\left([\boldsymbol{C}]\mathtt{m}+\boldsymbol{g}^{\mathscr{S}}+\boldsymbol{e}^{\mathscr{S}}\right)\,da = \boldsymbol{0};$$

thus, since $\mathscr{S}\cap P$ is an arbitrary (nice) subsurface of $\mathscr{S}$, this yields the local force balances

$$[\boldsymbol{S}]\mathtt{m}+\boldsymbol{b}^{\mathscr{S}} = \boldsymbol{0}, \tag{11–8a}$$

$$[\boldsymbol{C}]\mathtt{m}+\boldsymbol{g}^{\mathscr{S}}+\boldsymbol{e}^{\mathscr{S}} = \boldsymbol{0} \tag{11–8b}$$

at the interface. The moment balance (11–6b) localized to the interface yields no additional results.

d. Invariance under changes in velocity field for $\mathscr{S}(t)$. Normal configurational balance

The requirement that the theory be independent of the choice of velocity field $\boldsymbol{v}$ for $\mathscr{S}$ yields an important relation for the external forces $\boldsymbol{b}^{\mathscr{S}}$ and $\boldsymbol{e}^{\mathscr{S}}$. Let $\mathbf{t}$ denote the tangential component of $\boldsymbol{v}$. By (10–5) and the constraint $\boldsymbol{v}\cdot\mathbf{m}=V$, the integrand in (11–4) associated with the integral over $\mathscr{S}\cap P$ may be written in the form

$$\boldsymbol{b}^{\mathscr{S}}\cdot\langle\dot{\boldsymbol{y}}\rangle+\left\{\boldsymbol{e}^{\mathscr{S}}+\langle\boldsymbol{F}\rangle^{\top}\boldsymbol{b}^{\mathscr{S}}\right\}\cdot\boldsymbol{v}=\boldsymbol{b}^{\mathscr{S}}\cdot\left\{\langle\dot{\boldsymbol{y}}\rangle+\langle\boldsymbol{F}\rangle^{\top}\mathbf{m}V\right\}+\left\{\boldsymbol{e}^{\mathscr{S}}+\langle\boldsymbol{F}\rangle^{\top}\boldsymbol{b}^{\mathscr{S}}\right\}\cdot\mathbf{t}.$$

But because changes in $\boldsymbol{v}$ affect $\mathbf{t}$ but leave V unaltered, and because both P and $\mathbf{t}$ are arbitrary, $\boldsymbol{e}^{\mathscr{S}}+\langle\boldsymbol{F}\rangle^{\top}\boldsymbol{b}^{\mathscr{S}}$ must be normal to $\mathscr{S}$:

$$\mathbf{P}\boldsymbol{e}^{\mathscr{S}}=-\mathbf{P}\langle\boldsymbol{F}\rangle^{\top}\boldsymbol{b}^{\mathscr{S}}. \tag{11–9}$$

Further, by (10–2b), $\boldsymbol{F}\mathbf{P}$ is continuous across the interface, so that $\boldsymbol{F}^{\pm}\mathbf{P}=\langle\boldsymbol{F}\rangle\mathbf{P}$; hence (6–9), (11–8a), and (11–9) imply that

$$\mathbf{P}[\boldsymbol{C}]\mathbf{m}=-[(\boldsymbol{F}\mathbf{P})^{\top}\boldsymbol{S}]\mathbf{m}=-(\langle\boldsymbol{F}\rangle\mathbf{P})^{\top}[\boldsymbol{S}]\mathbf{m}=\mathbf{P}\langle\boldsymbol{F}\rangle^{\top}\boldsymbol{b}^{\mathscr{S}}=-\mathbf{P}\boldsymbol{e}^{\mathscr{S}}. \tag{11–10}$$

This identity and the configurational balance (11–8b) yield the important result

$$\mathbf{P}\boldsymbol{g}^{\mathscr{S}}=\mathbf{0},$$

and *the internal force on the interface is necessarily normal*:

$$\boldsymbol{g}^{\mathscr{S}}=g^{\mathscr{S}}\mathbf{m}. \tag{11–11}$$

Conversely, granted the Eshelby relation (6–9), the compatibility condition (10–2b), and the standard balance (11–8a), if $\boldsymbol{g}^{\mathscr{S}}$ is normal to the interface and if $\mathbf{P}\boldsymbol{e}^{\mathscr{S}}=-\mathbf{P}\langle\boldsymbol{F}\rangle^{\top}\boldsymbol{b}^{\mathscr{S}}$, then the tangential component of the configurational balance (11–8b) is satisfied automatically; this allows one to restrict attention to the **normal configurational balance**:

$$\mathbf{m}\cdot[\boldsymbol{C}]\mathbf{m}+\mathbf{m}\cdot\boldsymbol{e}^{\mathscr{S}}+g^{\mathscr{S}}=\mathbf{0}. \tag{11–12}$$

Next, since $\boldsymbol{e}^{\mathscr{S}}+\langle\boldsymbol{F}\rangle^{\top}\boldsymbol{b}^{\mathscr{S}}$ is normal to $\mathscr{S}$, the integral over $\mathscr{S}\cap P$ in (11–4) must have the form

$$\int_{\mathscr{S}\cap P}\{\boldsymbol{b}^{\mathscr{S}}\cdot\overset{\square}{\boldsymbol{y}}+\boldsymbol{e}^{\mathscr{S}}\cdot\mathbf{m}V\}da,$$

where here and henceforth we take $\boldsymbol{v}=V\mathbf{m}$, so that

$$\overset{\square}{\boldsymbol{y}}=\dot{\boldsymbol{y}}^{\pm}+V\boldsymbol{F}^{\pm}\mathbf{m}=\langle\dot{\boldsymbol{y}}\rangle+V\langle\boldsymbol{F}\rangle\mathbf{m}; \tag{11–13}$$

the steps leading to (5–15) therefore yield an intrinsic form for the working:

$$W(P)=\int_{\partial P}\boldsymbol{S}\boldsymbol{n}\cdot\dot{\boldsymbol{y}}\,da+\int_{P}\boldsymbol{b}\cdot\dot{\boldsymbol{y}}\,dv+\int_{\partial P}\pi U\,da+\int_{\mathscr{S}\cap P}\{\boldsymbol{b}^{\mathscr{S}}\cdot\overset{\square}{\boldsymbol{y}}+\boldsymbol{e}^{\mathscr{S}}\cdot\mathbf{m}V\}da. \tag{11–14}$$

e. Power balance. Internal working

$W(P)$ represents the rate at which work is performed on P by forces external to P. This *external* working is balanced by the working of forces acting internally to P, and it is this *internal* working that best characterizes forces such as the internal configurational force $\boldsymbol{g}^{\mathscr{S}}$.

Since $\operatorname{Div}\boldsymbol{S} = -\boldsymbol{b}$ in bulk, the identity (10–8d) with $\boldsymbol{T} = \boldsymbol{S}$ and $\boldsymbol{w} = \dot{\boldsymbol{y}}$ yields

$$\int_{\partial P} \boldsymbol{S}\boldsymbol{n}\cdot\dot{\boldsymbol{y}}\,da + \int_P \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\,dv = \int_P \boldsymbol{S}\cdot\dot{\boldsymbol{F}}\,dv + \int_{\mathscr{S}\cap P} [\boldsymbol{S}\mathfrak{m}\cdot\dot{\boldsymbol{y}}]da.$$

Further, using (10–2a), (10–5), (11–8a), and the identity $[\varphi\psi] = \langle\varphi\rangle[\psi]+[\varphi]\langle\psi\rangle$,

$$\begin{aligned}
[\boldsymbol{S}\mathfrak{m}\cdot\dot{\boldsymbol{y}}] &= [\boldsymbol{S}\mathfrak{m}]\cdot\langle\dot{\boldsymbol{y}}\rangle + \langle\boldsymbol{S}\mathfrak{m}\rangle\cdot[\dot{\boldsymbol{y}}]\\
&= -\boldsymbol{b}^{\mathscr{S}}\cdot\langle\dot{\boldsymbol{y}}\rangle + \langle\boldsymbol{S}\mathfrak{m}\rangle\cdot[\dot{\boldsymbol{y}}]\\
&= -\boldsymbol{b}^{\mathscr{S}}\cdot\langle\dot{\boldsymbol{y}}\rangle - \langle\boldsymbol{S}\mathfrak{m}\rangle\cdot[\boldsymbol{F}\mathfrak{m}]V\\
&= -\boldsymbol{b}^{\mathscr{S}}\cdot\langle\dot{\boldsymbol{y}}\rangle - [\boldsymbol{S}\mathfrak{m}\cdot\boldsymbol{F}\mathfrak{m}]V + [\boldsymbol{S}\mathfrak{m}]\cdot\langle\boldsymbol{F}\mathfrak{m}\rangle V\\
&= -\boldsymbol{b}^{\mathscr{S}}\cdot(\langle\dot{\boldsymbol{y}}\rangle + \langle\boldsymbol{F}\mathfrak{m}\rangle V) - [\boldsymbol{S}\mathfrak{m}\cdot\boldsymbol{F}\mathfrak{m}]V\\
&= -\boldsymbol{b}^{\mathscr{S}}\cdot\overset{\square}{\boldsymbol{y}} - \mathfrak{m}\cdot[\boldsymbol{F}^{\top}\boldsymbol{S}]\mathfrak{m}V,
\end{aligned} \tag{11–15}$$

a calculation that yields the following balances:

$$\begin{aligned}
&\int_{\partial P} \boldsymbol{S}\boldsymbol{n}\cdot\dot{\boldsymbol{y}}\,da + \int_P \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\,dv + \int_{\mathscr{S}\cap P} \boldsymbol{b}^{\mathscr{S}}\cdot\langle\dot{\boldsymbol{y}}\rangle da\\
&\qquad = \int_P \boldsymbol{S}\cdot\dot{\boldsymbol{F}}\,dv + \int_{\mathscr{S}\cap P} \langle\boldsymbol{S}\mathfrak{m}\rangle\cdot[\dot{\boldsymbol{y}}]da,
\end{aligned} \tag{11–16a}$$

$$\begin{aligned}
&\int_{\partial P} \boldsymbol{S}\boldsymbol{n}\cdot\dot{\boldsymbol{y}}\,da + \int_P \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\,dv + \int_{\mathscr{S}\cap P} \boldsymbol{b}^{\mathscr{S}}\cdot\overset{\square}{\boldsymbol{y}}\,da\\
&\qquad = \int_P \boldsymbol{S}\cdot\dot{\boldsymbol{F}}\,dv - \int_{\mathscr{S}\cap P} \mathfrak{m}[\boldsymbol{F}^{\top}\boldsymbol{S}]\mathfrak{m}V\,da.
\end{aligned} \tag{11–16b}$$

The identities (11–16a) represent power balances for P involving only standard forces. Of the two, (11–16a) is the more classical, because the relevant kinematical field is the motion velocity $\dot{\boldsymbol{y}}$. In contrast, (11–16b) accounts explicitly for the motion of the interface through the velocities V and $\overset{\square}{\boldsymbol{y}}$, and the force conjugate to V is the normal component of the *standard* part $-\boldsymbol{F}^{\top}\boldsymbol{S}$ of the *configurational* stress $\boldsymbol{C} = \pi\mathbf{1} - \boldsymbol{F}^{\top}\boldsymbol{S}$.

Of more use is the result obtained when we take configurational forces into account. Indeed, by (11–16b), the stress relation $\boldsymbol{C} = \pi\mathbf{1} - \boldsymbol{F}^{\top}\boldsymbol{S}$, and the normal configurational balance (11–12),

$$\begin{aligned}
&\int_{\partial P} \boldsymbol{S}\boldsymbol{n}\cdot\dot{\boldsymbol{y}}\,da + \int_P \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\,dv + \int_{\mathscr{S}\cap P} \left(\boldsymbol{b}^{\mathscr{S}}\cdot\overset{\square}{\boldsymbol{y}} + (\boldsymbol{e}^{\mathscr{S}}\cdot\mathfrak{m})V\right) da\\
&\qquad = \int_P \boldsymbol{S}\cdot\dot{\boldsymbol{F}}\,dv - \int_{\mathscr{S}\cap P} \left([\pi] + g^{\mathscr{S}}\right) V\,da,
\end{aligned} \tag{11–17}$$

or equivalently, adding $\int_{\partial P} \pi U \, da$ to both sides of this equation and appealing to (11–14),

$$W(P) = \int_P \boldsymbol{S} \cdot \dot{\boldsymbol{F}} \, dv + \int_{\partial P} \pi U \, da - \int_{\mathscr{S} \cap P} \left([\![\pi]\!] + g^{\mathscr{S}} \right) V \, da. \tag{11–18}$$

This identity is a **power balance** relating the **external working** $W(P)$, as given by, say, (11–14), to the **internal working** represented by the right side of (11–18). The terms with integrands $\boldsymbol{S} \cdot \dot{\boldsymbol{F}}$ and πU are discussed in the paragraphs following (3–8) and (5–24). Regarding the remaining terms:

- The term $-[\![\pi]\!] V = \pi^- V - \pi^+ V$ represents working associated with the exchange of bulk material between phases at the interface; its role is analogous to that of $\int_{\partial P} \pi U \, da$.
- The term $-g^{\mathscr{S}} V$ represents working needed to maintain the internal integrity of the material as the interface passes through it; the negative sign signifies that $g^{\mathscr{S}}$ performs positive work when and only when it opposes motion of the interface.

f. Second law. Internal dissipation inequality for the interface

The second law has the form (6–4) for each migrating control volume $P = P(t)$, with (bulk) free energy Ψ assumed smooth away from the interface and up to the interface from either side, and with $W(P(t))$ the working (11–4):

$$\frac{d}{dt} \left\{ \int_P \Psi \, dv \right\} \leq \int_{\partial P} (\boldsymbol{C}\boldsymbol{n} \cdot \boldsymbol{q} + \boldsymbol{S}\boldsymbol{n} \cdot \mathring{\boldsymbol{y}}) \, da + \int_P \boldsymbol{b} \cdot \dot{\boldsymbol{y}} \, dv + \int_{\mathscr{S} \cap P} (\boldsymbol{e}^{\mathscr{S}} \cdot \boldsymbol{v} + \boldsymbol{b}^{\mathscr{S}} \cdot \overset{\square}{\boldsymbol{y}}) \, da. \tag{11–19}$$

The right side of (11–19) may also be written in the intrinsic form (11–14); equivalently, the power balance (11–18) may be used to replace the external working by the internal working and hence to rewrite (11–19) as

$$\frac{d}{dt} \left\{ \int_P \Psi \, dv \right\} \leq \int_P \boldsymbol{S} \cdot \dot{\boldsymbol{F}} \, dv - \int_{\mathscr{S} \cap P} \left([\![\pi]\!] + g^{\mathscr{S}} \right) V \, da + \int_{\partial P} \pi U \, da.$$

By (6–8), $\pi = \Psi$; thus, using (10–8a), the dissipation $\mathscr{D}(P)$, which is the right side of (11–19) minus the left, has the form

$$\mathscr{D}(P) = -\int_P (\dot{\Psi} - \boldsymbol{S} \cdot \dot{\boldsymbol{F}}) \, dv - \int_{\mathscr{S} \cap P} g^{\mathscr{S}} V \, da \geq 0, \tag{11–20}$$

so that $-g^{\mathscr{S}} V$ is the *energy dissipated by the interface, per unit area*. In fact, shrinking P to the interface yields the **interfacial dissipation inequality**

$$g^{\mathscr{S}} V \leq 0 \tag{11–21}$$

(cf. (10–10b)). This inequality, requiring that the normal internal force oppose motion of the interface, is a central result of the theory.

g. Localizations using a pillbox argument

An alternative derivation of the local force balances (11–8) and the internal dissipation inequality (11–21) involves a classical pillbox argument.

Let $\mathscr{G}(t)$ denote a smoothly evolving subsurface of $\mathscr{S}(t)$ and, for all sufficiently small $\delta > 0$, let $\mathscr{G}_\delta(t)$ denote the *δ-pillbox* about $\mathscr{G}(t)$:

$$\mathscr{G}_\delta(t) = \{\boldsymbol{X} \in B \ : \ \boldsymbol{X} = \boldsymbol{Y} + \varepsilon \mathbf{m}(\boldsymbol{Y}, t), \boldsymbol{Y} \in \mathscr{G}(t), |\varepsilon| \le \delta\}. \tag{11–22}$$

Consider the migrating control volume $P(t) = \mathscr{G}_\delta(t)$ with δ small. Then $\partial\mathscr{G}_\delta(t)$ is the union of surfaces

$$\partial\mathscr{G}_\delta(t)^\pm = \{\boldsymbol{X} \ : \ \boldsymbol{X} = \boldsymbol{Y} \pm \delta \mathbf{m}(\boldsymbol{Y}, t), \boldsymbol{Y} \in \mathscr{G}(t)\} \tag{11–23}$$

and a surface whose area is $O(\delta)$; and the outward unit normal $\boldsymbol{n}(\boldsymbol{X}, t)$ and normal velocity $U(\boldsymbol{X}, t)$ for $\partial\mathscr{G}_\delta(t)$ satisfy

$$\boldsymbol{n}(\boldsymbol{X}, t) = \pm\mathbf{m}(\boldsymbol{Y}, t) \quad \text{and} \quad U(\boldsymbol{X}, t) = \pm V(\boldsymbol{Y}, t) \quad \text{for} \quad \boldsymbol{X} \in \partial\mathscr{G}_\delta(t)^\pm. \tag{11–24}$$

Thus, as $\delta \to 0$,

$$\int_{\partial\mathscr{G}_\delta(t)} \Phi U \, da \to \int_{\mathscr{G}(t)} [\Phi] V \, da, \tag{11–25a}$$

$$\int_{\partial\mathscr{G}_\delta(t)} \Phi \boldsymbol{n} \, da \to \int_{\mathscr{G}(t)} [\Phi] \mathbf{m} \, da, \tag{11–25b}$$

$$\frac{d}{dt}\left\{\int_{\mathscr{G}_\delta(t)} \Phi \, dv\right\} \to 0, \tag{11–25c}$$

where (11–25c) follows from (10–8a) and (11–25a). Further, for

$$\boldsymbol{q} = U\boldsymbol{n}, \qquad \mathring{\boldsymbol{y}} = \dot{\boldsymbol{y}} + U\boldsymbol{F}\boldsymbol{n}, \tag{11–26a}$$

$$\boldsymbol{v} = V\mathbf{m}, \qquad \overset{\square}{\boldsymbol{y}} = \dot{\boldsymbol{y}}^\pm + V\boldsymbol{F}^\pm\mathbf{m} = \langle\dot{\boldsymbol{y}}\rangle + V\langle\boldsymbol{F}\rangle\mathbf{m}, \tag{11–26b}$$

the intrinsic velocity fields and corresponding motion velocities for $\partial\mathscr{G}_\delta(t)$ and $\mathscr{S}(t)$, (11–24) yields

$$\int_{\partial\mathscr{G}_\delta(t)} \boldsymbol{C}\boldsymbol{n} \cdot \boldsymbol{q} \, da \to \int_{\mathscr{G}(t)} V\mathbf{m} \cdot [\boldsymbol{C}]\mathbf{m} \, da, \tag{11–27a}$$

$$\int_{\partial\mathscr{G}_\delta(t)} \boldsymbol{S}\boldsymbol{n} \cdot \mathring{\boldsymbol{y}} \, da \to \int_{\mathscr{G}(t)} [\boldsymbol{S}]\mathbf{m} \cdot \overset{\square}{\boldsymbol{y}} \, da. \tag{11–27b}$$

By (11–25b), the choice $P(t) = \mathscr{G}_\delta(t)$ in (11–6a) and (11–7) yields the local force balances (11–8a).

The localization of the second law is accomplished using (11–19) with velocity fields given by (11–26). Let $\mathscr{G}(t)$ denote a smoothly evolving subsurface of $\mathscr{S}(t)$,

and take $P(t) = \mathscr{G}_\delta(t)$, the δ-pillbox about $\mathscr{G}(t)$. Then, by (11–25c), (11–27), and the standard and configurational balances (11–8a) and (11–12),

$$0 \leq \int_{\mathscr{G}(t)} \left\{ \left(\mathbf{m} \cdot [\boldsymbol{C}]\mathbf{m} + \boldsymbol{e}^{\mathscr{S}} \cdot \mathbf{m} \right) V + \left([\boldsymbol{S}]\mathbf{m} + \boldsymbol{b}^{\mathscr{S}} \right) \cdot \overset{\square}{\boldsymbol{y}} \right\} da = - \int_{\mathscr{G}(t)} g^{\mathscr{S}} V \, da. \tag{11–28}$$

Since $\mathscr{G}(t)$ is arbitrary, the internal dissipation inequality (11–21) follows.

CHAPTER 12

Inertia. Basic Equations for the Interface

I assume throughout this chapter that the underlying observer is inertial and the external body forces $\boldsymbol{b}$ and (hence) $\boldsymbol{e}$ and the external interface forces $\boldsymbol{b}^{\mathscr{S}}$ and $\boldsymbol{e}^{\mathscr{S}}$ are inertial. Then $\boldsymbol{b}$ has the form (7–1). I now characterize $\boldsymbol{b}^{\mathscr{S}}$ through the equivalence of inertial force and temporal changes in momentum, and, knowing $\boldsymbol{b}^{\mathscr{S}}$, $\boldsymbol{e}^{\mathscr{S}}$ is determined through the equivalence of inertial working and temporal changes in kinetic energy.[1]

a. Relative kinetic energy

Throughout this section $\overset{\square}{\boldsymbol{y}}$ has the the intrinsic form $\overset{\square}{\boldsymbol{y}} = \dot{\boldsymbol{y}}^{\pm} + V\boldsymbol{F}^{\pm}\mathbb{m} = \langle\dot{\boldsymbol{y}}\rangle + V\langle\boldsymbol{F}\rangle\mathbb{m}$ (cf. 11–13). The field $k = \frac{1}{2}\rho|\dot{\boldsymbol{y}}|^2$ represents the kinetic energy and the interfacial fields

$$(k_{\text{rel}})^{\pm} = \frac{1}{2}\rho|\dot{\boldsymbol{y}}^{\pm} - \overset{\square}{\boldsymbol{y}}|^2$$

represent the kinetic energy at the two sides of the interface measured relative to the interface. Because $[\dot{\boldsymbol{y}} - \overset{\square}{\boldsymbol{y}}] = -V[\boldsymbol{F}\mathbb{m}]$ and $\langle\dot{\boldsymbol{y}} - \overset{\square}{\boldsymbol{y}}\rangle = -V\langle\boldsymbol{F}\mathbb{m}\rangle$ the identity

$$[|\varphi|^2] = 2[\varphi]\cdot\langle\varphi\rangle$$

[1] Following a procedure of Podio-Guidugli [1997] as applied by Gurtin and Podio-Guidugli [1996a,b, 1997].

may be used to show that $[|\dot{\boldsymbol{y}} - \overset{\square}{\boldsymbol{y}}|^2] = [|\boldsymbol{F}\mathbf{m}|^2]V^2$. Further, by (10–2) and (10–3), $|\boldsymbol{F}\mathbf{P}|^2 = |\boldsymbol{F}|^2 - |\boldsymbol{F}\mathbf{m}|^2$ and $[|\boldsymbol{F}\mathbf{P}|^2] = 0$. The **jump in relative kinetic energy**,

$$[k_{\text{rel}}] = \frac{1}{2}\,\rho[|\dot{\boldsymbol{y}} - \overset{\square}{\boldsymbol{y}}|^2], \tag{12–1}$$

may therefore be written alternatively as

$$[k_{\text{rel}}] = \frac{1}{2}\,\rho[|\boldsymbol{F}\mathbf{m}|^2]V^2 = \frac{1}{2}\,\rho[|\boldsymbol{F}|^2]V^2. \tag{12–2}$$

b. Determination of $\boldsymbol{b}^{\mathscr{S}}$ and $\boldsymbol{e}^{\mathscr{S}}$

The **production of momentum** in migrating control volume $P = P(t)$ is defined by

$$\mathscr{P}(P) = \frac{d}{dt}\left\{\int_P \rho\dot{\boldsymbol{y}}\,dv\right\} - \int_{\partial P} \rho\dot{\boldsymbol{y}}U\,da \tag{12–3}$$

and represents the temporal change in momentum of $P(t)$ minus the inflow of momentum due to the migration of $\partial P(t)$. By (10–8a),

$$\mathscr{P}(P) = \int_P \rho\ddot{\boldsymbol{y}}\,dv - \int_{\mathscr{S}\cap P} \rho[\dot{\boldsymbol{y}}]V\,da.$$

A basic premise of the theory is that $-\mathscr{P}(P)$ be equivalent to the total standard force on P due to inertia (cf. (7–6a)):

$$\int_P \boldsymbol{b}\,dv + \int_{\mathscr{S}\cap P} \boldsymbol{b}^{\mathscr{S}}\,da = -\mathscr{P}(P). \tag{12–4}$$

Thus, since $\boldsymbol{b} = -\rho\ddot{\boldsymbol{y}}$,

$$\int_{\mathscr{S}\cap P} \left(\boldsymbol{b}^{\mathscr{S}} - \rho[\dot{\boldsymbol{y}}]V\right)da = \mathbf{0},$$

and since P is arbitrary,

$$\boldsymbol{b}^{\mathscr{S}} = \rho[\dot{\boldsymbol{y}}]V. \tag{12–5}$$

Further, the kinetic energy $k = \frac{1}{2}\,\rho|\dot{\boldsymbol{y}}|^2$ satisfies $[k] = \frac{1}{2}[|\dot{\boldsymbol{y}}|^2] = [\dot{\boldsymbol{y}}]\cdot\langle\dot{\boldsymbol{y}}\rangle$; hence

$$\boldsymbol{b}^{\mathscr{S}}\cdot\langle\dot{\boldsymbol{y}}\rangle = [k]V.$$

Thus $\boldsymbol{b}^{\mathscr{S}}$ and $\boldsymbol{b}^{\mathscr{S}}\cdot\langle\dot{\boldsymbol{y}}\rangle$ represent rates at which momentum and kinetic energy are released by the interface (per unit area).

Similarly, the **production of kinetic energy** in a control volume $P = P(t)$ is given by

$$\mathscr{T}(P) = \frac{d}{dt}\left\{\int_P k\,dv\right\} - \int_{\partial P} kU\,da.$$

By (10–8a),

$$\mathscr{T}(P) = \int_P \rho \dot{\boldsymbol{y}} \cdot \ddot{\boldsymbol{y}}\, dv - \int_{\mathscr{S}\cap P} [k] V\, da;$$

hence

$$\int_P \boldsymbol{b} \cdot \dot{\boldsymbol{y}}\, dv + \int_{\mathscr{S}\cap P} \boldsymbol{b}^{\mathscr{S}} \cdot \langle \dot{\boldsymbol{y}} \rangle da = -\mathscr{T}(P). \tag{12–6}$$

A second basic premise of the theory is that the inertial working be equal to $-\mathscr{T}(P)$; (12–6) therefore establishes the average interfacial velocity $\langle \dot{\boldsymbol{y}} \rangle$ as the appropriate work-conjugate velocity for the inertial force $\boldsymbol{b}^{\mathscr{S}}$.

The results established thus far, that $\boldsymbol{b}^{\mathscr{S}} = \rho[\dot{\boldsymbol{y}}]V$ and the appropriate work-conjugate velocity for $\boldsymbol{b}^{\mathscr{S}}$ is $\langle \dot{\boldsymbol{y}} \rangle$, depend only on the kinematical assumption that $\boldsymbol{y}$ be continuous across the interface, smooth away from the interface, and smooth up to the interface from either side; these results are therefore also valid for shock waves.

Arguing as in Section 5e, the inertial working should also have the form expressed in (11–14), so that

$$\int_P \boldsymbol{b} \cdot \dot{\boldsymbol{y}}\, dv + \int_{\mathscr{S}\cap P} \left(\boldsymbol{b}^{\mathscr{S}} \cdot \overset{\square}{\boldsymbol{y}} + \boldsymbol{e}^{\mathscr{S}} \cdot \mathbf{m} V \right) da = -\mathscr{T}(P), \tag{12–7}$$

with $\overset{\square}{\boldsymbol{y}}$ given by (11–13). Thus, by (12–6),

$$\int_{\mathscr{S}\cap P} \left(\boldsymbol{b}^{\mathscr{S}} \cdot \overset{\square}{\boldsymbol{y}} + \boldsymbol{e}^{\mathscr{S}} \cdot \mathbf{m} V - \boldsymbol{b}^{\mathscr{S}} \cdot \langle \dot{\boldsymbol{y}} \rangle \right) da = 0,$$

and, because P is arbitrary,

$$\boldsymbol{b}^{\mathscr{S}} \cdot \langle \dot{\boldsymbol{y}} \rangle = \boldsymbol{b}^{\mathscr{S}} \cdot \overset{\square}{\boldsymbol{y}} + \boldsymbol{e}^{\mathscr{S}} \cdot \mathbf{m} V, \tag{12–8}$$

or equivalently, by (11–13), $\left(\boldsymbol{b}^{\mathscr{S}} \cdot \langle \boldsymbol{F} \rangle \mathbf{m} + \boldsymbol{e}^{\mathscr{S}} \cdot \mathbf{m} \right) V = 0$; hence

$$\boldsymbol{e}^{\mathscr{S}} \cdot \mathbf{m} = -\mathbf{m} \cdot \langle \boldsymbol{F} \rangle^{\top} \boldsymbol{b}^{\mathscr{S}},$$

at least for $V \neq 0$. On the other hand, (11–9) asserts that $\mathbf{P}\boldsymbol{e}^{\mathscr{S}} = -\mathbf{P}\langle \boldsymbol{F} \rangle^{\top} \boldsymbol{b}^{\mathscr{S}}$; therefore

$$\boldsymbol{e}^{\mathscr{S}} = -\langle \boldsymbol{F} \rangle^{\top} \boldsymbol{b}^{\mathscr{S}}, \tag{12–9}$$

a result that should be compared to the bulk relation $\boldsymbol{e} = -\boldsymbol{F}^{\top}\boldsymbol{b}$ (cf. (5–20)).

The two equivalent forms, $\boldsymbol{b}^{\mathscr{S}} \cdot \langle \dot{\boldsymbol{y}} \rangle$ and $\boldsymbol{b}^{\mathscr{S}} \cdot \overset{\square}{\boldsymbol{y}} + \boldsymbol{e}^{\mathscr{S}} \cdot \mathbf{m} V$, for the inertial working, underline the essential difference between the standard and configurational points of view. In the former the relevant velocity for the working is the motion velocity $\dot{\boldsymbol{y}}$, and since $\dot{\boldsymbol{y}}$ suffers a jump discontinuity across the interface, it seems appropriate that its average value $\langle \dot{\boldsymbol{y}} \rangle$ represent the work-conjugate velocity for $\boldsymbol{b}^{\mathscr{S}}$. On the other hand, if the interface is viewed as having no intrinsic material identity, then the exchange of material across the moving interface should be taken into account; this is the configurational view. Here the appropriate work-conjugate velocities are $V\mathbf{m}$ for $\boldsymbol{e}^{\mathscr{S}}$ and $\overset{\square}{\boldsymbol{y}}$ for $\boldsymbol{b}^{\mathscr{S}}$.

Next, by (10–2), (12–2), and (12–5),

$$\begin{aligned}\mathfrak{m}\cdot\langle \boldsymbol{F}\rangle^{\top}\boldsymbol{b}^{\mathscr{S}} &= \rho\langle \boldsymbol{F}\rangle\mathfrak{m}\cdot[\dot{\boldsymbol{y}}]V = -\rho\langle \boldsymbol{F}\mathfrak{m}\rangle\cdot[\boldsymbol{F}\mathfrak{m}]V^2\\ &= -\frac{1}{2}\,\rho[|\boldsymbol{F}\mathfrak{m}|^2]V^2 = -[k_{\mathrm{rel}}].\end{aligned}$$

Thus, by (12–9),

$$\boldsymbol{e}^{\mathscr{S}}\cdot\mathfrak{m} = [k_{\mathrm{rel}}]. \tag{12–10}$$

The results established thus far assert the equivalence of:

(i) $\boldsymbol{b}^{\mathscr{S}}$ and the (interfacial) release rate for momentum;

(ii) the total inertial working $\boldsymbol{b}^{\mathscr{S}}\cdot\langle\dot{\boldsymbol{y}}\rangle = \boldsymbol{b}^{\mathscr{S}}\cdot\overset{\square}{\boldsymbol{y}} + \boldsymbol{e}^{\mathscr{S}}\cdot\mathfrak{m}V$ and the release rate for kinetic energy;

(iii) the configurational inertial working $\boldsymbol{e}^{\mathscr{S}}\cdot\mathfrak{m}V$ and the release rate for the kinetic energy measured relative to the interface.

An alternative method of determining the inertial body force $\boldsymbol{e}^{\mathscr{S}}$ is to use, in place of (12–6), a hypothesis analogous to (12–4) for the pseudomomentum $\mathfrak{p} = -\rho\boldsymbol{F}^{\top}\dot{\boldsymbol{y}}$ (cf. (7–10)); guided by (7–14), such a hypothesis has the form

$$\int_P \boldsymbol{e}\,dv + \int_{\mathscr{S}\cap P}\boldsymbol{e}^{\mathscr{S}}\,da = -\frac{d}{dt}\left\{\int_P \mathfrak{p}\,dv\right\} + \int_{\partial P}\mathfrak{p}U\,da - \int_{\partial P} k\boldsymbol{n}\,da + \int_P \frac{1}{2}\,\dot{\boldsymbol{y}}^2\nabla\rho\,dv \tag{12–11}$$

and yields, as in the derivation of (12–5), the identity[2]

$$\boldsymbol{e}^{\mathscr{S}} = [\mathfrak{p}]V - [k]\mathfrak{m}. \tag{12–12}$$

By (11–13), $V\mathfrak{m}\cdot[\boldsymbol{F}^{\top}\dot{\boldsymbol{y}}] = [V\boldsymbol{F}\mathfrak{m}\cdot\dot{\boldsymbol{y}}] = [\dot{\boldsymbol{y}}\cdot(\overset{\square}{\boldsymbol{y}} - \dot{\boldsymbol{y}})]$, and the relation (12–10) for $\boldsymbol{e}^{\mathscr{S}}\cdot\mathfrak{m}$ is a consequence of (12–12) (cf. the sentence following (12–6)). The relations (12–9) and (12–12) are equivalent.

c. Standard and configurational balances with inertia

In view of (12–5), the standard force balance (11–8a) reduces to the **momentum balance**

$$[\boldsymbol{S}]\mathfrak{m} = -\rho[\dot{\boldsymbol{y}}]V, \tag{12–13}$$

while (6–9), (11–12), (12–2), and (12–10) yield the **normal configurational balance**

$$\mathfrak{m}\cdot[\Psi\mathbf{1} - \boldsymbol{F}^{\top}\boldsymbol{S}]\mathfrak{m} + \frac{1}{2}\,\rho[|\boldsymbol{F}\mathfrak{m}|^2]V^2 + g^{\mathscr{S}} = 0. \tag{12–14}$$

Further, by (11–8a) and (12–9), the term containing V^2, which is $\boldsymbol{e}^{\mathscr{S}}\cdot\mathfrak{m}$, may be written as $\langle\boldsymbol{F}\rangle\mathfrak{m}\cdot[\boldsymbol{S}]\mathfrak{m}$; thus the identity $[\varphi\psi] = \langle\varphi\rangle[\psi] + [\varphi]\langle\psi\rangle$ applied to

[2] An essentially equivalent relation, stated without proof for homogeneous elastic materials, is given by Maugin [1995, eq. (6–17a)].

$\mathbf{m}\cdot[\boldsymbol{F}^{\top}\boldsymbol{S}]\mathbf{m}=[\boldsymbol{F}\mathbf{m}\cdot\boldsymbol{S}\mathbf{m}]$ reduces (12–14) to

$$[\Psi]-\langle\boldsymbol{S}\rangle\mathbf{m}\cdot[\boldsymbol{F}]\mathbf{m}+g^{\mathscr{S}}=0. \tag{12–15}$$

This balance, a counterpart of the bulk relation (6–10), shows the internal interface force $g^{\mathscr{S}}$ to be affected by material variations in the free energy and deformation gradient across the interface via the terms $[\Psi]$ and $[\boldsymbol{F}]$.

d. Constitutive equation for the interface

The fields $\boldsymbol{S}$ and Ψ would generally be given by constitutive equations defining the material properties away from the interface. On the other hand, the quantities $g^{\mathscr{S}}$ and V, which characterize the mechanics and kinematics of the interface, require consitutive specification, because without further restriction the internal dissipation inequality (11–21) may be violated. The basic theory for the interface is therefore closed by relating $g^{\mathscr{S}}$ and V constitutively in a manner compatible with this inequality.

I allow the interface force $g^{\mathscr{S}}$ to depend on the kinetics and orientation of the interface through dependencies on V and $\mathbf{m}$, and on the deformation through dependencies on the limiting values of $\boldsymbol{F}$. I therefore consider **constitutive equations**

$$g^{\mathscr{S}}=\Phi(V,\mathbf{m},\boldsymbol{F}^{+},\boldsymbol{F}^{-}), \tag{12–16}$$

which reduce, via (11–21), to the specific form

$$g^{\mathscr{S}}=-bV \tag{12–17}$$

with $b=b(V,\mathbf{m},\boldsymbol{F}^{+},\boldsymbol{F}^{-})\geq 0$, the **kinetic modulus**, a constitutive quantity.

The relation (12–17) is the most general *smooth* constitutive equation of the form (12–16) that is consistent with the dissipation inequality (11–21) (cf. (9–25)). One might also consider a frictional-type constitutive assumption

$$V=0 \text{ for } |g^{\mathscr{S}}|<L,\qquad V=-\lambda\,\operatorname{sign}(g^{\mathscr{S}}) \text{ for } |g^{\mathscr{S}}|\geq L, \tag{12–18}$$

with $\lambda=\lambda(\mathbf{m},\boldsymbol{F}^{+},\boldsymbol{F}^{-})>0$. By (12–15) this allows for a dependence of λ on $g^{\mathscr{S}}$, granted bulk constitutive equations giving Ψ and $\boldsymbol{S}$ when $\boldsymbol{F}$ is known.

For specificity, I supplement the constitutive relation (12–17) for the interface with hyperelastic constitutive equations for the bulk phases α and β (cf. Section 9a):

$$\Psi=\Psi_{\alpha}(\boldsymbol{F}),\qquad \boldsymbol{S}=\boldsymbol{S}_{\alpha}(\boldsymbol{F})=\partial_{\boldsymbol{F}}\Psi_{\alpha}(\boldsymbol{F})\qquad \text{in phase } \alpha, \tag{12–19a}$$

$$\Psi=\Psi_{\beta}(\boldsymbol{F}),\qquad \boldsymbol{S}=\boldsymbol{S}_{\beta}(\boldsymbol{F})=\partial_{\boldsymbol{F}}\Psi_{\beta}(\boldsymbol{F})\qquad \text{in phase } \beta, \tag{12–19b}$$

where the subscripts α and β rather than the symbol $\hat{}$ are used to designate the associated constitutive functions. The local form of the second law in bulk is the inequality (6–11), and its satisfaction is ensured by (12–19).

e. Summary of basic equations

Assume that the underlying observer, the body force $\boldsymbol{b}$, and the interface forces $\boldsymbol{b}^{\mathscr{S}}$ and $\boldsymbol{e}^{\mathscr{S}}$ are inertial. The basic equations for the *bulk material* then consist of the momentum balance

$$\operatorname{Div} \boldsymbol{S} = \rho \ddot{\boldsymbol{y}} \tag{12–20}$$

supplemented by the constitutive equations (12–19a).

The basic equations for the *interface* are the **compatibility conditions**

$$[\dot{\boldsymbol{y}}] = -V[\boldsymbol{F}]\mathbf{m}, \tag{12–21a}$$

$$[\boldsymbol{F}]\mathbf{P} = \mathbf{0}, \tag{12–21b}$$

the **momentum balance**

$$[\boldsymbol{S}]\mathbf{m} = -\rho[\dot{\boldsymbol{y}}]V, \tag{12–22}$$

and the **normal configurational balance**

$$\mathbf{m} \cdot [\Psi \mathbf{1} - \boldsymbol{F}^{\top}\boldsymbol{S}]\mathbf{m} + [k_{\text{rel}}] = bV, \tag{12–23}$$

with $b = b(V, \mathbf{m}, \boldsymbol{F}^{+}, \boldsymbol{F}^{-}) \geq 0$, and $[k_{\text{rel}}] = \frac{1}{2}\rho[|\boldsymbol{F}\mathbf{m}|^2]V^2$ the jump in relative kinetic energy (cf. (12–10)). The balance (12–23) may be written in the alternative form

$$[\Psi] - \langle \boldsymbol{S} \rangle \mathbf{m} \cdot [\boldsymbol{F}]\mathbf{m} = bV. \tag{12–24}$$

(The relations (12–14), (12–15), and (12–17) are used in the derivation of (12–23) and (12–24).)

Heidug and Lehner,[3] Truskinovsky,[4] and Abeyaratne and Knowles[5] use the inequality $([\Psi] - \langle \boldsymbol{S} \rangle \mathbf{m} \cdot [\boldsymbol{F}]\mathbf{m})V \geq 0$, derived from the second law, to motivate a constitutive relation of the form (12–24) for the driving traction f *defined* by $f = -[\Psi] + \langle \boldsymbol{S} \rangle \mathbf{m} \cdot [\boldsymbol{F}]\mathbf{m}$. Their argument does not involve a configurational force balance and is hence simpler than that given here, which is due to Gurtin.[6] It does, however, involve postulating a constitutive relation for $[\Psi] - \langle \boldsymbol{S} \rangle \mathbf{m} \cdot [\boldsymbol{F}]\mathbf{m}$, which seems superfluous, since Ψ and $\boldsymbol{S}$ would typically be prescribed as functions of $\boldsymbol{F}$ through constitutive relations. In the development described here the configurational force balance provides an additional field $\boldsymbol{g}^{\mathscr{S}}$ with normal component available for constitutive prescription, thereby allowing for (12–24).

Given tensors $\boldsymbol{G}$ and $\boldsymbol{H}$,

$$\boldsymbol{H}\mathbf{P} = \mathbf{0} \quad \Rightarrow \quad \boldsymbol{G}\mathbf{m} \cdot \boldsymbol{H}\mathbf{m} = \boldsymbol{G} \cdot \boldsymbol{H}.$$

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[3] [1985].
[4] [1987, 1991].
[5] [1990, 1991].
[6] [1995] (cf. Gurtin and Struthers [1990]).

Indeed, assume that $\boldsymbol{H}\mathbf{P} = \mathbf{0}$. Then $\boldsymbol{G} = \boldsymbol{G}\mathbf{P} + \boldsymbol{G}(\mathbf{m} \otimes \mathbf{m})$, while $\boldsymbol{H} = \boldsymbol{H}(\mathbf{m} \otimes \mathbf{m})$, so that, because $(\boldsymbol{a} \otimes \boldsymbol{b}) \cdot (\boldsymbol{c} \otimes \boldsymbol{d}) = \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{d}$ and $\mathbf{P}(\mathbf{m} \otimes \mathbf{m}) = (\mathbf{P}\mathbf{m}) \otimes \mathbf{m} = \mathbf{0}$,

$$\begin{aligned}\boldsymbol{G} \cdot \boldsymbol{H} &= (\boldsymbol{G}\mathbf{P}) \cdot \{\boldsymbol{H}(\mathbf{m} \otimes \mathbf{m})\} + \{\boldsymbol{G}(\mathbf{m} \otimes \mathbf{m})\} \cdot \{\boldsymbol{H}(\mathbf{m} \otimes \mathbf{m})\} \\ &= \{\boldsymbol{G}\mathbf{P}(\mathbf{m} \otimes \mathbf{m})\} \cdot \boldsymbol{H} + \{(\boldsymbol{G}\mathbf{m}) \otimes \mathbf{m}\} \cdot \{(\boldsymbol{H}\mathbf{m}) \otimes \mathbf{m}\} = \boldsymbol{G}\mathbf{m} \cdot \boldsymbol{H}\mathbf{m}.\end{aligned}$$

Thus (12–24) may be written as

$$[\Psi] - \langle \boldsymbol{S} \rangle \cdot [\boldsymbol{F}] = bV. \tag{12–25}$$

f. Global energy inequality. Lyapunov relations

Assume that the body B is bounded, that $\mathscr{S} = \mathscr{S}(t)$ is a closed surface contained in the interior of B, and that the external forces are inertial. Then the second law (11–19), with right side in the intrinsic form (11–14) and (12–7) used to replace the inertial working by the production of kinetic energy, yields, for $P = B$,

$$\frac{d}{dt}\left\{\int_B (\Psi + k)\, dv\right\} - \int_{\partial B} \boldsymbol{S}\boldsymbol{n} \cdot \dot{\boldsymbol{y}}\, da = -\mathscr{D}(B) \leq 0. \tag{12–26}$$

Here, by (12–19a),

$$\Psi = \Psi_\alpha(\boldsymbol{F}) \text{ in } B_\alpha, \qquad \Psi = \Psi_\beta(\boldsymbol{F}) \text{ in } B_\beta, \tag{12–27}$$

where $B_\alpha = B_\alpha(t)$ and $B_\beta = B_\beta(t)$ are the bulk regions occupied by phases α and β; and $\mathscr{D}(B)$ is the dissipation (11–20) modified by (9–10); viz.,

$$\mathscr{D}(B) = -\int_{\mathscr{S}} g^{\mathscr{S}} V\, da = \int_{\mathscr{S}} bV^2 da, \tag{12–28}$$

with $b = b(V, \boldsymbol{m}, \boldsymbol{F}^+, \boldsymbol{F}^-) \geq 0$.

A consequence of (12–26) are Lyapunov relations of the form (7–17) (for a fixed boundary) and (7–19) (for a boundary under constant dead loads); in either case the inequality "≤ 0" may be replaced by the stronger assertion "$= -\mathscr{D}(B) \leq 0$."

Part C

An Equivalent Formulation of the Theory. Infinitesimal Deformations

In theories involving infinitesimal deformations it is customary to take the displacement $\boldsymbol{u}(\boldsymbol{X}, t) = \boldsymbol{y}(\boldsymbol{X}, t) - \boldsymbol{X}$ rather than the motion $\boldsymbol{y}(\boldsymbol{X}, t)$ as the basic kinematical field. With this in mind, I now reformulate the theory taking the displacement as the basic kinematical variable associated with the standard force system. This *displacement-based formulation* involves no approximations and is consistent with the *motion-based formulation* discussed in Parts A and B; its importance lies in its applicability to infinitesimal deformations. To simplify comparisons of the two formulations, the following abbreviations are convenient:

$$\boldsymbol{mbf} = \text{motion-based formulation},$$
$$\boldsymbol{dbf} = \text{displacement-based formulation}.$$

Because the discussion follows that of mbf, I will omit or simply sketch the most arguments.

CHAPTER 13

Formulation within a Classical Context

a. Background. Reasons for an alternative formulation in terms of displacements

Given a motion $\boldsymbol{y}$, the corresponding **displacement field** $\boldsymbol{u}$ is defined by

$$\boldsymbol{u}(\boldsymbol{X}, t) = \boldsymbol{y}(\boldsymbol{X}, t) - \boldsymbol{X} \tag{13–1}$$

and yields

$$\dot{\boldsymbol{u}} = \dot{\boldsymbol{y}}, \tag{13–2a}$$

$$\nabla \boldsymbol{u} = \boldsymbol{F} - \mathbf{1}. \tag{13–2b}$$

The problem with the application of mbf to small displacements arises from the approximation of the gradient $\boldsymbol{F} = \nabla \boldsymbol{y} = \mathbf{1} + \nabla \boldsymbol{u}$, because this approximation involves a term of $O(1)$ plus a term of $O(\delta)$ (assuming that $\boldsymbol{u}$ and its derivatives are $O(\delta)$, with δ small). That a theory based on $\boldsymbol{u}$ rather than $\boldsymbol{y}$ is more appropriate may be seen when comparing the formulas for time derivatives following the evolution of the boundary of a migrating control volume $P = P(t)$:

$$\mathring{\boldsymbol{u}} = \dot{\boldsymbol{u}} + (\nabla \boldsymbol{u})\boldsymbol{q}, \tag{13–3a}$$

$$\mathring{\boldsymbol{y}} = \dot{\boldsymbol{y}} + \boldsymbol{F}\boldsymbol{q}. \tag{13–3b}$$

Assuming that migrations of ∂P are associated with velocities $\boldsymbol{q}$ of $O(1)$, the terms in $\mathring{\boldsymbol{u}}$ are each $O(\delta)$; but

$$\mathring{\boldsymbol{y}} = \boldsymbol{q} + \mathring{\boldsymbol{u}},$$

so that $\mathring{\boldsymbol{y}}$ involves a term of $O(1)$ plus a term of $O(\delta)$, with lowest-order approximation $\mathring{\boldsymbol{y}} = \boldsymbol{q}$ yielding meaningless results.

Here I reformulate the general theory using $\boldsymbol{u}$ rather than $\boldsymbol{y}$ as the field that characterizes the kinematics of deformation.

b. Finite deformations. Modified Eshelby relation

Let $P = P(t)$ be a migrating control volume with $\boldsymbol{q}$ a velocity field for ∂P and $\mathring{\boldsymbol{u}}$ the associated time derivative of $\boldsymbol{u}$ following the evolution of ∂P. I now consider $\mathring{\boldsymbol{u}}$ and $\dot{\boldsymbol{u}}$ as work-conjugate velocities for $\boldsymbol{Sn}$ and $\boldsymbol{b}$, and therefore write the **working** in the form

$$W(P) = \int_{\partial P} (\boldsymbol{Cn}\cdot\boldsymbol{q} + \boldsymbol{Sn}\cdot\mathring{\boldsymbol{u}})\, da + \int_P \boldsymbol{b}\cdot\dot{\boldsymbol{u}}\, dv. \tag{13–4}$$

This expression should be compared with (5–5), the definition of $W(P)$ within the mbf. As we shall see, these apparently contradictory definitions are, in fact, consistent.

Consider the changes in spatial and material observer defined by (2–7) and (2–9). Then $\dot{\boldsymbol{u}}$ and $\mathring{\boldsymbol{u}}$ are invariant under material changes and transform according to

$$\dot{\boldsymbol{u}} \to \dot{\boldsymbol{u}} + \boldsymbol{w} + \boldsymbol{\omega}\times(\boldsymbol{y}-\boldsymbol{o}), \qquad \mathring{\boldsymbol{u}} \to \mathring{\boldsymbol{u}} + \boldsymbol{w} + \boldsymbol{\omega}\times(\boldsymbol{y}-\boldsymbol{o}) \tag{13–5}$$

under spatial changes. Thus, allowing material and spatial observers to act concurrently, a procedure that we shall henceforth follow, this yields the following expression for the working recorded by the new observers:

$$\begin{aligned} W(P) = &\int_{\partial P} \boldsymbol{Cn}\cdot(\boldsymbol{q}+\boldsymbol{a})\, da + \int_P (\boldsymbol{g}+\boldsymbol{e})\cdot\boldsymbol{a}\, dv \\ &+ \int_{\partial P} \boldsymbol{Sn}\cdot\big(\mathring{\boldsymbol{u}} + \boldsymbol{w} + \boldsymbol{\omega}\times(\boldsymbol{y}-\boldsymbol{o})\big)\, da \\ &+ \int_P \boldsymbol{b}\cdot(\dot{\boldsymbol{u}} + \boldsymbol{w} + \boldsymbol{\omega}\times(\boldsymbol{y}-\boldsymbol{o}))\, dv. \end{aligned} \tag{13–6}$$

Invariance under changes in observer requires the coincidence of (13–4) and (13–6); as before, this yields the standard force and moment balances (3–5) and the configurational balance (5–8); or equivalently,

$$\operatorname{Div}\boldsymbol{S} + \boldsymbol{b} = \boldsymbol{0}, \tag{13–7a}$$

$$\boldsymbol{S}(\boldsymbol{1}+\nabla\boldsymbol{u})^\top = (\boldsymbol{1}+\nabla\boldsymbol{u})\boldsymbol{S}^\top, \tag{13–7b}$$

$$\operatorname{Div}\boldsymbol{C} + \boldsymbol{g} + \boldsymbol{e} = \boldsymbol{0}. \tag{13–7c}$$

Next, arguing as in Section 5d, the requirement that the working (13–4) be independent of the choice of velocity field used to characterize the evolution of ∂P yields the relation

$$\boldsymbol{C} = \pi\boldsymbol{1} - \nabla\boldsymbol{u}^\top\boldsymbol{S} \tag{13–8}$$

with $\nabla\boldsymbol{u}^\top = (\nabla\boldsymbol{u})^\top$.

Consider a time-dependent change in reference configuration as discussed in (4–15)–(4–18), let $P(t)$ be the migrating control volume defined in (4–18), let $\boldsymbol{q}(\boldsymbol{X}, t)$ denote the velocity field for $\partial P(t)$ defined in (4–17), and let $\mathring{\boldsymbol{u}} = \dot{\boldsymbol{u}} + (\nabla \boldsymbol{u})\boldsymbol{q}$. The argument leading to (5–18) yields

$$W(P(t)) = \int_{\partial P(t)} \boldsymbol{C}\boldsymbol{n} \cdot \boldsymbol{q}\, da + \int_{\partial P(t)} \boldsymbol{S}\boldsymbol{n} \cdot \mathring{\boldsymbol{u}}\, da + \int_{P(t)} \boldsymbol{e} \cdot \boldsymbol{q}\, dv + \int_{P(t)} \boldsymbol{b} \cdot \mathring{\boldsymbol{u}}\, dv, \quad (13\text{–}9)$$

and because this expression should coincide with (13–4),

$$\int_P \boldsymbol{e} \cdot \boldsymbol{q}\, dv + \int_P \boldsymbol{b} \cdot \mathring{\boldsymbol{u}}\, dv = \int_P \boldsymbol{b} \cdot \dot{\boldsymbol{u}}\, dv, \quad (13\text{–}10)$$

which results in the **external** and **internal force relations**

$$\boldsymbol{e} = -\nabla \boldsymbol{u}^\top \boldsymbol{b}, \quad (13\text{–}11a)$$

$$\boldsymbol{g} = -\nabla \pi + \boldsymbol{S}{:}\nabla\nabla \boldsymbol{u}. \quad (13\text{–}11b)$$

The second law is presumed to be (6–4) with $W(P)$ given by (13–4):

$$\frac{d}{dt}\left\{\int_P \Psi\, dv\right\} \leq \int_{\partial P} (\boldsymbol{C}\boldsymbol{n} \cdot \boldsymbol{q} + \boldsymbol{S}\boldsymbol{n} \cdot \mathring{\boldsymbol{u}})\, da + \int_P \boldsymbol{b} \cdot \mathring{\boldsymbol{u}}\, dv. \quad (13\text{–}12)$$

The steps leading to (6–8) again yield $\pi = \Psi$, and this leads to the **modified Eshelby relation**

$$\boldsymbol{C} = \Psi \boldsymbol{1} - \nabla \boldsymbol{u}^\top \boldsymbol{S} \quad (13\text{–}13)$$

in conjunction with

$$\boldsymbol{g} = -\nabla \Psi + \boldsymbol{S}{:}\nabla\nabla \boldsymbol{u}, \quad (13\text{–}14)$$

a relation not different from (5–21), as $\nabla\nabla \boldsymbol{u} = \nabla \boldsymbol{F}$.

A comparison of (13–13) and (6–9) shows that the *configurational stress is different in the two formulations*. Specifically, if (13–11a), (13–13), and (13–14) are compared with (5–20), (6–9), and (6–10), the following relationship between mbf and dbf emerges:

$$\boldsymbol{C}(\text{mbf}) = \boldsymbol{C}(\text{dbf}) - \boldsymbol{S},$$
$$\boldsymbol{e}(\text{mbf}) = \boldsymbol{e}(\text{dbf}) - \boldsymbol{b},$$
$$\boldsymbol{g}(\text{mbf}) = \boldsymbol{g}(\text{dbf}).$$

A somewhat related result is that, under the transformation

$$\boldsymbol{C} \to \boldsymbol{C} - \boldsymbol{S}, \qquad \boldsymbol{e} \to \boldsymbol{e} - \boldsymbol{b}, \qquad \boldsymbol{g} \to \boldsymbol{g}, \quad (13\text{–}15)$$

the configurational force balance (5–10) is invariant, while the working transforms according to

$$\int_{\partial P} (\boldsymbol{C}\boldsymbol{n} \cdot \boldsymbol{q} + \boldsymbol{S}\boldsymbol{n} \cdot \mathring{\boldsymbol{y}})\, da + \int_P \boldsymbol{b} \cdot \dot{\boldsymbol{y}}\, dv + \int_{\partial P} (\boldsymbol{C}\boldsymbol{n} \cdot \boldsymbol{q} + \boldsymbol{S}\boldsymbol{n} \cdot \mathring{\boldsymbol{u}})\, da + \int_P \boldsymbol{b} \cdot \dot{\boldsymbol{u}}\, dv, \quad (13\text{–}16)$$

a transformation that demonstrates the consistency of the two formulations.

The term $\boldsymbol{C}-\boldsymbol{S}$ may, at first sight, seem incorrect because, at a point, $\boldsymbol{C}$ maps material vectors into material vectors, while $\boldsymbol{S}$ maps material vectors into spatial vectors. This inconsistency follows from the formula $\boldsymbol{F}=\mathbf{1}+\nabla\boldsymbol{u}$, because the identity $\mathbf{1}$ should there be considered a mapping of material vectors into spatial vectors. Granted this, $\boldsymbol{C}-\boldsymbol{S}$ might more appropriately be written as $\boldsymbol{C}-\mathbf{1}^{\top}\boldsymbol{S}$.

Finally, let $\boldsymbol{U}(\boldsymbol{x},t)=\boldsymbol{Y}(\boldsymbol{x},t)-\boldsymbol{x}$ denote the displacement corresponding to the inverse motion $\boldsymbol{Y}$ (cf. (2–4)), and let $\boldsymbol{U}'(\boldsymbol{x},t)=\partial\boldsymbol{U}(\boldsymbol{x},t)/\partial t=\boldsymbol{Y}'(\boldsymbol{x},t)$. Then, when the deformed control volume $\bar{P}$ corresponding to $P=P(t)$ is stationary,

$$W(P)=\int_{\partial P}(\boldsymbol{C}-\boldsymbol{S})n\cdot U'\,da+\int_{P}(\boldsymbol{e}-\boldsymbol{b})\cdot U'\,dv,$$

which is a counterpart of (5–23).

c. Infinitesimal deformations

I formulate the theory within an invariant thermodynamic framework in which the infinitesimal nature of the deformation is characterized by a restriction to infinitesimal changes in spatial observer. The working and the inequality representing the second law are then identical to their counterparts (13–4) and (13–12) in the finite theory; no approximation is needed. This is the chief advantage of dbf.

Infinitesimal changes in spatial observer transform the displacement by an additive infinitesimal rigid displacement at each time t and hence yield, at each t, transformation laws of the form

$$\dot{\boldsymbol{u}}(\boldsymbol{X},t)\to\dot{\boldsymbol{u}}(\boldsymbol{X},t)+\boldsymbol{w}+\boldsymbol{\omega}\times(\boldsymbol{X}-\boldsymbol{o}), \tag{13–17a}$$

$$\mathring{\boldsymbol{u}}(\boldsymbol{X},t)\to\mathring{\boldsymbol{u}}(\boldsymbol{X},t)+\boldsymbol{w}+\boldsymbol{\omega}\times(\boldsymbol{X}-\boldsymbol{o}), \tag{13–17b}$$

with $\boldsymbol{w}$ and $\boldsymbol{\omega}$ spatial vectors. These transformations differ from the transformations (13–5) of the finite theory, as the relative spin is $\boldsymbol{\omega}\times(\boldsymbol{X}-\boldsymbol{o})$ in (13–17) as opposed to $\boldsymbol{\omega}\times(\boldsymbol{y}-\boldsymbol{o})$ in (13–5).

Changes in material observer are, as in the finite theory, defined by (2–9).

In a theory of infinitesimal deformations the deformed and undeformed bodies are essentially indistinguishable, and it is not customary to account separately for material points and the places they occupy under a deformation. I shall follow this convention, but I shall continue to distinguish between the underlying spaces $\mathscr{E}_{\text{space}}$ and $\mathscr{E}_{\text{matter}}$ via separate classes of observers.

The working $W(P)$ is given by (13–4), but (13–6), which represents its form under a change in observer, now has $\boldsymbol{\omega}\times(\boldsymbol{y}-\boldsymbol{o})$ replaced by $\boldsymbol{\omega}\times(\boldsymbol{X}-\boldsymbol{o})$. Invariance under observer changes thus yields no alteration in the balances for standard and configurational forces, which remain (13–7a,c), but the relation (13–7b) for balance of moments is now replaced by the classical relation expression the symmetry of the standard stress. The basic balances are therefore

$$\operatorname{Div}\boldsymbol{S}+\boldsymbol{b}=0, \tag{13–18a}$$

$$\boldsymbol{S}=\boldsymbol{S}^{\top}, \tag{13–18b}$$

$$\mathrm{Div}\,\boldsymbol{C} + \boldsymbol{g} + \boldsymbol{e} = 0. \tag{13–18c}$$

Finally, the remaining arguments are unchanged, so that

$$\boldsymbol{C} = \Psi\mathbf{1} - \nabla\boldsymbol{u}^{\top}\boldsymbol{S}, \tag{13–19a}$$

$$\boldsymbol{e} = -\nabla\boldsymbol{u}^{\top}\boldsymbol{b}, \tag{13–19b}$$

$$\boldsymbol{g} = -\nabla\Psi + \boldsymbol{S}{:}\nabla\nabla\boldsymbol{u}. \tag{13–19c}$$

The basic field equations of the finite theory and those of the infinitesimal theory therefore differ only in the standard moment balance.

CHAPTER 14

Coherent Phase Interfaces

a. General theory

The ensuing discussion is valid for both finite and infinitesimal deformations. The compatibility conditions (10–2) take the form

$$[\dot{\boldsymbol{u}}] = -V[\nabla \boldsymbol{u}]\mathbf{m}, \tag{14–1a}$$

$$[\nabla \boldsymbol{u}]\mathbf{P} = \mathbf{0}, \tag{14–1b}$$

when expressed in terms of displacement. The working now has the form

$$W(P) = \int_{\partial P} (\boldsymbol{C}\boldsymbol{n}\cdot\boldsymbol{q} + \boldsymbol{S}\boldsymbol{n}\cdot\mathring{\boldsymbol{u}})\,da + \int_P \boldsymbol{b}\cdot\dot{\boldsymbol{u}}\,dv + \int_{\mathscr{S}\cap P} (\boldsymbol{e}^{\mathscr{S}}\cdot\boldsymbol{v} + \boldsymbol{b}^{\mathscr{S}}\cdot\overset{\square}{\boldsymbol{u}})\,da \tag{14–2}$$

with $\boldsymbol{q}$ and $\boldsymbol{v}$ velocity fields for ∂P and $\mathscr{S}$, and with $\mathring{\boldsymbol{u}}$ and $\overset{\square}{\boldsymbol{u}}$, respectively, corresponding time derivatives of $\boldsymbol{u}$ following the evolutions of ∂P and $\mathscr{S}$. (The field $\overset{\square}{\boldsymbol{u}}$ is defined by (10–4) with $\boldsymbol{y}$ replaced by $\boldsymbol{u}$ and transforms in the same manner as do $\dot{\boldsymbol{u}}$ and $\mathring{\boldsymbol{u}}$; cf. (10–4) and the paragraph containing (13–5).) The second law remains (6–4), but with $W(P)$ given by (14–2).

The development follows that of the theory described in Part B. The final results, which represent basic equations for the interface, consist of the compatibility conditions (14–1), the momentum balance

$$[\boldsymbol{S}]\mathbf{m} = -\rho[\dot{\boldsymbol{u}}]V, \tag{14–3}$$

and the normal configurational balance

$$\mathbf{m}\cdot[\Psi\mathbf{1} - \nabla\boldsymbol{u}^{\top}\boldsymbol{S}]\mathbf{m} + g^{\mathscr{S}} = -\frac{1}{2}\,\rho[|(\nabla\boldsymbol{u})\mathbf{m}|^2]V^2, \tag{14–4}$$

or equivalently,

$$[\Psi] - \langle \boldsymbol{S} \rangle \mathbf{m} \cdot [\nabla \boldsymbol{u}] \mathbf{m} + g^{\mathscr{S}} = 0. \tag{14–5}$$

There are supplemented by the internal dissipation inequality

$$g^{\mathscr{S}} V \leq 0, \tag{14–6}$$

which leads to constitutive equations of the form

$$g^{\mathscr{S}} = -bV \tag{14–7}$$

with $b = b(V, \mathbf{m}, \nabla \boldsymbol{u}^+, \nabla \boldsymbol{u}^-) \geq 0$. Granted (14–7), (14–5) may be written as

$$[\Psi] - \langle \boldsymbol{S} \rangle \mathbf{m} \cdot [\nabla \boldsymbol{u}] \mathbf{m} = bV, \tag{14–8}$$

or equivalently, arguing as in the verification of (12–25),

$$[\Psi] - \langle \boldsymbol{S} \rangle \cdot [\nabla \boldsymbol{u}] = bV. \tag{14–9}$$

b. Infinitesimal theory with linear stress-strain relations in bulk

Of importance in the infinitesimal theory is the **strain tensor**

$$\boldsymbol{E} = \frac{1}{2} (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top), \tag{14–10}$$

which is unaffected by infinitesimal rigid displacements. Since $\boldsymbol{S}$ is symmetric, $\langle \boldsymbol{S} \rangle \cdot [\nabla \boldsymbol{u}] = \langle \boldsymbol{S} \rangle \cdot [\boldsymbol{E}]$, and the interface condition (14–9) may be written in the form

$$[\Psi] - \langle \boldsymbol{S} \rangle \cdot [\boldsymbol{E}] = bV. \tag{14–11}$$

Assume that the bulk material is elastic, and, consistent with the assumption of infinitesimal deformations, consider constitutive equations of the form

$$\Psi = \frac{1}{2} \boldsymbol{E} \cdot \mathscr{L}_\alpha \boldsymbol{E}, \qquad \boldsymbol{S} = \mathscr{L}_\alpha \boldsymbol{E} \qquad \text{in phase } \alpha, \tag{14–12a}$$

$$\Psi = \frac{1}{2} (\boldsymbol{E} - \boldsymbol{E}_0) \cdot \mathscr{L}_\beta (\boldsymbol{E} - \boldsymbol{E}_0), \quad \boldsymbol{S} = \mathscr{L}_\beta (\boldsymbol{E} - \boldsymbol{E}_0) \quad \text{in phase } \beta, \tag{14–12b}$$

with **elasticity tensors** $\mathscr{L}_\alpha$ and $\mathscr{L}_\beta$ symmetric linear transformations of symmetric tensors into symmetric tensors, with $\boldsymbol{E}_0$ a symmetric tensor, and with $\mathscr{L}_\alpha$, $\mathscr{L}_\beta$, and $\boldsymbol{E}_0$ assumed constant. Underlying (14–12) is the tacit assumption that the stress-free states of the two phases differ by the **misfit strain** $\boldsymbol{E}_0$.

