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A Possible Way for Constructing Generators of the Poincaré Group in Quantum Field Theory

In memory of Mikhail Shirokov, the excellent scientist and modest person

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Abstract Starting from the instant form of relativistic quantum dynamics for a system of interacting fields, where amongst the ten generators of the Poincaré group only the Hamiltonian and the boost operators carry interactions, we offer an algebraic method to satisfy the Poincaré commutators. We do not need to employ the Lagrangian formalism for local fields with the Nöether representation of the generators. Our approach is based on an opportunity to separate in the primary interaction density a part which is the Lorentz scalar. It makes possible apply the recursive relations obtained in this work to construct the boosts in case of both local field models (for instance with derivative couplings and spins ≥ 1) and their nonlocal extensions. Such models are typical of the meson theory of nuclear forces, where one has to take into account vector meson exchanges and introduce meson-nucleon vertices with cutoffs in momentum space. Considerable attention is paid to finding analytic expressions for the generators in the clothed-particle representation, in which the so-called bad terms are simultaneously removed from the Hamiltonian and the boosts. Moreover, the mass renormalization terms introduced in the Hamiltonian at the very beginning turn out to be related to certain covariant integrals that are convergent in the field models with appropriate cutoff factors.

1 Introduction

After Dirac [1], any relativistic quantum theory may be so defined that the generator of time translations (Hamiltonian), the generators of space translations (linear momentum), space rotations (angular momentum) and Lorentz transformations (boost operator) satisfy the well-known commutations. Basic ideas, put forward by Dirac with his "front", "instant" and "point" forms of the relativistic dynamics, have been realized in many relativistic quantum mechanical models. In this context, the survey [2], being remarkable introduction to a subfield called the relativistic Hamiltonian dynamics, reflects various aspects and achievements of relativistic direct interaction theories. Among the vast literature on this subject we would like to note an exhaustive exposition in lectures [3,4] of the appealing features of the relativistic Hamiltonian dynamics with an emphasis on "light-cone quantization". Following a pioneering work [5], the term "direct" is related to a system with a fixed number of interacting particles, where interactions are rather direct than mediated through a field. In the approach it is customary to consider such interactions expressed in terms of the particle coordinates, momenta and spins.

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This notion supplemented by the principle of cluster separability (decomposition) was developed (see [2] and refs. therein) and applied to build up the so-called separable interactions and relativistic center-of-mass variables for composite systems [5,6]. There were assumed that the generators of the Poincaré group (Π) can be represented as expansions on powers of $1/c^2$ or, more exactly, $(v/c)^2$, where v is a typical nuclear velocity (cf. the (p/m) expansion, introduced in [7,8] in which m is the nucleon mass and p is a typical nucleon momentum). Afterwards, similar expansions were rederived and reexamined (with new physical inputs) in the framework of another approach [9] (sometimes called the Okubo–Glöckle–Müller method [10]). There, starting from a model Lagrangian for "scalar nucleons" interacting with a scalar meson field (cf. the Wentzel model [11]) the authors showed (to our knowledge first) how the Hamiltonian and the boost generator (the noncommuting operators), determined in a standard manner [12], can be blockdiagonalized by one and the same unitary transformation (UT) after Okubo [13]. The corresponding blocks derived in leading order in the coupling constant act in the subspace with a fixed nucleon number (the nucleon "sector" of the full Fock space R_F).

In general, the work [9] and its continuation [14,15] exemplify applications of local relativistic quantum field theory (QFT), where the generators of interest, being compatible with the basic commutation rules for fields, are constructed within the Lagrangian formalism using the Nöther theorem and its consequences. Although the available covariant perturbation theory and functional-integral methods are very successful when describing various relativistic and quantum effects in the world of elementary particles, the Hamilton method can be helpful too.

It is well known that it is the case, where one has to study properties of strongly interacting particles, e.g., as in nuclear physics with its problems of bound states for meson-nucleon systems. Of course, any Hamiltonian formulation of field theory, not being manifestly covariant, cannot be *ab initio* accepted as equivalent to the way after Feynman, Schwinger and Tomonaga. However, in order to overcome the obstacle starting from a field Hamiltonian H one can consider it as one of the ten infinitesimal operators (generators) of space-time translations and pure Lorentz transformations that act in a proper Hilbert space. Taken together they form a basis of the Poincaré–Lie algebra with the aforementioned commutation relations to ensure relativistic invariance (RI) in the Dirac sense, being referred to the RI as a whole. These relations will be recalled below to fix the notations and simplify the reference processing. Our main purpose is to meet the Poincaré commutators for a given interaction density which has the property to be a Lorentz scalar in the Dirac (D) picture. Such a possibility may be realized both in local and nonlocal models taking account their invariance with respect to space translations. It turns out that an algebraic method, which has been elaborated by us to get a recursive solution of the problem in question, works also in models (for instance, with derivative couplings and spin ≥ 1) where only some part of the interaction density in the D picture is the scalar.

As an illustration of our method, we will show its application for a nonlocal extension of the Wentzel model. At the point, let us remind of the nonlocal convergent field theory [16,17], where a conventional interaction Hamiltonian in the D picture (e.g., in quantum electrodynamics) is replaced by a nonlocal interaction with a formfactor (FF) to separate the field operators related to different points of the Minkowski space (cf. monograph [18] and refs. therein in which the same idea has been used directly for the initial action integral). Unlike this in what follows, where we are addressing the particle representation (see, for example, Chapter II in lectures [19] and Chapter IV of monograph [20]), the field concept has no its paramount importance, being only a departure point for an alternative consideration of the RI with particle creation. In the framework of the particle representation a nonlocal Hamiltonian for interacting particles can be built up by introducing some "cutoff" function (shortly g-factor) in every vertex which is associated with any particle creation and/or annihilation process. Such cutoffs in momentum space may be done either phenomenologically or with the aid of deeper physical reasonings as in case of the meson-nucleon vertices that can be calculated in different quark models (see, e.g., [21]).

As usually, the g-factors are needed, first of all, to carry out finite intermediate calculations trying to remove ultraviolet divergences inherent in local field models. One should emphasize that we include them in the Yukawa-type interactions in the "bare" particle representation (BPR) to derive or rather substantiate the corresponding regularized interactions between the so-called clothed particles (see Appendix C in [22]). Their falloff properties with the momenta increasing are also important to do convergent calculations of strong and electromagnetic FFs (see Sect. 5).

Second, we will show how within the three-dimensional formalism used here one can define a covariant generating function for the g-factors in case of a trilinear interaction. The function, being dependent on some Lorentz scalars composed of the particle three-momenta, plays a central role when integrating the Poincaré commutators and obtaining the analytical clothed-particle representation (CPR) expressions for the Hamiltonian, the boost operators, the mass renormalization terms, etc. [23,24].

Third, it is expected that by choosing appropriate g-factors (at least, as square integrable functions of the particle momenta) one can remove certain drawback of the initial local interaction not to have a dense domain in R_F, i.e., not to be self-adjoint and bounded below (the same is related to the boost operators). The delicate issue has been regarded in various papers devoted to the Nelson model [25] or "model with persistent vacuum" (see, e.g., [26,27] and refs. therein). It is true that the authors have confined themselves to the explorations with a sharp cutoff and antiparticles not included.

Along with a thorough option of the cutoffs for our nonlocal extensions of the conventional field models with the threelinear couplings the present research exemplifies one more realization of a fruitful idea put forward in relativistic QFT by Greenberg and Schweber [28] and developed by other authors, in particular, by Shirokov and his coworkers (see the survey [29] and refs. therein). First of all, we are keeping in mind their notion of "clothed" particles which points out a transparent way for including the so-called cloud or persistent effects in a system of interacting fields (to be definite, mesons and nucleons). It is achieved with the help of unitary clothing transformations (UCTs) (see article [30]) that implement the transition from the BPR to the CPR in the Hilbert space $\mathcal{H}$ of meson-nucleon states. In the course of the clothing procedure a large amount of virtual processes associated with the meson absorption/emission, the $N\bar{N}$-pair annihilation/production and other cloud effects turns out to be accumulated in the creation (destruction) operators for the clothed particles. The latter, being the quasiparticles of the method of UCTs, must have the properties (charges, masses etc.) of physical (observable) particles. Such a bootstrap reflects the most significant distinction between the concepts of clothed and bare particles.

As shown in [29] the total Hamiltonian H and the three boost operators $\mathbf{N} = (N^1, N^2, N^3)$ attain in the CPR one and the same sparse structure in $\mathcal{H}$ due to the elimination of the so-called bad terms that prevent the bare vacuum (the state without "bare" particles (following the terminology accepted in [29] (see also [30]) every time when we say bare particles the latter mean primary particles with physical masses)) and the bare one-particle states to be the H eigenstates. This result has been obtained within a conventional local model of the PS pion-nucleon coupling and, as we know, for local quantum theories usually one goes from a relativistically invariant Lagrangian to the corresponding Nöether integrals that satisfy the Lie algebra of the Poincaré group Π. Doing so, one can employ the Belinfante ansatz to express $\mathbf{N}$ through the Hamiltonian density (we recall it in Sect. 3). Here, trying to apply the UCT method to nonlocal field models we will move in the opposite direction, viz., from the fundamental Poincaré commutators towards the RI as a whole.

However, before to apply the UCT method (in particular, beyond the Lagrangian formalism with its local interaction densities) we will show and compare the two algebraic procedures to solve the basic commutator equations of Π (see Sect. 2). One of them, proposed here, has some touching points with the other developed in Refs. [31,32] and essentially repeated many years after by Chandler [33]. In paper [32] the author considers three kinds of neutral spinless bosons and nonlocal interaction between them in a relativistic version of the Lee model with a cutoff in momentum space. A similar model for two spinless particles has been utilized in [33] with a Yukawa-type interaction that belongs to the realm of the so-called models with persistent vacuum (see, for instance, [26]).

Certain resemblance between the present work and those explorations is that we prefer to proceed, as previously [23,24], within a corpuscular picture (see Chapter IV in monograph [20]), where each of the ten generators of the Poincaré group Π (and not only they) may be expressed as a sum of products of particle creation and annihilation operators $a^\dagger(n)$ and $a(n)$ $(n = 1, 2, \ldots)$ e.g., bosons and/or fermions. Some mathematical aspects of the corpuscular notion were formulated many years ago in [34] (Chapter III). As in [20], a label n is associated with all the necessary quantum numbers for a single particle: its momentum $\mathbf{p}_n$ (or the 4-momentum $p_n = (p_n^0, \mathbf{p}_n)$ on the mass shell $p_n^2 = p_n^{02} - \mathbf{p}_n^2 = m_n^2$ with the particle mass m_n), spin z-component (or for massless particles, helicity) μ_n, and species ξ_n. The operators $a^\dagger(n)$ and $a(n)$ satisfy the standard (canonical) commutation relations such as Eqs. (4.2.5)–(4.2.7) in [20].

In the framework of such a picture the Hamiltonian of a system of interacting mesons and nucleons can be written as

$$H = \sum_{C=0}^{\infty} \sum_{A=0}^{\infty} H_{CA}, \tag{1}$$

$$\begin{aligned} H_{CA} = \sum\!\!\!\!\!\!\int H_{CA}(1', 2', \ldots, n'_C; 1, 2, \ldots, n_A) \\ \times a^\dagger(1')a^\dagger(2')\ldots a^\dagger(n'_C)a(n_A)\ldots a(2)a(1), \end{aligned} \tag{2}$$

where the capital $C(A)$ denotes the particle-creation (annihilation) number for the operator substructure H_{CA}. Sometimes we say that the latter belongs to the class $[C.A]$ (cf. the terminology from [29]). Operation $\sum\!\!\!\!\!\!\int$ implies all necessary summations over discrete indices and covariant integrations over continuous spectra.

Further, it is proved [20] that the S-matrix meets the so-called cluster decomposition principle (see, e.g., [35]) if the coefficient functions H_{CA} embody a single three-dimensional momentum-conservation delta function, viz.,

$$H_{CA}(1', 2', \ldots, C; 1, 2, \ldots, A) = \delta(\mathbf{p}'_1 + \mathbf{p}'_2 + \cdots + \mathbf{p}'_C - \mathbf{p}_1 - \mathbf{p}_2 - \cdots - \mathbf{p}_A) \\ \times h_{CA}(p'_1\mu'_1\xi'_1, p'_2\mu'_2\xi'_2, \ldots, p'_C\mu'_C\xi'_C; p_1\mu_1\xi_1, p_2\mu_2\xi_2, \ldots, p_A\mu_A\xi_A), \tag{3}$$

where the c-number coefficients h_{CA} do not contain delta function.

Following the guideline "to free ourselves from any dependence on pre-existing field theories" (cit. from [20] on p. 175), the three boost operators $\mathbf{N} = (N^1, N^2, N^3)$ can be written as

$$\mathbf{N} = \sum_{C=0}^{\infty}\sum_{A=0}^{\infty} \mathbf{N}_{CA}, \tag{4}$$

$$\mathbf{N}_{CA} = \sum\!\!\!\!\!\!\int \mathbf{N}_{CA}(1', 2', \ldots, n'_C; 1, 2, \ldots, n_A) \\ \times a^\dagger(1')a^\dagger(2')\ldots a^\dagger(n'_C)a(n_A)\ldots a(2)a(1). \tag{5}$$

One of our purposes is to find some links between the coefficients in the r.h.s. of Eqs. (2) and (5), compatible with the fundamental relations of the Lie algebra for Π, that are given for convenience in Sect. 2.

In turn, the operator H, being divided into the no-interaction part H_F and the interaction H_I, owing to its translational invariance allows H_I to be written as

$$H_I = \int H_I(\mathbf{x})d\mathbf{x}. \tag{6}$$

Our consideration is focused upon various field models (local and nonlocal) in which the interaction density $H_I(\mathbf{x})$ consists of scalar $H_{sc}(\mathbf{x})$ and nonscalar $H_{nsc}(\mathbf{x})$ contributions,

$$H_I(\mathbf{x}) = H_{sc}(\mathbf{x}) + H_{nsc}(\mathbf{x}), \tag{7}$$

where the property to be a scalar means

$$U_F(\Lambda)H_{sc}(x)U_F^{-1} = H_{sc}(\Lambda x), \quad \forall x = (t, \mathbf{x}) \tag{8}$$

for all Lorentz transformations Λ. Henceforth, for any operator $O(\mathbf{x})$ in the Schrödinger (S) picture it is introduced its counterpart

$$O(x) = e^{iH_Ft}O(\mathbf{x})e^{-iH_Ft}$$

in the Dirac (D) picture.

In this context we would like to remind that in "... theories with derivative couplings or spins $j \geq 1$, it is not enough to take Hamiltonian as the integral over space of a scalar interaction density; we also need to add non-scalar terms to the interaction density to compensate non-covariant terms in the propagators" (quoted from Chapter 7 in [20]). Such a situation has been considered recently for interacting vector mesons and nucleons in the field-theoretical treatment [22,36] of nucleon-nucleon scattering. In any case, as will be shown, the existence of division (7) makes it possible to use and extend the available experience [29] in constructing the boost generators for a given $H_I(\mathbf{x})$.

As previously [24,29], special attention in our work is paid to the inclusion in H finite "mass-renormalization" terms that play an important role in ensuring the RI [32]. We stress "finite" since in what follows in order to get rid of the well known difficulties with divergences certain emphasis is made on nonlocal field models with a covariant cutoff. Thereby we prefer to deal with introducing cutoff functions in momentum space that is convenient for calculations of the S matrix (cf. a relativistic nonlocal field model proposed in [37] with cutoffs in coordinate space).

After this introduction we arrive to Sect. 2 which is devoted to some preliminaries concerning the underlying problem. In Sect. 3, by considering nonperturbative and perturbative recipes for ensuring the RI, we recall a number of relevant definitions from local QFT. Such a reminder enables us to set bridges between a traditional approach in QFT and direct algebraic means proposed here. By uniting our algebraic approach with the notion of clothed particles in QFT, in Sect. 4 we are seeking the boost operators in the CPR. Along the headline we introduce a nonstandard definition of the so-called mass renormalization terms and show their importance for ensuring the RI within a wide class of field models (local and nonlocal). Section 5 is contained explicit expressions for the interactions (quasipotentials) between the spinless scalar and charged bosons and the corresponding renormalization integrals.

The Appendices A, B and C are contained, respectively,

a) formulae for the Poincaré generators of free pions and nucleons in the corpuscular picture
b) equal-time commutators for the pion-nucleon interaction densities with a nonlocal trilinear coupling
c) evaluation of an integral that determines the mass renormalization term in case of a relativistic nonlocal model for interacting spinless neutral and charged bosons.

2 Basic Equations in Relativistic Theory with Particle Creation and Annihilation

For convenience, the Poincaré generators can be divided into the three kinds for: no-interaction generators

$$[P_i, P_j] = 0, \qquad [J_i, J_j] = i\varepsilon_{ijk} J_k, \qquad [J_i, P_j] = i\varepsilon_{ijk} P_k, \tag{9}$$

generators linear in H and $\mathbf{N}$

$$[\mathbf{P}, H] = 0, \qquad [\mathbf{J}, H] = 0, \qquad [J_i, N_j] = i\varepsilon_{ijk} N_k, \qquad [P_i, N_j] = i\delta_{ij} H, \tag{10}$$

and ones nonlinear in H and $\mathbf{N}$

$$[H, \mathbf{N}] = i\mathbf{P}, \qquad [N_i, N_j] = -i\varepsilon_{ijk} J_k, \qquad (i, j, k = 1, 2, 3), \tag{11}$$

where $\mathbf{P} = (P^1, P^2, P^3)$ and $\mathbf{J} = (J^1, J^2, J^3)$ are the linear momentum and angular momentum operators, respectively. In this context, let us remind that in the instant form of relativistic dynamics after Dirac [1] only the Hamiltonian and the boost operators carry interactions with conventional partitions

$$H = H_F + H_I \tag{12}$$

and

$$\mathbf{N} = \mathbf{N}_F + \mathbf{N}_I, \tag{13}$$

while $\mathbf{P} = \mathbf{P}_F$ and $\mathbf{J} = \mathbf{J}_F$. In short notations, we distinguish the set $G_F = \{H_F, \mathbf{P}_F, \mathbf{J}_F, \mathbf{N}_F\}$ for free particles and the set $G = \{H, \mathbf{P}_F, \mathbf{J}_F, \mathbf{N}\}$ for interacting particles.

In turn, every operator H_{CA} can be represented as

$$H_{CA} = \int H_{CA}(\mathbf{x}) d\mathbf{x}, \tag{14}$$

if one uses the formula

$$\delta(\mathbf{p} - \mathbf{p}') = \frac{1}{(2\pi)^3} \int e^{i(\mathbf{p}-\mathbf{p}')\mathbf{x}} d\mathbf{x}.$$

Thus, we come to the form well known from local field models,

$$H = \int H(\mathbf{x}) d\mathbf{x} \tag{15}$$

with the density

$$H(\mathbf{x}) = \sum_{C=0}^{\infty} \sum_{A=0}^{\infty} H_{CA}(\mathbf{x}). \tag{16}$$

For instance, in case with $C = A = 2$,

$$H_{22}(1', 2'; 1, 2) = \delta(\mathbf{p}_1' + \mathbf{p}_2' - \mathbf{p}_1 - \mathbf{p}_2) h(1', 2'; 1, 2) \tag{17}$$

and

$$H_{22}(\mathbf{x}) = \frac{1}{(2\pi)^3} \sum\!\!\!\!\!\!\int \exp[-i(\mathbf{p}_1' + \mathbf{p}_2' - \mathbf{p}_1 - \mathbf{p}_2)\mathbf{x}] \\ \times h(1', 2'; 1, 2) a^\dagger\left(1'\right) a^\dagger\left(2'\right) a(2) a(1). \tag{18}$$

Further, we will employ the transformation properties of the creation and annihilation operators with respect to Π. For example, in case of a massive particle with the mass m and spin j one considers that

$$U_F(\Lambda, b) a^\dagger(p, \mu) U_F^{-1}(\Lambda, b) = e^{i\Lambda pb} D^{(j)}_{\mu'\mu}(W(\Lambda, p)) a^\dagger(\Lambda p, \mu'), \\ \forall \Lambda \in L_+ \text{ and arbitrary spacetime shifts } \mathrm{b} = (\mathrm{b}^0, \mathbf{b}) \tag{19}$$

with D-function whose argument is the Wigner rotation $W(\Lambda, p)$, L_+ the homogeneous (proper) orthochronous Lorentz group. The correspondence $(\Lambda, b) \to U_F(\Lambda, b)$ between elements $(\Lambda, b) \in \Pi$ and unitary transformations $U_F(\Lambda, b)$ realizes an irreducible representation of Π on the Hilbert space $\mathcal{H}$ (to be definite) of meson-nucleon states. In this context, it is convenient to employ the operators $a(p, \mu) = a(\mathbf{p}, \mu)\sqrt{p_0}$ that meet the covariant commutation relations

$$[a(p', \mu'), a^\dagger(p, \mu)]_\pm = p_0 \delta(\mathbf{p} - \mathbf{p}') \delta_{\mu'\mu}, \\ [a(p', \mu'), a(p, \mu)]_\pm = [a^\dagger(p', \mu'), a^\dagger(p, \mu)]_\pm = 0. \tag{20}$$

Here $p_0 = \sqrt{\mathbf{p}^2 + m^2}$ is the fourth component of the 4-momentum $p = (p_0, \mathbf{p})$.

