

The Russian edition of the present book was published in June 2013. It just happened that it was the time between two significant dates: in 2011 the Nobel Prize was awarded ``for the discovery of the accelerated expansion of the Universe through observations of distant Supernovae" and in 2013 the Nobel Prize was awarded ``for the theoretical discovery of a mechanism that contributes to our understanding of the origin of the mass of subatomic particles". Both these formulations left the questions about the explanations of these phenomena in the framework of the fundamental principles open. Our book is devoted to attempts to explain the observed ``long distances to the Supernovae" and ``the small value of the Higgs particle mass" by the principles of affine and conformal symmetries and the vacuum postulate. Both these phenomena are described by quantum gravity in the form of joint irreducible unitary representations of the affine and conformal symmetry groups. These representations were used in our book to classify physical processes in the Universe, including its origin from the vacuum.

Principles of Quantum Universe

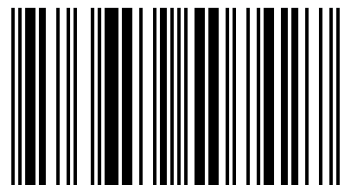


Victor Pervushin
Alexander Pavlov

Victor Pervushin

Victor Nikolaevich Pervushin - Advisor to the Directorate of the Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna. Alexander Egorovich Pavlov - Associate Professor, Russian State Agrarian University - Moscow Timiryazev Agricultural Academy. Their interests are particle physics, cosmology, and gravitation.

Principles of Quantum Universe



978-3-659-50742-7

Pervushin, Pavlov



**Victor Pervushin
Alexander Pavlov**

Principles of Quantum Universe

**Victor Pervushin
Alexander Pavlov**

Principles of Quantum Universe

LAP LAMBERT Academic Publishing

Impressum / Imprint

Bibliografische Information der Deutschen Nationalbibliothek: Die Deutsche Nationalbibliothek verzeichnet diese Publikation in der Deutschen Nationalbibliografie; detaillierte bibliografische Daten sind im Internet über <http://dnb.d-nb.de> abrufbar.

Alle in diesem Buch genannten Marken und Produktnamen unterliegen warenzeichen-, marken- oder patentrechtlichem Schutz bzw. sind Warenzeichen oder eingetragene Warenzeichen der jeweiligen Inhaber. Die Wiedergabe von Marken, Produktnamen, Gebrauchsnamen, Handelsnamen, Warenbezeichnungen u.s.w. in diesem Werk berechtigt auch ohne besondere Kennzeichnung nicht zu der Annahme, dass solche Namen im Sinne der Warenzeichen- und Markenschutzgesetzgebung als frei zu betrachten wären und daher von jedermann benutzt werden dürften.

Bibliographic information published by the Deutsche Nationalbibliothek: The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available in the Internet at <http://dnb.d-nb.de>.

Any brand names and product names mentioned in this book are subject to trademark, brand or patent protection and are trademarks or registered trademarks of their respective holders. The use of brand names, product names, common names, trade names, product descriptions etc. even without a particular marking in this works is in no way to be construed to mean that such names may be regarded as unrestricted in respect of trademark and brand protection legislation and could thus be used by anyone.

Coverbild / Cover image: www.ingimage.com

Verlag / Publisher:

LAP LAMBERT Academic Publishing

ist ein Imprint der / is a trademark of

OmniScriptum GmbH & Co. KG

Heinrich-Böcking-Str. 6-8, 66121 Saarbrücken, Deutschland / Germany

Email: info@lap-publishing.com

Herstellung: siehe letzte Seite /

Printed at: see last page

ISBN: 978-3-659-50742-7

Copyright © 2014 OmniScriptum GmbH & Co. KG

Alle Rechte vorbehalten. / All rights reserved. Saarbrücken 2014

Principles of Quantum Universe

V.N. Pervushin, A.E. Pavlov

Laboratory of Theoretical Physics

Joint Institute for Nuclear Research, Dubna

February 20, 2014

Contents

1	Introduction	17
1.1	What is this book about?	17
1.2	Program	23
1.3	Does the creation and evolution of the Universe depend on an observer?	30
1.4	Contents	58
2	Initial data and frames of reference	65
2.1	Units of measurement	65
2.2	Nonrelativistic mechanics of a particle	67
2.3	Foundations of Special Relativity	69
2.3.1	Action of a relativistic particle	69
2.3.2	Dynamics of a relativistic particle	71
2.3.3	Geometrodynamics of a relativistic particle	75
2.3.4	Reduction of geometrodynamics to the Planck's relativistic dynamics (1906)	78
2.3.5	Quantum anomaly of geometric interval	80

2.3.6	How does the invariant reduction differ from the choice of gauge?	83
2.4	Homogeneous approximation of General Relativity	85
2.4.1	Radiation-dominated cosmological model	85
2.4.2	Arrow of conformal time as quantum anomaly	90
2.5	Standard cosmological models	90
2.6	Summary	93
3	Principles of symmetry of physical theories	98
3.1	Irreducible representations of Lorentz group $SO(3, 1)$	98
3.2	Irreducible representations of Poincaré group $SO(3, 1) \dot{\otimes} T(4)$	101
3.3	Weyl group	106
3.4	Conformal group $SO(4, 2)$	109
3.5	Conformal invariant theories of gravitation	113
3.5.1	Weyl geometry	113
3.5.2	Deser's theory of gravitation	117
3.5.3	Scalar–tensor theory of Brans and Dicke	118
3.5.4	Dirac's dilaton theory	119
3.6	Affine group $A(4) = L(3, 1) \dot{\otimes} T(4)$	124
3.7	Group of general coordinate transformations	125

3.8	Fundamental elements of base Minkowskian space \mathbb{M}	127
3.9	Summary	128
4	Nonlinear realizations of symmetry groups	133
4.1	Differential forms of Cartan	133
4.1.1	Riemannian coordinates	137
4.2	Geometry of Lie groups	141
4.2.1	Structural equations	143
4.2.2	Exponential parametrization	146
4.3	Algebraic and dynamical principles of symmetry	147
4.4	Theory of gravitation as nonlinear realization of $A(4) \otimes SO(4, 2)$	152
4.4.1	Nonlinear realizations of the affine group	153
4.4.2	Dynamical conformal symmetry	157
4.4.3	Joint realization of affine and conformal symmetries	159
4.4.4	Explicitly conformal invariant General Relativity	162
4.5	Summary	165
5	Hamiltonian formulation of the theory of gravity	168
5.1	Foliation of space–time continuum	168
5.1.1	Derivation formulae	171
5.1.2	Gauss – Weingarten equations	174
5.1.3	Gauss – Codazzi equations	175

5.2	ADM coordinates	176
5.2.1	Kinematic subgroup	179
5.3	Hamiltonian formulation of GR	
	in terms of Cartan forms	182
5.4	Problems of Hamiltonian formulation	189
5.5	Exact solution of the Hamiltonian	
	constraint	194
5.5.1	Statement of the problem	194
5.5.2	Lagrangian formalism	195
5.5.3	Hamiltonian formalism	200
5.6	Summary	202
6	A model of an empty Universe	208
6.1	An empty Universe	208
6.2	The Supernovae data in	
	Conformal Cosmology	212
6.3	The hierarchy of cosmological scales	222
6.4	Special Relativity–General Relativity correspondence .	226
6.5	The arrow of time as a consequence of the postulate of	
	vacuum	229
6.6	Creation of the Universe	233
6.7	Summary	238
7	Quantization of gravitons in terms of Cartan forms	244
7.1	Affine gravitons	244
7.2	Comparison with metric gravitons	249
7.3	Vacuum creation of affine gravitons	253

7.4	Summary	259
8	Mathematical principles of description of the Universe	262
8.1	The classical theory of gravitation	262
8.2	Foundations of quantum theory of gravity	264
8.2.1	The irreducible unitary representation of the group $A(4) \otimes SO(4, 2)$	264
8.2.2	Casimir's vacuum	268
8.2.3	An approximation of a nearly empty Universe	270
8.3	Summary	275
9	Creation of matter in the Universe	279
9.1	The Big Bang or the vacuum creation?	279
9.1.1	Statement of the problem	279
9.1.2	Observational data on the CMB radiation origin	282
9.2	Vacuum creation of scalar bosons	285
9.3	Physical states of matter	289
9.4	QU modification of S-matrix in QFT	293
9.5	Summary	300
10	Reduced phase space QCD	308
10.1	Topological confinement	308
10.2	Quark-hadron duality	313
10.3	Chiral symmetry breaking in QCD	316

10.4 Summary	327
11 QU modification of the Standard Model	332
11.1 SM Lagrangian	332
11.2 The condensate mechanism of Higgs boson mass . . .	335
11.3 Estimation of the Higgs boson mass	342
11.4 Summary	345
12 Electroweak vector bosons	349
12.1 Cosmological creation of electroweak vector bosons	349
12.2 Sources of CMB radiation anisotropy	358
12.3 Baryon asymmetry of the Universe	361
12.4 Summary	366
13 Conformal cosmological perturbation theory	378
13.1 The equations of the theory of perturbations	378
13.2 The solution of the equations for small fluctuations	380
13.3 Summary	383
14 Cosmological modification of Newtonian dynamics	385
14.1 Free motion in conformally flat metric	385
14.2 The motion of a test particle in a central field	389

14.3	The Kepler problem in the Conformal theory	391
14.4	Capture of a particle by a central field	393
14.5	The problem of dark matter in Superclusters	395
14.6	The Kepler problem in the generalized Schwarzschild field	399
14.7	Quantum mechanics of a particle in Conformal cosmology	404
14.8	Summary	406
15	Afterword	410
15.1	Questions of Genesis	410
15.2	General discussion of results	413
15.2.1	Results of the work	413
15.2.2	Discussion	419
A	Reduced Abelian field theory	422
A.1	Reduced QED	422
A.1.1	Action and frame of reference	422
A.1.2	Elimination of time component	424
A.1.3	Elimination of longitudinal component	425
A.1.4	Static interaction	425
A.1.5	Comparison of radiation variables with the Lorentz gauge ones	426
A.2	Reduced vector boson theory	430
A.2.1	Lagrangian and reference frame	430

A.2.2	Elimination of time component	430
A.2.3	Quantization	433
A.2.4	Propagators and condensates	434
B	Quantum field theory for bound states	440
B.1	Ladder approximation	440
B.2	Bethe – Salpeter Equations	446
C	Abel – Plana formula	454
D	Functional Cartan forms	461
D.1	Dynamical model with high derivatives	461
D.2	Variational De Rham complex	466
E	Dynamics of the mixmaster model	471
E.1	Pseudo–Euclidean generalized Toda chain	471
E.2	Kovalevski exponents	473

Preface to the first English edition

The Russian edition of the present book was published in June 2013¹. It just happened that it was the time between two significant dates: in 2011 the Nobel Prize was awarded “*for the discovery of the accelerated expansion of the Universe through observations of distant Supernovae*” and in 2013 the Nobel Prize was awarded for “*the theoretical discovery of a mechanism that contributes to our understanding of the origin of the mass of subatomic particles*”. Both these formulations left the questions about the explanations of these phenomena in the framework of the fundamental principles open.

Our book is devoted to attempts to explain the observed *long distances to the Supernovae* and the *small value of the Higgs particle mass* by the principles of affine and conformal symmetries and the vacuum postulate. Both these phenomena are described by quantum gravity in the form of joint irreducible unitary representations of the affine and conformal symmetry groups. These representations were used in our book to classify physical processes in the Universe, including its origin from the vacuum. The representations of the Poincaré group were used in the same way by Wigner to classify particles and their bound states.

We are far from considering our understanding of the “distant Supernovae” and the “origin of the mass of subatomic particles” to be conclusive, but we do not abandon hope that the present revised and

¹Victor Pervushin, Alexander Pavlov: *Principles of Quantum Universe*. LAP LAMBERT Academic Publishing. 420 pp. (2013). Saarbrücken, Deutschland (in Russian)

enlarged English edition encourages a deeper and worthier investigation of these open questions in the future.

The authors express their appreciation and gratitude to the coauthors of the papers on which this book is based. The authors are grateful to I.V. Kronshtadtova and G.G. Sandukovskaya for proofreading the text of the book. The authors are grateful to Academician V.A. Matveev, Professors V.V. Voronov and M.G. Itkis for the support.

V.N. Pervushin

A.E. Pavlov

February, 2014

Dubna

Preface to the first Russian edition

This monograph is based on papers published during last 25 years by the authors and lectures delivered by one of the authors (V.P.) at the universities of Graz (Austria), Berlin, Heidelberg, Rostock (Germany), New Delhi (India), Fairfield, the Argonne National Laboratory (USA), the physical faculty of Moscow State University and in the Joint Institute for Nuclear Research. The main goal of the authors is to bring readers into the interesting and intriguing problem of description of modern experimental and observational data in the framework of ideas and methods elaborated until 1974 by the founders of the modern relativistic classical and quantum physics. The distinction of our approach from the standard ones consists in using conformal symmetry: everywhere, from the horizon of the Universe to quarks, we use scale-invariant versions of modern theories on the classical level with dimensionless coupling constants, breaking scale invariance only at the quantum level by normal ordering of products of field operators. The method of classification of novae data, obtained in the last fifteen years in cosmology and high-energy physics, essentially uses quantum theories and representations. From here the title of our book originated: “*Principles of Quantum Universe*”. Let us briefly present the content of the book. In Introduction (Chapter 1) we discuss the evolution of ideas and mathematical methods of theoretical physics during last five centuries of its development from Copernicus to Wheeler, focusing on the problem of classification of physical measurements and astrophys-

ical observations. In Chapter 2 we present the problems of choosing initial data and frames of reference in Newton's mechanics, relativistic theory of a particle, cosmological standard models of a miniverse. Chapter 3 is devoted to principles of symmetries, widely used in modern theoretical physics. In Chapter 4 we acquaint readers with the method of nonlinear realizations of groups of symmetries developed at the end of the sixtieth of the last century, which applied for derivation of the theory of gravitation by joint nonlinear realization of affine and conformal symmetries. In Chapter 5 the generally accepted Dirac – Bargmann's Hamiltonian formulation is presented; it is adapted to the gravitation theory, deduced in Chapter 4. In Chapter 6 a quantum cosmological model is studied which appeared in the empty Universe approximation with the Casimir energy dominance. In Chapter 7 the procedure of quantization of gravitons in terms of Cartan's forms is implemented and the vacuum creation of affine gravitons is considered. In Chapter 8 the operator of creation and evolution of the quantum Universe is constructed as a joint irreducible unitary representation of affine and conformal groups of symmetries. In Chapter 9 the creation of matter from vacuum is formulated in the considered model of the quantum Universe with a discussion of conformal modification of S-matrix as a consequence of solutions of constraint equations in the joint theory of gravitation and the Standard Model of elementary particles. In Chapter 10, within the frame of this model, we describe the spontaneous chiral symmetry breaking in QCD via normal ordering of products of operators of the gluon and quark fields, and also derive the quark–hadron duality and the parton model as one of the

consequences of conformal modification of S–matrix. In Chapter 11 a conformal modification of the Standard Model of elementary particles without the Higgs potential is presented. Chapter 12 is devoted to the vacuum creation of electroweak bosons; and the origins of anisotropy of temperature of CMB radiation and the baryon asymmetry of the Universe are discussed. In Chapter 13 a cosmological modification of the Schwarzschild solution and Newton’s potential is presented. In Chapter 14, in the framework of this cosmological modification of the Newton dynamics, the evolution of galaxies and their superclusters is discussed. In Chapter 15 (Postface), the list of the results is presented and the problems that arise in the model of the quantum Universe are discussed. In conclusion, the authors consider as a pleasant duty to express deep gratitude to Profs. A.B. Arbuzov, B.M. Barbashov, D. Blaschke, A. Borowiec, K.A. Bronnikov, V.V. Burov, M.A. Chavleishvili, A.Yu. Cherny, A.E. Dorokhov, D. Ebert, A.B. Efremov, P.K. Flin, N.S. Han, Yu.G. Ignatiev, E.A. Ivanov, E.A. Kuraev, J. Lukierski, V.N. Melnikov, R.G. Nazmitdinov, V.V. Nesterenko, V.B. Priezzhev, G. Roepke, Yu.P. Rybakov, S.I. Vinitzky, Yu.S. Vladimirov, M.K. Volkov, A.F. Zakharov, A.A. Zheltukhin for stimulating discussions of the problems which we tried to solve in this manuscript. One of the authors (V.P.) is particularly thankful to Profs. Ch. Isham and T. Kibble for discussions of the problems of the Hamiltonian approach to the General Relativity and for hospitality at the Imperial College, Prof. S. Deser, who kindly informed about his papers on Conformal theory of gravity, Prof. H. Kleinert for numerous discussions at the Free University of Berlin, Prof. M. McCallum for discussion of physi-

cal contents of solutions of the Einstein equations, Profs. H. Leutwyler and W. Plessas for discussions of mechanisms of chiral symmetry breaking in QCD, Prof. W. Thirring for discussion about predictions yielded by the General Relativity and the considered theory of gravity, concerning motions of bodies in celestial mechanics problems. V.P. is also thankful to his former post-graduate students and coauthors D. Behnke, I.A. Gogilidze, A.A. Gusev, N. Ilieva, Yu.L. Kalinovsky, A.M. Khvedelidze, D.M. Mladenov, Yu.P. Palij, H.-P. Pavel, M. Pawlowski, K.N. Pichugin, D.V. Proskurin, N.A. Sarikov, S. Schmidt, V.I. Shilin, S.A. Shuvalov, M.I. Smirichinski, N. Zarkevich, V.A. Zinchuk, A.G. Zorin for helpful collaboration. The authors are grateful to Profs. S. Dubnichka, M.G. Itkis, W. Chmielowski, V.A. Matveev, V.V. Voronov for support of collaboration with international scientific centers, also to B.M. Starchenko and Yu.A. Tumanov for presented photos. One of the authors (A.P.) is grateful to the Directorate of JINR for hospitality and possibility to work on the monograph. The results of the investigations, presented in the book, are implemented under partial support of the Russian Foundation of Basic Research (grants 96-01-01223, 98-01-00101), and also grants of the Heisenberg – Landau, the Bogoliubov – Infeld, the Blokhintsev – Votruba and the Max Planck society (Germany).

V.N. Pervushin

A.E. Pavlov

June, 2013

Dubna

Chapter 1

Introduction

1.1 What is this book about?

In his remarkable book¹ the Nobel laureate in Physics Steven Weinberg considers problems of Genesis according to the laws of classical cosmology. In the Epilogue he gives predictions of further life of the Universe resulted from these laws. *“However all these problems may be resolved, and whichever cosmological model proves correct, there is not much of comfort in any of this. It is almost irresistible for humans to believe that we have some special relation to the universe, that human life is not just a more-or-less farcical outcome of a chain of accidents reaching back to the first three minutes, but that we were somehow built in from the beginning. As I write this I happen to be in an aeroplane at 30,000 feet, flying over Wyoming en route home from San Francisco to Boston. Below, the earth looks very soft and comfortable –*

¹Weinberg, S.: *The First Three Minutes: A Modern View of the Origin of the Universe*. Basic Books, New York (1977).

fluffy clouds here and there, snow turning pink as the sun sets, roads stretching straight across the country from one town to another. It is very hard to realize that this all is just a tiny part of an overwhelmingly hostile universe. It is even harder to realize that this present universe has evolved from an unspeakably unfamiliar early condition, and faces a future extinction of endless cold or intolerable heat. The more the universe seems comprehensible, the more it also seems pointless. But if there is no solace in the fruits of our research, there is at least some consolation in the research itself. Men and women are not content to comfort themselves with tales of gods and giants, or to confine their thoughts to the daily affairs of life; they also build telescopes and satellites and accelerators, and sit at their desks for endless hours working out the meaning of the data they gather. The effort to understand the universe is one of the very few things that lifts human life a little above the level of farce, and gives it some of the grace of tragedy". One of these acts of the tragedy is dramatic events of last years in cosmology and physics of elementary particles: expanding of the Universe with acceleration and the intriguingly small value of the Higgs particle mass. These events throw discredit upon or leave without any hopefulness for success a lot of directions of modern theoretical investigations.

In recent years, two independent collaborations “High Supernova” and “Supernova Cosmology Project” obtained new unexpected data about cosmological evolution at very large distances – hundreds and thousands megaparsecs expressed in redshift values $z = 1 \div 1.7$ [1, 2, 3]. Surprisingly, it was found that the decrease of brightness with distance, on an average, happen noticeably faster than it is expected accord-

ing to the Standard cosmological model with the matter dominance. Supernovae are situated at distances further than it was predicted. Therefore, according to the Standard cosmological model, in the last period, the cosmological expansion proceeds with acceleration. Dynamics, by unknown reasons passes from the deceleration stage to an acceleration one of expansion. Observable data (see Fig.1.1) testify that the Universe is filled mainly not with massive dust that can not provide accelerating expansion but with unspecified enigmatic substance of other nature—“dark energy” [4]. Cosmic acceleration, at the present time, is provided by some hypothetic substance called as quintessence. This term is borrowed from ancient Greece when philosophers constructed their world view from five elements: earth, water, air, fire, and quintessence as a cosmic substance of celestial bodies. In the modern cosmology, this substance means a special kind of cosmic energy. Quintessence creates negative pressure (antigravitation) and leads to accelerating expansion. In classical cosmology it is necessary, once again, for rescue of the situation, to put Λ -term into the Einstein’s equations. The problem is that the energy density of accelerating expansion at the beginning of the Universe evolution differs 10^{57} times from the modern density. Up to now there is no such dynamical model that should be able to describe and explain the phenomenon of such *dynamical inflation* [5].

The crisis of the Standard cosmology enables us to reconceive its principles. In this critical situation these new observational data (see, for example, Figs. 1.1, 1.2, 1.3) look like a challenge for theoretical cosmology. In the present book this challenge is considered as a pos-

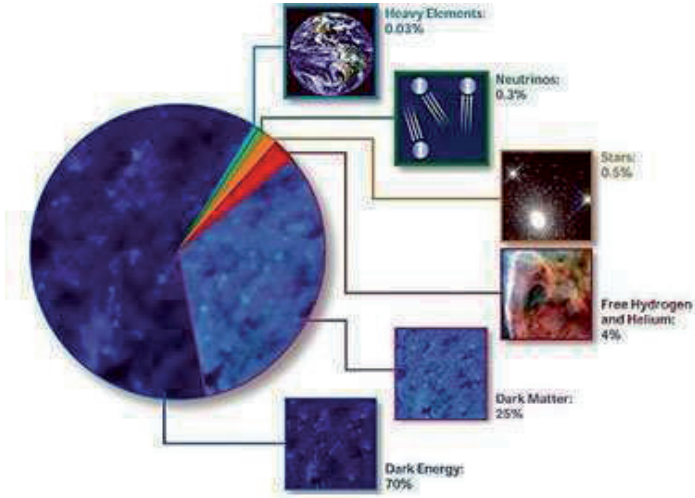


Figure 1.1: According to NASA diagram, 25% of the Universe is dark matter, 70% of the Universe is dark energy about which practically nothing is known.

sibility to construct the cosmological model that can explain modern observational facts at the level of well-known fundamental principles of relativity and symmetry without whatever dynamical inflation mechanism.

Let us recall that the theory of gravitation and corresponding cosmological models of the Universe are based on the classical papers of Einstein, Hilbert, Weyl, Dirac, Fock and other researchers, who postulated geometrical principles, including scale and conformal symmetries. In particular, the Lagrangian of Weyl's theory is an invariant with respect to conformal transformations [6]. P. Dirac in the year 1973 constructed a conformal gravitation theory where scale transformations of a scalar dilaton compensated scale transformations of other fields

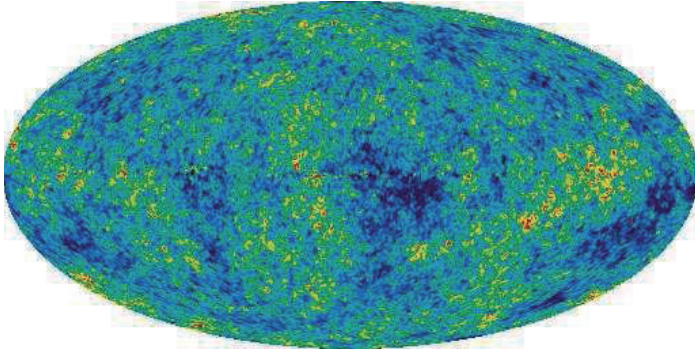


Figure 1.2: Map of Cosmic Microwave Background was lined up according to the data from the Wilkinson Microwave Anisotropy Probe (WMAP) apparatus. Previously, the first detailed map was done by data according to the COBE apparatus, however, its resolution is essentially (35 times) inferior to the data obtained by WMAP. The WMAP data show that the temperature distribution of Cosmic Microwave Background by the celestial sphere has definite structure, its fluctuations are not absolutely random. The angle anisotropy of Cosmic Microwave Background is presented, *id est* dependence of photon temperature of directions of their coming. The average photon temperature $T_0 = 2,725 \pm 0,001$, and the dipole component $\delta T_{\text{dipole}} = 3,346$ are subtracted. The picture of temperature variation is shown at the level $\delta T \sim 100\mu$, so $\delta T/T \sim 10^{-4} \div 10^{-5}$ (see <http://map.gsfc.nasa.gov>).

[7]. In the framework of this theory of gravitation, the volume of the Universe conserves during its evolution and the forthcoming collapse, inevitable in the Standard cosmology, does not occur. The Conformal gravitation theory with a scalar dilaton is derived from the finite group of symmetry of initial data via the method of Cartan's linear forms [8].

The Conformal gravitation theory in terms of Cartan's forms, keeping all achievements of the General Relativity for describing the solar

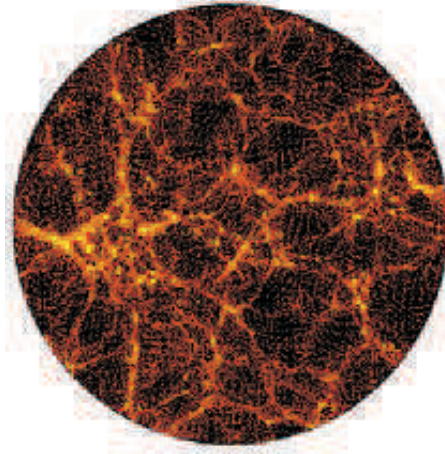


Figure 1.3: Large-scale structure of the Universe represents a complex of sufficient plane “leaves” divided by regions where the luminous matter is practically absent. These regions (voids) have sizes on the order of hundred megaparsecs. At the scales on the order of 300 megaparsecs the Universe is practically homogeneous.

system, admits a quantum formulation by quantization of initial data immediately for these linear forms. A remarkable possibility is given to test predictions of such a quantum theory of gravitation and its ability to describe the new data presented by observable cosmology and solutions of its vital problems.

The goal of the present book is a consistent treatment of groups of symmetries of initial data, Cartan’s method of linear forms, derivation of the Conformal gravitation theory, its Hamiltonian formulation and quantization, and also the description and interpretation of the new observational data in the framework of the quantum theory.

1.2 Program

One of the main problems of theoretical physics is a classification of observational and experimental data which form a space of all events (as an assembly of all measurements). Measurable and observational data have always the primary status everywhere. In the process of analysis of these data also theoretical concepts are arising, such as the Faraday – Maxwell fields and groups of symmetry of their equations, identified with the laws of nature. Classification of observational and experimental data, according to Copernicus, turned out to be sufficiently simple in some definite frame of reference. Indeed, classification of planet trajectories is appreciably simplified in the frame of reference, connected with the Sun, called heliocentric. Copernicus' simplification considerably helped Galileo, Kepler, and Newton in formulation of laws of the celestial mechanics².

In cosmology, there are also two privileged frames of reference: the cosmic one, where the Universe with matter is created which is memorized by temperature of Cosmic Microwave Background, and the other frame of reference of an observer (Earth frame) with its devices [9]. Let us remember a hierarchy of motions in which our planet is involved as

²In formal, all frames of reference are equal. By means of authors (Einstein, ., Infeld, L.: *The Evolution of Physics: From Early Concepts to Relativity and Quanta*. Touchstone, New York (1967)), if people understood relativity, there was not such dramatic, in the history of mankind, changing of world outlook, where the Earth was the center of the world. It is not hard to guess that the following, after heliocentric, was the galaxy-centric frame of reference. But, in every concrete problem, there is the privileged frame of reference, in which the contents of the problem are clarified.

it seems in our time³. In the galactic frame of reference⁴ $l = 90^0$, $b = 0^0$ the Earth rotates around the Sun with the velocity 30 km/sec; the Sun rotates with the velocity 220 km/sec around the center of our galaxy. In turn, the center of our galaxy (Milky Way) moves with the velocity 316 ± 11 km/sec to the center of the local group of galaxies⁵ in direction $l = (93 \pm 2)^0$, $b = (-4 \pm 1)^0$. Finally, we get the velocity of the galactic center relative to the center of the local group of galaxies is 91 km/sec in direction $l = 163^0$, $b = -19^0$.

The centers of our galaxy and Andromeda Nebula (galaxy 31) under action of mutual gravitational attraction come close with the velocity 120 km/sec. Suppose, that our galaxy and Andromeda contribute substantial loading to the common mass of the local group and the mass of our galaxy two times less than the mass of Andromeda, we get that our galaxy moves to Andromeda with the velocity 80 km/sec. A measurement of dipole anisotropy of Cosmic Microwave Background radiation (CMB radiation), implemented by the American cosmic apparatus COBE, established that the velocity of the Sun relative to CMB radiation is order (370 ± 3) km/sec in direction $l = (266, 4 \pm 0, 3)^0$,

³Chernin, .D.: *Cosmic vacuum*. Physics-Uspeski. **44**, 1099 (2001).

⁴Zereth latitude (b) in galactic system corresponds to galactic equatorial plane, and zeroth longitude (l) corresponds to direction to the center of the galaxy, located in Archer. Galactic latitude is measured from galactic equator to North (+) and to South (-), galactic latitude is measured in direction to West along galactic plane from galactic center.

⁵Local group of galaxies includes the Milky Way, Large and Small Magellanic Clouds, Giant galaxy Andromeda Nebula (31) and approximately 2-3 dozens of dwarf galaxies. For information: overall size of local groups is order $1 \text{ pc} = 3,0856 \times 10^{19}$ km. 1 parsec (pc) is a distance, with whom an object of size 1 astronomical unit (1 a.u. = $1,5 \times 10^{13}$ m is a mean distance from the Earth to the Sun) is seen at an angle 1 second: $1 \text{ pc} = 2,1 \times 10^5 \text{ a.u.} = 3,3$ (l. yr.). A light year (l. yr.) is a distance, traversed by photon per one Earth's year.

$b = (48, 4 \pm 0, 5)^0$. This anisotropy is responsible for motion of the observer relative to “global (absolute)” frame of reference. Inasmuch as movement of the Sun relative to the local group and its movement relative to the “absolute” frame of reference connected with the Cosmic Microwave Background radiation have, practically, opposite directions, the velocity of the center of the local group of galaxies relative to the CMB radiation happens to be sufficiently large: approximately (634 ± 12) km/sec in direction $l = (269 \pm 3)^0$, $b = (48, 4 \pm 0, 5)^0$.

In summary, the center of the local groups moves in the following directions:

a) in direction to Virgo $l = 274^0$, $b = 75^0$ with the velocity 139 km/sec;

b) in direction to the Great Attractor $l = 291^0$, $b = 17^0$, disposed at the distance 44 pc, with the velocity 289 km/sec;

c) in direction opposite to the local empty region, $l = 228^0$, $b = -10^0$ with the velocity 200 km/sec.

Taking into account all these movements, one can affirm that the local group moves with the velocity 166 km/sec in direction $l = 281^0$, $b = 43^0$. Inasmuch as errors of defining of individual velocities are in the order of 120 km/sec, the local group is able to be, practically, at the rest relative to far galaxies.

At the Beginning, at the instant of the Universe creation from vacuum, there were neither massive bodies, nor relic radiation, there was chosen a frame of reference, comoving to the velocity of the empty local element of volume. Such frame was introduced by Dirac in 1958 year, as the condition of a minimal three-dimensional surface, imbedded into

four-dimensional space-time [10].

In all cases in passing from the cosmic frames of reference to the frame of reference of devices of an observer, it is necessary to have transformations of physical observables, including, the interval itself. These transformations require the formulation of the General Relativity in the tetrad formalism.

In the present book we demonstrate that the choice of a frame of reference, co-moving to the velocity of the empty local element of volume, simplifies the classification of observational data on redshifts of spectral lines of far Supernovae and helps us to formulate the principles of symmetry of the unified theory of interactions and quantum mechanisms of their violations, as well as, in olden times, the Copernicus frame helped Newton to formulate the laws of celestial mechanics. The observable data on redshifts in the frame of reference of an empty volume testify about conformal symmetry of laws of gravitation and Maxwell electrodynamics⁶ and dominance of Casimir vacuum energy for empty space in the considered model of the Universe.

When we speak of the nature of dark energy and dark matter that is unknown, we mean that these quantities are not included into the classification of fields by irreducible representations of the Lorentz group and the Poincaré group in some frame of reference. The theme of

⁶Conformal invariance of the Maxwell equations was proved for the first time in papers: Bateman, H.: *The conformal transformations of a space of four dimensions and their applications to geometric optics*. Proc. London Math. Soc. **7**, 70 (1909); Cuninghame, E.: *The principle of relativity in electrodynamics and an extension of the theory*. Proc. London Math. Soc. **8**, 77 (1909). P.A.M. Dirac in his paper (Dirac, P.A.M.: *Wave equations in conformal space*. Ann. of Math. **37**, 429 (1936)) resulted alternative, more simple proof.

the present work is to describe observational and experimental data on redshifts of Supernovae, including dark energy and dark matter, in the framework of the well-known classification of fields by irreducible representations of the Lorentz, Poincaré [11] and Weyl [12] groups.

Fundamental physical equations (Newton, Maxwell, Einstein, Dirac, Yang – Mills, Weinberg – Salam – Glashow, *et al.*) are able to treat as invariant structural relations of the corresponding group of symmetry of initial data. The complete set of initial data includes all possible measurements in the field space of events [13]. The question now arises of what is more fundamental: equations of motion called the laws of nature that are independent on initial data, or finite-parametric groups of symmetry of frames of reference of initial data?

There is a point of view that is developed according to which all physical laws of nature can be obtained from the corresponding group of symmetry of initial data. The history of frames of reference of initial data, starting from Ptolemy and Copernicus, is considerably more ancient than the history of equations of motion. Let us overlook for the historical sequence of using in physics the groups of transformations of initial data with the finite number of parameters. Galileo group assigns transitions in the class of inertial frames of reference; the six-parametric group of Lorentz describes rotations and boosts in Minkowskian space; the Poincaré group, including Lorentz group as its subgroup, is complemented by four translations in the space-time; the affine group of all linear transformations consists of Poincaré group and ten symmetrical proper affine transformations; Weyl group includes the Poincaré group complemented by a scale transformation;

the fifteen-parametric group of conformal transformations includes the eleven-parametric Weyl group and four inversion transformations.

After creation of the special theory of relativity, for a decade, Albert Einstein searched for the formulation of the theory of relativity, extending the Poincaré group of symmetry of the Special Relativity to a group of general coordinate transformations. The searching for covariant description led Einstein to the tensor formulation of his theory. So, he named the theory of gravitation as the General Relativity. This name reflects the general heuristic principle of the relativistic theory of gravity. After the theory was constructed, in the following period of re-thinking of its foundation⁷, the group of general coordinate transformations gained a status of the gauge group of symmetry as in modern gauge theories. The group of general coordinate transformations in the General Relativity is used for descriptions of interactions, while the Poincaré group serves for classification of free fields.

For definition of the variables which are invariant relative to diffeomorphisms, and thereby, elimination of gauge arbitrariness in solutions of equations of the theory, it is necessary to separate general coordinate transformations (which play the role of gauge ones) from the Lorentzian ones. The solution of the problem of separation of general coordinate transformations from relativistic transformations of systems of reference was suggested by Fock [14] in his paper on introduction

⁷According to V.A. Fock (Fock, V.A.: *The Theory of Space, Time and Gravitation*. Pergamon press, London (1964)), principles, laid in the basis of the theory are following. The first basic idea is to unify space and time in one whole space. The second basic idea is to reject uniqueness of Minkowskian metrics and to pass to Riemannian metrics. The metric of space-time depends on events that take place in space-time, in first order, from the distribution and motion of masses.

of spinor fields in the Riemannian space. In fact, instead of a metric tensor, Fock introduced tetrads defined as “square root” of the metric tensor, with two indices. One index relates to Riemannian space, being the base space, and the second – to a tangent Minkowskian space. Tetrad components are coefficients of decomposition of Cartan’s forms via differentials of coordinate space. These differential forms, by definition, are invariants relative to general coordinate transformations, and have a meaning as measurable geometric values of physical space, and integrable non-invariant differentials of coordinate space considered as auxiliary mathematical values of the kind of electromagnetic potentials in electrodynamics.

There is a new approach based on *Ogievetsky’s theorem* [15] where the formulation of the theory of gravity on the basis of finite-parametric groups is essentially simpler than on the basis of the group of arbitrary coordinate transformations.

The novel approach can be based on some more elementary objects of the space–time. These elementary objects are fundamental representations of conformal transformations group which Roger Penrose associated with *twistors*. A space–time is constructed as adjoint representation of conformal group by means of twistors, just as pions are constructed as adjoint representations of the quark symmetry group in the theory of strong interactions. In physics of strong interactions there are energies, wherein pions are disassociated into elementary quarks. From this analogy it follows that a space–time also is able to be disassociated into elementary twistors under sufficient energies. In the next sections, on examples of Einstein’s General Relativity, we present the

derivation of physical laws from affine and conformal groups of symmetry and try to find a confirmation of the program by the last observable data both from cosmology and physics of elementary particles.

1.3 Does the creation and evolution of the Universe depend on an observer?

Interpretation of classical and quantum theories, in particular, the dependence of an object of observation on the observer, at all times, up to the present day, has been a subject of very fierce disputes. Albert Einstein asked a question⁸: “*When a person such as a mouse observes the universe, does that change the state of the universe?*” Let us show here some fragments of dramatic history of the Universe observers, including Copernicus, Tycho Brahe, Galileo, Kepler, Descartes, Newton, Lagrange, Faraday, Maxwell, Einstein, Weyl, Dirac, Fock, Wigner, Blokhintsev, and Wheeler.

Copernicus (1473 — 1543)

... Italy at the end of the 15th century, University in Bologna, tomes of ancient manuscripts put up for sale, and books of theories by Pythagoras of Samos, Eudoxus of Cnidus, Heraclitus of Pontica, Aristarchus of Samos, Hipparchus of Nicaea, Claudius Ptolemy and others, where breath-taking harmony of celestial spheres and Divus plan of the Uni-

⁸Wheeler, J.A.: *Albert Einstein (1879–1955). A Biographical Memoir by John Archibald Wheeler*. National Academy of Sciences. Washington. D.C. (1980).

verse were being opened. May be there, in Bologna⁹, the idea came to the young student Nicolaus Copernicus to give up the traditional concept of the Earth as the center of the Universe. In order to reveal the nature of visible motions of planets, Copernicus imaginatively placed his observer into the Sun and recalculated in heliocentric system of reference the trajectories of all planets. Copernicus' major theory was published in "*De revolutionibus orbium coelestium*" ("On the Revolutions of the Celestial Spheres"), in the year of his death, 1543, where he considered the Earth as one of ordinary planets rotating around the Sun. In the new Heliocentric frame of reference, the complicated character of planet motions described in the Geocentric frame of reference by Ptolemy epicycles, becomes essentially simpler. Just the mathematical simplicity of Copernicus' theory under description of motions of bodies of the Solar system opens the path to Kepler, Galileo, and Newton to creation of celestial mechanics, whose perfectibility has been proved by all practice of investigation of interplanetary environment and accuracy of predictions of celestial phenomena.

Tycho Brahe (1546 — 1601)

The King of Denmark and Norway Frederick *II*, by his decree, an island Hven disposed near Copenhagen, granted to Tycho Brahe in possession for life and also assigned great sums for building of an observatory and for its keeping. It was the first building in Europe specially constructed for astronomical observations. Tycho Brahe's observers

⁹In 1496, Copernicus took leave and travelled to Italy, where he enrolled in a religious law program at the University of Bologna.

were fishermen and sailors: his observatory was in existence of their duty. In the Universe of Tycho Brahe all planets, excluding the Earth, rotated round the Sun, while the Sun, together with these planets, rotated round the Earth. It is the very thing, that was and has been observed until the present days by all mariners. Tycho Brahe worked for his taxpayers, measured every day the position of Mars on the celestial sphere with great, even for our time, precision. Later on, Tycho Brahe leaved for Prague and served to the emperor Rudolf *II* as the palace astronomer and astrologer. The Geo-heliocentric system of the world had important advantage compared to Copernicus' one, especially after the trial of Galileo: it did not provoke any objections of the Inquisition.

Galileo (1564 — 1642)

It began with Galileo, the modern physics as a science of measurements. Galileo in his book about a would-be dialogue between Ptolemy and Copernicus introduced a plethora of observers with their inertial systems of reference. Coordinates of bodies and time in different systems of reference are connected by transformations of Galileo's group. Galileo's principle of relativity of constant motion was demonstrated by using of an imaginary experiment with systems of reference of two ships. Physical phenomena that happen inside the stationary ship do not differ from analogous phenomena inside the ship of constant motion, relative to the first one. Galileo introduced the main kinematic characteristics of a classical body moving rectilinear with constant velocity, and moving rectilinear with constant acceleration. Observa-

tions for falling bodies in the gravity field of the Earth led him to the conclusion that all bodies falling to the Earth had one and the same gravitational acceleration. According to Galileo's principle of relativity, all inertial systems of reference mathematically and physically are equivalent. Galileo's relativity means that all observers in one ship of the Universe measure the same *phenomena* (in our case, *trajectories of planets*), as the observers in the other ship of the Universe that moves with any constant velocity with regards to the first one. Observers of Ptolemy in non-inertial system of reference connected with the Earth, observe *trajectories of planets* which belong to another class of curves, in contrast to the observers of Copernicus who connect their system of reference with the Sun. In the "Dialogue Concerning the Two Chief World Systems" (1632) the Copernican system is compared with the traditional Ptolemaic system¹⁰. Ptolemy's and Copernicus' systems physically are not equivalent. Formally, in mechanics, all systems of reference are equivalent, and trajectories of bodies obtained in one system of reference is possible to be recalculated in the other system. Just the same recalculating was the main matter of work and scientific achievement of Nicolaus Copernicus. Copernicus singled out a system of reference where *equations of planet motions* have the first integrals of motion, called in Newton's celestial mechanics as a conserved energy and an angular momentum of system of bodies, that

¹⁰The "Dialogue" was published in Florence under a formal license from the Inquisition. In 1633, Galileo was convicted of "grave suspicion of heresy" based on the book, which was then placed on the Index of Forbidden Books, from which it was not removed until 1835 (after the theories it discussed had been permitted in print in 1822.) In an action that was not announced at the time, the publication of anything else he had written or ever might write was also banned.

was characteristic for central forces. In summary, analysing phenomena observed from different points of view, we come to a conclusion that the formal mathematical equivalence of systems of reference does not imply their physical equivalence.

Kepler (1571 — 1630)

A pupil of Tycho Brahe, Johannes Kepler got treasured data from his teacher; he recountered a trajectory of Mars on the celestial sphere for Copernicus' system and obtained later three laws of rotations of planets around the Sun. Kepler published these laws in his treatises "*Astronomia Nova*" (1609) and "*Harmonices Mundi*" (1619) ("The Harmony of the World"), and so promoted the establishment and development of Copernicus' doctrine of heliocentric system of reference. It became apparent that planets did not move by circles, as Copernicus thought, but along ellipses, in one focus of which the Sun was located. Galileo, in turn, decisively rejected Kepler's ellipses. In 1610 Galileo reported to Kepler about the discovery of Jupiter satellites. Kepler met this message with mistrust and in his polemical paper "*Dissertatio cum Nuncio Sidereo*" ("Conversation with the Starry Messenger") (1610) disagreed with humour: Logically, by his means, then Jupiter must be inhabited by intelligent beings. Why else would God have endowed Jupiter with the feature? Later, Kepler got his example of the telescope and confirmed the existence of satellites and was engaged in theory of lenses himself. The result was not only an improved telescope but a fundamental paper "*Dioptrice*" (1611). Kepler's system of the world meant not only to discovery laws of planet motions, but

much more. Analogous to Pythagoreans, in Kepler's mind, the world is the realization of some numerical harmony, simultaneously geometrical and musical; revelation of structure of this harmony could take the answers to the very deep questions: Kepler was convinced that¹¹

“Great is our God, and great is His excellence and there is no count of His wisdom. Praise Him heavens; praise Him, Sun, Moon, and Planets, with whatever sense you use to perceive, whatever tongue to speak of your Creator; praise Him, heavenly harmonies, praise him, judges of the harmonies which have been disclosed; and you also, my soul, praise the Lord your Creator as long as I shall live. For from Him and through Him and in Him are all things, “both sensible and intellectual”, both those of which we are entirely ignorant and those which we know, a very small part of them, as there is yet more beyond. To Him be the praise, honor and glory from age to age. Amen”.

Descartes (1596 — 1650)

The observer of Descartes thought logically (according to Aristotle) in some priori concepts of space and time, populating them with cosmic objects and leaving the concern of creation of these concepts to the Lord: *“Cogito ergo sum”*. In absolute space the *coordinate system* is set named as the Descartes one. He wrote about Galileo's condemnation to Mersenne¹²: *“But I have to say that I inquired in Leiden and Amsterdam whether Galileos World System was available, for I thought*

¹¹Kepler, Johannes: *The Harmony of the World*. American Philosophical Society (1997).

¹²Jonathan Bennett: *Selected Correspondence of Descartes*. (2013).

www.earlymoderntexts.com/pdfbits/deslet1.pdf

*Id heard that it was published in Italy last year. I was told that it had indeed been published but that all the copies had immediately been burnt at Rome, and that Galileo had been convicted and fined. I was so astonished at this that I almost decided to burn all my papers or at least to let no-one see them. For I could not imagine that he – an Italian and, as I understand, in the good graces of the Pope – could have been made a criminal for any reason except than that he tried, as he no doubt did, to establish that the earth moves. I know that some Cardinals had already censured this view, but I thought Id heard it said that it was nevertheless being taught publicly even in Rome. I must admit that if the view is false then so are the foundations of my philosophy, for it clearly follows from them; and its so closely interwoven in every part of my treatise that I could not remove it without damaging the whole work. But I utterly did not want to publish a discourse in which a single word would be disapproved of by the Church; so I preferred to suppress it rather than to publish it in a mutilated form”. In “*Principia Philosophiae*” (1644) there were formulated the main theses of Descartes:*

- God created the world and laws of nature, then the Universe acted as an independent mechanism.
- There is nothing in the world, beside moving matter of various kinds. Matter consists of elementary particles, local interactions of these execute all phenomena in nature.
- Mathematics is a powerful and universal method of studying nature, and an example for other sciences.

Newton (1643 — 1727)

Isaak Newton, using Copernicus' reference system, for the first time, formulated laws of nature in the form of differential equations and separated them from the initial data. Newton postulated the priority of laws of nature and reduced all mechanics to mathematical equations that are independent of the choice of initial data (and inertial reference systems). They predict evolution for all time of coordinates of a particle, if its initial position and initial velocity are set. For Newton's observer, to explain the world in terms of classical mechanics means to solve Newton's equations with initial data (Cauchy's problem). Here it is appropriate to remember Laplace's colorful expression¹³: *"We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes"*. *"Philosophiae Naturalis Principia Mathematica"* ("The Mathematical Principles of Natural Philosophy"¹⁴) of Newton, first published in 1687 year, absorbed all previous human experience of observations of motions of celestial and earth matter, demonstrated the same power of clarity, accuracy and efficiency of scientific methods of natural sci-

¹³Laplace, Pierre Simon: *A Philosophical Essay on Probabilities*. John Wiley & Sons. (1902).

¹⁴Newton, I.: *The Mathematical Principles of Natural Philosophy*. Encyclopedia Britannica (1952).

ence as Euclidean principles in geometry. The weak place in Newton's gravitation theory was, by opinions of scientists of that time, the absence of description of nature of the invisible force, that was able to act over vast distances. Newton stated only mathematical formalism, and left the questions of cause of gravitational attraction and its carrier open. On this occasion, Newton stated: "hypotheses non fingo", that became his famous expression. For scientific community educated on Descartes' philosophy, the approach was unusual and challenging, and only triumphal success of celestial mechanics in the 18th century forced physicists temporarily come to terms with the Newtonian theory. Physical basic concepts of the theory of gravitation were cleared only more than two centuries later, with appearance of the General Relativity. Newton's theory absolutized sharp differences of concepts of time, space, and matter, and the universal law of conservation of energy seemed to gain perpetual persistent status in philosophy. Newton introduced absolute space and time. They are the same for all observers. The first physical theory was constructed by Newton, based upon the name of its book, by analogy with Euclidean "Principles". Theological manuscripts of Isaak Newton tell us that Newton searched justification of principles of logical constructions of the first physical theory and concepts of absolute space and time, and, hence, absolute units of their measurements, in arduous discussions with gnoseology officially accepted in Trinity College where he was a professor¹⁵. New-

¹⁵Gnoseology, officially accepted in "Trinity College" and rejected by Newton, affirmed that a studied object must possess some realities, each described by their self consistent logics of Aristotle. According to the theory, the existence of two complementing each other confirmations was possible about one and the same object of cognition, under condition, that these confirmations refer to

ton's mechanics assigned the structure of mathematical formulation of modern fundamental physical theories, including Einstein's theory of gravitation, the Standard Model of elementary particles, *et cetera*, where equations as laws of nature were put as foundation.

Lagrange (1736 — 1813)

Joseph-Louis Lagrange re-wrote Newton's differential equations of motion in a covariant form, introducing generalized coordinates. He noticed in the Preface to his treatise "The Mécanique Analytique"¹⁶: "*The reader will find no figures in this work. The methods which I set forth do not require either constructions or geometrical or mechanical reasonings: but only algebraic operations, subject to a regular and uniform rule of procedure*". Lagrange was one of the creators of the calculus of variations, he derived the so-called Euler – Lagrange equations as the conditions for the extremum of functionals. Using the principle of the least action, he obtained equations of dynamics. He also extended the variational principle for systems with holonomic constraints, using the so-called method of Lagrange multipliers. Nonholonomic dynamics will be discovered later, only in the 20th century. The mathematical formalism of the calculus of variations will be necessary for theoretical physicists to formulate the equations of the gravitational field, at first, in the Lagrangian covariant form, then as generalized Hamiltonian dynamics.

different realities of this object (John Meyendorff: *Byzantine Theology. Trends and Doctrinal Themes*. N.Y. (1979); G. G. Florovsky: *Eastern Fathers of the 4th century*. Inter. Publishers Limited. (1972).

¹⁶Lagrange, Joseph-Louis: *Mécanique Analytique*. Cambridge University Press. (2009).

Faraday (1791 — 1867)

The first steps in creating the modern relativistic physics were taken by Michael Faraday. The great amount of scientific discoveries belong to him, such as a laboratory model of the electric motor that, in future, changed the life of the modern civilization. With impressive sequencing, Faraday demonstrated, by experiments, and developed his concept of field nature of matter and unity of all physical fields of nature – guiding ideas of physics of the 20th century where all particles are treated as excitations of physical fields. Faraday created the field concept of the theory of electricity and magnetism¹⁷. Before him, the presentation of a direct and instantaneous interaction between charges and currents through empty space dominated. Faraday experimentally proved that matter carrier of this interaction is the electromagnetic field. The fact that Faraday was unaware of the Newton mathematical formalism in mechanics was not a barrier on the way of the experimentalist, but helped him to formulate new basic concept of modern physics and predict the field nature of matter and unity of fields of nature, which physicists discovered in the 20th century.

Maxwell (1831 — 1879)

Maxwell had to “dress” (as Heinrich Rudolf Hertz picturesquely noted) Faraday’s theory into aristocratic clothes of mathematics. The first paper of Maxwell on the theory of electromagnetic field is entitled: “On

¹⁷In 1938 year, in an archive of the Royal Society there was found Faraday’s letter, written in 1832 year, which he asked to open after 100 years, where he predicated of electromagnetic nature of light (let us remember, that Maxwell was born in 1831 year).

Faraday's lines of force". Maxwell set a goal of translating the basic Faraday's treatise "Experimental Researches in Electricity"¹⁸ into the language of mathematical formulae. The Maxwell theory turned out to be universal in electromagnetic phenomena as Newton's theory in celestial phenomena. Electrodynamical formulae, written down in the language of mathematical field theory, became to live their own life, displaying their symmetric structure. The observer of Maxwell discovered the dependence of description of results of experimental measurements of electromagnetic phenomena from the definition of measured values in the field theory from the choice of a standard of their measurement. In Preface of his "Treatise on Electricity and Magnetism"¹⁹ Maxwell wrote: "*The most important aspect of any phenomenon from a mathematical point of view is that of a measurable quantity. I shall therefore consider electrical phenomena chiefly with a view to their measurement, describing the methods of measurement, and defining the standards on which they depend*". In Preliminary of his book he continued: "*Every expression of a Quantity consists of two factors or components. One of these is the name of a certain known quantity of the same kind is the quantity to be expressed, which is taken as a standard of reference. The other component is the number of times the standard is to be taken in order to make up the enquired quantity. The standard quantity is technically called the Unit, and the number is called the Numerical Value of the quantity*". The Maxwell theory, its symmetries and concepts

¹⁸Faraday, Michael: *Experimental Researches in Electricity*. J.M. Dent & Sons. Ltd. London. (1914).

¹⁹Maxwell, James Clerk: *Treatise on Electricity and Magnetism*. Clarendon Press, Oxford (1873).

are prototypes of all working relativistic quantum theories of the 20th century where all elementary particles are interpreted as oscillatory excitations of corresponding fields. The scientific works of Maxwell were not appreciated by his contemporaries. Only after Heinrich Hertz's experimental proof of the existence of electromagnetic waves predicted by Maxwell the theory of electromagnetism got the status of *consensus omnium*. It happened only ten years after Maxwell's death.

Einstein (1879 — 1955)

Geometries of Lobachevski and Riemann, field theory of Faraday and Maxwell disturbed confidence to the absolute space and time, and the 20th century became a century of relativity and principles of symmetries of quantized fields of matter. Einstein is a creator of two theories of relativity. The first one of these theories is the Special Relativity. It is based on the group of relativistic transformations of Maxwell's equations obtained by Lorentz and Poincaré. The Special Relativity is an adaptation of Newton's classical mechanics to relativistic transformations. The generally accepted form of the Special Relativity is the version of Einstein and Minkowski which opened a path to creation of modern quantum field theory. Any experimentalist of high energy physics knows that life-time of unstable particle, measured in the laboratory frame of reference, differs from life-time of the same particle measured in a frame moving together with the particle. If the particle is put into a train moving along the station, the Driver in the train and the Pointsman in the station measured different life-times of the particle. These times are connected by relativistic transforma-

tions obtained by Einstein. From the Newton mechanics point of view, these two different statements about life-time of the same particle are in contradiction. To avoid it, according to Trinity College doctrine, one should confirm that the particle has one reality for the Driver, and another – for the Pointsman; then one should construct two noncontradictory mechanics: the mechanics of the Driver and the mechanics of the Pointsman and the relation between them as a new element of the theory. Just by the very way of existence of two realities of one and the same particle, the development of relativistic quantum field theory went on. Einstein laid the foundation to this development, who understood that the Lorentzian symmetry of the theory of Maxwell meant equality of time and space coordinates of a relativistic particle. Such equality supposes that time and space form the unified spacetime named as the Minkowskian space of events. Hermann Minkowski proclaimed²⁰: “*The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality*”. Under its motion in this space, the particle depicts a world line, where the geometric interval plays a role of the parameter of evolution. The existence of two times of one and the same particle supposes, that for the complete description of motion of the particle in the world space of events, one needs as minimum, two observers to measure its initial data (see. Fig. 1.4). One of them is at rest, the other is co-moving with the

²⁰Minkowski, Hermann.: *Raum und Zeit*, Physikalische Zeitschrift. **10**, 75 (1908).

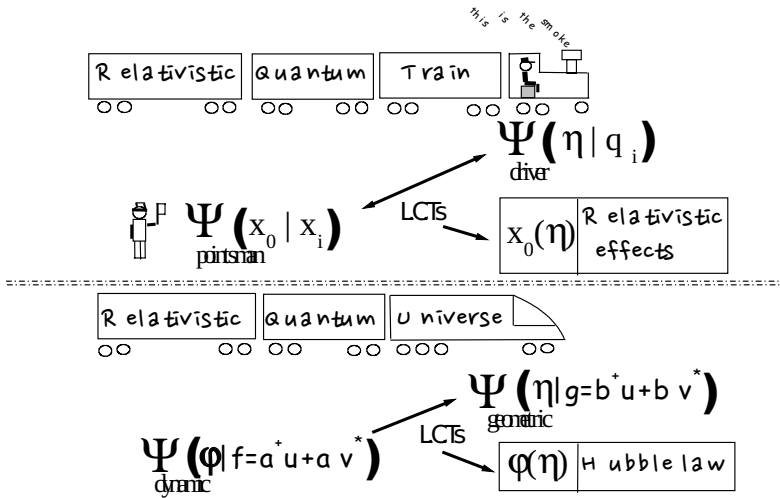


Figure 1.4: At the top of the Figure a relativistic train is depicted with an unstable particle, moving with velocity of 200 000 km per sec passing the Pointsman. If life-time of the particle, measured by the Driver, is 10 sec, then life-time of the same particle, measured by the Pointsman, is equal to $10/\sqrt{1 - (2/3)^2} \simeq 14$ sec. In the quantum field theory, that describes the process of creating a particle, these times are complementary, not contradictory. The Driver, being created together with the particle, could not be a twin to the Pointsman. The first measures the length of geometric interval (10 sec), and the second – dynamical parameter of evolution in the space of events (14 sec). At the bottom of the Figure a Universe is pictured, where a cosmological parameter of evolution φ plays the role of dynamical parameter of evolution in the space of events, and the conformal time η plays the role of the length of geometric interval. One and the same observer has two different measurement procedures of dynamical parameters of evolution (redshift) and the length of geometric interval (distance to cosmic objects). These two observers (the Pointsman and the Driver) of the relativistic object in *quantum geometrodynamics* do not contradict, but complement each other.

particle. The first one measures the time with his watch as a variable of the world space of events, and the second one measures the time with his watch as a geometric interval on the world line of the particle in this space of events. A new element of the theory appeared – an equation of constraint of four components of the vector of momentum, one of which is the energy. A solution of this equation of constraint for a particle at rest $E = mc^2$ lies in the base of the atomic energetics. The second of Einstein's theories generalizes the field paradigm of Faraday to gravitational interactions and it is named the General Relativity. The first attempts of generalizing of Faraday's field paradigm to other interactions were undertaken at the beginning of the last century. The searching of principles of symmetry was Einstein's underlying concept that differed him from other researchers. The basic ideas of the General Relativity were prepared by all history of development of non-Euclidean geometry of the 19th century by Gauss²¹, Bolyai²², Lobachevsky²³, Clifford²⁴, Riemann²⁵. Einstein declared that observational results of his theory did not depend on parameters of a very wide class of coordinate transformations. That is why Einstein named his theory the General Relativity. Einstein's dynamical equations are de-

²¹Gauss, C.F.: *General Investigations of Curved Surfaces of 1827 and 1825*. Princeton University (1902)

²²Bolyai, J.: *The Science Absolute of Space. Independent of the Truth or Falsity of Euclid's Axiom XI (which can never be decided a priori)*. Austin, Texas (1896).

²³Lobachevski, N.I.: *Complete Collected Works*. I-IV. Kagan, V.F. (Ed.). Moscow-Leningrad (1946)–(1951).

²⁴Clifford, W.K.: *Mathematical Papers*. MacMillan, New York-London (1968)

²⁵*Bernhard Riemann's gesammelte mathematische Werke und wissenschaftlicher Nachlass*. Teubner, Leipzig (1876).

rived from Hilbert's variational principle delivered in Hilbert's report "Foundations of Physics" presented on November 20th, 1915 to the Göttingen Mathematical Society²⁶.

Weyl (1885 — 1955)

The main goal of theoretical physics is to establish several physical principles to explain all observable effects just as a few Euclidean axioms and logical laws make it possible to prove many theorems in geometry. In modern physics, such fundamental principles are principles of symmetry. The following statement of Hermann Weyl is worth being reminded²⁷: *“What we learn from our whole discussion and what has indeed become a guiding principle in modern mathematics is this lesson: Whenever you have to do with a structure-endowed entity Σ try to determine its group of automorphisms, the group of those element – wise transformations which leave all structural relations undisturbed. You can expect to gain a deep insight into the constitution of Σ in this way”*. From this viewpoint, transformations of reference frames form an automorphism group in mechanics, while the equations of motion derived by variation of action are invariant structure relationships. The guiding principle of modern physical theories is to define

²⁶In addition, Hilbert first formulated the theorem that was later referred to as the second Noether theorem. This theorem leads to the interpretation of general coordinate transformations as gauge ones and, therefore, to all consequences concerning both a decrease in the number of independent degrees of freedom and the appearance of constraints imposed on initial data. The first Noether theorem states that any differentiable symmetry of the action of a physical system has a corresponding conservation law.

Noether, E.: *Invariante Variationsprobleme*. Nachr. D. König. Gesellsch. D. Wiss. Zu Göttingen, Math-phys. Klasse. 235 (1918).

²⁷Weyl, H.: *Symmetry*. Princeton University Press, Princeton. (1952).

the transformation groups of reference frames (treated as manifolds of initial data) that preserve the equations of motion. The Galileo group in Newtonian mechanics and the Poincaré group in the Special Relativity, groups of classification of elementary particles, gauge groups of symmetries led to equations of constraints of fields and their initial data. Weyl proposed a principle of scale symmetry of laws of nature: according to this proposition gravitation equations are independent of choice of measure units and differed from the General Relativity ones. In Weyl's geometry lengths of objects under motion over a closed contour are not integrable, and non-integrability is connected with the presence of electromagnetic field.

Fock (1898 — 1974)

Fock was the first to introduce a tangent space of Minkowski into the General Relativity. Now, all observers in the Universe are able to measure two parameters of evolution: a proper time interval, measured in the tangent space, and a parameter of evolution in the field space of events. The same two observers (we called them a "Pointsman" and a "Driver") could be also introduced into the General Relativity. In the Special Relativity hardly anybody is interested in what the fate of proper time of a particle should be, measured by a "Driver", after the causal quantization, which introduced the vacuum in the space of events, with the help of changing of the operator of creation of a particle with negative energy to the operator of annihilation of a particle with positive energy. We should see further that as a result of such change on the world line of a particle, the positive arrow and absolute

point of the beginning of geometrical interval of time measured by a “Driver”, appear. Such quantum anomaly of geometrical interval, in fact, means that if there is no particle there is no world line of a particle either nor the interval on this world line. Fock, solving the problem of particles in the General Relativity, introduced a privileged, the so-called, harmonic reference frame for solving Einstein’s equations. Fock understood that despite the formal point of view, all reference frames are equivalent, but investigating concrete problems, one should choose the most acceptable reference frame.

Dirac (1902 — 1979)

Dirac’s observer solves problems of consequent probability interpretation of the wave function that satisfy Dirac’s equations, and stability of the quantum system via the secondary quantization and filling all states with negative energies (Dirac’s sea). A solution of the Hamiltonian constraint both in the Special Relativity and the General Relativity has two signs. The negative sign of energy in the Special Relativity was associated with the existence of anti-particle – positron by Dirac. Theorists, headed by Dirac and Fock, solved the problem of negative energy via two quantizations of a particle: primary one, when generalized coordinates and conjugated momenta became operators in the equation of constraint, acting to the wave function, which was identified with Faraday-type field; and secondary one, when the same Faraday’s field, interpreted as a sum of operators of creation of a particle with positive energy (+) and annihilation of a particle with positive energy (–) as well. The most important element of Dirac’s

theory is a vacuum as a state with minimum of energy, which disappears if the operator of annihilation acts to it. This quantization in the modern field theory was called causal quantization, and the theory got the name: the quantum field theory. Dirac, following to Weyl, introduced an observer [7]: “*There are reasons for believing that the gravitational constant varies with time. Such a variation would force one to modify Einstein’s theory of gravitation. It is proposed that the modification should consist in the revival of Weyl’s geometry, in which lengths are non-integrable when carried around closed loops, the lack of integrability being connected with the electromagnetic field*”. Dirac formulated a new action principle [7]: “*A new action principle is set up, much simpler than Weyl’s, but requiring a scalar field function to describe the gravitational field, in addition to the $g_{\mu\nu}$* ”. This scalar field function got the name a *dilaton*. According to Weyl and Dirac, a standard of measurement of length is chosen being expanded with measured lengths together with its unit. If the standard is also expanding, the results of measurements of all lengths do not contain the cosmological scale factor. So, for an observer with the relative standard, the Universe is static, and all masses are proportional to the cosmological scale factor associated with the dilaton, and become vanishingly small at the beginning of the Universe origin. The modern cosmology, actually, uses a double standard under description of cosmic evolution of photons emitted by massive matter from the far cosmic object: *absolute* (world interval) and *relative* (conformal interval). Friedmann was the first who used geometric intervals with relative units of measurements (a coordinate distance to a cosmic object and a conformal time

of emitting of a photon), to solve his equations. These relative variables are used in the observable cosmology for description of motion of cosmic photons. They left their “footsteps” in the form of spectral lines in photographic plates put in the telescope. Comparing these photographic plates with those where spectral lines left photons of the Earth atoms at the present time, astrophysicists measure redshifts of spectra of energies of cosmic atoms. Energy spectrum of atoms, as known from quantum mechanics, is defined by masses of particles of which these atoms are combined. An acceptance of a *relative* standard leads to changing of masses, and an *absolute* standard — to changing of geometric intervals. Dirac’s observer, as, in olden times, Copernicus’ observer, could choose himself a standard of measurement, and define, according to Copernicus, what standard gives the most simplest classification of the observational data. All standards in Einstein’s theory are equivalent from the formal mathematical point of view, in the same way, as the frames of reference of Ptolemy and Copernicus in the celestial mechanics are, formally, equivalent. However, the phenomena are described by solutions of the equations of motion, where the initial data are demanded. For definition of the initial data in the General Relativity, it is necessary to pass to conformal variables. This fact was known to Andre Lichnerowicz yet in the year 1944²⁸. The transition to conformal variables in cosmology means recalculating of all observational data from an absolute unit to relative ones, multiplying these data to the cosmological scale factor in power equal to corre-

²⁸Lichnerowicz, Andre: *L’integration des equations de la gravitation relativiste et le probleme des n corps*. J. Math. Pures Appl. **23**, 37 (1944).

sponding conformal weights. Just this recalculating demonstrates, as we shall see further on, describing the data on Supernovae, physical nonequivalence of standard variables and the conformal ones. *Standard cosmology*, recalculated in terms of conformal variables is called *Conformal cosmology*. In particular, in *Conformal cosmology* measured lengths are always longer than in *Standard cosmology*. Data on Supernovae correspond to another equation of state of matter, identified to Casimir energy in an empty Universe, as we will show in the next Chapters.

Wigner (1902 — 1995)

Eugene Wigner showed that the principle of relativity of velocities offered by Copernicus for description of motion of planets, generalized by Poincaré and Einstein for motion of relativistic particles, turns out to be the principle of classification of all observable and measurable physical objects. As a result, physicists have got the classification of states of a particle according to its mass and spin. Such classification lays the basis of the quantum field theory. At the present time, physicists come to a conclusion to include the Universe itself to this scheme. Wigner explained [16]: *“The world is very complicated and it is clearly impossible for the human mind to understand it completely. It has therefore devised an artifice which permits the complicated nature of the world to be blamed on something which is called accidental and thus permits him to abstract a domain in which simple laws can be found. The complications are called initial conditions, the domain of regularities, laws of nature. Unnatural as such a division of the world’s*

structure may appear from a very detached point of view, and probable though it is that the possibility of such a division has own limits,²⁹ the underlying abstraction is probably one of the most fruitful ones that human mind has made. It has made science possible". An observer of Wigner divides three levels of development of a physical theory: *phenomena, laws of dynamics, and principles of symmetry* and prioritizes to the principles of symmetry, whereof the laws of dynamics and the description of nature can be deduced. It is wonderful that these principles have in their base the symmetry of the very same *accidental* initial data, of which the laws of nature are independent.

Blokhintsev (1908 — 1979)

Dmitry Blokhintsev's papers make us reflect about eternity of knowledge. How do the terms and concepts arise and how far are they absolute? How can we divide absolute and relative in learning of the world? What is "physical reality?" How can one define "boundaries of applicability of concepts". Especially, Blokhintsev's words amaze us about impossibility of simultaneous, with arbitrary accuracy, measurement of a coordinate and conjugated momentum of a particle: impossibil-

²⁹The artificial nature of the division of information into "initial conditions" and "laws of nature" is perhaps most evident in the realm of cosmology. Equations of motion which purport to be able to predict the future of a universe from an arbitrary present state clearly cannot have an empirical basis. It is, in fact, impossible to adduce reasons against the assumption that the laws of nature would be different even in small domains if the universe had a radically different structure. One cannot help agreeing to a certain degree with E.A. Milne who reminds us (*Kinematic relativity* Oxford Univ. Press, 1948, page 4) that, according to Mach, the laws of nature are a consequence of the universe. The remarkable fact is that this point of view could be so successfully disregarded and that the distinction between initial conditions and laws of nature has proved so fruitful.

ity means not restriction of our cognition, but limitation of the terms themselves (a coordinate and a corresponding momentum). Blokhintsev said³⁰: “*It is wrong to think that modern physical experiment has insufficient accuracy for measurements of “true” simultaneous values of momentum and coordinate of a microparticle. On the contrary, it is accurate enough to prove that for the microparticles the pair does not exist in nature*”. Separation of relative values from the objective ones and definitions of boundaries of their applicability, according to the quantum theory, gives amazing predictions of new effects and values of physical magnitudes that describe these effects. In such a manner, Dmitry Blokhintsev as a physicist can “explain in layman’s terms” values of physical magnitudes and predict fine effects, such as the Lamb shift discovered only 10 years later and inaugurated the birth of the quantum electrodynamics³¹. Blokhintsev’s observer treats a transition from a classical particle to a quantum one as quantization of initial data, but not of dynamical variables. Let us recall that Blokhintsev associated a set of all possible free initial data in quantum field theory with a statistical ensemble. The existence of the ensemble of quantum states of initial data is a reason of probabilistic interpretation of a wave function. For problems of cosmological particle creations, Blokhintsev’s quantization of initial data turns out more productive, from the point of view of classification of observational data than quantization of dynamical variables, because it forces us to search the complete set

³⁰Private communication.

³¹In 1938 year Blokhintsev delivered a lecture at the seminar on theoretical physics in Lebedev Physical Institute (Moscow), where he showed, that taking into consideration an interaction of electrons with the field of radiation, it is able to lead to shift of their energetic levels.

of free initial data as constants of motion in a definite frame of reference. In particular, in cosmology, where the particles are described as oscillators, squeezed by a cosmological scale factor, it is possible to quantize only the ensemble of initial data, as constants of motion, which are established via Bogoliubov's transformations. These transformations can be fulfilled at the level of classical theory, then it is possible to obtain the complete set of free initial data. For the theory with the quantized initial data, the Bogoliubov transformations provide a quantitative description of the Universe creation (mysterious for the variables without initial data), identified as the Big Bang.

Wheeler (1911 — 2008)

According to Wheeler's geometrodynamics, there are the same realities – dynamical and geometrical – in the modern theory of gravitation, where the Hubble's law appears as a pure relativistic relation between a cosmological scale factor played role as a time-like variable in the field space of events introduced by Wheeler in 1967 and a geometric interval of time on the world hypersurface in this space of events. A choice of a relative standard, as we already marked, transfers a cosmic evolution to masses, changing a fundamental parameter of Einstein's theory – Planck's mass into dynamical variable in Wheeler – De Witt's space of events, which has accidental initial data, as well as any variable in Newton's mechanics. So, the relative standard deprives Planck's mass of a fundamental status in Einstein's theory, as well as Copernicus' relativity deprived a fundamental status of the Earth position in Ptolemy's frame of reference. It could seem, that in the General

Relativity there is no absolute value in natural units of measurement. The scale symmetry of the classical theory can be broken via its quantization and normal ordering of operators, leading to Casimir's energy and Casimir's condensate. A secondary causal quantization of universes with a postulate of existence of vacuum as the state of minimal energy, in full analogy with quantum field theory, leads to a cosmological creation of universes and to a positive arrow of a geometric interval of time. This time has the absolute origin that we, the Universe's inhabitants, conceive as the point of the Universe's creation, with the equation of state established from the data obtained with the Hubble's telescope. In relativistic cosmology, the positive arrow of the geometric time and its origin are consequences of quantum vacuum postulate in the field space of events, *id est* they are evidences of quantum nature of our Universe. So, answering the question of what there was until the creation of the Universe, we are able to say, after Augustine of Hippo and Immanuel Kant, that there was not the time itself. Only in the limit of infinitely large Universe and infinitely large energy of motion of the Universe in the field space of events, the theory of early Universe becomes the Einstein classical theory and modern quantum field theory of elementary particles, accessible to our classical comprehension. The quantum theory confutes the Laplace's point of view: at some time, having the knowledge of locations and velocities of all particles, simultaneously, in the Universe. Wheeler asked³²: "*In short, whether the Universe is, in some strange sense, a sort of "self-induced circuit?"*"

³²Our free translation from "*Centenario di Einstein*" (1879-1979). *Astrofisica e Cosmologia Gravitazione Quanti e Relatività*. Giunti Barbèra. Firenze (1979).

Giving rise in some limited stage of its existence observers–members, whether the Universe acquires, in turn, through their observations that tangibility, which we call reality? Is not the mechanism of its existence? And whether from these reasonings to deduce the nature and necessity of the quantum principle?” Wheeler continued to discuss: *“Today textbooks on quantum mechanics tells us how to proceed in situations where one observer is involved. Scientific articles have dealt with the idealized experiments in the spirit of Einstein, Podolsky and Rosen, which involved two observers. We do not have an idea of what to do in an extreme situation, when a very large number of observers – the participants and a very large number of observations play a huge role”.*

A contemporary observer (1973 — till the present time)

We marked the date of birth of a contemporary observer by 1973³³, to bound him with ideas, models, and theories proposed till 1973³⁴ by the *giants* of physics and mathematics, *on whose shoulders he climbs*, to get the simple classification of all data on measurements and observations of physical and cosmological values obtained to the present time (2013). We deprive the observer of having the ability of using uncommitted in time ideas of his contemporaries, and, moreover, of having his own ideas. We assume that our observer knows everything from

³³In 1973 year the famous Dirac’s paper (Dirac, P.A.M.: *Long range forces and broken symmetries*. Proc. Roy. Soc. London. **A 333**, 403 (1973)) was appeared.

³⁴It turned, by the way, exactly 500 years, – half of the Millennium, – if you count the time interval from the year of birth of Copernicus.

Copernicus to Dirac, including three levels of classification of physical data marked by Wigner: *phenomena*, *laws of dynamics* and *principles of symmetry*, to the extent to which the knowledge has been outlined above. At all these levels, adequate choice of the reference frame of initial data and standards of their measure can essentially simplify the classification of physical data and, therefore, facilitate further development of our knowledge of the Universe and theories describing dynamics of processes, that occur therein. To explain the dynamics of the processes in the Universe, contemporary observer has the ability to use quantum theory of *phenomena*. The quantum theory of *phenomena*, described by unitary irreducible representations of a group of *symmetry* of initial data, may be much simpler than the classical theory that is based on solutions of classical *dynamical laws*, *id est* equations of motion. In other words, to describe *phenomena*, a classical observer has *laws of dynamics* as equations of Newton, Maxwell, Einstein, equations of the Standard Model of elementary particles and modern unified theories; at the same time, a contemporary observer has unitary irreducible representations of finite-parametric groups of *symmetry* of initial data for description of the same *phenomena*. Just therefore, we have a unique opportunity to further build our classification of data using unitary irreducible representations of these groups without applying to the classical *laws of dynamics* as the assumptions of the physical theory, or deduce classic *laws of dynamics* from the first principles of symmetry found up to 1973.