The basic equations then consist of the standard balance

$$\operatorname{Div} \boldsymbol{S} = \rho \ddot{\boldsymbol{u}}, \tag{14–13}$$

the constitutive equations (14–12), and the strain-displacement relation (14–10) in bulk, and the standard and configurational balances (14–3) and (14–11) at the interface.

The driving force $[\Psi] - \langle \boldsymbol{S} \rangle \cdot [\boldsymbol{E}]$ simplifies considerably when the two phases have the same elasticity tensor:

$$\mathscr{L}_\alpha = \mathscr{L}_\beta =: \mathscr{L}. \tag{14–14}$$

Granted this, since $\mathscr{L}$ is symmetric, $\boldsymbol{A} \cdot \mathscr{L}\boldsymbol{B} = \boldsymbol{B} \cdot \mathscr{L}A$ for all symmetric tensors $\boldsymbol{A}$ and $\boldsymbol{B}$. Thus, writing

$$\boldsymbol{\Sigma} = \mathscr{L}\boldsymbol{E},$$

it follows that

$$\Psi = \frac{1}{2}\boldsymbol{\Sigma} \cdot \boldsymbol{E}, \qquad \boldsymbol{S} = \boldsymbol{\Sigma} \qquad \text{in phase } \alpha,$$

$$\Psi = \frac{1}{2}\boldsymbol{\Sigma} \cdot \boldsymbol{E} - \boldsymbol{E}_0 \cdot \mathscr{L}\boldsymbol{E} + \frac{1}{2}\boldsymbol{E}_0 \cdot \mathscr{L}\boldsymbol{E}_0, \quad \boldsymbol{S} = \boldsymbol{\Sigma} - \mathscr{L}\boldsymbol{E}_0 \quad \text{in phase } \beta,$$

and, because $[\varphi]$ denotes the interfacial limit of φ from phase β minus that from α,

$$[\Psi] = \frac{1}{2}[\boldsymbol{\Sigma} \cdot \boldsymbol{E}] - \boldsymbol{E}_0 \cdot \mathscr{L}\boldsymbol{E}^+ + \frac{1}{2}\boldsymbol{E}_0 \cdot \mathscr{L}\boldsymbol{E}_0,$$

$$\langle \boldsymbol{S} \rangle \cdot [\boldsymbol{E}] = \langle \boldsymbol{\Sigma} \rangle \cdot [\boldsymbol{E}] - \frac{1}{2}[\boldsymbol{E}] \cdot \mathscr{L}\boldsymbol{E}_0.$$

Thus, appealing to the identity $[\boldsymbol{\Sigma} \cdot \boldsymbol{E}] = \langle \boldsymbol{\Sigma} \rangle \cdot [\boldsymbol{E}] + [\boldsymbol{\Sigma}] \cdot \langle \boldsymbol{E} \rangle$,

$$[\Psi] - \langle \boldsymbol{S} \rangle \cdot [\boldsymbol{E}] = \frac{1}{2}[\boldsymbol{\Sigma}] \cdot \langle \boldsymbol{E} \rangle - \frac{1}{2}\langle \boldsymbol{\Sigma} \rangle \cdot [\boldsymbol{E}] - \left(\langle \boldsymbol{E} \rangle - \frac{1}{2}\boldsymbol{E}_0 \right) \cdot \mathscr{L}\boldsymbol{E}_0.$$

But, by the symmetry of $\mathscr{L}$, $\langle \boldsymbol{\Sigma} \rangle \cdot [\boldsymbol{E}] = (\mathscr{L}\langle \boldsymbol{E} \rangle) \cdot [\boldsymbol{E}] = (\mathscr{L}[\boldsymbol{E}]) \cdot \langle \boldsymbol{E} \rangle = [\boldsymbol{\Sigma}] \cdot \langle \boldsymbol{E} \rangle$. Thus $[\Psi] - \langle \boldsymbol{S} \rangle \cdot [\boldsymbol{E}] = \left(\frac{1}{2}\boldsymbol{E}_0 - \langle \boldsymbol{E} \rangle \right) \cdot \boldsymbol{S}_0$, with

$$\boldsymbol{S}_0 = \mathscr{L}\boldsymbol{E}_0$$

the **misfit stress**, and the interface condition (14–11) has the simple form

$$\left(\frac{1}{2}\boldsymbol{E}_0 - \langle \boldsymbol{E} \rangle \right) \cdot \boldsymbol{S}_0 = bV. \tag{14–15}$$

Note that the driving force vanishes when the average interfacial strain has a value midway between the stress-free strains of the two phases.

For a cubic material the misfit strain is a dilation, while the misfit stress is a pressure:

$$\boldsymbol{E}_0 = \frac{1}{3}e_0\mathbf{1}, \qquad \boldsymbol{S}_0 = ke_0\mathbf{1},$$

with compressibility k a scalar constant. In this case, writing

$$e = \mathrm{tr}\boldsymbol{E},$$

the interface condition (14–15) becomes

$$k\left(\frac{1}{2}e_0 - \langle e \rangle \right) e_0 = bV.$$

Part D

Evolving Interfaces Neglecting Bulk Behavior

There are situations of physical interest in which the motion of a phase interface or grain boundary may be considered independent of transport processes, inertia, and deformation.[1] Granted this, standard forces are irrelevant; the underlying balance is a configurational force balance for the interface. I now consider behavior of this type, beginning with a discussion of evolving surfaces. Because deformation is neglected, *all vectors are material*; thus, without danger of confusion, the space of material vectors is here identified with $\mathbf{R}^3$.

[1] Cf. the introduction of Taylor, Cahn, and Handwerker [1992].

CHAPTER 15

Evolving Surfaces

a. Surfaces

Let $\mathscr{S}$ be a smooth surface oriented by a choice of unit normal field $\mathbf{m}(\boldsymbol{X})$. The space of all vectors perpendicular to $\mathbf{m}(\boldsymbol{X})$ is then the **tangent space** at $\boldsymbol{X} \in \mathscr{S}$ and a vector field $\mathbf{t}$ on $\mathscr{S}$ is **tangential** if $\mathbf{t}(\boldsymbol{X})$ lies in the tangent space at every $\boldsymbol{X} \in \mathscr{S}$.

Given a subsurface $\mathscr{G}$ of $\mathscr{S}$, the **outward unit normal** $\mathbf{n}$ to $\partial\mathscr{G}$ is the principle normal to the *curve* ∂G, directed outward from $\mathscr{G}$; for any $\boldsymbol{X} \in \partial\mathscr{G}$, $\mathbf{n}(\boldsymbol{X})$ is normal to $\partial\mathscr{G}$ but *tangent* to $\mathscr{S}$.

a1. Background. Superficial stress

In continuum mechanics, tensors arise from the notion of stress. Let P be a subregion of a stressed body, $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{X})$ the outward unit normal to ∂P, and $\boldsymbol{C} = \boldsymbol{C}(\boldsymbol{X})$ the configurational stress at a point $\boldsymbol{X}$ on ∂P. Then the vector $\boldsymbol{Cn}$ represents the force, per unit area, exerted on P across ∂P by the material outside of P. Thus $\boldsymbol{C}$ maps vectors $\boldsymbol{n}$ into vectors $\boldsymbol{Cn}$, and is hence a linear transformation from $\mathbf{R}^3$ into $\mathbf{R}^3$.

Stresses may also act *within* the surface $\mathscr{S}$, a classical example being surface tension described by a scalar field $\sigma(\boldsymbol{X})$ on $\mathscr{S}$. Suppose that P intersects the surface $\mathscr{S}$, with $\mathscr{G} = P \cap \mathscr{S}$ the corresponding surface of intersection. Let $\mathbf{n} = \mathbf{n}(\boldsymbol{X})$ denote the outward unit normal to $\partial\mathscr{G}$ at $\boldsymbol{X}$. The vector $\sigma\mathbf{n}$ represents a force exerted on $\mathscr{G}$ across $\partial\mathscr{G}$ by the portion of $\mathscr{S}$ that lies outside of $\mathscr{G}$. Thus, as with the standard notion of stress, surface stress at a point is a linear mapping that assigns a force, here $\sigma\mathbf{n}$, to a unit normal, here $\mathbf{n}$, which, rather than being a vector in $\mathbf{R}^3$, is a tangent vector. The natural notion of stress for a surface is therefore a linear transformation $\mathbf{C} = \mathbf{C}(\boldsymbol{X})$ from the tangent space at $\boldsymbol{X}$ into $\mathbf{R}^3$; $\mathbf{C}$ maps tangent

vectors $\mathbf{n}$ into (ordinary) vectors $\mathbf{Cn}$ in $\mathbf{R}^3$. (When $\mathbf{C}$ represents surface tension, $\mathbf{Cn}$ is also a tangent vector, but there are more general situations in which $\mathbf{Cn}$ need not be tangent.)

a2. Superficial tensor fields

As alluded to earlier, tensors are generally linear transformations from $\mathbf{R}^3$ into $\mathbf{R}^3$, but of interest here are tensor fields $\mathbf{T}$ on $\mathscr{S}$ with the property that, at each $\boldsymbol{X}$ on $\mathscr{S}$, $\mathbf{T}(\boldsymbol{X})$ is a linear transformation from the tangent space at $\boldsymbol{X}$ into $\mathbf{R}^3$. These two notions of a tensor field are reconciled by extending $\mathbf{T}(\boldsymbol{X})$ to all of $\mathbf{R}^3$ with the requirement that $\mathbf{T}(\boldsymbol{X})$ annihilate vectors normal to $\mathscr{S}$. Precisely, a **superficial tensor field** on $\mathscr{S}$ is a tensor field $\mathbf{T}$ on $\mathscr{S}$ that satisfies

$$\mathbf{T}\mathbf{m} = \mathbf{0}. \tag{15–1}$$

An example of a superficial tensor field is the **projection**

$$\mathbf{P}(\boldsymbol{X}) = \mathbf{1} - \mathbf{m}(\boldsymbol{X}) \otimes \mathbf{m}(\boldsymbol{X}) \tag{15–2}$$

onto the tangent space to $\mathscr{S}$ at $\boldsymbol{X}$; given any vector $\boldsymbol{a}$, $\mathbf{P} = \mathbf{P}(\boldsymbol{X})$ maps $\boldsymbol{a}$ into its component $\mathbf{P}\boldsymbol{a} = \boldsymbol{a} - (\boldsymbol{a} \cdot \mathbf{m})\mathbf{m}$ tangent to $\mathscr{S}$.

Similarly, a superficial 3-tensor field is a 3-tensor field $\boldsymbol{\lambda}$ on $\mathscr{S}$ that satisfies $\boldsymbol{\lambda}\mathbf{m} = \mathbf{0}$ (cf. the first paragraph of Subsection 1j3).

We will refer to a superficial tensor field as **tangential** or **normal** according as, given any vector $\boldsymbol{a}$, the vector $\mathbf{T}(\boldsymbol{X})\boldsymbol{a}$ is tangent or normal to $\mathscr{S}$ at each $\boldsymbol{X}$. If $\mathbf{T}$ is tangential, then $0 = \mathbf{m} \cdot \mathbf{T}\boldsymbol{a} = \boldsymbol{a} \cdot \mathbf{T}^\top \mathbf{m}$ for all $\boldsymbol{a}$, so that $\mathbf{T}^\top \mathbf{m} = \mathbf{0}$; if $\mathbf{T}$ is normal, then $\mathbf{P}\mathbf{T}\boldsymbol{a} = \mathbf{0}$ for all $\boldsymbol{a}$, so that $\mathbf{P}\mathbf{T} = \mathbf{0}$. The projection $\mathbf{P}$ is an example of a tangential tensor field. Given a tangential vector field $\mathbf{t}$ on $\mathscr{S}$,

$$\mathbf{T} = \mathbf{m} \otimes \mathbf{t} \tag{15–3}$$

is a normal tensor field (because $\mathbf{T}\boldsymbol{a} = (\mathbf{t} \cdot \boldsymbol{a})\mathbf{m}$ for all $\boldsymbol{a}$), and every normal superficial tensor field may be written in this form. More generally, each superficial tensor field $\mathbf{T}$ admits a unique decomposition into tangential and normal parts:

$$\mathbf{T} = \mathbf{T}_{\text{tan}} + \mathbf{m} \otimes \mathbf{t}; \tag{15–4}$$

further, $\mathbf{T}_{\text{tan}}$ and $\mathbf{t}$ have the explicit forms

$$\mathbf{T}_{\text{tan}} = \mathbf{P}\mathbf{T}, \qquad \mathbf{t} = \mathbf{T}^\top \mathbf{m}. \tag{15–5}$$

To verify these assertions, note first that $\mathbf{T}_{\text{tan}} = \mathbf{P}\mathbf{T}$ defines a tangential tensor, and if we define $\mathbf{t} = \mathbf{T}^\top \mathbf{m}$, then by (15–2),

$$\mathbf{T}_{\text{tan}} = \mathbf{P}\mathbf{T} = \mathbf{T} - (\mathbf{m} \otimes \mathbf{m})\mathbf{T} = \mathbf{T} - \mathbf{m} \otimes (\mathbf{T}^\top \mathbf{m}) = \mathbf{T} - \mathbf{m} \otimes \mathbf{t},$$

which is (15–4). Conversely, assume that (15–4) holds. Then premultiplying by $\mathbf{P}$ shows that $\mathbf{T}_{\text{tan}} = \mathbf{P}\mathbf{T}$, and operating with the transpose of (15–4) on $\mathbf{m}$ shows that $\mathbf{t} = \mathbf{T}^\top \mathbf{m}$. The decomposition (15–4) is therefore unique. The assertions regarding (15–4) and (15–5) also follow, after noting that the matrices of $\mathbf{T}$, $\mathbf{T}_{\text{tan}}$, and $\mathbf{t}$ relative

to an orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ with $\boldsymbol{e}_3 = \mathbf{m}$ have the form

$$\mathbf{T} = \begin{pmatrix} T_{11} & T_{12} & 0 \\ T_{21} & T_{22} & 0 \\ T_{31} & T_{32} & 0 \end{pmatrix} \qquad \mathbf{T}_{\tan} = \begin{pmatrix} T_{11} & T_{12} & 0 \\ T_{21} & T_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{t} = \begin{pmatrix} T_{31} \\ T_{32} \\ 0 \end{pmatrix}.$$

The following result is essential to what follows; if a superficial tensor field $\mathbf{T} = \mathbf{T}(\boldsymbol{X})$ satisfies

$$\mathbf{t} \cdot \mathbf{T}\mathbf{a} = 0 \tag{15–6}$$

for every pair of *orthogonal* vectors $\mathbf{t}$ and $\mathbf{a}$ *tangent* to $\mathscr{S}$ at $\boldsymbol{X}$, then[1]

$$\mathbf{T}_{\tan} = \sigma \mathbf{P} \tag{15–7}$$

at $\boldsymbol{X}$, with σ a scalar. To verify this assertion, assume that (15–6) holds. Choose an orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ at $\boldsymbol{X}$ with $\boldsymbol{e}_3 = \mathbf{m}$. Then $0 = \boldsymbol{e}_1 \cdot \mathbf{T}\boldsymbol{e}_2 = \boldsymbol{e}_2 \cdot \mathbf{T}\boldsymbol{e}_1$. Take $\mathbf{t} = \boldsymbol{e}_1 - \boldsymbol{e}_2$ and $\mathbf{a} = \boldsymbol{e}_1 + \boldsymbol{e}_2$. Then $0 = \mathbf{t} \cdot \mathbf{T}\mathbf{a} = \boldsymbol{e}_1 \cdot \mathbf{T}\boldsymbol{e}_1 - \boldsymbol{e}_2 \cdot \mathbf{T}\boldsymbol{e}_2$, so that $\boldsymbol{e}_1 \cdot \mathbf{T}\boldsymbol{e}_1 = \boldsymbol{e}_2 \cdot \mathbf{T}\boldsymbol{e}_2$. Thus, relative to this basis, the off-diagonal components of $\mathbf{T}$ in the tangent space are zero, while the diagonal components are equal. Thus $\mathbf{T}_{\tan}$ has the form (15–7).

The surface gradient $\nabla_{\mathscr{S}}$ on $\mathscr{S}$ may be defined through the chain rule. Let $\varphi, \boldsymbol{f}$, and $\boldsymbol{T}$ be smooth fields on $\mathscr{S}$, with φ scalar-valued, $\boldsymbol{f}$ vector-valued, and $\boldsymbol{T}$ tensor-valued. Then $\nabla_{\mathscr{S}}\varphi$, a tangential vector field, $\nabla_{\mathscr{S}}\boldsymbol{f}$, a superficial tensor field, and $\nabla_{\mathscr{S}}\boldsymbol{T}$ a superficial 3-tensor field, are defined as follows: Given any curve $\mathbf{z} = \mathbf{z}(\lambda)$ on $\mathscr{S}$,

$$\varphi(\mathbf{z})^{\cdot} = \nabla_{\mathscr{S}}\varphi(\mathbf{z}) \cdot \dot{\mathbf{z}}, \qquad \boldsymbol{f}(\mathbf{z})^{\cdot} = \big(\nabla_{\mathscr{S}}\boldsymbol{f}(\mathbf{z})\big)\dot{\mathbf{z}}, \qquad \boldsymbol{T}(\mathbf{z})^{\cdot} = (\nabla_{\mathscr{S}}\boldsymbol{T}(\mathbf{z}))\dot{\mathbf{z}}$$

(where here, but not elsewhere, the dot denotes differentiation with respect to λ). Note that, by definition,

$$\mathbf{m} \cdot \nabla_{\mathscr{S}}\varphi = \mathbf{0}, \qquad (\nabla_{\mathscr{S}}\boldsymbol{f})\mathbf{m} = \mathbf{0}, \qquad (\nabla_{\mathscr{S}}\boldsymbol{T})\mathbf{m} = \mathbf{0}.$$

(Were these relations not required, then, since $\dot{\mathbf{z}}$ is tangential, $\nabla_{\mathscr{S}}\boldsymbol{f}$, for example, would not be defined on vectors normal to $\mathscr{S}$.) If $\varphi, \boldsymbol{f}$, and $\boldsymbol{T}$ are smooth in a (three-dimensional) neighborhood of $\mathscr{S}$, then $\nabla_{\mathscr{S}}\varphi = \mathbf{P}\nabla\varphi$, $\nabla_{\mathscr{S}}\boldsymbol{f} = (\nabla\boldsymbol{f})\mathbf{P}$, and $\nabla_{\mathscr{S}}\boldsymbol{T} = (\nabla\boldsymbol{T})\mathbf{P}$; similar relations hold when $\varphi, \boldsymbol{f}$, and $\boldsymbol{T}$ are smooth up to $\mathscr{S}$ from either side, but then the limits $\Phi^{\pm}(\boldsymbol{X}) = \Phi(\boldsymbol{X} \pm 0\mathbf{m}(\boldsymbol{X}))$ are needed:

$$\nabla_{\mathscr{S}}\varphi^{\pm} = \mathbf{P}(\nabla\varphi)^{\pm}, \tag{15–8a}$$

$$\nabla_{\mathscr{S}}\boldsymbol{f}^{\pm} = (\nabla\boldsymbol{f})^{\pm}\mathbf{P}, \tag{15–8b}$$

$$\nabla_{\mathscr{S}}\boldsymbol{T}^{\pm} = (\nabla\boldsymbol{T})^{\pm}\mathbf{P}. \tag{15–8c}$$

To verify (15–8a), confine φ to one side of $\mathscr{S}$. Then by the chain rule and the smoothness of φ up to $\mathscr{S}$, $\varphi(\mathbf{z})^{\cdot} = \nabla\varphi(\mathbf{z}) \cdot \dot{\mathbf{z}}$. But because the curve $\mathbf{z} = \mathbf{z}(\lambda)$ lies on $\mathscr{S}$, $\dot{\mathbf{z}}$ is tangent to $\mathscr{S}$, so that $\dot{\mathbf{z}} = \mathbf{P}\dot{\mathbf{z}}$ and $\varphi(\mathbf{z})^{\cdot} = \nabla\varphi(\mathbf{z}) \cdot \mathbf{P}\dot{\mathbf{z}} = (\mathbf{P}\nabla\varphi(\mathbf{z})) \cdot \dot{\mathbf{z}}$, which establishes $\mathbf{P}\nabla\varphi$ as $\nabla_{\mathscr{S}}\varphi$, because $\mathbf{P}\nabla\varphi$ is tangential and the curve $\mathbf{z} = \mathbf{z}(\lambda)$ is arbitrary.

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[1] Gurtin and Struthers [1990, eq. (7–4)].

The **surface divergence** of a vector field $\boldsymbol{f}$ on $\mathscr{S}$ is defined by

$$\mathrm{Div}_{\mathscr{S}}\boldsymbol{f} = \mathrm{tr}(\nabla_{\mathscr{S}}\boldsymbol{f}), \tag{15–9}$$

while the surface divergence $\mathrm{Div}_{\mathscr{S}}\mathbf{T}$ of a superficial tensor field is defined through the identity

$$\boldsymbol{a}\cdot\mathrm{Div}_{\mathscr{S}}\mathbf{T} = \mathrm{Div}_{\mathscr{S}}(\mathbf{T}^{\top}\boldsymbol{a}) \tag{15–10}$$

for every constant vector $\boldsymbol{a}$.

Let $\mathscr{G}$ denote a subsurface of $\mathscr{S}$, and let $\mathbf{n}(\boldsymbol{X})$, a vector *tangent* to $\mathscr{S}$ at $\boldsymbol{X}\in\partial\mathscr{G}$, denote the outward unit normal to the boundary (curve) $\partial\mathscr{G}$ of $\mathscr{G}$. The **surface divergence theorem** then has the form

$$\int_{\partial\mathscr{G}}\mathbf{t}\cdot\mathbf{n}\,ds = \int_{\mathscr{G}}\mathrm{Div}_{\mathscr{S}}\mathbf{t}\,da, \tag{15–11a}$$

$$\int_{\partial\mathscr{G}}\mathbf{T}\mathbf{n}\,ds = \int_{\mathscr{G}}\mathrm{Div}_{\mathscr{S}}\mathbf{T}\,da \tag{15–11b}$$

for $\mathbf{t}$ a tangential vector field and $\mathbf{T}$ a superficial tensor field. To verify (15–11b), granted (15–11a), choose an arbitrary vector $\boldsymbol{a}$. Then

$$\boldsymbol{a}\cdot\int_{\partial\mathscr{G}}\mathbf{T}\mathbf{n}\,ds = \int_{\partial\mathscr{G}}(\mathbf{T}^{\top}\boldsymbol{a})\cdot\mathbf{n}\,ds = \int_{\mathscr{G}}\mathrm{Div}_{\mathscr{S}}(\mathbf{T}^{\top}\boldsymbol{a})\,da = \boldsymbol{a}\cdot\int_{\mathscr{G}}\mathrm{Div}_{\mathscr{S}}\mathbf{T}\,da,$$

which implies (15–11b).

The **curvature tensor** $\mathbf{L}$ and **total curvature** K (twice the mean curvature) are defined by

$$\mathbf{L} = -\nabla_{\mathscr{S}}\mathbf{m}, \tag{15–12a}$$

$$K = \mathrm{tr}\mathbf{L} = \mathbf{1}\cdot\mathbf{L} = \mathbf{P}\cdot\mathbf{L} = -\mathrm{Div}_{\mathscr{S}}\mathbf{m}. \tag{15–12b}$$

The curvature tensor is symmetric and (hence) tangential:

$$\mathbf{L} = \mathbf{L}^{\top}, \tag{15–13a}$$

$$\mathbf{L}^{\top}\mathbf{m} = \mathbf{0}, \tag{15–13b}$$

assertions that will be established at the end of this section.

The surfaces under consideration are smooth and may be represented near any of its points z as the zero-level set of a scalar function Φ; i.e., as the set of points $\boldsymbol{X}$ for which

$$\Phi(\boldsymbol{X}) = 0, \tag{15–14}$$

where Φ, a smooth scalar function on a three-dimensional neighborhood $\mathscr{N}$ of z, satisfies

$$\ell = |\nabla\Phi| \neq 0. \tag{15–15}$$

In this case, modulo a change in the sign of Φ, the normal $\mathbf{m}$ is given by

$$\mathbf{m} = \nabla\Phi/\ell, \tag{15–16}$$

so that $\mathbf{m}$ and $\mathbf{P}$ are well defined on $\mathscr{N}$ (even at points of $\mathscr{N}$ not on $\mathscr{S}$) and generate the curvature tensor and total curvature through

$$\mathbf{L} = -(\nabla \mathbf{m})\mathbf{P}, \qquad K = \mathrm{tr}\mathbf{L} = -\operatorname{Div}\mathbf{m} \tag{15–17}$$

(with ∇ and Div the three-dimensional gradient and divergence). Consequences of (15–14) and (15–15) are the identities

$$\nabla \ell = (\nabla\nabla\Phi)\mathbf{m}, \qquad \ell \nabla \mathbf{m} = \mathbf{P}\nabla\nabla\Phi, \tag{15–18}$$

and (hence) the relations

$$\mathbf{L} = -\ell^{-1}\mathbf{P}(\nabla\nabla\Phi)\mathbf{P}, \qquad K = -\ell^{-1}(\Delta\Phi - \mathbf{m}\cdot(\nabla\nabla\Phi)\mathbf{m}), \tag{15–19}$$

which, in particular, verify (15–13). Here $\Delta\Phi = \mathrm{tr}(\nabla\nabla\Phi)$ is the Laplacian.

Two important identities in which $\mathbf{t}$ is a tangential vector field are:

$$\operatorname{Div}_{\mathscr{S}}\mathbf{P} = K\mathbf{m}, \tag{15–20a}$$

$$\operatorname{Div}_{\mathscr{S}}(\mathbf{m}\otimes\mathbf{t}) = (\operatorname{Div}_{\mathscr{S}}\mathbf{t})\mathbf{m} - \mathbf{L}\mathbf{t}. \tag{15–20b}$$

b. Smoothly evolving surfaces

b1. Time derivative following $\mathscr{S}$. Normal time derivative

Let $\mathscr{S}(t)$ depend smoothly on the time t with $V(\boldsymbol{X}, t)$ the normal velocity of $\mathscr{S}(t)$. As in Chapter 10, an assignment of a vector $\boldsymbol{v}(\boldsymbol{X}, t)$ to each $\boldsymbol{X} \in \mathscr{S}(t)$ is a **velocity field for** $\mathscr{S}$ if $\boldsymbol{v}$ is a smooth field that satisfies $\boldsymbol{v}\cdot\mathbf{m} = V$ (cf. Chapter 4). The velocity field $\boldsymbol{v}$ for $\mathscr{S}$ may be viewed as a velocity field for evolving particles constrained to $\mathscr{S}$, with the path $\boldsymbol{Z}(\tau)$ traversed by the particle that passes through $\boldsymbol{X} \in \mathscr{S}(t)$ at time t the unique solution of (4–2). Then, given a scalar, vector, or tensor field $\varphi(\boldsymbol{X}, t)$ defined for $\boldsymbol{X} \in \mathscr{S}(t)$, the **time derivative of φ following** $\mathscr{S}$, as described by $\boldsymbol{v}$ is the time derivative along such paths:

$$\overset{\square}{\varphi}(\boldsymbol{X}, t) = \frac{d}{d\tau}\,\varphi(\boldsymbol{Z}(\tau), \tau)|_{\tau=t}. \tag{15–21}$$

If $\varphi(\boldsymbol{X}, t)$ is a scalar field, $\boldsymbol{f}(\boldsymbol{X}, t)$ a vector field, and $T(\boldsymbol{X}, t)$ a tensor field, each smooth in a three-dimensional neighborhood of $\mathscr{S}(t)$ for all t, then, by the chain rule $\overset{\square}{\varphi} = \dot{\varphi} + \nabla\varphi\cdot\boldsymbol{v}$, and similar formulas apply to $\overset{\square}{\boldsymbol{f}}$ and $\overset{\square}{\boldsymbol{T}}$. Similar relations hold when $\varphi, \boldsymbol{f}$ and $\boldsymbol{T}$ are smooth up to $\mathscr{S}$ from either side, but then the appropriate limits must be used:

$$\overset{\square}{\varphi}{}^{\pm} = \dot{\varphi}^{\pm} + (\nabla\varphi)^{\pm}\cdot\boldsymbol{v}, \tag{15–22a}$$

$$\overset{\square}{\boldsymbol{f}}{}^{\pm} = \dot{\boldsymbol{f}}^{\pm} + (\nabla\boldsymbol{f})^{\pm}\boldsymbol{v}, \tag{15–22b}$$

$$\overset{\square}{\boldsymbol{T}}{}^{\pm} = \dot{\boldsymbol{T}}^{\pm} + (\nabla\boldsymbol{T})^{\pm}\boldsymbol{v} \tag{15–22c}$$

(cf. (1–22)).

The derivative (15–21) is not intrinsic, because it depends on the choice of the velocity field $\boldsymbol{v}$ for $\mathscr{S}$; when $\boldsymbol{v}$ is normal,

$$\boldsymbol{v} = V\mathbf{m}, \tag{15–23}$$

then $\overset{\square}{\varphi}$ is *intrinsic* and represents the **normal time derivative** of φ following $\mathscr{S}$. Granted this,

$$\overset{\square}{\mathbf{m}} = -\nabla_{\mathscr{S}} V, \tag{15–24}$$

a result that will be used repeatedly; the verification of (15–24) follows from the discussion of the next paragraph.

Near any of its points and in a neighborhood of any time, $\mathscr{S}(t)$ may be considered as the set of $(\boldsymbol{X}, t)$ such that

$$\Phi(\boldsymbol{X}, t) = 0, \tag{15–25}$$

where Φ is a smooth function consistent with (15–15). Then granted (15–16),

$$V = -\dot{\Phi}/\ell \tag{15–26}$$

and the identities $\dot{\ell} = \mathbf{m} \cdot \nabla\dot{\Phi}$ and $\ell\dot{\mathbf{m}} = \mathbf{P}\nabla\dot{\Phi}$ yield

$$\overset{\square}{\mathbf{m}} = -\mathbf{P}\nabla V, \tag{15–27}$$

which is (15–24).

I close this subsection by constructing an evolving surface such that, at $\boldsymbol{X} = \mathbf{0}$ and $t = 0$, the fields $\mathbf{m}$, V, $\overset{\square}{\mathbf{m}}$, and $\overset{\square}{V}$ have arbitrarily prescribed values (consistent with the constraint $\mathbf{m} \cdot \overset{\square}{\mathbf{m}} = 0$). This result will be useful in deriving thermomechanical restrictions on constitutive relations. Consider the plane surface $\mathscr{S} = \mathscr{S}(t)$ defined by (15–25) with

$$\Phi(\boldsymbol{X}, t) = \mathbf{m}(t) \cdot (\boldsymbol{X} - \boldsymbol{Z}(t)). \tag{15–28}$$

Then $\mathbf{m}(t)$ represents a choice of unit normal to $\mathscr{S}(t)$ and

$$V(\boldsymbol{X}, t) = \dot{\boldsymbol{Z}}(t) \cdot \mathbf{m}(t) - \dot{\mathbf{m}}(t) \cdot (\boldsymbol{X} - \boldsymbol{Z}(t))$$

is the corresponding normal velocity. Let $\boldsymbol{Z}(0) = \mathbf{0}$ and define $\mathbf{m}_0 = \mathbf{m}(0)$; then because $\overset{\square}{V} = \dot{V} + V\mathbf{m} \cdot \nabla V$ and $\nabla V = -\dot{\mathbf{m}}$, it follows that

$$\begin{aligned} V(\mathbf{0}, 0) &= \mathbf{m}_0 \cdot \dot{\boldsymbol{Z}}(0), \\ \overset{\square}{V}(\mathbf{0}, 0) &= \mathbf{m}_0 \cdot \ddot{\boldsymbol{Z}}(0) + 2\dot{\mathbf{m}}(0) \cdot \dot{\boldsymbol{Z}}(0), \\ \overset{\square}{\mathbf{m}}(0) &= \dot{\mathbf{m}}(0). \end{aligned}$$

Further, given any vector $\boldsymbol{\tau}$ orthogonal to $\mathbf{m}_0$ and any function $\delta(t)$ with $\delta(0) = 0$ and $\dot{\delta}(0) = 1$, if $\mathbf{m}(t)$ satisfies

$$\mathbf{m}(t) = (\mathbf{m}_0 + \delta(t)\boldsymbol{\tau})/|\mathbf{m}_0 + \delta(t)\boldsymbol{\tau}|,$$

then $\mathbf{m}(0) = \mathbf{m}_0$, $\dot{\mathbf{m}}(0) = \boldsymbol{\tau}$. Thus appropriate choices for $\dot{\boldsymbol{Z}}(0)$, $\ddot{\boldsymbol{Z}}(0)$, $\mathbf{m}_0$, and $\boldsymbol{\tau}$ allow for arbitrary specification of $\mathbf{m}$, V, $\overset{\square}{\mathbf{m}}$, and $\overset{\square}{V}$ at $(\mathbf{0}, 0)$.

b2. Velocity fields for the boundary curve $\partial\mathcal{G}$ of a smoothly evolving subsurface of $\mathcal{S}$. Transport theorem

Let $\mathcal{G}(t)$ denote a smoothly evolving subsurface of $\mathcal{S}(t)$ with $\mathbf{n}(\boldsymbol{X}, t)$ the outward unit normal to the boundary curve $\partial\mathcal{G}(t)$. (Recall that $\mathbf{n}$ is *normal* to $\partial\mathcal{G}$ but *tangent* to $\mathcal{S}$; cf. the second paragraph of Section 15a.) The intrinsic motion of $\partial\mathcal{G}$ is in the plane spanned by $\mathbf{m}$ and $\mathbf{n}$ and is characterized by a velocity field of the form

$$V\mathbf{m} + V_{\partial G}\mathbf{n}, \tag{15–29}$$

where V is the normal velocity $\mathcal{S}$ and $V_{\partial\mathcal{G}}$, the **velocity of $\partial\mathcal{G}$ in the direction of $\mathbf{n}$**, is a uniquely defined field. More generally, an assignment of a vector $\boldsymbol{w}(\boldsymbol{X}, t)$ to each $\boldsymbol{X} \in \partial\mathcal{G}(t)$ is a **velocity field for** $\partial\mathcal{G}$ if $\boldsymbol{w}$ is a smooth field that satisfies

$$\boldsymbol{w} \cdot \mathbf{m} = V, \qquad \boldsymbol{w} \cdot \mathbf{n} = V_{\partial\mathcal{G}}, \tag{15–30}$$

so that the component of $\boldsymbol{w}$ tangent to $\partial\mathcal{G}$ is arbitrary. If the curve $\partial\mathcal{G}(t)$ is parametrized locally by functions $\boldsymbol{X} = \boldsymbol{r}(u, t)$, then $\boldsymbol{w}(\boldsymbol{X}, t) = \partial\boldsymbol{r}(u, t)/\partial t$ is a velocity field for $\partial\mathcal{G}(t)$.

For φ a superficial scalar field,

$$\frac{d}{dt}\left\{\int_{\mathcal{G}} \varphi\, da\right\} \quad \text{denotes} \quad \frac{d}{dt}\left\{\int_{\mathcal{G}(t)} \varphi(\boldsymbol{X}, t)\, da(\boldsymbol{X})\right\}.$$

The following **transport theorem**, in which $\overset{\square}{\varphi}$ is the normal time derivative of φ following $\mathcal{S}$, is basic to what follows:[2]

$$\frac{d}{dt}\left\{\int_{\mathcal{G}} \varphi\, da\right\} = \int_{\mathcal{G}} \left(\overset{\square}{\varphi} - \varphi K V\right) da + \int_{\partial\mathcal{G}} \varphi V_{\partial\mathcal{G}}\, ds. \tag{15–31}$$

Given a migrating control volume $P(t)$, let $\mathcal{G}(t)$ denote the intersection of the interface with P, and let $\mathbf{n}(\boldsymbol{X}, t)$ denote the outward unit normal to the boundary curve $\partial\mathcal{G}(t)$ (Figure 15.1). Then, because $\partial\mathcal{G}(t)$ lies on $\partial P(t)$ for all t, the inner product of (15–29) with $\boldsymbol{n}$ must yield the normal velocity of ∂P:

$$U = V(\mathbf{m} \cdot \boldsymbol{n}) + V_{\partial\mathcal{G}}(\mathbf{n} \cdot \boldsymbol{n}). \tag{15–32}$$

Thus a vector field $\boldsymbol{w}(\boldsymbol{X}, t)$ on $\partial\mathcal{G}(t)$ will be a velocity field for $\partial\mathcal{G}$ if and only if

$$\boldsymbol{w} \cdot \mathbf{m} = V, \qquad \boldsymbol{w} \cdot \boldsymbol{n} = U. \tag{15–33}$$

Indeed, $\boldsymbol{w} \cdot \boldsymbol{n} = (\boldsymbol{w} \cdot \mathbf{m})(\mathbf{m} \cdot \boldsymbol{n}) + (\boldsymbol{w} \cdot \mathbf{n})(\mathbf{n} \cdot \boldsymbol{n})$; hence, by (15–32), the conditions (15–30) and (15–33) are equivalent.

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[2]For the proof of (15–31), cf. Petryk and Mroz [1986], Gurtin, Struthers, and Williams [1989], Estrada and Kanwal [1991].

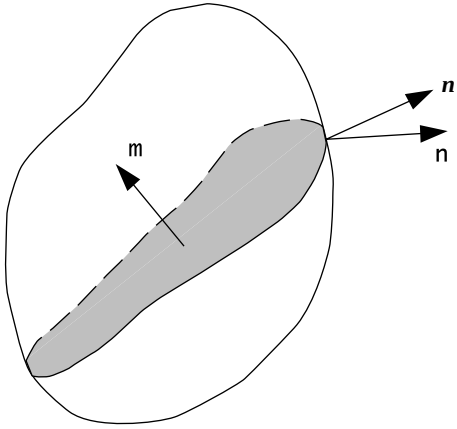

FIGURE 15.1. $\mathscr{G}$, shown shaded, is the portion of the interface in the control volume P; $\mathbf{n}$, the outward unit normal to $\partial\mathscr{G}$, is the principle normal to the curve $\partial\mathscr{G}$, directed outward from $\mathscr{G}$, and is hence normal to $\partial\mathscr{G}$, but tangent to the interface; $\boldsymbol{n}$ is the outward unit normal to ∂P; $\mathtt{m}$ is the unit normal to the interface.

b3. Transformation laws

If $\boldsymbol{v}$ and $\boldsymbol{w}$ are velocity fields for $\mathscr{S}$ and $\partial\mathscr{G}$, respectively, then $\boldsymbol{v}$ and $\boldsymbol{w}$ transform according to

$$\boldsymbol{v} \to \boldsymbol{v} + \boldsymbol{a}, \qquad \boldsymbol{w} \to \boldsymbol{w} + \boldsymbol{a} \tag{15–34}$$

under the change in material observer defined by (2–9).

CHAPTER 16

Configurational Force System. Working

a. Configurational forces. Working

I consider two phase (or grains) separated by a smoothly evolving surface $\mathscr{S}(t)$ and described by a configurational force system consisting of fields:

C	**bulk stress**
$\boldsymbol{g}$	**internal bulk force**
$\mathbf{C}$	**interfacial stress**
$\boldsymbol{g}^{\mathscr{S}}$	**internal interfacial force**
$\boldsymbol{e}^{\mathscr{S}}$	**external interfacial force**

The fields $\boldsymbol{C}, \boldsymbol{g}, \boldsymbol{g}^{\mathscr{S}}$, and $\boldsymbol{e}^{\mathscr{S}}$ are as discussed in Sections 5a and 11a, with all fields objective, and with $\boldsymbol{C}$ and $\boldsymbol{g}$ smooth away from the interface and up to the interface from either side. The interfacial stress $\mathbf{C}(\boldsymbol{X}, t)$, which represents forces such as surface tension that act within the interface, is a superficial tensor field on $\mathscr{S}(t)$. Given a migrating control volume $P = P(t)$, let $\mathscr{G} = \mathscr{G}(t)$ denote the portion of the interface in P,

$$\mathscr{G} = P \cap \mathscr{S}, \tag{16–1}$$

and let $\mathbf{n}(\boldsymbol{X}, t)$ denote the outward unit normal to $\partial\mathscr{G}(t)$ (Figure 15.1); then $\mathbf{Cn}$ represents the configurational force, per unit length, exerted on $\mathscr{G}$ (and hence P) across $\partial\mathscr{G}$ by the portion of $\mathscr{S}$ that lies outside of $\mathscr{G}$.

The neglect of inertia renders the external bulk force $\boldsymbol{e}$ irrelevant. The same is not true of $\boldsymbol{e}^{\mathscr{S}}$, which represents a force exerted on the interface by the external world. This force should be considered *virtual*, as it would seem difficult, if not impossible, to produce in a laboratory. The need for such forces is discussed in the introduction to Chapter 9.

Let $\boldsymbol{v}$ be a velocity field for $\mathscr{S}$. Consider a migrating control volume $P = P(t)$, with $\boldsymbol{q}$ a velocity field for ∂P and $\boldsymbol{w}$ a velocity field for $\partial\mathscr{G}$. Based on the discussion of Section 11b, the working on a migrating control volume $P = P(t)$ should have the form (11–4) with the terms involving $\boldsymbol{S}$, $\boldsymbol{b}$, and $\boldsymbol{b}^{\mathscr{S}}$ omitted but with an accounting of the working of the interfacial stress. Because $\mathbf{C}$ acts on P across $\partial\mathscr{G}$, it would seem appropriate to take as work-conjugate velocity for $\mathbf{C}$ the field $\boldsymbol{w}$, which describes the velocity of $\partial\mathscr{G}$. I therefore write the **working** $W(P)$ on a migrating control volume $P = P(t)$ in the form

$$W(P) = \int_{\partial P} \boldsymbol{C}\boldsymbol{n} \cdot \mathbf{q}\, da + \int_{\mathscr{G}} \boldsymbol{e}^{\mathscr{S}} \cdot \boldsymbol{v}\, da + \int_{\partial\mathscr{G}} \mathbf{C}\mathbf{n} \cdot \boldsymbol{w}\, ds. \tag{16–2}$$

b. Configurational force balance

An argument identical to that given in Section 11c yields, as a consequence of invariance under changes in material observer (cf. (15–34)), the **configurational force balance**

$$\int_{\partial P} \boldsymbol{C}\boldsymbol{n}\, da + \int_{P} \boldsymbol{g}\, dv + \int_{\mathscr{G}} (\boldsymbol{g}^{\mathscr{S}} + \boldsymbol{e}^{\mathscr{S}})\, da + \int_{\partial\mathscr{G}} \mathbf{C}\mathbf{n}\, ds = \mathbf{0}. \tag{16–3}$$

By the surface divergence theorem,

$$\int_{\partial\mathscr{G}} \mathbf{C}\mathbf{n}\, ds = \int_{\mathscr{G}} \mathrm{Div}_{\mathscr{S}}\, \mathbf{C}\, da;$$

the steps leading to (11–8b) therefore yield the **interfacial force balance**

$$[\boldsymbol{C}]\mathbf{m} + \boldsymbol{g}^{\mathscr{S}} + \boldsymbol{e}^{\mathscr{S}} + \mathrm{Div}_{\mathscr{S}}\mathbf{C} = \mathbf{0}. \tag{16–4}$$

On the other hand, restricting attention to P in (16–3) that do not intersect the interface yields the bulk relation

$$\mathrm{Div}\,\boldsymbol{C} + \boldsymbol{g} = \mathbf{0}.$$

c. Invariance under changes in velocity fields. Surface tension. Surface shear

Let $P = P(t)$ be a migrating control volume. I require that the working $W(P)$ be invariant under changes in the choice of velocity fields $\boldsymbol{q}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ for ∂P, $\mathscr{S}$, and $\partial\mathscr{G}$, respectively. Because the tangential component of *each* of these velocity fields may be chosen arbitrarily, and because there are no standard forces, the argument leading to (5–17) reduces $\boldsymbol{C}$ to a bulk tension, while that leading to (11–9) renders $\boldsymbol{e}^{\mathscr{S}}$ normal to the interface:

$$\boldsymbol{C} = \pi\mathbf{1}, \tag{16–5a}$$

$$\boldsymbol{e}^{\mathscr{S}} = e^{\mathscr{S}}\mathbf{m}, \tag{16–5b}$$

with $\boldsymbol{e}^{\mathscr{S}}(\boldsymbol{X}, t)$ a scalar field.

Thus we are left with the requirement that (16–2) be invariant under changes in the velocity field $\boldsymbol{w}$. Because the component of $\boldsymbol{w}$ tangent to $\partial\mathscr{G}$ is arbitrary, this invariance is equivalent to the requirement that, given any migrating control volume $P = P(t)$,

$$\int_{\partial\mathscr{G}} \mathbf{C}\mathbf{n} \cdot \mathbf{t}\, ds = 0 \tag{16–6}$$

for every vector field $\mathbf{t}$ tangential to $\partial\mathscr{G}$. Bearing in mind that P and hence $\partial\mathscr{G}$ is arbitrary, it follows[1] that, at any t and any point $\boldsymbol{X} \in \mathscr{S}(t)$,

$$\mathbf{t} \cdot \mathbf{C}\mathbf{a} = 0$$

for every pair of orthogonal vectors $\mathbf{t}$ and $\mathbf{a}$ tangent to $\mathscr{S}$ at $\boldsymbol{X}$, and (15–7) yields the conclusion that the tangential part of $\mathbf{C}$ has the form

$$\mathbf{C}_{\text{tan}} = \sigma\mathbf{P}, \tag{16–7}$$

where $\sigma(\boldsymbol{X}, t)$, a scalar, represents **surface tension**. Thus, by (15–4),

$$\mathbf{C} = \sigma\mathbf{P} + \mathbf{m} \otimes \boldsymbol{\tau}; \tag{16–8}$$

the vector $\boldsymbol{\tau}(\boldsymbol{X}, t)$, although tangential, represents, via the term $\mathbf{m} \otimes \boldsymbol{\tau}$, forces whose action is normal to $\mathscr{S}$; $\boldsymbol{\tau}$ is referred to as the **surface shear**. Given a control volume P with $\mathbf{n}$ the outward unit normal to $\partial\mathscr{G}$, the force, per unit length, within the interface applied to P across $\partial\mathscr{G}$, is given by

$$\mathbf{C}\mathbf{n} = \sigma\mathbf{n} + (\boldsymbol{\tau} \cdot \mathbf{n})\mathbf{m}, \tag{16–9}$$

with $\sigma\mathbf{n}$ tangential—and $(\boldsymbol{\tau} \cdot \mathbf{n})\mathbf{m}$ normal—to $\mathscr{S}$. Thus, by (15–30) and (16–9), the working of the interfacial stress can be written in the form

$$\int_{\partial\mathscr{G}} \mathbf{C}\mathbf{n} \cdot \boldsymbol{w}\, ds = \int_{\partial\mathscr{G}} (\sigma V_{\partial\mathscr{G}} + V\boldsymbol{\tau} \cdot \mathbf{n})\, ds, \tag{16–10}$$

showing that the work-conjugate velocities for the surface tension and surface shear, respectively, are the velocity of $\partial\mathscr{G}$ in the direction of its normal $\mathbf{n}$ and the normal velocity of $\mathscr{S}$

The isotropy of $\mathbf{C}_{\text{tan}}$ as a mapping of tangent vector into tangent vectors is basic; in no way is it related to material symmetry. As we shall see, material symmetry may indeed affect both σ and $\boldsymbol{\tau}$ through constitutive dependencies on $\mathbf{m}$. Note that relative to an orthonormal basis with $\boldsymbol{e}_3 = \mathbf{m}$,

$$\mathbf{C}_{\text{tan}} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

[1] Gurtin and Struthers [1990, eq. (7–4)].

d. Normal force balance. Intrinsic form for the working

A computation based on (15–20a) results in the identity

$$\mathrm{Div}_{\mathscr{S}}\,\mathbf{C} = (\sigma K + \mathrm{Div}_{\mathscr{S}}\,\boldsymbol{\tau})\mathbf{m} + \nabla_{\mathscr{S}}\sigma - \mathbf{L}\boldsymbol{\tau}, \tag{16–11}$$

thus, because $\nabla_{\mathscr{S}}\sigma$ and $\mathbf{L}\boldsymbol{\tau}$ are tangential, (16–5a) may be used to conclude that the normal and tangential components of the force balance (16–4) are

$$\sigma K + \mathrm{Div}_{\mathscr{S}}\boldsymbol{\tau} + [\pi] + g^{\mathscr{S}} + e^{\mathscr{S}} = 0, \tag{16–12a}$$

$$\nabla_{\mathscr{S}}\sigma - \mathbf{L}\boldsymbol{\tau} + \mathbf{P}\boldsymbol{g}^{\mathscr{S}} = 0, \tag{16–12b}$$

with

$$g^{\mathscr{S}} = \boldsymbol{g}^{\mathscr{S}} \cdot \mathbf{m} \tag{16–13}$$

the **normal internal force**. The **normal interfacial force balance** (16–12a) is a basic ingredient of the theory.

For the remainder of Chapter 16 the velocity fields are assumed to have the intrinsic forms (cf. (15–29))

$$\boldsymbol{q} = U\boldsymbol{n}, \qquad \boldsymbol{v} = V\mathbf{m}, \qquad \boldsymbol{w} = V\mathbf{m} + V_{\partial\mathscr{G}}\mathbf{n}. \tag{16–14}$$

Appealing to (16–5), the working (16–2) may be written intrinsically as

$$W(P) = \int_{\partial\mathscr{G}} \left(\sigma V_{\partial\mathscr{G}} + V\boldsymbol{\tau}\cdot\mathbf{n}\right) ds + \int_{\mathscr{G}} e^{\mathscr{S}} V\, da + \int_{\partial P} \pi U\, da. \tag{16–15}$$

e. Power balance. Internal working

By (15–24),

$$\mathrm{Div}_{\mathscr{S}}(V\boldsymbol{\tau}) = \boldsymbol{\tau}\cdot\nabla_{\mathscr{S}}V + V\,\mathrm{Div}_{\mathscr{S}}\,\boldsymbol{\tau} = -\boldsymbol{\tau}\cdot\overset{\square}{\mathbf{m}} + V\,\mathrm{Div}_{\mathscr{S}}\,\boldsymbol{\tau},$$

where $(\cdots)^{\square}$ denotes the *normal* time-derivative following $\mathscr{S}$. Thus, by (16–10) and the surface divergence theorem (15–11a), the working of the interfacial stress may be written in the form

$$\begin{aligned}\int_{\partial\mathscr{G}} \mathbf{C}\mathbf{n}\cdot\boldsymbol{w}\, ds &= \int_{\partial\mathscr{G}} \left(\sigma V_{\partial\mathscr{G}} + V\boldsymbol{\tau}\cdot\mathbf{n}\right) ds \\ &= \int_{\partial\mathscr{G}} \sigma V_{\partial\mathscr{G}}\, ds + \int_{\mathscr{G}} (-\boldsymbol{\tau}\cdot\overset{\square}{\mathbf{m}} + V\,\mathrm{Div}_{\mathscr{S}}\boldsymbol{\tau})\, da,\end{aligned} \tag{16–16}$$

and, eliminating the term $\mathrm{Div}_{\mathscr{S}}\boldsymbol{\tau}$ in (16–16) using the normal force balance (16–12a), the working (16–15) may be written as[2]

$$W(P) = -\int_{\mathscr{G}} \left\{\sigma K V + \boldsymbol{\tau}\cdot\overset{\square}{\mathbf{m}} + ([\pi] + g^{\mathscr{S}})V\right\} da + \int_{\partial\mathscr{G}} \sigma V_{\partial\mathscr{G}}\, ds + \int_{\partial P} \pi U\, da. \tag{16–17}$$

[2] Gurtin [1988], Gurtin and Struthers [1990].

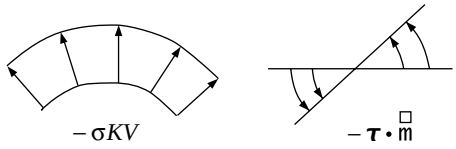

FIGURE 16.1. Contributions of the surface tension σ and surface shear $\boldsymbol{\tau}$ to the working at a phase interface.

The right side of (16–17) represents **internal working** on P, and the equivalence of (16–15) and (16–17) represents a **power balance** for P. The significance of the terms involving $[\pi]$ and $g^{\mathscr{S}}$ are discussed in the paragraph containing (11–18). Regarding the remaining terms (Figure 16.1):

- The term $-\sigma K V$ represents working associated with temporal changes in interfacial area due to the curvature of the interface.
- The term $-\boldsymbol{\tau}\cdot\overset{\square}{\mathbf{m}}$ represents working associated with temporal changes in the orientation of the interface.
- The terms

$$\int_{\partial\mathscr{G}} \sigma V_{\partial\mathscr{G}}\, ds, \qquad \int_{\partial P} \pi U\, da \tag{16–18}$$

represent working of the surface and bulk tensions *within* $\mathscr{G}$ and P at $\partial\mathscr{G}$ and ∂P as material is transferred to P. The same terms are also present in the *external* working (16–15), where they represent working by the agency exterior to P that is transferring material to P.

There is no expenditure of work associated with "tangential motion" of the interface. Consistent with constraint of this type, I leave as *indeterminate* the tangential component $\mathbf{P}\boldsymbol{g}^{\mathscr{S}}$ of the internal force, an assumption that renders the tangential balance (16–12b) unimportant and allows one to restrict attention to the normal balance (16–12a).

In view of (15–12a) and (15–24), for $\boldsymbol{v} = V\mathbf{m}$,

$$\nabla_{\mathscr{S}}\boldsymbol{v} = \mathbf{m}\otimes\nabla_{\mathscr{S}}V + V\nabla_{\mathscr{S}}\mathbf{m} = -(\mathbf{m}\otimes\overset{\square}{\mathbf{m}} + V\mathbf{L}) \tag{16–19}$$

and therefore

$$\mathbf{C}\cdot\nabla_{\mathscr{S}}\boldsymbol{v} = -(\sigma K V + \boldsymbol{\tau}\cdot\overset{\square}{\mathbf{m}}). \tag{16–20}$$

Thus the term $-(\sigma K V + \boldsymbol{\tau}\cdot\overset{\square}{\mathbf{m}})$ in (16–17) may be replaced by $\mathbf{C}\cdot\nabla_{\mathscr{S}}\boldsymbol{v}$, the internal working of the interfacial stress $\mathbf{C}$ over superficial variations of the vectorial normal velocity $\boldsymbol{v}$. The term $\mathbf{C}\cdot\nabla_{\mathscr{S}}\boldsymbol{v}$ should be compared to the stress power $\boldsymbol{S}\cdot\dot{\boldsymbol{F}} = \boldsymbol{S}\cdot(\nabla\dot{\mathbf{y}})$ that accompanies deformation (cf. (3–8)).

Equating (16–15) and (16–17) yields the identity

$$\int_{\partial\mathscr{G}} V(\boldsymbol{\tau}\cdot\mathbf{n})\, ds + \int_{\mathscr{G}} e^{\mathscr{S}} V\, da = -\int_{\mathscr{G}} \left\{\sigma K V + \boldsymbol{\tau}\cdot\overset{\square}{\mathbf{m}} + ([\pi] + g^{\mathscr{S}})V\right\} da, \tag{16–21}$$

which might be termed a **reduced power balance**, since the terms (16–18) are not present. In particular, the external working of the interfacial stress is incomplete; as it includes only the working $\int_{\partial\mathscr{G}} V(\boldsymbol{\tau}\cdot\mathbf{n})\,ds$ of the shear.

The stress in the form (16–8) and the balance (16–12a) are due to Gurtin and Struthers[3] and are independent of constitutive equations. For statical situations (with $\boldsymbol{C}=\Psi\mathbf{1}$, $\boldsymbol{g}^{\mathscr{S}}=\boldsymbol{e}^{\mathscr{S}}=\mathbf{0}$) related results were derived earlier using variational arguments based on a constitutive equation $\psi=\hat{\psi}(\mathbf{m})$ for the interfacial energy, with σ and $\mathbf{c}$ *defined* by $\sigma=\psi$ and $\boldsymbol{\tau}=-\partial_{\mathbf{m}}\hat{\psi}(\mathbf{m})$. In particular, Herring[4] derived the force balance appropriate to triple junctions, while Cahn and Hoffman[5] show variationally that the vector

$$\boldsymbol{\xi}=\sigma\mathbf{m}-\boldsymbol{\tau}$$

satisfies

$$\mathrm{Div}_{\mathscr{S}}\boldsymbol{\xi}=[\Psi].$$

Since $\mathbf{m}\cdot\mathrm{Div}_{\mathscr{S}}\mathbf{C}=-\mathrm{Div}_{\mathscr{S}}\boldsymbol{\xi}$, this is consistent with (16–12a). The **Cahn-Hoffman** vector $\boldsymbol{\xi}$ is widely used in materials science, which is not surprising, since it is only the normal component of $\mathrm{Div}_{\mathscr{S}}\mathbf{C}$ that generally appears in interface conditions. But the use of $\boldsymbol{\xi}$ as basic (rather than derived) masks the tensorial nature of stress, which is classical. In fact, $\boldsymbol{\xi}$ is apropos only when $\mathbf{C}$ has the specific form $\mathbf{C}=\sigma\mathbf{P}+\mathbf{m}\otimes\boldsymbol{\tau}$; but in situations that allow for a standard stress $\mathbf{S}$ within the interface, neither $\mathbf{C}$ nor $\mathbf{S}$ have this form (cf. Chapter 21).

To further relate the Cahn-Hoffman vector $\boldsymbol{\xi}$ to the interfacial stress $\mathbf{C}$, let $\mathscr{G}=\mathscr{G}(t)$ be a subsurface of the interface with $\mathbf{n}$ the outward unit normal to $\partial\mathscr{G}$, and let $\mathbf{t}=\mathbf{m}\times\mathbf{n}$, so that $\mathbf{t}$ is a unit tangent field on $\partial\mathscr{G}$. Then[6]

$$\mathbf{C}\mathbf{n}=\mathbf{t}\times\boldsymbol{\xi} \tag{16–22}$$

and the *external working* of the interfacial stress is given by

$$\int_{\partial\mathscr{G}}\mathbf{C}\mathbf{n}\cdot\boldsymbol{w}\,ds=\int_{\partial\mathscr{G}}(\mathbf{t}\times\boldsymbol{\xi})\cdot\boldsymbol{w}\,ds. \tag{16–23}$$

Further, the vectorial counterpart of (15–31) yields a transport theorem for the vector-area measure $\mathbf{m}\,da$,

$$\frac{d}{dt}\left\{\int_{\mathscr{G}}\mathbf{m}\,da\right\}=\int_{\mathscr{G}}\left(\overset{\square}{\mathbf{m}}-KV\mathbf{m}\right)da+\int_{\partial\mathscr{G}}\mathbf{m}V_{\partial\mathscr{G}}\,ds, \tag{16–24}$$

which identifies the vector field $\overset{\square}{\mathbf{m}}-KV\mathbf{m}$ as the rate at which the vector area is changing, measured per unit area. Using (16–25), the *internal working* of the interfacial stress may be written in the simple form

$$-\int_{\mathscr{G}}(\sigma KV+\boldsymbol{\tau}\cdot\overset{\square}{\mathbf{m}})\,da=\int_{\mathscr{G}}\boldsymbol{\xi}\cdot(\overset{\square}{\mathbf{m}}-KV\mathbf{m})\,da, \tag{16–25}$$

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[3][1990], eq. (7–5)] (cf. Gurtin [1988]).

[4][1951b].

[5][1972, 1974].

[6]Cahn ad Hoffman [1974, eq. (7)] note that $\mathbf{t}\times\boldsymbol{\xi}$ is the force, per unit length, exerted across an interfacial curve.

and hence represents working associated with temporal changes in the vector area of the interface. The verification of (16–22) and (16–25) is as follows:

$$\begin{aligned}\mathbf{C}\mathbf{n} &= \sigma\mathbf{n} + (\boldsymbol{\tau}\cdot\mathbf{n})\mathbf{m} = \sigma(\mathbf{t}\times\mathbf{m}) + \Big(\boldsymbol{\tau}\cdot(\mathbf{t}\times\mathbf{m})\Big)\mathbf{m}\\ &= \mathbf{t}\times(\sigma\mathbf{m}) - \Big(\mathbf{m}\cdot(\mathbf{t}\times\boldsymbol{\tau})\Big)\mathbf{m} = \mathbf{t}\times(\sigma\mathbf{m}) - \mathbf{t}\times\boldsymbol{\tau} = \mathbf{t}\times\boldsymbol{\xi},\end{aligned}$$

$$-\sigma KV - \boldsymbol{\tau}\cdot\overset{\square}{\mathbf{m}} = (\sigma\mathbf{m}-\boldsymbol{\tau})\cdot\Big(\overset{\square}{\mathbf{m}} - KV\mathbf{m}\Big) - \boldsymbol{\xi}\cdot\Big(\overset{\square}{\mathbf{m}} - KV\mathbf{m}\Big).$$

CHAPTER 17

Second Law

I now generalize the dissipation inequality to include interfacial energy. As before, I allow for a **bulk free energy** Ψ, but, in accord with the physical assumptions underlying the current development, I assume that Ψ is *constant* in each phase and write

$$F = [\Psi] (= \text{constant}). \tag{17–1}$$

In addition, I now allow for an **interfacial free energy** $\psi(\boldsymbol{X}, t)$, per unit area, and write the second law in the form:

$$\frac{d}{dt}\left\{\int_P \Psi \, dv + \int_{\mathscr{G}} \psi \, da\right\} \leq W(P) \tag{17–2}$$

for every migrating control volume $P = P(t)$, with $\mathscr{G} = \mathscr{G}(t) = \mathscr{S}(t) \cap P(t)$.

The argument leading to the relation $\pi = \Psi$ (cf. (16–12a)) is valid here also, so that, by (16–5a), the Eshelby relation takes the simple form

$$\boldsymbol{C} = \Psi \mathbf{1},$$

giving the configurational stress as a pure tension with value equal to the free energy. Further, because Ψ is constant in each phase, the balance $\operatorname{Div} \boldsymbol{C} + \boldsymbol{g} = \mathbf{0}$ yields

$$\boldsymbol{g} = \mathbf{0}. \tag{17–3}$$

Next, because Ψ is constant in each phase, (10–8a) yields

$$\frac{d}{dt}\left\{\int_P \Psi \, dv\right\} = -\int_{\mathscr{G}} [\Psi] V \, da + \int_{\partial P} \Psi U \, da,$$

so that using the transport theorem (15–31) and the power balance (16–17) with $\pi = \Psi$, (17–2) becomes

$$\int_{\mathscr{G}} \left(\overset{\square}{\psi} - \psi K V \right) da + \int_{\partial\mathscr{G}} (\psi - \sigma) V_{\partial\mathscr{G}}\, ds \leq - \int_{\mathscr{G}} \left\{ \sigma K V + \boldsymbol{\tau} \cdot \overset{\square}{\mathbf{m}} + g^{\mathscr{S}} V \right\} da. \tag{17–4}$$

Given a time τ, it is possible to find a second referential control volume $P'(t)$ with $P'(\tau) = P(\tau)$, but with $V_{\partial\mathscr{G}'}(\boldsymbol{X}, \tau)$, the velocity of $\partial\mathscr{G}'(\tau)$ in the direction of its normal, an *arbitrary* scalar field on $\partial\mathscr{G}'(\tau)$; satisfaction of (17–4) for all such $V_{\partial\mathscr{G}'}$ implies that

$$\sigma = \psi \tag{17–5}$$

and the *surface tension and surface free energy coincide*.[1] Thus, because $\mathscr{G}$ is arbitrary, (17–4) reduces to the **interfacial dissipation inequality**

$$\overset{\square}{\psi} + \boldsymbol{\tau} \cdot \overset{\square}{\mathbf{m}} + g^{\mathscr{S}} V \leq 0, \tag{17–6}$$

with $(\cdots)^{\square}$ the normal time derivative following $\mathscr{S}$.

It is worth noting the similarities between the bulk tension π and the surface tension σ:

- Bulk tension works to increase the volume of bulk material, surface tension works to increase the area of the interface.
- The configurational stresses $\boldsymbol{C} = \pi \mathbf{1}$ and $\mathbf{C}_{\text{tan}} = \sigma \mathbf{P}$ have isotropic forms; these are not consequences of material symmetry, but are general results that follow from invariance under changes in velocity fields describing the migrations of $\partial P(t)$ and $\partial\mathscr{G}(t)$.
- Both π and σ are related to energy: π to bulk free energy, σ to interfacial free energy.