The relativistic invariance (RI) implies

$$U_F(\Lambda, b) H_{22}(x) U_F^{-1}(\Lambda, b) = H_{22}(\Lambda x + b), \quad \forall x = (t, \mathbf{x}). \tag{21}$$

Accordingly this definition we have

$$H_{22}(x) = \frac{1}{(2\pi)^3} \sum\!\!\!\!\!\!\int \exp[i(p_1' + p_2' - p_1 - p_2)x] \\ \times h(1', 2'; 1, 2) a^\dagger\left(1'\right) a^\dagger(2') a(2) a(1). \tag{22}$$

With the aid of Eq. (19) it is easily seen that condition (21) imposes the following constraint upon the h-coefficients in the r.h.s. of Eq. (22):

$$D^{(j_1')}_{\eta_1'\mu_1'}(W(\Lambda, p_1')) D^{(j_2')}_{\eta_2'\mu_2'}(W(\Lambda, p_2')) D^{(j_1)*}_{\eta_1\mu_1}(W(\Lambda, p_1)) D^{(j_2)*}_{\eta_2\mu_2}(W(\Lambda, p_2)) \\ \times h(p_1'\mu_1', p_2'\mu_2'; p_1\mu_1, p_2\mu_2) = h(\Lambda p_1'\eta_1', \Lambda p_2'\eta_2'; \Lambda p_1\eta_1, \Lambda p_2\eta_2). \tag{23}$$

Of course, summations over all dummy labels are implied.

3 Nonperturbative and Perturbative Recipes for Ensuring Relativistic Invariance

We will find an effective way to meet the commutation relations of the Lie algebra for the Poincaré group in terms of the creation (annihilation) operators of particles in momentum space with the concept of fields not to be used. Our algebraic approach is aimed at the ensuring of RI *as a whole* unlike the Lagrangian formalism, where requirements of relativistic symmetry are *manifestly* provided at the beginning. Meanwhile, we strive to go out beyond the traditional QFT with local Lagrangian densities via a special regularization of interactions in a total initial Hamiltonian.

3.1 Definitions of the Poincaré Generators in a Local QFT. Application to Interacting Pion and Nucleon Fields

It is well known that within the Lagrangian formalism the 4-vector $P^{\mu} = (H, \mathbf{P})$ is determined by the Nöether integrals

$$P^{\nu} = \int \mathcal{T}^{0\nu}(\mathbf{x})d\mathbf{x} \quad (\nu = 0, 1, 2, 3), \tag{24}$$

where $\mathcal{T}^{0\nu}(\mathbf{x})$ are the components of the energy-momentum tensor density $\mathcal{T}^{\mu\nu}(x)$ at $t = 0$.

Other Nöether integrals are expressed through the angular-momentum tensor density

$$\mathcal{M}^{\beta[\mu\nu]}(x) = x^{\mu}\mathcal{T}^{\beta\nu}(x) - x^{\nu}\mathcal{T}^{\beta\mu}(x) + \Sigma^{\beta[\mu\nu]}(x), \tag{25}$$

that contains, in general, so-called polarization part $\Sigma^{\beta[\mu\nu]}$ (henceforth, the symbol $[\alpha, \beta]$ for any labels α and β means the property $f^{[\beta,\alpha]} = -f^{[\alpha,\beta]}$ for its carrier f) associated with spin degrees of freedom. Namely, the six independent integrals

$$M^{\mu\nu} = \int \mathcal{M}^{0[\mu\nu]}(x)d\mathbf{x}\bigg|_{t=0} \tag{26}$$

are considered as the generators of space rotations

$$J^{i} = \varepsilon_{ikl}M^{kl} \quad (i, k, l = 1, 2, 3) \tag{27}$$

and the boosts

$$N^{k} \equiv M^{0k} = -\int x^{k}\mathcal{T}^{00}(\mathbf{x})d\mathbf{x} + \int \Sigma^{0[0k]}(\mathbf{x})d\mathbf{x},\ (k = 1, 2, 3). \tag{28}$$

As an illustration, for interacting pion and nucleon fields with the PS coupling starting from the Lagrangian density after [38] (cf. model (13.42) in [39] with its non-hermitian Lagrangian density)

$$\begin{aligned} \mathcal{L}_{SCH}(x) = {} & \frac{1}{2}\bar{\psi}_{H}(x)(i\gamma^{\mu}\overrightarrow{\partial}_{\mu} - m_{0})\psi_{H}(x) + \frac{1}{2}\bar{\psi}_{H}(x)(-i\gamma^{\mu}\overleftarrow{\partial}_{\mu} - m_{0})\psi_{H}(x) \\ & + \frac{1}{2}\left[\partial_{\mu}\varphi_{H}(x)\partial^{\mu}\varphi_{H}(x) - \mu_{0}^{2}\varphi_{H}^{2}(x)\right] - ig_{0}\bar{\psi}_{H}(x)\gamma_{5}\psi_{H}(x)\varphi_{H}(x), \end{aligned} \tag{29}$$

one has (omitting argument x):

(i) Euler–Lagrange equations

$$\frac{\partial\mathcal{L}_{SCH}}{\partial\bar{\psi}_{H}} - \partial_{\mu}\frac{\partial\mathcal{L}_{SCH}}{\partial\bar{\psi}_{H\,\mu}} = 0, \quad \frac{\partial\mathcal{L}_{SCH}}{\partial\psi_{H}} - \partial_{\mu}\frac{\partial\mathcal{L}_{SCH}}{\partial\psi_{H\,\mu}} = 0, \tag{30}$$

or

$$\begin{aligned} & \frac{1}{2}(i\gamma^{\mu}\overrightarrow{\partial}_{\mu} - m_{0})\psi_{H} = ig_{0}\gamma_{5}\psi_{H}\varphi_{H}, \\ & \frac{1}{2}\bar{\psi}_{H}(-i\gamma^{\mu}\overleftarrow{\partial}_{\mu} - m_{0}) = ig_{0}\bar{\psi}_{H}\gamma_{5}\varphi_{H} \end{aligned} \tag{31}$$

with "bare" nucleon mass m_0, pion mass μ_0 and coupling constant g_0,

(ii) energy-momentum tensor density

$$\begin{aligned} \mathcal{T}_{SCH}^{\mu\nu} = {} & \frac{\partial\mathcal{L}_{SCH}}{\partial\bar{\psi}_{H\,\mu}}\bar{\psi}_{H}^{\nu} + \frac{\partial\mathcal{L}_{SCH}}{\partial\psi_{H\,\mu}}\psi_{H}^{\nu} + \frac{\partial\mathcal{L}_{SCH}}{\partial\varphi_{H\,\mu}}\varphi_{H}^{\nu} - g^{\mu\nu}\mathcal{L}_{SCH} \\ \equiv {} & \mathcal{T}_{N}^{\mu\nu} + \mathcal{T}_{\pi}^{\mu\nu} + \mathcal{T}_{I}^{\mu\nu}, \end{aligned} \tag{32}$$

where

$$\mathcal{T}_N^{\mu\nu} = \frac{i}{2}\bar{\psi}_H\gamma^\mu\partial^\nu\psi_H - \frac{i}{2}\gamma^\mu\psi_H\partial^\nu\bar{\psi}_H - g^{\mu\nu}\mathcal{L}_N, \quad (33)$$
$$\mathcal{T}_\pi^{\mu\nu} = \partial^\mu\varphi_H\partial^\nu\varphi_H - g^{\mu\nu}\mathcal{L}_\pi, \quad (34)$$
$$\mathcal{T}_I^{\mu\nu} = ig_0 g^{\mu\nu}\bar{\psi}_H\gamma_5\psi_H\varphi_H, \quad (35)$$

(iii) polarization contribution

$$\Sigma_{SCH}^{\beta[\mu\nu]} = \frac{1}{2}i\bar{\psi}_H\{\gamma^\beta\Sigma^{\mu\nu} + \Sigma^{\mu\nu}\gamma^\beta\}\psi_H, \quad (36)$$

where

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu].$$

In formulae (29)–(36) unlike operators $O(x)$ in the D picture, we have operators

$$O_H(x) = e^{iHt}O(\mathbf{x})e^{-iHt},$$

in the Heisenberg (H) picture. As before we prefer to employ the definitions:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0, \{\gamma_\mu, \gamma_5\} = 0, \gamma_5^\dagger = \gamma_0\gamma_5\gamma_0 = -\gamma_5.$$

The corresponding Hamiltonian density is given by

$$H_{SCH}(\mathbf{x}) = \mathcal{T}_{SCH}^{00}(\mathbf{x}) = H_{ferm}^0(\mathbf{x}) + H_\pi^0(\mathbf{x}) + V_{ps}^0(\mathbf{x}), \quad (37)$$

where

$$H_{ferm}^0(\mathbf{x}) = \frac{1}{2}\bar{\psi}(\mathbf{x})[-i\overrightarrow{\gamma}\overrightarrow{\partial} + m_0]\psi(\mathbf{x}) + \frac{1}{2}\bar{\psi}(\mathbf{x})[+i\overleftarrow{\gamma}\overleftarrow{\partial} + m_0]\psi(\mathbf{x}), \quad (38)$$
$$H_\pi^0(\mathbf{x}) = \frac{1}{2}\left[\pi^2(\mathbf{x}) + \nabla\varphi(\mathbf{x})\nabla\varphi(\mathbf{x}) + \mu_0^2\varphi^2(\mathbf{x})\right], \quad (39)$$
$$V_{ps}^0(\mathbf{x}) = ig_0\bar{\psi}(\mathbf{x})\gamma_5\psi(\mathbf{x})\varphi(\mathbf{x}). \quad (40)$$

Following a common recipe (see, e.g., Sect. 7.5 in [20]) we have introduced the canonical conjugate variable

$$\pi(\mathbf{x}) \equiv \dot{\varphi}(x)|_{t=0} \quad (41)$$

for the pion field. One should note that the second integral in the r.h.s. of Eq. (28) does not contribute to the model boost since operator (36) is identically equal zero. In fact,

$$\gamma^0\Sigma^{0k} + \Sigma^{0k}\gamma^0 = \frac{i}{4}\{\gamma^0[\gamma^0\gamma^k - \gamma^k\gamma^0] + [\gamma^0\gamma^k - \gamma^k\gamma^0]\gamma^0\}$$
$$= \frac{i}{4}\{\gamma^k - \gamma^0\gamma^k\gamma^0 + \gamma^0\gamma^k\gamma^0 - \gamma^k\} = 0.$$

Thus we have

$$\mathbf{N}_{SCH} = -\int \mathbf{x}\mathcal{T}_{SCH}^{00}(\mathbf{x})d\mathbf{x} = -\int \mathbf{x}H_{SCH}(\mathbf{x})d\mathbf{x}. \quad (42)$$

The relation (42) exemplifies the so-called Belinfante ansatz:

$$\mathbf{N} = -\int \mathbf{x}H(\mathbf{x})d\mathbf{x}, \quad (43)$$

which, as it has first been shown in [41], holds for any local field model with a symmetrized density tensor $\mathcal{T}^{\mu\nu}(x) = \mathcal{T}^{\nu\mu}(x)$. Such a representation helps [29] to get a sparse structure simultaneously for blockdiagonalization of the Hamiltonian and the generators of Lorentz boosts in the CPR. The relation (43) also has turned out to be useful when formulating a local analog of the Siegert theorem in the covariant description of electromagnetic interactions with nuclei [40]. We shall come back to this point later.

By passing, we would like to note that the tensor (32) being symmetrized after Belinfante can be written in the form

$$\mathcal{T}^{\mu\nu}_{sym} = \mathcal{T}^{\mu\nu}_{N,sym} + \mathcal{T}^{\mu\nu}_{\pi} + \mathcal{T}^{\mu\nu}_{I}, \tag{44}$$

$$\mathcal{T}^{\mu\nu}_{N,sym} = \frac{i}{4}(\bar{\psi}_H(x)\gamma^{\mu}\partial^{\nu}\psi_H(x) + \bar{\psi}_H(x)\gamma^{\nu}\partial^{\mu}\psi_H(x) - \partial^{\nu}\bar{\psi}_H(x)\gamma^{\mu}\psi_H(x) - \partial^{\mu}\bar{\psi}_H(x)\gamma^{\nu}\psi_H(x)) - g^{\mu\nu}\mathcal{L}_N.$$

Further, the Hamiltonian density can be represented as

$$H_{SCH}(\mathbf{x}) = H_F(\mathbf{x}) + H_I(\mathbf{x}) \tag{45}$$

with the free part

$$H_F(\mathbf{x}) = H_{\pi}(\mathbf{x}) + H_{ferm}(\mathbf{x}) \tag{46}$$

and the interaction density

$$H_I(\mathbf{x}) = V_{ps}(\mathbf{x}) + H_{ren}(\mathbf{x}), \;\; V_{ps}(\mathbf{x}) = ig\bar{\psi}(\mathbf{x})\gamma_5\psi(\mathbf{x})\varphi(\mathbf{x}), \tag{47}$$

where we have introduced the mass and vertex counterterms:

$$H_{ren}(\mathbf{x}) = M^{mes}_{ren}(\mathbf{x}) + M^{ferm}_{ren}(\mathbf{x}) + H^{int}_{ren}(\mathbf{x}), \tag{48}$$

$$M^{mes}_{ren}(\mathbf{x}) = \frac{1}{2}(\mu_0^2 - \mu_{\pi}^2)\varphi^2(\mathbf{x}),$$

$$M^{ferm}_{ren}(\mathbf{x}) = (m_0 - m)\bar{\psi}(\mathbf{x})\psi(\mathbf{x})$$

and

$$H^{int}_{ren}(\mathbf{x}) = i(g_0 - g)\bar{\psi}(\mathbf{x})\gamma_5\psi(\mathbf{x})\varphi(\mathbf{x}).$$

One should note that the densities in Eqs. (46)–(47) are obtained from Eqs. (38)–(39) replacing the bare values m_0, μ_0 and g_0, respectively, by the “physical” values m, μ_{π} and g. Such a transition can be done via the mass-changing Bogoliubov-type unitary transformations (details in [30]). In particular, the fields involved can be expressed through the set $\alpha = a^{\dagger}(a), b^{\dagger}(b), d^{\dagger}(d)$ of the creation (destruction) operators for the bare pions and nucleons with the physical masses,

$$\varphi(\mathbf{x}) = (2\pi)^{-3/2}\int(2\omega_{\mathbf{k}})^{-1/2}[a(\mathbf{k}) + a^{\dagger}(-\mathbf{k})]exp(i\mathbf{kx})d\mathbf{k}, \tag{49}$$

$$\pi(\mathbf{x}) = -i(2\pi)^{-3/2}\int(\omega_{\mathbf{k}}/2)^{1/2}[a(\mathbf{k}) - a^{\dagger}(-\mathbf{k})]exp(i\mathbf{kx})d\mathbf{k}, \tag{50}$$

$$\psi(\mathbf{x}) = (2\pi)^{-3/2}\int(m/E\mathbf{p})^{1/2}\sum_{\mu}[u(\mathbf{p}\mu)b(\mathbf{p}\mu) + v(-\mathbf{p}\mu)d^{\dagger}(-\mathbf{p}\mu)]exp(i\mathbf{px})d\mathbf{p}. \tag{51}$$

Substituting (45) into (42), we find

$$\mathbf{N} = \mathbf{N}_F + \mathbf{N}_I$$

with

$$\mathbf{N}_F = \mathbf{N}_{ferm} + \mathbf{N}_{\pi} = -\int\mathbf{x}H_{ferm}(\mathbf{x})d\mathbf{x} - \int\mathbf{x}H_{\pi}(\mathbf{x})d\mathbf{x}$$

and

$$\mathbf{N}_I = -\int\mathbf{x}H_I(\mathbf{x})d\mathbf{x}.$$

Now, taking into account the transformation properties of the fermion field $\psi(x)$ and the pion field $\varphi(x)$ with respect to Π, it is readily seen that in the D picture density (45) is a scalar, i.e.,

$$U_F(\Lambda, b) H_{SCH}(x) U_F^{-1}(\Lambda, b) = H_{SCH}(\Lambda x + b), \tag{52}$$

so

$$U_F(\Lambda, b) H_I(x) U_F^{-1}(\Lambda, b) = H_I(\Lambda x + b). \tag{53}$$

Just such a property has been used for that example on p.7.

It is well known (see, e.g., Sect. 5.1 in [20]) that for a large class of theories the property (53) with the corresponding interaction densities $H_I(x)$, being supplemented by the condition

$$[H_I(x'), H_I(x)] = 0 \quad for \quad (x' - x)^2 \leq 0, \tag{54}$$

plays a crucial role in ensuring the RI of the S-matrix. Appendix A is contained explicit expressions of all free generators for the πN system and tests for them to be satisfied the Poincaré algebra.

3.2 An Algebraic Approach within the Hamiltonian Formalism

As mentioned above, we are addressing those theories that start from a total Hamiltonian (12) with the interaction $H_I = \int H_I(\mathbf{x}) d\mathbf{x}$ whose density is sum (7) so

$$H_I = H_{sc} + H_{nsc} \equiv \int H_{sc}(\mathbf{x}) d\mathbf{x} + \int H_{nsc}(\mathbf{x}) d\mathbf{x}. \tag{55}$$

It means that only the density in the first integral has the property (53), i.e.,

$$U_F(\Lambda, b) H_{sc}(x) U_F^{-1}(\Lambda, b) = H_{sc}(\Lambda x + b). \tag{56}$$

Then, taking into account that the first relation (11) is equivalent to the equality

$$[\mathbf{N}_F, H_I] = [H, \mathbf{N}_I], \tag{57}$$

we will evaluate its l.h.s.. In this connection, let us regard the operator

$$H_{sc}(t) = \int H_{sc}(x) d\mathbf{x} \tag{58}$$

and its similarity transformation

$$e^{i\beta \mathbf{N}_F} H_{sc}(t) e^{-i\beta \mathbf{N}_F} = \int H_{sc}(L(\beta)x) d\mathbf{x}, \tag{59}$$

where $L(\beta)$ is any Lorentz boost with the parameters $\beta = (\beta^1, \beta^2, \beta^3)$.

From (59) it follows that

$$i e^{i\beta^1 N_F^1} [N_F^1, H_{sc}(t)] e^{-i\beta^1 N_F^1} = \frac{\partial}{\partial \beta^1} \int H_{sc}(L(\beta^1)x) d\mathbf{x}, \tag{60}$$

whence, for instance,

$$\begin{aligned} i[N_F^1, H_{sc}(t)] &= \lim_{\beta^1 \to 0} \frac{\partial}{\partial \beta^1} \int H_{sc}(t - \beta^1 x^1, x^1 - \beta^1 t, x^2, x^3) d\mathbf{x} \\ &= -\int \left(t \frac{\partial}{\partial x^1} H_{sc}(x) + x^1 \frac{\partial}{\partial t} H_{sc}(x) \right) d\mathbf{x}, \end{aligned} \tag{61}$$

since for the infinitesimal boost

$$L(\beta)x = (t - \beta \mathbf{x}, \mathbf{x} - \beta t).$$

In turn, from (61) we get

$$[N_F^1, H_{sc}] = i \lim_{t\to 0} \int (-it[P^1, H_{sc}(x)] + ix^1[H_F, H_{sc}(x)])d\mathbf{x},$$

so

$$[\mathbf{N}_F, H_{sc}] = -\int \mathbf{x}[H_F, H_{sc}(\mathbf{x})]d\mathbf{x}. \tag{62}$$

By using Eq. (62) equality (57) can be written as

$$-\int \mathbf{x}[H_F, H_{sc}(\mathbf{x})]d\mathbf{x} = [H_F, \mathbf{N}_I] + [H_I, \mathbf{N}_I] + [H_{nsc}, \mathbf{N}_F]. \tag{63}$$

Evidently, this equation is fulfilled if we put

$$\mathbf{N}_I = \mathbf{N}_B \equiv -\int \mathbf{x} H_{sc}(\mathbf{x})d\mathbf{x} \tag{64}$$

and

$$[H_{sc}, \mathbf{N}_I] = -\int \mathbf{x}d\mathbf{x} \int d\mathbf{x}'[H_{sc}(\mathbf{x}'), H_{sc}(\mathbf{x})] = [\mathbf{N}_F + \mathbf{N}_I, H_{nsc}] \tag{65}$$

or

$$\int d\mathbf{x} \int d\mathbf{x}'(\mathbf{x}' - \mathbf{x})[H_{sc}(\mathbf{x}'), H_{sc}(\mathbf{x})] = \int \mathbf{x}d\mathbf{x} \int d\mathbf{x}'[H_{nsc}(\mathbf{x}'), H_F(\mathbf{x}) + H_{sc}(\mathbf{x})]. \tag{66}$$

In a model with $H_{nsc} = 0$ the latter reduces to

$$\int e^{-i\mathbf{PX}}\mathbf{I}e^{i\mathbf{PX}}d\mathbf{X} = 0, \tag{67}$$

where

$$\mathbf{I} = \frac{1}{2}\int \mathbf{r}d\mathbf{r}\left[H_{sc}\left(\frac{1}{2}\mathbf{r}\right), H_{sc}\left(-\frac{1}{2}\mathbf{r}\right)\right]. \tag{68}$$

By running again the way from Eq. (57) to Eqs. (67)–(68) we see that the nonlinear commutation (11)

$$[H, \mathbf{N}] = i\mathbf{P}$$

will take place once along with the Belinfante-type relation (64) the interaction density meets the condition

$$\int \mathbf{r}d\mathbf{r}\left[H_{sc}\left(\frac{1}{2}\mathbf{r}\right), H_{sc}\left(-\frac{1}{2}\mathbf{r}\right)\right] = 0. \tag{69}$$

One should note that we have arrived to Eq. (64) being inside the Poincarè algebra itself without addressing the Nöether integrals, these stepping stones of the Lagrangian formalism. In the context, we would like to stress that the condition (69) is weaker compared to the constraint

$$\left[H_{sc}\left(\frac{1}{2}\mathbf{r}\right), H_{sc}\left(-\frac{1}{2}\mathbf{r}\right)\right] = 0 \tag{70}$$

imposed for all $\mathbf{r}$ excepting, may be, the point $\mathbf{r} = 0$. But we recall it as a special case of the microcausality requirement that is realized in local field models. Beyond such models, as it will be shown in Appendix B, Eqs. (64) and (57) may be incompatible. It makes us seek an alternative to assumption (64) in our attempts to meet Eq. (63).