1.4 Contents

Since the time of Newton’s physical theory, the dynamic processes have been described by solutions of differential equations and some unique initial data. The initial data are set by fitting of the results obtained via the observations or direct measurements with devices that are identified with a certain frame of reference³⁵. The question is what are the criteria and principles for the selection of the initial data for the Universe? To answer these questions, in this paper we consider a cosmological model in which the initial state of the Universe is given by vacuum of particles. To such a state of vacuum the concept of “temperature” can not even be applied as it arises when describing the motion of the particles after the creation from the cosmological vacuum.

In this case, the starting point of the observable Universe is the instance of the creation of primary particle-like irregularities, the size of which is determined by their masses in the Standard Model of elementary particles. Since the Compton wavelength of the particle can not be larger than the horizon of the Universe at the time of its creation, the very instance of creation of a particle can be estimated by equating its Compton wavelength and the horizon of the Universe. Thus, the initial value of the cosmological scale factor is given by the condition

³⁵Any measurement or observation suggests the existence of two selected frames of reference – the first is connected with the instruments of measurement, while the second (co-moving) is associated with the object whose parameters are measured. In particular, in modern cosmology the rest frame of instruments uniquely associated with the Earth, and the accompanying reference system of the Universe – with the Cosmic Microwave Background radiation. The rest frame of reference differs from the co-moving one by non-zero dipole component of temperature fluctuations 10^{-3} .

that the Compton wavelength of the primary particles is equal to the horizon of the Universe (or equal its mass to Hubble's parameter) at the time of creation of the primary cosmological particles from the vacuum. This condition follows from the uncertainty principle, which limits changing of energy in the Universe by finite time of its existence.

In the Standard Model of elementary particles candidates for the role of the primary particles are massive vector particles and minimally-interacting scalar particles. Cosmological creation of these primary particles is described in the Conformal model of a homogeneous Universe, which differs from the Standard cosmological model in the principles of relativity of initial data, relativity of time, and relativity of units that should be set forth below.

Let us outline conclusions of this Chapter.

1. The initial value of the cosmological scale factor given by the condition of coincidence of Compton wavelength of the primary particles with the horizon of the Universe (or their mass with the Hubble parameter) at the instance of cosmological creation of the primary particles from the vacuum. This condition follows from the uncertainty principle, which limits the change of energy in the Universe by its finite time of existence.
2. The principle of relativity of time is the second difference from the accepted Standard cosmological model. Recall that the relativity of time in Special Relativity suggests that time coordinate is also a degree of freedom of the particle, so that a complete set of degrees of freedom forms the space of events introduced by

Minkowski. Instead of a single Newton's time there are three: the time-like variable in the space of events, time-geometric interval on the trajectory of the particles in the space of events, and a coordinate evolution parameter, reparametrization of which leads to a Hamiltonian constraint of momenta in Minkowski space. The resolution of this constraint regarding the time-variable momentum gives the energy of the particle in the Special Relativity (namely, the one used in the calculation of the energy in nuclear reactions). The primary and secondary quantization of the Hamiltonian constraint in the Special Relativity allows us to formulate a quantum field theory with the postulate of the vacuum as the state with the lowest energy. In this book the quantum model of the Universe is obtained in the theory of gravity as a generalization of the above construction of the quantum field theory from the Special Relativity using primary and secondary quantization of the Hamiltonian constraint with the postulate of a vacuum.

3. The third difference from the accepted Standard cosmological model is the Weyl's relativity of units (or scales) of measurements, which means that the physical devices measure a dimensionless ratio of the interval of time or space to the measurement unit, defined by the standard mass. Weyl's measured quantities (mass, density, temperature and so on) associated with the measured values of the Standard cosmology by multiplying the latter on a cosmological scale factor in power defined by confor-

mal weight of each of these units. *Standard cosmology, expressed in terms of measured values, is called the Conformal cosmology.* Since in Conformal cosmology a distance identified with measurable one is always greater than the distance in the Standard cosmology, the same Supernovae data correspond to different equations of states in Conformal cosmology and the Standard cosmology.

Thus, the above-mentioned principles of relativity explain the origin of all matter in the Universe as a decay product of the primary scalar and vector bosons created from vacuum, and the arrow of time as an inevitable consequence of the primary and secondary quantization of the Hamiltonian constraint. To describe such creation of particles from the vacuum we construct the creation operator of evolution of the quantum Universe as a joint and irreducible unitary representation of the affine and conformal symmetry groups. Expected average values of this operator between the states of matter (according to their classification of the Poincaré group in the tangent Minkowskian space) are used to describe modern experimental and observational data.

Bibliography

- [1] Riess, A.G., *et al.* [Supernova Search Team Collaboration]: *Observational evidence from supernovae for an accelerating universe and a cosmological constant.* *Astron. J.* **116**, 1009 (1998).
[arXiv: astro-ph/9805201]
- [2] Perlmutter, S., *et al.* [The Supernova Cosmology Project]: *Constraining dark energy with SN Ia and large-scale structure.* *Astrophys. J.* **517**, 565 (1999)
- [3] Riess, A.G., *et al.* [Supernova Search Team Collaboration]: *The farthest known supernova: support for an accelerating universe and a glimpse of the epoch of deceleration.* *Astrophys. J.* **560**, 49 (2001).
[arXiv: astro-ph/0104455]
- [4] Weinberg, S.: *Cosmology.* Oxford University Press, Oxford (2008)
- [5] Ijjas, A., Steinhardt, P., Loeb, A.: *Inflationary paradigm in trouble after Planck 2013.* *Phys. Lett. B* **723**, 261 (2013)
- [6] Weyl, H.: *Gravitation und elektrizität.* *Sitzungsber. d. Berl. Akad.*, 465 (1918)

-
- [7] Dirac, P.A.M.: *Long range forces and broken symmetries*. Proc. Roy. Soc. London. **A 333**, 403 (1973)
- [8] Borisov, A.B., Ogievetsky, V.I.: *Theory of dynamical affine and conformal symmetries as gravity theory*. Theor. Math. Phys. **21**, 1179 (1975)
- [9] Barbashov, B.M., Pervushin V.N., and Proskurin D.V.: *An excursus into modern cosmology*. Phys. of Particles & Nuclei. **34**, 68 (2003)
- [10] Dirac, P.A.M.: *The theory of gravitation in Hamiltonian form*. Proc. Roy. Soc. London. **A 246**, 333 (1958)
- [11] Schweber, S.: *Introduction to Relativistic Quantum Field Theory*. Dover Publications. USA (2005)
- [12] Ramond, P.: *Field Theory: A Modern Primer*. The Benjamin/Cummings Publishing Company, Inc. London (1981)
- [13] Wheeler, J.A.: *In Batelle Rencontres: 1967, Lectures in Mathematics and Physics*. De Witt, S., Wheeler, J.A. (eds.). New York (1968)
- [14] Fock, V.: *Geometrisierung der Diracschen theorie des electrons*. Zs. f. Fiz. **57**, 261 (1929)
- [15] Ogievetsky, V.I.: *Infinite-dimensional algebra of general covariance group as the closure of finite-dimensionsal algebra of conformal and linear groups*. Lett. Nuovo Cimento. **8**, 988 (1973)

- [16] Wigner, E.: *Invariance in physical theory*. Proc. Amer. Phil. Soc. **93**, 521 (1949)

Chapter 2

Initial data and frames of reference

2.1 Units of measurement

All units of measurement can be expressed in terms of three basic: units of length, mass and time. Two fundamental physical theories of the last century have reduced the number of basic units from three to one. In the Special Relativity, it was found that there is a fundamental limiting speed of propagating of physical processes which is equal to the speed of light in the vacuum c . In the quantum theory, it was appeared a new fundamental constant – the quantum of action \hbar . If we choose a system of units in which $c = 1$ and $\hbar = 1$ all three main units – length, mass and time can be expressed via any one of them. Several examples of various physical quantities expressed in terms of mass M are shown in Table (2.1) [1]. Once we choose the unit of measurement of mass M , then all other units are identified.

Magnitude	Units M, L, T	Units of mass M
Length	L	M^{-1}
Velocity	LT^{-1}	M^0
Force	MLT^{-2}	M^2
Electric charge	$M^{1/2}L^{3/2}T^{-1}$	M^0
Magnetic field	$M^{1/2}L^{-1/2}T^{-1}$	M^2
Angular momentum	ML^2T^{-1}	M^0
Gravitational constant	$M^{-1}L^3T^{-2}$	M^{-2}

Table 2.1: Physical quantities in units of mass ($c = 1, \hbar = 1$).

In theoretical physics, the natural units of measurement [2] are the Planck time

$$T_{Pl} = \sqrt{\frac{\hbar G}{c^5}} \approx 5.4 \times 10^{-44} \text{sec},$$

the Planck length

$$L_{Pl} = \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \times 10^{-33} \text{cm},$$

and the Planck mass

$$M_{Pl} = \sqrt{\frac{\hbar c}{G}} \approx 2.2 \times 10^{-5} \text{g}.$$

In the future we will use the natural system of measurements $c = \hbar = 1$.

2.2 Nonrelativistic mechanics of a particle

First, we shall review the initial notions by using a simple example of one-dimensional nonrelativistic mechanics, in Lagrange formulation, defined by the action functional

$$S_L = \int dt L(X(t), dX(t)/dt) \quad (2.1)$$

with the Lagrangian

$$L(X(t), dX(t)/dt) = \frac{m}{2} \left[\frac{dX(t)}{dt} \right]^2.$$

Here $X(t)$ is a variable, describing a particle trajectory, t is a time coordinate, and m is a mass of a particle treated as a fundamental parameter of the theory. A condition of extremum of the action (2.1)

$$\delta S_L = 0,$$

under fixed boundary conditions

$$\delta X(t_0) = \delta X(t_1) = 0,$$

yields the differential equation of motion of the particle

$$m \frac{d^2 X(t)}{dt^2} = 0. \quad (2.2)$$

The general solution

$$X(t) = X_I + \frac{P_I}{m}(t - t_I) \quad (2.3)$$

of this equation depends on the initial data of the particle: its position X_I and momentum P_I

$$X(t_I) = X_I, \quad \frac{dX(t)}{dt} \equiv \frac{P_I}{m},$$

given at the initial time t_I . The initial data are measured with a set of physical devices (in this example, with a ruler and a watch in a fixed space-time point), associated with a reference frame. The reference frames, moving with constant relative velocities, are referred to as *inertial frames*. The transformation

$$X \mapsto \tilde{X} = X + X_g + v_g(t - t_I),$$

turns a fixed reference frame with its origin at the point

$$X(t_I) = X_I,$$

into the reference frame, moving with the velocity v_g and with its origin at a point

$$X_g(t_I) = X_I + X_g.$$

This transformation group for reference frames in Newtonian mechanics is referred to as the *Galileo group*. The differential equation (2.2) is independent of initial data and, therefore, of a frame of reference. The independence of the equations treated as laws of nature on initial data is referred to as the *principle of relativity* [3]. In the Hamiltonian approach, the action (2.1) takes the form

$$S_H = \int dt \left[P(t) \frac{dX(t)}{dt} - H \right], \quad (2.4)$$

where $P(t)$ is the momentum of a particle, and the canonical variables $\{P, X\}$ are coordinates of the *phase space*. *Hamiltonian function*

$$H(P) = \frac{P^2}{2m} \quad (2.5)$$

is the generator of the *phase flow*, and its value on a phase trajectory is the energy of the particle

$$E = H(P_I).$$

The equations of motion of the particle obtained by variation of the action (2.4) by the canonical variables are the first-order differential equations:

$$P(t) = m \frac{dX(t)}{dt}, \quad \frac{dP(t)}{dt} = 0, \quad (2.6)$$

rather than the second-order differential equation (2.2). According to Newtonian mechanics, all observers in different frames of reference use the same absolute time t .

2.3 Foundations of Special Relativity

2.3.1 Action of a relativistic particle

As it was shown above, the notion of the spatial coordinates $X_{(i)}$, $i = 1, 2, 3$ in Newtonian mechanics as dynamical variables is clearly separated from the absolute time t , treated as an *evolution parameter*.

The relativistic mechanics was constructed after the Maxwell electrodynamics. Symmetry group of the electrodynamics was obtained

by Lorentz¹ and Poincaré²[4]. The time $t = X_{(0)}$ and spatial coordinates $X_{(i)}, i = 1, 2, 3$ are treated in this group as coordinates $X_{(\alpha)}, \alpha = 0, 1, 2, 3$ of unified space of events or the Minkowskian space-time³ [5] with the scalar product of any pair of vectors

$$A_{(\alpha)}B_{(\alpha)} \equiv A_{(0)}B_{(0)} - A_{(i)}B_{(i)}.$$

Relativistic particles are described in the Special Relativity by the action

$$S_{\text{SR}} = -m \int d\tau \sqrt{\left(\frac{dX_{(\alpha)}}{d\tau}\right)^2}. \quad (2.7)$$

This action is invariant with respect to transformations of the Poincaré group

$$\bar{X}_{(\alpha)} = X_{I(\alpha)} + \Lambda_{(\alpha)(\beta)}X_{(\beta)},$$

which is the transformation group of reference frames. Its subgroup of rotations $\Lambda_{(\alpha)(\beta)}X_{(\beta)}$ is referred to as the Lorentz group.

Fixing the indices $(0), (i)$ in this *space of events* $[X_{(0)}|X_{(i)}]$ implies the choice of a specific Lorentz reference frame. It should be noted that the Special Relativity contains a new symmetry with respect to

¹Lorentz, H.A. Versl. Kon. Akad. v. Wet. Amsterdam. S. 809 (1904).

²In 1905 (published 1906) it was noted by Henri Poincaré that the Lorentz transformations can be regarded as rotations of coordinates in a four-dimensional Euclidean space with three real space coordinates and one imaginary coordinate representing time as $\sqrt{-1}ct$. Poincaré presented the Lorentz transformations in terms of the familiar Euclidean rotations.

³Hermann Minkowski reformulated the Special Relativity in four dimensions. His concept of space of events as a unified four-dimensional space-time continuum arose. He did not use the imaginary time coordinate, but represented the four variables (x, y, z, t) of space and time as coordinates of four dimensional affine space. Points in this space correspond to events in the space-time. In this space, there is a defined light-cone associated with each point, and events beyond the light-cone are classified as space-like or time-like.

the transformations that do not change the initial data; namely, the action (2.7) is invariant with respect to the reparametrization of the coordinate evolution parameter

$$\tau \longrightarrow \tilde{\tau} = \tilde{\tau}(\tau), \quad (2.8)$$

It results in originating a constraint between the variables. This transformation group is referred to as a *gauge group*, while the quantities invariant with respect to gauge transformations are called *observables*.

The geometric time interval

$$s(\tau) = \int_0^\tau d\tilde{\tau} \sqrt{\left(\frac{dX_{(\alpha)}}{d\tilde{\tau}}\right)^2} \quad (2.9)$$

on the world line of a particle in the space of events $X_{(\alpha)}$ can be taken as an observable that is invariant with respect to the time reparametrization. This interval is measured by a co-moving observer. *The time variable* of the space of events $X_{(0)}$ is the time measured by an external observer. The goal of the theory is to solve the equations describing trajectories in the space of events in terms of *gauge invariants*.

The non-covariant variational principle for a relativistic particle was proposed by Max Planck [6]. He delivered a lecture “The principle of relativity and the fundamental equations of mechanics” to the Deutsche Physical Society in 1906.

2.3.2 Dynamics of a relativistic particle

Any reference frame in SR is defined by a unit time-like vector $l_{(\mu)}$: with

$$l_{(\mu)}^2 = l_{(0)}^2 - l_{(i)}^2 = 1,$$

which will be referred to as a time axis. These vectors form a complete set of the Lorentz reference frames. The time in each frame is defined in Minkowskian space $X_{(\mu)}$ as a scalar product of the time axis vector and the coordinate:

$$X_{(0)} = l_{(\mu)} X_{(\mu)}.$$

The spatial coordinates are defined on the three-dimensional hypersurface

$$X_{(\mu)}^\perp = X_{(\mu)} - l_{(\mu)}(l_{(\nu)} X_{(\nu)}),$$

perpendicular to the time axis $l_{(\nu)}$.

Without the loss of generality, the time axis can be chosen in the form

$$l_{(\mu)} = (1, 0, 0, 0),$$

defining the observer rest reference frame. After solving the equations, the arbitrary Lorentz frame can be introduced. Taking out the factor $dX_{(0)}/d\tau$ from the radical in Eq. (2.7), we arrive at the action integral in the Planck's non-covariant formulation:

$$\begin{aligned} S_{\text{SR}} &= -m \int d\tau \frac{dX_{(0)}}{d\tau} \sqrt{1 - \sum_i \left[\frac{dX_{(i)}}{dX_{(0)}} \right]^2} = & (2.10) \\ &= -m \int dX_{(0)} \sqrt{1 - \sum_i \left[\frac{dX_{(i)}}{dX_{(0)}} \right]^2}. \end{aligned}$$

Expressing the momentum

$$P_{(i)} = \frac{\partial L}{\partial V_{(i)}} = \frac{mV_{(i)}}{\sqrt{1 - V_{(k)}^2}}, \quad (2.11)$$

in terms of the velocity $V_{(i)} = dX_{(i)}/dX_{(0)}$ entering into the variation of the Lagrangian (2.10)

$$L = -m\sqrt{1 - V_{(i)}^2}$$

one can obtain the Hamiltonian function

$$H(P_{(i)}) = P_{(i)}V_{(i)} - L = \sqrt{m^2 + P_{(i)}^2(X_{(0)})} \quad (2.12)$$

and rewrite action (2.10) in the Hamiltonian form

$$S_{\text{SR}} = \int dX_{(0)} \left[P_{(i)} \frac{dX_{(i)}}{dX_{(0)}} - H(P_{(i)}) \right]. \quad (2.13)$$

The energy of a particle is defined as a value of the Hamiltonian function on the trajectory:

$$E = H(P_{I(i)}) = \sqrt{m^2 + P_{I(i)}^2}.$$

The famous formula $E = mc^2$ (with $c = 1$) is a consequence of the definition of physical observables from the correspondence to the classical mechanics and follows from the low energy expansion of the Hamiltonian function in powers of dynamical variables:

$$H(P_{(i)}) = \sqrt{m^2 + P_{(i)}^2} = m + \frac{P_{(i)}^2}{2m} + \dots \quad (2.14)$$

Variation of action (2.13) with respect to canonical momenta $P_{(i)}$ and variables $X_{(i)}$ yields, respectively, the velocity in terms of the momenta,

$$V_{(i)} = \frac{P_{(i)}}{\sqrt{m^2 + P_{(i)}^2}}, \quad (2.15)$$

and the momentum conservation law:

$$\frac{dP_{(i)}}{dX_{(0)}} = 0.$$

The solution of these equations determines the particle trajectory in the space of events:

$$X_{(i)}(X_{(0)}) = X_{(i)}(X_{I(0)}) + V_{(i)}[X_{(0)} - X_{I(0)}], \quad (2.16)$$

where $X_{I(0)}$ is the initial time relative to the observer rest frame.

The transformation to any reference frame is described by the corresponding Lorentz transformation and equivalent to an appropriate choice of a time axis. Every reference frame has its proper time, energy, and momentum. The relationship between dynamical variables and times in different reference frames is treated as the relativity principle formulated, most clearly, by Einstein [7]. According to the Einstein relativity principle, the Lorentz transformations contain extra information on *relativistic effects*, as compared to solutions (2.16) of the dynamical equations derived by variation of the action (2.13). Therefore, the appearance of relativistic effects due to the Lorentz kinematic transformations (*id est*, transformations of reference frames) means that the Einstein theory significantly differs from the Newtonian mechanics. In the latter, all the physical effects are to be deduced from the equations of motion by variational method with due regard to the initial data. In this case, the Galileo group in Newtonian mechanics contains nothing new beyond the solutions of the equations of motion.

The following question arises: Can a relativistic particle theory be formulated in such a way that all physical consequences, including relativistic effects, are described by a variational equation? We will prove that such a relativistic particle theory can be formulated with perfect analogy to Hilberts “Foundations of Physics” [8], *id est*, as *geometrody-*

namics. According to this theory, the description of a physical system is based on the action functional, geometric interval, symmetry of reference frames, gauge symmetry, equations of motion, and constraint equations for initial data.

2.3.3 Geometrodynamics of a relativistic particle

Following to the ideas of Hilbert [8], one can construct the *geometrodynamics* of a relativistic particle. The covariant approach is based on two principles: the action [9, 10]

$$S_{\text{SR}} = -\frac{m}{2} \int d\tau e(\tau) \left[\left(\frac{dX_{(\alpha)}}{e(\tau)d\tau} \right)^2 + 1 \right] \quad (2.17)$$

for the variables $X_{(\alpha)} = [X_{(0)}|X_{(i)}]$ forming the space of events of the moving particle and the *geometric interval*

$$ds = e(\tau)d\tau \quad (2.18)$$

in the Riemannian one-dimensional space on the world line of the particle in this space (see Fig. 2.1). Here, $e(\tau)$ is the only metric component, the so-called lapse-function of the *coordinate evolution parameter* (ein-bein).

Variation of the action with respect to the function $e(\tau)$ yields the equations of geometrodynamics

$$[e(\tau)d\tau]^2 = dX_{(\alpha)}^2 \equiv dX_{(0)}^2 - dX_{(1)}^2 - dX_{(2)}^2 - dX_{(3)}^2. \quad (2.19)$$

Solving these equations in $e(\tau)$, we arrive at

$$e(\tau) = \pm \sqrt{\left(\frac{dX_{(\alpha)}}{d\tau} \right)^2}. \quad (2.20)$$

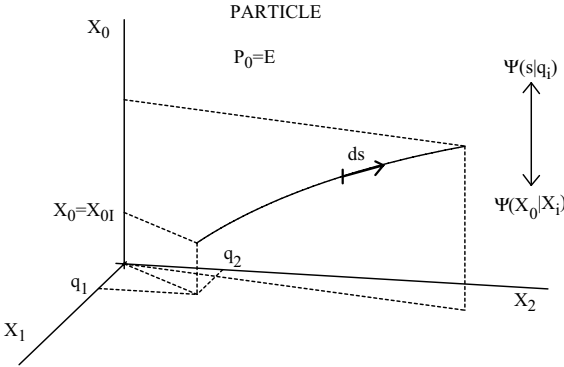


Figure 2.1: Motion of an unstable relativistic particle in a world line in the space of events. The motion is completely described by the two Newtonian-like sets of observables, *dynamical* and *geometric*. Each has its proper time and wave function Ψ . The two measured life-times of the particle (the time as either a *dynamical* variable X^0 or a *geometric* interval s) are interrelated by the equations of motion following from the action of Hilbert-type *geometrostatics* rather than by Lorentz transformations.

It is seen that the action (2.17) coincides in these solutions with the initial action (2.7) of the relativistic particle up to a sign. The negative sign of $e(\tau)$ in Eq. (2.20) implies the change of the mass sign in the action (2.7) for an antiparticle. Equation (2.19) is referred to as a constraint equation. For the Hamiltonian relativistic-particle theory with constraints, the corresponding action can be derived from (2.17) by introducing the canonical momenta

$$P_{(\alpha)} = \partial L_{SR} / \partial \dot{X}_{(\alpha)} :$$

$$S_{SR} = \int_{\tau_1}^{\tau_2} d\tau \left[-P_{(\alpha)} \frac{dX_{(\alpha)}}{d\tau} + \frac{e(\tau)}{2m} \left(P_{(\alpha)}^2 - m^2 \right) \right]. \quad (2.21)$$

Here, the lapse-function $e(\tau)$ of the coordinate evolution parameter τ defines geometric interval (2.18):

$$ds = e(\tau)d\tau \quad \mapsto \quad s(\tau) = \int_0^\tau d\bar{\tau}e(\bar{\tau}). \quad (2.22)$$

The action (2.21) and the interval (2.22) are invariant with respect to the reparametrization of the coordinate evolution parameter τ :

$$\tau \quad \longrightarrow \quad \tilde{\tau} = \tilde{\tau}(\tau). \quad (2.23)$$

Therefore, SR could be referred to as a one-dimensional GR, with the reparametrization group of coordinate evolution parameter (2.23) serving as a group of gauge (general coordinate) transformations. The equation for the auxiliary lapse-function $\delta S_{\text{SR}}/\delta e = 0$ determines the Hamiltonian constraint imposed on the particle momenta $P_{(0)}, P_{(i)}$:

$$P_{(0)}^2 - P_{(i)}^2 = m^2, \quad (2.24)$$

the so-called a mass surface equation.

The equations

$$P_{(\alpha)} = m \frac{dX_{(\alpha)}}{ed\tau} \equiv m \frac{dX_{(\alpha)}}{ds}, \quad \frac{dP_{(\alpha)}}{ds} = 0, \quad (2.25)$$

for the variables $P_{(\alpha)}, X_{(\alpha)}$ derived by a variation of action (2.21) are gauge-invariant.

The solution

$$X_{(\alpha)}(s) = X_{I(\alpha)} + \frac{P_{I(\alpha)}}{m}s, \quad (2.26)$$

of these equations in terms of geometric interval (2.22) is a generalization of the solution of the Newtonian equations (2.3) in the Minkowskian

space. In this case, the geometric time interval serves as an evolution parameter, while $P_{I(\alpha)}, X_{I(\alpha)}$ are initial data for the four variables at the point $s = 0$:

$$X_{(\alpha)}(s = 0) = X_{I(\alpha)}. \quad (2.27)$$

These equations contain three new features, as compared to Newtonian mechanics, namely, the momentum constraint (2.24), the time component in solution (2.26) of the equation of motion, and the initial value $X_{I(0)}$ of time as a variable.

2.3.4 Reduction of geometrodynamics to the Planck's relativistic dynamics (1906)

The action (2.21) and the interval (2.22) were above referred to as geometrodynamics of a particle. The geometrodynamics of a particle is characterized by two times in every reference frames namely, the time as a *geometric interval* measured by an observer on the world line and the time as a *dynamical variable* measured by an fixed observer. The physical interpretation of solutions (2.24) and (2.26) of geometrodynamics is determined by the choice of a specific Lorentz reference frame $P_\mu = (P_{(0)}, P_{(i)})$, the so called observer rest frame. The solution $P_{(0)}$

$$P_{(0)\pm} = \pm \sqrt{P_{(i)}^2 + m^2} = \pm H \quad (2.28)$$

of constraint equation (2.24) in the zero momentum component $P_{(0)}$ in this reference frame is the Hamiltonian function in the spatial dynamical variables $[P_{(i)}, X_{(i)}]$. According to the principle of correspondence

to Newtonian mechanics, these variables belong to the so-called *reduced phase space* [11]. The variable $X_{(0)}$ is the evolution time relative to the observer rest frame.

In a given Lorentz reference frame, the time component of solution (2.26),

$$X_{(0)}(s) - X_{I(0)} = \frac{P_{(0)\pm}}{m} s. \quad (2.29)$$

of geometrodynamics has no analogy in Newtonian mechanics. In this case, the formula (2.29) is a pure kinematic relation between the two times noted above, namely, the dynamical variable $X_{(0)}$ and the geometric interval s :

$$s = [X_{(0)} - X_{I(0)}] \frac{m}{P_{(0)\pm}}. \quad (2.30)$$

This equation will be referred to as a *geometric ratio* of the two times of a relativistic particle, namely, the time $[X_{(0)}]$ as a variable and the time s as an interval.

The substitution of geometric ratio (2.30) into spatial part

$$X_{(i)}(s) = X_{I(i)} + \frac{P_{(i)}}{m} s \quad (2.31)$$

of solution (2.26) gives the relativistic equation of motion in the reduced phase space $[P_{(i)}, X_{(i)}]$,

$$X_{(i)} = X_{I(i)} + \frac{P_{I(i)}}{P_{(0)+}} [X_{(0)} - X_{I(0)}]. \quad (2.32)$$

with the time $[X_{(0)}]$ as a variable.

Thus, *geometrodynamics* in a specific reference frame consists of constraint-free “particle dynamics” (2.32) and “geometry” (2.31) describing purely relativistic effects by the equations of motion in the same reference frame [9, 10].

The formula (2.13) for the action describing a moving particle can be derived by substitution of solution (2.28) into *geometrodynamic* action (2.21). Such a substitution also gives the action for a particle with negative energy (2.28):

$$S_{\text{SR}}|_{P_{(0)}=P_{(0)-}} = \int_{X_{(0)}}^{X_{(0)I}} d\bar{X}_0 \left[-P_{(i)} \frac{dX_{(i)}}{d\bar{X}_{(0)}} - \sqrt{P_{(i)}^2 + m^2} \right]. \quad (2.33)$$

The equations corresponding to this action have the solutions

$$\begin{aligned} X_{(i)} &= X_{I(i)} + \frac{P_{(i)}}{P_{(0)-}} [X_{I(0)} - X_{(0)}(s)] \\ &= X_{(i)} + \frac{P_{(i)}}{P_{(0)+}} [X_{(0)}(s) - X_{I(0)}]. \end{aligned} \quad (2.34)$$

The problem of the negative energy was solved later, under construction of the relativistic quantum field theory [12].

2.3.5 Quantum anomaly of geometric interval

It is known that quantum relativistic mechanics is defined as the quantization of energy constraint (2.24)

$$P_{(0)}^2 - P_{(i)}^2 = m^2,$$

by substituting the particle momentum $P_{(\alpha)} = (P_{(0)}, P_{(i)})$ by its operator $\hat{P}_{(\alpha)} = -i\partial_{(\alpha)}$. The quantization yields the Klein – Gordon – Fock equation for a wave function

$$\left[\hat{P}_{(\alpha)}^2 - m^2 \right] \Psi[P_{(\alpha)}|X_{(\alpha)}] = 0 \quad (2.35)$$

as a quantum analog of constraint equation (2.24). This equation has the normalized solution

$$\Psi[P_{(\alpha)}|X_{(\alpha)}] = \frac{1}{\sqrt{2|P_{(0)}|}} \times \quad (2.36)$$

$$\times \left[a^+ \Psi_{P_{(0)+}} \theta(X_{(0)} - X_{I(0)}) + a^- \Psi_{P_{(0)-}}^* \theta(X_{I(0)} - X_{(0)}) \right],$$

where θ is the Heaviside step function. Two linear-independent terms with the coefficients a^+ and a^- correspond to two classical solutions of constraint equation (2.24) with positive and negative energies (2.28). Quantum field theory is known to be formulated as the quantization of the coefficients a^+ , a^- , *id est* as the second quantization of relativistic particles [12]. In this case, to exclude negative energies, $-|P_{(0)}|$ and therefore to ensure the stability of quantum systems, the coefficients a^+ and a^- are to be treated as creation and annihilation operators, respectively, for particles with positive energy ⁴. This treatment is equivalent to the postulate of the existence of the vacuum as a lowest-energy state in the space of events. The postulate imposes a constraint on the motion of a classical particle in the space of events, namely, the particle with the energy $P_{(0)+}$ ($P_{(0)-}$) which moves forward (backward):

$$P_{(0)+} \rightarrow X_{I(0)} \leq X_{(0)}; \quad P_{(0)-} \rightarrow X_{I(0)} \geq X_{(0)}. \quad (2.37)$$

The following question arises: How does the *causal quantization* (2.36) with the restriction (2.37) influence geometric interval s (2.22)?

To answer this question we perform the Lorentz transformation from the rest reference frame to the comoving frame: $[\bar{X}_{(0)}|\bar{X}_{(i)}]$, where

⁴Moreover, the initial data $X_{I(0)}$ is treated in the quantum theory as a point of creation or annihilation of a particle.

$\bar{P}_{(i)} = 0$ and $\bar{P}_{(0)\pm} = \pm m$. It follows from (2.30) and (2.37) that the time $\bar{X}_{(0)}$ in the comoving frame is related to the geometric interval s by the equation

$$\begin{aligned} s(\bar{X}_{(0)}|\bar{X}_{I(0)}) &= \tag{2.38} \\ &= (\bar{X}_{(0)} - \bar{X}_{I(0)})\theta(\bar{X}_{(0)} - \bar{X}_{I(0)})\theta(\bar{P}_{(0)}) + (\bar{X}_{I(0)} - \bar{X}_{(0)})\theta(\bar{X}_{I(0)} - \bar{X}_{(0)})\theta(-\bar{P}_{(0)}). \end{aligned}$$

This expression for the geometric interval s in quantum field theory looks like the Greens causal function of the comoving time:

$$\frac{d^2 s(\bar{X}_{(0)}|\bar{X}_{I(0)})}{d\bar{X}_{(0)}^2} = \delta(\bar{X}_{(0)} - \bar{X}_{I(0)}). \tag{2.39}$$

Therefore, the *positive geometric arrow of time* $s \geq 0$ is a consequence of the postulate of existence of the vacuum as a lowest-energy state of the system, which leads to the existence of the absolute time origin $s = 0$. The positive arrow of time implies breaking of classical symmetry with respect to the transformation s to $-s$. In contrast to the classical symmetry, breaking of symmetry in quantum theory is referred to a *quantum anomaly*⁵. Under the assumption of the existence of the vacuum as a physical lowest-energy state, the second quantization of an arbitrary relativistic system leads to the absolute geometric-time origin $s = 0$ in this system. The question on what was before the creation of a relativistic particle or a universe has no physical sense for an observer measuring time because time $s = 0$ is created

⁵The anomaly associated with Dirac fields also follows from the vacuum existence postulate. This fact was first pointed out by Jordan [13] and then rediscovered by the authors [14, 15]. The vacuum existence postulate is verified by a number of experimental effects, in particular, anomalous decays of pseudoscalar bound states (neutral pion and para-positronium) into two photons.

together with the quantum relativistic universe as a consequence of the universe stability.

We have seen that in quantum theory in each frame of reference there are two measurable times: *the proper time* (2.38) at the world line and *the relative time* $[X_{(0)}]$ (2.38) in the space of events $[X_{(0)}, X_{(j)}]$. In quantum theory, these two times are supplementary, and they cannot be identified. Therefore, the so-called *twin paradox* that appears in classical relativistic mechanics does not take place in the quantum theory.

2.3.6 How does the invariant reduction differ from the choice of gauge?

We now compare the gauge invariant method of describing the field dynamics [9, 10] to the gauge-noninvariant method, assuming that the coordinate time τ becomes observed⁶. In the case of SR under consideration, this assumption implies the use of the synchronous gauge $e(\tau) = 1$ in action (2.21):

$$S_{\text{SR}} = \int_{\tau_1}^{\tau_2} d\tau \left[-P_{(\alpha)} \frac{dX_{(\alpha)}}{d\tau} + \frac{e(\tau)}{2m} \left(P_{(\alpha)}^2 - m^2 \right) \right]. \quad (2.40)$$

This yields a constraint-free theory:

$$S_{\text{SR}}|_{[e=1]} = \quad (2.41)$$

⁶The assumption of the coordinate time x^0 as an observable was used in GR under description of the island gravitational objects [16].

$$= \int_{\tau_1}^{\tau_2} d\tau \left[P_{(i)} \frac{dX_{(i)}}{d\tau} - P_{(0)} \frac{dX_{(0)}}{d\tau} - \frac{1}{2m} \left(-P_{(0)}^2 + P_{(i)}^2 + m^2 \right) \right].$$

From the viewpoint of quantization, Eq. (2.41) describes an unstable system because it contains the variable $X_{(0)}$ making a negative contribution to the energy

$$E = \frac{1}{2m} (-P_{(0)}^2 + P_{(i)}^2 + m^2),$$

which is defined in the interval $(-\infty < E < \infty)$. The particle action on the three-dimensional hypersurface defined by the condition $P_{(0)} = 0$ (the similar constraint in GR is referred to as a minimal surface [16]) coincides with the Newtonian action

$$S_{\text{SR}}|_{[e=1, P_{(0)}=0]} = S_{\text{Newton}} = \int_{\tau_1}^{\tau_2} d\tau \left[P_{(i)} \dot{X}_{(i)} - \frac{P_{(i)}^2}{2m} \right]. \quad (2.42)$$

up to a constant factor, as well as with the Einstein action (2.13) in the nonrelativistic limit, where the time $X(0)$ in the rest frame coincides with the time interval s . Theory (2.41) under the constraint

$$P_{(0)}^2 = P_{(i)}^2 + m^2$$

reduces to SR because the gauge symmetry is restored.

Einstein [7] found that, in contrast to classical mechanics, just two observers are needed to completely describe the motion of a relativistic particle: the first is at rest, and the second moves with the particle. For example, every Einstein's observer measures its proper lifetime of an unstable particle. Therefore, time is a relative quantity. Einstein described this time relativity as a pure kinematic effect by using the

Lorentz transformations from a fixed reference frame to a moving one. As was shown above, there exists a geometrodynamical generalization of the Einstein dynamics of 1905 (2.13) to a gauge theory with the constraint (2.21). This generalization allows us to describe the two-time relativity as a consequence of the equations of motion rather than the Lorentz kinematic transformations. This geometrodynamical description defines the new two-time relativity as a ratio of the dynamical particle evolution parameter $X_{(0)}$ to geometric interval s (2.22). We now illustrate this inference with a mini-universe. In this case, purely relativistic effects can not be described kinematically by transformations of the Lorentz type variables.

2.4 Homogeneous approximation of General Relativity

2.4.1 Radiation-dominated cosmological model

As was shown in the preceding Section, relativistic effects in variational equations can be dynamically described within the framework of SR formulated by analogy to the Hilberts variational description of GR [8]. According to Hilbert, GR geometrodynamics is based on two basic notions: the action

$$W_H = - \int d^4x \sqrt{-g} \frac{R^{(4)}(g)}{6} \quad (2.43)$$

in the units

$$\sqrt{\frac{3}{8\pi}} M_{\text{Pl}} = c = \hbar = 1$$

and the geometric interval of the Riemannian coordinate manifold

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \quad (2.44)$$

Both the action (2.43) and the interval (2.44) are invariant with respect to the general coordinate transformations

$$x^\mu \rightarrow \tilde{x}^\mu = \tilde{x}^\mu(x^0, x^1, x^2, x^3). \quad (2.45)$$

They serve as a generalization of the action considered above and as an interval for a relativistic particle, invariant with respect to the reparametrization group of coordinate time.

In the case of the homogeneous approximation,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \simeq \left\{ \underbrace{g_{00}(x^0)}_{(dt)^2} [dx^{0j}]^2 - \underbrace{|g^{(3)}(x^0)|^{1/3}}_{a^2(t)} [dx^j]^2 \right\} \quad (2.46)$$

one can keep only two metric components:

$$g_{00} = |g^{(3)}|^{1/3} N_0^2$$

and the spatial metric determinant $|g^{(3)}(x^0)|$. In Friedmann's notations this metric takes the following form

$$ds^2 = dt^2 - a^2(t)(d\mathbf{r})^2. \quad (2.47)$$

Here t is the *world time*, $a(t)$ is the *cosmological scale factor*, and

$$r \equiv \sqrt{x_1^2 + x_2^2 + x_3^2} \quad (2.48)$$

is the *coordinate distance* to a considered cosmic object. They are invariant with respect to reparametrization of the coordinate evolution parameter

$$x^0 \rightarrow \tilde{x}^0 = \tilde{x}^0(x^0). \quad (2.49)$$

From the light cone interval (2.47)

$$dt = a(t)dr$$

one gets a relation between the *coordinate distance* and the *conformal time* η :

$$r(\eta) = \int_{t_I}^{t_0} \frac{dt}{a(t)} \equiv \eta_0 - \eta. \quad (2.50)$$

Here η_0 is the present day value of the conformal time, for which the cosmological scale factor is equal to unit $a(\eta_0) = 1$, and η is the time of emission of a photon by an atom at a cosmic object, that is at the coordinate distance r to the Earth. In other words, this coordinate distance r is equal to a difference between η_0 and η

$$r = \eta_0 - \eta, \quad \text{or} \quad \eta = \eta_0 - r. \quad (2.51)$$

In the case of the homogeneous approximation, (2.47) the GR action (2.43) reduces to the cosmology action [19, 18]

$$W_H = -V_0 \int dx^0 N_0 \left[\left(\frac{da}{N_0 dx^0} \right)^2 + \rho_{\text{rad}} \right] = \int dx^0 L, \quad (2.52)$$

that can contain an additional matter term, in particular, the energy density of radiation $\rho_{\text{rad}} = \text{constant}$. Here V_0 is the volume, L is the Lagrangian, and

$$N(x^0) = a^{-1} \sqrt{g_{00}}$$

is the *lapse function*. This action keeps the reparametrization time invariance. As shown above, the reparametrization group of the coordinate parameter means that one of the variables (here, the only

variable is a) is identified with time as a variable, while its canonical momentum

$$P_a = \frac{\partial L}{\partial(da/dx^0)} = -2V_0 \frac{da}{N_0 dx^0} \equiv -2V_0 \frac{da}{d\eta}, \quad (2.53)$$

is taken as the corresponding Hamiltonian function, with its value on the equations of motion becoming the energy of events. The action (2.52) reduces to the Hamiltonian cosmology action

$$W_H = \int dx^0 \left(P_a \frac{da}{dx^0} - N_0 \left[-\frac{P_a^2}{4V_0} + V_0 \rho_{\text{rad}} \right] \right). \quad (2.54)$$

Variation of the action (2.54) with respect to the lapse function N_0 :

$$\frac{\delta W_H}{\delta N_0} = 0,$$

yields the energy constraint equation

$$\frac{P_a^2}{4V_0} = V_0 \rho_{\text{rad}}. \quad (2.55)$$

Solutions of this constraint take the form

$$P_a = \pm E; \quad E = 2V_0 \sqrt{\rho_{\text{rad}}}. \quad (2.56)$$

The Hubble law

$$\left(\frac{da}{d\eta} \right)^2 = \rho_{\text{rad}} \quad (2.57)$$

follows from Eqs. (2.53) and (2.55), and it yields the relation between the two times in the form of the Friedmann differential equation:

$$\eta_0 - \eta_I = \int_{a_I}^{a_0} \frac{da}{\sqrt{\rho_{\text{rad}}}} = \frac{(a_0 - a_I)}{\sqrt{\rho_{\text{rad}}}}. \quad (2.58)$$

This relation describes classical cosmology, *id est* the Hubble law, and is an auxiliary relation to the Wheeler – De Witt quantum cosmology,

provided that this cosmology is defined as a quantization of the constraint equation (2.55). The quantization is defined by the substitution of variables by operators, $\hat{P} = \imath d/da$ acting on the Wheeler – De Witt wave function Ψ :

$$\left[\frac{d^2}{da^2} + 4V_0^2 \rho_{\text{rad}} \right] \Psi_U(a) = 0. \quad (2.59)$$

As we have shown above, both the classical and the quantum cosmologies followed from the Hilbert geometrodynamics. This allows us to combine them to settle their troubles, namely, the quantization of classical cosmology and the description of the Hubble law in quantum cosmology. The wave function can be presented in the form of the sum of two terms

$$\begin{aligned} \Psi_U(a) = & \frac{1}{\sqrt{2|E|}} \times \\ & \times \left[A_U^+ \exp(\imath Ea) \theta(a_{(0)} - a_{I(0)}) + A_U^- \exp(-\imath Ea) \theta(a_{I(0)} - a_{(0)}) \right], \end{aligned} \quad (2.60)$$

where A_U^+ and A_U^- are treated as the creation and annihilation operators of the Universe with the positive energy in accordance with the vacuum postulate $A_U^-|0\rangle = 0$, by analogy with the wave function of a relativistic particle (2.36).

In the Standard cosmology this wave function describes the Universe in the epoch of the radiation dominance, where the energy density corresponds to the number of the CMB photons $\sim 10^{87}$ and their mean energy and wave length ~ 1 mm. The question is: How can we derive these quantities from the first principles?

2.4.2 Arrow of conformal time as quantum anomaly

The vacuum existence postulate $A_U^-|0\rangle = 0$ restricts the motion of the universe in the field space of events, and it means that the universe moves forward ($a > a_I$) or backward ($a < a_I$) if the energy of events is positive ($P_a \geq 0$) or negative ($P_a \leq 0$), respectively, where a_I is the initial value. In quantum theory, the quantity a_I is considered as a creation point of the universe with positive energy $P_a \geq 0$ or as an annihilation point of the anti-universe with negative energy $P_a \leq 0$. We can assume that the singular point $a = 0$ belongs to the anti-universe: $P_a \leq 0$. A universe with positive energy of events has no cosmological singularity $a = 0$. According to the vacuum existence postulate, the conformal time (5.1) as a solution of the Hubble law is positive for both the universe and anti-universe

$$\eta_0 - \eta_I \geq 0. \quad (2.61)$$

2.5 Standard cosmological models

We now consider the equation (2.57) of the universe evolution

$$\left[\frac{da}{d\eta} \right]^2 = \rho_c(a). \quad (2.62)$$

The universe is filled with homogeneous matter with the density $\rho_c(a)$. In the Standard cosmology, the conformal density is the sum of those depending on the scale a

$$\rho_c(a) = \rho_{\text{rigid}} a^{-2} + \rho_{\text{rad}} + \rho_M a + \rho_\Lambda a^4. \quad (2.63)$$

Here, ρ_{rigid} is the contribution of the equation of rigid state for which the density is equal to pressure

$$\rho_{\text{rigid}} = p_{\text{rigid}}, \quad (2.64)$$

The densities ρ_{rad} , ρ_M , and ρ_Λ describe the contributions of radiation, baryon matter, and Lambda-term, respectively. For each of these states, the Eq.(2.62) can be solved, under the initial conditions

$$a(\eta_0) = 1, \quad a'(\eta_0) = H_0 :$$

in terms of the conformal time η :

$$a_{\text{rigid}}(\eta) = \sqrt{1 - 2H_0 r}, \quad a_{\text{rad}}(\eta) = 1 - H_0 r, \quad (2.65)$$

$$a_M(\eta) = \left[1 - \frac{1}{2}H_0 r\right]^2, \quad a_\Lambda(\eta) = \frac{1}{1 + H_0 r}. \quad (2.66)$$

The conformal time η is defined in observational cosmology as the instant of time at which the photon is radiated by an atom at a cosmic object moving with the velocity $c = 1$ in a geodetic line on the world cone

$$ds^2 = a^2[(\eta)^2 - dr^2] = 0.$$

This allows us to find the relation of the distance

$$r = \eta_0 - \eta. \quad (2.67)$$

In other words, η_0 is the present-day conformal time of the photon measured by a terrestrial observer with $a(\eta_0) = 1$, and η is the time instant at which the photon is radiated by an atom at the distance

r from the Earth. Therefore, η is the difference between the present-day conformal time η_0 and the time of a photon flight to the Earth, coinciding with the distance (2.67). Equation (2.67) gives

$$\eta = \eta_0 - r. \quad (2.68)$$

In an observational cosmology, the density (2.63) can be expressed in terms of the present-day critical density ρ_{cr} :

$$\rho_c(a) = \rho_{\text{cr}}\Omega(a), \quad (2.69)$$

$$\Omega(a) = \Omega_{\text{rigid}}a^{-2} + \Omega_{\text{rad}} + \Omega_M a + \Omega_\Lambda a^4 \quad (2.70)$$

and the relative ones Ω_{rigid} , Ω_{rad} , Ω_M , Ω_Λ satisfying the condition [17]

$$\Omega_{\text{rigid}} + \Omega_{\text{rad}} + \Omega_M + \Omega_\Lambda = 1.$$

The Classical cosmology describes the redshift of the radiation spectrum $E(\eta)$ at a cosmic object as relative to the spectrum $E(\eta_0)$ at the Earth. The redshift is defined as the scale factor versus the coordinate distance (given by conformal time (2.68)) to the object.

Taking these relations into account and substituting $a = 1/(1+z)$ and $\eta = \eta_0 - r$, we can write down the scale evolution equation (2.63) at the light ray geodetic line $dr/d\eta = -1$ in the form

$$\frac{1}{H_0} \frac{dz}{dr} = (1+z)^2 \sqrt{\rho_{\text{cr}} [\Omega_{\text{rigid}}(1+z)^2 + \Omega_{\text{rad}} + \Omega_M(1+z)^{-1} + \Omega_\Lambda(1+z)^{-4}]},$$

where $H_0 = \sqrt{\rho_{\text{cr}}}$.

The solution

$$H_0 r(z) = \int_1^{1+z} \frac{dx}{\sqrt{\Omega_{\text{rigid}}x^6 + \Omega_{\text{rad}}x^4 + \Omega_M x^3 + \Omega_\Lambda}}, \quad (2.71)$$

coinciding with solution (2.67) of this equation, determines the coordinate distance as a function of redshift z and gives formulae (2.65), (2.66) for every state. The formula (2.71), being a basis of the observational cosmology (for example, see [18]), is used to determine the equation of state of matter in the Universe according to the astrophysical measurements of the redshift in assumption of flat space. The formula is universal for all standards of measurement. The Friedmannian distance $R(z)$ in the Standard cosmology is connected with conformal distance $r(z)$ in Conformal cosmology by the relation

$$R(z) = a(z)(\eta_0 - \eta) = a(z)r(z), \quad a = \frac{1}{1+z}, \quad (2.72)$$

following from the definition of the metric

$$ds^2 = a^2(\eta) [d\eta^2 - (d\mathbf{r})^2]$$

and the relation (2.51). Thus, different standards for the same data on the dependence of the redshift of distances correspond to different equations of state of matter in the Universe.

2.6 Summary

Thus, the unified geometrodynamical formulation of both theories (SR and GR), which is based on the Hilbert variational principle [8], makes it possible to quantize the cosmological models similarly to the first and second quantization of a relativistic particle. The latter is a basis of the modern quantum field theory [12], which is verified by a great number of high-energy experiments. A similar approach to the quantization within the framework of GR was first formulated by Wheeler

[19] and De Witt [20]. They assumed that cosmological time, treated as a variable, is identical with the cosmological scale factor. Moreover, they introduced into GR the concept of a field space of events, in which the relativistic universe moves, by analogy to the motion of a relativistic particle of the Minkowskian space. However, the Wheeler – De Witt formulation [19, 20] does not contain time as a geometric interval and, therefore, its scale-factor dependence (interpreted in the Friedmann cosmology as the Hubble law). Thus, as noted above, classical cosmology fails to quantize [18], while quantum cosmology fails to describe the Hubble law [19, 20]. In this Section, we use the invariant reduction of the Wheeler – De Witt cosmology, considered as a relativistic universe geometrodynamics, to restore the relation of observational cosmology (*id est* the Hubble law) to the first and second quantization of the universe and calculate the distribution of created universes. This reduction allows us to solve a series of problems, namely, Hubble evolution, universe creation from the vacuum, arrow of time, initial data, and elimination of the cosmological singularity, under the assumption that the Hamiltonian is diagonal and the universe is stable. In what follows, we consider a similar invariant reduction in the General Relativity in order to define physical observables, quantize gravity, and formulate a low-energy perturbation theory.

Bibliography

- [1] Narlikar, J.V.: *Violent Phenomena in the Universe*. Oxford University Press. Oxford, New York (1984)
- [2] Planck, M.: *Über irreversible Strahlungsvorgänge*. Sitzungsber. Deut. Akad. Wiss. Berlin. Kl. Math.-Phys. Tech., 440 (1899)
- [3] Einstein, A.: *Die relativitätstheorie*. 1, **56**, Naturforsch. Gesellschaft, Vierteljahresschrift, Zürich, Jahrg. (1911)
- [4] Poincaré, H.: *Sur la dynamique de l'électron*. Rendiconti del Circolo matematico di Palermo. **21**, 129 (1906)
- [5] Minkowski, H.: *Space and Time. Minkowski's Papers on Relativity*. Minkowski Institute Press (2012)
- [6] Planck, M.: *Das prinzip der relativität und die grundgleichungen der mechanik*. Verhandlungen Deutsche Physikalische Gesellschaft. **8**, 136 (1906)
- [7] Einstein, A.: *Zur elektrodynamik der bewegter körper*. Anal. d. Phys. **17**, 891 (1905)

- [8] Hilbert, D.: *Die gründlungen der physik*, Nachrichten von der Kön. Gesellschaft der Wiss. Göttingen. Math.-phys. Kl. **3**, 395 (1915)
- [9] Pawlowski, M., Pervushin, V.N.: *Reparametrization-invariant path integral in GR and “Big Bang” of quantum Universe*. Int. J. Mod. Phys. **16**, 1715 (2001)
- [10] Barbashov, B.M., Pervushin, V.N., Proskurin, D.V.: *Physical coordinates as dynamic variables in relativistic theories*. Theor. and Math. Phys. **132**, 1045 (2002)
- [11] Lanczos, Cornelius: *The Variational Principles of Mechanics*. University of Toronto Press, Toronto (1962)
- [12] Bogoliubov, N.N., Shirkov, D.V.: *Introduction to the Theory of Quantized Fields*. John Wiley & Sons, New York (1980)
- [13] Jordan, P.: *Zur neutrinotheorie des lichtes*. Z. Phys. **93**, 464 (1935)
- [14] Pervushin, V.N.: *The vacuum in gauge theories*. Riv. del Nuovo Cimento. **8**, 1 (1985)
- [15] Ilieva, N.P., Pervushin, V.N.: *Minimal quantization of two-dimensional gauge theories*. Particles & Nuclei. **22**, 573 (1991)
- [16] Faddeev, L.D., Popov, V.N.: *Covariant quantization of the gravitational field*. Physics–Uspekhi. **16**, 777 (1974)

-
- [17] Bahcall, N.A., Ostriker, J.P., Perlmutter, S., and Steinhardt, P.J.: *The cosmic triangle: revealing the state of the Universe*. Science. **284**, 1481 (1999)
- [18] Narlikar, J.V.: *An Introduction to Cosmology*. Jones and Bartlett, Boston (1983)
- [19] Wheeler, J.A.: *Lecture in Mathematics and Physics*. Benjamin, New York (1968)
- [20] De Witt, B.S.: *Quantum theory of gravity. I. The canonical theory*. Phys. Rev. **160**, 1113 (1967)

Chapter 3

Principles of symmetry of physical theories

3.1 Irreducible representations of Lorentz group $SO(3, 1)$

The Lorentz group is determined by the requirement of invariance of the speed of light in all inertial reference systems. It is a generalization of the Galilean transformations, and includes those that are mixed up spatial and time coordinates of a particle. The set of linear transformations, preserving the invariant form of the interval

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \equiv (dx_0)^2 - (dx_1)^2 - (dx_2)^2 - (dx_3)^2,$$

is called the Lorentz group. Transformations of the group are defined as

$$x'_\mu = \Lambda_{\mu\nu} x_\nu, \tag{3.1}$$

where $\Lambda \in O(3, 1)$. We introduce the Hermitian generators of the Lorentz transformations

$$L_{\mu\nu} = \imath(x_\mu\partial_\nu - x_\nu\partial_\mu).$$

The generators $L_{\mu\nu}$ form a Lie algebra $so(3, 1)$:

$$[L_{\mu\nu}, L_{\rho\tau}] = \imath(g_{\mu\rho}L_{\nu\tau} - g_{\mu\tau}L_{\nu\rho} - g_{\nu\rho}L_{\mu\tau} + g_{\nu\tau}L_{\mu\rho}). \quad (3.2)$$

The most common representation of operators, satisfying the commutation relations (3.2), has the form

$$M_{\mu\nu} \equiv \imath(x_\mu\partial_\nu - x_\nu\partial_\mu) + \Sigma_{\mu\nu},$$

where the spin operators $\Sigma_{\mu\nu}$ form the same Lie algebra (3.2) and commute with operators $L_{\mu\nu}$. The Hermitian generators M_{ij} form an algebra of rotations $su(2)$:

$$[M_{ij}, M_{kl}] = -\imath\delta_{jk}M_{il} + \imath\delta_{ik}M_{jl} + \imath\delta_{jl}M_{ik} - \imath\delta_{il}M_{jk}. \quad (3.3)$$

We introduce the operators of space rotations

$$J_i \equiv \frac{1}{2}\varepsilon_{ijk}L_{jk},$$

where ε_{ijk} is the Levi-Civita symbol, antisymmetric in all indices, and boost operators

$$K_i \equiv L_{0i}.$$

From the algebra (3.2) we get

$$[J_i, J_j] = \imath\varepsilon_{ijk}J_k, \quad [K_i, K_j] = -\imath\varepsilon_{ijk}J_k, \quad [J_i, K_j] = \imath\varepsilon_{ijk}K_k. \quad (3.4)$$

The commutation relations (3.4) is possible to dissociate by introducing the linear combinations

$$N_i \equiv \frac{1}{2}(J_i + \imath K_i), \quad N_i^+ \equiv \frac{1}{2}(J_i - \imath K_i)$$

with the algebra

$$[N_i, N_j^+] = 0, \quad [N_i, N_j] = \imath \varepsilon_{ijk} N_k, \quad [N_i^+, N_j^+] = \imath \varepsilon_{ijk} N_k^+. \quad (3.5)$$

Therefore, in the new generators, the Lie algebra (3.2) is represented as the direct sum of complex-conjugated spin algebras:

$$su(2) \oplus su(2).$$

There are two Casimir operators $N_i N_i$, $N_i^+ N_i^+$, belonging to a universal enveloping algebra [1, 2] with eigenvalues $n(n+1)$, $m(m+1)$. States within a considered representation differ by eigenvalues of the operators N_3 , N_3^+ of the corresponding algebras. According to the Schur's lemma, the operators, commuting with all the generators of an algebra, are proportional to the unit. Therefore, the obtained representations can be numbered by pairs of numbers (n, m) that take integer and half-integer values:

$$n, m = 0, 1/2, 1, 3/2, 2, \dots$$

For example, let us consider the following representations combined by a pair of integer and half-integer numbers:

1. $(0, 0)$: spin is equal to zero—scalar or pseudo-scalar particle;
2. $(1/2, 0)$: spin is equal to $1/2$, left Weyl's spinor;
3. $(0, 1/2)$: spin is equal to $1/2$, right Weyl's spinor;

4. $(0, 1/2) \oplus (1/2, 0)$: Dirac's spinor;
5. $(1/2, 0) \oplus (0, 1/2) = (0, 0) \oplus (1, 0)$: In this case, the inner product is given by the antisymmetric product. A new representation $(1,0)$ is described by anti-symmetric self-dual second-rank tensor. The representation $(0,1)$ corresponds to the anti-selfdual tensor;
6. $(0, 1) \oplus (1, 0)$: Maxwell's tensor of the electromagnetic field.

3.2 Irreducible representations of Poincaré group $SO(3,1) \dot{\otimes} T(4)$

An additional requirement of invariance of an isolated physical system with respect to uniform translations in space and time leads to a generalization of the six-parameter Lorentz group (3.1) to a ten-parameter Poincaré group [1]

$$x'_\mu = \Lambda_{\mu\nu} x_\nu + a_\mu, \tag{3.6}$$

where $\Lambda_{\mu\nu} \in SO(3,1)$, $a_\mu \in \mathbb{R}$.

Hermitian generators of translations $P_\mu = i\partial_\mu$ commute with each other:

$$[P_\mu, P_\nu] = 0, \tag{3.7}$$

but do not commute with the generators of the Lorentz group:

$$[M_{\mu\nu}, P_\rho] = -ig_{\mu\rho}P_\nu + ig_{\nu\rho}P_\mu. \tag{3.8}$$

Algebra of Poincaré is the semidirect sum of an ideal¹ (3.7) and the Lorentz algebra $so(3,1)$. As mentioned above, all irreducible represen-

¹In the theory of Lie algebras, the ideal is a maximal commutative subalgebra.

tations are characterized by the eigenvalues of the Casimir operators that commute with all the generators of the algebra of the group.



Eugene Wigner (November 17, 1902, Budapest – January 1, 1995, Princeton, USA) was a Hungarian American theoretical physicist and mathematician. He received a share of the Nobel Prize in Physics in 1963 “for his contributions to the theory of the atomic nucleus and the elementary particles, particularly through the discovery and application of fundamental symmetry principles”; the other half of the award was shared between Maria Goeppert–Mayer and J. Hans D. Jensen. Wigner is important for having laid the foundation for the theory of symmetries in quantum mechanics as well as for his research into the structure of the atomic nucleus. Wigner developed the theory of irreducible representations of the Poincaré group as the theory of the classification of elementary particles.

The Casimir operator is the square of the four-momentum operator $P_\mu P^\mu$, because of its invariance with respect to the Lorentz transformations. The second Casimir operator is constructed from the pseudovector Pauli – Lubański W^μ :

$$W^\mu \equiv \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}, \quad (3.9)$$

where $\varepsilon^{\mu\nu\rho\sigma}$ is the antisymmetric tensor of Levi–Civita. It is the generator of the little group of the Poincaré group, that is the maximal subgroup (with four generators), leaving the eigenvalues of the four-momentum vector P_μ invariant. Considering (3.7) and (3.8), we obtain the commutation relations for the vector:

$$[W_\mu, P_\nu] = 0, \quad [M_{\mu\nu}, W_\rho] = -\varepsilon_{\mu\nu\rho} W_\nu + \varepsilon_{\nu\rho\mu} W_\mu. \quad (3.10)$$

Consequently,

$$[W_\mu, W_\nu] = -\varepsilon_{\mu\nu\rho\sigma} W^\rho P^\sigma.$$

From here we see the square of the length of the vector $W^\mu W_\mu$ is the Casimir operator. Wigner’s representations are infinite-dimensional that corresponds to an unlimited momenta. From physical point of view, the following representations of the group are of special interest.

1. An eigenvalue of the operator

$$P_\mu P^\mu \equiv m^2$$

is a real positive number. An eigenvalue of the operator $W^\mu W_\mu$ is

$$W^2 = -m^2 s(s+1),$$

where s is a spin, with values $s = 0, 1/2, 1, \dots$. States within the representation differ by the third component of the spin

$$s_3 = -s, -s+1, \dots, s-1, s$$

and continuous eigenvalues p_i . The state corresponds to a particle with a mass m , a spin s , three-dimensional momentum p_i and the projection

of the spin s_3 . Massive particles with the spin s possess $2s + 1$ degrees of freedom.

2. An eigenvalue of the operator $P^\mu P_\mu$ is equal to zero, that corresponds to a particle with zero rest mass. An eigenvalue of the operator $W^\mu W_\mu$ is equal to zero. The scalar product of operators P^μ and W^μ is equal to zero:

$$P^\mu W_\mu = 0.$$

The coefficient of proportionality is called helicity and equals $\pm s$, where

$$s = 0, 1/2, 1, \dots$$

is a spin of the representation. Examples of particles: a photon with spin 1 and two states with helicity ± 1 , a neutrino with helicity $\pm 1/2$ and a metric graviton with two states of helicity ± 2 .

3. An eigenvalue of the operator

$$P_\mu P^\mu \equiv m^2 < 0$$

is a real negative number. Hypothetical particles with imaginary mass are called *tachyons* [3]. They are widely met in the physical world, appearing as quasi-particles in complex systems, having lost stability at phase transitions. In the theory of elementary particles tachyons make the vacuum state of the system unstable that leads to its restructuring, providing the appearance of mass of the elementary particles. In the Standard cosmology, the tachyon unstable vacuum state of the scalar field is used in an inflationary scenario of the expansion of the Universe. Further, as we shall see in Chapter 7, in cosmological models of the Universe, the gravitons acquire tachyon mass, equal to a Hubble

parameter. Classification of fields by the Poincaré group differs from the classification of fields by the Lorentz group, primarily because that involves the selection of the reference frame, where the time coordinate is separated from the space coordinates.

In quantum electrodynamics, under the classification by irreducible representations of the Poincaré group, the time and spatial components of the field are not equal, and satisfy different equations and describe different physical phenomena. In particular, the time component of the field is treated as a Coulomb potential of charges, forming simultaneously quantum states. And only the transverse field spatial components are treated as independent electromagnetic waves (photons), which give the radiative corrections to the spectrum of bound states. In the case of free massless photons, one can select such a frame of reference where conditions are imposed: the velocity of the longitudinal component is zero and the longitudinal component is zero itself. This last condition is called Coulomb gauge, or the choice of radiation variables.

In a certain frame of reference a massive field is divided into a time component and three spatial components. Time component is non-dynamic and plays a role of the Yukawa potential, comoving to an appropriate charge. Three spatial components of the massive vector field are divided into two perpendicular to the direction of the wave vector, and one is a longitudinal. All three components are independent dynamical variables describing the degree of freedom with some initial data.

In Einstein's theory of gravity a separation of the time coordinate

is called $4 = 3 + 1$ splitting of the space–time [4]. At the same time, at every point of a pseudo–Riemannian space it is possible to construct an appropriate tangent Minkowskian space where there are transformations of the Poincaré group. In the General Relativity, from ten components of the metric only two spatial components of the metric describe independent degrees of freedom of gravitons, while all the others give Newton’s potentials and their generalizations in the General Relativity.

3.3 Weyl group

Weyl group [2] includes, along with the Poincaré group, an Abelian group of scale transformations². The theory is scale – invariant if its classical action does not contain dimensional constants. If coordinates of a space transform under a scale transformation

$$x_\mu \rightarrow x'_\mu = e^\lambda x_\mu, \quad \lambda > 0, \quad (3.11)$$

then a scalar field transforms as:

$$\varphi(x) \rightarrow \varphi'(x') = e^{\lambda n} \varphi(x), \quad (3.12)$$

²Scale transformations were experienced by Alice (Carroll, Lewis: *Alice’s Adventures in Wonderland*. Macmillan and Co., London (1865)) in order to penetrate through a small door. Scientists have thought about this, as seen that every country adopted its measure lengths, weights. Thus, Galileo, reflecting on the invariance of the laws of nature for a change of scale, wrote in the book “*Dialogue Concerning Two New Sciences*” (1638) the following considerations. If you increase the size of the animal two times, its weight will increase eight times, proportional to the volume. The same cross-sectional size of its bones grow four times the square of the resolution. Consequently, they can only withstand four times the load.

where n is a conformal weight of the field.

In the infinitesimal form, under $\exp(\lambda) \approx 1 + \lambda$ the law of transformation takes a form

$$\varphi'(e^\lambda x) \approx \varphi'(x + \lambda x) \approx \varphi'(x) + \lambda x^\mu \frac{\partial \varphi'}{\partial x^\mu}(x),$$

consequently,

$$\delta\varphi \equiv \varphi'(x) - \varphi(x) = \lambda x^\mu \frac{\partial \varphi'}{\partial x^\mu}(x). \quad (3.13)$$

From here we get a generator of scale transformation (*generator of dilatation*) D :

$$D \equiv -ix^\mu \frac{\partial}{\partial x^\mu}.$$



Hermann Klaus Hugo Weyl, (9 November 1885 – 8 December 1955) was a German mathematician, theoretical physicist and philosopher. Although much of his working life was spent in Zürich, Switzerland and then Princeton, the USA he is associated with the University of Göttingen tradition of mathematics, represented by David Hilbert and Hermann Minkowski. After the creation of the General Theory of Relativity by Einstein, he turned to the unified field theory. Although the unified theory of gravity and electromagnetism failed, his theory of gauge invariance assumed great significance. Weyl is also well known for the use of the group theory to quantum mechanics.

Commutation relations of the generator of dilation D with the generators of the Poincaré algebra:

$$[D, P_\mu] = -iP_\mu, \quad [D, M_{\mu\nu}] = 0.$$

Representations of the Weyl group are characterized by integer and half-integer numbers – conformal weights. Conformal weights n of various fields (scalar $n = -1$, spinor $n = -3/2$, vector $n = 0$, tensor $n = 2$) are possible to be calculated, assuming the sum of conformal weights of all the factors in the action for these fields to be zero.

Free actions of these fields and their interaction with each other may include dimensional parameters, such as the mass. Then one talks about the *hard* violation of the scale symmetry of the theory. Breaking of a scale symmetry of the theory is called *soft* if such violation has occurred as a result of quantization of the original scale-invariant classical theory. Then one talks about the quantum anomalies. An example of such anomalies is the quantum vacuum fluctuations due to creation and annihilation of particles of quantum fields. The source of the anomaly can be taken for the separation of the field on the positive and negative frequency parts and the subsequent interpretation of the coefficient of the wave function particles with negative energy as the annihilation operator of the particles with positive energy. This treatment of the particles with negative energy, nowadays, in quantum field theory is the only unique way to build a vacuum state with the lowest energy [5].

In this way, the separation of a field to the positive and negative frequency parts, led to anomal meson decays, the above-mentioned

Casimir energy, and dimensional condensates of fields in quantum field theory (QFT), classical versions of which did not contain all of these effects, experimentally validated. One can say that the very construction of a stable vacuum in quantum field theory, as the state of the lowest energy, and even a hypothesis about its existence, suggests the possibility of quantum anomalies.

3.4 Conformal group $SO(4, 2)$

A conformal transformation is an invertible mapping of the space-time coordinates $x^\mu \rightarrow x'^\mu(x)$ such that it leaves the metric tensor invariant up to a local scale factor

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \Omega^2(x)g_{\mu\nu}(x).$$

A classical theorem of Liouville [6] states that any conformal, or angle-preserving map, between open subsets of \mathbb{R}^n , for $n \geq 3$, is a composition of an inversion, dilatation, and isometries. So, the aforementioned Weyl's group, supplemented by special conformal transformations

$$x_\mu \rightarrow \frac{x_\mu + \beta_\mu x^2}{1 + 2(\beta_\lambda x^\lambda) + \beta^2 x^2}, \quad x^2 \equiv x_\mu x^\mu, \quad \beta_\mu \in \mathbb{R}, \quad (3.14)$$

defines the group of conformal transformations. The special conformal transformation consists of an inversion on a unit hyperboloid

$$x^\mu \rightarrow \frac{x^\mu}{x^2},$$

followed by a translation

$$x^\mu \rightarrow x^\mu + \beta^\mu,$$

followed by a second inversion on a unit hyperboloid

$$x^\mu \rightarrow \frac{x^\mu}{x^2}.$$

An infinitesimal special conformal transformation

$$x^\mu \rightarrow x'^\mu = x^\mu - 2(\beta_\lambda x^\lambda)x^\mu + \beta^\mu x^2, \quad |\beta^\mu| \ll |x^\mu|,$$

such that

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta_\nu^\mu - 2(\beta_\lambda x^\lambda)\delta_\nu^\mu + 2(x_\nu\beta^\mu - x^\mu\beta_\nu).$$

A generator of dilatation D , generators of special conformal transformations K_μ

$$K_\mu = -i(x^2\partial_\mu - 2x_\mu(x_\lambda\partial^\lambda)), \quad (3.15)$$

and the above presented generators of the Poincaré group P_μ , $M_{\mu\nu}$, besides the algebra of Poincaré, satisfy the following commutation relations [7]:

$$\begin{aligned} [D, P_\mu] &= -iP_\mu, & [D, M_{\mu\nu}] &= 0, \\ [D, K_\mu] &= iK_\mu, & [K_\mu, K_\nu] &= 0, \end{aligned} \quad (3.16)$$

$$[K_\mu, P_\nu] = -2i(g_{\mu\nu}D + M_{\mu\nu}), \quad [K_\rho, M_{\mu\nu}] = i(g_{\rho\mu}K_\nu - g_{\rho\nu}K_\mu).$$

In order to identify the structure of algebra we introduce the following notations for the generators

$$J_{\mu\nu} = M_{\mu\nu}, \quad J_{65} = D, \quad J_{5\mu} = \frac{1}{2}(P_\mu - K_\mu), \quad J_{6\mu} = \frac{1}{2}(P_\mu + K_\mu).$$

Then we obtain the commutation relations

$$[J_{KL}, J_{MN}] = i(g_{KN}J_{LM} + g_{LM}J_{KN} - g_{KM}J_{LN} - g_{LN}J_{KM})$$

with the diagonal 6-dimensional metric tensor

$$g_{AA} = (+ - - -, -+), \quad A = 0, 1, 2, 3, 5, 6.$$

This shows that the commutation relations define the algebra $so(4, 2)$ of the group of orthogonal rotations in the pseudo-Euclidean space, which is isomorphic to the algebra $su(2, 2)$ of the fundamental representation of a twistor space \mathbb{C}^4 :

$$so(4, 2) \approx su(2, 2).$$

The generators of the six-dimensional self-representation are given by [8]

$$(J_{AB})_{CD} = \iota(g_{AC}g_{BD} - g_{AD}g_{BC}).$$

The little group which leaves $x = 0$ invariant is given by special conformal transformations, dilatations and Lorentz transformations. In accordance with the standard theory of induced representations [9], if we determine any representation $S(g, 0)$ of the little group, we can determine the complete action of the generators of the conformal group on the field $\varphi(x)$. Let $\Sigma_{\mu\nu}, \Delta, \kappa_\mu$ be the infinitesimal generators of the little group. They satisfy

$$\begin{aligned} [\kappa_\mu, \kappa_\nu] &= 0, & [\Delta, \kappa_\mu] &= \iota\kappa_\mu, \\ [\kappa_\rho, \Sigma_{\mu\nu}] &= \iota(g_{\rho\mu}\kappa_\nu - g_{\rho\nu}\kappa_\mu), & (3.17) \\ [\Sigma_{\rho\sigma}, \Sigma_{\mu\nu}] &= \iota(g_{\sigma\mu}\Sigma_{\rho\nu} - g_{\rho\mu}\Sigma_{\sigma\nu} - g_{\sigma\nu}\Sigma_{\rho\mu} + g_{\rho\nu}\Sigma_{\sigma\mu}). \end{aligned}$$

For every element X of the conformal algebra we can define

$$X\varphi(x) = \exp(-\iota P_\mu x^\mu) X' \varphi(0),$$

where

$$\begin{aligned} X' &\equiv \exp(iP_\mu x^\mu) X \exp(-iP_\mu x^\mu) = \\ &= \sum_{n=0}^{\infty} x^{\nu_1} \cdots x^{\nu_n} [P_{\nu_1}, [\cdots [P_{\nu_n}, X] \cdots]]. \end{aligned}$$

Using the commutation relations (3.16) it is found that there are at most three non-vanishing terms in this sum [7]. For example, for $X = K_\mu$, we get

$$\exp(iP_\nu x^\nu) K_\mu \exp(-iP_\nu x^\nu) = K_\mu - 2x^\nu (g_{\mu\nu} D + M_{\mu\nu}) + 2x_\mu x^\nu P_\nu - x^2 P_\mu.$$

From this we deduce the action of the generators $K_\mu, D, M_{\mu\nu}$ on $\varphi(x)$, since the action on $\varphi(0)$ is known by hypothesis, for example

$$K_\mu \varphi(0) = \kappa_\mu \varphi(0).$$

Finally,

$$P_\mu \varphi(x) = i\partial_\mu \varphi(x),$$

$$M_{\mu\nu} \varphi(x) = (i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \Sigma_{\mu\nu}) \varphi(x),$$

$$D \varphi(x) = (ix^\mu \partial_\mu + \Delta) \varphi(x),$$

$$K_\mu \varphi(x) = (i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu - 2ix^\nu (g_{\mu\nu} \Delta + \Sigma_{\mu\nu})) + \kappa_\mu) \varphi(x).$$

Let us consider here only a little group of finite-dimensional representations for which

$$\kappa_\mu = 0.$$

There are nilpotent finite-dimensional representations else, for which $\kappa_\mu \neq 0$, but $\kappa_\mu^n = 0$ for some value n , and infinite-dimensional representations. The matrices $\Sigma_{\mu\nu}$ form the irreducible representation of

the Lorentz algebra, and for $\kappa_\mu = 0$, according to the Schur's lemma, the matrix Δ is proportional to a unit matrix:

$$\Delta = \iota d,$$

where d is the conformal weight of the field φ .

3.5 Conformal invariant theories of gravitation

The equations of free massless fields – Maxwell, Klein – Gordon, Dirac, Yang – Mills are conformal invariant. Let us consider some attempts of generalizations the General Relativity.

