ON LANDAU–KHALATNIKOV–FRADKIN TRANSFORMATION IN QED

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Abstract

We present the results of studies of the gauge dependence of the massless fermion propagator in QED in the framework of dimensional regularization. The results were obtained using the Landau–Khalatnikov–Fradkin transformation between massless charged particle propagators interacting with gauge fields in two different gauges. In the $d = 4 - 2\varepsilon$ case, we present the exact results obtained in [1], which relate the hatted and standard ζ values and are valid in all orders of perturbation theory. In three-dimensional quenched QED assuming the finiteness of the perturbative expansion coefficients, it was shown in [2] that exactly for $d = 3$ all odd perturbative coefficients, starting from the third order, must be equal to zero in any gauge.To test this, in Ref. [3] we calculated three- and four-loop corrections to the massless fermionic propagator. Three-loop corrections are finite and gauge-invariant, while four-loop corrections have singularities.

О ПРЕОБРАЗОВАНИИ ЛАНДАУ–ХАЛАТНИКОВА–ФРАДКИНА В КЭД

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Аннотация

Представлены результаты исследования калибровочной зависимости безмассового фермионного пропагатора в КЭД в рамках размерной регуляризации. Результаты были получены с помощью преобразования Ландау–Халатникова–Фрадкина между пропагаторами безмассовых заряженных частиц, взаимодействующими с калибровочными полями в двух разных калибровках. В случае $d = 4 - 2\varepsilon$ мы представляем точные результаты, полученные в [1], которые связывают стандартные ζ-функции и ζ-функции со шляпкой и справедливы во всех порядках теории возмущений. В трехмерной замороженной КЭД в предположении конечности коэффициентов пертурбативного разложения, в [2] показано, что при $d = 3$ все нечетные пертурбативные коэффициенты, начиная с третьего порядка, должны быть равны нулю в любой калибровке. Чтобы проверить это, в [3] мы рассчитали трех- и четырехпетлевые поправки к безмассовому фермионному пропагатору. Трехпетлевые поправки конечны и калибровочно-инвариантны, а четырехпетлевые поправки содержат сингулярности.

1 Introduction

The Landau–Khalatnikov–Fradkin transformation (LKFT) [4, 5] elegantly links the QED fermion propagator in two different gauges (and similarly for the fermion-photon vertex but it is beyond the present consideration). This transformation has a simple form in representing a coordinate space and allows us to compute Green functions in an arbitrary covariant gauge if we know their value at any particular gauge.

Here we show the LKFT applications around three and four dimensions in the framework of dimensional regularization.

In the case of the usual four-dimensional QED (QED_4) let us consider the multiloop structure of propagator-type functions. It was recently noticed that contributions for various massless Euclidean physical quantities proportional to even values of the ζ function, ζ_{2n}, mysteriously often cancel out (see, *e.g.*, [6–13]). Such puzzling facts have recently given rise to the "absence of π theorem". The latter is based on [14, 15] observation that ε-dependent transformation of ζ values:

$$\hat{\zeta}_3 \equiv \zeta_3 + \frac{3\varepsilon}{2}\zeta_4 - \frac{5\varepsilon^3}{2}\zeta_6, \quad \hat{\zeta}_5 \equiv \zeta_5 + \frac{5\varepsilon}{2}\zeta_6, \quad \hat{\zeta}_7 \equiv \zeta_7\,, \tag{1}$$

eliminates even zetas. The reason for the appearance of the hatted ζ-values is not clear and requires additional investigations.

A generalization (1) is available in Refs. [16–19]. The results (1) and their generalizations make it possible to predict the terms $\sim \pi^{2n}$ in higher orders of perturbation theory (PT) (see their estimate in [16–20]). Also note that [16–19] results contain multi-zeta values, which are beyond the scope of this paper.

Here we consider the LKFT results obtained in [1] (see also [21,22]) to study the general properties of the PT expansion of the fermion propagator. We show how transformation naturally reveals the existence of the hatted transcendental basis. Moreover, this allows the results shown in (1) to be extended to any ε order.

Quantum electrodynamics in three space-time dimensions of (QED_3) with N flavors of four-component massless Dirac fermions has been under continuous study for the past forty years as a useful field-theoretical model. QED_3 served as a toy model for exploring several key problems in quantum field theory, such as infrared singularities in low-dimensional massless particle theories, coupling constant nonanalyticity within PT, dynamic symmetry breaking and fermion mass generation, phase transition and the relationship between chiral symmetry breaking and confinement.