At this point, we put $\mathbf{N}_I = \mathbf{N}_B + \mathbf{D}$ to get the relationship

$$[H_F, \mathbf{D}] = [\mathbf{N}_B + \mathbf{D}, H_{sc}] + [\mathbf{N}_F + \mathbf{N}_B + \mathbf{D}, H_{nsc}], \tag{71}$$

that replaces the commutator $[H, \mathbf{N}] = i\mathbf{P}$ and determines the displacement $\mathbf{D}$.

Assuming that the scalar density $H_{sc}(\mathbf{x})$ is of the first order in coupling constants involved and putting

$$H_{nsc}(\mathbf{x}) = \sum_{p=2}^{\infty} H_{nsc}^{(p)}(\mathbf{x}), \tag{72}$$

we will search the operator $\mathbf{D}$ in the form

$$\mathbf{D} = \sum_{p=2}^{\infty} \mathbf{D}^{(p)}, \tag{73}$$

i.e., as a perturbation expansion in powers of the interaction H_{sc}. Here the label (p) denotes the pth order in these constants. By the way, one should keep in mind that the terms in the r.h.s. of Eq. (72) are usually associated with perturbation series for mass and vertex counterterms. Evidently, their incorporation may affect the corresponding higher-order contributions with $p \geq 2$ to the boost. In this context, to comprise different situations of practical interest let us consider field models in which

$$H_{nsc}(\mathbf{x}) = V_{nsc}(\mathbf{x}) + V_{ren}(\mathbf{x})$$

with a nonscalar interaction

$$V_{nsc} = \int V_{nsc}(\mathbf{x}) d\mathbf{x}$$

and some "renormalization" contribution

$$V_{ren} = \int V_{ren}(\mathbf{x}) d\mathbf{x}.$$

The latter may be scalar or not. Of course, such a division of $H_{nsc}(\mathbf{x})$ can be done at the beginning in Eq. (55). But the scheme, introduced here, seems to us more flexible.

By substituting the expansions (72) and (73) into Eq. (71) we get the chain of relations

$$[H_F, \mathbf{D}^{(2)}] = [\mathbf{N}_F, H_{nsc}^{(2)}] + [\mathbf{N}_B, H_{sc}], \tag{74}$$

$$[H_F, \mathbf{D}^{(3)}] = [\mathbf{N}_F, H_{nsc}^{(3)}] + [\mathbf{D}^{(2)}, H_{sc}] + [\mathbf{N}_B, H_{nsc}^{(2)}], \tag{75}$$

$$[H_F, \mathbf{D}^{(p)}] = [\mathbf{N}_F, H_{nsc}^{(p)}] + [\mathbf{N}_B, H_{nsc}^{(p-1)}] + [\mathbf{D}^{(p-1)}, H_{sc}] + [\mathbf{D}, H_{nsc}]^{(p)}, \quad (p = 4, 5, \ldots) \tag{76}$$

for a recursive finding of the operators $\mathbf{D}^{(p)}$ $(p = 2, 3, \ldots)$.

Further, after such substitutions into the commutators

$$[P_k, N_j] = i\delta_{kj} H,$$
$$[J_k, N_j] = i\varepsilon_{kjl} N_l$$

and

$$[N_k, N_j] = -i\varepsilon_{kjl} J_l$$

we deduce, respectively, the following relations (the remaining Poincaré commutations are fulfilled once one deals with any rotationally and translationally invariant theory):

$$[P_k, D_j^{(p)}] = i\delta_{kj} H_{nsc}^{(p)} \quad (p = 2, 3, \ldots) \tag{77}$$

from

$$[P_k, D_j] = i\delta_{kj} H_{nsc}, \tag{78}$$

$$[J_k, D_j^{(p)}] = i\varepsilon_{kjl} D_l^{(p)} \tag{79}$$

from

$$[J_k, D_j] = i\varepsilon_{kjl} D_l \tag{80}$$

and

$$[N_{Fk}, N_{Bj}] + [N_{Bk}, N_{Fj}] = 0, \tag{81}$$

$$[N_{Fk}, D_j^{(2)}] + [D_k^{(2)}, N_{Fj}] + [N_{Bk}, N_{Bj}] = 0, \tag{82}$$

$$[N_{Fk}, D_j^{(3)}] + [D_k^{(3)}, N_{Fj}] + [N_{Bk}, D_j^{(2)}] + [D_k^{(2)}, N_{Bj}] = 0, \tag{83}$$

$$[N_{Fk}, D_j^{(p)}] + [D_k^{(p)}, N_{Fj}] + [N_{Bk}, D_j]^{(p)} + [D_k, N_{Bj}]^{(p)} + [D_k, D_j]^{(p)} = 0, \quad (p = 4, 5, \ldots) \tag{84}$$

from

$$[N_{Fk}, N_{Bj} + D_j] + [N_{Bk} + D_k, N_{Fj}] + [N_{Bk} + D_k, N_{Bj} + D_j] = 0. \tag{85}$$

Now, keeping in mind an elegant method by Chandler [33], we invoke on the property (see [34]) of a formal solution Y to the equation

$$[H_F, Y] = X \tag{86}$$

to be any linear functional $F(X)$ of a given operator $X \neq 0$. In other words, it means that

$$[H_F, F(X)] = X \tag{87}$$

with $F(\lambda_1 X_1 + \lambda_2 X_2) = \lambda_1 F(X_1) + \lambda_2 F(X_2)$, where λ_1 and λ_2 are arbitrary c-numbers. In addition, one can see that

$$[H_F, F(X)] = F([H_F, X]). \tag{88}$$

Moreover, it turns out that

$$[\mathbf{P}, F(X)] = F([\mathbf{P}, X]), \tag{89}$$

$$[\mathbf{J}, F(X)] = F([\mathbf{J}, X]), \tag{90}$$

$$[\mathbf{N}_F, F(X)] = F([\mathbf{N}_F, X]) + iF(F([\mathbf{P}, X])). \tag{91}$$

In order to prove the relations let us employ the Jacobi identity

$$[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0 \tag{92}$$

and write

$$[\mathcal{O}, [H_F, F(X)]] = -[F(X), [\mathcal{O}, H_F]] + [H_F, [\mathcal{O}, F(X)]]$$

with some operator $\mathcal{O}$. Then

$$[\mathcal{O}, F(X)] = F([\mathcal{O}, X]) + F([F(X), [\mathcal{O}, H_F]]). \tag{93}$$

The formulae (89)–(91) follow from Eq. (93) if one takes into account the Poincaré commutators for the free generators G_F. We derive them here after [33] when moving from the nonlinear commutation (71) to ensuring the RI as a whole. After this let us verify all commutations (77)–(85) when one uses the solution $Y = F(X)$ to Eq. (86).

First, with the help of (89) we find from Eqs. (74)–(75),

$$\begin{aligned}[P_k, D_j^{(2)}] &= F([P_k, [N_{Fj}, H_{nsc}^{(2)}]]) = F([[P_k, N_{Fj}], H_{nsc}^{(2)}]) \\ &= i\delta_{kj} F([H_F, H_{nsc}^{(2)}]) = i\delta_{kj} H_{nsc}^{(2)}, \end{aligned} \tag{94}$$

$$\begin{aligned}[P_k, D_j^{(3)}] &= F([P_k, [N_{Fj}, H_{nsc}^{(3)}]]) = F([[P_k, N_{Fj}], H_{nsc}^{(3)}]) \\ &= i\delta_{kj} F([H_F, H_{nsc}^{(3)}]) = i\delta_{kj} H_{nsc}^{(3)}. \end{aligned} \tag{95}$$

We have used the formulae $[P_k, N_{Bj}] = i\delta_{kj} H_{sc}$ and $[P_k, H_{sc}] = [P_k, H_{nsc}] = 0$. Analogously, one can verify Eqs. (79) with $p = 2, 3$. Second, Eq. (81) is trivial.

Third,

$$\begin{aligned}&[N_{Fk}, D_j^{(2)}] + [D_k^{(2)}, N_{Fj}] + [N_{Bk}, N_{Bj}] \\ &\quad = F([N_{Fk}, [N_{Bj}, H_{sc}]]) - F([N_{Fj}, [N_{Bk}, H_{sc}]]) + [N_{Bk}, N_{Bj}] \\ &\quad = -F([N_{Bj}, [N_{Bk}, H_F]]) + F([N_{Bk}, [N_{Bj}, H_F]]) + [N_{Bk}, N_{Bj}] \\ &\quad = F([H_F, [N_{Bj}, N_{Bk}]]) + [N_{Bk}, N_{Bj}] = 0 \end{aligned} \tag{96}$$

and

$$\begin{aligned}&[N_{Fk}, D_j^{(3)}] + [D_k^{(3)}, N_{Fj}] + [N_{Bk}, D_j^{(2)}] + [D_k^{(2)}, N_{Bj}] \\ &\quad = -F([H_{sc}, [N_{Fk}, D_j^{(2)}]]) - F([N_{Bk}, [D_j^{(2)}, H_F]]) \\ &\qquad - [N_{Bk}, D_j^{(2)}] + F([N_{Fk}, [N_{Bj}, H_{nsc}^{(2)}]]) \\ &\qquad + F([H_{sc}, [N_{Fj}, D_k^{(2)}]]) + F([N_{Bj}, [D_k^{(2)}, H_F]]) \\ &\qquad + [N_{Bj}, D_k^{(2)}] - F([N_{Fj}, [N_{Bk}, H_{nsc}^{(2)}]]) \\ &\qquad + [N_{Bk}, D_j^{(2)}] + [D_k^{(2)}, N_{Bj}] \\ &\quad = F([H_{sc}, [N_{Bk}, N_{Bj}]]) + F([N_{Bk}, [N_{Bj}, H_{sc}]]) \\ &\qquad + F([N_{Bj}, [H_{sc}, N_{Bk}]]) + F([H_{nsc}^{(2)}, [N_{Fk}, N_{Bj}]]) \\ &\qquad + F([N_{Fk}, [N_{Bj}, H_{nsc}^{(2)}]]) + F([N_{Bj}, [H_{nsc}^{(2)}, N_{Fk}]]) = 0. \end{aligned} \tag{97}$$

At last, Eqs. (77) and Eqs. (79) with $p \geq 3$ and Eqs. (84) with $p \geq 4$ can be proved inductively. One should emphasize that for these derivations we have again addressed the strategy chosen in [33]. Unfortunately, that approach by Chandler is either well forgotten or little known. Therefore, we are trying to present an entire picture. However, to be more constructive one needs to have a definite realization of the functional $F(X)$. In this connection, we will use the representation

$$Y = -i \lim_{\eta \to 0+} \int_0^\infty X(t) e^{-\eta t} dt \tag{98}$$

of the operator Y that enters the equation (86). The existence proof for such a solution is sufficiently delicate (see discussion in Appendix A of Ref. [29]). Of course, it depends on the operator X. We shall come back to the point in Sect. 3.3 for a situation, where $[H_F, X] = 0$.

Henceforth, the ensuring of RI via Eqs. (71)–(76) calls the way I.

3.3 Comparison with Other Approaches: Application to a Nonlocal Field Model

There are different perturbative schemes to meet the Poincaré algebra (at least, in its instant form after Dirac). One of them, elaborated in [14], is based upon a simultaneous blockdiagonalization of the field Hamiltonian and the boost operators by using a development of the Okubo idea [9] and constructing the corresponding unitary transformation in a perturbative way (see also Sect. 6 in [29]).

An entirely algebraic approach [32] (see also [33] and a private communication to A.S.) is most close to that exposed in Sect. 3.2. In fact, its departure point is to apply a perturbation expansion of the commutation relations (9)–(11) inserting into them the series

$$H_I = \sum_{p=1}^{\infty} H_I^{(p)}$$

and

$$\mathbf{N}_I = \sum_{p=1}^{\infty} \mathbf{N}_I^{(p)},$$

$$[H_F, \mathbf{N}_I^{(1)}] = [\mathbf{N}_F, H_I^{(1)}], \tag{99}$$

$$[H_F, \mathbf{N}_I^{(2)}] = [\mathbf{N}_F, H_I^{(2)}] + [\mathbf{N}_I^{(1)}, H_I^{(1)}], \tag{100}$$

$$[H_F, \mathbf{N}_I^{(3)}] = [\mathbf{N}_F, H_I^{(3)}] + [\mathbf{N}_I^{(1)}, H_I^{(2)}] + [\mathbf{N}_I^{(2)}, H_I^{(1)}], \tag{101}$$

$$[P^i, N_I^{(p)j}] = \delta_{ij} H_I^{(p)} \qquad (p = 1, 2, \ldots) \tag{102}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

The recursive procedure based upon Eqs. (99)–(101) will be referred to as the way II.

Now, making a comparison between I and II we obtain with the aid of formula (98) the lowest-order terms:

$$\mathbf{N}_I^{(1)} = \mathbf{N}_B = -\int \mathbf{x} H_I^{(1)}(\mathbf{x}) d\mathbf{x} = -\int \mathbf{x} H_{sc}(\mathbf{x}) d\mathbf{x} \tag{103}$$

and

$$\mathbf{N}_I^{(2)} = \mathbf{D}^{(2)} = \Phi([\mathbf{N}_F, H_{nsc}^{(2)}]) - i \lim_{\eta \to 0+} \int_0^{\infty} [\mathbf{N}_B(t), H_{sc}(t)] e^{-\eta t} dt \tag{104}$$

from Eq. (64) and Eq. (74) vs

$$\mathbf{N}_I^{(1)} = -i \lim_{\eta \to 0+} \int_0^{\infty} [\mathbf{N}_F(t), H_{sc}(t)] e^{-\eta t} dt \tag{105}$$

and

$$\mathbf{N}_I^{(2)} = \Phi([\mathbf{N}_F, H_{nsc}^{(2)}]) - i \lim_{\eta \to 0+} \int_0^{\infty} [\mathbf{N}_I^{(1)}(t), H_{sc}(t)] e^{-\eta t} dt \tag{106}$$

from Eq. (99) and Eq. (100), respectively. The first terms in the r.h.s. of Eq. (104) and Eq. (106) have been expressed through a linear functional $\Phi(X)$ since its argument $X = [\mathbf{N}_F, H_{nsc}^{(2)}]$, in general, can embody a part that commutes with H_F (see that note below the recipe (98)).

It is easily seen that these relations give rise to identical results since the commutator in the r.h.s. Eq. (105) can be written as (see Eq. (62))

$$[\mathbf{N}_F(t), H_{sc}(t)] = -\int \mathbf{x} d\mathbf{x} [H_F, H_{sc}(t)] = [H_F, \mathbf{N}_B(t)] \tag{107}$$

or

$$[\mathbf{N}_F(t), H_{sc}(t)] = -i \frac{d}{dt} \mathbf{N}_B(t),$$

so

$$\mathbf{N}_I^{(1)} = -i \lim_{\eta\to 0+} \int_0^\infty [\mathbf{N}_F(t), H_{sc}(t)] e^{-\eta t} dt$$
$$= \mathbf{N}_B(0) - \lim_{\eta\to 0+} \eta \int_0^\infty \mathbf{N}_B(t) e^{-\eta t} dt = \mathbf{N}_B.$$

By assumption,

$$\lim_{\eta\to 0+} \eta \int_0^\infty \mathbf{N}_B(t) e^{-\eta t} dt = 0,$$

that should be verified every time for a given model interaction.

Besides, if the condition (69) takes place, the approach II enables us to arrive to the same result as our approach does, i.e., the Belinfante ansatz by Eq. (64).

As mentioned, the latter is inherent in some local field theories. Therefore, we would like to employ the way I when handling nonlocal field models. Let us consider a system of "scalar nucleons" (more precisely, charged spinless bosons) and neutral scalar bosons (see, e.g., Chapter 1 in [42]) with the following interaction density (cf. [9,37]):

$$H_I(\mathbf{x}) = V_{loc}(\mathbf{x}) + V_{ren}(\mathbf{x}), \tag{108}$$
$$V_{loc}(\mathbf{x}) = g\varphi_s(\mathbf{x}) : \psi_b^\dagger(\mathbf{x})\psi_b(\mathbf{x}) : \tag{109}$$

and

$$V_{ren}(\mathbf{x}) = \delta\mu_s : \varphi_s^2(\mathbf{x}) : + \delta\mu_b : \psi_b^\dagger(\mathbf{x})\psi_b(\mathbf{x}) : \tag{110}$$

with the mass shifts $\delta\mu_s = \frac{1}{2}(\mu_{0s}^2 - \mu_s^2)$ ($\delta\mu_b = (\mu_{0b}^2 - \mu_b^2)$). In order to regard a nonlocal extension of this local model let us substitute the expansions

$$\varphi_s(\mathbf{x}) = [2(2\pi)^3]^{-1/2} \int \frac{d\mathbf{k}}{\omega_{\mathbf{k}}} [a(k) + a^\dagger(k_-)] e^{i\mathbf{kx}},$$
$$\psi_b(\mathbf{x}) = [2(2\pi)^3]^{-1/2} \int \frac{d\mathbf{p}}{E_{\mathbf{p}}} [b(p) + d^\dagger(p_-)] e^{i\mathbf{px}}$$

into Eqs. (109) and (110) to get

$$V_{loc}(\mathbf{x}) = g[2(2\pi)^3]^{-3/2} \int \frac{d\mathbf{p}'}{E_{\mathbf{p}'}} \int \frac{d\mathbf{p}}{E_{\mathbf{p}}} \int \frac{d\mathbf{k}}{\omega_{\mathbf{k}}} exp[i(\mathbf{p} + \mathbf{k} - \mathbf{p}')\mathbf{x}]$$
$$\times : [a(k) + a^\dagger(k_-)][b^\dagger(p') + d(p'_-)][b(p) + d^\dagger(p_-)] : \tag{111}$$

and

$$V_{ren}(\mathbf{x}) = \delta\mu_s(\mathbf{x}) + \delta\mu_b(\mathbf{x}) \tag{112}$$

with

$$\delta\mu_s(\mathbf{x}) = \frac{\delta\mu_s}{2(2\pi)^3} \int \frac{d\mathbf{k}'}{\omega_{\mathbf{k}'}} \int \frac{d\mathbf{k}}{\omega_{\mathbf{k}}} : [a(k') + a^\dagger(k'_-)] e^{i(\mathbf{k}'+\mathbf{k})\mathbf{x}} [a(k) + a^\dagger(k_-)] :; \tag{113}$$
$$\delta\mu_b(\mathbf{x}) = \frac{\delta\mu_b}{2(2\pi)^3} \int \frac{d\mathbf{p}'}{E_{\mathbf{p}'}} \int \frac{d\mathbf{p}}{E_{\mathbf{p}}} : [b^\dagger(p') + d(p'_-)] e^{i(\mathbf{p}-\mathbf{p}')\mathbf{x}} [b(p) + d^\dagger(p_-)] : . \tag{114}$$

The interaction operator itself

$$H_I = \int H_I(\mathbf{x})d\mathbf{x} = V_{loc} + V_{ren},$$
$$V_{loc} = \int V_{loc}(\mathbf{x})d\mathbf{x} = \frac{g}{2[2(2\pi)^3]^{1/2}} \int \frac{d\mathbf{p}'}{E_{\mathbf{p}'}} \int \frac{d\mathbf{p}}{E_{\mathbf{p}}} \int \frac{d\mathbf{k}}{\omega_{\mathbf{k}}} \delta(\mathbf{p}' - \mathbf{p} - \mathbf{k})$$
$$\times a(k) : [b^\dagger(p')b(p) + b^\dagger(p')d^\dagger(p_-) + d(p'_-)b(p) + d(p'_-)d^\dagger(p_-)] : +H.c., \quad (115)$$
$$V_{ren} = \int [\delta\mu_s(\mathbf{x}) + \delta\mu_b(\mathbf{x})]d\mathbf{x}. \quad (116)$$

Let us consider its nonlocal extension

$$H_I = V_{nloc} + M_s + M_b, \quad (117)$$

where in accordance with the representation (3) we introduce the following normally-ordered structures:

$$V_{nloc} = \int V_{nloc}(\mathbf{x})d\mathbf{x} = \int \frac{d\mathbf{p}'}{E_{\mathbf{p}'}} \int \frac{d\mathbf{p}}{E_{\mathbf{p}}} \int \frac{d\mathbf{k}}{\omega_{\mathbf{k}}}$$
$$\times \{\delta(\mathbf{p}' - \mathbf{p} - \mathbf{k})g_{11}(p', p, k)b^\dagger(p')b(p) + \delta(\mathbf{p}' + \mathbf{p} - \mathbf{k})g_{12}(p', p, k)b^\dagger(p')d^\dagger(p)$$
$$+ \delta(\mathbf{p}' + \mathbf{p} + \mathbf{k})g_{21}(p', p, k)d(p')b(p)$$
$$+ \delta(\mathbf{p}' - \mathbf{p} - \mathbf{k})g_{22}(p', p, k)d^\dagger(p')d(p)\}a(k) + H.c. \quad (118)$$

or in more compact form

$$V_{nloc} = V_b + V_b^\dagger,$$
$$V_b = \int V_b(\mathbf{x})d\mathbf{x} = \int \frac{d\mathbf{k}}{\omega_{\mathbf{k}}} : F_b^\dagger G(k) F_b : a(k), \quad (119)$$

where

$$V_b(\mathbf{x}) = \int \frac{d\mathbf{k}}{\omega_{\mathbf{k}}} e^{i\mathbf{k}\mathbf{x}} : F_b^\dagger G_k(\mathbf{x}) F_b : a(k)$$

with

$$\{G_k(\mathbf{x})\}_{\varepsilon'\varepsilon} = \frac{1}{(2\pi)^3} \bar{g}_{\varepsilon'\varepsilon}(p', p, k) exp[i((-1)^{\varepsilon'}\mathbf{p}' - (-1)^{\varepsilon}\mathbf{p})\mathbf{x}],$$

while the operators M_s and M_b will be given below.