3.5.1 Weyl geometry

In Weyl geometry there is no absolute way to compare elements of length at points spaced of each other, but it preserves the angles between the vectors during the conformal mapping. A comparison can be held for infinitely close points [10]. Let us consider a vector of length s in a point with coordinates x^μ . We transfer it parallel to itself to a point with coordinates $x^\mu + \delta x^\mu$. The change of its length is proportional to s and δx^μ :

$$\delta s = s \kappa_\mu \delta x^\mu, \tag{3.18}$$

where κ_μ is some vector. Suppose that the standard of length is changed so that the length is multiplied by $\lambda(x)$, depending of coordinates. Then s becomes equal to $s' = \lambda(x)s$, and $s + \delta s$ changes

as

$$s' + \delta s' = (s + \delta s)\lambda(x + \delta x) = (s + \delta s)\lambda(x) + s \frac{\partial \lambda}{\partial x^\mu} \delta x^\mu,$$

where we neglect the values of the second order. One obtains

$$\delta s' = \lambda \delta s + s \frac{\partial \lambda}{\partial x^\mu} \delta x^\mu = \lambda s \left(\kappa_\mu + \frac{\partial \phi}{\partial x^\mu} \right) \delta x^\mu,$$

where

$$\phi \equiv \ln \lambda. \quad (3.19)$$

Thus, we obtain

$$\delta s' = s' \kappa_\mu' \delta x^\mu,$$

where

$$\kappa_\mu' = \kappa_\mu + \frac{\partial \phi}{\partial x^\mu}. \quad (3.20)$$

If the vector is parallelly carried around the closed loop, the change in its length will be expressed by the following formula:

$$\delta s = s F_{\mu\nu} \delta S^{\mu\nu},$$

where

$$F_{\mu\nu} \equiv \frac{\partial \kappa_\mu}{\partial x^\nu} - \frac{\partial \kappa_\nu}{\partial x^\mu}, \quad (3.21)$$

and $\delta S^{\mu\nu}$ is an element of area enclosed by the loop. The antisymmetric tensor (3.21) is invariant under gauge transformations of the form (3.20). The vector, carried along the contour, changes its length, so the geometry underlying in the base of the theory, is non-Riemannian.

From the point of view of the analytical description of the geometry, quadratic and linear differential forms

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad \omega^1 = \kappa_\mu dx^\mu$$

equivalent to the corresponding forms

$$\lambda g_{\mu\nu} dx^\mu dx^\nu, \quad \kappa_\mu dx^\mu + d \ln \lambda.$$

In the Weyl theory, field quantities κ_μ , appeared in (3.18) taken as the electromagnetic potentials. They are subject to gauge transformations (3.20), not associated with changes in geometry, but only a change in length standards. These quantities (3.21) have a geometrical meaning, independent of length standards, and comply with the electromagnetic field tensor. Thus, the Weyl geometry, in the author's opinion, describes the electromagnetic field by the geometric language. Weyl's introduced the conformal-invariant tensor

$$C_{\lambda\mu\nu\kappa} \equiv \\ \equiv R_{\lambda\mu\nu\kappa} - \frac{1}{2}(g_{\lambda\nu}R_{\mu\kappa} - g_{\lambda\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu}) + \frac{1}{6}(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu})R.$$

The dynamical equations are constructed from the variational principle of minimal action. The Lagrangian density of the gravitational field should be a magnitude with a conformal weight -2 . Weyl chose it as the square Riemannian curvature similar to the electromagnetic field

$$\mathcal{L} = R^\mu_{\nu\alpha\beta} R^\nu_{\mu}{}^{\alpha\beta}.$$

Einstein criticized the Weyl's theory. The field equations are rather complex. Despite the remarkable properties of the theory [11], it was not accepted by physicists because it contradicts the quantum theory – quantum phenomena give us an absolute standard of length. The terminology, however, is caught in physics: gauge, gauge transformations, gauge invariants.

In the case of cosmology, there are only two ways to select the units of lengths of geometric intervals – absolute units when the length of interval

$$dl^2 = g_{ij}^{(3)} dx^i dx^j$$

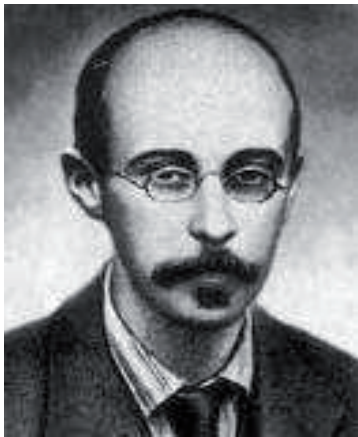
measured by the energy scale, and the relative, where such similar units are measured by similar, *id est*, intervals

$$\tilde{dl}^2 = \tilde{g}_{ij}^{(3)} dx^i dx^j$$

– by intervals, and energies – by energies. In case of choosing the absolute units, the space expands, and the sizes of space objects remain unchanged. In case of choosing the relative units, the space remains unchanged, and the sizes of space objects are reduced. Both of these features are discussed in the book of A. A. Friedmann [12], dedicated to the cosmology of the Universe, that binds a second chance with the principle of scale invariance of the laws of nature. A. A. Friedmann finds the following remarkable words about the principle of scale invariance³: “...*moving from country to country, we have to change the scale, id est, measured in Russia — by arshins, Germany — meters, England — feet. Imagine that such a change of scale we had to do from point to point, and then we got the above operation of changing of scale. Scale changing in the geometric world corresponds, in the physical world, to different ways of measuring of the length. . . . Properties of the world are divided into two classes: some are independent of the above said change of scale, better to say, do not change their shape under any changes of scale, while others under changing of the scale, will*

³Our free translation from Russian.

change their shape. Let us agree on their own properties of the world, belonging to the first class, and call scale invariant. Weyl expands the invariance postulate, adding to it the requirements that all physical laws were scale-invariant properties of the physical world. Consistent with such an extension of the postulate of invariance, we have to demand that the world equations would be expressed in a form satisfactory to not only coordinate, but the scale invariance”.



Alexander A. Friedmann (16 June 1888 – 16 September 1925), an outstanding Russian theoretical physicist. In 1923 his book “The World as Space and Time” [in Russian] was issued; it informed the public about the new physics. Friedmann predicted the expansion of the Universe. The first non-stationary solutions of Einstein’s equations received by him in 1922–1924 in the study of relativistic models of the Universe began the development of the theory of non-stationary Universe. The scientist studied the non-stationary homogeneous isotropic models with space, of first, positive and then negative curvature filled with dust matter (zero pressure).

3.5.2 Deser’s theory of gravitation

Conformal invariant scalar - tensor theory of gravitation was built by S. Deser in 1970 [14]. Let us present his further arguments. Under the

conformal transformation

$$g_{\mu\nu} = \bar{g}_{\mu\nu}\phi^{-2}, \quad \sqrt{-g} = \sqrt{-\bar{g}}\phi^{-4}$$

with some conformal factor ϕ we have

$$\frac{1}{6}\sqrt{-g}R(g)\phi^2 = \frac{1}{6}\sqrt{-\bar{g}}R(\bar{g}) - \sqrt{-\bar{g}}\bar{g}^{\mu\nu}\phi_{;\mu}\phi_{;\nu}\phi^{-2}$$

and

$$\sqrt{-g}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} = \phi^{-2}\sqrt{-\bar{g}}\bar{g}^{\mu\nu}\phi_{;\mu}\phi_{;\nu}.$$

Consequently, in the expression

$$\frac{1}{2}\int d^4x\sqrt{-g}\left[\phi_{;\mu}\phi_{;\nu}g^{\mu\nu} + \frac{1}{6}R\phi^2\right] = \frac{1}{12}\int d^4x\sqrt{-\bar{g}}R(\bar{g}),$$

the scalar field has been removed from the degrees of freedom.

A scalar field, added to the theory, is coupled non-minimally with the metric field of gravity

$$W(\phi) = -\frac{1}{2}\int d^4x\sqrt{-g}\left(g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} + \frac{1}{6}R\phi^2\right).$$

3.5.3 Scalar–tensor theory of Brans and Dicke

The theory of Brans – Dicke modifies the Einstein’s theory of gravity by introducing a scalar field ϕ [15], which is related to the density of mass in the Universe. The authors of the new theory proceeded from Mach’s principle, which states that the phenomenon of inertia is a consequence of accelerations of bodies relative to the total mass distribution in the Universe. A variation of the action

$$\delta\int d^4x\sqrt{-g}\left(\phi R - \xi\frac{\phi_{;\alpha}\phi^{;\alpha}}{\phi}\right) = 0,$$

where ξ is some dimensionless constant, leads to the following fields equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{\xi}{\phi^2} \left(\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi_{;\alpha}\phi^{;\alpha} \right) + \frac{1}{\phi}(\phi_{;\mu;\nu} - g_{\mu\nu}\square\phi).$$

The model of Deser is obtained from Brans – Dicke model at $\xi = -3/2$.

”Is the earth affected by its cosmological setting in the Universe;– asked Dicke,– ”As the Universe expands, as distant matter moves away from us, are there effects upon the earth of this changing distribution of matter;“



Paul Adrien Maurice Dirac (8 August 1902 – 20 October 1984) was an English theoretical physicist who made fundamental contributions to the early development of both quantum mechanics and quantum electrodynamics. Among other discoveries, he formulated the Dirac equations which describe the behavior of fermions and predicted the existence of antimatter. Dirac shared the Nobel Prize in Physics for 1933 with Erwin Schrödinger, “for the discovery of new productive forms of atomic theory”. He also did work that forms the basis of modern attempts to reconcile General Relativity with quantum mechanics.

3.5.4 Dirac’s dilaton theory

Fundamental physical constants allow us to set the system of absolute units of distance, time, mass, *et cetera*. These constants are more than

it is necessary for that purpose, and one can build a dimensionless combination. The ratio of the electrical and gravitational forces, acting between the electron and proton

$$\frac{e^2}{Gm_em_p},$$

is order $\sim 10^{39}$, the ratio of the mass of the Universe to the mass of the proton is order $\sim 10^{78}$. If we express the age of the Universe $\sim 10^9$ years in atomic units

$$\frac{e^2}{m_e c^3},$$

then we get a numeric value close to $\sim 10^{39}$. This leads to the idea that large numbers should not be viewed as constants, but the functions of time expressed in atomic units, *id est*, up to prime factors t, t^2 and so on, where t is time in the modern era in atomic units. P.A.M. Dirac expressed the new principle named by him as *the Hypothesis of Large Numbers*:

“Any two of the very large dimensionless numbers occurring in Nature are connected by a simple mathematical relation, in which the coefficients are of the order of magnitude unity” [13].

The gravitational constant changes simultaneously with time t of aging epoch inversely to t .

The scale-invariant theory of gravity, which retains all the achievements of Einstein’s theory, was formulated by Dirac in the famous paper [16]. For this purpose, he developed the analysis in the conformal geometry. Under any change of scale a length ds is multiplied by a factor $\lambda(x)$: $ds' = \lambda ds$. If the local value φ is transformed by the law $\varphi' = \lambda^n \varphi$, one says, that its conformal weight is n . From the expres-

sion for the interval $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ it follows that the metric tensor $g_{\mu\nu}$ has a conformal weight 2, because dx^μ are not affected by a scale transformation. Contravariant tensor $g^{\mu\nu}$ has a conformal weight -2 , and $\sqrt{-g}$ has a conformal weight 4. Following Dirac, let us obtain generalized covariant derivatives. Take first a scalar S of power n . Under scale changing its covariant derivative (which is the usual derivative) $S_{,\mu}$ is transformed by the formula

$$S'_\mu = (\lambda^n S)_{,\mu} = \lambda^n S_{,\mu} + n\lambda^{n-1}\lambda_{,\mu}S = \lambda^n \left[S_{,\mu} + n(\kappa'_\mu - \kappa_\mu)S \right],$$

where we used (3.19), (3.20). From here we get

$$(S_\mu - n\kappa_\mu S)' = \lambda^n (S_\mu - n\kappa_\mu S), \quad (3.22)$$

and the definition of the covariant derivative of a scalar:

$$S_{*\mu} = S_\mu - n\kappa_\mu S. \quad (3.23)$$

Note that it, according to (3.22), has a conformal weight n .

For getting covariant derivatives of vectors and tensors, we introduce modified symbols of Christoffel ${}^*\Gamma_{\mu\nu}^\alpha$, which are determined through the usual symbols $\Gamma_{\mu\nu}^\alpha$ by the following way:

$${}^*\Gamma_{\mu\nu}^\alpha = \Gamma_{\mu\nu}^\alpha - g_\mu^\alpha \kappa_\nu - g_\nu^\alpha \kappa_\mu + g_{\mu\nu} \kappa^\alpha. \quad (3.24)$$

Symbols ${}^*\Gamma_{\mu\nu}^\alpha$ are invariant with respect to gauge transformations. Let A_μ be a vector with conformal weight n . An expression

$$A_{\mu,\nu} - {}^*\Gamma_{\mu\nu}^\alpha A_\alpha$$

is a tensor. Under the gauge transformations it transforms as follows:

$$(A_{\mu,\nu} - {}^*\Gamma_{\mu\nu}^\alpha A_\alpha)' = \lambda^n A_{\mu,\nu} + n\lambda^{n-1}\lambda_{,\nu}A_\mu - {}^*\Gamma_{\mu\nu}^\alpha \lambda^n A_\alpha =$$

$$= \lambda^n \left(A_{\mu;\nu} + n(\kappa'_\nu - \kappa_\nu)A_\mu - {}^*\Gamma_{\mu\nu}^\alpha \lambda^n A_\alpha \right).$$

Consequently, the covariant derivative of the vector has the form:

$$A_{\mu*\nu} = A_{\mu;\nu} - n\kappa_\nu A_\mu - {}^*\Gamma_{\mu\nu}^\alpha A_\alpha,$$

or, using the definition (3.24), we rewrite it as

$$A_{\mu*\nu} = A_{\mu;\nu} - (n-1)\kappa_\nu A_\mu + \kappa_\mu A_\nu - g_{\mu\nu} \kappa^\alpha A_\alpha. \quad (3.25)$$

In a similar way to a contravariant vector B^μ of power n we get

$$B_{*\nu}^\mu = B_{;\nu}^\mu - (n+1)\kappa_\nu B^\mu + \kappa^\mu B_\nu - g_\nu^\mu \kappa_\alpha B^\alpha. \quad (3.26)$$

Then you can form a covariant derivative for tensors with different upper and lower indices by the same rules. The covariant derivative has the same degree as the initial value. The Leibnitz rule for the product of two tensors is also performed

$$(TU)_{*\alpha} = T_{*\alpha}U + TU_{*\alpha},$$

as well as the consistency condition:

$$g_{\mu\nu*\alpha} = 0, \quad g_{*\alpha}^{\mu\nu} = 0.$$

Now we find the second covariant derivative of a scalar S of a power n

$$S_{*\mu*\nu} = S_{*\mu;\nu} - (n-1)\kappa_\nu S_{*\mu} + \kappa_\mu S_{*\nu} - g_{\mu\nu} \kappa^\sigma S_{*\sigma}.$$

Substituting here the formula for the first covariant derivative (3.23), we get the following expression

$$S_{*\mu*\nu} = S_{\mu;\nu} - n\kappa_{\mu;\nu}S - n\kappa_\mu S_\nu - n\kappa_\nu(S_\mu - n\kappa_\mu S) + \kappa_\nu S_{*\mu} + \kappa_\mu S_{*\nu} - g_{\mu\nu} \kappa^\sigma S_{*\sigma}.$$

So far as $S_{\mu;\nu} = S_{\nu;\mu}$, then

$$S_{*\mu*\nu} - S_{*\nu*\mu} = -n(\kappa_{\mu;\nu} - \kappa_{\nu;\mu})S = -nF_{\mu\nu}S.$$

For a vector A_μ of power n we have

$$\begin{aligned} & A_{\mu*\nu*\sigma} = \\ & = A_{\mu*\nu;\sigma} - n\kappa_\sigma A_{\mu*\nu} + (g_\mu^\rho \kappa_\sigma + g_\sigma^\rho \kappa_\mu - g_{\mu\sigma} \kappa^\rho) A_{\rho*\nu} + (g_\nu^\rho \kappa_\sigma + g_\sigma^\rho \kappa_\nu - g_{\sigma\nu} \kappa^\rho) A_{\mu*\rho}. \end{aligned}$$

To get a curvature tensor, we calculate the difference between the derivatives of the vector A_μ

$$\begin{aligned} & A_{\mu*\nu*\sigma} - A_{\mu*\sigma*\nu} = \\ & = \left({}^*B_{\mu\nu\sigma\rho} + \frac{1}{2}(g_{\rho\nu} F_{\mu\sigma} + g_{\mu\sigma} F_{\rho\nu} - g_{\rho\sigma} F_{\mu\nu} - g_{\mu\nu} F_{\rho\sigma}) \right) A^\rho - (n-1)F_{\nu\sigma} A_\mu. \end{aligned}$$

The tensor ${}^*B_{\mu\nu\sigma\rho}$ has the conformal weight 2 and symmetries under permutations of the indices

$${}^*B_{\mu\nu\sigma\rho} = -{}^*B_{\mu\sigma\nu\rho} = -{}^*B_{\rho\nu\sigma\mu} = {}^*B_{\nu\mu\rho\sigma},$$

and also

$${}^*B_{\mu\nu\sigma\rho} + {}^*B_{\mu\sigma\rho\nu} + {}^*B_{\mu\rho\nu\sigma} = 0.$$

It can be called the Riemann tensor of the Weyl space. The Ricci tensor is obtained by contraction of the Riemann tensor by indices

$${}^*B_{\mu\nu} = {}^*B_{\mu\sigma\nu}^\sigma = R_{\mu\nu} + \kappa_{\mu;\nu} + \kappa_{\nu;\mu} + g_{\mu\nu} \kappa_{;\sigma}^\sigma + 2\kappa_\mu \kappa_\nu - 2g_{\mu\nu} \kappa^\sigma \kappa_\sigma.$$

It has conformal weight equal to zero. Contracting once more, we obtain a curvature

$${}^*R = {}^*R_\sigma^\sigma = R + 6\kappa_{;\sigma}^\sigma - 6\kappa^\sigma \kappa_\sigma,$$

which is the scalar of power -2 .

The action of scalar - tensor theory of gravitation is proposed to be taken as a

$$W = \int d^4x \sqrt{-g} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \beta^2 R + 6\beta^{;\mu} \beta_{;\mu} + c\beta^4 \right),$$

where β is a scalar field, c is a constant, and the first term is the contribution from the electromagnetic field. Introduced by Dirac, the scalar field has been called a *dilaton* [17] that means *expansion* because of the dilaton D plays the role of a very cosmological scale factor as a parameter of evolution in the space of field degrees of freedom, where the motion of the Universe is given. Unlike the standard General Relativity, the Dirac's dilaton does not *expand* lengths, but increases masses.

3.6 Affine group $A(4) = L(3, 1) \dot{\otimes} T(4)$

Affine group $A(4) = L(3, 1) \dot{\otimes} T(4)$ consists of all linear transformations of the space - time:

$$x'_\mu = a_{\mu\nu} x_\nu + c_\mu.$$

Affine group is a semidirect product of the group $L(3, 1)$ and the translation group and contains the Poincaré group as a subgroup. Algebra of generators of the affine group consists of four translations P_μ , six generators of the Lorentz group $M_{\mu\nu}$ and ten generators of properly affine transformations $R_{\mu\nu}$

$$R_{\mu\nu} = -i(x_\mu \partial_\nu - x_\nu \partial_\mu),$$

together with dilatations, it has a form:

$$\begin{aligned}
 [M_{\mu\nu}, M_{\rho\tau}] &= \imath(g_{\mu\rho}M_{\nu\tau} - g_{\mu\tau}M_{\nu\rho} - g_{\nu\rho}M_{\mu\tau} + g_{\nu\tau}M_{\mu\rho}), \\
 [M_{\mu\nu}, R_{\rho\tau}] &= \imath(g_{\mu\rho}R_{\nu\tau} + g_{\mu\tau}R_{\nu\rho} - g_{\nu\rho}R_{\mu\tau} - g_{\nu\tau}R_{\mu\rho}), \\
 [R_{\mu\nu}, R_{\rho\tau}] &= \imath(g_{\mu\rho}M_{\nu\tau} + g_{\mu\tau}M_{\nu\rho} + g_{\nu\rho}M_{\mu\tau} + g_{\nu\tau}M_{\mu\rho}), \\
 [M_{\mu\nu}, P_\rho] &= \imath(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu), \\
 [R_{\mu\nu}, P_\rho] &= \imath(g_{\mu\rho}P_\nu + g_{\nu\rho}P_\mu).
 \end{aligned} \tag{3.27}$$

In the vector representation, the generators $M_{\mu\nu}$ and $R_{\mu\nu}$ are defined as

$$(M_{\mu\nu})_{\alpha\beta} = -\imath(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}), \quad (R_{\mu\nu})_{\alpha\beta} = -\imath(g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha}).$$

The self-linear and self-conformal transformations do not correspond to the main conservation laws. Therefore, these symmetries must be dynamic ones, spontaneously broken.

3.7 Group of general coordinate transformations

The Einstein's theory of general relativity is covariant under general coordinate transformations

$$x'_\mu = f_\mu(x_0, x_1, x_2, x_3),$$

where $f_\mu(x)$ are arbitrary smooth functions of coordinates. The group of transformations is an infinite-parameter one. The action of the

group can be reduced to alternating actions of its two finite-parameter subgroups: the special linear group $SL(3, 1)$ and the conformal group $SO(4, 2)$.

According to *Ogievetsky's theorem* [18], the *invariance, under the infinite-parameter generally covariant group, is equivalent to simultaneous invariance under the affine and the conformal group.*

The proof of the theorem is based on the note, that infinite - dimensional algebra of general transformation of coordinates is the closure of the finite-dimensional algebras: $sl(3, 1)$ and $so(4, 2)$. The generator of special conformal transformations in the coordinate space (3.15) is quadratic in the coordinates. The result of its commuting with the generator $-ix_\mu\partial_\nu$ is again quadratic in x . Then, commuting the resulting operators with one another, we arrive at operators of the third degree in x , *et cetera*. In this way, step by step, we get all generators of the group of arbitrary smooth transformations of coordinates

$$\delta x_\mu = f_\mu(x),$$

the parameters of which are coefficients of expanding of functions $f_\mu(x)$ in series by powers of coordinates. The algebra of this group has infinite number of generators

$$L^{n_0 n_1 n_2 n_3} = -ix_0^{n_0} x_1^{n_1} x_2^{n_2} x_3^{n_3} \partial_\mu,$$

where

$$n = n_0 + n_1 + n_2 + n_3.$$

Thereby, there is a new approach where the formulation of the theory of gravity on the basis of finite-parametric groups is essentially simpler

than on the basis of the group of arbitrary coordinate transformations.

3.8 Fundamental elements of base Minkowskian space \mathbb{M}

We associate a vector $x = (x^0, x^1, x^2, x^3)$ of the space \mathbb{M} with the Hermitian matrix (2×2) using quaternions:

$$X = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} = x^0 I_2 + \sum_{i=1,2,3} x^i \sigma_i,$$

where I_2 is a unit matrix (2×2), and σ_i are matrices of Pauli. On the light cone, where $\det X = 0$, this matrix can be presented as a direct product of a two-dimensional column $Q = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ to a complex - conjugated line $Q^+ = (\bar{\xi}, \bar{\eta})$

$$\frac{X}{\sqrt{2}} = Q \otimes Q^+ = \begin{pmatrix} \xi \bar{\xi} & \xi \bar{\eta} \\ \eta \bar{\xi} & \eta \bar{\eta} \end{pmatrix},$$

where ξ, η are two complex numerics. Thus, the Lorentz group can be described by the spinor language.

Analogously, fundamental elements of the base Minkowskian space-time of \mathbb{M} , on which relativistic fields were built, were introduced by Roger Penrose and named by him *twistors* [19]. Points of the space-time are represented by two-dimensional linear subspaces of the four-dimensional complex vector (twistor) space, on which Hermitian form of signature $(+ + --)$ is defined.

Then the matrix X can be associated with the matrix (4×2) : $\begin{pmatrix} \iota X \\ I_2 \end{pmatrix}$, where I_2 is a unit matrix. Now we consider a two-dimensional plane in a complex space \mathbb{C}^4 , spanned on two four-dimensional column - vectors of the matrix. The obtained two-dimensional complex plane is the image of the point $x \in \mathbb{M}$ in the complexified space-Grassmannian \mathbb{CM} . Twistors themselves are the elements of the fundamental representation of the group $SU(2, 2)$. A twistor Z^α with components (Z^0, Z^1, Z^2, Z^3) belongs to \mathbb{C}^4 :

$$(Z^0, Z^1, Z^2, Z^3) \in \mathbb{C}^4.$$

3.9 Summary

Is it possible for the classification of modern observational data (within the concept of quantum relativistic Universe) to identify the wave function of the Universe with some unitary irreducible representation of any finite-parametric symmetry group? To answer this question in this Chapter candidates for the role of such symmetry were considered: 15-parametric group of conformal transformations and 20-parametric group of affine transformations as natural extensions of the Poincaré group. We would like to recall that the 20-parametric group of affine transformations of the coordinates of the Minkowskian space includes 4 shifts, 6 Lorentz (antisymmetric) transformations and 10 properly affine (symmetric) transformations. The fundamental representation of the conformal group, called twistors, allows us to suggest that the space-time on the light cone, as the adjoint representation of the con-

formal group, consists of more elementary elements – twistors, just as in the theory of strong interactions mesons consist of quarks. In the following chapters we will assume that the analogy of the theory of gravity to the theory of strong interactions has deeper roots, and construct a theory of gravitation as nonlinear realization of affine and conformal symmetries in the image and likeness of building of chiral phenomenological Lagrangians, that were successfully operated for description of the experimental low-energy data of meson physics.

Bibliography

- [1] Wigner, E.P.: *On unitary representations of the inhomogeneous Lorentz group*. Ann. of Math. **40**, 149 (1939)
- [2] Barut, A., Rączka, R.: *Theory of Group Representations and Applications*. World Scientific Publishing Co. (1986)
- [3] Andreev, A.Yu., Kirzhnits, D.A.: *Tachyons and the instability of physical systems*. Physics–Uspekhi. **39**, 1071 (1996)
- [4] Dirac, P.A.M.: *Fixation of coordinates in the Hamiltonian theory of gravitation*. Phys. Rev. **114**, 924 (1959)
- [5] Pervushin, V.N.: *On the vacuum in the gauge theories*. Riv. del Nuovo Cimento. **8**, 1 (1985)
- [6] Dubrovin, B.A., Novikov, S.P., Fomenko, A.T.: *Modern Geometry: Methods and Applications*. Nauka, Moscow (1986)
- [7] Mack, G., Salam, A.: *Finite-component field representations of the conformal group*. Ann. of Phys. **53**, 174 (1969)

-
- [8] Salam, A., Strathdee, J.: *Non-linear realization – II: conformal symmetry*. Int. Center for Theor. Phys. IC/68/107. Miramare – Trieste (1968)
- [9] Hermann, R.: *Lie Groups for Physicists*. W.A. Benjamin, New York (1966)
- [10] Weyl, H.: *Gravitation und elektrizität*. Sitzungsber. d. Berl. Akad., 465 (1918)
- [11] Bergmann, P.G.: *Introduction to the Theory of Relativity*. Dover Publications (1976)
- [12] Friedmann, A.A.: *World as Space and Time*. Nauka, Moscow (1965)
- [13] Dirac, P.A.M.: *A new basis for cosmology*. Proc. Roy. Soc. London. **A 165**, 199 (1938)
- [14] Deser, S.: *Scale invariance and gravitational coupling*. Annals Phys. **59**, 248 (1970)
- [15] Brans, C., Dicke, R.H.: *Mach's principle and a relativistic theory of gravitation*. Phys. Rev. **124**, 925 (1961)
- [16] Dirac, P.A.M.: *Long range forces and broken symmetries*. Proc. Roy. Soc. London. **A 333**, 403 (1973)
- [17] Pawlowski, M., Papoyan, V.V., Pervushin, V.N., Smirichinski, V.I.: *Conformal unification of General Relativity and Standard*

Model for strong and electroweak interactions. Phys. Lett. **B 418**, 263 (1998)

[18] Ogievetsky, V.I.: *Infinite-dimensional algebra of general covariance group as the closure of finite-dimensional algebra of conformal and linear groups.* Lett. Nuovo Cimento. **8**, 988 (1973)

[19] Penrose, R.: *The twistor programme.* Reports on Math. Phys. **12** (1), 65 (1977)

Chapter 4

Nonlinear realizations of symmetry groups

4.1 Differential forms of Cartan

A space of affine connection is built in the following way [1]. Let us consider n -dimensional manifold. In every point

$$M(x^1, x^2, \dots, x^n)$$

we define an affine frame by n linear independent vectors $\mathbf{I}_i(M)$, $i = 1, \dots, n$ and consider it to be imbedded to n -dimensional affine space A_n . The space has with our manifold a mutual point M and mutual vectors in the point M . Any vector ξ in point M can be expanded by vectors of the frame $\xi = \xi^k \mathbf{I}_k(M)$. The manifold is said to be a *space of affine connection* if the affine correspondence between local affine spaces A_n and A'_n , attached to infinitely close points

$$M(x^1, x^2, \dots, x^n), \quad M'(x^1 + dx^1, x^2 + dx^2, \dots, x^n + dx^n)$$

of our manifold is set.



Élie Joseph Cartan (9 April 1869–6 May 1951), a famous French mathematician. The main theme of his works was the theory of Lie groups. He worked over the foundational material on the complex simple Lie algebras. This proved definitive, as far as the classification went, with the identification of the four main families and the five exceptional cases. He defined the general notion of anti-symmetric differential form, in the style used now. Cartan added the exterior derivative, as an entirely geometric and coordinate-independent operation. With these basics—Lie groups and differential forms—he went on to produce a very large body of work, and also some general techniques, such as moving frames that were gradually incorporated into the mathematical mainstream.

This correspondence can be set by pointing out the place for the frame of space A'_n to be placed after mapping of A'_n to A_n . The vector of displacement of point $\overrightarrow{MM'} = d\mathbf{r}$ we expand by basis

$$d\mathbf{r} = \omega^k \mathbf{I}_k(M). \quad (4.1)$$

The mapped basis vectors $\mathbf{I}_i(M')$ differ infinitely from $\mathbf{I}_i(M)$, so

$$\mathbf{I}_i(M') = \mathbf{I}_i(M) + d\mathbf{I}_i,$$

and also $d\mathbf{I}_i$ we expand by basis vectors:

$$d\mathbf{I}_i = \omega_i^k \mathbf{I}_k(M). \quad (4.2)$$

Coefficients of expanding ω^k and ω_i^k define the affine connection. They depend on the choice of points M, M' , so they are expressed as linear differential forms $\omega^k(d)$ and $\omega_i^k(d)$ of coordinates. If in the space of affine connection a curve $x^i = x^i(t)$ is set, the coefficients ω^k, ω_i^k are represented as functions of the parameter t , multiplied by dt . Let us integrate the system of differential equations (4.1) for unknown vector functions \mathbf{r} and \mathbf{I}_i . Initial conditions in the initial point of the path under $t = 0$: $\mathbf{r} = 0, \mathbf{I}_i = \mathbf{I}_i^0$. As a result of integration, \mathbf{r}, \mathbf{I}_i will be vector functions of t in the space A_n^0 of the point M_0 and give the affine mapping of the space A_n at the arbitrary point of the path $M(t)$ to A_n^0 .

An especially interesting case occurs when a path is closed and we return to the initial point M_0 . Then we get a mapping of the space A_n^0 to itself. The group derived at the A_n^0 by these affine mappings is called a *holonomy group* of the space of affine connection. For recognition of characteristics of the considered geometry, let us consider an infinitely small transformation of holonomy group, corresponding to infinitely small closed path of going around. Let the cycle be as a small parallelogram. There are two differential Cartan forms

$$\omega(d) = a_i dx^i, \quad \theta(\delta) = b_i \delta x^i.$$

An external product of these forms ω and θ is called the anti-symmetric product

$$\omega(d) \wedge \theta(\delta) \equiv \omega(d)\theta(\delta) - \theta(d)\omega(\delta) = (a_i b_k - b_i a_k) dx_i \delta x_k.$$

An external differential of the form ω is the following expression

$$\omega' \equiv d\omega(\delta) - \delta\omega(d) \equiv da_i \delta x^i - \delta a_i dx^i = \left(\frac{\partial a_i}{\partial x^k} - \frac{\partial a_k}{\partial x^i} \right) dx^k \delta x^i.$$

An ordinary differentiation corresponds to the shift along one of coordinate axes, whereas the external differentiation corresponds to passing along the closed infinitely small cycle. In case the Cartan form is full differential, the external differentiation of this form is zero identically (Poncaré's lemma).

Cartan's equations of structure are obtained by external differentiation of equations (4.1) and (4.2). For the Euclidean space

$$0 = d\mathbf{I}_i \wedge \omega^i(\delta) + \mathbf{I}_i(\omega^i)',$$

$$0 = d\mathbf{I}_j \wedge \omega_j^j(\delta) + \mathbf{I}_j(\omega_j^j)'.$$

Substituting (4.2) into the obtained equations, we find

$$0 = \mathbf{I}_i \left((\omega^i)' + \omega_j^i \wedge \omega^j \right),$$

$$0 = \mathbf{I}_j \left((\omega_j^j)' + \omega_k^j \wedge \omega_i^k \right).$$

From the linear independence of \mathbf{I}_i the structural equations of Euclidean space are the following:

$$(\omega^i)' + \omega_j^i \wedge \omega^j = 0,$$

$$(\omega_i^j)' + \omega_k^j \wedge \omega_i^k = 0.$$

In the Euclidean space, the frame connected with point M is not changed in the contour path.

In the general case of the Riemannian geometry, a frame under the infinitely small contour undergoes displacement

$$(\omega^i)' + \omega_j^i \wedge \omega^j = \Omega^i,$$

$$(\omega_i^j)' + \omega_k^j \wedge \omega_i^k = \Omega_i^j.$$

Additional movement to return the frame into the initial place defines a torsion and a curvature of the Riemannian space. The torsion is set by the shift

$$\Omega^i = -\frac{1}{2}T_{jk}^i \omega^j \wedge \omega^k,$$

in order to return the origin of the frame into the initial placement, and the curvature of the Riemannian space – by additional rotation of the frame into the initial placement by the value

$$\Omega_i^j = -\frac{1}{2}R_{ikl}^j \omega^k \wedge \omega^l.$$

Here T_{jk}^i —the tensor of torsion, and R_{ikl}^j —the tensor of curvature.

4.1.1 Riemannian coordinates

Let us consider the Riemannian space with zero torsion $T = 0$, $R \neq 0$. Any point M in sufficiently small area of the point O lies in the definite geodesic line with origin O . Let a^i be the direction cosines of its tangent line in O and t – a length of geodesic line OM . Then normal coordinates of point M are called n values, defined by equations

$$x^i = a^i t, \quad 0 \leq t \leq 1.$$

In point O the orthogonal frame is set. In every point M in the area of point O the orthogonal frame is set by parallel transfer along the geodesic curve OM . We find the forms ω^i, ω_i^j , setting infinitesimal displacement and rotation under transition from the frame in point M to the frame connected to infinitesimally close point M' . We use variables a^i, t , assuming finally $t = 1, a^i = x^i$. If we put $a^i = const$ and change t , the frame transfers in a parallel way

$$\omega^i = a^i dt; \quad \omega_i^j \equiv 0.$$

We define $\bar{\omega}^i, \bar{\omega}_i^j$ as values of the form ω^i and ω_i^j under $dt = 0$ and changing a^i . Then

$$\omega^i(t, a^i; dt, da^i) = a^i dt + \bar{\omega}^i(t, a^i; da^i); \quad (4.3)$$

$$\omega_i^j(t, a; dt, da) = \bar{\omega}_i^j(t, a^i; da^i). \quad (4.4)$$

Now let us define the form $\bar{\omega}^i, \bar{\omega}_i^j$ as functions of t , consider a^i, da^i as parameters. The initial point of the following reasonings are equations of structure of space with zero torsion, which Cartan named as *fundamental*:

$$(\omega^i)' = \omega^k \wedge \omega_k^i; \quad (4.5)$$

$$(\omega_i^j)' = \omega_i^k \wedge \omega_k^j - \frac{1}{2} R_{ikl}^j \omega^k \wedge \omega^l. \quad (4.6)$$

Substituting here ω^i, ω_i^j with expressions (4.3), (4.4) and separating terms containing dt , one gets

$$[da^i, dt] + [dt, \frac{\partial \bar{\omega}^i}{\partial t}] + d_a \bar{\omega}^i = [a^k dt + \bar{\omega}^k, \bar{\omega}_k^i];$$

$$[dt, \frac{\partial \bar{\omega}_i^j}{\partial t}] + d_a \bar{\omega}_i^j = [\bar{\omega}_i^k, \bar{\omega}_k^j] - \frac{1}{2} R_{ikh}^j [a^k dt + \bar{\omega}^k, a^h dt + \bar{\omega}^h],$$

where d_a denotes differentiation by all a^i under condition $t = const.$ Comparing terms with multiplier dt , we get equations:

$$\frac{\partial \bar{\omega}^i}{\partial t} = da^i + a^k \bar{\omega}_k^i \quad (4.7)$$

$$\frac{\partial \bar{\omega}_i^j}{\partial t} = -\frac{1}{2} R_{ikh}^j (a^k \bar{\omega}^h - a^h \bar{\omega}^k) \quad (4.8)$$

The equations (4.7), (4.8) solved by iterations in the parameter t with the boundary conditions. We obtain the solutions with an accuracy $o(t^3)$

$$\bar{\omega}^i = t da^i - \frac{1}{6} t^3 R_{jkl}^i a^j a^k da^l,$$

$$\bar{\omega}_j^i = -\frac{1}{2} t^2 R_{jkl}^i a^k da^l.$$

Under $t = 1$: $\bar{\omega}^i = \omega^i$, $\bar{\omega}_j^i = \omega_j^i$, so the interval

$$ds^2 = \eta_{ij} \omega^i \omega^j = g_{ij} dx^i dx^j,$$

where the metric

$$g_{ij} = \eta_{ij} + \frac{1}{3} R_{ijkl} x^k x^l,$$

and

$$\omega_j^i = -\frac{1}{2} R_{jkl}^i x^k dx^l.$$

Then we can continue iterations and obtain the solutions with an accuracy $o(t^5)$

$$\bar{\omega}^i = t da^i - \frac{1}{3!} t^3 (R_{nlk}^i a^n a^l) da^k + t^5 \frac{1}{5!} (R_{nlk_1}^i a^n a^l) (R_{n_1 l_1 k_2}^{k_1} a^{n_1} a^{l_1}) da^{k_2},$$

$$\bar{\omega}_j^i = -\frac{1}{2!} t^2 R_{jkl}^i a^k da^l + t^4 \frac{1}{4!} (R_{jlk_1}^i a^l) (R_{n_1 l_1 k_2}^{k_1} a^{n_1} a^{l_1}) da^{k_2}.$$

The solutions of these equations under $t = 1$, $a^i = x^i$ have the following form:

$$\omega^i(x, dx)|_{t=1} = \sum_1^{\infty} (m^n)_k^i \frac{(-1)^n}{(2n+1)!} dx^k; \quad (4.9)$$

$$\omega_j^i(x, dx)|_{t=1} = - \sum_0^{\infty} (m^n)_p^l \frac{(-1)^n}{(2n+2)!} dx^p R_{jkl}^i x^k, \quad (4.10)$$

where

$$m_k^i \equiv R_{mlk}^i x^n x^l; \quad (m^2)_j^i \equiv m_{k_1}^i m_j^{k_1}.$$

Symbolically, the expressions (4.9), (4.10) are able to be written down shorter:

$$\omega^i(x, dx) = (\sin \sqrt{m}/\sqrt{m})_k^i dx^k; \quad (4.11)$$

$$\omega_j^i(x, dx) = R_{jkl}^i x^k [(1 - \cos \sqrt{m})/m]_p^l dx^p. \quad (4.12)$$

For the Euclidean space $R_{jkl}^i \equiv 0$ and the Cartan forms in the normal coordinates:

$$\omega^i(x, dx) = dx^i; \quad \omega_j^i \equiv 0.$$

The square interval of length between infinitely close points is defined by the expression [2]

$$ds^2 = \omega^i(x, dx)\omega_i(x, dx) \equiv g_{ab}(x)dx^a dx^b \quad (4.13)$$

in accordance with geometric sense of the form ω^i . The group of transformations of space that keeps the quadratic form (4.13) invariant, is called the *movement group of the Riemannian space*.

Let us state connection with standard concepts of differential geometry—metric tensor and symbols of Christoffel. For this, we should pass to

natural frames \mathbf{N}_a :

$$(\mathbf{N}_a, \mathbf{N}_b) = g_{ab}; \quad \mathbf{I}_i = e_i^a(x)\mathbf{N}_a; \quad e_i^a e_a^j = \delta_{ij}; \quad e^{ai} = e_i^a,$$

where

$$d\mathbf{r} = \mathbf{N}_a(M)dx^a, \tag{4.14}$$

$e_i^a(x)$ are coefficients of decompositions of Cartan's forms $\omega^i(x, dx)$ by differentials dx^a :

$$\omega^i(x, dx) = e_a^i(x)dx^a.$$

Laws of changing of an arbitrary vector \mathbf{A} on the natural basis have form

$$\begin{aligned} d(A^i \mathbf{I}_i) &= (dA^i + A^j \omega_j^i) \equiv d(A^a \mathbf{N}_a) = dA^a N_a + A^a d(e_a^i I_i) = \\ &= [dA^b + A^a (de_a^i e_i^b + e_a^i \omega_j^i e_i^b)] N^j = [dA^b + A^a \Gamma_{ac}^b dx^c] N^j, \end{aligned}$$

where

$$\Gamma_{ac}^b dx^c = (e_i^b de_a^i + e_a^b \omega_j^i e_i^j).$$

4.2 Geometry of Lie groups

Let us consider some finite continuous group G , dependent of $n + r$ parameters $a_1, a_2, \dots, a_n; \eta_1, \eta_2, \dots, \eta_r$. One can consider parameters a, η as coordinates of point A in $n + r$ -dimensional space called *group space*. Latin indices are used for notifications of values connected with generators X_i , and Greek ones—with generators Y_α . Here a^k, η^α are parameters of the group; Y_α are generators of transformations that belong to subgroup H ; X_k are generators that complement H to the

full group G , *id est* generators of factor-space G/H . These generators obey the algebraic commutation conditions

$$[Y_\alpha, Y_\beta] = \imath C_{\alpha\beta}^\gamma Y_\gamma; \quad [X_k, Y_\alpha] = \imath C_{k\alpha}^i X_i; \quad [X_i, X_k] = \imath C_{ik}^\alpha Y_\alpha$$

Generators of the group can be considered as analog of basis vectors of Cartesian frame placed to the origin of coordinates. Definition of equality of vectors in the group space allows one to introduce a transformation corresponding to infinitesimal vector with origin in the arbitrary point of the space (a, η) .

Every point of this space $A(a, \eta)$ assigns a transformation of group $G_{a,\eta} = G_A$ and vice versa. Let us agree to name a point corresponding to identical transformation as the initial point of space. A pair of points set a vector. A base point for investigation of geometry of group space is the definition of equality of vectors. We shall say that two vectors A_1A_2 and B_1B_2 are equal, if an element G_{A_1} maps to G_{A_2} and an element G_{B_1} maps to G_{B_2} by one and the same transformation $G(g)$, acting by the rule:

$$G(g)G_{A_1} = G_{A_2}; \quad G(g)G_{B_1} = G_{B_2}.$$

From here we get

$$G_{A_2}G_{A_1}^{-1} = G_{B_2}G_{B_1}^{-1}.$$

Any point infinitely close to the initial point is analytically defined by infinitesimal transformation of the group. Any infinitesimal transformation is expressed linearly by $n + r$ generators X_k, Y_α :

$$G_{(da^k, d\eta^\alpha)} = I + dG_{(a^k, \eta^\alpha)}$$

$$G_{(da^k, \eta^\alpha)} = \iota[da^k X_k + d\eta^\alpha Y_\alpha]; \quad k = 1, \dots, n; \quad \alpha = 1, \dots, r. \quad (4.15)$$

A vector $(0, 0; da', d\eta')$ is equal to a vector $(a, \eta; a + da, \eta + d\eta)$, if a point $(0, 0)$ maps to a point $da', d\eta'$, a point (a, η) maps to a point $(a + da, \eta + d\eta)$ by one and the same transformation

$$G(g) \equiv G(a + da, \eta + d\eta)G^{-1}(a, \eta) = G(d\eta', da')G^{-1}(0, 0) = G(d\eta', da').$$

It means that with every point of the space (a, η) it is possible to connect such Cartesian frame, equal (in the group sense) to the frame connected with the point of origin. A vector $(a, \eta; a + da, \eta + d\eta)$ has the same analytic expression as (4.15). Denoting

$$da'^i = \omega^i(a, \eta; da, d\eta), \quad d\eta'^\alpha = \theta^\alpha(a, \eta; da, d\eta),$$

we get

$$G_{(a, \eta)} dG_{(a, \eta)}^{-1} = \iota(\omega^i X_i + \theta^\alpha Y_\alpha). \quad (4.16)$$

4.2.1 Structural equations

Let f be a function of variables in the space of irreducible representations of some group. An infinitesimal action of elements of the group on the function f has a form

$$df = \iota[\omega^i(d)X_i + \theta^\alpha(d)Y_\alpha]f. \quad (4.17)$$

Let us build a bilinear differential

$$\delta df = \iota[\delta\omega^i(d)X_i + \delta\theta^\alpha(d)Y_\alpha]f + \iota[\omega^i(d)\delta(X_i f) + \theta^\alpha(d)\delta(Y_\alpha f)],$$

where the differential of functions $X_i f$ and $Y_\alpha f$ is defined according to (4.17). An action of external differential to the left side of the equality

(4.17) results to zero:

$$(df)' = \delta df - d\delta f = 0.$$

Equating the coefficients of similar linear independent generators in (4.17) leads to the system of the structural equations:

$$\begin{aligned} (\omega^i)' &= C_{k\beta}^i \omega^k \wedge \theta^\beta; \\ (\theta^\gamma)' &= \frac{1}{2} C_{\alpha\beta}^\gamma \theta^\alpha \wedge \theta^\beta + \frac{1}{2} C_{ki}^\gamma \omega^k \wedge \omega^i; \end{aligned}$$

where

$$\begin{aligned} (\omega^k)' &= -\delta \omega^k(d) + d\omega^k(\delta); \\ \omega^k \wedge \theta^\beta &= \omega^k(d)\theta^\beta(\delta) - \omega^k(\delta)\theta^\beta(d). \end{aligned}$$

Having obtained the equations, one can pass to such structural equations where there is dependence only on Latin indices connected with the space of parameters a^i of coset G/H . For this, we define new differential forms

$$\omega_k^i = C_{k\beta}^i \theta^\beta \quad (4.18)$$

and use the Jacobi identities leading to

$$C_{\alpha\beta}^k C_{jk}^l = C_{\beta k}^l C_{\alpha j}^k - C_{\alpha k}^l C_{\beta j}^k.$$

Finally we get for the differential forms ω^i , ω_k^i the following equations

$$(\omega^i)' = \omega^k \wedge \omega_k^i, \quad (4.19)$$

$$(\omega_j^i)' = -\frac{1}{2} R_{jkl}^i \omega^k \wedge \omega^l + \omega_j^k \wedge \omega_k^i, \quad (4.20)$$

where

$$-R_{jki}^i = C_{j\gamma}^i C_{kl}^\gamma;$$

and dependence is left only on the Latin indices of the coset space.

The equations (4.19), (4.20) coincide in form with structural equations of Cartan for the Riemannian n -dimensional space with nonzero curvature. Further, we consider only a group space of parameters a^i , setting the parameters equal to zero: $\eta_\alpha = 0$. One can treat the differential forms ω^i as components of infinitesimal shift of the frame origin relatively from the frame in the point a , and ω_j^i as changing of the frame components. According to such geometric interpretation of the forms ω^i and θ^γ as shift and rotation, it is natural to consider that transformation of group G is a rotation if it belongs to subgroup H , and a shift if it is originated by infinitesimal transformation $\omega^i X_i$. The subgroup H of transformations leaves fixed the origin of the frame of group space and is named as *stationary subgroup of the space*. One can get realizations of the transformations representing a common group transformation G in the form of multiplication

$$G = K(a)H(\eta), \quad (4.21)$$

where $K(a)$ is a transformation that belongs to the left coset G/H of group G by subgroup H . Acting from the left on the element of group G by arbitrary $G(g)$ and factorizing the obtained element according to (4.21):

$$G(g)K(a)H(\eta) = K(a'(a, g))H(\eta'(\eta, a, g)), \quad (4.22)$$

one can define in which manner the parameters a, η are being transformed. A parametrization of $K(a)$, or, in other words, the explicit form of finite group transformations, can be quite arbitrary. It corresponds to arbitrary motions of frames in the differential geometry of

Cartan. Every parametrization K is equivalent to definite choice of coordinates in the coset G/H .

4.2.2 Exponential parametrization

Let us consider the exponential parametrization of groups

$$K(a) = \exp(\iota a^j X_j). \quad (4.23)$$

We define explicitly the Cartan forms and infinitesimal transformations in this case. The equations for the differential forms are the following

$$\exp(-\iota X_k a^k) d [\exp(\iota X_k a^k)] = \iota [\omega^i(a, da) X_i + \theta^\alpha(a, da) Y_\alpha].$$

Let us introduce a parameter t to (4.23) using a substitution $a^k \rightarrow a^k t$:

$$\exp(-\iota X_k a^k t) d [\exp(\iota X_k a^k t)] = \iota [\omega^i(ta, tda) X_i + \theta^\alpha(ta, tda) Y_\alpha].$$

Differentiating by t , the left and the right sides of the obtained equality, we get the system of equations:

$$\begin{aligned} \frac{\partial \omega^i}{\partial t} &= da^i + a^k \theta^\beta C_{k\beta}^i \\ \frac{\partial \theta^\alpha}{\partial t} &= a^i \omega^l C_{il}^\alpha. \end{aligned}$$

After substitution (4.18) in terms of the forms ω^i, ω_j^i these equations coincide with the fundamental Cartan's equations (4.5), (4.6), which describe the motion of the frame along the geodesic lines and define Cartan's forms in the normal coordinates. Consequently, the exponential parametrization of the finite group transformations are equivalent to the choice of normal coordinates in coset G/H :

$$\omega^i(a, da) = (\sin \sqrt{m} / \sqrt{m})_k^i da^k;$$

$$\theta^\alpha(a, da) = [(1 - \cos \sqrt{m}/m)]_k^i da^k C_{ii}^\alpha a^l;$$

$$m_i^j = -C_{j\alpha}^i C_{ki}^\alpha a^j a^k.$$

4.3 Algebraic and dynamical principles of symmetry

According to Wigner [3], all groups of symmetry are divided into two classes: *algebraic symmetries* that reflect laws of conservations and are used for classification of free physical objects – particles and fields, universes and their quantum analogous, and *dynamical symmetries*¹, which allow one to define interactions between these objects, and also constrains of initial data and their quantization. Progress in understanding of the role and the essence of dynamical symmetries is connected to studying of spontaneous symmetry breaking phenomenon of vacuum. Firstly, the effects of spontaneous symmetry breaking phenomena were considered in the theory of many particles by N.N. Bogoliubov [4], in the relativistic theory — by Nambu [5] and Goldstone [6].

Symmetry under a group is said spontaneously broken if the vacuum of the system with an invariant Lagrangian as a state with minimal energy is stable only under transformations of subgroup H of the full group G . In such a case the subgroup H is an *algebraic* group of classifications of fields and particles of the theory. Spontaneous break-

¹By vivid expression of E. Wigner, (Wigner, E.: *Symmetries and Reflections*. Indiana University press, Bloomington – London (1970)), the algebraic symmetries belong to the area of *terra cognita*, and dynamical symmetries—to the area of *terra incognita*.

ing of symmetry of the vacuum is accompanied by creation of separate fields with zero mass, called as *Goldstone* fields (Bogoliubov's theorem in statistical physics and Goldstone's theorem in field theory).

In particular, in the theory of strong interactions in the capacity of *dynamical symmetry* the *chiral symmetry*² takes place. According to this symmetry, the strong interactions are invariant under actions of transformation groups, including those with isotopic transformations with algebra of generators

$$[I_i, I_j] = \imath \varepsilon_{ijk} I_k \quad (4.24)$$

also generators K_j with algebra

$$[I_i, K_j] = \imath \varepsilon_{ijk} K_k, \quad [K_i, K_j] = \imath \varepsilon_{ijk} I_k, \quad (4.25)$$

changing states with different parity. An example of linear representation of *chiral symmetry* is right and left neutrino. There is nonlinear realization of *chiral symmetry* – chiral phenomenological Lagrangians, with which the low-energy results in QCD during 1967–72 were obtained, before formulation of the theory of QCD in 1973–74. In the method of nonlinear realization of *chiral symmetry* with six parameters, three isotopic parameters belong to the subgroup H of vacuum stability, and three rest are proper chiral transformations, changing states with different parity. The latter three chiral parameters are identified with three *Goldstone fields*.

Shifts $\omega^i(\partial_\mu)$ and rotations $\theta^\alpha(\partial_\mu)$ describe various movements of orthogonal frames in the coset space. Chiral phenomenological La-

²Chira – in Greek is a hand which is traditionally used for illustration of right and left particle helicity.

grangians of fields interactions are built univalently in the coset $K = G/H$ out of these linear forms. These Lagrangians allow one to describe numerous processes in low-energy physics of hadrons in satisfactory agreement with experimental data [7, 8, 9, 10, 11]. In the method of phenomenological Lagrangians a dynamical group is a complete group G , an algebraic subgroup is its stationary subgroup H . Parameters a^i of coset space are identified with fields of Goldstone particles. Cartan forms ω^i, θ^α can be used to build invariants of groups, *id est* dynamical Lagrangians. It is natural to identify with the Lagrangian of Goldstone particles a^i a squared geodesic interval between two infinitely close points a^i and $a^i + da^i$. The invariant with respect to translations and rotations will be the expression

$$\mathcal{L} = \frac{1}{2} \omega^i(a, \partial_\mu a) \omega^j(a, \partial_\mu a) C_{il}^\alpha C_{\alpha j}^l,$$

where

$$C_{il}^\alpha C_{\alpha j}^l = g_{ij}$$

is the metric tensor of the group space.

Forms, describing the change of the basis, are used for definition the covariant differentiation of particle fields Ψ which interact with Goldstone fields

$$\mathcal{L}(\Psi, \partial_\mu \Psi + \theta^\alpha(a, \partial_\mu a) T_\alpha \Psi).$$

The fields Ψ are classified by irreducible representations T_α of a subgroup H .

We consider nonlinear realizations of group $A(4)$, which become linear on its subgroup – Poincaré group. Let us have a look more

closely at realization in coset space $A(4)/L$, where L is the Lorentz group. We take a symmetric tensor field $h_{\mu\nu}$ and define an action of an element of the group g :

$$g \exp(ix_\mu P_\mu) \exp\left(\frac{i}{2} h_{\mu\nu} R_{\mu\nu}\right) = \\ \exp(ix'_\mu P_\mu) \exp\left(\frac{i}{2} h'_{\mu\nu}(x') R_{\mu\nu}\right) \exp\left(\frac{i}{2} U_{\mu\nu}(x') L_{\mu\nu}\right),$$

where $x'_\mu, h'_{\mu\nu}(x')$ and $U_{\mu\nu}(x')$ depend of parameters of transformation g and field $h_{\mu\nu}$. Let Ψ be an arbitrary field which is a linear representation of the Lorentz group. Then, an action of the group $A(4)$ to the field Ψ is defined as

$$g\Psi = \Psi'(x) = \exp\left(\frac{i}{2} U_{\mu\nu}(h(x), g) L_{\mu\nu}^\Psi\right) \Psi,$$

where $L_{\mu\nu}^\Psi$ is a matrix generator in linear representation of the Lorentz group. Then arbitrary frame movements (shifts and rotations) in coset space $A(4)/L$ are described by Cartan's forms ω as coefficients of expansion of infinitesimal transformations of generators of algebra $A(4)$ (3.27):

$$\left[\exp\left(-\frac{i}{2} h_{\alpha\beta} R_{\alpha\beta}\right) \exp(-ix_\mu P_\mu) \right] d \left[\exp(ix_\mu P_\mu) \exp\left(\frac{i}{2} h_{\alpha\beta} R_{\alpha\beta}\right) \right] = \\ = GdG^{-1} = \underbrace{i[P_{(\alpha)} \cdot \omega_{(\alpha)}^P + R_{(\alpha)(\beta)} \cdot \omega_{(\alpha)(\beta)}^R]}_{\text{shifts } K=A(4)/L} + \underbrace{L_{(\alpha)(\beta)} \cdot \omega_{(\alpha)(\beta)}^L}_{\text{rotations } K=A(4)/L},$$

Forms

$$\omega_{(\alpha)}^P(d) = e_{(\alpha)\mu} dx^\mu, \quad (4.26)$$

$$\omega_{(\alpha)(\beta)}^R(d) = \frac{1}{2} \left(e_{(\alpha)}^\mu de_{(\beta)\mu} + e_{(\beta)}^\mu de_{(\alpha)\mu} \right), \quad (4.27)$$

$$\omega_{(\alpha)(\beta)}^L(d) = \frac{1}{2} \left(e_{(\alpha)}^\mu de_{(\beta)\mu} - e_{(\beta)}^\mu de_{(\alpha)\mu} \right). \quad (4.28)$$

define covariant differentials of coordinates and Goldstone fields and are used for definition of covariant differential of fields Ψ . Here $e_{(\alpha)\mu}$ are components of tetrads with two indexes. One index belongs to Riemann space μ , and the second (α) – to tangent Minkowski space. Components of tetrads are coefficients of expansion of Cartan forms by differentials of coordinate space.

For descriptions of fermions in Riemann space the Fock frame in tetrad formalism is used [12]. The action of fermion field is set as

$$W_{\text{matter}}[g, \Psi] = \int d^4x \sqrt{-g} [-\bar{\Psi} v_{\gamma(\beta)} D_{(\beta)} \Psi - m_0 \bar{\Psi} \Psi], \quad (4.29)$$

where

$$\gamma_{(\beta)} = \gamma^\mu e_{(\beta)\mu}$$

– Dirac γ -matrices, summarized with tetrads $e_{(\beta)\nu}$, and m_0 is a fermion mass at the present time. Covariant differentials of set of fields Ψ are defined by the formula

$$D_{(\gamma)} \Psi = \frac{D\Psi}{\omega_{(\gamma)}^P} = \left[\partial_{(\gamma)} + \frac{i}{2} v_{(\alpha)(\beta),(\gamma)} L_{(\alpha)(\beta)}^\Psi \right] \Psi, \quad (4.30)$$

where

$$\partial_{(\gamma)} = (e^{-1})_{\mu(\gamma)} \partial_\mu,$$

and

$$L_{(\alpha)(\beta)}^\Psi = [\gamma_{(\alpha)}, \gamma_{(\beta)}]$$

– are generators of the Lorentz group, a linear form $v_{(\alpha)(\beta),(\gamma)}$ is built by Cartan forms (4.27) (4.28):

$$v_{(\alpha)(\beta),(\gamma)} = \left[\omega_{(\alpha)(\beta)}^L(\partial_{(\gamma)}) + \omega_{(\alpha)(\gamma)}^R(\partial_{(\beta)}) - \omega_{(\beta)(\gamma)}^R(\partial_{(\alpha)}) \right]. \quad (4.31)$$

4.4 Theory of gravitation as nonlinear realization of $A(4) \otimes SO(4, 2)$



At the end of the 1950s - beginning of the 1960s of the last century, in close collaboration with I.V. Polubarinov, V.I. Ogievetsky obtained a number of pioneer results in the range of theory of fields treatment of gauge theories and gravitation. The very bright achievement was the new comprehension of the theory of gravitation as nonlinear realization of two spontaneously broken space-time symmetries – conformal and affine ones, and a graviton as corresponding to the Goldstone particle. Until the last years of his life he was a chief of the “Supersymmetry” sector in the Laboratory of Theoretical Physics named after of N.N. Bogoliubov, JINR.

In this section we follow the paper [13]. Using an analogy with phenomenological chiral Lagrangians [11], it is possible to obtain phenomenological affine Lagrangian as nonlinear joint realization of affine and conformal symmetry groups. A nonlinear realization of the affine group leads to a symmetric tensor field as a Goldstone field. The requirement that the theory correspond simultaneously to a realization of the conformal group as well leads uniquely to the theory of a tensor field whose equations are Einstein’s. Such nonlinear joint realization

was constructed in [13].

4.4.1 Nonlinear realizations of the affine group

We consider nonlinear realizations of group $A(4)$, which become linear on its subgroup – Poincaré group. Let us have a look more closely at realization in coset space $A(4)/L$, where L is the Lorentz group. We take a symmetric tensor field $h_{\mu\nu}$ and define an action of an element of the group g :

$$\begin{aligned} g \exp(ix_\mu P_\mu) \exp\left(\frac{i}{2} h_{\mu\nu} R_{\mu\nu}\right) &= \\ &= \exp(ix'_\mu P_\mu) \exp\left(\frac{i}{2} h'_{\mu\nu}(x') R_{\mu\nu}\right) \exp\left(\frac{i}{2} u_{\mu\nu}(h, g) L_{\mu\nu}\right), \end{aligned}$$

where x'_μ are the transformed coordinates, $h'_{\mu\nu}(x')$ is the transformed field $h_{\mu\nu}(x)$, and $u_{\mu\nu}(h(x), g)$ depend of parameters of transformation g and the field $h_{\mu\nu}$. Let Ψ be an arbitrary field which is a linear representation of the Lorentz group. Then, an action of the group $A(4)$ to the field Ψ is defined as

$$g : \Psi = \Psi'(x') = \exp\left(\frac{i}{2} u_{\mu\nu}(h(x), g) L_{\mu\nu}^\Psi\right) \Psi, \quad (4.32)$$

where $L_{\mu\nu}^\Psi$ is a matrix generator in linear representation of the Lorentz group. The generator is for a scalar

$$L_{\mu\nu}^\Psi = 0,$$

for a spinor

$$L_{\mu\nu}^\Psi = \frac{1}{2} \sigma_{\mu\nu},$$

for a vector

$$(L_{\mu\nu}^{\Psi})_{\alpha\beta} = -\iota(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\nu\alpha}\delta_{\mu\beta}),$$

et cetera. The corresponding infinitesimal transformations are given by

$$\delta\Psi(x) = \Psi'(x) = \Psi'(x') - \Psi(x),$$

and for three above considered cases:

$$\delta\varphi(x) = 0,$$

$$\delta\Psi(x) = -\frac{\iota}{4}u_{\mu\nu}(h(x), g)\sigma_{\mu\nu}\Psi(x),$$

$$\delta a_{\mu}(x) = u_{\mu\nu}(h(x), g)a_{\nu}(x).$$

The Poincaré group is presented by standard linear transformations. For shifts we have

$$g = e^{\iota c_{\mu}P_{\mu}}, \quad u_{\mu\nu} = 0, \quad x'_{\mu} = x_{\mu} + c_{\mu}, \quad h'_{\mu\nu}(x') = h_{\mu\nu}, \quad \Psi'(x') = \Psi(x).$$

We introduce in the vector representation the quantity

$$r_{\mu\nu}(x) = \left(\exp\left(\frac{\iota}{2}h_{\alpha\beta}R_{\alpha\beta}\right) \right)_{\mu\nu} = (e^h)_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu} + \frac{1}{2}h_{\mu\sigma}h_{\sigma\nu} + \dots \quad (4.33)$$

and its inverse

$$r_{\mu\nu}^{-1}(x) = (e^{-h})_{\mu\nu}. \quad (4.34)$$

Then we construct functions of the field $h_{\mu\nu}$ as squares of the tensors $r_{\mu\nu}$ and $r_{\mu\nu}^{-1}$

$$g_{\mu\nu} = r_{\mu\sigma}(x)r_{\sigma\nu}(x) = (e^{2h})_{\mu\nu}, \quad g^{\mu\nu} = r_{\mu\sigma}^{-1}r_{\sigma\nu}^{-1} = (e^{-2h})_{\mu\nu}.$$

In the theory of gravitation, these quantities correspond to the co-variant and contra-variant metric tensors. One can introduce linearly transforming quantities for all fields of integer spin. For example

$$A_\mu(x) = r_{\mu\nu}(x)a_\nu(x), \quad A^\mu(x) = r_{\mu\nu}^{-1}(x)a_\nu(x), \quad A^\nu = g^{\mu\nu}A_\nu.$$

For field of half-integer spin, the transition to linearly transforming quantities is impossible, because of absence of finite-dimensional spinor representations of the affine group.

Let us introduce the notation

$$C = \exp(ix_\mu P_\mu) \exp\left(\frac{\imath}{2} h_{\alpha\beta} R_{\alpha\beta}\right)$$

and consider the transformation properties of the expression $C^{-1}dC$

$$\begin{aligned} C'^{-1}dC' &= \exp\left(\frac{\imath}{2} u_{\mu\nu} L_{\mu\nu}\right) C^{-1}gd\left(g^{-1}C \exp\left(\frac{\imath}{2} u_{\mu\nu} L_{\mu\nu}\right)\right) = \\ &= \exp\left(\frac{\imath}{2} u_{\mu\nu} L_{\mu\nu}\right) (C^{-1}dC) \exp\left(-\frac{\imath}{2} u_{\mu\nu} L_{\mu\nu}\right) + \\ &\quad + \exp\left(\frac{\imath}{2} u_{\mu\nu} L_{\mu\nu}\right) d \exp\left(-\frac{\imath}{2} u_{\mu\nu} L_{\mu\nu}\right). \end{aligned}$$

Expanding $C^{-1}dC$ with respect to the generators of the affine group

$$C^{-1}dC = \imath\omega_\mu^P(d)P_\mu + \frac{\imath}{2}\omega_{\mu\nu}^R(d)R_{\mu\nu} + \frac{\imath}{2}\omega_{\mu\nu}^L(d)L_{\mu\nu},$$

we obtain the Cartan forms

$$\omega_\mu^P(d) = r_{\mu\nu}dx_\nu, \tag{4.35}$$

$$\omega_{\mu\nu}^R(d) = \frac{1}{2}\{r^{-1}(x), dr(x)\}_{\mu\nu}, \tag{4.36}$$

$$\omega_{\mu\nu}^L(d) = \frac{1}{2}[r^{-1}(x), dr(x)]_{\mu\nu}, \tag{4.37}$$

where we have used matrix notation, for example,

$$[r^{-1}(x), dr(x)]_{\mu\nu} \equiv r_{\mu\sigma}^{-1} dr_{\sigma\nu} - dr_{\mu\sigma} r_{\sigma\nu}^{-1}.$$

Forms ω^L are used to define the covariant differentials of fields Ψ :

$$D\Psi = \left(d + i\omega_{\mu\nu}^L(d) \frac{L_{\mu\nu}^\Psi}{2} \right) \Psi.$$

The covariant differentiation with respect to coordinate of space-time is determined by dividing by the differential $\omega_\lambda^P(d)$:

$$\nabla_\lambda \Psi(x) = \frac{D\Psi}{\omega_\lambda^P(d)} = r_{\lambda\tau}^{-1} \partial_\tau \Psi(x) + \frac{i}{2} v_{\mu\nu,\lambda} L_{\mu\nu}^\Psi \Psi(x),$$

where

$$v_{\mu\nu,\lambda}^{min} = \frac{1}{2} r_{\lambda\tau}^{-1}(x) [r^{-1}(x), \partial_\tau r(x)]_{\mu\nu},$$

The transformation properties of the covariant derivative $\nabla_\lambda \Psi$ are not changed if $v_{\mu\nu,\lambda}^{min}$ is replaced by

$$\begin{aligned} V_{\mu\nu,\lambda} &= v_{\mu\nu,\lambda}^{min} + c_1(\nabla_\mu h_{\nu\lambda} - \nabla_\nu h_{\mu\lambda}) + \\ &+ c_2(\delta_{\mu\lambda} \nabla_\nu h_{\sigma\sigma} - \delta_{\nu\lambda} \nabla_\mu h_{\sigma\sigma}) + c_3(\delta_{\mu\lambda} \nabla_\tau h_{\nu\tau} - \delta_{\nu\lambda} \nabla_\tau h_{\mu\tau}). \end{aligned}$$

Therefore, the general form of the covariant derivative

$$\nabla_\lambda \Psi(x) = r_{\lambda\tau}^{-1} \partial_\tau \Psi(x) + \frac{i}{2} V_{\mu\nu,\lambda} L_{\mu\nu}^\Psi \Psi(x) \quad (4.38)$$

is not fixed and contains the nonminimal constants c_1, c_2, c_3 . Below we shall show that the values of these constants are determined by the requirement of conformal invariance.

The invariant element of volume is given by the exterior product

$$dV = -i\omega_0^P(d) \wedge \omega_1^P(d) \wedge \omega_2^P(d) \wedge \omega_3^P(d) = \det ||r_{\mu\nu}|| d^4x. \quad (4.39)$$

The action $\int \mathcal{L} \det ||r_{\mu\nu}|| d^4x$ is invariant under the affine group if the Lagrangian density

$$\mathcal{L}(\Psi(x), \nabla_\lambda \Psi(x), \nabla_\lambda h_{\mu\nu}(x))$$

is a scalar with respect to the Lorentz group. A further restriction on the theory is achieved by imposing the requirement of conformal symmetry.

4.4.2 Dynamical conformal symmetry

Let us present an element of the conformal group \bar{g} in the form

$$\bar{g} = \exp(\imath c_\mu P_\mu) \exp(\imath \beta_\mu K_\mu) \exp(\imath \beta D) \exp\left(\frac{\imath}{2} \beta_{\mu\nu} L_{\mu\nu}\right), \quad (4.40)$$

where $c_\mu, \beta_\mu, \beta, \beta_{\mu\nu}$ are parameters of the transformation. We introduce the Goldstone fields $\varphi_\mu(x)$ and $\sigma(x)$ and denote

$$\bar{C}(x) = \exp(\imath x_\mu P_\mu) \exp(\imath \varphi_\mu(x) K_\mu) \exp(\imath \sigma(x) D),$$

The action of the conformal group is determined from

$$\bar{g} : \bar{g} \bar{C}(x) = \bar{C}'(x') \exp\left(\frac{\imath}{2} u_{\mu\nu}(x, \varphi_\mu, \sigma) L_{\mu\nu}\right).$$

The infinitesimal self conformal and scale transformations of $x, \varphi_\mu(x)$, and $\sigma(x)$ have the form

$$\delta x_\mu = x^2 \beta_\mu - 2(\beta x) x_\mu - \lambda x_\mu,$$

$$\delta \sigma(x) = 2(\beta x) + \lambda,$$

$$\delta \varphi_\mu(x) = [1 + 2(x\varphi(x))] \beta_\mu + 2(x\beta) \varphi_\mu(x) - 2(\beta\varphi(x)) x_\mu + \lambda \varphi_\mu.$$

Note that $\varphi_\mu(x)$ transforms in accordance with the same law as $(1/2)\sigma_\mu(x)$. All the remaining fields transform in accordance with their representations of the Lorentz group (see the formula (4.32) but with the parameters $\bar{u}_{\mu\nu}$). An infinitesimal transformation with \bar{g} in the general form (4.40) leads to

$$\bar{u}_{\mu\nu} = \beta_{\mu\nu} + 2(\beta_\mu x_\nu - \beta_\nu x_\mu),$$

for example, for the vector field

$$\begin{aligned} \delta a_\mu(x) &= a'_\mu(x') - a_\mu(x) = \\ &= \bar{u}_{\mu\nu} a_\nu(x) = \beta_{\mu\nu} a_\nu(x) + 2\beta_\mu(x a(x)) - 2x_\mu(\beta a(x)). \end{aligned}$$

We expand $\bar{C}^{-1}d\bar{C}(x)$ with respect to the generators P_μ , $M_{\mu\nu}$, D , and K_μ and find the Cartan forms:

$$\begin{aligned} \bar{\omega}_\mu^P(d) &= e^{\sigma(x)} dx^\mu, \\ \bar{\omega}_\mu^K(d) &= e^{-\sigma(x)}(d\varphi_\mu(x) + \varphi^2(x)dx_\mu - 2(\varphi(x)dx)\varphi_\mu(x)), \\ \bar{\omega}^D(d) &= d\sigma(x) - 2dx_\mu\varphi_\mu(x), \\ \bar{\omega}_{\mu\nu}^L(d) &= 2(dx_\mu\varphi_\nu(x) - dx_\nu\varphi_\mu(x)). \end{aligned}$$

The covariant derivative of the field $\sigma(x)$ is defined in accordance with

$$\bar{\nabla}_\mu\sigma(x) = \frac{\bar{\omega}^D(d)}{\bar{\omega}_\mu^P(d)} = e^{-\sigma(x)}(\partial_\mu\sigma(x) - 2\varphi_\mu(x)).$$

The covariant derivative of the arbitrary field $\Psi(x)$ that transforms in accordance with a representation of the Lorentz group with generators $L_{\mu\nu}^\Psi$ is constructed on the basis of the Cartan form:

$$\nabla_\lambda\Psi(x) = \frac{d\Psi(x) + (\imath/2)\bar{\omega}_{\mu\nu}^L L_{\mu\nu}^\Psi\Psi(x)}{\omega_\lambda^P(d)} = e^{-\sigma(x)}(\partial_\lambda\Psi(x) + 2\imath\varphi_\nu L_{\lambda\nu}^\Psi\Psi).$$

After substitution $\varphi_\nu = (1/2)\partial_\nu\sigma$, we obtain

$$\bar{\nabla}_\lambda \Psi(x) = e^{-\sigma(x)}(\partial_\lambda \Psi(x) + i\partial_\nu \sigma L_{\lambda\nu}^\Psi \Psi(x)).$$

For the tensor field $h_{\alpha\beta}$

$$\bar{\nabla}_\lambda h_{\alpha\beta}(x) = e^{-\sigma(x)}*$$

$$*[\partial_\lambda h_{\alpha\beta} + \partial_\tau \sigma(x)(\delta_{\alpha\lambda} h_{\tau\beta}(x) + \delta_{\beta\lambda} h_{\alpha\tau}(x) - \delta_{\alpha\tau} h_{\lambda\beta}(x) - \delta_{\beta\tau} h_{\alpha\lambda}(x))].$$

The scalar element of volume, $d\bar{V}'(x') = dV(x)$, is written as

$$d\bar{V}(x) = -i\bar{\omega}_0^P \wedge \bar{\omega}_1^P \wedge \bar{\omega}_2^P \wedge \bar{\omega}_3^P = e^{i\sigma(x)} d^4x. \quad (4.41)$$

4.4.3 Joint realization of affine and conformal symmetries

The authors of the paper [13] proved that the theory coincides with the Einstein General Relativity with Hilbert action, if one chooses Lorentz subgroups as capacity of subgroup of vacuum stability, and ten gravitons identify with ten parameters of coset space of proper affine transformations

$$G = e^{iP \cdot x} e^{iR \cdot h}.$$

The trace of the affine generator $R_{\mu\nu}$ is related to the generator D of scale transformations by the equation $R_{\mu\mu} = 2D$. For example, in space-time $R_{\mu\nu} = -i(x_\mu \partial_\nu + x_\nu \partial_\mu)$, $D = -ix_\lambda \partial_\lambda$. Therefore, we must identify the trace of the affine Goldstone field $h_{\mu\nu}$ with the conformal Goldstone field:

$$\sigma(x) = \frac{1}{4} h_{\mu\nu}$$

and set

$$h_{\mu\nu}(x) = \bar{h}_{\mu\nu}(x) + \delta_{\mu\nu}\sigma(x).$$

The affine element of volume (4.39) coincides with the conformal one (4.41).