Moreover, QED_3 has found many applications also in condensed matter physics, in particular, in superconductivity with high T_c [23–25], in planar antiferromagnets [26], as well as in studies of graphene [27], where excitations of quasiparticles have linear dispersion at low energies and are described by the massless Dirac equation in $2 + 1$ dimensions (see reviews on graphene studies in [28–31]).

Massless QED_3 plays an important role in investigating the problems of dynamic symmetry breaking and fermion mass generation in gauge theories. The main question that has been debated for a long time is whether there is a critical fermion flavor number, N_{cr}, where the separation of the chiral symmetric phase and the phase with broken chiral symmetry occurs (see [32–48]). Analytical studies of chiral symmetry breaking and mass generation in QED_3 are usually based on the use of the Schwinger–Dyson equations with some ansatzes for the full fermion-photon vertex.

In the recent paper [2], we studied the gauge-covariance of the massless fermion propagator in quenched QED_3 in a covariant gauge. We recall here that the quenched limit of QED is the approximation in which we can neglect the effects of closed fermion loops. This approximation arose in the study of the lattice representation of QED_4 (see [49–52]), which showed that a reasonable estimate of the hadron spectrum can be obtained by eliminating all internal quark loops. Moreover, the quenching approximation in QED_4 is now used to include QED effects in lattice QCD calculations (see the recent paper [53] and discussions therein).

Immediately after its introduction in the study of the lattice representation QED_4, the quenched approximation in QED_4 was also used in Refs. [54–60] within the framework of the formalism based on the study of the Schwinger-Dyson equations.

In Ref. [2], following [1, 60, 61] we applied dimensional regularization and studied the LKFT self-consistency in quenched QED_3 in a covariant gauge. Analysis of [2] led to the conclusion that in exactly three dimensions, $d = 3$, all odd perturbative coefficients, starting from the third order, must be equal to zero in any gauge if QED_3 does not have (infrared) singularities, as discussed in [62–64] To test this, in Ref. [3] we calculated the three- and four-loop orders and found that the three-loop corrections are finite and gauge-invariant, while the four-loop corrections have singularities.

2 LKFT in x space with $d = 4 - 2\varepsilon$

Further, we will consider QED in the Euclidean space of dimension d (in the first two sections $d = 4 - 2\varepsilon$). The general form of the fermion propagator in the momentum and x-space representations, $S_F(p,\xi)$ and $S_F(x,\xi)$, in some gauge ξ is as follows:

$$S_F(p,\xi) = \frac{i}{\hat{p}}\, P(p,\xi)\,, \quad S_F(x,\xi) = \hat{x}\, X(x,\xi)\,, \tag{2}$$

where the tensor structure is distinguished by the factors $\hat{p}$ and $\hat{x}$ containing the γ-Dirac matrices. The two representations, $S_F(x,\xi)$ and $S_F(p,\xi)$, are related by a Fourier transform, which is defined as:

$$S_F(p,\xi) = \int \frac{\mathrm{d}^d x}{(2\pi)^{d/2}}\, e^{\mathrm{i}px}\, S_F(x,\xi)\,, \quad S_F(x,\xi) = \int \frac{\mathrm{d}^d p}{(2\pi)^{d/2}}\, e^{-\mathrm{i}px}\, S_F(p,\xi)\,. \tag{3}$$

The famous LKFT very simply relates the fermionic propagator in two different gauges, *e.g.*, ξ and η. In dimensional regularization, it looks like this [1]:

$$S_F(x,\xi) = S_F(x,\eta)\, e^{\mathrm{i}D(x)},\; D(x) = -ie^2\, \Delta\, \mu^{2\varepsilon} \int \frac{\mathrm{d}^d q}{(2\pi)^d}\, \frac{e^{-\mathrm{i}qx}}{q^4},\quad \Delta = \xi - \eta\,. \tag{4}$$

Now we can proceed to the calculation of $D(x)$. To do this, you can use the following simple formulas for the Fourier transform of massless propagators (see, *e.g.*, [65]):

$$\int \mathrm{d}^d x\, \frac{e^{\mathrm{i}px}}{x^{2\alpha}} = \frac{2^{2\tilde{\alpha}}\pi^{d/2} a_0(\alpha)}{p^{2\tilde{\alpha}}}, \quad \int \mathrm{d}^d p\, \frac{e^{-\mathrm{i}px}}{p^{2\alpha}} = \frac{2^{2\tilde{\alpha}}\pi^{d/2} a_0(\alpha)}{x^{2\tilde{\alpha}}},$$
$$a_n(\alpha) = \frac{\Gamma(d/2 - \alpha + n)}{\Gamma(\alpha)}\,. \tag{5}$$