Adopting the convention

$$[b^\dagger(p'), d(p')] \begin{bmatrix} X_{11}(p', p) & X_{12}(p', p) \\ X_{21}(p', p) & X_{22}(p', p) \end{bmatrix} \begin{bmatrix} b(p) \\ d^\dagger(p) \end{bmatrix}$$
$$= F_{\varepsilon'}^\dagger(p') X_{\varepsilon'\varepsilon}(p', p) F_\varepsilon(p) \equiv F_b^\dagger(p') X(p', p) F_b(p) \quad (120)$$

for any 2×2 matrix $X(p', p)$ and the column

$$F_b(p) = \begin{bmatrix} b(p) \\ d^\dagger(p) \end{bmatrix} \equiv \begin{bmatrix} F_1(p) \\ F_2(p) \end{bmatrix}$$

(cf. formula (A.8) in [29]), sometimes it is convenient to proceed with

$$\int \frac{d\mathbf{p}'}{E_{\mathbf{p}'}} \int \frac{d\mathbf{p}}{E_{\mathbf{p}}} F_b^\dagger(p') X(p', p) F_b(p) \equiv F_b^\dagger X F_b.$$

In this context the matrix $G(k)$ in Eq. (119) is composed of the elements

$$G_{\varepsilon'\varepsilon}(p', p, k) = \bar{g}_{\varepsilon'\varepsilon}(p', p, k)\delta(\mathbf{k} + (-1)^{\varepsilon'}\mathbf{p}' - (-1)^{\varepsilon}\mathbf{p}),$$
$$(\varepsilon', \varepsilon = 1, 2) \quad (121)$$

where $\bar{g}_{\varepsilon'\varepsilon}(p', p, k)$ coincide with $g_{\varepsilon'\varepsilon}(p', p, k)$ except $\bar{g}_{22}(p', p, k) = g_{22}(p, p', k)$.

It is implied that the operators $a(a^\dagger)$, $b(b^\dagger)$ and $d(d^\dagger)$ meet the commutation relations

$$[a(k), a^\dagger(k')] = k_0\delta(\mathbf{k}-\mathbf{k}'), \tag{122}$$

$$[b(p), b^\dagger(p')] = [d(p), d^\dagger(p')] = p_0\delta(\mathbf{p}-\mathbf{p}') \tag{123}$$

with all the remaining ones being zero. Here $k_0 = \omega_\mathbf{k} = \sqrt{\mathbf{k}^2+\mu_s^2}$ ($p_0 = E_\mathbf{p} = \sqrt{\mathbf{p}^2+\mu_b^2}$) is the energy of the neutral (charged) particle with the mass $\mu_s(\mu_b)$. By the way, from (123) it follows that

$$[F_{\varepsilon'}(p'), F_\varepsilon^\dagger(p)] = p_0\delta(\mathbf{p}'-\mathbf{p})\sigma_{\varepsilon'\varepsilon}, \tag{124}$$

where $\sigma_{\varepsilon'\varepsilon} = (-1)^{\varepsilon-1}\delta_{\varepsilon'\varepsilon}$.

Furthermore, the creation/destruction operators have the transformation properties like (19). For example,

$$U_F(\Lambda)a(k)U_F^{-1}(\Lambda) = a(\Lambda k). \tag{125}$$

Therefore in the D picture

$$U_F(\Lambda)V_{loc}(x)U_F^{-1}(\Lambda) = V_{loc}(\Lambda x), \tag{126}$$

i.e., the interaction density $V_{loc}(x)$ is a Lorentz scalar.

For our nonlocal model we will retain the property assuming that

$$U_F(\Lambda)V_{nloc}(x)U_F^{-1}(\Lambda) = V_{nloc}(\Lambda x). \tag{127}$$

It is readily seen that this relation holds if the coefficients $g_{\varepsilon'\varepsilon}$ meet the condition

$$g_{\varepsilon'\varepsilon}(\Lambda p', \Lambda p, \Lambda k) = g_{\varepsilon'\varepsilon}(p', p, k). \tag{128}$$

On the mass shells with $p'^2 = p^2 = \mu_b^2$ and $k^2 = \mu_s^2$ the latter means that the functions $g_{\varepsilon'\varepsilon}(p', p, k)$ can depend only upon the invariants $p'p$, $p'k$ and pk.

The transition from V_{loc} to V_{nloc} can be interpreted as an endeavor to regularize the theory. In the context, the introduction of some cutoff functions $g_{\varepsilon'\varepsilon}$ in momentum space is aimed at removing ultraviolet divergences typical of local field models with interactions like expression (109).

One should keep in mind that along with the requirement (128) these cutoffs are subject to other constraints imposed by different symmetries. For example, the tacit invariance of the hermitian operator (118) with respect to: (i) space inversion $\mathcal{P}$; (ii) time reversal $\mathcal{T}$ and (iii) charge conjugation $\mathcal{C}$ yields the relations

$$g_{\varepsilon'\varepsilon}(p', p, k) = g_{\varepsilon'\varepsilon}(p, p', k), \quad \varepsilon' \neq \varepsilon \tag{129}$$

$$g_{\varepsilon'\varepsilon}(p', p, k) = g_{\varepsilon'\varepsilon}(p'_-, p_-, k_-), \tag{130}$$

$$g_{11}(p', p, k) = g_{22}(p', p, k), \tag{131}$$

which can be derived assuming (see, e.g., Sect. 5.2 in [20]) the following properties

$$\mathcal{P}a(\mathbf{k})\mathcal{P}^{-1} = a(-\mathbf{k}), \quad \mathcal{P}b(\mathbf{p})\mathcal{P}^{-1} = b(-\mathbf{p}), \quad \mathcal{P}d(\mathbf{p})\mathcal{P}^{-1} = d(-\mathbf{p}), \tag{132}$$

$$\mathcal{T}a(\mathbf{k})\mathcal{T}^{-1} = a(-\mathbf{k}), \quad \mathcal{T}b(\mathbf{p})\mathcal{T}^{-1} = b(-\mathbf{p}), \quad \mathcal{T}d(\mathbf{p})\mathcal{T}^{-1} = d(-\mathbf{p}), \tag{133}$$

$$\mathcal{C}a(\mathbf{k})\mathcal{C}^{-1} = a(\mathbf{k}), \quad \mathcal{C}b(\mathbf{p})\mathcal{C}^{-1} = d(\mathbf{p}), \quad \mathcal{C}d(\mathbf{p})\mathcal{C}^{-1} = b(\mathbf{p}), \tag{134}$$

$\forall$ **p** and **k**.

As to constructing the "mass renormalization terms" M_s and M_b we note that within the clothing procedure exposed in the next section they can be represented in the form:

$$M_s = \int \frac{d\mathbf{k}}{\omega_\mathbf{k}^2}\{m_1(k)a^\dagger(k)a(k) + m_2(k)[a^\dagger(k)a^\dagger(k_-) + a(k)a(k_-)]\} \tag{135}$$

and

$$M_b = \int \frac{d\mathbf{p}}{E_\mathbf{p}^2}\{m_{11}(p)b^\dagger(p)b(p) + m_{12}(p)b^\dagger(p)d^\dagger(p_-) + m_{21}(p)b(p)d(p_-) + m_{22}(p)d^\dagger(p)d(p)\}, \tag{136}$$

where the coefficients $m_{1,2}(k)$ and $m_{\varepsilon'\varepsilon}(p)$, being for the time unknown, may be momentum dependent. Of course, the latter (for simplicity, real) should be symmetrical, i.e. $m_{12}(p) = m_{21}(p)$, to ensure the hermiticity of M_b. We will confine ourselves to the consideration of such terms. Of course, the so-called charge and wave function counterterms can be included too to be cancelled then by the g^3-order contributions (cf. [43]) starting from the commutator $\frac{1}{3}[R, [R, V_{bad}]]$ in expansion (159).

Now, in order to derive the corresponding lowest-order contributions to the boost operator for our nonlocal model, we find using Eqs. (103)–(104),

$$\mathbf{N}_I^{(1)} = \mathbf{N}_B = -\int \mathbf{x} V_{nloc}(\mathbf{x}) d\mathbf{x}, \tag{137}$$

$$\mathbf{N}_I^{(2)} = \mathbf{D}^{(2)} = \Phi([\mathbf{N}_F, H_{nsc}^{(2)}]) - i \lim_{\eta\to 0+} \int_0^\infty [\mathbf{N}_B(t), V_{nloc}(t)] e^{-\eta t} dt, \tag{138}$$

$$\begin{aligned}\mathbf{N}_I^{(3)} = \mathbf{D}^{(3)} &= -i \lim_{\eta\to 0+} \int_0^\infty [\mathbf{N}_B(t), H_{nsc}^{(2)}(t)] e^{-\eta t} dt \\ &\quad -i \lim_{\eta\to 0+} \int_0^\infty [\mathbf{D}^{(2)}(t), V_{nloc}(t)] e^{-\eta t} dt - i \lim_{\eta\to 0+} \int_0^\infty [\mathbf{N}_F(t), H_{nsc}^{(3)}(t)] e^{-\eta t} dt, \\ &\quad \ldots \quad \ldots \quad \ldots \quad \ldots\end{aligned} \tag{139}$$

where $H_{nsc}^{(2)}(t) = exp(iH_F t)(M_s^{(2)} + M_b^{(2)})exp(-iH_F t)$ with the leading-order contributions $M_s^{(2)}$ and $M_b^{(2)}$ to the operators M_s and M_b that will be given explicitly below. We will confine ourselves to the evaluation of contributions $\mathbf{N}_I^{(1)}$ and $\mathbf{N}_I^{(2)}$. It suffices to conceive of some manifestations of the model nonlocality.

Thus, by handling relation (138), we encounter commutator

$$\begin{aligned}[\mathbf{N}_B(t), V_{nloc}(t)] &= -\int \mathbf{x}'[V_{nloc}(t, \mathbf{x}'), V_{nloc}(t, \mathbf{x})] d\mathbf{x} d\mathbf{x}' \\ &= -\frac{1}{2} \int d\mathbf{x}' \int d\mathbf{x} (\mathbf{x}' - \mathbf{x}) [V_{nloc}(t, \mathbf{x}'), V_{nloc}(t, \mathbf{x})]\end{aligned} \tag{140}$$

with

$$[V_{nloc}(t, \mathbf{x}'), V_{nloc}(t, \mathbf{x})] = exp(iH_F t)[V_{nloc}(\mathbf{x}'), V_{nloc}(\mathbf{x})] exp(-iH_F t),$$

where

$$[V_{nloc}(\mathbf{x}'), V_{nloc}(\mathbf{x})] = [V_b(\mathbf{x}'), V_{nloc}(\mathbf{x})] - H.c. \tag{141}$$

and

$$[V_b(\mathbf{x}'), V_{nloc}(\mathbf{x})] = [V_b(\mathbf{x}'), V_b(\mathbf{x})] + [V_b(\mathbf{x}'), V_b^\dagger(\mathbf{x})]. \tag{142}$$

The first term in the r.h.s. of (142) is equal to

$$\begin{aligned}[V_b(\mathbf{x}'), V_b(\mathbf{x})] = [(2\pi)^6]^{-1} &\int \frac{d\mathbf{k}}{\omega_{\mathbf{k}}} \int \frac{d\mathbf{k}_1}{\omega_{\mathbf{k}_1}} \int \frac{d\mathbf{p}'}{E_{\mathbf{p}'}} \int \frac{d\mathbf{p}}{E_{\mathbf{p}}} \int \frac{d\mathbf{q}}{E_{\mathbf{q}}} \\ &\times [exp[i((-1)^{\varepsilon'}\mathbf{p}' - (-1)^{\rho'}\mathbf{q} + \mathbf{k})\mathbf{x}'] exp[i((-1)^{\rho}\mathbf{q} - (-1)^{\varepsilon}\mathbf{p} + \mathbf{k}_1)\mathbf{x}] \\ &- exp[i((-1)^{\varepsilon'}\mathbf{p}' - (-1)^{\rho'}\mathbf{q} + \mathbf{k})\mathbf{x}] exp[i((-1)^{\rho}\mathbf{q} - (-1)^{\varepsilon}\mathbf{p} + \mathbf{k}_1)\mathbf{x}]] \\ &\times F_{\varepsilon'}^\dagger(p')\bar{g}_{\varepsilon'\rho'}(p', q, k)\sigma_{\rho'\rho}\bar{g}_{\rho\varepsilon}(q, p, k_1)F_{\varepsilon}(p)a(k)a(k_1).\end{aligned} \tag{143}$$

This matrix form can be derived using commutations

$$[F_{\varepsilon'}(p'), F_{\varepsilon}^\dagger(p)] = p_0 \delta(\mathbf{p}' - \mathbf{p})\sigma_{\varepsilon'\varepsilon}, \tag{144}$$

where $\sigma_{\varepsilon'\varepsilon} = (-1)^{\varepsilon - 1}\delta_{\varepsilon'\varepsilon}$.

In turn, we have

$$
\begin{aligned}
&F^{\dagger}_{\varepsilon'}(p')\bar{g}_{\varepsilon'\rho'}(p',q,k)\sigma_{\rho'\rho}\bar{g}_{\rho\varepsilon}(q,p,k_1)F_{\varepsilon}(p)\\
&\quad = F^{\dagger}_1(p')[\bar{g}_{11}(p',q,k)\bar{g}_{11}(q,p,k_1) - \bar{g}_{12}(p',q,k)\bar{g}_{21}(q,p,k_1)]F_1(p)\\
&\qquad + F^{\dagger}_1(p')[\bar{g}_{11}(p',q,k)\bar{g}_{12}(q,p,k_1) - \bar{g}_{12}(p',q,k)\bar{g}_{22}(q,p,k_1)]F_2(p)\\
&\qquad + F^{\dagger}_2(p')[\bar{g}_{21}(p',q,k)\bar{g}_{11}(q,p,k_1) - \bar{g}_{22}(p',q,k)\bar{g}_{21}(q,p,k_1)]F_1(p)\\
&\qquad + F^{\dagger}_2(p')[\bar{g}_{21}(p',q,k)\bar{g}_{12}(q,p,k_1) - \bar{g}_{22}(p',q,k)\bar{g}_{22}(q,p,k_1)]F_2(p).
\end{aligned}
$$

When $\bar{g}_{11}(p',p,k) = \bar{g}_{12}(p',p,k) = \bar{g}_{21}(p',p,k) \equiv \bar{g}(p',p,k)$, we get

$$[V_b(\mathbf{x}'), V_b(\mathbf{x})] = 0 \tag{145}$$

and

$$[V_b(\mathbf{x}'), V_b^{\dagger}(\mathbf{x})] - H.c. = 0 \tag{146}$$

so

$$[V_{nloc}(\mathbf{x}'), V_{nloc}(\mathbf{x})] = 0. \tag{147}$$

Then we obtain from Eq. (138)

$$\mathbf{D}^{(2)} = \Phi([\mathbf{N}_F, M_s^{(2)} + M_b^{(2)}]) \tag{148}$$

and we see that even with relation (147) reminiscent of the well-known microcausality condition (cf. Eq. (54)) one has to evaluate the displacement operator $\mathbf{D}$, if the mass renormalization terms are inequal to zero. But the latter is the case. Otherwise, we would come to some contradiction with Eq. (74) and Eq. (77) (details see in Sect. 4.2).

By using the formulae (234) and (235) and taking into account that to an accuracy of adding an arbitrary function of H_F the solution Y to $[H_F, Y] = X$ repeats the operator structure of X, we arrive to the division

$$\mathbf{D}^{(2)} = \mathbf{D}^{(2)}_{con} + \mathbf{D}^{(2)}_{ncon}, \tag{149}$$

where the particle-number-conserving and -nonconserving contributions $\mathbf{D}^{(2)}_{con}$ and $\mathbf{D}^{(2)}_{ncon}$ are determined by

$$
\begin{aligned}
\mathbf{D}^{(2)}_{con} = {} & \frac{i}{2}\int\frac{d\mathbf{k}'}{\omega_{\mathbf{k}'}}\int\frac{d\mathbf{k}}{\omega_{\mathbf{k}}}(\omega_{\mathbf{k}'}\omega_{\mathbf{k}} + \mathbf{k}'\mathbf{k} + \mu_s^2)\left(\frac{m_1^{(2)}(k)}{\omega_{\mathbf{k}}} - \frac{m_1^{(2)}(k')}{\omega_{\mathbf{k}'}}\right)\\
&\times\frac{a^{\dagger}(k')a(k)}{\omega_{\mathbf{k}'} - \omega_{\mathbf{k}}}\frac{\partial}{\partial\mathbf{k}}\delta(\mathbf{k}-\mathbf{k}')\\
&+\frac{i}{2}\int\frac{d\mathbf{p}'}{E_{\mathbf{p}'}}\int\frac{d\mathbf{p}}{E_{\mathbf{p}}}(E_{\mathbf{p}'}E_{\mathbf{p}} + \mathbf{p}'\mathbf{p} + \mu_b^2)\left(\frac{m_{11}^{(2)}(p)}{E_{\mathbf{p}}} - \frac{m_{11}^{(2)}(p')}{E_{\mathbf{p}'}}\right)\\
&\times\frac{b^{\dagger}(p')b(p)}{E_{\mathbf{p}'} - E_{\mathbf{p}}}\frac{\partial}{\partial\mathbf{p}}\delta(\mathbf{p}-\mathbf{p}')\\
&+\frac{i}{2}\int\frac{d\mathbf{p}'}{E_{\mathbf{p}'}}\int\frac{d\mathbf{p}}{E_{\mathbf{p}}}(E_{\mathbf{p}'}E_{\mathbf{p}} + \mathbf{p}'\mathbf{p} + \mu_b^2)\left(\frac{m_{22}^{(2)}(p)}{E_{\mathbf{p}}} - \frac{m_{22}^{(2)}(p')}{E_{\mathbf{p}'}}\right)\\
&\times\frac{d^{\dagger}(p')d(p)}{E_{\mathbf{p}'} - E_{\mathbf{p}}}\frac{\partial}{\partial\mathbf{p}}\delta(\mathbf{p}-\mathbf{p}')
\end{aligned}
\tag{150}
$$

and

$$\mathbf{D}^{(2)}_{ncon} = i\int \frac{d\mathbf{k}'}{\omega_{\mathbf{k}'}}\int \frac{d\mathbf{k}}{\omega_{\mathbf{k}}} m_2^{(2)}(k)\frac{\omega_{\mathbf{k}'}\omega_{\mathbf{k}} + \mathbf{k}'\mathbf{k} + \mu_s^2}{\omega_{\mathbf{k}}} \times \frac{a^\dagger(k')a^\dagger(k_-) - a(k')a(k_-)}{\omega_{\mathbf{k}'} + \omega_{\mathbf{k}}}\frac{\partial}{\partial \mathbf{k}}\delta(\mathbf{k}-\mathbf{k}') + i\int \frac{d\mathbf{p}'}{E_{\mathbf{p}'}}\int \frac{d\mathbf{p}}{E_{\mathbf{p}}} m_{12}^{(2)}(p)\frac{E_{\mathbf{p}'}E_{\mathbf{p}} + \mathbf{p}'\mathbf{p} + \mu_b^2}{E_{\mathbf{p}}} \times \frac{b^\dagger(p')d^\dagger(p_-) - b(p')d(p_-)}{E_{\mathbf{p}'} + E_{\mathbf{p}}}\frac{\partial}{\partial \mathbf{p}}\delta(\mathbf{p}-\mathbf{p}'). \qquad (151)$$

One should point out that the operator $\mathbf{D}^{(2)}_{con}$ stems from the structure

$$[\mathbf{N}_F, M_s^{(2)} + M_b^{(2)}] \sim a^\dagger a + b^\dagger b + d^\dagger d,$$

which commutes with H_F.

4 Boost Operators for Clothed Particles

As shown in [29], the Belinfante ansatz turns out to be useful when constructing the Lorentz boosts in the CPR. Their generator $\mathbf{N} \equiv \mathbf{N}(\alpha)$, being a function of the primary operators $\{\alpha\}$ (such as $a^\dagger(a)$, $b^\dagger(b)$ and $d^\dagger(d)$ for the examples regarded above) in the BPR, is expressed through the corresponding operators $\{\alpha_c\}$ for particle creation and annihilation in the CPR. The transition $\{\alpha\} \Longrightarrow \{\alpha_c\}$ is implemented via the special unitary transformations $W(\alpha) = W(\alpha_c)$, viz.,

$$\alpha = W(\alpha_c)\alpha_c W^\dagger(\alpha_c), \qquad (152)$$

satisfying certain physical requirements (details can be also found in Refs. [24,30]).

