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ЭЛЕМЕНТАРНЫЕ ЧАСТИЦЫ И ПОЛЯ

NEW REPRESENTATION FOR ENERGY–MOMENTUM AND ITS APPLICATIONS TO RELATIVISTIC DYNAMICS

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In this paper we introduce the concept *counterpart of rapidity* and define energy and momentum of the relativistic particle as functions of the counterpart of rapidity. Formulae of the relativistic mechanics defined in such a way are regular near the zero-mass and speed of light state. This representation admits to attain a correct limit of the formulae of the relativistic mechanics, including the Dirac equation, at zero-mass point and explains violation of the parity at this state. On the other hand, the representation for energy–momentum can be realized as a mapping from the massless state onto the massive one which looks like a "q deformation". Hypothesis on quantization of the energy–momentum and the velocity near the light speed is suggested. The group of transformations using the counterpart of rapidity as a parameter of transformation is constructed.

1. INTRODUCTION

In the present paper we elaborate new expressions for the energy, momentum, and velocity of a relativistic particle regular at the point $m = 0$, $v = c$. The formulae for the energy–momentum are presented as functions of some hyperbolic angle χ dual to the rapidity ψ, which forms a *counterpart of the rapidity*.

In one-dimensional case the hyperbolic angles χ and ψ are reciprocal quantities. However, in general, the rapidity and its counterpart physically and geometrically are quite different each from an other. The rapidity, ψ, is equal to zero at the rest state, $v = 0$, and goes to infinity when the velocity tends to $v = c$, whereas its counterpart, the hyperbolic angle χ, is equal to zero at the point $v = c$ and becomes infinity at the rest state, $v = 0$. Another important property of the counterpart of rapidity is its dependence of the proper mass: $\chi = mc/\pi_0$. The quantity $c\pi_0$ is interpreted as an energy of the massless state. Thus, the formulae for energy–momentum can be realized as a mapping from an energy of the massless state onto the energy of a particle with mass. These formulae look like as formulae of "q deformation". An analysis of this observation prompts to introduce a hypothesis on quantization of the velocity near the speed of light.

In a similar manner as translations of the rapidity form a part of the Lorentz group of transformations, translations of the counter-rapidity form some group of transformations.

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Usefulness of the present theory we demonstrate exploring the Dirac equation at the limit $m = 0$. The representation for the energy–momentum via counter-rapidity allows to reach a correct limit at $m = 0$ explicitly displaying violation of the parity in the Dirac equation at this limit.

2. REPRESENTATIONS OF ENERGY–MOMENTUM AS FUNCTIONS OF RAPIDITY AND ITS COUNTERPART

2.1. Elements of Relativistic Dynamics of Charged Particle

Consider a motion of the relativistic particle with charge e in the external electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$. The relativistic equations of motion with respect to the proper time τ are given by the Lorentz-force equations:

$$\frac{d\mathbf{p}}{d\tau} = \frac{e}{mc}\mathbf{E}p_0 + \frac{e}{m}[\mathbf{p}\times\mathbf{B}], \quad \frac{dp_0}{d\tau} = \frac{e}{mc}(\mathbf{E}\cdot\mathbf{p}), \tag{1}$$

$$\frac{d\mathbf{r}}{d\tau} = \frac{\mathbf{p}}{m}, \quad \frac{dt}{d\tau} = \frac{p_0}{mc}. \tag{2}$$

These equations imply the first integral of motion, the "mass-shell" equation:

$$p_0^2 - (\mathbf{p}\cdot\mathbf{p})^2 = M^2c^2. \tag{3}$$

In the case of stationary potential field, i.e. when $e\mathbf{E} = -\nabla V(r)$, the equations imply the other constant of motion, the energy of the relativistic particle

$$\mathcal{E} = cp_0 + V(r). \tag{4}$$

If the external electromagnetic field strengths are given in the covariant form, $F_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$, then Eqs. (1), (2) are written in the form of Minkowski force equations [1]:

$$\frac{d}{d\tau}u^\mu = \frac{e}{mc}F^\mu{}_\nu u^\nu, \quad u^\mu = \frac{dx^\mu}{d\tau}. \tag{5}$$