A reversal of the steps leading to (17–6) shows that the dissipation, defined as the right side of (17–2) minus the left, is given by

$$\mathscr{D}(P) = - \int_{\mathscr{G}} \left(\overset{\square}{\psi} + \boldsymbol{\tau} \cdot \overset{\square}{\mathbf{m}} + g^{\mathscr{S}} V \right) da \geq 0. \tag{17–7}$$

Assume that $e^{\mathscr{S}} = 0$. Then, by (16–15) with $\sigma = \psi$, the second law (17–2) for a stationary control volume P takes a form

$$\frac{d}{dt} \left\{ \int_P \Psi\, dv + \int_{\mathscr{G}} \psi\, da \right\} \leq \int_{\partial\mathscr{G}} \psi V_{\partial\mathscr{G}}\, ds + \int_{\partial\mathscr{G}} V \boldsymbol{\tau} \cdot \mathbf{n}\, ds \tag{17–8}$$

in which the working of the surface tension is replaced by an accounting of the flow of interfacial energy across $\partial\mathscr{G}$.

[1] Gurtin and Struthers [1990] (cf. Gurtin [1991]).

CHAPTER 18

Constitutive Equations. Evolution Equation for the Interface

Guided by the interfacial dissipation inequality, I allow the free energy (and hence the surface tension), the shear, and the normal internal force to depend constitutively on the orientation and kinetics of the interface through dependencies on the interface normal and normal velocity. The second law in the form of the interfacial dissipation inequality is then used to restrict these constitutive equations.

a. Functions of orientation

Let $\varphi(\mathbf{m})$ be a scalar function and $\mathbf{f}(\mathbf{m})$ a vector function of the (unit) interface normal $\mathbf{m}$. The derivatives $\partial_{\mathbf{m}}\varphi(\mathbf{m})$ and $\partial_{\mathbf{m}}\mathbf{f}(\mathbf{m})$ are defined by the chain rule. Given any curve $\mathbf{m}(\lambda)$ on the unit sphere,

$$\varphi(\mathbf{m})^{\cdot} = \{\partial_{\mathbf{m}}\varphi(\mathbf{m})\}\cdot\dot{\mathbf{m}}, \qquad \mathbf{f}(\mathbf{m})^{\cdot} = \left\{\partial_{\mathbf{m}}\mathbf{f}(\mathbf{m})\right\}\dot{\mathbf{m}} \tag{18–1}$$

(with the dot here the derivative with respect to λ); $\partial_{\mathbf{m}}\varphi(\mathbf{m})$ is tangent to the unit sphere, while $\partial_{\mathbf{m}}\mathbf{f}(\mathbf{m})$ is defined by (18–1) only on vectors perpendicular to $\mathbf{m}$, but is extended by requiring that $\left\{\partial_{\mathbf{m}}\mathbf{f}(\mathbf{m})\right\}\mathbf{m} = 0$. Then for $\mathbf{m}(\boldsymbol{X}, t)$ the unit normal field on $\mathscr{S}(t)$, a calculation using the chain rule and (15–12a) yields the identities

$$\nabla_{\mathscr{S}}\varphi(\mathbf{m}) = -\mathbf{L}\partial_{\mathbf{m}}\varphi(\mathbf{m}), \tag{18–2a}$$

$$\nabla_{\mathscr{S}}\mathbf{f}(\mathbf{m}) = -\left\{\partial_{\mathbf{m}}\mathbf{f}(\mathbf{m})\right\}\mathbf{L}, \tag{18–2b}$$

$$\mathrm{Div}_{\mathscr{S}}\mathbf{f}(\mathbf{m}) = -\left\{\partial_{\mathbf{m}}\mathbf{f}(\mathbf{m})\right\}\cdot\mathbf{L}. \tag{18–2c}$$

b. Constitutive equations

I base the theory on constitutive equations of the form

$$\psi = \hat{\psi}(\mathbf{m}, V), \tag{18–3a}$$

$$\boldsymbol{\tau} = \hat{\boldsymbol{\tau}}(\mathbf{m}, V), \tag{18–3b}$$

$$g^{\mathscr{S}} = \hat{g}^{\mathscr{S}}(\mathbf{m}, V). \tag{18–3c}$$

Given an arbitrary *constitutive process* (i.e., an evolution of the interface consistent with the constitutive equations (18–3a)), the force balances (16–12) give the external force $e^{\mathscr{S}}$ and the indeterminate internal force $\mathbf{P}\boldsymbol{g}^{\mathscr{S}}$ needed to support the process; these balances in no way restrict the class of processes possible for the material. On the other hand, unless the constitutive equations are suitably restricted, not all constitutive processes will be compatible with the second law in the form (17–6). A basic hypothesis of the theory is that *all constitutive processes be consistent with the dissipation inequality* (17–6).[1]

This hypothesis has strong consequences. Granted (18–3), (17–6) is equivalent to the inequality

$$\partial_V \hat{\psi}(\mathbf{m}, V)\, \overset{\square}{V} + \left\{ \partial_{\mathbf{m}} \hat{\psi}(\mathbf{m}, V) + \hat{\boldsymbol{\tau}}(\mathbf{m}, V) \right\} \cdot \overset{\square}{\mathbf{m}} + \hat{g}^{\mathscr{S}}(\mathbf{m}, V) V \leq 0, \tag{18–4}$$

with $(\cdots)^{\square}$ the normal time derivative following $\mathscr{S}$. Since one can always find an evolution of the interface such that $\mathbf{m}$, V, $\overset{\square}{V}$, and $\overset{\square}{\mathbf{m}}$ have arbitrary values at some given point and time (cf. the paragraph containing (15–28)), and since the left side of (18–4) is *linear* in $\overset{\square}{V}$ and $\overset{\square}{\mathbf{m}}$, the coefficients of these two fields must vanish, because otherwise $\overset{\square}{V}$ and $\overset{\square}{\mathbf{m}}$ could be chosen to violate (18–4). Thus $\partial_V \hat{\psi} = 0$, so that $\hat{\psi}$ is independent of V, and $\boldsymbol{\tau} = \partial_{\mathbf{m}} \hat{\psi}$, so that $\boldsymbol{\tau}$ is also. The constitutive equations must therefore be consistent with the following restrictions:

(i) the free energy ψ and the shear $\boldsymbol{\tau}$ must be independent V and related through

$$\psi = \hat{\psi}(\mathbf{m}), \qquad \boldsymbol{\tau} = -\partial_{\mathbf{m}} \hat{\psi}(\mathbf{m}); \tag{18–5}$$

(ii) the normal internal force must obey the following inequality for all values of its arguments:

$$\hat{g}^{\mathscr{S}}(\mathbf{m}, V) V \leq 0. \tag{18–6}$$

A consequence of (18–6) is that the constitutive equation $g^{\mathscr{S}} = \hat{g}^{\mathscr{S}}(\mathbf{m}, V)$, when smooth in V, must have the form

$$g^{\mathscr{S}} = -b(\mathbf{m}, V) V, \qquad b(\mathbf{m}, V) \geq 0, \tag{18–7}$$

with $b(\mathbf{m}, V)$ a constitutive quantity called the **kinetic modulus** (cf. (9–25)).[2] The reduced relations (18–5) and (18–7) are the most general smooth constitutive

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[1] This extension of the procedure of Coleman and Noll [1963] to two-phase materials is due to Gurtin [1988] (cf. Angenent and Gurtin [1989], Gurtin [1993b]).

[2] For $b(\mathbf{m}, V)$ independent of V, $b(\mathbf{m})^{-1}$ is referred to as the mobility of the interface.

equations of the form (18–3a) that are consistent with the dissipation inequality (17–6). A consequence of the reduced relations is that the constitution of the material is determined by two functions: the interfacial free energy $\hat{\psi}(\mathbf{m})$ and the kinetic modulus $b(\mathbf{m}, V)$. I now add the assumption that, for all values of their arguments,

$$\hat{\psi}(\mathbf{m}) > 0, \qquad b(\mathbf{m}, V) > 0. \tag{18–8}$$

For an **isotropic body** the interfacial energy $\hat{\psi}(\mathbf{m})$ and the kinetic modulus $b(\mathbf{m}, V)$ are independent of the orientation $\mathbf{m}$, so that, in particular,

$$\psi = \text{constant}, \qquad \boldsymbol{\tau} = \mathbf{0}. \tag{18–9}$$

A standard assumption is **linear kinetics** for which $b(\mathbf{m}, V)$ is independent of V:

$$g^{\mathscr{S}} = -b(\mathbf{m})V. \tag{18–10}$$

By (18–5), the dissipation (17–7) takes the form

$$\mathscr{D}(P) = -\int_{\mathscr{G}} g^{\mathscr{S}} V \, da = \int_{\mathscr{G}} b(\mathbf{m}, V)V^2 \, da \geq 0, \qquad \mathscr{G} = P \cap \mathscr{S}, \tag{18–11}$$

and is quadratic in V when the kinetics is linear.

Anisotropy of the interface manifests itself in a nontrivial dependence of $\hat{\psi}(\mathbf{m})$ on $\mathbf{m}$. An interesting and important consequence of (18–5) is that for *for an anisotropic interface the surface shear cannot generally vanish*.[3] This demonstrates the non-intuitive nature of configurational forces; the interface is infinitesimally thin, yet it supports shear.

Note that, because $\sigma = \psi$, (18–2a) and (18–5) imply that $\nabla_{\mathscr{S}}\sigma = \mathbf{L}\boldsymbol{\tau}$ and therefore that *the tangential force balance* (16–12b) *is satisfied identically with*

$$\mathbf{P}\boldsymbol{g}^{\mathscr{S}} = \mathbf{0}. \tag{18–12}$$

The surface shear must be balanced by configurational couples exerted by the bulk material, although such couples, being indeterminate, need not be made explicit. This furnishes an additional argument in support of the separate treatment of configurational forces when discussing deformation. If the variational treatment of Section 1b (for an elastic body) is generalized to include an anisotropic interfacial energy, then the resulting Euler-Lagrange equation *in bulk* remains (1–7) supplemented by (1–2). As is well known, granted invariance of the energy under changes in spatial observer, this classical equation supports neither bulk internal couples nor bulk couple stresses; a configurational system is needed to balance the couples induced by surface shear.

[3]This is clear from the work of Herring [1951b], Hoffman and Cahn [1972], Cahn and Hoffman [1974], who discuss the equilibrium theory within a variational framework.

c. Evolution equation for the interface

Assume that the external force $\boldsymbol{e}^{\mathscr{S}}$ vanishes. By (18–2c) and the relations $\sigma = \psi$, $\boldsymbol{\tau} = -\partial_{\mathbf{m}}\hat{\psi}$, and $K = \mathbf{1} \cdot \mathbf{L}$,

$$\begin{aligned} \sigma K + \mathrm{Div}_{\mathscr{S}} \boldsymbol{\tau} &= \hat{\psi}(\mathbf{m})K + \left(\partial_{\mathbf{m}}\partial_{\mathbf{m}}\hat{\psi}(\mathbf{m})\right) \cdot \mathbf{L} \\ &= \left\{\hat{\psi}(\mathbf{m})\mathbf{1} + \partial_{\mathbf{m}}\partial_{\mathbf{m}}\hat{\psi}(\mathbf{m})\right\} \cdot \mathbf{L}, \end{aligned} \tag{18–13}$$

and the evolution equation for the interface follows from the normal force balance (16–12a) and the relations $\boldsymbol{C} = \Psi\mathbf{1}$, $F = [\Psi]$, and $g^{\mathscr{S}} = -b(\mathbf{m}, V)V$:

$$b(\mathbf{m}, V)V = \left\{\hat{\psi}(\mathbf{m})\mathbf{1} + \partial_{\mathbf{m}}\partial_{\mathbf{m}}\hat{\psi}(\mathbf{m})\right\} \cdot \mathbf{L} + F.^{4} \tag{18–14}$$

Consider an isotropic body with linear kinetics, modulo a rescaling that yields $b = \psi = 1$; and assume that there is no difference in free energy between the bulk phases, so that $F = 0$. The evolution equation (18–14) then has the sample form[5]

$$V = K, \tag{18–15}$$

a parabolic partial differential equation with a large literature.[6]

The linearity in (18–14) in $\mathbf{L}$ is a consequence of the geometry; this linearity is not constitutive.

The derivatives in (18–14) must respect the constraint $|\mathbf{m}| = 1$. A simpler form of the equation follows if $\hat{\psi}(\mathbf{m})$ is extended from the unit sphere to all nonzero vectors by defining $\hat{\psi}(z) = |z|\hat{\psi}(z/|z|)$, because then the term $\{\ldots\}$ in (18–14) reduces to $\partial_z\partial_z\tilde{\psi}(z)$ evaluated at $z = \mathbf{m}$, the derivative ∂_z being in $\mathbf{R}^3$.

For *nonsmooth* interfaces—which are possible when $\tilde{\psi}(z)$ is nonconvex—the evolution equation (18–14) is not, by itself, sufficient to describe the motion of the interface; the weaker form (16–3) of the configurational force balance must be used. For example, across a curve $\mathscr{C}$ defined by a jump in the interface normal $\mathbf{m}$, (16–3) leads to the balance $[\mathbf{Cn}] = \mathbf{0}$, where $[\mathbf{Cn}]$ denotes the jump in superficial traction across such a curve.

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[4]Proposed by Uwaha [1987, eq. (2)] (in $\mathbf{R}^2$ with $b = b(\mathbf{m})$) and independently (in $\mathbf{R}^3$) by Gurtin [1988, eq. (8–3)]. (Cf. also Angenent and Gurtin [1989] (in $\mathbf{R}^2$).) Evolution according to (18–14) with $b = b(\mathbf{m})$ is studied by Angenent [1991], Chen, Giga, and Goto [1991], and Soner [1993]. The special case $V = b(\mathbf{m})^{-1}F$ was introduced by Frank [1958]. A formulation of (18–14) using a variational definition of the curvature term (Taylor [1992]) is given by Taylor, Cahn, and Handwerker [1992], who give extensive references.

[5]Burke and Turnbull [1952] and Mullins [1956] introduced $V = K$ to study the motion of grain boundaries.

[6]Cf. Brakke [1978], Sethian [1985], Abresch and Langer [1986], Gage and Hamilton [1986], Grayson [1987], Osher and Sethian [1988], Evans and Spruck [1991, 1992], Chen, Giga and Goto [1991], Giga and Sato [1991], Taylor, Cahn, and Handwerker [1992], Almgren, Taylor, and Wang [1993]).

d. Lyapunov relations

Assume that the body B is bounded, that $\mathscr{S} = \mathscr{S}(t)$ is a closed surface contained in the interior of B, and that $\boldsymbol{e}^{\mathscr{S}} = \mathbf{0}$. Let Ψ_α and Ψ_β denote the constant values of the bulk free energy in the regions $B_\alpha = B_\alpha(t)$ and $B_\beta = B_\beta(t)$ occupied by phases α and β, so that $F = \Psi_\beta - \Psi_\alpha$. Then, by (17–1),

$$\frac{d}{dt}\left\{\int_B \Psi\, dv\right\} = \frac{d}{dt}\left\{\int_B (\Psi - \Psi_\beta)\, dv\right\} = -\frac{d}{dt}\{F \text{ vol}(B_\alpha)\}; \tag{18–16}$$

hence (16–2) (with $\boldsymbol{q} = \mathbf{0}$, $\mathscr{G} = \mathscr{S}$, and $\partial\mathscr{S} = \emptyset$), (17–2), and (17–7) imply that *the total free energy decreases with time*:

$$\frac{d}{dt}\left\{\int_{\mathscr{S}} \hat{\psi}(\mathbf{m})\, da - F \text{ vol}(B_\alpha)\right\} = -\int_{\mathscr{S}} b(\mathbf{m}, V) V^2\, da \leq 0. \tag{18–17}$$

In particular, for an isotropic body with linear kinetics, (18–9) and (18–10) imply that

$$\frac{d}{dt}\{\psi \text{ area}(\mathscr{S}) - F \text{ vol}(B_\alpha)\} = -\int_{\mathscr{S}} bV^2\, da \leq 0. \tag{18–18}$$

A consequence of (18–18) is that, for $F = 0$, the area of the interface decreases with time. The inequality $F > 0$ occurs when B_α has a lower bulk energy than B_β; in this instance (18–18) indicates a tendency for the more stable α-phase to grow, at least in those situations where volume dominates area.

CHAPTER 19

Two-Dimensional Theory[1]

Because of its geometric simplicity, I shall develop the two-dimensional theory from scratch, rather than as a special case of the three-dimensional theory.

Notation with a direct counterpart in the three-dimensional theory will be used without explanation, and arguments that following directly from their analogs in that theory will be left to the reader.

a. Kinematics

The interface between phases is presumed to be a smoothly evolving *closed curve* $\mathscr{C}(t)$. Further, $\mathbf{t}(\boldsymbol{X}, t)$ and $\mathbf{m}(\boldsymbol{X}, t)$, respectively, denote **tangent** and **normal** fields for $\mathscr{C}(t)$ such that, in components,

$$\mathbf{t} = (\cos\vartheta, \sin\vartheta), \qquad \mathbf{m} = (-\sin\vartheta, \cos\vartheta), \tag{19–1}$$

with $\vartheta(\boldsymbol{X}, t)$ the counterclockwise angle from the (1, 0) axis to $\mathbf{t}(\boldsymbol{X}, t)$; ds denotes the arc-length differential, with $ds > 0$ in the direction of $\mathbf{t}$; and *the subscript s denotes partial differentiation with respect to arc length*. Then (19–1) yield the **Frenet formulas**

$$\mathbf{t}_s = K\mathbf{m}, \qquad \mathbf{m}_s = -K\mathbf{t}, \tag{19–2}$$

with

$$K = \vartheta_s \tag{19–3}$$

the **curvature**.

[1] Cf. Angenent and Gurtin [1989] and Gurtin [1993b] for a more complete discussion, with proofs.

The normal time derivative $\overset{\square}{\vartheta}$ and the arc-length derivative of the normal velocity V of $\mathscr{C}$ are related through

$$\overset{\square}{\vartheta} = V_s, \tag{19–4}$$

which represents a two-dimensional analog of (15–24).

Attention is restricted to migrating control volumes $P(t)$ for which $\mathscr{C}(t)$ intersects $\partial P(t)$ at exactly two points. I will consistently use the following notation for such control volumes: $\mathscr{G}(t)$ denotes the portion of the interface in $P(t)$,

$$\mathscr{G}(t) = \mathscr{C}(t) \cap P(t);$$

$\boldsymbol{X}_A(t)$ and $\boldsymbol{X}_B(t)$, respectively, denote the initial and terminal points of $\mathscr{G}(t)$ (in the sense of arc length); for any function $\Phi(\boldsymbol{X}, t)$,

$$\Phi_A(t) = \Phi(\boldsymbol{X}_A(t), t), \qquad \Phi_B(t) = \Phi(\boldsymbol{X}_B(t), t); \tag{19–5}$$

$u_A(t)$ and $u_B(t)$ defined by

$$u_A = \mathbf{t}_A \cdot \dot{\boldsymbol{X}}_A, \qquad u_B = \mathbf{t}_B \cdot \dot{\boldsymbol{X}}_B \tag{19–6}$$

are the **tangential endpoint velocities** of $\mathscr{G}(t)$. Then, because V_A and V_B are the normal components of the velocities $\dot{\boldsymbol{X}}_A$ and $\dot{\boldsymbol{X}}_B$ (cf. (15–29)),

$$\dot{\boldsymbol{X}}_A = u_A \mathbf{t}_A + V_A \mathbf{m}_A, \qquad \dot{\boldsymbol{X}}_B = u_B \mathbf{t}_B + V_B \mathbf{m}_B. \tag{19–7}$$

Given a smooth field $\varphi(\boldsymbol{X}, t)$ on $\mathscr{C}(t)$ and a smoothly evolving connected subcurve $\mathscr{G} = \mathscr{G}(t)$ of $\mathscr{C}(t)$,

$$\frac{d}{dt}\left\{\int_{\mathscr{G}} \varphi \, ds\right\} = \int_{\mathscr{G}} \left(\overset{\square}{\varphi} - \varphi K V\right) ds + \varphi_B u_B - \varphi_A u_A, \tag{19–8}$$

with $\overset{\square}{\varphi}$ the normal time derivative of φ; (19–8) is the analog of the transport theorem (15–31).

b. Configurational forces. Working. Second law

The bulk stress $\boldsymbol{C}$ and the internal and external interfacial forces $\boldsymbol{g}^{\mathscr{C}}$ and $\boldsymbol{e}^{\mathscr{C}}$ are direct counterparts of the corresponding fields introduced in Section 16a, and, anticipating (17–3), I omit mention of the internal bulk force $\boldsymbol{g}$ from the outset.

Because the interface is a curve, interfacial stress is most simply described by a **vector stress** $\mathbf{C}(\boldsymbol{X}, t)$. Let S denote the arc length of a fixed point $\boldsymbol{X} \in \mathscr{C}(t)$. Then suppressing t, $\mathbf{C}(\boldsymbol{X})$ represents the force exerted across $\boldsymbol{X}$ by the portion of the interface with $s > S$ on the portion with $s < S$. The expansion

$$\mathbf{C} = \sigma \mathbf{t} + \tau \mathbf{m} \tag{19–9}$$

represents a counterpart of (16–8), with $\sigma = \mathbf{C} \cdot \mathbf{t}$ **surface tension** and $\tau = \mathbf{C} \cdot \mathbf{m}$ **surface shear**.

Let $P = P(t)$ be a migrating control volume. Because $\dot{\boldsymbol{X}}_A$ and $\dot{\boldsymbol{X}}_B$ represent the velocities of the endpoints of the portion of the interface in P, $\mathbf{C}$ performs work on P of amount $\mathbf{C}_B \cdot \dot{\boldsymbol{X}}_B - \mathbf{C}_A \cdot \dot{\boldsymbol{X}}_A$; by (19–7) and (19–9),

$$\mathbf{C}_B \cdot \dot{\boldsymbol{X}}_B - \mathbf{C}_A \cdot \dot{\boldsymbol{X}}_A = \sigma_B u_B + \tau_B V_B - (\sigma_A u_A + \tau_A V_A). \tag{19–10}$$

Thus, for $\boldsymbol{v}$ a velocity field for $\mathscr{C}$ and $\boldsymbol{q}$ a velocity field for ∂P, the **working** $W(P)$ has the form

$$W(P) = \int_{\partial P} \boldsymbol{C}\boldsymbol{n} \cdot \boldsymbol{q}\, ds + \int_{\mathscr{G}} \boldsymbol{e}^{\mathscr{C}} \cdot \boldsymbol{v}\, ds + \mathbf{C}_B \cdot \dot{\boldsymbol{X}}_B - \mathbf{C}_A \cdot \dot{\boldsymbol{X}}_A. \tag{19–11}$$

Under a change in material observer as defined by (2–9), $\dot{\boldsymbol{X}}_A$ transforms to $\dot{\boldsymbol{X}}_A + \boldsymbol{a}$, and similarly for $\dot{\boldsymbol{X}}_B$; invariance of the working under such changes yields the **configurational force balance** (cf. (16–3))

$$\int_{\partial P} \boldsymbol{C}\boldsymbol{n}\, ds + \int_{\mathscr{G}} (\boldsymbol{g}^{\mathscr{C}} + \boldsymbol{e}^{\mathscr{S}})\, ds + \mathbf{C}_B - \mathbf{C}_A = 0. \tag{19–12}$$

Consider the δ-pillbox $\mathscr{G}_\delta(t)$ about an arbitrary connected subcurve $\mathscr{G}(t)$ of the interface. Then (19–12) applied to $\mathscr{G}_\delta(t)$ yields, in the limit $\delta \to 0$,

$$\int_{\mathscr{G}} ([\boldsymbol{C}]\mathtt{m} + \boldsymbol{g}^{\mathscr{C}} + \boldsymbol{e}^{\mathscr{C}})\, ds + \mathbf{C}_B - \mathbf{C}_A = \mathbf{0}; \tag{19–13}$$

dividing this relation by the length ℓ of $\mathscr{G}$ and passing to the limit $\ell \to 0$ yields

$$\mathbf{C}_s + [\boldsymbol{C}]\mathtt{m} + \boldsymbol{g}^{\mathscr{C}} + \boldsymbol{e}^{\mathscr{C}} = 0. \tag{19–14}$$

The argument leading to (16–5a) is valid here also; hence $\boldsymbol{C} = \pi \mathbf{1}$, $\boldsymbol{e}^{\mathscr{C}} = e^{\mathscr{C}}\mathtt{m}$. Thus, by (19–9) and the Frenet formulas, the normal and tangential components of (19–14) are

$$\sigma K + \tau_s + [\pi] + g^{\mathscr{C}} + e^{\mathscr{C}} = 0, \tag{19–15a}$$

$$\sigma_s - \tau K + (\boldsymbol{g}^{\mathscr{C}} \cdot \mathtt{t})\mathtt{t} = 0, \tag{19–15b}$$

with $g^{\mathscr{C}} = \boldsymbol{g}^{\mathscr{C}} \cdot \mathtt{m}$ the **normal internal force**.

Next, by (19–4),

$$(V\tau)_s = V\tau_s + \tau \overset{\square}{\vartheta}, \tag{19–16}$$

and therefore, using (19–10),

$$\mathbf{C}_B \cdot \dot{\boldsymbol{X}}_B - \mathbf{C}_A \cdot \dot{\boldsymbol{X}}_A = \sigma_B u_B - \sigma_A u_A + \int_{\mathscr{G}} \left(V\tau_s + \tau \overset{\square}{\vartheta} \right) ds.$$

Thus, eliminating the term τ_s using (19–15a), the working becomes (cf. (16–17))

$$W(P) = -\int_{\mathscr{G}} \left\{ \sigma K V - \tau \overset{\square}{\vartheta} + \left([\pi] + g^{\mathscr{C}} \right) V \right\} ds + \int_{\partial P} \pi U\, ds + \sigma_B u_B - \sigma_A u_A. \tag{19–17}$$

The second law has the form

$$\frac{d}{dt} \left\{ \int_P \Psi\, da + \int_{\mathscr{G}} \psi\, ds \right\} \leq W(P) \tag{19–18}$$

for every migrating control volume $P = P(t)$, with $\psi(\boldsymbol{X}, t)$ the interfacial free energy per unit length and Ψ the bulk free energy per unit area. As before, Ψ is presumed to be constant in each phase.

The argument leading to the relations $\pi = \Psi$ and $\sigma = \psi$ are as given in Chapter 17 and make use of (19–17), (19–18), and the transport theorem (19–8); these, in turn lead to the interfacial dissipation inequality (cf. (17–6))

$$\overset{\square}{\psi} - \tau\, \overset{\square}{\vartheta} + g^{\mathscr{C}} V \leq 0. \tag{19–19}$$

c. Constitutive theory

I consider constitutive equations of the form

$$\psi = \hat{\psi}(\vartheta, V), \tag{19–20a}$$

$$\tau = \hat{\tau}(\vartheta, V), \tag{19–20b}$$

$$g^{\mathscr{C}} = \hat{g}^{\mathscr{C}}(\vartheta, V), \tag{19–20c}$$

and, as before, require that all constitutive processes be consistent with the dissipation inequality (19–19); this yields the following restrictions (cf. (18–5) and (18–6)):

(i) the free energy ψ and the shear τ are independent of V and related through

$$\psi = \hat{\psi}(\vartheta), \qquad \tau = \hat{\psi}'(\vartheta); \tag{19–21}$$

(ii) the normal internal force must obey the inequality

$$\hat{g}^{\mathscr{C}}(\vartheta, V) V \leq 0. \tag{19–22}$$

Granted smoothness, the most general form of the constitutive equation for $g^{\mathscr{C}}$ consistent with (19–22) is

$$g^{\mathscr{C}} = -b(\vartheta, V) V, \qquad b(\vartheta, V) \geq 0. \tag{19–23}$$

I henceforth assume that

$$\hat{\psi}(\vartheta) > 0, \qquad b(\vartheta, V) > 0. \tag{19–24}$$

Using (19–1), the tangent and normal may be considered functions $\mathbf{t} = \mathbf{t}(\vartheta)$ and $\mathbf{m} = \mathbf{m}(\vartheta)$, which renders the stress vector (19–9) also a function of ϑ:

$$\mathbf{C} = \hat{\mathbf{C}}(\vartheta) = \hat{\psi}(\vartheta)\mathbf{t}(\vartheta) + \hat{\psi}'(\vartheta)\mathbf{m}(\vartheta). \tag{19–25}$$

Further, by (19–1)

$$\mathbf{t}'(\vartheta) = \mathbf{m}(\vartheta), \qquad \mathbf{m}'(\vartheta) = -\mathbf{t}(\vartheta), \tag{19–26}$$

and this yields

$$\hat{\mathbf{C}}'(\vartheta) = a(\vartheta)\mathbf{m}(\vartheta), \tag{19–27}$$

with $a(\vartheta)$ the **stress-angle modulus**

$$a(\vartheta) = \hat{\psi}(\vartheta) + \hat{\psi}''(\vartheta). \tag{19–28}$$

d. Evolution equation for the interface

By (19–3) and (19–21), $\tau_s = \hat{\psi}''(\vartheta)\vartheta_s = \hat{\psi}''(\vartheta)K$; thus, assuming that $\boldsymbol{e}^{\mathscr{C}}$ vanishes, (19–23), (19–28), and (19–15a) with $\pi = \Psi$, $\sigma = \psi$, and

$$F = [\Psi](= \text{constant})$$

yield the evolution equation (cf. Footnote 4, Chapter 18)

$$b(\vartheta, V)V = a(\vartheta)K + F, \tag{19–29}$$

or, granted linear kinetics,

$$b(\vartheta)V = a(\vartheta)K + F. \tag{19–30}$$

For an isotropic body modulo a rescaling that yields $b = \psi = 1$, and for $F = 0$, (19–30) reduces to the **curve-shortening equation**

$$V = K;$$

as shown by Gage and Hamilton[2] and Grayson,[3] a simple closed curve in $\mathbf{R}^2$ of arbitrary initial shape with evolution governed by $V = K$ shrinks to a point in finite time, with its asymptotic shape a circle.

Locally, an evolving curve may be represented as the graph of a function $y = h(x, t)$, provided the $x = x_1$ and $y = x_2$ axes are chosen appropriately. Consider a choice in which arc length increases with increasing x (Figure 19.1). Then, denoting partial differentiation by a subscript,

$$h_x = \tan\vartheta, \qquad V = h_t \cos\vartheta, \qquad K = h_{xx}(1 + \tan^2\vartheta)^{-3/2}, \tag{19–31}$$

$-\pi/2 < \vartheta < \pi/2$, and, assuming linear kinetics, the evolution equation (19–30) takes the form

$$B(\vartheta)h_t = A(\vartheta)h_{xx} + F, \tag{19–32}$$

where

$$B(\vartheta) = b(\vartheta)\cos\vartheta, \qquad A(\vartheta) = (1 + \tan^2\vartheta)^{-3/2}a(\vartheta). \tag{19–33}$$

The equations (19–30) and (19–32) are equivalent for $-\pi/2 < \vartheta < \pi/2$; for other values of ϑ a different set of axes must be chosen.

By (19–24) and (19–33), $B(\vartheta) > 0$, while $\operatorname{sgn} A(\vartheta) = \operatorname{sgn} a(\vartheta)$ with $A(\vartheta) = 0 \Leftrightarrow a(\vartheta) = 0$. This yields the following important remark.

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[2][1986].
[3][1987].

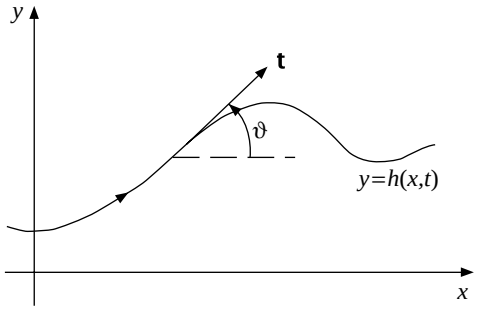

FIGURE 19.1. Sign conventions when the curve is a graph $y = h(x, t)$.

Remark. The evolution equation (19–30) is:

(i) parabolic on any angle interval over which the stress-angle modulus $a(\vartheta)$ is strictly positive;
(ii) backward parabolic on any angle interval over which $a(\vartheta)$ is strictly negative;[4]
(iii) degenerate parabolic at any angle ϑ_0 for which $a(\vartheta_0) = 0$, but $a(\vartheta) > 0$ for all $\vartheta \neq \vartheta_0$ sufficiently close to ϑ_0.

e. Corners

If the interface has a corner $\boldsymbol{Z}$, then ϑ has a jump discontinuity and the curvature $K = \vartheta_s$ is undefined. The partial differential equation is then not meaningful, at least in a classical sense, but the balance of (19–13) is, and passing to the limit as $\mathscr{G}$, with initial and terminal points on opposite sides of $\boldsymbol{Z}$, shrinks to $\boldsymbol{Z}$, yields the requirement that $\hat{\mathbf{C}}$ be continuous across $\boldsymbol{Z}$, or equivalently, by (19–25), that

$$\hat{\mathbf{C}}(\vartheta^-) = \hat{\mathbf{C}}(\vartheta^+), \tag{19–34}$$

where $\vartheta^\pm$ denote the limits of the angle ϑ from the two sides $\boldsymbol{Z}$. Thus *the set of corners consistent with balance of configurational forces is a constitutive property of the material*.

f. Angle-convexity. The Frank diagram

The parabolicity of the evolution equation (19–32) is related to the convexity of the interfacial free energy $\hat{\psi}(\vartheta)$. The definition of convexity for such a function is not obvious; the usual definition is inapplicable because $\hat{\psi}(\vartheta)$ is periodic. A

[4]Gjostein [1963] and Cahn and Hoffman [1974] give strong arguments in support of interfacial energies that satisfy $a(\vartheta) < 0$ for certain values of ϑ.

natural notion of convexity for a scalar function of ϑ can be phrased in terms of its Frank diagram.[5] As this notion will play a major role in discussing the fracture of anisotropic materials, the basic results concerning convexity are discussed for an arbitrary smooth scalar function $\gamma(\vartheta)$.

The **Frank diagram** of $\gamma(\vartheta)$ is the *curve* defined in polar coordinates (r, ϑ) by

$$\text{Frank}(\gamma) = \left\{(r, \vartheta) : r = \gamma(\vartheta)^{-1}, \gamma(\vartheta) > 0\right\}; \tag{19–35}$$

Frank(γ) is therefore the locus of $\gamma(\vartheta)^{-1}\mathbf{t}(\vartheta)$ as ϑ, the angle to $\mathbf{t}(\vartheta)$, traverses the set of angles with $\gamma(\vartheta) > 0$.

Assume that, as for the interfacial energy $\hat{\psi}(\vartheta)$,

$$\gamma(\vartheta) > 0 \qquad \text{for all } \vartheta,$$

so that Frank(γ) is a simple closed curve. Then $\gamma(\vartheta)$ will be termed **angle convex** if the Frank diagram of $\gamma(\vartheta)$ is strictly convex; that is, if given any angle ϑ, the tangent line $\mathscr{L}$ to Frank(γ) at the point $\boldsymbol{x}$ with angle ϑ intersects Frank(γ) only at $\boldsymbol{x}$. Here the **angle ϑ of a point $\boldsymbol{x} \neq 0$** is defined by $\mathbf{t}(\vartheta) = \boldsymbol{x}/|\boldsymbol{x}|$.

More generally, a **convexifying tangent** $\mathscr{L}$ is defined to be a straight line $\mathscr{L}$ tangent to Frank(γ) at one or more points but disjoint from the region interior to Frank(γ). The angles of the points of intersection of $\mathscr{L}$ with Frank(γ) will then be referred to as **tangency angles** of $\mathscr{L}$ (Figure 19.2); γ will be termed **regular** if each of its convexifying tangents has at most a finite number of tangency angles; an arbitrary angle ϑ will be termed **globally stable** if ϑ is a tangency angle of some convexifying tangent. (The set of globally stable angles is then the set of all angles at which the boundary of the convex hull of Frank(γ) coincides with Frank(γ).) Note that if γ is not angle convex, then Frank(γ) has at least one **convexifying tangent** $\mathscr{L}$ with two or more tangency angles.

The following remarks should underline the importance of angle convexity.[6]

An alternative but equivalent notion of convexity is phrased in terms of the function $\tilde{\gamma}(\boldsymbol{x})$ defined for $|\boldsymbol{x}| \neq 0$ by

$$\tilde{\gamma}(\boldsymbol{x}) = |\boldsymbol{x}|\gamma(\vartheta), \tag{19–36}$$

with ϑ the angle of $\boldsymbol{x}$. Then the angle convexity of $\gamma(\vartheta)$ is equivalent to the requirement that

$$\tilde{\gamma}(\boldsymbol{x}) - \tilde{\gamma}(\boldsymbol{z}) < (\boldsymbol{x} - \boldsymbol{z}) \cdot \nabla\tilde{\gamma}(\boldsymbol{x}) \tag{19–37}$$

for all nonzero vectors $\boldsymbol{x}$ and $\boldsymbol{z}$ whose angles are unequal. That angle convexity should be related to the more standard convexity expressed by (19–37) becomes somewhat more transparent upon recalling that the level sets of a convex function are convex, and noting that, by (19–36), the level set $\tilde{\gamma}(\boldsymbol{x}) \equiv 1$ has the equation $r = \gamma(\vartheta)^{-1}$.

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[5] Frank [1963]. Frank [1963], Angenent and Gurtin [1989], and Gurtin [1993b] define the Frank diagram using the angle to $\mathbf{m}$ rather than the angle to $\mathbf{t}$; this yields only minor differences (e.g., $\hat{\mathbf{C}}(\vartheta)$ is normal to Frank$(\hat{\psi})$ here, but tangent in Frank's theory). The tangent angle is used here because it is more natural in discussing fracture (Part G).

[6] Cf., e.g., Gurtin [1993], §7.

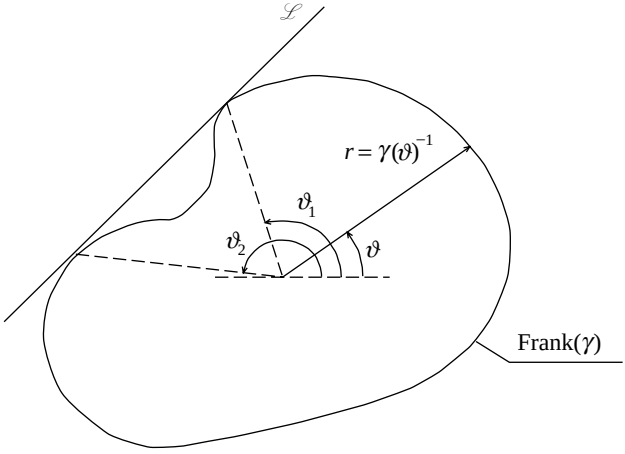

FIGURE 19.2. Frank diagram [the curve Frank(γ)] of a function $\gamma(\vartheta)$ that is not angle-convex. $\mathscr{L}$ is a convexifying tangent; ϑ_1 and ϑ_2 are tangency angles.

A result demonstrating the physical significance of angle convexity concerns the energy

$$\int_{\mathscr{G}} \hat{\psi}(\vartheta(s))\, ds$$

of an oriented curve $\mathscr{G}$ and is stated as follows: $\hat{\psi}(\vartheta)$ is angle convex if and only if, among all oriented curves from one *arbitrarily* prescribed point to another, the straight line segment has (strictly) least energy.

The *Wulff shape* (or Wulff crystal) corresponding to $\hat{\psi}(\vartheta)$ is the region W that minimizes

$$\int_{\partial W} \hat{\psi}(\vartheta(s))\, ds$$

over the set of all regions W of unit area. The set of globally stable angles of $\hat{\psi}(\vartheta)$ then consists of the angles of tangents to ∂W. (If $\hat{\psi}(\vartheta)$ is not angle convex, then ∂W will have corners, because some angles will be missing.)

Of importance is the Frank diagram of functions of the form

$$J(\vartheta) = \boldsymbol{j} \cdot \mathbf{t}(\vartheta) \tag{19–38}$$

where $\boldsymbol{j} \neq \mathbf{0}$ is a prescribed vector. The angle interval on which Frank(J) is defined consists of those angles ϑ for which $J(\vartheta) > 0$ and hence is of the form $(\vartheta_0, \vartheta_0 + \pi)$; not all angles are included.

Lemma 19.1.

(i) *Frank*(J) *is the* straight line *consisting of all* $\boldsymbol{x}$ *such that* $\boldsymbol{x} \cdot \boldsymbol{j} = 1$.

(ii) $\boldsymbol{j}$ *is orthogonal to Frank*(J) *and* $|\boldsymbol{j}|^{-1}$ *is the perpendicular distance from Frank*(J) *to the origin.*

(iii) *Let $\boldsymbol{j}_1$ and $\boldsymbol{j}_2$ be nonzero vectors and define $J_1(\vartheta) = \boldsymbol{j}_1 \cdot \mathbf{t}(\vartheta)$, $J_2(\vartheta) = \boldsymbol{j}_2 \cdot \mathbf{t}(\vartheta)$. Then*

$$\mathit{Frank}(J_1) = \mathit{Frank}(J_2) \quad \Rightarrow \quad \boldsymbol{j}_1 = \boldsymbol{j}_2.$$

PROOF. Because Frank(J) is the curve

$$r = \{\mathbf{t}(\vartheta) \cdot \boldsymbol{j}\}^{-1}, \tag{19–39}$$

and because $\boldsymbol{x} = r\,\mathbf{t}(\vartheta)$, (i) follows. Next, $\boldsymbol{j}$ is the gradient of the function $\boldsymbol{x} \cdot \boldsymbol{j} - 1$; thus $\boldsymbol{j}$ is orthogonal to Frank(J). Further $\boldsymbol{x} = |\boldsymbol{j}|^{-2}\boldsymbol{j} \in$ Frank(J); hence $|\boldsymbol{j}|^{-1}$ is the perpendicular distance from Frank(J) to the origin. To verify (iii), assume that the Frank diagrams of J_1 and J_2 coincide. Then, by (ii), $\boldsymbol{j}_1 = \pm\boldsymbol{j}_2$. But for $\boldsymbol{j}_1 = -\boldsymbol{j}_2$ the two Frank diagrams would not coincide, since their angle intervals would differ by the angle π. □

Lemma 19.2. *Let $\gamma(\vartheta)$ be a smooth, strictly positive function and let $\boldsymbol{j}$ be a nonzero vector. Then*

(i) *Given an angle φ with $J(\varphi) > 0$,*

$$J(\varphi) = \gamma(\varphi) \quad \Leftrightarrow \quad \mathit{Frank}(\gamma) \text{ intersects } \mathit{Frank}(J) \text{ at a point } \boldsymbol{x} \text{ with angle } \varphi; \tag{19–40a}$$

$$J(\varphi) < \gamma(\varphi) \quad \Leftrightarrow \quad \text{the point } \boldsymbol{x} \text{ on } \mathit{Frank}(\gamma) \text{ with angle } \varphi \text{ lies strictly outside of } \mathit{Frank}(\gamma). \tag{19–40b}$$

(ii) *Frank(J) is tangent to Frank(γ) at the point $\boldsymbol{x}$ with angle φ if and only if*

$$\boldsymbol{j} = \gamma(\varphi)\mathbf{t}(\varphi) + \gamma'(\varphi)\mathbf{m}(\varphi). \tag{19–41}$$

PROOF. Assertion (i) is a direct consequence of the definition of the Frank diagram.

To verify (ii), assume that Frank(J) intersects Frank(γ) at a point $\boldsymbol{x}$ with angle φ, or equivalently, by (19–40a), that

$$J(\varphi) = \mathbf{t}(\varphi) \cdot \boldsymbol{j} = \gamma(\varphi). \tag{19–42}$$

Then Frank(J) and Frank(γ) are tangent at $\boldsymbol{x}$ if and only if the derivatives, with respect to ϑ, of the curves $r = \{\mathbf{t}(\vartheta) \cdot j\}^{-1}$ and $r = \gamma(\vartheta)^{-1}$ coincide at $\vartheta = \varphi$. Thus, since $\mathbf{t}'(\vartheta) = \mathbf{m}(\vartheta)$, Frank$(J)$ and Frank(γ) are tangent at $\boldsymbol{x}$ if and only if

$$\mathbf{m}(\varphi) \cdot \boldsymbol{j} = \gamma'(\varphi). \tag{19–43}$$

Therefore, if Frank(J) and Frank(γ) are tangent at $\boldsymbol{x}$, then (19–42) and (19–43), and hence (19–41), hold. Conversely, (19–41) yields (19–42) and (19–43), and these imply that the Frank diagrams are tangent at $\boldsymbol{x}$. □

Lemma 19.3. *Let $\gamma(\vartheta)$ be a smooth, strictly positive function.*

(i) *Given angles $\vartheta_1 < \vartheta_2$, Frank$(\gamma)$ is flat over the interval $[\vartheta_1, \vartheta_2]$ if and only if*

$$\gamma(\vartheta) + \gamma''(\vartheta) = 0 \qquad \text{for all } \vartheta \in [\vartheta_1, \vartheta_2]. \tag{19–44}$$

(ii) *If $\gamma(\vartheta)$ is angle convex, then*

$$\gamma(\vartheta) + \gamma''(\vartheta) \geq 0 \qquad \textit{for all } \vartheta, \tag{19–45}$$

with $\gamma(\vartheta) + \gamma''(\vartheta) = 0$ at isolated angles, if anywhere.

(iii) *Assume that $\gamma(\vartheta)$ is regular. If $\gamma(\vartheta)$ is not angle convex, then*

$$\gamma(\vartheta) + \gamma''(\vartheta) < 0 \qquad \textit{for some } \vartheta. \tag{19–46}$$

PROOF. Let

$$\tilde{\boldsymbol{J}}(\varphi) = \gamma(\varphi)\mathbf{t}(\varphi) + \gamma'(\varphi)\mathbf{m}(\varphi) \tag{19–47}$$

for all φ. Then, because $\mathbf{t}' = \mathbf{m}$ and $\mathbf{m}' = -\mathbf{t}$,

$$\tilde{\boldsymbol{J}}'(\varphi) = [\gamma(\varphi) + \gamma''(\varphi)]\mathbf{m}(\varphi). \tag{19–48}$$

Further, Frank(γ) is flat over $[\vartheta_1, \vartheta_2]$ if and only if it has a single tangent line over the entire interval, or, by Lemma 19.1(iii) and Lemma 19.2(ii), if and only if $\tilde{\boldsymbol{J}}(\varphi)$ is constant on $[\vartheta_1, \vartheta_2]$, or by (19–48), if and only if (19–44) is satisfied. Thus (i) is valid.

Assume that $\gamma(\vartheta)$ is angle convex. Choose an arbitrary angle φ. Let $\boldsymbol{j}$ be given by (19–41) and J by (19–38), so that, by Lemma 19.2(ii), Frank(J) is tangent to Frank(γ) at the point with angle φ. Thus, by (19–40), $\Phi(\varphi)$ defined near $\vartheta = \varphi$ by

$$\Phi(\vartheta) = J(\vartheta) - \gamma(\vartheta) = \boldsymbol{j} \cdot \mathbf{t}(\vartheta) - \gamma(\vartheta)$$

has a maximum at $\vartheta = \varphi$. Thus $\Phi''(\varphi) \leq 0$. But, because $\mathbf{t}' = \mathbf{m}$ and $\mathbf{m}' = -\mathbf{t}$,

$$\Phi''(\vartheta) = -\boldsymbol{j} \cdot \mathbf{t}(\vartheta) - \gamma''(\vartheta);$$

(19–38) and (19–40a) therefore imply that

$$\Phi''(\varphi) = -\gamma(\varphi) - \gamma''(\varphi) \leq 0,$$

which is (19–45). Further, angle convexity requires that Frank(γ) be *strictly* convex. Thus Frank(γ) can have no flat portions, which, by (i), implies the final assertion of (ii).

Assume that $\gamma(\vartheta)$ is not angle convex. Then Frank(γ) has a convexifying tangent $\mathscr{L}$ with (at least two) tangency angles ϑ_1 and ϑ_2, $\vartheta_1 < \vartheta_2$. Let $\boldsymbol{j}_1 = \tilde{\boldsymbol{j}}(\vartheta_1)$, $\boldsymbol{j}_2 = \tilde{\boldsymbol{j}}(\vartheta_2)$, with $\tilde{\boldsymbol{j}}$ defined in (19–47), and let J_1 and J_2 denote J in (19–38) with $\boldsymbol{j} = \boldsymbol{j}_1$ and $\boldsymbol{j} = \boldsymbol{j}_2$. Then, by Lemma 19.2, $\mathscr{L} = \text{Frank}(J_1) = \text{Frank}(J_2)$, so that, by Lemma 19.1(iii), $\boldsymbol{j}_1 = \boldsymbol{j}_2$. Since $\vartheta_2 < \vartheta_1 + \pi$, it is always possible to find a vector $\boldsymbol{a}$ such that $\Phi(\varphi) := \boldsymbol{a} \cdot \mathbf{m}(\varphi) > 0$ for all $\varphi \in [\vartheta_1, \vartheta_2]$. Then, by (19–48), integrating $\boldsymbol{a} \cdot \tilde{\boldsymbol{j}}'(\varphi)$ from ϑ_1 to ϑ_2 yields

$$\int_{\vartheta_1}^{\vartheta_2} \left[\gamma(\varphi) + \gamma''(\varphi)\right] \Phi(\varphi)\, d\varphi = 0,$$

so that either: (1) $\gamma(\vartheta) + \gamma''(\vartheta) = 0$ for all $\vartheta \in [\vartheta_1, \vartheta_2]$; or (2) $\gamma''(\vartheta) + \gamma(\vartheta) < 0$ for some $\vartheta \in [\vartheta_1, \vartheta_2]$. The assumption of regularity for γ rules out flat portions on Frank(γ); hence, by (i) of this lemma, regularity also rules out (1). □

g. Convexity of the interfacial energy and evolution of the interface

The results of the last section have important consequences regarding an evolving interface.

Theorem on Convexity and Evolution

(i) *Assume that* $\hat{\psi}(\vartheta)$ *is angle convex. Then the evolution equation* (19–30) *is parabolic at all but possibly an isolated set of angles; at these angles* (19–30) *is degenerate parabolic.*

(ii) *Assume that* $\hat{\psi}(\vartheta)$ *is regular, but not angle convex. Then there are angles* ϑ *at which the evolution equation* (19–30) *is backward parabolic.*

(iii) *A corner between globally stable angles* ϑ_1 *and* ϑ_2 *is possible only if* $\hat{\psi}(\vartheta)$ *is not angle convex, and then only for* ϑ_1 *and* ϑ_2 *tangency angles of the same convexifying tangent of Frank*$(\hat{\psi})$.

PROOF. The results (i) and (ii) follow from the Remark at the end of Section 19d and (ii) and (iii) of Lemma 19.3. To verify (iii), assume that a corner between globally stable angles ϑ_1 and and ϑ_2 exists, so that, by (19–34), $\hat{\mathbf{C}}(\vartheta_1) = \hat{\mathbf{C}}(\vartheta_2) =: \boldsymbol{j}$. Then, by (19–25) and Lemma 19.2(ii), Frank(J) is tangent to Frank$(\hat{\psi})$ at the point with angle ϑ_1 and with angle ϑ_2, so that, by definition, Frank(J) is a convexifying tangent of Frank(γ). Thus ϑ_1 and ϑ_2 are tangency angles of the same convexifying tangent. □

The initial-value problem for the evolution equation (19–30) may be stated as follows: given an initial interfacial curve $\mathscr{C}_0$, find an interfacial curve that evolves according to (19–30) and satisfies the iniial condition $\mathscr{C}(0) = \mathscr{C}_0$. When $a(\vartheta) > 0$ for all ϑ, so that (19–30) is parabolic, this problem is locally well posed.[7]

When $a(\vartheta) < 0$ on certain angle intervals, the problem is more difficult because of the backward parabolicity of (19–30). A method of overcoming this difficulty is to allow the interface to contain corners that exclude the backward-parabolic ranges of ϑ.[8] In the presence of a corner (19–30) does not by itself characterize the motion of the interface; there is the additional condition (19–34) requiring that the stress be continuous across each corner. If ϑ_1 and ϑ_2, each globally stable, are the angles of such a corner, then ϑ_1 and ϑ_2 must be tangency angles of the same convexifying tangent of Frank$(\hat{\psi})$. Thus, restricting attention to evolutions for which ϑ is globally stable, the possible corner angles are known in advance.[9]

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[7] Angenent [1991], Chen, Giga, and Goto [1991], Barles, Soner, and Souganidis [1993], Soner [1993].

[8] Angenent and Gurtin [1989].

[9] Local well-posedness of evolutions consistent with (19–30) and (19–34)—ensuing from initial curves consisting of sections with $a(\vartheta) > 0$ separated by appropriate corners—is established by Angenent and Gurtin [1994].

Part E

Coherent Phase Interfaces wtih Interfacial Energy and Deformation

A more general theory that accounts for both deformation and interfacial energy is complicated. For that reason, I begin with a simple theory that neglects standard stresses within the interface, an assumption tantamount to neglecting stresses associated with stretching the interface.

The kinematics relevant to this chapter are as discussed in Chapter 10.

CHAPTER 20

Theory Neglecting Standard Interfacial Stress

a. Standard and configurational forces. Working

The theory is based on a **standard force system**

$\boldsymbol{S}$	**bulk stress**
$\boldsymbol{b}$	**external bulk force**
$\boldsymbol{b}^{\mathscr{S}}$	**external interfacial force**

and a **configurational force system**

$\boldsymbol{C}$	**bulk stress**
$\boldsymbol{g}$	**internal bulk force**
$\boldsymbol{e}$	**external bulk force**
C	**interfacial stress**
$\boldsymbol{g}^{\mathscr{S}}$	**internal interfacial force**
$\boldsymbol{e}^{\mathscr{S}}$	**external interfacial force**

The fields are assumed objective with bulk fields smooth away from the interface and up to the interface from either side (cf. the discussion of Sections 4a, 5a, 11a, and 16a).

> The external forces $\boldsymbol{b}, \boldsymbol{b}^{\mathscr{S}}, \boldsymbol{e}$, and $\boldsymbol{e}^{\mathscr{S}}$ should be considered as having inertial and *virtual* components, with the virtual components not specified constitutively, but instead assignable in any way compatible with the basic laws (cf. the second paragraph of Chapter 9).

Let $\boldsymbol{v}$ be a velocity field for $\mathscr{S}$. Consider a migrating control volume $P = P(t)$. Let $\mathscr{G} = P \cap \mathscr{S}$, choose velocity fields $\boldsymbol{q}$ for ∂P and $\boldsymbol{w}$ for $\partial\mathscr{G}$, and let

$$\mathring{\boldsymbol{y}} = \dot{\boldsymbol{y}} + \boldsymbol{F}\boldsymbol{q}, \qquad \overset{\square}{\boldsymbol{y}} = \langle \dot{\boldsymbol{y}} \rangle + \langle \boldsymbol{F} \rangle \boldsymbol{v}, \tag{20–1}$$

respectively, denote motion velocities following ∂P and $\mathscr{S}$ as described by $\boldsymbol{q}$ and $\boldsymbol{v}$. Then, arguing as in Sections 5b, 11b, and 16a, the **working** $W(P)$ on a migrating control volume $P = P(t)$ is written in the form (cf. (11–4), (16–2))

$$W(P) = \int_{\partial P} (\boldsymbol{C}\boldsymbol{n}\cdot\boldsymbol{q} + \boldsymbol{S}\boldsymbol{n}\cdot\mathring{\boldsymbol{y}})\, da + \int_P \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\, dv + \int_{\mathscr{S}} (\boldsymbol{e}^{\mathscr{S}}\cdot\boldsymbol{v} + \boldsymbol{b}^{\mathscr{S}}\cdot\overset{\square}{\boldsymbol{y}})\, da + \int_{\partial\mathscr{S}} \mathbf{C}\mathbf{n}\cdot\boldsymbol{w}\, ds. \quad (20\text{–}2)$$

Invariance of the working under changes in material and spatial observer yields the **standard force** and **moment balances**

$$\int_{\partial P} \boldsymbol{S}\boldsymbol{n}\, da + \int_P \boldsymbol{b}\, dv + \int_{\mathscr{S}} \boldsymbol{b}^{\mathscr{S}}\, da = 0, \quad (20\text{–}3a)$$

$$\int_{\partial P} (\boldsymbol{y} - \boldsymbol{o}) \times \boldsymbol{S}\boldsymbol{n}\, da + \int_P (\boldsymbol{y} - \boldsymbol{o}) \times \boldsymbol{b}\, dv + \int_{\mathscr{S}} (\boldsymbol{y} - \boldsymbol{o}) \times \boldsymbol{b}^{\mathscr{S}}\, da = \underline{0}, \quad (20\text{–}3b)$$

and the **configurational force balance**

$$\int_{\partial P} \boldsymbol{C}\boldsymbol{n}\, da + \int_P (\boldsymbol{g} + \boldsymbol{e})\, dv + \int_{\partial\mathscr{S}} \mathbf{C}\mathbf{n}\, ds + \int_{\mathscr{S}} (\boldsymbol{g}^{\mathscr{S}} + \boldsymbol{e}^{\mathscr{S}})\, da = \mathbf{0}; \quad (20\text{–}4)$$

these lead to the bulk relations (3–6) and (5–10) and to the local force balances

$$[\boldsymbol{S}]\mathbf{m} + \boldsymbol{b}^{\mathscr{S}} = \mathbf{0}, \quad (20\text{–}5a)$$

$$[\boldsymbol{C}]\mathbf{m} + \boldsymbol{g}^{\mathscr{S}} + \boldsymbol{e}^{\mathscr{S}} + \mathrm{Div}_{\mathscr{S}}\mathbf{C} = 0, \quad (20\text{–}5b)$$

for the interface (cf. (11–8), (16–4)).

Invariance of the working under changes in the choice of velocity fields for $\partial P(t)$, $\mathscr{S}(t)$, and $\partial\mathscr{G}(t)$ yields (5–17), (11–9), and (16–7); viz.,

$$\boldsymbol{C} = \pi\mathbf{1} - \boldsymbol{F}^{\top}\boldsymbol{S}, \quad (20\text{–}6a)$$

$$\mathbf{C}_{\tan} = \sigma\mathbf{P}, \quad (20\text{–}6b)$$

$$\mathbf{P}\boldsymbol{e}^{\mathscr{S}} = -\mathbf{P}\langle\boldsymbol{F}\rangle^{\top}\boldsymbol{b}^{\mathscr{S}}. \quad (20\text{–}6c)$$

Thus, as before, the tangential part of the configurational interfacial stress is a surface tension σ, so that, by (15–5) (cf. (16–8)),

$$\mathbf{C} = \sigma\mathbf{P} + \mathbf{m}\otimes\boldsymbol{\tau}. \quad (20\text{–}7)$$

Further, (20–5a), (20–6a,b) and the compatibility relation $[\boldsymbol{F}]\mathbf{P} = \mathbf{0}$ again yield (11–10), and using this relation and (16–11), the normal and tangential configurational force balances are given by

$$\sigma K + \mathrm{Div}_{\mathscr{S}}\boldsymbol{\tau} + \mathbf{m}\cdot[\boldsymbol{C}]\mathbf{m} + g^{\mathscr{S}} + e^{\mathscr{S}} = 0, \quad (20\text{–}8a)$$

$$\nabla_{\mathscr{S}}\sigma - \mathbf{L}\boldsymbol{\tau} + \mathbf{P}\boldsymbol{g}^{\mathscr{S}} = 0, \quad (20\text{–}8b)$$

with

$$g^{\mathscr{S}} = \boldsymbol{g}^{\mathscr{S}}\cdot\mathbf{m}, \qquad e^{\mathscr{S}} = \boldsymbol{e}^{\mathscr{S}}\cdot\mathbf{m}, \quad (20\text{–}9)$$

and differ from (16–12) in the form of the configurational stress $\boldsymbol{C}$, which here includes the deformational contribution $-\boldsymbol{F}^{\top}\boldsymbol{S}$.

Choosing the intrinsic forms $\boldsymbol{q} = U\boldsymbol{n}$, $\boldsymbol{v} = V\mathbf{m}$, and $\boldsymbol{w} = V\mathbf{m} + V_{\partial\mathscr{G}}\mathbf{n}$ for the velocity fields, so that $\mathring{\boldsymbol{y}} = \dot{\boldsymbol{y}} + U\boldsymbol{F}\boldsymbol{n}$ and $\overset{\square}{\boldsymbol{y}} = \langle\dot{\boldsymbol{y}}\rangle + V\langle\boldsymbol{F}\rangle\mathbf{m}$, and appealing to (16–10), (20–9), and the stress relation $\boldsymbol{C} = \pi\mathbf{1} - \boldsymbol{F}^{\top}\boldsymbol{S}$, the working may be written in the intrinsic form (cf. (11–14), (16–15))

$$W(P) = \int_{\partial P} \boldsymbol{S}\boldsymbol{n}\cdot\dot{\boldsymbol{y}}\,da + \int_{P}\boldsymbol{b}\cdot\dot{\boldsymbol{y}}\,dv + \int_{\partial P}\pi U\,da + \int_{\mathscr{G}}\left\{\boldsymbol{b}^{\mathscr{S}}\cdot\overset{\square}{\boldsymbol{y}} + e^{\mathscr{S}}V\right\}da + \int_{\partial\mathscr{G}}\left(\sigma V_{\partial\mathscr{G}} + V\boldsymbol{\tau}\cdot\mathbf{n}\right)ds. \quad (20\text{–}10)$$

b. Power balance. Internal working

The identities (11–16b) and (16–16) remain valid within the current more general theory; thus, since $\boldsymbol{C} = \pi\mathbf{1} - \boldsymbol{F}^{\top}\boldsymbol{S}$,

$$\int_{\partial P}\boldsymbol{S}\boldsymbol{n}\cdot\dot{\boldsymbol{y}}\,da + \int_{P}\boldsymbol{b}\cdot\dot{\boldsymbol{y}}\,dv + \int_{\mathscr{G}}\boldsymbol{b}^{\mathscr{S}}\cdot\overset{\square}{\boldsymbol{y}}\,da = \int_{P}\boldsymbol{S}\cdot\dot{\boldsymbol{F}}\,dv + \int_{\mathscr{G}}(\mathbf{m}\cdot[\boldsymbol{C}]\mathbf{m} - [\pi])\,V\,da, \quad (20\text{–}11a)$$

$$\int_{\partial\mathscr{G}}\left(\sigma V_{\partial\mathscr{G}} + V\boldsymbol{\tau}\cdot\mathbf{n}\right)ds = \int_{\partial\mathscr{G}}\sigma V_{\partial\mathscr{G}}\,ds + \int_{\mathscr{G}}(-\boldsymbol{\tau}\cdot\overset{\square}{\mathbf{m}} + V\ \mathrm{Div}_{\mathscr{S}}\boldsymbol{\tau})\,da. \quad (20\text{–}11b)$$

Adding these relations and using the normal configurational force balance (20–8a) to eliminate the term $(\mathbf{m}\cdot[\boldsymbol{C}]\mathbf{m} + \mathrm{Div}_{\mathscr{S}}\boldsymbol{\tau})V$ yields

$$W(P) = \int_{P}\boldsymbol{S}\cdot\dot{\boldsymbol{F}}\,dv - \int_{\mathscr{G}}\left\{\sigma KV + \boldsymbol{\tau}\cdot\overset{\square}{\mathbf{m}} + \left([\pi] + g^{\mathscr{S}}\right)V\right\}da + \int_{\partial\mathscr{G}}\sigma V_{\partial\mathscr{G}}\,ds + \int_{\partial P}\pi U\,da. \quad (20\text{–}12)$$

The right side of (20–12) represents **internal working** on P, and the equivalence of (20–10) and (20–12) represents a **power balance** for P. Equating (20–10) and (20–12) yields, upon canceling the terms $\int_{\partial P}\pi U\,da$ and $\int_{\partial\mathscr{G}}\sigma V_{\partial\mathscr{G}}\,ds$, the **reduced power balance**

$$\int_{\partial P}\boldsymbol{S}\boldsymbol{n}\cdot\dot{\boldsymbol{y}}\,da + \int_{P}\boldsymbol{b}\cdot\dot{\boldsymbol{y}}\,dv + \int_{\mathscr{G}}\left\{\boldsymbol{b}^{\mathscr{S}}\cdot\overset{\square}{\boldsymbol{y}} + e^{\mathscr{S}}V\right\}da + \int_{\partial\mathscr{G}}V\boldsymbol{\tau}\cdot\mathbf{n}\,ds = \int_{P}\boldsymbol{S}\cdot\dot{\boldsymbol{F}}\,dv - \int_{\mathscr{G}}\left\{\sigma KV + \boldsymbol{\tau}\cdot\overset{\square}{\mathbf{m}} + \left([\pi] + g^{\mathscr{S}}\right)V\right\}da. \quad (20\text{–}13)$$

The balances (20–12) and (20–13) should be compared to (16–17) and (16–21). The term $\boldsymbol{b}^{\mathscr{S}}\cdot\overset{\square}{\boldsymbol{y}} + e^{\mathscr{S}}V$ in (20–10) and (20–13) may be replaced by $\boldsymbol{b}^{\mathscr{S}}\cdot\langle\dot{\boldsymbol{y}}\rangle$ if the external forces are inertial (cf. (12–8)).

c. Second law

c1. Second law. Interfacial dissipation inequality

The **second law** takes the form

$$\frac{d}{dt}\left\{\int_P \Psi\, dv + \int_{\mathscr{G}} \psi\, da\right\} \le W(P) \tag{20–14}$$

with $W(P)$ given by (20–2). Here $\psi(\boldsymbol{X}, t)$ is the **interfacial free energy**, assumed smooth on $\mathscr{S}(t)$, while $\Psi(\boldsymbol{X}, t)$ is the **bulk free energy**, assumed smooth away from the interface and up to the interface from either side.