4.1 Elimination of Bad Terms in Generators of the Poincaré Group

A key point of the clothing procedure exposed in [29] is to remove the so-called bad terms from the Hamiltonian

$$H \equiv H(\alpha) = H_F(\alpha) + H_I(\alpha) = W(\alpha_c)H(\alpha_c)W^\dagger(\alpha_c) \equiv K(\alpha_c), \qquad (153)$$

more exactly, from a primary interaction $V(\alpha)$ that enters $H_I(\alpha) = V(\alpha) + V_{ren}(\alpha)$ (cf., e.g., our nonlocal model with $V_{nloc} = V(\alpha)$ and $V_{ren} = V_{ren}(\alpha) = M_s(\alpha) + M_b(\alpha)$). For example, such terms $b_c^\dagger b_c a_c^\dagger$, $b_c^\dagger d_c^\dagger a_c$, $b_c^\dagger d_c^\dagger a_c^\dagger$, $d_c d_c^\dagger a_c^\dagger$ enter $V(\alpha_c)$ determined by Eq. (115) after the replacement of the bare operators in it by the clothed ones. These terms are removed together with their Hermitian conjugate counterterms to retain the hermiticity of the similarity transformation (153). By definition, such terms prevent the physical vacuum $|\Omega\rangle$ (the H lowest eigenstate) and the one-clothed-particle states $|n\rangle_c = a_c^\dagger(n)|\Omega\rangle$ to be the H eigenvectors for all n included. Here creation operators $a_c^\dagger(n)$ are clothed counterparts of those operators $a^\dagger(n)$ that are contained in expansion (2). The bad terms (a recursive scheme for successive eliminations of such terms has been regarded in [30]) occur every time when any normally ordered product

$$a^\dagger(1')a^\dagger(2')\ldots a^\dagger(n'_C)a(n_A)\ldots a(2)a(1)$$

of the class [C.A] embodies, at least, one substructure which belongs to one of the classes $[k.0]$ ($k = 1, 2, \ldots$) and $[k.1]$ ($k = 0, 1, \ldots$).

Therefore, in correspondence with the decomposition (55) we have

$$H_I(\alpha) = \int H_I(\mathbf{x})d\mathbf{x} = H_{sc}(\alpha) + H_{nsc}(\alpha), \qquad (154)$$

$$H_{sc(nsc)}(\alpha) = \int H_{sc(nsc)}(\mathbf{x})d\mathbf{x},$$

assuming that

$$H_{sc}(\alpha) = V_{bad}(\alpha) + V_{good}(\alpha)$$

to remove the bad part V_{bad} from the similarity transformation

$$\begin{aligned} K(\alpha_c) &= W(\alpha_c)[H_F(\alpha_c) + H_I(\alpha_c)]W^\dagger(\alpha_c) \\ &= W(\alpha_c)[H_F(\alpha_c) + V_{bad}(\alpha_c) + V_{good}(\alpha_c) + H_{nsc}(\alpha_c)]W^\dagger(\alpha_c). \end{aligned} \tag{155}$$

Remind that term "good", as an antithesis of "bad", is applied here to those operators (e.g., of the class [k.2] with $k \geq 2$) which destroy both the no-clothed-particle state Ω and the one-clothed-particle states. For the unitary clothing transformation (UCT) $W = expR$ with $R = -R^\dagger$ (sometimes, for brevity, we omit evident arguments) it is implied that we will eliminate the bad terms V_{bad} in the r.h.s. of

$$\begin{aligned} K(\alpha_c) = H_F(\alpha_c) + V_{bad}(\alpha_c) + [R, H_F] + [R, V_{bad}] + \frac{1}{2}[R, [R, H_F]] \\ + \frac{1}{2}[R, [R, V_{bad}]] + \cdots + e^R V_{good} e^{-R} + e^R H_{nsc} e^{-R} \end{aligned} \tag{156}$$

(cf. Eq. (2.19) in [29]) by requiring that

$$[H_F, R] = V_{bad} \tag{157}$$

for the operator R of interest.

One should note that unlike the original clothing procedure exposed in [29,30] we eliminate here the bad terms only from H_{sc} interaction in spite of such terms can appear in the nonscalar interaction as well. This preference is relied upon the previous experience [36] and [22] when applying the method of UCTs in the theory of nucleon-nucleon scattering. Now we get the division

$$H = K(\alpha_c) = K_F + K_I \tag{158}$$

with a new free part $K_F = H_F(\alpha_c) \sim a_c^\dagger a_c$ and interaction

$$\begin{aligned} K_I = V_{good}(\alpha_c) + H_{nsc}(\alpha_c) + [R, V_{good}] \\ + \frac{1}{2}[R, V_{bad}] + [R, H_{nsc}] + \frac{1}{3}[R, [R, V_{bad}]] + \cdots, \end{aligned} \tag{159}$$

where the r.h.s. involves along with good terms other bad terms to be removed via subsequent UCTs described in Subsect. 2.4 of [29] and Sect. 3 of [30].

In parallel, we have

$$\mathbf{N} \equiv \mathbf{N}(\alpha) = \mathbf{N}_F(\alpha) + \mathbf{N}_I(\alpha) = W(\alpha_c)\mathbf{N}(\alpha_c)W^\dagger(\alpha_c) \equiv \mathbf{B}(\alpha_c) \tag{160}$$

or

$$\mathbf{B}(\alpha_c) = \mathbf{N}_F(\alpha_c) + \mathbf{N}_I(\alpha_c) + [R, \mathbf{N}_F] + [R, \mathbf{N}_I] + \cdots, \tag{161}$$

where accordingly the division

$$\begin{aligned} \mathbf{N}_I &= \mathbf{N}_B + \mathbf{D}, \\ \mathbf{N}_B &= -\int \mathbf{x} H_{sc}(\mathbf{x}) d\mathbf{x} = \mathbf{N}_{bad} + \mathbf{N}_{good}, \end{aligned} \tag{162}$$

Equation (161) can be rewritten as

$$\begin{aligned} \mathbf{B}(\alpha_c) = \mathbf{N}_F(\alpha_c) + \mathbf{N}_{bad}(\alpha_c) + [R, \mathbf{N}_F] + [R, \mathbf{N}_{bad}] + \frac{1}{2}[R, [R, \mathbf{N}_F]] \\ + \frac{1}{2}[R, [R, \mathbf{N}_{bad}]] + \cdots + e^R \mathbf{N}_{good} e^{-R} + e^R \mathbf{D} e^{-R}. \end{aligned} \tag{163}$$

But it turns out (see the proof of Eq. (3.26) in [29]) that if R meets the condition (157), then

$$[\mathbf{N}_F, R] = \mathbf{N}_{bad} = -\int \mathbf{x} V_{bad}(\mathbf{x}) d\mathbf{x} \tag{164}$$

so the boost generators in the CPR can be written likely Eq. (158),

$$\mathbf{N} = \mathbf{B}(\alpha_c) = \mathbf{B}_F + \mathbf{B}_I, \tag{165}$$

where $\mathbf{B}_F = \mathbf{N}_F(\alpha_c)$ is the boost operator for noninteracting clothed particles while $\mathbf{B}_I$ includes the contributions induced by interactions between them

$$\begin{aligned} \mathbf{B}_I &= \mathbf{N}_{good}(\alpha_c) + \mathbf{D}(\alpha_c) + [R, \mathbf{N}_{good}] \\ &\quad + \frac{1}{2}[R, \mathbf{N}_{bad}] + [R, \mathbf{D}] + \frac{1}{3}[R, [R, \mathbf{N}_{bad}]] + \cdots \end{aligned} \tag{166}$$

One should note that in formulae (159) and (166) we are focused upon the R-commutations with the first-eliminated interaction V_{bad}. As shown in [29], the brackets, on the one hand, yield new interactions responsible for different physical processes and, on the other hand, cancel (as a recipe) the mass and other counterterms that stem from $H_{nsc}(\alpha_c)$ and $\mathbf{D}(\alpha_c)$. Such a cancellation will be regarded in the next subsection.

But at this point we will come back to our model with $V_{bad} = V_{nloc}$, $V_{good} = 0$ and $R = R_{nloc}$ to calculate the simplest commutator $[R_{nloc}, V_{nloc}]$ in which accordingly condition (157) the clothing operator R_{nloc} is determined by

$$[H_F, R_{nloc}] = V_{nloc}. \tag{167}$$

From the equation it follows (cf. Appendix A in [29]) that its solution can be given by

$$R_{nloc} = \int \frac{d\mathbf{k}}{\omega_{\mathbf{k}}} : F_b^\dagger R(k) F_b : a(k) - H.c. = \mathcal{R}_{nloc} - \mathcal{R}_{nloc}^\dagger. \tag{168}$$

The matrix $R(k)$ is composed of the elements

$$\begin{aligned} R_{\varepsilon'\varepsilon}(p', p, k) &= -\frac{\bar{g}_{\varepsilon'\varepsilon}(p', p, k)}{\omega_{\mathbf{k}} + (-1)^{\varepsilon'} E_{\mathbf{p}'} - (-1)^{\varepsilon} E_{\mathbf{p}}} \delta(\mathbf{k} + (-1)^{\varepsilon'} \mathbf{p}' - (-1)^{\varepsilon} \mathbf{p}). \\ &\quad (\varepsilon', \varepsilon = 1, 2) \end{aligned} \tag{169}$$

Such a solution is valid if $\mu_s < 2\mu_b$. In other words, under such an inequality the operator R_{nloc} has the same structure as V_{nloc} itself. Then, all we need is to evaluate

$$[R_{nloc}, V_{nloc}] = [\mathcal{R}_{nloc} - \mathcal{R}_{nloc}^\dagger, V_{nloc}] = [\mathcal{R}_{nloc}, V_{nloc}] + H.c., \tag{170}$$

where accordingly (119)

$$[\mathcal{R}_{nloc}, V_{nloc}] = [\mathcal{R}_{nloc}, V_b] + [\mathcal{R}_{nloc}, V_b^\dagger]. \tag{171}$$

Further, using Eqs. (119), (168) and identity (245) we find

$$[\mathcal{R}_{nloc}, V_b] = \int \frac{d\mathbf{k}'}{\omega_{\mathbf{k}'}} \int \frac{d\mathbf{k}}{\omega_{\mathbf{k}}} F_b^\dagger [R(k'), G(k)] F_b a(k') a(k) \tag{172}$$

and

$$\begin{aligned} [\mathcal{R}_{nloc}, V_b^\dagger] &= \int \frac{d\mathbf{k}'}{\omega_{\mathbf{k}'}} \int \frac{d\mathbf{k}}{\omega_{\mathbf{k}}} \{F_b^\dagger [R(k'), G(k_-)] F_b a^\dagger(k) a(k') \\ &\quad + \delta(\mathbf{k}' - \mathbf{k}) : F_b^\dagger R(k') F_b :: F_b^\dagger G(k_-) F_b :\}, \end{aligned} \tag{173}$$

where the matrix $G(k)$ is determined by Eq. (121) and it is implied that

$$\begin{aligned} &[R(k'), G(k)](p', p) \\ &\quad = \int \frac{d\mathbf{q}}{E_{\mathbf{q}}} [R(p', q, k') G(q, p, k) - G(p', q, k) R(q, p, k')]. \end{aligned} \tag{174}$$

After the normal ordering of meson and boson operators in commutator $[R_{nloc}, V_{nloc}]$ one can obtain the $2 \to 2$ interactions of the type $b^\dagger a^\dagger ba$, $d^\dagger a^\dagger da$, $b^\dagger d^\dagger aa$, $a^\dagger a^\dagger bd$ and $b^\dagger b^\dagger bb$, $b^\dagger d^\dagger bd$, $d^\dagger d^\dagger dd$ in the r.h.s. of Eqs. (172) and (173)and their H.c..

For example, the boson-boson interaction operator can be represented as

$$\frac{1}{2}[R_{nloc}, V_{nloc}](bb \to bb)$$
$$= -\frac{1}{4}\int \frac{d\mathbf{p}_2'}{E_{\mathbf{p}_2'}}\int \frac{d\mathbf{p}_2}{E_{\mathbf{p}_2}}\int \frac{d\mathbf{p}_1'}{E_{\mathbf{p}_1'}}\int \frac{d\mathbf{p}_1}{E_{\mathbf{p}_1}}\delta(\mathbf{p}_1' + \mathbf{p}_2' - \mathbf{p}_1 - \mathbf{p}_2) g_{11}(p_1', p_1, k) g_{11}(p_2', p_2, k)$$
$$\times \left\{\frac{1}{(p_1 - p_1')^2 - \mu_s^2} + \frac{1}{(p_2 - p_2')^2 - \mu_s^2}\right\} b_c^\dagger(p_2') b_c^\dagger(p_1') b_c(p_2) b_c(p_1) \tag{175}$$

with $\mathbf{k} = \mathbf{p}_1' - \mathbf{p}_1$. Simultaneously, we get the pair-production interaction operator

$$\frac{1}{2}[R_{nloc}, V_{nloc}](aa \to b\bar{b}) = \frac{1}{2}\int \frac{d\mathbf{p}'}{E_{\mathbf{p}'}}\int \frac{d\mathbf{p}}{E_{\mathbf{p}}}\int \frac{d\mathbf{k}'}{\omega_{\mathbf{k}'}}\int \frac{d\mathbf{k}}{\omega_{\mathbf{k}}}\delta(\mathbf{p}' + \mathbf{p} - \mathbf{k}' - \mathbf{k})$$
$$\times \left(\frac{1}{E_{\mathbf{q}'}}\frac{g_{11}(p', q', k') g_{12}(p, q', k)}{E_{\mathbf{p}'} - E_{\mathbf{q}'} - \omega_{\mathbf{k}'}} - \frac{1}{E_{\mathbf{q}'}}\frac{g_{11}(p, q_-', k) g_{12}(p', q_-', k')}{E_{\mathbf{p}'} + E_{\mathbf{q}'} - \omega_{\mathbf{k}'}}\right.$$
$$\left. + \frac{1}{E_{\mathbf{q}}}\frac{g_{12}(p', q', k') g_{11}(p, q', k)}{E_{\mathbf{p}} - E_{\mathbf{q}} - \omega_{\mathbf{k}}} - \frac{1}{E_{\mathbf{q}}}\frac{g_{11}(p', q_-', k') g_{12}(p, q_-', k)}{E_{\mathbf{p}} + E_{\mathbf{q}} - \omega_{\mathbf{k}}}\right)$$
$$\times b_c^\dagger(p') d_c^\dagger(p) a_c(k') a_c(k), \tag{176}$$

where $\mathbf{q}' = \mathbf{p}' - \mathbf{k}'$, $\mathbf{q} = \mathbf{p} - \mathbf{k}$ with the 4-momenta $q' = (E_{\mathbf{q}'}, \mathbf{q}')$ and $q = (E_{\mathbf{q}}, \mathbf{q})$.

In parallel, taking into account that in our model with $\mathbf{N}_{bad} = \mathbf{N}_B$ we find the respective contributions to $\mathbf{B}_I$,

$$\frac{1}{2}[R_{nloc}, \mathbf{N}_B](bb \to bb)$$
$$= \frac{i}{4}\int \frac{d\mathbf{p}_2'}{E_{\mathbf{p}_2'}}\int \frac{d\mathbf{p}_2}{E_{\mathbf{p}_2}}\int \frac{d\mathbf{p}_1'}{E_{\mathbf{p}_1'}}\int \frac{d\mathbf{p}_1}{E_{\mathbf{p}_1}}\frac{\partial}{\partial \mathbf{p}_1'}\delta(\mathbf{p}_1' + \mathbf{p}_2' - \mathbf{p}_1 - \mathbf{p}_2)$$
$$\times g_{11}(p_1', p_1, k) g_{11}(p_2', p_2, k)$$
$$\times \left\{\frac{1}{(p_1 - p_1')^2 - \mu_s^2} + \frac{1}{(p_2 - p_2')^2 - \mu_s^2}\right\} b_c^\dagger(p_2') b_c^\dagger(p_1') b_c(p_2) b_c(p_1) \tag{177}$$

and

$$\frac{1}{2}[R_{nloc}, \mathbf{N}_B](aa \to b\bar{b})$$
$$= -\frac{i}{2}\int \frac{d\mathbf{p}'}{E_{\mathbf{p}'}}\int \frac{d\mathbf{p}}{E_{\mathbf{p}}}\int \frac{d\mathbf{k}'}{\omega_{\mathbf{k}'}}\int \frac{d\mathbf{k}}{\omega_{\mathbf{k}}}\frac{\partial}{\partial \mathbf{p}}\delta(\mathbf{p} + \mathbf{p}' - \mathbf{k}' - \mathbf{k})$$
$$\times \left(\frac{1}{E_{\mathbf{q}'}}\frac{g_{11}(p', q', k') g_{12}(p, q', k)}{E_{\mathbf{p}'} - E_{\mathbf{q}'} - \omega_{\mathbf{k}'}} - \frac{1}{E_{\mathbf{q}'}}\frac{g_{11}(p, q_-', k) g_{12}(p', q_-', k')}{E_{\mathbf{p}'} + E_{\mathbf{q}'} - \omega_{\mathbf{k}'}}\right.$$
$$\left. + \frac{1}{E_{\mathbf{q}}}\frac{g_{12}(p', q', k') g_{11}(p, q', k)}{E_{\mathbf{p}} - E_{\mathbf{q}} - \omega_{\mathbf{k}}} - \frac{1}{E_{\mathbf{q}}}\frac{g_{11}(p', q_-', k') g_{12}(p, q_-', k)}{E_{\mathbf{p}} + E_{\mathbf{q}} - \omega_{\mathbf{k}}}\right)$$
$$\times b_c^\dagger(p') d_c^\dagger(p) a_c(k') a_c(k). \tag{178}$$

In Eqs. (175) and (177) we meet a covariant (Feynman-like) “propagator”

$$\frac{1}{2}\left\{\frac{1}{(p_1 - p_1')^2 - \mu_s^2} + \frac{1}{(p_2 - p_2')^2 - \mu_s^2}\right\}, \tag{179}$$

which on the energy shell

$$E_{\mathbf{p}_1} + E_{\mathbf{p}_1} = E_{\mathbf{p}_1'} + E_{\mathbf{p}_2'} \tag{180}$$

is converted into the genuine Feynman propagator for the corresponding S matrix (cf. discussions in [29,44]).

4.2 Mass Renormalization and Relativistic Invariance

We have seen how in the framework of the nonlocal meson-boson model one can build the $2 \to 2$ interactions between the clothed mesons and bosons. They appear in a natural way from the commutator $\frac{1}{2}[R_{nloc}, V_{nloc}]$ as the operators $b^\dagger a^\dagger ba$, $d^\dagger a^\dagger da$, $b^\dagger b^\dagger bb$, $b^\dagger d^\dagger bd$, $d^\dagger d^\dagger dd$, $b^\dagger d^\dagger aa$, $a^\dagger a^\dagger bd$ of the class [2.2]. Moreover, this commutator is a spring of the good operators $a^\dagger a$, $b^\dagger b$ and $d^\dagger d$ of the class [1.1] together with the bad operators aa and bd of the class [0.2] (henceforth, for brevity, we omit the subscript c) and their hermitian conjugates $a^\dagger a^\dagger$ and $b^\dagger d^\dagger$ of the class [2.0]. These operators may be cancelled by the respective counterterms from

$$H_{nsc}(\alpha) = M_s(\alpha) + M_b(\alpha) \tag{181}$$

in the r.h.s. of Eq. (159). Let us show that such a cancellation gives rise to certain definitions of the mass coefficients in Eqs. (135) and (136).

Indeed, with the help of the same technique as in [29] one can show

$$\frac{1}{2}[R_{nloc}, V_{nloc}](a^\dagger a) = -\frac{1}{2}\int \frac{d\mathbf{k}}{\omega_{\mathbf{k}}^2}\int \frac{d\mathbf{p}}{E_{\mathbf{p}}E_{\mathbf{p}-\mathbf{k}}}\Big[\frac{g_{21}^2(p, q_-, k_-)}{E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}} + \omega_{\mathbf{k}}} + \frac{g_{12}^2(p, q_-, k)}{E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}} - \omega_{\mathbf{k}}}\Big]a^\dagger(k)a(k), \tag{182}$$

where $q = (E_{\mathbf{p}-\mathbf{k}}, \mathbf{p} - \mathbf{k})$. In the same way we obtain

$$\frac{1}{2}[R_{nloc}, V_{nloc}](aa) = -\frac{1}{2}\int \frac{d\mathbf{k}}{\omega_{\mathbf{k}}^2}\int \frac{d\mathbf{p}}{E_{\mathbf{p}}E_{\mathbf{p}-\mathbf{k}}} g_{12}(p, q_-, k)g_{21}(p, q_-, k_-) \times \left[\frac{1}{E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}} + \omega_{\mathbf{k}}} + \frac{1}{E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}} - \omega_{\mathbf{k}}}\right] a(k)a(k_-) \tag{183}$$

or

$$\frac{1}{2}[R_{nloc}, V_{nloc}](aa) = \int \frac{d\mathbf{k}}{\omega_{\mathbf{k}}^2}\int \frac{d\mathbf{p}}{E_{\mathbf{p}}} g_{12}(p, q_-, k)g_{21}(p, q_-, k_-) \times \left[\frac{1}{\mu_s^2 + 2p_-k} + \frac{1}{\mu_s^2 - 2pk}\right] a(k)a(k_-). \tag{184}$$

Recall that the last transition can be done by means of some trick considered in Appendix A from [29].