This gives:

$$D(x) = \frac{\mathrm{i}\,\Delta\, A}{\varepsilon}\, \Gamma(1-\varepsilon)\, (\pi\mu^2 x^2)^{\varepsilon}, \quad A = \frac{\alpha_{\mathrm{em}}}{4\pi} = \frac{e^2}{(4\pi)^2}\,. \tag{6}$$

3 LKFT in momentum space with $d = 4 - 2\varepsilon$

Suppose that for some gauge parameter η the fermionic propagator $S_F(p,\eta)$ with external momentum p has the form (2) with $P(p,\eta)$ as:

$$P(p,\eta) = \sum_{m=0}^{\infty} a_m(\eta)\, A^m \left(\frac{\tilde{\mu}^2}{p^2}\right)^{m\varepsilon}, \quad \tilde{\mu}^2 = 4\pi\mu^2\,, \tag{7}$$

where $a_m(\eta)$ are the coefficients of the loop expansion of the propagator, and $\tilde{\mu}$ is the renormalization scale, which lies somewhere between the MS-scale μ and the $\overline{\text{MS}}$-scale $\overline{\mu}$.

Using the Fourier transform (5), we have

$$\int \frac{d^d q}{(2\pi)^d} \frac{e^{-iqx}}{(q^2)^\alpha} q_\mu = \left(\frac{i\,\partial}{\partial x_\mu}\right) \int \frac{d^d q}{(2\pi)^d} \frac{e^{-iqx}}{(q^2)^\alpha} = \frac{1}{(4\pi)^{d/2}} \frac{2^{2\tilde{\alpha}+1}\, a_1(\alpha)\, x_\mu}{i\,(x^2)^{\tilde{\alpha}+1}} \,. \tag{8}$$

Then, using (3), we obtain that:

$$S_F(x,\eta) = \frac{2^{d-1}\,\hat{x}}{(4\pi\, x^2)^{d/2}} \sum_{m=0}^{\infty} b_m(\eta)\, A^m \left(\pi\mu^2 x^2\right)^{m\varepsilon}, \quad b_m(\eta) = a_m(\eta)\, \frac{\Gamma(d/2 - m\varepsilon)}{\Gamma(1+m\varepsilon)} \,. \tag{9}$$

With the help of (9) together with an expansion of the LKFT exponent, we have

$$S_F(x,\xi) = \frac{2^{d-1}\hat{x}}{(4\pi x^2)^{d/2}} \sum_{m=0}^{\infty} b_m(\eta)\, A^m \left(\pi\mu^2 x^2\right)^{m\varepsilon} \sum_{l=0}^{\infty} \left(-\frac{A\Delta}{\varepsilon}\right)^l \frac{\Gamma^l(1-\varepsilon)}{l!} \,(\pi\mu^2 x^2)^{l\varepsilon}. \tag{10}$$

Factorizing all x-dependence yields:

$$\begin{aligned} S_F(x,\xi) &= \frac{2^{d-1}\,\hat{x}}{(4\pi\, x^2)^{d/2}} \sum_{p=0}^{\infty} b_p(\xi)\, A^p \left(\pi\mu^2 x^2\right)^{p\varepsilon} \,, \\ b_p(\xi) &= \sum_{m=0}^{p} \frac{b_m(\eta)}{(p-m)!} \left(-\frac{\Delta}{\varepsilon}\right)^{p-m} \Gamma^{p-m}(1-\varepsilon) \,. \end{aligned} \tag{11}$$

Hence, taking the correspondence between the results for propagators $P(p,\eta)$ and $S_F(x,\eta)$ in (7) and (9), respectively, together with the result (11) for $S_F(x,\eta)$, we have for $P(p,\xi)$:

$$P(p,\xi) \;=\; \sum_{m=0}^{\infty} a_m(\xi)\, A^m \left(\frac{\tilde{\mu}^2}{p^2}\right)^{m\varepsilon} \,, \tag{12}$$

where

$$\begin{aligned} a_m(\xi) &= b_m(\xi)\, \frac{\Gamma(1+m\varepsilon)}{\Gamma(d/2-m\varepsilon)} \qquad (13) \\ &= \sum_{l=0}^{m} \frac{a_l(\eta)}{(m-l)!}\, \frac{\Gamma(d/2-l\varepsilon)\Gamma(1+m\varepsilon)}{\Gamma(1+l\varepsilon)\Gamma(d/2-m\varepsilon)} \left(-\frac{\Delta}{\varepsilon}\right)^{m-l} \Gamma^{m-l}(1-\varepsilon) \,. \end{aligned}$$

In this way, we have derived the expression of $a_m(\xi)$ using a simple expansion of the LKFT exponent in x space. From this LKFT representation, we see that the magnitude $a_m(\xi)$ is determined by $a_l(\eta)$ with $0 \leq l \leq m$.