The theory in bulk is no different from that presented in Part A; in particular, $\pi = \Psi$; thus applying the transport theorems (10–8a) and (15–31) to the left side of (20–14) and using the power balance (20–12),

$$\begin{aligned}\int_{\mathscr{G}} \left(\overset{\square}{\psi} - \psi K V\right) da + \int_{\partial\mathscr{G}} (\psi - \sigma) V_{\partial\mathscr{G}}\, ds + \int_P \left(\dot{\Psi} - \boldsymbol{S}\cdot\dot{\boldsymbol{F}}\right) dv \\ \le -\int_{\mathscr{G}} \left\{\sigma K V + \boldsymbol{\tau}\cdot\overset{\square}{\mathbf{m}} + g^{\mathscr{S}} V\right\} da.\end{aligned} \tag{20–15}$$

The argument leading to (17–5) is also valid here and, as before, it implies that

$$\sigma = \psi.$$

Shrinking P to the interface therefore yields, with the aid of (10–10b), the **interfacial dissipation inequality** (17–6); viz.

$$\overset{\square}{\psi} + \boldsymbol{\tau}\cdot\overset{\square}{\mathbf{m}} + g^{\mathscr{S}} V \le 0, \tag{20–16}$$

with $(\ldots)^{\square}$ the normal time derivative following $\mathscr{S}$.

Finally, the dissipation, defined as the right side of (20–14) minus the left, is given by

$$\mathscr{D}(P) = -\int_P \left(\dot{\Psi} - \boldsymbol{S}\cdot\dot{\boldsymbol{F}}\right) dv - \int_{\mathscr{G}} \left(\overset{\square}{\psi} + \boldsymbol{\tau}\cdot\overset{\square}{\mathbf{m}} + g^{\mathscr{S}} V\right) da \ge 0. \tag{20–17}$$

c2. Derivation of the interfacial dissipation inequality using a pillbox argument

Let $\mathscr{G}(t)$ denote a smoothly evolving subsurface of $\mathscr{S}(t)$, and take $P(t) = \mathscr{G}_\delta(t)$, the δ-pillbox about $\mathscr{G}(t)$. Then, granted (11–26), a consequence of (11–25c), (11–27), the transport theorem (15–31),and the balance $[\boldsymbol{S}]\mathbf{m} + \boldsymbol{b}^{\mathscr{S}} = \mathbf{0}$ is that the second law (20–14) may be written in the form

$$\begin{aligned}\int_{\mathscr{G}} \left(\overset{\square}{\psi} - \psi K V\right) da + \int_{\partial\mathscr{G}} \psi V_{\partial\mathscr{G}}\, ds \\ \le \int_{\mathscr{G}} \left\{(\mathbf{m}\cdot[\boldsymbol{C}]\mathbf{m} + e^{\mathscr{S}})V\right\} da + \int_{\partial\mathscr{G}} \mathbf{C}\mathbf{n}\cdot\boldsymbol{w}\, ds.\end{aligned}$$

On the other hand, using (20–8a) to eliminate the term $\mathrm{Div}_{\mathscr{S}}\boldsymbol{\tau}$ from (16–16),

$$\int_{\partial\mathscr{G}} \mathbf{C}\mathbf{n}\cdot\boldsymbol{w}\,ds = \int_{\partial\mathscr{G}} \sigma V_{\partial\mathscr{G}}\,ds - \int_{\mathscr{G}} \left\{\boldsymbol{\tau}\cdot\overset{\square}{\mathbf{m}} + V(\sigma K + \mathbf{m}\cdot[\boldsymbol{C}]\mathbf{m} + g^{\mathscr{S}} + e^{\mathscr{S}})\right\} da,$$

and, arguing as above, we may use the last two relations to conclude that $\sigma = \psi$, and hence that the **interfacial dissipation inequality** (20–16) is satisfied.

d. Constitutive equations

The bulk phases α and β are presumed to be hyperelastic as defined by constitutive equations (12–19).

Regarding the interface, I consider constitutive equations giving ψ, $\boldsymbol{\tau}$, and $g^{\mathscr{S}}$ when $\mathbf{m}$, V, and the interfacial limits $\boldsymbol{F}^{\pm}$ of the deformation gradient are known. In view of the compatibility condition (10–2b), $\mathbf{0} = [\boldsymbol{F}]\mathbf{P} = [\boldsymbol{F}](\mathbf{1} - \mathbf{m}\otimes\mathbf{m})$, and hence $[\boldsymbol{F}] = ([\boldsymbol{F}]\mathbf{m})\otimes\mathbf{m}$. The limits $\boldsymbol{F}^{\pm}$ are therefore determined by $\mathbf{m}$, the average

$$\boldsymbol{A} := \langle\boldsymbol{F}\rangle = \frac{1}{2}(\boldsymbol{F}^{+} + \boldsymbol{F}^{-}),$$

and the jump

$$\boldsymbol{j} := [\boldsymbol{F}]\mathbf{m}$$

through the identity

$$\boldsymbol{F}^{\pm} = \boldsymbol{A} \pm \frac{1}{2}\boldsymbol{j}\otimes\mathbf{m}.$$

Therefore, replacing the variables $(\mathbf{m}, V, \boldsymbol{F}^{+}, \boldsymbol{F}^{-})$ by $(\mathbf{m}, V, \boldsymbol{A}, j)$, I consider constitutive equations of the form

$$\psi = \hat{\psi}(\mathbf{m}, V, \boldsymbol{A}, \boldsymbol{j}), \tag{20–18a}$$

$$\boldsymbol{\tau} = \hat{\boldsymbol{\tau}}(\mathbf{m}, V, \boldsymbol{A}, \boldsymbol{j}), \tag{20–18b}$$

$$g^{\mathscr{S}} = \hat{g}^{\mathscr{S}}(\mathbf{m}, V, \boldsymbol{A}, \boldsymbol{j}). \tag{20–18c}$$

Consider an arbitrary *constitutive process*; that is, an evolution $\mathscr{S} = \mathscr{S}(t)$ of the interface together with a motion $\boldsymbol{y}$ related to $\mathscr{S}$ through the constitutive equations (12–19) and (20–18) and through the compatibility conditions (10–2). Then Ψ and $\boldsymbol{S}$ (and hence $\boldsymbol{C}$) are determined in bulk, while $\sigma = \psi$, $\boldsymbol{\tau}$ and (and hence $\mathbf{C}$), and $g^{\mathscr{S}}$ are determined on the interface. Given these fields, the force balances and basic identities can be satisfied using the external and indeterminate forces; that is, using the bulk body forces $\boldsymbol{b}$, $\boldsymbol{e}$, and $\boldsymbol{g}$ and the interfacial forces $\boldsymbol{b}^{\mathscr{S}}$, $\boldsymbol{e}^{\mathscr{S}}$, and $\mathbf{P}\boldsymbol{g}^{\mathscr{S}}$, which are arbitrarily assignable (since $\boldsymbol{b}$, $\boldsymbol{e}$, $\boldsymbol{b}^{\mathscr{S}}$,and $\boldsymbol{e}^{\mathscr{S}}$ have virtual components, while $\boldsymbol{g}$ and $\mathbf{P}\boldsymbol{g}^{\mathscr{S}}$ are indeterminate). The force balances consist of the bulk balances (3–6a) and (5–10), the standard interfacial balance (20–5a), and the normal and tangential configurational balances (20–8a) and (20–8b); the basic identities are the external force relation (5–20) and the invariance requirement (20–6c). The relations (3–6a), (5–10), and (5–20) give the body forces $\boldsymbol{b}$, $\boldsymbol{e}$, and $\boldsymbol{g}$; (20–5a) gives $\boldsymbol{b}^{\mathscr{S}}$; (20–6c) and (20–8a) give $\boldsymbol{e}^{\mathscr{S}}$; and (20–8b) gives $\mathbf{P}\boldsymbol{g}^{\mathscr{S}}$. Thus, all of the relevant balances and

identities are satisfied, and all that remains to be satisfied is the second law in the form of the interfacial dissipation inequality (20–16). (The local form of the second law in bulk is (6–11), and its satisfaction is ensured by (12–19).)

Note that there are neither too many nor two few external and indeterminate forces available to ensure satisfaction of the relevant balances and identities, an inflexibility that demonstrates the lack of capriciousness in the use of external forces.

To ensure compatibility of all constitutive processes with the second law, *all constitutive processes are required to be consistent with the dissipation inequality* (20–16). Granted (20–18), (20–16) is equivalent to the inequality

$$\partial_V \hat{\psi}(\mathfrak{m}, V, \boldsymbol{A}, \boldsymbol{j})\, \overset{\square}{V} + \left\{\partial_{\boldsymbol{A}} \hat{\psi}(\mathfrak{m}, V, \boldsymbol{A}, \boldsymbol{j})\right\} \cdot \overset{\square}{\boldsymbol{A}} + \left\{\partial_{\boldsymbol{j}} \hat{\psi}(\mathfrak{m}, V, \boldsymbol{A}, \boldsymbol{j})\right\} \cdot \overset{\square}{\boldsymbol{j}}$$
$$+ \left\{\partial_{\mathfrak{m}} \hat{\psi}(\mathfrak{m}, V, \boldsymbol{A}, \boldsymbol{j}) + \hat{\boldsymbol{\tau}}(\mathfrak{m}, V, \boldsymbol{A}, \boldsymbol{j})\right\} \cdot \overset{\square}{\mathfrak{m}} + \hat{g}^{\mathscr{S}}(\mathfrak{m}, V, \boldsymbol{A}, \boldsymbol{j}) V \leq 0, \quad (20\text{–}19)$$

with $(\ldots)^{\square}$ the normal time derivative following $\mathscr{S}$.

As will be shown in Section 20d, one can always find an evolution of the interface together with a motion (consistent with (10–2)) such that $\mathfrak{m}$, V, $\boldsymbol{A}$, $\overset{\square}{\mathfrak{m}}$, $\overset{\square}{V}$, $\overset{\square}{\boldsymbol{A}}$, and $\overset{\square}{\boldsymbol{j}}$ have arbitrary values at some given point and time. Thus, because the left side of (20–19) is *linear* in $\overset{\square}{\mathfrak{m}}$, $\overset{\square}{V}$, $\overset{\square}{\boldsymbol{A}}$, and $\overset{\square}{\boldsymbol{j}}$, the coefficients of these fields must vanish:

$$\partial_V \hat{\psi}(\mathfrak{m}, V, \boldsymbol{A}, \boldsymbol{j}) = \boldsymbol{0},$$
$$\partial_{\boldsymbol{A}} \hat{\psi}(\mathfrak{m}, V, \boldsymbol{A}, \boldsymbol{j}) = \boldsymbol{0},$$
$$\partial_{\boldsymbol{j}} \hat{\psi}(\mathfrak{m}, V, \boldsymbol{A}, \boldsymbol{j}) = \boldsymbol{0},$$
$$\hat{\boldsymbol{\tau}}(\mathfrak{m}, V, \boldsymbol{A}, \boldsymbol{j}) = -\partial_{\mathfrak{m}} \hat{\psi}(\mathfrak{m}, V, \boldsymbol{A}, \boldsymbol{j}).$$

The general constitutive equations (20–18) must therefore be consistent with the following set of restrictions:

(i) the free energy ψ and the shear $\boldsymbol{\tau}$ must be independent of V, $\boldsymbol{A}$, and $\boldsymbol{j}$ (and hence the limits $\boldsymbol{F}^{\pm}$) and must be related through

$$\psi = \hat{\psi}(\mathfrak{m}), \qquad \boldsymbol{\tau} = -\partial_{\mathfrak{m}} \hat{\psi}(\mathfrak{m}); \quad (20\text{–}20)$$

(ii) the normal internal force must obey the inequality

$$\hat{g}^{\mathscr{S}}(\mathfrak{m}, V, \boldsymbol{A}, \boldsymbol{j}) V \leq 0; \quad (20\text{–}21)$$

for $\hat{g}^{\mathscr{S}}$ smooth this inequality yields the constitutive equation (cf. (9–25))

$$g^{\mathscr{S}} = -b(\mathfrak{m}, V, \boldsymbol{A}, \boldsymbol{j}) V, \qquad b(\mathfrak{m}, V, \boldsymbol{A}, \boldsymbol{j}) \geq 0. \quad (20\text{–}22)$$

The relations (20–20) and (20–22) are the most general constitutive equations of the form (20–18) that are smooth and consistent with the dissipation inequality (20–16). Note that the **kinetic modulus** $b(\mathfrak{m}, V, \boldsymbol{A}, \boldsymbol{j})$ may depend not only on $\mathfrak{m}$ and V, but also on the interfacial limits $\boldsymbol{F}^{\pm}$.

Note that the argument leading to (18–12) is also valid here; thus the tangential force balance (20–8b) is satisfied identically with $\mathbf{P}\boldsymbol{g}^{\mathscr{S}} = \boldsymbol{0}$.

e. Construction of the process used in restricting the constitutive equations

The domain $\mathscr{D}$ of the constitutive equations (20–18) consists of all unit vectors $\mathbf{m}$, all scalars V, and all tensors $\boldsymbol{A}$ and vectors $\boldsymbol{j}$ such that the tensors $\boldsymbol{F}^{\pm} = \boldsymbol{A} \pm \frac{1}{2}\boldsymbol{j} \otimes \mathbf{m}$ have strictly positive determinant. Choose $(\mathbf{m}_0, V_0, \boldsymbol{A}_0, \boldsymbol{j}_0) \in \mathscr{D}$ arbitrarily.

Consider the plane surface $\mathscr{S} = \mathscr{S}(t)$ discussed in the paragraph containing (15–28), which I will refer to as Paragraph C. For this choice of $\mathscr{S}$, the fields $\mathbf{m}$, V, $\overset{\square}{\mathbf{m}}$, and $\overset{\square}{V}$ can be chosen to have arbitrarily prescribed values at $\boldsymbol{X} = \mathbf{0}$ and $t = 0$. Next, choose, arbitrarily, a tensor $\boldsymbol{A}_1$ and a vector $\boldsymbol{j}_1$, and let

$$\boldsymbol{A}(t) = \boldsymbol{A}_0 + \delta(t)\boldsymbol{A}_1, \qquad \boldsymbol{j}(t) = \boldsymbol{j}_0 + \delta(t)\boldsymbol{j}_1, \tag{20–23}$$

where $\delta(t)$ is a scalar function such that

$$\delta(0) = 0, \qquad \dot{\delta}(0) = 1, \qquad |\delta(t)| \leq \varepsilon \quad \text{for all } t \tag{20–24}$$

(which is consistent with, but more restrictive than, the conditions imposed on $\delta(t)$ in Paragraph C). Because the set of all tensors with strictly positive determinant is open in the set of all tensors, it is possible to choose ε small enough that $\big(\mathbf{m}(t), V(t), \boldsymbol{A}(t), \boldsymbol{j}(t)\big) \in \mathscr{D}$ for all t. Let

$$\boldsymbol{F}^{\pm}(t) = \boldsymbol{A}(t) \pm \frac{1}{2}\boldsymbol{j}(t) \otimes \mathbf{m}(t) \tag{20–25}$$

and define $\boldsymbol{y}$ by

$$\boldsymbol{y}(\boldsymbol{X}, t) = \boldsymbol{F}^{-}(t)(\boldsymbol{X} - \boldsymbol{Z}(t)) \qquad \text{for} \quad \mathbf{m}(t) \cdot (\boldsymbol{X} - \boldsymbol{Z}(t)) < 0, \tag{20–26a}$$

$$\boldsymbol{y}(\boldsymbol{X}, t) = \boldsymbol{F}^{+}(t)(\boldsymbol{X} - \boldsymbol{Z}(t)) \qquad \text{for} \quad \mathbf{m}(t) \cdot (\boldsymbol{X} - \boldsymbol{Z}(t)) > 0. \tag{20–26b}$$

Then (20–25), (20–26), and the results of Paragraph C imply that

$$[\dot{\boldsymbol{y}}] = (\dot{\boldsymbol{j}} \otimes \mathbf{m} + \boldsymbol{j} \otimes \dot{\mathbf{m}})(\boldsymbol{X} - \boldsymbol{Z}) - (\boldsymbol{j} \otimes \mathbf{m})\dot{\boldsymbol{Z}} = -V\boldsymbol{j} = -V[\boldsymbol{F}]\mathbf{m};$$

hence $\boldsymbol{y}$ satisfies the compatibility conditions (10–2). Further,

$$\boldsymbol{A}(0) = \boldsymbol{A}_0, \qquad \boldsymbol{j}(0) = \boldsymbol{j}_0, \qquad \overset{\square}{\boldsymbol{A}}(\mathbf{0}, 0) = \boldsymbol{A}_1, \qquad \overset{\square}{\boldsymbol{j}}(\mathbf{0}, 0) = \boldsymbol{j}_1; \tag{20–27}$$

thus, because $\boldsymbol{A}_1$ and $\boldsymbol{j}_1$ are arbitrary, these results and those of Paragraph C yield an evolution $\mathscr{S} = \mathscr{S}(t)$ of the interface and a motion $\boldsymbol{y}$—with $\mathscr{S}$ and $\boldsymbol{y}$ related through the compatibility conditions (10–2)—such that $\mathbf{m}$, V, $\boldsymbol{A}$, $\boldsymbol{j}$, $\overset{\square}{\mathbf{m}}$, $\overset{\square}{V}$, $\overset{\square}{\boldsymbol{A}}$, and $\overset{\square}{\boldsymbol{j}}$ have arbitrary values at the point $\boldsymbol{X} = \mathbf{0}$ and the time $t = 0$.

f. Basic equations with inertial external forces

f1. Standard and configurational balances

Assume that the underlying observer, the body force $\boldsymbol{b}$, and the interface forces $\boldsymbol{b}^{\mathscr{S}}$ and $\boldsymbol{e}^{\mathscr{S}}$ are inertial. The argument used to derive (12–13)–(12–15), here used in

conjunction with (20–5a) and (20–8a), then yields the momentum balance (12–13) and, because $\sigma = \psi$, the normal configurational balance

$$\psi K + \mathrm{Div}_{\mathscr{S}} \boldsymbol{\tau} + \mathbf{m} \cdot [\Psi \mathbf{1} - \boldsymbol{F}^{\top} \boldsymbol{S}] \mathbf{m} + [k_{\mathrm{rel}}] + g^{\mathscr{S}} = 0, \tag{20–28}$$

with $[k_{\mathrm{rem}}] = \frac{1}{2}\rho[|\dot{\boldsymbol{y}} - \overset{\square}{\boldsymbol{y}}|^2]$ (cf. (12–1)), or equivalently,

$$\psi K + \mathrm{Div}_{\mathscr{S}} \boldsymbol{\tau} + [\Psi] - \langle \boldsymbol{S} \rangle \mathbf{m} \cdot [\boldsymbol{F}] \mathbf{m} + g^{\mathscr{S}} = 0. \tag{20–29}$$

These relations are independent of constitutive equations. (The indeterminacy of the tangential component of $\boldsymbol{g}^{\mathscr{S}}$ renders the tangential configurational balance superfluous.)

f2. Summary of basic equations

The basic equations for the *bulk material*, assumed elastic, consist of the momentum balance (12–20) supplemented by the constitutive equations (12–19).

The basic equations for the *interface* are the *compatibility conditions*

$$[\dot{\boldsymbol{y}}] = -V[\boldsymbol{F}]\mathbf{m}, \tag{20–30a}$$

$$[\boldsymbol{F}]\mathbf{P} = \mathbf{0}, \tag{20–30b}$$

the *momentum balance*

$$[\boldsymbol{S}]\mathbf{m} = -\rho[\dot{\boldsymbol{y}}]V, \tag{20–31}$$

and the *normal configurational balance*

$$\left\{\hat{\psi}(\mathbf{m})\mathbf{1} + \partial_{\mathbf{m}}\partial_{\mathbf{m}}\hat{\psi}(\mathbf{m})\right\} \cdot \mathbf{L} + \mathbf{m} \cdot [\Psi \mathbf{1} - \boldsymbol{F}^{\top}\boldsymbol{S}]\mathbf{m} + [k_{\mathrm{rel}}] = bV \tag{20–32}$$

or equivalently,

$$\left\{\hat{\psi}(\mathbf{m})\mathbf{1} + \partial_{\mathbf{m}}\partial_{\mathbf{m}}\hat{\psi}(\mathbf{m})\right\} \cdot \mathbf{L} + [\Psi] - \langle \boldsymbol{S} \rangle \mathbf{m} \cdot [\boldsymbol{F}]\mathbf{m} = bV, \tag{20–33}$$

with $b = b(V, \mathbf{m}, \boldsymbol{A}, \boldsymbol{j}) \geq 0$. The relations (20–32) and (20–33) follow from (20–28) and (20–29) with ψ, $\boldsymbol{\tau}$, and $g^{\mathscr{S}}$ specified by the constitutive relations (20–20) and (20–22), using the identity $\psi K + \mathrm{Div}_{\mathscr{S}} \boldsymbol{\tau} = \left\{\hat{\psi}(\mathbf{m})\mathbf{1} + \partial_{\mathbf{m}}\partial_{\mathbf{m}}\hat{\psi}(\mathbf{m})\right\} \cdot \mathbf{L}$ (cf. (18–13)).

If the interface is isotropic with linear kinetics, then both ψ and b are constant and (20–32) reduces to

$$\psi K + \mathbf{m} \cdot [\Psi \mathbf{1} - \boldsymbol{F}^{\top}\boldsymbol{S}]\mathbf{m} + [k_{\mathrm{rel}}] = bV, \tag{20–34}$$

while (20–33) takes the form

$$\psi K + [\Psi] - \langle \boldsymbol{S} \rangle \mathbf{m} \cdot [\boldsymbol{F}]\mathbf{m} = bV. \tag{20–35}$$

The system of equations discussed in this section represents the simplest correction to the system of Section 12e, which was derived neglecting interfacial energy; the only difference between the two systems is the presence of the capillarity term ψK in the last two equations.

g. Global energy inequality. Lyapunov relations

Assume that the body B is bounded, that $\mathscr{S} = \mathscr{S}(t)$ is a closed surface contained in the interior of B, and that the external forces are inertial. Then the second law (20–14), with right side in the intrinsic form (20–10) and with (12–7) used to replace the inertial working by the production of kinetic energy, yields, for $P = B$,

$$\frac{d}{dt}\left\{\int_B (\Psi + k)\, dv + \int_{\mathscr{S}} \psi\, da\right\} - \int_{\partial B} \boldsymbol{S}\boldsymbol{n}\cdot\dot{\boldsymbol{y}}\, da = -\mathscr{D}(B) \leq 0. \tag{20–36}$$

Here the bulk free energy is given by (12–19) in the subregions of B occupied by α and β; the interfacial free energy is given by $\psi = \hat{\psi}(\mathfrak{m})$; and the dissipation $\mathscr{D}(B)$ is given by (20–17) subject to the restrictions $\dot{\Psi} = \boldsymbol{S}\cdot\dot{\boldsymbol{F}}$ and $\overset{\square}{\psi} = -\boldsymbol{\tau}\cdot\overset{\square}{\mathfrak{m}}$, so that

$$\mathscr{D}(B) = -\int_{\mathscr{S}} g^{\mathscr{S}} V\, da = \int_{\mathscr{S}} bV^2 da, \tag{20–37}$$

with $b = b(V, \mathfrak{m}, \boldsymbol{A}, \boldsymbol{j}) \geq 0$.

A consequence of (20–36) and the definitions (7–15) and (7–16) are the following Lyapunov relations: for a *fixed boundary*

$$\frac{d}{dt}\left\{\int_B (\Psi + k)\, dv + \int_{\mathscr{S}} \psi\, da\right\} = -\mathscr{D}(B) \leq 0; \tag{20–38}$$

for a *boundary under constant dead loads*

$$\frac{d}{dt}\left\{\int_B (\Psi - \boldsymbol{S}_0\cdot\boldsymbol{F} + k)\, dv + \int_{\mathscr{S}} \psi\, da\right\} = -\mathscr{D}(B) \leq 0. \tag{20–39}$$

CHAPTER 21

General Theory with Standard and Configurational Stress within the Interface[1]

I now generalize the theory to include standard stress within the interface. The basic ingredients of the general theory are a superficial tensor field $\mathbf{S}$ that represents this stress and a superficial tensor field $\mathbf{F}$ that represents the action of the deformation gradient within the interface.

a. Kinematics. Tangential deformation gradient

The basic kinematics of motion in the presence of an evolving interface $\mathscr{S} = \mathscr{S}(t)$ is as presented in Chapter 10, and the discussion here begins where that chapter ends. Let $\boldsymbol{y}$ be a motion of B. Because $\boldsymbol{y}$ is continuous across $\mathscr{S}$ and smooth up to $\mathscr{S}$ from either side, the surface gradient $\nabla_{\mathscr{S}}\boldsymbol{y}$ may be computed using the second of (15–8) applied to $\boldsymbol{y}^+$ or $\boldsymbol{y}^-$; the result is the **tangential deformation gradient**

$$\mathbf{F} = \nabla_{\mathscr{S}}\boldsymbol{y} = \boldsymbol{F}^+\mathbf{P} + \boldsymbol{F}^-\mathbf{P} = \langle \boldsymbol{F} \rangle \mathbf{P} \tag{21–1}$$

(cf. (10–2b)). The tensor $\mathbf{F}(\boldsymbol{X}, t)$ maps vectors tangent to $\mathscr{S}(t)$ at $\boldsymbol{X}$ into vectors tangent to the deformed interface at $\boldsymbol{y}(\boldsymbol{X}, t)$.[2] Indeed, $\langle \boldsymbol{F} \rangle^{-\top}\mathbf{m}$ divided by its magnitude is the unit normal to the deformed interface, and for $\mathbf{t}$ a vector field tangent to $\mathscr{S}$,

$$\big(\langle \boldsymbol{F} \rangle^{-\top}\mathbf{m}\big) \cdot \mathbf{F}\mathbf{t} = \big(\langle \boldsymbol{F} \rangle^{-\top}\mathbf{m}\big) \cdot \langle \boldsymbol{F} \rangle \mathbf{P}\mathbf{t} = \mathbf{m} \cdot \mathbf{t} = 0.$$

[1] Cf. Gurtin and Struthers [1990] and Gurtin [1993a, 1995] for more complete discussions.

[2] Gurtin and Murdoch [1975].

An identity basic to the discussion of internal working concerns the surface gradient of the normal time derivative $\overset{\square}{y} = \langle \dot{y} \rangle + V \langle F \rangle \mathbf{m}$ following $\mathscr{S}$.[3]

$$\nabla_{\mathscr{S}} \overset{\square}{y} = \langle F \rangle^{\square} \mathbf{P} - \langle F \rangle \left(\mathbf{m} \otimes \overset{\square}{\mathbf{m}} \right) - V \langle F \rangle \mathbf{L} . \tag{21–2}$$

To verify (21–2), consider y restricted to one side of the interface, so that $\overset{\square}{y} = \dot{y} + Fv$, with $v = V\mathbf{m}$, where, for convenience, the plus and minus signs that signify interfacial limits are suppressed. Then, by (15–8b), $\nabla_{\mathscr{S}} \dot{y} = \dot{F}\mathbf{P}$. Further, by the product rule, $\nabla_{\mathscr{S}}(Fv)$ equals $\nabla_{\mathscr{S}}$ applied to Fv holding F fixed plus $\nabla_{\mathscr{S}}$ applied to Fv holding v fixed. By (16–19), the derivative holding F fixed is

$$F \nabla_{\mathscr{S}} v = -F(\mathbf{m} \otimes \overset{\square}{\mathbf{m}}) - VF\mathbf{L} .$$

On the other hand, by (1–23c) and (15–8b), for v fixed,

$$\nabla_{\mathscr{S}}(Fv) = \big(\nabla(Fv)\big)\mathbf{P} = \big((\nabla F)v\big)\mathbf{P}.$$

Thus, since $\overset{\square}{F} = \dot{F} + (\nabla F)v$ (cf. (15–22c)),

$$\nabla_{\mathscr{S}} \overset{\square}{y} = \big(\dot{F} + (\nabla F)v\big)\mathbf{P} - F \left(\mathbf{m} \otimes \overset{\square}{\mathbf{m}} \right) - VF\mathbf{L} = \overset{\square}{F}\mathbf{P} - F \left(\mathbf{m} \otimes \overset{\square}{\mathbf{m}} \right) - VF\mathbf{L} . \tag{21–3}$$

Finally, because (21–3) holds with y (and hence F) restricted to either side of the interface, (21–3) holds with y and F replaced by their averages; hence (21–2) is satisfied.

Let $\mathscr{G}(t)$ denote a smoothly evolving subsurface of $\mathscr{S}(t)$ with $\mathbf{n}(X, t)$ the outward unit normal to $\partial\mathscr{G}(t)$. As noted in the paragraph containing (15–29), the motion of $\partial\mathscr{G}(t)$ is characterized *intrinsically* by the velocity field $V\mathbf{m} + V_{\partial\mathscr{G}}\mathbf{n}$, with $V_{\partial\mathscr{G}}$, the velocity of $\partial\mathscr{G}$ in the direction of $\mathbf{n}$. Further, a field w is referred to as a **velocity field for** $\partial\mathscr{G}$ if $w \cdot \mathbf{m} = V$ and $w \cdot \mathbf{n} = V_{\partial\mathscr{G}}$, with no restriction placed on the component of w tangent to $\partial\mathscr{G}$. Given such a field w, the **motion velocity** $\overset{\triangle}{y}$ **following** $\partial\mathscr{G}$, as described by w, is defined as in (4–3), and this results in the expression

$$\overset{\triangle}{y} = \langle \dot{y} \rangle + \langle F \rangle w. \tag{21–4}$$

The transformation laws for w and $\overset{\triangle}{y}$ under observer changes are analogous to those specified for v and $\overset{\square}{y}$ in (10–6) and (10–7).

The field $\overset{\triangle}{y}$ depends on the choice of w; when $w = V\mathbf{m} + V_{\partial\mathscr{G}}\mathbf{n}$ then $\overset{\triangle}{y}$ is **intrinsic** and, since $\langle F \rangle \mathbf{n} = \langle F \rangle \mathbf{P}\mathbf{n} = \mathbf{F}\mathbf{n}$, may be written in the form

$$\overset{\triangle}{y} = \overset{\square}{y} + V_{\partial\mathscr{G}} \mathbf{F}\mathbf{n}, \tag{21–5}$$

with $\overset{\square}{y}$ the normal velocity following $\mathscr{S}$.

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[3] Gurtin and Struthers [1990, eq. (3–29)].

b. Standard and configurational forces. Working

The force systems are as discussed in Section 20a, the only change being the addition of a superficial tensor field $\mathbf{S}(\boldsymbol{X}, t)$ on $\mathscr{S}(t)$ that represents standard forces within the interface. Let $P = P(t)$ be a migrating control volume with $\mathscr{G} = \mathscr{G}(t)$ the portion of the interface in P and $\mathbf{n}(\boldsymbol{X}, t)$ the outward unit normal to $\partial\mathscr{G}(t)$; then $\mathbf{Sn}$ represents standard forces within the interface applied to P across $\partial\mathscr{G}$.

Let $P = P(t)$ be a migrating control volume, with $\boldsymbol{q}$ a velocity field for ∂P and $\boldsymbol{w}$ a velocity field for $\partial\mathscr{G}$, let $\boldsymbol{v}$ be a velocity field for $\mathscr{S}$, and consider the motion velocities

$$\mathring{\boldsymbol{y}} = \dot{\boldsymbol{y}} + \boldsymbol{F}\boldsymbol{q}, \qquad \overset{\square}{\boldsymbol{y}} = \langle\dot{\boldsymbol{y}}\rangle + \langle\boldsymbol{F}\rangle\boldsymbol{v}, \qquad \overset{\triangle}{\boldsymbol{y}} = \langle\dot{\boldsymbol{y}}\rangle + \langle\boldsymbol{F}\rangle\boldsymbol{w}$$

following ∂P, $\mathscr{S}$, and $\partial\mathscr{G}$. Then, arguing as in the paragraph containing (5–2), I consider $\overset{\triangle}{\boldsymbol{y}}$ as the appropriate work-conjugate velocity for $\mathbf{S}$ and write the **working** $W(P)$ on $P = P(t)$ in the form

$$W(P) = \int_{\partial P} (\boldsymbol{C}\boldsymbol{n}\cdot\boldsymbol{q} + \boldsymbol{S}\boldsymbol{n}\cdot\mathring{\boldsymbol{y}})\,da + \int_P \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\,dv$$
$$+ \int_{\mathscr{G}} (\boldsymbol{e}^{\mathscr{S}}\cdot\boldsymbol{v} + \boldsymbol{b}^{\mathscr{S}}\cdot\overset{\square}{\boldsymbol{y}})\,da + \int_{\partial\mathscr{G}} (\mathbf{C}\boldsymbol{n}\cdot\boldsymbol{w} + \mathbf{S}\boldsymbol{n}\cdot\overset{\triangle}{\boldsymbol{y}})\,ds. \quad (21\text{–}6)$$

The requirement that $W(P)$ be invariant under changes in material and spatial observer yields the **standard force** and **moment balances**

$$\int_{\partial P} \boldsymbol{S}\boldsymbol{n}\,da + \int_P \boldsymbol{b}\,dv + \int_{\partial\mathscr{G}} \mathbf{S}\boldsymbol{n}\,ds + \int_{\mathscr{G}} \boldsymbol{b}^{\mathscr{S}}\,da = \boldsymbol{0},$$
$$\int_{\partial P} (\boldsymbol{y}-\boldsymbol{o})\times\boldsymbol{S}\boldsymbol{n}\,da + \int_P (\boldsymbol{y}-\boldsymbol{o})\times\boldsymbol{b}\,dv + \int_{\partial\mathscr{G}} (\boldsymbol{y}-\boldsymbol{o})\times\mathbf{S}\boldsymbol{n}\,ds \quad (21\text{–}7a)$$
$$+ \int_{\mathscr{G}} (\boldsymbol{y}-\boldsymbol{o})\times\boldsymbol{b}^{\mathscr{S}}\,da = \boldsymbol{0}, \quad (21\text{–}7b)$$

and the configurational force balance (20–4) and these yield, for the interface, the local force balances

$$[\boldsymbol{S}]\mathbf{m} + \mathrm{Div}_{\mathscr{S}}\mathbf{S} + \boldsymbol{b}^{\mathscr{S}} = \boldsymbol{0}, \quad (21\text{–}8a)$$
$$[\boldsymbol{C}]\mathbf{m} + \mathrm{Div}_{\mathscr{S}}\mathbf{C} + \boldsymbol{g}^{\mathscr{S}} + \boldsymbol{e}^{\mathscr{S}} = \boldsymbol{0}, \quad (21\text{–}8b)$$

and the moment balance

$$\mathbf{S}\mathbf{F}^{\top} = \mathbf{F}\mathbf{S}^{\top}. \quad (21\text{–}9)$$

The derivation of (21–8) is identical to that of (16–4). The proof of (21–9) follows that of the classical relation (3–6b). Thus, consider the tensor $\boldsymbol{M}(P)$ defined as the left side of (21–7b) with the operation "$\times$" replaced by "$\otimes$." Then (21–7b) is equivalent to the assertion that $\boldsymbol{M}(P)$ be symmetric: $\boldsymbol{M}(P) = \boldsymbol{M}(P)^{\top}$. Since $\mathbf{F} = \nabla_{\mathscr{S}}\boldsymbol{y}$, the surface divergence theorem implies that

$$\int_{\partial\mathscr{G}} (\boldsymbol{y}-\boldsymbol{o})\times\mathbf{S}\boldsymbol{n}\,ds = \int_{\mathscr{G}} (\boldsymbol{y}-\boldsymbol{o})\otimes\mathrm{Div}_{\mathscr{S}}\mathbf{S}\,da + \int_{\mathscr{G}} \mathbf{F}\mathbf{S}^{\top}\,da.$$

Thus, if $P(t)$ is taken to be the δ-pillbox $\mathscr{G}_\delta(t)$ about an arbitrary subsurface $\mathscr{G}(t)$ of $\mathscr{S}(t)$, then by (11–25b),

$$\int_{\partial\mathscr{G}_\delta} (\boldsymbol{y}-\boldsymbol{o})\otimes \boldsymbol{S}\boldsymbol{n}\, da \to \int_{\mathscr{G}} (\boldsymbol{y}-\boldsymbol{o})\otimes [\boldsymbol{S}]\mathbf{m}\, da$$

as $\delta \to 0$, so that, by (21–8a), in this limit,

$$\boldsymbol{M}(\mathscr{G}_\delta(t)) \to \int_{\mathscr{G}} \mathbf{F}\mathbf{S}^\top da.$$

Because $\boldsymbol{M}(\mathscr{G}_\delta(t))$ is symmetric and $\mathscr{G}$ is arbitrary, (21–9) follows.

By (15–4), the configurational stress $\mathbf{C}$ may be decomposed into tangential and normal parts,

$$\mathbf{C} = \mathbf{C}_{\text{tan}} + \mathbf{m}\otimes\boldsymbol{\tau};$$

thus, because $\mathbf{m}\cdot \text{Div}_{\mathscr{S}}(\mathbf{m}\otimes\boldsymbol{\tau}) = \text{Div}_{\mathscr{S}}\boldsymbol{\tau}$ and $(\mathbf{C}_{\text{tan}})^\top\mathbf{m} = \mathbf{C}^\top\mathbf{P}\mathbf{m} = 0$,

$$\mathbf{m}\cdot \text{Div}_{\mathscr{S}}\mathbf{C}_{\text{tan}} = \text{Div}_{\mathscr{S}}\left\{(\mathbf{C}_{\text{tan}})^\top\mathbf{m}\right\} - \mathbf{C}_{\text{tan}}\cdot\nabla_{\mathscr{S}}\mathbf{m} = \mathbf{C}_{\text{tan}}\cdot\mathbf{L},$$

and the normal component of the configurational force balance (21–8b) takes the form

$$\mathbf{m}\cdot[\boldsymbol{C}]\mathbf{m} + \mathbf{C}_{\text{tan}}\cdot\mathbf{L} + \text{Div}_{\mathscr{S}}\boldsymbol{\tau} + g^{\mathscr{S}} + e^{\mathscr{S}} = 0, \tag{21–10a}$$

$$g^{\mathscr{S}} = \mathbf{m}\cdot\boldsymbol{g}^{\mathscr{S}}, \qquad e^{\mathscr{S}} = \mathbf{m}\cdot\boldsymbol{e}^{\mathscr{S}}. \tag{21–10b}$$

As before, invariance of the working $W(P)$ under changes in the choice of velocity fields for $\partial P(t)$ and $\mathscr{S}(t)$ yields $\boldsymbol{C} = \pi\mathbf{1} - \boldsymbol{F}^\top\boldsymbol{S}$ and $\mathbf{P}\boldsymbol{e}^{\mathscr{S}} = -\mathbf{P}\langle\boldsymbol{F}\rangle^\top\boldsymbol{b}^{\mathscr{S}}$. On the other hand, invariance under changes in the velocity field $\boldsymbol{w}$ for $\partial\mathscr{G}$ no longer renders the tangential part of the configurational interfacial stress a surface tension. To verify this, note that, because $\overset{\triangle}{\boldsymbol{y}} = \langle\dot{\boldsymbol{y}}\rangle + \langle\boldsymbol{F}\rangle\boldsymbol{w}$ and the tangential component of $\boldsymbol{w}$ is arbitrary, invariance of (21–6) under the choice of $\boldsymbol{w}$ is equivalent to the requirement that

$$\int_{\partial\mathscr{G}} (\mathbf{C}\mathbf{n} + \langle\boldsymbol{F}\rangle^\top\mathbf{S}\mathbf{n})\cdot\mathbf{t}\, ds = 0$$

for every vector field $\mathbf{t}$ tangent to the curve $\partial\mathscr{G}$. Thus, arguing as in the proof of (16–7), there is a scalar field σ, the **surface tension**, such that $(\mathbf{C}+\langle\boldsymbol{F}\rangle^\top\mathbf{S})_{\text{tan}} = \sigma\mathbf{P}$, and, since $\mathbf{P}\langle\boldsymbol{F}\rangle^\top = \langle\boldsymbol{F}\mathbf{P}\rangle^\top = \mathbf{F}^\top$,

$$\mathbf{C}_{\text{tan}} = \sigma\mathbf{P} - \mathbf{F}^\top\mathbf{S}. \tag{21–11}$$

Thus, *in the presence of standard stresses within the interface, the tangential part of the configurational interfacial stress is no longer a surface tension*; *instead* $\mathbf{C}_{\text{tan}}$ *has a form comparable to its bulk counterpart* $\boldsymbol{C} = \pi\mathbf{1} - \boldsymbol{F}^\top\boldsymbol{S}$.

Also important is the normal part $\mathbf{d}$ of the tensor $\mathbf{C} + \langle\boldsymbol{F}\rangle^\top\mathbf{S}$; by the second of (15–5), $\mathbf{d}$ and the normal part $\boldsymbol{\tau}$ of $\mathbf{C}$ are related through

$$\boldsymbol{\tau} = \mathbf{d} - \mathbf{S}^\top\langle\boldsymbol{F}\rangle\mathbf{m}. \tag{21–12}$$

Note the similarity between this expression and the formula (21–11) for the surface tension σ; in fact,

$$\mathbf{C} + \langle \boldsymbol{F} \rangle^{\top} \mathbf{S} = \sigma \mathbf{P} + \mathbf{m} \otimes \mathbf{d},$$

an expansion that motivates our referring to $\mathbf{d}$, rather than to $\boldsymbol{\tau}$, as the **surface shear**.

Finally, choosing the intrinsic forms

$$\begin{aligned} &\boldsymbol{q} = U\boldsymbol{n}, && \boldsymbol{v} = V\mathbf{m}, && \boldsymbol{w} = \boldsymbol{v} + V_{\partial\mathscr{G}}\mathbf{n}, \\ &\mathring{\boldsymbol{y}} = \dot{\boldsymbol{y}} + \boldsymbol{F}\boldsymbol{q}, && \mathring{\boldsymbol{y}} = \langle \dot{\boldsymbol{y}} \rangle + \langle \boldsymbol{F} \rangle \boldsymbol{v}, && \overset{\triangle}{\boldsymbol{y}} = \overset{\square}{\boldsymbol{y}} + V_{\partial\mathscr{G}}\mathbf{F}\mathbf{n} \end{aligned}$$

for the velocity fields and using (21–4),

$$\begin{aligned} \mathbf{C}\mathbf{n} \cdot \boldsymbol{w} + \mathbf{S}\mathbf{n} \cdot \overset{\triangle}{\boldsymbol{y}} &= \mathbf{C}\mathbf{n} \cdot \boldsymbol{v} + \mathbf{C}\mathbf{n} \cdot \mathbf{n} V_{\partial\mathscr{G}} + \mathbf{S}\mathbf{n} \cdot (\overset{\square}{\boldsymbol{y}} + V_{\partial\mathscr{G}}\mathbf{F}\mathbf{n}) \\ &= V_{\partial\mathscr{G}}\mathbf{n} \cdot (\mathbf{C} + \mathbf{F}^{\top}\mathbf{S})\mathbf{n} + \mathbf{C}\mathbf{n} \cdot \boldsymbol{v} + \mathbf{S}\mathbf{n} \cdot \overset{\square}{\boldsymbol{y}} \\ &= \sigma V_{\partial\mathscr{G}} + \mathbf{C}\mathbf{n} \cdot \boldsymbol{v} + \mathbf{S}\mathbf{n} \cdot \overset{\square}{\boldsymbol{y}}. \end{aligned}$$

Thus, arguing as in the derivation of (20–10), the working may be written in the intrinsic form (cf. (11–14), (16–15))

$$\begin{aligned} W(P) = &\int_{\partial P} \boldsymbol{S}\boldsymbol{n} \cdot \dot{\boldsymbol{y}}\, da + \int_{P} \boldsymbol{b} \cdot \dot{\boldsymbol{y}}\, dv + \int_{\partial P} \pi U\, da \\ &+ \int_{\mathscr{G}} (\boldsymbol{b}^{\mathscr{S}} \cdot \overset{\square}{\boldsymbol{y}} + e^{\mathscr{S}} V)\, da + \int_{\partial\mathscr{G}} \left(\sigma V_{\partial\mathscr{G}} + \mathbf{C}\mathbf{n} \cdot \boldsymbol{v} + \mathbf{S}\mathbf{n} \cdot \overset{\square}{\boldsymbol{y}} \right) ds. \end{aligned} \tag{21–13}$$

Note that, since $\mathbf{C} + \langle \boldsymbol{F} \rangle^{\top}\mathbf{S} = \sigma\mathbf{P} + \mathbf{m} \otimes \mathbf{d}$, the term $\mathbf{C}\mathbf{n} \cdot \boldsymbol{v} + \mathbf{S}\mathbf{n} \cdot \overset{\square}{\boldsymbol{y}}$ may be written as $\mathbf{S}\mathbf{n} \cdot \langle \dot{\boldsymbol{y}} \rangle + (\mathbf{d} \cdot \mathbf{n})V$, and, granted inertial external forces, the term $\boldsymbol{b}^{\mathscr{S}} \cdot \overset{\square}{\boldsymbol{y}} + e^{\mathscr{S}} V$ may be replaced by $\boldsymbol{b}^{\mathscr{S}} \cdot \langle \dot{\boldsymbol{y}} \rangle$ (cf. (12–8)).

c. Power balance. Internal working

Using the surface divergence theorem,

$$\int_{\partial\mathscr{G}} \mathbf{S}\mathbf{n} \cdot \overset{\square}{\boldsymbol{y}}\, ds = \int_{\mathscr{G}} \left\{ \overset{\square}{\boldsymbol{y}} \cdot \mathrm{Div}_{\mathscr{S}}\mathbf{S} + \mathbf{S} \cdot \nabla_{\mathscr{S}} \overset{\square}{\boldsymbol{y}} \right\} da, \tag{21–14a}$$

$$\int_{\partial\mathscr{G}} \mathbf{C}\mathbf{n} \cdot \boldsymbol{v}\, ds = \int_{\mathscr{G}} \left\{ V\mathbf{m} \cdot \mathrm{Div}_{\mathscr{S}}\mathbf{C} + \mathbf{C} \cdot \nabla_{\mathscr{S}} \boldsymbol{v} \right\} da. \tag{21–14b}$$

In view of (21–8a), the calculation (11–15) remains valid provided $\boldsymbol{b}^{\mathscr{S}}$ is replaced by $\boldsymbol{b}^{\mathscr{S}} + \mathrm{Div}_{\mathscr{S}}\mathbf{S}$, and this yields

$$[\boldsymbol{S}\mathbf{m} \cdot \dot{\boldsymbol{y}}] = -(\boldsymbol{b}^{\mathscr{S}} + \mathrm{Div}_{\mathscr{S}}\mathbf{S}) \cdot \overset{\square}{\boldsymbol{y}} - \mathbf{m} \cdot [\boldsymbol{F}^{\top}\boldsymbol{S}]\mathbf{m}V$$

in place of (11–15), and hence

$$\int_{\partial P} \boldsymbol{S}\boldsymbol{n}\cdot\dot{\boldsymbol{y}}\,da + \int_P \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\,dv + \int_{\mathscr{G}} \boldsymbol{b}^{\mathscr{S}}\cdot\overset{\square}{\boldsymbol{y}}\,da + \int_G \overset{\square}{\boldsymbol{y}}\cdot\mathrm{Div}_{\mathscr{S}}\mathbf{S}\,da$$
$$= \int_P \boldsymbol{S}\cdot\dot{\boldsymbol{F}}\,dv - \int_{\mathscr{G}} \mathbf{m}\cdot[\boldsymbol{F}^\top\boldsymbol{S}]\mathbf{m}V\,da$$

in place of (11–16b). Thus, by (21–14a),

$$\int_{\partial P} \boldsymbol{S}\boldsymbol{n}\cdot\dot{\boldsymbol{y}}\,da + \int_P \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\,dv + \int_{\mathscr{G}} \boldsymbol{b}^{\mathscr{S}}\cdot\overset{\square}{\boldsymbol{y}}\,da + \int_{\partial\mathscr{G}} \mathbf{S}\boldsymbol{n}\cdot\overset{\square}{\boldsymbol{y}}\,ds$$
$$= \int_P \boldsymbol{S}\cdot\dot{\boldsymbol{F}}\,dv + \int_{\mathscr{G}} \mathbf{S}\cdot\nabla_{\mathscr{S}}\overset{\square}{\boldsymbol{y}}\,da - \int_{\mathscr{G}} \mathbf{m}\cdot[\boldsymbol{F}^\top\boldsymbol{S}]\mathbf{m}V\,da. \qquad (21\text{–}15)$$

Next, $\mathbf{C} = \mathbf{C}_{\tan} + \mathbf{m}\otimes\boldsymbol{\tau}$ and $\nabla_{\mathscr{S}}\boldsymbol{v} = -\mathbf{m}\otimes\overset{\square}{\mathbf{m}} - V\mathbf{L}$ (cf. (16–19)); thus, by (21–11) and (21–12),

$$\mathbf{C}\cdot\nabla_{\mathscr{S}}\boldsymbol{v} = -V\mathbf{C}_{\tan}\cdot\mathbf{L} - \boldsymbol{\tau}\cdot\overset{\square}{\mathbf{m}} = -\sigma KV + V(\mathbf{F}^\top\mathbf{S})\cdot\mathbf{L} - \mathbf{d}\cdot\overset{\square}{\mathbf{m}} + \big(\mathbf{S}^\top\langle\boldsymbol{F}\rangle\mathbf{m}\big)\cdot\overset{\square}{\mathbf{m}}.$$

On the other hand, since $\mathbf{S}\mathbf{m} = \mathbf{0}$, it follows that $\mathbf{S}\mathbf{P}^\top = \mathbf{S}\mathbf{P} = \mathbf{S}$, and since $\mathbf{L}$ is tangential, $\mathbf{P}\mathbf{L} = \mathbf{L}$ and $\langle\boldsymbol{F}\rangle\mathbf{L} = \langle\boldsymbol{F}\rangle\mathbf{P}\mathbf{L} = \mathbf{F}\mathbf{L}$; hence (21–2) yields

$$\mathbf{S}\cdot\nabla_{\mathscr{S}}\overset{\square}{\boldsymbol{y}} = \mathbf{S}\cdot(\langle\boldsymbol{F}\rangle^{\square}\mathbf{P}) - (\langle\boldsymbol{F}\rangle\mathbf{m})\cdot(\mathbf{S}\overset{\square}{\mathbf{m}}) - V\mathbf{S}\cdot(\langle\boldsymbol{F}\rangle\mathbf{L})$$
$$= \mathbf{S}\cdot\langle\boldsymbol{F}\rangle^{\square} - (\mathbf{S}^\top\langle\boldsymbol{F}\rangle\mathbf{m})\cdot\overset{\square}{\mathbf{m}} - V(\mathbf{F}^\top\mathbf{S})\cdot\mathbf{L}.$$

Adding the last two relations yields an expression for the stress power of the configurational and standard forces within the interface,

$$\mathbf{C}\cdot\nabla_{\mathscr{S}}\boldsymbol{v} + \mathbf{S}\cdot\nabla_{\mathscr{S}}\overset{\square}{\boldsymbol{y}} = -\sigma KV - \mathbf{d}\cdot\overset{\square}{\mathbf{m}} + \mathbf{S}\cdot\langle\boldsymbol{F}\rangle^{\square}; \qquad (21\text{–}16)$$

thus adding (21–14b) and (21–15) yields

$$\int_{\partial P} \boldsymbol{S}\boldsymbol{n}\cdot\dot{\boldsymbol{y}}\,da + \int_P \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\,dv + \int_{\mathscr{G}} \boldsymbol{b}^{\mathscr{S}}\cdot\overset{\square}{\boldsymbol{y}}\,da + \int_{\partial\mathscr{G}} (\mathbf{S}\boldsymbol{n}\cdot\overset{\square}{\boldsymbol{y}} + \mathbf{C}\boldsymbol{n}\cdot\boldsymbol{v})\,ds$$
$$= \int_P \boldsymbol{S}\cdot\dot{\boldsymbol{F}}\,dv + \int_{\mathscr{G}} \Big\{\mathbf{S}\cdot\langle\boldsymbol{F}\rangle^{\square} + \big(\mathbf{m}\cdot\mathrm{Div}_{\mathscr{S}}\mathbf{C} - \mathbf{m}\cdot[\boldsymbol{F}^\top\boldsymbol{S}]\mathbf{m}\big)V - \sigma KV - \mathbf{d}\cdot\overset{\square}{\mathbf{m}}\Big\}\,da.$$

Because $\boldsymbol{C} = \pi\mathbf{1} - \boldsymbol{F}^\top\boldsymbol{S}$, the configurational force balance (21–8b) may be used to eliminate the term $\mathbf{m}\cdot\mathrm{Div}_{\mathscr{S}}\mathbf{C} - \mathbf{m}\cdot[\boldsymbol{F}^\top\boldsymbol{S}]\mathbf{m}$; the result is

$$\int_{\partial P} \boldsymbol{S}\boldsymbol{n}\cdot\dot{\boldsymbol{y}}\,da + \int_P \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\,dv + \int_{\mathscr{G}} \Big(\boldsymbol{b}^{\mathscr{S}}\cdot\overset{\square}{\boldsymbol{y}} + e^{\mathscr{S}}V\Big)\,da + \int_{\partial\mathscr{G}} \Big(\mathbf{S}\boldsymbol{n}\cdot\overset{\square}{\boldsymbol{y}} + \mathbf{C}\boldsymbol{n}\cdot\boldsymbol{v}\Big)\,ds$$
$$= \int_P \boldsymbol{S}\cdot\dot{\boldsymbol{F}}\,dv + \int_{\mathscr{G}} \Big\{\mathbf{S}\cdot\langle\boldsymbol{F}\rangle^{\square} - \big(\sigma K + [\pi] + g^{\mathscr{S}}\big)V - \mathbf{d}\cdot\overset{\square}{\mathbf{m}}\Big\}\,da, \qquad (21\text{–}17)$$

with $g^{\mathscr{S}} = \mathbf{m}\cdot\boldsymbol{g}^{\mathscr{S}}$ and $e^{\mathscr{S}} = \mathbf{m}\cdot\boldsymbol{e}^{\mathscr{S}}$.

The identity (21–17) represents a **reduced power balance** for P, because its left side differs from $W(P)$ as given by (21–13) in the absence of the integrals $\int_{\partial P} \pi U\,da$

and $\int_{\partial\mathscr{G}} \sigma V_{\partial\mathscr{G}}\, ds$; adding these integrals to the right side of (21–17) yields the **power balance**

$$W(P) = \int_P \boldsymbol{S}\cdot\dot{\boldsymbol{F}}\, dv + \int_{\mathscr{G}} \left\{ \mathbf{S}\cdot\langle \boldsymbol{F}\rangle^{\square} - \sigma K V - \mathbf{d}\cdot\overset{\square}{\mathbf{m}} - ([\pi] + g^{\mathscr{S}})V \right\} da + \int_{\partial\mathscr{G}} \sigma V_{\partial\mathscr{G}}\, ds + \int_{\partial P} \pi U\, da. \tag{21–18}$$

The right side of this balance represents **internal working**; this working differs in two respects from its counterpart (20–12) (of the theory that neglects $\mathbf{S}$):

The internal working includes a term, the **interfacial stress power** $\mathbf{S}\cdot\langle \boldsymbol{F}\rangle^{\square}$, that represents working associated with stretching of the interface.

The surface tension σ and the surface shear $\mathbf{d}$ no longer represent the tangential and normal parts of the configurational stress $\mathbf{C}$. But although these fields are combinations of standard and configurational terms, they are, in a sense, *internally configurational*, since they perform work, internally, over temporal changes of interfacial area and orientation.

d. Second law. Interfacial dissipation inequality

The **second law** takes the form

$$\frac{d}{dt}\left\{ \int_P \Psi\, dv + \int_{\mathscr{G}} \psi\, da \right\} \le W(P), \tag{21–19}$$

where $W(P)$ is given by (21–6), while ψ and Ψ are the interfacial and bulk free energies as described following (20–14).

Using the power balance (21–18) for $W(P)$, an argument similar to that leading to (20–15) yields (20–15) with the right side replaced by

$$\int_{\mathscr{G}} \left\{ \mathbf{S}\cdot\langle \boldsymbol{F}\rangle^{\square} - (\sigma K + g^{\mathscr{S}})V - \mathbf{d}\cdot\overset{\square}{\mathbf{m}} \right\} da,$$

and this leads to the identity

$$\sigma = \psi \tag{21–20}$$

and the interfacial dissipation inequality

$$\overset{\square}{\psi} - \mathbf{S}\cdot\langle \boldsymbol{F}\rangle^{\square} + \mathbf{d}\cdot\overset{\square}{\mathbf{m}} + g^{\mathscr{S}} V \le 0, \tag{21–21}$$

with $(\ldots)^{\square}$ the normal time derivative following $\mathscr{S}$.

Here it is important to note that (21–11) and (21–20) yield an **Eshelby relation for the interface**:

$$\mathbf{C}_{\tan} = \psi\mathbf{P} - \mathbf{F}^{\top}\mathbf{S}. \tag{21–22}$$

Finally, note that for a stationary control volume P and inertial external forces, (12–6), (21–13) with $\sigma = \psi$, the sentence following (21–13), and (21–19) yield the following version of the second law:

$$\frac{d}{dt}\left\{\int_P (\Psi + k)\,dv + \int_{\mathscr{G}} \psi\,da\right\} = \int_{\partial\mathscr{G}} \psi V_{\partial\mathscr{G}}\,ds + \int_{\partial P} \boldsymbol{S}\boldsymbol{n}\cdot\dot{\boldsymbol{y}}\,da + \int_{\partial\mathscr{G}} \left(\mathbf{S}\mathbf{n}\cdot\langle\dot{\mathbf{y}}\rangle + (\mathbf{d}\cdot\mathbf{n})V\right)\,ds.$$

e. Constitutive equations

The bulk phases α and β are again presumed to be elastic as defined by (12–19).

Regarding the interface, I consider constitutive equations of the form (20–18), with $\boldsymbol{\tau}$ replaced by $\mathbf{d}$ and with an additional relation for $\mathbf{S}$:

$$\psi = \hat{\psi}(\mathbf{m}, V, \boldsymbol{A}, \boldsymbol{j}), \tag{21–23a}$$

$$\mathbf{d} = \hat{\mathbf{d}}(\mathbf{m}, V, \boldsymbol{A}, \boldsymbol{j}), \tag{21–23b}$$

$$\mathbf{S} = \hat{\mathbf{S}}(\mathbf{m}, V, \boldsymbol{A}, \boldsymbol{j}), \tag{21–23c}$$

$$g^{\mathscr{S}} = \hat{g}^{\mathscr{S}}(\mathbf{m}, V, \boldsymbol{A}, \boldsymbol{j}). \tag{21–23d}$$

Here, as before, $\boldsymbol{A} = \langle \boldsymbol{F}\rangle$ and $\boldsymbol{j} = [\boldsymbol{F}]\mathbf{m}$.

An argument similar to that following (20–18) shows that there are sufficient external and indeterminate forces available to ensure satisfaction of all relevant balances and identities.

The requirement that *all constitutive processes be consistent with the dissipation inequality* (21–21) is equivalent to the inequality

$$\begin{aligned}&\partial_V \hat{\psi}(\mathbf{m}, V, \boldsymbol{A}, \boldsymbol{j})\,\overset{\square}{V}\\ &\quad + \left\{\partial_{\boldsymbol{A}} \hat{\psi}(\mathbf{m}, V, \boldsymbol{A}, \boldsymbol{j}) - \hat{\mathbf{S}}(\mathbf{m}, V, \boldsymbol{A}, \boldsymbol{j})\right\}\cdot\overset{\square}{\boldsymbol{A}} + \left\{\partial_{\boldsymbol{j}} \hat{\psi}(\mathbf{m}, V, \boldsymbol{A}, \boldsymbol{j})\right\}\cdot\overset{\square}{\boldsymbol{j}}\\ &\quad + \left\{\partial_{\mathbf{m}} \hat{\psi}(\mathbf{m}, V, \boldsymbol{A}, \boldsymbol{j}) + \hat{\mathbf{d}}(\mathbf{m}, V, \boldsymbol{A}, \boldsymbol{j})\right\}\cdot\overset{\square}{\mathbf{m}} + \hat{g}^{\mathscr{S}}(\mathbf{m}, V, \boldsymbol{A}, \boldsymbol{j})V \le 0,\end{aligned} \tag{21–24}$$

with $(\ldots)^{\square}$ the normal time derivative following $\mathscr{S}$. An argument identical to that used to prove (20–20) and (20–21) then yields the following conclusions:

(i) the free energy ψ, the shear $\mathbf{d}$, and the standard interfacial stress $\mathbf{S}$ must be independent of V and $\boldsymbol{j}$, and must be related through

$$\psi = \hat{\psi}(\mathbf{m}, \boldsymbol{A}) \tag{21–25a}$$

$$\mathbf{d} = -\partial_{\mathbf{m}} \hat{\psi}(\mathbf{m}, \boldsymbol{A}), \tag{21–25b}$$

$$\mathbf{S} = \partial_{\boldsymbol{A}} \hat{\psi}(\mathbf{m}, \boldsymbol{A}); \tag{21–25c}$$

(ii) the normal internal force must obey the inequality

$$\hat{g}^{\mathscr{S}}(\mathbf{m}, V, \boldsymbol{A}, \boldsymbol{j})V \leq 0, \tag{21–26}$$

which, for $\hat{g}^{\mathscr{S}}$ smooth, results in the constitutive equation

$$g^{\mathscr{S}} = -b(\mathbf{m}, V, \boldsymbol{A}, \boldsymbol{j})V, \qquad b(\mathbf{m}, V, \boldsymbol{A}, \boldsymbol{j}) \geq 0. \tag{21–27}$$

By (21–25c) and because ψ can depend on $\boldsymbol{F}^{\pm}$ at most through $\boldsymbol{A} = \langle \boldsymbol{F} \rangle$, the interfacial stress $\mathbf{S}$ vanishes if and only if the interfacial energy ψ is independent of the deformation gradient.

Because $\mathbf{S}$ is a superficial tensor field, $\mathbf{S}\mathbf{m} = \mathbf{0}$. Thus a second consequence of the stress relation (21–25c) is that the response function $\hat{\psi}$ satisfy

$$\left(\partial_{\boldsymbol{A}} \hat{\psi}(\mathbf{m}, \boldsymbol{A})\right) \mathbf{m} = \mathbf{0}. \tag{21–28}$$

Further, since $\mathbf{F} = \boldsymbol{A}\mathbf{P}$, the identity $\boldsymbol{A} = \boldsymbol{A}\mathbf{P} + (\boldsymbol{A}\mathbf{m}) \otimes \mathbf{m}$ yields the decomposition

$$\boldsymbol{A} = \mathbf{F} + \boldsymbol{a} \otimes \mathbf{m}, \qquad \boldsymbol{a} = \boldsymbol{A}\mathbf{m};$$

this allows $\hat{\psi}(\mathbf{m}, \boldsymbol{A})$ to be considered as a function

$$\tilde{\psi}(\mathbf{m}, \mathbf{F}, \boldsymbol{a}) = \hat{\psi}(\mathbf{m}, \boldsymbol{A}) = \hat{\psi}(\mathbf{m}, \mathbf{F} + \boldsymbol{a} \otimes \mathbf{m}).$$

Thus, by (21–28), $\partial_{\boldsymbol{a}} \tilde{\psi} = (\partial_{\boldsymbol{A}} \hat{\psi})\mathbf{m} = \mathbf{0}$, $\tilde{\psi}(\mathbf{m}, \mathbf{F}, \boldsymbol{a})$ is therefore independent of $\boldsymbol{a}$. Further, $\partial_{\mathbf{F}} \tilde{\psi} = \partial_{\boldsymbol{A}} \hat{\psi}$; hence

$$\psi = \tilde{\psi}(\mathbf{m}, \mathbf{F}), \qquad \mathbf{S} = \partial_{\mathbf{F}} \tilde{\psi}(\mathbf{m}, \mathbf{F})$$

and the interfacial energy ψ and standard stress $\mathbf{S}$ depend on $\boldsymbol{A} = \langle \boldsymbol{F} \rangle$ through the tangential deformation gradient $\mathbf{F} = \langle \boldsymbol{F} \rangle \mathbf{P}$.

Remark. When deciding on possible energies $\hat{\psi}(\mathbf{m}, \boldsymbol{A})$ for an admissible theory, the condition (21–28) is crucial: granted (21–28), the relations (21–25b,c) may be used as *defining relations* for $\mathbf{S}$ and $\mathbf{d}$.

Finally, I sketch an argument showing that the tangential force balance (21–8b) is satisfied identically with $\mathbf{P}\boldsymbol{g}^{\mathscr{S}} = \mathbf{0}$. The steps in the argument are as follows:[4]

1. Restrict $\boldsymbol{F}$ to one side of the interface. Then, for $\boldsymbol{G}$ a constant tensor,

$$\begin{aligned} \boldsymbol{a} \cdot \mathrm{Div}_{\mathscr{S}}(\boldsymbol{F}^{\top}\boldsymbol{G}) = \mathrm{Div}_{\mathscr{S}}(\boldsymbol{G}^{\top}\boldsymbol{F}\boldsymbol{a}) &= tr\left\{\left(\nabla(\boldsymbol{G}^{\top}\boldsymbol{F}\boldsymbol{a})\right)\mathbf{P}\right\} = tr\left\{\boldsymbol{G}^{\top}\left(\nabla(\boldsymbol{F}\boldsymbol{a})\right)\mathbf{P}\right\} \\ &= (\boldsymbol{G}\mathbf{P}) \cdot \left(\nabla(\boldsymbol{F}\boldsymbol{a})\right) = \{(\boldsymbol{G}\mathbf{P}){:}\nabla\boldsymbol{F}\} \cdot \boldsymbol{a}, \end{aligned}$$

so that

$$\mathrm{Div}_{\mathscr{S}}(\boldsymbol{F}^{\top}\boldsymbol{G}) = (\boldsymbol{G}\mathbf{P}){:}\nabla\boldsymbol{F}.$$

2. Let $\boldsymbol{A} = \langle \boldsymbol{F} \rangle$. Then using the result 1,

$$\mathbf{P}\,\mathrm{Div}_{\mathscr{S}}\left\{\mathbf{C} + \boldsymbol{A}^{\top}\mathbf{S}\right\} = \mathbf{P}\,\mathrm{Div}_{\mathscr{S}}\,\mathbf{C} + \mathbf{P}\boldsymbol{A}^{\top}\,\mathrm{Div}_{\mathscr{S}}\,\mathbf{S} + \mathbf{S}{:}\nabla_{\mathscr{S}}\boldsymbol{A}.$$

[4] A much simpler proof is possible in two space dimensions.