Correspondence with nonrelativistic equations gives an interpretation of the constant of motion M^2 as a squared mass of the particle, so that $M^2 = m^2$. For the massless particle $M^2 = 0$ [2]. It is important to emphasize that the relativistic dynamics of charged particle is formed by the pair of energies [3]:

$$q_1 := cp_0 - mc^2, \quad q_2 := cp_0 + mc^2. \tag{6}$$

In the nonrelativistic limit the former is transformed into kinetic energy of the Newtonian particle

$$cp_0 - mc^2 \to \frac{p^2}{2m}. \tag{7}$$

Next, we shall restrict ourselves by considering only length of the momentum $p = |\mathbf{p}|$. For that purpose let us use the project of the Lorentz-force equation (1) on the direction of motion, that is

$$\frac{dp}{d\psi} = p_0, \quad \frac{dp_0}{d\psi} = p, \tag{8}$$

$$\frac{d\psi}{d\tau} = \frac{e}{mc}E, \quad E = (\mathbf{n} \cdot \mathbf{E}), \quad \mathbf{n} = \frac{\mathbf{p}}{p}.$$

From the first two equations of (8) by taking into account (3), we find

$$p_0 = mc\cosh(\psi), \quad p = mc\sinh(\psi), \tag{9}$$

where $\psi = 0$ corresponds to the rest state with $p = 0$, $p_0 = mc$. Velocity with respect to coordinate time is defined by

$$\frac{v}{c} = \frac{p}{p_0} = \tanh(\psi). \tag{10}$$

The same expression is used for the mapping between rapidity ψ and velocity v in the Lorentz kinematics [4].

2.2. Hyperbolic Angle Dual to Rapidity

Conventionally formulae (9) are considered as formulae of parametrization of the mass-shell equation (3). In this context notice, however, that this is not unique form of parametrization. In fact, we can satisfy (3) by taking

$$p_0 = mc\coth(\chi), \quad p = \frac{mc}{\sinh(\chi)}.$$

The parametrization as an objective does not give any interpretation of the parameter, however. Now, let us consider the procedure of parametrization from another point of view.

Consider the quantities q_2, q_1 as solutions of a quadratic equation

$$X^2 - 2p_0X + p^2 = 0, \tag{11}$$

where

$$2p_0 = q_1 + q_2, \quad p^2 = q_1q_2, \quad 2mc = q_2 - q_1. \tag{12}$$

Notice, this quadratic equation is another form of the mass-shell equation (3). In fact, translation of X by $X = mc + p_0$ leads to (3). According to Hamilton–Cayley theorem general solution of Eq. (11) can be represented by the following matrix

$$E := \begin{pmatrix} 0 & -p^2 \\ 1 & 2p_0 \end{pmatrix}, \tag{13}$$

eigenvalues of which are roots of polynomial equation (11). This matrix generates some evolution along parameter ϕ [3, 5, 6]. Write the Euler formula for exponential function of the matrix $E\phi$

$$\exp(E\phi) = g_0(\phi; p_0, p^2) + Eg_1(\phi; p_0, p^2). \tag{14}$$

In terms of the eigenvalues this equation is decoupled into a pair of equations

$$\exp(q_1\phi) = g_0(\phi; p_0, p^2) + q_1g_1(\phi; p_0, p^2), \tag{15}$$
$$\exp(q_2\phi) = g_0(\phi; p_0, p^2) + q_2g_1(\phi; p_0, p^2).$$

Theorem [7].
The following equation holds true

$$\frac{q_2}{q_1} = \exp(2mc\phi_0), \tag{16}$$

where $2mc = q_2 - q_1$.

Proof:

$$\exp(2mc\phi) = \tag{17}$$
$$= \frac{\exp(q_2\phi)}{\exp(q_1\phi)} = \frac{g_1q_2 + g_0}{g_1q_1 + g_0} = \frac{q_2 - U}{q_1 - U},$$

where

$$U = -\frac{g_0}{g_1}.$$

Notice, simultaneous translation of the eigenvalues by $q_i = q_i + U$, $i = 1, 2$ remains unchanged the difference between them $2mc = q_2 - q_1$, or, in other words, it remains the mass unchanged. Hence, this translation induces a corresponding translation of the hyperbolic argument:

$$\exp(2mc(\phi + \delta(U))) = \frac{q_2}{q_1} = \exp(2mc\phi_0). \tag{18}$$

Inversely, translation of ϕ_0 by $\phi = \phi_0 + \delta$ has to remain $2m$ unchanged because m does not depend on ϕ, whereas q_1, q_2 will undergo some translation simultaneously by $q_2 = q_2 + \Delta$, $q_1 = q_1 + \Delta$.

End of proof.