Very often, however, the subject of the study is not the magnitude $a_m(\xi)$ but the p- and Δ-dependencies of each magnitude $a_l(\eta)$ as it evolves from the η to the ξ gauge. The corresponding result for the p- and Δ-dependencies of $\hat{a}_m(\xi, p)$ can be obtained interchanging the order of the sums in the r.h.s. of (12). Performing such interchange yields:

$$P(p,\xi) \; = \; \sum_{m=0}^{\infty} \hat{a}_m(\xi,p)\, A^m \left(\frac{\tilde{\mu}^2}{p^2}\right)^{m\varepsilon}, \tag{14}$$

where

$$\hat{a}_m(\xi,p) = a_m(\eta) \sum_{l=0}^{\infty} \frac{\Gamma(d/2-m\varepsilon)\Gamma(1+(l+m)\varepsilon)}{\Gamma(1+m\varepsilon)\Gamma(d/2-(l+m)\varepsilon)} \left(-\frac{A\Delta}{\varepsilon}\right)^l \frac{\Gamma^l(1-\varepsilon)}{l!} \left(\frac{\tilde{\mu}^2}{p^2}\right)^{l\varepsilon}. \tag{15}$$

3.1 Scale fixing

Following [1], we consider only the case of so-called MS-like schemes. In such schemes, we need to fix some terms resulting from the application of dimensional regularization. This procedure will be called *scale fixing* and will play a decisive role in our analysis.

It is well known that when calculating two-point massless diagrams, the final results do not contain ζ_2. [1] Therefore, it is convenient to choose a certain scale in which ζ_2 disappears already at intermediate steps of the calculations. To do this, in [1] we also introduced a new scale based on the old calculations of massless diagrams, performed by Vladimirov who added (see [69]) an additional factor $\Gamma(1-\varepsilon)$ to the contribution of each loop. This corresponds to adding the factor $\Gamma^{-1}(1-\varepsilon)$ to the corresponding scale. In [1] we called this scale the *minimal Vladimirov* scale, or the MV-scale, and gave the following definition:

$$\mu_{\rm MV}^{2\varepsilon} \; = \; \frac{\tilde{\mu}^{2\varepsilon}}{\Gamma(1-\varepsilon)}\,. \tag{16}$$

[1] Strictly speaking, ζ_2 may appear in some formulas, such as the sum rules in deep inelastic scattering. They come from analytic continuation [66–68] but we will not consider this case in this paper.

In the scale, we can rewrite the result (15) in the following general form:

$$\hat{a}_m(\xi,p) \;=\; a_m(\eta)\sum_{l=0}^{\infty}\frac{1-(m+1)\varepsilon}{1-(m+l+1)\varepsilon}\,\Phi_{\rm MV}(m,l,\varepsilon)\,\frac{(\Delta\,A)^l}{(-\varepsilon)^l l!}\left(\frac{\mu^2_{\rm MV}}{p^2}\right)^{l\varepsilon}, \quad (17)$$

where

$$\Phi_{\rm MV}(m,l,\varepsilon) = \frac{\Gamma(1-(m+1)\varepsilon)\Gamma(1+(m+l)\varepsilon)\Gamma^{2l}(1-\varepsilon)}{\Gamma(1+m\varepsilon)\Gamma(1-(m+l+1)\varepsilon)}\,. \quad (18)$$

The factor $(1-(m+1)\varepsilon)/(1-(m+l+1)\varepsilon)$ was specially extracted from $\Phi_{\rm MV}(m,l,\varepsilon)$, to insure the same transcendental level, *i.e.*, the same value s for ζ_s for each order of the ε expansion $\Phi_{\rm MV}(m,l,\varepsilon)$ (see below).

3.2 MV scale

The Γ function $\Gamma(1+\beta\varepsilon)$ has the following expansion:

$$\Gamma(1+\beta\varepsilon) \;=\; \exp\left[-\gamma\beta\varepsilon+\sum_{s=2}^{\infty}(-1)^s\,\eta_s\beta^s\varepsilon^s\right], \quad \eta_s = \frac{\zeta_s}{s}\,. \quad (19)$$

Substituting Eq. (19) in Eq. (18), we obtain for the factor $\Phi_{\rm MV}(m,l,\varepsilon)$:

$$\Phi_{\rm MV}(m,l,\varepsilon) \;=\; \exp\left[\sum_{s=2}^{\infty}\eta_s\,p_s(m,l)\,\varepsilon^s\right], \quad (20)$$

where

$$p_s(m,l) \;=\; (m+1)^s-(m+l+1)^s+2l+(-1)^s\Big\{(m+l)^s-m^s\Big\}, \quad (21)$$

and, as expected by the MV scale, we have $p_1(m,l)=0, p_2(m,l)=0$.