Furthermore, assuming that

$$M_s^{(2)}(\alpha) + \frac{1}{2}[R_{nloc}, V_{nloc}]_{2mes} = 0 \tag{185}$$

with

$$[R_{nloc}, V_{nloc}]_{2mes} = [R_{nloc}, V_{nloc}](a^\dagger a) + [R_{nloc}, V_{nloc}](aa) + [R_{nloc}, V_{nloc}](a^\dagger a^\dagger),$$

we find

$$m_1^{(2)}(k) = \frac{1}{2}\int \frac{d\mathbf{p}}{E_{\mathbf{p}}E_{\mathbf{p}-\mathbf{k}}}\left[\frac{g_{21}^2(p, q_-, k_-)}{E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}} + \omega_{\mathbf{k}}} + \frac{g_{12}^2(p, q_-, k)}{E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}} - \omega_{\mathbf{k}}}\right] \tag{186}$$

and

$$m_2^{(2)}(k) = -\int \frac{d\mathbf{p}}{E_{\mathbf{p}}} g_{12}(p, q_-, k)g_{21}(p, q_-, k_-) \times \left[\frac{1}{\mu_s^2 + 2p_-k} + \frac{1}{\mu_s^2 - 2pk}\right]. \tag{187}$$

The operators that conserve the boson (antiboson) number can be written as (details see in [24]):

$$\frac{1}{2}[R_{nloc}, V_{nloc}](b^\dagger b) = \int \frac{d\mathbf{k}}{\omega_\mathbf{k}} \int \frac{d\mathbf{p}}{E_\mathbf{p}^2 E_{\mathbf{p}-\mathbf{k}}} \left[\frac{g_{11}^2(p,q,k)}{E_\mathbf{p} - E_{\mathbf{p}-\mathbf{k}} - \omega_\mathbf{k}} - \frac{g_{21}^2(p,q_-,k_-)}{E_\mathbf{p} + E_{\mathbf{p}-\mathbf{k}} + \omega_\mathbf{k}} \right] b^\dagger(p) b(p), \tag{188}$$

$$\frac{1}{2}[R_{nloc}, V_{nloc}](d^\dagger d) = \int \frac{d\mathbf{k}}{\omega_\mathbf{k}} \int \frac{d\mathbf{p}}{E_\mathbf{p}^2 E_{\mathbf{p}-\mathbf{k}}} \left[\frac{g_{22}^2(p,q,k)}{E_\mathbf{p} - E_{\mathbf{p}-\mathbf{k}} - \omega_\mathbf{k}} - \frac{g_{21}^2(p,q_-,k_-)}{E_\mathbf{p} + E_{\mathbf{p}-\mathbf{k}} + \omega_\mathbf{k}} \right] d^\dagger(p) d(p). \tag{189}$$

One can show that from the condition

$$M_b^{(2)}(\alpha) + \frac{1}{2}[R_{nloc}, V_{nloc}]_{2bos} = 0, \tag{190}$$

where

$$[R_{nloc}, V_{nloc}]_{2bos} = [R_{nloc}, V_{nloc}](b^\dagger b) + [R_{nloc}, V_{nloc}](b^\dagger d^\dagger) + [R_{nloc}, V_{nloc}](db) + [R_{nloc}, V_{nloc}](d^\dagger d)$$

it follows

$$m_{11}^{(2)}(p) = -\int \frac{d\mathbf{k}}{\omega_\mathbf{k} E_{\mathbf{p}-\mathbf{k}}} \left[\frac{g_{11}^2(p,q,k)}{E_\mathbf{p} - E_{\mathbf{p}-\mathbf{k}} - \omega_\mathbf{k}} - \frac{g_{21}^2(p,q_-,k_-)}{E_\mathbf{p} + E_{\mathbf{p}-\mathbf{k}} + \omega_\mathbf{k}} \right], \tag{191}$$

$$m_{22}^{(2)}(p) = -\int \frac{d\mathbf{k}}{\omega_\mathbf{k} E_{\mathbf{p}-\mathbf{k}}} \left[\frac{g_{11}^2(p,q,k)}{E_\mathbf{p} - E_{\mathbf{p}-\mathbf{k}} - \omega_\mathbf{k}} - \frac{g_{21}^2(p,q_-,k_-)}{E_\mathbf{p} + E_{\mathbf{p}-\mathbf{k}} + \omega_\mathbf{k}} \right]. \tag{192}$$

Similarly one can obtain the non-diagonal coefficients

$$m_{12}^{(2)}(p) = m_{21}^{(2)}(p) = -\int \frac{d\mathbf{k}}{\omega_\mathbf{k} E_{\mathbf{p}-\mathbf{k}}} g_{11}(p,q,k) g_{21}(p,q_-,k_-) \times \left[\frac{1}{E_\mathbf{p} - E_{\mathbf{p}-\mathbf{k}} - \omega_\mathbf{k}} - \frac{1}{E_\mathbf{p} + E_{\mathbf{p}-\mathbf{k}} + \omega_\mathbf{k}} \right] \tag{193}$$

or

$$m_{12}^{(2)}(p) = m_{21}^{(2)}(p) = -\int \frac{d\mathbf{k}}{\omega_\mathbf{k}} g_{11}(p,q,k) g_{21}(p,q_-,k_-) \left[\frac{1}{\mu_s^2 - 2pk} + \frac{1}{\mu_s^2 + 2p_-k} \right] - \int \frac{d\mathbf{q}}{E_\mathbf{q}} g_{11}(p,q,u) g_{21}(p,q_-,u_-) \left(\frac{1}{2[\mu_b^2 - pq] - \mu_s^2} + \frac{1}{2[\mu_b^2 + pq_-] - \mu_s^2} \right), \tag{194}$$

where $u = (E_{\mathbf{p}-\mathbf{q}}, \mathbf{p} - \mathbf{q})$.

The integrands in Eqs. (187) and (194) are contained the covariant denominators that have already occurred in [24] and [29]. Thus the clothing procedure has allowed us to get analytical expressions for the interaction operators between the clothed particles. Moreover, we have obtained some prescriptions when finding the coefficients in the "mass renormalization" operators.

Unlike the momentum-independent mass shifts obtained in [24,29] and [14] these coefficients, as mentioned below Eq. (136), may be momentum dependent. But the most significant property of the integrals (186), (187) and (191)–(193) is to take on finite values. In the context, those divergent integrals from [24,29], being coincident with the Feynman one-loop ones for the pion and nucleon mass shifts, are of interest as a prelude to the present exploration.

At last, one should emphasize that if one starts from expansion (72) with the second-order contribution $H_{nsc}^{(2)} = 0$, then the RI would be violated at the beginning because of the obvious discrepancy between Eqs. (74) and (77).

5 Discussion: Towards Working Formulae

We see that the way I in combination with the UCTs method makes our consideration more and more appropriate for practical applications (in particular, as one has to work with the vertex cutoffs). It is well known that the role of such cutoffs may be twofold, viz., first, as mentioned in Introduction to get rid of ultraviolet divergences in the course of all intermediate calculations and, second, to introduce the particle finite-size effects. In this context, we will proceed with the g-factors, which allow us, on the one hand, to do comparatively simple calculations and, on the other hand, to preserve the basic premises. In addition, of interest are their properties that could provide the momentum independence of the particle mass shifts.

5.1 The Leading Order Mass Shifts and Their Momentum Dependence

The formulae for the $2 \to 2$ interactions in Sect. 4.1 and for the mass coefficients in Sect. 4.2 become more tractable if we assume that

$$g_{\varepsilon'\varepsilon}(p', p, k) = v_{\varepsilon'\varepsilon}([k + (-1)^{\varepsilon'} p' - (-1)^{\varepsilon} p][k - (-1)^{\varepsilon'} p' + (-1)^{\varepsilon} p]). \tag{195}$$

One can verify the nonlocal model with such cutoffs possesses necessary properties (128)–(131). In terms of the $v_{\varepsilon'\varepsilon}$ functions we get

$$m_1^{(2)}(k) = \frac{1}{2}\int \frac{d\mathbf{p}}{E_{\mathbf{p}}E_{\mathbf{p}-\mathbf{k}}}\left[\frac{v_{21}^2(\omega_{\mathbf{k}}^2 - (E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}})^2)}{E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}} + \omega_{\mathbf{k}}} + \frac{v_{12}^2(\omega_{\mathbf{k}}^2 - (E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}})^2)}{E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}} - \omega_{\mathbf{k}}}\right] \tag{196}$$

and

$$m_2^{(2)}(k) = -\int \frac{d\mathbf{p}}{E_{\mathbf{p}}} v_{21}(\omega_{\mathbf{k}}^2 - (E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}})^2) v_{12}(\omega_{\mathbf{k}}^2 - (E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}})^2) \times \left[\frac{1}{\mu_s^2 + 2p_{-}k} + \frac{1}{\mu_s^2 - 2pk}\right]. \tag{197}$$

Now, by handling the charge-independent cutoffs,

$$v_{12}(x) = v_{21}(x) = f(x), \tag{198}$$

we obtain

$$\begin{aligned} m_1^{(2)}(k) &= m_2^{(2)}(k) \\ &= \int \frac{d\mathbf{p}}{E_{\mathbf{p}}E_{\mathbf{p}-\mathbf{k}}}(E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}})\frac{f^2(\omega_{\mathbf{k}}^2 - (E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}})^2)}{(E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}})^2 - \omega_{\mathbf{k}}^2} \\ &= -\int \frac{d\mathbf{p}}{E_{\mathbf{p}}} f^2(\omega_{\mathbf{k}}^2 - (E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}})^2)\left[\frac{1}{\mu_s^2 + 2p_{-}k} + \frac{1}{\mu_s^2 - 2pk}\right] \\ &= -\int \frac{d\mathbf{p}}{E_{\mathbf{p}}} \frac{f^2(\omega_{\mathbf{k}}^2 - (E_{\mathbf{p}} + E_{\mathbf{p}+\mathbf{k}})^2)}{\mu_s^2 + 2pk} - \int \frac{d\mathbf{p}}{E_{\mathbf{p}}} \frac{f^2(\omega_{\mathbf{k}}^2 - (E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}})^2)}{\mu_s^2 - 2pk}. \end{aligned} \tag{199}$$

The second form of these coefficients has been prompted by the trick [29] with

$$\frac{E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}}}{(E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}})^2 - \omega_{\mathbf{k}}^2} = -E_{\mathbf{p}-\mathbf{k}}\left(\frac{1}{\mu_s^2 - 2pk} + \frac{1}{\mu_s^2 + 2p_{-}k}\right) + \frac{E_{\mathbf{p}} - E_{\mathbf{p}-\mathbf{k}}}{(E_{\mathbf{p}} - E_{\mathbf{p}-\mathbf{k}})^2 - \omega_{\mathbf{k}}^2}$$

and using the properties (129)–(131). In other words, the option (198) yields the momentum-independent coefficients $m_1^{(2)}(k) = m_2^{(2)}(k) \equiv m_s^{(2)}$. Indeed, along with the Lorentz invariant denominators the integrand in the r.h.s. of (199) is contained function $f(I)$ whose argument

$$I(\mathbf{p}, \mathbf{k}) \equiv \omega_{\mathbf{k}}^2 - (E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}})^2 = \mu_s^2 - 2\mu_b^2 - 2E_{\mathbf{p}}E_{\mathbf{p}-\mathbf{k}} - 2\mathbf{p}(\mathbf{p}-\mathbf{k})$$

does not change under the simultaneous transformation $\mathbf{p} \Rightarrow \mathbf{p}' = \Lambda\mathbf{p}$ and $\mathbf{p}-\mathbf{k} \Rightarrow \Lambda(\mathbf{p}-\mathbf{k})$ on the mass shells $p^2 = \mu_b^2$ and $k^2 = \mu_s^2$. Similar combinations have been considered in [47], where the author, handling the mass renormalization problem within noncovariant perturbation theory for a nonlocal extension of the Wentzel model, gives some reasonings in favor of the momentum independence of such integrals as (199). In particular, he has addressed earlier works [48,49] in which similar evaluations have been carried out by means of a cumbersome procedure with so-called w-transformation of integration variables. By invoking those results, one can reduce the triple integral to the simple one,

$$m_s^{(2)} = 8\pi \int_0^{\infty} \frac{t^2 dt}{\sqrt{t^2 + \mu_b^2}} \frac{f^2(\mu_s^2 - 4t^2 - 4\mu_b^2)}{4t^2 + 4\mu_b^2 - \mu_s^2}. \tag{200}$$

For our purposes it suffices to use alternate derivation of this result, given in Appendix C.

Furthermore, from Eqs. (191)–(193) it follows

$$\begin{aligned} m_{11}^{(2)}(p) &= m_{22}^{(2)}(p) \\ &= -\int \frac{d\mathbf{k}}{\omega_{\mathbf{k}} E_{\mathbf{p}-\mathbf{k}}} \left[\frac{v_{11}^2(\omega_{\mathbf{k}}^2 - (E_{\mathbf{p}} - E_{\mathbf{p}-\mathbf{k}})^2)}{E_{\mathbf{p}} - E_{\mathbf{p}-\mathbf{k}} - \omega_{\mathbf{k}}} - \frac{v_{21}^2(\omega_{\mathbf{k}}^2 - (E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}})^2)}{E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}} + \omega_{\mathbf{k}}} \right], \end{aligned} \tag{201}$$

$$\begin{aligned} m_{12}^{(2)}(p) &= m_{21}^{(2)}(p) \\ &= -\int \frac{d\mathbf{k}}{\omega_{\mathbf{k}} E_{\mathbf{p}-\mathbf{k}}} v_{11}(\omega_{\mathbf{k}}^2 - (E_{\mathbf{p}} - E_{\mathbf{p}-\mathbf{k}})^2) v_{21}(\omega_{\mathbf{k}}^2 - (E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}})^2) \\ &\quad \times \left[\frac{1}{E_{\mathbf{p}} - E_{\mathbf{p}-\mathbf{k}} - \omega_{\mathbf{k}}} - \frac{1}{E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}} + \omega_{\mathbf{k}}} \right]. \end{aligned} \tag{202}$$

Evaluation of these coefficients is simplified once we put

$$\begin{aligned} v_{11}(\omega_{\mathbf{k}}^2 - (E_{\mathbf{p}} - E_{\mathbf{p}-\mathbf{k}})^2) &= v_{21}(\omega_{\mathbf{k}}^2 - (E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}})^2) \\ &= f(\omega_{\mathbf{k}}^2 - (E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}})^2). \end{aligned} \tag{203}$$

$$\begin{aligned} m_b^{(2)}(p) \equiv m_{11}^{(2)}(p) = m_{21}^{(2)}(p) &= -\int \frac{d\mathbf{k}}{\omega_{\mathbf{k}} E_{\mathbf{p}-\mathbf{k}}} f^2(\omega_{\mathbf{k}}^2 - (E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}})^2) \\ &\quad \times \left[\frac{1}{E_{\mathbf{p}} - E_{\mathbf{p}-\mathbf{k}} - \omega_{\mathbf{k}}} - \frac{1}{E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}} + \omega_{\mathbf{k}}} \right] \\ &= 2\int \frac{d\mathbf{k}}{\omega_{\mathbf{k}}} \frac{f^2(\omega_{\mathbf{k}}^2 - (E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}})^2)}{E_{\mathbf{p}-\mathbf{k}}^2 - (E_{\mathbf{p}} - \omega_{\mathbf{k}})^2} + 2\int \frac{d\mathbf{k}}{E_{\mathbf{p}-\mathbf{k}}} \frac{f^2(\omega_{\mathbf{k}}^2 - (E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}})^2)}{\omega_{\mathbf{k}}^2 - (E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}})^2} \end{aligned} \tag{204}$$

or

$$\begin{aligned} m_b^{(2)}(p) &= C_1(p) + C_2(p), \\ C_1(p) &= 2\int \frac{d\mathbf{k}}{\omega_{\mathbf{k}}} \frac{f^2(\omega_{\mathbf{k}}^2 - (E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{k}})^2)}{2pk - \mu_s^2} \end{aligned}$$

and

$$C_2(p) = 2\int \frac{d\mathbf{q}}{E_{\mathbf{q}}} \frac{f^2(\mu_s^2 - 2\mu_b^2 - 2pq)}{\mu_s^2 - 2\mu_b^2 - 2pq}.$$

Evidently, the second integral does not depend upon p so

$$C_2(p) = C_2(0) = 2\int \frac{d\mathbf{q}}{E_\mathbf{q}} \frac{f^2(\mu_s^2 - 2\mu_b^2 - 2\mu_b E_\mathbf{q})}{\mu_s^2 - 2\mu_b^2 - 2\mu_b E_\mathbf{q}}$$
$$= 8\pi \int_0^\infty \frac{q^2 dq}{E_\mathbf{q}} \frac{f^2(\mu_s^2 - 2\mu_b^2 - 2\mu_b E_\mathbf{q})}{\mu_s^2 - 2\mu_b^2 - 2\mu_b E_\mathbf{q}}. \tag{205}$$

It is not the case for integral $C_1(p)$. Thus under the link (203) the boson "mass renormalization" coefficients may be momentum dependent (cf. our comment below Eq. (136)).

At the point one should realize that within our approach, where we are trying to do without any fantom such as the bare masses and coupling constants, the one-meson and one-boson operators M_s and M_b cannot appear in the new form $K(\alpha_c)$ of the initial Hamiltonian. Their main destination is to provide the RI as whole (see the note below Eq. (148)) and we have seen how the second-order displacement $\mathbf{D}^{(2)}$ by Eqs. (149)–(151) and higher-order contributions to the boost operator can be evaluated in the CPR. It is important that the integrals $m_s^{(2)}$, $C_1(p)$ and $C_2(p)$ are convergent at proper choice of the cutoff function. Moreover, as shown in Appendix C, the $m_s^{(2)}$ value considerably decreases when moving from the large Λ values (smeared cutoffs) to smaller Λ's, i.e., cutoffs more localized in momentum space. It is equivalent to an effective weakening of the initial nonlocal interaction with its coupling constant g. A similar trend takes place for other "renormalization" integrals $C_1(p)$ and $C_2(0)$ when the former changes very slowly with the p increase starting from p values comparable to a fixed Λ. These results give us a spring of inspiration for future explorations of the convergence of the recursive procedure proposed here.

Of course, the introduction of a unique cutoff factor $f(x)$ simplifies the interpretation of the integrals obtained in Sect. 4.1. In fact, under the conditions (198) and (203) we find, for example,

$$\frac{1}{2}[R_{nloc}, V_{nloc}](aa \to b\bar{b}) = \int \frac{d\mathbf{p}'}{E_{\mathbf{p}'}} \int \frac{d\mathbf{p}}{E_\mathbf{p}} \int \frac{d\mathbf{k}}{\omega_\mathbf{k}} \int \frac{d\mathbf{k}'}{\omega_{\mathbf{k}'}} \delta(\mathbf{p}' + \mathbf{p} - \mathbf{k}' - \mathbf{k})$$
$$\times f(\omega_{\mathbf{k}'}^2 - (E_{\mathbf{p}'} + E_{\mathbf{p}'-\mathbf{k}'})^2) f(\omega_\mathbf{k}^2 - (E_\mathbf{p} + E_{\mathbf{p}-\mathbf{k}})^2)$$
$$\times \left[\frac{1}{(p'-k')^2 - \mu_b^2} + \frac{1}{(p-k)^2 - \mu_b^2}\right] b_c^\dagger(p') d_c^\dagger(p) a_c(k') a_c(k). \tag{206}$$

Again, we encounter the Feynman-like "propagator", which on the energy shell is converted into the true Feynman propagator for the corresponding S matrix. Moreover, it turns out that the commutator

$$[V_{nloc}(t, \mathbf{x}'), V_{nloc}(t, \mathbf{x})] = 0$$

under the constraints (198) and (203) too (cf. Eq. (147)). Thus, we see that the correction $\mathbf{D}^{(2)}$ is determined by

$$\mathbf{D}^{(2)} = \mathbf{D}_{con}^{(2)} + \mathbf{D}_{ncon}^{(2)}, \tag{207}$$

$$\mathbf{D}_{con}^{(2)} = \frac{i}{2} m_s^{(2)} \int \frac{d\mathbf{k}}{\omega_\mathbf{k}^2} (\frac{\partial a^\dagger(k)}{\partial \mathbf{k}} a(k) - a^\dagger(k) \frac{\partial a(k)}{\partial \mathbf{k}})$$
$$+ \frac{i}{2} C_2(0) \int \frac{d\mathbf{p}}{E_\mathbf{p}^2} \left(\frac{\partial b^\dagger(p)}{\partial \mathbf{p}} b(p) - b^\dagger(p) \frac{\partial b(p)}{\partial \mathbf{p}} + \frac{\partial d^\dagger(p)}{\partial \mathbf{p}} d(p) - d^\dagger(p) \frac{\partial d(p)}{\partial \mathbf{p}}\right)$$
$$+ \frac{i}{2} \int \frac{d\mathbf{p}'}{E_{\mathbf{p}'}} \int \frac{d\mathbf{p}}{E_\mathbf{p}} (E_{\mathbf{p}'} E_\mathbf{p} + \mathbf{p}'\mathbf{p} + \mu_b^2) \left(\frac{C_1(p)}{E_\mathbf{p}} - \frac{C_1(p')}{E_{\mathbf{p}'}}\right)$$
$$\times \frac{b^\dagger(p') b(p) + d^\dagger(p') d(p)}{E_{\mathbf{p}'} - E_\mathbf{p}} \frac{\partial}{\partial \mathbf{p}} \delta(\mathbf{p} - \mathbf{p}') \tag{208}$$

and

$$\begin{aligned}\mathbf{D}_{ncon}^{(2)} &= \frac{i}{2} m_s^{(2)} \int \frac{d\mathbf{k}}{\omega_\mathbf{k}} \frac{\mathbf{k}}{\omega_\mathbf{k}^3} (a^\dagger(k) a^\dagger(k_-) - a(k) a(k_-)) \\ &+ i m_s^{(2)} \int \frac{d\mathbf{k}}{\omega_\mathbf{k}^2} \left(\frac{\partial a^\dagger(k)}{\partial \mathbf{k}} a^\dagger(k_-) - a(k_-) \frac{\partial a(k)}{\partial \mathbf{k}} \right) \\ &+ \frac{i}{2} C_2(0) \int \frac{d\mathbf{p}}{E_\mathbf{p}} \frac{\mathbf{p}}{E_\mathbf{p}^3} (b^\dagger(p) d^\dagger(p_-) - b(p) d(p_-)) \\ &+ i C_2(0) \int \frac{d\mathbf{p}}{E_\mathbf{p}^2} \left(\frac{\partial b^\dagger(p)}{\partial \mathbf{p}} d^\dagger(p_-) - b^\dagger(p) \frac{\partial d^\dagger(p_-)}{\partial \mathbf{p}} - \frac{\partial b(p)}{\partial \mathbf{p}} d(p_-) + b(p) \frac{\partial d(p_-)}{\partial \mathbf{p}} \right) \\ &+ i \int \frac{d\mathbf{p}'}{E_{\mathbf{p}'}} \int \frac{d\mathbf{p}}{E_\mathbf{p}} C_1(p) \frac{E_{\mathbf{p}'} E_\mathbf{p} + \mathbf{p}'\mathbf{p} + \mu_b^2}{E_\mathbf{p}} \\ &\times \frac{b^\dagger(p') d^\dagger(p_-) - b(p') d(p_-)}{E_{\mathbf{p}'} + E_\mathbf{p}} \frac{\partial}{\partial \mathbf{p}} \delta(\mathbf{p} - \mathbf{p}'). \end{aligned} \tag{209}$$

Being compared with the free boosts by Eqs. (234) and (235) (of course, both in the CPR) the correction (207) reduces to replacements of

$$\frac{1}{\sqrt{\omega_\mathbf{k} \omega_{\mathbf{k}'}}} \longrightarrow \frac{1}{\sqrt{\omega_\mathbf{k} \omega_{\mathbf{k}'}}} \left(1 + \frac{m_s^{(2)}}{\omega_\mathbf{k} \omega_{\mathbf{k}'}} \right)$$

and

$$\frac{1}{\sqrt{E_\mathbf{p} E_{\mathbf{p}'}}} \longrightarrow \frac{1}{\sqrt{E_\mathbf{p} E_{\mathbf{p}'}}} \left(1 + \frac{C_2(0)}{E_\mathbf{p} E_{\mathbf{p}'}} + \frac{1}{E_{\mathbf{p}'} - E_\mathbf{p}} \left[\frac{C_1(p)}{E_\mathbf{p}} - \frac{C_1(p')}{E_{\mathbf{p}'}} \right] \right),$$

respectively, in the integrands for the meson boost and the boson boost. It turns out that at moderate Λ values $\sim$1 GeV (typical of the theory of meson-nucleon interactions) in the cutoff function (266) the respective numerical deviations from the free boosts can be small.