3. The identity $\mathbf{C} + \langle \boldsymbol{F} \rangle^{\top} \mathbf{S} = \sigma \mathbf{P} + \mathbf{m} \otimes \mathbf{d}$ and the argument leading to (11–10), imply that $\mathbf{P}[\boldsymbol{C}]\mathbf{m} = -\mathbf{P}\boldsymbol{e}^{\mathscr{S}} + \mathbf{P}\boldsymbol{A}^{\top} \operatorname{Div}_{\mathscr{S}} \mathbf{S}$.
4. $\mathbf{P} \operatorname{Div}_{\mathscr{S}} \{\mathbf{C} + \boldsymbol{A}^{\top} \mathbf{S}\} = \nabla_{\mathscr{S}} \sigma - \mathbf{L}\mathbf{d}$, which is an analog of (16–11).
5. Using the results 1–4, the tangential component of the force balance (21–8b) may be expressed in the form

$$\nabla_{\mathscr{S}} \sigma - \mathbf{L}\mathbf{d} - \mathbf{S} : \nabla_{\mathscr{S}} \boldsymbol{A} + \mathbf{P}\boldsymbol{g}^{\mathscr{S}} = \mathbf{0},$$

and, since $\sigma = \psi$, this result and (21–15) yield $\mathbf{P}\boldsymbol{g}^{\mathscr{S}} = \mathbf{0}$.

f. Basic equations with inertial external forces

Assume that the underlying observer, the body force $\boldsymbol{b}$, and the interface forces $\boldsymbol{b}^{\mathscr{S}}$ and $\boldsymbol{e}^{\mathscr{S}}$ are inertial.

The basic equations for the *bulk material*, assumed elastic, consist of the momentum balance (12–20) supplemented by the constitutive equations (12–19).

The basic equations for the *interface* are then the *compatibility conditions* (20–30a,b), the *momentum balance*

$$[\boldsymbol{S}]\mathbf{m} + \operatorname{Div}_{\mathscr{S}} \mathbf{S} = -\rho[\dot{\boldsymbol{y}}]V, \tag{21–29}$$

and the *normal configuration balance*

$$\mathbf{m} \cdot [\Psi \mathbf{1} - \boldsymbol{F}^{\top} \boldsymbol{S}]\mathbf{m} + (\psi \mathbf{P} - \mathbf{F}^{\top} \mathbf{S}) \cdot \mathbf{L} + \operatorname{Div}_{\mathscr{S}} \boldsymbol{\tau} + [k_{\mathrm{rel}}] = bV, \tag{21–30}$$

with $[k_{\mathrm{rel}}] = \frac{1}{2} \rho [|\dot{\boldsymbol{y}} - \overset{\square}{\boldsymbol{y}}|^2]$ (cf. (12–1)), $\boldsymbol{\tau} = \mathbf{d} - \mathbf{S}^{\top} \langle \boldsymbol{F} \rangle \mathbf{m}$, and $b = b(\mathbf{m}, V, \boldsymbol{A}, \boldsymbol{j})$;[5] these relations are supplemented by the constitutive equations

$$\psi = \hat{\psi}(\mathbf{m}, \boldsymbol{A}), \qquad \mathbf{d} = -\partial_{\mathbf{m}} \hat{\psi}(\mathbf{m}, \boldsymbol{A}), \qquad \mathbf{S} = \partial_{\boldsymbol{A}} \hat{\psi}(\mathbf{m}, \boldsymbol{A}). \tag{21–31}$$

Note that (21–30), which represents the normal configurational force balance, may be written in the more suggestive form

$$\mathbf{m} \cdot [\text{bulk Eshelby tensor}]\mathbf{m} + (\text{interfacial Eshelby tensor}) \cdot \mathbf{L} + \operatorname{Div}_{\mathscr{S}} \boldsymbol{\tau} + [k_{\mathrm{rel}}] = bV. \tag{21–32}$$

The relation (21–29) follows from (12–5) and (21–8a), while (21–30) makes use of (12–10), (21–11), (21–22), and (21–27).

g. Lyapunov relations

If the body B is bounded, if $\mathscr{S}$ is a closed surface contained in the interior of B, and if the external forces are inertial, then the global energy inequality (20–36)

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[5] Gurtin and Struthers [1990], Gurtin [1993a, 1995]. See also Lusk [1994]. For *statics* (21–29) was derived by Gurtin and Murdoch [1975] from a force balance, while (21–30) was derived by Leo and Sekerka [1989] as a Euler-Lagrange equation for stable equilibria (cf. Alexander and Johnson [1985], Johnson and Alexander [1986], and Fonseca [1989]).

remains valid, but with $\psi = \hat{\psi}(\mathfrak{m}, \boldsymbol{A})$, and, granted this, the Lyapunov relations (20–38) and (20–39) are valid without change.

CHAPTER 22

Two-Dimensional Theory with Standard and Configurational Stress within the Interface

The three-dimensional theory with *both* standard and configurational forces within the interface is complicated; for that reason I now develop its simpler two-dimensional counterpart.

a. Kinematics

The notation and terminology of Chapter 19 will be used throughout: the interface is a smoothly evolving closed curve $\mathscr{C}(t)$; $\mathbf{t}(\boldsymbol{X}, t)$ and $\mathbf{m}(\boldsymbol{X}, t)$ are tangent and normal fields for $\mathscr{C}(t)$ such that $\mathbf{t} = (\cos\vartheta, \sin\vartheta)$ and $\mathbf{m} = (-\sin\vartheta, \cos\vartheta)$; $\vartheta(\boldsymbol{X}, t)$ is the counterclockwise angle from the (1, 0) axis to $\mathbf{t}(\boldsymbol{X}, t)$; and $K = \vartheta_s$ is the curvature.

Let $\boldsymbol{y}$ be a motion. Within this two-dimensional framework the compatibility condition (10–2) is unchanged; but, since $\mathbf{P} = \mathbf{t} \otimes \mathbf{t}$, (10–2b) may be written as $[\boldsymbol{F}]\mathbf{t} = \mathbf{0}$. Moreover,

$$\boldsymbol{y}_s = \boldsymbol{F}^{\pm}\mathbf{t} = \langle \boldsymbol{F} \rangle \mathbf{t} \tag{22–1}$$

and hence

$$\mathbf{F} = \langle \boldsymbol{F} \rangle \mathbf{t} \otimes \mathbf{t} = \boldsymbol{y}_s \otimes \mathbf{t}.$$

Because $\boldsymbol{F}$ is invertible, the **interfacial stretch**

$$\lambda = |\boldsymbol{y}_s|$$

is strictly positive; trivially,

$$\boldsymbol{y}_s = \lambda \bar{\mathbf{t}}, \qquad \bar{\mathbf{t}} = \frac{\boldsymbol{y}_s}{|\boldsymbol{y}_s|}, \tag{22–2}$$

with $\bar{\mathbf{t}}$ tangent to the deformed interface.

The basic identity (21–2) here has the simple form

$$(\overset{\square}{\boldsymbol{y}})_s = (\boldsymbol{y}_s)^{\square} - KV\boldsymbol{y}_s, \tag{22–3}$$

where $(\ldots)^{\square}$ is the normal time derivative following the interface. The verification of (22–3) proceeds as follows: Let $\boldsymbol{v} = V\mathbf{m}$. Then, since $V_s = \overset{\square}{\vartheta}$, $\overset{\square}{\mathbf{t}} = \overset{\square}{\vartheta}\,\mathbf{m}$, and $\mathbf{m}_s = -K\mathbf{t}$,

$$\boldsymbol{v}_s = \overset{\square}{\vartheta}\,\mathbf{m} - KV\mathbf{t} = \overset{\square}{\mathbf{t}} - KV\mathbf{t}. \tag{22–4}$$

Thus, as $\overset{\square}{\boldsymbol{y}} = \langle\dot{\boldsymbol{y}}\rangle + \langle\boldsymbol{F}\rangle\boldsymbol{v}$, $\langle\boldsymbol{F}\rangle^{\square} = \langle\dot{\boldsymbol{y}}\rangle + \langle\nabla\boldsymbol{F}\rangle\boldsymbol{v}$, and, by (1–23b), $(\langle\nabla\boldsymbol{F}\rangle\mathbf{t})\boldsymbol{v} = (\langle\nabla\boldsymbol{F}\rangle\boldsymbol{v})\mathbf{t}$,

$$\begin{aligned}(\overset{\square}{\boldsymbol{y}}) &= \langle\dot{\boldsymbol{y}}\rangle_s + \langle\boldsymbol{F}\rangle_s\boldsymbol{v} + \langle\boldsymbol{F}\rangle\boldsymbol{v}_s = \langle\dot{\boldsymbol{F}}\rangle\mathbf{t} + (\langle\nabla\boldsymbol{F}\rangle\mathbf{t})\boldsymbol{v} + \langle\boldsymbol{F}\rangle\overset{\square}{\mathbf{t}} - KV\langle\boldsymbol{F}\rangle\mathbf{t}\\ &= \langle\boldsymbol{F}\rangle^{\square}\mathbf{t} + \langle\boldsymbol{F}\rangle\overset{\square}{\mathbf{t}} - KV\boldsymbol{y}_s = (\langle\boldsymbol{F}\rangle\mathbf{t})^{\square} - KV\boldsymbol{y}_s = (\boldsymbol{y}_s)^{\square} - KV\boldsymbol{y}_s.\end{aligned}$$

Let $P(t)$ denote an (arbitrary) migrating control volume whose intersection with the interface is a connected curve $\mathscr{G}(t)$; let $\boldsymbol{X}_A(t)$ and $\boldsymbol{X}_B(t)$, respectively, denote the initial and terminal points of $\mathscr{G}(t)$; and let $u_A(t)$ and $u_B(t)$, defined by (19–6), denote the tangential endpoint velocities of $\mathscr{G}(t)$. The corresponding endpoint velocities $\dot{\boldsymbol{X}}_A$ and $\dot{\boldsymbol{X}}_B$ then obey (19–7), so that, by (21–4) with $\boldsymbol{w}$ replaced by $\dot{\boldsymbol{X}}_A$, the motion velocity $\overset{\triangle}{\boldsymbol{y}}_A$ follow the initial point of $\partial\mathscr{G}$ is given by

$$\overset{\triangle}{\boldsymbol{y}}_A = \langle\dot{\boldsymbol{y}}\rangle_A + \langle\boldsymbol{F}\rangle_A\dot{\boldsymbol{X}}_A = (\langle\dot{\boldsymbol{y}}\rangle + V\langle\boldsymbol{F}\rangle\mathbf{m})_A + (\boldsymbol{y}_s)_A u_A = \overset{\square}{\boldsymbol{y}}_A + (\boldsymbol{y}_S)_A u_A \tag{22–5}$$

and similarly for $\overset{\triangle}{\boldsymbol{y}}_B$, where $\overset{\square}{\boldsymbol{y}}$ is the normal time derivative following $\mathscr{C}$.

b. Forces. Working

The force systems are as discussed in Sections 20a and 21b, but the configurational and standard forces *within* the interface are now described, respectively, by *vector* stress fields $\mathbf{c}(\boldsymbol{X}, t)$ and $\mathbf{s}(\boldsymbol{X}, t)$. These fields represent forces exerted across $\boldsymbol{X}$ *by* the material "into which $\mathbf{t}(\boldsymbol{X}, t)$ points" *on* the material "from which $\mathbf{t}(\boldsymbol{X}, t)$ points" (cf. Section 19b).

Let $P = P(t)$ be a migrating control volume. The rates at which $\mathbf{c}$ and $\mathbf{s}$ perform work on P are given by $\mathbf{c}_B\cdot\dot{\boldsymbol{X}}_B - \mathbf{c}_A\cdot\dot{\boldsymbol{X}}_A$ and $\mathbf{s}_B\cdot\overset{\triangle}{\boldsymbol{y}}_B - \mathbf{s}_A\cdot\overset{\triangle}{\boldsymbol{y}}_A$. Thus, for $\boldsymbol{v}$ a velocity field for $\mathscr{C}$, $\boldsymbol{q}$ a velocity field for ∂P, and $\mathring{\boldsymbol{y}}$ and $\overset{\square}{\boldsymbol{y}}$, the corresponding motion velocities following ∂P and $\mathscr{C}$, the **working** $W(P)$ has the form

$$\begin{aligned}W(P) = &\int_{\partial P}\left(\boldsymbol{C}\boldsymbol{n}\cdot\boldsymbol{q} + \boldsymbol{S}\boldsymbol{n}\cdot\mathring{\boldsymbol{y}}\right)ds + \int_P \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\,da + \int_{\mathscr{G}}\left(\boldsymbol{e}^{\mathscr{C}}\cdot\boldsymbol{v} + \boldsymbol{b}^{\mathscr{C}}\cdot\overset{\square}{\boldsymbol{y}}\right)ds\\ &+ \mathbf{c}_B\cdot\dot{\boldsymbol{X}}_B - \mathbf{c}_A\cdot\dot{\boldsymbol{X}}_A + (\mathbf{s}\cdot\overset{\triangle}{\boldsymbol{y}})_B - (\mathbf{s}\cdot\overset{\triangle}{\boldsymbol{y}})_A.\end{aligned} \tag{22–6}$$

The requirement that $W(P)$ be invariant under changes in material and spatial observer yields the **configurational force balance**

$$\int_{\partial P} \boldsymbol{C}\boldsymbol{n}\, ds + \int_P \boldsymbol{g}\, da + \int_{\mathscr{G}} (\boldsymbol{g}^{\mathscr{C}} + \boldsymbol{e}^{\mathscr{C}})\, ds + \mathbf{c}_B - \mathbf{c}_A = \mathbf{0} \tag{22–7}$$

and the **standard force** and **moment balances**

$$\int_{\partial P} \boldsymbol{S}\boldsymbol{n}\, ds + \int_P \boldsymbol{b}\, da + \int_{\mathscr{G}} \boldsymbol{b}^{\mathscr{C}}\, ds + \mathbf{s}_B - \mathbf{s}_A = 0, \tag{22–8a}$$

$$\int_{\partial P} (\boldsymbol{y} - \boldsymbol{o}) \times \boldsymbol{S}\boldsymbol{n}\, ds + \int_P (\boldsymbol{y} - \boldsymbol{o}) \times \boldsymbol{b}\, da + \int_{\mathscr{G}} (\boldsymbol{y} - \boldsymbol{o}) \times \boldsymbol{b}^{\mathscr{C}}\, ds + (\boldsymbol{y}_B - \boldsymbol{o}) \times \mathbf{s}_B - (\boldsymbol{y}_A - \boldsymbol{o}) \times \mathbf{s}_A = \mathbf{0}, \tag{22–8b}$$

and these yield, for the interface, the local force balances

$$[\boldsymbol{S}]\mathbf{m} + \mathbf{s}_s + \boldsymbol{b}^{\mathscr{C}} = \mathbf{0}, \tag{22–9a}$$

$$[\boldsymbol{C}]\mathbf{m} + \mathbf{c}_s + \boldsymbol{g}^{\mathscr{C}} + \boldsymbol{e}^{\mathscr{C}} = \mathbf{0}, \tag{22–9b}$$

and a moment balance

$$\boldsymbol{y}_s \times \mathbf{s} = \mathbf{0} \tag{22–10}$$

requiring that the *standard stress* $\mathbf{s}$ *be tangent to the deformed interface*.[1]

The derivation of (22–9) is no different than that of (19–14). To verify (22–10), consider the δ-pillbox $\mathscr{G}_\delta(t)$ about an arbitrary connected subcurve $\mathscr{G}(t)$ of the interface. Then, because

$$(\boldsymbol{y}_B - \boldsymbol{o}) \times \mathbf{s}_B - (\boldsymbol{y}_A - \boldsymbol{o}) \times \mathbf{s}_A = \int_{\mathscr{G}} \{(\boldsymbol{y} - \boldsymbol{o}) \times \mathbf{s}\}_s\, ds,$$

(22–8b) applied to $\mathscr{G}_\delta(t)$ yields, in the limit $\delta \to 0$,

$$\mathbf{0} = \int_{\mathscr{G}} (\boldsymbol{y} - \boldsymbol{o}) \times ([\boldsymbol{S}]\mathbf{m} + \mathbf{s}_s = \boldsymbol{b}^{\mathscr{C}})\, ds + \int_{\mathscr{G}} \boldsymbol{y}_s \times \mathbf{s}\, ds.$$

Thus, by (22–9a), $\int_{\mathscr{G}} \boldsymbol{y}_s \times \mathbf{s}\, ds = \mathbf{0}$; since $\mathscr{G}$ is arbitrary, this implies (22–10).

An important consequence of the moment balance (22–10) is the existence of a **scalar stress** ζ such that

$$\mathbf{s} = \zeta \bar{\mathbf{t}}. \tag{22–11}$$

Invariance of the working under changes in the choice of velocity fields for $\partial P(t)$ and $\mathscr{C}(t)$ yields $\boldsymbol{C} = \pi \mathbf{1} - \boldsymbol{F}^\top \boldsymbol{S}$ and $\mathbf{P}\boldsymbol{e}^{\mathscr{C}} = -\mathbf{P}\langle \boldsymbol{F} \rangle^\top \boldsymbol{b}^{\mathscr{C}}$, and, since $\mathbf{P} = \mathbf{t} \otimes \mathbf{t}$, the latter may be rewritten as

$$\mathbf{t} \cdot \boldsymbol{e}^{\mathscr{C}} = -\boldsymbol{y}_s \cdot \boldsymbol{b}^{\mathscr{C}}. \tag{22–12}$$

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[1]The symbol "$\times$" here denotes the (scalar) two-dimensional cross product $\boldsymbol{a} \times \boldsymbol{b} = a_1 b_2 - a_2 b_1$.

Further, by (19–7) and (22–5),

$$\mathbf{c}_A \cdot \dot{\boldsymbol{X}}_A + (\mathbf{s} \cdot \overset{\triangle}{\boldsymbol{y}})_A = \mathbf{c}_A \cdot (u_A \mathbf{t}_A + V_A \mathbf{m}_A) + \mathbf{s}_A \cdot \left\{ \overset{\square}{\boldsymbol{y}}_A + (\boldsymbol{y}_s)_A u_A \right\}$$
$$= (\mathbf{c} \cdot \mathbf{t} + \mathbf{s} \cdot \boldsymbol{y}_s)_A u_A + (\mathbf{c} \cdot \mathbf{m} V)_A + (\mathbf{s} \cdot \overset{\square}{\boldsymbol{y}})_A .$$

and similarly for the terminal point of $\mathscr{G}$. The **surface tension** σ and **surface shear** τ are defined by

$$\sigma = \mathbf{c} \cdot \mathbf{t} + \mathbf{s} \cdot \boldsymbol{y}_s = (\mathbf{c} + \langle \boldsymbol{F} \rangle^\top \mathbf{s}) \cdot \mathbf{t} = \mathbf{c} \cdot \mathbf{t} + \zeta \lambda, \tag{22–13a}$$
$$\tau = \mathbf{c} \cdot \mathbf{m} \tag{22–13b}$$

(cf. (21–11), (21–12)), and thus

$$\mathbf{c}_A \cdot \dot{\boldsymbol{X}}_A + (\mathbf{s} \cdot \overset{\triangle}{\boldsymbol{y}})_A = \sigma_A u_A + \tau_A V_A + (\mathbf{s} \cdot \overset{\square}{\boldsymbol{y}})_A .$$

The choice of τ as surface shear, which might seem inconsistent with its three-dimensional counterpart defined through (21–12), will be discussed at the end of the next subsection.

By (22–13) and the Frenet formulas (19–2), the normal component of (22–9b) is

$$(\sigma - \zeta \lambda) K + \tau_s + \mathbf{m} \cdot [\boldsymbol{C}] \mathbf{m} + g^{\mathscr{C}} + e^{\mathscr{C}} = 0, \tag{22–14a}$$
$$g^{\mathscr{C}} = \mathbf{m} \cdot \boldsymbol{g}^{\mathscr{C}}, \qquad e^{\mathscr{C}} = \mathbf{m} \cdot \boldsymbol{e}^{\mathscr{C}}. \tag{22–14b}$$

c. Power balance. Internal working. Second law

Assume now that the velocity fields have the intrinsic forms

$$\boldsymbol{q} = U \boldsymbol{n}, \qquad \boldsymbol{v} = V \mathbf{m},$$
$$\mathring{\boldsymbol{y}} = \dot{\boldsymbol{y}} + \boldsymbol{F} \boldsymbol{q}, \qquad \overset{\square}{\boldsymbol{y}} = \langle \dot{\boldsymbol{y}} \rangle + \langle \boldsymbol{F} \rangle \boldsymbol{v}.$$

Then, arguing as in the derivation of (20–10), the working may be written intrinsically as (cf. (21–13))

$$W(P) = \int_{\partial P} \boldsymbol{S} \boldsymbol{n} \cdot \dot{\boldsymbol{y}} \, ds + \int_P \boldsymbol{b} \cdot \dot{\boldsymbol{y}} \, da + \int_{\partial P} \pi U \, ds + \int_{\mathscr{G}} (\boldsymbol{b}^{\mathscr{C}} \cdot \overset{\square}{\boldsymbol{y}} + e^{\mathscr{C}} V) \, ds$$
$$+ \sigma_B u_B - \sigma_A u_A + (\tau V)_B - (\tau V)_A + (\mathbf{s} \cdot \overset{\square}{\boldsymbol{y}})_B - (\mathbf{s} \cdot \overset{\square}{\boldsymbol{y}})_A . \tag{22–15}$$

Next, by (22–4) and (22–13),

$$\mathbf{c} \cdot \boldsymbol{v}_S = -(\mathbf{c} \cdot \mathbf{t}) K V + (\mathbf{c} \cdot \mathbf{m}) \overset{\square}{\vartheta} = -\sigma K V + \tau \overset{\square}{\vartheta} + K V \mathbf{s} \cdot \boldsymbol{y}_s . \tag{22–16}$$

Thus, by (22–3),

$$\mathbf{c} \cdot \boldsymbol{v}_s + \mathbf{s} \cdot (\overset{\square}{\boldsymbol{y}})_s = -\sigma K V + \tau \overset{\square}{\vartheta} + \mathbf{s} \cdot (\boldsymbol{y}_s)^{\square}, \tag{22–17}$$

which represents the stress power of the standard and configurational forces within the interface (cf. (21–16)). The counterparts of (21–14a,b) and (21–15) in the current two-dimensional theory are

$$(\mathbf{s}\cdot\overset{\square}{\boldsymbol{y}})_B-(\mathbf{s}\cdot\overset{\square}{\boldsymbol{y}})_A=\int_G\left\{\overset{\square}{\boldsymbol{y}}\cdot\mathbf{s}_s+\mathbf{s}\cdot(\overset{\square}{\boldsymbol{y}})_s\right\}\,ds, \tag{22–18a}$$

$$(\mathbf{c}\cdot\boldsymbol{v})_B-(\mathbf{c}\cdot\boldsymbol{v})_A=\int_G\{V\mathbf{m}\cdot\mathbf{c}_s+\mathbf{c}\cdot\boldsymbol{v}_s\}\,ds. \tag{22–18b}$$

and

$$\begin{aligned}\int_{\partial P}\boldsymbol{S}\boldsymbol{n}\cdot\dot{\boldsymbol{y}}\,ds+\int_P\boldsymbol{b}\cdot\dot{\boldsymbol{y}}\,da+\int_{\mathscr{G}}\boldsymbol{b}^{\mathscr{C}}\cdot\overset{\square}{\boldsymbol{y}}\,ds+(\mathbf{s}\cdot\overset{\square}{\boldsymbol{y}})_B-(\mathbf{s}\cdot\overset{\square}{\boldsymbol{y}})_A\\=\int_P\boldsymbol{S}\cdot\dot{\boldsymbol{F}}\,da+\int_{\mathscr{G}}\mathbf{s}\cdot(\overset{\square}{\boldsymbol{y}})_s ds-\int_{\mathscr{G}}\mathbf{m}\cdot[\boldsymbol{F}^\top\boldsymbol{S}]\mathbf{m}V\,ds.\end{aligned} \tag{22–19}$$

By (22–17), adding (22–18b) and (22–19) yields, after using the configurational balance (22–9b) and the identity $\boldsymbol{C}=\pi\mathbf{1}-\boldsymbol{F}^\top\boldsymbol{S}$ to eliminate the term $(\mathbf{m}\cdot\mathbf{c}_s-\mathbf{m}\cdot[\boldsymbol{F}^\top\boldsymbol{S}]\mathbf{m})$,

$$\begin{aligned}&\int_{\partial P}\boldsymbol{S}\boldsymbol{n}\cdot\dot{\boldsymbol{y}}\,ds+\int_P\boldsymbol{b}\cdot\dot{\boldsymbol{y}}\,da+\int_{\mathscr{G}}(\boldsymbol{b}^{\mathscr{C}}\cdot\overset{\square}{\boldsymbol{y}}+e^{\mathscr{C}}V)\,ds\\&\quad+(\mathbf{s}\cdot\overset{\square}{\boldsymbol{y}})_B-(\mathbf{s}\cdot\overset{\square}{\boldsymbol{y}})_A+(\mathbf{c}\cdot\boldsymbol{v})_B-(\mathbf{c}\cdot\boldsymbol{v})_A\\&=\int_P\boldsymbol{S}\cdot\dot{\boldsymbol{F}}\,da+\int_{\mathscr{G}}\left\{\mathbf{s}\cdot(\boldsymbol{y}_s)^{\square}-\sigma KV+\tau\,\overset{\square}{\vartheta}-([\pi]+g^{\mathscr{C}})V\right\}\,ds.\end{aligned} \tag{22–20}$$

Thus, by (22–15), because $\mathbf{c}\cdot\boldsymbol{v}=\tau V$,

$$\begin{aligned}W(P)=&\int_P\boldsymbol{S}\cdot\dot{\boldsymbol{F}}\,da+\int_{\mathscr{C}}\left\{\mathbf{s}\cdot(\boldsymbol{y}_s)^{\square}-\sigma KV+\tau\,\overset{\square}{\vartheta}-([\pi]+g^{\mathscr{C}})V\right\}\,ds\\&+\int_{\partial P}\pi U\,ds+\sigma_B u_B-\sigma_A u_A.\end{aligned} \tag{22–21}$$

The relations (22–20) and (22–21) should be compared with their three-dimensional analogs (21–17) and (21–18).

The **second law** has the form

$$\frac{d}{dt}\left\{\int_P\Psi\,da+\int_{\mathscr{G}}\psi\,ds\right\}\le W(P) \tag{22–22}$$

for each migrating control volume $P=P(t)$, with $\psi(\boldsymbol{X},t)$ the interfacial free energy per unit length and $\Psi(\boldsymbol{X},t)$ the bulk free energy per unit area, and with $W(P)$ given by (22–6). Thus, appealing to (22–21),

$$\begin{aligned}\frac{d}{dt}\left\{\int_P\Psi\,da+\int_{\mathscr{G}}\psi\,ds\right\}\le&\int_P\boldsymbol{S}\cdot\dot{\boldsymbol{F}}\,dv\\&+\int_{\mathscr{G}}\left\{\mathbf{s}\cdot(\boldsymbol{y}_s)^{\square}-\sigma KV+\tau\,\overset{\square}{\vartheta}-([\pi]+g^{\mathscr{C}})V\right\}\,ds\end{aligned}$$

$$+ \int_{\partial P} \pi U \, ds + \sigma_B u_B - \sigma_A u_A .$$

The argument leading to the relations

$$\pi = \Psi, \qquad \sigma = \psi, \tag{22–23}$$

are as given in Chapter 17 and use (22–21) and the transport theorems (10–8a) and (19–8); these, in turn, lead to the interfacial dissipation inequality (cf. (19–19), (21–21))

$$\overset{\square}{\psi} - \mathbf{s} \cdot (\boldsymbol{y}_s)^{\square} - \tau \, \overset{\square}{\vartheta} + g^{\mathscr{E}} V \leq 0. \tag{22–24}$$

Since $\lambda = |\boldsymbol{y}_s|$, (22–11) yields $\mathbf{s} \cdot (\boldsymbol{y}_s)^{\square} = \zeta \, \bar{\mathbf{t}} \cdot (\overset{\square}{\lambda} \, \bar{\mathbf{t}} + \lambda \bar{\mathbf{t}}^{\square}) = \zeta \, \overset{\square}{\lambda}$; (22–24) may therefore be expressed equivalently as

$$\overset{\square}{\psi} - \zeta \, \overset{\square}{\lambda} - \tau \, \overset{\square}{\vartheta} + g^{\mathscr{E}} V \leq 0. \tag{22–25}$$

Remark. Within the three-dimensional theory the surface shear is defined by (21–12), whose analog here, namely,

$$\mathbf{c} \cdot \mathbf{m} + \langle \boldsymbol{F} \rangle^{\top} \mathbf{s} \cdot \mathbf{m},$$

differs from the choice $\tau = \mathbf{c} \cdot \mathbf{m}$ (cf. (22–13b)). The reason for this difference is best explained in terms of the *internal* working. In the three-dimensional theory the stress is a *tensor* $\mathbf{s}$ and its internal working is given by $\mathbf{s} \cdot \langle \boldsymbol{F} \rangle^{\square}$; in the two-dimensional theory the stress is a vector $\mathbf{s}$ with $\mathbf{s} \cdot (\boldsymbol{y}_s)^{\square}$ its working. Within the two-dimensional theory the *tensorial* stress $\mathbf{s}$ corresponding to $\mathbf{s}$ is defined by $\mathbf{S} = \mathbf{s} \otimes \mathbf{t}$, so that $\mathbf{S}\mathbf{t} = \mathbf{s}$ and $\mathbf{S}\mathbf{m} = \mathbf{0}$, and the terms $\mathbf{S} \cdot \langle \boldsymbol{F} \rangle^{\square}$ and $\mathbf{s} \cdot (\boldsymbol{y}_s)^{\square}$ may be easily related. Because $\overset{\square}{\mathbf{t}} = \overset{\square}{\vartheta} \, \mathbf{m}$,

$$(\boldsymbol{y}_s)^{\square} = (\langle \boldsymbol{F} \rangle \mathbf{t})^{\square} = \langle \boldsymbol{F} \rangle^{\square} \mathbf{t} + \langle \boldsymbol{F} \rangle \, \overset{\square}{\mathbf{t}} = \langle \boldsymbol{F} \rangle^{\square} \mathbf{t} + \langle \boldsymbol{F} \rangle (\mathbf{m} \cdot \overset{\square}{\mathbf{t}}) \mathbf{m} = \langle \boldsymbol{F} \rangle^{\square} \mathbf{t} + \overset{\square}{\vartheta} \langle \boldsymbol{F} \rangle \mathbf{m},$$

and

$$\mathbf{s} \cdot (\boldsymbol{y}_s)^{\square} = \mathbf{S}\mathbf{t} \cdot \langle \boldsymbol{F} \rangle^{\square} \mathbf{t} + \overset{\square}{\vartheta} \, \mathbf{s} \cdot \langle \boldsymbol{F} \rangle \mathbf{m}.$$

Further, for any tensor $\boldsymbol{G}$, $\boldsymbol{G} = \boldsymbol{G}\mathbf{t} \otimes \mathbf{t} + \boldsymbol{G}\mathbf{m} \otimes \mathbf{m}$; hence

$$\mathbf{S} \cdot \langle \boldsymbol{F} \rangle^{\square} = (\mathbf{S}\mathbf{t} \otimes \mathbf{t}) \cdot (\langle \boldsymbol{F} \rangle^{\square} \mathbf{t} \otimes \mathbf{t} + \langle \boldsymbol{F} \rangle^{\square} \mathbf{m} \otimes \mathbf{m}) = \mathbf{S}\mathbf{t} \cdot \langle \boldsymbol{F} \rangle^{\square} \mathbf{t}.$$

Thus, because $\tau = \mathbf{c} \cdot \mathbf{m}$,

$$\mathbf{s} \cdot (\boldsymbol{y}_s)^{\square} + \tau \, \overset{\square}{\vartheta} = \mathbf{S} \cdot \langle \boldsymbol{F} \rangle^{\square} + \left(\mathbf{c} \cdot \mathbf{m} + \langle \boldsymbol{F} \rangle^{\top} \mathbf{s} \cdot \mathbf{m} \right) \overset{\square}{\vartheta} . \tag{22–26}$$

The surface shear is defined as that field whose working accompanies temporal changes in orientation. The identity (22–26) shows this definition to be dependent on whether the stress is represented as a vector or as a tensor: the shear is τ for a vector stress and $\mathbf{c} \cdot \mathbf{m} + \langle \boldsymbol{F} \rangle^{\top} \mathbf{s} \cdot \mathbf{m}$ for a tensor stress.

d. Constitutive equations

Guided by (21–23a), I posit constitutive equations of the form

$$\psi = \hat{\psi}(\vartheta, V, \boldsymbol{y}_s, \boldsymbol{j}), \tag{22–27a}$$
$$\tau = \hat{\tau}(\vartheta, V, \boldsymbol{y}_s, \boldsymbol{j}), \tag{22–27b}$$
$$\zeta = \hat{\zeta}(\vartheta, V, \boldsymbol{y}_s, \boldsymbol{j}), \tag{22–27c}$$
$$g^{\mathscr{C}} = \hat{g}^{\mathscr{C}}(\vartheta, V, \boldsymbol{y}_s, \boldsymbol{j}), \tag{22–27d}$$

with $\boldsymbol{j} = [\boldsymbol{F}]\mathbf{m}$. It is convenient to consider the dependence on $\boldsymbol{y}_s$ as a dependence on the interfacial stretch λ and the unit tangent $\bar{\mathbf{t}}$ to the deformed interface and, with a minor abuse of notation, to write $(\ldots)$ for either $(\vartheta, V, \boldsymbol{y}_s, \boldsymbol{j})$ or $(\vartheta, V, \lambda, \bar{\mathbf{t}}, \boldsymbol{j})$.

The requirement that *all consitutive processes be consistent with the dissipation inequality* (22–25) is equivalent to the requirement that

$$\Big\{\partial_\vartheta \hat{\psi}(\ldots) - \hat{\tau}(\ldots)\Big\} \overset{\square}{\vartheta} + \Big\{\partial_\lambda \hat{\psi}(\ldots) - \hat{\zeta}(\ldots)\Big\} \overset{\square}{\lambda} + \partial_V \hat{\psi}(\ldots) \overset{\square}{V} + \Big\{\partial_{\bar{\mathbf{t}}} \hat{\psi}(\ldots)\Big\} \cdot \bar{\mathbf{t}}^{\square} + \Big\{\partial_{\boldsymbol{j}} \hat{\psi}(\ldots)\Big\} \cdot \overset{\square}{\boldsymbol{j}} + \hat{g}^{\mathscr{C}}(\ldots) V \leq 0 \tag{22–28}$$

for all migrations of the interface and all motions of the body. Arguing as in the proof of (20–20) and (20–21),[2] this leads to the following conclusions:

(i) the free energy ψ, the shear τ, and the scalar stress ζ must be independent of V, $\bar{\mathbf{t}}$, and $\boldsymbol{j}$, and must be related through

$$\psi = \hat{\psi}(\vartheta, \lambda), \tag{22–29a}$$
$$\tau = \partial_\vartheta \hat{\psi}(\vartheta, \lambda), \tag{22–29b}$$
$$\zeta = \partial_\lambda \hat{\psi}(\vartheta, \lambda); \tag{22–29c}$$

(ii) the normal internal force must obey the inequality

$$\hat{g}^{\mathscr{C}}(\vartheta, V, \boldsymbol{y}_s, \boldsymbol{j}) V \leq 0, \tag{22–30}$$

which, for $\hat{g}^{\mathscr{C}}$ smooth, results in the constitutive equation

$$g^{\mathscr{C}} = -b(\vartheta, V, \boldsymbol{y}_s, \boldsymbol{j}) V, \qquad b(\vartheta, V, \boldsymbol{y}_s, \boldsymbol{j}) \geq 0. \tag{22–31}$$

A consequence of the restriction (22–29c) is that

$$\mathbf{s} = \partial_{\boldsymbol{y}_s} \hat{\psi}(\vartheta, \lambda). \tag{22–32}$$

Indeed, since $\lambda = |\boldsymbol{y}_s|$, the derivative of λ with respect to $\boldsymbol{y}_s$ is $\boldsymbol{y}_s/|\boldsymbol{y}_s| = \bar{\mathbf{t}}$. Thus $\partial_{\boldsymbol{y}_s} \hat{\psi}(\vartheta, \lambda) = \partial_\lambda \hat{\psi}(\vartheta, \lambda) \bar{\mathbf{t}}$ and (22–32) follows from (22–11) and (22–29c).

A second consequence of the restrictions (22–29) is that, granted the standard balance (22–9a), the tangential component of the configurational balance (22–9b)

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[2] The requirement that $\bar{\mathbf{t}}^{\square}$ be orthogonal to $\bar{\mathbf{t}}$ causes no problem as $\partial_{\bar{\mathbf{t}}} \hat{\psi}(\ldots)$ is also orthogonal to $\bar{\mathbf{t}}$.

is satisfied identically without the need for tangential internal forces:

$$\mathbf{t}\cdot\boldsymbol{g}^{\mathscr{C}}=\mathbf{0}. \tag{22–33}$$

To see this, let $\boldsymbol{f}=\boldsymbol{y}_s$ and note that, by (22–12) and the Eshelby relation,

$$\begin{aligned}\mathbf{t}\cdot\mathbf{c}_s&=(\psi-\mathbf{s}\cdot\boldsymbol{f})_s-\tau K=(\partial_\vartheta\hat\psi)K+(\partial_f\hat\psi)\boldsymbol{f}_s-\mathbf{s}\cdot\boldsymbol{f}_s-\mathbf{s}_s\cdot\boldsymbol{f}-\tau K\\&=-\boldsymbol{f}\cdot\mathbf{s}_s=\boldsymbol{f}\cdot[\boldsymbol{S}]\mathbf{m}+\boldsymbol{f}\cdot\boldsymbol{b}^{\mathscr{C}}=\langle\boldsymbol{F}\rangle\mathbf{t}\cdot[\boldsymbol{S}]\mathbf{m}-\mathbf{t}\cdot\boldsymbol{e}^{\mathscr{C}}\\&=\mathbf{t}\cdot[\boldsymbol{F}^{\top}\boldsymbol{S}]\mathbf{m}-[\boldsymbol{F}]\mathbf{t}\cdot\langle\boldsymbol{S}\rangle\mathbf{m}-\mathbf{t}\cdot\boldsymbol{e}^{\mathscr{C}}=\mathbf{t}\cdot[\boldsymbol{F}^{\top}\boldsymbol{S}]\mathbf{m}-\mathbf{t}\cdot\boldsymbol{e}^{\mathscr{C}}\\&=-\mathbf{t}\cdot[\boldsymbol{C}]\mathbf{m}-\mathbf{t}\cdot\boldsymbol{e}^{\mathscr{C}}.\end{aligned} \tag{22–34}$$

Thus, granted (22–33), the tangential component of the balance (22–9b) is satisfied identically, and conversely.

e. Evolution equations for the interface

Assume that the external forces are purely inertial. The basic equations for the *bulk material*, assumed elastic, then consist of the momentum balance (12–20) supplemented by the constitutive equations (12–19). The basic equations for the *interface* are the *compatibility conditions*

$$[\dot{\boldsymbol{y}}]=-V[\boldsymbol{F}]\mathbf{m},\qquad[\boldsymbol{F}]\mathbf{t}=\mathbf{0}, \tag{22–35}$$

the *momentum balance*

$$[\boldsymbol{S}]\mathbf{m}+\mathbf{s}_s=-\rho[\dot{\boldsymbol{y}}]V, \tag{22–36}$$

and the *normal configurational balance*

$$\mathbf{m}\cdot[\Psi\mathbf{1}-\boldsymbol{F}^{\top}\boldsymbol{S}]\mathbf{m}+[k_{\rm rel}]+(\psi-\zeta\lambda)K+\tau_s=bV, \tag{22–37}$$

with $\mathbf{s}=\zeta\bar{\mathbf{t}},\lambda=|\boldsymbol{y}_s|,\bar{\mathbf{t}}=\boldsymbol{y}_s/|\boldsymbol{y}_s|,[k_{\rm rel}]=\frac{1}{2}\rho[|\dot{\boldsymbol{y}}-\overset{\square}{\boldsymbol{y}}|^2]$ and $b=b(\vartheta,V,\boldsymbol{y}_s,\boldsymbol{j})$ (cf. (12–5), (12–10)); these relations are supplemented by the constitutive equations

$$\psi=\hat\psi(\vartheta,\lambda),\qquad\tau=\partial_\vartheta\hat\psi(\vartheta,\lambda),\qquad\zeta=\partial_\lambda\hat\psi(\vartheta,\lambda). \tag{22–38}$$

The theory simplifies considerably when the free energy of the interface is independent of the stretch λ and the kinetics linear in the sense that $b=b(\vartheta)$; then

$$\psi=\hat\psi(\vartheta),\qquad\tau=\hat\psi'(\vartheta),\qquad\zeta=0, \tag{22–39}$$

and the balances reduce to

$$[\boldsymbol{S}]\mathbf{m}=-\rho[\dot{\boldsymbol{y}}]V, \tag{22–40a}$$

$$\mathbf{m}\cdot[\Psi\mathbf{1}-\boldsymbol{F}^{\top}\boldsymbol{S}]\mathbf{m}+[k_{\rm rel}]+a(\vartheta)K=b(\vartheta)V, \tag{22–40b}$$

with $a(\vartheta)=\hat\psi(\vartheta)+\hat\psi''(\vartheta)$; (22–40) are two-dimensional counterparts of (20–28) and (20–31).

Part F

Solidification

To demonstrate the role of configurational forces in situations not purely mechanical, I turn now to two-phase heat flow, neglecting deformation.

CHAPTER 23

Solidification. The Stefan Condition as a Consequence of the Configurational Force Balance

a. Single-phase theory

I begin with a summary of the basic results of Section 6c, but with deformation neglected. The basic thermodynamical laws, **balance of energy** and **growth of entropy**, have the form

$$\frac{d}{dt}\left\{\int_P \varepsilon\, dv\right\} = -\int_{\partial P} \boldsymbol{h}\cdot\boldsymbol{n}\, da + \int_P r\, dv, \tag{23–1a}$$

$$\frac{d}{dt}\left\{\int_P \eta\, dv\right\} \geq -\int_{\partial P} (\boldsymbol{h}/T)\cdot\boldsymbol{n}\, da + \int_P (r/T)dv, \tag{23–1b}$$

for *P stationary*. These are equivalent to the local equations

$$\dot{\varepsilon} = -\operatorname{Div}\boldsymbol{h} + r, \tag{23–2a}$$

$$\dot{\eta} \geq -\operatorname{Div}(\boldsymbol{h}/T) + r/T, \tag{23–2b}$$

which combine to form the free-energy inequality

$$\dot{\Psi} + \eta\dot{T} + T^{-1}\boldsymbol{h}\cdot\nabla T \leq 0, \tag{23–3}$$

with

$$\Psi = \varepsilon - T\eta \tag{23–4}$$

the **free energy**.

To the classical laws just described, I add the **configurational force balance**

$$\int_{\partial P} \boldsymbol{C}\boldsymbol{n}\, da + \int_P \boldsymbol{g}\, dv = 0, \tag{23–5}$$

which has the local form

$$\operatorname{Div}\boldsymbol{C} + \boldsymbol{g} = \boldsymbol{0}. \tag{23–6}$$

(Since inertial forces are not considered, there is no need to introduce an external force $\boldsymbol{e}$.) The configurational stress $\boldsymbol{C}$ and internal configurational force have the specific forms

$$\boldsymbol{C} = \Psi \mathbf{1}, \tag{23–7a}$$

$$\boldsymbol{g} = -\nabla\Psi, \tag{23–7b}$$

which guarantee satisfaction of (23–6).

The constitutive equations (derived in Subsection 9b2) consist of a relation between free energy and temperature, a relation giving the entropy as the negative of the derivative of the free energy with respect to temperature, and a Fourier law for heat conduction,

$$\Psi = \hat{\Psi}(T), \tag{23–8a}$$

$$\eta = -\hat{\Psi}'(T), \tag{23–8b}$$

$$\boldsymbol{h} = -\boldsymbol{K}(T)\nabla T, \tag{23–8c}$$

with **conductivity tensor** $\mathbf{K}(T)$ assumed positive-definite. The relations (23–4) and (23–8a) yield an auxiliary constitutive relation for the internal energy, viz.,

$$\varepsilon = \hat{\varepsilon}(T) = \hat{\Psi}(T) - T\hat{\Psi}'(T), \tag{23–9}$$

whose derivative is the **specific heat**

$$c(T) = \hat{\varepsilon}'(T). \tag{23–10}$$

The partial differential equation of the theory, the **heat equation**, is balance of energy supplemented by the constitutive equations (23–8c) and (23–10):

$$c(T)\dot{T} = \mathrm{Div}\big(\boldsymbol{K}(T)\nabla T\big) + r. \tag{23–11}$$

Note that, by (23–7b) and (23–8b),

$$\boldsymbol{g} = \eta\nabla T, \tag{23–12}$$

which I take as a defining relation for $\boldsymbol{g}$.

b. The classical two-phase theory revisited. The Stefan condition as a consequence of the configurational balance

I now consider phases, α and β, with $\Psi_\alpha(T)$ and $\Psi_\beta(T)$, $\boldsymbol{K}_\alpha(T)$ and $\boldsymbol{K}_\beta(T)$, $\varepsilon_\alpha(T)$ and $\varepsilon_\beta(T)$, and $c_\alpha(T)$ and $c_\beta(T)$, the corresponding free energies, conductivity tensors, internal energies, and specific heats, with resulting constitutive relations of the form (23–8)–(23–10). I also assume that there is a unique temperature, the **melting temperature** T_M, at which the free energies of the individual phases coincide:

$$\Psi_\alpha(T_M) = \Psi_\beta(T_M). \tag{23–13}$$

The classical theory neglects interfacial structure and therefore begins with basic laws in the form (23–1) and (23–5). A further assumption is that the *temperature be continuous*, so that

$$[T] = 0, \tag{23–14}$$

but all the other fields are allowed to suffer jump discontinuities across the interface.

Balance of energy then yields the interfacial balance

$$[\varepsilon]V = [\boldsymbol{h}] \cdot \mathbf{m}, \tag{23–15}$$

which is the first of the classical interface conditions for the Stefan problem. (This condition is derived as a consequence of (23–1a) and (10–8a), with $\Phi = \varepsilon$ and P stationary, by shrinking P to the interface.)

Next, the configurational balance (23–5) yields

$$[\boldsymbol{C}]\mathbf{m} = \mathbf{0}, \tag{23–16}$$

which, by (23–7a), has the alternative form

$$[\Psi] = 0, \tag{23–17}$$

or, in view of the hypothesis ending in (23–13),

$$T = T_M \qquad \text{on the interface.} \tag{23–18}$$

Thus, granted (23–13), *the classical Stefan condition equating the temperature at the interface to the melting temperature is equivalent to the configurational force balance applied across the interface*.[1]

The **Stefan problem** consists of the *bulk equations*

$$c_\alpha(T)\dot{T} = \operatorname{Div}(\boldsymbol{K}_\alpha(T)\nabla T) + r \qquad \text{in phase } \alpha, \tag{23–19a}$$

$$c_\beta(T)\dot{T} = \operatorname{Div}(\boldsymbol{K}_\beta(T)\nabla T) + r \qquad \text{in phase } \beta, \tag{23–19b}$$

and the *interface conditions*

$$[\varepsilon]V = [\boldsymbol{h}] \cdot \mathbf{m}, \tag{23–20a}$$

$$T = T_M, \tag{23–20b}$$

supplemented by suitable initial and boundary conditions. (In (23–20a), $\varepsilon = \varepsilon_\alpha(T)$ and $\boldsymbol{h} = -\boldsymbol{K}_\alpha(T)\nabla T$ in phase α, and similarly in phase β.) By (23–4), (23–13), and (23–18), the interfacial energy balance (23–20a) may be written equivalently as

$$T[\eta]V = [\boldsymbol{h}] \cdot \mathbf{m}. \tag{23–21}$$

For solidification, in which one of the phases is solid and the other liquid, one generally adds the restrictions

$$T \geq T_M \quad \text{in the liquid,} \qquad T \leq T_M \qquad \text{in the solid,}$$

[1] Cf. Gurtin [1988].

but these conditions are not a consequence of the hypotheses which the theory is based. There are important physical situations with liquid at temperatures below T_M (supercooling) and situations in which the solid is at temperatures above T_M (superheating).

Not only does the configurational balance allow for a derivation of the classical Stefan condition, it allows for a weak formulation of the Stefan problem by replacing the condition $T = T_M$ on the interface (which, being local, is inappropriate to a weak formulation) with a partial differential equation. In particular, (23–2) and the configurational balance (23–6) with $\boldsymbol{C}$ given by (23–7a) and $\boldsymbol{g}$ by (23–12) yield, for $r = 0$,

$$\dot{\varepsilon} = -\operatorname{Div} \boldsymbol{h}, \tag{23–22a}$$

$$\nabla \Psi = -\eta \nabla T, \tag{23–22b}$$

$$\dot{\eta} \leq -\operatorname{Div}(\boldsymbol{h}/T), \tag{23–22c}$$

to be interpreted in a weak sense, for example, in the sense of distributions. The distribution form of (23–22a) gives that partial differential equation classically in bulk and the balance (23–15) at the interface. The configurational balance (23–22b) is satisfied automatically in bulk; its only contribution is at the interface, where $\nabla \Psi$ is a distribution, because Ψ suffers a jump discontinuity, while $\eta \nabla T$ does not contribute, because η and ∇T are bounded. In fact, (23–22b) formally yields $[\![\Psi]\!] = 0$ and hence the Stefan condition (23–18). Finally, (23–22c) is satisfied automatically in bulk and across the interface. To verify this latter assertion, note that (23–22c) yields $T[\![\eta]\!]V \leq [\![\boldsymbol{h}]\!] \cdot \mathbf{m}$, or equivalently, by (23–4) and (23–15), $[\![\Psi]\!]V \geq 0$, an inequality satisfied by virtue of (23–17). It might therefore appear that the entropy inequality (23–22c) is superfluous, which is true when the interface moves smoothly, because the constitutive equations are compatible with the second law, but there are situations involving large amounts of supercooling or superheating in which the interface moves "infinitely fast" resulting in an instantaneous change in phase for entire subregions of the body:[2] the entropy inequality is then needed to ensure that such instantaneous changes be consistent with the second law.[3]

[2] Sherman [1970], Fasano and Primicerio [1977], Götz and Zaltzman [1993], Gurtin [1994].

[3] Gurtin [1994].

CHAPTER 24

Solidification with Interfacial Energy and Entropy[1]

I now generalize the classical Stefan theory to include surface structure, retaining the requirement that *the temperature be continuous across the interface*. The resulting theory is complicated, and that is why I also develop, from the general theory, well-known approximate theories that have been successful in applications.

a. General theory

I consider the basic laws for which migrating control volume $P = P(t)$ in the form (cf. (6–13))

$$\frac{d}{dt}\left\{\int_P \varepsilon\, dv + \int_{\mathscr{G}} \bar{\varepsilon}\, da\right\} = -\int_{\partial P} \boldsymbol{h}\cdot\boldsymbol{n}\, da + \int_P r\, dv + \int_{\mathscr{G}} r^{\mathscr{S}}\, da + \int_{\partial P} QU\, da + \int_{\partial\mathscr{G}} \bar{Q} V_{\partial\mathscr{G}}\, ds + W(P), \tag{24–1a}$$

$$\frac{d}{dt}\left\{\int_P \eta\, dv + \int_{\mathscr{G}} \bar{\eta}\, da\right\} \geq -\int_{\partial P} (\boldsymbol{h}/T)\cdot\boldsymbol{n}\, da + \int_P (r/T) dv + \int_{\mathscr{G}} (r^{\mathscr{S}}/T) da + \int_{\partial P} (Q/T)U\, da + \int_{\partial\mathscr{G}} (\bar{Q}/T) V_{\partial\mathscr{G}}\, ds, \tag{24–1b}$$

$$\int_{\partial P} \boldsymbol{C}\boldsymbol{n}\, da + \int_P \boldsymbol{g}\, dv + \int_{\partial\mathscr{G}} \mathbf{c}\mathbf{n}\, ds + \int_{\mathscr{G}} (\boldsymbol{g}^{\mathscr{S}} + \boldsymbol{e}^{\mathscr{S}}) da = \mathbf{0}, \tag{24–1c}$$

[1]This chapter, which represents major conceptual improvements of Gurtin [1988], is based on ideas presented in Gurtin [1995].

with $\mathscr{G} = \mathscr{G}(t)$ the portion of the interface in P and $\mathbf{n}$ the outward unit normal to $\partial\mathscr{G}$. Here $W(P)$ is given by (16–2) and, as such, *yields all the results of* Chapter 16, including the configurational force balance (24–1c), the decomposition

$$\mathbf{C} = \sigma\mathbf{p} + \mathbf{m} \otimes \boldsymbol{\tau}$$

(cf. (16–8)), and the normal force balance

$$\sigma K + \mathrm{Div}_{\mathscr{S}}\boldsymbol{\tau} + [\pi] + g^{\mathscr{S}} + e^{\mathscr{S}} = 0 \tag{24–2}$$

(cf. (16–12a)). Further, Q is as discussed in Section 6c; $\bar{\varepsilon}$ is the interfacial energy; $\bar{\eta}$ is the interfacial entropy; $r^{\mathscr{S}}$ is the heat supplied directly to the interface; $\bar{Q}$, a configurational heating, is a superficial analog of Q in the sense that

$$\int_{\partial\mathscr{G}} \bar{Q} V_{\partial\mathscr{G}}\, ds, \qquad \int_{\partial\mathscr{G}} (\bar{Q}/T) V_{\partial\mathscr{G}}\, ds$$

represent flows of heat and entropy into $\mathscr{G}$ induced by the motion of the boundary curve $\partial\mathscr{G}$.[2]

The power balance (16–17), the transport theorem (15–31) with $\varphi = \varepsilon$ and with $\varphi = \eta$, and the argument leading to (17–5) here yield the interface relations

$$\bar{\eta} = \bar{Q}/T, \qquad \sigma = \psi,$$

with interfacial free energy ψ given by

$$\psi = \bar{\varepsilon} - T\bar{\eta}. \tag{24–3}$$

The interfacial forms of balance of energy and growth of entropy are more complicated than before. The results $\eta = Q/T$ and $\Psi = \pi$ remain valid (cf. Section 6c), and, since $\bar{\eta} = \bar{Q}/T$ and $\sigma = \psi$, a pillbox argument applied to (24–1a,b) using (11–25), (15–31), (16–17), and the continuity of T results in the conditions

$$T[\eta]V = [\boldsymbol{h}] \cdot \mathbf{m} + \overset{\square}{\bar{\varepsilon}} - T\bar{\eta}KV + \boldsymbol{\tau} \cdot \overset{\square}{\mathbf{m}} + g^{\mathscr{S}}V - r^{\mathscr{S}}, \tag{24–4a}$$

$$[\eta]V \leq T^{-1}[\boldsymbol{h}] \cdot \mathbf{m} + \overset{\square}{\bar{\eta}} - \bar{\eta}KV - T^{-1}r^{\mathscr{S}}. \tag{24–4b}$$

These yield the **interfacial dissipation inequality**

$$\overset{\square}{\psi} + \bar{\eta}\,\overset{\square}{T} + \boldsymbol{\tau} \cdot \overset{\square}{\mathbf{m}} + g^{\mathscr{S}}V \leq 0, \tag{24–5}$$

with $(\ldots)^{\square}$ the normal time derivative following $\mathscr{S}$.

Guided by (24–5), I consider constitutive equations of the form (18–3), but with $(T, \mathbf{m}, V)$ as independent variables and an additional constitutive equation, of the

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[2]The paragraph containing (2–3) contains the phrase: "It is convenient to denote by an overbar a quantity that has been transported, via the motion, to the deformed configuration." Here, instead, an overbar is used to designate certain interfacial fields; e.g., ε denotes the internal energy, per unit volume, in bulk, while $\bar{\varepsilon}$ denotes the internal energy of the interface, per unit area. There should be no danger of confusion because motion of the body is not considered.

same type, for the interfacial entropy $\bar{\eta}$. The most general constitutive equations of this form consistent with the dissipation inequality (24–5) are

$$\psi = \hat{\psi}(T, \mathbf{m}), \tag{24–6a}$$

$$\bar{\eta} = -\partial_T \hat{\psi}(T, \mathbf{m}), \tag{24–6b}$$

$$\boldsymbol{\tau} = -\partial_{\mathbf{m}} \hat{\psi}(T, \mathbf{m}), \tag{24–6c}$$

$$g^{\mathscr{S}} = -b(T, \mathbf{m}, V)V, \tag{24–6d}$$

with $b(T, \mathbf{m}, V) \geq 0$.

I omit the proof of (24–6), which is similar to that of (18–5)–(18–7), and I leave it to the reader to verify that there are sufficient external and indeterminate forces and supplies to satisfy all balances.

I henceforth restrict attention to linear kinetics, so that $b = b(T, \mathbf{m})$, and to situations in which the external fields vanish:

$$r = e^{\mathscr{S}} = r^{\mathscr{S}} = 0.$$

The interface conditions consist of the energy balance (24–4a) and the configurational balance (16–12a) with $\pi = \Psi$, $\sigma = \psi$:

$$[\Psi] = -\psi K - \mathrm{Div}_{\mathscr{S}} \boldsymbol{\tau} - g^{\mathscr{S}}. \tag{24–7}$$

These interface conditions supplemented by the constitutive equations are the basic free-boundary conditions of the theory; the condition (24–7) replaces the classical Stefan condition. For an isotropic body, the interfacial free energy and the kinetic modulus are independent of $\mathbf{m}$, so that, in particular, $\boldsymbol{\tau} \equiv \mathbf{0}$; in this case (24–6a) and (24–7) yield

$$\Psi_\alpha(T) - \Psi_\beta(T) = \hat{\psi}(T)K - b(T)V, \tag{24–8}$$

where here and in what follows, $\mathbf{m}$ is assumed to point outward from the phase α region.

As before, I assume that there is a unique temperature T_M that satisfies $\Psi_\alpha(T_M) = \Psi_\beta(T_M)$; even so, it is clear from (24–8) that one should not expect $T = T_M$ at the interface. Indeed, granted isotropy, generally $T \neq T_M$ whenever the interface is curved and/or moving. Thus curvature and motion of the interface generally induce supercooling or superheating.

Consequences of the constitutive restrictions (24–6a) are the Gibbs relations

$$\overset{\square}{\psi} = -\bar{\eta}\,\overset{\square}{T} - \boldsymbol{\tau} \cdot \overset{\square}{\mathbf{m}}, \qquad \overset{\square}{\bar{\varepsilon}} = T\,\overset{\square}{\bar{\eta}} - \boldsymbol{\tau} \cdot \overset{\square}{\mathbf{m}}.$$

Further, the energy balance (24–4a) is equivalent to the relation

$$\begin{aligned} T[\eta]V &= [\boldsymbol{h}] \cdot \mathbf{m} + \lambda, \\ \lambda &= -T \left\{ \partial_T \partial_T \hat{\psi}(T, \mathbf{m})\, \overset{\square}{T} + \partial_{\mathbf{m}} \partial_T \hat{\psi}(T, \mathbf{m})\, \overset{\square}{\mathbf{m}} - \partial_T \hat{\psi}(T, \mathbf{m}) K V \right\} \\ &\quad - b(T, \mathbf{m}, V)V^2. \end{aligned}$$

The classical form of this balance is $T[\eta]V = [\boldsymbol{h}] \cdot \mathbf{m}$ (cf. (23–21)); the interfacial term λ is generally neglected. The next section will address this issue.

b. Approximate Theory. The Gibbs-Thomson condition as a consequence of the configurational balance

The interface conditions (24–4a) and (24–7) are complicated. I now *formally* derive an approximate theory for an interface whose free energy and kinetic modulus have constitutive response functions which with their derivatives are small, say $O(\delta)$ with δ small. This renders the corresponding response functions for the interfacial entropy and shear also $O(\delta)$. In contrast, I assume that all bulk fields are constitutively $O(1)$.

Let

$$u = (T - T_M)/T_M, \tag{24–9}$$

and define the **latent heat** ℓ, assumed nonzero, by

$$\ell = \varepsilon_\beta(T_M) - \varepsilon_\alpha(T_M), \tag{24–10}$$

or, in view of (23–13), by

$$\ell = T_M\big(\eta_\beta(T_M) - \eta_\alpha(T_M)\big). \tag{24–11}$$

Here and in what follows, $\varepsilon_\alpha(T)$ and $\varepsilon_\beta(T)$ and $\eta_\alpha(T)$ and $\eta_\beta(T)$ are the bulk constitutive functions for the internal energy and entropy computed from the free energies $\Psi_\alpha(T)$ and $\Psi_\beta(T)$ using (23–8b) and (23–9). As before, phases are labeled so that $\mathbf{m}$ is outward from the phase α region, so that jumps of bulk fields across the interface are "β minus α."

Let

$$F(T) = \Psi_\beta(T) - \Psi_\alpha(T), \quad E(T) = \varepsilon_\beta(T) - \varepsilon_\alpha(T), \quad N(T) = \eta_\beta(T) - \eta_\alpha(T)$$

so that, by (23–4) and (23–13),

$$N(T) = -F'(T), \quad F(T) = E(T) - TN(T), \quad F(T_M) = 0, \quad \ell = T_M N(T_M).$$

Then, expanding $F(T)$ about $T = T_M$ yields $F(T) = F'(T_M)(T - T_M) + O(u^2) = -\ell u + O(u^2)$; similarly, $E(T) = T_M N(T_M) + O(u) = \ell + O(u)$. Thus

$$\Psi_\beta(T) - \Psi_\alpha(T) = -\ell u + O(u^2), \tag{24–12a}$$

$$\varepsilon_\beta(T) - \varepsilon_\alpha(T) = \ell + O(u), \tag{24–12b}$$

$$T\big(\eta_\beta(T) - \eta_\alpha(T)\big) = \ell + O(u). \tag{24–12c}$$

Assume that V and the derivatives of T and $\mathbf{m}$ are bounded; then, differentiating the constitutive relation $\boldsymbol{\tau} = \hat{\boldsymbol{\tau}}(T, \mathbf{m})$,

$$\mathrm{Div}_{\mathscr{S}}\boldsymbol{\tau} = \partial_T\hat{\boldsymbol{\tau}}(T, \mathbf{m}) \cdot \nabla_{\mathscr{S}}T + \partial_{\mathbf{m}}\hat{\boldsymbol{\tau}}(T, \mathbf{m}) \cdot \nabla_{\mathscr{S}}\mathbf{m} = O(\delta); \tag{24–13}$$

thus, by (24–7) and (24–12a),

$$u = O(\delta).$$

For any constitutive response function $\hat{\varphi}(T, \mathbf{m})$ for the interface, define

$$\varphi_M(\mathbf{m}) = \hat{\varphi}(T_M, \mathbf{m}). \tag{24–14}$$

Then $\hat{\varphi}(T, \mathfrak{m}) = \varphi_M(\mathfrak{m}) + O(u^2)$ and, granted the additional assumption $\nabla_{\mathscr{S}} u = O(\delta)$, (24–13) yields the estimate

$$\mathrm{Div}_{\mathscr{S}} \boldsymbol{\tau} = \partial_{\mathfrak{m}} \boldsymbol{\tau}_M(\mathfrak{m}) \cdot \nabla_{\mathscr{S}} \mathfrak{m} + O(\delta^2) = \mathrm{Div}_{\mathscr{S}} \boldsymbol{\tau}_M(\mathfrak{m}) + O(\delta^2).$$

Thus, by (24–12a,c), dropping terms of $O(\delta)$ in (24–4a) and terms of $O(\delta^2)$ in (24–7) yields the approximate interface conditions:

$$\ell u = \psi_M(\mathfrak{m}) K + \mathrm{Div}_{\mathscr{S}} \boldsymbol{\tau}_M(\mathfrak{m}) - b_M(\mathfrak{m}) V, \tag{24–15a}$$

$$\ell V = [\![\boldsymbol{h}]\!] \cdot \mathfrak{m}, \tag{24–15b}$$

with

$$\boldsymbol{\tau}_M(\mathfrak{m}) = -\partial_{\mathfrak{m}} \psi_M(\mathfrak{m}). \tag{24–16}$$

Note that, by (15–12a), (15–17) and (24–15a) can be written alternatively as

$$\ell u = \{\psi_M(\mathfrak{m}) \mathbf{1} + \partial_{\mathfrak{m}} \partial_{\mathfrak{m}} \psi_M(\mathfrak{m})\} \cdot \mathbf{L} - b_M(\mathfrak{m}) V. \tag{24–17}$$

Within this framework the **Stefan problem** consists of the bulk equations (23–19) in conjunction with two interface conditions: the energy balance (24–15b), which is classical, and the **generalized Stefan condition** (24–17), which includes effects of curvature and kinetics. For an isotropic body, ψ_M and b_M are constants, which I write as ψ and b, and (24–17) reduces to[3]

$$\ell u = \psi K - bV, \tag{24–18}$$

which is the **Gibbs-Thomson condition**

$$\ell u = \psi K \tag{24–19}$$

augmented by the term bV, which accounts for interface kinetics.

c. Free-boundary problems for the approximate theory. Growth theorems

c1. The quasilinear and quasistatic problems

I now consider free-boundary problems based on the approximate interface conditions (24–15b) and (24–17) in conjunction with the bulk equations (23–19) with $r = 0$, linearized about the melting temperature T_M:

$$c_\alpha \dot{u} = \mathrm{Div}(\boldsymbol{K}_\alpha \nabla u) \qquad \text{in phase } \alpha, \tag{24–20a}$$

$$c_\beta \dot{u} = \mathrm{Div}(\boldsymbol{K}_\beta \nabla u) \qquad \text{in phase } \beta, \tag{24–20b}$$

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[3] $\ell u = -b(\mathfrak{m}) V$ was introduced by Frank [1958] and was used by Chernov [1963, 1964]; $\ell u = \psi K$ was introduced by Mullins [1960] (in the context of mass transport) and was used by Mullins and Sekerka [1963, 1964]; $\ell u = \psi K - bV$ was used by Voronkov [1964]. Cf. also Seidensticker [1966], Tarshis and Tiller [1966], and the review articles by Sekerka [1968, 1973, 1984], Chernov [1971, 1974], Delves [1974], and Langer [1980].

with $c_\alpha = c_\alpha(T_M)$ and $\boldsymbol{K}_\alpha = \boldsymbol{K}_\alpha(T_M)$, and similarly for phase β. Assume without loss in generality that at the melting temperature phase β has higher internal energy than α:

$$\ell > 0. \tag{24–21}$$

Further, to avoid an unnecessary constant, rescale by defining $\tilde{c}_\alpha$, $\tilde{c}_\beta$, $\tilde{\boldsymbol{K}}_\alpha$, $\tilde{\boldsymbol{K}}_\beta$, $\tilde{\boldsymbol{h}}$, $\tilde{\psi}$, and $\tilde{b}$ via division by ℓ of c_α, c_β, $\boldsymbol{K}_\alpha$, $\boldsymbol{K}_\beta$, $\boldsymbol{h}$, ψ, and b. Dropping the "wave," this yields the **quasilinear system** consisting of the *bulk equations*

$$c_\alpha \dot{u} = \mathrm{Div}(\boldsymbol{K}_\alpha \nabla u) \qquad \text{in phase } \alpha, \tag{24–22a}$$

$$c_\beta \dot{u} = \mathrm{Div}(\boldsymbol{K}_\beta \nabla u) \qquad \text{in phase } \beta, \tag{24–22b}$$

and the *interface conditions*

$$V = [\boldsymbol{h}] \cdot \mathbf{m}, \tag{24–23a}$$

$$u = \mathbf{B}(\mathbf{m}) \cdot \mathbf{L} - b(\mathbf{m})V, \tag{24–23b}$$

with

$$\mathbf{B}(\mathbf{m}) = \psi(\mathbf{m})\mathbf{1} + \partial_{\mathbf{m}}\partial_{\mathbf{m}}\psi(\mathbf{m}),$$

with $\boldsymbol{h} = -\boldsymbol{K}_\alpha \nabla u$ in phase α and $\boldsymbol{h} = -\boldsymbol{K}_\beta \nabla u$ in phase β, and with

$$b(\mathbf{m}) \geq 0, \qquad \boldsymbol{K}_\alpha \text{ and } \boldsymbol{K}_\beta \text{ positive semidefinite.} \tag{24–24}$$

The quasilinear problem for a body B consists of (24–22) and (24–23) supplemented by initial conditions prescribing $u(\boldsymbol{X}, 0)$ and $\mathscr{S}(0)$ for all $\boldsymbol{X} \in B$ and by boundary conditions giving u on a portion of ∂B and $\boldsymbol{h} \cdot \boldsymbol{n}$ on the remainder of ∂B for all time (≥ 0).