As can be seen from the Eq. (20), $\Phi_{\rm MV}(m,l,\varepsilon)$ contains the values of the ζ_s function of a given weight (or transcendental level) s in the factor ε^s. This property severely limits the coefficients of the ε series, which simplifies our analysis. It resembles the one found earlier in Ref. [70]. When used wisely, this property sometimes allows you to get results without any calculations (as in Ref. [71–74]). In other cases, this simplifies the structure of the results, which can then be very easily predicted in the form of some ansatz (see Refs. [75–81]).

3.3 Solution of the recurrence relations

Now let us focus on the polynomial $p_s(m,l)$ in Eq. (21), which can be conveniently divided into even and odd values of s. Then we see that the following recursive relations hold:

$$p_{2k} \;=\; p_{2k-1}+Lp_{2k-2}+p_3, \quad p_{2k-1}=p_{2k-2}+Lp_{2k-3}+p_3, \quad L=l(l+1)\,. \tag{22}$$

For the MV scheme, these relations depend only on L, which leads to strong simplifications.

Taking the results for p_{2k} in the following form

$$p_{2k} \;=\; \sum_{s=2}^{k} p_{2s-1}\, C_{2k,2s-1} \;=\; \sum_{m=1}^{k-1} p_{2k-2m+1}\, C_{2k,2k-2m+1}\,, \tag{23}$$

one can determine the exact k dependency $C_{2k,2s-1}$, which has the following structure:

$$C_{2k,2k-2m+1} = b_{2m-1}\,\frac{(2k)!}{(2m-1)!\,(2k-2m+1)!}, \quad b_{2m-1} \;=\; \frac{(2^{2m}-1)}{m}\,B_{2m}\,, \tag{24}$$

where B_m are the well-known Bernoulli numbers.

3.4 Hatted ζ values

At this point it is convenient to represent the exponent argument in the r.h.s. of (20) as follows:

$$\sum_{s=3}^{\infty} \eta_s\, p_s\, \varepsilon^s \;=\; \sum_{k=2}^{\infty} \eta_{2k}\, p_{2k}\, \varepsilon^{2k} + \sum_{k=2}^{\infty} \eta_{2k-1}\, p_{2k-1}\, \varepsilon^{2k-1}\,. \tag{25}$$

Using Eq. (23) the first term in the r.h.s. of Eq. (25) can be expressed as:

$$\sum_{k=2}^{\infty} \eta_{2k}\, p_{2k}\, \varepsilon^{2k} = \sum_{k=2}^{\infty} \eta_{2k}\, \varepsilon^{2k} \sum_{s=2}^{k} p_{2s-1}\, C_{2k,2s-1} = \sum_{s=2}^{\infty} p_{2s-1} \sum_{k=s}^{\infty} \eta_{2k}\, C_{2k,2s-1}\, \varepsilon^{2k}\,. \tag{26}$$

Then, Eq. (25) can be written as $\sum_{s=2}^{\infty} \hat{\eta}_{2s-1}\, p_{2s-1}\, \varepsilon^{2s-1}$, where

$$\begin{aligned} \hat{\eta}_{2s-1} \;&=\; \eta_{2s-1} + \sum_{k=s}^{\infty} \eta_{2k}\, C_{2k,2s-1}\, \varepsilon^{2(k-s)+1}\,, \\ C_{2k,2s-1} \;&=\; b_{2k-2s+1}\,\frac{(2k)!}{(2s-1)!\,(2k-2s+1)!}\,. \end{aligned} \tag{27}$$

and, correspondingly, for $\zeta_s = s\eta_s$ (see Eq. (19))

$$\hat{\zeta}_{2s-1} \;=\; \zeta_{2s-1} + \sum_{k=s}^{\infty} \zeta_{2k}\, \hat{C}_{2k,2s-1}\, \varepsilon^{2(k-s)+1} \tag{28}$$

with

$$\hat{C}_{2k,2s-1} \;=\; \frac{2s-1}{2k}\, C_{2k,2s-1} \;= b_{2k-2s+1}\, \frac{(2k-1)!}{(2s-2)!\,(2k-2s+1)!}\,. \tag{29}$$

Together with (29) and (24), Eq. (28) provides an exact expression for the hatted ζ values in terms of standard ζ, which is holds for all orders of ε expansion.

4 LKFT with $d = 3 - 2\varepsilon$

In Sections 4 and 5, we will consider a Euclidean space of dimension $d = 3 - 2\varepsilon$. The general form of the fermionic propagators $S_F(p,\xi)$ and $S_F(x,\xi)$ in some gauge ξ is shown in Eq.(2).