5.2 Deuteron Properties in the CPR

Besides, we would like to outline the basic elements of another our exploration that is in progress. It is the case, where relying upon the available experience of relativistic calculations of the deuteron static moments in [50–52] and the deuteron FFs (see reviews [53–55] and refs. therein) one has to deal with the matrix elements $\langle \mathbf{P}', M' | J^\mu(0) | \mathbf{P} = 0, M \rangle$ (to be definite in the laboratory frame). Here the operator $J^\mu(0)$ is the Nöther current density $J^\mu(x)$ at $x = 0$, sandwiched between the eigenstates of a "strong" field Hamiltonian H (cf., discussion in Sect. 5 of lecture [23]). In the CPR with $H = K(\alpha_c)$ (Eq. (158)) and $\mathbf{N} = \mathbf{B}(\alpha_c)$ (Eq. (160)) the deuteron state $|\mathbf{P} = 0, M\rangle$ ($|\mathbf{P}' = \mathbf{q}, M'\rangle$) in the rest (the frame moving with the velocity $\mathbf{v} = \mathbf{q}/m_d$) meets the eigenvalue equation

$$P^\mu |\mathbf{P}, M\rangle = P_d^\mu |\mathbf{P}, M\rangle \tag{210}$$

with the three-momentum transfer $\mathbf{q}$, four-momentum $P_d^\mu = (E_d, \mathbf{P})$, $E_d = \sqrt{\mathbf{P}^2 + m_d^2}$, $m_d = m_p + m_n - \varepsilon_d$ and the deuteron binding energy $\varepsilon_d > 0$.

We know that such observables as the charge, magnetic and quadrupole moments of the deuteron can be expressed through the matrix elements in question (e.g., within the Bethe-Salpeter (BS) formalism [53–55]), where, according to the original contribution [56], one introduces the corresponding covariant FFs. With the aid of cumbersome numerical methods the latter have been evaluated in terms of the Mandelstam current sandwiched between the deuteron BS amplitudes. Some results in the subfield one can find in [57,58].

Unlike this, following [23] and [59], we consider the expansion in the R-commutators

$$J^\mu(0) = W J_c^\mu(0) W^\dagger = J_c^\mu(0) + [R, J_c^\mu(0)] + \frac{1}{2}[R, [R, J_c^\mu(0)]] + \cdots, \tag{211}$$

where $J_c^\mu(0)$ is the initial current in which the bare operators $\{\alpha\}$ are replaced by the clothed ones $\{\alpha_c\}$. Decomposition (211) involves one-body, two-body and more complicated interaction currents, if one uses the terminology customary in the theory of meson exchange currents (MEC) [60]. Further, to the approximation

$$K_I = K(NN \to NN) \sim b_c^\dagger b_c^\dagger b_c b_c \tag{212}$$

and

$$\mathbf{B}_I = \mathbf{B}(NN \to NN) \sim b_c^\dagger b_c^\dagger b_c b_c \tag{213}$$

(see, respectively, (175) and (177)) the eigenvalue problem (210) becomes simpler so its solution acquires the form

$$|\mathbf{P}, M\rangle = \int d\mathbf{p}_1 \int d\mathbf{p}_2 D_M([\mathbf{P}]; \mathbf{p}_1\mu_1; \mathbf{p}_2\mu_2) b_c^\dagger(\mathbf{p}_1\mu_1) b_c^\dagger(\mathbf{p}_2\mu_2)|\Omega\rangle. \tag{214}$$

In this connection, let us recall the relation

$$|\mathbf{q}, M\rangle = exp[i\beta\mathbf{B}(\alpha_c)]|\mathbf{0}, M\rangle \tag{215}$$

with $\beta = \beta\mathbf{n}$, $\mathbf{n} = \mathbf{n}/n$ and $\tanh\beta = v$, that takes place owing to the property

$$e^{i\beta\mathbf{B}} P^\mu e^{-i\beta\mathbf{B}} = P^\nu L_\nu^\mu(\beta), \tag{216}$$

where $L(\beta)$ is the matrix of the corresponding Lorentz transformation. Note also that the label $M = (\pm 1, 0)$ denotes the eigenvalue of the third component of the total (field) angular-momentum operator in the deuteron center-of-mass (details can be found in [22]). The c-coefficients D_M in Eq. (214) are calculated by solving the homogeneous Lippmann–Schwinger equation with the quasipotentials taken from [22] (see formulae (67)–(69) therein). Numerical results can be obtained either using the angular-momentum decomposition (as in [22]) or without it (as in [61,62]). In other words, we are able to do without a semirelativistic treatment, where only lowest order relativistic contributions are included (see [63] and refs. therein).

In its turn, the operator (211) being between the clothed two-nucleon states contributes as

$$\eta_c J^\mu(0)\eta_c = J^\mu_{one\text{-}body} + J^\mu_{two\text{-}body}, \tag{217}$$

where the operator

$$J^\mu_{one\text{-}body} = \int d\mathbf{p}' d\mathbf{p} F^\mu_{p,n}(\mathbf{p}', \mathbf{p}) b_c^\dagger(\mathbf{p}) b_c(\mathbf{p}) \tag{218}$$

with

$$\begin{aligned} &F^\mu_{p,n}(\mathbf{p}', \mathbf{p}) \\ &\quad = e\bar{u}(\mathbf{p}')(F_1^{p,n}[(p'-p)^2]\gamma^\mu + i\sigma^{\mu\nu}(p'-p)_\nu F_2^{p,n}[(p'-p)^2])u(\mathbf{p}) \end{aligned} \tag{219}$$

that describes the virtual photon interaction with the clothed proton (neutron). In Eqs. (217) η_c is the projection operator on the subspace $\mathcal{H}_{2N} \in \mathcal{H}$ spanned on the two-clothed-nucleon states $|2N\rangle = b_c^\dagger b_c^\dagger|\Omega\rangle$.

Its appearance follows from the observation, in which the primary Nöther current operator, being between the physical (clothed) states $|\Psi_N\rangle = b_c^\dagger|\Omega\rangle$, yields the usual on-mass-shell expression

$$\langle\Psi_{p,n}(\mathbf{p}')|J^\mu(0)|\Psi_{p,n}(\mathbf{p})\rangle = F^\mu_{p,n}(\mathbf{p}', \mathbf{p})$$

in terms of the Dirac and Pauli nucleon FFs. Of course, all nucleon polarization labels are implied here together with necessary summations over them in Eq. (218) and so on.

By keeping in the r.h.s. of Eq. (217) only the one-body contribution we arrive to certain off-energy-shell extrapolation of the so-called relativistic impulse approximation (RIA) in the theory of e.m. interactions with nuclei (bound systems). In a recent work by Dubovyk and Shebeko the deuteron magnetic and quadrupole moments have been calculated, using the RIA, to be submitted to Few Body Systems, where the previous paper [22] has been published.

Of course, the RIA results should be corrected including more complex mechanisms of e-d scattering, that are contained in

$$J^{\mu}_{two-body} = \int d\mathbf{p}'_1 d\mathbf{p}'_2 d\mathbf{p}_1 d\mathbf{p}_2 F^{\mu}_{MEC}(\mathbf{p}'_1, \mathbf{p}'_2; \mathbf{p}_1, \mathbf{p}_2) b^{\dagger}_c(\mathbf{p}'_1) b^{\dagger}_c(\mathbf{p}'_1) b_c(\mathbf{p}_1) b_c(\mathbf{p}_2). \tag{220}$$

Analytic (approximate) expressions for the coefficients F^{μ}_{MEC} stem from the R-commutators (beginning with the third one) in the expansion (211), which, first, belong to the class [2.2], as in Eq. (218), and, second, depend on even numbers of mesons involved. It requires a separate consideration aimed at finding a new family of MEC, as we hope not only for the e-d scattering.

At last, one should note that, as before, we prefer to handle the explicitly gauge-independent (GI) representation of photonuclear reaction amplitudes with one-proton absorption or emittance [64,65]. This representation is an extension of the Siegert theorem, in which, the amplitude of interest is expressed through the Fourier transforms of electric (magnetic) field strengths and the generalized electric (magnetic) dipole moments of hadronic system. It allows us to retain the GI in the course of inevitably approximate calculations.

6 Summary

We propose a constructive way of ensuring the RI in QFT with cutoffs in momentum space. In contrast to the traditional approach, where the generators of Π are determined as the Nöether integrals of the energy-momentum tensor density, we do not utilize the Lagrangian formalism so fruitful in case of local field models. Our purpose is to find these generators as elements of the Lie algebra of Π starting from the total Hamiltonian whose interaction density in the Dirac picture includes a Lorentz-scalar part $H_{sc}(x)$. Respectively, the algebraic aspect of the RI as a whole for the present exploration with the so-called instant form of relativistic dynamics is of paramount importance.

In the context, using purely algebraic means the boost generators can be decomposed into the Belinfante operator built of H_{sc} and the operator which accumulates the chain of recursive relations in the second and higher orders in H_{nsc}. Thereby, it becomes clear that Poincaré commutations are not fulfilled if the Hamiltonian does not contain some additional ingredients, which we call the mass renormalization terms, though beyond local field models such a terminology looks rather conventional. We have shown how the method of UCTs enables us to determine the corresponding operators for a given model. Moreover, it can be done using its nonlocal extensions satisfying the requirements of special relativity and preserving certain continuity with local QFTs.

We see that our approach is sufficiently flexible being applied not merely to local field models including ones with derivative couplings and spin $j \geq 1$. Its realization, shown here for the nonlocal extensions of the well-known Yukawa-type couplings, gives us an encouraging impetus when constructing the interactions between the clothed particles simultaneously in the Hamiltonian and the corresponding boost operator. In the course of such a work that is under way (see Sect. 5.2) we are trying to understand to what extent the deuteron quenching in flight affects the deuteron electromagnetic form factors. In our opinion, the present exploration may be also helpful for a field-theoretical treatment of particle decays in flight.

The RI of the S-matrix, that follows from the RI as a whole, can be employed in future calculations, first, in the Dirac picture owing to a unitary equivalence of the CPR to the BPR and, second, in the Heisenberg picture after finding certain links between the in (out) states and the clothed-particle ones (see our talk in Durham [45]). It is known that the latter is most appropriate for describing collisions with the bound systems. We are ready to show our results in these directions somewhere else.

At last, we have tried to offer not only a fresh look at constructing the generators in question but also a nonstandard renormalization procedure in relativistic quantum field theory. In this context, let us remind the prophetic words by Dirac [46]: “I am inclined to suspect that renormalization theory is something that will not survive in the future, and the remarkable agreement between theory and experiment should be looked on as a fluke”.

Appendix A: Generators of the Poincaré Group in the BPR for Free Pion and Nucleon Fields

Replacing in the free densities $H_{\pi}(\mathbf{x})$ and $H_{ferm}(\mathbf{x})$ the fields and their conjugates by expansions (49)–(51) we arrive to the operators of the no-interaction Hamiltonian

$$H_F = H_{\pi} + H_{ferm}$$

with

$$H_\pi = \int \omega_{\mathbf{k}} a^\dagger(\mathbf{k}) a(\mathbf{k}) d\mathbf{k} \tag{221}$$

and

$$H_{ferm} = \sum\!\!\!\!\!\!\int E_{\mathbf{p}} (b^\dagger(\mathbf{p}\mu) b(\mathbf{p}\mu) + d^\dagger(\mathbf{p}\mu) d(\mathbf{p}\mu)) d\mathbf{p}, \tag{222}$$

the linear momentum $\mathbf{P} = \mathbf{P}_F = \mathbf{P}_\pi + \mathbf{P}_{ferm}$ with

$$\mathbf{P}_\pi = \int \mathbf{k} a^\dagger(\mathbf{k}) a(\mathbf{k}) d\mathbf{k} \tag{223}$$

and

$$\mathbf{P}_{ferm} = \sum\!\!\!\!\!\!\int \mathbf{p} (b^\dagger(\mathbf{p}\mu) b(\mathbf{p}\mu) + d^\dagger(\mathbf{p}\mu) d(\mathbf{p}\mu)) d\mathbf{p}, \tag{224}$$

the angular momentum $\mathbf{J} = \mathbf{J}_F = \mathbf{J}_\pi + \mathbf{J}_{ferm}$ with

$$\mathbf{J}_\pi = \frac{i}{2} \int d\mathbf{k} \mathbf{k} \times \left(\frac{\partial a^\dagger(\mathbf{k})}{\partial \mathbf{k}} a(\mathbf{k}) - a^\dagger(\mathbf{k}) \frac{\partial a(\mathbf{k})}{\partial \mathbf{k}} \right) \tag{225}$$

and $\mathbf{J}_{ferm} = \mathbf{L}_{ferm} + \mathbf{S}_{ferm}$, where

$$\begin{aligned} \mathbf{L}_{ferm} = \frac{i}{2} \sum\!\!\!\!\!\!\int d\mathbf{p}\mathbf{p} \times \bigg(& \frac{\partial b^\dagger(\mathbf{p}\mu)}{\partial \mathbf{p}} b(\mathbf{p}\mu) - b^\dagger(\mathbf{p}\mu) \frac{\partial b(\mathbf{p}\mu)}{\partial \mathbf{p}} \\ & + \frac{\partial d^\dagger(\mathbf{p}\mu)}{\partial \mathbf{p}} d(\mathbf{p}\mu) - d^\dagger(\mathbf{p}\mu) \frac{\partial d(\mathbf{p}\mu)}{\partial \mathbf{p}} \bigg), \end{aligned} \tag{226}$$

$$\mathbf{S}_{ferm} = \frac{1}{2} \sum\!\!\!\!\!\!\int d\mathbf{p} \chi^\dagger(\mu') \sigma \chi(\mu) (b^\dagger(\mathbf{p}\mu') b(\mathbf{p}\mu) - d^\dagger(\mathbf{p}\mu') d(\mathbf{p}\mu)), \tag{227}$$

the boosts $\mathbf{N}_F = \mathbf{N}_\pi + \mathbf{N}_{ferm}$ with

$$\mathbf{N}_\pi = \frac{i}{2} \int d\mathbf{k} \omega_{\mathbf{k}} \left(\frac{\partial a^\dagger(\mathbf{k})}{\partial \mathbf{k}} a(\mathbf{k}) - a^\dagger(\mathbf{k}) \frac{\partial a(\mathbf{k})}{\partial \mathbf{k}} \right) \tag{228}$$

and $\mathbf{N}_{ferm} = \mathbf{N}_{ferm}^{orb} + \mathbf{N}_{ferm}^{spin}$, where

$$\begin{aligned} \mathbf{N}_{ferm}^{orb} = \frac{i}{2} \sum\!\!\!\!\!\!\int d\mathbf{p} E_{\mathbf{p}} \bigg(& \frac{\partial b^\dagger(\mathbf{p}\mu)}{\partial \mathbf{p}} b(\mathbf{p}\mu) - b^\dagger(\mathbf{p}\mu) \frac{\partial b(\mathbf{p}\mu)}{\partial \mathbf{p}} \\ & + \frac{\partial d^\dagger(\mathbf{p}\mu)}{\partial \mathbf{p}} d(\mathbf{p}\mu) - d^\dagger(\mathbf{p}\mu) \frac{\partial d(\mathbf{p}\mu)}{\partial \mathbf{p}} \bigg), \end{aligned} \tag{229}$$

$$\mathbf{N}_{ferm}^{spin} = -\frac{1}{2} \sum\!\!\!\!\!\!\int d\mathbf{p}\mathbf{p} \times \frac{\chi^\dagger(\mu') \sigma \chi(\mu)}{E_{\mathbf{p}} + m} \big(b^\dagger(\mathbf{p}\mu') b(\mathbf{p}\mu) + d^\dagger(\mathbf{p}\mu') d(\mathbf{p}\mu) \big). \tag{230}$$

In these formulae $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + \mu_\pi^2}$ ($E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$) the pion (nucleon) energy and $\chi(\mu)$ the Pauli spinor. When deriving Eqs. (227) and (230) we have used the relations

$$\begin{aligned} & u^\dagger(\mathbf{p}\mu') \frac{\partial u(\mathbf{p}\mu)}{\partial \mathbf{p}} - \frac{\partial u^\dagger(\mathbf{p}\mu')}{\partial \mathbf{p}} u(\mathbf{p}\mu) \\ & \quad = v^\dagger(\mathbf{p}\mu') \frac{\partial v(\mathbf{p}\mu)}{\partial \mathbf{p}} - \frac{\partial v^\dagger(\mathbf{p}\mu')}{\partial \mathbf{p}} v(\mathbf{p}\mu) = i \frac{\chi^\dagger(\mu') \sigma \chi(\mu)}{m(E_{\mathbf{p}} + m)} \times \mathbf{p}. \end{aligned} \tag{231}$$

with the orthonormalization conditions

$$u^{\dagger}(\mathbf{p}\mu')u(\mathbf{p}\mu) = v^{\dagger}(-\mathbf{p}\mu')v(-\mathbf{p}\mu) = \frac{E_{\mathbf{p}}}{m}\delta_{\mu\mu'},$$
$$u^{\dagger}(\mathbf{p}\mu')v(-\mathbf{p}\mu) = v^{\dagger}(-\mathbf{p}\mu')u(\mathbf{p}\mu) = 0.$$

such as in [39].