Generally, one expects the interface to move slowly in comparison to the time scale for heat conduction. With this in mind, consider the **quasistatic system** that neglects the terms $c_\alpha \dot{u}$ and $c_\beta \dot{u}$ in the bulk equations,

$$\mathrm{Div}(\boldsymbol{K}_\alpha \nabla u) = 0 \qquad \text{in phase } \alpha, \tag{24–25a}$$

$$\mathrm{Div}(\boldsymbol{K}_\beta \nabla u) = 0 \qquad \text{in phase } \beta, \tag{24–25b}$$

but retains the interface conditions (24–23). The quasistatic problem consists of this system supplemented by the boundary conditions of the quasilinear system in conjunction with the initial specification of the interface.

c2. Growth theorems

I now establish Lyapunov functions for solutions of the quasilinear and quasistatic systems. Consider a bounded body B, write $B_\alpha(t)$ and $B_\beta(t)$ for the complementary subregions of B occupied by phases α and β, assume that the interface $\mathscr{S}(t)$ is a *closed surface* that never intersects ∂B, and restrict attention to the following two types of boundary conditions:

(i) **isolated boundary**

$$\boldsymbol{h} \cdot \boldsymbol{n} = \mathbf{0} \qquad \text{on } \partial B \text{ for all time;} \tag{24–26}$$

(ii) **thermally uniform boundary**

$$u = U \qquad \text{on } \partial B \text{ for all time.} \tag{24–27}$$

In (ii), $U = U(t)$, a function of time alone, is the prescribed **boundary temperature**.

By (24–14),

$$\mathscr{T}(\mathscr{S}) = \int_{\mathscr{S}} \psi(\mathfrak{m}) da \tag{24–28}$$

is the **total interfacial free energy** at the melting temperature, while

$$\mathscr{D}(u) = \int_{B_\alpha} \nabla u \cdot \boldsymbol{K}_\alpha \nabla u \, dv + \int_{B_\beta} \nabla u \cdot \boldsymbol{K}_\beta \nabla u \, dv + \int_{\mathscr{S}} b(\mathfrak{m}) V^2 da \tag{24–29}$$

is, to within the approximations inherent in the quasilinear system, proportional to the **total production of entropy**.

Growth Theorem.[4] Let u be a solution of the quasilinear system with $c_\alpha = c_\beta = c$.

(i) If the boundary is isolated,

$$\frac{d}{dt}\left\{ \operatorname{vol}(B_\alpha) - c \int_B u \, dv \right\} = 0, \tag{24–30a}$$

$$\frac{d}{dt}\left\{ \mathscr{T}(\mathscr{S}) + \frac{1}{2} c \int_B u^2 dv \right\} = -\mathscr{D}(u) \le 0. \tag{24–30b}$$

(ii) If the boundary is thermally uniform,

$$\frac{d}{dt}\left\{ \mathscr{T}(\mathscr{S}) + \frac{1}{2} c \int_B (u - U)^2 dv \right\} + U \frac{d}{dt} \operatorname{vol}(B_\alpha) = -\mathscr{D}(u) \le 0. \tag{24–31}$$

Let u be a solution of the quasistatic system.

(iii) If the boundary is isolated,

$$\frac{d}{dt} \operatorname{vol}(B_\alpha) = 0, \tag{24–32a}$$

$$\frac{d}{dt} \mathscr{T}(\mathscr{S}) = -\mathscr{D}(u) \le 0. \tag{24–32b}$$

(iv) If the boundary is thermally uniform,

$$\frac{d}{dt} \mathscr{T}(\mathscr{S}) + U \frac{d}{dt} \operatorname{vol}(B_\alpha) = -\mathscr{D}(u) \le 0. \tag{24–33}$$

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[4] Gurtin [1988, p. 211]; vol(D) denotes the volume of a region D.

PROOF. The proof is based on three identities. The first,

$$\int_{B_\alpha} \operatorname{Div} \boldsymbol{h}\, dv + \int_{B_\beta} \operatorname{Div} \boldsymbol{h}\, dv = \int_{\partial B} \boldsymbol{h} \cdot \boldsymbol{n}\, da + \int_{\mathscr{S}} [\boldsymbol{h}] \cdot \mathfrak{m}\, da, \tag{24–34}$$

is a consequence of the divergence theorem and the fact that the heat flux $\boldsymbol{h}$, here defined in the sentence containing (24–24), is smooth away from the interface and up to the interface from either side. The other identities are:

$$\int_{\mathscr{S}} V\, da = \frac{d}{dt} \operatorname{vol}(B_\alpha), \tag{24–35a}$$

$$\int_{\mathscr{S}} uV\, da = -\frac{d}{dt} \mathscr{F}(\mathscr{S}) - \int_{\mathscr{S}} b(\mathfrak{m}) V^2\, da. \tag{24–35b}$$

The result (24–35a) follows from (6–6) with $\Psi \equiv 1$ and $P = B_\alpha$, because $\mathscr{S}$ is the only portion of ∂B_α that is migrating. The verification of (24–35b) is based on the interface condition

$$u = \psi(\mathfrak{m})K + \partial_{\mathfrak{m}} \partial_{\mathfrak{m}} \psi(\mathfrak{m}) \cdot \mathbf{L} - b(\mathfrak{m})V. \tag{24–36}$$

By (15–12a), (15–24), and (18–2c),

$$\begin{aligned} \psi(\mathfrak{m})^{\square} &= \partial_{\mathfrak{m}} \psi(\mathfrak{m}) \cdot \overset{\square}{\mathfrak{m}} = -\partial_{\mathfrak{m}} \psi(\mathfrak{m}) \cdot \nabla_{\mathscr{S}} V \\ &= -\operatorname{Div}_{\mathscr{S}} \{V \partial_{\mathfrak{m}} \psi(\mathfrak{m})\} - V \partial_{\mathfrak{m}} \partial_{\mathfrak{m}} \psi(\mathfrak{m}) \cdot \mathbf{L}; \end{aligned}$$

thus, since $\mathscr{S}$ is a closed surface, (15–31) yields

$$\begin{aligned} \frac{d}{dt} \mathscr{F}(\mathscr{S}) &= \int_{\mathscr{S}} \left(\psi(\mathfrak{m})^{\square} - \psi(\mathfrak{m}) K V \right) da \\ &= -\int_{\mathscr{S}} \{ \partial_{\mathfrak{m}} \partial_{\mathfrak{m}} \psi(\mathfrak{m}) \cdot \mathbf{L} + \psi(\mathfrak{m}) K \}\, V\, da, \end{aligned}$$

and (24–35b) follows from (24–36) .

Let u be a solution of the quasilinear system with $c_\alpha = c_\beta = c$. Because u is continuous across the interface,

$$\frac{d}{dt} \int_B \{u^p\, dv\} = \int_B (u^p)^{\cdot}\, dv \tag{24–37}$$

for $p = 1, 2$. Assume that the boundary is isolated in the sense (24–26). Then (24–30a) follows from (24–34) in conjunction with (24–22a), (24–23a), (24–35a), and (24–37); while (24–30b) follows from (24–34) with $\boldsymbol{h}$ replaced by $u\boldsymbol{h}$, using the continuity of u in conjunction with (24–22a), (24–23a), (24–29), (24–35b), and (24–37). Assume, on the other hand, that the boundary is thermally uniform. Then, because $U(t)$ is independent of position,

$$\int_{\partial B} u\boldsymbol{h} \cdot \boldsymbol{n}\, da = U \int_{\partial B} \boldsymbol{h} \cdot \boldsymbol{n}\, da,$$

and the proof of (24–31) follows using (24–34), both as is and with $\boldsymbol{h}$ replaced by $u\boldsymbol{h}$. The proof is left to the reader.

Finally, (24–32) and (24–33) follow from (24–30a) and (24–31) with $c = 0$. □

For a solid-liquid system, our agreement that phase β have higher internal energy at T_M renders α the solid phase. If ∂B is supercooled, then $U\,\mathrm{vol}(B_\alpha) < 0$; (24–33) would then indicate a tendency of the solid phase to grow, at least when bulk effects dominate.

For an isotropic body $\boldsymbol{h} = -k_\alpha \nabla u$ in phase α and $\boldsymbol{h} = -k_\beta \nabla u$ in phase β with k_α and k_β scalar constants, while ψ and b are constant; in this case the quasistatic system reduces to the **Mullins-Sekerka system**[5] for which

$$\Delta u = 0 \tag{24–38}$$

in bulk and

$$u = \psi K - bV, \tag{24–39a}$$

$$V = [\boldsymbol{h}] \cdot \mathfrak{m} \tag{24–39b}$$

on the interface, with Δ the Laplacian: $\Delta u = \mathrm{Div}\,\nabla u$. In this case (24–32) become

$$\frac{d}{dt}\mathrm{vol}(B_\alpha) = 0, \tag{24–40a}$$

$$\psi \frac{d}{dt}\mathrm{area}(\mathscr{S}) = -\mathscr{D}(u) \leq 0, \tag{24–40b}$$

while (24–33) take the form

$$\psi \frac{d}{dt}\,\mathrm{area}(\mathscr{S}) + U\,\frac{d}{dt}\,\mathrm{vol}(B_\alpha) = -\mathscr{D}(u) \leq 0. \tag{24–41}$$

An analogous simplification holds for the quasilinear problem.

[5]Mullins and Sekerka [1963, 1964], although they take $b = 0$.

Part G

Fracture

The goal of this part[1] is a framework for fracture that uses the notion of configurational forces. Away from the crack the theory is as discussed previously;[2] here the emphasis is on deriving results for the crack and, especially, for the tip.

The following notation is used throughout:

$\boldsymbol{v}$	**vector velocity of the tip**,
V	**scalar velocity of the tip**,
$\Phi^{\square}$	**time derivative of a bulk field Φ following the tip**.

In previous chapters, $\boldsymbol{v}$ and V designated vector and scalar velocity fields for an evolving interface, while $\Phi^{\square}$ signified the time derivative following the interface. This should not be a source of confusion, as the crack faces, although endowed with structure similar to that of an interface, are immobile.

To avoid geometric complexities, the discussion is restricted to *two space dimensions*. For convenience, I adopt the following conventions:

1. Even though the crack is a curve, I refer to its faces as *crack surfaces*.
2. The crack faces are treated in unison, so that a term such as *surface tension* refers to the sum of the surface tensions of the individual faces.
3. The crack is considered internal to all control volumes,[3] and hence such control volumes account only *implicitly* for forces and working that result when the crack faces are in contact.

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[1] This chapter is taken from Gurtin and Podio-Guidugli [1996, 1997].

[2] E.g., the Eshelby relation (6–9), the standard and configurational balances (3–6) and (5–10), and the dissipation inequality (6–11).

[3] I do not allow for *external* tractions applied to the crack faces. Because the theory applies to an arbitrarily small neighborhood of the tip, this involves no essential loss in generality.

CHAPTER 25

Cracked Bodies

The discussion begins with *smooth* cracks. In future sections the results will be applied to crack kinking, an application that involves no inconsistency, as the evolution of the tip is governed by *local* physical laws that apply away from—although arbitrarily close to—points at which the crack kinks.

a. Smooth cracks. Control volumes

For each t in some open time interval, let $\mathscr{C}(t)$ be a smooth, connected, oriented curve in B with one end, $\boldsymbol{Z}_0$, fixed at the boundary ∂B, with the remainder of $\mathscr{C}(t)$—including the other endpoint $\boldsymbol{Z}(t)$—contained in the interior of B, and with $\mathscr{C}(t_1) \subset \mathscr{C}(t_2)$ for all $t_2 \geq t_1$. B is viewed as a referential neighborhood of a **growing crack** $\mathscr{C}(t)$ with $\boldsymbol{Z}(t)$ the **crack tip**. The phrase **in bulk** will be used to signify *away from the crack*. Note that if at some time a subcurve $\mathscr{G}$ of the crack does not contain the tip, then $\mathscr{G}$ is stationary at all subsequent times.

Arc length s is measured from $\boldsymbol{Z}_0$ with $s(\boldsymbol{X})$ the arc length to a point $\boldsymbol{X} \in \mathscr{C}(t)$. Let $\mathbf{t}(\boldsymbol{X})$ denote the unit tangent to $\mathscr{C}(t)$ in the direction of increasing s. Because $\mathbf{t}(\boldsymbol{Z}(t))$ represents the direction of (possible) propagation, the **tip velocity**

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{Z}(t)}{dt} \tag{25–1}$$

may be written in the form

$$\boldsymbol{v}(t) = V(t)\mathbf{t}(\boldsymbol{Z}(t)), \qquad V(t) \geq 0, \tag{25–2}$$

with V the **tip speed**.

Throughout, $\mathbf{m}(\boldsymbol{X})$ denotes a continuous unit normal field for $\mathscr{C}(t)$. Note that, because $\mathscr{C}(t)$ is smooth up to $\boldsymbol{Z}(t)$,

$$\mathbf{m}(\boldsymbol{X}) \cdot \boldsymbol{\nu}(t) \to 0 \qquad \text{as } \boldsymbol{X} \to \boldsymbol{Z}(t). \tag{25–3}$$

The functions $s(\boldsymbol{X})$, $\mathbf{t}(\boldsymbol{X})$, and $\mathbf{m}(\boldsymbol{X})$ actually depend on t, because their common domain $\mathscr{C}(t)$ depends on t; more precisely, one should write $\mathbf{t}(\boldsymbol{X}, t)$, say, with the understanding that $\mathbf{t}(\boldsymbol{X}, t_1) = \mathbf{t}(\boldsymbol{X}, t_2)$ for all $\boldsymbol{X} \in \mathscr{C}(t_1)$ whenever $t_2 \geq t_1$.

A **migrating control volume** $P(t)$ is here (restricted to be) a closed subregion of B for which $\mathscr{C}(t)$ does not intersect $\partial P(t)$ at more than two points and for which $\boldsymbol{Z}(t) \notin \partial P(t)$. Then $P(t)$ must be one of the following:

(i) a **bulk control volume** (a control volume that does not intersect on the crack),
(ii) a **crack control volume** (a control volume that contains a portion of the crack, but not the tip), or
(iii) a **tip control volume** (a control volume that contains the tip in its interior).

For bulk control volumes the basic laws as discussed in Section A; I therefore here *restrict attention to crack and tip control volumes*.

As before, for $P = P(t)$ a control volume, $\boldsymbol{n}$ designates the outward unit normal to ∂P, while U is the (scalar) normal velocity of the *curve* ∂P in the direction $\boldsymbol{n}$.

The following notation is convenient. Let $\varphi(\boldsymbol{X}, t)$ be defined on the crack. For $P(t)$ a tip control volume, $\mathscr{C}(t)$ intersects $\partial P(t)$ at a single point, $\boldsymbol{X}_A(t)$, and

$$\varphi_A(t) = \varphi(\boldsymbol{X}_A(t), t), \qquad \varphi_{\text{tip}}(t) = \varphi(\boldsymbol{Z}(t), t).$$

For $P(t)$ a crack control volume, $\mathscr{C}(t)$ intersects $\partial P(t)$ at two points, $\boldsymbol{X}_A(t)$ and $\boldsymbol{X}_B(t)$, and

$$\varphi_A(t) = \varphi(\boldsymbol{X}_A(t), t), \qquad \varphi_B(t) = \varphi(\boldsymbol{X}_B(t), t);$$

here, for definiteness,

$$s(\boldsymbol{X}_B(t)) > s(\boldsymbol{X}_A(t)). \tag{25–4}$$

Further, $u_A(t)$, for a crack or tip control volume, and $u_B(t)$, for a crack control volume, are the (scalar) velocities defined by

$$\dot{\boldsymbol{X}}_A(t) = u_A(t)\mathbf{t}_A(t), \qquad \dot{\boldsymbol{X}}_B(t) = u_B(t)\mathbf{t}_B(t). \tag{25–5}$$

An important example of a tip control volume is a **tip disc** $D_\delta(t)$, which is a disc of radius δ centered at the tip $\boldsymbol{Z}(t)$; here, for $\boldsymbol{n}(\boldsymbol{X}, t)$ the outward unit normal to $\partial D_\delta(t)$ and δ sufficiently small that $\mathbf{t}_A \cdot \boldsymbol{n}_A \neq 0$,

$$U = \boldsymbol{v} \cdot \boldsymbol{n}, \qquad u_A = \frac{U}{\mathbf{t}_A \cdot \boldsymbol{n}_A}; \tag{25–6}$$

moreover, as $\delta \to 0$,

$$\mathbf{t}_A(t) \to \mathbf{t}(\boldsymbol{Z}(t)), \qquad u_A(t) \to V(t). \tag{25–7}$$

Let $\mathscr{G}(t)$ denote an arbitrary connected subcurve of the crack, with $\boldsymbol{Z}(t) \notin \mathscr{G}(t)$. An example of a crack control volume is the δ-**pillbox** $\mathscr{G}_\delta(t)$ about $\mathscr{G}(t)$ as defined

in (11–22). Writing $\boldsymbol{X}_A(t)$ and $\boldsymbol{X}_B(t)$, $s(\boldsymbol{X}_A(t)) < s(\boldsymbol{X}_B(t))$, for the endpoints of $\mathscr{G}(t)$, $\partial\mathscr{G}_\delta = \partial\mathscr{G}_\delta(t)$ consists of:

(i) two curves, each parallel to—and a distance δ from—$\mathscr{G}$; on these curves the normal velocity of $\partial\mathscr{G}_\delta$ vanishes;
(ii) two end faces of length 2δ perpendicular to $\mathscr{G}$, one at $\boldsymbol{X}_A$, the other at $\boldsymbol{X}_B$.

b. Derivatives following the tip. Tip integrals. Transport theorems

A **bulk field** $\Phi(\boldsymbol{X}, t)$ (i.e., a field defined away from the crack) is **smooth away from the tip** if, away from $\boldsymbol{Z}(t)$, $\Phi(\boldsymbol{X}, t)$ and its derivatives have limits up to the crack from either side. (For $\boldsymbol{X} \in \mathscr{C}(t)$, $\boldsymbol{X} \notin \boldsymbol{Z}(t)$, the jump $[\![\Phi]\!](\boldsymbol{X}, t)$ and the interfacial limits $\Phi^{\pm}(\boldsymbol{X}, t)$ are then defined as in (10–1).)

Let $\Phi(\boldsymbol{X}, t)$ be a bulk field that is smooth away from the tip. Consider the corresponding field $\hat{\Phi}(\boldsymbol{Y}, t)$ in which $\boldsymbol{Y}$ represents the position of the material point $\boldsymbol{X}$ relative to the tip $\boldsymbol{Z}(t)$:

$$\hat{\Phi}(\boldsymbol{Y}, t) = \Phi(\boldsymbol{X}, t), \qquad \boldsymbol{Y} = \boldsymbol{X} - \boldsymbol{Z}(t). \tag{25–8}$$

The partial derivative

$$\overset{\square}{\Phi}(\boldsymbol{X}, t) = \frac{\partial}{\partial t}\, \hat{\Phi}(\boldsymbol{Y}, t)\Big|_{\boldsymbol{Y}=\boldsymbol{X}-\boldsymbol{Z}(t)} \tag{25–9}$$

with respect to t holding $\boldsymbol{Y}$ fixed, but considered as a function of $(\boldsymbol{X}, t)$, represents the **time derivative of** $\Phi(\boldsymbol{X}, t)$ **following** $\boldsymbol{Z}(t)$; by the chain rule,

$$\overset{\square}{\Phi} = \dot{\Phi} + \nabla\Phi \cdot \boldsymbol{v} \tag{25–10}$$

away from the crack.

Essential to the theory are limits such as $\lim_{\delta\to 0} \int_{\partial D_\delta} \Phi \boldsymbol{n}\, ds$. When meaningful, such limits, termed **tip integrals**, will be written in the form

$$\oint_{\text{tip}} \Phi \boldsymbol{n}\, ds = \lim_{\delta\to 0} \int_{\partial D_\delta} \Phi \boldsymbol{n}\, ds.$$

For Ψ the bulk free energy and $\boldsymbol{S}$ the bulk stress, $\oint_{\text{tip}} \Psi(\boldsymbol{n} \cdot \boldsymbol{V}) ds$ and $\oint_{\text{tip}} \boldsymbol{S}\boldsymbol{n}\, ds$, respectively, represent the flow of free energy in—and the net traction on—an infinitesimal neighborhood of the tip.

Let $P(t)$ be a control volume. The notation

$$P_\delta(t) = P(t) \backslash D_\delta(t), \tag{25–11a}$$

$$\mathscr{G}(t) = \mathscr{C}(t) \cap P(t) \tag{25–11b}$$

will be used consistently; in (25–11a), $\delta > 0$ is presumed to be sufficiently small that

$$\partial P_\delta(t) = \partial P(t) \cup \partial D_\delta(t).$$

Let $\Phi(\boldsymbol{X}, t)$ be smooth away from the tip, and let $P = P(t)$ be a crack control volume. Then, by (10–8a,b),

$$\frac{d}{dt}\left\{\int_P \Phi\, da\right\} = \int_P \dot{\Phi}\, da + \int_{\partial P} \Phi U\, ds, \tag{25–12a}$$

$$\int_P \nabla\Phi\, da = \int_{\partial P} \Phi \boldsymbol{n}\, ds - \int_{\mathscr{J}} [\Phi]\mathbf{m}\, ds \tag{25–12b}$$

(because the crack is stationary away from the tip).

Next, let $P = P(t)$ be a tip control volume and consider the crack control volume $P_\delta = P_\delta(t)$, using the same letter $\boldsymbol{n}$ for the outward unit normal on both ∂P and ∂D_δ, so that the outward unit normal and normal velocity for ∂P_δ are $-\boldsymbol{n}$ and $U = -\boldsymbol{v}\cdot\boldsymbol{n}$ on that portion of ∂P_δ coincident with ∂D_δ; then (25–12) hold with $P = P(t)$ replaced by $P_\delta = P_\delta(t)$, so that

$$\frac{d}{dt}\left\{\int_{P_\delta} \Phi\, da\right\} = \int_{P_\delta} \dot{\Phi}\, da + \int_{\partial P} \Phi U\, ds - \int_{\partial D_\delta} \Phi(\boldsymbol{v}\cdot\boldsymbol{n}) ds, \tag{25–13a}$$

$$\int_{P_\delta} \nabla\Phi\, da = \int_{\partial P} \Phi\boldsymbol{n}\, ds - \int_{\mathscr{C}\cap P_\delta} [\Phi]\mathbf{m}\, ds - \int_{\partial D_\delta} \Phi\boldsymbol{n}\, ds. \tag{25–13b}$$

Taking the inner product of (25–13b) with $\boldsymbol{v}$ and subtracting the resulting relation from (25–13a) yields, by (25–10), an identity,

$$\frac{d}{dt}\left\{\int_{P_\delta} \Phi\, da\right\} = \int_{P_\delta} \overset{\square}{\Phi}\, da + \int_{\partial P} \Phi(U - \boldsymbol{v}\cdot\boldsymbol{n}) ds + \int_{\mathscr{C}\cap P_\delta} [\Phi]\mathbf{m}\cdot\boldsymbol{v}\, ds, \tag{25–14}$$

that will form a basis for the derivation of relations appropriate to the limit $\delta \to 0$.

Let $P = P(t)$ be a tip control volume. For $\Phi(\boldsymbol{X}, t)$ smooth away from the tip the behavior of $\Phi(\boldsymbol{X}, t)$ at $\boldsymbol{Z}(t)$ is not specified and the integral $\int_P \Phi\, da$ may not exist. Here it seems most convenient to define such an integral in terms of its **Cauchy principle value**; i.e., as the limit of the integral over $P_\delta(t)$ as $\delta \to 0$.

The next definition allows for a succinct statement of hypotheses concerning momenta and energies. A field Φ will be termed a **regular bulk field** if, in addition to being smooth away from the tip:

(R1) for each tip control-volume $P = P(t)$ the limits

$$\int_P \Phi\, da = \lim_{\delta\to 0} \int_{P_\delta} \Phi\, da, \tag{25–15a}$$

$$\int_P \overset{\square}{\Phi}\, da = \lim_{\delta\to 0} \int_{P_\delta} \overset{\square}{\Phi}\, da, \tag{25–15b}$$

exist, with (25–15b) uniform in time;

(R2) $[\Phi]\mathbf{m}\cdot\boldsymbol{v}$ is integrable on $\mathscr{C}$, uniformly in time (cf. (25–3)).

In actual solutions of crack problems, the underlying fields are generally singular in the distance $r = |\boldsymbol{X} - \boldsymbol{Z}(t)|$ from $\boldsymbol{Z}(t)$; for that reason hypotheses seem best worded

in terms of the derivative $\overset{\square}{\Phi}$, which is taken holding r fixed. Indeed, typical estimates[1] for a crack in a *linearly elastic* body are that the displacement $\boldsymbol{u}$ satisfy $\nabla u \sim r^{-1/2}$ and $\dot{u} \sim r^{-1/2}$ as $r \to 0$, which yield, for the energy Ψ,

$$\Psi \sim r^{-1}, \qquad \overset{\square}{\Phi} \sim r^{-1}, \tag{25–16a}$$

$$\dot{\Psi} \sim r^{-2}, \qquad \nabla\Phi \sim r^{-2}, \tag{25–16b}$$

Granted uniformity in t, (25–16a) would imply (R1) and (R2) (cf. (25–3)). On the other hand, the estimate for $\dot{\Psi}$ would render the assumption $\int\limits_P \dot{\Psi}\, da = \lim_{\delta\to 0} \int\limits_{P_\delta} \dot{\Psi}\, da$ of uncertain *general* value.

An important consequence of the assumption of regularity for a bulk field Φ is that, given any migrating control volume $P = P(t)$,

$$\int\limits_{P(t)} \Phi\, da \text{ is a differentiable function of } t; \tag{25–17}$$

a second and equally important consequence is the **transport identity**

$$\frac{d}{dt}\left\{\int\limits_{P(t)} \Phi\, da\right\} = \int\limits_{P(t)} \overset{\square}{\Phi}\, da + \int\limits_{\partial P(t)} \Phi(U - \boldsymbol{v}\cdot\boldsymbol{n})ds + \int\limits_{\mathscr{J}(t)} [\![\Phi]\!]\mathfrak{m}\cdot\boldsymbol{v}\, ds, \tag{25–18}$$

valid for any migrating control volume $P = P(t)$. This identity expresses $d/dt \int\limits_P \Phi\, da$ in terms of quantities—the temporal change $\overset{\square}{\Phi}$ and the inflows $\Phi(U - \boldsymbol{n}\cdot\boldsymbol{v})$ and $[\![\Phi]\!]\mathfrak{m}\cdot\boldsymbol{v}$—measured in a frame moving with the tip.

To verify (25–17) and (25–18), let

$$\varphi(t) = \int\limits_{P(t)} \Phi\, da, \qquad \varphi_\delta(t) = \int\limits_{P_\delta(t)} \Phi\, da;$$

then (25–14) and properties (R1) and (R2) of regular fields yield the conclusion that, as $\delta \to 0$, $d\varphi_\delta/dt$ tends to the right side of (25–18) *uniformly*, while $\varphi_\delta \to \varphi$. Thus $\varphi(t) = \int\limits_{P(t)} \Phi\, da$ is a differentiable function of t and $d\varphi_\delta/dt \to d\varphi/dt$, or equivalently,

$$\frac{d}{dt}\left\{\int\limits_{P_\delta} \Phi\, da\right\} \to \frac{d}{dt}\left\{\int\limits_{P} \Phi\, da\right\}, \tag{25–19}$$

and (25–18) holds.

The next result is central to the localization of the second law to crack tips.

Tip Transport Theorem for Bulk Fields. Let Φ be a regular bulk field. Then

$$\frac{d}{dt}\left\{\int\limits_{D_\delta} \Phi\, da\right\} \to 0 \qquad \text{as } \delta \to 0, \tag{25–20}$$

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[1] Cf. Freund [1990, p. 43]. Even within the infinitesimal theory, this estimate is generally not valid beyond linear elasticity (cf. Rice and Rosengren [1968], Hutchinson [1968]).

uniformly in time.

PROOF. The first step is to show that, for F an arbitrary field,

$$\text{if } \int_P F\,da = \lim_{\delta\to 0}\int_{P_\delta} F\,da \text{ exists for some (and hence every) control volume } P, \text{ then } \lim_{\delta\to 0}\int_{D_\delta} F\,da = 0, \text{ with uniformity in time in the second limit if in the first.} \tag{25–21}$$

To verify (25–21), given $\lambda > \tau > 0$, let $D_{\lambda\tau}(t)$ denote the annulus

$$D_{\lambda\tau}(t) = D_\lambda(t)\backslash D_\tau(t).$$

Granted $\int_P F\,da = \lim_{\delta\to 0}\int_{P_\delta} F\,da$ for all P $(*)$, let $\int_{D_{\rho\delta}} F\,da = \int_{D_{\rho\varepsilon}} F\,da - \int_{D_{\delta\varepsilon}} F\,da$, $\rho > \delta > \varepsilon$. Then, if we let $\varepsilon \to 0$ and $\delta \to 0$, in that order, the result is $\lim_{\delta\to 0}\int_{D_\delta} F\,da = 0$, with this limit uniform if $(*)$ is uniform. Hence (25–21) is valid.

Because $U = \boldsymbol{v}\cdot\boldsymbol{n}$ is the normal velocity of ∂D_δ, (25–18) with $P = D_\delta$ yields

$$\frac{d}{dt}\left\{\int_{D_\delta}\Phi\,da\right\} = \int_{D_\delta}\overset{\square}{\Phi}\,da + \int_{\mathscr{C}\cap D_\delta}[\Phi]\mathfrak{m}\cdot\boldsymbol{v}\,ds. \tag{25–22}$$

Thus, by (25–15b) and (25–21)

$$\lim_{\delta\to 0}\int_{D_\delta}\overset{\square}{\Phi}\,da = 0, \tag{25–23}$$

uniformly. Also, by property (R2), the integral in (25–22) over $\mathscr{C}\cap D_\delta$ goes to zero with δ, uniformly. Thus (25–20) is valid. □

Let Φ be regular. Then, as a consequence of (25–13a) and (25–19), $\oint_{\text{tip}}\Phi(\boldsymbol{v}\cdot\boldsymbol{n})ds$ exists if and only if $\int_P\dot{\Phi}\,da = \lim_{\delta\to 0}\int_{P_\delta}\dot{\Phi}\,da$ exists for some (and hence every) control volume P, and, granted either,

$$\frac{d}{dt}\left\{\int_{P(t)}\Phi\,da\right\} = \int_{P(t)}\dot{\Phi}\,da + \int_{\partial P(t)}\Phi U\,ds - \oint_{\text{tip}}\Phi(\boldsymbol{v}\cdot\boldsymbol{n})\,ds.$$

Further, for a δ-pillbox $\mathscr{G}_\delta = \mathscr{G}_\delta(t)$, (25–12a) and (i) and (ii) of the paragraph following (25–7) imply that

$$\frac{d}{dt}\left\{\int_{\mathscr{G}_\delta}\Phi\,da\right\} \to 0, \tag{25–24a}$$

$$\int_{\partial\mathscr{G}_\delta}\Phi U\,ds \to 0, \tag{25–24b}$$

as $\delta \to 0$.

Also important are transport relations for fields $\varphi(\boldsymbol{X})$ (independent of t) that are continuous on the crack, up to the tip. Let $\mathscr{G}(t)$, defined in (25–11b), be the portion of the crack in a control volume $P(t)$. Then, by (19–8),[2] using the notation specified in the paragraph containing (25–4),

$$\frac{d}{dt}\left\{\int_{\mathscr{G}(t)} \varphi\, ds\right\} = \varphi_{\text{tip}} V - \varphi_A u_A \qquad \text{for a tip control volume,} \tag{25–25a}$$

$$\frac{d}{dt}\left\{\int_{\mathscr{G}(t)} \varphi\, ds\right\} = \varphi_B u_B - \varphi_A u_A \qquad \text{for a crack control volume.} \tag{25–25b}$$

Thus, (25–7) and (25–25a), as $\delta \to 0$

$$\frac{d}{dt}\left\{\int_{\mathscr{C}\cap D_\delta} \varphi\, ds\right\} \to 0 \quad \text{as } \delta \to 0, \tag{25–26}$$

which is an interfacial counterpart of (25–20).

[2]Bearing in mind that V in (19–8) is the normal velocity of the interface, while V here is the scalar velocity of the tip.

CHAPTER 26

Motions

Motions y are defined as in Section 2a, except that $\boldsymbol{y}$ is not required to be continuous across the crack. Precisely, $\boldsymbol{y}(\boldsymbol{X}, t)$ is assumed to be smooth away from the tip, to satisfy the **impenetrability condition**

$$[\boldsymbol{y}] \cdot \mathbf{m} \geq 0, \tag{26–1}$$

and to have a limiting value $\boldsymbol{y}(\boldsymbol{Z}(t), t)$ at the tip,

$$\boldsymbol{y}(\boldsymbol{X}, t) \to \boldsymbol{y}(\boldsymbol{Z}(t), t) \qquad \text{as } \boldsymbol{X} \to \boldsymbol{Z}(t) \tag{26–2}$$

from bulk or from points of the crack, so that the *deformed tip* is well defined. The deformation gradient $\boldsymbol{F} = \nabla \boldsymbol{y}$ and the material velocity $\dot{\boldsymbol{y}}$ are then smooth away from the tip, although these fields are generally singular at the tip.

By (25–9), the **motion velocity $\overset{\square}{\boldsymbol{y}}$ following the tip** is the derivative

$$\overset{\square}{\boldsymbol{y}}(\boldsymbol{X}, t) = \frac{\partial}{\partial t} \boldsymbol{y}(\boldsymbol{Z}(t) + \boldsymbol{Y}, t)\big|_{\boldsymbol{Y} = \boldsymbol{X} - \boldsymbol{X}(t)} \tag{26–3}$$

holding the distance $\boldsymbol{Y}$ from the tip fixed; equivalently,

$$\overset{\square}{\boldsymbol{y}} = \dot{\boldsymbol{y}} + \boldsymbol{F}\boldsymbol{v} \tag{26–4}$$

in bulk. The field $\overset{\square}{\boldsymbol{y}}(\boldsymbol{X}, t)$ is assumed to have a limiting value $\bar{\boldsymbol{v}}(t)$ at the tip:

$$\overset{\square}{\boldsymbol{y}}(\boldsymbol{X}, t) \to \bar{\boldsymbol{v}}(t) \qquad \text{as } \boldsymbol{X} \to \boldsymbol{Z}(t), \tag{26–5}$$

(cf. the paragraph containing (25–16a)); if the limit is uniform in t, then, by (26–2), $\boldsymbol{y}(\boldsymbol{Z}(t), t)$ is differentiable in t and, as would be expected from (26–3),

$$\bar{\boldsymbol{v}}(t) = \frac{d}{dt}(\boldsymbol{Z}(t), t) \tag{26–6}$$

so that $\bar{\boldsymbol{v}}$ is the velocity of the deformed tip. The results (26–5) and (26–6) establish the consistency of the current definition of $\overset{\square}{\boldsymbol{y}}$ (via (25–9)) and the definition (10–4) for an evolving interface. A direct consequence of (26–4) and (26–5) is that

$$\text{for } \boldsymbol{v} = \boldsymbol{0}, \quad \dot{\boldsymbol{y}}(\boldsymbol{X}, t) \to \bar{\boldsymbol{v}}(t) \text{ as } \boldsymbol{X} \to \boldsymbol{Z}(t). \tag{26–7}$$

Let $P = P(t)$ be a control volume. As before, $\boldsymbol{q}$ denotes an arbitrary velocity field for the boundary curve ∂P, and $\mathring{\boldsymbol{y}}$, defined by (4–3), the motion velocity following ∂P as described by $\boldsymbol{q}$, so that

$$\mathring{\boldsymbol{y}} = \dot{\boldsymbol{y}} + \boldsymbol{F}\boldsymbol{q}. \tag{26–8}$$

The tip velocity $\boldsymbol{v}(t)$ is admissible as a velocity field for the boundary of the tip disc $D_\delta(t)$; the motion velocity $\overset{\square}{\boldsymbol{y}}$ following the tip therefore coincides on ∂D_δ with the motion velocity $\mathring{y}$ following ∂D_δ as described by $\boldsymbol{v}$.

CHAPTER 27

Forces. Working

As in the treatment of coherent interfaces presented in Part B, the constitutive theory for the tip involves a simple dissipation inequality of the form (11–21), and that is why the introduction of (virtual) external forces to ensure satisfaction of the balance laws in all processes would seem excessive. I therefore assume that (the underlying observer and) all external body forces are inertial.

As before, I characterize inertia using *inertial forces*, with the momentum and kinetic energy produced at the crack tip accounted for by *concentrated forces* within the standard and configurational systems. As the crack produces neither momentum nor kinetic energy away from the tip, there is no need to consider inertial forces distributed over the crack.

a. Forces

To the standard forces $\boldsymbol{S}$ and $\boldsymbol{b}$ and configurational forces $\boldsymbol{C}$, $\boldsymbol{g}$, and $\boldsymbol{e}$ introduced in Chapters 3 and 5, assumed here to be objective and smooth away from the tip, with $\boldsymbol{b}$, $\boldsymbol{g}$, and $\boldsymbol{e}$ integrable (as Cauchy principal values), I add two fields *distributed over the crack*:

$\boldsymbol{g}^{\mathscr{C}}$	**internal configurational force**
$\mathbf{c}$	**surface stress**

and three forces *concentrated at the tip*:

$\boldsymbol{b}_{\text{tip}}$	**external standard force**
$\boldsymbol{g}_{\text{tip}}$	**internal configurational force**
$\boldsymbol{e}_{\text{tip}}$	**external configurational force**

Precisely, $\boldsymbol{b}_{\text{tip}}(t)$ is a spatial vector; $\boldsymbol{g}^{\mathscr{C}}(\boldsymbol{X}, t)$, $\mathbf{C}(\boldsymbol{X})$, $\boldsymbol{g}_{\text{tip}}(t)$ are material vectors; all five functions are objective; $\boldsymbol{g}^{\mathscr{C}}(\boldsymbol{X}, t)$ is integrable over $\mathscr{C}(t)$; $\mathbf{C}(\boldsymbol{X})$ is smooth (up to the tip). As the crack faces are treated in unison, $\mathbf{C}$ and $\boldsymbol{g}^{\mathscr{C}}$ represent forces associated with the *union* of the crack faces.

In classical theories of fracture the strength of the tip singularity is insufficient to induce a release of momentum and this, in turn, yields $\boldsymbol{b}_{\text{tip}} \equiv \mathbf{0}$. But even within this restricted framework $\boldsymbol{e}_{\text{tip}}$, the configurational counterpart of $\boldsymbol{b}_{\text{tip}}$, does not vanish; for that reason, we consider the general theory with $\boldsymbol{b}_{\text{tip}} \neq \mathbf{0}$, because this leads to a parallel treatment of inertia within the two force systems.

The configurational surface stress $\mathbf{C}(\boldsymbol{X})$, which acts within the free surfaces of the crack, is the analog of the configurational surface stress (tensor) $\mathbf{C}(\boldsymbol{X}, t)$ within an interface (cf. Part E); the current two-dimensional treatment allows $\mathbf{C}(\boldsymbol{X})$ to be identified with a vector (cf. Chapter 19). Fix $\boldsymbol{X}$ and t, and let $\mathscr{C}^+ = \mathscr{C}^+(t)$ denote the portion of the crack with arc-length values greater than $s(\boldsymbol{X})$, and $\mathscr{C}^- = \mathscr{C}^-(t)$ the portion with values less than $s(\boldsymbol{X})$. The surface stress $\mathbf{C} = \mathbf{C}(\boldsymbol{X})$ then represents the force exerted across s by the material in $\mathscr{C}^+$ on the material in $\mathscr{C}^-$, with $\mathbf{C} \cdot \mathbf{t}$ a **surface tension** and $\mathbf{C} \cdot \mathbf{m}$ a **surface shear**

Neither $\mathbf{C}$ nor the surface energy ψ (cf. Section 28a) are allowed to depend on t. Such dependencies, while not difficult to accommodate, seem unimportant to the characterization of real materials, for which $\mathbf{C}$ and ψ typically depend constitutively on the normal $\mathbf{m}$ to $\mathscr{C}$, and $\mathbf{m} = \mathbf{m}(\boldsymbol{X})$ is independent of t. As in Chapter 18, one generally expects a configurational shear $\mathbf{C} \cdot \mathbf{m}$ whenever the surface energy is anisotropic.

The internal configurational force $\boldsymbol{g}_{\text{tip}}$ is associated with the breaking of bonds during crack growth or, more generally, to phenomena occurring at the tip at length scales that are small compared to the gross length scales of the body;[1] as such this force is important in discussing the kinetics of crack growth. Of less importance is the internal configurational force $\boldsymbol{g}^{\mathscr{C}}$ distributed over the crack surfaces. Because the crack surfaces do not migrate, $\boldsymbol{g}^{\mathscr{C}}$, like its bulk counterpart $\boldsymbol{g}$, is *indeterminate*.[2]

I assume that the tip singularity is not too strong in the sense that

$$\int_{\partial D_\delta(t)} |\boldsymbol{S}\boldsymbol{n}| ds \qquad \text{is bounded as } \delta \to 0.^{3} \tag{27–1}$$

This assumption, (26–2), (26–5), and the definition of the tip integral yield two useful results in which $\boldsymbol{y}_{\text{tip}}(t) = \boldsymbol{y}(\boldsymbol{Y}(t), t)$:

$$\oint_{\text{tip}} \boldsymbol{S}\boldsymbol{n} \cdot (\overset{\square}{\boldsymbol{y}} - \bar{\boldsymbol{v}}) ds = 0, \tag{27–2a}$$

$$\oint_{\text{tip}} (\boldsymbol{y} - \boldsymbol{y}_{\text{tip}}) \times \boldsymbol{S}\boldsymbol{n}\, ds = \mathbf{0}. \tag{27–2b}$$

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[1] Cf. the discussion of Freund [1990, pp. 10–11]. An example of such phenomena is "small-scale yielding" associated with a crack-tip plastic zone (cf. Rice [1968]).

[2] The internal force $\boldsymbol{g}$ is discussed in the paragraph following (5–1).

[3] For a crack in a *linearly elastic body*, $s \sim r^{-1/2}$ as $r \to 0$, which yields the vanishing of the integral in (27–1).

b. Working

The working of the bulk forces is as discussed in Part A. If $P = P(t)$ is a tip control volume, then $\mathbf{c}$ performs work on P of amount $-\mathbf{c}_A \cdot \dot{\boldsymbol{X}}_A = -u_A \mathbf{t}_A \cdot \mathbf{c}_A$, while $\mathbf{c}_B \cdot \dot{\boldsymbol{X}}_B - \mathbf{c}_A \cdot \dot{\boldsymbol{X}}_A = u_B \mathbf{t}_B \cdot \mathbf{c}_B - u_A \mathbf{t}_A \cdot \mathbf{c}_A$ represents the corresponding work performed on a crack control volume (cf. the paragraph containing (25–4)). The motion of the tip is accompanied by working of the inertial forces $\boldsymbol{e}_{\text{tip}}$ and $\boldsymbol{b}_{\text{tip}}$, with the tip velocities $\boldsymbol{v}$ and $\bar{\boldsymbol{v}}$ as appropriate conjugate velocities. Finally, internal forces perform no work.

The **working** $W(P)$ on a migrating control volume $P = P(t)$ therefore takes the form

$$W(P) = \int_{\partial P} (\boldsymbol{C}\boldsymbol{n} \cdot \boldsymbol{q} + \boldsymbol{S}\boldsymbol{n} \cdot \mathring{\boldsymbol{y}})\, ds + \int_P \boldsymbol{b} \cdot \dot{\boldsymbol{y}}\, da + W_c(P), \tag{27–3}$$

where $\int_P \boldsymbol{b} \cdot \dot{\boldsymbol{y}}\, da$ denotes $\lim_{\delta \to 0} \int_{P_\delta} \boldsymbol{b} \cdot \dot{\boldsymbol{y}}\, da$, which is assumed to exist, and where $W_c(P(t))$, the working associated with the crack, is given by

$$W_c(P) = u_B \mathbf{t}_B \cdot \mathbf{c}_B - u_A \mathbf{t}_A \cdot \mathbf{c}_A \tag{27–4}$$

for a crack control volume, and

$$W_c(P) = \boldsymbol{b}_{\text{tip}} \cdot \bar{\boldsymbol{v}} + \boldsymbol{e}_{\text{tip}} \cdot \boldsymbol{v} - u_A \mathbf{t}_A \cdot \mathbf{c}_A \tag{27–5}$$

for a tip control volume. Arguing as in the proof of (5–15), $W(P)$ may be written intrinsically as

$$W(P) = \int_{\partial P} \boldsymbol{S}\boldsymbol{n} \cdot \dot{\boldsymbol{y}}\, ds + \int_P \boldsymbol{b} \cdot \dot{\boldsymbol{y}}\, da + \int_{\partial P} \pi U\, ds + W_c(P) \tag{27–6}$$

with π given by (5–17).

c. Standard and configurational force balances

The velocities $\boldsymbol{v}$, $u_A \mathbf{t}_A$, $u_B \mathbf{t}_B$, and $\bar{\boldsymbol{v}}$ transform according to

$$\boldsymbol{v} \to \boldsymbol{v} + \boldsymbol{a}, \qquad u_A \mathbf{t}_A \to u_A \mathbf{t}_A + \boldsymbol{a}, \qquad u_B \mathbf{t}_B \to u_B \mathbf{t}_B + \boldsymbol{a}, \qquad \bar{\boldsymbol{v}} \to \bar{\boldsymbol{v}}$$

under the change in material observer defined by (2–9), and according to

$$\boldsymbol{v} \to \boldsymbol{v}, \qquad u_A \mathbf{t}_A \to u_A \mathbf{t}_A, \qquad u_B \mathbf{t}_B \to u_B \mathbf{t}_B, \qquad \bar{\boldsymbol{v}} \to \bar{\boldsymbol{v}} + \boldsymbol{w} + \boldsymbol{\omega} \times (\boldsymbol{y}_{\text{tip}} - \boldsymbol{o})$$

under the change in spatial observer defined by (2–7) (cf. the consistency requirement as stated in Section 2d). Thus, by the transformation laws for $\boldsymbol{q}, \mathring{\boldsymbol{y}}, \dot{\boldsymbol{y}}, \boldsymbol{v}$ and $\dot{\boldsymbol{y}}$ specified in (2–8), (2–12), (4–6), and (4–7), the working $W(P)$ as recorded by the new observers has the form

$$W(P) = \int_{\partial P} \boldsymbol{C}\boldsymbol{n} \cdot (\boldsymbol{q} + \boldsymbol{a}) ds + \int_P (\boldsymbol{g} + \boldsymbol{e}) \cdot \boldsymbol{a}\, da + \int_{\partial P} \boldsymbol{S}\boldsymbol{n} \cdot \big(\mathring{\boldsymbol{y}} + \boldsymbol{w} + \boldsymbol{\omega} \times (\boldsymbol{y} - \boldsymbol{o})\big) ds$$

$$+ \int_P \boldsymbol{b} \cdot (\dot{\boldsymbol{y}} + \boldsymbol{w} + \boldsymbol{\omega} \times (\boldsymbol{y} - \boldsymbol{o}))da + W_c(P) \qquad (27\text{–}7)$$

with

$$W_c(P) = (u_B \mathbf{t}_B + \boldsymbol{a}) \cdot \mathbf{c}_B - (u_A \mathbf{t}_A + \boldsymbol{a}) \cdot \mathbf{c}_A + \int_{P \cap \mathscr{C}} \boldsymbol{g}^{\mathscr{C}} \cdot \boldsymbol{a}\, ds$$

for a crack control volume and

$$W_c(P) = \boldsymbol{g}_{\text{tip}} \cdot \boldsymbol{a} + \boldsymbol{e}_{\text{tip}} \cdot (\boldsymbol{v} + \boldsymbol{a}) - (u_A \mathbf{t}_A + \boldsymbol{a}) \cdot \mathbf{c}_A + \boldsymbol{b}_{\text{tip}} \big(\bar{\boldsymbol{v}} + \boldsymbol{w} + \boldsymbol{\omega} \times (\boldsymbol{y}_{\text{tip}} - \boldsymbol{o})\big)$$

for a tip control volume.

The requirement that the working be invariant under changes in material and spatial observer requires the equivalence of (27–3) and (27–7) for all $\boldsymbol{a}$, $\boldsymbol{w}$, and $\boldsymbol{\omega}$. This yields:

(i) **standard force** and **moment balances**

$$\int_{\partial P} \boldsymbol{S}\boldsymbol{n}\, ds + \int_P \boldsymbol{b}\, da = \boldsymbol{0}, \qquad (27\text{–}8a)$$

$$\int_{\partial P} (\boldsymbol{y} - \boldsymbol{o}) \times \boldsymbol{S}\boldsymbol{n}\, ds + \int_P (\boldsymbol{y} - \boldsymbol{o}) \times \boldsymbol{b}\, da = \boldsymbol{0} \qquad (27\text{–}8b)$$

for a (bulk or) crack control volume and

$$\int_{\partial P} \boldsymbol{S}\boldsymbol{n}\, ds + \int_P \boldsymbol{b}\, da + \boldsymbol{b}_{\text{tip}} = \boldsymbol{0}, \qquad (27\text{–}9a)$$

$$\int_{\partial P} (\boldsymbol{y} - \boldsymbol{o}) \times \boldsymbol{S}\boldsymbol{n}\, ds + \int_P (\boldsymbol{y} - \boldsymbol{o}) \times \boldsymbol{b}\, da + (\boldsymbol{y}_{\text{tip}} - \boldsymbol{o}) \times \boldsymbol{b}_{\text{tip}} = \boldsymbol{0} \qquad (27\text{–}9b)$$

for a tip control volume;

(ii) **configurational force balances**

$$\int_{\partial P} \boldsymbol{C}\boldsymbol{n}\, ds + \int_P \boldsymbol{g}\, da + \int_{\mathscr{S}} \boldsymbol{g}^{\mathscr{C}}\, ds + \mathbf{c}_B - \mathbf{c}_A = \boldsymbol{0} \qquad (27\text{–}10)$$

for a crack control volume and

$$\int_{\partial P} \boldsymbol{C}\boldsymbol{n}\, ds + \int_P \boldsymbol{g}\, da + \int_{\mathscr{S}} \boldsymbol{g}^{\mathscr{C}}\, ds + \boldsymbol{g}_{\text{tip}} + \boldsymbol{e}_{\text{tip}} - \mathbf{c}_A = \boldsymbol{0} \qquad (27\text{–}11)$$

for a tip control volume

Since $\boldsymbol{b}, \boldsymbol{g}$, and $\boldsymbol{g}^{\mathscr{C}}$ are integrable, (27–9a) and (27–11) applied to a tip disc $D_\delta(t)$ yields, after passing to the limit $\delta \to 0$, the following balances at the tip:

$$\oint_{\text{tip}} \boldsymbol{S}\boldsymbol{n}\, ds + \boldsymbol{b}_{\text{tip}} = \boldsymbol{0}, \qquad (27\text{–}12a)$$

$$\oint_{\text{tip}} \boldsymbol{C}\boldsymbol{n}\, ds + \boldsymbol{g}_{\text{tip}} + \boldsymbol{e}_{\text{tip}} - \mathbf{c}_{\text{tip}} = \boldsymbol{0}. \qquad (27\text{–}12b)$$

(In view of (27–12a), $(\boldsymbol{y} - \boldsymbol{o})$ in (27–9b) may be replaced by $(\boldsymbol{y}_{\text{tip}} - \boldsymbol{o})$; thus, by (27–2b), the limit $\delta \to 0$ in the standard moment balance (27–9b) yields no additional information.)

To derive local relations for the crack, consider the δ-pillbox $\mathscr{G}_\delta(t)$ about an arbitrary subcurve $\mathscr{G}(t)$ of the crack, with $\boldsymbol{Z}(t) \notin \mathscr{G}(t)$. Then (27–8a) and (27–10) applied to $\mathscr{G}(t)$ yield, in the limit $\delta \to 0$,

$$\int_{s_A}^{s_B} [\boldsymbol{S}]\mathfrak{m}\, ds = \mathbf{0}, \qquad \int_{s_A}^{s_B} ([\boldsymbol{C}]\mathfrak{m} + \boldsymbol{g}^{\mathscr{C}})ds + \mathbf{c}_B - \mathbf{c}_A = \mathbf{0} \tag{27–13}$$

where $s_B > s_A$, are the arc-length values that mark the endpoints of $\mathscr{G}(t)$. Let $s_B \to s_A$ after dividing by $s_B - s_A$; the result is

$$[\boldsymbol{S}\mathfrak{m}] = \mathbf{0}, \tag{27–14a}$$

$$[\boldsymbol{C}\mathfrak{m}] + \mathbf{c}_s + \boldsymbol{g}^{\mathscr{C}} = \mathbf{0} \tag{27–14b}$$

on the crack away from the tip.

d. Inertial forces. Kinetic energy

As before, let $\rho(\boldsymbol{X}) \geq 0$ denote the referential mass density, assumed smooth; let

$$\boldsymbol{p} = \rho\dot{\boldsymbol{y}}, \qquad k = \frac{1}{2}\,\rho|\dot{\boldsymbol{y}}|^2 \tag{27–15}$$

denote the densities of momentum and kinetic energy; and define the productions of momentum and kinetic energy in a migrating control volume $P = P(t)$ by

$$\mathscr{P}(P) = \frac{d}{dt}\left\{\int_P \boldsymbol{p}\, da\right\} - \int_{\partial P} \boldsymbol{p}U\, ds, \tag{27–16a}$$

$$\mathscr{T}(P) = \frac{d}{dt}\left\{\int_P k\, da\right\} - \int_{\partial P} kU\, ds. \tag{27–16b}$$

To ensure that these definitions have meaning, k and $\boldsymbol{p}$ are assumed to be regular.

The bulk inertial force $\boldsymbol{b}$ has the form $\boldsymbol{b} = -\rho\ddot{\boldsymbol{y}}$ (cf. (7–1)). The inertial forces at the tip, which are not so obvious, are characterized through the following two relations involving inertial forces, inertial working, and the productions of momentum and kinetic energy for an arbitrary tip control volume $P = P(t)$ (cf. (7–6)):

$$\int_P \boldsymbol{b}\, da + \boldsymbol{b}_{\text{tip}} = -\mathscr{P}(P); \tag{27–17a}$$

$$\int_P \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\, da + \boldsymbol{b}_{\text{tip}}\cdot\bar{\boldsymbol{v}} + \boldsymbol{e}_{\text{tip}}\cdot\boldsymbol{v} = -\mathscr{T}(P). \tag{27–17b}$$

Consequences of these balances are the relations

$$\boldsymbol{b}_{\text{tip}} = \oint_{\text{tip}} \boldsymbol{p}(\boldsymbol{v}\cdot\boldsymbol{n})ds, \tag{27–18a}$$

$$\boldsymbol{b}_{\text{tip}}\cdot\bar{\boldsymbol{v}} + \boldsymbol{e}_{\text{tip}}\cdot\boldsymbol{v} = \oint_{\text{tip}} k(\boldsymbol{v}\cdot\boldsymbol{n})ds, \tag{27–18b}$$

$$\boldsymbol{e}_{\text{tip}}\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\oint_{\text{tip}} k_{\text{rel}}\boldsymbol{n}\, ds, \qquad k_{\text{rel}} = \frac{1}{2}\rho|\dot{\boldsymbol{y}} - \bar{\boldsymbol{v}}|^2; \tag{27–18c}$$

(27–18a) asserts the equivalence of $\boldsymbol{b}_{\text{tip}}$ and the release rate for momentum, (27–18b) the equivalence of the total inertial working and the release rate for kinetic energy, (27–18c) the equivalence of the configurational inertial working and the release rate for the kinetic energy measured relative to the tip.

To establish (27–18a), note first that, by (25–21) and the integrability (in the sense of a Cauchy principal value) of $\boldsymbol{b}$ and $\boldsymbol{b}\cdot\dot{\boldsymbol{y}}$,

$$\int_{D_\delta} \boldsymbol{b}\, da \to \boldsymbol{0}, \qquad \int_{D_\delta} \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\, da \to 0 \tag{27–19}$$

as $\delta \to 0$. Bearing this in mind, the identities (27–18a,b) and the existence of $\oint_{\text{tip}} \boldsymbol{p}(\boldsymbol{v}\cdot\boldsymbol{n})ds$ and $\oint_{\text{tip}} k(\boldsymbol{v}\cdot\boldsymbol{n})ds$ follow from (27–16a,b), (27–17a,b), the regularity of $\boldsymbol{p}$ and k, and (25–20) with $\Phi = \boldsymbol{p}$ and $\Phi = k$. The result (27–18c) follows from (27–18a,b), the identity

$$k - \boldsymbol{p}\cdot\boldsymbol{v} = \frac{1}{2}\,\rho|\dot{\boldsymbol{y}} - \bar{\boldsymbol{v}}|^2 - \frac{1}{2}\,\rho|\bar{\boldsymbol{v}}|^2, \tag{27–20}$$

the continuity of ρ, and the spatial independence of $\bar{v}$.

An alternative method of determining $\boldsymbol{e}_{\text{tip}}$ is to use, in place of (27–17b), a hypothesis analogous to (27–17a) for the pseudomomentum $\mathfrak{p} = -\rho\boldsymbol{F}^\top\dot{\boldsymbol{y}}$ (cf. (7–10), (12–11)):[4]

$$\int_P \boldsymbol{e}\, da + \boldsymbol{e}_{\text{tip}} = -\frac{d}{dt}\left\{\int_P \mathfrak{p}\, da\right\} + \int_{\partial P} \mathfrak{p}U\, ds - \int_{\partial P} k\boldsymbol{n}\, ds + \int_P \frac{1}{2}\,\dot{\boldsymbol{y}}^2\nabla\rho\, da. \tag{27–21}$$

Then, as in the derivation of (27–18a), (27–21) applied to a tip disc $D_\delta(t)$ yields, in the limit $\delta \to 0$, an identity of Dascalu and Maugin[5]

$$\boldsymbol{e}_{\text{tip}} = \oint_{\text{tip}} \mathfrak{p}(\boldsymbol{v}\cdot\boldsymbol{n})ds - \oint_{\text{tip}} k\boldsymbol{n}\, ds. \tag{27–22}$$

Since $\boldsymbol{v} = V\mathbf{t}$, $\mathfrak{p} = -F^\top\boldsymbol{p}$, and $\bar{\boldsymbol{v}} = \dot{\boldsymbol{y}} + \boldsymbol{F}\boldsymbol{v}$,

$$\begin{aligned}\mathbf{t}\cdot\{\mathfrak{p}(\boldsymbol{v}\cdot\boldsymbol{n}) - k\boldsymbol{n}\} &= -\{\mathbf{t}\cdot(\boldsymbol{F}^\top\boldsymbol{p})V + k\}\mathbf{t}\cdot\boldsymbol{n} = -\{\boldsymbol{p}\cdot\boldsymbol{F}\boldsymbol{v} + k\}\mathbf{t}\cdot\boldsymbol{n}\\ &= -\{\boldsymbol{p}\cdot(\bar{\boldsymbol{v}} - \dot{\boldsymbol{y}}) + k\}\mathbf{t}\cdot\boldsymbol{n} = -\{\boldsymbol{p}\cdot\bar{\boldsymbol{v}} - k\}\mathbf{t}\cdot\boldsymbol{n};\end{aligned}$$

thus (27–20) and (27–22) imply that

$$\boldsymbol{e}_{\text{tip}}\cdot\mathbf{t} = \mathbf{t}\cdot\oint_{\text{tip}} k_{\text{rel}}\boldsymbol{n}\, ds, \tag{27–23}$$

and hence that $\boldsymbol{e}_{\text{tip}}\cdot\boldsymbol{v}$ is given by (27–18c).

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[4]Assume that $\mathfrak{p} = -\rho\boldsymbol{F}^\top\dot{\boldsymbol{y}}$ is regular, $\oint_{\text{tip}} \mathfrak{p}(\boldsymbol{v}\cdot\boldsymbol{n})ds$ and $\oint_{\text{tip}} k\boldsymbol{n}\, ds$ exist, and $\rho > 0$, which, with k regular, ensures the integrability of $\frac{1}{2}\dot{\boldsymbol{y}}^2\nabla\rho = k\nabla(\ln\rho)$.

[5][1993]; they formally derive an equivalent relation (their eq. (6)) for homogeneous elastic materials. The hypothesis (27–21) has stronger consequences than (27–17b), as the latter does not imply (27–22), nor does it yield (27–23) when $V = 0$.

CHAPTER 28

The Second Law

a. Statement of the second law

The free energy of the body is represented by a bulk free energy $\Psi(\boldsymbol{X}, t)$ distributed over B and a surface free energy $\psi(\boldsymbol{X})$ distributed over $\mathscr{C}(t)$, with $\Psi(\boldsymbol{X}, t)$ regular, $\psi(\boldsymbol{X})$ smooth,

$$\psi > 0, \tag{28–1}$$

and ψ independent of time (cf. the paragraph in petite type preceding (27–1)).