Simultaneously for the affine and conformal symmetry the covariant derivative of any field $\Psi(x)$ is

$$\nabla_\lambda \Psi(x) = r_{\lambda\tau}^{-1} \partial_\tau \Psi(x) + \frac{i}{2} V_{\mu\nu,\lambda} L_{\mu\nu}^\Psi \Psi(x), \quad (4.42)$$

where the connection $V_{\mu\nu,\lambda}$ is uniquely defined as

$$V_{\mu\nu,\lambda} = \frac{1}{2} \left(r_{\lambda\gamma}^{-1} [r^{-1}, \partial_\gamma r]_{\mu\nu} - r_{\mu\gamma}^{-1} \{r^{-1}, \partial_\gamma r\}_{\lambda\nu} + r_{\nu\gamma}^{-1} \{r^{-1}, \partial_\gamma r\}_{\lambda\mu} \right). \quad (4.43)$$

Covariant expression for action of Goldstone fields can be obtained with the aid of commutator of covariant differentiation of a field Ψ (4.30)

$$[\nabla_\delta \nabla_\gamma - \nabla_\gamma \nabla_\delta] \Psi = \frac{i}{2} R_{\alpha\beta,\delta\gamma}^{(4)} L_{\alpha\beta}^\Psi \Psi, \quad (4.44)$$

where

$$R_{\mu\nu,\lambda\rho}^{(4)} = r_{\lambda\gamma}^{-1} \partial_\gamma V_{\mu\nu,\rho} + V_{\mu\nu,\gamma} V_{\rho\gamma,\lambda} + V_{\mu\gamma,\rho} V_{\nu\gamma,\lambda} - (\lambda \leftrightarrow \rho) \quad (4.45)$$

is a tensor of curvature. Under the action of the affine and conformal groups, $R_{\mu\nu\lambda\rho}$ transforms as a tensor under Lorentz transformations with the parameter $u_{\mu\nu}$ and $\bar{u}_{\mu\nu}$. Its contraction

$$R^{(4)} = R_{\mu\nu,\mu\nu}^{(4)} = 2r_{\mu\gamma}^{-1} \partial_\gamma V_{\mu\nu,\nu} + V_{\mu\nu,\gamma} V_{\nu\gamma,\mu} - V_{\mu\gamma,\mu} V_{\nu\gamma,\nu}$$

is a scalar with respect to the affine and conformal groups.

Let us show that the theory constructed in this way is identical with Einstein's theory of gravitation. We have seen above that linearly transforming co- and contra-variant quantities are constructed by multiplying by $r_{\mu\nu}$ or $r_{\mu\nu}^{-1}$ with respect to each index, respectively. Thus, $A_\mu = r_{\mu\bar{\nu}}a_{\bar{\nu}}$ is a covariant vector, $A_{\mu\nu} = r_{\mu\bar{\mu}}r_{\nu\bar{\nu}}a_{\bar{\mu}\bar{\nu}}$ is a covariant tensor, $A_\mu^\nu = r_{\mu\bar{\mu}}r_{\nu\bar{\nu}}^{-1}a_{\mu\nu}$ is a mixed tensor, *et cetera*. A similar operation can also be performed with the covariant derivative of a covariant vector A_μ as

$$D_\lambda A_\mu = r_{\lambda\bar{\lambda}}r_{\mu\bar{\mu}}\nabla_{\bar{\lambda}}a_{\bar{\mu}},$$

of a contra-variant vector as

$$D_\lambda A^\mu = r_{\lambda\bar{\lambda}}r_{\mu\bar{\mu}}^{-1}\nabla_{\bar{\lambda}}a_{\bar{\mu}},$$

of a covariant tensor of second rank as

$$D_\lambda A_{\mu\nu} = r_{\lambda\bar{\lambda}}r_{\mu\bar{\mu}}r_{\nu\bar{\nu}}\nabla_{\bar{\lambda}}a_{\bar{\mu}\bar{\nu}},$$

and so forth. These definitions are identical to the standard definitions in the theory of gravitation; for example,

$$\begin{aligned} D_\lambda A_\mu &= r_{\lambda\bar{\lambda}}r_{\mu\bar{\mu}}\nabla_{\bar{\lambda}}a_{\bar{\mu}} = r_{\lambda\bar{\lambda}}r_{\mu\bar{\mu}} \left[r_{\bar{\lambda}\bar{\tau}}^{-1}\partial_{\bar{\tau}}(r_{\bar{\mu}\bar{\sigma}}A_{\bar{\sigma}}) + V_{\bar{\mu}\bar{\beta},\bar{\lambda}}r_{\bar{\beta}\bar{\sigma}}^{-1}A_{\bar{\sigma}} \right] = \\ &= \partial_\lambda A_\mu - \Gamma_{\lambda\mu}^\sigma A_\sigma, \end{aligned}$$

where $\Gamma_{\mu\nu}^\sigma$ is the Christoffel symbol

$$\begin{aligned} \Gamma_{\mu\lambda}^\sigma &= - (r\partial_\lambda r^{-1})_{\mu\sigma} - r_{\mu\mu}r_{\lambda\bar{\lambda}}V_{\mu\sigma,\bar{\lambda}}r_{\bar{\sigma}\bar{\sigma}}^{-1} \equiv \\ &\equiv \frac{1}{2}g^{\sigma\gamma}(\partial_\mu g_{\gamma\lambda} + \partial_\lambda g_{\mu\gamma} - \partial_\gamma g_{\mu\lambda}). \end{aligned}$$

The covariant curvature tensor is expressed as

$$R_{\mu\nu\lambda\rho} = r_{\mu\bar{\mu}}r_{\nu\bar{\nu}}r_{\lambda\bar{\lambda}}r_{\rho\bar{\rho}}R_{\bar{\mu}\bar{\nu}\bar{\lambda}\bar{\rho}}.$$

For the Ricci tensor we have

$$R_{\mu\lambda} = r_{\mu\bar{\mu}}r_{\lambda\bar{\lambda}}R_{\bar{\mu}\bar{\nu}\bar{\lambda}\bar{\nu}},$$

and the curvature R is obtained by complete contraction:

$$R = R_{\mu\nu,\mu\nu}.$$

Then one can get the Hilbert action for the General Relativity

$$W_{\text{H}}(g) = - \int d^4x \left[\sqrt{-g} \frac{R(g)}{6} \right] \quad (4.46)$$

with the interval

$$ds^2 = \omega_{\lambda}^P(d)\omega_{\lambda}^P(d) = r_{\mu\lambda}dx^{\mu}r_{\nu\lambda}dx^{\nu} = g_{\mu\nu}dx^{\mu}dx^{\nu}. \quad (4.47)$$

We may point out that we obtained the theory of gravitation in the tetrad formalism.

4.4.4 Explicitly conformal invariant General Relativity

However, the action of the General Relativity (4.46) unobviously invariant of conformal group of symmetry: the conformal invariance is hidden. The explicitly conformal invariant version of the General Relativity is obtained from the action (4.46), if we implement substitution of variables

$$g_{\mu\nu} = e^{-2D\tilde{}}g_{\mu\nu}$$

and choose another definition of measured interval. In this case the curvature takes a form

$$R^{(4)}(g = e^{-2D}\tilde{g}) = e^{-D} \left(R^{(4)}(\tilde{g}) - 6\tilde{\square} \right) e^{-D},$$

where

$$\tilde{\square} \equiv \frac{1}{\sqrt{-\tilde{g}}} \frac{\partial}{\partial x^\mu} \left(\sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} \frac{\partial}{\partial x^\nu} \right)$$

is the operator of D'Alembert in metric \tilde{g} with interval

$$\tilde{ds}^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu. \tag{4.48}$$

After this substitution the action (4.46) takes a form

$$\begin{aligned} W_C(\tilde{g}, D) &= \tag{4.49} \\ &= - \int d^4x \left[\frac{\sqrt{-\tilde{g}}}{6} R^{(4)}(\tilde{g}) e^{-2D} - e^{-D} \frac{\partial}{\partial x^\mu} \left(\sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} \frac{\partial}{\partial x^\nu} e^{-D} \right) \right], \end{aligned}$$

where D is a scalar dilaton field, scale transformation of which compensates transformations of another fields.

Let us prove the conformal invariance of the action (4.49) explicitly. For this, we take once more conformal transformation

$$\tilde{g}_{\mu\nu} = e^{-2\lambda} \hat{g}_{\mu\nu}.$$

Since the Ricci scalar transforms under the conformal transformations as [14]

$$\frac{1}{6} \sqrt{-\tilde{g}} \tilde{R}^{(4)} = \frac{1}{6} e^{-2\lambda} \sqrt{-\hat{g}} \hat{R}^{(4)} - e^{-\lambda} \frac{\partial}{\partial x^\mu} \left(\sqrt{-\hat{g}} \hat{g}^{\mu\nu} \frac{\partial}{\partial x^\nu} e^{-\lambda} \right),$$

the action (4.49) takes the form:

$$W_C(\hat{g}, D, \lambda) = \tag{4.50}$$

$$\begin{aligned}
&= - \int d^4x \left[\frac{\sqrt{-\widehat{g}}}{6} R^{(4)}(\widehat{g}) e^{-2(D+\lambda)} - e^{-(2D+\lambda)} \frac{\partial}{\partial x^\mu} \left(\sqrt{-\widehat{g}} \widehat{g}^{\mu\nu} \frac{\partial}{\partial x^\nu} e^{-\lambda} \right) \right] - \\
&\quad - \int d^4x e^{-D} \frac{\partial}{\partial x^\mu} \left(e^{-2\lambda} \sqrt{-\widehat{g}} \widehat{g}^{\mu\nu} \frac{\partial}{\partial x^\nu} e^{-D} \right).
\end{aligned}$$

We transform the last two terms in the resulting expression, after selecting a common factor $\exp(-(D+\lambda))$:

$$\begin{aligned}
&e^{-D} \frac{\partial}{\partial x^\mu} \left(\sqrt{-\widehat{g}} \widehat{g}^{\mu\nu} \frac{\partial}{\partial x^\nu} e^{-\lambda} \right) + \frac{\partial}{\partial x^\mu} \left(\sqrt{-\widehat{g}} \widehat{g}^{\mu\nu} e^{-\lambda} \frac{\partial}{\partial x^\nu} e^{-D} \right) + \\
&+ \sqrt{-\widehat{g}} \widehat{g}^{\mu\nu} \frac{\partial}{\partial x^\mu} e^{-\lambda} \frac{\partial}{\partial x^\nu} e^{-D} = \frac{\partial}{\partial x^\mu} \left(\sqrt{-\widehat{g}} \widehat{g}^{\mu\nu} \frac{\partial}{\partial x^\nu} e^{-(D+\lambda)} \right).
\end{aligned}$$

Now, we finally obtain

$$W_C(\widehat{g}, D, \lambda) = \tag{4.51}$$

$$= - \int d^4x \left[\frac{\sqrt{-\widehat{g}}}{6} R^{(4)}(\widehat{g}) e^{-2(D+\lambda)} - e^{-(D+\lambda)} \frac{\partial}{\partial x^\mu} \left(\sqrt{-\widehat{g}} \widehat{g}^{\mu\nu} \frac{\partial}{\partial x^\nu} e^{-(D+\lambda)} \right) \right].$$

The requirement of invariance of the action (4.49) determines the dilation field transformation:

$$D + \lambda = \widehat{D}.$$

Thus we proved the conformal invariance of the action (4.49)

$$W_C(\widehat{g}, D, \lambda) = W_C(\widehat{g}, \widehat{D}). \tag{4.52}$$

In the action of conformal invariant theory (4.49) a number of variables is the same as in the theory of Einstein (4.46). Moreover, in the infinite volume all solutions of classical equations of the theory (4.49) correspond to the solutions of classical equations of the theory (4.46).

However, the observed data irrefutably testify the finite space volume and finite time interval of life of the Universe, finite energy and finite energy density. All these finite values can be defined in a concrete system of reference. In conformal theory there is a system of reference with unit determinant of space metric with conformal interval (4.48). It is just the system we use for the classification of observed data.

4.5 Summary

For the purpose of construction of the quantum operator of creation and evolution of the Universe as unitary irreducible representation of conformal and affine groups of symmetry, one represents general elements of the theory of nonlinear realizations of groups of symmetry developed by Élie Cartan [1]. Then the derivation of classical theory of gravitation as nonlinear joint realization of conformal and affine symmetries [13] is presented by analogy with the derivation of the chiral phenomenological Lagrangian for pions [11].

The derived theory of gravitation contains, besides known physical effects of the General Relativity for the Solar system, all elements of further development of Einstein's ideas suggested by his contemporaries and followers, including Hilbert's variational principle of action (1915), Fock's frames [12] in the tangent space of Minkowski, Dirac's conformal interval, where the determinant of metrics is identified with the scalar dilaton.

Bibliography

- [1] Cartan, É.: *Lecons sur la Geometric des Espaces de Riemann*. Gauthier–Villars, Paris (1946)
- [2] Helgason, S.: *Differential Geometry, Lie Groups, and Symmetric Spaces*. Providence, R.I. AMS (2001)
- [3] Wigner, E.: *Symmetries and Reflections*. Indiana University press, Bloomington – London (1970)
- [4] Bogoliubov, N.N.: *On some problems of the theory of superconductivity*. Physica. **26**, 56 (1960)
- [5] Nambu, Y.: *Axial vector current conservation in weak interactions*. Phys. Rev. Lett. **4**, 380 (1960)
- [6] Goldstone, J.: *Field theories with superconductor solutions*. Nuovo Cimento. **19**, 54 (1961)
- [7] Weinberg, S.: *Dynamical approach to current algebra*. Phys. Rev. Lett. **18**, 188 (1967)
- [8] Coleman, S., Wess, G., Zumino, B.: *Structure of phenomenological Lagrangians. I*. Phys. Rev. **177**, 2239 (1969)

-
- [9] Callan, C.G., Coleman, S., Wess, G., Zumino, B.: *Structure of phenomenological Lagrangians. II*. Phys. Rev. **177**, 2247 (1969)
- [10] Volkov, D.V.: *Phenomenological Lagrangians*. Phys. Part. & Nuclei. **4**, 1 (1973)
- [11] Volkov, M.K., Pervushin, V.N.: *Essentially Nonlinear Quantum Theories, Dynamical Symmetries, and Physics of Pions*. Atomizdat, Moscow (1978)
- [12] Fock, V.: *Geometrisierung der Diracschen theorie des electrons*. Zs. f. Fiz. **57**, 261 (1929)
- [13] Borisov, A.B., Ogievetsky, V.I.: *Theory of dynamical affine and conformal symmetries as gravity theory*. Theor. Math. Phys. **21**, 1179 (1975)
- [14] Eisenhart, L.P.: *Riemannian Geometry*. Princeton University Press (1926)
- [15] Dirac, P.A.M.: *Long range forces and broken symmetries*. Proc. Roy. Soc. London. **A 333**, 403 (1973)

Chapter 5

Hamiltonian formulation of the theory of gravity

5.1 Foliation of space–time continuum

Let us consider a family of space-like hypersurfaces Σ_t , immersed in four-dimensional space-time, where t is some parameter, fixing hypersurface, and x^μ are some local coordinates of the space-time. At each point M of the hypersurface

$$x^\mu = x^\mu(t, u^1, u^2, u^3)$$

we construct a frame, consisting of the vectors [1]

$$\mathbf{n}, \quad \mathbf{e}_1 = \frac{\partial}{\partial u^1}, \quad \mathbf{e}_2 = \frac{\partial}{\partial u^2}, \quad \mathbf{e}_3 = \frac{\partial}{\partial u^3}. \quad (5.1)$$

The unit vectors

$$\mathbf{e}_i = \frac{\partial x^\mu}{\partial u^i} \frac{\partial}{\partial x^\mu} \equiv e_i^\mu \partial_\mu \quad (5.2)$$

are tangent to the coordinate lines $u = \text{const}$ on the hypersurface, and $\mathbf{n} = n^\mu \partial_\mu$ is the vector, normal to the tangent plane in M .

The induced metric

$$\gamma_{ij} = \frac{\partial x^\mu}{\partial u^i} \frac{\partial x^\nu}{\partial u^j} g_{\mu\nu} \quad (5.3)$$

sets an invariant first quadratic form on the hypersurface

$$\gamma_{ij}(u^1, u^2, u^3) du^i du^j. \quad (5.4)$$

The metric γ_{ij} is negative definite. Let us consider now the quantities from (5.2)

$$e_i^\mu = \frac{\partial x^\mu}{\partial u^i}, \quad \mu = 0, 1, 2, 3; \quad i = 1, 2, 3 \quad (5.5)$$

in some point M of the hypersurface. The Greek index refers to a containing space-time and responds to coordinate transformations in it as a contravariant index:

$$\frac{\partial x^\mu}{\partial u^i} = \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial u^i}, \quad \text{so} \quad e_i^\mu = \frac{\partial x^\mu}{\partial x^\nu} e_i^\nu.$$

The Latin index refers to the hypersurface and responds to coordinate transformations u^i in it as a covariant index. The variables e_i^μ (5.5) will be called as a mixed tensor, one-contravariant in the space-time and one-covariant in the space. Similarly, mixed tensors of any rank can be defined in the space. For example, suppose we have a mixed tensor field

$$Z_j^{\mu i} = Z_j^{\mu i}(u^1, u^2, u^3).$$

For an infinitesimal displacement along the hypersurface we define an absolute differential of this tensor by the formula

$$DZ_j^{\mu i} = dZ_j^{\mu i} + \Gamma_{\nu\alpha}^{\mu} Z_j^{\alpha i} dx^{\nu} + {}^* \Gamma_{lk}^i Z_j^{\mu k} du^l - {}^* \Gamma_{lj}^k Z_k^{\mu i} du^l. \quad (5.6)$$

Terms, corresponding to Latin indices, are compiled using the connection coefficients ${}^* \Gamma_{lk}^i$ computed in the space with the metric tensor γ_{ij} (5.3).

From the absolute differential constructed in (5.6) it is not difficult to pass to the covariant derivative of the mixed tensor with u^i . Producing replacements

$$dZ_j^{\mu i} = \frac{\partial Z_j^{\mu i}}{\partial u^k} du^k, \quad dx^{\mu} = \frac{\partial x^{\mu}}{\partial u^i} du^i = e_i^{\mu} du^i,$$

one obtains

$$DZ_j^{\mu i} = \left(\frac{\partial Z_j^{\mu i}}{\partial u^p} + e_p^{\nu} \Gamma_{\nu\epsilon}^{\mu} Z_j^{\epsilon i} + {}^* \Gamma_{pk}^i Z_j^{\mu k} - {}^* \Gamma_{pj}^k Z_k^{\mu i} \right) du^p. \quad (5.7)$$

Coefficients under du^p in (5.7) will be called covariant derivatives of the mixed tensor on u^p :

$${}^* \nabla_p Z_j^{\mu i} = \frac{\partial Z_j^{\mu i}}{\partial u^p} + e_p^{\nu} \Gamma_{\nu\epsilon}^{\mu} Z_j^{\epsilon i} + {}^* \Gamma_{pk}^i Z_j^{\mu k} - {}^* \Gamma_{pj}^k Z_k^{\mu i}. \quad (5.8)$$

Now we turn to the second alternating absolute differential. On the hypersurface we take an arbitrary two-dimensional surface $u^i = u^i(\alpha, \beta)$. Differentials d and D correspond to infinitesimal displacements along a coordinate line α , and \tilde{d} , \tilde{D} correspond to infinitesimal displacements along a coordinate line β . Let us consider some vector fields U_{μ} and W^{ν} with Greek indices in the space-time. Then we yield

$$\left(\tilde{D}D - D\tilde{D} \right) U_{\mu} = R_{\mu\alpha\beta}^{\nu} U_{\nu} dx^{\alpha} \tilde{d}x^{\beta}, \quad (5.9)$$

$$\left(\tilde{D}D - D\tilde{D}\right)W^\mu = -R^\nu_{\mu\alpha\beta}W^\nu dx^\alpha \widetilde{dx}^\beta, \quad (5.10)$$

and for the second alternating covariant derivatives:

$$\left(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha\right)U_\mu = U_\nu R^\nu_{\mu\beta\alpha}, \quad (5.11)$$

$$\left(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha\right)W^\mu = -W^\nu R^\mu_{\nu\beta\alpha}. \quad (5.12)$$

Let us consider some vector fields U_i and W^j in the space. The differentials D and \tilde{D} have sense of absolute differentials in the space. Analogously to (5.9), (5.10), we obtain

$$\left(\tilde{D}D - D\tilde{D}\right)U_i = U_j {}^*R^j_{ikl} du^k \widetilde{du}^l, \quad (5.13)$$

$$\left(\tilde{D}D - D\tilde{D}\right)W^i = -W^j {}^*R^i_{jkl} du^k \widetilde{du}^l. \quad (5.14)$$

Here ${}^*R^i_{jkl}$ denotes the curvature tensor of the space. If now an arbitrary mixed tensor field $Z_j^{\mu i}$ is set in the space, then we get

$$\begin{aligned} &\left(\tilde{D}D - D\tilde{D}\right)Z_j^{\mu i} = \\ &= -R^\mu_{\nu\alpha\beta}Z_j^{\nu i} dx^\alpha \widetilde{dx}^\beta - {}^*R^i_{mkl}Z_j^{\mu m} dx^k \widetilde{dx}^l + {}^*R^m_{jkl}Z_m^{\mu i} dx^k \widetilde{dx}^l. \end{aligned}$$

The formula for the alternating second absolute derivative of a mixed tensor is:

$$\left({}^*\nabla_l {}^*\nabla_k - {}^*\nabla_k {}^*\nabla_l\right)Z_j^{\mu i} = -R^\mu_{\nu\alpha\beta}Z_j^{\nu i} e_k^\alpha e_l^\beta - {}^*R^i_{mkl}Z_j^{\mu m} + {}^*R^m_{jkl}Z_m^{\mu i}. \quad (5.15)$$

5.1.1 Derivation formulae

To study the space it is important to look how the selected frame (5.1) changes from a point to another point. Let us make an infinitesimal

displacement of a given point M along the hypersurface. The covariant derivative of the mixed tensor

$${}^*\nabla_j e_i^\mu = \frac{\partial}{\partial u^j} e_i^\mu + e_j^\nu \Gamma_{\nu\alpha}^\mu e_i^\alpha - {}^*\Gamma_{ji}^k e_k^\mu.$$

So far as

$$\frac{\partial e_i^\mu}{\partial u^j} = \frac{\partial^2 x^\mu}{\partial u^i \partial u^j},$$

and $\Gamma_{\nu\alpha}^\mu$, and ${}^*\Gamma_{ji}^k$ are symmetric on low indices, it follows

$${}^*\nabla_j e_i^\mu = {}^*\nabla_i e_j^\mu. \quad (5.16)$$

The contravariant vector ${}^*\nabla_j e_i^\mu$ is orthogonal to all vectors e_k^μ , so it is directed by normal to the hypersurface. Indeed, according to (5.3) the induced metric tensor is

$$\gamma_{ij} = e_i^\mu e_j^\nu g_{\mu\nu}.$$

Let us calculate term by term its absolute differential

$$D\gamma_{ij} = (De_i^\mu) e_j^\nu g_{\mu\nu} + e_i^\mu (De_j^\mu) g_{\mu\nu} + e_i^\mu e_j^\nu (Dg_{\mu\nu}).$$

Since $D\gamma_{ij}$ coincides with the absolute differential in the space, and $Dg_{\mu\nu}$ with the absolute differential in the space-time, they both are equal to zero. Substituting De_i^μ by ${}^*\nabla_i e_i^\mu du^j$, and taking account that du^j are arbitrary, we obtain

$$({}^*\nabla_k e_i^\mu) e_j^\nu + e_i^\mu ({}^*\nabla_k e_j^\mu) g_{\mu\nu} = 0. \quad (5.17)$$

We add two more equations derived from (5.17) by circular permutation of the indices

$$({}^*\nabla_i e_j^\mu) e_k^\nu + e_j^\mu ({}^*\nabla_i e_k^\mu) g_{\mu\nu} = 0, \quad (5.18)$$

$$((^*\nabla_j e_k^\mu) e_i^\nu + e_k^\mu (^*\nabla_j e_i^\mu)) g_{\mu\nu} = 0. \quad (5.19)$$

Taking into account (5.16), we notice that here three pairwise sum of three values (5.17), (5.18), (5.19) are equal to zero, so that each of them is equal to zero:

$$(^*\nabla_k e_i^\mu) e_j^\nu g_{\mu\nu} = 0.$$

So, we can define

$$\boxed{^*\nabla_k e_i^\mu = K_{ki} n^\mu}. \quad (5.20)$$

The symmetric tensor $K_{ki} = K_{ik}$ is called the *tensor of extrinsic curvature of the hypersurface*. The corresponding invariant quadratic form

$$K_{ij} du^i du^j \quad (5.21)$$

is called the second quadratic form on the hypersurface.

We now express $^*\nabla_i n^\mu$ using tensor K_{ik} . The normal vector n^μ is orthogonal to any tangent vector e_i^μ

$$g_{\mu\nu} n^\mu e_i^\nu = 0. \quad (5.22)$$

Taking the covariant derivative of (5.22) and taking into account

$$^*\nabla_k g_{\mu\nu} = 0,$$

we obtain

$$g_{\mu\nu} (^*\nabla_k n^\mu) e_i^\nu + g_{\mu\nu} n^\mu (^*\nabla_k e_i^\nu) = 0.$$

Hence, since

$$g_{\mu\nu} n^\mu n^\nu = 1, \quad (5.23)$$

one yields

$$g_{\mu\nu}(*\nabla_k n^\mu) e_i^\nu = -K_{ki}. \quad (5.24)$$

Differentiating (5.23), we get

$$g_{\mu\nu}(*\nabla_k n^\mu) n^\nu + g_{\mu\nu} n^\mu (*\nabla_k n^\nu) = 0.$$

Since both terms of the left side are equal, then

$$g_{\mu\nu}(*\nabla_k n^\mu) n^\nu = 0. \quad (5.25)$$

So the vector $*\nabla_k n^\mu$ is orthogonal to the normal vector n^ν , placed in the tangential plane and can be decomposed by vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$\boxed{*\nabla_k n^\mu = -K_k^l e_l^\mu} \quad (5.26)$$

Coefficients of decomposition in (5.26) are obtained by comparing with equality (5.24). The obtained formulae (5.20) and (5.26) are named *derivation formulae* of the theory of hypersurfaces. They express the covariant derivatives of the vectors \mathbf{e}_i, \mathbf{n} through these vectors themselves.

5.1.2 Gauss – Weingarten equations

The tensor of extrinsic curvature measures a curvature of 3-space in the space-time. The covariant derivative of the normal vector in the i – th coordinate direction is

$$\nabla_i \mathbf{n} = -K_i^j \mathbf{e}_j. \quad (5.27)$$

Take the scalar product of both sides of (5.27) with the basic vector \mathbf{e}_k .

$$K_{ij} = K_i^{l(3)} g_{lj} = K_i^l (\mathbf{e}_l \cdot \mathbf{e}_j) = -(\mathbf{e}_j \cdot \nabla_i \mathbf{n}) =$$

$$= (\mathbf{n} \cdot \nabla_i \mathbf{e}_j) = (\mathbf{n} \cdot \mathbf{e}_0) \Gamma_{ji}^0 = (\mathbf{n} \cdot \nabla_j \mathbf{e}_i) = K_{ji}.$$

The finding the covariant derivative

$$\nabla_i \mathbf{e}_j = {}^* \Gamma_{ji}^k \mathbf{e}_k + \Gamma_{ji}^0 \mathbf{e}_0$$

it is necessary to know the scalar products with every of four independent vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{n}$. Thus one arrives at the Gauss – Weingarten equations, well-known in the geometry of imbedded hypersurfaces

$$\boxed{\nabla_i \mathbf{e}_j = K_{ij} \mathbf{n} + {}^* \Gamma_{ij}^l \mathbf{e}_l = \nabla_j \mathbf{e}_i.} \quad (5.28)$$

The coefficients of $\nabla_i \mathbf{e}_j$ decomposition can be obtained by scalar multiplication of the both sides of (5.28) by \mathbf{n} and \mathbf{e}_i . From this formula, knowing the change of each basis vectors, one can find change of any vector \mathbf{A} that lies in Σ .

$$\begin{aligned} \nabla_i \mathbf{A} &= \nabla_i (\mathbf{e}_j A^j) = \mathbf{e}_j \frac{\partial A^j}{\partial x^i} + A^j \nabla_i \mathbf{e}_j = \\ &= \mathbf{e}_j \frac{\partial A^j}{\partial x^i} + {}^* \Gamma_{ji}^k A^j \mathbf{e}_k + A^j K_{ij} \mathbf{n} = \\ &= \mathbf{e}_j \left(\frac{\partial A^j}{\partial x^i} + {}^* \Gamma_{ki}^j A^k \right) + A^j K_{ij} \mathbf{n} = \mathbf{e}_j A_{j|i} + K_{ij} A^j \mathbf{n}. \end{aligned}$$

5.1.3 Gauss – Codazzi equations

We now derive formulae, connecting the first (5.4) and second (5.21) quadratic forms on the hypersurface. Let us consider the derivation formulae (5.20), (5.26) as a system of differential equations for the unknown functions $e_i^\mu(u^1, u^2, u^3)$, $n^\mu(u^1, u^2, u^3)$ and obtain the equations of integrability. An alternating second covariant derivatives of

the vectors e_i

$$\begin{aligned} (*\nabla_l^* \nabla_k - * \nabla_k^* \nabla_l) e_i^\mu &= -R_{\nu\alpha\beta}^\mu e_i^\nu e_k^\alpha e_l^\beta + *R_{ikl}^m e_m^\mu = \\ &= (-*\nabla_k K_{li} + *\nabla_l K_{ki}) n^\mu - (K_{ki} K_l^m - K_{li} K_k^m) e_m^\mu \end{aligned} \quad (5.29)$$

take us the conditions of integrability of the system of equations (5.20). Analogously, we get the conditions of integrability of differential equations (5.26):

$$(*\nabla_l^* \nabla_k - * \nabla_k^* \nabla_l) n^\mu = -R_{\nu\alpha\beta}^\mu n^\nu e_k^\alpha e_l^\beta = -(*\nabla_l K_k^m - *\nabla_k K_l^m) e_m^\mu. \quad (5.30)$$

A scalar product of both sides of the (5.29) with e_j^ν :

$$\boxed{*R_{ijkl} = R_{\alpha\beta\gamma\delta} e_i^\alpha e_j^\beta e_k^\gamma e_l^\delta + (K_{ik} K_{jl} - K_{il} K_{jk})}. \quad (5.31)$$

They are called the *equations of Gauss*. A scalar product of both sides of the (5.29) with n^ν :

$$\boxed{R_{\alpha\beta\gamma\delta} n^\alpha e_i^\beta e_j^\gamma e_k^\delta = -(*\nabla_k K_{ij} - *\nabla_j K_{ik})}. \quad (5.32)$$

They are called the *equations of Codazzi*.

Let us consider now the equations (5.30). Contraction them with $g_{\mu\nu} e_i^\nu$, we get the relations (5.32). Then, contracting them with $g_{\mu\nu} n^\nu$, we get identities. Thus, integrability conditions of the derivation equations exhaust by the conditions (5.31) and (5.32).

5.2 ADM coordinates

The interval is presented in the following form

$$ds^2 = (Ndt)^2 - {}^{(3)}g_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \quad (5.1)$$

Here N is a lapse function, N^i is a shift-vector, $\gamma_{ij} \equiv -^{(3)}g_{ij}$ are metric coefficients of 3-geometry (see Fig. 5.2). The metric tensor of space-time

$$(g_{\mu\nu}) = \begin{pmatrix} N^2 - N_i N^i & -N_i \\ -N_j & -^{(3)}g_{ij} \end{pmatrix}, \quad (5.2)$$

and the inverse one is

$$(g^{\mu\nu}) = \begin{pmatrix} 1/N^2 & -N^i/N^2 \\ -N^j/N^2 & -(^{(3)}g^{ij} - N^i N^j/N^2) \end{pmatrix}. \quad (5.3)$$

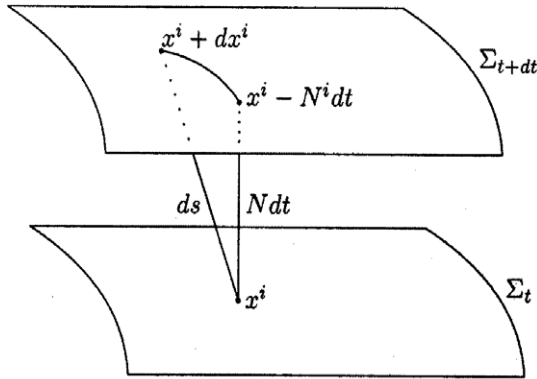


Figure 5.2 shows a line of time and two space-like three-dimensional hypersurfaces through which this line of time is spent in the General Relativity. A transition from a Σ_t to a hypersurface Σ_{t+dt} is described by a lapse function N and a shift-vector N_i . The family of all the space-like three-dimensional hypersurfaces is called a congruence, and the appropriate parametrization of the metric component is called $4 = 3 + 1$ bundle of the space-time.

The covariant components $N_i = {}^{(3)}g_{ij}N^j$ are calculated using 3-metrics. An element of 4-volume is calculated as

$$(-g)^{1/2} dx^0 dx^1 dx^2 dx^3 = N \left({}^{(3)}g \right)^{1/2} dt dx^1 dx^2 dx^3.$$

The unit time-like normal vector has the components

$$n_\mu = (N, 0, 0, 0).$$

Raising the indices via (5.3), we obtain its contravariant components

$$n^\mu = \left(\frac{1}{N}, -\frac{N^i}{N} \right).$$

The tensor K_{ij} has the values

$$\begin{aligned} K_{ij} &= (\mathbf{n} \cdot \nabla_i \mathbf{e}_j) = (\mathbf{n} \cdot \mathbf{e}_0) \Gamma_{ji}^0 = N \Gamma_{ji}^0 = \\ &= N (g^{00} \Gamma_{0,ij} + g^{0k} \Gamma_{k,ij}) = \frac{1}{N} \left(\Gamma_{0,ij} - N^k {}^{(3)}\Gamma_{k,ij} \right) = \\ &= \frac{1}{2N} \left(\frac{\partial N_i}{\partial x^j} + \frac{\partial N_j}{\partial x^i} + \frac{\partial}{\partial t} {}^{(3)}g_{ij} - 2N^k \Gamma_{k,ij} \right), \end{aligned}$$

or, finally

$$K_{ij} = \frac{1}{2N} \left(\frac{\partial}{\partial t} {}^{(3)}g_{ij} - {}^{(3)}\nabla_j N_i - {}^{(3)}\nabla_i N_j \right). \quad (5.4)$$

We are able now to obtain the Ricci scalar using (5.31) and (5.32).

$$R = {}^{(3)}R + \text{Tr} (K^2) - (\text{Tr} K)^2 \quad (5.5)$$

The difference between the obtained formula and the real one is not sufficient for dynamics of the gravitation field: we omitted only a time derivative term plus divergence part. They do not influence on the equations of its motion.

5.2.1 Kinematic subgroup

Metric components are objects of arbitrary general coordinate transformations. In particular, the group of general coordinate transformations (diffeomorphisms) of the Hamiltonian approach contains the following coordinate transformations

$$x^0 \rightarrow \tilde{x}^0 = \tilde{x}^0(x^0); \quad (5.6)$$

$$x_i \rightarrow \tilde{x}_i = \tilde{x}_i(x^0, x_1, x_2, x_3). \quad (5.7)$$

This group of transformations preserves the family (congruence) of hypersurfaces $x^0 = \text{const}$, and is called a *kinematic subgroup* [3] of the group of general coordinate transformations

$$x_\mu \rightarrow \tilde{x}_\mu = \tilde{x}_\mu(x^0, x_1, x_2, x_3).$$

The group of kinematic transformations contains reparametrizations of a *coordinate time* (5.6) in class of functions, depending only on a coordinate time, which we call as global. Whereas, the transformations (5.7) we call as local. Thus, the subgroup of diffeomorphisms of the Hamiltonian formulation of the General Relativity (5.6) and (5.7) is composed of one global and three local transformations, *id est*, the structure of the kinematic subgroup is of the form $1G \oplus 3L$.

Identification and extraction of physical degrees of freedom is one of the most important problems of the theory of gravity, which stimulated Dirac to create a generalized Hamiltonian formulation of systems with constraints [4], and, later, to develop this formulation by many authors [5, 6, 7]. The solution to this problem consists in extraction of true

evolution of the observed dynamical and geometrical magnitudes from general coordinate (gauge) transformation (5.6) and (5.7).

Introduced above, the formalism of Cartan allows us to formulate the theory of gravity in terms of invariants relative to general coordinate transformations through the transition to invariant components of Fock's frames. A foliation of the space-time $4=3+1$ (see Fig.5.2) involves the introduction of components of Fock frames $\omega_{(\alpha)}$ in the following form

$$\omega_{(0)} = e^{-2D} N dx^0, \quad (5.8)$$

$$\omega_{(b)} = \mathbf{e}_{(b)i} dx^i + N_{(b)} dx^0. \quad (5.9)$$

Here N is the lapse function in the theory (4.49),

$$N_{(b)} = N^j \mathbf{e}_{(b)j}$$

are components of the shift-vector; $\mathbf{e}_{(b)i}$ are orthonormal triad components with unit determinant:

$$\mathbf{e}_{(b)i} \mathbf{e}_{(b)}^j = \delta_i^j; \quad \mathbf{e}_{(a)j} \mathbf{e}_{(b)}^j = \delta_{(a)(b)}.$$

In the action of affine - conformal theory of gravity (4.49), expressed through the Maurer - Cartan forms, differentials of coordinates of the Riemannian space are not directly measurable quantities dx^0 and dx^i , but invariant under general coordinate transformations of orthogonal components of the frame in the tangent space (5.8) and (5.9). These components are, in general, non-integrable linear forms. The dependence of the linear forms from the coordinates of the tangent space

$$X_{(b)} = x^i \mathbf{e}_{(b)i}$$



V.A. Fock was born in Saint - Petersburg. After graduating from high school in Petrograd (1916), he entered the faculty of physics and mathematics of the University of Petrograd. His primary scientific contribution lies in the development of quantum physics, although he also contributed significantly to the fields of mechanics, theoretical optics, theory of gravitation, physics of continuous media. In 1926 he derived the Klein – Gordon equation. He gave his name to Fock space, the Fock representation and the Fock state, and developed the Hartree – Fock method in 1930. He wrote the first textbook on quantum mechanics “Foundations of Quantum Mechanics” (1931) and a very influential monograph “The Theory of Space, Time and Gravitation” (1955).

can be found by using the Leibnitz rule

$$AdB = d[AB] - [AB]d \ln A$$

and the condition of orthogonality of triads

$$\mathbf{e}_{(a)i} \mathbf{e}_{(a)}^j = \delta_i^j.$$

Substituting these expressions to the linear form

$$\bar{\omega}_{(b)}(d) = \mathbf{e}_{(b)i} dx^i,$$

we get

$$d[x^i]_{\mathbf{e}_{(b)i}} = d[x^i \mathbf{e}_{(b)i}] - x^i d[\mathbf{e}_{(b)i}] = d[x^i \mathbf{e}_{(b)i}] - [x^i \mathbf{e}_{(a)i}] [\mathbf{e}_{(a)}^j] d[\mathbf{e}_{(b)j}].$$

Then, using the definition of the observables $X_{(b)} = x^i \mathbf{e}_{(b)i}$, it is possible to find the sought-for dependence:

$$\begin{aligned} \bar{\omega}_{(b)}(d) &= \mathbf{e}_{(b)i} dx^i = dX_{(b)} - X_{(c)} \mathbf{e}_{(c)}^i d\mathbf{e}_{(b)i} \\ &= dX_{(b)} - X_{(c)} [\omega_{(c)(b)}^R(d) + \omega_{(c)(b)}^L(d)], \end{aligned} \quad (5.10)$$

where

$$\begin{aligned} \omega_{(c)(b)}^R(d) &= \frac{1}{2} (\mathbf{e}_{(c)}^i d\mathbf{e}_{(b)i} + \mathbf{e}_{(b)}^i d\mathbf{e}_{(c)i}), \\ \omega_{(c)(b)}^L(d) &= \frac{1}{2} (\mathbf{e}_{(c)}^i d\mathbf{e}_{(b)i} - \mathbf{e}_{(b)}^i d\mathbf{e}_{(c)i}) \end{aligned}$$

are the Cartan's forms (coefficients of the spin connection), describing strong gravitational waves. The factor $X_{(c)}$ in Eq.(5.10) means that a hypersurface, perpendicular to the wave vector of a gravitational wave experiences an expansion or a contraction of the Hubble type [8], known in the Standard cosmology.

5.3 Hamiltonian formulation of GR in terms of Cartan forms

Let us reformulate the standard description of the General Relativity in terms of Cartan's forms. The Hilbert action with the electromagnetic

field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the scalar field Q has a form¹:

$$W[g, A, Q] = \tag{5.11}$$

$$= - \int d^4x \sqrt{-g} \left(\frac{1}{6} R^{(4)}(g) - \frac{1}{4} F_{\mu\alpha} F_{\nu\beta} g^{\mu\nu} g^{\alpha\beta} + \partial_\mu Q \partial_\nu Q g^{\mu\nu} \right).$$

Passing to conformal variables $g_{\mu\nu} = e^{-2D} \tilde{g}_{\mu\nu}$, we get the action

$$W[\tilde{g}, A, Q] = - \int d^4x \sqrt{-\tilde{g}} \left[e^{-D} \left(\frac{1}{6} R(\tilde{g}) - \square \right) e^{-D} \tag{5.12}$$

$$- \frac{1}{4} F_{\mu\alpha} F_{\nu\beta} \tilde{g}^{\mu\nu} \tilde{g}^{\alpha\beta} + \partial_\mu Q \partial_\nu Q \tilde{g}^{\mu\nu} \right], \tag{5.13}$$

where

$$\square \equiv \frac{1}{\sqrt{-\tilde{g}}} \partial_\mu \left(\sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} \partial_\nu \right) \tag{5.14}$$

is the operator of D'Alembert. By determining the tetrad components (5.8) and (5.9), the action (5.11) is rewritten in the following form

$$W = \int d^4x N [\mathcal{L}_D + \mathcal{L}_g + \mathcal{L}_A + \mathcal{L}_Q]. \tag{5.15}$$

Here

$$\begin{aligned} \mathcal{L}_D &= -v_D^2 - \frac{4}{3} e^{-7D/2} \Delta e^{-D/2}, \\ \mathcal{L}_g &= \frac{1}{6} \left[v_{(a)(b)} v_{(a)(b)} - e^{-4D(3)} R(\mathbf{e}) \right], \\ \mathcal{L}_A &= \frac{1}{2} \left[v_{(b)(A)}^2 - F_{ij} F^{ij} \right], \\ \mathcal{L}_Q &= e^{-2D} (v_Q)^2 - e^{-2D} (\partial_{(b)} Q)^2; \end{aligned}$$

¹Remember that we use the natural system of units

$$\hbar = c = M_{\text{Planck}} \sqrt{3/(8\pi)} = 1.$$

Lagrangian densities and

$$\begin{aligned} v_Q &= \frac{1}{N} [(\partial_0 - N^l \partial_l)Q + \partial_l N^l / 3], \\ v_D &= \frac{1}{N} [(\partial_0 - N^l \partial_l)D + \partial_l N^l / 3], \end{aligned} \quad (5.16)$$

$$v_{(a)(b)} = \frac{1}{N} \left[\omega_{(a)(b)}^R (\partial_0 - N^l \partial_l) + \partial_{(a)} N_{(b)}^\perp + \partial_{(b)} N_{(a)}^\perp \right], \quad (5.17)$$

$$v_{(b)(A)} = \frac{1}{N} \mathbf{e}_{(a)}^i [\partial_0 A_i - \partial_i A_0 + F_{ij} N^j]$$

are velocities of the metric components and fields, and ${}^{(3)}R(\mathbf{e})$ is a three-dimensional spatial curvature, expressed in terms of triads $\mathbf{e}_{(a)i}$

$${}^{(3)}R = {}^{(3)}R(\mathbf{e}) - \frac{4}{3} e^{7D/2} \Delta e^{-D/2}, \quad (5.18)$$

$${}^{(3)}R(\mathbf{e}) = \quad (5.19)$$

$$= -2\partial_i [\mathbf{e}_{(b)}^i \sigma_{(c)|(b)(c)}] - \sigma_{(c)|(b)(c)} \sigma_{(a)|(b)(a)} + \sigma_{(c)|(d)(f)} \sigma_{(f)|(d)(c)},$$

where

$$\sigma_{(c)|(a)(b)} = [\omega_{(a)(b)}^L (\partial_{(c)}) + \omega_{(a)(c)}^R (\partial_{(b)}) - \omega_{(b)(c)}^R (\partial_{(a)})],$$

$$\omega_{(a)(b)}^R (\partial_{(c)}) = \frac{1}{2} [\mathbf{e}_{(a)}^j \partial_{(c)} \mathbf{e}_{(b)}^j + \mathbf{e}_{(b)}^i \partial_{(c)} \mathbf{e}_{(a)}^i], \quad (5.20)$$

$$\omega_{(a)(b)}^L (\partial_{(c)}) = \frac{1}{2} [\mathbf{e}_{(a)}^j \partial_{(c)} \mathbf{e}_{(b)}^j - \mathbf{e}_{(b)}^i \partial_{(c)} \mathbf{e}_{(a)}^i], \quad (5.21)$$

and

$$\Delta \equiv \partial_i [\mathbf{e}_{(a)}^i \mathbf{e}_{(a)}^j \partial_j]$$

is the operator of Beltrami – Laplace.

With the help of Legendre transformation $v^2/N = pv - Np^2/4$ we define momenta

$$p_{(a)(b)} = \frac{v_{(a)(b)}}{3}, \tag{5.22}$$

$$p_D = 2v_D, \tag{5.23}$$

$$p_Q = 2v_Q, \tag{5.24}$$

$$p_{A(b)} = v_{A(b)}. \tag{5.25}$$

As a result, the action (5.15) takes the Hamiltonian form

$$W = \tag{5.26}$$

$$= \int d^4x \left[p_Q \partial_0 Q + p_{(a)(b)} \omega_{(a)(b)}^R(\partial_0) + p_{A(b)} \partial_0 A_{(b)} - p_D \partial_0 D - \mathcal{C} \right],$$

$$\mathcal{C} = \tag{5.27}$$

$$= N\mathcal{H} + N_{(b)}T_{(b)} + A_{(0)}\partial_{(b)}p_{A(b)} + \lambda_{(0)}p_D + \lambda_{(b)}\partial_k \mathbf{e}_{(b)}^k + \lambda_A \partial_{(b)}A_{(b)},$$

where N , $N_{(b)}$ and $A_{(0)}$ are Lagrangian multipliers, variation in which yields the first kind constraints on the classification of Dirac [4], and $\lambda_{(0)}$, $\lambda_{(b)}$ and λ_A are Lagrangian multipliers for the second kind of constraints

$$\partial_k \mathbf{e}_{(b)}^k = 0, \tag{5.28}$$

$$p_D = 0. \tag{5.29}$$

First three constraints (5.28) fix space coordinates, and the constraint (5.29) is known as the minimum condition of three-dimensional hypersurface embedded in a four-dimensional pseudo-Riemannian space. In the Lagrangian formulation the constraint (5.29) looks like an equation

on the divergence of the shift-vector

$$\partial_0(e^{-3D}) + \partial_l(N^l e^{-3D}) = 0. \quad (5.30)$$

Magnitudes

$$\mathcal{H} = -\frac{\delta W}{\delta N} = \mathcal{H}_D + \mathcal{H}_g + \mathcal{H}_A + \mathcal{H}_Q, \quad (5.31)$$

where

$$\mathcal{H}_D = -\frac{p_D^2}{4} - \frac{4}{3}e^{-7D/2}\Delta e^{-D/2}, \quad (5.32)$$

$$\mathcal{H}_g = \left[6p_{(a)(b)}^2 + \frac{e^{-4D}}{6} {}^{(3)}R(\mathbf{e}) \right], \quad (5.33)$$

$$\mathcal{H}_A = \frac{e^{-2D}}{2} \left[p_{i(A)} p_{(A)}^i + F_{ij} F^{ij} \right], \quad (5.34)$$

$$\mathcal{H}_Q = e^{-2D} \left[e^{2D} \frac{p_Q^2}{4} + e^{-2D} (\partial_{(b)} Q)^2 \right], \quad (5.35)$$

and

$$T_{(0)(a)} = -\mathbf{e}_{(a)}^i \frac{\delta W}{\delta N_i} = -\partial_{(b)} p_{(b)(a)} + \tilde{T}_{(0)(a)}, \quad (5.36)$$

where

$$\tilde{T}_{(0)(a)} = \sum_{F=\bar{\phi}, \bar{Q}, \bar{F}} p_F \partial_{(a)} F \quad (5.37)$$

are components of the tensor of energy – momentum.

Conditions of the components of the energy – momentum are equal to zero

$$\mathcal{H} = 0, \quad (5.38)$$

$$T_{(0)(a)} = 0 \quad (5.39)$$

and were called by Dirac as primary constraints of the first kind. In accordance with it, the first of these conditions (5.38) is called a *Hamiltonian constraint*, by analogy with the corresponding condition for a relativistic particle. Recall that the Hamiltonian of a relativistic particle is a solution of the Hamiltonian constraint, relative to the momentum canonically conjugated to the evolution parameter in the space of events. In this way of explicit solutions of the primary constraints of the first kind there is one of the central problems of relativistic theories of gravity – a choice of a parameter of evolution in the field space of events.

As for the explicit solution of the second constraint (5.39), it is convenient to use the expansion

$$N_{(b)} = N_{(b)}^{\parallel} + N_{(b)}^{\perp}, \quad (5.40)$$

$$\partial_{(b)} N_{(b)}^{\parallel} = \partial_j N^j, \quad (5.41)$$

$$\partial_{(b)} N_{(b)}^{\perp} = 0, \quad (5.42)$$

$$p_{(b)(a)} = p_{(b)(a)}^{\perp} + \partial_{(a)} f_{(b)}^{\perp} + \partial_{(b)} f_{(a)}^{\perp}. \quad (5.43)$$

Square of the momentum in the equation (5.33) is possible to be represented as

$$p_{(b)(a)}^2 = (p_{(a)(b)}^{\perp})^2 + [\partial_{(a)} f_{(b)}^{\perp} + \partial_{(b)} f_{(a)}^{\perp}]^2, \quad (5.44)$$

where $f_{(a)}^{\perp}$ satisfies the equation

$$\left[\Delta f_{(a)}^{\perp} + \partial_{(a)} \partial_{(b)} f_{(a)}^{\perp} \right] = \tilde{T}_{(0)(a)}, \quad (5.45)$$

which follows from (5.39) after a substitution of (5.43).

The constraint of the second class (5.29) leads to one more sec-

ondary constraint

$$\frac{\delta W}{\delta D} = -T_D = 0,$$

namely,

$$(\partial_\tau - \mathcal{N}_{(b)}\partial_{(b)})p_D = T_D, \quad (5.46)$$

where

$$T_D = \frac{4}{3} \left[7\mathcal{N}e^{-7D/2}\Delta e^{-D/2} + e^{-D/2}\Delta[\mathcal{N}e^{-7D/2}] \right] - \\ -\mathcal{N}\partial_D[\mathcal{H}_g + \mathcal{H}_A + \mathcal{H}_Q].$$

We here just adapted the standard Hamiltonian formulation [2] of the theory of gravitation to Cartan's forms. In terms of these forms, the curvature takes the bilocal form. The action of such a theory describes a physical system as a squeezed oscillator [9]. It gives hope to construct a quantum theory of such a system, if we can solve the problems of the standard Hamiltonian formulation at the level of Cartan's forms.

5.4 Problems of Hamiltonian formulation

Let us list these problems.

1. The first of them is a problem of unique determination of non-zero Hamiltonian as the generator of evolution. The fact that the General Relativity is a singular theory with primary and secondary first-class constraints. The Hamiltonian as a constraint is equal to zero. As a result, under the constraints $\mathcal{C} = 0$ the action takes a form

$$\begin{aligned}
 W_{C=0} &= & (5.47) \\
 &= \int d^4x \left[p_{(a)(b)} \omega_{(a)(b)}^R(\partial_0) + p_Q \partial_0 Q + p_{A(b)} \partial_0 A_{(b)} - p_D \partial_0 D \right],
 \end{aligned}$$

where all of the canonical momenta and velocities satisfy the conditions of constraints [10]. This fact complicates the unambiguous definition of the generator of the evolution for a quantum state in a corresponding quantum theory.

2. The second problem is the self-consistency of the perturbation theory. As it was noticed as early as by K. Kuchař [11], the lapse function N generally is not included in the linearized constraint equations. That is non-self-consistency that in turn greatly impedes formulation of a perturbative quantum theory. Indeed, the metric representation of the functional of the state is based on the assumption that the components of the metric tensor are taken as independent variables. In the classical theory this hypothesis was formulated as a “thin sandwich theorem”, according to the

initial values of the metric tensor together with its derivatives that uniquely (under suitable boundary conditions) determine the space-time metric. It is supposed that by setting the initial hypersurface the metric tensor together with its derivatives and using four constraint equations, one can determine the four unknown ones – the lapse function and the shift-vector, that is to define a fully 4-metric of space-time. In the linear approximation, this theorem is violated and it is necessary somehow to fix the lapse function and the shift-vector. From here one can conclude that in the linear approximation there is not enough information to determine, for example, the lapse function by a given metrics and its time derivative.

3. One more problem is the problem of reduction. It means a separation of the dynamical variables of the theory on the constraints surfaces of the excess parameters of gauge transformations. Of course, this problem is related to the previous two. There are two ways to solve this problem. The first consists of imposition of additional gauge conditions to exclude extra variables. The second way is solving of the constraints. To the advantages of the first method one should notice its convenience and simplicity, because, as usual, such conditions are chosen that substantially enable calculations, but its drawback is in quite narrow applicability of such gauge and lack of confidence that the particular gauge does not spoil the “true” dynamics. The method of solving of the constraints, if it could be carried out, should be ideal for

researchers [12, 13, 14].

4. Structure of the four local constraints of the first kind (5.28) and (5.29) does not reflect the structure of diffeomorphisms of the Hamiltonian formulation of the theory of gravity. Recall that in the Hamiltonian formulation we have one global (5.6) and three local diffeomorphisms (5.7).
5. Of course, on the way of constructing the quantum gravity there are a number of other problems of fundamental as well as technical character. Among them we mention nonrenormalizability of the theory associated with the dimension of Newton's constant, and the questions of interpretation of the vector of quantum state [15] describing the quantum Universe. For the latter, there are no outside classic instruments.
6. The local condition of the minimal surface $v_D = 0$ (5.29) leads to the absence of any dynamics including cosmological².
7. The class of functions of standard perturbation theory [16]

$$g_{\mu\nu}(x^0, \mathbf{x}) = \eta_{\mu\nu} + O(1/|\mathbf{x}|),$$

where $\eta_{\mu\nu} = \text{Diag}:(1, -1, -1, -1)$, eliminates the cosmological evolution. Recall that since the pioneering results of Friedmann

²In the General Relativity, this statement can be summarized as follows: in non static space of the General Relativity with a closed family of hypersurfaces $t = \text{const}$ and non-zero energy – momentum tensor of matter, there is no global time-like congruence (*id est*, a continuous family of time-like lines) that a field of unit tangent vectors to this congruence would satisfy to the following properties: 1) tensor of the angular velocity is equal to zero, 2) trace of the tensor of velocities of deformations is also equal to zero [12, 13].

and, continuing their modern development [17, 18, 19, 20], the cosmological evolution is introduced into the theory of gravity by nonperturbative infrared dynamics of the metric tensor $g_{\mu\nu}(x^0) \neq \eta_{\mu\nu}$ with a finite time interval, and a finite volume of space.

In other words, the above list of problems is proposed to be solved with the introduction of the zero harmonic of the dilaton to the action (4.49). In the future, as we said above, we call the equation (5.38) not the zero Hamiltonian, but the *Hamiltonian constraint*, and solve the Hamiltonian constraint relative to one of the canonical momenta, in complete analogy with solution of the equation of mass surface in the Special Theory of Relativity. Then, the canonical momentum must be associated with the Hamiltonian of the reduced system (which on the solutions of the classical equations will be identified with the energy of the system). Canonically conjugated magnitude to the Hamiltonian must be a scalar (or scalar density) with respect to kinematic transformations. Zero harmonic of the dilaton as a conformal multiplier, extracted from the metric, is precisely such a quantity [20].

Here we list the solutions to these problems that have been given at the level of a model miniverse:

1. The information capacity of a relativistic theory with constraints is much more than the information contained in a non-relativistic theory. It is sufficient to say that the relativistic theory of gravitation (as well as the General Relativity and other theories) has *three spaces*:
 - 1) *pseudo-Riemannian*, introduced by Einstein,

- 2) *tangent*, introduced by Fock, and
- 3) *space of events*, introduced by Bryce De Witt.

In all of these spaces there is their own evolution parameter:

- 1) a coordinate time as an object of the general coordinate transformations,
- 2) a geometric interval (or components of the Fock's frame), and
- 3) dynamic evolution parameter in the space of events, respectively.

A non-zero Hamiltonian as the generator of evolution in the space of events is uniquely determined, if we specify the dynamic evolution parameter in this space, solve the equation of the Hamiltonian constraint and implement the primary and secondary quantization to establish a stable vacuum.

2. A lapse function N is included into the number of the observed ones only in the form of a factor to the differentials of the coordinate time. (In other words, only the invariants as components of the Fock's frame are measured).
3. A way of resolving constraints, if it is fully implemented, is ideal for the identification of the true dynamics of relativistic systems with constraints [12, 13]. This is the method we use hereinafter.
4. Next, we will distinguish the Hamiltonian constraint from a non-zero Hamiltonian, which is a solution to that constraint.

5. Questions of interpretation of the vector of quantum state, describing the quantum Universe, are solved taking into account the fact that the role of external classical instruments was played by Casimir vacuum.

A solving of the constraints is called a reduction of the extended phase space on the space of physical variables. The problem of solving of the constraint equations in terms of linear forms will be discussed in the next Section of this Chapter.

5.5 Exact solution of the Hamiltonian constraint

5.5.1 Statement of the problem

From the theory of nonlinear finite-parametric representations of the symmetry groups there was derived the action of the Conformal Theory of Gravity that contains all consequences of the General Relativity for the Solar system. However, the Conformal Theory of Gravity is significantly different from the General Relativity under descriptions of cosmological data.

- The action functional of the theory is defined on three spaces — Riemannian x^μ , tangent $\omega_{(\alpha)}$ and field $[D|F]$, each of them has its own evolution parameter: x^0 , $\omega_{(0)}$, and $\langle D \rangle$.

- The second difference is the identification of the observed distances with the conformal geometric interval. In contrast to the stan-

dard interval of the General Relativity, the geometric interval can describe all the observed data at different periods of the evolution of the Universe by the dominant Casimir vacuum energy of the empty Universe.

- The action of the Conformal Theory of Gravity becomes bilocal in terms of the Cartan forms and allows the quantization of gravitons directly in terms of Cartan.

- The fourth difference is that the observed values of the theory are the components of the Fock's frame in the tangent space of Minkowski $\omega_{(\alpha)}$, linear Cartan's forms and the field variables of the space of events $[D|F]$, so the solutions of the equations of the theory, including the constraints, can be expressed only in terms of these linear forms.

5.5.2 Lagrangian formalism

Recall that in the Hamiltonian formulation we have one global (5.6) and three local diffeomorphisms (5.7). In this Chapter we show that the Hamiltonian formulation of the theory of gravity also contains three local constraints in full accordance with the structure of diffeomorphisms (5.7) and one global constraint, as a consequence of the invariance of the theory relative to reparametrization of the coordinate time (5.6).

Invariance of the theory with respect to the reparametrization of the coordinate time (5.6) means that a time-like evolution parameter in the field space of events is identified with the zero harmonic of the dilaton field $\langle D \rangle$ [8]. Recall that the zero harmonic is determined by

“averaging” over the final volume $V_0 = \int_{V_0} d^3x$

$$\langle D \rangle(x^0) \equiv \frac{1}{V_0} \int_{V_0} d^3x D(x^0, x^1, x^2, x^3). \quad (5.48)$$

In astrophysics and cosmology, the zero harmonic of the dilaton (5.48) describes *luminosity*, defining it (with sign minus) as the logarithm of the cosmological scale factor

$$\langle D \rangle = -\ln a = \ln(1+z), \quad (5.49)$$

where $z = (1-a)/a$ is a redshift. In theories of gravity, indeed, the zero harmonic of the dilaton plays the role of a parameter of evolution in the field space of events.

The non-zero harmonics of the dilaton, which we have denoted above as \bar{D} , satisfy an orthogonality condition with zero harmonic:

$$\int_{V_0} d^3x \bar{D} = 0.$$

In force of the condition of orthogonality of harmonics, nonzero harmonics from this zero harmonics are independent. This means that the non-zero harmonics \bar{D} have zero velocities (5.30)

$$v_{\bar{D}} = \frac{1}{N} [\partial_0(e^{-3D}) + \partial_l(N^l e^{-3D})] = 0 \quad (5.50)$$

and momenta [8]

$$p_{\bar{D}} = 2v_{\bar{D}} = 0$$

(see Eqs. (5.16), (5.23) and (5.29)). A condition of zero velocities (5.50) in the Lagrangian formalism looks like an equation of the divergence of the shift-vector.

We can choose the shift-vector divergence so that non-zero harmonics of the dilaton, as we already mentioned above, will be Newtonian-like potentials just those that increase twice the angle of deflection of light by the gravitational field of the Sun in comparison with the theory of Newton. Thus, the action (4.49) becomes the sum of two terms

$$W = W_G + \overline{W}, \quad (5.51)$$

where

$$W_G = - \int dx^0 \left[\frac{d\langle D \rangle(x^0)}{dx^0} \right]^2 \int d^3x \frac{1}{N} \equiv \int dx^0 L_G \quad (5.52)$$

is the kinetic part of the action for the zero harmonics of the dilaton and an expression \overline{W} coincides with the action (5.15), where the speed of a local volume element (5.16) is equal to zero. Thus, the equation of the theory of gravity, obtained by variation of the action (5.51) on lapse function

$$N \frac{\delta W_G}{\delta N} = -N \frac{\delta \overline{W}}{\delta N} \equiv N \tilde{\mathcal{H}},$$

takes a form

$$\frac{1}{N} \left[\frac{d\langle D \rangle(x^0)}{dx^0} \right]^2 = N \tilde{\mathcal{H}}; \quad (5.53)$$

here

$$\tilde{\mathcal{H}} = -\frac{4}{3} e^{-7D/2} \Delta e^{-D/2} + \overline{\mathcal{H}}, \quad (5.54)$$

$$\overline{\mathcal{H}} = \mathcal{H}_g + \mathcal{H}_A + \mathcal{H}_Q, \quad (5.55)$$

$$\mathcal{H}_g = \frac{1}{6} \left[v_{(a)(b)} v_{(a)(b)} + e^{-4D} R^{(3)}(\mathbf{e}) \right], \quad (5.56)$$

$$\mathcal{H}_A = \frac{1}{2} \left[v_{(b)(A)}^2 + F_{ij} F^{ij} \right], \quad (5.57)$$

$$\mathcal{H}_Q = e^{-2D} (v_Q)^2 + e^{-2D} (\partial_{(b)} Q)^2 \quad (5.58)$$

– Hamiltonian densities at zero speed of the local volume element in the expression (5.32). Averaging the equation (5.53) on three-dimensional volume (see (5.48)) and using the definitions

$$\left\langle \frac{1}{N} \right\rangle \equiv \frac{1}{N_0}; \quad N \equiv N_0 \mathcal{N} \quad \implies \quad \left\langle \frac{1}{\mathcal{N}} \right\rangle = 1 \quad (5.59)$$

and $N_0 dx^0 = d\tau$, we get the diffeoinvariant global constraint equation

$$\left[\frac{d\langle D \rangle(x^0)}{N_0 dx^0} \right]^2 \equiv \left[\frac{d\langle D \rangle(\tau)}{d\tau} \right]^2 = \langle \mathcal{N} \tilde{\mathcal{H}} \rangle. \quad (5.60)$$

Substituting it into equation (5.53), we get surprisingly simple equation for diffeoinvariant local lapse function

$$\left[\frac{d\langle D \rangle(\tau)}{d\tau} \right]^2 = \mathcal{N}^2 \tilde{\mathcal{H}}. \quad (5.61)$$

Since the left side of the equality does not depend on the spatial coordinates, a normalization condition $\langle \mathcal{N}^{-1} \rangle = 1$ allows us to express the diffeoinvariant local lapse function (5.59) in the explicit form

$$\mathcal{N} = \frac{\langle \sqrt{\tilde{\mathcal{H}}} \rangle}{\sqrt{\tilde{\mathcal{H}}}}. \quad (5.62)$$

Substituting (5.62) into Eq.(5.61), we obtain the required global constraint equation

$$\left[\frac{d\langle D \rangle(\tau)}{d\tau} \right]^2 = \langle \sqrt{\tilde{\mathcal{H}}} \rangle^2. \quad (5.63)$$

The solution of the equation (5.60) gives the cosmological dependence of the zero harmonic of the dilaton on the time interval of luminosity:

$$\tau = \int_{\langle D \rangle_I}^{\langle D \rangle_0} d\langle D \rangle \langle \sqrt{\tilde{\mathcal{H}}} \rangle^{-1}, \quad (5.64)$$

where $\langle D \rangle_I$, $\langle D \rangle_0$ are the initial and final data, correspondingly. The dependence of the zero harmonic of the dilaton from the time interval of luminosity, in the exact theory of gravitation, is an analogue of the Hubble law in cosmology.

The equation of motion for the zero harmonic of the dilaton

$$\frac{\delta W}{\delta \langle D \rangle} \equiv -T_{\langle D \rangle} = 0$$

coincides with the equation obtained by differentiating on τ of the global constraint equation (5.60):

$$\frac{d^2 \langle D \rangle}{(d\tau)^2} = \frac{d\langle \sqrt{\tilde{\mathcal{H}}} \rangle}{d\tau} = \frac{1}{\sqrt{\tilde{\mathcal{H}}}} \frac{d\langle \tilde{\mathcal{H}} \rangle}{d\tau} = \frac{d\langle \tilde{\mathcal{H}} \rangle}{d\langle D \rangle}.$$

In the case of dominance of the vacuum energy, the right hand side is equal to zero, and we get an empty model of the Universe, which is considered in detail in Chapter 6. For non-zero harmonics an equation of motion

$$\frac{\delta W}{\delta \bar{D}} = -T_{\bar{D}} = 0$$

takes a form

$$T_{\bar{D}} = T_D - \langle T_D \rangle = 0, \quad (5.65)$$

$$T_D = \frac{4}{3} \left[7\mathcal{N}e^{-7D/2} \Delta e^{-D/2} + e^{-D/2} \Delta [\mathcal{N}e^{-7D/2}] \right] - \mathcal{N} \frac{\partial \bar{\mathcal{H}}}{\partial D}, \quad (5.66)$$

where $\bar{\mathcal{H}}$ is given by the equations (5.55) – (5.58). Thus, solving the constraints, we expressed all components of the metric through the components of the energy-momentum tensor and linear forms of Cartan (5.10)

$$\tilde{ds}^2 = e^{-4D} \frac{(\sqrt{\tilde{\mathcal{H}}})^2}{\tilde{\mathcal{H}}} d\tau^2 - \quad (5.67)$$

$$-\left(dX_{(b)} - X_{(c)}[\omega_{(c)(b)}^R(d) + \omega_{(c)(b)}^L(d)] - \mathcal{N}_{(b)}d\tau\right)^2.$$

Square interval in diffeoinvariant form on the surface of constraint (5.59) depends only on the indices of the tangent space.

5.5.3 Hamiltonian formalism

For Hamiltonian formulation of the theory, we introduce momenta of the fields according to the definitions (5.22) – (5.25). The momentum of the global component of the dilaton

$$P_{\langle D \rangle} = \frac{\partial L_G}{\partial \langle d \langle D \rangle / dx^0 \rangle} = -2V_0 \frac{d \langle D \rangle}{N_0 dx^0} \equiv V_0 p_{\langle D \rangle}, \quad (5.68)$$

the momentum of the scalar field

$$p_Q = 2v_Q = \frac{2}{N} \left[(\partial_0 - N^l \partial_l) Q + \frac{1}{3} \partial_l N^l \right], \quad (5.69)$$

momenta of the photon field

$$p_{A(b)} = v_{A(b)} = \frac{1}{N} \mathbf{e}_{(a)}^i \left[\partial_0 A_i - \partial_i A_0 + F_{ij} N^j \right], \quad (5.70)$$

and momenta of the gravitational field

$$p_{(a)(b)} = \frac{v_{(a)(b)}}{3} \equiv p_{(b)(a)}^\perp + \partial_{(a)} f_{(b)}^\perp + \partial_{(b)} f_{(a)}^\perp, \quad (5.71)$$

$$v_{(a)(b)} = \frac{1}{N} \left[\omega_{(a)(b)}^R (\partial_0 - N^l \partial_l) + \partial_{(a)} N_{(b)}^\perp + \partial_{(b)} N_{(a)}^\perp \right]. \quad (5.72)$$

In the Hamiltonian formalism, the equation for the shift-vector has the form (5.36)

$$\mathbf{e}_{(b)}^i \frac{\delta W}{\delta N_i} = -T_{(0)(a)} = \partial_{(b)} p_{(b)(a)} - \tilde{T}_{(0)(a)} = 0, \quad (5.73)$$

where

$$\tilde{T}_{(0)(a)} = \sum_{F=A^T, \bar{Q}} p_F \partial_{(a)} F \quad (5.74)$$

– components of the tensor of energy - momentum of the photon and the scalar field. The condition of the transverseness of the graviton

$$\partial_{(a)} \omega_{(a)(b)}^R = 0$$

allows us to express the transverse part of the shift-vector through the components of the tensor of energy-momentum of the photon and the scalar field, while the divergence of the shift-vector (*id est*, its longitudinal part) is given by the condition of zero momentum of the local dilaton (5.50)

$$p_{\bar{D}} = 2v_{\bar{D}} = \frac{2}{N} [\partial_0(e^{-3D}) + \partial_l(N^l e^{-3D})] = 0. \quad (5.75)$$

Thus, we determine all these components of the photon and the gravitational field, except for the longitudinal component of the photon and the anti-symmetric linear form of the gravitational field $\omega_{(c)(b)}^L(d)$. However, these components in the kinetic terms of the action are absent, and they are determined by the distribution of the external currents and the matter, respectively.

Thus, on the level of constraints $\mathcal{C} = 0$ the action is

$$W_{\mathcal{C}=0} = \quad (5.76)$$

$$= \int d^3x \left[\int [p_{(a)(b)} \omega_{(a)(b)}^R(d) + p_Q dQ + p_{A(b)} dA_{(b)}] \right] - \int P_{\langle D \rangle} d\langle D \rangle,$$

where the canonical momentum of the dilaton $P_{\langle D \rangle}$ satisfies the Hamiltonian constraint

$$P_{\langle D \rangle}^2 = \left[2 \int d^3x \frac{d\langle D \rangle(\tau)}{d\tau} \right]^2 = \left[2 \int d^3x \sqrt{\tilde{\mathcal{H}}} \right]^2 \quad (5.77)$$

and plays a role of the generator of evolution. The value of the momentum of the zero harmonics on the solutions of the equations of motion is the energy of the Universe in this space of events. This is one of ways to solve the problem of non-zero energy as well in General Relativity [8].

Thus, if we leave the zero harmonic of the dilaton $\langle D(x^0) \rangle$, the homogeneous lapse function $N_0(x^0)$ and the vacuum energy of quantum oscillators, we obtain a simple dynamical system, known in literature [21] as a *miniuniverse* (see Appendix E).

In the next two chapters *miniuniverse* will be considered as an example, to demonstrate the ability of solving of most of the problems listed above.

5.6 Summary

The quantization of any dynamical system involves a Hamiltonian description of the system. This chapter is focused on adapting Dirac's Hamiltonian approach to the General Relativity for the affine Conformal Theory of Gravity, presented in Chapter 4. All the problems of a unique definition of the energy and time, inherent in the Hamiltonian description of the General Relativity, are inherited by the Conformal Theory of Gravity. However, as it has been shown, these problems

have a unique solution by introducing the concept of zero harmonic of the dilaton and postulating of the existence of vacuum as the state with the lowest energy, in full accordance with the dimension of diffeomorphisms of the Hamiltonian evolution. The Hamilton equation of constraint in presence of the zero harmonic of the dilaton becomes algebraic and exactly resolved relative to the canonical momentum of the zero harmonic. This canonical momentum becomes a generator of evolution of the Universe in a field space of events and defines the energy of the Universe on the solutions of the classical equations. The zero harmonic of the dilaton and the energy of the vacuum set the empty Universe model, which Chapter 6 discusses.

Bibliography

- [1] Rashevsky, P.K.: *Riemannian Geometry and Tensorial Analysis*. Nauka, Moscow (1967)
- [2] Arnowitt, R., Deser, S., Misner, C .W.: *The dynamics of general relativity*. In Gravitation: an Introduction to Current Research. Witten, L. (ed.). Wiley, New York (1963)
- [3] Zelmanov, A.L., Agakov, V.G.: *Elements of General Relativity*. Nauka, Moscow (1989)
- [4] Dirac, P.A.M.: *On generalized Hamiltonian dynamics*. Can. J. Math. **2**, 129 (1950).
- Dirac, P.A.M.: *Generalized Hamiltonian dynamics*. Proc. Roy. Soc. London. **A 246**, 326 (1958).
- Dirac, P.A.M.: *The theory of gravitation in Hamiltonian form*. Proc. Roy. Soc. London. **A 246**, 333 (1958).
- Dirac, P.A.M.: *Fixation of coordinates in the Hamiltonian theory of gravitation*. Phys. Rev. **144**, 924 (1959).
- Dirac, P.A.M.: *The Principles of Quantum Mechanics*. Oxford University Press (1982)

-
- [5] Sundermeyer, K.: *Constrained Dynamics*. Lecture Notes in Physics. Springer Verlag, Berlin – Heidelberg – New York (1982)
- [6] Faddeev, L.D., Slavnov, A.A.: *Gauge Fields. Introduction to Quantum Theory*. Addison – Wesley Publishing Co. (1991)
- [7] Henneaux, M., Teitelboim, C.: *Quantization of Gauge Systems*. Princeton University Press, Princeton (1992)
- [8] Pervushin, V.N., Arbuzov, A.B., Barbashov, B.M., Nazmitdinov, R.G., Borowiec, A., Pichugin, K.N., and Zakharov, A.F.: *Conformal and affine Hamiltonian dynamics of general relativity*. Gen. Relativ. & Gravit., **44**, 2745 (2012)
- [9] Arbuzov, A.B., Barbashov, B.M., Nazmitdinov, R.G., Pervushin, V.N., Borowiec, A., Pichugin, K.N., Zakharov, A.F.: *Conformal Hamiltonian dynamics of General Relativity*, Phys. Lett. **B 691**, 230 (2010).
[arXiv:1007.0293 [gr-qc]]
- [10] Isham, C.J., Kuchař, K.V.: *Representations of spacetime diffeomorphisms. I. Canonical parametrized field theories*. Annals of Physics. **164**, 288 (1985).
Isham, C.J., Kuchař, K.V.: *Representations of spacetime diffeomorphisms. II. Canonical geometrodynamics*. Annals of Physics. **164**, 316 (1985)
- [11] Kuchař, K.V.: *Ground state functional of the linearized gravitational field*. Journ. f Math. Phys. **11**, 3322 (1970)

- [12] Pervushin, V.N., Smirichinski, V.I.: *Dynamics of Einstein's equations in terms of the eigenvalues of the first and second curvature*. Phys. of Part. & Nuclei. **61**, 142 (1998)
- [13] Pervushin, V.N., Smirichinski, V.I.: *The trace of external curvature and dynamics of Einstein equations*. Phys. of Part. & Nuclei. **61**, 142 (1998)
- [14] Pervushin, V.N., Smirichinski, V.I.: *Conformal symmetry and Higgs effect in quantum cosmology*. Mod. Phys. Lett. **A 13**, 119 (1998)
- [15] Kuchař, K.V.: *Time and interpretations of quantum gravity*. Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics. (Eds. Kunstatter, G., Vincent, D., and Williams, J.). World Scientific (1992)
- [16] Faddeev, L.D., Popov, V.N.: *Covariant quantization of the gravitational field*. Physics–Uspekhi. **16**, 777 (1974)
- [17] Lifshitz, E.M., Khalatnikov, I.M.: *Problems of relativistic cosmology*. Physics–Uspekhi. **6**, 495 (1964)
- [18] Mukhanov, V.F., Feldman, H.A., and Brandenberger, R.H.: *Theory of cosmological perturbations*. Phys. Rep. **215**, 203 (1992)
- [19] Giovannini, M.: *Theoretical tools for the physics of CMB anisotropies*. Int. J. Mod. Phys. **D 14**, 363 (2005).
- [arXiv: astro-ph/0412601]

[20] Barbashov, B.M., Pervushin, V.N., Zakharov, A.F., Zinchuk, V.A.: *Hamiltonian cosmological perturbation theory*. Phys. Lett. **B 633**, 458 (2006).