As in Eq.(7), the fermion propagator can be represented in the form

$$P(p,\eta) \;= \sum_{m=0}^{\infty} a_m(\eta)\, B^m \left(\frac{\overline{\mu}^2}{p^2}\right)^{m\varepsilon}, \quad B \;=\; \frac{\alpha}{2\sqrt{\pi}\, p}\,, \tag{30}$$

where $\overline{\mu}$ is the $\overline{MS}$ scale.

Repeating the evaluation done in Eqs. (8)–(14) with the replacement $\varepsilon \to 1/2 + \varepsilon$ and $A \to B$, we obtain the results

$$P(p,\xi) \;=\; \sum_{m=0}^{\infty} a_m(\xi)\, B^m \left(\frac{\tilde{\mu}^2}{p^2}\right)^{m\varepsilon}, \tag{31}$$

where

$$a_k(\xi) \;= \sum_{m=0}^{k} (-2\Delta)^{k-m}\, a_m(\eta)\, \hat{\Phi}(m,k,\varepsilon)\; \phi(k-m,\varepsilon) \tag{32}$$

and

$$\begin{aligned} \hat{\Phi}(m,k,\varepsilon) \;&=\; \frac{\Gamma(3/2 - m/2 - (m+1)\varepsilon)\Gamma(1 + k/2 + k\varepsilon)}{\Gamma(1 + m/2 + m\varepsilon)\Gamma(3/2 - k/2 - (k+1)\varepsilon)}\,, \\ \phi(l,\varepsilon) \;&=\; \frac{\Gamma^l(1/2 - \varepsilon)}{l!\,(1+2\varepsilon)^l\Gamma^l(1+\varepsilon)}\,. \end{aligned} \tag{33}$$

Now consider $a_m(\xi)$ with $m \leq 4$:

$$
\begin{aligned}
&a_0(\xi)=a_0(\eta)=1,\ \ a_1(\xi) = a_1(\eta)-\frac{\pi}{2}\delta\Big(1+2\varepsilon(l_2-1)\Big)a_0(\eta),\ \ (\delta=\sqrt{\pi}\Delta),\\
&a_2(\xi) = a_2(\eta) - \frac{4}{\pi}\delta\Big(1-2\varepsilon(l_2+1)\Big)\,a_1(\eta) + \delta^2\,\Big(1-4\varepsilon\Big)\,a_0(\eta)\,,\\
&a_3(\xi) = a_3(\eta) + 6\pi\varepsilon\,\delta\,a_2(\eta) - 12\varepsilon\,\delta^2\,a_1(\eta) + 2\pi\varepsilon\,\delta^3\,a_0(\eta)\,, \qquad (34)\\
&a_4(\xi)=a_4(\eta)-\frac{2\delta}{3\pi\varepsilon}\Big(1+2\varepsilon(3-l_2)\Big)a_3(\eta) - 2\delta^2 a_2(\eta) + \frac{8\delta^3}{3\pi}a_1(\eta) - \frac{\delta^4}{3}a_0(\eta).
\end{aligned}
$$

Setting $\eta = 0$, i.e. choosing the initial Landau gauge, we can represent the results (35) for $a_m(\xi)$ in the form

$$
a_m(\xi) \ = \ a_m(0) + \xi\,\tilde{a}_m(\xi) \qquad (35)
$$

and verify that our results for $\tilde{a}_m(\xi)$ are completely determined by $a_l(0)$, $(l < m)$, i.e. coefficients of lower orders in accordance with the LKF transformation.

It is clearly seen in (35) that, starting from the third, the contributions of odd orders to even ones are accompanied by singularities ε^{-1}. In turn, the even orders contribute to the odd ones, starting from the third, $\sim \varepsilon$. Assuming the perturbative finiteness of the massless quenched QED_3 [62–64] and the existence of a finite limit at $\varepsilon \to 0$, we showed in [2] that exactly in $d = 3$ all odd PT terms except a_1 must be equal to zero in any gauge. This required verification, and in Ref. [3] we calculated the three- and four-loop corrections shown in the next section.