Thus, we got some interrelation between the fraction and the hyperbolic exponential function, or between simultaneous translation of two quantities of the fraction and the hyperbolic rotation. Write (18) as follows

$$\frac{p_0 + mc}{p_0 - mc} = \exp(2mc\phi), \tag{19}$$

where the mass m is a constant of the evolution with respect to parameter ϕ, whereas p_0, p depend of ϕ and this dependence is given by hyperbolic trigonometry

$$p_0 = mc\coth(mc\phi), \quad p = \frac{mc}{\sinh(mc\phi)}. \tag{20}$$

It is interesting to apply the present theorem to the following fraction

$$\frac{p_0 + p}{p_0 - p} = \frac{c + v}{c - v}, \tag{21}$$

where p and p_0 are variables of the evolution. Let $p > 0$, then (21) we have to re-write as follows

$$\frac{p_0 + p}{p_0 - p} = \frac{\frac{p_0}{p} + 1}{\frac{p_0}{p} - 1}.$$

On making use of the theorem we come to the following equality

$$\frac{\frac{p_0}{p} + 1}{\frac{p_0}{p} - 1} = \exp(2\psi). \tag{22}$$

From (22) by taking into account (3) we get

$$p_0 = mc\cosh(\psi), \tag{23}$$
$$p = mc\sinh(\psi), \quad \frac{v}{c} = \tanh(\psi),$$

which is nothing else but Eqs. (9), (10).

We possess now with two different representations for the energy and momentum via hyperbolic trigonometry:

$$\text{(I)} \quad p_0 = mc\cosh(\psi), \quad p = mc\sinh(\psi);$$
$$\text{(II)} \quad p_0 = mc\coth(mc\phi), \quad p = \frac{mc}{\sinh(mc\phi)}.$$

An essential feature of the latter is its regularity at the point $m = 0$, $v = c$. At this point the hyperbolic argument is equal to the energy–momentum of a massless particle:

$$p(m = 0) = p_0(m = 0) = \frac{1}{\phi} = \pi_0. \tag{24}$$

Thus, the value $c\pi_0$ is the energy of the relativistic system at the point $m = 0$, $v = c$. We should emphasize some difference between two hyperbolic angles. At the rest, $\psi = 0$, but $\phi = \infty$, and vice versa, when $v = c$, $\phi = 0$ but $\psi = \infty$. The particle with $m > 0$ is not able to attain the state of speed of light, conversely, the particle possessing $\pi_0 > 0$ cannot fall to the rest state. In the rest state the energy is equal to the proper inertial mass (in energy units) and, in the same manner, in the state of the light speed the energy is equal to π_0. Thus, the kinetic energy of the relativistic particle is governed, besides the inertial mass m, with some internal energy we denoted by π_0. The massive particle moving with velocity less than light velocity possesses both parameters, m and π_0. The parameter π_0 is, in some sense, counterpart of the inertial mass which determines the value of the kinetic energy of the motion. This quantity corresponds to the energy of the particle in its massless state [8].

Let v be the velocity of a particle with respect to coordinate time. This velocity is essentially less than the light velocity, $v < c$. Besides v let us introduce some complementary velocity $\bar{v}$ obeying the equation

$$v^2 + \bar{v}^2 = c^2. \tag{25}$$

Now let us express $\bar{v}$ via the parameter ($\chi = mc\phi$). We get

$$\bar{v}^2 = c^2 - v^2 = c^2\left(1 - \frac{p^2}{p_0^2}\right) = c^2\tanh^2(\chi). \tag{26}$$

Substitute (23) and (26) into (25), this gives

$$c^2\tanh^2(\psi) + c^2\tanh^2(\xi) = c^2. \tag{27}$$

Notice that $\bar{v}$ is expressed via hyperbolic angle (χ) in a manner quite similar as v is expressed via rapidity (ψ). Interrelation between χ and ψ can be also expressed by the following formulae of reciprocity

$$\exp(\psi) = \coth\left(\frac{\chi}{2}\right), \quad \exp(\chi) = \coth\left(\frac{\psi}{2}\right). \tag{28}$$

Hence, complementary velocity $\bar{v}$ and counterpart of rapidity χ are reciprocal with velocity v and rapidity ψ.

3. DIRAC EQUATION NEAR ZERO-MASS POINT

The right- and left-two-component spinors under Lorentz-boost transformations are transformed as follows [9]

$$\xi_R(P) = \frac{p_0 + mc + (\boldsymbol{\sigma} \cdot \mathbf{p})}{\sqrt{2m(p_0 + mc)}}\xi_R(0), \tag{29}$$
$$\xi_L(P) = \frac{p_0 + mc - (\boldsymbol{\sigma} \cdot \mathbf{p})}{\sqrt{2m(p_0 + mc)}}\xi_L(0),$$

where $\xi_R(0)$, $\xi_L(0)$ mean “right” and “left” spinors, correspondingly, at the rest state. When a particle stays at rest, it is impossible to define its spin is "right", or is "left". Hence, $\xi_R(0) = \xi_L(0)$. From (29) it follows

$$mc\xi_R(P) = (p_0 + (\boldsymbol{\sigma}\cdot\mathbf{p}))\xi_L(P),$$
$$mc\xi_L(P) = (p_0 - (\boldsymbol{\sigma}\cdot\mathbf{p}))\xi_R(P).$$