[arXiv: hep-th/0501242]

[21] Misner, Ch.: *Quantum cosmology*. I. Phys. Rev. **186**, 1319 (1969)

Chapter 6

A model of an empty Universe

6.1 An empty Universe

In the first five chapters, we set out the elements of an alternative physics program which appeared in the period from 1915 till 1974 to describe and classify the experimental data. This program is based on the principles of symmetry of the initial data. The essence of this program is as follows.

1. There are elementary objects (such as quarks or twistors) as fundamental representations of the group G : $(SU(2) \otimes SU(2))$ or $(A(4) \otimes C)$.

2. Mesons or space - time are formed from these elementary objects as adjoint representations of the group G , that allows the determination of the vacuum stability subgroup H and the corresponding coset $K = G/H$.

3. Linear Cartan forms, describing an arbitrary motion (translation and rotation) of the frame in this coset, are derived on the algebra of G .

4. Lagrangians of the chiral theory and the theory of gravity are constructed with the help of these forms.

5. Cartan forms are invariant under gauge transformations.

Next, we demonstrate the ability of feasibility of this program to describe the observational data on the example of *miniuniverse*. Here we will understand the *miniuniverse* as the theory of gravitation developed above, in which we keep only the zero harmonics of the dilaton $\langle D(x^0) \rangle$, the homogeneous lapse function $N_0(x^0)$, and the vacuum energy of the quantum oscillators

$$\rho_{\text{Cas}}^{\tau}(a) = \frac{1}{V_0} \sum_f H_{(f)\text{Cas}}^{\tau} = \frac{1}{V_0} \sum_{q,f} \frac{\omega_{q,f}}{2}, \quad (6.1)$$

where the vacuum energy of a finite universe in this model appears as a sum of vacuum energies of all fields f . In quantum field theory, this amount is called the Casimir energy [1]. Recall that the vacuum energy arises under the normal ordering of the field operators after separation of them to positive and negative frequency parts. In particular, the energy of the sum of the oscillators has the form

$$\begin{aligned} \frac{1}{2} \sum_n (p_n^2 + \omega_n^2 q_n^2) &= \frac{1}{2} \sum_n \omega_n (a_n^+ a_n^- + a_n^- a_n^+) = \\ &= \sum_n \omega_n a_n^+ a_n^- + \sum_n \frac{\omega_n}{2}. \end{aligned}$$

The latter term is called the vacuum energy, defined as the state of a set of oscillators with the lowest energy.

Let us leave the vacuum energy in the action defined by the formulae (5.51)–(5.52), instead of matter fields. Then we get the cosmological model of the homogeneous empty Universe, described by the action (up to a total derivative)

$$W_{\text{Universe}} = -V_0 \int_{\tau_I}^{\tau_0} \underbrace{dx^0 N_0}_{=d\tau} \left[\left(\frac{d\langle D \rangle}{N_0 dx^0} \right)^2 + \rho_{\text{Cas}}^\tau(\langle D \rangle) \right]. \quad (6.2)$$

The value of the Casimir energy of field oscillators $\rho_{\text{Cas}}^\tau(\langle D \rangle)$ is inversely proportional to the size of the spatial volume. Therefore, in the classical limit of an infinite volume, the action (6.2) is equal to zero

$$\lim_{V_0 \rightarrow \infty} W_{\text{Universe}} = 0.$$

Varying the action (6.2) by variables $\langle D \rangle$ and N_0 , we get two equations

$$\frac{\delta W_{\text{Universe}}}{\delta \langle D \rangle} = 0 \quad \Rightarrow \quad 2 \frac{d}{d\tau} \left[\frac{d\langle D \rangle}{d\tau} \right] = \frac{d\rho_{\text{Cas}}^\tau}{d\langle D \rangle}, \quad (6.3)$$

$$\frac{\delta W_{\text{Universe}}}{\delta N_0} = 0 \quad \Rightarrow \quad \left[\frac{d\langle D \rangle}{d\tau} \right]^2 = \rho_{\text{Cas}}^\tau. \quad (6.4)$$

The second equation is the integral of the first one and is treated as a constraint equation of the initial data (initial momentum) of the dilaton. According to the second Noether's theorem, the second equation is a consequence of the invariance of the action (6.2), relatively to reparametrization of the coordinate evolution parameter:

$$x^0 \rightarrow \tilde{x}^0 = \tilde{x}^0(x^0).$$

The second equation, rewritten in terms of the cosmological scale factor

$a = \exp(-\langle D \rangle)$ and conformal density

$$\rho_{\text{Cas}}^\eta(a) = \frac{\rho_{\text{Cas}}^\tau}{a^2} \equiv \frac{H_0}{d_{\text{Cas}}(a)}, \quad (6.5)$$

coincides with the Friedmann equation

$$\left[\frac{da}{d\eta} \right]^2 = \rho_{\text{Cas}}^\eta(a), \quad (6.6)$$

where $d_{\text{Cas}}(a)$ in (6.5) is the conformal size of the Universe and H_0 is the Hubble parameter, and $a = (1+z)^{-1}$ is a scale cosmological factor and z is a redshift.

The solution of the Friedmann equation (6.6) gives a conformal horizon

$$d_{\text{horison}}(a) = 2r_{\text{horison}}(a) = 2 \int_0^a d\bar{a} [\rho_{\text{Cas}}^\eta(\bar{a})]^{-1/2}. \quad (6.7)$$

The horizon is defined as the distance that the photon runs on the light cone $d\eta^2 - dr^2 = 0$ during the life - time of the Universe. In this case, a conformal horizon coincides with apparent size of the Universe $d_{\text{Cas}}(a)$ in (6.5):

$$d_{\text{Cas}}(a) = d_{\text{horison}}(a). \quad (6.8)$$

Solutions of the equations (6.5), (6.6), (6.7), and (6.8)

$$d_{\text{horison}}(a) = \frac{a^2}{H_0} \Rightarrow \rho_{\text{Cas}}^\tau = H_0^2 \equiv \rho_{\text{cr}} \quad (6.9)$$

give the Hubble diagram of the description of Supernovae [2, 3] in Conformal Cosmology [4, 5, 6, 7, 8, 9], obtained as a consequence of the Dirac definition of measured intervals in the approximation of the

empty space. In terms of conformal variables, the solution corresponds to the equation of rigid state of an *empty space* (6.8)

$$\left[\frac{da}{d\eta} \right]^2 = \frac{\rho_{\text{cr}}}{a^2}. \quad (6.10)$$

In terms of the values of luminosity, where ρ_{Cas}^{τ} is a constant,

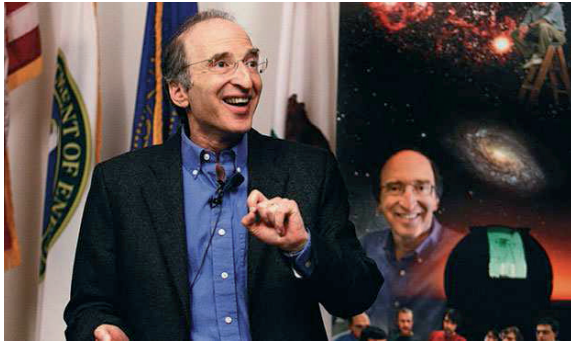
$$\frac{d\rho_{\text{Cas}}^{\tau}}{d\langle D \rangle} = 0,$$

we get the inertial motion of the dilaton with an acceleration equal to zero. For an empty space, the obtained solution (6.10) describes the Supernova data in the Conformal Cosmology, where the measured distances are *longer* than those, used in the Standard Cosmology. Thus, according to the principles of conformal and affine symmetries, namely, this *remoteness* of Supernovae were discovered by observers [2, 3].

6.2 The Supernovae data in Conformal Cosmology

The Nobel Prize in Physics of 2011 was awarded to S. Perlmutter, A. Riess, and B. Schmidt for their work [2, 10, 11, 12] related to studying of Supernovae Type Ia to determine the parameters of cosmological models. Thus it was assumed that the maximum luminosity of Supernovae does not depend on the distance to them, but depends on the rate of change of the luminosity according to the so-called law of the Pskovski — Phillips [2], that is, they are the so-called “standard candles”. Studying distant Supernovae from Earth, observers found that these stars are at least a quarter fainter than predicted by the

theory which means that the stars are too far away. Thus, calculating parameters of expansion in cosmological models of Friedmann – Robertson – Walker [13] with an arbitrary equation of state of matter, the researchers found that in frames of the Standard Cosmology, this process is proceeded with acceleration.



Saul Perlmutter (born September 22, 1959) is an American astrophysicist and Nobel Prize in Physics 2011 (together with Brian Schmidt and Adam Riess) “for the discovery of the accelerated expansion of the Universe through observations of distant Supernovae”. Perlmutter grew up in the Philadelphia area Mount Airy, where he studied in an elementary school Greene Street Friends School and Germantown Friends School. In 1981 he graduated with honors from Harvard University. In 1986 in the University of California at Berkeley, Perlmutter received his PhD. His thesis was devoted to the problem of detection of objects – candidates for the role of Nemesis. Currently, Perlmutter heads the project “Supernova Cosmology Project” at the Lawrence Berkeley National Laboratory. His team together with a group of Brian Schmidt proved the existence of accelerated expansion of the Universe. Perlmutter is also a lead investigator in the Supernova/Acceleration Probe project, which aims to build a satellite dedicated to finding and studying more Supernovae in the dis-

tant Universe. This will help to better constrain the rate at which the Universe is accelerating. He is also a participant in the Berkeley Earth Surface Temperature project, which aims to increase our understanding of recent global warming through improved analyses of climate data.

It corresponds to the presence of a non-zero Lambda term¹. In this case, one speaks about the so-called dark energy. There is still an unresolved within the Standard cosmology, problem of the origin of matter with similar properties. This form of matter is not predicted even by representations of the Poincaré group.

On the other hand, there is a Conformal cosmological model [4], which allows us to describe the Supernova data without lambda-term, because in this model the observed distances are identified with conformal longer intervals. The authors of the discovery in paper [2] recognize the fact of existence of both alternative explanations, and compare interpretations of the results of observations with the Conformal cosmological model [4].

As we saw above, in the Conformal cosmological model, to explain far distances to Supernovae there are enough assumptions about the

¹Albert Einstein in paper (A. Einstein: *Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie*. Sitzungsber. d. Berl. Akad. **1**, 142 (1917)) was forced to introduce a universal λ -term in the equations of the theory from static requirements of the cosmological solutions, defensively only in that supplement where its covariance equations are not violated. Later, Einstein, closely acquainted with the work of Alexander Friedmann (Friedmann, A: *Über die Krümmung des Raumes*. Zs. für Phys. **10**, 377 (1922)) admitted the erroneousness of the introduction of λ -term, thereby, opening the way for the study of non-stationary models. After exploring the world with positive curvature, Alexander Friedmann in paper (Friedmann, A: *Über die Möglichkeit einer Welt mit konstanter negativer Krümmung des Raumes*. Zs. für Phys. **21**, 326 (1924)) obtained a cosmological solution with negative curvature.

dominance of the zero energy of the vacuum. According to quantum mechanics, in microworld, each particle has a zero energy of fluctuations of vacuum, which is called the Casimir energy [1].



Brian Paul Schmidt (born February 24, 1967) is a Distinguished Professor, Australian Research Council Laureate Fellow and astrophysicist at The Australian National University Mount Stromlo Observatory and Research School of Astronomy and Astrophysics. Schmidt, along with Riess and Perlmutter, jointly won the 2011 Nobel Prize in Physics for their observations which led to the discovery of the accelerating Universe. Schmidt attended Bartlett High School in Anchorage, Alaska, and graduated from it in 1985. He earned his BS (Physics) and BS (Astronomy) from the University of Arizona in 1989. He received his MA (Astronomy) in 1992 and then PhD (Astronomy) in 1993 from Harvard University. Schmidt was a Postdoctoral Fellow at the Harvard – Smithsonian Center for Astrophysics (1993-1994) before moving on to Mount Stromlo Observatory in 1995. Schmidt led the High-Z Supernova Search Team from Australia, and in 1998 with Adam Riess the first evidence was presented that the Universe's expansion rate is accelerating. The observations were contrary to the current theory that the expansion of the Universe should be slowing down; on the contrary, by monitoring the brightness and measuring the redshift of the Super-

novae, they discovered that these billion-year old exploding stars and their galaxies were accelerating away from our reference frame. Schmidt is currently leading the SkyMapper telescope Project and the associated Southern Sky Survey.



Adam Guy Riess (born 16 December 1969) is an American astrophysicist at the Johns Hopkins University and the Space Telescope Science Institute and is known for his research in using Supernovae as cosmological probes. Riess shared the 2011 Nobel Prize in Physics with Saul Perlmutter and Brian P. Schmidt for providing evidence that the expansion of the Universe is accelerating. Riess was born in Washington, D.C., grew up in Warren, New Jersey. Riess graduated from the Massachusetts Institute of Technology in 1992. He received his PhD from Harvard University in 1996; it resulted in measurements of over twenty new type Ia Supernovae and a method to make Type Ia Supernovae into accurate distance indicators by correcting for intervening dust and intrinsic inhomogeneities. Riess jointly led the study with Brian Schmidt in 1998 for the High- z Supernova Search Team. The team's observations were contrary to the current theory that the expansion of the universe was slowing down; instead, by monitoring the color shifts in the light from supernovas from Earth, they discovered that these billion-year old novae were still accelerating. This result was also found nearly simultaneously by the Supernova Cosmology Project, led by

Saul Perlmutter. The corroborating evidence between the two competing studies led to the acceptance of the accelerating Universe theory, and initiated new research to understand the nature of the Universe, such as the existence of dark matter.

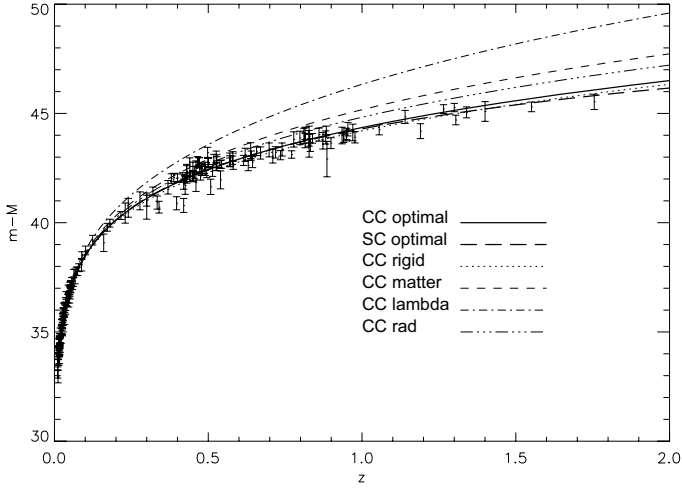


Figure 6.1: Hubble diagram, based on 73 data obtained by the collaboration SNLS. For the theoretical analysis, the model of a flat Universe in the Standard cosmology (SC) and the Conformal cosmology (CC) is used. The better agreement with the data demands the value of cosmological constant: $\Omega_{\Lambda} = 0,755$ and cold matter: $\Omega_m = 0,245$ in case of the Standard cosmology, while in the case of the Conformal cosmology, these data are consistent with the regime of nucleosynthesis and domination of the rigid state condition $\Omega_{\text{rigid}} = 0,755$.

In papers [8, 14] it was shown that with account of the great number of data of Supernovae, the interpretation of observational data with the Conformal cosmological model (solid curve in Fig. 6.1) is almost as good as the interpretation in the framework of models of Friedmann

– Robertson – Walker with a non-zero Lambda member (the dashed line in Fig. 6.1). According to the Conformal model, S. Perlmutter, A. Riess and B. Schmidt discovered the physical vacuum which is the Universe. For comparative analysis, data of the collaboration “Supernova Legacy Survey” (SNLS) are used [15]. Detection of Supernova explosions was carried out with the telescope CFHT (Canada – France – Hawaii), whereupon with modern telescopes, photometric and spectroscopic study of Supernovae were conducted. It is appropriate to remind the correct statement of the Nobel laureate in Physics Steven Weinberg² about interpretation of experimental data on redshifts. *“I do not want to give the impression that everyone agrees with this interpretation of the red shift. We do not actually observe galaxies rushing away from us; all we are sure of is that the lines in their spectra are shifted to the red, i.e. towards longer wavelengths. There are eminent astronomers who doubt that the red shifts have anything to do with Doppler shifts or with expansion of the universe”.*

If we identify the observed values with conformal variables (conformal time, conformal density, conformal temperature and Planck running mass), the evolution of the lengths in cosmology is replaced by the evolution of the masses. This identification means choosing the equations of General Relativity (GR) and the Standard Model (SM) in conformally invariant form, where the space scale factor scales all masses including the Planck mass. The initial values of the masses are much less than modern ones. In this case, the Planck epoch in the

²Weinberg, S.: *The First Three Minutes: A Modern View of the Origin of the Universe*. Basic Books, New York (1977).

early Universe loses its absolute predestination. It was shown [7, 16] that in the case of regime of rigid state equation the early Universe is a factory of the cosmological creation of massive vector bosons from the vacuum, when the Compton wavelength of these bosons coincides with the event horizon of the early Universe, so the conformal-invariant versions of the Standard Model and the General Relativity can, in principle, explain the origin of the observed matter as the final product of decay of primary bosons. In an evolving Universe as opposed to a stationary Universe, a part of photons is lost during their flight to the Earth. This is due to an increase in the angular size of the cone of light of emitted photons (absolute standard) or because of reduced angular size of the cone of light of absorbed photons (the relative standard), as shown in Fig. 6.2 for both cases.

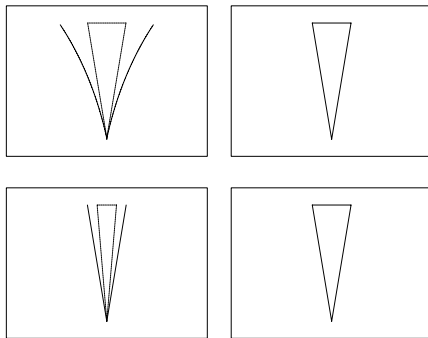


Figure 6.2: Comparison of case of a stationary Universe (right panel) to the case of evolving Universe with an absolute standard (top left panel), and with the case of the evolving Universe with the relative standard (bottom left panel).

To restore full luminosity for both standards (both absolute and relative) we have to multiply by a factor of coordinate distance $(1+z)^2$ [4], therefore the observational cosmology uses the luminosity distance ℓ , which is defined as the measured distance (r (2.71) or R (2.72)), multiplied by a factor $(1+z)^2$ for both standards

$$\ell_{\text{abs}}(z) = (1+z)^2 R(z) = (1+z)r(z), \quad (6.11)$$

$$\ell_{\text{rel}}(z) = (1+z)^2 r(z). \quad (6.12)$$

In the literature, the first case corresponds to the Standard cosmology (SC), the second – to the Conformal Cosmology (CC). So, for the relative standard we have an additional factor $(1+z)$, and relations (6.11) and (6.12) mean, that observational data are described by different regimes for different standards of measurement. In Fig. 6.1 [4] the results of the Standard and Conformal cosmologies for the relation between the effective magnitude and redshift are compared:

$$m(z) = 5 \log[H_0 \ell(z)] + \mathcal{M},$$

where \mathcal{M} is a constant, according to latest data for Supernovae [2, 10]. As we see from Fig. 6.1, in the region $0 \leq z \leq 2$ observational data, including the last point (SN 1997ff) with $z = 1,7$ [2], are interpolated in the Standard cosmology with the absolute standard and parameters

$$\Omega_{\text{rigid}} = 0, \quad \Omega_M \geq 0,245, \quad \Omega_\Lambda \leq 0,755 \quad (6.13)$$

as well as in the Conformal Cosmology with the relative standard and parameters

$$\Omega_{\text{rigid}} \geq 0,755, \quad \Omega_M \leq 0,245, \quad \Omega_\Lambda = 0. \quad (6.14)$$

Further, in case of the relative standard of measurements, the evolution of Supernovae does not contradict to the equation of state of primordial nucleosynthesis with the dependence of the cosmological factor from the observed conformal time

$$\tilde{a}(\eta) = [z + 1]^{-1}(\eta) = \sqrt{1 + 2H_0(\eta - \eta_0)}, \quad \Omega_{\text{rigid}} = 1. \quad (6.15)$$

From this relation it is easy to find the coordinate distance $r = \eta_0 - \eta$ as function of z :

$$H_0 r(z) = \frac{1}{2} \left[1 - \frac{1}{(1+z)^2} \right] = \frac{1}{(1+z)^2} \left(z + \frac{z^2}{2} \right), \quad (6.16)$$

which implies the dependence of the luminosity distance (6.12) from the redshift z :

$$\ell_{\text{rel}}(z) = (1+z)^2 r(z) = \frac{1}{H_0} \left[z + \frac{z^2}{2} \right]. \quad (6.17)$$

In Fig. 6.1 the function (6.17), arisen as a result of the solution of the equation of rigid state, is shown by the solid line and we see that the astrophysical data on Supernovae and primordial nucleosynthesis, recalculated in terms of the relative standard, testify that all evolution of the Universe takes place in the regime of dominance of rigid state (6.15) with relative density

$$\rho_{\text{rigid}}(a) = \frac{\rho_{\text{cr}}}{a^2} = \frac{H_0^2}{a^2}. \quad (6.18)$$

It is singular at the zero value of the scale factor. If this density is dominant in the modern era, it also dominated in the primary era of the early Universe, for which the solution (6.15)

$$a(\eta) = a_I \sqrt{1 + 2H_I(\eta - \eta_I)} \quad (6.19)$$

is expressed in terms of the initial data

$$a_I = a(\eta_I), \quad H_I = \frac{a'(\eta_I)}{a(\eta_I)}, \quad (6.20)$$

which are associated with modern values

$$a_0 = a(\eta_0), \quad H_0 = \frac{a'(\eta_0)}{a(\eta_0)}$$

by the following relations

$$a(\eta) = a_I \sqrt{1 + 2H_I(\eta - \eta_I)} = a_0 \sqrt{1 + 2H_0(\eta - \eta_0)}. \quad (6.21)$$

6.3 The hierarchy of cosmological scales

Let us consider the beginning of the Universe, assuming dominance of the Casimir vacuum energy. If in the Beginning, the Universe was quantum, we can apply the postulate of Planck's least action to determine the initial value of the cosmological factor and consider the hierarchy (classification) of cosmological scales according to their conformal weights.

A hypothetical observer measures the conformal horizon (6.7)

$$d_{\text{horizon}}(a) = 2r_{\text{horizon}}(z) = 2 \int_{a_I \rightarrow 0}^a d\bar{a} \frac{\bar{a}}{\sqrt{\rho_{\text{cr}}}} = \frac{a^2}{H_0}. \quad (6.22)$$

as the distance that the photon runs on its light cone $d\eta^2 - dr^2 = 0$ during the lifetime of the Universe. In accordance with the formula (6.22), four-dimensional volume of the early Universe, limited by the horizon

$$\eta_{\text{horizon}} = r_{\text{horizon}}(z) = \frac{1}{2H_0(1+z)^2},$$

is equal to

$$V_{\text{horizon}}^{(4)}(z) = \frac{4\pi}{3} r_{\text{horizon}}^3(z) \cdot \eta_{\text{horizon}}(z) = \frac{4\pi}{3 \cdot 16H_0^4(1+z)^8}. \quad (6.23)$$

It is natural to assume that at the time of its creation, the Universe was quantum. In this case, values of the action of the Universe are *quantized*. A minimal *quant* of the action of the Universe under initial scale factor $a_{\text{Pl}} = (1 + z_{\text{Pl}})^{-1}$ is given by the Planck postulate

$$W_{\text{Universe}} = \rho_{\text{cr}} V_{\text{horizon}}^{(4)}(z_{\text{Pl}}) = \frac{M_{\text{Pl}}^2}{H_0^2} \frac{1}{32(1+z_{\text{Pl}})^8} = 2\pi. \quad (6.24)$$

Using the current data for the Planck mass and the Hubble parameter³ at $(\tau = \tau_0)$ and $h \simeq 0.7$

$$M_{\text{C}} e^{\langle D \rangle(\tau_0)} = M_{\text{Pl}} = 1.2211 \times 10^{19} \text{GeV}, \quad \langle D \rangle(\tau_0) = 0, \quad (6.25)$$

$$\frac{d}{d\tau} \langle D \rangle(\tau_0) = H_0 = 2.1332 \times 10^{-42} \text{GeV} \times h = 1.4332 \times 10^{-42} \text{GeV}, \quad (6.26)$$

we obtain from (6.24) the primordial redshift value

$$a_{\text{Pl}}^{-1} = (1 + z_{\text{Pl}}) \approx \left[\frac{M_{\text{Pl}}}{H_0} \right]^{1/4} \times \left[\frac{4}{\pi} \right]^{1/8} \times \frac{1}{2} \simeq 0.85 \times 10^{15}. \quad (6.27)$$

In other words, the Planck mass and the present value of the Hubble parameter are related to each other as the age of the Universe, expressed in terms of the primary redshift in the fourth order

$$\frac{M_{\text{Pl}}}{H_0} = (1 + z_{\text{Pl}})^4 \simeq z_{\text{Pl}}^4.$$

We can say that the conformal weight of the Planck mass is four in the class frames of reference associated with the time interval of luminosity

³ $h = 0,71 \pm 0,02$ (stat) $\pm 0,06$ (syst) is the Hubble parameter in units 100 (km/sec)/Mps [17].

$d\tau$, where the relativistic energy of the particle is

$$\omega_\tau = a^2 \sqrt{\mathbf{k}^2 + a^2 M_0^2}.$$

From the expansion of the energy in powers of the cosmological scale factor there is a classification of energies by irreducible representations of the Weyl group [18]. According to these representations, conformal weights $n = 0, 2, 3, 4$ correspond to velocity of dilaton $v_D = H_0$, massless energy $a^2 \sqrt{\mathbf{k}^2}$, massive energy $M_0 a^3$, Newtonian coupling constant $M_{\text{Pl}} a^4$ (6.24), respectively. In this classification it is possible also to include a non-relativistic particle

$$H_0 \times a_{\text{Pl}}^{-1} = 10^{-13} \text{ cm}^{-1}$$

with conformal weight $n = 1$ of its energy

$$\omega_\tau^{\text{nonr}} = \frac{a^1 \mathbf{k}^2}{M_0}.$$

The cosmological evolution of all these energies is given by the Hubble parameter and can be written by a unified formula of the kind

$$\langle \omega \rangle^{(n)}(a) = \left(\frac{a}{a_{\text{Pl}}} \right)^{(n)} H_0, \quad (6.28)$$

According to this formula in the Beginning of the Universe, values of all these energies coincide with the Hubble parameter. In modern time the values of all these energies are determined by the product of the Hubble parameter to the primary (Planck) value of the redshift (6.27) in power equal to the corresponding conformal weight:

$$\langle \omega \rangle_0^{(0)} = H_0, \quad \langle \omega \rangle_0^{(1)} = R_{\odot}^{-1},$$

n	n=0	n=1	n= 2	n=3	n=4
$\omega_0^{(n)}$	$H_0 \simeq 1,4 \cdot 10^{-42}$	$\simeq 1,2 \cdot 10^{-27}$	$\simeq 10^{-12}$	$\simeq 3 \cdot 10^2$	$\simeq 4 \cdot 10^{18}$

Table 6.1: The hierarchy of cosmological scales in GeV.

$$\langle \omega \rangle_0^{(2)} = k_{CMB}, \quad \langle \omega \rangle_0^{(3)} = M_{EW}, \quad \langle \omega \rangle_0^{(4)} = M_{0P1}.$$

As a result, Planck's postulate of minimal action leads to a hierarchy of cosmological scales to the present days ($a = 1$)

$$\omega_0^{(n)} \equiv \langle \omega \rangle^{(n)}(a) \Big|_{(a=1)} = (1/a_{P1})^{(n)} H_0, \quad (6.29)$$

shown in Table 6.1.

Table 6.1 contains scales corresponding to the inverse size of the Solar System for the conformal weight ($n = 1$), the average momentum of the CMB ($n = 2$), electroweak scale of SM ($n = 3$) and Planck's mass ($n = 4$). We conclude that the observational data suggest that the cosmic evolution (6.28) of all of these energy scales with conformal weights ($n = 0, 1, 2, 3, 4$) has a common origin, which can be the Casimir energy of an empty space.

Thus, the use of minimum action postulate leads to the primary value of the cosmological scale factor a_{P1} , given by the equation (6.27) in the considered Conformal model of the Universe. The classification of the different states of matter, in accordance with their conformal weights, reveals a hierarchy of energy scales, according to the observations.

Why is an empty Universe filled with particles? And why are these particles just enough as we observe in the Universe? The answers to these questions will be discussed in the next chapters.

6.4 Special Relativity–General Relativity correspondence

The problem of *understanding creation and evolution of the Universe* is probably not in the exact solution of the equations of the General Relativity, but in an ontology, *id est*, in the adequate application of the concepts of modern relativistic and quantum physics to observational cosmology. To demonstrate this assertion, let us consider according to Wheeler and De Witt, a quantum theory of the Universe for a model of an empty Universe (6.2), that can be solved exactly in classical and quantum cases.

Above, there were presented arguments and evidence that the classical exact solution provided a description of the data on the dependence of the redshift from the distance to cosmic objects in all ages of the evolution of the Universe, including the latest data on the Supernovae. As we show below, the quantum solution gives a positive arrow of time interval and allows to describe the creation of the Universe from a vacuum, defined as the state with the lowest energy, according to the postulates of quantum field theory. We make in the action (6.2) the change of variables (neglecting the total derivative)

$$\sqrt{2V_0} \langle D \rangle = X_U, \quad (6.30)$$

$$\sqrt{2V_0} \rho_{cr} = M_U. \quad (6.31)$$

Then the expression (6.2) takes the form of the action for a relativistic

particle at rest in the Special Relativity

$$W_{\text{Universe}} = -\frac{1}{2} \int_{\tau_I}^{\tau_0} dx^0 N_0 \left[\left(\frac{dX_U}{N_0 dx^0} \right)^2 + M_U^2 \right] = \int_{\tau_I}^{\tau_0} dx^0 L, \quad (6.32)$$

$$dx^0 N_0 = d\tau, \quad (6.33)$$

where the speed of light is set equal to unity, $c = 1$. Here, X_U plays a role of an evolution parameter in a space of measurements, x^0 is a coordinate evolution parameter in the one-dimensional Riemannian manifold as the object of coordinate transformations $x^0 \rightarrow \tilde{x}^0 = \tilde{x}^0(x^0)$ with *unmeasured parameters*, $N_0(x^0)$ is a lapse function that has sense as a metrics for a geometric time interval $N_0(x^0)dx^0 = d\tau$ for classical equations of motion.

By introducing the canonical momentum variable X_U :

$$P_U = \frac{\partial L}{\partial(\partial_0 X_U)},$$

the action (6.32) we rewrite in the canonical form

$$W = \int dx^0 \left[P_U \frac{dX_U}{dx^0} + \frac{N_0}{2} (P_U^2 - M_U^2) \right]. \quad (6.34)$$

The equations of motion take a form

$$\frac{dP_U}{d\tau} = 0, \quad \frac{dX_U}{d\tau} = P_U.$$

The solutions of these equations

$$X_U(\tau) = X_{IU} + P_{IU} (\tau - \tau_I) \quad (6.35)$$

depend on initial data

$$X_U(\tau = \tau_I) = X_{IU}, \quad P_{IU} = E_U.$$

A variation of the action by the metric $N_0(x^0)$ yields a formula for Hamiltonian constraint of an initial momentum E_U

$$E_U^2 - M_U^2 = 0. \quad (6.36)$$

In the General Relativity, this formula is traditionally identified with the zero-point energy of the system. In our monograph, the *energy of relativistic Universe* is the solution of the constraint (6.36) in respect to E_U

$$E_U = \pm M_U. \quad (6.37)$$

A formula for the Hamiltonian constraint (6.36) for the *energy of relativistic Universe* is an analog of the Hamiltonian constraint for the *energy of relativistic particle* at rest

$$E^2 - m^2 = 0.$$

For the *relativistic particle* a solution of the Hamiltonian constraint gives two values of energy: positive and negative $E = \pm m$. A negative value of energy means that the classical relativistic particle is unstable. To get rid of negative energies in the relativistic theory and enter a stable vacuum as a state of the lowest energy, one makes two quantizations of the Hamiltonian constraint (6.36): the primary, when the constraint is converted into the equation for the wave function, and the secondary, where the wave function itself becomes the operator in Fock's *space of numbers of occupation*. Let us consider these quantizations in our model of the Universe.

6.5 The arrow of time as a consequence of the postulate of vacuum

For the first time such an idea on the level of quantization of the General Relativity by analogy with the quantization of the Special Relativity was formulated by Bryce De Witt [19], where he identified a *parameter of evolution* in cosmology with the cosmological scale factor and introduced in the General Relativity the concept of *field space of events*, where the *relativistic Universe* moved, by analogy with the concept of Minkowskian space of events, where the *relativistic particle* moved.

The primary quantization of the constraint $P_U^2 = M_U^2$, by replacement of a particle momentum $P_{(0)}$ to an operator

$$\hat{P}_U = -i \frac{d}{dX_U}$$

leads to an equation of the Klein – Gordon for the wave function [20]

$$(\hat{P}_U^2 - M_U^2)\Psi_U = 0, \quad (6.38)$$

which in cosmology is called the equation of Wheeler – De Witt (WDW).

Its solution is a sum of two terms

$$\begin{aligned} \Psi_U &= \frac{1}{\sqrt{2E_U}} \times & (6.39) \\ &\times \left[A_I^+ e^{iE(X_U - X_{IU})} \theta(X_U - X_{IU}) + A_I^- e^{-iE(X_U - X_{IU})} \theta(X_{IU} - X_U) \right] \end{aligned}$$

with coefficients A_I^+ , A_I^- , according to two classical solutions of the constraint equation with positive and negative energy. The secondary

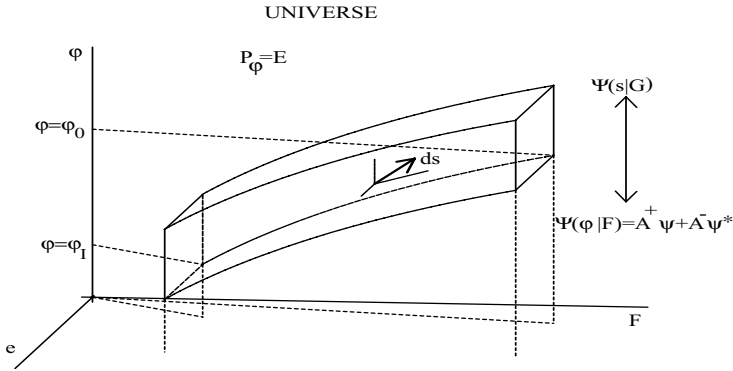


Figure 6.3: The figure shows the motion of a relativistic Universe in its field space of events. Full description of the motion is given by the two sets of observables: *dynamical* in the space of events and *geometrical* in the tangent Minkowskian space. Each of these sets has its parameter of evolution and its wave function Ψ . Two measured parameters of evolution (*dynamical* parameter $\varphi = M_{Pl}e^{-D}$ and time as *geometrical* interval s) are connected by the Hubble law.

quantization of the initial data

$$[\hat{A}_I^-, \hat{A}_I^+] = 1$$

leads to the vacuum $A_I^-|0\rangle = 0$ as state with the lowest energy if the coefficient A_I^+ is interpreted as an operator of creation of a particle with positive energy, which flies ahead of the initial data

$$X_{IU} < X_U,$$

and the coefficient A_I^- as an operator of annihilation of a particle also with positive energy, which flies to the initial data

$$X_U < X_{IU}.$$

Substituting these solutions into the expression for the trajectory of the Universe (6.35), we obtain, that the geometrical interval $\tau - \tau_I > 0$ is always greater than zero. This is the arrow of time. Thus, the existence of a stable vacuum leads to the arrow of time.

The WDW equation (6.38) is obtained by variation of the action of the corresponding classical theory of field type Klein – Gordon [19]⁴:

$$W_U = \frac{1}{2} \int dX_U \left[\left(\frac{d\Psi_U}{dX_U} \right)^2 - E_U^2 \Psi_U^2 \right] \equiv \int dX_U L_U. \quad (6.40)$$

Such an approach could be called a field theory for the universes [21, 22].

The negative energy solutions in (6.37) means that the relativistic system has not a minimum of its energy and arbitrarily small interaction makes the system unstable. The system can be made stable in a quantum field theory, arising from the second quantization of WDW field Ψ_U , if further the existence of a vacuum state with the lowest energy is postulated. Introducing the canonical momenta

$$P_\Psi = \frac{\partial L_U}{\partial(\partial_{X_U} \Psi_U)},$$

one can yield the Hamiltonian formulation of the theory with an action (6.40)

$$W_U = \int dX_U \left(P_\Psi \frac{d\Psi_U}{dX_U} - H_U \right), \quad (6.41)$$

⁴Unlike the original relativistic system (6.2) with three spaces, the formulation of the Wheeler and De Witt (6.40) loses *time as geometrical interval* and, consequently, its dependence on the scale factor, which is interpreted in the classical Friedmann cosmology as Hubble's law. As a result, in classical cosmology one does not know how to quantize, and in quantum cosmology [19] – how to describe the Hubble law.

where

$$H_U = \frac{1}{2} [P_\Psi^2 + E_U^2 \Psi_U^2] \quad (6.42)$$

is the Hamiltonian. Determination of the energy E_U for one particular universe gives us the opportunity to present the Hamiltonian H_U in standard form of product of the energy E_U and the number of occupation of excitations of Wheeler – De Witt field, which can be identified with the number of created *universes*

$$\hat{N}_U = A^+ A^-, \quad (6.43)$$

$$H_U = \frac{1}{2} E_U [A^+ A^- + A^- A^+] = E_U \left[N_U + \frac{1}{2} \right] \quad (6.44)$$

by passing to holomorphic variables [23]

$$\Psi_U = \frac{1}{\sqrt{2E_U}} (A^+ + A^-), \quad P_\Psi = i\sqrt{\frac{E_U}{2}} (A^+ - A^-), \quad (6.45)$$

where A^+ , A^- are operators of creation and annihilation of *universes*, correspondingly.

To eliminate the negative energy one would have to postulate that the A^- is the operator of annihilation of *universe* with positive energy, it assumes the existence of a vacuum state as the state with the lowest energy:

$$A^- |0\rangle_A = 0. \quad (6.46)$$

A number of *universes* $N_U = A^+ A^-$ (6.43) can not be saved if the energy E_U depends on X_U . In this case, the vacuum state (6.46) becomes unstable, since a dependence of the energy E_U of the dynamic evolution parameter X_U leads to an additional term in the action (6.40), if

it is rewritten in terms of holomorphic variables in the functional space

$$\begin{aligned}
 P_\Psi \frac{d\Psi_U}{dX_U} &= \tag{6.47} \\
 &= \left[\frac{\imath}{2} \left(A^+ \frac{dA^-}{dX_U} - A^- \frac{dA^+}{dX_U} \right) - \frac{\imath}{2} (A^+ A^+ - A^- A^-) \Delta(X_U) \right],
 \end{aligned}$$

where

$$\Delta(X_U) = \frac{1}{2E_U} \frac{dE_U}{dX_U}. \tag{6.48}$$

The last term in the expression (6.47) describes a cosmological creation of *universes* from the *vacuum*. The method of describing of such cosmological creation is the Bogoliubov transformations [23, 24].

6.6 Creation of the Universe

To determine a *vacuum* and a set of conserving numbers, called integrals of motion, we can use (as in the case of cosmological creation of particles [23]) the Bogoliubov transformations [24] of variables (A^+, A^-)

$$A^+ = \alpha B^+ + \beta^* B^-, \quad A^- = \alpha^* B^- + \beta B^+ \quad (|\alpha|^2 - |\beta|^2 = 1), \tag{6.49}$$

so corresponding equations, expressed in terms of *universes* (A^+, A^-) :

$$\left(\imath \frac{d}{dX_U} + E_U \right) A^+ = \imath A^- \Delta, \quad \left(\imath \frac{d}{dX_U} - E_U \right) A^- = \imath A^+ \Delta, \tag{6.50}$$

take a diagonal form in terms of *quasiuniverses* B^+, B^- :

$$\left(\imath \frac{d}{dX_U} + E_B \right) B^+ = 0, \quad \left(\imath \frac{d}{dX_U} - E_B \right) B^- = 0. \tag{6.51}$$

This means that coefficients of the Bogoliubov transformations satisfy to equations

$$\left(i \frac{d}{dX_U} + E_U \right) \alpha = i\beta\Delta, \quad \left(i \frac{d}{dX_U} - E_U \right) \beta^* = i\alpha^*\Delta. \quad (6.52)$$

If we express the coefficients of the Bogoliubov transformations in the form of

$$\alpha = e^{i\theta} \cosh r, \quad \beta^* = e^{i\theta} \sinh r, \quad (6.53)$$

where magnitudes r, θ are called parameters of shear and rotation, respectively, these equations take the following form

$$\left(\frac{d\theta}{dX_U} - E_U \right) \sinh 2r = -\Delta \cosh 2r \sin 2\theta, \quad \frac{dr}{dX_U} = \Delta \cos 2\theta, \quad (6.54)$$

while the energy of the *quasiuniverses* in equations (6.51) is given by

$$E_B = \frac{E_U - \partial_{X_U} \theta}{\cosh 2r}. \quad (6.55)$$

By these equations (6.51), a number of *quasiuniverses* $\mathcal{N}_B = (B^+ B^-)$ is conserved

$$\frac{d\mathcal{N}_B}{dX_U} \equiv \frac{d(B^+ B^-)}{dX_U} = 0. \quad (6.56)$$

Hence, we get the definition of *vacuum* as state without *quasiuniverses*:

$$B^- |0\rangle_U = 0. \quad (6.57)$$



N.N. Bogoliubov (8 (21) August 1909, Nizhni Novgorod – 13 February 1992, Moscow). The outstanding Russian mathematician and physicist, Academician of the Russian Academy of Sciences, founder of scientific schools in nonlinear mechanics and theoretical physics. Since 1956 - Director of the Laboratory of Theoretical Physics, JINR, Dubna, since 1965 till 1988 Director of JINR. Head of quantum field theory and statistical physics department of Moscow State University from 1966 to 1992. The main works are devoted to asymptotic methods of nonlinear mechanics, quantum field theory, statistical mechanics, calculus of variations, the approximate methods of mathematical analysis, differential equations and mathematical physics, the theory of stability, dynamical systems, and other areas of theoretical physics.

A number of created *universes* from this Bogoliubov vacuum can be found by calculating the average of the operator of *number of universes* (6.43) by Bogoliubov vacuum. It can be seen that this number is proportional to square of the coefficient, given in equation (6.49)

$$N_U(X_U) = \langle 0|A^+A^-|0 \rangle_U \equiv |\beta|^2. \quad (6.58)$$

This value can be called as *number of universes* $N_U(X_U)$, while the

N ^o	UNIVERSE	PARTICLE
1.	$x^0 \rightarrow \tilde{x}^0 = \tilde{x}^0(x^0)$	$\tau \rightarrow \tilde{\tau} = \tilde{\tau}(\tau)$
2.	$N(x^0) dx^0 = d\tau = \frac{d\eta}{a^2} = \frac{dt}{a^3}$	$ds = e(\tau) d\tau$
3.	$[\langle D \rangle \tilde{F}]$	$[X_0 X_k]$
4.	$P_{\langle D \rangle}^2 - E_U^2 = 0$	$P_0^2 - E_0^2 = 0$
5.	$\tau_{(\pm)} = \pm \int_{\langle D \rangle_I}^{\langle D \rangle_0} d\langle D \rangle \langle (\tilde{H})^{-1/2} \rangle \geq 0$	$s_{\pm} = \pm \frac{m}{E} [X_0^0 - X_I^0] \geq 0$
6.	$E_U = \pm 2 \int d^3x (\tilde{H})^{1/2}$	$E_p = \pm \sqrt{m^2 + \mathbf{p}^2}$
7.	$[\hat{P}_{\langle D \rangle}^2 - E_U^2] \Psi_{\text{WDW}} = 0$	$[\hat{P}_0^2 - E_0^2] \Psi_{\text{KG}} = 0$
8.	$\Psi_U = \frac{A^+ + A^-}{\sqrt{2E_U}}$	$\Psi_{\text{KG}} = \frac{a^+ + a^-}{\sqrt{2E_0}}$
9.	$A^+ = \alpha B^+ + \beta^* B^-$	$a^+ = \alpha b^+ + \beta^* b^-$
10.	$B^- 0\rangle_B = 0$	$b^- 0\rangle_b = 0$
11.	$\langle_B 0 A^+ A^- 0\rangle_B \neq 0$	$\langle_b 0 a^+ a^- 0\rangle_b \neq 0$

value of

$$R_U(X_U) = \left(\frac{i}{2}\right) \langle_U 0 | [A^+ A^+ - A^- A^-] |0\rangle_U = \quad (6.59)$$

$$= i(\alpha^* \beta^* - \alpha \beta) = -\sinh 2r \sin 2\theta$$

as *Bogoliubov condensate*, correspondingly. Bogoliubov equations, expressed in terms of quantities of *number of universes* $N_U(X_U)$ and

Correspondence Universe – particle.

N^o1 – group of diffeomorphisms,

N^o2 – diffeoinvariant integrals,

N^o3 – space of events with a parameter of evolution,

N^o4 – Hamiltonian constraint in the space of events,

N^o5 – Hubble law,

N^o6 – energy in the space of events,

N^o7 – primary quantization,

N^o8 – secondary quantization,

N^o9 – Bogoliubov transformations,

N^o10 – vacuum of quasiparticles,

N^o11 – occupation numbers of *universes* and particles.

Bogoliubov condensate $R_U(X_U)$ take the form

$$\begin{cases} \frac{dN_U}{dX_U} = \Delta(X_U) \sqrt{4N_U(N_U + 1) - R_U^2}, \\ \frac{dR_U}{dX_U} = -2E_U(X_U) \sqrt{4N_U(N_U + 1) - R_U^2} \end{cases} \quad (6.60)$$

with initial data

$$N_U(X_{UI}) = R_U(X_{UI}) = 0.$$

We see that the vacuum postulate leads to a positive value of conformal time as for the Universe

$$E_U > 0, \quad X_U > X_{UI},$$

and for an anti-Universe

$$E_U < 0, \quad X_{UI} < X_U,$$

that is leading to an arrow of conformal time. Time has a beginning and the Quantum Universe is created with his time.

6.7 Summary

We have considered in this Chapter an empty Universe model, leaving only a zero harmonic of the dilaton and experimentally measured Casimir vacuum energy. Casimir vacuum energy is given by the size of the space. If the size of space is equal to the horizon, which in turn is determined by Casimir energy in the *empty* Universe, there is a self-consistent equation of state of the *empty* space. The solution of this equation gives the dependence of the density of the cosmological scale factor. In the Conformal cosmology, with longer intervals than in the Standard one, the resulting dependence of the density of empty space of the cosmological scale factor describes the recent observational data on the Supernovae. Thus, data on the Supernovae show that we still live in a nearly empty Universe.

The Planck principle of minimal action of gravity (*id est* the quantum of action) for the space limited by the size of the horizon, leads to the primary value of the cosmological scale factor (and the corresponding redshift) $\sim 0.8 \times 10^{15}$. Representations of the Weyl group for massive and massless particles give (together with modern values of Hubble's parameter and the Planck mass) a hierarchy of cosmological scales in a surprising agreement with the values of the CMB temperature (for massless particles) and the scale of the electroweak interaction (for massive particles).

From a mathematical point of view, the Conformal theory of gravity in the approximation of the empty Universe is a model of squeezed oscillator of the dilaton. Primary and secondary quantization of this model with the postulate of the existence of the vacuum leads to the arrow of the geometric interval of time and to a certain wave function of the Universe, such as the irreducible unitary representations of the Poincaré group for a relativistic particle. Now we are ready to consider the irreducible unitary representations of affine and conformal symmetry groups in the exact theory, that will be done in Chapter 8.

Bibliography

- [1] Bordag, M., Klimchitskaya, G.L., Mohideen, U., Mostepanenko, V.M.: *Advances in the Casimir Effect*. Oxford University Press, New York (2009)
- [2] Riess, A.G., *et al.* [Supernova Search Team Collaboration]: *The farthest known supernova: support for an accelerating universe and a glimpse of the epoch of deceleration*. *Astrophys. J.* **560**, 49 (2001).
[arXiv: astro-ph/0104455]
- [3] Zhu, Z.H., Fujimoto, M.K.: *Constraints on Cardassian expansion from distant type Ia supernovae*. *Astrophys. J.* **585**, 52 (2003).
[arXiv: astro-ph/0303021]
- [4] Behnke, D., Blaschke, D.B., Pervushin, V.N., Proskurin, D.: *Description of Supernova data in Conformal cosmology without cosmological constant*. *Phys. Lett.* **B 530**, 20 (2002).
[arXiv: gr-qc/0102039]
- [5] Barbashov, B.M., Pervushin, V.N., Zakharov, A.F., Zinchuk, V.A.: *Hamiltonian cosmological perturbation theory*. *Phys. Lett.* **B 633**, 458 (2006).

[arXiv: hep-th/0501242]

- [6] Behnke, D.: *Conformal Cosmology Approach to the Problem of Dark Matter*. PhD Thesis, Rostock Report MPG-VT-UR 248/04 (2004)

- [7] Blaschke, D.B., Vinitzky, S.I., Gusev, A.A., Pervushin, V.N., Proskurin, D.V.: *Cosmological production of vector bosons and Cosmic Microwave Background radiation*. Physics of Part. & Nuclei. **67**, 1074 (2004).

[arXiv: hep-ph/0504225]

- [8] Zakharov, A.F., Pervushin, V.N.: *Conformal cosmological model parameters with distant SNe Ia data: 'gold' and 'silver'*. Int. J. Mod. Phys. **D 19**, 1875 (2010).

[arXiv: 1006.4745 [gr-qc]]

- [9] Arbuzov, A.B., Barbashov, B.M., Nazmitdinov, R.G., Pervushin, V.N., Borowiec, A., Pichugin, K.N., Zakharov, A.F.: *Conformal Hamiltonian dynamics of general relativity*. Phys. Lett. **B 691**, 230 (2010).

[arXiv: 1007.0293 [gr-qc]]

- [10] Riess, A.G., *et al.* [Supernova Search Team Collaboration]: *Observational evidence from supernovae for an accelerating universe and a cosmological constant*. Astron. J. **116**, 1009 (1998).

[arXiv: astro-ph/9805201]

- [11] Perlmutter, S., *et al.* [The Supernova Cosmology Project]: *Constraining dark energy with SN Ia and large-scale structure*. *Astrophys. J.* **517**, 565 (1999)
- [12] Riess, A.G., *et al.* [Supernova Search Team Collaboration]: *Type Ia supernova discoveries at $z \lesssim 1$ from the Hubble Space Telescope: Evidence for past deceleration and constraints on dark energy evolution*. *Astrophys. J.* **607**, 665 (2004)
- [13] Misner, Ch., Thorne, K., Wheeler, J.: *Gravitation*. Freeman & Co. (1973)
- [14] Zakharov, A.F., Pervushin, V.N.: *Conformal cosmological model and SNe Ia data*. *Physics of Atomic Nuclei.* **75**, 1492 (2012)
- [15] Astier, P., *et al.* [The Supernova Legacy Survey]: *Ω_M , Ω_Λ , and w from the first year data set*. *Astronomy and Astrophysics*, **447**, 31 (2006)
- [16] Pervushin, V.N., Proskurin, D.V.: *Conformal general relativity*. *Gravitation & Cosmology.* **8**, 161 (2002).
[arXiv: gr-qc/0106006]
- [17] Weinberg, S.: *Cosmology*. Oxford University Press (2008)
- [18] Ramond, P.: *Field Theory: A Modern Primer*. The Benjamin/Cummings Publishing Company, Inc. London (1981)
- [19] De Witt, B.S.: *Quantum theory of gravity. I. The canonical theory*. *Phys. Rev.* **160**, 1113 (1967)

-
- [20] Vilenkin, A.: *Interpretation of the wave function of the Universe*. Phys. Rev. **D 39**, 1116 (1989)
- [21] Pavlov, A.: *Quantum theory of a Friedmann field*. Int. J. Theor. Phys. **34**, 961 (1995)
- [22] Pavlov, A.: *A quantized open homogeneous isotropic cosmological model*. Phys. Lett. **A 165**, 211 (1992).
Pavlov, A.: *A quantized flat homogeneous isotropic cosmological model*. Phys. Lett. **A 165**, 215 (1992).
Pavlov, A.: *Dynamics of a compact hyperbolic cosmological model with dustlike matter and radiation*. Int. J. Theor. Phys. **35**, 2169 (1996)
- [23] Pervushin, V.N., Smirichinski, V.I.: *Bogoliubov quasiparticles in constrained systems*. J. Phys. A: Math. Gen. **32**, 6191 (1999)
- [24] Bogoliubov, N.N.: *On the theory of superfluidity*. J. Phys. USSR **2**, 23 (1947)

Chapter 7

Quantization of gravitons in terms of Cartan forms

7.1 Affine gravitons

It is well known that the General Relativity in terms of the metric components is nonrenormalizable theory. A renormalizable quantum gravitation theory does not exist [1]. Here we show that GR in terms of Cartan forms becomes not only renormalizable theory, but describes free gravitons, far from the matter sources, where the Newton-type potentials can be neglected:

$$\bar{D} = 0, \quad \mathcal{N}_i = 0, \quad \mathcal{N} = 1$$

(like as, in QED, photons become free far from the charges and currents) [2].

Let us consider the graviton action (5.47), where one keeps only the simplex components

$$\bar{\omega}_{(a)}(d) = \tilde{\mathbf{e}}_{i(a)} dx^i.$$

They obey the condition of the diffeo-invariance. It is one of the main differences of the diffeo-invariant Conformal GR from the metric GR. The choice of the diffeo-invariant symmetry condition in the GR leads to the result that follows from the theorem [3]: *any arbitrary two-dimensional space metric*

$$dl^2 = h_{AB} dx^A dx^B, \quad (A, B = 1, 2)$$

can be represented by diffeomorphisms

$$x^A \rightarrow \tilde{x}^A = \tilde{x}^A(x^1, x^2)$$

in a diagonal form. The result is in the fact that a kinematic-invariant nonlinear plane wave moving in the direction \mathbf{k} with the unit determinant $\det h = 1$ contains only a single metric component.

In particular, in the frame of reference $\mathbf{k} = (0, 0, k_3)$ one has

$$\tilde{\mathbf{e}}_{(1)}^1 = e^{g(x_{(3)}, \tau)}, \quad \tilde{\mathbf{e}}_{(2)}^2 = e^{-g(x_{(3)}, \tau)}, \quad \tilde{\mathbf{e}}_{(3)}^3 = 1;$$

all other (non-diagonal) components $\tilde{\mathbf{e}}_{(a)}^i$ are equal to zero.

Thus, we obtain

$$\bar{\omega}_{(1)} = dX_{(1)} - [X_{(1)}]dg, \quad (7.1)$$

$$\bar{\omega}_{(2)} = dX_{(2)} + [X_{(2)}]dg, \quad (7.2)$$

$$\bar{\omega}_{(3)} = dx_3 = dX_{(3)}, \quad (7.3)$$

where a single-component affine graviton $g = g(X_{(3)}, \tau)$ is a function depending on the time and a single spatial coordinate $X_{(3)}$ in the tangent space $X_{(b)}$. The solutions of the equation

$$\frac{\delta W}{\delta g} = 0 \quad \rightarrow \quad g = g(\eta, X)$$

can be expressed via the tangent coordinates:

$$X_{(1)} = e^{g(x_{(3)}, \tau)} \bar{x}^1 \quad (7.4)$$

$$X_{(2)} = e^{-g(x_{(3)}, \tau)} \bar{x}^2. \quad (7.5)$$

Eqs. (7.1) and (7.2) mean an expansion (or contraction) of the hypersurface $X_{(A)}$ ($A = 1, 2$) perpendicular to the direction of the gravitational wave propagation $X_{(3)}$. A gravitation wave changes the particle velocity via the Hubble like law: the more is the base, the more is the additional velocity induced by the graviton. The exact local Hamiltonian density for the affine graviton is given by (5.33)

$$\mathcal{H}_g = \left[6p_{(a)(b)}^2 + \frac{1}{6} R^{(3)}(\tilde{\mathbf{e}}) \right], \quad (7.6)$$

where $R^{(3)}(\mathbf{e})$ and $p_{(a)(b)}^2$ are defined from Eqs. (5.19)–(5.22), respectively. For the frame of reference $\mathbf{k} = (0, 0, k_3)$, we have [4]:

$$R^{(3)}(\tilde{\mathbf{e}}) = (\partial_{(3)} g)^2, \quad p_{(a)(b)}^2 = \frac{1}{9} [\partial_\tau g]^2. \quad (7.7)$$

There is a difference of the diffeo-invariant affine graviton from the metric one

$$g_{ij}^{TT} = g_{ji}^{TT}$$

in GR [4]. While the affine graviton has a single degree of freedom, the metric graviton has two traceless and transverse components that

satisfy four constraints

$$g_{ii}^{TT} = 0, \quad (7.8)$$

$$g_{i3}^{TT} = g_{3i}^{TT} = 0. \quad (7.9)$$

In the general case of the Conformal GR $\tilde{\mathbf{e}}_{(b)i} = \mathbf{e}_{(b)i}^T$, both the transverse constraint

$$\partial_i \mathbf{e}_{(b)i}^T = 0 \quad (7.10)$$

and the unit determinant one

$$|\mathbf{e}_{(b)i}^T| = 1 \quad (7.11)$$

(as the analog of the Lichnerowicz gauge in the metric formalism [5]) admit to generalize Eqs. (7.1), (7.2), and (7.3) for the linear forms

$$\begin{aligned} \bar{\omega}_{(b)}(d) &= \mathbf{e}_{(b)i}^T dx^i & (7.12) \\ &= d[\mathbf{e}_{(b)i}^T x^i] - x^j d\mathbf{e}_{(b)j}^T \\ &= dX_{(b)} - X_{(c)} \mathbf{e}_c^{Ti} d\mathbf{e}_{(b)i}^T \\ &= dX_{(b)} - X_{(c)} \left[\omega_{(b)(c)}^R + \omega_{(b)(c)}^L \right] \end{aligned}$$

in the tangent coordinate space. Here $X_{(b)}$ can be obtained by the formal generalization of Eqs. (7.1), (7.2), and (7.3) by means of the Leibnitz rule

$$\mathbf{e}_{(b)i}^T d[x^i] = d[\mathbf{e}_{(b)i}^T x^i] - x^i d\mathbf{e}_{(b)i}^T.$$

The diffeomorphism-invariance admits the choice of the gauge in Eq. (7.12)

$$\omega_{(b)(c)}^L = 0. \quad (7.13)$$

Similar result is valid for a general case of arbitrary wave vector

$$\mathbf{k} = \frac{2\pi}{V_0^{1/3}} \mathbf{1},$$

where $X_{(3)}$ is replaced by

$$X_{(k)} = \frac{(\mathbf{k} \cdot \mathbf{X})}{\sqrt{\mathbf{k}^2}}.$$

The single-component graviton $g(\tau, \mathbf{X})$ considered as the tensor massless representation of the Wigner classification of the Poincaré group can be decomposed into a series of strong waves (in natural units)

$$\begin{aligned} \omega_{(a)(b)}^R(\partial_{(c)}) &= i \sum_{\mathbf{k}^2 \neq 0} \frac{e^{i\mathbf{k}\mathbf{X}}}{\sqrt{2\omega_{\mathbf{k}}}} \mathbf{k}_c \times \\ &\times [\varepsilon_{(a)(b)}^R(k) g_{\mathbf{k}}^+(\eta) + \varepsilon_{(a)(b)}^R(-k) g_{-\mathbf{k}}^-(\eta)]. \end{aligned} \quad (7.14)$$

Here $\varepsilon_{(a)(b)}^R(\mathbf{k})$ satisfies the constraints

$$\varepsilon_{(a)(a)}^R(k) = 0, \quad (7.15)$$

$$\mathbf{k}_{(a)} \varepsilon_{(a)(b)}^R(k) = 0, \quad (7.16)$$

similar to (7.8), (7.9). The variable $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2}$ is the graviton energy and the affine graviton

$$\bar{g}_{\mathbf{k}} = \frac{\sqrt{8\pi}}{M_{\text{Planck}} V_0^{1/2}} g_{\mathbf{k}} \quad (7.17)$$

is normalized to the units of volume and time (like a photon in QED [4]).

In the mean field approximation

$$\mathcal{N}(x^0, x^j) = 1, \quad N^j = 0, \quad \bar{D} = 0, \quad (7.18)$$

$$\tilde{d}s^2 = [d\eta]^2 - [\omega_{(b)} \otimes \omega_{(b)}], \quad (7.19)$$

when one neglects all Newtonian-type interactions, the action of an affine graviton reduces to the form of the exact action for the strong gravitational wave [4]

$$W_{\text{lin}}^g = \int d\tau L_{\tau}^g,$$

$$\begin{aligned}
\mathbb{L}_\tau^g &= \frac{v_{(a)(b)}^2 - e^{-4D}R^{(3)}}{6} = \sum_{\mathbf{k}^2 \neq 0} \frac{v_{\mathbf{k}}^g v_{-\mathbf{k}}^g - e^{-4D} \mathbf{k}^2 \bar{g}_{\mathbf{k}} \bar{g}_{-\mathbf{k}}}{2} = \\
&= \left[\sum_{\mathbf{k}^2 \neq 0} p_{-\mathbf{k}}^g v_{\mathbf{k}}^g \right] - \mathbb{H}_\tau^g, \tag{7.20}
\end{aligned}$$

where $v_{\mathbf{k}}^g = \partial_\tau \bar{g}_{\mathbf{k}}$ is the derivative with respect to the luminosity time interval and

$$\mathbb{H}_\tau^g = \sum_{\mathbf{k}^2 \neq 0} \frac{p_{\mathbf{k}}^g p_{-\mathbf{k}}^g + e^{-4(D)} \mathbf{k}^2 \bar{g}_{\mathbf{k}} \bar{g}_{-\mathbf{k}}}{2} \tag{7.21}$$

is the corresponding Hamiltonian.

Thus, in the mean field approximation (7.18) the diffeo-invariant sector of the strong gravitational plane waves coincides with a bilinear theory given by Eqs. (7.20)–(7.21). In this approximation our model is reduced to a rather simple theory which is bilinear with respect to the single-component graviton field as discussed also in Ref. [2]. Note that we consider here the tangential space, and the chosen variables allow us to obtain the simple solutions. The main postulated condition here was the requirement of the diffeo-invariance of the graviton equation of motion. While in the standard GR the symmetry properties are required only for the interval, we impose the symmetry with respect to diffeomorphisms also on the Maurer – Cartan forms.

7.2 Comparison with metric gravitons

It is instructive to compare the properties of the affine and metric gravitons, which was done first in Ref. [6]).

The action of metric gravitons in the accepted GR [7] coincides with the affine one (7.20) in the lowest order of the decomposition over $\mathbf{k}^2/M_{\text{Pl}}^2$

$$W_{\text{non-lin}}^{\text{GR}} = W_{\text{lin}}^g + W_{\text{non-lin}}, \quad (7.22)$$

if we keep only diagonal graviton components. It is well-known [1] that the accepted action (7.22) is highly nonlinear even in the approximation (7.18).

In the approximation (7.18), we keep only the dynamical part $\omega_{(cb)}^R$ (which enters into the action (7.20)) and the present day value of the cosmological scale factor $a = e^{-(D)} = 1$. Let us compare the affine gravitons (7.12) with the commonly accepted metric gravitons, given by the decomposition [7]

$$\tilde{ds}_h^2 = (d\eta)^2 - dx^i dx^j (\delta_{ij} + 2h_{ij}^{TT} + \dots). \quad (7.23)$$

In the accepted case, the graviton moves in the direction of vector \mathbf{k} , its wave amplitude $\cos(\omega_{\mathbf{k}} x_{(k)})$ depends on the scalar product

$$x_{(k)} = \frac{(\mathbf{k} \cdot \mathbf{x})}{\omega_{\mathbf{k}}}.$$

The graviton changes the squared test particle velocity

$$\left(\frac{ds}{d\eta}\right)^2 \sim \frac{dx^i dx^j}{d\eta d\eta} \varepsilon_{ij}^\alpha$$

in the plane, orthogonal to the direction of motion. Here ε_{ij}^α is the traceless transverse tensor:

$$\varepsilon_{ii}^\alpha = 0, \quad k_i \varepsilon_{ij}^\alpha = 0.$$

All these effects are produced by the first order of series (7.23)

$$\begin{aligned} dl_h^2 &= 2dx^i dx^j h_{ij}^{TT}(t, \mathbf{x}) = \\ &= dx^i dx^j \varepsilon_{ij}^\alpha \sqrt{6} \cos(\omega_{\mathbf{k}} x_{(k)}) (H_0/\omega_{\mathbf{k}}) \Omega_{\text{kh}}^{1/2} + O(h^2), \end{aligned} \quad (7.24)$$

where H_0 is the Hubble parameter,

$$\Omega_{\text{kh}} = \frac{\omega_{\mathbf{k}} N_{\text{kh}}}{V_0 \rho_{\text{cr}}}$$

is the energy density of the gravitons in units of the cosmological critical energy density. One observes that in the accepted perturbation theory the contribution of a single gravitational wave to the geometrical intervals, Eq. (7.23), is suppressed by the factor $H_0/\omega_{\mathbf{k}}$.

In our version the linear term of the spacial part of Eq. (7.12) takes the form

$$dl_g^2 = 2dX_{(b)} X_{(c)} \omega_{(c)(b)}^R = dX_{(b)} X_{(c)} \varepsilon_{(c)(b)}^\alpha \sqrt{6} \cos\{\omega_{\mathbf{k}} X_{(k)}\} H_0 \Omega_{\text{kh}}^{1/2}.$$

Evidently, two models (the GR and the CGR) differ by an additional factor which can be deduced from the ratio

$$\left| \frac{dl_h^2}{dl_g^2} \right| = \left| \frac{dx^i dx^j (h_{ij}^{TT})}{(dX_{(b)} X_{(c)} \omega_{(c)(b)}^R)} \right| \simeq \frac{1}{r_{\perp} \omega_{\mathbf{k}}} \sim \frac{\lambda_g}{r_{\perp}}. \quad (7.25)$$

Here

$$r_{\perp} = \sqrt{|\vec{X}_{\perp}|^2}$$

is the coordinate distance between two test particles in the plane perpendicular to the wave motion direction and λ_g is the graviton wave length. Therefore, in the CGR there is the effect of the *expansion* of the plane perpendicular to the affine wave motion direction.

As a result, in the CGR the total velocity of a test classical particle in the central gravitational field of a mass M and of a strong gravitational wave is the sum of three velocities at the cosmic evolution $a \neq 1$. The first term is the standard Newtonian (N) velocity, the second is the velocity of the *graviton expansion* (g) in the field of a gravitational wave, and the third one is the velocity of the Hubble evolution (H):

$$|\vec{v}|^2 = \left| \frac{dl_g}{d\eta} \right|^2 = \tag{7.26}$$

$$\left[\underbrace{\vec{n}_N \sqrt{\frac{r_g}{2R_\perp}}}_{\text{Newtonian velocity}} + \underbrace{\vec{n}_g \sqrt{R_\perp H_0} \sqrt{\Omega_g}}_{\text{graviton expansion}} + \underbrace{\vec{n}_H \gamma H_0 R_\perp}_{\text{Hubble evolution}} \right]^2.$$

Here,

$$R_\perp = r_\perp a(\eta)$$

is the Friedman distance from the central mass, H_0 is the Hubble parameter,

$$r_g(R_\perp) = \frac{M}{M_{\text{Pl}}^2}$$

is a constant gravitational radius, and

$$\begin{cases} \vec{n}_N = (0, -1, 0), \\ \vec{n}_g = (+1/\sqrt{2}, -1/\sqrt{2}, 0), \\ \vec{n}_H = (1, 0, 0) \end{cases} \tag{7.27}$$

are the unit velocity vectors. Their scalar products are

$$(\vec{n}_N \cdot \vec{n}_g) \neq 0, \quad (\vec{n}_N \cdot \vec{n}_H) = 0, \quad (\vec{n}_g \cdot \vec{n}_g) \neq 0, \quad (\vec{n}_N \cdot \vec{n}_H) = 0.$$

The graviton energy density Ω_g is given in units of the cosmological critical energy density ρ_{cr} .

The last two terms provide possible sources of a modified Newtonian dynamics. One observes that the interference of the Newtonian and the graviton-induced velocities in (7.26)

$$v_{n-g \text{ interf}} \simeq \sqrt[4]{\Omega_g r_g H_0}$$

does not depend on the radius R_\perp . This term can play a role of “dark” matter. In this case it is enough $\Omega_g \simeq 0.1$, in order to get the velocity of the Sun in our Galaxy $v_{n-g \text{ interf}} \simeq 200$ km/sec.

The third term could imitate the Dark Matter effect in COMA-type clusters with $|R| \sim 10^{25}$ cm, in accordance with the validity limit of the Newtonian dynamics,

$$\frac{r_g}{R_{\text{limit}}} < 2(R_{\text{limit}} H_0)^2,$$

discussed in [8, 9]. The factor $\gamma = \sqrt{2}$ is defined by the cosmological density [10].

Thus, in our model strong gravitational waves possess peculiar properties which can be tested by observations and experiments.

7.3 Vacuum creation of affine gravitons

Here we are going to study the effect of intensive creation of affine gravitons. We will briefly recapitulate the derivation given in Ref. [6] and further, using the initial data of the hierarchy of cosmological scales obtained in Sect. 6.3, estimate the number of created particles.

The approximation defined by Eqs. (7.20)–(7.21) can be rewritten

by means of the conformal variables and coordinates, where the action

$$W_{\text{lin}}^g = \int_{\eta_I}^{\eta_0} d\eta \left[-V_0 (\partial_\eta \langle D \rangle)^2 e^{-2\langle D \rangle} + \mathbb{L}_\eta^g \right] \quad (7.28)$$

is given in the interval $\eta_I \leq \eta \leq \eta_0$ and spatial volume V_0 . Here the Lagrangian and Hamiltonian

$$\mathbb{L}_\eta^g = \sum_{\mathbf{k}^2 \neq 0} e^{-2\langle D \rangle} \frac{v_{\mathbf{k}}^g v_{-\mathbf{k}}^g - \mathbf{k}^2 \bar{g}_{\mathbf{k}} \bar{g}_{-\mathbf{k}}}{2} = \left[\sum_{\mathbf{k}^2 \neq 0} p_{-\mathbf{k}}^g v_{\mathbf{k}}^g \right] - H_\eta^g, \quad (7.29)$$

$$H_\eta^g = \sum_{\mathbf{k}^2 \neq 0} \frac{e^{2\langle D \rangle} p_{\mathbf{k}}^g p_{-\mathbf{k}}^g + e^{-2\langle D \rangle} \omega_{0\mathbf{k}}^2 \bar{g}_{\mathbf{k}} \bar{g}_{-\mathbf{k}}}{2} \quad (7.30)$$

are defined in terms of the variables $\bar{g}_{\mathbf{k}}$, their momenta, and one-particle conformal energy

$$p_{\mathbf{k}}^g = e^{-2\langle D \rangle} v_{\mathbf{k}}^g = e^{-2\langle D \rangle} \partial_\eta \bar{g}_{\mathbf{k}}, \quad \omega_{0\mathbf{k}}^g = \sqrt{\mathbf{k}^2}, \quad (7.31)$$

respectively. The transformation (squeezing)

$$p_{\mathbf{k}}^g = \tilde{p}_{\mathbf{k}}^g e^{-\langle D \rangle} [\omega_{0\mathbf{k}}^g]^{-1/2}, \quad \bar{g}_{\mathbf{k}} = \tilde{g}_{\mathbf{k}} e^{\langle D \rangle} [\omega_{0\mathbf{k}}^g]^{1/2} \quad (7.32)$$

leads to the canonical form

$$H_\eta^g = \sum_{\mathbf{k}^2 \neq 0} \omega_{0\mathbf{k}}^g \frac{\tilde{p}_{\mathbf{k}}^g \tilde{p}_{-\mathbf{k}}^g + \tilde{g}_{\mathbf{k}} \tilde{g}_{-\mathbf{k}}}{2} = \sum_{\mathbf{k}} \mathcal{H}_{\mathbf{k}}^g, \quad (7.33)$$

$$\mathcal{H}_{\mathbf{k}}^g = \frac{\omega_{0\mathbf{k}}^g}{2} [\tilde{g}_{\mathbf{k}}^+ \tilde{g}_{-\mathbf{k}}^- + \tilde{g}_{\mathbf{k}}^- \tilde{g}_{-\mathbf{k}}^+], \quad (7.34)$$

where

$$\tilde{g}_{\mathbf{k}}^\pm = [\tilde{g}_{\mathbf{k}} \mp i\tilde{p}_{\mathbf{k}}] / \sqrt{2} \quad (7.35)$$

are the conformal-invariant classical variables in the holomorphic representation [10].

In virtue of Eqs. (7.31)–(7.35), the action (7.28) takes the form

$$\begin{aligned}
 W_{\text{lin}}^g &= \int_{\eta_I}^{\eta_0} d\eta \left[-V_0(\partial_\eta \langle D \rangle)^2 e^{-2\langle D \rangle} - \mathbf{H}_\eta^g \right] \\
 &+ \int_{\eta_I}^{\eta_0} d\eta \sum_{\mathbf{k}^2 \neq 0} \tilde{p}_{-\mathbf{k}} [\partial_\eta \tilde{g}_{\mathbf{k}} + \partial_\eta \langle D \rangle \tilde{g}_{\mathbf{k}}].
 \end{aligned} \tag{7.36}$$

The evolution equations for this action are

$$\partial_\eta \tilde{g}_{\mathbf{k}}^\pm = \pm i\omega_{0\mathbf{k}}^g \tilde{g}_{\mathbf{k}}^\pm + H_\eta \tilde{g}_{\mathbf{k}}^\mp, \tag{7.37}$$

where $H_\eta = \partial_\eta(\ln a) = -\partial_\eta \langle D \rangle$ is the conformal Hubble parameter (in our model $H_\eta = H_0/a^2$).