5 Fermion propagator: three- and four-loop coefficients

For the calculations, it is convenient to use

$$
P(p,\xi) \ = \ \frac{1}{1-\sigma(p,\xi)}\,, \qquad (36)
$$

where the 1PI-part $\sigma(p,\xi)$ can be represented (similary to (30)) as

$$
\sigma(p,\xi) \ = \ \sum_{m=1}^{\infty} \sigma_m(\xi)\,B^m \left(\frac{\overline{\mu}^2}{p^2}\right)^{m\varepsilon}. \qquad (37)
$$

Some details of the calculations can be found in [3]. Here we present the results for $\sigma_m(\xi)$, which can be represented in the form similar to (35):

$$
\sigma_m(\xi) \ = \ \sigma_m(0) + \xi\,\tilde{\sigma}_m(\xi)\,. \qquad (38)
$$

Taking into account the first two orders of the ε expansion, we have for $\sigma_m(0)$

$$\sigma_1(0) = 0; \quad \sigma_2(0) = \pi \left[\frac{3\pi^2}{4} - 7 - \Big((1-3l_2)\pi^2 + 12\Big)\varepsilon\right],$$
$$\sigma_3(0) = \pi^{5/2}\left[\frac{43\pi^2}{4} - 105 + \varepsilon\left\{2\Big(185 - 105l_2 + 137\zeta_3\Big) - \frac{\pi^2}{6}\Big(451 - 171l_2\Big)\right\}\right],$$
$$\sigma_4(0) = \pi^2\left[\left(\frac{43}{6}\pi^2 - 70\right)\frac{1}{\varepsilon} + \overline{\sigma}_4 + \frac{5954}{3} + \frac{173}{18}\pi^2 - \frac{513}{10}\pi^4\right], \tag{39}$$

where $\overline{\sigma}_4$ contains the most complicated part

$$\overline{\sigma}_4 \;=\; 209l_2^4 + 5016a_4 + 4264\mathrm{Cl}_4(\pi/2) + \left(\frac{533}{3}\mathrm{C} - 930\,l_2\right)\pi^2 + \frac{2078}{3}\zeta_3\,, \tag{40}$$

and

$$l_2 = \ln 2, \quad a_4 = \mathrm{Li}_4(1/2), \quad \zeta_n = \mathrm{Li}_n(1)\,, \tag{41}$$

C is Catalan, Li_n are polylogarithms and Cl_4 is Clausen number.

With the same accuracy, we have for the coefficients $\tilde{\sigma}_m(\xi)$

$$\tilde{\sigma}_1(\xi) = -\frac{\pi^{3/2}}{2}\Big(1 - 2(1-l_2)\varepsilon\Big), \quad \tilde{\sigma}_2(\xi) = \pi\,\xi\left[1 - \frac{\pi^2}{4} - \big(4 - (1-l_2)\pi^2\big)\varepsilon\right],$$
$$\tilde{\sigma}_3(\xi) = \pi^{5/2}\left[\frac{3\pi^2}{4} - 7 + \left(1 - \frac{\pi^2}{8}\right)\xi^2 + \varepsilon\left\{-40 - 14l_2 + \frac{\pi^2}{2}(4+9l_2)\right.\right.$$
$$\left.\left. + \left(2l_2 - 4 + \frac{3\pi^2}{4}(1-l_2)\right)\xi^2\right\}\right];$$
$$\tilde{\sigma}_4(\xi) = \pi^2\left[\left(70 - \frac{43\pi^2}{6}\right)\frac{1}{\varepsilon} + \frac{520}{3} - \frac{\pi^2}{9}\Big(881 + 42l_2\Big) + \frac{129\pi^4}{27} - \frac{548}{3}\zeta_3\right.$$
$$\left. + \xi\left(28 - \frac{33\pi^2}{4} + \frac{9\pi^4}{16}\right) + \xi^3\left(-\frac{4}{3} + \frac{3\pi^2}{4} - \frac{\pi^4}{16}\right)\right]. \tag{42}$$

Note that the finite parts of the coefficients $\sigma_1(\xi)$ and $\sigma_2(\xi)$ coincide with the corresponding ones in Ref. [82]. So, we see that

$$\sigma_4(\xi) \;=\; \pi^2\left(\frac{43}{6}\pi^2 - 70\right)\frac{(1-\xi)}{\varepsilon} + O(\varepsilon^0)\,, \tag{43}$$

i.e. the four-loop results are finite in Feynman gauge.