I write the **second law** for a control volume $P = P(t)$ in the form

$$\frac{d}{dt}\left\{\int_P \Psi\, da + \int_{\mathscr{S}} \psi\, ds\right\} \leq W(P), \tag{28–2}$$

with working $W(P)$ given by (27–3)–(27–5). The difference

$$\Gamma(P) = W(P) - \frac{d}{dt}\left\{\int_P \Psi\, da + \int_{\mathscr{S}} \psi\, ds\right\} \geq 0 \tag{28–3}$$

then represents the **energy dissipated** in P, per unit time. Using (27–6) and the identity $\pi = \Psi$ (cf. (6–8)), the second law may be written intrinsically as

$$\frac{d}{dt}\left\{\int_P \Psi\, da + \int_{\mathscr{S}} \psi\, ds\right\} \leq \int_{\partial P} (\boldsymbol{S}\boldsymbol{n}\cdot\dot{\boldsymbol{y}} + \Psi U) ds + \int_P \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\, da + W_c(P). \tag{28–4}$$

b. The second law applied to crack control volumes

Consider a δ-pillbox $\mathscr{G}_\delta(t)$ about an arbitrary subsurface $\mathscr{G}(t)$ of the crack. Assume that $\boldsymbol{Z}(t) \notin \mathscr{G}$. Then, because ψ is independent of time, (11–25b), (25–24), and (25–25b) imply that

$$\frac{d}{dt}\left\{\int_{\mathscr{G}_\delta} \Psi\, da + \int_{\mathscr{G}} \psi\, ds\right\} \to \psi_B u_B - \psi_A u_A,$$

$$\int_{\partial\mathscr{G}_\delta} (\boldsymbol{S}\boldsymbol{n}\cdot\dot{\boldsymbol{y}} + \Psi U) ds + \int_{\mathscr{G}_\delta} \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\, da \to \int_{\mathscr{G}} [\boldsymbol{S}\mathbf{m}\cdot\dot{\boldsymbol{y}}] ds,$$

as $\delta \to 0$. Thus, by (27–4), (28–4) implies that

$$(\psi_B - \mathbf{t}_B\cdot\mathbf{c}_B)u_B - (\psi_A - \mathbf{t}_A\cdot\mathbf{c}_A)u_A \le \int_{\mathscr{G}} [\boldsymbol{S}\mathbf{m}\cdot\dot{\boldsymbol{y}}] ds,$$

and, because at any prescribed time u_A and u_B may be specified independently of $\mathscr{G}$ and of each other, this yields the *equivalence of surface tension and surface free energy*,

$$\mathbf{c}\cdot\mathbf{t} = \psi. \tag{28–5}$$

Further, because $\mathscr{G}$ is arbitrary, $[\boldsymbol{S}\mathbf{m}\cdot\dot{\boldsymbol{y}}] \ge 0$; therefore, as $[\boldsymbol{S}]\mathbf{m} = \mathbf{0}$,

$$\boldsymbol{S}^{\pm}\mathbf{m}\cdot[\dot{\boldsymbol{y}}] \ge 0.$$

If the crack faces are in contact, then

$$[\boldsymbol{y}]\cdot\mathbf{m} = 0, \qquad (\mathbf{t}\cdot\boldsymbol{S}^{\pm}\mathbf{m})\mathbf{t}\cdot[\dot{\boldsymbol{y}}] \ge 0,$$

and at these faces the working of the bulk shears over the tangential slip must be nonnegative. On the other hand, because external tractions on the crack faces are not considered, the crack surfaces when not in contact are traction-free:

$$[\boldsymbol{y}]\cdot\mathbf{m} > 0, \qquad \boldsymbol{S}^{\pm}\mathbf{m} = 0.$$

c. The second law applied to tip control volumes. Standard form of the second law

Consider (28–2) applied to the tip disc $D_\delta = D_\delta(t)$ with $\boldsymbol{v}$ the velocity field for ∂D_δ and (hence) $\overset{\square}{\boldsymbol{y}}$ the corresponding motion velocity following ∂D_δ (cf. the sentence following (26–8)). By (25–20) and (25–26),

$$\frac{d}{dt}\left\{\int_{D_\delta} \Psi\, da + \int_{\mathscr{C}\cap D_\delta} \psi\, ds\right\} \to 0$$

as $\delta \to 0$; thus (27–3) and (27–5) yield

$$\int_{\partial D_\delta} (\boldsymbol{Sn} \cdot \overset{\square}{\boldsymbol{y}} + \boldsymbol{Cn} \cdot \boldsymbol{v}) ds \tag{28–6}$$
$$+ \boldsymbol{b}_{\text{tip}} \cdot \bar{\boldsymbol{v}} + \boldsymbol{e}_{\text{tip}} \cdot \boldsymbol{v} - u_A \mathbf{t}_A \cdot \mathbf{c}_A + o(1) = \Gamma(D_\delta) \geq 0$$

(where the symbol $o(1)$ signifies a term that approaches zero with δ). Next, by (27–2a) and (27–12a), as $\delta \to 0$,

$$\int_{\partial D_\delta} \boldsymbol{Sn} \cdot \overset{\square}{\boldsymbol{y}}\, ds = \bar{\boldsymbol{v}} \cdot \int_{\partial D_\delta} \boldsymbol{Sn}\, ds + o(1) \to -\boldsymbol{b}_{\text{tip}} \cdot \bar{\boldsymbol{v}}, \tag{28–7}$$

while (27–12b) yields

$$\int_{\partial D_\delta} \boldsymbol{Cn} \cdot \boldsymbol{v}\, ds = \boldsymbol{v} \cdot \int_{\partial D_\delta} \boldsymbol{Cn}\, ds \to \boldsymbol{v} \cdot (\mathbf{c}_{\text{tip}} - \boldsymbol{g}_{\text{tip}} - \boldsymbol{e}_{\text{tip}}). \tag{28–8}$$

Thus, by (25–2) and (25–7), passing to the limit $\delta \to 0$ in (28–6) yields two important results:

$$\boldsymbol{g}_{\text{tip}} \cdot \boldsymbol{v} \leq 0, \tag{28–9}$$

which represents an **internal dissipation inequality** for the crack tip; and

$$\Gamma_{\text{tip}} = -\boldsymbol{g}_{\text{tip}} \cdot \boldsymbol{v}, \qquad \Gamma_{\text{tip}} = \lim_{\delta \to 0} \Gamma(D_\delta), \tag{28–10}$$

establishing $\boldsymbol{g}_{\text{tip}} \cdot \boldsymbol{v}$, and hence the breaking of bonds, as the sole source of dissipation at the tip.

By (27–5), (27–16b), (27–17b), and (28–5), the second law (28–4) for a tip control volume $P = P(t)$ may be written in the form

$$\frac{d}{dt} \left\{ \int_P (\Psi + k) da + \int_{\mathscr{S}} \psi\, ds \right\} \leq \int_{\partial P} \boldsymbol{Sn} \cdot \dot{\boldsymbol{y}}\, ds + \int_{\partial P} (\Psi + k) U\, ds - \psi_A u_A \tag{28–11}$$

and $\Gamma(P)$, defined by (28–3), may be expressed as the right side of (28–11) minus the left. Since $\int_{\partial P} (\Psi + k) U\, ds$ and $-\psi_A u_A$, respectively, represent net flows of bulk and surface energy into P across ∂P, (28–11) is consistent with more standard views concerning the formulation of basic laws for control volumes. (For a crack control volume there would be an additional term $\psi_B u_B$ on the right side of (28–11).

By (28–11) with $P = D_\delta$, (25–20) with $\Phi = \Psi + k$, which is regular, (25–25a) with $\varphi = \psi$, and the sentence containing (28–11),

$$\oint_{\text{tip}} \{\boldsymbol{Sn} \cdot \dot{\boldsymbol{y}} + (\Psi + k)(\boldsymbol{v} \cdot \boldsymbol{n})\}\, ds - \psi_{\text{tip}} V = \Gamma_{\text{tip}}, \tag{28–12}$$

which represents an energy balance for the crack tip. (The tip integral in (28–12) exists because the other limits resulting in (28–12) exist.)

d. Tip traction. Energy release rate. Driving force

Let $\mathbf{t}(t) = \mathbf{t}_{\text{tip}}(t) = \mathbf{t}(\boldsymbol{Z}(t))$. The following quantities are essential to our discussion:

$$\boldsymbol{j} = \oint_{\text{tip}} \left\{(\Psi + k_{\text{rel}})\mathbf{1} - \boldsymbol{F}^{\top}\boldsymbol{S}\right\}\boldsymbol{n}\, ds \qquad \textbf{tip traction}, \tag{28–13a}$$

$$J = \mathbf{t}\cdot\boldsymbol{j} = \mathbf{t}\cdot\oint_{\text{tip}} \left\{(\Psi + k_{\text{rel}})\mathbf{1} - \boldsymbol{F}^{\top}\boldsymbol{S}\right\}\boldsymbol{n}\, ds \quad \textbf{energy release rate}, \tag{28–13b}$$

$$f = J - \psi_{\text{tip}} \qquad \textbf{driving force}. \tag{28–13c}$$

I now discuss these definitions in more detail.

The vector $\boldsymbol{j}$ represents the configurational traction $\oint_{\text{tip}} \boldsymbol{C}\boldsymbol{n}\, ds$ on the material in an infinitesimal neighborhood of the tip, augmented by the "inertial traction" $\oint_{\text{tip}} k_{\text{rel}}\boldsymbol{n}\, ds$.

With a view toward discussing f and J, assume for the remainder of this section that the crack is growing:

$$\boldsymbol{v} = V\mathbf{t}, \qquad V > 0. \tag{28–14}$$

Since $\mathbf{t}\cdot\mathbf{c}_{\text{tip}} = \psi_{\text{tip}}$, (27–18c) divided by V, (28–5), and (28–13c) imply that

$$f = \mathbf{t}\cdot\left[\oint_{\text{tip}} \left\{\Psi\mathbf{1} - \boldsymbol{F}^{\top}\boldsymbol{S}\right\}\boldsymbol{n}\, ds - \mathbf{c}_{\text{tip}} + \boldsymbol{e}_{\text{tip}}\right]. \tag{28–15}$$

The stress $\boldsymbol{C} = \Psi\mathbf{1} - \boldsymbol{F}^{\top}\boldsymbol{S}$, the surface stress $\mathbf{c}$, and the inertial force $\boldsymbol{e}_{\text{tip}}$ give rise to a net *noninternal* configurational force on the material in an infinitesimal neighborhood of the tip; f represents the component of that force in the direction of propagation. Using the Eshelby relation (6–9), the configurational balance (27–12b) may be written as

$$\oint_{\text{tip}} \left\{\Psi\mathbf{1} - \boldsymbol{F}^{\top}\boldsymbol{S}\right\}\boldsymbol{n}\, ds - \mathbf{c}_{\text{tip}} + \boldsymbol{g}_{\text{tip}} + \boldsymbol{e}_{\text{tip}} = \mathbf{0}. \tag{28–16}$$

The last two relations and (28–13c) yield the **tangential configurational balance**

$$J - \psi_{\text{tip}} + \mathbf{t}\cdot\boldsymbol{g}_{\text{tip}} = 0, \tag{28–17}$$

or equivalently,

$$f = -\mathbf{t}\cdot\boldsymbol{g}_{\text{tip}}, \tag{28–18}$$

a balance between the driving force f and $-\mathbf{t}\cdot\boldsymbol{g}_{\text{tip}}$, the internal force that opposes motion of the tip. By (28–9) and (28–10),

$$\Gamma_{\text{tip}} = fV \geq 0, \tag{28–19}$$

and f is conjugate to the scalar velocity V.

By (28–12), (28–19), and (28–13c),

$$J = V^{-1}\oint_{\text{tip}} \{\boldsymbol{S}\boldsymbol{n}\cdot\dot{\boldsymbol{y}} + (\Psi + k)(\boldsymbol{v}\cdot\boldsymbol{n})\}\, ds; \tag{28–20}$$

JV therefore represents the working on—and bulk energy flow into—an infinitesimal neighborhood of the tip; J itself measures this quantity per unit crack length rather than per unit time. Also, by (28–13b), J is the component of $\boldsymbol{j}$ in the direction of propagation; for $k_{\text{rel}} = 0$, J is the limiting value of the Eshelby-Rice integral.[1]

Consequences of the tangential balance (28–18) and the second law, as manifested by the internal dissipation inequality (28–9), are the following necessary conditions for crack growth:

(i) *the driving force must be non-negative*,

$$f \geq 0; \tag{28–21}$$

(ii) *the tip traction must form an acute angle with the direction of propagation*,

$$\mathbf{t} \cdot \boldsymbol{j} \geq \psi_{\text{tip}} > 0. \tag{28–22}$$

The *Griffith criterion* asserts that a crack will run when and only when $\mathbf{t} \cdot \boldsymbol{j} > \psi_{\text{tip}}$ and hence whenever (28–21) is satisfied strictly. Within the current framework (28–21) represents only a necessary condition for crack propagation; in fact for the class of constitutive equations to be considered, (28–21) may be satisfied strictly without motion of the tip. The results (28–21) and (28–22) are, however, independent of constitutive assumptions.

Remark. For a straight crack ($\mathbf{t} =$ constant) in a homogeneous elastic body, neglecting inertia, with the crack faces traction-free ($\boldsymbol{S}^{\pm}\mathbf{m} = \mathbf{0}$), the energy release rate may be computed via an integration along a path away from the tip. Let $\mathscr{Y} = \mathscr{Y}(t)$ denote any smooth, closed, nonintersecting path that begins and ends on the crack and surrounds the tip; let $\boldsymbol{n}$ denote the outward unit normal to $\mathscr{Y}$; let

$$\mathscr{J}(\mathscr{Y}) = \mathbf{t} \cdot \int_{\mathscr{Y}} (\Psi \mathbf{1} - \boldsymbol{F}^{\top}\boldsymbol{S})\boldsymbol{n}\, ds;$$

bear in mind that $\operatorname{Div}\boldsymbol{C} = \mathbf{0}$, because $\boldsymbol{g} = \boldsymbol{e} = \mathbf{0}$ (cf. (9–3), (9–11)), and that $\mathbf{t} \cdot [\boldsymbol{C}]\mathbf{m} = \Psi\, \mathbf{t} \cdot \mathbf{m} = 0$. Apply the tensorial version of (25–13b) to $\operatorname{Div}\boldsymbol{C}$ with P the region P_δ between $\mathscr{Y}$ and ∂D_δ and then pass to the limit as $\delta \to 0$; the result is:[2]

$$J = \mathscr{J}(\mathscr{Y}) \qquad \text{for any choice of path } \mathscr{Y}.$$

e. The standard momentum condition

The quantity $\boldsymbol{b}_{\text{tip}} = \oint_{\text{tip}} \boldsymbol{p}(\boldsymbol{v}\cdot\boldsymbol{n})ds = [\oint_{\text{tip}} \boldsymbol{p}\otimes\boldsymbol{n}\, ds]\boldsymbol{v}$ represents the momentum flow into an infinitesimal neighborhood of the crack tip. Theories of crack propagation

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[1] The notion of an energy release rate was introduced by Atkinson and Eshelby [1968] and justified by Freund [1972]; cf. Freund [1990, pp. 221–235]. Within the framework of quasistatic elasticity the basic ideas are inherent in the work of Eshelby [1956] and Rice [1968]; there the energy release rate coincides with the path-independent J-integral.

[2] Cf. Eshelby [1956], Rice [1968].

for specific materials are generally consistent with the hypothesis[3]

$$\oint_{\text{tip}} \boldsymbol{p} \otimes \boldsymbol{n}\, ds = 0, \tag{28–23}$$

which I will refer to as the **standard momentum condition**. Granted (28–23), $\boldsymbol{b}_{\text{tip}} = \boldsymbol{0}$, so that, by (27–12a),

$$\oint_{\text{tip}} \boldsymbol{S}\boldsymbol{n}\, ds = \boldsymbol{0}. \tag{28–24}$$

Further, by (28–23), $\oint_{\text{tip}} (\boldsymbol{p} \cdot \bar{\boldsymbol{v}})\boldsymbol{n}\, ds = \boldsymbol{0}$; thus, since $k_{\text{rel}} = \frac{1}{2}\,\rho|\dot{\boldsymbol{y}} - \bar{\boldsymbol{v}}|^2$, (27–20) yields

$$\oint_{\text{tip}} k_{\text{rel}}\boldsymbol{n}\, ds = \oint_{\text{tip}} k\boldsymbol{n}\, ds. \tag{28–25}$$

The importance of (28–25) is that it results in relations

$$\boldsymbol{j} = \oint_{\text{tip}} \left\{(\Psi + k)\boldsymbol{1} - \boldsymbol{F}^{\top}\boldsymbol{S}\right\}\boldsymbol{n}\, ds, \tag{28–26a}$$

$$J = \mathbf{t} \cdot \boldsymbol{j} = \mathbf{t} \cdot \oint_{\text{tip}} \left\{(\Psi + k)\boldsymbol{1} - \boldsymbol{F}^{\top}\boldsymbol{S}\right\}\boldsymbol{n}\, ds, \tag{28–26b}$$

in which the tip traction $\boldsymbol{j}$ does not depend explicitly on the speed or direction of the crack. (Without the standard momentum condition, $\boldsymbol{j}$ and hence J are dependent on k_{rel} and hence, by (26–4) and (26–5), on $\boldsymbol{v}$.)

It is important to bear in mind that the dependence of J on k actually represents a dependence on k_{rel}, with the reduction of k_{rel} to k a consequence of the standard momentum condition. Here (and throughout the literature) the energy release rate J plays an essential role in the constitutive theory for the tip, and while k_{rel} is invariant under Galilean changes in spatial observer, k is not.

By (26–7), for a *stationary crack* the standard momentum condition is satisfied automatically; in fact, $\oint_{\text{tip}} k\boldsymbol{n}\, ds = \oint_{\text{tip}} k_{\text{rel}}\boldsymbol{n}\, ds = \boldsymbol{0}$, so that

$$\boldsymbol{j} = \oint_{\text{tip}} \left\{\Psi\boldsymbol{1} - \boldsymbol{F}^{\top}\boldsymbol{S}\right\}\boldsymbol{n}\, ds, \qquad J = \mathbf{t} \cdot \oint_{\text{tip}} \left\{\Psi\boldsymbol{1} - \boldsymbol{F}^{\top}\boldsymbol{S}\right\}\boldsymbol{n}\, ds. \tag{28–27}$$

[3]This is satisfied when $\dot{\boldsymbol{y}} = O(r^{-p})$, $p < 1$. In linear elasticity, $p = \frac{1}{2}$ (cf., e.g., Freund [1990, §1.4.3, §4]).

CHAPTER 29

Basic Results for the Crack Tip

The basic equations for the crack tip consist of the *standard force balance*

$$\oint_{\text{tip}} \boldsymbol{S}\boldsymbol{n}\, ds + \boldsymbol{b}_{\text{tip}} = \boldsymbol{0} \tag{29–1}$$

(or $\oint_{\text{tip}} \boldsymbol{S}\boldsymbol{n}\, ds = \boldsymbol{0}$, granted the standard momentum condition) and the *tangential configurational balance*

$$f + \mathbf{t} \cdot \boldsymbol{g}_{\text{tip}} = 0 \tag{29–2}$$

(cf. (28–18)). These balances are supplemented by a relation

$$\mathbf{t} \cdot \mathbf{c}_{\text{tip}} = \psi_{\text{tip}} > 0 \tag{29–3}$$

establishing the equivalence of surface tension and surface free energy, and an *internal dissipation inequality*

$$\mathbf{t} \cdot \boldsymbol{g}_{\text{tip}} \leq 0 \qquad \text{for } V > 0, \tag{29–4}$$

which is the second law localized to the crack tip; in this regard,

$$\Gamma_{\text{tip}} = -(\mathbf{t} \cdot \boldsymbol{g}_{\text{tip}})V = fV \tag{29–5}$$

represents the energy dissipated at the tip, per unit time.

The surface shear $\mathbf{m} \cdot \mathbf{c}$ and the normal internal force $\mathbf{m} \cdot \boldsymbol{g}_{\text{tip}}$ perform no work, because there is no motion of the crack normal to itself; and the internal configurational forces $\boldsymbol{g}$ and $\boldsymbol{g}^{\mathscr{C}}$ perform no work, because structural changes in the material occur only at the tip. That is why these forces are considered *indeterminate* and the balances (5–10), (27–14b), and the normal part of (28–16) are viewed as equations for $\boldsymbol{g}$, $\boldsymbol{g}^{\mathscr{C}}$, and $\mathbf{m} \cdot (\mathbf{c}_{\text{tip}} - \boldsymbol{g}_{\text{tip}})$. (Contrast this to a phase interface, whose migration results in the working of internal configurational forces distributed over it and surface stresses acting within it (cf. Parts B and E).) On the other hand, the surface tension $\mathbf{t} \cdot \mathbf{c}$

and the tangential force $\mathbf{t} \cdot \boldsymbol{g}_{\text{tip}}$ perform work, but only when the crack tip advances. Configurational forces therefore play no role away from the tip,[1] while at the tip the sole operative forces are those involved in the tangential part of the balance (28–17), namely, J, $\mathbf{t} \cdot \mathbf{c}_{\text{tip}} = \psi_{\text{tip}}$, and $\mathbf{t} \cdot \boldsymbol{g}_{\text{tip}}$. Bulk constitutive equations for Ψ and $\boldsymbol{S}$ yield, via (28–13b), an auxiliary constitutive specficiation for J; the next chapter will discuss constitutive equations for both ψ_{tip} and the internal force $\mathbf{t} \cdot \boldsymbol{g}_{\text{tip}}$.

It is important to differentiate among the roles played by the surface energy ψ_{tip}, the energy release-rate J, and the tangential component $\mathbf{t} \cdot \boldsymbol{g}_{\text{tip}}$ of the internal configurational force. Throughout the literature one finds constitutive prescriptions for J, but no configurational force balance. The view here[2] is that $\mathbf{t} \cdot \boldsymbol{g}_{\text{tip}}$ and ψ_{tip} are constitutive, with J a defined quantity related to $\mathbf{t} \cdot \boldsymbol{g}_{\text{tip}}$ and ψ_{tip} through the tangential configurational balance (28–17). J is typically represented by bulk quantities that already have constitutive prescriptions; to write an *additional* constitutive equation for J would seem inappropriate. In the theory described here the configurational force balance provides a quantity $\boldsymbol{g}_{\text{tip}}$ with tangential component available for constitutive prescription. The physical consistency of this view is underlined by the fact that the second law yields the single inequality $\Gamma_{\text{tip}} = -V\mathbf{t} \cdot \boldsymbol{g}_{\text{tip}} \geq 0$, involving the same variable, whose satisfaction indicates the need for additional constitutive assumptions involving V and $\mathbf{t} \cdot \boldsymbol{g}_{\text{tip}}$. In short, the prescription of a constitutive equation for J masks:

(i) the presence of a fundamental balance law, the configurational force balance;
(ii) the existence of a physically significant quantity, the internal configurational force, which acts at the tip, with tangential component $\mathbf{t} \cdot \boldsymbol{g}_{\text{tip}}$ a direct response to the breaking of bonds during fracture.

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[1] Maugin [1993, eq. (7–61)] prescribes a condition for the *free surfaces* of the crack that in two dimensions takes the form $\mathbf{m} \cdot [\boldsymbol{C}]\mathbf{m} = -\sigma K$ (σ = constant surface tension, K = curvature). I believe this condition to be erroneous; I believe that when written correctly this condition represents the normal part of (27–14b), viz. $\mathbf{m} \cdot [\boldsymbol{C}]\mathbf{m} + \sigma K = -\mathbf{m} \cdot \boldsymbol{g}^{\mathscr{C}}$, and is satisfied trivially, because $\boldsymbol{g}^{\mathscr{C}}$ is *indeterminate*.

[2] Gurtin and Podio-Guidugli [1997].

CHAPTER 30

Constitutive Theory for Growing Cracks

This chapter and the next are based on the assumption that the standard momentum condition is satisfied.

a. Constitutive relations at the tip

It is convenient to characterize the direction of propagation $\mathbf{t}$ by its counterclockwise angle ϑ from the (1, 0) axis:

$$\mathbf{t} = \mathbf{t}(\vartheta) = (\cos\vartheta, \sin\vartheta), \qquad \mathbf{m} = \mathbf{m}(\vartheta) = (-\sin\vartheta, \cos\vartheta). \tag{30–1}$$

Two constitutive relations are considered for the tip: the first gives the surface energy ψ_{tip} as a smooth function of ϑ,

$$\psi_{\text{tip}} = \hat{\psi}(\vartheta), \tag{30–2}$$

with

$$\hat{\psi}(\vartheta) > 0; \tag{30–3}$$

the second gives the speed V as a function of ϑ and the component $-\mathbf{t}\cdot\boldsymbol{g}_{\text{tip}}$ of the internal configurational force opposing propagation. The force $-\mathbf{t}\cdot\boldsymbol{g}_{\text{tip}}$ represents a response to the breaking of bonds at the tip, and it seems reasonable to suppose that propagation is possible only when this force is sufficiently large. For notational convenience, the balance law (29–2) is used to write the constitutive equation for V in terms of ϑ and the driving force $f(= -\mathbf{t}\cdot\boldsymbol{g}_{\text{tip}})$. I therefore consider, for V, a constitutive equation consisting of two parts: a **fracture limit**

$$V = 0 \qquad \text{for } f \le F(\vartheta), \tag{30–4}$$

with

$$F(\vartheta) > 0 \tag{30–5}$$

the **limit force**; and a **kinetic equation**

$$V = \hat{V}(\vartheta, f) > 0 \qquad \text{for } f > F(\vartheta), \tag{30–6}$$

where $\lim_{f \to F(\vartheta)} \hat{V}(\vartheta, f) = 0$, where $\hat{V}(\vartheta, f)$ is smooth up to $f = F(\vartheta)$, and where

$$\partial_f \hat{V}(f, \vartheta) \geq 0 \qquad \text{for } f > F. \tag{30–7}$$

The constitutive assumptions (30–4)–(30–7) ensure that $V \geq 0$, that V is a nondecreasing function of f, and, most important, that the dissipation inequality (29–4) is satisfied.

Materials scientists often model grain boundaries, phase boundaries, and free surfaces as sharp surfaces endowed with energy densities dependent on surface orientation.[1] Within the current framework, surface physics of this type is characterized by the constitutive function $\hat{\psi}(\vartheta)$ for the free energy. For a *phase boundary*, an energy of the form $\hat{\psi}(\vartheta)$ gives rise to a vector surface-stress $\mathbf{c}$, which, as a result of thermomechanical arguments (cf. (19–25)), has the form

$$\mathbf{c} = \hat{\psi}(\vartheta)\mathbf{t}(\vartheta) + \hat{\psi}'(\vartheta)\mathbf{m}(\vartheta), \tag{30–8}$$

with $\hat{\psi}(\vartheta)$ the surface tension and $\hat{\psi}'(\vartheta)$ the *surface shear*. Here, unlike phase boundaries, the sole kinetics associated with the crack surfaces is that associated with the tip, a "constraint" that allows for $\mathbf{c} \cdot \mathbf{t} = \hat{\psi}(\vartheta)$ but renders the surface shear $\mathbf{c} \cdot \mathbf{m}$ indeterminate (cf. the remark in petite type following (29–5)).

b. The Griffith-Irwin function

The energy release rate is henceforth considered as a function

$$J(\boldsymbol{j}, \vartheta) = \mathbf{t}(\vartheta) \cdot \boldsymbol{j} \tag{30–9}$$

of the angle of propagation ϑ and the tip traction $\boldsymbol{j}$. For a body characterized by (30–2)–(30–7), crack propagation occurs when and only when $f > F(\vartheta)$, or equivalently, by (28–13c),

$$J(\boldsymbol{j}, \vartheta) > \gamma(\vartheta), \tag{30–10}$$

with

$$\gamma(\vartheta) = \hat{\psi}(\vartheta) + F(\vartheta) > 0. \tag{30–11}$$

The function $\gamma(\vartheta)$ will be referred to as the **Griffith-Irwin function**.

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[1] Cf. Herring [1951a,b], Frank [1963], Gjostein [1963].

The Griffith criterion, $J(\boldsymbol{j}, \vartheta) > \hat{\psi}(\vartheta)$, represents only a conservative estimate for propagation. That surface energy is not the sole limiting factor to crack initiation was noted by Irwin,[2] who proposed that ψ be augmented by a quantity γ_p representing the "plastic work dissipated in the surrounding material per unit surface area created."[3] The role of γ_p is here played by F.

Crack propagation is related to the convexity of $\gamma(\vartheta)$. Note that, if $\hat{\psi}(\vartheta)$ and $F(\vartheta)$ are angle-convex, then so is $\gamma(\vartheta)$, an assertion that follows from the equivalence of angle convexity and the condition expressed by (19–37).

c. Constitutively isotropic crack tips. Tips with constant mobility

The crack tip will be referred to as **constitutively isotropic** if $\hat{\psi}(\vartheta)$, $\hat{V}(\vartheta, f)$, and $F(\vartheta)$ are independent of the orientation ϑ of the crack. Granted this, the constitutive equations (30–2)–(30–7) become

$$\psi_{\text{tip}} = \text{constant} > 0, \tag{30–12a}$$

$$V = 0 \quad \text{for } f \leq F, \qquad V = \hat{V}(f) > 0 \quad \text{for } f > F, \tag{30–12b}$$

$$\hat{V}'(f) \geq 0 \quad \text{for } f > F, \qquad F = \text{constant} > 0, \tag{30–12c}$$

so that γ is constant and hence angle-convex.

A simple constitutive assumption, but one that accounts for anisotropy in the free energy and fracture limit, is that of a **crack tip with constant mobility**. Here the constitutive equations are (30–2)–(30–7) with the kinetic relation (30–6) in the specific form

$$\hat{V}(\vartheta, f) = M[f - F(\vartheta)] \qquad \text{for } f > F(\vartheta), \tag{30–13}$$

where

$$M > 0 \tag{30–14}$$

is a constitutive constant that represents the **mobility** of the tip.

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[2][1948].

[3]Cf. Freund [1990, pp. 8–10], from whom the quote is taken, and who gives a complete discussion of these ideas with relevant references.

CHAPTER 31

Kinking and Curving of Cracks. Maximum Dissipation Criterion

The discussion of kinking will use local results derived for smooth cracks, but these results will be applied only on the smooth portions of the crack.

Consider a *stationary* crack and a program of continuously increasing loads. Let ϑ^- denote the angle of the tangent $\mathbf{t}(\vartheta^-)$ at the tip.[1] In certain circumstances one might expect crack propagation to initiate at an angle ϑ^+ different from ϑ^-, indicating an initial *kink*. Once the crack has begun to run, the surface energy and speed V are given by (30–2) and (30–6) subject to (30–3), (30–5), and (30–7); however an additional constitutive relation, specifying the direction of propagation, is needed. This relation is derived under the assumption that the crack propagates in a direction that maximizes the rate at which it dissipates energy.

The following terminology is useful: given a stationary crack, *crack initiation* indicates the onset of a *running crack*, and to emphasize the possibility of kinking, the term *kink angle* is used for the angle ϑ^+ immediately after initiation, a phrase not meant to rule out the case $\vartheta^+ = \theta^-$.

I assume that the standard momentum condition is satisfied and, without loss in generality, that $J(\boldsymbol{j}, \vartheta) > 0$ (cf. (28–21)).

[1] This conflicts with the notation $\Phi^{\pm}$ for the limiting values of a bulk field Φ at the crack. Because ϑ is not a bulk field and the remainder of this chapter is restricted to the tip, this ambiguity should not cause confusion.

a. Criterion for crack initiation. Kink angle

By (30–2) and (30–9), the driving force (28–13c) may be considered a function of ϑ and the tip traction $\boldsymbol{j}$;

$$f = f(\boldsymbol{j}, \vartheta) = J(\boldsymbol{j}, \vartheta) - \hat{\psi}(\vartheta). \tag{31–1}$$

Attention is henceforth restricted to situations in which $\boldsymbol{j}$ is a continuous function of time at the instant of crack initiation, whether or not the crack develops an initial kink.[2] This stipulation renders the tip traction $\boldsymbol{j}$ a useful parameter for describing the loading in an arbitrarily small neighborhood of the crack tip. Also, because (30–4) and (30–6) are presumed to describe the dynamics of the crack, ϑ before initiation must be consistent with $J(\boldsymbol{j}, \vartheta) \leq \gamma(\vartheta)$, while ϑ after initiation must satisfy $J(\boldsymbol{j}, \vartheta) > \gamma(\vartheta)$. With this in mind, the tip traction $\boldsymbol{j}$ (or, more simply, the loading) is referred to as:

(i) **subcritical** if $J(\boldsymbol{j}, \vartheta) < \gamma(\vartheta)$ for all ϑ;
(ii) **critical** if $J(\boldsymbol{j}, \vartheta) \leq \gamma(\vartheta)$ for all ϑ, but $J(\boldsymbol{j}, \vartheta) = \gamma(\vartheta)$ for some ϑ (so that the loading is not subcritical);
(iii) **supercritical** if, for some ϑ, $J(\boldsymbol{j}, \vartheta) > \gamma(\vartheta)$.

Subcritical loading is then a necessary condition for a crack to remain stationary; supercritical loading is a necessary condition for crack propagation; and critical loading is a necessary condition for crack initiation. The next theorem, a direct consequence of (19–40) and the foregoing definitions, shows the intimate relation between these conditions and the geometry of the Frank diagrams of J and γ. In this regard, recall that, by Lemma 19.1(iii), there is a one-to-one correspondence between $\boldsymbol{j}$ and Frank(J).

Criticality Theorem. The tip traction $\boldsymbol{j}$ is:

(i) subcritical if and only if Frank(J) does not intersect Frank(γ);
(ii) critical if and only if Frank(J) is a convexifying tangent to Frank(γ);
(iii) supercritical if and only if Frank(J) intersects the region interior to Frank(γ).

If the loading is critical, then those angles ϑ^+ that satisfy $J(\boldsymbol{j}, \vartheta^+) = \gamma(\vartheta^+)$ will be referred to as **possible kink angles**, because, by (30–4) and (30–6), such angles mark the transition between $f < F(\vartheta)$ and $f > F(\vartheta)$ and hence between $V = 0$ and $V > 0$.

Initiation Theorem. Assume that the tip traction $\boldsymbol{j}$ is critical. Then ϑ^+ is a possible kink angle if and only if ϑ^+ is a tangency angle of the convexifying tangent Frank(J) to Frank(γ). Granted this, $\boldsymbol{j}$ is related to ϑ^+ through

$$\boldsymbol{j} = \gamma(\vartheta^+)\mathbf{t}(\vartheta^+) + \gamma'(\vartheta^+)\mathbf{m}(\vartheta^+). \tag{31–2}$$

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[2] Continuity of $\boldsymbol{j}$ may *preclude* the possibility of a kink. I am unaware of any rigorous result related to this important issue, even for isotropic, linearly elastic, antiplane shear, where the underlying partial differential equation is Laplace's equation.

Conversely, if for some convexifying tangent $\mathscr{L}$ to $\gamma(\vartheta)$ and some tangency angle ϑ^+ of $\mathscr{L}$, (31–2) is satisfied, then $\boldsymbol{j}$ is critical.

PROOF. Assume that $\boldsymbol{j}$ is critical. Then, because the set of all possible kink angles ϑ^+ coincides with the set of ϑ^+ that satisfy $J(\boldsymbol{j}, \vartheta^+) = \gamma(\vartheta^+)$, (ii) of the Criticality Theorem and (19–40a) imply that ϑ^+ is a possible kink angle if and only if ϑ^+ is a tangency angle of Frank(J). Granted this, (31–2) follows from Lemma 19.2(ii).

Conversely, if for some convexifying tangent $\mathscr{L}$ to Frank(γ) and some tangency angle ϑ^+ of $\mathscr{L}$, (31–2) is satisfied, then Lemma 19.2(ii) (and the tacit smoothness of γ) imply that $\mathscr{L} =$ Frank(J); the criticality of $\boldsymbol{j}$ then follows from (ii) of the Criticality Theorem. □

Corollaries. Assume that $\boldsymbol{j}$ is critical.

(i) If $\gamma(\vartheta)$ is angle-convex, then there is at most one possible kink angle.

(ii) For a constitutively isotropic tip there is at most one kink angle ϑ^+, and

$$\mathbf{t}(\vartheta^+) = \boldsymbol{j}/|\boldsymbol{j}|, \tag{31–3}$$

so that the direction of the kink coincides with the direction of $\boldsymbol{j}$.

(iii) If the tip is constitutively anisotropic and ϑ^+ is a possible kink angle, then

$$\mathbf{m}(\vartheta^+) \cdot \boldsymbol{j}^+ = \gamma'(\vartheta^+), \tag{31–4}$$

so that, for $\gamma'(\vartheta) \neq 0$, the direction of propagation corresponding to ϑ^+ is not parallel to $\boldsymbol{j}$.

PROOF. The result (i) is a consequence of the Initiation Theorem and the fact that for $\gamma(\vartheta)$ angle-convex, each tangent to Frank(γ) intersects Frank(γ) at exactly one point. For an isotropic tip, γ is constant and hence angle-convex, so that the kink angle is uniquely determined by $\boldsymbol{j}$. Further, (31–2) with $\gamma =$ constant implies (31–3). For an anisotropic tip, (31–4) follows from (31–2). □

Interestingly, the formulas (30–8) for $\mathbf{c}$ and (31–2) for $\boldsymbol{j}$ are identical, granted the replacements

$$\mathbf{c} \text{ and } \hat{\psi}(\vartheta) \to \boldsymbol{j}^+ \text{ and } \gamma(\vartheta^+). \tag{31–5}$$

The right side of (31–2) thus has the form of a "surface stress," with the Griffith-Irwin function $\gamma(\vartheta)$ playing the role of the free energy; in particular, $\gamma(\vartheta)$ and $\gamma'(\vartheta)$ represent analogs of the surface tension and surface shear, and the relation (31–2) represents a balance between this surface stress and the tip traction.

The results of this section give a geometric picture (Figure 31.1) of the qualitative aspects of the fracture process. A stationary crack will remain stationary as long as $\boldsymbol{j}$ is such that the line Frank(J) remains strictly outside the closed curve Frank(γ). Initiation of a running crack begins at a time for which Frank(J) passes across Frank(γ) with a portion of Frank(J) entering the open region Λ, say, interior to Frank(γ), and the crack will continue to run as long as a portion of Frank(J) remains within Λ. At the time of initiation, Frank(J) touches Frank(γ) but has no intersection with Λ; hence Frank(J) must be a convexifying tangent, and the possible kink angles are those angles that mark the intersection of Frank(γ) with Frank(J).

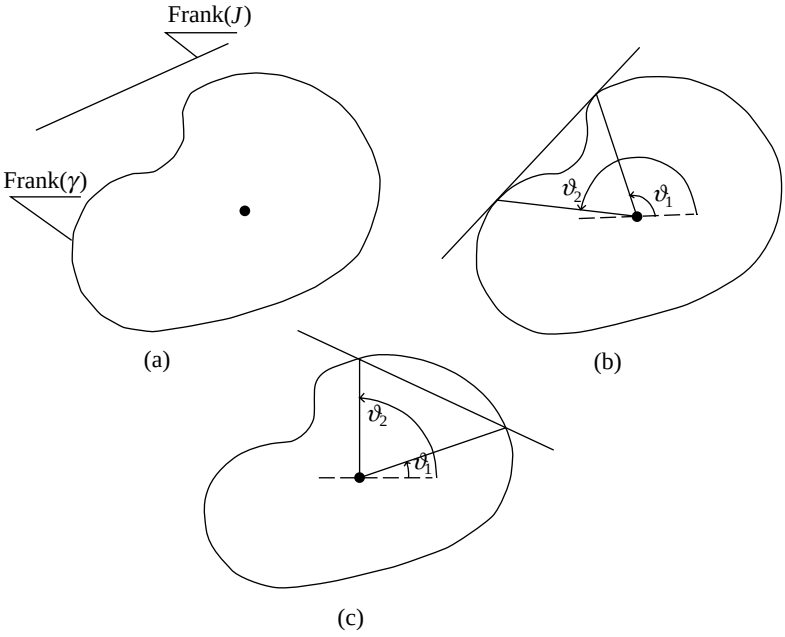

FIGURE 31.1. Frank diagrams of γ and J: (a) for a stationary crack; (b) for possible initiation of a running crack (ϑ_1 and ϑ_2 are possible kink angles); (c) for a running crack (the angle describing the direction of propagation lies between ϑ_1 and ϑ_2).

b. Maximum dissipation criterion for crack propagation

I now restrict attention to a running crack. Then, by (30–6) and (31–1), the dissipation rate (29–5) may be considered a function of the tip traction $\boldsymbol{j}$ and the angle ϑ at which the crack advances:

$$\Gamma_{\text{tip}} = \Gamma_{\text{tip}}(\boldsymbol{j}, \vartheta) = f\hat{V}(\vartheta, f), \qquad f = J(\boldsymbol{j}, \vartheta) - \hat{\psi}(\vartheta). \tag{31–6}$$

A major hypothesis of the theory is that at each time t the angle $\vartheta = \vartheta(t)$ satisfy the **maximum dissipation criterion**:

$$\Gamma_{\text{tip}}(\boldsymbol{j}, \vartheta) = \max_{\alpha \in \mathscr{G}(\boldsymbol{j})} \Gamma_{\text{tip}}(\boldsymbol{j}, \alpha)\,, \tag{31–7a}$$

$$\mathscr{G}(\boldsymbol{j}) = \{\text{set of angles } \vartheta \text{ such that } J(\boldsymbol{j}, \vartheta) > \gamma(\vartheta)\} \tag{31–7b}$$

Then, because

$$\boldsymbol{j} \text{ is necessarily supercritical,} \tag{31–8}$$

the set $\mathscr{G}(\boldsymbol{j})$ is a nonempty open set whose boundary consists of angles ϑ that satisfy $J(\boldsymbol{j}, \vartheta) = \gamma(\vartheta)$; furthermore,

$$\Gamma_{\text{tip}}(\boldsymbol{j}, \alpha) > 0 \quad \text{for } \alpha \in \mathscr{G}(\boldsymbol{j}), \qquad \Gamma_{\text{tip}}(\boldsymbol{j}, \alpha) = 0 \quad \text{for } \alpha \in \partial\mathscr{G}(\boldsymbol{j}). \tag{31–9}$$

Granted smoothness, the maximum problem (31–7a) *has a solution*, and any such solution ϑ must satisfy

$$\partial\Gamma_{\text{tip}}(\boldsymbol{j},\vartheta)/\partial\vartheta = 0. \tag{31–10}$$

It is important to note that *the maximum dissipation criterion may not define a unique angle of propagation for a given value of* $\boldsymbol{j}$. The next lemma will be useful in determining conditions under which this angle is unique.

Lemma 31.1.

(i) *Assume that* $\gamma(\vartheta)$ *is angle-convex. Then* $\mathcal{G}(\boldsymbol{j})$ *is connected.*
(ii) *Assume that* $\mathcal{G}(\boldsymbol{j})$ *is connected. If*

$$\partial^2\Gamma_{\text{tip}}(\boldsymbol{j},\vartheta)/\partial\vartheta^2 < 0 \qquad \textit{for all solutions } \vartheta \in \mathcal{G}(\boldsymbol{j}) \textit{ of } (31\text{–}10), \tag{31–11}$$

then (31–7a) *has a unique solution.*

PROOF. Let $J(\vartheta) = J(\boldsymbol{j},\vartheta) = \boldsymbol{j}\cdot\mathbf{t}(\vartheta)$. To establish (i) it suffices to show that $\partial\mathcal{G}(\boldsymbol{j})$ contains *exactly* two angles. $\partial\mathcal{G}(\boldsymbol{j})$ consists of angles ϑ such that $\gamma(\vartheta) = J(\vartheta)$; thus, by (19–40a) $\Omega := \text{Frank}(\gamma)\cap\text{Frank}(J)$ contains a point $\boldsymbol{x}$ for each such angle. Since $\gamma(\vartheta)$ is angle-convex, Frank(γ) is strictly convex; thus, because Frank(J) is a straight line, Ω contains at most two points, so that $\partial\mathcal{G}(\boldsymbol{j})$ contains at most two angles. But $\mathcal{G}(\boldsymbol{j})$ is open; thus $\partial\mathcal{G}(\boldsymbol{j})$ contains exactly two angles.

Assume that $\mathcal{G}(\boldsymbol{j})$ is connected. If $\Gamma_{\text{tip}}(\boldsymbol{j},\vartheta)$ were to have more than one maximum on $\mathcal{G}(\boldsymbol{j})$, then it would also have a minimum on $\mathcal{G}(\boldsymbol{j})$, and this would violate (31–11). Thus (ii) is valid. □

It is possible to obtain specific results for the direction of a running crack when the material is one of the specific types discussed in Section 30c.

Theorem on the Direction of a Running Crack.

(i) For a constitutively isotropic tip the crack will propagate in the direction of the tip traction $\boldsymbol{j}$:

$$\mathbf{t}(\vartheta) = \boldsymbol{j}/|\boldsymbol{j}|. \tag{31–12}$$

(ii) For a tip with constant mobility there is at least one angle ϑ at which the crack will propagate, and any such ϑ will satisfy the identity

$$\{2f(\boldsymbol{j},\vartheta) - F(\vartheta)\}\{\mathbf{m}(\vartheta)\cdot\boldsymbol{j} - \hat{\psi}'(\vartheta)\} = f(\boldsymbol{j},\vartheta)F'(\vartheta). \tag{31–13}$$

If γ is also angle-convex, then ϑ is unique.

(iii) For a tip with constant mobility and F = constant,

$$\mathbf{m}(\vartheta)\cdot\boldsymbol{j} = \hat{\psi}'(\vartheta). \tag{31–14}$$

PROOF. The proof begins with the identities:

$$f(\boldsymbol{j},\vartheta) = \mathbf{t}(\vartheta)\cdot\boldsymbol{j} - \hat{\psi}(\vartheta) > F(\vartheta) \geq 0 \qquad \text{for all } \vartheta \in \mathcal{G}(\boldsymbol{j}), \tag{31–15a}$$

$$f'(\boldsymbol{j},\vartheta) = \mathbf{m}(\vartheta)\cdot\boldsymbol{j} - \hat{\psi}'(\vartheta), \tag{31–15b}$$

$$f''(\boldsymbol{j},\vartheta) = -\mathbf{t}(\vartheta)\cdot\boldsymbol{j} - \hat{\psi}''(\vartheta), \tag{31–15c}$$

where $f'(\boldsymbol{j}, \vartheta) = \partial f(\boldsymbol{j}, \vartheta)/\partial\vartheta$.

To establish (i) set the derivative of $\Gamma_{\text{tip}}(\vartheta) = f(\vartheta)\hat{V}(f(\vartheta))$ equal to zero; the result is

$$f'\{\hat{V}'(f)f + \hat{V}(f)\} = 0. \tag{31–16}$$

Thus, since the loading is supercritical, (31–15a) and (30–12b,c) imply that the term $\{\cdots\} > 0$, so that $f'(\vartheta) = 0$, which, by (31–15b) and the fact that $\hat{\psi}(\vartheta)$ is constant, yields

$$\mathbf{m}(\vartheta)\cdot\boldsymbol{j} = 0. \tag{31–17}$$

Also, since $f = \mathbf{t}(\vartheta)\cdot\boldsymbol{j} - \psi_{\text{tip}} > 0$, $\mathbf{t}(\vartheta)\cdot\boldsymbol{j} > 0$; thus (31–12) holds.

Consider (ii). Assume that the loading is supercritical. To establish the uniqueness of the propagation angle, fix $\boldsymbol{j}$, write $f(\vartheta) = f(\boldsymbol{j}, \vartheta)$, and let $\Phi(\vartheta) = \Gamma_{\text{tip}}(\boldsymbol{j}, \vartheta)/M = f(\vartheta)[f(\vartheta) - F(\vartheta)]$; then

$$\Phi' = f'(f - F) + f(f' - F'), \tag{31–18a}$$

$$\Phi'' = f''(f - F) + 2f'(f' - F') + f(f'' - F''). \tag{31–18b}$$

The formula (31–13) follows upon setting (31–18a) to zero, with the use of (31–15b).

Assume that $\gamma(\vartheta)$ is angle-convex. Then (19–45), (30–11), and (30–15a,c) yield

$$f'' + f = -\hat{\psi}'' - \hat{\psi}, \qquad f'' - F'' + f - F = -\gamma'' - \gamma \le 0.$$

Assume that $\Phi' = 0$ at some angle. Then, by (31–18b), at that angle,

$$\begin{aligned}\Phi'' &= f''(f - F) + f(f'' - F'') - 2(f')^2(f - F)/f \\ &\le -2f(f - F) - 2(f')^2(f - F)/f \le 0.\end{aligned}$$

Thus $\Phi''(\vartheta) \le 0$ for any ϑ that satisfies $\Phi'(\vartheta) = 0$; in view of Lemma 31.1, this yields a unique solution of (31–7) and hence results in a unique angle of propagation.

Finally (iii) is a direct consequence of (31–13) and (31–15a). □

An alternative fracture criterion, due to Cotterell,[3] asserts that the crack will propagate in a direction ϑ that maximizes the energy release rate $J(\boldsymbol{j}, \vartheta)$. Since $J(\boldsymbol{j}, \vartheta) = \boldsymbol{j}\cdot\mathbf{t}(\vartheta)$, this yields $\mathbf{t}(\vartheta) = \boldsymbol{j}/|\boldsymbol{j}|$ and hence coincides with the prediction (31–12) of the maximum dissipation criterion for a constitutively isotropic tip, but not generally for one that is anisotropic (cf. (31–13), (31–14)). If for an anisotropic tip the Cotterell criterion is interpreted to signify crack propagation in a direction that maximizes the *total* energy release rate

$$J(\boldsymbol{j}, \vartheta) - \hat{\psi}(\vartheta), \tag{31–19}$$

which includes the release of surface energy, then the predictions of the two criteria coincide provided both the limit force F and the mobility M are constant, but when this is not so the predictions of the two criteria differ.

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[3][1965]. Cf. Hussain, Pu, and Underwood [1974], Palaniswamy and Kanuss [1978], Cotterell and Rice [1980], Le [1989a,b], Stumpf and Le [1990, 1992].

Of the two criteria, that of maximum dissipation seems more firmly rooted in thermodynamics: under isothermal conditions the maximum dissipation criterion is equivalent to the requirement that the crack propagate in a direction that maximizes the entropy production at the tip. Further, for a conservatively loaded elastic body, the maximum dissipation criterion ensures that the total energy decrease at a maximal rate.

Regarding initiation, the kink angles predicted here are direct consequences of the theory; no extraneous criterion is used. Again the prediction coincides with that of the Cotterell criterion for a constitutively isotropic tip, and if the total energy release rate (31–19) is used in the Cotterell criterion, also for an anisotropic tip with F constant, but otherwise the predictions of the two theories differ.

CHAPTER 32

Fracture in Three Space Dimensions (Results)[1]

The theory developed thus far may be extended to planar cracks in three-dimensional bodies. Here the crack is a *plane surface* in B and the tip is a (smooth) curve $\mathscr{L}(t)$. The chief ingredients of the configurational force system consist of a bulk stress $\boldsymbol{C} = \Psi\mathbf{1} - \boldsymbol{F}^{\top}\boldsymbol{S}$, an internal force $\boldsymbol{g}_{\text{tip}}$ distributed over the crack-tip curve, a surface tension σ that acts within the free surfaces of the crack, and a line tension λ that acts within the tip curve. (Inertia is neglected, as are surface and line shears.) As in the two-dimensional theory, the force $\boldsymbol{g}_{\text{tip}}$ is associated with the breaking of bonds during crack growth. As a consequence of the second law the tensions σ and λ, here taken to be constant and strictly positive, are shown to coincide with the corresponding surface and line energies.

The central results of the theory are a balance

$$\mathbf{n} \cdot \oint_{\text{tip}} (\Psi\mathbf{1} - \boldsymbol{F}^{\top}\boldsymbol{S})\boldsymbol{n}\, ds - \sigma - \lambda K + \mathbf{n} \cdot \boldsymbol{g}_{\text{tip}} = 0 \tag{32–1}$$

and an inequality

$$V\mathbf{n} \cdot \boldsymbol{g}_{\text{tip}} \leq 0 \tag{32–2}$$

that follow from the limiting forms of the configurational force balance and the second law at the crack tip. Here $\mathbf{n}$ is the unit normal to the tip curve in the plane of the crack; $\oint_{\text{tip}}(\cdots)\boldsymbol{n}\, ds$ represents an integral around an infinitesimal circular loop surrounding the tip and perpendicular to it, with $\boldsymbol{n}$ the outward unit normal

[1]Gurtin and Shvartsman [1997]. Equivalent results, but with $\sigma = \lambda = 0$, based on configurational forces within a somewhat different framework, and without a discussion of the constitutive behavior of the tip, were derived earlier by Dascalu and Maugin [1993]. Cf. also Le [1989a,b], Stumpf and Le [1990, 1992].

to the loop; K is the curvature and V the normal velocity of the tip curve $\mathscr{L}$, with $K > 0$ when $\mathscr{L}$ is a circle.

The results (32–1) and (32–2) are independent of constitution; when constitutive assumptions involving V and $\mathbf{n} \cdot \boldsymbol{g}_{\text{tip}}$ are prescribed in a manner consistent with (32–2), then (32–1) represents an evolution equation for the crack tip.

Writing $f = -\mathbf{n} \cdot \boldsymbol{g}_{\text{tip}}$ for the driving force, and arguing as in Chapter 30, possible constitutive equations are

$$V = 0 \quad \text{for } f \leq F, \qquad V = \hat{V}(f) > 0 \quad \text{for } f > F, \tag{32–3}$$

with $F \geq 0$, as before, a **limit force** for crack propagation. As in the two-dimensional theory, the **Griffith-Irwin modulus** is defined by

$$\gamma = \sigma + F. \tag{32–4}$$

A consequence of (32–1) and (32–3) is then the **generalized Griffith criterion**

$$\mathbf{n} \cdot \oint_{\text{tip}} (\Psi \mathbf{1} - \boldsymbol{F}^{\top} \boldsymbol{S}) \boldsymbol{n} \, ds > \gamma + \lambda K, \tag{32–5}$$

a condition both necessary and sufficient for crack growth. What is interesting is the appearance of the curvature of the tip curve through the term λK, a term that provides an impediment to crack growth that increases with diminishing crack size.

Consider a *penny-shaped crack* with tip curve $\mathscr{L}$ a circle of radius R, so that $K = 1/R > 0$. Then for a body of prescribed constitution under prescribed boundary conditions, if the asymptotics associated with the limit $R \to 0$ yield

$$\mathbf{n} \cdot \oint_{\text{tip}} (\Psi \mathbf{1} - \boldsymbol{F}^{\top} \boldsymbol{S}) \boldsymbol{n} \, ds = o(R^{-1}), \tag{32–6}$$

then *a penny-shaped crack of sufficiently small radius will not grow*.

Part H

Two-Dimensional Theory of Corners and Junctions Neglecting Inertia[1]

Junctions are formed by the intersection of two or more interfaces. They, like crack tips, are point singularities, and many of the concepts and results for junctions have antecedents within the theory of fracture. To stress this commonality, the discussion follows that of Part G on fracture, although here standard stress within the interface is considered but inertia is neglected.

> Junctions involving three or more interfaces are important in the study of grain boundaries, but their evolution generally takes place on a time scale far longer than that related to inertial effects. Inertial contributions at junctions could be of possible importance when discussing dynamical twinning.

To avoid complicated topological considerations, the discussion is restricted to two space dimensions, and, because the theory away from a junction is as presented in Chapter 22, only the junction itself is treated in detail.

[1] Simha and Bhattacharya [1998].

CHAPTER 33

Preliminaries. Transport Theorems

a. Terminology

Let R be a closed subregion of the body, and let $\mathscr{C}(t)$ denote the union,

$$\mathscr{C}(t) = \bigcup_{n=1}^{N} \mathscr{C}_n(t), \tag{33–1}$$

of smoothly evolving, connected, oriented, simple curves $\mathscr{C}_1(t), \mathscr{C}_2(t), \ldots, \mathscr{C}_N(t)$, $N \geq 2$, that intersect a at a single point $\boldsymbol{Z}(t)$. Assume that each curve $\mathscr{C}_n(t)$ has one endpoint at $\boldsymbol{Z}(t)$ and crosses ∂R exactly once, and that arc length is measured from $\boldsymbol{Z}(t)$. The following notation and terminology are useful: $\boldsymbol{Z}(t)$ is a **junction**, with the special case $N = 2$, also referred to as a **corner**;

$$\boldsymbol{v}(t) = \dot{\boldsymbol{Z}}(t)$$

is the **junction velocity**; the curves $\mathscr{C}_n(t)$ are **interfaces**; the set $\mathscr{C}(t)$ is the **interface system**; on $\mathscr{C}(t)$ away from $\boldsymbol{Z}(t)$, $\mathbf{t}(\boldsymbol{X}, t)$ is the unit tangent in the direction of increasing arc length, while $\mathbf{m}(\boldsymbol{X}, t)$ is a continuous unit normal field.

Tip discs and tip integrals, basic to the discussion of fracture, have obvious counterparts here: a **junction disc** $D_\delta(t)$ is a disc of radius δ centered at $\boldsymbol{Z}(t)$, and when meaningful, limits such as $\lim_{\delta\to 0} \int_{\partial D_\delta} \Phi \boldsymbol{n}\, ds$, termed **junction integrals**, are written in the form

$$\oint_{\text{jun}} \Phi \boldsymbol{n}\, ds = \lim_{\delta\to 0} \int_{\partial D_\delta} \Phi \boldsymbol{n}\, ds.$$

Consider a particular interface $\mathscr{C}_n(t)$ with n arbitrary but fixed, and let $\boldsymbol{X} = \hat{\boldsymbol{X}}(s, t)$ denote the **arc-length parametrization** of $\mathscr{C}_n(t)$. For all sufficiently small r, $\partial D_r(t)$ intersects the curve $\mathscr{C}_n(t)$ exactly once, and this induces a one-to-one

relation

$$r = \hat{r}(s,t) = |\hat{\boldsymbol{X}}(s,t) - \boldsymbol{Z}(t)|, \qquad s = \hat{s}(r,t), \tag{33–2}$$

between arc length s on $\mathscr{C}_n(t)$ and the distance r from the junction to the point on $\mathscr{C}_n(t)$ with arc length s. Further, by (33–2), $\partial\hat{r}(s,t)/\partial s = 1$ and $\partial\hat{r}(s,t)/\partial t = 0$ at $s = 0$. Because only a small neighborhood of the junction is relevant to the subsequent analysis, I further restrict the region R to be small enough that on each interface within R this one-to-one relation between arc length and distance is satisfied with $\partial\hat{r}(s,t)/\partial s > 0$, so that $s = \hat{s}(r,t)$ is smooth. The mapping (33–2) then allows for a **distance parametrization**

$$\boldsymbol{X} = \tilde{\boldsymbol{X}}(r,t) = \hat{\boldsymbol{X}}(\hat{s}(r,t),t)$$

of $\mathscr{C}_n(t)$ in terms of (r,t). The notation $\boldsymbol{X} = \hat{\boldsymbol{X}}_n(s,t)$ and $\boldsymbol{X} = \tilde{\boldsymbol{X}}_n(r,t)$ will be used for the arc length and distance parametrizations of $\mathscr{C}_n(t)$ when it is necessary to distinguish between interfaces.

On $\mathscr{C}_n(t)$ both s and r are measured from $\boldsymbol{Z}(t)$; thus,

$$\hat{\boldsymbol{X}}_n(0,t) = \tilde{\boldsymbol{X}}_n(0,t) = \boldsymbol{Z}(t),$$

and, as each interface is smooth up to $\boldsymbol{Z}(t)$,

$$\partial_t\hat{\boldsymbol{X}}_n(0,t) = \partial_t\tilde{\boldsymbol{X}}_n(0,t) = \boldsymbol{v}(t). \tag{33–3}$$

The field $\boldsymbol{v}_{\mathscr{C}}(\boldsymbol{X},t)$ on $\mathscr{C}(t)$ defined on each interface $\mathscr{C}_n(t)$ by

$$\boldsymbol{v}_{\mathscr{C}}(\boldsymbol{X},t) = \partial_t\hat{\boldsymbol{X}}_n(s,t), \qquad \boldsymbol{X} = \hat{\boldsymbol{X}}_n(s,t)$$

represents a **velocity field for** $\mathscr{C}(t)$, as it is a velocity field for each $\mathscr{C}_n(t)$. By (33–3), $\boldsymbol{v}_{\mathscr{C}}(\boldsymbol{X},t)$ is continuous across the junction, with

$$\boldsymbol{v}_{\mathscr{C}}(\boldsymbol{Z}(t),t) = \boldsymbol{v}(t). \tag{33–4}$$

b. Transport theorems

A **migrating control volume** $P(t)$ is (here restricted to be) a smoothly evolving, closed subregion of R with the following properties: $P(t)$ contains $\boldsymbol{Z}(t)$ in its interior; each interface $\mathscr{C}_n(t)$ intersects $\partial P(t)$ at a single point. As before, $\boldsymbol{n}(\boldsymbol{X},t)$ designates the outward unit normal to $\partial P = \partial P(t)$, while U is the normal velocity of ∂P in the direction $\boldsymbol{n}$.

b1. Bulk fields

The discussion parallels that of fracture. A **bulk field** $\Phi(\boldsymbol{X},t)$ is a field defined and smooth away from $\mathscr{C}(t)$.

Given a bulk field $\Phi(\boldsymbol{X},t)$, the **time derivative of** $\Phi(\boldsymbol{X},t)$ **following** $\boldsymbol{Z}(t)$ is defined by (25–9) and has the explicit form

$$\overset{\square}{\Phi} = \dot{\Phi} + \nabla\Phi\cdot\boldsymbol{v} \tag{33–5}$$

away from $\mathscr{C}$.

A bulk field Φ will be termed **regular** if, away from the junction, $\Phi(\boldsymbol{X}, t)$ is smooth up to each interface from either side, and if

(S1) for each migrating control volume $P = P(t)$, the limits

$$\int_P \Phi \, da = \lim_{\delta \to 0} \int_{P_\delta} \Phi \, da, \qquad (P_\delta(t) = P(t) \backslash D_\delta(t)) \tag{33–6a}$$

$$\int_P \overset{\square}{\Phi} \, da = \lim_{\delta \to 0} \int_{P_\delta} \overset{\square}{\Phi} \, da \tag{33–6b}$$

exist, with (33–6b) uniform in time;

(S2) $[\Phi]\mathbf{m} \cdot (\boldsymbol{v}_{\mathscr{C}} - \boldsymbol{v})$ is integrable on $\mathscr{C}$, uniformly in time.

There is apparently little known about the singularity at a corner in an interface between two materials, let alone at a junction of three or more interfaces, although the singularity at a boundary corner of a linearly elastic body without surface structure is, in all cases with which I am familiar, less severe than that of a crack in the same body. In any case, it would seem that, for the bulk energy Ψ, the time derivative $\overset{\square}{\Psi}$ following the junction would have a singularity weaker than that of $\dot{\Psi}$ (cf. (25–16)).

The arguments given in the paragraphs containing (25–17)–(25–23) here yield analogous results for junctions; namely, the temporal differentiability of $\int_{P(t)} \Phi \, da$, the transport identity

$$\frac{d}{dt} \left\{ \int_{P(t)} \Phi \, da \right\} = \int_{P(t)} \overset{\square}{\Phi} \, da + \int_{\partial P(t)} \Phi (U - \boldsymbol{v} \cdot \boldsymbol{n}) ds + \int_{\mathscr{G}(t)} [\Phi]\mathbf{m} \cdot (\boldsymbol{v} - \boldsymbol{v}_{\mathscr{C}}) ds,$$

$\mathscr{G} = \mathscr{C} \cap P$, and the following theorem.

Junction Transport Theorem for Bulk Fields. Let Φ be a regular bulk field. Then

$$\frac{d}{dt} \left\{ \int_{D_\delta} \Phi \, da \right\} \to 0 \qquad \text{as } \delta \to 0, \tag{33–7}$$

uniformly in time.

b2. Interfacial fields

Central to the proof of the transport theorem, (33–7), is the field $\overset{\square}{\Phi}$. To establish an analogous theorem for (density) fields $\varphi(\boldsymbol{X}, t)$ on $\mathscr{C}(t)$ requires an interfacial counterpart of $\overset{\square}{\Phi}$. This is not straightforward, because for $\boldsymbol{Y}$ fixed and $\boldsymbol{v} \neq \mathbf{0}$ the set of t at which $\varphi(\boldsymbol{Z}(t) + \boldsymbol{Y}, t)$ is defined would generally be discrete. But close to $\boldsymbol{Z}(t)$, φ can be considered as a function of t and the distance r from $\boldsymbol{Z}(t)$; this function, when converted to a density $\tilde{\varphi}$ measured with respect to r and then differentiated with respect to t, furnishes the desired counterpart of $\overset{\square}{\Phi}$.

Let $\varphi(\boldsymbol{X}, t)$ be smooth on $\mathscr{C}_n(t)$ away from the junction. Then, choosing the distance parametrization $\boldsymbol{X} = \tilde{\boldsymbol{X}}(r, t)$ for $\mathscr{C}_n(t)$, let

$$\tilde{\varphi}(r, t) = \mathscr{J}(r, t)\varphi(\tilde{\boldsymbol{X}}(r, t), t), \qquad \mathscr{J}(r, t) = \partial\hat{s}(r, t)/\partial r; \tag{33–8}$$

$\tilde{\varphi}(r, t)$ expresses $\varphi(\boldsymbol{X}, t)$ on $\mathscr{C}_n(t)$ in terms of its distance from the junction, with $\tilde{\varphi}(r, t)$ considered as a density *measured per unit distance* rather than per unit arc length. The partial derivative

$$(\partial_t\tilde{\varphi})(r, t) = \partial\tilde{\varphi}(r, t)/\partial t \tag{33–9}$$

holding the distance r from $\boldsymbol{Z}$ *fixed* represents an interfacial counterpart of $\dot{\Phi}$. When it becomes necessary to make explicit the interface $\mathscr{C}_n(t)$ in question, $\tilde{\varphi}^{(n)}(r, t)$ will be used in place of $\tilde{\varphi}(r, t)$.