It is generally accepted to solve these equations by means of the Bogoliubov transformations

$$\tilde{g}_{\mathbf{k}}^+ = \alpha_{\mathbf{k}} b_{\mathbf{k}}^+ + \beta_{-\mathbf{k}}^* b_{-\mathbf{k}}^-, \tag{7.38}$$

$$\tilde{g}_{\mathbf{k}}^- = \alpha_{\mathbf{k}}^* b_{\mathbf{k}}^- + \beta_{-\mathbf{k}} b_{-\mathbf{k}}^+, \tag{7.39}$$

$$\alpha_{\mathbf{k}} = \cosh r_{\mathbf{k}}^g e^{i\theta_{\mathbf{k}}^g}, \quad \beta_{\mathbf{k}}^* = \sinh r_{\mathbf{k}}^g e^{i\theta_{\mathbf{k}}^g}, \tag{7.40}$$

where $r_{\mathbf{k}}^g$ and $\theta_{\mathbf{k}}^g$ are the squeezing parameter and the rotation one, respectively. These transformations preserve the Heisenberg algebra $O(2|1)$ [12] and diagonalize Eqs. (7.37) in the form of:

$$\partial_\eta b_{\mathbf{k}}^\pm = \pm i\bar{\omega}_{B\mathbf{k}}^g b_{\mathbf{k}}^\pm, \tag{7.41}$$

if the parameters of squeezing $r_{\mathbf{k}}^g$ and rotation $\theta_{\mathbf{k}}^g$ satisfy the following

equations [10]:

$$\partial_\eta r_{\mathbf{k}}^g = H_\eta \cos 2\theta_{\mathbf{k}}^g, \quad (7.42)$$

$$\omega_{0\mathbf{k}}^g - \partial_\eta \theta_{\mathbf{k}}^g = H_\eta \coth 2r_{\mathbf{k}}^g \sin 2\theta_{\mathbf{k}}^g, \quad (7.43)$$

$$\omega_{B\mathbf{k}}^g = \frac{\omega_{0\mathbf{k}}^g - \partial_\eta \theta_{\mathbf{k}}^g}{\coth 2r_{\mathbf{k}}^g}. \quad (7.44)$$

A general solution of the classical equations can be written with the aid of a complete set of the initial data $b_{0\mathbf{k}}^\pm$:

$$b_{\mathbf{k}}^\pm(\eta) = \exp \left\{ \pm i \int_{\eta_0}^{\eta} d\bar{\eta} \omega_{B\mathbf{k}}^g(\bar{\eta}) \right\} b_{0\mathbf{k}}^\pm. \quad (7.45)$$

On the other hand, quantities $b_{0\mathbf{k}}^+(b_{0\mathbf{k}}^-)$ can be considered as the creation (annihilation) operators, which satisfy the commutation relations:

$$[b_{0\mathbf{k}}^-, b_{0\mathbf{k}'}^+] = \delta_{\mathbf{k},-\mathbf{k}'}, \quad [b_{0\mathbf{k}}^-, b_{0\mathbf{k}'}^-] = 0, \quad [b_{0\mathbf{k}}^+, b_{0\mathbf{k}'}^+] = 0, \quad (7.46)$$

if one introduces the vacuum state as $b_{0\mathbf{k}}^-|0\rangle = 0$. Indeed, relations (7.46) are the results of: i) the classical Poisson bracket $\{P_{\tilde{F}}, \tilde{F}\} = 1$ which transforms into

$$[\tilde{g}_{\mathbf{k}}^-, \tilde{g}_{-\mathbf{k}}^+] = \delta_{\mathbf{k},\mathbf{k}'}; \quad (7.47)$$

ii) the solution (7.45) for the initial data; iii) the Bogoliubov transformations (7.38), (7.39).

With the aid of Eqs. (7.38)–(7.40) and (7.45)–(7.47) we are able to calculate the vacuum expectation value of the total energy (7.33), (7.34)

$$\langle 0 | H_\eta^g(a) | 0 \rangle = \sum_{\mathbf{k}} \omega_{0\mathbf{k}}^g |\beta_{\mathbf{k}}|^2 = \sum_{\mathbf{k}} \omega_{0\mathbf{k}}^g \frac{\cosh\{2r_{\mathbf{k}}^g(a)\} - 1}{2}. \quad (7.48)$$

The numerical analysis [6] of Eqs. (7.42)–(7.43) for unknown variables $(r_{\mathbf{k}}^g, \theta_{\mathbf{k}}^g)$ with the zero boundary conditions at $a = a_I$ (at the beginning of creation)

$$r_{\mathbf{k}}^g(a_I) = 0, \quad \theta_{\mathbf{k}}^g(a_I) = 0 \quad (7.49)$$

enables us to suggest an approximate analytical solution for the evolution equations.

Our approximation consists in the following. It arises, if instead of $r_{\mathbf{k}}$ one substitutes an approximate value r_{appr} in the vicinity of the soft mode of the Bogoliubov energy (7.44) $\omega_{0\text{appr}} = \partial_\eta \theta_{\text{appr}}^g$,

$$r_{\text{appr}} = \frac{1}{2} \int_{X_I=2\theta_{\text{appr}}^g(a_I)}^{X=2\theta_{\text{appr}}^g(a)} \frac{d\bar{X}}{\bar{X}} \cosh \bar{X} \simeq 2\langle D \rangle_I, \quad (7.50)$$

$$X(a) = 2\theta_{\text{appr}}^g(a) = 2 \int_{\eta(a_I)}^{\eta(a)} d\eta \omega_{0\mathbf{k}}. \quad (7.51)$$

This soft mode provides a transition [6] at the point $a_{\text{relax}}^2 \simeq 2a_{\text{Pl}}^2$ from the unstable state of the particle creation to the stable state with almost a constant occupation number during the relaxation time

$$\eta_{\text{relax}} \simeq \frac{2e^{-2\langle D \rangle_I}}{2H_0} \equiv \frac{2a_I^2}{2H_0}. \quad (7.52)$$

At the point of the relaxation, the determinant of Eqs. (7.37) changes its sign and becomes positive [13]. Finally, we obtain

$$\langle 0 | \mathcal{H}_{\mathbf{k}}^g | 0 \rangle \Big|_{(a > a_{\text{relax}})} = \omega_{0\mathbf{k}}^g \frac{\cosh[2r_{\mathbf{k}}^g] - 1}{2} \approx \frac{\omega_{0\mathbf{k}}^g}{4a_I^4}. \quad (7.53)$$

We have verified that the deviation of the results obtained with the aid of this formula from the numerical solutions of Eqs. (7.42)–(7.43) (see Ref. [6]) does not exceed 7%.

In virtue of this result, we obtain the total energy

$$\langle 0 | \mathbf{H}_\eta^g | 0 \rangle \Big|_{(a > a_{\text{relax}})} \approx \frac{1}{2a_I^4} \sum_{\mathbf{k}} \frac{\omega_{0\mathbf{k}}^g}{2} \equiv \frac{H_\eta^g \text{Cas}(a)}{2a_I^4}, \quad (7.54)$$

where $H_\eta^g \text{Cas}(a)$ is the Casimir vacuum energy.

Thus, the total energy of the created gravitons is

$$\langle 0 | \mathbf{H}_\eta^g | 0 \rangle \simeq \frac{\tilde{\gamma} H_0}{4a^2 a_I^4}. \quad (7.55)$$

It appeared that the dilaton initial data $a_I = e^{-(D)t}$ and H_0 determine both the total energy (7.54) of the created gravitons and their occupation number N_g at the relaxation time (7.52):

$$N_g(a_{\text{relax}}) \simeq \frac{\langle 0 | \mathbf{H}_\eta^g | 0 \rangle}{\langle \omega_k^g \rangle} \simeq \frac{\tilde{\gamma}^{(g)}}{16a_I^6} \simeq 10^{87}, \quad (7.56)$$

where we divided the total energy by the mean one-particle energy

$$\langle \omega_k^g \rangle \approx \langle \omega^{(2)} \rangle(a_I)$$

defined in Eq. (6.28). For numerical estimations we use $\tilde{\gamma}^{(g)} \approx 0.03$. The number of the primordial gravitons is compatible with the number of the CMB photons as it was predicted in Ref. [7].

The main result of this Section consists in the evaluation of the primordial graviton number (7.56). We suppose that the Casimir energy is defined by the total ground state energy of created excitations, see Eq. (7.54).

7.4 Summary

We developed a Hamiltonian approach to the gravitational model, formulated as the nonlinear realization of joint affine and conformal symmetries. With the aid of the Dirac – ADM foliation, the conformal and affine symmetries provide a natural separation of the dilaton and gravitational dynamics in terms of the Maurer – Cartan forms. As a result, the exact solution of the energy constraint yields the diffeo-invariant evolution operator in the field space.

In the CGR, the conformal symmetry breaking happens due to the Casimir vacuum energy. This energy is obtained as a result of the quantization scheme of the Hamiltonian dynamics. The diffeo-invariant dynamics in terms of the Maurer – Cartan forms with application of the affine symmetry condition leads to the reduction of the graviton representation to the one-component field. The affine graviton strong wave yields the effect of expansion (or contraction) in the hypersurface perpendicular to the direction of the wave propagation. We demonstrated that the Planck least action postulate applied to the Universe limited by its horizon provides the value of the cosmological scale factor at the Planck epoch. A hierarchy of cosmological energy scales for the states with different conformal weights is found. The intensive creation of primordial gravitons and Higgs bosons is described assuming that the Casimir vacuum energy is the source of this process. We have calculated the total energy of the created particles, Eq. (7.54), and their occupation numbers, Eq. (7.56).

Bibliography

- [1] Faddeev, L.D., Popov, V.N.: *Covariant quantization of the gravitational field*. Physics–Uspekhi. **16**, 777 (1974)
- [2] Pervushin, V.N., Arbuzov, A.B., Barbashov, B.M., Nazmitdinov, R.G., Borowiec, A., Pichugin, K.N., and Zakharov, A.F.: *Conformal and affine Hamiltonian dynamics of general relativity*. Gen. Relativ. & Gravit., **44**, 2745 (2012)
- [3] Tod, K.P.: *Three-dimensional Einstein –Weyl geometry. Geometry of low-dimensional manifolds*. Cambridge University Press (1990)
- [4] Landau, L.D., Lifshitz, E.M.: *The Classical Theory of Fields*. Pergamon Press (1971)
- [5] Lichnerowicz, A.: *L'integration des equations de la gravitation relativiste et le probleme des n corps*. Journ. Math. Pures and Appl. **B 37**, 23 (1944).
- York, J.W. (Jr.): *Gravitational degrees of freedom and the initial-value problem*. Phys. Rev. Lett. **26**, 1658 (1971).
- York, J.W. (Jr.): *Role of conformal three-geometry in the dynamics of gravitation*. Phys. Rev. Lett. **28**, 1082 (1972).

-
- [6] Arbuzov, A.B., Barbashov, B.M., Nazmitdinov, R.G., Pervushin, V.N., Borowiec, A., Pichugin, K.N., and Zakharov, A.F.: *Conformal Hamiltonian dynamics of General Relativity*. Phys. Lett. **B 691**, 230 (2010)
- [7] Babak, S.V., Grishchuk, L.P.: *The energy – momentum tensor for the gravitational field*. Phys. Rev. **D 61**, 024038 (2000)
- [8] Einstein, A., Straus, E.G.: *The influence of the expansion of space on the gravitation fields surrounding the individual stars*. Rev. Mod. Phys. **17**, 120 (1945)
- [9] Flin, P., Gusev, A.A., Pervushin, V.N., Vinitzky, S.I., Zorin, A.G.: *Cold dark matter as cosmic evolution of galaxies in relative units*. Astrophysics. **47**, 242 (2004)
- [10] Zakharov, A.F., Zinchuk, V.A., Pervushin, V.N.: *Tetrad formalism and frames of references in General Relativity*. Phys. Part. & Nuclei. **37**, 104 (2006)
- [11] Grib, A.A., Mamaev, S.G., Mostepanenko, V.M.: *Vacuum Quantum Effects in Strong Fields*. Friedmann Lab. Publ. (1994)
- [12] Jordan, T.F., Mukunda, N., Pepper, S.V.: *Irreducible representations of generalized oscillator operators*. Math. Phys. **4**, 1089 (1963)
- [13] Andreev, A.Yu., Kirzhnits, D.A.: *Tachyons and the instability of physical systems*. Physics–Uspekhi. **39**, 1071 (1996)

Chapter 8

Mathematical principles of description of the Universe

8.1 The classical theory of gravitation

The classical theory of gravitation, presented in our monograph is based on the following three principles:

1. The joint nonlinear realization of *affine and conformal symmetry groups* via Cartan's forms, described in Chapter 4.
2. 3+1 foliation of a pseudo-Riemannian space with *kinematic* subgroup of the group of general coordinate transformations, described in Chapter 5.
3. *The reduction of the phase space* by solving of all constraints.

The solving of all constraints, including the Hamiltonian one, which were presented in Chapter 5, reveals a diffeoinvariant physical content of the considered conformal and affine theory of gravitation. The diffeoinvariant content of the conformal theory of gravity includes in itself:

- dynamics at the surface of all the constraints, which is described by the action (5.47)

$$W_{C=0} = \int d^3x \left[\int \left(p_{(a)(b)} \omega_{(a)(b)}^R(d) + p_Q dQ + p_{A(b)} dA_{(b)} \right) \right] - \int P_{\langle D \rangle} d\langle D \rangle; \tag{8.1}$$

- square of the geometric interval (5.67) as a sum of squares of the components of the Fock’s frame in terms of the observed values

$$\tilde{d}s^2 = e^{-4D} \frac{\langle \sqrt{\tilde{\mathcal{H}}} \rangle^2}{\tilde{\mathcal{H}}} d\tau^2 - \left(dX_{(b)} - X_{(c)} \omega_{(c)(b)}^R(d) - \mathcal{N}_{(b)} d\tau \right)^2; \tag{8.2}$$

- geometrodynamics (such as Hubble’s law)

$$\tau = \int_{\langle D \rangle_t}^{\langle D \rangle_0} \frac{d\langle D \rangle}{\langle \sqrt{\tilde{\mathcal{H}}} \rangle}, \tag{8.3}$$

as cosmological relationship between geometry and dynamics as a function of the geometric luminosity interval from zero harmonic of the dilaton.

The action (8.1) contains an operator of evolution of the Universe

$$P_{\langle D \rangle} = \pm \mathbf{E}_U, \tag{8.4}$$

$$\mathbf{E}_U = 2 \int d^3x \sqrt{\tilde{\mathcal{H}}}, \tag{8.5}$$

determined from the exact solution of the constraint (5.77)

$$P_{\langle D \rangle}^2 - \mathbf{E}_U^2 = 0. \quad (8.6)$$

The role of the evolution parameter in the field space of events performs a value $\langle D \rangle$ called in observational cosmology as luminosity (or brightness), and $P_{\langle D \rangle}$ is its canonical momentum. The value of the generator of evolution of the Universe (8.5) on the equations of motion is

$$\frac{\delta W_{C=0}}{\delta F} = 0, \quad \frac{\delta W_{C=0}}{\delta P_F} = 0, \quad (8.7)$$

where F are field variables, we call the *energy of the Universe* in the field space of events by analogy with the *energy of a particle* in Minkowskian space in the Special Relativity.

8.2 Foundations of quantum theory of gravity

8.2.1 The irreducible unitary representation of the group $A(4) \otimes SO(4, 2)$

The theory of gravity was presented above as a nonlinear realization of finite-dimensional affine and conformal groups of *symmetry*, that close the group of general coordinate transformations. Therefore, as mentioned above, there is a unique opportunity to build further classification of experimental and observational data, using the unitary irreducible representations of these groups, without resorting to the

classical *laws of dynamics* as initial statements of the physical theory, or concluding classical *laws of dynamics* from the first principles of symmetry.

In the quantum theory of the Universe at the level of operator quantization in the field space of events $[\langle D \rangle | F]$ Hamiltonian constraint equation (8.4) becomes the equation of Wheeler – De Witt type (2.35)

$$\left[\hat{P}_{\langle D \rangle}^2 - \mathbf{E}_{\mathbb{U}}^2 \right] \hat{\Psi}_{\langle D \rangle_I, \langle D \rangle_0} = 0, \quad (8.8)$$

corresponding to the dimension of the kinematic subgroup of invariance of the Hamiltonian formulation. In the quantum theory, the canonical variables $\hat{P}_{\langle D \rangle}, \langle D \rangle$ become operators with a commutation relation

$$[\hat{P}_{\langle D \rangle}, \langle D \rangle] = i.$$

The general solution of this Wheeler – De Witt equation in the approximation of the empty Universe with Casimir vacuum energy obtained in Section 6.6 by the Bogoliubov transformation.

By analogy with the unitary irreducible representation of the Poincaré group (see Chapter 2 (2.36)) in quantum field theory we get a general operator solution of the Wheeler – De Witt equation (8.8) for the Universe as a sum of two -ordered with respect to parameter $\langle D \rangle$ exponents:

$$\hat{\Psi}_{\langle D \rangle_I, \langle D \rangle_0} = \hat{A}_{\langle D \rangle_I}^+ \hat{\mathbb{U}}_I^0 \frac{1}{\sqrt{2\mathbf{E}_{0\mathbb{U}}}} + \hat{A}_{\langle D \rangle_I}^- \frac{1}{\sqrt{2\mathbf{E}_{0\mathbb{U}}}} \hat{\mathbb{U}}_0^{I\dagger}, \quad (8.9)$$

describing the creation of the Universe at the time of $\langle D \rangle_I$, its evolution from $\langle D \rangle_I$ till the moment $\langle D \rangle_0$ and a state at the modern epoch $\langle D \rangle_0$. The two terms correspond to the positive and negative energy, where

$\hat{A}_{\langle D \rangle_I}^+$ can be interpreted as an operator of creation of the Universe at the moment $\langle D \rangle_I$ from a state of vacuum, and $\hat{A}_{\langle D \rangle_I}^-$ is an operator of annihilation of the Universe, correspondingly, with a commutation relation

$$[\hat{A}_{\langle D \rangle_I}^-, \hat{A}_{\langle D \rangle_I}^+] = 1 : \quad \hat{U}_I^0 = T_{\langle D \rangle} \exp \left\{ -i \int_{\langle D \rangle_I}^{\langle D \rangle_0} d\langle D \rangle \mathbf{E}_U \right\}; \quad \hat{U} \cdot \hat{U}^\dagger = \hat{I} \quad (8.10)$$

is an operator of evolution in the space of events, or space of measurements $[\langle D \rangle|F]$, relatively to the evolution parameter $[\langle D \rangle]$. A vacuum state $B^-|_{\langle D \rangle_I} 0\rangle = 0$ is set by actions of the Bogoliubov operators B^\pm , which diagonalize the evolution equations, as was shown above in Section 6.6. Negative energy is removed by the second quantization of the Universe and all the fields.

Thus, the reduction of the extended phase space to the subspace of physical variables gives the corresponding reduced action (8.1), which *is rejected* in the Standard Hamiltonian formulation of the General Relativity [1] as trivial. This action here is at the *forefront*, as the basic element of constructing of a quantum operator of creation and evolution of the Universe in the field space of events, by analogy with dynamic formulation of the Special Theory of Relativity.

Using a direct correspondence of Wheeler – De Witt between a particle in the Special Relativity and the Universe in the General Relativity (see Table at the end of Section 6.6), and the definition of irreducible unitary representations of the Poincaré group in the space

of events $[P_{(\alpha)}|X_{(\alpha)}]$:

$$\begin{aligned} \Psi[P_{(\alpha)}|X_{(\alpha)}] &= & (8.11) \\ &= \frac{1}{\sqrt{2|P_{(0)}|}} \left[a^+ \Psi_{P_{(0)+}} \theta(X_{(0)} - X_{I(0)}) + a^- \Psi_{P_{(0)-}}^* \theta(X_{I(0)} - X_{(0)}) \right], \end{aligned}$$

we can interpret the functional $\hat{\Psi}_{\langle D \rangle_I, \langle D \rangle_0}$ (8.9) as an unitary representation of the group $A(4) \otimes C$ in the field space of events $[\langle D \rangle|F]$. In the quantum geometrodynamics of the Universe for the relativistic theory of gravitation, we shall not forget also the geometric interval (8.2) and the relation (8.3) between the geometric interval and the dynamic parameter of evolution. This relation is the Hubble law in the exact theory, which includes quantum effects, such as the arrow of time, appearing in the quantum description of the Universe as a consequence of the postulate of the existence of the vacuum.

The unitary of representation of the (8.9)

$$\hat{U} \cdot \hat{U}^\dagger = \hat{I}$$

follows from the assumption of a positive definite metric in the Hilbert space of states. In the future, we show that the theory used to describe the matter after the constraints have been solved, indeed, contains only self-conjugated fields with a positive probability, for which the energy of the Universe (8.5) is positive and has not an imaginary part

$$\mathbf{E}_U = 2 \int d^3x \sqrt{\tilde{\mathcal{H}}} \geq 0; \quad \text{Im} \sqrt{\mathbf{H}} = 0. \quad (8.12)$$

For the construction of the irreducible representations (8.9) we introduce a complete set of orthogonal states

$$\langle Q|Q' \rangle = \delta_{Q,Q'}, \quad \sum_Q |Q\rangle \langle Q| = \hat{I}. \quad (8.13)$$

Here \hat{I} is a unit operator, and Q are quantum numbers, which characterize this representation of orthonormal states of the Universe, arisen out of the vacuum as the state with the lowest energy by an action of the operator of creation. The set Q includes numbers of occupation of particles and their one-particle energies, spins and other quantum numbers. All these definitions are within a *framework of the axiomatic approach in Quantum Field Theory*, including the postulate of the existence of vacuum [2] and representations of the Poincaré group in the tangent space of Minkowski.

The new fact is only that we are expanding representations of the Poincaré group in the tangent space of Minkowski by zero harmonic of the dilaton in full compliance with two classes of functions of kinematic subgroup of diffeomorphisms of Hamiltonian describing of the evolution of the Universe. It should describe the physical excitations of quantum gravity by the two classes of functions. Thus, all physical excitations in the reduced phase space can be classified by the homogeneous dilaton (*zero harmonic*) and localized *field-particles*, and the Newtonian-like *potentials* with zero momenta. Two independent variables: *dilaton and graviton* are squeezed oscillators, that allow the quantization and the Casimir vacuum energy specified in Chapter 6.

8.2.2 Casimir's vacuum

The canonical momentum of the zero harmonic of the dilaton is an evolution operator in the field space of events. The canonical momentum of the dilaton is not equal to zero if there is a non-zero Casimir energy

of all the other fields in the empty space, as was shown in Chapter 6. One method of measuring of this homogeneous dilaton is the redshift of spectral lines of atoms. Moreover, the occurrence of atoms of matter is also described by the *operator of creation of the Universe* (8.9). In this regard, we will consider the homogeneous dilaton $\langle D \rangle$ as a form of matter, along with non-homogeneous particles and their bound states, if matter is understood as all that is measured and independent of an observer. In any case, the separation of the dilaton from the metric of a space which was offered by Dirac, allows us to include the homogeneous dilaton in a field space of events as a parameter of evolution. The key idea of creation of the Universe (and, as we shall see later, its matter) is that the homogeneous dilaton in Conformal quantum theory of gravity is a squeezed oscillator. Therefore, the statement *that at the beginning of the Universe there was the homogeneous dilaton and the Casimir vacuum energy of all fields*¹ is physically correct in the context of solving the problem of creation of the Universe and particles of matter from the vacuum in the early Universe by standard quantum theory methods, as it has been done in Chapter 7 in describing creation of gravitons from the vacuum.

¹A similar phrase in the Standard cosmology “*in the early Universe there was a redshift of spectral lines of atoms of matter, and then there appeared the very atoms of matter*” recalls rather a fabulous statement about the Cheshire Cat: *at first there was a smile of a Cat, and then himself* (Carroll, Lewis: *Alice’s Adventures in Wonderland*. Macmillan and Co., London (1865).)

8.2.3 An approximation of a nearly empty Universe

According to the conformal scenario, the Universe was empty and cold at the time of its creation from vacuum. At this moment, the Casimir vacuum energy dominated. The Universe is almost empty throughout its evolution, including the modern era, according to the latest data on the Supernovae. As was shown in Chapter 6, in the framework of the Conformal cosmology, a cosmological scale factor of the empty Universe depends on the measured time interval as the square root. This is consistent with the description of the chemical evolution, which indicates that there are only a few percents of the baryonic matter in the Universe. In other words, the Universe is *cold and almost empty* all the time of its existence, since the conformal temperature is a constant equal to three Kelvin. This constant appeared under the normal ordering of the field operators in the Hamiltonian as a spontaneous breaking of conformal symmetry. The operator of creation and evolution of the Universe also contains additional quantum anomalies, such as the arrow of time. After the procedure of normal ordering the Hamiltonian takes the form

$$\tilde{\mathcal{H}} = \rho_{\text{Cas}} + : \tilde{\mathcal{H}} :, \quad (8.14)$$

where

$$\rho_{\text{Cas}} = \sum_{f,Q} \frac{\omega_{f,Q}}{2} \quad (8.15)$$

is the density of the Casimir energy of all the particles, as discussed in Chapter 6. The Casimir energy is appear as a result of normal

ordering the product of the field operators in the free Hamiltonian and the interaction Hamiltonian. According to the observational data on the Supernovae, the Casimir energy

$$\mathbf{E}_U^I = 2 \int d^3x \sqrt{\rho_{\text{Cas}}} \quad (8.16)$$

of the Universe is dominated. Dominance of the Casimir energy is the second cornerstone of our construction.

Let us consider further an expansion of the generator of the evolution (8.12) regarding this vacuum expectation

$$\begin{aligned} \mathbf{E}_U &= 2 \int d^3x \sqrt{\rho_{\text{Cas}} + : \tilde{\mathcal{H}} :} = \\ &= 2 \int d^3x \sqrt{\rho_{\text{Cas}}} + \frac{\int d^3x : \tilde{\mathcal{H}} :}{\sqrt{\rho_{\text{Cas}}}} + \dots = \\ &= \mathbf{E}_U^I + \frac{\mathbf{H}_{\text{QFT}}}{\sqrt{\rho_{\text{Cas}}}} + \dots \end{aligned} \quad (8.17)$$

where in the expression

$$\mathbf{H}_{\text{QFT}} = \int d^3x : \tilde{\mathcal{H}} : \quad (8.18)$$

it is easy to recognize the Hamiltonian of all fields of matter, including the field of gravitons. All of these fields have a positive definite metric after the explicit solving of all constraints in the frame of reference, selected by a unit time-like vector [2] (see Chapter 7).

In the case of approaching nearly empty space, the evolution operator (8.10) is presented as the product of three factors

$$\frac{\hat{\mathbf{U}}}{\sqrt{2\mathbf{E}_U}} = \frac{\mathbf{U}_0}{\sqrt{2\mathbf{E}_U^I}} \cdot \left[1 - \frac{1}{4} \hat{\Omega}_{\text{creation}} \right] \cdot T_{\tilde{t}} \exp \left\{ -i \int_{\tilde{t}_I}^{\tilde{t}_0} d\tilde{t} \tilde{\mathbf{H}}_{\text{QFT}} \right\}. \quad (8.19)$$

In the first factor on the left you can recognize the cosmological wave function of the empty Universe

$$\frac{\mathbb{U}_0}{\sqrt{2\mathbf{E}_U^I}},$$

earlier discussed in Section 6.6. The second factor in the form of square brackets, containing the operator of the relative density of matter creation in the Universe, includes the ratio of the Hamiltonian of QFT to the vacuum energy

$$\hat{\Omega}_{\text{creation}} = \frac{\mathbf{H}_{\text{QFT}}}{V_0 \rho_{\text{Cas}}}. \quad (8.20)$$

The third factor

$$T_{\tilde{t}} \exp \left\{ -i \int_{\tilde{t}_I}^{\tilde{t}_0} d\tilde{t} \mathbf{H}_{\text{QFT}} \right\} \equiv \hat{\mathbb{U}}_{\tilde{t}_I}^{\tilde{t}_0} \quad (8.21)$$

is a standard evolution operator in quantum field theory with respect to time

$$d\tilde{t} = \frac{d\langle D \rangle}{\sqrt{\rho_{\text{Cas}}}}, \quad (8.22)$$

which is given by the effective parameter of evolution in the field space of events. We shall see later that this time (8.22) coincides with the conformal time.

$$d\tilde{t} = d\eta. \quad (8.23)$$

The third factor can be presented as the product of N factors, breaking up the entire time interval of evolution of the Universe on N parts.

$$\hat{\mathbb{U}}_{\tilde{t}_I}^{\tilde{t}_0} = \prod_{n=1}^{n=N} \hat{\mathbb{U}}_{\tilde{t}_I}^{\tilde{t}_0 - n\Delta t}. \quad (8.24)$$

Inserting an identity operator between the factors as a sum over a complete set of all possible states

$$\hat{I} = \sum_Q |Q\rangle\langle Q|,$$

one can get the elements of the S - matrix in the representation of interaction [2]

$$\left\langle Q' \left| T_t \exp \left\{ -i \int_{\tilde{t}_0 - n\Delta t}^{\tilde{t}_0 - (n-1)\Delta t} dt \tilde{\mathbf{H}}_{\text{QFT}} \right\} \right| Q'' \right\rangle \equiv \left\langle Q'_{\text{int}} \left| \hat{S} \right| Q''_{\text{int}} \right\rangle. \quad (8.25)$$

A time interval Δt is determined by the energy resolution of physical devices and characteristic time of the processes in high-energy physics in modern accelerators².

Thus, the Hamiltonian formulation of the theory of gravity in the *reduced* phase space leads to a modification of a well-defined theory of the S - matrix, which will be discussed in the next chapter. The *reduced* Hamiltonian approach is the primary method of learning the theory of gauge fields, starting with the pioneering work of Dirac [3, 4], Heisenberg and Pauli [5, 6] and papers of Schwinger on the quantization of non-Abelian fields [7] (see details in [8, 9, 10, 11] and Appendix A).

²In this case, the interval Δt is a moment of time life of physicists, and the lifetime of physicists is a moment of the lifetime of the Universe.



Igor V. Polubarinov (1928, Moscow 1998, Dubna) is a Russian physicist. He is known by his pioneering results in the field - theoretical interpretation of gauge theories and gravity, obtained in close cooperation with V.I. Ogievetsky, and Hamiltonian formulation of the S-matrix for physical gauge fields, left after solving of all constraints. One of the most valuable results of I.V. Polubarinov is construction of an explicit form relativistic transformations of the physical fields from one frame to another. He was at the forefront of the fundamental operator quantization of gauge fields, on which the present monograph is based.

In all these papers, time components of the vector field with negative contributions to the energy are excluded, as was accepted in the Dirac approach to quantum electrodynamics [3, 4]. Dirac's Hamiltonian approach to QED in 1927 was based on a gauge-invariant action on the surface of constraint

$$W_{\text{QED}}^{\text{Dirac}} = W_{\text{QED}} \Big|_{\substack{\delta W_{\text{QED}} = 0 \\ \delta A_0^\ell}}, \quad (8.26)$$

where a component $A_0^\ell = (A \cdot \ell)$ is defined as the scalar product of the vector A_μ and a unit time-like vector ℓ_μ .

Such an elimination of the time component results into static interactions that form simultaneous bound states in QED, described by the Schrödinger equation, and in QCD, described by Salpeter equation

(see Appendix B). It was shown that the Dirac Hamiltonian approach leads to the correct relativistic transformations of observed quantized fields in non-Abelian gauge theories and theories of massive vector fields [7, 10, 11]. The Hamiltonian formulation [9] is considered as a justification of modern methods of quantization of gauge theories³, including the method of Faddeev – Popov [12], which used to describe the Standard Model of elementary particles [13].

The operator of creation of the Universe in the approximation (8.19) describes the three classes of processes: vacuum creation of matter, given in the previous chapter on the example of gravitons; scattering and decay of elementary particles described by the S-matrix and the interference of the S-matrix and vacuum creation. Below, we describe the physical content of the operator of creation of the Universe (8.19), having considered the two extreme cases: when the cosmological scale factor tends to zero (as a statement of the problem of origin of matter in the Universe), and when the cosmological scale factor tends to unity (as a modified description of the S-matrix elements of the scattering, decay, and formation of bound states of elementary particles).

8.3 Summary

Classical and quantum theory of the Universe appear as a result of solving of all constraint equations in the conformal-invariant theory. Herewith, the only source of the violation of the conformal symme-

³There is a stronger statement of Julian Schwinger “*we rejected all Lorentz gauge formulations as unsuited to the role of providing the fundamental operator quantization*” (see [7] p.324).

try is quantum anomalies such as the Casimir energy type, or Casimir condensates, arising via normal ordering of products of field operators in this theory. Dominance of the Casimir energy, confirmed, as was shown in Chapter 6, by the present data on Supernovae, leads to the approximation of a nearly empty Universe. This approximation means the factorization constructed above of the operator of creation and evolution of the Universe, on the wave function of the Universe, describing data on the Supernova by the Casimir energy and modified by the operator of evolution of fields of matter in QFT. As a result, we have a well-defined *cosmological modification* of the operator of evolution of fields, under their quantization in the phase space of field variables that remain after the solution of the constraint equations in the considered theory of gravity. Thus, the Hamiltonian approach provides an adequate formalism for unification of the theory of gravitational field with the Standard Model of elementary particles, in which both theories are considered at the quantum level, after solving all constraint equations in a certain frame of reference. In the future, we will consider in detail the above mentioned *cosmological modification* of the operator of evolution of fields in QFT, and also the creation of particle-like perturbations of these fields from the vacuum of the Universe.

Bibliography

- [1] Arnowitt, R., Deser, S., Misner, C.W.: *The dynamics of general relativity*. In *Gravitation: an Introduction to Current Research*. Witten, L. (ed.). Wiley, New York (1963)
- [2] Bogoliubov, N.N., Logunov, A.A., Oksak, A.I., Todorov, I.T.: *General Principles of Quantum Field Theory*. Kluwer Academic Publishers (1989)
- [3] Dirac, P.A.M.: *The physical interpretation of the quantum dynamics*. Proc. Roy. Soc. London. **A 113**, 621 (1927)
- [4] Dirac, P.A.M.: *Gauge-invariant formulation of quantum electrodynamics*. Can. J. Phys. **33**, 650 (1955)
- [5] Heisenberg, W., Pauli, W.: *Zur quantendynamik der wellenfelder*. Z. Phys. **56**, 1 (1929)
- [6] Heisenberg, W., Pauli, W.: *Zur quantentheorie der wellenfelder. II*. Z. Phys. **59**, 168 (1930)
- [7] Schwinger, J.: *Non-abelian gauge fields. Relativistic invariance*. Phys. Rev. **127**, 324 (1962)

- [8] Polubarinov, I.V.: *Equations of quantum electrodynamics*. Phys. Part. & Nuclei. **34**, 738 (2003)
- [9] Faddeev, L.D.: *The Feynman integral for singular Lagrangians*. Theor. Math. Phys. **1**, 1 (1969)
- [10] Pavel, H.-P., Pervushin, V.N.: *Reduced phase space space quantization of massive vector theory*. Int. J. Mod. Phys. **A 14**, 2885 (1999)
- [11] Pervushin, V.N.: *Dirac variables in gauge theories*. Phys. Part. & Nuclei. **34**, 679 (2003)
- [12] Faddeev, L.D., Popov, V.N.: *Feynman diagrams for the Yang – Mills field*. Phys. Lett. **B 25**, 29 (1967)
- [13] Bardin, D., Passarino, G.: *The Standard Model in the Making: Precision Study of the Electroweak Interactions*. Clarendon, Oxford (1999)

Chapter 9

Creation of matter in the Universe

9.1 The Big Bang or the vacuum creation?

9.1.1 Statement of the problem

Recall that quantum field theory in the Early Universe describes a set of particles as oscillators interacting with the cosmological scale factor. The scale factor squeezes the phase space of these oscillators. This squeezing is a source of the cosmological creation of the particles from vacuum. There is the following classification of these squeezed oscillators. Massless particles (fermions and photons) are not squeezed and created; massive particles (fermions and transversal components of massive vector W -, Z - bosons) are created very weakly; and strongly

squeezed oscillators (gravitons, scalar particles, and longitudinal components of electroweak massive vector bosons) suffer the intensive cosmological creation from vacuum due to their strong dependence on the cosmological scale factor. The Cosmic Microwave Background radiation and the baryon matter in the Early Universe can be the decay products of such primordial electroweak bosons and their annihilations.

The question: *Is modern theory able to explain the origin of observed matter in the Universe by its cosmological production from a vacuum?* was considered at the end of the 60s – early 70s of the last century in the set of papers [1, 2, 3, 4, 5]. As it is well known, the answer to this question is associated with the problem of particle creation in the vicinity of a cosmological singularity. Thus far, it has been common practice to assume that the number of product pairs is by far insufficient for explaining the total amount of observed matter [6]. We recall that the cosmological creation of massive particles is calculated by going over to conformal variables [7, 8], for which the limit of zero scale factor (point of a cosmic singularity) means vanishing of masses. Massive vector and scalar bosons are the only particles of the Standard Model that have a singularity at zero mass [9, 10]. In this limit, the normalization of the wave function for massive bosons is singular in mass [9, 10]. The absence of the massless limit in the theory of massive vector bosons is well known [11, 12]. In calculations in the lowest order of perturbation theory, this leads to a divergence of the number of produced longitudinal bosons [6, 8]. There exist two opinions concerning the removal of this singularity. In [6], the divergence of the number of particles is removed by means of a standard renormalization of the

gravitational constant. However, it is also indicated in the monograph of Grib *et al.*, [6] that the number of produced particles is determined by the imaginary part of loop Feynman diagrams; since, in quantum field theory, it is the real parts of these diagrams that are subjected to renormalization, this means that the above mentioned divergence of the number of particles does not belong to the class of divergences in quantum field theory that are removed by means of a conventional renormalization of physical quantities. Indeed, the physical origin of this divergence is that the problem of a cosmological creation of particles from a vacuum is treated within an idealized formulation. The point is that the quantum production of particles in a finite volume for a system featuring interaction and exchange effects may lead to a set of Bose particles having a specific statistical distribution with respect to energy, so that it is able to ensure the convergence of the respective integral of the momentum distribution. In the present study, we analyze physical conditions and models for which the number of product vector bosons may be quite sufficient for explaining the origin of matter in the Universe. Such cosmological models include the Conformal cosmology [13, 14], where conformal quantities of the General Theory of Relativity and of the Standard Model are defined as observables for which there are relative reference units of intervals.

The ensuing exposition of this Chapter is organized as follows. The next Subsection 9.1.2 is devoted to discussion of the problem of an origin of the Cosmic Microwave Background (CMB) radiation in the light of our classification of observational data in order to determine the initial data of the CMB origin. Section 9.2 describes the cosmological

creation of the primordial scalar bosons that are a source of the origin of all matter in the Universe. In Section 9.3 the physical states of the matter are classified in the context of the irreducible unitary representation of the Poincaré group. In Section 9.4 we give the generation and modification of the Faddeev – Popov path integral method in the light of this classification at the present day value of the cosmological scale factor $a = 1$.

9.1.2 Observational data on the CMB radiation origin

Observational data on the CMB testify about the first instances of creation and evolution of the matter in the Universe. According to these data obtained by means of satellites, air-balloons, and in observatories, the picture of temperature distribution of the CMB radiation is asymmetrical. This asymmetry is treated in the modern cosmology [15] as a signal of the motion of the Earth with reference to the CMB radiation. This motion has a velocity 368 km/sec towards the constellation Leo. To remove this asymmetry one should pass to a co-moving frame of reference, where the CMB radiation takes the symmetric form. Such a co-moving frame of reference by no means can be associated with any heavy body, as it was accepted in the celestial mechanics. In physics a new situation arises, when a frame of reference is connected with the parameters of the photon gas. In this case the choice of the co-moving frame of reference allows us to remove the dipole component of CMB radiation, separating out the motion of the very observer. As it was

shown in the previous chapters, the choice of the conformal etalons allows us to separate out the cosmic evolution of massive devices of the very observer. In terms of conformal etalons [18] observational Supernovae data [16, 17] allows us to obtain dependence of the horizon $r(z)$ on the redshift $1 + z$

$$r(z) = (1 + z)^{-2} \cdot H_0^{-1} \simeq (1 + z)^{-2} \times 10^{29} \text{ mm},$$

that is almost universal for all epoch, including the Beginning of the Universe. We propose that this Hubble law was valid at the instances of creation of the primordial particles and the origin of the CMB temperature. What does an origin of the concept of temperature mean? And how to define the range of the validity of this concept? According to D.I. Blokhintsev [19], the definition of the ranges of the validity of concepts can predict new effects and determine the values of physical magnitudes describing these effects, as we wrote in Chapter 1.

The concept of the temperature begins arising when the mean wavelength of a CMB photon coincides with the horizon (*id est* with the visible size of the Universe)

$$r(z_I) \simeq (1 + z_I)^{-2} \times 10^{29} \text{ mm} = 1 \text{ mm}.$$

From this it follows that at this instance the redshift is equal to

$$1 + z_I \simeq 3 \times 10^{14}. \quad (9.1)$$

The second observational fact consists in the present day values of masses of the primordial particles, the decay products of which give the CMB photons at the Beginning

$$M_0 = (1 + z_I) \cdot T_0 \sim 100 \text{ GeV}, \quad (9.2)$$



Dmitry I. Blokhintsev (29 December 1907 (11 January 1908), Moscow – 27 January 1979, Dubna), Russian physicist. Independently studied the basics of differential and integral calculus. Corresponded with K.E. Tsiolkovsky. From Tsiolkovsky Dmitry Ivanovich took the spirit of Russian science of the early 20-th century, which is not expressed so much in the quest to achieve concrete results, but how to create a holistic harmonious world. He graduated from the Physics Department of Moscow State University (1930). He taught there (since 1936 - professor, then head of the Department of Theoretical Nuclear Physics). He was the founder of the Division of Nuclear Physics at the Physics Department of Moscow State University. In 1935-1947 he worked in the Lebedev Physics Institute of the USSR Academy of Sciences. Since 1947 he was the Director of the Research Laboratory in Obninsk, on the basis of which under his leadership the Physics – Energy Institute was created. Together with I.V. Kurchatov, Blokhintsev initiated the establishment of the world's first nuclear power electrostation. Blokhintsev initiated the establishment of the Joint Institute for Nuclear Research (JINR) in Dubna. In 1956, he was elected as first director of the Institute. In the years 1956–1965 — JINR Director, since 1965 – Director of the Laboratory of Theoretical Physics, JINR.

that are in the region of the electroweak scale defined by the values of W - Z - masses.

Here

$$T_0 = 2.35 \times 10^{-13} \text{ GeV}$$

is the mean energy of the CMB photons

$$(1 + z_I) \simeq 3 \times 10^{14}$$

is the critical value of the redshift obtained above. The third observational fact is that the time of the CMB origin $(1+z_I) \simeq 3 \cdot 10^{14}$ coincides with the time when the Higgs particle Compton size becomes of order of the visible size of the Universe H_0/a_I^2 . The equality of such times yields the same region for e-w boson mass values.

$$M_0 = \sqrt{\frac{T_0^3}{H_0}} \sim 100 \text{ GeV}.$$

Just at this time one can introduce the concepts of these particles, and at this instance they are created from the vacuum [14, 20, 21]. This approximal coincidence of the instances of origin of the CMB temperature and the creation of the primordial e-w bosons points out that the CMB is the decay product of these bosons. In the next Section we give the direct calculation of the occupation number of the primordial Higgs particles created from the vacuum.

9.2 Vacuum creation of scalar bosons

In our model the interaction of scalar bosons and gravitons with the dilaton may be considered on an equal footing [22]. Using this fact, we

can consider the intensive creation of the Higgs scalar particles from the vacuum using the results of Section 7.3 devoted to the creation of gravitons.

To proceed we have to add the SM sector to the theory under construction. In order to preserve the common origin of the conformal symmetry breaking by the Casimir vacuum energy, we have to exclude the unique dimensional parameter from the SM Lagrangian, *id est* the Higgs term with a negative squared tachyon mass. However, following Kirzhnits [23], we can include the vacuum expectation of the Higgs field ϕ_0 , so that:

$$\phi = \phi_0 + \bar{h}/[a\sqrt{2}], \quad \int d^3x \bar{h} = 0.$$

The origin of this vacuum expectation value ϕ_0 can be associated with the Casimir energy arising as a certain external initial data at $a = a_{P1}$. In fact, let us apply the Planck least action postulate to the Standard Model action:

$$\phi_0 \approx a_{P1}^3 H_0$$

in agreement with its value in Table 6.1.

The standard vacuum stability conditions at $a = 1$

$$\langle 0|0 \rangle_{\phi=\phi_0} = 1, \quad \left. \frac{d \langle 0|0 \rangle}{d\phi} \right|_{\phi=\phi_0} = 0 \quad (9.3)$$

$$V_{\text{eff}}(\phi_0) = 0, \quad \frac{dV_{\text{eff}}(\phi_0)}{d\phi_0} = 0. \quad (9.4)$$

It results in a zero contribution of the Higgs field vacuum expectation into the Universe energy density. In other words, the SM mechanism

of a mass generation can be completely repeated in the framework of our approach to the spontaneous symmetry breaking.

In particular, one obtains that the Higgs boson mass is determined from the equation $V''_{\text{eff}}(\langle\phi\rangle) = M_h^2$. Note that in our construction the Universe evolution is provided by the dilaton, without making use of any special potential and/or any inflaton field. In this case we have no reason to spoil the renormalizability of the SM by introducing the non-minimal interaction between the Higgs boson and the gravity. In the middle field approximation (far from heavy masses), our gravitation theory supplemented by the Standard Model the Higgs bosons are described by the action

$$W_h = \int d\tau \sum_{\mathbf{k}^2 \neq 0} \frac{v_{\mathbf{k}}^h v_{-\mathbf{k}}^h - h_{\mathbf{k}} h_{-\mathbf{k}} a^2 \omega_{0\mathbf{k}}^h{}^2}{2} = \sum_{\mathbf{k}^2 \neq 0} p_{-\mathbf{k}}^h v_{\mathbf{k}}^h - H_\tau^h, \quad (9.5)$$

where

$$\omega_{0\mathbf{k}}^h(a) = \sqrt{\mathbf{k}^2 + a^2 M_{0h}^2} \quad (9.6)$$

is the massive one-particle energy with respect to the conformal time interval.

There are values of the scale factor a , when the mass term in one-particle energy is less than the conformal Hubble parameter value

$$aM_{0h} < H_0 a^{-2}.$$

As a result, the Casimir energy for the Higgs particles coincides with the graviton one at the considered epoch:

$$H_{\text{Cas}}^h \simeq \sum_{\mathbf{k}} \frac{\sqrt{\mathbf{k}^2}}{2} = H_{\text{Cas}}^g.$$

In this case the calculation of the scalar particle creation energy completely repeats the scheme for the graviton creation discussed in Chapter 7. Assuming thermalization in the primordial epoch, we expect that the occupation number of the primordial Higgs bosons is of the order of the known CMB photon one

$$N_h \sim N_\gamma = 411\text{mm}^{-3} \times \frac{4\pi r_h^3}{3} \simeq 10^{87}. \quad (9.7)$$

Thus, the CGR provides a finite occupation number of the produced primordial particles. Note that in other approaches [6] a subtraction is used to achieve a finite result. Moreover, the number of produced particles happens to be of the order of the known CMB photon number. To our opinion this coincidence supports our model, since the number of photons can naturally inherit the number of primordial Higgs bosons (if one considers the photons as one of the final decay products of the bosons). According to our model, the relativistic matter has been created very soon after the Planck epoch at $z_{\text{Pl}} \simeq 10^{15}$. Later on it cooled down and at $z_{\text{CMB}} \simeq 1000$ the CMB photons decoupled from recombined ions and electrons as discussed by Gamow. In our model the CMB temperature is defined directly from the Hubble parameter and the Planck mass (related to the Universe age a_{Pl}).

Note that the obtained occupation number corresponds to the thermalized system of photons with the mean wave length $T \simeq 3^\circ \text{ K}$ in finite volume $V_0 \sim H_0^{-3}$:

$$(N_\gamma)^{1/3} \simeq 10^{29} \simeq \lambda_{\text{CMB}} H_0^{-1}. \quad (9.8)$$

As concerns vacuum creation of spinor and vector SM particles, it

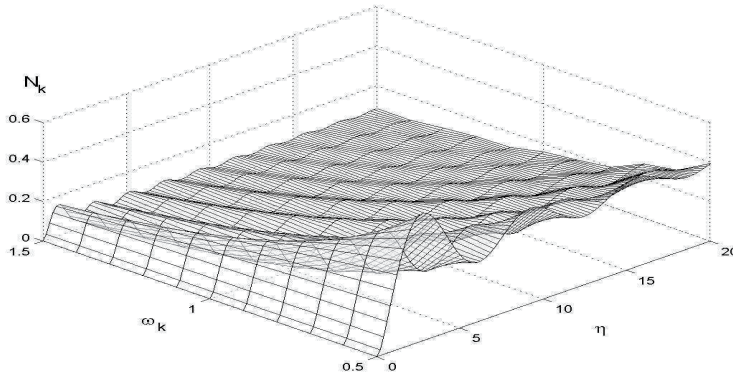


Figure 9.1: The process of the vacuum creation of $N_h \sim 10^{88}$ Higgs particles during the first 10^{-12} sec is shown. Here \nearrow is the time axis, \uparrow is the number of scalar bosons and longitudinal components of vector ones, and \searrow is the axis of their momenta [21].

is known [6] to be suppressed very much with respect to the one of scalars and gravitons.

The intensive creation of primordial gravitons and Higgs bosons is described assuming that the Casimir vacuum energy is the source of this process [21].

The vacuum creation of massive spinors and transversal components of vector bosons [6] is suppressed with respect to the creation of scalar particles and gravitons.

9.3 Physical states of matter

According to the general principles of quantum field theory (QFT), physical states of the lowest order of perturbation theory are completely covered by local fields as particle-like representations of the

Poincaré group of transformations of four-dimensional space-time. The existence of each elementary particle is associated with a quantum field ψ . These fields are operators defined in all space-time and acting on states $|\mathcal{P}, s\rangle$ in the Hilbert space with positively defined scalar product. The states correspond to the wave functions $\Psi_\alpha(x) = \langle 0|\psi_\alpha(x)|\mathcal{P}, s\rangle$ of free particles.

Its algebra is formed by generators of the four translations $\hat{\mathcal{P}}_\mu = i\partial_\mu$ and six rotations $\hat{M}^{\mu\nu} = i[x_\mu\partial_\nu - x_\nu\partial_\mu]$. The unitary and irreducible representations are eigen-states of the Casimir operators of mass and spin, given by

$$\hat{\mathcal{P}}^2|\mathcal{P}, s\rangle = m_\psi^2|\mathcal{P}, s\rangle, \quad (9.9)$$

$$-\hat{w}_p^2|\mathcal{P}, s\rangle = s(s+1)|\mathcal{P}, s\rangle, \quad (9.10)$$

$$\hat{w}_\rho = \frac{1}{2}\varepsilon_{\lambda\mu\nu\rho}\hat{\mathcal{P}}^\lambda\hat{M}^{\mu\nu}. \quad (9.11)$$

The unitary irreducible Poincaré representations describe wave-like dynamical local excitations of two transverse photons in QED

$$A_{(b)}^T(t, \mathbf{x}) = \quad (9.12)$$

$$= \int \frac{d^3k}{(2\pi)^3} \sum_{\alpha=1,2} \frac{1}{\sqrt{2\omega(\mathbf{k})}} \varepsilon_{(b)\alpha} \left[e^{i(\omega_{\mathbf{k}}t - \mathbf{k}\mathbf{x})} A_{\mathbf{k},\alpha}^+ + e^{-i(\omega_{\mathbf{k}}t - \mathbf{k}\mathbf{x})} A_{\mathbf{k},\alpha}^- \right].$$

Two independent polarizations $\varepsilon_{(b)\alpha}$ are perpendicular to the wave vector and to each other, and the photon dispersion is given by $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2}$. The creation and annihilation operators of photon obey the commutation relations

$$[A_{\mathbf{k},\alpha}^-, A_{\mathbf{k}',\beta}^+] = \delta_{\alpha,\beta}\delta(\mathbf{k} - \mathbf{k}').$$

The bound states of elementary particles (fermions) are associated with bilocal quantum fields formed by the instantaneous potentials (see [24, 25, 26])

$$\begin{aligned} \mathcal{M}(x, y) = \mathcal{M}(z|X) &= \sum_H \int \frac{d^3\mathcal{P}}{(2\pi)^3 \sqrt{2\omega_H}} \int \frac{d^4 q e^{iq \cdot z}}{(2\pi)^4} \times \\ &\times [e^{i\mathcal{P} \cdot X} \Gamma_H(q^\perp | \mathcal{P}) a_H^+(\mathcal{P}, \mathbf{q}^\perp) + e^{-i\mathcal{P} \cdot X} \bar{\Gamma}_H(q^\perp | \mathcal{P}) a_H^-(\mathcal{P}, \mathbf{q}^\perp)], \end{aligned} \quad (9.13)$$

where

$$\mathcal{P} \cdot X = \omega_H X_0 - \mathcal{P}\mathbf{X}, \quad q_\mu^\perp = q_\mu - \frac{\mathcal{P} \cdot q}{M_H^2} \mathcal{P}_\mu,$$

$\mathcal{P}_\mu = (\omega_H, \mathcal{P})$ are the momentum components on the mass shell,

$$\omega_H = \sqrt{M_H^2 + \mathcal{P}^2},$$

and

$$X = \frac{x + y}{2}, \quad z = x - y, \quad (9.14)$$

are the total coordinate and the relative one, respectively. The functions Γ belong to the complete set of orthonormalized solutions of the BS equation [27] in a specific gauge theory, $a_H^\pm(\mathcal{P}, \mathbf{q}^\perp)$ are coefficients treated in quantum theory as the creation (+) and annihilation (-) operators (see Appendix B).

The irreducibility constraint, called the Markov – Yukawa constraint, is imposed on the class of instantaneous bound states [24]

$$z^\mu \hat{\mathcal{P}}_\mu \mathcal{M}(z|X) \equiv \imath z^\mu \frac{d}{dX^\mu} \mathcal{M}(z|X) = 0. \quad (9.15)$$

In Ref. [28] the in- and out- asymptotical states are the “rays” defined as a product of these irreducible representations of the Poincaré group

$$\langle \text{out} | = \langle \prod_J \mathcal{P}_J, s_J |, \quad | \text{in} \rangle = | \prod_J \mathcal{P}_J, s_J \rangle. \quad (9.16)$$

This means that all particles (elementary and composite) are far enough from each other to neglect their interactions in the in-, out- states. All their asymptotical states $\langle \text{out} |$ and $|\text{in} \rangle$ including the bound states are considered as the irreducible representations of the Poincaré group.



Moisei A. Markov (30 April (13 May) 1908, Rasskazovo, Tambov province 1 October 1994, Moscow), Russian physicist - theorist, Academician. Markov graduated from Moscow State University in 1930. He is a member of the Presidium of the Academy of Sciences of the USSR. He has been chairman of the Interdepartmental Commission for Nuclear Physics since 1971. He was one of the organizers of the Joint Institute for Nuclear Research. He made a pioneer contribution into the development of neutrino investigations, demonstrated the expediency of conducting neutrino experiments at great depths underground and the possibility of conducting such experiments in accelerators. He studied fundamental problems of elementary particle physics and quantum gravitation on the boundary of particle physics and cosmology.

These irreducible representations form a complete set of states, and the reference frames are distinguished by the eigenvalues of the appropriate time operator $\hat{\ell}_\mu = \frac{\hat{\mathcal{P}}_\mu}{M_J}$

$$\hat{\ell}_\mu |\mathcal{P}, s\rangle = \frac{\mathcal{P}_{J\mu}}{M_J} |\mathcal{P}_J, s\rangle, \quad (9.17)$$

where the Bogoliubov – Logunov – Todorov rays (9.16) can include bound states.

9.4 QU modification of S-matrix in QFT

The S-matrix elements are defined as the evolution operator expectation values between in- and out- states

$$\underbrace{\mathcal{M}_{\text{in,out}}}_{P\text{-inv},G\text{-inv}} = \underbrace{\langle \text{out} |}_{P\text{-covariant}} \underbrace{\hat{S}[\hat{\ell}]}_{P\text{-covariant},G\text{-inv}} \underbrace{|\text{in}\rangle}_{P\text{-covariant}}, \quad (9.18)$$

where the abbreviation “ $G - inv$ ”, or “gauge-invariant”, assumes the invariance of S-matrix with respect to the gauge transformations, and “ $P - covariant$ ” means relativistic covariance with respect to the Poincaré group transformations. The conformal modification of S - matrix, in this case, means that the conformal symmetry can be violated by the quantization procedure with the vacuum postulate. in the reduced phase space after resolution of all constraints. Such reduced phase space quantization coincides with the Dirac approach to the gauge invariant theories. The Dirac approach to gauge-invariant S-matrix was formulated at the rest frame $\ell_\mu^0 = (1, 0, 0, 0)$ [29, 30, 31]. The Dirac Hamiltonian approach to QED of 1927 was based on the constraint-shell action [29]

$$W_{\text{QED}}^{\text{Dirac}} = W_{\text{QED}} \Big|_{\frac{\delta W_{\text{QED}}}{\delta A_0^\ell} = 0}, \quad (9.19)$$

where the component A_0^ℓ is defined by the scalar product

$$A_0^\ell = (A \cdot \ell)$$

of vector field A_μ and the unit time-like vector ℓ_μ . The gauge was established by Dirac as the first integral of the Gauss constraint

$$\int^t dt \frac{\delta W_{\text{QED}}}{\delta A_0^\ell} = 0, \quad t = (x \cdot \ell). \quad (9.20)$$

In this case, the S-matrix elements (9.18) are relativistic invariant and independent of the frame reference provided the condition (9.17) is fulfilled [26].

Dirac introduced the radiation variables

$$\imath e A_k^*[A_j] = u^*[A_j] (\imath e A_k - \partial_k) (u^*)^{-1}[A_j], \quad (9.21)$$

$$\psi^*[A_j, \psi] = u^*[A_j] \psi, \quad (9.22)$$

where the phase factors $u^*[A_j]$ satisfy the equation

$$u^*[A_j] (\imath e a_0[A_j] - \partial_0) (u^*)^{-1}[A_j] = 0; \quad (9.23)$$

here $a_0[A_j]$ is a solution of the Gauss constraint equation

$$\Delta a_0[A_j] = \partial_j \partial_0 A_j. \quad (9.24)$$

One can be convinced that the radiation variables (9.21) are gauge-invariant functionals.

Then the problem arises how to construct a gauge-invariant S-matrix in an arbitrary frame of reference. It was Heisenberg and Pauli's question to von Neumann [30]: "How to generalize the Dirac Hamiltonian approach to QED of 1927 [29] to any frame?" [30, 31, 32, 33]. The reply of von Neumann was to go back to the initial Lorentz-invariant formulation

$$\imath e A_k = (u^*)^{-1}[A_j] (\imath e A_k^*[A_j] - \partial_k) u^*[A_j], \quad (9.25)$$

$$\psi = (u^*)^{-1}[A_j] \psi^*[A_j, \psi], \quad (9.26)$$

and to choose the co-moving frame

$$\ell_\mu^0 = (1, 0, 0, 0) \rightarrow \ell_\mu^{\text{co-moving}} = \ell_\mu, \quad \ell_\mu \ell^\mu = (\ell \cdot \ell) = 1 \quad (9.27)$$

and to repeat the gauge-invariant Dirac scheme in this frame for calculation of the spectrum and S -matrix elements (9.18). In the following we call this gauge-invariant approach the von Neumann – Polubarinov formulation because Polubarinov constructed the corresponding gauge transformations (9.27) in the manifest form [31, 32, 33]. In this approach the S -matrix elements (9.18) are relativistic invariant and do not contain nonphysical states with indefinite metrics provided by the constraint (9.17) [26, 34]. Therefore, relativistic bound states can be successfully included in the relativistic covariant unitary perturbation theory [34]. They satisfy the Markov – Yukawa constraint (9.15), where the time axis ℓ^0 is the eigenvalue of the total momentum operator of instantaneous bound states.

In QED this framework yields the observational spectrum of bound states [35], and leads to the Schrödinger equation (see Appendix B), and paves the way for constructing a bound state generating functional in QCD. The functional construction is based on the Poincaré group representations with ℓ^0 being the eigenvalue of the total momentum operator of instantaneous bound states. In order to demonstrate the Lorentz-invariant version of the Dirac method [29] given by Eq. (9.19) in a non-Abelian theory, we consider the simplest example of the Lorentz-invariant formulation of the naive path integral without any ghost fields and FP-determinant. Recall that the simplest example of the Lorentz formulation that uses the naive path integral without

any ghost fields and FP-determinant is described by the generating functional

$$Z[J, \eta, \bar{\eta}] = \int \left[\prod_{\mu, a} dA_{\mu}^a \right] d\psi d\bar{\psi} e^{iW[A, \psi, \bar{\psi}] + iS[J, \eta, \bar{\eta}]}. \quad (9.28)$$

We use standard the QCD action $W[A, \psi, \bar{\psi}]$ and the source terms

$$W = \int d^4x \left[-\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \bar{\psi} (\gamma^{\mu} (\partial_{\mu} + \hat{A}_{\mu}) - m) \psi \right], \quad (9.29)$$

$$F_{0k}^a = \partial_0 A_k^a - \partial_k A_0^a + g f^{abc} A_0^b A_k^c \equiv \dot{A}_k^a - \nabla_k^{ab} A_0^b, \quad (9.30)$$

$$S[A_{\mu}] = \int d^4x [A_{\mu} J^{\mu} + \bar{\eta} \psi + \bar{\psi} \eta], \quad \hat{A}_{\mu} = g \frac{\lambda^a A_{\mu}^a}{2i}. \quad (9.31)$$

There are a lot of drawbacks of this path integral (9.28) from the point of view of QFT and the Faddeev – Popov functional [37]. They are the following:

1. The time component A_0 has indefinite metric.
2. The integral (9.28) contains the infinite gauge factor.
3. The bound state spectrum contains tachyons.
4. The analytical properties of field propagators are gauge dependent.
5. Operator foundation is absent [36].
6. Low-energy region of hadronization is not separated from the high-energy one.

All these defects can be removed by the integration over the indefinite metric time component $A_{\mu} \ell^{\mu} \equiv (A \cdot \ell)$, where ℓ^{μ} is an arbitrary unit

time-like vector: $\ell^2 = 1$. If $\ell^0 = (1, 0, 0, 0)$ then $A_\mu \ell^\mu = A_0$. In this case the generational functional (9.28) takes the form

$$\begin{aligned}
Z[\ell^0] &= \int \left[\prod_{x,j,a} dA_j^{a*}(x) \right] e^{iW_{\text{YM}}^*} \delta(L^a) [\det(\nabla_j(A^*))^2]^{-1/2} Z_\psi, \\
L^a &= \int d\bar{t} \nabla_i^{ab}(A^*) \dot{A}_i^{*b} = 0, \\
W_{\text{YM}}^* &= \int d^4x \frac{(\dot{A}_j^{a*})^2 - (B_j^a)^2}{2}, \\
Z_\psi[J^*, \eta^*, \bar{\eta}^*] &= \int d\psi^* d\bar{\psi}^* e^{-\frac{i}{2}(\psi^* \bar{\psi}^*, \mathcal{K} \psi^* \bar{\psi}^*) - (\psi^* \bar{\psi}^*, G_A^{-1}) + iS[J^*, \eta^*, \bar{\eta}^*]} \quad (9.32) \\
(\psi^* \bar{\psi}^*, G_A^{-1}) &= \int d^4x \bar{\psi}^* \left[v\gamma_0 \partial_0 - \gamma_j (\partial_j + \hat{A}_j^*) - m \right] \psi^*, \\
(\psi^* \bar{\psi}^*, \mathcal{K} \psi^* \bar{\psi}^*) &= \int d^4x d^4y j_0^a(x) \left[\frac{1}{(\nabla_j(A^*))^2} \delta^4(x-y) \right]^{ab} j_0^b(y).
\end{aligned}$$

The infinite factor is removed by the gauge fixing (9.20) treated as an antiderivative function of the Gauss constraint. A_i^{*a} denotes fields A_i^a under gauge fixing condition (9.20). It becomes homogeneous

$$\nabla_i^{ab}(A^*) \dot{A}_i^{*b} = 0,$$

because A_0^* is determined by the interactions of currents. It is just the non-Abelian generalization [32, 33] of the Dirac approach to QED [29].

In the case of QCD there is a possibility to include the nonzero condensate of transverse gluons

$$\langle A_j^{*a} A_i^{*b} \rangle = 2C_{\text{gluon}} \delta_{ij} \delta^{ab}$$

as a source of the conformal symmetry breaking. The Lorentz-invariant bound state matrix elements can be obtained, if we choose the time-axis ℓ of the Dirac Hamiltonian dynamics as the operator acting in

the complete set of bound states (9.17) and given by Eqs. (9.14) and (9.15). This scheme enables us to restore the Lorentz-invariance, if the time-axis is taken as the operator $\hat{\ell}$ proper frame of reference of each bound state. This means the von Neumann substitution (9.27) given in [30]

$$Z[\ell^0] \rightarrow Z[\ell] \rightarrow Z[\hat{\ell}]. \quad (9.33)$$

Thus, we shown how to use the Dirac Hamiltonian formulation in order to describe bound states by a relativistic-invariant manner with the Feynman rules depending on the Markov – Yukawa frame of reference

In the modern literature such dependence is treated as a defect that complicates the perturbation theory. In order to remove this dependence, in the Faddeev – Popov approach [37] one passes to the variables of the type of gauge transformation,

$$\hat{A}_k^*[A_j^{Lb}] = u^*[A_j^{Lb}] \left(\hat{A}_k^L + \partial_k \right) u^{*-1}[A_j^{Lb}], \quad (9.34)$$

$$\psi^*[A_j^{Lb}, \psi^L] = u^*[A_j^{Lb}] \psi^L, \quad (9.35)$$

where A_μ^L obeys to condition that does not depend on a frame of reference, in particular, the Lorentz constraint $\partial_\mu A_\mu^L = 0$, and the phase factors $u^*[A_j^L]$ satisfy the equation

$$u^*[A_j^L] \left(\hat{a}_0[A_j^L] + \partial_0 \right) (u^*)^{-1}[A_j^L] = 0. \quad (9.36)$$

Here $a_0^c[A_j]$ is a solution of the Gauss constraint

$$[[\nabla_j(A^L)]^2]^{cb} a_0^b = \nabla_i^{cb}(A^L) \dot{A}_i^{Lb}. \quad (9.37)$$

A solution of Eq. (9.36) takes the form

$$u^*[A_j^{Lb}] = v(\mathbf{x})T \exp \left\{ \int^t d\bar{t} \hat{a}_0[A_j^{Lb}] \right\}, \quad (9.38)$$

where the symbol T means the time ordering of matrix under the exponent sign, $v(\mathbf{x})$ are initial data of Eq. (9.36). These gauge transformations keep the action due to its gauge invariance.

$$W^* = W[A_\mu^L]. \quad (9.39)$$

Such variable change is known as the *choice of a gauge* [37]. The choice of gauge changes the Feynman rules. One can choose a *gauge* for which the Feynman rules completely do not depend on initial data [38]. However, in the generating functional (9.32), there are sources of the transversal fields $S[J^*, \eta^*, \bar{\eta}^*]$ that depend on initial data $v(\mathbf{x})$ of Eq. (9.36).

The assertion about the *gauge*-independence of the physical contents of the generating functional that removes any initial data is called the Faddeev theorem [38]. To prove this theorem one has to be convinced also that the sources of the radiation variables can be replaced by the sources of the fields in the Lorentz *gauge* formulation

$$\begin{aligned} S[J^*, \eta^*, \bar{\eta}^*] &= \int d^4x [J_k^{*c} A_k^{*c}[A_j^{Lb}] + \bar{\eta}^* \psi^*[A_j^{Lb}, \psi^L] + \bar{\psi}^*[A_j^{Lb}, \psi^L] \eta^*] \\ &\rightarrow S[A_\mu^L, \psi^L, \bar{\psi}^L], \end{aligned} \quad (9.40)$$

where $S[A_\mu^L]$ is given by Eq. (9.31). Actually, this theorem was proved in paper [38] only for the scattering processes of the elementary particles in QED.

However, as we saw above, this *change of a gauge* can disturb the bound state spectrum. In any case, questions arise about the range of validity of the Faddeev theorem [38]. In the next chapters we shall return to these questions and discuss the status of the primordial initial data and their physical effects.

9.5 Summary

The gravitation theory treated as the joint nonlinear realization of affine and conformal symmetry groups in terms of the diffeoinvariant Cartan forms allows us to quantize these forms. This quantum gravitation theory is unified with the Standard Model of the elementary particles on the equal footing in the form of the wave function of the Universe as a joint irreducible unitary representation (IUR) of the affine and conformal symmetry groups constructed in the previous Chapter 8. Results given in Chapter 9 testify that this wave function of the Universe yields the classification of real physical processes in the Universe, including its creation from vacuum together with its matter content in agreement with observational and experimental data.

In this Chapter we present theoretical and observational arguments in favor of the fact that there was the Beginning and there arose from vacuum 10^{88} primordial Higgs particles at the Beginning during the first 10^{-12} sec. All matter content of the present day Universe can be the decay products of these primordial bosons. In this case, one can expect that the wave function of the Universe yields classification of the real processes in the Universe at present time. In any case, we

reminded the Poincaré classification of the particle states including the bound ones and obtained relation of our approach to QFT with the accepted Faddeev – Popov path integral. The next Chapter will be devoted to the status of the initial data in QCD.

Bibliography

- [1] Chernikov, N.A., Tagirov, E.A.: *Quantum theory of scalar fields in de Sitter space-time*. Annales Poincaré Phys. Theor. **A9**, 109 (1968)
- [2] Parker, L.: *Particle creation in expanding universes*. Phys. Rev. Lett. **21**, 562 (1968).
- Parker, L.: *Quantized fields and particle creation in expanding universes. I*. Phys. Rev. **183**, 1057 (1969).
- Parker, L.: *Quantized fields and particle creation in expanding universes. II*. Phys. Rev. **D 3**, 346 (1971)
- [3] Sexl, R.U., Urbantke, H.K.: *Production of particles by gravitational fields*. Phys. Rev. **179**, 1247 (1969)
- [4] Zel'dovich, Ya.B.: *Particle production in cosmology*. JETP Letters. **12**, 307 (1971)
- [5] Zel'dovich, Ya.B., Starobinskii, A.A.: *Particle production and vacuum polarization in an anisotropic gravitational field*. JETP. **34**, 1159 (1972)

-
- [6] Grib, A.A., Mamaev, S.G., Mostepanenko, V.M.: *Vacuum Quantum Effects in Strong Fields*. Friedmann Lab. Publ. (1994)
- [7] Pervushin, V.N., Smirichinski, V.I.: *Bogoliubov quasiparticles in constrained systems*. J. Phys. **A 32**, 6191 (1999)
- [8] Pervushin, V.N., Proskurin, D.V., Gusev, A.A.: *Cosmological particle origin in Standard Model*. Gravitation & Cosmology. **8**, 181 (2002)
- [9] Wentzel, G.: *Quantum Theory of Fields*. Courier Dover Publns. (2003)
- [10] Pavel, H.-P, Pervushin, V.N.: *Reduced phase space quantization of massive vector theory*. Int. J. Mod. Phys. **A**, 2285 (1999)
- [11] Ogievetski, V.I., Polubarinov, I.V.: *Gauge-invariant formulation of the theory of neutral vector field*. ZhETF. **41**, 246 (1961)
- [12] Slavnov, A.A., Faddeev, L.D.: *Massless and massive Yang – Mills fields*. Theor. Math. Phys. **3**, 312 (1971)
- [13] Arbuzov, A.B., Barbashov, B.M., Nazmitdinov, R.G., Pervushin, V.N., Borowiec, A., Pichugin, K.N., and Zakharov, A.F.: *Conformal Hamiltonian dynamics of General Relativity*. Phys. Lett. **B 691**, 230 (2010).
- [14] Pervushin, V.N.: *Early Universe as a W^- , Z^- factory*. Acta Physica Slovaca. **53**, 237 (2003).

- Blaschke, D.B., Vinitzky, S.I., Gusev, A.A., Pervushin, V.N., Proskurin, D.V.: *Cosmological production of vector bosons and Cosmic Microwave Background radiation*. Physics of Atomic Nuclei. **67**, 1074 (2004)
- [15] Giovannini, M.: *Theoretical tools for the physics of CMB anisotropies*. Int. J. Mod. Phys. **D 14**, 363 (2005).
[arXiv: astro-ph/0412601]
- [16] Riess, A.G., *et al.* [Supernova Search Team Collaboration]: *Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant*. Astron. J. **116**, 1009 (1998).
[astro-ph/9805201]
- Perlmutter, S., *et al.* [The Supernova Cosmology Project]: *Measurements of Omega and Lambda from 42 High-Redshift Supernovae*. Astrophys. J. **517**, 565 (1999).
[astro-ph/9812133]
- [17] Riess, A.G., *et al.* [Supernova Search Team Collaboration]: *Type Ia Supernova discoveries at $z \lesssim 1$ from the Hubble space telescope: evidence for past deceleration and constraints on dark energy evolution*. Astrophys. J., **607**, 665 (2004)
- [18] Behnke, D., Blaschke, D.B., Pervushin, V.N., Proskurin, D.: *Description of Supernova data in Conformal cosmology without cosmological constant*. Phys. Lett. **B 530**, 20 (2002).
[arXiv: gr-qc/0102039]

-
- [19] Blokhintsev, D.I.: *Principles of Quantum Mechanics*. Allyn and Bacon, Boston (1964)
- [20] Arbuzov, A.B., Barbashov, B.M., Pervushin, V.N., Shuvalov, S.A., Zakharov, A.F.: *Is it possible to estimate the Higgs Mass from the CMB Power Spectrum?* Physics of Atomic Nuclei. **72**, 744 (2009).
[arXiv:0802.3427 [hep-ph]]
- [21] Pervushin, V.N., Arbuzov, A.B., Barbashov, B.M., Nazmitdinov, R.G., Borowiec, A., Pichugin, K.N., and Zakharov, A.F.: *Conformal and affine Hamiltonian dynamics of general relativity*. Gen. Relativ. & Gravit., **44**, 2745 (2012)
- [22] Barbashov, B.M., Pervushin, V.N., Zakharov, A.F., Zinchuk, V.A.: *Hamiltonian cosmological perturbation theory*. Phys. Lett. **B 633**, 458 (2006).
[arXiv: hep-th/0501242]
- [23] Kirzhnits, D. A. *The hot Universe and the Weiberg model*, JETP Lett. **15** 529 (1972)
- [24] Markov, M.A.: *On the "four - dimensional long stretched" electron in a relativistic quantum region*. ZhETF. **10**, 1311 (1940).
Yukawa, H.: *Quantum theory of non-local fields. Part I. Free fields*. Phys. Rev. **77**, 219 (1950)
- [25] A.A. Logunov, A.N. Tavkhelidze: *Quasi-optical approach to quantum field theory*. Nuovo Cim. **29**, 380 (1963).