5.1 $a_m(\xi)$

The coefficients $a_m(\xi)$ and $\sigma_m(\xi)$ are related each other as

$$a_1 = \sigma_1, \;\; a_2 = \sigma_2 + \sigma_1^2, \;\; a_3 = \sigma_3 + 2\sigma_2\sigma_1 + \sigma_1^3, \;\; a_4 = \sigma_4 + 2\sigma_3\sigma_1 + \sigma_2^2 + 3\sigma_2\sigma_1^2 + \sigma_1^4 \,. \tag{44}$$

Since $\sigma_1(\xi) \sim \xi$, we see in (44) that $a_i(0) = \sigma_i(0)$ for $i \le 3$ and thus so $a_i(0)$ with $i \le 3$ can be found in Eq. (39). According to (44) we have for $a_4(0)$:

$$a_4(0) = \sigma_4(0) + \pi^2 \left(\frac{3\pi^2}{4} - 7\right)^2 = \pi^2 \left[\left(\frac{43}{6}\pi^2 - 70\right)\frac{1}{\varepsilon} + \overline{\sigma}_4 + \frac{6101}{3} - \frac{8}{9}\pi^2 - \frac{4059}{80}\pi^4\right]. \tag{45}$$

With the same accuracy, we have for the coefficients $\tilde{a}_m(\xi)$

$$\begin{aligned}
\tilde{a}_1(\xi) &= \tilde{\sigma}_1(\xi) = -\frac{\pi^{3/2}}{2}\Big(1 - 2(1-l_2)\varepsilon\Big), \quad \tilde{a}_2(\xi) = \pi\,\xi\Big(1 - 4\varepsilon\Big), \\
\tilde{a}_3(\xi) &= \pi^{5/2}\,\varepsilon\left(\frac{43\pi^2}{4} - 105 + 2\xi^2\right), \\
\tilde{a}_4(\xi) &= \frac{\pi^2}{3}\left[\left(210 - \frac{43\pi^2}{2}\right)\frac{1}{\varepsilon} + 520 + \frac{2\pi^2}{3}\Big(32 - 21 l_2\Big) - 548\zeta_3 + 6\xi\left(7 - \frac{3\pi^2}{4}\right) - \xi^3\right].
\end{aligned} \tag{46}$$

Note that the finite parts of the coefficients $a_1(\xi)$ and $a_2(\xi)$ coincide with the corresponding ones in [83–85] (see also the Ref. [2] and discussions therein).

We see that the coefficients $\tilde{a}_m(\xi)$ $(m = 2, 3, 4)$ have a simpler form than the corresponding coefficients $\tilde{\sigma}_m(\xi)$. Moreover, as in the case of $\sigma_4(\xi)$, we also see that

$$a_4(\xi) \;=\; \sigma_4(\xi) + O(\varepsilon^0) \;=\; \pi^2\left(\frac{43}{6}\pi^2 - 70\right)(1-\xi)\,\frac{1}{\varepsilon} + O(\varepsilon^0)\,, \tag{47}$$

i.e. the four-loop results are finite in Feynman gauge.

6 Conclusion

In the framework of QED$_4$, based on the LKFT results (17) for the fermionic propagator, we have shown the specific recurrence relations (22) found in [1] between even and odd values of the polynomial associated with a uniformly transcendental factor $\Phi_{\rm MV}(m, l, \varepsilon)$ (18). These relations are simple in the MV scheme introduced in (16). So, they terms of hatted ζ values and thus lead to Eq. (28) relating the hatted and standard ζ values for all PT orders. The coefficients in Eq. (28) are expressed in terms of the well-known Bernoulli numbers, B_{2m} (see (29) and (24)).

These results impose constraints on the results of multi-loop calculations in any PT order, which have already been used in the recent paper [20]. However, the reason for the appearance of hatted ζ values, i.e. the appearance of even ζ-values along with additional ε powers is not clear and requires additional explanation.

In the case of QED_3, in our recent paper [2] (see also [86]) we studied the LKFT for the massless fermionic propagator in the quenched approximation. Studying this transformation in dimensional regularization, we found that the contributions of odd orders, starting from the third, to even ones, are accompanied by singularities that look like ε^{-1} in dimensional regularization. In turn, the even orders produce contributions to the odd ones, starting from the third, $\sim \varepsilon$.

Following the arguments in favor of the perturbative finiteness of the massless quenched QED_3 [62–64] and assuming the existence of a finite limit at $\varepsilon \to 0$, in [2] we have shown exactly in $d = 3$ that *all odd terms of $a_{2t+1}(\xi)$ in perturbation theory except a_1 must be exactly zero in any gauge.*

This statement was very strong and needed verification, which was done exactly in Ref. [3]. We calculated three- and four-loop corrections, i.e. terms $a_3(\xi)$ and $a_4(\xi)$, directly in the PT framework. We found that $a_3(\xi)$ is finite and gauge independent when $\varepsilon \to 0$. The coefficient $a_4(\xi)$ has singularities, which violates the status of the infrared perturbative finiteness of the massless quenched QED_3.

Moreover, in Ref. [3] we found that the singularities contributing to the coefficient $a_4(\xi)$, $\sim (1 - \xi)$ and thus $a_4(\xi)$ is finite in the Feynman gauge. The reason for this effect is not clear and more research is needed to elucidate it.

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