Strictly speaking the fundamental relations (9)–(11) should be verified for every field theory. In this connection, let us check that

$$\left[P^j, N_F^l\right] = i\delta_{jl}H_F. \tag{232}$$

In fact, we find step by step

$$\begin{aligned}
\left[P^j, N_F^l\right] &= [P_\pi^j, N_\pi^l] + [P_{ferm}^j, N_{ferm}^l],\\
\left[P_\pi^j, N_\pi^l\right] &= -i\frac{\partial}{\partial u^j}\left\{e^{i\mathbf{P}_\pi\mathbf{u}}N_\pi^l e^{-i\mathbf{P}_\pi\mathbf{u}}\right\}|_{\mathbf{u}=0}\\
&= \frac{1}{2}\frac{\partial}{\partial u^j}\int d\mathbf{k}\omega_{\mathbf{k}}\left(\frac{\partial}{\partial k^l}[e^{i\mathbf{uk}}a^{\dagger}(\mathbf{k})]a(\mathbf{k})e^{-i\mathbf{uk}} - e^{i\mathbf{uk}}a^{\dagger}(\mathbf{k})\frac{\partial}{\partial k^l}[e^{-i\mathbf{uk}}a(\mathbf{k})]\right)|_{\mathbf{u}=0}\\
&= \frac{\partial}{\partial u^j}\left\{i\int \omega_{\mathbf{k}}d\mathbf{k}u^l a^{\dagger}(\mathbf{k})a(\mathbf{k}) - i\mathbf{N}_\pi^l\right\}_{\mathbf{u}=0} = i\delta_{jl}H_\pi,\\
\left[P_{ferm}^j, N_{ferm}^l\right] &= \left[P_{ferm}^j, N_{ferm}^{orb,l}\right]\\
&= -\frac{\partial}{\partial u^j}\sum\!\!\!\!\!\!\int d\mathbf{p}E_{\mathbf{p}}\left(\frac{\partial}{\partial p^l}\left[e^{i\mathbf{up}}b^{\dagger}(\mathbf{p}\mu)\right]b(\mathbf{p}\mu)e^{-i\mathbf{up}} - e^{i\mathbf{up}}b^{\dagger}(\mathbf{p}\mu)\frac{\partial}{\partial p^l}\left[e^{-i\mathbf{up}}b(\mathbf{p}\mu)\right]\right.\\
&\quad \left. + b^{\dagger}(\mathbf{p}\mu) \to d^{\dagger}(\mathbf{p}\mu), b(\mathbf{p}\mu) \to d(\mathbf{p}\mu)\right)|_{\mathbf{u}=0}\\
&= \frac{\partial}{\partial u^j}\left\{iu^l\sum\!\!\!\!\!\!\int E_{\mathbf{p}}d\mathbf{p}(b^{\dagger}(\mathbf{p}\mu)b(\mathbf{p}\mu) + d^{\dagger}(\mathbf{p}\mu)d(\mathbf{p}\mu)) - i\mathbf{N}_{ferm}^l\right\}_{\mathbf{u}=0}\\
&= i\delta_{jl}H_{ferm}.
\end{aligned}$$

Analogously,

$$[H_F, \mathbf{N}_F] = -i\frac{d}{d\lambda}\left\{e^{iH_F\lambda}N_F e^{-iH_F\lambda}\right\}|_{\lambda=0} = i\mathbf{P}. \tag{233}$$

We also need the expression

$$\mathbf{N}_{mes} = \frac{i}{2}\int d\mathbf{k}'d\mathbf{k}a^{\dagger}(\mathbf{k}')a(\mathbf{k})\frac{\omega'_{\mathbf{k}}\omega_{\mathbf{k}} + \mathbf{k}'\mathbf{k} + \mu_s^2}{\sqrt{\omega'_{\mathbf{k}}\omega_{\mathbf{k}}}}\frac{\partial}{\partial\mathbf{k}}\delta(\mathbf{k}-\mathbf{k}'), \tag{234}$$

equivalent to (228) and the free boost

$$\mathbf{N}_{bos} = \frac{i}{2}\int d\mathbf{p}'d\mathbf{p}(b^{\dagger}(\mathbf{p}')b(\mathbf{p}) + d^{\dagger}(\mathbf{p}')d(\mathbf{p}))\frac{E'_{\mathbf{p}}E_{\mathbf{p}} + \mathbf{p}'\mathbf{p} + \mu_b^2}{\sqrt{E'_{\mathbf{p}}E_{\mathbf{p}}}}\frac{\partial}{\partial\mathbf{p}}\delta(\mathbf{p}-\mathbf{p}'), \tag{235}$$

for the spinless charged bosons.

Appendix B: Evaluation of Commutator $[V_{\pi N}(\mathbf{x}'), V_{\pi N}(\mathbf{x})]$ with a Nonlocal πN Interaction

Let us rewrite the πN interaction density in the r.h.s of Eq. (47) as

$$V_{ps}(\mathbf{x}) \equiv V_{loc}(\mathbf{x}) = \varphi_{ps}(\mathbf{x}) f_{loc}(\mathbf{x}), \tag{236}$$

$$f_{loc}(\mathbf{x}) = ig\frac{m}{(2\pi)^3}\int\frac{d\mathbf{p}'}{E_{\mathbf{p}'}}\int\frac{d\mathbf{p}}{E_{\mathbf{p}}}[\bar{B}(p'), \bar{D}(p')]\gamma_5\begin{bmatrix}B(p)\\ D(p)\end{bmatrix}e^{-i(\mathbf{p}'-\mathbf{p})\mathbf{x}} \tag{237}$$

with notations

$$B(p) = \sum_{\mu} u(p\mu)b(p\mu),$$

$$D(p) = \sum_{\mu} v(p_-\mu)d^{\dagger}(p_-\mu)$$

and commutations

$$\{B_a(p'), \bar{B}_b(p)\} = p_0\delta(\mathbf{p}'-\mathbf{p})(a|P_+(p)|b), \tag{238}$$

$$\{D_a(p'), \bar{D}_b(p)\} = p_0\delta(\mathbf{p}'-\mathbf{p})(a|P_-(p_-)|b), \tag{239}$$

where a and b spinor indices and

$$P_{\pm}(p) = \frac{\hat{p}\pm m}{2m}$$

the standard projection operators.

Here we will consider a nonlocal extension of the Yukawa-type density (236) by introducing

$$V_{nloc}(\mathbf{x}) = \varphi_{ps}(\mathbf{x}) f_{nloc}(\mathbf{x}), \tag{240}$$

$$f_{nloc}(\mathbf{x}) = i\frac{m}{(2\pi)^3}\int\frac{d\mathbf{p}'}{E_{\mathbf{p}'}}\int\frac{d\mathbf{p}}{E_{\mathbf{p}}}g(p',p)[\bar{B}(p'), \bar{D}(p')]\gamma_5\begin{bmatrix}B(p)\\ D(p)\end{bmatrix}e^{-i(\mathbf{p}'-\mathbf{p})\mathbf{x}} \tag{241}$$

with a real and permutably symmetric cutoff function $g(p', p)$. Henceforth, such an occurrence in $V_{nloc}(\mathbf{x})$ of the "coordinate" $\mathbf{x}$ and the subscript *nloc* does not contradict each other. The former originates from translational invariance (cf. the transition from Eq. (6) to Eq. (14)) while the latter allows us to work with the interaction density not being constructed from fields (in our case $\bar{\psi}$ and ψ) which are taken at one and the same point. A similar nonlocal interaction one can find in [37] (see Eq. (4.45) therein). One more condition,

$$g(\Lambda p', \Lambda p) = g(p', p) \tag{242}$$

allows for the operator $V_{nloc}(x)$ to be the Lorentz scalar. Since

$$[V_{nloc}(\mathbf{x}), V_{nloc}(\mathbf{y})] = \varphi(\mathbf{x})\varphi(\mathbf{y})[f_{nloc}(\mathbf{x}), f_{nloc}(\mathbf{y})],$$

the requirement in question

$$[V_{nloc}(\mathbf{x}), V_{nloc}(\mathbf{y})] = 0 \tag{243}$$

is equivalent to

$$[f_{nloc}(\mathbf{x}), f_{nloc}(\mathbf{y})] = 0. \tag{244}$$

At this point, using that technique from Appendix A of [29] with the aid of the identities

$$\begin{aligned} &= A\{B,C\}D - \{A,C\}BD - C\{D,A\}B + CA\{D,B\},\\ [AB, CD] &= A[B,C]D + [A,C]DB + AC[B,D] + C[A,D]B. \end{aligned} \tag{245}$$

for four operators A, B, C and D, one can show that

$$[f_{nloc}(\mathbf{x}), f_{nloc}(\mathbf{y})] = \frac{m}{(2\pi)^3}\int\frac{d\mathbf{p}'}{E_{\mathbf{p}'}}\int\frac{d\mathbf{p}}{E_{\mathbf{p}}}e^{-i\mathbf{p}'\mathbf{x}+i\mathbf{p}\mathbf{y}} f_{nloc}(\mathbf{x}-\mathbf{y}; p', p) - H.c. \tag{246}$$

with

$$f_{nloc}(\mathbf{x}-\mathbf{y};\, p',p) = -\frac{m}{(2\pi)^3}\int \frac{d\mathbf{q}}{E_{\mathbf{q}}} e^{i\mathbf{q}(\mathbf{x}-\mathbf{y})}$$
$$\times g(p',q)g(p,q)[\bar{B}(p')+\bar{D}(p')]\gamma_5[P_+(q_+)+P_-(q_-)]\gamma_5[B(p)+D(p)]$$

or

$$f_{nloc}(\mathbf{x}-\mathbf{y};\, p',p) = g(\mathbf{x}-\mathbf{y};\, p',p)[B^\dagger(p')+D^\dagger(p')][B(p)+D(p)], \tag{247}$$

where

$$g(\mathbf{x}-\mathbf{y};\, p',p) = \frac{1}{(2\pi)^3}\int d\mathbf{q} e^{i\mathbf{q}(\mathbf{x}-\mathbf{y})} g(p',q)g(p,q). \tag{248}$$

Putting $g(p',p)\equiv g$ that yields $g(\mathbf{z};\, p',p)=\delta(\mathbf{z})$, we come back to the initial local model with its property

$$[f_{loc}(\mathbf{x}),\, f_{loc}(\mathbf{y})] = 0. \tag{249}$$

In order to go out beyond the model one can regard the two options,

$$g(p',p) = g exp\left[\frac{(p'-p)^2}{2\Lambda^2}\right] = g(\Lambda) exp\left[-\frac{p'p}{\Lambda^2}\right] \tag{250}$$

and

$$g(p',p) = g\frac{\Lambda^2-\mu_\pi^2}{\Lambda^2-(p'-p)^2}. \tag{251}$$

Here we will restrict ourselves to the first using the second for other applications. In the context, it is convenient to deal with the invariants

$$I^{(\pm)}(z;\, p',p) = \int \frac{d\mathbf{q}}{E_{\mathbf{q}}} e^{\mp iqz} g(p',q)g(p,q) = I^{(\pm)}(\Lambda z;\, \Lambda p',\Lambda p). \tag{252}$$

In case of the factor (250) we encounter the integrals

$$I^{(\pm)}(x'-x;\, p',p) = g^2(\Lambda)\int \frac{d\mathbf{q}}{E_{\mathbf{q}}} e^{\mp iq(x'-x)} e^{-\lambda uq}, \tag{253}$$

where $u=p'+p$, $\lambda=\Lambda^{-2}$ and $g(\Lambda)=g exp(\lambda m^2)$. Thus, since $I^{(-)*}=I^{(+)}$, our task is to evaluate

$$\Delta^{(+)}(x'-x+i\lambda u;\, m) = \int \frac{d\mathbf{q}}{E_{\mathbf{q}}} exp[i(x'-x+i\lambda u)q]. \tag{254}$$

But from $\Delta^{(+)}(\Lambda z;\, m)=\Delta^{(+)}(z;\, m)$ it follows that

$$\Delta^{(+)}(x'-x+i\lambda u;\, m) = \Delta^{(+)}(v+i\lambda r;\, m), \tag{255}$$

where the Lorentz transformation $L=L(\mathbf{u})$ is such that $Lu=(r_0,\mathbf{0})$ with $r_0>0$. Recall that $u^2=(p'+p)^2=2m^2+2p'p>0$. In turn, $v=L(x'-x)$.

Furthermore, it is well known (see, e.g., formula (3.961.1) in [66]) that

$$\int_0^\infty e^{-\beta\sqrt{\gamma^2+y^2}} \sin ay \frac{ydy}{\sqrt{\gamma^2+y^2}} = \frac{ay}{\sqrt{\gamma^2+a^2}} K_1\left(\gamma\sqrt{a^2+\beta^2}\right),$$
$$[Re\beta>0,\, Re\gamma>0,\, a>0]$$

where $K_1(z)$ the modified Bessel function.

Using the result we find

$$\begin{aligned}\Delta^{(+)}(v+i\lambda r;m) &= \int \frac{d\mathbf{q}}{E_{\mathbf{q}}} e^{-E_{\mathbf{q}}(\lambda r_0 - i v_0)} e^{i\mathbf{q}\mathbf{v}} \\ &= 4\pi \int_0^\infty \frac{q dq}{\sqrt{q^2+m^2}} e^{-\sqrt{q^2+m^2}(\lambda y_0 - i v_0)} \frac{\sin(q|\mathbf{v}|)}{|\mathbf{v}|} \\ &= 4\pi m^2 \frac{K_1(z_0)}{z_0} \end{aligned} \tag{256}$$

with $z_0 = m\sqrt{\lambda^2 u^2 - (x'-x)^2 - 2i\lambda v_0 \sqrt{u^2}}$. In the case of interest $x'-x = (0, \mathbf{x}-\mathbf{y})$ and $v_0 = L^0_j(\mathbf{x}-\mathbf{y})^j = \frac{\mathbf{u}(\mathbf{x}-\mathbf{y})}{\sqrt{u^2}}$, so $z_0 = m\sqrt{\lambda^2 u^2 + (\mathbf{x}-\mathbf{y})^2 - 2i\lambda \mathbf{u}(\mathbf{x}-\mathbf{y})} = \varsigma_0$ and

$$g(\mathbf{x}-\mathbf{y};\, p', p) = 4\pi i m^4 \lambda u^0 \frac{K_2(\varsigma_0)}{\varsigma_0^2}. \tag{257}$$

Here we have employed the matrix

$$L^\mu_\nu(u) = \begin{bmatrix} \frac{u^0}{\sqrt{u^2}} & \frac{u^j}{\sqrt{u^2}} \\ -\frac{u^i}{\sqrt{u^2}} & \delta^i_j - \frac{u^i u_j}{u^2 + \sqrt{u^2} u^0} \end{bmatrix}.$$

Formula (257) suffices for the statement below (70).

Appendix C: Evaluation of Integral $m_s^{(2)}(k)$

The alternative in question is prompted by Pauli and Rose [67] with their refined trick to be applied to

$$m_s^{(2)}(k) = \int d\mathbf{p} \frac{E_{\mathbf{p}+\frac{\mathbf{k}}{2}} + E_{\mathbf{p}-\frac{\mathbf{k}}{2}}}{E_{\mathbf{p}+\frac{\mathbf{k}}{2}} E_{\mathbf{p}-\frac{\mathbf{k}}{2}}} \frac{f^2\left(\omega_{\mathbf{k}}^2 - \left(E_{\mathbf{p}+\frac{\mathbf{k}}{2}} + E_{\mathbf{p}-\frac{\mathbf{k}}{2}}\right)^2\right)}{\left(E_{\mathbf{p}+\frac{\mathbf{k}}{2}} + E_{\mathbf{p}-\frac{\mathbf{k}}{2}}\right)^2 - \omega_{\mathbf{k}}^2}. \tag{258}$$

In order to go on let us introduce the new variables w, v and φ, where φ is the azimuthal angle around the axis parallel to $\mathbf{k}$, so

$$\frac{1}{2}\left(E_{\mathbf{p}+\frac{\mathbf{k}}{2}} + E_{\mathbf{p}-\frac{\mathbf{k}}{2}}\right) = w, \quad \frac{1}{2}\left(E_{\mathbf{p}+\frac{\mathbf{k}}{2}} - E_{\mathbf{p}-\frac{\mathbf{k}}{2}}\right) = v. \tag{259}$$

Using the corresponding Jacobian, we obtain

$$\frac{1}{4} \frac{E_{\mathbf{p}+\frac{\mathbf{k}}{2}} + E_{\mathbf{p}-\frac{\mathbf{k}}{2}}}{E_{\mathbf{p}+\frac{\mathbf{k}}{2}} E_{\mathbf{p}-\frac{\mathbf{k}}{2}}} d\mathbf{p} = \frac{w}{k} dv dw d\varphi. \tag{260}$$

From (259) we get

$$\frac{1}{2}\left(E^2_{\mathbf{p}+\frac{\mathbf{k}}{2}} + E^2_{\mathbf{p}-\frac{\mathbf{k}}{2}}\right) = w^2 + v^2 = \mathbf{p}^2 + \frac{\mathbf{k}^2}{4} + \mu_b^2, \tag{261}$$

$$\frac{1}{4}\left(E^2_{\mathbf{p}+\frac{\mathbf{k}}{2}} - E^2_{\mathbf{p}-\frac{\mathbf{k}}{2}}\right) = wv = \frac{1}{2} pk \cos\varphi. \tag{262}$$

The limits of integration over the new variables are

$$
\begin{aligned}
&0 \leq \varphi \leq 2\pi, \\
&v_- = -\frac{k}{2}\sqrt{\frac{w^2 - \frac{k^2}{4} - \mu_b^2}{w^2 - \frac{k^2}{4}}} \leq v \leq v_+ = \frac{k}{2}\sqrt{\frac{w^2 - \frac{k^2}{4} - \mu_b^2}{w^2 - \frac{k^2}{4}}}, \\
&w_0 \equiv \sqrt{\frac{k^2}{4} + \mu_b^2} \leq w \leq \infty.
\end{aligned} \tag{263}
$$

By integrating in (258) we arrive to

$$
\begin{aligned}
m_s^{(2)}(k) &= 4\int_{w_0}^{\infty} dw \int_{v_-}^{v_+} dv \int_0^{2\pi} d\varphi \frac{1}{k}\frac{w}{4w^2 - \omega_{\mathbf{k}}^2} f^2(\omega_{\mathbf{k}}^2 - 4w^2) \\
&= 8\pi \int_{w_0}^{\infty} \sqrt{\frac{w^2 - \frac{k^2}{4} - \mu_b^2}{w^2 - \frac{k^2}{4}}} \frac{w dw}{4w^2 - \omega_{\mathbf{k}}^2} f^2(\omega_{\mathbf{k}}^2 - 4w^2) \\
&= 8\pi \int_{\mu_b}^{\infty} d\epsilon \frac{\sqrt{\epsilon^2 - \mu_b^2}}{4\epsilon^2 - \mu_s^2} f^2(\mu_s^2 - 4\epsilon^2) \\
&= 8\pi \int_0^{\infty} \frac{t^2 dt}{\sqrt{t^2 + \mu_b^2}} \frac{f^2(\mu_s^2 - 4t^2 - 4\mu_b^2)}{4t^2 + 4\mu_b^2 - \mu_s^2}
\end{aligned} \tag{264}
$$

that coincides with the formula (200). Using the popular form (251) we have the cutoff function

$$
g_{12}(p, q_-, k) = g\frac{\Lambda^2 - \mu_s^2}{\Lambda^2 - (p - q_-)^2} \tag{265}
$$

that in combination with assumption (198) is equivalent to the relation

$$
f(I) = g\frac{\Lambda^2 - \mu_s^2}{\Lambda^2 + \mu_s^2 - 4\mu_b^2 - I} \tag{266}
$$

and gives the expression

$$
\begin{aligned}
m_s^{(2)} &= 2\pi g^2 \frac{(\Lambda^2 - \mu_s^2)^2}{\Lambda^4} \\
&\times \left[\frac{\Lambda^2(4\mu_b^2 - \mu_s^2)}{(\Lambda^2 - 4\mu_b^2 + \mu_s^2)^2} \left(\frac{\Lambda}{\sqrt{4\mu_b^2 - \Lambda^2}} \arctan\frac{\sqrt{4\mu_b^2 - \Lambda^2}}{\Lambda} - \frac{\Lambda^2}{\mu_s\sqrt{4\mu_b^2 - \mu_s^2}} \arctan\frac{\mu_s}{\sqrt{4\mu_b^2 - \mu_s^2}} \right) \right. \\
&\left. + \frac{\Lambda^2}{\Lambda^2 - 4\mu_b^2 + \mu_s^2} \left(\frac{\Lambda(2\mu_b^2 - \Lambda^2)}{(4\mu_b^2 - \Lambda^2)^{3/2}} \arctan\frac{\sqrt{4\mu_b^2 - \Lambda^2}}{\Lambda} + \frac{1}{2}\frac{\Lambda^2}{4\mu_b^2 - \Lambda^2} \right) \right],
\end{aligned} \tag{267}
$$

if $\Lambda < 2\mu_b$ and

$$m_s^{(2)} = 2\pi g^2 \frac{(\Lambda^2-\mu_s^2)^2}{\Lambda^4} \times \left[\frac{\Lambda^2(4\mu_b^2-\mu_s^2)}{(\Lambda^2-4\mu_b^2+\mu_s^2)^2} \left(\frac{\Lambda}{2\sqrt{\Lambda^2-4\mu_b^2}} \ln \frac{\Lambda+\sqrt{\Lambda^2-4\mu_b^2}}{\Lambda-\sqrt{\Lambda^2-4\mu_b^2}} - \frac{\Lambda^2}{\mu_s\sqrt{4\mu_b^2-\mu_s^2}} \arctan \frac{\mu_s}{\sqrt{4\mu_b^2-\mu_s^2}} \right) + \frac{\Lambda^2}{\Lambda^2-4\mu_b^2+\mu_s^2} \left(\frac{\Lambda(\Lambda^2-2\mu_b^2)}{2(\Lambda^2-4\mu_b^2)^{3/2}} \ln \frac{\Lambda+\sqrt{\Lambda^2-4\mu_b^2}}{\Lambda-\sqrt{\Lambda^2-4\mu_b^2}} + \frac{1}{2}\frac{\Lambda^2}{4\mu_b^2-\Lambda^2} \right) \right], \tag{268}$$

if $\Lambda > 2\mu_b$.

By putting $\mu_s = \mu_\pi = 0.6994\,fm^{-1}$ and $\mu_b = m = 4.7583\,fm^{-1}$ we find the following sequence

$$m_s^{(2)} 10^4/2\pi g^2 = 1.853,\quad 34.69,\quad 109.84,\quad 224.74,\quad 335.05, \ldots$$

at $\Lambda = 1,\ 2,\ 3,\ 4,\ \mu_b, \ldots$. All values in fm^{-1}.

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