Let $\varphi(\boldsymbol{X}, t)$ be defined and smooth on $\mathscr{C}(t)$ away from the junction. For any migrating control volume $P(t)$ and δ sufficiently small, let $P_\delta(t) = P(t)\backslash D_\delta(t)$, so that $\mathscr{C}(t) \cap P_\delta(t)$ represents the portion of $\mathscr{C}(t)$ that lies inside $P(t)$ yet outside $D_\delta(t)$. Define

$$\int_{\mathscr{C}\cap P_\delta} \tilde{\varphi}\, dr = \sum_n \left\{ \int_{C_n\cap P_\delta} \tilde{\varphi}^{(n)}(r, t)dr \right\},$$

$$\int_{\mathscr{C}\cap P_\delta} \partial_t\tilde{\varphi}\, dr = \sum_n \left\{ \int_{C_n\cap P_\delta} \partial_t\tilde{\varphi}^{(n)}(r, t)dr \right\},$$

where $\sum_n$ denotes the sum from $n = 1$ to $n = N$; then

$$\int_{\mathscr{C}\cap P_\delta} \tilde{\varphi}\, dr = \int_{\mathscr{C}\cap P_\delta} \varphi\, ds. \tag{33–10}$$

Granted these definitions, φ will be termed a **regular interfacial field** if, in addition to being smooth away from the junction, the limits

$$\int_{\mathscr{C}\cap P} \tilde{\varphi}\, dr = \lim_{\delta\to 0} \int_{\mathscr{C}\cap P_\delta} \tilde{\varphi}\, dr, \tag{33–11a}$$

$$\int_{\mathscr{C}\cap P} \partial_t\tilde{\varphi}\, dr = \lim_{\delta\to 0} \int_{\mathscr{C}\cap P_\delta} \partial_t\tilde{\varphi}\, dr \tag{33–11b}$$

exist for each migrating control volume $P = P(t)$, with (33–11b) uniform in time. A consequence of (33–11a) is that

$$\begin{aligned} &\text{given any migrating control volume } P = P(t), \\ &\int_{\mathscr{C}(t)\cap P(t)} \varphi\, ds \text{ is a differentiable function of } t. \end{aligned} \tag{33–12}$$

Because $P(t) = P_\lambda(t)\cup D_\lambda(t)$ and $\int_{P_\lambda(t)} \varphi\, ds$ is differentiable, it suffices to establish (33–12) for $P(t)$ an arbitrary junction disc $D_\lambda(t)$. In this case $P_\delta(t)$ $(\lambda > \delta > 0)$ is the annulus $D_{\lambda\delta}(t) = D_\lambda(t)\backslash D_\delta(t)$, so that

$$\int_{\mathscr{C}\cap P_\delta} \tilde{\varphi}\, dr = \sum_n \left\{ \int_\delta^\lambda \tilde{\varphi}^{(n)}(r, t)dr \right\} \tag{33–13}$$

and therefore

$$\frac{d}{dt}\left\{\int_{\mathscr{C}\cap P_\delta}\tilde{\varphi}\,dr\right\}=\int_{\mathscr{C}\cap P_\delta}\partial_t\tilde{\varphi}\,dr. \tag{33–14}$$

Thus, by (33–11b), passing to the limit $\delta\to 0$ yields two conclusions: that (33–12) is valid for $P(t)=D_\lambda(t)$ and hence for any migrating control volume; and that

$$\frac{d}{dt}\left\{\int_{\mathscr{C}\cap D_\lambda}\tilde{\varphi}\,dr\right\}=\int_{\mathscr{C}\cap D_\lambda}\partial_t\tilde{\varphi}\,dr. \tag{33–15}$$

Finally, replacing λ by δ in (33–15) and passing to the limit $\delta\to 0$ using the argument following (25–21) yields the following important result.

Junction Transport Theorem for Interfacial Fields. Let φ be a regular interfacial field. Then

$$\frac{d}{dt}\left\{\int_{\mathscr{C}\cap D_\delta}\varphi\,ds\right\}\to 0\qquad\text{as }\delta\to 0, \tag{33–16}$$

uniformly in time.

CHAPTER 34

Thermomechanical Theory of Junctions and Corners

Throughout this chapter R, a closed subregion of the body, contains an interface system $\mathscr{C}(t)$ with $\boldsymbol{Z}(t)$ its junction, as described in the previous chapter. Away from this junction the basic fields and equations for the bulk material and for the interface are as discussed in Chapter 22.

As for fracture, external forces are not considered.

a. Motions

Motions $\boldsymbol{y}(\boldsymbol{X}, t)$ are assumed to be continuous on R, and, away from $\boldsymbol{Z}(t)$, are assumed to be smooth up to each interface from either side. Then, in particular,

$$\boldsymbol{y}(\boldsymbol{X}, t) \to \boldsymbol{y}(\boldsymbol{Z}(t), t) \qquad \text{as } \boldsymbol{X} \to \boldsymbol{Z}(t), \tag{34–1}$$

and the *deformed junction* is well defined.

As with fracture, the *motion velocity* $\overset{\square}{\boldsymbol{y}}$ *following the junction* satisfies

$$\overset{\square}{\boldsymbol{y}} = \dot{\boldsymbol{y}} + \boldsymbol{F}\boldsymbol{v} \tag{34–2}$$

in bulk (cf. (33–5)) and is assumed to have a junction limit

$$\overset{\square}{\boldsymbol{y}}(\boldsymbol{X}, t) \to \bar{\boldsymbol{v}}(t) \qquad \text{as } \boldsymbol{X} \to \boldsymbol{Z}(t) \tag{34–3}$$

from the bulk material. If this limit is uniform in t, then

$$\bar{\boldsymbol{v}}(t) = \frac{d}{dt}\,\boldsymbol{y}(\boldsymbol{Z}(t), t);$$

$\bar{\boldsymbol{v}}(t)$ represents the velocity of the junction in the deformed configuration.

Fix $\delta > 0$ and consider the junction disc $D_\delta(t)$. Because $D_\delta(t)$ is transported rigidly with $\boldsymbol{Z}(t)$, the junction velocity $\boldsymbol{v}(t)$ represents an admissible choice of

velocity field for $\partial D_\delta(t)$, and, by (34–2), the motion velocity $\overset{\square}{\boldsymbol{y}}(\boldsymbol{X}, t)$ following the function coincides with the motion velocity following $\partial D_\delta(t)$ as described by $\boldsymbol{v}(t)$ (cf. (4–4)).

Let $\mathscr{C}^\delta = \mathscr{C}^\delta(t) = \mathscr{C} \cap D_\delta$, so that

$$\partial C^\delta = \mathscr{C} \cap \partial D_\delta \quad \text{consists of the endpoints of the subset of } \mathscr{C} \text{ in } D_\delta.$$

A velocity field for ∂C^δ is easily constructed. For each n, let $\boldsymbol{X} = \tilde{\boldsymbol{X}}_n(r, t)$ denote the distance parametrization on the interface $\mathscr{C}_n(t)$ and consider the field on $\mathscr{C}(t)$ defined by

$$\boldsymbol{w}(\boldsymbol{X}, t) = \partial_t \tilde{\boldsymbol{X}}_n(r, t), \qquad \boldsymbol{X} = \tilde{\boldsymbol{X}}_n(r, t).$$

Then, for any $\boldsymbol{X}$ in $\partial\mathscr{C}^\delta(t)$, $\boldsymbol{w}(\boldsymbol{X}, t)$ represents a **velocity field for** $\partial\mathscr{C}^\delta(t)$ at $\boldsymbol{X}$, because $\boldsymbol{X} = \tilde{\boldsymbol{X}}_n(\delta, t)$ for some n and $\boldsymbol{w}(\boldsymbol{X}, t) = \partial_t \tilde{\boldsymbol{X}}_n(\delta, t)$; in addition, by (33–3),

$$\boldsymbol{w}(\boldsymbol{X}, t) \to \boldsymbol{v}(t) \qquad \text{as } \boldsymbol{X} \to \boldsymbol{Z}(t). \tag{34–4}$$

Further, in view of (21–4) (cf. (22–5)), the field $\overset{\triangle}{\boldsymbol{y}}$ defined on $\mathscr{C}(t)$ away from the junction by

$$\overset{\triangle}{\boldsymbol{y}} = \langle \dot{\boldsymbol{y}} \rangle + \langle \boldsymbol{F} \rangle \boldsymbol{w}, \tag{34–5}$$

when restricted to $\partial\mathscr{C}^\delta(t)$, represents the **motion velocity following** $\partial\mathscr{C}^\delta(t)$. By (34–2), $\overset{\square}{\boldsymbol{y}} = \langle \dot{\boldsymbol{y}} \rangle + \langle \boldsymbol{F} \rangle \boldsymbol{v}$; consistent with this and (34–3)–(34–5), I assume that

$$\overset{\triangle}{\boldsymbol{y}}(\boldsymbol{X}, t) \to \bar{\boldsymbol{v}}(t) \qquad \text{as } \boldsymbol{X} \to \boldsymbol{Z}(t). \tag{34–6}$$

b. Notation

Consider a field $\varphi(\boldsymbol{X}, t)$ defined on $\mathscr{C}(t)$ away from the junction. Suppressing the argument t, let

$$\varphi|_{\partial\mathscr{C}^\delta} = \sum_{X \in \partial\mathscr{C}^\delta} \varphi(\boldsymbol{X}); \tag{34–7a}$$

$$\sum_{\text{jun}} \varphi = \lim_{\delta \to 0} \varphi|_{\partial\mathscr{C}^\delta} = \sum_n \left\{ \lim_{\substack{X \to Z \\ X \in \mathscr{C}_n}} \varphi(\boldsymbol{X}) \right\}, \tag{34–7b}$$

provided the limits exist. The **junction sum** $\sum_{\text{jun}} \varphi$ represents the sum of limiting values of φ at the junction, while $\varphi|_{\partial\mathscr{C}^\delta}$ represents the sum of values of φ over the set $\partial\mathscr{C}_\delta$ of endpoints of the subset of $\mathscr{C}$ in D_δ.

c. Forces. Working

The standard stress $\boldsymbol{S}$, the configurational stress $\boldsymbol{C}$, and the internal body force $\boldsymbol{g}$ are as introduced in Chapters 3 and 5. Away from the junction, these fields are

presumed to be smooth away from and up to each interface from either side, with $\boldsymbol{g}$ integrable over R and

$$\int_{\partial D_\delta(t)} |\boldsymbol{S}\boldsymbol{n}|\,ds \qquad \text{bounded as } \delta \to 0. \tag{34–8}$$

In view of (34–1) and (34–3), (34–8) implies that

$$\oint_{\text{jun}} \boldsymbol{S}\boldsymbol{n} \cdot (\overset{\square}{\boldsymbol{y}} - \bar{\boldsymbol{v}})\,ds = 0. \tag{34–9}$$

In addition to these bulk fields, I consider three fields *distributed over the interface system* $\mathscr{C}$:

$\boldsymbol{g}^{\mathscr{C}}$	**internal configurational force**;
$\mathbf{C}$	**configurational stress**;
$\mathbf{S}$	**standard stress**;

and add one force *concentrated at the junction*:

$$\boldsymbol{g}_{\text{jun}} \qquad \textbf{internal configurational force}.$$

The basic hypotheses are that $\boldsymbol{g}^{\mathscr{C}}(\boldsymbol{X}, t)$ is integrable over $\mathscr{C}(t)$, while $\mathbf{C}(\boldsymbol{X}, t)$ and $\mathbf{S}(\boldsymbol{X}, t)$ are smooth away from and up to the junction from each interface (although the corresponding limits will generally vary from interface to interface). The force $\boldsymbol{g}_{\text{jun}}(t)$, being concentrated at the junction, is a function of time alone; $\boldsymbol{g}_{\text{jun}}$ is associated with rearrangements of atoms at the junction or, more generally, to phenomena occurring at the junction at length scales that are small compared to the gross length scales of the body.

The goal of this chapter is a set of basic equations for the junction, so attention is restricted to junction discs $D_\delta = D_\delta(t)$. The working of the bulk forces on D_δ is as discussed in Part A. The interfacial forces $\mathbf{C}$ and $\mathbf{S}$ perform work on D_δ at points at which the interface system $\mathscr{C}$ intersection ∂D_δ. Because the velocity of such points and their motion velocity are described, respectively, by the fields $\boldsymbol{w}$ and $\overset{\triangle}{\boldsymbol{y}}$, the rate at which $\mathbf{C}$ and $\mathbf{S}$ perform work on D_δ is given by $(\mathbf{C} \cdot \boldsymbol{w} + \mathbf{S} \cdot \overset{\triangle}{\boldsymbol{y}})|_{\partial \mathscr{C}^\delta}$. Thus, choosing $\boldsymbol{v}$ as the velocity field for ∂D_δ with $\overset{\square}{\boldsymbol{y}}$ the corresponding motion velocity following ∂D_δ, the **working** $W(D_\delta)$ has the form

$$W(D_\delta) = \int_{\partial D_\delta} (\boldsymbol{C}\boldsymbol{n} \cdot \boldsymbol{v} + \boldsymbol{S}\boldsymbol{n} \cdot \overset{\square}{\boldsymbol{y}})\,ds + (\mathbf{C} \cdot \boldsymbol{w} + \mathbf{S} \cdot \overset{\triangle}{\boldsymbol{y}})|_{\partial \mathscr{C}^\delta}. \tag{34–10}$$

The requirement that $W(D_\delta)$ be invariant under changes in material and spatial observer yields the **configurational force balance**

$$\int_{\partial D_\delta} \boldsymbol{C}\boldsymbol{n}\,ds + \int_{D_\delta} \boldsymbol{g}\,da + \int_{\mathscr{C} \cap D_\delta} \boldsymbol{g}^{\mathscr{C}}\,ds + \mathbf{C}|_{\partial \mathscr{C}^\delta} + \boldsymbol{g}_{\text{jun}} = \mathbf{0}, \tag{34–11}$$

the **standard force balance**

$$\int_{\partial D_\delta} \boldsymbol{S}\boldsymbol{n}\,ds + \mathbf{S}|_{\partial \mathscr{C}^\delta} = \mathbf{0}, \tag{34–12}$$

and a moment balance that is unimportant. Because the fields involved in the integrals over D_δ and $\mathscr{C} \cap D_\delta$ are integrable, (34–11) and (34–12) yield, upon passing to the limit $\delta \to 0$, the **junction balances**

$$\sum_{\text{jun}} \mathbf{s} + \oint_{\text{jun}} \boldsymbol{S}\boldsymbol{n}\, ds = \mathbf{0}, \tag{34–13a}$$

$$\sum_{\text{jun}} \mathbf{c} + \oint_{\text{jun}} \boldsymbol{C}\boldsymbol{n}\, ds + \boldsymbol{g}_{\text{jun}} = \mathbf{0} \tag{34–13b}$$

(cf. Subsection 27c).

d. Second law

The free energy of the body is represented by a regular bulk free energy $\Psi(\boldsymbol{X}, t)$ distributed over B and a regular interfacial free energy $\psi(\boldsymbol{X}, t)$ distributed over $\mathscr{C}(t)$, and the **second law** for the junction disc $D_\delta = D_\delta(t)$ has the form

$$\frac{d}{dt}\left\{ \int_{D_\delta} \Psi\, da + \int_{\mathscr{C}\cap D_\delta} \psi\, ds \right\} \leq W(D_\delta), \tag{34–14}$$

with working $W(D_\delta)$ given by (34–10). By (33–7) and (33–16),

$$\frac{d}{dt}\left\{ \int_{D_\delta} \Psi\, da + \int_{\mathscr{C}\cap D_\delta} \psi\, ds \right\} \to 0 \tag{34–15}$$

as $\delta \to 0$. On the other hand, note that, by (34–9), since $\boldsymbol{w} \to \boldsymbol{v}$ and $\overset{\triangle}{\boldsymbol{y}} \to \bar{\boldsymbol{v}}$,

$$W(D_\delta) \to \boldsymbol{v} \cdot \oint_{\text{jun}} \boldsymbol{C}\boldsymbol{n}\, ds + \bar{\boldsymbol{v}} \cdot \oint_{\text{jun}} \boldsymbol{S}\boldsymbol{n}\, ds + \sum_{\text{jun}} (\mathbf{c} \cdot \boldsymbol{v} + \mathbf{s} \cdot \bar{\boldsymbol{v}});$$

thus, appealing to (34–13a),

$$W(D_\delta) \to -\boldsymbol{g}_{\text{jun}} \cdot \boldsymbol{v}. \tag{34–16}$$

The results (34–15) and (34–16), when combined with the imbalance (34–14), result in an **internal dissipation inequality** for the junction:

$$\boldsymbol{g}_{\text{jun}} \cdot \boldsymbol{v} \leq 0. \tag{34–17}$$

e. Basic results for the junction

The basic results consist of the *standard force balance*

$$\sum_{\text{jun}} \mathbf{s} + \oint_{\text{jun}} \boldsymbol{S}\boldsymbol{n}\, ds = \mathbf{0}, \tag{34–18}$$

the *configurational force balance*

$$\sum_{\text{jun}} \mathbf{C} + \oint_{\text{jun}} \boldsymbol{C}\boldsymbol{n}\, ds + \boldsymbol{g}_{\text{jun}} = \mathbf{0}, \tag{34–19}$$

and the *internal dissipation inequality*

$$\boldsymbol{g}_{\text{jun}} \cdot \boldsymbol{v} \leq 0 \tag{34–20}$$

representing the second law localized to the junction.

f. Weak singularity conditions. Nonexistence of corners

Consider the following hypothesis, which I will refer to as the **standard weak singularity condition**:

$$\oint_{\text{jun}} \boldsymbol{S}\boldsymbol{n}\, ds = \mathbf{0}. \tag{34–21}$$

A consequence of (34–21) is that the standard interfacial stress be balanced at the junction:

$$\sum_{\text{jun}} \mathbf{s} = \mathbf{0}. \tag{34–22}$$

Because a corner is a junction between *two* interfaces, the limiting values $\mathbf{s}_1$ and $\mathbf{s}_2$ of the standard stress vector at a corner must satisfy

$$\mathbf{s}_1 + \mathbf{s}_2 = \mathbf{0}.$$

On the other hand, in view of the sentence containing (22–10), the standard stress is necessarily tangent to the deformed interface. Thus either $\mathbf{s}_1$ and $\mathbf{s}_2$ vanish, or $\mathbf{s}_1$ and $\mathbf{s}_2$ are nonzero and parallel, a condition possible only when the deformed interface has a continuously turning tangent. This proves the following theorem.

Smooth Corner Theorem. If the standard weak singularity condition holds, then a (nontrivial) corner is possible only if one of the following two conditions is satisfied:

(i) At the corner, the union of the deformed interfaces has a continuously turning tangent.
(ii) The standard stress within each of the two interfaces vanishes at the corner.

A second possible restriction is the **configurational weak singularity condition**:

$$\oint_{\text{jun}} \boldsymbol{C}\boldsymbol{n}\, ds = \mathbf{0}. \tag{34–23}$$

A condition analogous to the standard weak singularity condition is satisfied at a crack tip in a linearly elastic body, where

$$\boldsymbol{S} \sim r^{-1/2}, \qquad \boldsymbol{C} \sim r^{-1}, \tag{34–24}$$

(cf. (28–24) and the paragraph containing (25–16)). Further, violation of the standard weak singularity condition at a junction would require $\boldsymbol{S} \sim r^{-p}$, $p \geq 1$, and since one would expect the singularity in $\boldsymbol{C}$ to be worse than that in $\boldsymbol{S}$, as is the case in (34–24), this could result in $\oint_{\text{jun}} |\boldsymbol{C}\boldsymbol{n}| ds = \infty$[1] and hence render the theory of dubious value. For these reasons, *I would expect the standard weak singularity condition to be generally applicable. In fact, I would conjecture that in most cases of interest the configurational weak singularity condition is also satisfied, even though its counterpart is not satisfied at a crack tip.*

Note that when both weak singularity conditions are satisfied,

$$\sum_{\text{jun}} \mathbf{s} = \mathbf{0}, \tag{34–25a}$$

$$\sum_{\text{jun}} \mathbf{c} + \boldsymbol{g}_{\text{jun}} = \mathbf{0}. \tag{34–25b}$$

g. Constitutive equations

Away from the junction, the constitutive equations for $\mathscr{C}(t)$ are as derived in Chapter 22; viz.

$$\psi = \hat{\psi}(\vartheta, \lambda), \tag{34–26a}$$

$$\mathbf{s} = \partial_{\boldsymbol{y}_s} \hat{\psi}(\vartheta, \lambda), \tag{34–26b}$$

$$\mathbf{c} = (\psi - \mathbf{s} \cdot \boldsymbol{y}_s)\mathbf{t} + \tau \mathbf{m}, \tag{34–26c}$$

$$\tau = \partial_{\vartheta} \hat{\psi}(\vartheta, \lambda) \tag{34–26d}$$

where

$$\lambda = |\boldsymbol{y}_s|, \tag{34–27}$$

while ϑ is defined in $\mathscr{C}$, away from the junction, as the angle from the $(1, 0)$ axis to $\mathbf{t}$ (cf. (22–13), (22–23), (22–29), (22–32)).

I consider a constitutive relation for the junction giving the internal force $\boldsymbol{g}_{\text{jun}}$ when the junction velocity $\boldsymbol{v}$ and the list

$$\boldsymbol{\vartheta}_{\text{jun}} = (\vartheta_1, \vartheta_2, \ldots, \vartheta_N), \qquad \vartheta_n(t) = \lim_{\substack{\boldsymbol{X} \to \boldsymbol{Z}(t) \\ \boldsymbol{X} \in \mathscr{C}_n(t)}} \vartheta(\boldsymbol{X}, t) \tag{34–28}$$

of limiting values of the tangent angle ϑ are known. Specifically,

$$\boldsymbol{g}_{\text{jun}} = -\boldsymbol{B}(\boldsymbol{v}, \boldsymbol{\vartheta}_{\text{jun}})\boldsymbol{v}, \tag{34–29}$$

with **kinetic tensor** $\boldsymbol{B}(\boldsymbol{v}, \boldsymbol{\vartheta}_{\text{jun}})$ consistent with $\boldsymbol{v} \cdot \boldsymbol{B}(\boldsymbol{v}, \boldsymbol{\vartheta}_{\text{jun}})\boldsymbol{v} \geq 0$ so that $\boldsymbol{g}_{\text{jun}} \cdot \boldsymbol{v} \leq 0$. For $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{\vartheta}_{\text{jun}})$ (linear kinetics), this condition reduces to the requirement that

---

[1] But need not. For $ds = r\, d\gamma$ with (r, γ) polar coordinates at the junction, the γ-dependence of $\boldsymbol{C}$ near the junction could render $\oint_{\text{jun}} \boldsymbol{C}\boldsymbol{n}\, ds$ finite; this occurs in the study of dislocations (cf. e.g., Cermelli and Gurtin [1999]).

$\boldsymbol{B}(\vartheta_{\text{jun}})$ be positive semidefinite. The field $\boldsymbol{y}_s$ is not included as an independent variable in (34–29), because it could be unbounded at the junction.

h. Final junction conditions

Accounting for the constitutive equation (34–29), the final junction conditions are

$$\sum_{\text{jun}} \mathsf{s} + \oint_{\text{jun}} \boldsymbol{S}\boldsymbol{n}\, ds = \boldsymbol{0}, \tag{34–30a}$$

$$\sum_{\text{jun}} \mathsf{c} + \oint_{\text{jun}} \boldsymbol{C}\boldsymbol{n}\, ds = \boldsymbol{B}(\boldsymbol{v}, \vartheta_{\text{jun}})\boldsymbol{v}, \tag{34–30b}$$

which simplify to

$$\sum_{\text{jun}} \mathsf{s} = \boldsymbol{0}, \tag{34–31a}$$

$$\sum_{\text{jun}} \mathsf{c} = \boldsymbol{B}(\boldsymbol{v}, \vartheta_{\text{jun}})\boldsymbol{v} \tag{34–31b}$$

when both weak singularity conditions are satisfied.

For the theory discussed in Chapter 19, in which bulk behavior (and hence deformation) was neglected, the appropriate junction condition would be (34–31b), generalizing the usual balance[2]

$$\sum_{\text{jun}} \mathsf{c} = \boldsymbol{0}. \tag{34–32}$$

A consequence of (34–32) is that a corner in an interface with constitution governed by an angle-convex free-energy is not possible (cf. (iii) of the theorem on convexity and evolution in Subsection 19g). Interestingly, the more general balance (34–31b) *allows for such corners*, although they would disappear at equilibrium.

[2]Cf. Herring [1951b, eq. (19)].

Part I

Appendices on the Principle of Virtual Work for Coherent Phase Interfaces

Balance laws for force are often derived as consequences of a principle of virtual work[1] a paradigm I now use to discuss configurational forces.

I consider *weak* and *strong* versions of this principle. The **weak principle** is based on a virtual kinematics that allows for virtual motions and virtual migrations of the interface. In this principle the external and internal work expenditures are presumed balanced for all choices of the virtual kinematics. These work expenditures are for the body as a whole; no use is made of control volumes.

The strong principle of virtual work allows also for virtually migrating control volumes. In this principle the external and internal work expenditures—on and within each such control volume—are presumed balanced for all virtual motions and all virtual migrations of the interface.

For specificity, I limit the discussion to coherent phase interfaces, but the general ideas are applicable to the study of other defects such as cracks. Throughout the discussion the external forces tacitly account for inertia.

[1] Also called the *principle of virtual power* and the *principal of virtual velocities*. There is a large literature on the application of this principle to continua: cf., e.g., Truesdell and Toupin [1960, §232], Germain [1972, 1973a,b, 1976], Casal [1973], Breuneval [1973], Antman and Osborne [1979], Maugin [1980], Antman [1995, §II.5, §XII.5], DiCarlo [1996].

A1. Weak Principle of Virtual Work

a. Virtual kinematics

Consider a two-phase body at a prescribed time t, assumed fixed throughout this discussion. At time t the phases α and β are assumed to occupy closed complementary subregions B_α and B_β of the reference body B, with the interface $\mathscr{S} = B_\alpha \cap B_\beta$ a smooth *closed*[1] surface whose unit normal $\mathbf{m}(\boldsymbol{X})$ points outward from B_α. The deformation gradient $\boldsymbol{F}(\boldsymbol{X})$ at time t, considered as *prescribed*, is presumed smooth away from $\mathscr{S}$ and up to $\mathscr{S}$ from either side, with

$$\boldsymbol{F}^+\mathbf{P} = \boldsymbol{F}^-\mathbf{P} = \langle \boldsymbol{F} \rangle \mathbf{P} = \mathbf{F} \qquad \text{on } \mathscr{S}, \tag{A1–1}$$

a relation considered as defining for $\mathbf{F}$ (cf. (21–1)).

At time t the body is allowed to undergo a **virtual kinematics** $\boldsymbol{K} = (V, \dot{\boldsymbol{y}})$ consisting of a virtual motion with velocity $\dot{\boldsymbol{y}}(\boldsymbol{X})$ in conjunction with a virtual migration of the interface in which $\mathscr{S}$ has (scalar) normal velocity $V(\boldsymbol{X})$; here $\dot{\boldsymbol{y}}$ and V are arbitrary functions on B and $\mathscr{S}$, respectively, with $\dot{\boldsymbol{y}}$ assumed smooth away from $\mathscr{S}$ and up to $\mathscr{S}$ from either side, and with $\dot{\boldsymbol{y}}$ and V subject to the compatibility condition

$$[\![\dot{\boldsymbol{y}}]\!] = -V[\![\boldsymbol{F}]\!]\mathbf{m} \qquad \text{on } \mathscr{S} \tag{A1–2}$$

(cf. (10–2a)). Further, writing

$$\boldsymbol{v} = V\mathbf{m}, \tag{A1–3}$$

[1] $\partial\mathscr{S} = \emptyset$. I do not wish to discuss conditions at the intersection of $\mathscr{S}$ with ∂B.

the fields $\dot{\boldsymbol{F}}$, $\overset{\square}{\mathbf{m}}$, $\overset{\square}{\boldsymbol{y}}$, and $\langle \boldsymbol{F} \rangle^{\square}$ are *defined* in terms of the velocity fields V and $\dot{\boldsymbol{y}}$ through

$$\dot{\boldsymbol{F}} = \nabla \dot{\boldsymbol{y}} \tag{A1–4a}$$

$$\overset{\square}{\mathbf{m}} = -\nabla_{\mathscr{S}} V, \tag{A1–4b}$$

$$\overset{\square}{\boldsymbol{y}} = \langle \dot{\boldsymbol{y}} \rangle + \langle \boldsymbol{F} \rangle \boldsymbol{v}, \tag{A1–4c}$$

$$\langle \boldsymbol{F} \rangle^{\square} = \langle \dot{\boldsymbol{F}} \rangle + \langle \nabla \boldsymbol{F} \rangle \boldsymbol{v}. \tag{A1–4d}$$

It is important to emphasize that B, $\mathscr{S}$, and $\boldsymbol{F}$ are prescribed, while V and $\dot{\boldsymbol{y}}$ are arbitrary fields consistent with (A1–2). Further, as the time t is fixed, the only variable under consideration is $\boldsymbol{X}$; thus $\dot{\boldsymbol{y}}$, $\dot{\boldsymbol{F}}$, $\overset{\square}{\mathbf{m}}$, $\overset{\square}{\boldsymbol{y}}$, and $\langle \boldsymbol{F} \rangle^{\square}$ do not explicitly represent temporal derivatives of fields, although they are consistent with identities that would be obtained were there an actual motion and an actual migration of the interface (cf. (10–5), (15–22), (15–24)). The fields $\dot{\boldsymbol{y}}$ and $\dot{\boldsymbol{F}}$ might be identified with the variations $\delta \boldsymbol{y}$ and $\delta \boldsymbol{F}$ of the classical theory.

A consequence of (15–12a) and (A1–4) are the identities

$$\nabla_{\mathscr{S}} \boldsymbol{v} = -\mathbf{m} \otimes \overset{\square}{\mathbf{m}} - V \mathbf{L}, \tag{A1–5a}$$

$$\nabla_{\mathscr{S}} (\overset{\square}{\boldsymbol{y}}) = \langle \boldsymbol{F} \rangle^{\square} \mathbf{P} - \langle \boldsymbol{F} \rangle (\mathbf{m} \otimes \overset{\square}{\mathbf{m}}) - V \langle \boldsymbol{F} \rangle \mathbf{L}, \tag{A1–5b}$$

whose proofs are almost identical to those of (16–19) and (21–2).

b. Forces. Weak principle of virtual work

The standard and configurational force systems are represented by the following fields:

standard force system

$\boldsymbol{S}$	**bulk stress** (tensor);
$\mathbf{S}$	**surface stress** (superficial tensor);
$\boldsymbol{b}$	**external body force** (vector);
$\boldsymbol{b}^{\mathscr{S}}$	**external surface force** (vector);

configurational force system

σ	**surface tension** (scalar);
$\mathbf{d}$	**surface shear** (tangent vector);
$h^{\mathscr{S}}$	**effective internal interface force** (scalar);
$e^{\mathscr{S}}$	**external interface force** (scalar).

Here $\boldsymbol{S}$ and $\boldsymbol{b}$ are fields on B that are continuous away from $\mathscr{S}$ and up to $\mathscr{S}$ from either side, while $\mathbf{S}$, $\boldsymbol{b}^{\mathscr{S}}$, σ, $\mathbf{d}$, $h^{\mathscr{S}}$, and $e^{\mathscr{S}}$ are continuous on $\mathscr{S}$.

Given a virtual kinematics $\boldsymbol{K} = (V, \dot{\boldsymbol{y}})$, the **virtual external working** $W_{\text{ext}}(\boldsymbol{K})$ and the **virtual internal working** $W_{\text{int}}(\boldsymbol{K})$ are defined by

$$W_{\text{ext}}(\boldsymbol{K}) = \int_{\partial B} \boldsymbol{S}\boldsymbol{n} \cdot \dot{\boldsymbol{y}}\, da + \int_B \boldsymbol{b} \cdot \dot{\boldsymbol{y}}\, dv + \int_{\mathscr{S}} (\boldsymbol{e}^{\mathscr{S}} V + \boldsymbol{b}^{\mathscr{S}} \cdot \overset{\square}{\boldsymbol{y}}) da, \tag{A1–6a}$$

$$W_{\text{int}}(\boldsymbol{K}) = \int_{\partial B} \boldsymbol{S} \cdot \dot{\boldsymbol{F}}\, dv + \int_{\mathscr{S}} \left\{ \mathbf{S} \cdot \langle \boldsymbol{F} \rangle^{\square} - \sigma K V - \mathbf{d} \cdot \overset{\square}{\mathbf{m}} - h^{\mathscr{S}} V \right\} da \tag{A1–6b}$$

(cf. the left and right sides of (21–17)), and the **weak principle of virtual work** is the assertion that, for any choice of $\boldsymbol{K}$,

$$W_{\text{ext}}(\boldsymbol{K}) = W_{\text{int}}(\boldsymbol{K}). \tag{A1–7}$$

Weak Theorem of Virtual Work. The weak principle of virtual work holds if and only if the following force balances are satisfied:

(i) the standard bulk balance

$$\operatorname{Div} \boldsymbol{S} + \boldsymbol{b} = \boldsymbol{0}, \tag{A1–8}$$

(ii) the standard interfacial balance

$$[\boldsymbol{S}]\mathbf{m} + \operatorname{Div}_{\mathscr{S}} \mathbf{S} + \boldsymbol{b}^{\mathscr{S}} = \boldsymbol{0}, \tag{A1–9}$$

(iii) the normal configurational balance

$$\sigma K - (\mathbf{F}^{\top}\mathbf{S}) \cdot \mathbf{L} - \mathbf{m} \cdot [\boldsymbol{F}^{\top} \boldsymbol{S}]\mathbf{m} + \operatorname{Div}_{\mathscr{S}} (\mathbf{d} - \mathbf{S}^{\top} \langle \boldsymbol{F} \rangle \mathbf{m}) + h^{\mathscr{S}} + e^{\mathscr{S}} = 0. \tag{A1–10}$$

c. Proof of the weak theorem of virtual work

Assume that the weak principle of virtual work holds, so that

$$\int_{\partial B} \boldsymbol{S}\boldsymbol{n} \cdot \dot{\boldsymbol{y}}\, da + \int_B \boldsymbol{b} \cdot \dot{\boldsymbol{y}}\, dv + \int_{\mathscr{S}} (\boldsymbol{b}^{\mathscr{S}} \cdot \overset{\square}{\boldsymbol{y}} + e^{\mathscr{S}} V) da$$
$$= \int_B \boldsymbol{S} \cdot \dot{\boldsymbol{F}}\, dv + \int_{\mathscr{S}} \left\{ \mathbf{S} \cdot \langle \boldsymbol{F} \rangle^{\square} - \sigma K V - \mathbf{d} \cdot \overset{\square}{\mathbf{m}} - h^{\mathscr{S}} V \right\} da \tag{A1–11}$$

for all choices of the virtual kinematics $(V, \dot{\boldsymbol{y}})$. Choose a virtual kinematics with $V \equiv 0$ and $\dot{\boldsymbol{y}}$ smooth everywhere and consistent with $\dot{\boldsymbol{y}} = 0$ in a neighborhood of $\mathscr{S}$ (so that $\langle \boldsymbol{F} \rangle^{\square} = \langle \dot{\boldsymbol{F}} \rangle = \boldsymbol{0}$ on $\mathscr{S}$). Then, by (A1–11) with the divergence theorem applied to the integral over ∂B,

$$\int_B (\operatorname{Div} \boldsymbol{S} + \boldsymbol{b}) \cdot \dot{\boldsymbol{y}}\, dv = 0;$$

because this must hold for all virtual velocities $\dot{\boldsymbol{y}}$ with the properties described above, the standard bulk balance (A1–8) must be satisfied.

Consider next an arbitrary virtual kinematics $(V, \dot{\boldsymbol{y}})$. By (A1–2), (A1–4c), and an argument similar to (11–15),

$$[\boldsymbol{S}\mathbf{m} \cdot \dot{\boldsymbol{y}}] = [\boldsymbol{S}\mathbf{m}] \cdot \overset{\square}{\boldsymbol{y}} - \mathbf{m} \cdot [\boldsymbol{F}^{\top} \boldsymbol{S}]\mathbf{m} V.$$

Thus, since $\operatorname{Div} \boldsymbol{S} = -\boldsymbol{b}$, the identity (10–8d) with $\boldsymbol{T} = \boldsymbol{S}$ and $\boldsymbol{w} = \dot{\boldsymbol{y}}$ yields

$$\int_{\partial B} \boldsymbol{S}\boldsymbol{n} \cdot \boldsymbol{y}\, da + \int_B \boldsymbol{b} \cdot \dot{\boldsymbol{y}}\, dv = \int_B \boldsymbol{S} \cdot \dot{\boldsymbol{F}}\, dv + \int_{\mathscr{S}} ([\boldsymbol{S}\mathbf{m}] \cdot \overset{\square}{\boldsymbol{y}} - \mathbf{m} \cdot [\boldsymbol{F}^\top \boldsymbol{S}]\mathbf{m} V) da. \quad \text{(A1–12)}$$

Further, since $\mathbf{S}\mathbf{m} = \mathbf{0}$, it follows that $\mathbf{S}\mathbf{P} = \mathbf{S}$, and, since $\mathbf{L}$ is tangential, $\mathbf{P}\mathbf{L} = \mathbf{L}$ and $\langle \boldsymbol{F} \rangle \mathbf{L} = \langle \boldsymbol{F} \rangle \mathbf{P}\mathbf{L} = \mathbf{F}\mathbf{L}$; hence (A1–5b) and the symmetry of $\mathbf{P}$ yield

$$\begin{aligned} \mathbf{S} \cdot \nabla_{\mathscr{S}} \overset{\square}{(\boldsymbol{y})} &= \mathbf{S} \cdot (\langle \boldsymbol{F} \rangle^{\square} \mathbf{P}) - (\langle \boldsymbol{F} \rangle \mathbf{m}) \cdot (\mathbf{S} \overset{\square}{\mathbf{m}}) - V \mathbf{S} \cdot (\langle \boldsymbol{F} \rangle \mathbf{L}) \\ &= \mathbf{S} \cdot \langle \boldsymbol{F} \rangle^{\square} - (\mathbf{S}^\top \langle \boldsymbol{F} \rangle \mathbf{m}) \cdot \overset{\square}{\mathbf{m}} - V (\mathbf{F}^\top \mathbf{S}) \cdot \mathbf{L}. \end{aligned} \quad \text{(A1–13)}$$

Note, for future use, that given any smooth vector field $\boldsymbol{k}$ on $\mathscr{S}$, $\mathbf{m} \cdot \mathbf{S}^\top \boldsymbol{k} = \boldsymbol{k} \cdot \mathbf{S}\mathbf{m} = 0$, since $\mathbf{S}\mathbf{m} = 0$; thus $\mathbf{S}^\top \boldsymbol{k}$ is a tangential vector field and we may use the surface divergence theorem and the fact that $\partial \mathscr{S} = \emptyset$ to conclude that

$$\int_{\mathscr{S}} \operatorname{Div}_{\mathscr{S}} (\mathbf{S}^\top \boldsymbol{k}) da = 0,$$

a result that will be used repeatedly without mention. In particular, because

$$\mathbf{S} \cdot \nabla_{\mathscr{S}} \overset{\square}{(\boldsymbol{y})} = \operatorname{Div}_{\mathscr{S}} \left\{ \mathbf{S}^\top \overset{\square}{\boldsymbol{y}} \right\} - \overset{\square}{\boldsymbol{y}} \cdot \operatorname{Div}_{\mathscr{S}} \mathbf{S}, \quad \text{(A1–14)}$$

(A1–13) yields

$$\int_{\mathscr{S}} \mathbf{S} \cdot \langle \boldsymbol{F} \rangle^{\square} da = \int_{\mathscr{S}} \left\{ (\mathbf{S}^\top \langle \boldsymbol{F} \rangle \mathbf{m}) \cdot \overset{\square}{\mathbf{m}} + V (\mathbf{F}^\top \mathbf{s}) \cdot \mathbf{L} - \overset{\square}{\boldsymbol{y}} \cdot \operatorname{Div}_{\mathscr{S}} \mathbf{S} \right\} da. \quad \text{(A1–15)}$$

Also, since $\mathbf{d} \cdot \overset{\square}{\mathbf{m}} = -\mathbf{d} \cdot \nabla_{\mathscr{S}} V = -\operatorname{Div}_{\mathscr{S}} (V \mathbf{d}) + V \operatorname{Div}_{\mathscr{S}} \mathbf{d}$,

$$\int_{\mathscr{S}} \mathbf{d} \cdot \overset{\square}{\mathbf{m}}\, da = \int_{\mathscr{S}} V \operatorname{Div}_{\mathscr{S}} \mathbf{d}\, da. \quad \text{(A1–16)}$$

Similarly,

$$\int_{\mathscr{S}} (\mathbf{S}^\top \langle \boldsymbol{F} \rangle \mathbf{m}) \cdot \overset{\square}{\mathbf{m}}\, da = \int_{\mathscr{S}} V \operatorname{Div}_{\mathscr{S}} (\mathbf{S}^\top \langle \boldsymbol{F} \rangle \mathbf{m}) da. \quad \text{(A1–17)}$$

Combining (A1–11), (A1–12), and (A1–15)–(A1–17),

$$\begin{aligned} &\int_{\mathscr{S}} \left\{ \sigma K - (\mathbf{F}^\top \mathbf{S}) \cdot \mathbf{L} - \mathbf{m} \cdot [\boldsymbol{F}^\top \boldsymbol{S}]\mathbf{m} + \operatorname{Div}_{\mathscr{S}} (\mathbf{d} - \mathbf{S}^\top \langle \boldsymbol{F} \rangle \mathbf{m}) + h^{\mathscr{S}} + e^{\mathscr{S}} \right\} V\, da \\ &\quad + \int_{\mathscr{S}} (\operatorname{Div}_{\mathscr{S}} \mathbf{S} + [\boldsymbol{S}\mathbf{m}] + \boldsymbol{b}^{\mathscr{S}}) \cdot \overset{\square}{\boldsymbol{y}}\, da = 0 \end{aligned} \quad \text{(A1–18)}$$

on $\mathscr{S}$ for any choice of the virtual kinematics. Take $V \equiv 0$ and let $\dot{\boldsymbol{y}}$ be an arbitrary *smooth* vector field; (A1–18) then reduces to

$$\int_{\mathscr{S}} (\operatorname{Div}_{\mathscr{S}} \mathbf{S} + [\boldsymbol{S}\mathbf{m}] + \boldsymbol{b}^{\mathscr{S}}) \cdot \dot{\boldsymbol{y}}\, da = 0$$

for all such $\dot{\boldsymbol{y}}$, which implies (A1–9). Thus (A1–18) yields

$$\int_{\mathscr{S}} \left\{ \sigma K - (\mathbf{F}^\top \mathbf{S}) \cdot \mathbf{L} - \mathbf{m} \cdot [\boldsymbol{F}^\top \boldsymbol{S}]\mathbf{m} + \operatorname{Div}_{\mathscr{S}} (\mathbf{d} - \mathbf{S}^\top \langle \boldsymbol{F} \rangle \mathbf{m}) + h^{\mathscr{S}} + e^{\mathscr{S}} \right\} V\, da = 0$$

for all smooth scalar fields V, and this implies (A1–10). The weak principle is virtual work therefore implies the force balances (A1–8)–(A1–10). The converse assertion, that (A1–8)–(A1–10) imply the weak principle of virtual work, is left to the reader.

A2. Strong Principle of Virtual Work

The interface, the deformation gradient, and the virtual kinematics are as described in Appendix A1. Here, additional structure is introduced through the consideration of arbitrary control volumes that undergo virtual migrations.

a. Virtually migrating control volumes

Let a virtual kinematics $\boldsymbol{K} = (V, \dot{\boldsymbol{y}})$ be given. A virtually migrating control volume compatible with $\boldsymbol{K}$ is a triplet $\boldsymbol{P} = (P, \boldsymbol{q}, \boldsymbol{w})$ consisting of a *fixed* subregion P of B together with a virtual velocity field $\boldsymbol{q}$ for ∂P and a virtual velocity field $\boldsymbol{w}$ for $\partial\mathscr{G}$, where

$$\mathscr{G} = \mathscr{S} \cap P,$$

and $\boldsymbol{q}$ and $\boldsymbol{w}$ are consistent with the compatibility conditions

$$\boldsymbol{q} \cdot \mathbf{m} = \boldsymbol{w} \cdot \mathbf{m} = V, \tag{A2–1a}$$

$$\boldsymbol{q} \cdot \boldsymbol{n} = \boldsymbol{w} \cdot \boldsymbol{n}, \qquad (\boldsymbol{n} = \text{outward unit normal to } \partial P) \tag{A2–1b}$$

(cf. (15–33)). Note that the component of $\boldsymbol{q}$ tangent to ∂P is unconstrained, and because the plane spanned by $\mathbf{m}$ and $\boldsymbol{n}$ is perpendicular to the curve $\partial\mathscr{G}$, the component of $\boldsymbol{w}$ tangent to $\partial\mathscr{G}$ is also unconstrained. Given a virtually migrating control volume $(P, \boldsymbol{q}, \boldsymbol{w})$, the motion velocities $\mathring{\boldsymbol{y}}$ and $\overset{\triangle}{\boldsymbol{y}}$ following ∂P and $\partial\mathscr{G}$, respectively, are defined by (cf. (4–4), (21–4))

$$\mathring{\boldsymbol{y}} = \dot{\boldsymbol{y}} + \boldsymbol{F}\boldsymbol{q}, \qquad \overset{\triangle}{\boldsymbol{y}} = \langle \dot{\boldsymbol{y}} \rangle + \langle \boldsymbol{F} \rangle \boldsymbol{w}. \tag{A2–2}$$

Let $\mathbf{n}$ denote the outward unit normal to $\partial\mathscr{G}$. When $\boldsymbol{q}$ and $\boldsymbol{w}$ have the explicit forms

$$\boldsymbol{q} = U\boldsymbol{n}, \qquad \boldsymbol{w} = \boldsymbol{v} + V_{\partial\mathscr{G}}\mathbf{n}, \tag{A2–3}$$

with U arbitrary and $V_{\partial\mathscr{G}}$ defined by $U = V(\mathbf{m}\cdot\boldsymbol{n}) + V_{\partial\mathscr{G}}(\mathbf{n}\cdot\boldsymbol{n})$, so that (A2–1) are satisfied (cf. Subsection 15b2), then (A1–1) and (A1–4c), (A2–2) take the form

$$\mathring{\boldsymbol{y}} = \dot{\boldsymbol{y}} + U\boldsymbol{F}\boldsymbol{n}, \qquad \overset{\triangle}{\boldsymbol{y}} = \overset{\square}{\boldsymbol{y}} + V_{\partial\mathscr{G}}\mathbf{F}\mathbf{n}\,. \tag{A2–4}$$

The relations (A2–3) and (A2–4) are virtual counterparts of the intrinsic velocities specified in the paragraphs containing (21–4) and (21–5).

b. Forces. Strong principle of virtual work

The standard and configurational force systems are represented by the fields $\boldsymbol{S}$, $\boldsymbol{b}$, $\mathbf{S}$, $\boldsymbol{b}^{\mathscr{S}}$, σ, $\mathbf{d}$ and $e^{\mathscr{S}}$ discussed in Appendix A1, but with the *configurational* system supplemented by the following fields:

$\boldsymbol{C}$	**bulk stress** (scalar);
$\mathbf{C}$	**surface stress** (superficial tensor);
π	**bulk tension** (scalar);
$g^{\mathscr{S}}$	**internal interface force** (scalar).

Here $\boldsymbol{C}$ and π are fields on B that are continuous away from $\mathscr{S}$ and up to $\mathscr{S}$ from either side, while $\mathbf{C}$ and $g^{\mathscr{S}}$ are continuous on $\mathscr{S}$. As before, $\mathbf{C}_{\text{tan}}$ and $\boldsymbol{\tau}$ denote the tangential and normal parts of $\mathbf{C}$:

$$\mathbf{C} = \mathbf{C}_{\text{tan}} + \mathbf{m}\otimes\boldsymbol{\tau}. \tag{A2–5}$$

The **virtual external working** $W_{\text{ext}}(\boldsymbol{K},\boldsymbol{P})$ and the **virtual internal working** $W_{\text{int}}(\boldsymbol{K},\boldsymbol{P})$, corresponding to a virtual kinematics $\boldsymbol{K} = (V,\dot{\boldsymbol{y}})$ and a virtually migrating control volume $\boldsymbol{P} = (P,\boldsymbol{q},\boldsymbol{w})$ compatible with $\boldsymbol{K}$, are defined by

$$W_{\text{ext}}(\boldsymbol{K},\boldsymbol{P}) = \int_{\partial P}(\boldsymbol{C}\boldsymbol{n}\cdot\boldsymbol{q} + \boldsymbol{S}\boldsymbol{n}\cdot\mathring{\boldsymbol{y}})da + \int_P \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\,dv$$
$$+ \int_{\mathscr{G}}(e^{\mathscr{S}}V + \boldsymbol{b}^{\mathscr{S}}\cdot\overset{\square}{\boldsymbol{y}})da + \int_{\partial\mathscr{G}}(\mathbf{C}\mathbf{n}\cdot\boldsymbol{w} + \mathbf{S}\mathbf{n}\cdot\overset{\triangle}{\boldsymbol{y}})ds, \tag{A2–6a}$$

$$W_{\text{int}}(\boldsymbol{K},\boldsymbol{P}) = \int_P \boldsymbol{S}\cdot\dot{\boldsymbol{F}}\,dv + \int_G \left\{\mathbf{S}\cdot\langle\boldsymbol{F}\rangle^{\square} - \sigma K V - \mathbf{d}\cdot\overset{\square}{\mathbf{m}} - ([\pi] + g^{\mathscr{S}})V\right\} da$$
$$+ \int_{\partial\mathscr{G}}\sigma(\boldsymbol{w}\cdot\mathbf{n})ds + \int_{\partial P}\pi(\boldsymbol{q}\cdot\boldsymbol{n})da, \tag{A2–6b}$$

where $\mathscr{G} = \mathscr{S}\cap P$ and $\mathbf{n}$ is the outward unit normal to $\partial\mathscr{G}$ (cf. (21–6) and (21–18)). Note that the surface tension σ and the surface shear $\mathbf{d}$ are simply fields that perform work internally during virtual changes of interfacial area and orientation, while π is a field that performs work during virtual changes in volume; at this point in the discussion these fields bear no relation to the standard and configurational stresses $\boldsymbol{S}$, $\mathbf{S}$, $\boldsymbol{C}$, and $\mathbf{C}$.

The **strong principle of virtual work** is the assertion that, for any choice of virtual kinematics $\boldsymbol{K} = (V,\dot{\boldsymbol{y}})$, and any virtually migrating control volume $\boldsymbol{P} =$

$(P, \boldsymbol{q}, \boldsymbol{w})$ compatible with $\boldsymbol{K}$,

$$W_{\text{ext}}(\boldsymbol{K}, \boldsymbol{P}) = W_{\text{int}}(\boldsymbol{K}, \boldsymbol{P}). \tag{A2–7}$$

Strong Theorem of Virtual Work. The strong principle of virtual work holds if and only if the relations

$$\boldsymbol{C} = \pi \mathbf{1} - \boldsymbol{F}^{\top} \boldsymbol{S}, \tag{A2–8a}$$

$$\mathbf{C}_{\text{tan}} = \sigma \mathbf{P} - \mathbf{F}^{\top} \mathbf{S}, \tag{A2–8b}$$

$$\boldsymbol{\tau} = \mathbf{d} - \mathbf{S}^{\top} \langle \boldsymbol{F} \rangle \mathbb{m}; \tag{A2–8c}$$

the standard bulk balance

$$\operatorname{Div} \boldsymbol{S} + \boldsymbol{b} = \mathbf{0}; \tag{A2–9}$$

the standard interfacial balance

$$[\boldsymbol{S}]\mathbb{m} + \operatorname{Div}_{\mathscr{S}} \mathbf{S} + \boldsymbol{b}^{\mathscr{S}} = \mathbf{0}; \tag{A2–10}$$

and the normal configuration balance

$$\mathbb{m} \cdot [\boldsymbol{C}]\mathbb{m} + \mathbf{C}_{\text{tan}} \cdot \mathbf{L} + \operatorname{Div}_{\mathscr{S}} \boldsymbol{\tau} + g^{\mathscr{S}} + e^{\mathscr{S}} = \mathbf{0} \tag{A2–11}$$

are satisfied.

c. Proof of the strong theorem of virtual work

Assume that the strong principle of virtual work holds. Then, because $W_{\text{int}}(\boldsymbol{K}, \boldsymbol{P})$ depends on $\boldsymbol{q}$ and $\boldsymbol{w}$ at most through $\boldsymbol{w} \cdot \mathbb{n}$ and $\boldsymbol{q} \cdot \boldsymbol{n}$, this must also be true for $W_{\text{ext}}(\boldsymbol{K}, \boldsymbol{P})$. By (A2–2), the portion of $W_{\text{ext}}(\boldsymbol{K}, \boldsymbol{P})$ that depends on $\boldsymbol{q}$ and $\boldsymbol{w}$ is

$$\int_{\partial P} \boldsymbol{q} \cdot (\boldsymbol{C} + \boldsymbol{F}^{\top} \boldsymbol{S}) \boldsymbol{n} \, da + \int_{\partial \mathscr{G}} \boldsymbol{w} \cdot (\mathbf{C} + \langle \boldsymbol{F} \rangle^{\top} \mathbf{S}) \mathbb{n} \, ds, \tag{A2–12}$$

and arguments identical to those used to verify (5–14) and (21–11) yield the existence of scalar fields ω and ξ such that $\boldsymbol{C} + \boldsymbol{F}^{\top} \boldsymbol{S} = \omega \mathbf{1}$ and $\mathbf{C}_{\text{tan}} + \mathbf{F}^{\top} \mathbf{S} = \xi \mathbf{P}$. Thus (A2–12) reduces to

$$\int_{\partial P} \omega (\boldsymbol{q} \cdot \boldsymbol{n}) da + \int_{\partial \mathscr{G}} \xi (\boldsymbol{w} \cdot \mathbb{n}) ds, \tag{A2–13}$$

and (A2–13) must be equal to the portion of $W_{\text{int}}(\boldsymbol{K}, \boldsymbol{P})$ that depends on $\boldsymbol{q}$ and $\boldsymbol{w}$, viz.

$$\int_{\partial P} \pi (\boldsymbol{q} \cdot \boldsymbol{n}) da + \int_{\partial \mathscr{G}} \sigma (\boldsymbol{w} \cdot \mathbb{n}) ds. \tag{A2–14}$$

Therefore, $\omega = \pi$ and $\xi = \sigma$; this yields (A2–8a,b).

Next, taking the virtual fields $\boldsymbol{q}$ and $\boldsymbol{w}$ in the "intrinsic forms" (A2–3), so that (A2–4) are satisfied, and appealing to (A2–8a,b), the balance (A2–7) may be

written in the reduced form

$$\int_{\partial P} \boldsymbol{S}\boldsymbol{n}\cdot\dot{\boldsymbol{y}}\,da + \int_P \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\,dv + \int_{\mathscr{G}} (\boldsymbol{b}^{\mathscr{S}}\cdot\overset{\square}{\boldsymbol{y}} + e^{\mathscr{S}}V)da + \int_{\partial\mathscr{G}} (\mathbf{S}\boldsymbol{n}\cdot\overset{\square}{\boldsymbol{y}} + \mathbf{C}\boldsymbol{n}\cdot\boldsymbol{v})ds$$
$$= \int_P \boldsymbol{S}\cdot\dot{\boldsymbol{F}}\,dv + \int_{\mathscr{G}} \left\{ \mathbf{S}\cdot\langle\boldsymbol{F}\rangle^{\square} - \sigma KV - \mathbf{d}\cdot\overset{\square}{\mathbf{m}} - ([\pi] + g^{\mathscr{S}})V \right\} da \quad \text{(A2–15)}$$

and is to be satisfied for any choice of the virtual kinematics $(V, \dot{\boldsymbol{y}})$ and all fixed control volumes P. By (A1–5a) and (A2–5),

$$\mathbf{C}\cdot\nabla_{\mathscr{S}}\boldsymbol{v} = -V\mathbf{C}_{\tan}\cdot\mathbf{L} - \boldsymbol{\tau}\cdot\overset{\square}{\mathbf{m}};$$

hence, by (A1–13) and (A2–8b),

$$\begin{aligned}\mathbf{C}\cdot\nabla_{\mathscr{S}}\boldsymbol{v} + \mathbf{S}\cdot\nabla_{\mathscr{S}}(\overset{\square}{\boldsymbol{y}}) &= -V(\mathbf{C}_{\tan} + \mathbf{F}^{\top}\mathbf{S})\cdot\mathbf{L} - (\boldsymbol{\tau} + \mathbf{S}^{\top}\langle\boldsymbol{F}\rangle\mathbf{m})\cdot\overset{\square}{\mathbf{m}} + \mathbf{S}\cdot\langle\boldsymbol{F}\rangle^{\square} \\ &= -\sigma KV - (\boldsymbol{\tau} + \mathbf{S}^{\top}\langle\boldsymbol{F}\rangle\mathbf{m})\cdot\overset{\square}{\mathbf{m}} + \mathbf{S}\cdot\langle\boldsymbol{F}\rangle^{\square}. \qquad \text{(A2–16)}\end{aligned}$$

Further, as in (21–14),

$$\int_{\partial\mathscr{G}} (\mathbf{S}\boldsymbol{n}\cdot\overset{\square}{\boldsymbol{y}} + \mathbf{C}\boldsymbol{n}\cdot\boldsymbol{v})ds = \int_{\mathscr{G}} \left\{ \overset{\square}{\boldsymbol{y}}\cdot\mathrm{Div}_{\mathscr{S}}\mathbf{S} + \mathbf{S}\cdot\nabla_{\mathscr{S}}(\overset{\square}{\boldsymbol{y}}) + \boldsymbol{v}\cdot\mathrm{Div}_{\mathscr{S}}\mathbf{C} + \mathbf{C}\cdot\nabla_{\mathscr{S}}\boldsymbol{v} \right\} da.$$

Thus, by (A2–15), (A2–14) takes the form

$$\int_{\partial P} \boldsymbol{S}\boldsymbol{n}\cdot\dot{\boldsymbol{y}}\,da + \int_P \boldsymbol{b}\cdot\dot{\boldsymbol{y}}\,dv + \int_{\mathscr{G}} \left\{ (\mathrm{Div}_{\mathscr{S}}\mathbf{S} + \boldsymbol{b}^{\mathscr{S}})\cdot\overset{\square}{\boldsymbol{y}} + (\mathbf{m}\cdot\mathrm{Div}_{\mathscr{S}}\mathbf{C} + e^{\mathscr{S}})V \right\} da$$
$$= \int_P \boldsymbol{S}\cdot\dot{\boldsymbol{F}}\,dv + \int_{\mathscr{G}} \left\{ (\boldsymbol{\tau} + \mathbf{S}^{\top}\langle\boldsymbol{F}\rangle\mathbf{m} - \mathbf{d})\cdot\overset{\square}{\mathbf{m}} - ([\pi] + g^{\mathscr{S}})V \right\} da. \quad \text{(A2–17)}$$

Choosing a control volume that does not intersect the interface yields, after applying the divergence theorem to the integral over ∂P, $\int_P (\mathrm{Div}\,\boldsymbol{S} + \boldsymbol{b})\cdot\dot{\boldsymbol{y}}\,dv = 0$; because this must hold for all virtual velocities $\dot{\boldsymbol{y}}$, the bulk relation (A2–9) must be satisfied.

Next, the argument used to establish (A2–12) holds with B and $\mathscr{S}$ replaced by P and $\mathscr{G}$, respectively, and since (A2–8a) implies that $\mathbf{m}\cdot[\boldsymbol{F}^{\top}\boldsymbol{S}]\mathbf{m} = [\pi] - \mathbf{m}\cdot[\boldsymbol{C}]\mathbf{m}$, (A2–16) reduces to

$$\int_{\mathscr{G}} \left\{ (\mathrm{Div}_{\mathscr{S}}\mathbf{S} + [\boldsymbol{S}]\mathbf{m} + \boldsymbol{b}^{\mathscr{S}})\cdot\overset{\square}{\boldsymbol{y}} + (\mathbf{m}\cdot\mathrm{Div}_{\mathscr{S}}\mathbf{C} + \mathbf{m}\cdot[\boldsymbol{C}]\mathbf{m} + g^{\mathscr{S}} + e^{\mathscr{S}})V \right\} da$$
$$= \int_{\mathscr{G}} \left\{ (\boldsymbol{\tau} + \mathbf{S}^{\top}\langle\boldsymbol{F}\rangle\mathbf{m} - \mathbf{d})\cdot\overset{\square}{\mathbf{m}} \right\} da. \quad \text{(A2–18)}$$

Because $\mathscr{G}$ may be chosen arbitrarily,

$$(\mathrm{Div}_{\mathscr{S}}\mathbf{S} + [\boldsymbol{S}]\mathbf{m} + \boldsymbol{b}^{\mathscr{S}})\cdot\overset{\square}{\boldsymbol{y}} + (\mathbf{m}\cdot\mathrm{Div}_{\mathscr{S}}\mathbf{C} + \mathbf{m}\cdot[\boldsymbol{C}]\mathbf{m} + g^{\mathscr{S}} + e^{\mathscr{S}})V$$
$$- (\boldsymbol{\tau} + \mathbf{S}^{\top}\langle\boldsymbol{F}\rangle\mathbf{m} - \mathbf{d})\cdot\overset{\square}{\mathbf{m}} = 0 \quad \text{(A2–19)}$$

on $\mathscr{S}$ for any choice of the virtual kinematics. Take $V \equiv 0$ (so that $\overset{\square}{\mathbf{m}} \equiv 0$) and let $\dot{\boldsymbol{y}}$ be an arbitrary *smooth* vector field. Then, by (A1–4c), (A2–17) takes the form

$$(\mathrm{Div}_{\mathscr{S}} \mathbf{S} + [\boldsymbol{S}]\mathbf{m} + \boldsymbol{b}^{\mathscr{S}}) \cdot \dot{\boldsymbol{y}} = 0;$$

since $\dot{\boldsymbol{y}}$ is arbitrary, this yields (A2–10). Thus, by (A1–4b),

$$(\mathbf{m} \cdot \mathrm{Div}_{\mathscr{S}} \mathbf{C} + \mathbf{m} \cdot [\boldsymbol{C}]\mathbf{m} + g^{\mathscr{S}} + e^{\mathscr{S}})V + (\boldsymbol{\tau} + \mathbf{S}^{\top}\langle \boldsymbol{F} \rangle \mathbf{m} - \mathbf{d}) \cdot \nabla_{\mathscr{S}} V = 0.$$

Given any point $\boldsymbol{X}_0$ of $\mathscr{S}$ it is possible to choose a smooth field V such that $V(\boldsymbol{X}_0)$ and $\nabla_{\mathscr{S}} V(\boldsymbol{X}_0)$ have arbitrarily prescribed values. This yields (A2–8c) and (A2–11). (The vector $\nabla_{\mathscr{S}} V(\boldsymbol{X}_0)$ is necessarily tangent to $\mathscr{S}$ at $\boldsymbol{X}_0$, but, in view of the sentence following (A1–13), so is $\boldsymbol{\tau} + \mathbf{S}^{\top}\langle \boldsymbol{F} \rangle \mathbf{m} - \mathbf{d}$.)

The converse assertion, that the relations (A2–8) and the force balances (A2–9)–(A2–11) imply the strong principle of virtual work, is left to the reader.

d. Comparison of the strong and weak principles

The weak principle of virtual work is equivalent to the standard bulk and interfacial balances

$$\mathrm{Div}\, \boldsymbol{S} + \boldsymbol{b} = \boldsymbol{0}, \tag{A2–20a}$$

$$[\boldsymbol{S}]\mathbf{m} + \mathrm{Div}_{\mathscr{S}} \mathbf{S} + \boldsymbol{b}^{\mathscr{S}} = \boldsymbol{0}, \tag{A2–20b}$$

and the normal configurational balance

$$\sigma K - (\mathbf{F}^{\top}\mathbf{S}) \cdot \mathbf{L} - \mathbf{m} \cdot [\boldsymbol{F}^{\top}\boldsymbol{S}]\mathbf{m} + \mathrm{Div}_{\mathscr{S}}(\mathbf{d} - \mathbf{S}^{\top}\langle \boldsymbol{F} \rangle \mathbf{m}) + h^{\mathscr{S}} + e^{\mathscr{S}} = 0. \tag{A2–21}$$

Note that neither the principle nor the balances involve the configurational stresses $\boldsymbol{C}$ and $\mathbf{C}$ or the bulk tension π, and $h^{\mathscr{S}}$ is not the internal configurational force but instead the sum of all fields internally work-conjugate to V.

The strong principle of virtual work makes use of the configurational stresses $\boldsymbol{C}$ and $\mathbf{C}$ as well as the bulk tension π, and it delivers explicit relations for the stresses:

$$\boldsymbol{C} = \pi \mathbf{1} - \boldsymbol{F}^{\top}\boldsymbol{S}, \tag{A2–22a}$$

$$\mathbf{C}_{\mathrm{tan}} = \sigma \mathbf{P} - \mathbf{F}^{\top}\mathbf{S}, \tag{A2–22b}$$

$$\boldsymbol{\tau} = \mathbf{d} - \mathbf{S}^{\top}\langle \boldsymbol{F} \rangle \mathbf{m}. \tag{A2–22c}$$

In fact, the strong principle is equivalent to (A2–22) together with the standard balances (A2–20) and the normal configurational balance

$$\mathbf{m} \cdot [\boldsymbol{C}]\mathbf{m} + \mathbf{C}_{\mathrm{tan}} \cdot \mathbf{L} + \mathrm{Div}_{\mathscr{S}} \boldsymbol{\tau} + g^{\mathscr{S}} + e^{\mathscr{S}} = \boldsymbol{0}. \tag{A2–23}$$

Granted (A2–22), the configurational balances (A2–21) and (A2–23) are equivalent provided

$$h^{\mathscr{S}} = g^{\mathscr{S}} + [\pi]. \tag{A2–24}$$

A chief difference between the two principles is that the weak principle is written for the body as a whole, while the strong principle is written for arbitrary control volumes, which may undergo virtual migrations. The structure of the strong principle is far more detailed than that of the weak principle and as such embodies more physics. The strong principle may be combined with other physical laws such as the second law in the form (21–19). On the other hand, the simplicity of the weak principle makes it appropriate as a weak statement of the force balances, and as such it may be useful for analysis, granted a knowledge of the more detailed structure needed to formulate boundary- or initial-value problems.

The weak and strong principles of virtual work are equivalent to the set of force balances not rendered irrelevant by the presence of indeterminate forces.[2] That is both the strength and the weakness of the principle of virtual work. Because it yields only the relevant balances, it uses only those fields that enter those balances: no more, no less. But this simplicity is at the expense of a physical framework in which configurational forces obey a balance that is well defined at and away from defects, a balance that has the classical form common to most basic laws of continuum physics.

One could enlarge the theory slightly and get all balances from the virtual form of invariance under changes in observer. But then the theory would be essentially the same as the theory in the main body of the book, where force balances follow from invariance of the working. Moreover, within that theory, internal working is a derived quantity, not an independent notion.

[2] E.g., for a coherent interface the only relevant configurational balance is the scalar interfacial balance (A2–11).

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