These two equations can be written in a matrix form

$$\begin{pmatrix} -mc & p_0 + (\boldsymbol{\sigma}\cdot\mathbf{p}) \\ p_0 - (\boldsymbol{\sigma}\cdot\mathbf{p}) & -mc \end{pmatrix} \begin{pmatrix} \xi_R \\ \xi_L \end{pmatrix} = 0. \quad (30)$$

This is Dirac equation for the massive particle with spin one-half in *chiral (or spinor, or Weyl)* representation. The corresponding system of differential equations is

$$\begin{pmatrix} -mc^2 & i\hbar\partial_t - i\hbar c\boldsymbol{\sigma}\cdot\nabla \\ i\hbar\partial_t + i\hbar c\boldsymbol{\sigma}\cdot\nabla & -mc^2 \end{pmatrix} \times$$
$$\times \begin{pmatrix} \xi_R \\ \xi_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

ξ_R and ξ_L respectively correspond to the irreducible representations $(1/2, 0)$ and $(0, 1/2)$ of the Lie algebra $SU(2)\otimes SU(2)$, which is isomorphic to the Lie algebra of the proper Lorentz group. Under parity ($\mathbf{x}\to-\mathbf{x}$), ξ_R and ξ_L transform into each other. So the four-component spinor $\Psi(chiral)$ is an irreducible representation of the Lorentz algebra extended by parity. Also $\begin{pmatrix} \xi_R \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \xi_L \end{pmatrix}$ are eigenstates of the matrix $\gamma_5 := i\gamma_0\gamma_1\gamma_2\gamma_3$ with eigenvalues $+1$ and -1, respectively. In the Weyl representation,

$$\gamma_5(chiral) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For massless particles, the equations for the two-component spinors ξ_R and ξ_L decouple, and plane-wave solutions satisfy

$$\widehat{\lambda}\xi_R = \xi_R, \quad \widehat{\lambda}\xi_L = -\xi_L,$$

where $\widehat{\lambda}$ is the helicity operator. These equations are called the Weyl equations and historically served as principal wave equations for "right" and "left" neutrino, respectively. Both equations correspond to the massless particles and the systems of these equations are inter-related with operation of parity. Thus, violation of symmetry of parity does not follow from the original equations, violation of parity in this approach is an additional hypothesis. Now, let us explore the limit $m = 0$ of the Dirac equation on making use of the representation of energy–momentum via counter-rapidity.

In order to give a main idea, firstly, let us concentrate our attention on Eq. (30) written in one-dimensional space:

$$mc\xi_R(P) = (p_0 + p)\xi_L(P), \quad (31)$$
$$mc\xi_L(P) = (p_0 - p)\xi_R(P);$$

here ξ_R, ξ_L are some scalar quantities. Now tend the mass parameter to zero. We get

$$0 = (p_0 + p)\xi_L(P), \quad 0 = (p_0 - p)\xi_R(P). \quad (32)$$

From this system we obtain $\xi_L(P) = 0$, because at the state $m = 0$, $p_0 = p \neq 0$. The second equation is reduced to identity. This formal consideration of the present equations is not valid, of course, firstly, because we did not explore dependence of the quantities ξ_R, ξ_L of the mass parameter. In this context, the formulae (29) also are useless for us, because at the point $m = 0$ we do not possess more with the concept of "rest state".

To Eqs. (31) we come easily making use of (19), (20). Let us start from formulae (20) for the components of momentum as functions of the parameter ϕ. We have

$$\exp(2mc\phi) = \frac{p_0 + mc}{p_0 - mc},$$
$$p_0 = mc\coth(mc\phi), \quad p = \frac{mc}{\sinh(mc\phi)},$$

which can be re-written as follows:

$$(p_0 + mc)\exp(-mc\phi/2) = p\exp(mc\phi/2), \quad (33)$$
$$(p_0 - mc)\exp(mc\phi/2) = p\exp(-mc\phi/2).$$

This system can be cast into matrical form

$$\begin{pmatrix} p_0 + mc & p \\ p & p_0 - mc \end{pmatrix} \begin{pmatrix} \exp(-mc\phi/2) \\ \exp(mc\phi/2) \end{pmatrix} = 0. \quad (34)$$

Another equivalent form of these equations is given by the following system

$$\begin{pmatrix} -mc & p_0 + p \\ p_0 - p & -mc \end{pmatrix} \begin{pmatrix} \cosh(mc\phi/2) \\ \sinh(mc\phi/2) \end{pmatrix} = 0. \quad (35)$$

These equations are invariant with respect to transformation of parity.