- Kadyshevsky, V.G., Mir-Kasimov, R.M., and Skachkov, N.B.: *Three-dimensional formulation of the relativistic two-body problem*. Phys. Part. & Nuclei. **2**, 69 (1973)
- [26] Lukierski, J., Oziewicz, M.: *Relative time dependence as gauge freedom and bilocal models of hadrons*. Phys. Lett. **B 69**, 339 (1977)
- [27] Salpeter, E.E., Bethe, H.A.: *A relativistic equation for bound-state problem*. Phys. Rev. **84**, 1232 (1951)
- [28] Bogoliubov, N.N., Logunov, A.A., Oksak, A.I., Todorov, I.T.: *General Principles of Quantum Field Theory*. Kluwer Academic Publishers (1989)
- [29] Dirac, P.A.M.: *The quantum theory of the emission and absorption of radiation*. Proc. Roy. Soc. London. **A 114**, 243 (1927).
Dirac, P.A.M.: *Gauge-invariant formulation of quantum electrodynamics*. Can. J. Phys. **33**, 650 (1955)
- [30] Heisenberg, W., Pauli, W.: *Zur quantendynamik der wellenfelder*. Z. Phys. **56**, 1 (1929).
Heisenberg, W., Pauli, W.: *Zur quantentheorie der wellenfelder. II*. Z. Phys. **59**, 168 (1930)
- [31] Polubarinov, I.V.: *Equations of quantum electrodynamics*. Phys. Part. & Nuclei. **34**, 741 (2003).
- [32] Pervushin, V.N.: *Hadron QCD bound states in gauge theories*. Nucl. Phys. **B 15** (Proc. Supp.), 197 (1990)

-
- [33] Pervushin, V.N.: *Dirac variables in gauge theories*. Phys. Part. & Nuclei. **34**, 679 (2003)
- [34] Kalinovsky, L.Y., Kallies, W., Kaschlunn, L., Münchow, L., Pervushin, V.N., Sarikov, N.A.: *Relativistic bound states in QCD*. Few-Body Systems. **10**, 87 (1991)
- [35] Salpeter, E.E.: *Mass corrections to the fine structure of hydrogen-like atoms*. Phys. Rev. **87**, 328 (1952)
- [36] Schwinger, J.: *Non-abelian gauge fields. Relativistic invariance*. Phys. Rev. **127**, 324 (1962).
- [37] Faddeev, L.D., Popov, V.N.: *Feynman diagrams for the Yang – Mills field*. Phys. Lett. **B 25**, 29 (1967)
- [38] Faddeev, L.D.: *The Feynman integral for singular Lagrangians*. Theor. Math. Phys. **1**, 1 (1969)

Chapter 10

Reduced phase space

QCD

10.1 Topological confinement

At the beginning of the sixties of the twentieth century Feynman found that the naive generalization of his path integral method (9.28) of construction of QED fails in the non-Abelian theories. The unitary S-matrix in the non-Abelian theory was obtained in the form of the Faddeev – Popov (FP) path integral by the brilliant application of the theory of connections in vector bundle (see [37] in the previous Chapter). Many physicists are of opinion that the FP path integral is the highest level of quantum description of the gauge constrained systems. Really, the FP integral at least allows us to prove both renormalizability of the unified theory of electroweak interactions and asymptotic freedom of the non-Abelian theory. However, the FP integral still remains serious and challenging problems of the quark confinement (in the form

of the Feynman quark-hadron duality), spontaneous chiral symmetry breaking and hadronization. These problems require generalization of the FP path integral to the bound states in the non-Abelian theories in their frame of reference, including corresponding initial data.

In this Chapter, we consider these problems in the context of the Quantum Universe (QU) modification of QCD described in the previous Chapter. The QU modification of QCD means QCD in the reduced phase space obtained by the manifest resolution of all constraints in a concrete frame of reference with initial data. The choice of the vacuum and normal ordering of the field products leads to a source of spontaneous conformal symmetry breaking in the quantum theory.

The role of constraints is played by the equations of the time component. In the YM theory ([1], §16) the time component of a Yang-Mills field occupies a particular place, since it has no canonical momentum. Dirac [2] and other authors of the first classical theories on quantization of gauge fields [3, 4], who followed him, removed the time component of gauge fields by gauge transformation. In our case, the similar transformation is

$$\hat{A}_k^*[A_j^b] = u^*[A_j^b] \left(\hat{A}_k + \partial_k \right) u^{*-1}[A_j^b], \quad (10.1)$$

$$\psi^*[A_j^b, \psi] = u^*[A_j^b]\psi, \quad (10.2)$$

$$u^*[A_j^b] = v(\mathbf{x})T \exp \left\{ \int^t d\bar{t} \hat{a}_0[A_j^b] \right\}, \quad (10.3)$$

where the symbol T denotes the time ordering of the matrices under the exponent sign. It determines a non-Abelian analogue of the Dirac variables (see also [5], Section 2.2) within arbitrary stationary matrices

$v(\mathbf{x})$ considered as initial data of the solution to the equation (9.36)

$$[(\nabla_j(A^*))^2]^{cb} a_0^b = \nabla_i^{cb}(A^*) \dot{A}_i^{*b} = 0 \quad (10.4)$$

at the time instant t_0 . Here $\hat{a}_0[A_j^b]$ is the solution of the Gauss constraint. One can see that at the level of Dirac variables the Lorentz transformations of initial fields become nonlinear (formula (28) in [5]), and the group of gauge transformations reduces to the group of stationary transformations, which set the degeneration of the initial data of physical fields (including the classical vacuum $A_0 = A_i = 0$, determined as a state of zero energy). We imply as a gauge fixation, in the given case, the putting of the initial data in perturbation theory as a transversality condition [5, 6].

In non-Abelian theories a set of stationary gauge transformations is the set of three-dimensional paths in the group space of the Lie group $SU_c(2)$ subdivided into topological manifolds. These manifolds are determined by integers, *degrees of the map* [7]:

$$n = -\frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} \times \quad (10.5)$$

$$\times Tr[v^{(n)}(\mathbf{x}) \partial_i v^{(n)}(\mathbf{x})^{-1} v^{(n)}(\mathbf{x}) \partial_j v^{(n)}(\mathbf{x})^{-1} v^{(n)}(\mathbf{x}) \partial_k v^{(n)}(\mathbf{x})^{-1}].$$

The degree of the map shows how many times a three-dimensional path $v(\mathbf{x})$ turns around the $SU_c(3)$ manifold when the coordinate x_i runs over the space where it is given. The condition (10.5) means that all the sets of three-dimensional paths have the homotopy group

$$\pi_3(SU_c(3)) = \mathbf{Z},$$

and all the fields $v^{(n)}\partial_i v^{(n)-1}$ are given in the class of functions for which the integral (10.5) has a finite (or countable) value

$$\hat{A}_i^{(n)} = v^{(n)}(\hat{A}_i^{(0)} + \partial_i)v^{(n)-1}, \quad v^{(n)}(\mathbf{x}) = \exp[n\Phi_0(\mathbf{x})]. \quad (10.6)$$

Due to the gauge invariance of the action, the phase factors of the topological degeneration disappear. However, these phase factors remain at the sources of physical fields in the generating functional. The theory with the topological degeneration of the initial data, where the sources contain the phase factors of the topological degeneration,

$$\text{tr}[\hat{J}^i v^{(n)} \bar{\hat{A}}_i^{(0)} v^{(n)-1}],$$

differs from the theory without the degeneration and with the sources $\text{tr}[\hat{J}^i \bar{\hat{A}}_i^{(0)}]$. In the theory with the degeneration of the initial data it is necessary to average the amplitudes over the degeneration parameters. Such averaging can lead to the disappearance of a series of physical states.

It has been shown in [8, 9] that the topology can be the origin of color confinement as complete destructive interference of the phase factors of the topological degeneration of initial data.

The mechanical analogy of the topological degeneration of initial data is the free rotator $N(t)$ with the action of free particle

$$W(N_{out}, N_{in}|t_1) = \int_0^{t_1} dt \frac{\dot{N}^2}{2} I, \quad p = \dot{N}I, \quad H_0 = \frac{p^2}{2I} \quad (10.7)$$

given on a ring where the points $N(t) + n$ (n is integer) are physically equivalent. Instead of a initial date $N(t=0) = N_{in}$ in the mechanics

in the space with the trivial topology, the observer of the rotator has the manifold of initial data

$$N^{(n)}(t=0) = N_{in} + n; \quad n = 0, \pm 1, \pm 2, \dots$$

An observer does not know where is the rotator. It can be at points

$$N_{in}, \quad N_{in} \pm 1, \quad N_{in} \pm 2, \quad N_{in} \pm 3, \dots$$

Therefore, he should to average a wave function

$$\Psi(N) = e^{ipN} .$$

over all values of the topological degeneration with the θ -angle measure $\exp(i\theta n)$. In the result we obtain the wave function

$$\Psi(N)_{\text{observable}} = \lim_{L \rightarrow \infty} \frac{1}{2L} \sum_{n=-L}^{n=+L} e^{i\theta n} \Psi(N+n) = \exp\{i(2\pi k + \theta)N\} , \quad (10.8)$$

where k is integer. In the opposite case $p \neq 2\pi k + \theta$ the corresponding wave function (*id est*, the probability amplitude) disappears

$$\Psi(N)_{\text{observable}} = 0$$

due to the complete destructive interference.

The consequence of this topological degeneration is that a part of values of the momentum spectrum becomes unobservable in the comparison with the trivial topology.

This fact can be treated as confinement of those values which do not coincide with the discreet ones

$$p_k = 2\pi k + \theta, \quad 0 \leq \theta \leq \pi . \quad (10.9)$$

The observable spectrum follows also from the constraint of the equivalence of the point N and $N + 1$

$$\Psi(N) = e^{i\theta}\Psi(N + 1), \quad \Psi(N) = e^{ipN}. \quad (10.10)$$

In the result we obtain the spectral decomposition of the Green function of the free rotator (10.7) (as the probability amplitude of transition from the point N_{in} to N_{out}) over the observable values of spectrum (10.9)

$$\begin{aligned} G(N_{out}, N_{in}|t_1) &\equiv \langle N_{out} | \exp(-i\hat{H}t_1) | N_{in} \rangle = \\ &= \frac{1}{2\pi} \sum_{k=-\infty}^{k=+\infty} \exp \left[-i\frac{p_k^2}{2I}t_1 + ip_k(N_{out} - N_{in}) \right]. \end{aligned} \quad (10.11)$$

Using the connection with the Jacobian theta-functions [10]

$$\Theta_3(Z|\tau) = \sum_{k=-\infty}^{k=+\infty} \exp[i\pi k^2\tau + 2ikZ] = (-i\tau)^{-1/2} \exp\left[\frac{Z^2}{i\pi\tau}\right] \Theta_3\left(\frac{Z}{\tau} \middle| -\frac{1}{\tau}\right)$$

we can represent expression (10.11) in a form of the sum over all paths

$$G(N_{out}, N_{in}|t_1) = \sqrt{\frac{I}{i4\pi t_1}} \sum_{n=-\infty}^{n=+\infty} \exp[i\theta n] \exp[+iW(N_{out}, N_{in} + n|t_1)], \quad (10.12)$$

where

$$W(N_{out} + n, N_{in}|t_1) = \frac{(N_{out} + n - N_{in})^2 I}{2t_1}$$

is the rotator action (10.7).

10.2 Quark–hadron duality

All physical states and the Green functions should be averaged over all topological copies in the group space. Averaging over all parameters of

the degenerations can lead to a complete destructive interference of all color amplitudes [8, 9]. In this case, only colorless (“hadron”) states have to form a complete set of physical states. Using the example of a free rotator, we have seen that the disappearance of a part of physical states due to the topological degeneration (confinement) does not violate the composition law for Green functions

$$G_{ij}(t_1, t_3) = \sum_h G_{ih}(t_1, t_2) G_{hj}(t_2, t_3) \quad (10.13)$$

defined as the amplitude of the probability to find a system with the Hamiltonian H in a state j at the time t_3 , if at the time t_1 this system was in a state i , where (i, j) belongs to a complete set of all states $\{h\}$:

$$G_{ij}(t_1, t_3) = \langle i | \exp \left(-i \int_{t_1}^{t_3} H dt \right) | j \rangle$$

The particular case of this composition law (10.13) is the unitarity of S-matrix

$$SS^+ = I \Rightarrow \sum_h \langle i | S | h \rangle \langle h | S^+ | j \rangle = \langle i | j \rangle$$

known as the law of probability conservation for S-matrix elements

$$S = I + iT :$$

$$\sum_h \langle i | T | h \rangle \langle h | T^* | j \rangle = 2\text{Im} \langle i | T | j \rangle . \quad (10.14)$$

The left side of this law is the analogy of the spectral series of the free rotator (10.11). The destructive interference keeps only colorless “hadron” states. Whereas, the right side of this law far from resonances

can be presented in a form of the perturbation series over the Feynman diagrams that follow from the Hamiltonian. Due to gauge invariance $H[A^{(n)}, q^{(n)}] = H[A^{(0)}, q^{(0)}]$ this Hamiltonian does not depend on the topological phase factors and it contains the perturbation series in terms of only the zero - map fields (id est, in terms of constituent color particles) that can be identified with the Feynman partons. The Feynman path integral as the generation functional of this perturbation series is the analogy of the sum over all path of the free rotator (10.12).

Therefore, confinement in the spirit of the complete destructive interference of color amplitudes [8, 9] and the law of probability conservation for S-matrix elements (10.14) leads to Feynman quark-hadron duality, that is foundation of all the parton model [11] and QCD application [12]. The quark-parton duality gives the method of direct experimental measurement of the quark and gluon quantum numbers from the deep inelastic scattering cross-section [11]. For example, according to Particle Data Group the ratio of the sum of the probabilities of τ -decay hadron modes to the probability of τ -decay muon mode is

$$\frac{\sum_h w_{\tau \rightarrow h}}{w_{\tau \rightarrow \mu}} = 3.3 \pm 0.3 .$$

This is the left-hand side of Eq. (10.14) normalized to the value of the lepton mode probability of τ -decay. On the right-hand side of Eq. (10.14) we have the ratio of the imaginary part of the sum of quark-gluon diagrams (in terms of constituent fields free from the topological phase factors) to the one of the lepton diagram. In the lowest order of QCD perturbation on the right-hand side we get the number of colors

N_c and, therefore,

$$3.3 \pm 0.3 = N_c .$$

Thus in the constraint-shell QCD we can understand not only “why we do not see quarks”, but also “why we can measure their quantum numbers”. This mechanism of confinement due to the quantum interference of phase factors (revealed by the explicit resolving the Gauss law constraint [8, 9]) disappears after the change of “physical” sources $A^*J^* \Rightarrow AJ$ that is called the transition to another gauge in the gauge-fixing method.

10.3 Chiral symmetry breaking in QCD

Instantaneous QCD interactions are described by the non-Abelian generalization of the Dirac gauge in QED. The relativistic invariant bilocal effective action obtained in [5], takes the form for the quark sector, in the color singlet channel

$$W_{\text{instant}} = \int d^4x \bar{q}(x) (i\rlap{/}\partial - \hat{m}^0)q(x) - \quad (10.15)$$

$$-\frac{1}{2} \int d^4x d^4y j_0^a(x) \left[\frac{1}{(\nabla_j(A^*))^2} \delta^4(x-y) \right]^{ab} j_0^b(y),$$

where

$$j_0^a(x) = \bar{q}(x) \frac{\lambda^a}{2} \gamma_0 q(x)$$

is the 4-th component of the quark current, with the Gell-Mann color matrices λ^a (see the notations in Appendix A). For simplicity indices $(1, 1'|2, 2')$ denote in (1) all spinor, color and flavor ones. The symbol

$$\hat{m}^0 = \text{diag}(m_u^0, m_d^0, m_s^0)$$

denotes the bare quark mass matrix. The normal ordering of the transverse gluons in the nonlinear action (10.15)

$$\nabla^{db} A_0^b \nabla^{dc} A_0^c$$

leads to the condensate of gluons

$$g^2 f^{ba_1d} f^{da_2c} \langle A_i^{a_1*} A_j^{a_2*} \rangle = 2g^2 N_c \delta^{bc} \delta_{ij} C_{\text{gluon}} = M_g^2 \delta^{bc} \delta_{ij}, \quad (10.16)$$

where

$$\langle A_j^{*a} A_i^{*b} \rangle = 2C_{\text{gluon}} \delta_{ij} \delta^{ab}. \quad (10.17)$$

This condensate yields the squared effective gluon mass in the squared covariant derivative

$$\nabla^{db} A_0^b \nabla^{dc} A_0^c =: \nabla^{db} A_0^b \nabla^{dc} A_0^c : + M_g^2 A_0^d A_0^d.$$

The constant

$$C_{\text{gluon}} = \int \frac{d^3k}{(2\pi)^3 2\sqrt{\mathbf{k}^2 + M_g^2}}$$

is finite after subtraction of the infinite volume contribution, and its value is determined by the hadron size like the Casimir vacuum energy. Finally, in the lowest order of perturbation theory, this gluon condensation yields the effective Yukawa potential in the colorless meson sector [13]

$$\underline{V}(\mathbf{k}) = \frac{4}{3} \frac{g^2}{\mathbf{k}^2 + M_g^2} \quad (10.18)$$

and the NJL type model with the effective gluon mass M_g^2 . While deriving the last equation, we use the relation

$$\left[\sum_{a=1}^{a=N_c^2-1} \frac{\lambda_{1,1'}^a}{2} \frac{\lambda_{2,2'}^a}{2} \right] = \frac{1}{2} \delta_{1,2'} \delta_{2,1'} - \frac{1}{6} \delta_{1,1'} \delta_{2',2'}.$$

The product of this expression by the unit matrix $\delta_{1,2'}$ and summation yield the coefficient $3/2 - 1/6 = 4/3$ in front of the Yukawa potential (10.18). Below we consider the potential model (10.18) in the form

$$\begin{aligned} W_{\text{instant}}[q, \bar{q}] &= \int d^4x \bar{q}(x) (\not{\partial} - \hat{m}^0) q(x) - & (10.19) \\ & - \frac{1}{2} \int d^4x d^4y j_\ell^a(x) V(x^\perp - y^\perp) \delta((x - y) \cdot \ell) j_\ell^a(y) \equiv \\ & \equiv \imath (q\bar{q}, G_0^{-1}) - \frac{1}{2} (q\bar{q}, K q\bar{q}), \end{aligned}$$

with the choice of the time axis as the eigenvalues of the bound state total momentum. This model can formulate the effective action in terms of bound state bilocal fields given in Appendix B. In this case the semiclassical approach repeats the ladder approximation.

In particular, the equation of stationarity (B.8) coincides with the Schwinger – Dyson (SD) equation

$$\Sigma(x - y) = m^0 \delta^{(4)}(x - y) + \imath \mathcal{K}(x, y) G_\Sigma(x - y). \quad (10.20)$$

It describes the spectrum of Dirac particles in bound states. In the momentum space with

$$\underline{\Sigma}(k) = \int d^4x \Sigma(x) e^{ik \cdot x}$$

for the Coulomb type kernel, we obtain the following equation for the mass operator $\underline{\Sigma}$

$$\underline{\Sigma}(k) = m^0 - \imath \int \frac{d^4q}{(2\pi)^4} V(k^\perp - q^\perp) \not{\ell} \underline{G}_\Sigma(q) \not{\ell}, \quad (10.21)$$

where

$$\underline{G}_\Sigma(q) \equiv (\not{q} - \underline{\Sigma}(q))^{-1}$$

is the Fourier representation of the potential,

$$k_\mu^\perp = k_\mu - \ell_\mu(k \cdot \ell)$$

is the relative transverse momentum. The quantity $\underline{\Sigma}$ depends only on the transverse momentum $\underline{\Sigma}(k) = \underline{\Sigma}(k^\perp)$, because of the instantaneous form of the potential $\underline{V}(k^\perp)$. We can put

$$\underline{\Sigma}_a(q) = E_a(\mathbf{q}) \cos 2v_a(\mathbf{q}) \equiv M_a(\mathbf{q}). \quad (10.22)$$

Here $M_a(\mathbf{q})$ is the constituent quark mass and

$$\cos 2v_a(\mathbf{q}) = \frac{M_a(\mathbf{q})}{\sqrt{M_a^2(\mathbf{q}) + \mathbf{q}^2}} \quad (10.23)$$

determines the Foldy – Wouthuysen type matrix

$$S_a(\mathbf{q}) = \exp[i(\mathbf{q}\boldsymbol{\gamma}/q)v_a(\mathbf{q})] = \cos v_a(\mathbf{q}) + i(\mathbf{q}\boldsymbol{\gamma}/q) \sin v_a(\mathbf{q}), \quad (10.24)$$

with the vector of Dirac matrices $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ lying in the range $0 \leq v_a(q) \leq \pi/2$, and $v_a(\mathbf{q})$ is the Foldy – Wouthuysen angle. The fermion spectrum can be obtained by solving the SD equation (10.21). It can be integrated over the longitudinal momentum $q_0 = (q \cdot \ell)$ in the reference frame $\ell^0 = (1, 0, 0, 0)$, where $q^\perp = (0, \mathbf{q})$. By using Eq. (10.24), the quark Green function can be presented in the form

$$\begin{aligned} \underline{G}_{\Sigma_a} &= [q_0 \not{\ell} - E_a(q^\perp) S_a^{-2}(q^\perp)]^{-1} = \\ &= \left[\frac{\Lambda_{(+)\mathbf{a}}^{(\ell)}(q^\perp)}{q_0 - E_a(q^\perp) + i\epsilon} + \frac{\Lambda_{(-)\mathbf{a}}^{(\ell)}(q^\perp)}{q_0 + E_a(q^\perp) + i\epsilon} \right] \not{\ell}, \quad (10.25) \end{aligned}$$

where

$$\Lambda_{(\pm)\mathbf{a}}^{(\ell)}(q^\perp) = S_a(q^\perp) \Lambda_{(\pm)}^{(\ell)}(0) S_a^{-1}(q^\perp), \quad \Lambda_{(\pm)}^{(\ell)}(0) = (1 \pm \not{\ell})/2 \quad (10.26)$$

are the operators separating the states with positive ($+E_a$) and negative ($-E_a$) energies. As a result, we obtain the following equations for the one-particle energy E and the angle v (10.23) with the potential given by Eq. (10.18)

$$E_a(k^\perp) \cos 2v_a(k^\perp) = m_a^0 + \frac{1}{2} \int \frac{d^3q^\perp}{(2\pi)^3} V(k^\perp - q^\perp) \cos 2v_a(q^\perp).$$

In the rest frame $\ell^0 = (1, 0, 0, 0)$ this equation takes the form

$$M_a(\mathbf{k}) = m_a^0 + \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} V(\mathbf{k} - \mathbf{q}) \cos 2v_a(\mathbf{q}). \quad (10.27)$$

By using the integral over the solid angle

$$\begin{aligned} \int_0^\pi d\vartheta \sin \vartheta \frac{2\pi}{M_g^2 + (\mathbf{k} - \mathbf{q})^2} &= \int_{-1}^{+1} d\xi \frac{2\pi}{M_g^2 + k^2 + q^2 - 2kq\xi} = \\ &= \frac{\pi}{kq} \ln \frac{M_g^2 + (k+q)^2}{M_g^2 + (k-q)^2} \end{aligned}$$

and the definition of the QCD coupling constant $\alpha_s = 4\pi g^2$, it can be rewritten as

$$M_a(k) = m_a^0 + \frac{\alpha_s}{3\pi k} \int_0^\infty dq \frac{q M_a(q)}{\sqrt{M_a^2(q) + q^2}} \ln \frac{M_g^2 + (k+q)^2}{M_g^2 + (k-q)^2}. \quad (10.28)$$

The suggested scheme allows us to consider the SD equation (10.27) in the limit when the bare current mass m_a^0 equals to zero [14]. Then the ultraviolet divergence is absent, and, hence, the renormalization procedure can be successfully avoided. This kind of nonlinear integral equations was considered in the paper [15] numerically.

The solution for $M_a(q)$ in the separable approximation [16] in the form of the step function was used for the estimation of the quark and

meson spectra in agreement with the experimental data. Currently, numerical solutions of the nonlinear equation (10.28) are under way, and the details of computations will be published elsewhere.

As discussed in the Appendix B, the SD equation (10.27) can be rewritten in the form (10.28). Once we know the solution of Eq. (10.28) for $M_a(q)$, we can determine the Foldy – Wouthuysen angles v_a , ($a = u, d$) for u-,d- quarks with the help of relation (10.23). Then the BS equations in the form

$$\begin{aligned}
 M_\pi L_2^\pi(\mathbf{p}) &= [E_u(\mathbf{p}) + E_d(\mathbf{p})]L_1^\pi(\mathbf{p}) - & (10.29) \\
 &\int \frac{d^3q}{(2\pi)^3} V(\mathbf{p}-\mathbf{q})L_1^\pi(\mathbf{q})[c^-(\mathbf{p})c^-(\mathbf{q}) + \xi s^-(\mathbf{p})s^-(\mathbf{q})], \\
 M_\pi L_1^\pi(\mathbf{p}) &= [E_u(\mathbf{p}) + E_d(\mathbf{p})]L_2^\pi(\mathbf{p}) - \\
 &\int \frac{d^3q}{(2\pi)^3} V(\mathbf{p}-\mathbf{q})L_2^\pi(\mathbf{q})[c^+(\mathbf{p})c^+(\mathbf{q}) + \xi s^+(\mathbf{p})s^+(\mathbf{q})]
 \end{aligned}$$

yield the pion mass M_π and wave functions $L_1^\pi(\mathbf{p})$ and $L_2^\pi(\mathbf{p})$. Here m_u, m_d are the current quark masses,

$$E_a = \sqrt{\mathbf{p}^2 + M_a^2(\mathbf{p})}, \quad (a = u, d)$$

are the u-,d- quark energies, $\xi = (\mathbf{p} \mathbf{q})/pq$, and we use the notations

$$E(\mathbf{p}) = E_a(\mathbf{p}) + E_b(\mathbf{p}), \quad (10.30)$$

$$c^\pm(\mathbf{p}) = \cos[v_a(\mathbf{p}) \pm v_b(\mathbf{p})], \quad (10.31)$$

$$s^\pm(\mathbf{p}) = \sin[v_a(\mathbf{p}) \pm v_b(\mathbf{p})]. \quad (10.32)$$

The model is simplified in some limiting cases. Once the quark masses m_u and m_d are small and approximately equal, Eqs. (10.27) and (10.29)

take the form

$$m_a = M_a(\mathbf{p}) - \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} V(\mathbf{p} - \mathbf{q}) \cos 2v_u(\mathbf{q}), \quad (10.33)$$

$$\frac{M_\pi L_2^\pi(\mathbf{p})}{2} = E_u(\mathbf{p}) L_1^\pi(\mathbf{p}) - \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} V(\mathbf{p} - \mathbf{q}) L_1^\pi(\mathbf{q}). \quad (10.34)$$

Solutions of equations of this type are considered in the numerous papers [17, 18, 19, 20, 21] (see also review [15]) for different potentials. One of the main results of these papers was the pure quantum effect of spontaneous chiral symmetry breaking. In this case, the instantaneous interaction leads to rearrangement of the perturbation series and strongly changes the spectrum of elementary excitations and bound states in contrast to the naive perturbation theory.

In the limit of massless quarks $m_u = 0$ the left-hand side of Eq. (10.33) is equal to zero. The nonzero solution of Eq. (10.33) implies that there exists a mode with zero pion mass $M_\pi = 0$ in accordance with the Goldstone theorem. This means that the BS equation (10.34), being the equation for the wave function of the Goldstone pion, coincides with the the SD equation (10.33) in the case of $m_u = M_\pi = 0$. Comparison of the equations yields

$$L_1^\pi(\mathbf{p}) = \frac{M_u(\mathbf{p})}{\sqrt{2F_\pi E_u(\mathbf{p})}} = \frac{\cos 2v_u(\mathbf{p})}{\sqrt{2F_\pi}}, \quad (10.35)$$

where the constant of the proportionality F_π in Eq. (10.35) is called the weak decay constant. In the more general case of massive quark $m_u \neq 0$, $M_\pi \neq 0$, this constant is determined from the normalization condition (B.40)

$$1 = \frac{4N_c}{M_\pi} \int \frac{d^3q}{(2\pi)^3} L_2 L_1 = \frac{4N_c}{M_\pi} \int \frac{d^3q}{(2\pi)^3} L_2 \frac{\cos 2v_u(\mathbf{p})}{F_\pi} \quad (10.36)$$

with $N_c = 3$. In this case the wave function $L_1^\pi(p)$ is proportional to the Fourier component of the quark condensate

$$\begin{aligned} C_{\text{quark}} &= \sum_{n=1}^{n=N_c} \langle q_n(t, \mathbf{x}) \bar{q}_n(t, \mathbf{y}) \rangle = \\ &= 4N_c \int \frac{d^3 p}{(2\pi)^3} \frac{M_u(\mathbf{p})}{2\sqrt{\mathbf{p}^2 + M_u^2(\mathbf{p})}}. \end{aligned} \quad (10.37)$$

Using Eqs. (10.23) and (10.35), one can rewrite the definition of the quark condensate (10.37) in the form

$$C_{\text{quark}} = 4N_c \int \frac{d^3 q}{2(2\pi)^3} \cos 2v_u(\mathbf{q}). \quad (10.38)$$

Let us assume that the representation for the wave function L_1 (10.35) is still valid for non-zero but small quark masses. Then the subtraction of the BS equation (10.34) from the SD one (10.33) multiplied by the factor $1/F_\pi$ determines the second meson wave function L_2

$$\frac{M_\pi}{2} L_2^\pi(\mathbf{p}) = \frac{m_u}{\sqrt{2} F_\pi}. \quad (10.39)$$

The wave function $L_2^\pi(p)$ is independent of the momentum in this approximation. Substituting the equation

$$L_2 = \frac{2m_u}{\sqrt{2} M_\pi F_\pi} = \text{const}$$

into the normalization condition (10.36), and using Eqs. (10.35) and (10.38), we arrive at the Gell-Mann – Oakes – Renner (GMOR) relation [22]

$$M_\pi^2 F_\pi^2 = 2m_u C_{\text{quark}}. \quad (10.40)$$

Thus, in the framework of instantaneous interaction we prove the Goldstone theorem in the bilocal variant, and the GMOR relation directly

results from the existence of the gluon and quark condensates. Strictly speaking, the postulate that the finiteness of the quark condensate is finite implies that in QCD the ultraviolet divergence can be removed. Here one can remind that the ultraviolet divergence in the observational Casimir energy of photons between two metal plates in one dimensional space is removed by the Bose - distribution function with the temperature as the inverse free length [23]. In this case, it is well known that the spectrum of the vacuum oscillations coincides with the spectrum of an absolute black body with the energy density

$$\rho_{\text{Cas}}^1 = \frac{1}{2L} \sum_{n=0}^{\infty} \omega_n = \frac{1}{\pi} \int_0^{\infty} \frac{d\omega \omega}{1 - e^{2L\omega}} = -\frac{\pi}{24L^2} \quad (10.41)$$

where $\omega_n = \pi n/L$, $n = 1, 2, \dots$ is the complete set of the one-photon energy in the space between two metal plates of a size L .

We suppose that the energy of the vacuum oscillations of fermions in the Dirac sea is suppressed by the Fermi - distribution function (see Appendix C)

$$f_{(+)}(q) = \frac{1}{\exp\{(\omega(q) - 1)L\} + 1}, \quad (10.42)$$

$$\omega(q) = \sqrt{1 + (q^2/M^2(0))},$$

where L is the inverse effective temperature in the unit of the constituent mass. The substitution of this Fermi - distribution function under the sign of the integral in Eq. (10.38) and the change of variable of integration

$$dp p^2 = d\omega \omega \sqrt{\omega^2 - 1}$$

leads to the expression of the Casimir condensate in the units of the

mass $M(0) = 1$

$$C_{\text{quark}}(L) = \frac{3}{\pi^2} \int_1^\infty \frac{d\omega \sqrt{\omega^2 - 1}}{1 + e^{(\omega-1)L}} \Big|_{L=1} \simeq 0.39, \quad (10.43)$$

where L is the inverse effective temperature and the constituent mass is the unit. In the chiral massless limit ($m^0 \rightarrow 0$) the solution of the Schwinger–Dyson equation and the Salpeter one yield a meson spectrum via the constituent quark masses $M_{\text{const}} \simeq 320$ GeV [24]. Using the GMOR relation (10.40) and the constituent quark mass value ~ 320 MeV we can define a *conformal invariant* as the ratio of the condensate value to the cubed constituent mass

$$\frac{\langle u\bar{u} \rangle}{M_u^3} = \frac{M_\pi^2 F_\pi^2}{2m_u M_u^3} \simeq 0.41 \pm 0.08. \quad (10.44)$$

The comparison of the theoretical value (10.43) with the experimental one (10.44) shows us that the inverse effective temperature L coincides with the Compton length ($L = M_u^{-1} = 1$).

Thus, the breakdown of the chiral symmetry in QCD can be characterized by the quark condensate (10.44). It is clear that the substitution of the effective vacuum temperature factor (10.42) $f_{(+)}(q)$ at ($L = M_u^{-1} = 1$) in the integral (10.27) at the large momenta allows us to neglect in the potential its dependence on the momentum of integration \mathbf{q}

$$\begin{aligned} M_a(\mathbf{k}) &= m_a^0 + \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} V(\mathbf{k} - \mathbf{q}) \cos 2v_a(\mathbf{q}) f_{(+)}(q) = \\ &\simeq m_a^0 + \underline{V}(\mathbf{k}) \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \cos 2v_a(\mathbf{q}) f_{(+)}(q) = \\ &= m_a^0 + \frac{g^2}{3(\mathbf{k}^2 + M_g^2)} \langle u\bar{u} \rangle, \end{aligned} \quad (10.45)$$

where $\underline{V}(\mathbf{k})$ is given by Eq. (10.18). This expression is in agreement with the short distance operator product expansion applied to quark fields [26].

10.4 Summary

We considered the status of the initial data in QCD in the context of the Quantum Universe wave function classification of physical processes in the Universe given in Chapters 8 and 9. This classification proposes the reduced space phase QCD Hamiltonian. One of the main results is the topological degeneration of initial data that leads to the color confinement in the form of quark-hadron duality. In the constraint-shell QCD we can understand not only “why we do not see quarks”, but also “why we can measure their quantum numbers”. The normal ordering of the field products in the reduced space phase QCD Hamiltonian leads to both the QCD inspired model of the low energy meson physics [16] with the Gell-Mann – Oakes – Renner (GMOR) relation [22] and the short distance operator product expansion applied to quark fields [26].

Bibliography

- [1] Gitman, D.M., Tyutin, I.V.: *Quantization of Fields with Constraints*. Springer (1990)
- [2] Dirac, P.A.M.: *The quantum theory of the emission and absorption of radiation*. Proc. Roy. Soc. London. **A 114**, 243 (1927).
- Dirac, P.A.M.: *Gauge-invariant formulation of quantum electrodynamics*. Can. J. Phys. **33**, 650 (1955)
- [3] Heisenberg, W., Pauli, W.: *Zur quantendynamik der wellenfelder*. Z. Phys. **56**, 1 (1929).
- Heisenberg, W., Pauli, W.: *Zur quantentheorie der wellenfelder. II*. Z. Phys. **59**, 168 (1930)
- [4] Fermi, E.: *Quantum theory of radiation*. Rev. Mod. Phys. **4**, 87 (1932)
- [5] Pervushin, V.N.: *Dirac variables in gauge theories*. Lecture Notes in DAAD Summerschool on Dense Matter in Particle - and Astrophysics, JINR, Dubna, Russia, August 20 - 31, 2001.
- [arXiv: hep-th/ 0109218]

-
- Pervushin, V.N.: *Dirac variables in gauge theories*. Phys. Particles & Nuclei. **34**, 348 (2003)
- [6] Zumino, B.: *Gauge properties of propagators in quantum electrodynamics*. J. Math. Phys. **1**, 1 (1960)
- [7] Belavin, A.A., Polyakov, A.M., Schwartz, A.S., Tyupkin, Yu.S.: *Pseudoparticle solutions of the Yang – Mills equations*. Phys. Lett. **59 B**, 85 (1975)
- [8] Pervushin, V.N.: *The vacuum in gauge theories*. Riv. Nuovo Cimento. **8**, 1 (1985)
- [9] Pervushin, V.N., Nguyen Suan Han: *Hadronization and confinement in quantum chromodynamics*. Can. J. Phys. **69**, 684 (1991)
- [10] Pollard, B.R.: *An Introduction to Algebraic Topology*. Notes of lectures given during the session 1976-1977. University of Bristol.
- [11] Feynman, R.P.: *Photon – Hadron Interactions*. W.A. Benjamin, Inc. (1972)
- [12] Efremov A.V., Radyushkin, A.V.: *Hard processes, parton model and QCD*. Riv. Nuovo Cimento. **3**, 1 (1980)
- [13] Belkov, A.A., Pervushin, V.N., Ebert, D.: *The low-energy predictions of nonlinear chiral Lagrangians based on the quark dynamics*. Phys. Part. & Nuclei. **22**, 5 (1991)

- [14] Cherny, A.Yu., Dorokhov, A.E., Nguyen Suan Han, Pervushin, V.N., Shilin, V.I.: *Bound states in gauge theories as the Poincaré group representations*. Physics of Atomic Nuclei. **76**, 382 (2013).
[arXiv:1112.5856 [hep-th]]
- [15] Puzynin, I.V., Amirkhanov, I.V., Zemlyanaya, E.V., Pervushin, V.N., Puzynina, T.P., Strizh, T.A., Lakhno, V.D.: *The generalized continuous analog of Newton's method for the numerical study of some nonlinear quantum-field models*. Phys. Part. & Nuclei. **30**, 87 (1999)
- [16] Kalinovsky, Yu.L., Kaschlunn, L., Pervushin, V.N.: *A new QCD inspired version of the Nambu – Jona–Lasinio model*. Phys. Lett. **B 231**, 288 (1989).
Kalinovsky, Yu.L., Kaschlunn, L., Pervushin, V.N.: *Mesons in the low-energy limit of QCD*. Fortsch. Phys. **38**, 353 (1990)
- [17] Kunihiro, T., Hatsuda, T.: *Effects of flavour mixing induced by axial anomaly on the quark condensates and meson spectra*. Phys. Lett. **B 206**, 385 (1988)
- [18] Bernard, V., Jaffe, R.L., Meißner, U.-G.: *Strangeness mixing and quenching in the Nambu – Jona–Lasinio model*. Nucl. Phys. **B 308**, 753 (1988)
- [19] Bernard, V., Meißner, U.-G.: *Properties of vector and axial-vector mesons from a generalized Nambu – Jona–Lasinio model*. Nucl. Phys. **A 489**, 647 (1988)

-
- [20] Reinhardt, H., Alkofer, R.: *Instanton-induced flavour mixing in mesons*. Phys. Lett. **B 207**, 482 (1988)
- [21] Yaouanc, A. Le, Oliver, L., Ono, S., Pène, O., Raynal, J.-C.: *Quark model of light mesons with dynamically broken chiral symmetry*. Phys. Rev. **D 31**, 137 (1985)
- [22] Langfeld, K., Kettner, Ch.: *The quark condensate in the GMOR relation*. Mod. Phys. Lett. **A 11**, 1331 (1996)
- [23] Bordag, M., Klimchitskaya, G.L., Mohideen, U., Mostepanenko, V.M.: *Advances in the Casimir Effect*. Oxford University Press, New York (2009)
- [24] Kalinovsky, L.Y., Kallies, W., Kaschluhn, L., Münchow, L., Pervushin, V.N., Sarikov, N.A.: *Relativistic bound states in QCD*. Few-Body Systems. **10**, 87 (1991)
- [25] J. Beringer *et al.* [Particle Data Group Collaboration]: *Review of Particle Physics*. Phys. Rev. **D 86**, 010001(2012)
- [26] Politzer, H.D.: *Effective quark masses in the chiral limit*. Nuclear Physics. **B 117**, 397 (1976)

Chapter 11

QU modification of the Standard Model

11.1 SM Lagrangian

The Standard Model (SM) known as the Weinberg – Salam – Glashow minimal electroweak theory was constructed on the basis of the Yang – Mills theory [1] with the symmetry group $SU(2) \otimes U(1)$ [2] in two steps. The first step was the choice of the Lagrangian \mathcal{L}_G and physical variables. The second step was the choice of a mechanism of the mass generation. Let us consider the gauge-invariant Lagrangian

$$\begin{aligned} \mathcal{L}_G = & -\frac{1}{4}G_{\mu\nu}^a G_a^{\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ & + \sum_s \bar{s}_1^R \not{\nu} \gamma^\mu \left(D_\mu^{(-)} + \not{\nu} g' B_\mu \right) s_1^R + \sum_s \bar{L}_s \not{\nu} \gamma^\mu D_\mu^{(+)} L_s, \end{aligned} \quad (11.1)$$

where

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\varepsilon_{abc}A_\mu^b A_\nu^c$$

is the tension of the non-Abelian $SU(2)$ fields and

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

is the tension of the Abelian $U(1)$ field,

$$D_\mu^{(\pm)} = \partial_\mu - ig \frac{\tau_a}{2} A_\mu^a \pm \frac{i}{2} g' B_\mu$$

are covariant derivatives, and $\bar{L}_s = (\bar{s}_1^L \bar{s}_2^L)$ are the fermion duplets, g and g' are the Weinberg coupling constants.

The physical variables as measurable bosons W_μ^+ , W_μ^- , Z_μ are determined by the relation

$$W_\mu^\pm \equiv A_\mu^1 \pm A_\mu^2 = W_\mu^1 \pm W_\mu^2, \quad (11.2)$$

$$Z_\mu \equiv -B_\mu \sin \theta_W + A_\mu^3 \cos \theta_W, \quad (11.3)$$

$$\tan \theta_W = \frac{g'}{g}, \quad (11.4)$$

where θ_W is the Weinberg angle. In terms of these variables the Lagrangian (11.1) takes the form

$$\begin{aligned} \mathcal{L}_G = & -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{4}(\partial_\mu Z_\nu - \partial_\nu Z_\mu)^2 - \\ & - \frac{1}{2}|D_\mu W_\nu^+ - D_\nu W_\mu^+|^2 - ie(\partial_\mu A_\nu - \partial_\nu A_\mu)W^{+\mu}W^{-\nu} - \\ & - g^2 \cos^2 \theta_W [Z^2(W^+W^-) - (W^+Z)(W^-Z)] + \\ & + ig \cos \theta_W (\partial_\mu Z_\nu - \partial_\nu Z_\mu)W^{+\mu}W^{-\nu} + \\ & + \frac{1}{2}ig \cos \theta_W [(D_\mu W_\nu^+ - D_\nu W_\mu^+)(W^{-\mu}Z^\nu - W^{-\nu}Z^\mu) - h.c.], \end{aligned}$$

where $D_\mu = \partial_\mu + ieA_\mu$ is covariant derivative, A_μ is a photon field, e is the electromagnetic interaction coupling. According to the *principles of quantum Universe*, the conformal symmetry of the theory can be

broken by only the normal ordering of the quantum field products in the reduced phase space after resolution of all constraint equations. In the previous Chapter 10, it was shown that the non-Abelian fields have the topological degeneration of initial data. This degeneration can be removed by the interaction of non-Abelian fields with an elementary scalar field. The conformal-invariant Lagrangian of the scalar field h interacting with vector bosons and fermions f is chosen in the form

$$\begin{aligned} \mathcal{L}_h = & \frac{1}{2}(\partial_\mu h)^2 - \frac{\lambda^2}{8}h^4 + \sum_{f=s_1, s_2} \bar{f} [\gamma \partial - g_f h] f \\ & + \frac{1}{8}h^2 g^2 [(W^+ W^- + W^- W^+) + Z_\mu^2 / \cos^2 \theta_W] \end{aligned}$$

where W^\pm , Z - are vector fields with the Weinberg coupling $g = 0.645$; θ_W is the Weinberg angle, and $\sin^2 \theta_W = 0.22$. Masses of the vector bosons arise if the scalar field h has the zero harmonics

$$h = v + H, \quad \int d^3 x H = 0. \quad (11.5)$$

The value of the zero harmonics v is determined by the Casimir condensate. They automatically arise after the normal ordering procedure for all field operators. The normal ordering of the interaction Hamiltonian of the scalar field yields the condensate density $\langle HH \rangle_{\text{Cas}}$

$$\langle HH \rangle_{\text{Cas}} = \frac{1}{V_0} \sum_p \frac{1}{2\sqrt{p^2 + m^2}}. \quad (11.6)$$

This magnitude is connected with the Casimir energy [3, 4]

$$E_{\text{Cas}} = \frac{1}{2} \sum_{\mathbf{p}} \sqrt{p^2 + m^2} \quad (11.7)$$

by the relation

$$\langle HH \rangle_{\text{Cas}} = \frac{2}{V_0} \frac{\partial}{\partial m^2} E_{\text{Cas}}. \quad (11.8)$$

In the continual limit of the QFT one has

$$\begin{aligned} \frac{1}{V_0} \sum_{\mathbf{p}} \frac{1}{2\sqrt{p^2 + m_t^2}} &\Rightarrow \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\sqrt{p^2 + m^2}} = \\ &= m^2 \int \frac{d^3x}{(2\pi)^3} \frac{1}{2\sqrt{x^2 + 1}} \equiv \gamma_0 \cdot m^2. \end{aligned} \quad (11.9)$$

Thus, the Casimir condensate density of a massive scalar field in the absence of any additional scale is proportional to its squared mass

$$\langle HH \rangle_{\text{Cas}} = \gamma_0 \cdot m^2 \Rightarrow \frac{\langle HH \rangle_{\text{Cas}}}{m^2} \equiv \gamma_0, \quad (11.10)$$

where γ_0 is a *dimensionless conformal parameter* with a zero conformal weight (see discussion on conformal weights in [5]). The normal ordering of a fermion pair (we intentionally interchange the order of fermion fields to deal with positive condensates)

$$f\bar{f} =: f\bar{f} : + \langle f\bar{f} \rangle$$

yields the condensate density of the fermion field $\langle f\bar{f} \rangle$ in the Yukawa interaction term in Eq. (11.13). In virtue of the above given results, we have for the top quark Casimir condensate density

$$\langle t\bar{t} \rangle_{\text{Cas}} = 4N_c \frac{m_t}{V_0} \sum_{\mathbf{p}} \frac{1}{2\sqrt{p^2 + m_t^2}} = 4N_c \cdot \gamma_0 \cdot m_t^3, \quad (11.11)$$

where $N_c = 3$ is the color number.

11.2 The condensate mechanism of Higgs boson mass

Recently a few research groups have reported upon the discovery of scalar particles with almost similar masses around 125 – 126 GeV

[6, 7, 8]. The experimentalists express extreme caution in the identification of these particles with the long-waiting Higgs boson. Indeed, the literature contains a plethora of predictions on lower and upper limits of the Higgs mass based on many different ideas, models and numerical techniques which are close to the observed values. The question on a genuine mechanism which provides an unambiguous answer about the Higgs mass is a real challenge to high energy physics and is crucial for the base of the Standard Model (SM) ¹. It is especially noteworthy that in the SM the Higgs mass is introduced *ad hoc*.

According to the general wisdom, all SM particles (may be except neutrinos) own masses due to the spontaneous symmetry breaking (SSB) of the electroweak gauge symmetry [9, 10, 11]. In particular, one deals with the potential (in notation of Ref. [12]):

$$V_{\text{Higgs}}(\phi) = \frac{\lambda^2}{2}(\phi^\dagger\phi)^2 + \mu^2\phi^\dagger\phi, \quad (11.12)$$

where one component of the complex scalar doublet field

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$$

acquires a non-zero vacuum expectation value

$$\langle\phi^0\rangle = \frac{v}{\sqrt{2}}.$$

Note that the tachyon-like mass term in the potential is critical for this construction. In contrast to the SSB, it breaks the conformal symmetry explicitly being the only *fundamental* dimension-full parameter

¹*Rencontres de Moriond*. La Thuile, Italy (2013).

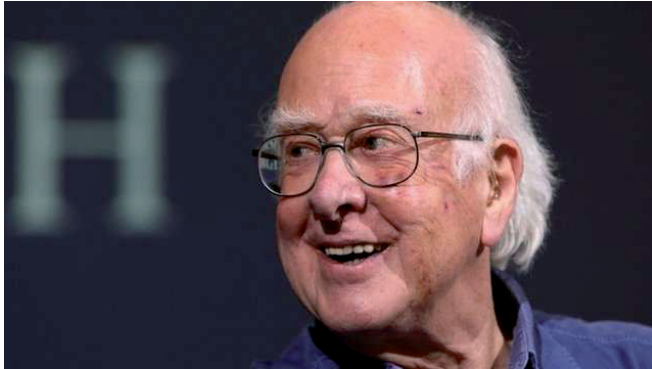
in the SM. We recall that the explicit conformal symmetry breaking in the Higgs sector gives rise to the unsolved problem of fine tuning in the renormalization of the Higgs boson mass. That is certainly one of the most unpleasant features of the SM. In the classical approximation, from the condition of the potential minimum one obtains the relation between the vacuum expectation value and the primary parameters μ and λ in the form $v = \sqrt{-2\mu^2}/\lambda$. This quantity can be defined as well with the aid of the Fermi coupling constant derived from the muon life time measurements:

$$v = (\sqrt{2}G_{\text{Fermi}})^{-1/2} \approx 246.22 \text{ GeV}.$$

The experimental studies at the LHC [6, 7] and Tevatron [8] observe an excess of events in the data compared with the background in the mass range around ~ 126 GeV. Taking into account radiative corrections, such a mass value makes the SM being stable up to the Planck mass energy scale [13]. Nevertheless, the status of the SM and the problem of the mechanism of elementary particle mass generation are still unclear.

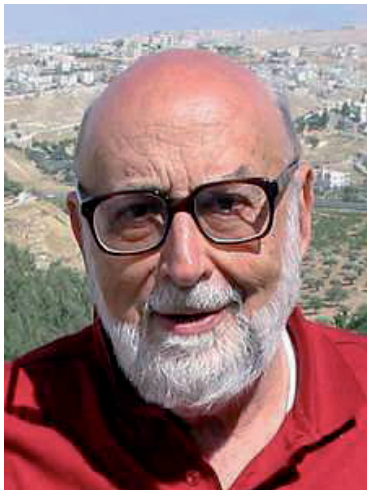
The idea on dynamical breaking of the electroweak gauge symmetry with the aid of the top quark condensate has been continuously discussed in literature since the pioneering papers [14, 15, 16] (see also review [17] and references therein). Such approaches suffer, however, from formal quadratic divergences in tadpole loop diagrams leading, in particular, to the naturalness problem (or fine tuning) in the renormalization of the Higgs boson mass. All mentioned facts suggest that it might be good to examine if the SSB is also responsible for the Higgs field. To begin with, we suppose that there is a general mechanism

of the SSB, which is responsible for the appearance of all SM field condensates.



Peter Ware Higgs, (born 29 May 1929, Newcastle Upon Tyne, England) is a British theoretical physicist, Nobel Prize laureate (shared with Francois, Baron Englert) and emeritus professor at the University of Edinburgh. He is best known for his 1960s proposal of broken symmetry in electroweak theory, explaining the origin of mass of elementary particles in general and of the W- and Z- bosons in particular. This so-called Higgs mechanism, which was proposed by several physicists besides Higgs at about the same time, predicts the existence of a new particle, the Higgs boson. CERN announced on 4 July 2012 that they had experimentally established the existence of a Higgs-like boson, but further work is needed to analyze its properties and see if it has the properties expected from the Standard Model Higgs boson. Higgs himself said on this occasion, that he did not expect the experimental confirmation of his theory in his life. On 14 March 2013, the newly discovered particle was tentatively confirmed to be “+” parity and zero spin, two fundamental criteria of a Higgs boson, making it the first known scalar particle to be discovered in nature. The Higgs mechanism is generally accepted as an important ingredient in the Standard Model of particle

physics, without which certain particles would have no mass.



Francois, Baron Englert (born 6 November 1932) is a Belgian theoretical physicist and 2013 Nobel prize laureate (shared with Peter Higgs). He is Professor emeritus at the Université libre de Bruxelles where he is member of the Service de Physique Théorique. He is also a Sackler Professor by Special Appointment in the School of Physics and Astronomy at Tel Aviv University and a member of the Institute for Quantum Studies at Chapman University in California. He was awarded the 2010 J. J. Sakurai Prize for Theoretical Particle Physics (with Gerry Guralnik, C. R. Hagen, Tom Kibble, Peter Higgs, and Robert Brout), the Wolf Prize in Physics in 2004 (with Brout and Higgs) and the High Energy and Particle Prize of the European Physical Society (with Brout and Higgs) in 1997 for the mechanism which unifies short and long range interactions by generating massive gauge vector bosons. He is the recipient of the 2013 Prince of Asturias Award in technical and scientific research, together with Peter Higgs and the CERN.

The main feature of our approach is the assumption about the underlying (softly broken) conformal symmetry which protects the jump

of the Higgs boson mass to a cut-off scale. We will call this mechanism the spontaneous conformal symmetry breaking (SCSB) [18]. Evidently, in this case one should require the conservation of the conformal symmetry of the genuine theory fundamental Lagrangian. It will be shown that the SCSB provides the breaking of the gauge, chiral, and conformal symmetries on equal footing. Therefore, it allows one also to introduce the universal relation between different condensates determined relative to the corresponding mass power depending on quantum statistics (see Eq. (11.19)). Our basic assumption is that this relation is not violated by the SCSB.

Following the ideas of Nambu [14, 15], we generate the SCSB of the Higgs potential, using the top quark condensate. It is assumed that the general construction of the SM should remain unchanged. Let us start with the conformal invariant Lagrangian of Higgs boson interactions (11.12)

$$\mathcal{L}_{\text{int}} = -\frac{\lambda^2}{8}h^4 - g_t h \bar{t}t. \quad (11.13)$$

Here, for the sake of discussion, we consider only the most intensive terms: the self-interaction and the Yukawa ones of the top quark coupling constant g_t . From the beginning, we assume that the $O(4)$ symmetry of the Higgs sector is spontaneously broken to the $O(3)$ symmetry. Contributions of other interaction terms will be considered below as well.

Keeping in mind all these results, we are ready to treat the contribution of the top quarks to the effective potential [4] generated by the

term Eq. (11.13):

$$V_{\text{cond}}(h) = \frac{\lambda^2}{8} h^4 - g_t \langle t\bar{t} \rangle h. \quad (11.14)$$

The extremum condition for the potential

$$\left. \frac{dV_{\text{cond}}}{dh} \right|_{h=v} = 0$$

yields the relation

$$v^3 \frac{\lambda^2}{2} = g_t \langle t\bar{t} \rangle. \quad (11.15)$$

This relation follows from the fact that the Higgs field has a zero harmonic v in the standard decomposition of the field h over harmonics

$$h = v + H,$$

where H is the sum of all nonzero harmonics with a condition

$$\int d^3x H = 0.$$

Here, the Yukawa coupling of the top quark $g_t \approx 1/\sqrt{2}$ is known from the experimental value of top quark mass

$$m_t = v g_t \simeq 173.4 \text{ GeV}.$$

The spontaneous symmetry breaking yields the potential minimum which results in the nonzero vacuum expectation value v and Higgs boson mass. The substitution $h = v + H$ into the potential (11.14) leads to the result

$$V_{\text{cond}}(h) = V_{\text{cond}}(v) + \frac{m_H^2}{2} H^2 + \frac{\lambda^2 v}{2} H^3 + \frac{\lambda^2}{8} H^4, \quad (11.16)$$

which defines the scalar particle mass as

$$m_H^2 \equiv \frac{\lambda^2}{2} 3v^2. \quad (11.17)$$

We stress that this relation is different from the one ($m_H = \lambda v$) which emerges in the SM with the Higgs potential (11.12).

With the aid of Eqs. (11.15), (11.17), the squared scalar particle mass can be expressed in terms of the t quark condensate:

$$m_H^2 = \frac{3g_t \langle t\bar{t} \rangle}{v}. \quad (11.18)$$

The hypothesis on universality of the conformal invariant ratios of the field condensate densities and the corresponding mass powers (see Eqs. (11.9)–(11.11)) allows us to determine the t quark condensate density with the aid of the light quark one.

11.3 Estimation of the Higgs boson mass

The supposition about the Casimir condensate density of the light quarks in Chapter 10 (see (10.44), (11.9)–(11.11)) allows us to determine the Casimir condensate density of the t -quark

$$\frac{\langle t\bar{t} \rangle}{m_t^3} = \frac{\langle q\bar{q} \rangle}{m_q^3}, \quad (11.19)$$

and estimate the Higgs boson mass. We consider the left and right hand sides as scale invariants. However, their numerators and denominators are scale variables. Therefore, we have to choose the proper scales. For the left hand side, the scale is naturally defined by the known t quark mass. We define the scale of the right hand side by the light quark

condensate density $\langle q\bar{q} \rangle$. It is quite accurately determined in the chiral limit of the QCD low-energy phenomenology [12]:

$$\langle q\bar{q} \rangle \simeq (250 \text{ eV})^3. \quad (11.20)$$

At this scale, the light quark possesses the constituent mass

$$m_q \approx 330 \text{ MeV}$$

estimated in the QCD-inspired model [19]. With the aid of Eq. (11.19) one determines the top quark condensate value

$$\langle t\bar{t} \rangle \approx (126 \text{ GeV})^3. \quad (11.21)$$

Such a large value of the top quark condensate does not affect the low energy QCD phenomenology, since its contribution is very much suppressed by the ratio of the corresponding energy scales (squared).

By means of Eqs. (11.19), (11.20), in the tree approximation we obtain for the scalar particle mass

$$(m_H^0)^2 = (130 \pm 15 \text{ GeV})^2. \quad (11.22)$$

Here, we have assigned 10% uncertainty into the ratio light quark condensate and its constituent mass.

The tentative estimate of the Higgs boson mass given above is rather preliminary. In order to improve this value we consider below the contributions of other condensates at the tree level. The mass can be also affected by radiative corrections which will be analyzed elsewhere. Under the assumption of γ_0 universality, the normal ordering of the field operators

$$HH =: HH : + \langle HH \rangle$$

yields

$$\frac{\langle HH \rangle}{m_H^2} = \gamma_0. \quad (11.23)$$

The normal ordering of the vector fields $V_i V_j$ defines the vector field condensates normalized on each degree of freedom

$$\langle VV \rangle = M_V^2 \cdot \gamma_0, \quad V = W^\pm, Z, \quad (11.24)$$

calculated in the gauge $V_0 = 0$. Here, M_V is a corresponding mass of the vector field. Transverse and longitudinal components are considered on equal footing in the reduced phase space quantization of the massive vector theory [20]. As a result, one obtains the upper limit of the vector field condensate contributions for the mass formula (11.22) at the tree level for the SM

$$\Delta m_H^2 = \frac{3\lambda^2}{4} \langle HH \rangle + \frac{3}{8} g^2 \left(2 \langle WW \rangle + \frac{\langle ZZ \rangle}{\cos^2 \theta_W} \right), \quad (11.25)$$

where g and θ_W are the Weinberg coupling constant and the mixing angle. In Eq. (11.25) the first term is a contribution to the square mass due to the very scalar field condensate $\langle HH \rangle$. Taking into account the values of coupling constants, mixing angle, masses, and condensates, we arrive to the following result

$$m_H = m_H^0 \left[1 + 4 \frac{\Delta m_H^2}{v^2} \right]^{1/2} \approx m_H^0 \times (1 + 0.02), \quad (11.26)$$

where m_H^0 is given by Eq. (11.22). If there exist additional heavy fields interacting with the SM Higgs boson, their condensates would contribute to the Higgs boson mass.

11.4 Summary

In conclusion, we suggest the condensate mechanism of Spontaneous Conformal Symmetry Breaking in the Standard Model of strong and electroweak interactions. We suppose that this mechanism is related to the vacuum Casimir energy in the Standard Model. Our key assumption is that the condensates of all fields normalized to their masses and degrees of freedom represent a conformal invariant. This idea enables us to avoid efficiently the problem of the regularization of the divergent tadpole loop integral. The top quark condensate supersedes the phenomenological negative square mass term in the Higgs potential. In contrast to the standard Higgs mechanism, the condensate mechanism allows one to establish relations between all condensates and masses including the Higgs one. According to our results, the latter is of the order 130 ± 15 GeV, if the universality relation (11.19) holds true.

Bibliography

- [1] Yang C.N., Mills R.L.: *Conservation of isotopic spin and isotopic gauge invariance*. Phys. Rev., **96**, 191 (1954)
- [2] Weinberg, S.: *The Quantum Theory of Fields*. Cambridge University Press (2005)
- [3] Kirsten, K.: *Spectral Functions in Mathematics and Physics*. Chapman & Hall/CRC, London–New York (2002)
- [4] Pervushin, V.N., Arbuzov, A.B., Nazmitdinov, R.G., Pavlov, A.E., Zakharov, A.F.: *Condensate mechanism of conformal symmetry breaking and the Higgs boson*.
[arXiv: hep-ph/1209.4460]
- [5] Pervushin, V.N., Arbuzov, A.B., Barbashov, B.M., Nazmitdinov, R.G., Borowiec, A., Pichugin, K.N., and Zakharov, A.F.: *Conformal and affine Hamiltonian dynamics of general relativity*. Gen. Relativ. & Gravit., **44**, 2745 (2012)
- [6] Aad, G. *et al.* [ATLAS Collaboration]: *Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC*. Phys. Lett. **B 716**, 1 (2012)

-
- [7] Chatrchyan, S. *et al.* [CMS Collaboration]: *Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC.* Phys. Lett. **B 716**, 30 (2012)
- [8] Aaltonen, T. *et al.* [CDF and D0 Collaborations]: *Evidence for a particle produced in association with weak bosons and decaying to a bottom-antibottom quark pair in Higgs boson searches at the Tevatron.* Phys. Rev. Lett. **109**, 071804 (2012)
- [9] Englert, F., Brout, R.: *Broken symmetry and the mass of gauge vector mesons.* Phys. Rev. Lett. **13**, 321 (1964)
- [10] Higgs, P.: *Broken symmetries and the masses of gauge bosons.* Phys. Rev. Lett. **13**, 508 (1964)
- [11] Guralnik, G.S., Hagen, C.R., Kibble, T.W.B.: *Global conservation laws and massless particles.* Phys. Rev. Lett. **13**, 585 (1964)
- [12] Beringer, J. *et al.* [Particle Data Group Collaboration]: *Review of Particle Physics.* Phys. Rev. **D 86**, 010001 (2012)
- [13] Bezrukov, F., Kalmykov, M.Y., Kniehl, B.A., Shaposhnikov, M.: *Higgs boson mass and new physics.*
[arXiv: hep-ph/1205.2893]
- [14] Nambu, Y.: *Dynamical symmetry breaking.* In: *Evolutionary trends in the physical sciences.* Tokyo, 51 (1990) in *Proceedings of the 1989 Workshop on Dynamical Symmetry Breaking*, (Eds. T. Muta, K. Yamawaki). Nagoya University. Japan, 1990.

- [15] Nambu, Y.: *in Oxford 1990, Proceedings Plots, quarks and strange particles* EFI 90-046 (90,rec.Oct.) 12 p.
- [16] Bardeen, W.A., Hill, C.T., Lindner, M.: *Minimal dynamical symmetry breaking of the standard model*. Phys. Rev. **D 41**, 1647 (1990)
- [17] Cvetič, G.: *Top-quark condensation*. Rev. Mod. Phys. **71**, 513 (1999)
- [18] Pervushin, V., Arbuzov, A., Barbashov, B., Cherny, A., Dorokhov, A., Borowiec, A., Nazmitdinov, R., Pavlov, A., Shilin, V., Zakharov, A.: *Condensate mechanism of conformal symmetry breaking*. PoS (Baldin ISHEPP XXI) 023. International Baldin Seminar on High Energy Physics Problems (2012). <http://pos.sissa.it>
[arXiv: hep-ph/1211.4386]
- [19] Kalinovsky, Yu.L., Kaschluhn, L., Pervushin, V.N.: *A new QCD inspired version of the Nambu–Jona–Lasinio model*. Phys. Lett. **B 231**, 288 (1989).
Ebert, D., Reinhard, H., and Volkov, M.K.: *Effective hadron theory of QCD*. Prog. Part. Nucl. Phys. **33**, 1 (1994).
Cherny, A.Yu., Dorokhov, A.E., Nguyen Suan Han, Pervushin, V.N., and Shilin, V.I.: *Bound states in gauge theories as the Poincaré group representations*. Phys. Atomic Nuclei. **76**, 382 (2013).
[arXiv: hep-th/1112:5856]
- [20] Pavel, H.P., Pervushin, V.N.: *Reduced phase-space quantization of massive vector theory*. Int. J. Mod. Phys. **A 14**, 2885 (1999)

Chapter 12

Electroweak vector bosons

12.1 Cosmological creation of electroweak vector bosons

Let us consider the vector massive particles in the conformal flat metrics

$$\tilde{d}s^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = (d\eta)^2 - (dx^i)^2, \quad (12.1)$$

in the cosmological model, with the time interval

$$d\eta = \bar{N}_0(x^0) dx^0 \quad (12.2)$$

that follows from the homogeneous approximation of our gravitation theory developed in the previous chapters. The field motion equations including the dilaton one are derived from the action

$$W = W_{\text{Cas}} + W_v . \quad (12.3)$$

Here

$$W_{\text{Cas}} = -V_0 \int_{x_1^0}^{x_2^0} dx^0 \bar{N}_0 \left[\left(\frac{da}{\bar{N}_0 dx^0} \right)^2 + \rho_{\text{Cas}}(a) \right] \quad (12.4)$$

is the action of the cosmological scale factor in the supposition of the Casimir energy dominance

$$\rho_{\text{Cas}}(a) = \frac{H_0^2}{a^2};$$

$$W_v = \int d^4x \sqrt{-g} \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} M_v^2 v_\mu v^\mu \right] \quad (12.5)$$

is the vector boson action.

The classification of observational data in the framework of the considered model of the quantum Universe as representations of the $A(4) \otimes C$ group supposes that the concept of a particle in QFT can be associated with the fields that have positive energy and the positive probability. The negative energies are removed by resolution of constraints and the causal quantization in the reduced phase space. According to the causal quantization the creation operator of a particle with a negative energy is replaced by the annihilation operator of a particle with a positive energy. The results of such quantization in the metrics (12.1) are given in Appendix A.

The Quantum Universe model supposes the identification of real observational magnitudes with the conformal variables. This identification yields the Universe evolution that differs from the Standard model. In the following we shall use the Hamiltonian form of the field Fourier harmonics

$$v_k^I = \int d^3x e^{i\mathbf{k}\cdot\mathbf{x}} v^I(\mathbf{x}).$$

The action takes the form

$$\begin{aligned}
 W = & \int_{x_1^0}^{x_2^0} dx^0 \sum_k \left[\mathbf{p}_k^\perp \partial_0 v_k^\perp + \mathbf{p}_k^\parallel \partial_0 v_k^\parallel \right] + \\
 & + \int_{x_1^0}^{x_2^0} dx^0 \left(-P_a \frac{da}{dx^0} + N_0 \left[\frac{P_a^2}{4V_0} - (H^\perp + H^\parallel) \right] \right), \tag{12.6}
 \end{aligned}$$

where $\mathbf{p}_k^\perp, \mathbf{p}_k^\parallel$ are transversal and longitudinal momenta and

$$\begin{aligned}
 H^\perp &= \sum_k \frac{1}{2} \left[\mathbf{p}_k^{\perp 2} + \omega^2 v_k^{\perp 2} \right], \\
 H^\parallel &= \sum_k \frac{1}{2} \left[\left(\frac{\omega(a, k)}{M_v a} \right)^2 \mathbf{p}_k^{\parallel 2} + (M_v a)^2 v_k^{\parallel 2} \right] \tag{12.7}
 \end{aligned}$$

are the free Hamiltonian with one-particle energy

$$\omega(a, k) = \sqrt{\mathbf{k}^2 + (M_v a)^2};$$

here we introduced the notions

$$\mathbf{p}_k^{\parallel 2} \equiv (\mathbf{p}_k^\parallel \cdot \mathbf{p}_{-k}^\parallel).$$

Let us consider the case of the rigid equation state with initial data

$$\begin{aligned}
 a(\eta) &= a_I \sqrt{1 + 2H_I(\eta - \eta_I)}, & (a_I^2 H_I = H_0), \\
 a_I &= a(\eta = \eta_I) : \\
 \tau = 2\eta H_I &= \frac{\eta}{\eta_I}, & x = \frac{q}{M_I}, & \gamma_v = \frac{M_I}{H_I}, \\
 M_I &= M_v(\eta = \eta_I). \tag{12.8}
 \end{aligned}$$

In terms of the conformal variables the one particle energy takes the form

$$\omega_v = H_I \gamma_v \sqrt{1 + \tau + x^2}.$$

Then the Bogoliubov equations are

$$\left[\frac{\gamma_v}{2} \sqrt{(1+\tau) + x^2} - \frac{d\theta_v^{\parallel}}{d\tau} \right] \tanh(2r_v^{\parallel}) \quad (12.9)$$

$$= - \left[\frac{1}{2(1+\tau)} - \frac{1}{4[(1+\tau) + x^2]} \right] \sin(2\theta_v^{\parallel}),$$

$$\frac{d}{d\tau} r_v^{\parallel} = \left[\frac{1}{2(1+\tau)} - \frac{1}{4[(1+\tau) + x^2]} \right] \cos(2\theta_v^{\parallel}),$$

$$\left[\frac{\gamma_v}{2} \sqrt{(1+\tau) + x^2} - \frac{d}{d\tau} \theta_v^{\perp} \right] \tanh(2r_v^{\perp}) = - \left[\frac{1}{4[(1+\tau) + x^2]} \right] \sin(2\theta_v^{\perp}),$$

$$\frac{d}{d\tau} r_v^{\perp} = \left[\frac{1}{4[(1+\tau) + x^2]} \right] \cos(2\theta_v^{\perp}). \quad (12.10)$$

We solved these equations numerically [1, 2, 3] at positive values of the momentum $x = q/M_I$, considering that the asymptotic behavior of the solutions is given by

$$r(\tau) \rightarrow \text{const} \cdot \tau, \quad \theta(\tau) = O(\tau), \quad \tau \rightarrow +0.$$

The distributions of longitudinal $\mathcal{N}^{\parallel}(x, \tau)$ and transverse $\mathcal{N}^{\perp}(x, \tau)$ vector bosons are given in Fig. 12.1 for the initial data $H_I = M_I$, $\gamma_v = 1$.

From the figure, it can be seen that for $x > 1$ the longitudinal component of the boson distribution is everywhere much greater than the transverse component, demonstrating a more copious cosmological creation of longitudinal bosons in relation to transverse bosons. A slow decrease in the longitudinal component as a function of momentum leads to a divergence of the integral for the density of product particles

$$n_v(\eta) = \frac{1}{2\pi^2} \int_0^{\infty} dq q^2 \left[\mathcal{N}^{\parallel}(q, \eta) + 2\mathcal{N}^{\perp}(q, \eta) \right] \rightarrow \infty. \quad (12.11)$$

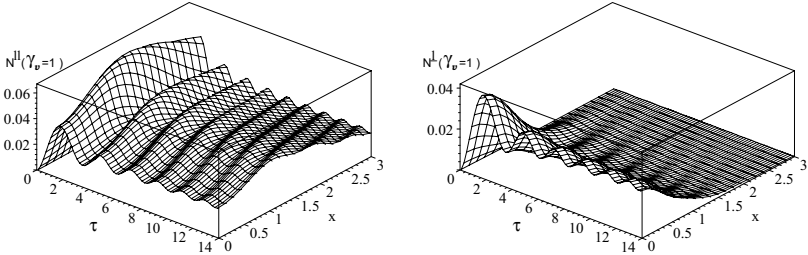


Figure 12.1: Longitudinal ($\mathcal{N}^{\parallel}(q, \eta)$) and transverse ($\mathcal{N}^{\perp}(q, \eta)$) components of the boson distribution versus the dimensionless time $\tau = 2\eta H_I$ and the dimensionless momentum $x = q/M_I$ at the initial data $M_I = H_I$ ($\gamma_v = 1$) [1, 2, 3].

The divergence of the integral in (12.11) stems from idealizing the problem of the production of a pair of particles in a finite volume for a system where there are simultaneous interactions associated with the removal of fields having a negative probability and where identical particles affect one another (so-called exchange effects). It is well-known [4, 5] that in this case one deals with the production of a set (rather than a pair) of particles, which acquires, owing to the aforementioned interactions, the properties of a statistical system. As a model of such a statistical system, we consider here a degenerate Bose – Einstein gas with the Boltzmann – Chernikov distribution function that has the form

$$\mathcal{F}(T_v, q, M_v(\eta), \eta) = \left\{ \exp \left[\frac{\omega_v(\eta) - M_v(\eta)}{T_v} \right] - 1 \right\}^{-1}, \quad (12.12)$$

(we use the system of units where the constant is $k_B = 1$), where T_v is the boson temperature. We set apart the problem of theoretically validating such a statistical system and its thermodynamic exchange,

only assuming fulfillment of specific conditions ensuring its existence. In particular, we can introduce the notion of the temperature T_v only in an equilibrium system. A thermal equilibrium is thought to be stable if the time within which the vector-boson temperature T_v is established, that is, the relaxation time [6, 7]

$$\eta_{\text{rel}} = [n(T_v)\sigma_{\text{scat}}]^{-1} \quad (12.13)$$

[as expressed in terms of their density $n(T_v)$ and the scattering cross section $\sigma_{\text{scat}} \sim 1/M_I^2$], does not exceed the time of vector-boson-density formation owing to cosmological creation, the latter time being controlled by the primordial Hubble parameter, $\eta_v = 1/H_I$. From (12.13), it follows that the particle-number density is proportional to the product of the Hubble parameter and the mass squared (this product being an integral of the motion in the present example); that is,

$$n(T_v) = n(T_v, \eta_v) \simeq C_H H_I M_I^2, \quad (12.14)$$

where C_H is a constant. The expression for the density $n(T_v, \eta)$ in Eq. (12.14) assumes the form

$$n_v(T_v, \eta) = \frac{1}{2\pi^2} \int_0^\infty dq q^2 \mathcal{F}(T_v, q, M(\eta), \eta) \left[\mathcal{N}^{\parallel}(q, \eta) + 2\mathcal{N}^{\perp}(q, \eta) \right]. \quad (12.15)$$

Here, the probability of the production of a longitudinal and a transverse boson with a specific momentum in an ensemble featuring exchange interaction is given (in accordance with the multiplication law for probabilities) by the product of two probabilities, the probability of their cosmological creation, $\mathcal{N}^{\parallel, \perp}$, and the probability of a single-particle state of vector bosons obeying the Bose – Einstein distribution

in (12.12). A dominant contribution to the integral in (12.15) from the region of high momenta (in the above idealized analysis disregarding the Boltzmann factor this resulted in a divergence) implies the relativistic temperature dependence of the density,

$$n(T_v, \eta_v) = C_T T_v^3, \quad (12.16)$$

where C_T is a coefficient. A numerical calculation of the integral in (12.15) for the values $T_v = M_I = H_I$, which result from the assumption about the choice of initial data ($C_T = C_H$), reveals that this integral is weakly dependent on time in the region $\eta \geq \eta_v = H_I^{-1}$ and, for the constant C_T , yields the value

$$C_T = \frac{n_v}{T_v^3} = \frac{1}{2\pi^2} \left\{ [1.877]^{||} + 2[0.277]^\perp = 2.431 \right\}, \quad (12.17)$$

where the contributions of longitudinal and transverse bosons are labeled with the superscripts ($||$) and (\perp), respectively.

On the other hand, the lifetime η_L of boson product in the early Universe in dimensionless units, $\tau_L = \eta_L/\eta_I$, where $\eta_I = (2H_I)^{-1}$, can be estimated by using the equation of state and the W-boson lifetime within the Standard Model. Specifically, we have

$$1 + \tau_L = \frac{2H_I \sin^2 \theta_{(W)}}{\alpha_{\text{QED}} M_W(\eta_L)} = \frac{2 \sin^2 \theta_{(W)}}{\alpha_{\text{QED}} \gamma_v \sqrt{1 + \tau_L}}, \quad (12.18)$$

where $\theta_{(W)}$ is the Weinberg angle, $\alpha_{\text{QED}} = 1/137$ and $\gamma_v = M_I/H_I \geq 1$. From the solution to Eq. (12.18),

$$\tau_L + 1 = \left(\frac{2 \sin^2 \theta_{(W)}}{\gamma_v \alpha_{\text{QED}}} \right)^{2/3} \simeq \frac{16}{\gamma_v^{2/3}} \quad (12.19)$$

it follows that, at $\gamma_v = 1$, the lifetime of bosons product is an order of magnitude longer than the Universe relaxation time:

$$\tau_L = \frac{\eta_L}{\eta_I} \simeq \frac{16}{\gamma_v^{2/3}} - 1 = 15. \quad (12.20)$$

Therefore, we can introduce the notion of the vector-boson temperature T_v , which is inherited by the final vector-boson-decay products (photons). According to currently prevalent concepts, these photons form Cosmic Microwave Background radiation in the Universe. Indeed, suppose that one photon comes from the annihilation of the products of W^\pm -boson decay and that the other comes from Z bosons. In view of the fact that the volume of the Universe is constant within the evolution model being considered, it is then natural to expect that the photon density coincides with the boson density [1]

$$n_\gamma = T_\gamma^3 \frac{1}{\pi^2} \times 2.404 \simeq n_v. \quad (12.21)$$

On the basis of (12.14), (12.16), (12.17), and (12.21), we can estimate the temperature T_γ of Cosmic Microwave Background radiation arising upon the annihilation and decay of W - and Z - bosons. This yields

$$T_\gamma \simeq \left[\frac{2.431}{2.404 \times 2} \right]^{1/3} T_v = 0.8 \times T_v, \quad (12.22)$$

where the vector-boson temperature

$$T_v = [H_I M_I^2]^{1/3}$$

is an invariant quantity within the model being considered. This invariant can be estimated at

$$T_v = [H_I M_I^2]^{1/3} = [H_0 M_W^2]^{1/3} = 2.73/0.8 \text{ K} = 3.41 \text{ K}, \quad (12.23)$$

which is a value that is astonishingly close to the observed temperature of Cosmic Microwave Background radiation. In the present case, this directly follows, as is seen from the above analysis of our numerical calculations, from the dominance of longitudinal vector bosons with high momenta and from the fact that the relaxation time is equal to the inverse Hubble parameter. The inclusion of physical processes, like the heating of photons owing to electronpositron annihilation [8], amounts to multiplying the photon temperature (12.22) by $(11/4)^{1/3} = 1.4$; therefore, we have

$$T_\gamma(e^+ e^-) \simeq (11/4)^{1/3} \times 0.8 T_v = 2.77 K . \quad (12.24)$$

We note that in other models [9] the fluctuations of the product-particle density are related to primary fluctuations of Cosmic Microwave Background radiation [10].