Compare Eq. (35) with Eq. (31), both written in one-dimensional form. The difference is that the state vector in (35) has an explicit form with respect to the mass parameter m. Evidently, Eqs. (35) have a correct behavior when $m \to 0$.

Write now Eqs. (35) in the following form:

$$(p_0 + p) = mc\coth\left(\frac{mc\phi}{2}\right), \quad (36)$$

$$(p_0 - p) = mc \tanh\left(\frac{mc\phi}{2}\right).$$

These equations are written in one-dimensional form, where p means a projection on the direction of motion:

$$p = (\mathbf{n}\cdot\mathbf{p}), \quad (\mathbf{n}\cdot\mathbf{p})^2 = p^2.$$

Within the formalism of quantum mechanics of particle with spin one-half we write

$$p\xi(P) = (\boldsymbol{\sigma}\cdot\mathbf{p})\xi(P), \quad (\boldsymbol{\sigma}\cdot\mathbf{p})^2\xi(P) = p^2\xi(P),$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli sigma matrices, $\xi(P)$ — two-component Pauli spinors. On the basis of algebraic system (36) we postulate the following system of quantum equations for two-component spinors ς_L, ς_R:

$$(p_0 + (\boldsymbol{\sigma}\cdot\mathbf{p}))\varsigma_L(m,\phi) = \tag{37a}$$
$$= mc \coth\left(\frac{1}{2}mc\phi\right)\varsigma_R(m,\phi),$$
$$(p_0 - (\boldsymbol{\sigma}\cdot\mathbf{p}))\varsigma_R(m,\phi) = \tag{37b}$$
$$= mc \tanh\left(\frac{1}{2}mc\phi\right)\varsigma_L(m,\phi).$$

Equations (37a), (37b) possess correct limit at the point $m = 0$. It should be emphasized, in Dirac equation (31) one cannot put $m = 0$ directly because the Dirac equations are written for the particles which admit a rest position. In particularly, this possibility was used in order to establish the equation $\xi_R(0) = \xi_L(0)$ [9]. Moreover, in Eqs. (33) one cannot put $m = 0$, too. Unlike this case, Eqs. (37a), (37b) admit regular behavior at the limit $m = 0$. In that formulation in the massless state $\varsigma_L(0, 1/\pi_0) = \varsigma_R(0, 1/\pi_0) = \varsigma_0$. Thus, when in Eqs. (37) we tend the mass m to zero, Eq. (37a) becomes

$$(p_0 + (\boldsymbol{\sigma}\cdot\mathbf{p}))\varsigma_0 = 2\pi_0\varsigma_0, \tag{38}$$

whereas Eq. (37b) is reduced to identity. From this observation we come to the following conclusions:

At the point $m = 0$ four-component spinor is reduced to two-component spinor, which describes a motion of only one particle with $m = 0$. This particle is identified with the "left" neutrino.

The counterpart of rapidity is related with the mass of the "left" neutrino which is experimentally observed [10]. Since the equations are reduced to one two-component spinor field, there is no sense to speak about the "right" neutrino. The symmetry of parity at the limit $m = 0$ is, obviously, violated.

4. MAPPING FROM THE MASSLESS STATE ONTO THE MASSIVE ONE AS A q DEFORMATION

At the beginning of the previous section we have noted that $\phi = 1/\pi_0$, where the quantity π_0 can be interpreted as an energy of the massless state. In this notation formulae for the energy–momentum (20) are written as follows

$$p = \frac{mc}{\sinh(\frac{mc}{\pi_0})}, \quad p_0 = mc \coth\left(\frac{mc}{\pi_0}\right). \tag{39}$$

These formulae can be considered as a mapping from one kind of the energy onto the other, or, from the massless state onto the state with finite mass. In other words, the quantity π_0 is not a fixed value.

Now let us make some modification of formulae (39). For that purpose introduce some parameter in units of mass and label this parameter by κ. Define a dimensionless variable α by

$$\alpha = \frac{\kappa c}{\pi_0}, \quad \pi_0 = \frac{\kappa c}{\alpha}. \tag{40}$$

α runs from $\alpha = 0$ (which corresponds to velocity $v = c$) till $\alpha = \infty$ (which corresponds to the rest state with $v = 0$).

Re-write formulae for the energy–momentum (39) in these variables

$$p = \frac{mc}{\sinh(\frac{m}{\kappa}\alpha)}, \quad p_0 = mc\coth\left(\frac{m}{\kappa}\alpha\right). \tag{41}$$

For velocity we get

$$\frac{v}{c} = \frac{1}{\cosh(\frac{m}{\kappa}\alpha)}. \tag{42}$$

This formula can be considered as some mapping between dimensionless parameters v/c and α. Let us examine physical meaning of the new parameter κ.

Here it is useful to notice that the formula for the momentum admits the following integral representation

$$\frac{\kappa}{p} = \frac{\sinh(\frac{m}{\kappa}\alpha)}{\frac{m}{\kappa}} = \int_{-\alpha/2}^{\alpha/2} \exp\left(2\frac{m}{\kappa}x\right)dx. \tag{43}$$

In [2] it has been shown that κ^{-1} geometrically can be interpreted as a curvature of a hyperbolic space. In this space the length of the circle with radius m is defined by formulae [11]:

$$L(\alpha = 1) := 2\pi\kappa \sinh\left(\frac{m}{\kappa}\right).$$

Correspondingly, the length of the circle with radius $m\alpha$ is equal to

$$L(\alpha) = 2\pi\kappa \sinh\left(\frac{m}{\kappa}\alpha\right). \tag{44}$$

Taking into account this correspondence let us perform the following modifications in the formulae for energy–momentum

$$\frac{mc}{p} = \sinh\left(\frac{m}{\kappa}\alpha\right) \to \frac{c\kappa}{p} = \frac{\sinh(\frac{m}{\kappa}\alpha)}{\sinh(\frac{m}{\kappa})}. \tag{45}$$

$$\frac{p_0}{mc} = \coth\left(\frac{m}{\kappa}\alpha\right) \to \frac{p_0}{c\kappa} = \sinh\left(\frac{m}{\kappa}\right)\coth\left(\frac{m}{\kappa}\alpha\right). \tag{46}$$

Now remember the formula of q deformation of an integer quantity N:

$$(N)_q := \frac{q^N - q^{-N}}{q - q^{-1}}. \tag{47}$$

From this point of view the fraction in (45) is q deformation of α with parameter of deformation $q = \exp(m/\kappa)$. In notation of (47) Eq. (45) can be written as follows

$$\frac{c\kappa}{p} = (\alpha)_q, \quad q = \exp\left(\frac{m}{\kappa}\right). \tag{48}$$

Notice, $(\alpha = 1)_q = 1$ for any q. Therefore

$$p(m \neq 0, \alpha = 1) = \kappa c, \quad \text{for any} \quad m > 0.$$

On the other hand, if $m = 0$ and $q = 1$, then from (48) it follows

$$p(m = 0) = \pi_0 = \frac{c\kappa}{\alpha}. \tag{49}$$

Hence at the point $\alpha = 1$, momenta of the particles with different masses, including the massless particle, are equal to $c\kappa$:

$$\begin{aligned} p(m \neq 0, \alpha = 1) = p(m = 0, \alpha = 1) = \\ = \pi_0(\alpha = 1) = c\kappa. \end{aligned} \tag{50}$$

In fact, these formulae imply existence of a point on the axis of momentum where the momenta of the massive and massless particles are equal. Notice, however, the velocity of the particle at this point is not equal to the light velocity. At this point the energy and the velocity are given by

$$cp_0(\alpha = 1) = c^2\kappa\cosh\left(\frac{m}{\kappa}\right), \tag{51}$$

$$v = \frac{c}{\cosh(\frac{m}{\kappa})}.$$

Thus, the point $\alpha = 1$ is a peculiar point of the relativistic dynamics where the constant κ now has to be understood as an universal constant.

The procedure of q deformation is usually used in order to extend formulae obtained for integer number to the field of real numbers. Seemingly, α is represented by integer numbers. Here, let us give an additional argumentation to this hypothesis. For that purpose remember integral representation (43). Now, instead of the fraction in (43) we deal with the fraction

$$\frac{c\kappa}{p} = \frac{\sinh(\frac{m}{\kappa}\alpha)}{\sinh(\frac{m}{\kappa})}. \tag{52}$$

It is interesting to observe, for the fraction in (52) we shall put an equivalent sum instead of the integral because now we assumed that α is an integer number. Let J be a half-integer number with $J = 0, 1/2, 1, 3/2, 2, \ldots,$ and $\alpha = 2J + 1 = 1, 2, 3, \ldots$ Then the following equation holds true

$$\frac{c\kappa}{p} = \frac{\sinh(\frac{m}{\kappa}(2J+1))}{\sinh(\frac{m}{\kappa})} = \sum_{n=-J}^{J} \exp\left(n\frac{m}{\kappa}\right). \tag{53}$$

This equality prompts us to introduce a hypothesis on quantization of α. Experimentally the quantization can be observed near the light velocity where the velocity of the massive particle brings nearer the light velocity spasmodically according to law

$$v = \frac{c}{\cosh(\frac{m}{\kappa}(2J+1))}. \tag{54}$$

5. COVARIANT FORM OF THE COUNTER-RAPIDITY AND ITS GROUP OF TRANSFORMATION

In the Lorentz group of transformations the rapidity performs a role of parameter of transformation corresponding to the Lorentz-boosts. In this form the rapidity is presented as a vector with three components. In this section we shall construct the group which uses the counter-rapidity as a parameter of transformation.

Firstly, let us consider transformation for four-velocity $\{u_0, u_1, u_2, u_3\}$ in the plane (u_0, u_1) with respect to translation of hyperbolic angle χ: $\chi' = \chi + \delta$. Introduce the following variables

$$V_0 = \cosh\delta, \quad V = \frac{c}{\sinh\delta}.$$

Then, by using hyperbolic trigonometry we get

$$u_0' = c\coth(\chi + \delta) = \frac{u_0V_0 + c^2}{u_0 + V_0},$$

$$u_1' = \frac{c}{\sinh(\chi + \delta)} = u_1 V\frac{1}{u_0 + V_0}.$$

The purpose of this section is to generalize this transformation to the case of four-vectors. Let us start with evolution equations for momenta $\{p_0, p_1, p_2 = 0, p_3 = 0\}$ with respect to rapidity

$$\frac{dp}{d\phi} = -pp_0, \quad \frac{dp_0}{d\phi} = -p^2. \tag{55}$$

When the other coordinates of the momentum are not trivial, these equations are extended as follows

$$\frac{d\mathbf{p}}{d\phi} = -\mathbf{p}p_0, \quad \frac{dp_0}{d\phi} = -(\mathbf{p}\cdot\mathbf{p}). \tag{56}$$

However, it is easily seen that this equation is not Lorentz-covariant equation.

In order to extend this equation to the Lorentz-covariant equation we have to suppose that our parameter of transformation, the hyperbolic angle ϕ, is only one of the components of some four-component vector $\xi^\nu, \nu = 0,1,2,3$. With respect to these parameters the evolution of four-vector $x_\mu, \mu = 0,1,2,3$, is governed by the following equations

$$\frac{\partial x_\nu}{\partial \xi^\mu} = \rho^2\eta_{\nu\mu} - x_\nu x_\mu, \quad \rho^2 = (x^\mu x_\mu), \tag{57}$$

where $\eta_{\nu\mu} = \mathrm{diag}(1,-1,-1,-1)$ is the Minkowski metric. From these equations we return to Eqs. (56) by taking $x_0 = p_0, x_1 = p, x_2 = x_3 = 0, \xi^0 = \phi, \xi^1 = \xi^2 = \xi^3 = 0$. From this point of view, evolution Eqs. (56) are zeroth component of the general form of equations written in the following form

$$\frac{\partial p_\nu}{\partial \xi^\mu} = (mc)^2\eta_{\nu\mu} - p_\nu p_\mu.$$

The generators of transformation with respect to parameters $\xi^\nu, \nu = 0,1,2,3$, are defined by derivatives

$$G_\nu = \frac{\partial}{\partial \xi^\nu}, \quad \nu = 0,1,2,3. \tag{58}$$

Express these generators in terms of space–time coordinates x^μ as follows

$$G_\nu = \frac{\partial x^\mu}{\partial \xi^\nu}\frac{\partial}{\partial x^\mu} = \rho^2\frac{\partial}{\partial x^\nu} - x_\nu\left(x^\mu\frac{\partial}{\partial x^\mu}\right), \quad \nu,\mu = 0,1,2,3. \tag{59}$$

Now let us construct commutation relations containing the generators $G_\mu, \mu = 0,1,2,3$, as elements of the group which we are seeking. Let us denominate this group by "Γ group". First of all, calculate commutators between G generators. They are

$$[G_\mu, G_\nu] = \rho^2\, M_{\mu\nu}, \quad M_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu. \tag{60}$$

Thus, in surplus Γ group contains elements of the Lorentz group $M_{\mu\nu}$ and the factor ρ^2 which is also the element of the Γ group. The length is invariant under the action of G generator and the generators of the Lorentz group,

$$[G_\mu, \rho^2] = 0, \quad [\rho^2, M_{\mu\nu}] = 0. \tag{61}$$

The operator G_μ is a sum of two operators, namely, the differential operator in Minkowski space ∂_μ and generator of dilatation $D = (x^\mu\partial_\mu)$: $G_\mu = \rho^2\partial_\mu - x_\mu D$. By taking into account $[M_{\mu\nu}, D] = 0$ and $[\rho^2, M_{\mu\nu}] = 0$, we get

$$[M_{\mu\nu}, G_\lambda] = \eta_{\nu\lambda}G_\mu - \eta_{\mu\lambda}G_\nu. \tag{62}$$

The generators of the Lorentz group obey ordinary commutation relations

$$[M_{\mu\nu}, M_{\lambda\eta}] = (\eta_{\mu\lambda}M_{\nu\eta} - \eta_{\nu\lambda}M_{\mu\eta} + \eta_{\mu\eta}M_{\lambda\nu} - \eta_{\nu\eta}M_{\lambda\mu}). \tag{63}$$

The Γ group possesses two Casimir operators:

$$C_1 = G^2, \tag{64}$$
$$C_2 = M_{\mu\nu}M_{\mu\nu}G^2/2 - M_{\mu\lambda}M_{\mu\lambda}G_\mu G_\nu.$$

The Γ group can be extended by introducing the dilation operator $D = (x^\mu\partial_\mu)$. In this case we must add two additional commutators

$$[D, G_\mu] = G_\mu, \quad [D, \rho^2] = 2\rho^2.$$

This extension is similar to the extension of Poincaré group to the Weyl group in adding the dilation operator [12].

It is worth to note an analogy between elements of special conformal group $K_\mu, \mu = 0,1,2,3$, and the generators of Γ group. A representation of the special conformal elements via differential operators in Minkowski space is given by

$$K_\mu = \rho^2\partial_\mu - 2x_\mu D.$$

Thus, this operator differs from the operator G_μ only by factor 2 at the second term.

Also, let us emphasize similarity of the generator G_μ to the generator of translation on the surface with constant curvature [13]. In fact, if the factor $\rho^2 = \mathrm{const}$, the operator G is transformed into the operator of translation along hyperbolic surface imbedded into four-dimensional Minkowski space.

It is important that the commutation relations of the Γ group can be realized by the set of finite-dimensional matrices formed by Dirac *gamma*-matrices. These matrices satisfy anti-commutation relations

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}\cdot 1,$$

where $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ [14].

In terms of the γ matrices we can construct the following commutation relations

$$[\gamma_\mu, \gamma_\nu] = \Sigma_{\mu\nu}, \tag{65}$$
$$[\Sigma_{\mu\nu}\Sigma_{\lambda\eta}] = (\eta_{\mu\lambda}\Sigma_{\nu\eta} - \eta_{\nu\lambda}\Sigma_{\mu\eta} + \eta_{\mu\eta}\Sigma_{\lambda\nu} - \eta_{\nu\eta}\Sigma_{\lambda\mu}),$$
$$[\Sigma_{\mu\nu}, \gamma_\lambda] = \eta_{\nu\lambda}\gamma_\mu - \eta_{\mu\lambda}\gamma_\nu.$$

With the spin matrices $\Sigma_{\mu\nu}$ one may organize finite-dimensional nonunitary representation of the Lorentz group. In this context the set of matrices γ_μ, $\Sigma_{\mu\nu}$ will realize nonunitary finite-dimensional representation of the Γ group. Notice that only compact groups admit finite nontrivial unitary representations [12].

6. CONCLUSIONS

In order to complete our knowledge it is indispensable to make observations of the physical objects from different systems of references. Up to now, the relativistic physics dealt only with observers installed upon inertial systems of references with velocities less than light velocity. This velocity can be expressed via some hyperbolic angle, the rapidity, with undefined physical sense of the latter and with formulae for the energy and momentum singular at the point of zero mass and speed of light. In this paper, we suggested to go in opposite direction: from the light speed to the rest state. For that purpose we introduced complementary velocity, which in the light-speed state equals zero, and the rapidity related with this velocity. An important feature of the rapidity, we have introduced, is the following: the hyperbolic angle to this rapidity is proportional to the mass of the particle. This fact provides regularity of the representation at the zero-mass point. Two definitions of the rapidity (the new and the old one) are reciprocal to each other, but this is true only in one-dimensional (with respect to space coordinates) case. In general, in covariant formulation, the counterpart of rapidity (new) is quite distinct from the (old) rapidity. In the Lorentz group of transformations three components of the boosts correspond to the rapidity, whereas in group of transformations with respect to the counterpart of rapidity the parameter of transformation is represented by four-vector. Noteworthy, the finite-dimensional representation of the generators of this group is given by Dirac's gamma-matrices.

The counterpart of rapidity is proportional to the mass of particle and inverse value of some energy, which has been interpreted as an energy of the massless state. The latter does not depend on the mass of the particle, thus this is some universal value. The formulae in (39) are reflection of the universal energy onto the energy of the particle with certain mass.

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