One can find the relation of the energy density of the vector bosons

$$\rho_v(\eta_I) \sim T^4 \sim H_I^4 \sim M_I^4$$

to the Universe density:

$$\frac{\rho_v(\eta_I)}{\rho_{\text{tot}}(\eta_I)} = \frac{M_I^2}{(M_{\text{Pl}} a_I)^2} = \frac{M_W^2}{M_{\text{Pl}}^2} = 10^{-34} . \quad (12.25)$$

This value indicates that the inverse effect of product particles on the evolution of the Universe is negligible.

Thus, the quantum version of General Relativity and the Standard Model, considered as the result of a spontaneous breakdown of the scale symmetry of a conformal-invariant theory in a specific frame and initial data can explain the origin of the Universe and its matter from vacuum.

12.2 Sources of CMB radiation anisotropy

In Sections 7.3 and 9.2 we discussed two different methods of the estimation of the primordial boson number. The first of them is the direct solution of the equations for Bogoliubov transformation coefficients with initial data determined by the Planck principle of minimal action for the Universe filled in the Casimir vacuum energy. The second is the cut of the momentum integral by means of the Boltzmann – Chernikov distribution function [5, 11], where the temperature parameter value (12.23), (12.24) is determined from the quantum uncertainty principle, and this conformal temperature is the cosmological motion integral of the vacuum equation of state with the dominance of the same Casimir energy.

The coincidence of these two different methods of calculation of the primordial boson number testifies to the early thermalization of the primordial bosons before the instance of formation of their decay products that include the Cosmic Microwave Background radiation.

Really, the arguments considered before, mean that the CMB photons can inform us about the parameters of electroweak interactions and masses, including the Higgs particle mass [12], and a possibility to estimate the magnitude of the CMB anisotropy.

Its observational value about $\alpha_{\text{QED}}^2 \sim 10^{-5}$ [13] testifies to the dominance of the two photon processes. Therefore, the CMB anisotropy

revealed in the region of the three peaks

$$220 \pm 20, \quad 546 \pm 50, \quad 800 \pm 80$$

can reflect parameters of the primordial bosons and their decay processes, in particular

$$h \rightarrow \gamma\gamma, \quad W^+W^- \rightarrow \gamma\gamma, \quad \text{and} \quad ZZ \rightarrow \gamma\gamma.$$

These values were known until quite recently with accuracy as minimum 10%. For their description by means of the Standard cosmological model one uses the metric scalar perturbations with the negative probability [14], that are forbidden in the considered Conformal General Relativity with the vacuum postulate [15], as we have seen above in Chapter 8.

In this case, two-photon decays of primordial Higgs particles and annihilation processes of primordial W , and Z bosons can explain three clear peaks in the CMB power spectrum without any acoustic waves with negative energy [16], if the values for multipole momenta ℓ_P of their processes are proportional to number of emitters at the horizon length

$$\ell_P = \tilde{H}^{-1}(z_P) \tilde{M}(z_P) \sim (1 + z_P)^{-3}$$

(in accordance with the new CC analysis of Supernovae type Ia data) and the energy of any photon is proportional to the mean photon energy in CMB multiplied by the effective scale factor

$$(1 + z_P)^{-1} \sim \ell_P^{1/3}$$

in accordance with the experience of description of the recombination epoch and the primordial helium abundance [17, 18].

One can be convinced that first three peaks

$$\ell_1 = 220, \quad \ell_2 = 546, \quad \ell_3 = 800$$

reflect the ratio of W and Z masses

$$M_Z/M_W = 1.134 \approx (800/546)^{1/3} = 1.136, \quad (12.26)$$

consequently,

$$\sin^2 \theta_W \approx 0.225,$$

and the value of Higgs particle mass as

$$\begin{aligned} m_h &= 2M_W (\ell_1/\ell_2)^{1/3} = & (12.27) \\ &= 2M_W (220 \pm 20/546 \pm 50)^{1/3} \simeq 120 \pm 8 \text{ GeV}. \end{aligned}$$

The Higgs boson mass is close to the present fit of the LEP experimental data supporting rather low values just above the experimental limit

$$114.4 < m_h < 128 \text{ GeV}.$$

To get a more accurate estimate of the Higgs mass and a better description of the CMB power spectrum within the model under consideration, one has to perform an involved analysis of the kinetic equations for nonequilibrium Universe [6] with primordial particle creation and subsequent decays.

For a reader who wishes to describe these peaks as acoustic disturbances according to the formulas given in the book [14], we will make the following remarks. The first is the choice of *conformal* magnitudes, as the real observables, instead of *worlds*. The second is the choice of

the rigid state equation, instead of dominance of radiation. The third is the history of evolution of masses, instead of temperature history of the Universe. In this case, all the characteristic resonance processes of the history of evolution of masses (like the transition from plasma to the atoms) occur at the same values of the redshift $z \simeq 1100$, as in the temperature history of the Universe. However, in the Conformal cosmology the constant temperature (12.23), (12.24) plays the role of a fundamental parameter, determined from the microscopic quantum theory.

12.3 Baryon asymmetry of the Universe

It is well known that, because of a triangle anomaly, W - and Z -boson interaction with lefthanded fermion doublets

$$\psi_L^{(i)}, \quad i = 1, 2, \dots, n_L,$$

leads to a nonconservation of the number of fermions of each type (i) [19, 20, 21, 22],

$$\partial_\mu j_{L\mu}^{(i)} = \frac{1}{32\pi^2} \text{Tr} \hat{F}_{\mu\nu}^* \hat{F}_{\mu\nu}, \quad (12.28)$$

where

$$\hat{F}_{\mu\nu} = -iF_{\mu\nu}^a g_W \tau_a / 2$$

is the strength of the vector fields

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc} A_\mu^b A_\nu^c.$$

In each of three generations of leptons (μ, τ) and color quarks, we have four fermion doublets, in all there are: $n_L = 12$ of them. Each

of the 12 fermions doublets interacts with the triplet of non-Abelian fields

$$A^1 = (W^{(-)} + W^{(+)})/\sqrt{2}, \quad A^2 = i(W^{(-)} - W^{(+)})/\sqrt{2}, \quad A^3 = Z/\cos\theta_{(W)}$$

with constant $g = e/\sin\theta_{(W)}$.

Taking the integral of the quality in (12.28) with respect to conformal time and three-dimensional variables d^4x , we can find a relation between the change

$$\Delta F^{(i)} = \int d^4x \partial_\mu j_{L\mu}^{(i)}$$

of the fermionic number

$$F^{(i)} = \int d^3x j_0^{(i)}$$

and the Chern class functional:

$$N_{CS} = \frac{1}{32\pi^2} \int d^4x \text{Tr} \hat{F}_{\mu\nu} * \hat{F}_{\mu\nu}.$$

The difference is equal to

$$\Delta F^{(i)} = N_{CS} \neq 0, \quad i = 1, 2, \dots, n_L. \quad (12.29)$$

The equality (12.29) is considered as a selection rule: that, the fermionic number changes identically for all fermion types [21]:

$$N_{CS} = \Delta L^e = \Delta L^\mu = \Delta L^\tau = \Delta B/3,$$

at the same time, the change in the baryon charge B and the change in the lepton charge

$$L = L^e + L^\mu + L^\tau$$

are related to each other in such a way that $B - L$ is conserved, while $B + L$ is not invariant. Upon taking the sum of the equalities in (12.29) over all doublets, one can obtain

$$\Delta(B + L) = 12N_{CS}.$$

One can evaluate the expectation value of the Chern functional (12.29) (in the lowest order of perturbation theory in the coupling constant) in the Bogoliubov vacuum $b|0\rangle_{\text{sq}} = 0$. Specifically, we have

$$N_{CS} = N_W \equiv -\frac{1}{32\pi^2} \int_0^{\eta_L} d\eta \int d^3x \langle 0 | \text{Tr} \hat{F}_{\mu\nu}^W * \hat{F}_{\mu\nu}^W | 0 \rangle, \quad (12.30)$$

where η_{L_W} is the W- boson lifetime, and N_W is the contribution of the primordial W-bosons. $\eta = 0$ and $\eta = \eta_L$ is given by

$$N_W = 2\alpha_W V_0 \int_0^{\eta_L} d\eta \int_0^\infty dk |k|^3 R_W(k, \eta),$$

where

$$\alpha_W = \alpha_{\text{QED}} / \sin^2 \theta_W$$

and

$$R_W = \frac{i}{2} b < 0 | b^+ b^+ - b^- b^- | 0 \rangle_b = -\sinh(2r(\eta_L)) \sin(2\theta(\eta_L))$$

is the Bogoliubov condensate, that is specified by relevant solutions to the Bogoliubov equations.

By calculating the integral, with time values of life of vector bosons

$$\tau_{L_W} = 15, \quad n_\gamma \simeq n_\nu,$$

we obtain the estimate of the average value of the functional of Chern – Simons on states of the primary bosons [22]

$$\frac{N_{CS}}{V_0} = \frac{(N_W)}{V_0} = \frac{\alpha_{\text{QED}}}{\sin^2 \theta_{(W)}} T^3 4 \times 1.44 = 0.8 n_\gamma. \quad (12.31)$$

Hence we obtain the following estimate of the value of violation of density of fermion number in the considered cosmological model [22]

$$\frac{\Delta F^{(i)}}{V_0} = \frac{N_{CS}}{V_0} = 0.8 n_\gamma, \quad (12.32)$$

where

$$n_\gamma = 2.402 \times T^3 / \pi^2$$

is the density of number of relic photons. According to Sakharov [23], the violation of the fermion number is frozen by CP - non-conservation, which leads to the density of baryon numbers

$$n_b = X_{\text{CP}} \frac{\Delta F^{(i)}}{V_{(r)}} \simeq X_{\text{CP}} n_\gamma. \quad (12.33)$$

where a multiplier X_{CP} is defined by electroweak interaction between d and s quarks ($d + s \rightarrow s + d$), responsible for CP -violation, experimentally observed in the decay of K -mesons [24].

From the ratio of the number of baryons to number of photons it is possible to estimate the constant of ultra-weak interaction: $X_{\text{CP}} \sim 10^{-9}$. Thus, the evolution of the Universe, the primary vector bosons and ultra-weak interaction [24], responsible for the violation of CP -symmetry with the constant $X_{\text{CP}} \sim 10^{-9}$, lead to the baryonic asymmetry of the Universe with a density

$$\rho_b(\eta = \eta_L) \simeq 10^{-9} \times 10^{-34} \rho_{\text{cr}}(\eta = \eta_L), \quad (12.34)$$

where for η_L it is possible to assess the future evolution of the density of baryons, select the lifetime of the W-boson.

After the decay of bosons their temperature is inherited by the Cosmic Microwave Background radiation. All subsequent evolution of matter in a constant cold Universe scenario repeats the known hot Universe one [25], because this evolution is defined as a conformally invariant relation of mass and temperature m/T .

Formulae (12.18), (12.25), and (12.34) provide an opportunity to assess the ratio of the present values of the baryon density and Casimir energy density played the role of primary quintessence of the considered model:

$$\Omega_b(\eta_0) = \frac{\rho_b(\eta_0)}{\rho_{cr}(\eta_0)} = \left[\frac{a_0}{a_L} \right]^3 = \left[\frac{a_0}{a_I} \right]^3 \left[\frac{a_I}{a_L} \right]^3. \quad (12.35)$$

It is take into account that the baryon density increases as mass, and the density of Casimir energy decreases as the inverse square of the mass. Recall the value of relationships

$$\left[\frac{a_0}{a_I} \right]^3 \sim 10^{43},$$

and relation $[a_I/a_L]^3$ given by a lifetime of bosons (12.19) and by the equation of state of matter $a(\eta) \sim \sqrt{\eta}$, hence we obtain $\Omega_b(\eta_0)$: taking into account the delay of baryon production for the life of the vector bosons

$$\Omega_b(\eta_0) = \left[\frac{a_0}{a_L} \right]^3 10^{-43} \sim 10^{43} \left[\frac{\eta_I}{\eta_L} \right]^{3/2} 10^{-43} \sim \left[\frac{\alpha_{QED}}{\sin^2 \theta_{(W)}} \right] \sim 0.03, \quad (12.36)$$

which is consistent with observations [26].

12.4 Summary

General Relativity and Standard Model are considered as a theory of dynamical scale symmetry with definite initial data compatible with the accepted Higgs mechanism. In this theory the Early Universe behaves like a factory of electroweak bosons and Higgs scalars, and it gives a possibility to identify three peaks in the Cosmic Microwave Background power spectrum with the contributions of photonic decays and annihilation processes of primordial Higgs, W and Z bosons in agreement with the QED coupling constant, Weinbergs angle, and Higgs particle mass of about 120 ± 6 GeV.

The key points of our construction are the choice of the conformal variables and conformal interval. Thus the Standard Model supplemented with the Casimir vacuum energy does not contradict with the following scenario of the evolution of the Universe within the Conformal cosmology [27]:

$\eta \sim 10^{-12}s$, creation of vector bosons from a “vacuum”;

$10^{-12}s < \eta < 10^{-10}s$, formation of baryon-antibaryon asymmetry;

$\eta \sim 10^{-10}s$, decay of vector bosons;

$10^{-10}s < \eta < 10^{11}s$, primordial chemical evolution of matter;

$\eta \sim 10^{11}s$, recombination or separation of Cosmic Microwave Background radiation;

$\eta \sim 10^{15}s$, formation of galaxies;

$\eta > 10^{17}s$, terrestrial experiments and evolution of Supernovae.

Our description is not complete, but it gives us a clear consistent statement of the problems in the framework of the well established

principles of classification of observational and experimental facts in physics and astrophysics. There are tendencies in modern cosmology [16, 28] to ignore these principles, in particular, to replace the initial data with the fundamental parameters of the equations of motion. However, this replacement leads to contradictions in modern models in applying mathematical tools of the type of the classification of relativistic states, or the Hamiltonian method because the latter were developed especially to solve equations with initial data. The scale-invariant cosmological model has several features quite different from those in the widely accepted Λ CDM model.

In particular, the model considered here does not need

1) considering acoustic waves $P_D \neq 0$ and their creation from *vacuum*, when the *vacuum* postulate excludes these waves by the Dirac constraint $P_D = 0$;

2) proposing “a dynamical inflation” $V_0 \neq V_I$ and $K \neq 0$, when the inflation equation $\rho = K + V = -p = -K + V$ is valid only if $K = 0$;

3) proclaiming the dominance of the potential term $V_I/K_I \sim a^{-6}$ at the Planck epoch, when $a_I^{-1} \sim 10^{61}$ is valid, and the kinetic term K has a huge enhancement factor of $a_I^{-6} \sim 10^{366}$ times with respect to the potential one V ;

4) use of the Planck epoch initial data in the dynamical theory, where these initial data should be free from any fundamental parameters of equations including the Planck one.

Thus, here we propose to construct Cosmology as a theory of *a quantum relativistic Universe* in a direct analogy to the quantum field theory. In the model the Universe evolves in the field super-space of

events [29] with respect to four-dimension interval in the Minkowskian space-time. The latter is defined as the tangent space-time of invisible Riemannian manifold as an object of general coordinate transformations that lead to constraints including the energy one.

Within the conformal formulation of the General Theory of Relativity and the Standard Model, we have investigated conditions under which the origin of matter can be explained by its cosmological creation from a vacuum. We have presented some arguments in support to the statement that the number of product vector-boson pairs is sufficient for explaining the total amount of observed matter and its content, provided that the Universe is considered as a conventional physical object that is characterized by a finite volume and a finite lifetime and which is described by a conformal invariant version of the General Theory of Relativity and the Standard Model featuring scale invariant equations where all masses, including the Planck mass, are replaced by the dilaton variable and where the spatial volume is replaced by a constant. In this case, the energy of the entire Universe in the field space of events is described by analogy with the description of the energy of a relativistic quantum particle in the Minkowski space: one of the variables (dilaton in the case being considered) becomes an evolution parameter, while the corresponding canonically conjugate momentum assumes the role of energy. This means that measured quantities are identified with conformal variables that are used in observational cosmology and in quantum field theory in calculating cosmological particle creation from a vacuum. Within the errors of observation, this identification of conformal variables with observables is compatible with data on the

chemical evolution of matter and data on Supernovae, provided that cosmic evolution proceeds via the regime dominated by the density of the vacuum. Thus, the identification of conformal coordinates and variables used in observational cosmology and in quantum field theory with measured quantities is the first condition under which the origin of matter can be explained by its cosmological creation from the vacuum. This is possible within a conformal invariant unified theory, where the Planck mass, which is an absolute quantity in the General Theory of Relativity, becomes an ordinary present-day value of the dilaton and where the Planck era loses its absolute meaning. The construction of a stable vacuum of perturbation theory by eliminating (through the choice of gauge-invariant variables) unphysical fields whose quantization leads to a negative normalization of the wave function in this reference frame is the second condition.

Finally, the elimination of divergences in summing the probabilities of product particles over their momenta by thermalizing these particles in the region where the Boltzmann H-theorem is applicable is the third condition. Under these conditions, it has been found in the present study that, in describing the creation of vector bosons from a vacuum in terms of conformal variables, one arrives at the temperature of Cosmic Microwave Background radiation as an integral of the motion of the Universe and at the baryon antibaryon asymmetry of the Universe with the superweak-interaction coupling constant and the baryon density of these results being in satisfactory agreement with the corresponding observed values and being compatible with the most recent data on Supernovae and nucleosynthesis.

In Chapter 9, it was shown that only gravitons, Higgs bosons (and the corresponding longitudinal components of the vector bosons) can be created from the vacuum in an empty Universe. One of the central results is the creation from the vacuum of 10^{88} particles of boson Higgs field, decays of which form all the material content of the Universe in accordance with modern observational facts on Supernovae and primordial nucleosynthesis. We have presented arguments in favor of the fact that the data on Supernovae, primordial nucleosynthesis, and cosmological particle production converted to units of relative standard length, can be described in conformal evolution scenario of the Universe by a single regime of the Casimir vacuum energy dominance. This chapter shows that transversal bosons in the course of their lifetimes, form baryon asymmetry of the Universe as a consequence of “polarization” of these bosons the vacuum Dirac sea of left fermions according to the selection rules of the Standard Model [21]. In this case the difference between the number of baryons and leptons remains, and their sum is not preserved. The experimentally observed ultra-weak interaction [24], responsible for the violation of CP-symmetry with the constant $X_{CP} \sim 10^{-9}$, freezes the baryon asymmetry of the Universe with a density (12.34). After the decay of bosons, their temperature is inherited by Cosmic Microwave Background radiation.

All subsequent evolution of matter in the constant cold Universe follows the known scenario of the hot Universe [25], because this evolution is defined as conformally invariant relations of mass and temperature m/T . Baryon density increases as the mass, and density of the Casimir energy decreases as the inverse square of the mass. As a

result, the present value of the relative baryon density is equal, up to a factor of order unity, to the Weinberg constant

$$\Omega_b(\eta_0) \simeq \frac{\alpha_{QED}}{\sin^2 \theta_W} \sim 0.03$$

in agreement with observations.

In this Chapter we have shown that the standard definition of the temperature of the primary boson (a fundamental constant in the conformal evolution of the Universe) as the square root of the product of the density of the particles and their scattering cross sections is not contrary to the direct calculation of number of produced particles with a constant temperature. In other words, the temperature of the primary vector W-, Z-bosons can be estimated from the same formula which is used to describe the chemical evolution of matter [25]: time of setting of temperature, equal to the inverse of the product of the particle density on the cross section of scattering can not exceed the lifetime of the Universe, which is proportional to the inverse of the Hubble parameter. The temperature history of the hot Universe, rewritten in conformal variables, looks like the history of evolution of the masses of elementary particles in the Universe at a constant cold temperature of the CMB.

Independence of the temperature of the background radiation $T_{\text{CMB}} \sim 2.725$ K from the redshift z , at first glance, is in contrast to the observation 6.0 K ; $T_{\text{CMB}} (z = 2.3371)$; 14 K. The relative occupancy of the various energy levels, where the temperature was brought to this observation, follows from the Boltzmann statistics. However, the argument of the Boltzmann factor as the weight ratio of the temperature has the

same dependence on factor z and in the cold Universe. Therefore, this ratio can be interpreted as the dependence of the energy levels, *id est* mass, of redshift at constant temperature. Distribution of chemical elements as well, determined mainly by Boltzmann factors, depend on conformal invariant relationship of temperature to mass.

Bibliography

- [1] Pervushin, V.N., Proskurin, D.V., Gusev, A.A.: *Cosmological particle origin in Standard Model*. Grav. & Cosmology. **8**, 181 (2002)
- [2] Blaschke, D. B., Vinitzky, S.I., Gusev, A.A., Pervushin, V.N., Proskurin, D.V.: *Cosmological production of vector bosons and Cosmic Microwave Background radiation*. Physics of Atomic Nuclei. **67**, 1050 (2004).
- [arXiv: gr-qc/0103114]
- [3] Gusev, A., Pervushin, V., Vinitzky, S., Zinchuk, V., Zorin, A.: *Cosmological creation of W-, Z- bosons and large-scale structure of the Universe*. In: *Problems of Gauge Theories*. Barbashov, B.M., Nesterenko, V.V. (Eds). JINR. Dubna (2004)
- [4] Kittel, Ch.: *Thermal Physics*. John Willey & Sons, Inc. New York (1969)
- [5] Muller, Ingo: *A History of Thermodynamics: The Doctrine of Energy and Entropy*. Springer (2007)

- [6] Ignatyev, Yu.G., Ignatyev, D.Yu.: *Kinetics of the nonequilibrium Universe. I. Kinetics of local thermodynamic equilibrium recovery*. Gravitation & Cosmology. **13**, 101 (2007)
- [7] Bernstein, J.: *Kinetic Theory in the Expanding Universe*. Cambridge University Press (1985)
- [8] Kolb, E.W., Turner, M.S.: *The Early Universe*. Addison-Wesley, Reading (1993)
- [9] Niemeyer, J.C.: *Inflation with a Planck-scale frequency cutoff*. Phys. Rev. **D 63**, 123502 (2001)
- [10] Bond, J.R., *et al.* (MaxiBoom Collab.): *CMB analysis of boomerang & maxima & the cosmic parameters $\Omega_{tot}, \Omega_b h^2, \Omega_{cdm} h^2, \Omega_\Lambda, n_s$* . In: Proceedings of the IAU Symposium 201 (PASP), CITA-2000-65
- [11] Chernikov, N.A.: *Derivation of the equations of relativistic hydrodynamics from the relativistic transport equation*. Phys. Lett. **5**, 115 (1963).
- Chernikov, N.A.: *The relativistic gas in the gravitational field*. Acta Phys. Polonica. **23**, 629 (1963).
- Chernikov, N.A.: *Equilibrium distribution of the relativistic gas*. Acta Phys. Polonica. **26**, 1069 (1964).
- Chernikov, N.A.: *Microscopic foundation of relativistic hydrodynamics*. Acta Phys. Polonica. **27**, 465 (1964)

-
- [12] Arbuzov, A.B., Barbashov, B.M., Borowiec, A., Pervushin, V.N., Shuvalov, S.A., Zakharov, A.F.: *General relativity and Standard Model in scale-invariant variables*. Grav. & Cosmology. **15**, 199 (2009)
- [13] Spergel, D.N., *et al.* *First - Year Wilkinson Microwave Anisotropy Probe (WMAP): Observations: determination of cosmological parameters*. Astrophys. J. Suppl. **148**, 175 (2003).
[arXiv: astro-ph/0302209]
- [14] Weinberg, S.: *Cosmology*. Oxford University Press (2008)
- [15] Barbashov, B.M., Pervushin, V.N., Zakharov, A.F., Zinchuk, V.A.: *Hamiltonian cosmological perturbation theory*. Phys. Lett. **B 633**, 458 (2006)
- [16] Giovannini, M.: *Theoretical tools for CMB physics*. Int. Jour. Mod. Phys., **D14**, 363 (2005).
[arXiv: astro-ph/0412601]
- [17] Behnke, D.: *Conformal Cosmology Approach to the Problem of Dark Matter*. PhD Thesis, Rostock Report MPG-VT-UR 248/04 (2004)
- [18] Cyburt, R.H., Fields, B.D., Olive, K.A.: *Primordial nucleosynthesis in light of WMAP*. Phys. Lett. **B 567**, 227 (2003).
Olive, K.A., Steigman, G., Walker, T.P.: *Primordial nucleosynthesis: theory and observations*. Phys. Rep. **333**, 389 (2000)

- [19] Adler, S.: *Axial-vector vertex in spinor electrodynamics*. Phys. Rev. **177**, 2426 (1969).
- Bell, J.S., Jackiw, R.: *A PCAC puzzle: $\pi^0 \rightarrow \gamma\gamma$ in the σ -model*. Nuovo Cimento. **60 A**, 47 (1969).
- Bardeen, W.A.: *Anomalous Ward identities in spinor field theories*. Phys. Rev. **184**, 1848 (1969)
- [20] 't Hooft, G.: *Symmetry breaking through Bell – Jackiw anomalies*. Phys. Rev. Lett. **37**, 8 (1976).
- 't Hooft, G.: *Computation of the quantum effects due to a four-dimensional pseudoparticle*. Phys. Rev. **D 14**, 3432 (1976)
- [21] Shaposhnikov, M.E. *Baryon asymmetry of the Universe in Standard electroweak theory*. Nucl. Phys. **B 287**, 757 (1987).
- Matveev, V.A., Rubakov, V.A., Tavkhelidze, A.N., Shaposhnikov, M.E.: *Nonconservation of baryon number under extremal conditions*. Physics – Uspekhi. **31**, 916 (1988).
- Rubakov, V.A., Shaposhnikov, M.E.: *Electroweak baryon number non-conservation in the early Universe and high-energy collisions*. Physics – Uspekhi. **39**, 461 (1996)
- [22] Blaschke, D., Behnke, D., Pervushin, V., Proskurin, D.: *Relative standard of measurement and Supernova data*. Report-no: MPG-VT-UR 240/03 (2003).
- [arXiv: astro-ph/0302001]

-
- [23] Sakharov, A.D.: *Violation of CP invariance, C asymmetry, and baryon asymmetry of the Universe*. JETP Lett. **5**, 24 (1967)
- [24] Okun, L.B.: *Leptons and Quarks*. Elsevier Science Publ. Co. North-Holland (1982)
- [25] Weinberg, S.: *The First Three Minutes: A Modern View of the Origin of the Universe*. Basic Books, New York (1977)
- [26] Fukugita, M., Hogan, C.J., Peebles, P.J.E.: *The cosmic baryon budget*. Astrophysical J. **503**, 518 (1998)
- [27] Pervushin, V.N.: *Early Universe as a W^- , Z^- Factory*. Acta Physica Slovaca. **53**, 237 (2003).
- Blaschke, D.B., Prozorkevich, A.V., Reichel, A.V., Smolyansky, S.A.: *Kinetic description of vacuum creation of massive vector bosons*. Physics of Atomic Nuclei. **68**, 1046 (2005)
- [28] Mukhanov, V.F., Feldman, H.A., Brandenberger, R.H.: *Theory of cosmological perturbations*. Phys. Repts. **215**, 203 (1992)
- [29] Wheeler, J.A.: In *Batelle Rencontres: 1967, Lectures in Mathematics and Physics*, De Witt, S., Wheeler, J.A. (eds.). New York (1968).
- De Witt, B.S.: *Quantum theory of gravity. I. The canonical theory*. Phys. Rev. **160**, 1113 (1967)

Chapter 13

Conformal cosmological perturbation theory

13.1 The equations of the theory of perturbations

In this Chapter, the conformal cosmological theory of perturbations will be considered for calculation of the lapse function \mathcal{N} and nonzerth harmonics of dilaton \bar{D} , with a certain geometric interval (5.67)

$$\begin{aligned} \tilde{d}s^2 &= e^{-4\bar{D}} \mathcal{N}^2 d\eta^2 - \\ &- \left(dX_{(b)} - X_{(c)} [\omega_{(c)(b)}^R(d) + \omega_{(c)(b)}^L(d)] - \mathcal{N}_{(b)} d\tau \right)^2, \end{aligned} \quad (13.1)$$

where

$$\mathcal{N} = \frac{\langle \sqrt{\tilde{\mathcal{H}}} \rangle^2}{\tilde{\mathcal{H}}}.$$

Recall that in the general case the local energy density (5.31) is

$$\tilde{\mathcal{H}} = -\frac{4}{3}e^{-7D/2}\Delta e^{-D/2} + \sum_{J=0,2,3,4,6} e^{-JD}\mathcal{T}_J(\tilde{F}), \quad (13.2)$$

$$\Delta = \partial_i[\mathbf{e}_{(b)}^i \mathbf{e}_{(b)}^j \partial_j]$$

is the Beltrami – Laplace operator. The sum is over of the densities of states: rigid radiation ($J = 2$), matter ($J = 3$), curvature ($J = 4$), Λ -type term ($J = 6$), respectively, in terms of the conformal fields

$$\tilde{F}^{(n)} = e^{nD}F^{(n)}, \quad (13.3)$$

where is the conformal weight.

In this case, the equation of the nonzeroth harmonics (5.65) and (5.66) takes the form [1]

$$T_D - \langle T_D \rangle = 0, \quad (13.4)$$

where

$$T_D = \frac{2}{3} \left\{ 7\mathcal{N}e^{-7D/2}\Delta e^{-D/2} + e^{-D/2}\Delta \left[\mathcal{N}e^{-7D/2} \right] \right\} + \quad (13.5)$$

$$+\mathcal{N} \sum_{J=0,2,3,4,6} J e^{-JD}\mathcal{T}_J.$$

One can solve all Hamiltonian equations (13.2), and (13.4) to define simplex components

$$\tilde{\omega}_{(0)} = e^{-2D}\mathcal{N}d\tau, \quad \mathcal{N} = \frac{\langle \sqrt{\tilde{\mathcal{H}}} \rangle}{\sqrt{\tilde{\mathcal{H}}}}, \quad (13.6)$$

$$\tilde{\omega}_{(b)} = dX_{(b)} - X_{(c)}\omega_{(c)(b)}^R + \mathcal{N}_{(b)}d\tau. \quad (13.7)$$

Recall that in the lowest order of perturbation theory, $\omega_{(c)(b)}^R$ describes the free one-component transverse strong gravitational wave considered

in Section 3. The longitudinal component of the shift-vector $\mathcal{N}_{(b)}$, unambiguously determined by the constraint (5.50), becomes equal to

$$\partial_\eta e^{-3\bar{D}} + \partial_{(b)} \left(e^{-3\bar{D}} \mathcal{N}_{(b)} \right) = 0. \quad (13.8)$$

13.2 The solution of the equations for small fluctuations

For small fluctuations

$$\mathcal{N} e^{-7\bar{D}/2} = 1 - \nu_1, \quad e^{-\bar{D}/2} = 1 + \mu_1 + \dots \quad (13.9)$$

the first order Eqs. of (13.2) and (13.5) take the form

$$[-\hat{\Delta} + 14\rho_{(0)} - \rho_{(1)}]\mu_1 + 2\rho_{(0)}\nu_1 = \bar{\mathcal{T}}_{(0)},$$

$$[7 \cdot 14\rho_{(0)} - 14\rho_{(1)} + \rho_{(2)}]\mu_1 + [-\hat{\Delta} + 14\rho_{(0)} - \rho_{(1)}]\nu_1 = 7\bar{\mathcal{T}}_{(0)} - \bar{\mathcal{T}}_{(1)},$$

where

$$\rho_{(n)} = \langle \mathcal{T}_{(n)} \rangle \equiv \sum_{J=0,2,3,4,6} (2J)^n (1+z)^{2-J} \langle \mathcal{T}_J \rangle, \quad (13.10)$$

$$\mathcal{T}_{(n)} = \sum_{J=0,2,3,4,6} (2J)^n (1+z)^{2-J} \mathcal{T}_J. \quad (13.11)$$

In the first order of perturbation with respect to the Newton coupling constant the lapse function and the dilaton takes the form [1]

$$e^{-\bar{D}/2} = 1 + \frac{1}{2} \int d^3y \left[G_{(+)}(x, y) \bar{T}_{(+)}^{(D)}(y) + G_{(-)}(x, y) \bar{T}_{(-)}^{(D)}(y) \right], \quad (13.12)$$

$$\mathcal{N} e^{-7\bar{D}/2} = 1 - \frac{1}{2} \int d^3y \left[G_{(+)}(x, y) \bar{T}_{(+)}^{(N)}(y) + G_{(-)}(x, y) \bar{T}_{(-)}^{(N)}(y) \right], \quad (13.13)$$

where $G_{(\pm)}(x, y)$ are the Green functions satisfying the equations

$$[\pm m_{(\pm)}^2 - \Delta]G_{(\pm)}(x, y) = \delta^3(x - y).$$

Here

$$m_{(\pm)}^2 = H_0^2 \frac{3(1+z)^2}{4} \left[14(\beta \pm 1)\Omega_{(0)}(a) \mp \Omega_{(1)}(a) \right],$$

$$\beta = \sqrt{1 + [\Omega_{(2)}(a) - 14\Omega_{(1)}(a)]/[98\Omega_{(0)}(a)]},$$

and

$$\bar{T}_{(\pm)}^{(D)} = \bar{T}_{(0)} \mp 7\beta[7\bar{T}_{(0)} - \bar{T}_{(1)}], \tag{13.14}$$

$$\bar{T}_{(\pm)}^{(N)} = [7\bar{T}_{(0)} - \bar{T}_{(1)}] \pm (14\beta)^{-1}\bar{T}_{(0)}, \tag{13.15}$$

are the local currents, and

$$\Omega_{(n)}(a) = \sum_{J=0,2,3,4,6} (2J)^n (1+z)^{2-J} \Omega_J, \tag{13.16}$$

where

$$\Omega_{J=0,2,3,4,6} = \frac{\langle \mathcal{T}_J \rangle}{H_0^2}$$

are partial densities of states: rigid, radiation, matter, curvature, Λ -term, respectively;

$$\Omega_{(0)}(a = 1) = 1, \quad 1 + z = a^{-1}$$

is the Hubble parameter.

In the context of these definitions, a full family of solutions (13.12), (13.13) for the lapse function and the nonzeroth dilaton harmonics of the Hamiltonian constraints (5.63) – (5.64) yields a Newton-type

potential. In particular, for a point mass distribution in a finite volume which corresponds to the nonzero terms with

a) $J = 0, 3$ in Eq. (13.10);

b) $J = 3$ in Eq. (13.11);

c) $J = 0, 3$ in Eq. (13.16)

(otherwise zero), we have

$$\bar{\mathcal{T}}_{(0)}(x) = \frac{\bar{\mathcal{T}}_{(1)}(x)}{6} \equiv \frac{3}{4a^2} M \left[\delta^3(x-y) - \frac{1}{V_0} \right]. \quad (13.17)$$

As a result, solutions (13.12) and (13.13) are transformed to the Schwarzschild-type form

$$e^{-\bar{D}/2} = 1 + \frac{r_g}{4r} \left[\frac{1+7\beta}{2} e^{-m_{(+)a}r} + \frac{1-7\beta}{2} \cos m_{(-)a}r \right], \quad (13.18)$$

$$\mathcal{N}e^{-7\bar{D}/2} = 1 - \frac{r_g}{4r} \left[\frac{14\beta+1}{28\beta} e^{-m_{(+)a}r} + \frac{14\beta-1}{28\beta} \cos m_{(-)a}r \right], \quad (13.19)$$

where

$$r_g = M/M_{\text{Pl}}^2, \quad \beta = 5/7, \quad m_{(+)} = 3m_{(-)}, \\ m_{(-)} = H_0 \sqrt{3(1+z)\Omega_{\text{Matter}}}/2.$$

These solutions describe the Jeans-like spatial oscillations of the scalar potentials (13.18) and (13.19) even for the case of zero pressure. These spatial oscillations can determine the clustering of matter in the recombination epoch, when the redshift is close to the value $z_{\text{recomb.}} \simeq 1100$. Indeed, if we use for the matter clustering parameter (that follows from spatial oscillations of the modified Newton law (13.18), (13.19)) the observational value [2]

$$r_{\text{clustering}} \simeq 130 \text{ Mps} \simeq \frac{1}{m_{(-)}} = \frac{1}{H_0 [\Omega_{\text{Matter}}(1+z_{\text{recomb}})]^{1/2}}, \quad (13.20)$$

one obtains $\Omega_{\text{Matter}} \sim 0.2$. This estimation is in agreement with the one recently discovered in the quest of the large scale periodicity distribution (see for details in [3]).

Constraint (5.50) yields the shift of the coordinate origin in the process of the evolution

$$\mathcal{N}^i = \left(\frac{x^i}{r} \right) \left(\frac{\partial_\eta V}{\partial_r V} \right), \quad V(\eta, r) = \int^r d\tilde{r} \tilde{r}^2 e^{-3\bar{D}(\eta, \tilde{r})}. \quad (13.21)$$

In the limit $H_0 = 0$ at $a_0 = 1$, the solutions (13.18) and (13.19) coincide with the isotropic Schwarzschild solutions:

$$e^{-\bar{D}/2} = 1 + \frac{r_g}{4r}, \quad \mathcal{N} e^{-\bar{D}/2} = 1 - \frac{r_g}{4r}, \quad \mathcal{N}_i = 0.$$

Solution (13.18) doubles the angle of the photon beam deflection by the Sun field. Thus, the CGR provides also the Newtonian limit in our variables.

13.3 Summary

Chapter 13 was developed to the conformal diffeoinvariant version of the cosmological perturbation theory. We have received a modification of the Schwarzschild solutions for the evolution of the Universe. It was shown that the non zero harmonics of the dilaton lead to the Jeans oscillations even in the case of a massive dust.

Bibliography

- [1] Barbashov, B.M., Pervushin, V.N., Zakharov, A.F., Zinchuk, V.A.: *Hamiltonian cosmological perturbation theory*. Phys. Lett. **B 633**, 458 (2006).

[arXiv: hep-th/0501242]

- [2] Bajan, K., Biernaska, M., Flin, P., Godłowski, W., Pervushin, V., Zorin, A.: *Large scale periodicity in redshift distribution*. Spacetime & Substance **4**, 225 (2003).

[arXiv: astro-ph/0408551]

- [3] Zakharov, A.F., Pervushin, V.N.: *Conformal cosmological model parameters with distant SNe Ia data: 'gold' and 'silver'*. Int. J. Mod. Phys. **D 19**, 1875 (2010).

[arXiv: 1006.4745 [gr-qc]]

Chapter 14

Cosmological modification of Newtonian dynamics

14.1 Free motion in conformally flat metric

The considered above joint unitary irreducible representations of the affine and conformal groups added by the SM fields contain both the Newtonian dynamics of a massive classical particle and the Friedmann cosmological metrics. The problem of the validity of the Newtonian dynamics arises when the Newtonian velocity value of a cosmic object becomes the order of the Hubble velocity value of this object [1]. Another problem is the choice of a frame of reference where initial data are given.

In the cosmological models considered above, we faced with the three classes of frames of reference. First of them is the world time-space interval in the Friedmann – Lemâître – Robertson – Walker (FLRW) metrics

$$ds^2 = dt^2 - a^2(t)dx_i^2 = a^2(\eta)\tilde{ds}^2 = a^2(\eta) [d\eta^2 - dx_i^2], \quad (14.1)$$

associated with heavy massive body. Here dt is the world time, η is the conformal time, x_1, x_2, x_3 are the conformal coordinates, and $a(\eta)$ is the conformal scale factor. The second class differs from the first one by the conformal long interval

$$\tilde{ds}^2 = [d\eta^2 - dx_i^2]. \quad (14.2)$$

and varying mass

$$m(\eta) = m_0 a(\eta). \quad (14.3)$$

The third class of frames of reference is associated with luminosity interval

$$ds_L^2 = a^{-6}(\eta)ds^2 = a^{-4}\tilde{ds}^2 = a^{-4} [d\eta^2 - dx_i^2]. \quad (14.4)$$

This class of frames of reference comoves the void local volume element.

In this Chapter we consider the dynamics of a classical particle in both the world interval (14.1) and the conformal one (14.2) [2, 3, 4, 5].

The one-particle energy $E = p_0$ is defined by the constraint

$$p_\mu p_\nu - m^2(\eta) = 0, \quad (14.5)$$

which implies that

$$p_0 = \sqrt{p^2 + m^2(\eta)} \simeq m(\eta) + \frac{p^2}{2m(\eta)}, \quad (14.6)$$

where $m(\eta) = m_0 a(\eta)$ is a running mass (14.3).

The action of a relativistic particle in a conformally flat metric (14.2) in the non-relativistic limit leads to the classical action for a particle

$$S_0 = \int_{\eta_I}^{\eta_0} d\eta [p_i x'_i - p_0 + m], \quad (14.7)$$

where $x'_i = dx^i/d\eta$, p_0 is given by expression (14.6).

Let us consider action (14.7) for the radial motion in the non-relativistic limit:

$$S_N = \int_{\eta_I}^{\eta_0} d\eta \frac{r'^2 m(\eta)}{2}; \quad (14.8)$$

here $r = \sqrt{x_i x^i}$ and $r' = dr/d\eta$. In this case, the equation of motion is

$$[r'(\eta)m(\eta)]' = 0 \quad (14.9)$$

with initial data

$$r_I = r(\eta_I), \quad r'_I = p_I/m_0, \quad m_I = m(\eta_I).$$

This equation has the following solution:

$$r(\eta) = r_I + p_I \int_{\eta_I}^{\eta} \frac{d\bar{\eta}}{m(\bar{\eta})}. \quad (14.10)$$

The Friedmann world time $dt = d\eta a(\eta)$ and the absolute coordinate

$$R(t) = a(\eta)r(\eta) \quad (14.11)$$

are determined by the conformal transformation with the scale factor

$$a(\eta(t)) = a(t),$$

which is usually chosen to be unity for the modern age

$$\eta = \eta_0 : \quad a(\eta_0) = 1,$$

the scale factor in the initial time moment $\eta = \eta_I$ is defined by the z -factor:

$$a(\eta_I) = a_I = \frac{1}{(1 + z_I)},$$

where $z(\eta_I) = z_I$. Since Friedmannian variables are tied to the modern era $\eta = \eta_0$, the time η_I is convenient to replace by η_0 . Then the world spatial interval

$$R(t) = a(\eta)r(\eta)$$

is set by the expression

$$R(t) = a(t) \left[r_0 + \frac{p_I}{m_0} \int_{t_I}^t \frac{d\bar{t}}{a^2(\bar{t})} \right] \quad (14.12)$$

and satisfies the equation of motion

$$\ddot{R}(t) - (H^2 + \dot{H})R = 0, \quad (14.13)$$

where $H(t) = \dot{a}(t)/a(t)$ is the Hubble parameter. The equation of motion follows from the action

$$S_N(t) = \int_{t_I}^{t_0} dt \frac{(\dot{R} - HR)^2 m_0}{2}. \quad (14.14)$$

The same action can be obtained geometrically using the definitions of the measured intervals in the Standard cosmology

$$dl = a(t)dr = d[ra(t)] - r\dot{a}(t)dt = [\dot{R} - HR]dt,$$

including the world space interval $R(t) = r a(t)$ in a space-time with the Friedmann – Lemâitre – Robertson – Walker (FLRW) metric

$$(ds^2) = (dt)^2 - a^2(t)(dx^i)^2. \quad (14.15)$$

The observable coordinates X^i of the expanding Universe can be written as

$$X^i = a(t)x^i, \quad dX^i = a(t)dx^i + x^i da(t), \quad (14.16)$$

and instead of the Euclidean differentials dx^i the covariant ones are used

$$a(t)dx^i = d[a(t)x^i] - x^i da(t) = dX^i - X^i \frac{da(t)}{a(t)}. \quad (14.17)$$

In the Standard cosmology a particle mass is constant.

The interval (14.15) in terms of variables (14.16) becomes equal to

$$(ds^2) = (dt)^2 - \sum_{i=1,2,3} (dX^i - H(t)X^i dt)^2, \quad (14.18)$$

where $H(t)$ is the world Hubble parameter. All these equations by conformal transformations are reduced to the equations given in the book of Peebles [6].

14.2 The motion of a test particle in a central field

The energy of a particle that moves along a geodesic line in a space with a given metric can be found by solving the equation of the mass

shell. Equating the square of 4-momentum $p_\mu p^\mu$ to the square of the mass in the metric (14.2):

$$p^2 = g^{\mu\nu} p_\mu p_\nu = m^2 \quad (14.19)$$

we find an expression for the energy p_0

$$p_0 \approx \pm \left[\left(1 - \frac{r_g}{2r}\right) m + \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} \right]. \quad (14.20)$$

From the condition of positive energy $p_0 > 0$ in the right-hand side (14.20) we choose a positive sign; as a result, in the non-relativistic limit, we arrive to the action¹

$$S_{\text{classic}} = \int_{\eta_I}^{\eta_0} d\eta [p_r r' + p_\theta \theta' - E_{\text{classic}}], \quad (14.21)$$

where

$$E_{\text{classic}} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{r_g m}{2r}, \quad (14.22)$$

and $m = m(\eta)$ is the conformal mass of a test body, which depends on the time (evolution) and is determined by (14.3). The product $r_g m$ is a conformal invariant independent of time. For a constant mass $m = m_0$ one obtains the classical action.

In the case of a particle with a constant mass moving along a circle ($r = r_0$) the Newtonian speed

$$w_0 = \sqrt{\frac{r_g}{2r_0}}$$

coincides with the orbital one

$$v_0 = \frac{p_\theta}{m_0 r_0}.$$

¹The equations of motion for a free particle, taking into account the expansion of the Universe, do not differ from the ones given in the monograph of Peebles [6].

The equality $w_0 = v_0$ is the basis of the analysis of observational data on dark matter in the Universe [7].

To determine the region of applicability of the Newton theory with a constant mass and the status of the circular trajectories, we shall investigate the Kepler problem for variable masses (14.54), the dependence on time of which is determined by astrophysical data on the Supernovae [2].

14.3 The Kepler problem in the Conformal theory

Taking into consideration the dependence of the coordinate distance from the conformal time (14.2) and cosmic evolution in a rigid state condition one can move from the evolution parameter η to a monotonically increasing function $a(\eta)$

$$a(\eta) = \sqrt{1 + 2H_0(\eta - \eta_0)}. \quad (14.23)$$

Then from the equation of motion for the Newtonian action (14.21) taking into account the dependence of the mass from the conformal time (14.54) and relation (14.23) we obtain an explicit parametric solution $a(\tau)$ and $r(\tau)$ with a parameter τ introduced in [4, 5, 8]:

$$a(\tau) = c_1 \frac{N_1(\tau)}{\tau^{2/3} N(\tau)}, \quad \frac{r(\tau)}{r_0} = c_2 \tau^{2/3} N(\tau), \quad (14.24)$$

where

$$N(\tau) = \alpha_1 U(\tau)^2 + \beta_1 U(\tau)V(\tau) + \gamma_1 V(\tau)^2, \quad (14.25)$$

$$N_1(\tau) = \left(\tau \frac{dN(\tau)}{d\tau} + \frac{2}{3}N(\tau) \right)^2 \pm 4\tau^2 N(\tau)^2 + \omega^2 \Delta, \quad (14.26)$$

$$\Delta = 4\alpha_1\gamma_1 - \beta_1^2 > 0, \quad c_1, c_2, \alpha_1, \beta_1, \gamma_1 = \text{const}, \quad (14.27)$$

$$c_1 = \left(\frac{3w_0^2}{4c_0^2} \right)^{1/3} \frac{c_0 v_0}{2w^2 |\omega| \Delta^{1/2}}, \quad c_2 = \left(\frac{4c_0^2}{3w_0^2} \right)^{1/3} \frac{v_0}{|\omega| \Delta^{1/2}}. \quad (14.28)$$

Here

$$w_0^2 = \frac{rg}{2r_0}, \quad v_0 = \frac{p\theta}{m_0 r_0}, \quad c_0 = H_0 r_0 \quad (14.29)$$

are the Newtonian, orbital and cosmic velocity, respectively.

For the upper sign in (14.26)

$$U(\tau) = J_{1/3}(\tau), \quad V(\tau) = Y_{1/3}(\tau), \quad \omega = \frac{2}{\pi}, \quad (14.30)$$

where $J_{1/3}(\tau)$ and $Y_{1/3}(\tau)$ are the Bessel functions of the first and the second kind. For the lower sign

$$U(\tau) = I_{1/3}(\tau), \quad V(\tau) = K_{1/3}(\tau), \quad \omega = -1, \quad (14.31)$$

where $I_{1/3}(\tau)$ and $K_{1/3}(\tau)$ are the modified Bessel functions of the first and the second kind.

The solution (14.24) – (14.31) includes five independent constants which can be found from the following algebraic system of equations:

$$\left. \frac{r}{r_0} \right|_{\tau=\tau_0} = 1, \quad \left. \frac{dr}{da} \right|_{\tau=\tau_0} = 0, \quad a|_{\tau=\tau_0} = 1, \quad (14.32)$$

$$\frac{9}{64} \left(\frac{c_2^2}{c_1} \right)^2 \omega^2 \Delta = \frac{v_0^2}{c_0^2}, \quad \frac{9}{128} \left(\frac{c_2}{c_1} \right)^3 = \frac{w_0^2}{c_0^2}$$

in the region of their solvability. For example, for

$$\tau_0 = 1, \quad v_0^2 = 0,25, \quad w_0^2 = 0,05, \quad c_0 = 1$$

the system has the following solution:

$$c_1 = -0,48, \quad c_2 = -0,32, \quad \alpha_1 = -0,78, \quad \beta_1 = 0, \quad \gamma_1 = -0,48,$$

with the condition $\Delta > 0$.

The solution corresponding to the lower sign in (14.26) is restricted to zero and is not limited in infinity because of the properties of the function $K_{1/3}(\tau)$. The solution corresponding to the upper sign in (14.26) is limited in infinity and describes a finite motion along an ellipse. A character of the motion at short times can be regarded as the determination of a periodic regime after some initial perturbation.

These two types of solutions correspond to two different signs of energy (14.22): positive energy corresponds to a free motion of a particle and its negative energy corresponds to a bound state.

14.4 Capture of a particle by a central field

From the equations of motion resulting from (14.21) and determination of the energy (14.22) one can find the rate of the energy change of the object:

$$\frac{dE_{\text{classic}}}{d\eta} = -H(\eta) \left[\frac{p_r^2 + p_\theta^2/r^2}{2m} \right], \quad (14.33)$$

where $H(\eta) = da/d\eta/a$ is the Hubble parameter. From (14.33) it follows that the derivative of the energy is always negative and tends

to zero, so the energy itself asymptotically decreases to a negative value, and the reason of the non-conservation of energy is the cosmic evolution of the masses (see Fig. 14.1) [9, 10].

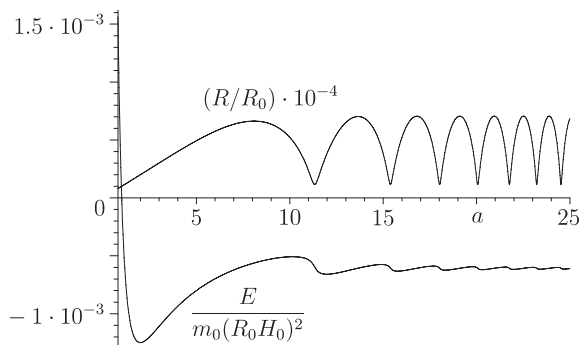


Figure 14.1: The upper part of the graph shows the solution of the equations for action (14.21) in dimensionless variables $y(x) = R/R_I$ and $x = H_I(t - t_I)$ with the boundary conditions $y(x = 0) = 1$ and $y'(x = 0) = 0$. The curve in the lower part of the graph shows the evolution of the total energy (14.22) in variables $R = ar$ and $P = p/a$.

Thus, the cosmic evolution of the mass reduces the energy of the test particle to negative asymptotic values under the condition $E = 0$ which, in particular, takes place under the initial data $v_I^2 = 2w_I^2$; particle transfers to a bound state and its trajectory is an ellipse. The described mechanism of capturing particles can be applied to the dynamics of stars and galaxies and should lead to the formation of galaxies and clusters with an anisotropic distribution of the Hubble flows in the Local Group, which is supported by the observations [11, 12].

14.5 The problem of dark matter in Superclusters

In modern cosmological investigations, the effects of dark matter are analyzed by using characteristics of the Newtonian motion in the gravitational fields or clusters of galaxies [7, 13, 14, 15, 16]; however, the following discrepancy occurs: The Newtonian motion of galaxies is described in a flat space-time

$$ds^2 = dt^2 - \sum_i (dx^i)^2,$$

and analysis of the observational data is carried out in terms of the metric FLRW (14.15).

Let us consider the Newtonian motion of a particle in a gravitational field of a space with the FLRW metric in which for the observed coordinates in an expanding Universe the coordinates (14.16) are adopted and instead of differentials of the Euclidean space dX^i the covariant differentials of the FLRW space are used (14.17). In this case, the Kepler problem is defined by the equation

$$\ddot{R}(t) - (H^2 + \dot{H})R - \frac{(m_0 R^2 \dot{\theta})^2}{m_0^2 R^3} + \frac{r_g}{2R^2} = 0. \quad (14.34)$$

This equation reduces to the equation that has been solved by transition to conformal variables.

The law of conservation of energy in the flat space ($H(t) = 0$) leads to the next dependence of the radius from the orbital speed:

$$R\dot{\theta} = \sqrt{\frac{r_g}{2R}}, \quad (14.35)$$

where $r_g = 2\alpha/m_I \simeq 3 \cdot 10^5 M$ cm is the gravitational radius of the object, M is its mass expressed in solar masses. In the considered case of the rigid state equation (14.34) in the class of solutions $R = R_I$, $\dot{R}_I = 0$ one has the expression

$$R_I \dot{\theta} = \sqrt{\frac{r_g}{2R_I} + 2(H_I R_I)^2}, \quad (14.36)$$

or

$$v_I = \sqrt{w_I^2 + 2c_I^2},$$

where

$$v_I = R_I \dot{\theta}, \quad w_I^2 = \frac{r_g}{2R_I}, \quad c_I = R_I H_I. \quad (14.37)$$

In a more general case, for the metric (14.2)

$$(ds^2) = a^2(\eta) [d\eta^2 - (dx^i)^2]$$

with the equation of state (2.69)

$$\frac{1}{H_0^2} \left(\frac{da}{d\eta} \right)^2 = \Omega(a),$$

where $\Omega(a)$ is defined in (2.70) one has

$$\dot{H} + H^2 = -H^2 \left(1 - \frac{a}{2\Omega} \frac{d\Omega}{da} \right),$$

or, substituting it into (14.34), we get

$$\ddot{R} + H^2 \left(1 - \frac{a}{2\Omega} \frac{d\Omega}{da} \right) R - \frac{(m_0 R^2 \dot{\theta})^2}{m_0^2 R^3} + \frac{r_g}{2R^2} = 0.$$

In this case, the equality $v_I^2 = w_I^2$ becomes the following relation:

$$v_I^2 = w_I^2 + (H R_I)^2 \left(1 - \frac{a}{2\Omega} \frac{d\Omega}{da} \right).$$

$\Omega(a) =$	$\Omega_{\text{rigid}} a^{-2}$	Ω_{rad}	$\Omega_M a$	$\Omega_\Lambda a^4$
$\left(1 - \frac{a}{2\Omega} \frac{d\Omega}{da}\right) =$	2	1	1/2	-1

The Table shows that the lowest deficit of the dark matter of the four “pure” states is related to the rigid state Ω_{rigid} which corresponds to the case under consideration (14.36). Note that in the Standard cosmology, the cosmic evolution increases the deficit of dark matter:

$$R_I \dot{\theta} = \sqrt{\frac{r_g}{2R_I} - \frac{(H_I R_I)^2}{2}}. \tag{14.38}$$

From (14.38) it follows that the conventional Newtonian characteristics to describe the behavior of the orbital velocities are not applicable to radial distances when the double square of cosmic speed is comparable in magnitude to square Newtonian velocities²: $2c_I^2 \geq w_I^2$.

For evaluation of the radial distance, in this case one can get the distance (we will call it a critical distance) R_{cr} , wherein $2c_I^2 = w_I^2$, hence

$$R_{\text{cr}} = \left(\frac{r_g}{2H_I^2}\right)^{1/3}. \tag{14.39}$$

The current value of the Hubble parameter $H_0^{-1} \simeq 10^{28}$ cm leads to the value of the critical distance

$$R_{\text{cr}} \simeq 10^{20} \left(\frac{M}{M_\odot}\right)^{1/3} \text{ cm}. \tag{14.40}$$

²This fact was known to Einstein and Straus [1] (see also [3, 4, 9, 10]).

The critical radius for the Coma cluster ($M \simeq 10^{15} M_{\odot}$ [7]) is comparable with the size of the cluster:

$$R_{\text{size}} \sim 3 \cdot 10^{25} \text{ cm} > R_{\text{cr}} \sim 10^{25} \text{ cm}, \quad (14.41)$$

and our arguments are applicable. For our galaxy, ($M \simeq 10^{12} M_{\odot}$) the corresponding estimate gives

$$R_{\text{size}} \sim 10^{23} \text{ cm} < R_{\text{cr}} \sim 10^{24} \text{ cm}, \quad (14.42)$$

that is the critical radius of our galaxy, an order of magnitude larger than its size.

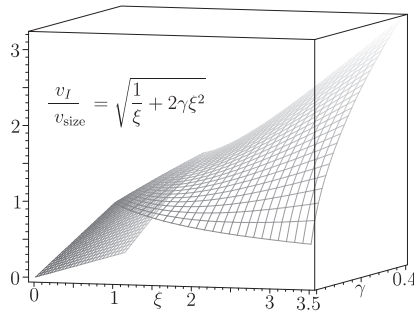


Figure 14.2: The dependence of the orbital velocity of the “particle” v_I on its radius *id est* the distance from the center of the object, $\xi = R/R_{\text{size}}$, where R_{size} is a radius of the object, $\gamma = (R_{\text{size}}/R_{\text{cr}})^3$ and $R_{\text{cr}} = [r_g/H^2]^{1/3} = 10^{20} M^{1/3} \text{ cm}$ is a value of radius for which Newtonian velocity coincides with the Hubble one, M is a mass of the object in units of solar masses. Under $\gamma = 0$ a rotation curve coincides with the curve obtained in Newtonian mechanics.

It is convenient to consider the rotational curve of the circular speed $v_I = R_I \dot{\theta}$ (14.36) in dimensionless terms $\xi = R/R_{\text{size}}$ and γ :

$$\frac{v_I}{v_{\text{size}}} = \sqrt{\frac{1}{\xi} + 2\gamma\xi^2}, \quad (14.43)$$

where

$$v_{\text{size}} = \sqrt{\frac{r_g}{2R_{\text{size}}}}, \quad \gamma = \left(\frac{R_{\text{size}}}{R_{\text{cr}}}\right)^3,$$

R_{size} is a size of the object and

$$R_{\text{cr}} = \left(\frac{r_g}{H^2}\right)^{1/3} = 10^{20} M^{1/3} \text{cm}$$

is the value of radius in cm, for which the Newtonian velocity coincides with the Hubble one, M is the mass of the object in units of solar masses (Fig. 14.2). The dependence (14.43) at $\gamma = 0$ corresponds to the Newtonian case, and the curve at $\gamma \neq 0$ deviates from the Newtonian curve. This deviation cannot be explained by the introduction of the halo of dark matter [13, 14, 15, 16], but rather a cosmological modification of the Newtonian dynamics described in this monograph. Therefore, the violation of the virial theorem for $R \geq R_{\text{cr}}$, found in clusters of galaxies and interpreted as evidence for the existence of dark matter, in the Conformal cosmology is considered as the result of evolution of the Universe [3, 4, 9, 10], as was predicted by Einstein and Strauss in [1].

14.6 The Kepler problem in the generalized Schwarzschild field

Let us consider the general case of the motion of a test body or a particle in a spherically-symmetric gravitational field of the heavy mass. We generalize the Schwarzschild metric in the synchronous reference frame by replacing the ordinary mass m_0 by its conformal analogue

$m_0 a(\eta) = m(\eta)$:

$$ds^2 = \left(1 - \frac{2\alpha}{mr}\right) dt^2 - \frac{dr^2}{1 - 2\alpha/(mr)} - r^2 \sin^2(\theta) d\theta^2, \quad (14.44)$$

where

$$m = m(\eta), \quad r = \sqrt{x_i x^i}, \quad a(\eta) = \sqrt{1 + 2H_I(\eta - \eta_I)},$$

and consider the motion in the cylindrical coordinates

$$X^1 = R \cos \Theta, \quad X^2 = R \sin \Theta, \quad R = ar. \quad (14.45)$$

Here H_I is the initial value of the Hubble velocity in the space with the rigid state equation of matter [2] when the density of energy and pressure are equal. In terms of the conformal time $d\eta = dt/a$ and conformal values $r = R/a$ let us write the action for a particle in the form

$$S_{\text{Schw}} = \int_{\eta_I}^{\eta_0} d\eta \left[P_r \frac{dr}{d\eta} + P_\theta \frac{d\theta}{d\eta} - E_{\text{Schw}} \right], \quad (14.46)$$

where

$$Q_{\text{Schw}} = \left(1 - \frac{r_g m_I}{r m}\right)^{1/2}, \quad r_g = M_O G,$$

P_r, P_θ are conjugated momenta of the corresponding coordinates and E_{Schw} is the energy of the system

$$E_{\text{Schw}} = Q_{\text{Schw}} \sqrt{P_r^2 Q_{\text{Schw}}^2 + P_\theta^2 / r^2 + m^2} - m. \quad (14.47)$$

The trajectory of the test particle is shown in Fig. 14.3, and the Newtonian limit of the action (14.46) takes the form

$$S_A = \int_{\eta_I}^{\eta_0} d\eta \left[P_r \frac{dr}{d\eta} + P_\theta \frac{d\theta}{d\eta} - \frac{P_r^2 + P_\theta^2 / r^2}{2m} - \frac{\alpha}{r} \right], \quad (14.48)$$

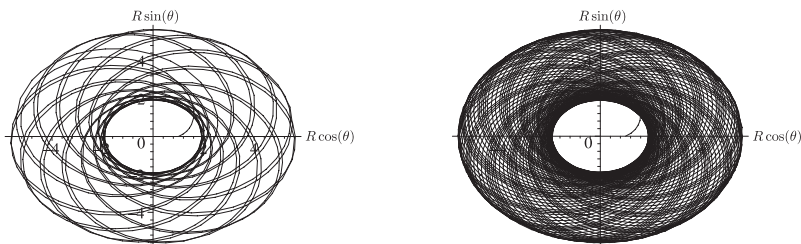


Figure 14.3: The solution of the equations of motion for the action (14.46) at $c_I = 1$, $v_I = 1$, and $w_I^2 = 0,25$. In both figures there is shown the trajectory of the same object from the starting point $(1,0)$ for different intervals of time in the generalized Schwarzschild field (14.44).

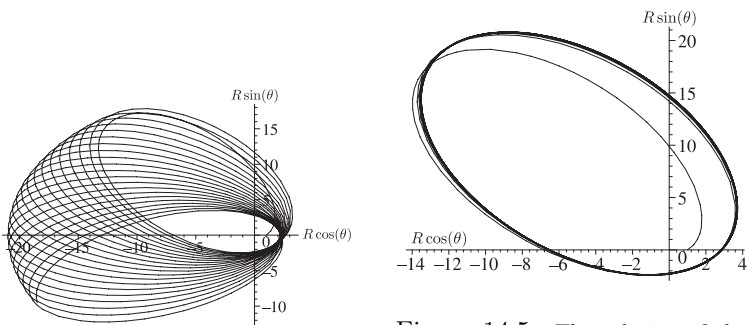


Figure 14.4: The solution of the equations of motion for (14.46) at $c_I = 0,25$, $v_I = 0,25$, and $w_I^2 = 0,015625$. These values of parameters correspond to the relativistic limit of equations for (14.46) in which the classical ellipse begins to turn counterclockwise.

Figure 14.5: The solution of the equations of motion for (14.46) at $c_I = 0,01$, $v_I = 0,01$, and $w_I^2 = 2,5 \cdot 10^{-5}$. These values of parameters correspond to the classical limit and the classical ellipse on relatively large times since the beginning of motion. Just as in the original case $c_I = 1$ (generalized Schwarzschild field), the particle at small times is “captured” by an ellipse.

where

$$\alpha = M_{\text{O}}m_I G$$

is the Newton constant of interaction of the galaxy with a mass m_I in the central gravitational field with a central mass M_{O} .

Let us consider three velocities:

$$w_I = \sqrt{\frac{r_g}{2r_I}}, \quad v_I = \frac{P_{\theta}}{m_I r_I}, \quad c_I = H_I r_I \quad (14.49)$$

Newtonian, orbital and cosmic, respectively. The limit of small velocities $w_I, v_I, c_I \rightarrow 0$ corresponds to the classical approximation (see Fig. 14.5) — classical Kepler's problem with the expansion of the Universe. In this limit, we obtain the action (14.46), where instead of the Schwarzschild Hamiltonian (14.47) its Newtonian limit is:

$$E_{\text{Schw}} \sim E_{\text{classic}} = \frac{P_r^2}{2am_I} + \frac{P_{\theta}^2}{2am_I r^2} - \frac{r_g m_I}{2r}. \quad (14.50)$$

It is convenient to study the solution of the problem in terms of dimensionless magnitudes

$$x = H_I(\eta - \eta_I), \quad r = r_I y, \quad P_r = m_I p, \quad (14.51)$$

in terms of which the effective action for the radial motion takes the form

$$S_{\text{eff}} = r_I m_I \int_{x_I}^{x_0} dx \left(p \frac{dy}{dx} - \frac{1}{c_I} E_{\text{eff}} \right), \quad (14.52)$$

where

$$E_{\text{eff}} = \frac{E_{\text{Schw}}}{m_I} = \sqrt{1 - 2w_I^2/(ay)} \sqrt{a^2 + (1 - 2w_I^2/(ay))p^2 + v_I^2/y^2} - a \simeq$$

$$\simeq \frac{p^2 + v_I^2/y^2}{2a} - \frac{w_I^2}{y}, \quad (14.53)$$

$a = \sqrt{1 + 2x}$. The approximate equality holds here for small velocities, whereas if we put $a = 1$, we obtain the classical orbital motion $y = 1$, $p = 0$, where the Newtonian velocity w_I coincides with the orbital v_I . This equality, rather its violation, is the basis for theoretical analysis of observational data on dark matter in the Universe [7, 13, 14, 16].

In Fig. 14.3, there is shown a numerical solution in dimensionless magnitudes (14.51) of the Schwarzschild equations of motion which begins in the state of zero energy (14.47) and zero radial velocity $P_I = 0$. It can be seen that the particle is being trapped in a bound state and this is true for all space velocities. In Figs. 14.3, 14.4, and 14.5, there are shown solutions of equations (14.46) following under the initial conditions

$$y(0) = 1, \quad \frac{dy}{dx}(0) = 0$$

and parameters

$$v_I = c_I, \quad w_I^2 = 0, 25c_I^2, \quad (c_I = 1, 0, 25, 0, 01).$$

In all the figures a trajectory starts from the point $(1, 0)$. It can be seen that a trajectory of a test object is removed at some distance from the starting point and then becomes a periodic (“capture” of an object) in both time and space (Fig.14.5). In decreasing velocities of particles, their trajectories gradually pass into the classical ellipses of the Kepler problem. Thus, the exact solution of a modified Kepler problem with Hamiltonian (14.50) and numerical solutions in the case of Hamiltonian (14.47) show that the cosmic evolution of mass reduces

the energy of a test particle (stars and galaxies). Cosmic evolution reduces the energy of free stars and galaxies causing them to form bound states such as galaxies or their clusters, respectively.

14.7 Quantum mechanics of a particle in Conformal cosmology

Let us consider a quantum mechanics of a particle in the Conformal cosmology, where the masses of elementary particles also become dynamic [4]

$$m(\eta) = m_0 \cdot \tilde{a}(\eta). \quad (14.54)$$

These masses determine the emission spectrum of atoms at time moment η ; their change $m'/m = a'/a \sim 10^{-42}$ GeV is significantly less than the energy levels of the atom for $\tilde{a}(\eta_0) = 1$ with quantum number k

$$E_k^0 = -\frac{m\alpha^2}{2k^2} \sim 10^{-8} \text{ GeV}, \quad (14.55)$$

being the eigenvalues of the Schrödinger stationary equation

$$\hat{\mathbf{E}}(p, x)\Psi^0 \equiv \left[\frac{\hat{p}^2}{2m_0} - \frac{\alpha}{r} \right] \Psi^0 = E_k^0 \Psi^0. \quad (14.56)$$

The spectrum of the hydrogen atom with a mass-dependent time (14.54), at any other instant $\eta = \eta_0 - r$ can be found by solving the quasistationary Schrödinger equation

$$\hat{\mathbf{E}}_c(p, x)\Psi \equiv \left[\frac{\hat{p}^2}{2m_0\tilde{a}(\eta)} - \frac{\alpha}{r} \right] \Psi = E_k(\eta)\Psi. \quad (14.57)$$

Its solution is the spectrum

$$E_k(\eta) = \tilde{a}(\eta)E_k^0, \quad (14.58)$$

where E_k^0 are the levels of the atom with constant mass (14.55). A rigorous derivation of (14.58) is based on the canonical transformation to the Friedmann variables [4, 5]

$$(p, x) \longrightarrow (P = p/a, X = xa),$$

and the non-stationary Schrödinger equation with the variable mass

$$\hat{\mathbf{E}}_c(p, x)\Psi_c = -i\frac{\partial}{\partial\eta}\Psi_c$$

results into the Schrödinger equation

$$\hat{\mathbf{E}}(P, X)\Psi = -i\frac{\partial}{\partial t}\Psi - H(t)PX\Psi \quad (14.59)$$

with a constant mass and an additional term disappearing at $H \rightarrow 0$, where $H(t)$ is the Hubble parameter.

From (14.58) follows the definition of the redshift $z(r)$

$$z(r) + 1 = \frac{E_k(\eta_0)}{E_k(\eta_0 - r)} = \frac{1}{\tilde{a}(\eta_0 - r)} \quad (14.60)$$

of the spectral lines on a space object at *coordinate* distance r from the Earth, relative to the spectral lines of the Earth atoms

$$E_k^0 = E_k(\eta_0)$$

when photons are being detected under the condition $\tilde{a}(\eta_0) = 1$.

14.8 Summary

In this Chapter, there were considered equations of dynamics of a test particle in a central gravitational field taking into account the evolution of the Universe and found the exact analytical solution for the Conformal cosmological model, compatible with the latest data on the Supernovae. These equations were used to describe the effect of the capture of a test particle in a gravitational field of the expanding Universe. It was shown that the capture effect can lead to the formation of galaxies and their clusters with anisotropic radial vector field of velocities. Such a velocity field could explain the anisotropy of the Hubble flow of velocities in the Local Group of galaxies observed by I. Karachentsev with the colleagues.

In the framework of the model the limits of applicability of the Newtonian approximation commonly used in the literature to describe the dark matter were assessed. The formula to describe the orbital speeds with the cosmological evolution of the Universe predicted by Einstein and Straus in 1945 [1] is obtained. According to this formula, the evolution of the Universe can imitate the effect of dark matter for Superclusters of galaxies.

Bibliography

- [1] Einstein, A., Straus, E.G.: *The influence of the expansion of space on the gravitation fields surrounding the individual stars*. Rev. Mod. Phys. **17**, 120 (1945)
- [2] Behnke, D., Blaschke, D.B., Pervushin, V.N., and Proskurin, D.V.: *Description of Supernova data in conformal cosmology without cosmological constant*. Phys. Lett. **B 530**, 20 (2002).
[arXiv: gr-qc/0102039]
- [3] Gusev, A., Flin, P., Pervushin, V., Vinitzky, S., Zorin, A.: *The Universe evolution as a possible mechanism of formation of galaxies and their clusters*. Astrophysics. **47**, 242 (2004).
[arXiv: astro-ph/0301543]
- [4] Barbashov, B.M., Zorin, A.G., Pervushin, V.N., Flin, P.: *Cosmic evolution of galaxies in relative units*. In: *Problems of Gauge Theories*. Barbashov, B.M., Nesterenko, V.V. (Eds). JINR. Dubna (2004)
- [5] Flin, P., Pervushin, V., Zorin, A.: *Capture of cosmic objects by central gravitational field of a galaxy cluster*.

[arXiv: astro-ph/0406051]

- [6] Peebles, P.J.E.: *The Large-Scale Structure of the Universe*. Princeton Series in Physics, Princeton University Press, Princeton, New Jersey (1980)
- [7] Gurevich, L.E., Chernin, A.D.: *Introduction into Cosmology*. Nauka, Moscow (1978)
- [8] Polyanin, A.D., Zaitsev, V.F.: *Handbook of Exact Solutions for Ordinary Differential Equations*. Chapman & Hall/CRC (2003)
- [9] Biernacka, M., Flin, P., Pervushin, V., Zorin, A.: *Newtonian motion as origin of anisotropy of the local velocity field of galaxies*. Part. & Nuclei Lett. 2[119], 64 (2004).
[arXiv: astro-ph/0206114]
- [10] Bajan, K., Flin, P., Pervushin, V.: *On the alignment of cosmic anisotropies*. In: Proc. Joint Intern. Conf. New Geometry of Nature. Kazan State University, Kazan, Russia, **III**, pp.9-11 (2003)
- [11] Karachentsev, I.D.: *Hidden mass in the Local Universe*. Physics – Uspekhi. **44**, 818 (2001)
- [12] Longair, Malcolm S.: *Galaxy Formation*. Springer. Berlin – Heidelberg – New York (2008)
- [13] Einasto, J., Kaasik, A., and Saar, E.: *Dynamic evidence on massive coronas of galaxies*. Nature. **250**, 309 (1974)

-
- [14] Einasto, J., Saar, E., Kaasik, A., and Chernin, A.D.: *Missing mass around galaxies: morphological evidence*. *Nature*. **252**, 111 (1974)
- [15] Rubin, V.C.: *Dark matter in spiral galaxies*. *Scientific American*, **248**, 96 (1983)
- [16] Primack, J.R.: *Status of cold dark matter cosmology*. Proceedings of 5th International UCLA Symposium on Sources and Detection of Dark Matter, Marina del Rey, (ed. D. Cline) (2002).
[arXiv: astro-ph/0205391]

Chapter 15

Afterword

15.1 Questions of Genesis

In the far away future descendants will certainly define our time as the time of great astrophysical discoveries comparable in importance to the era of great geographical discoveries of the late fifteenth and early sixteenth centuries. On unknown vastness of the Universe, scientists have discovered traces of hitherto unseen physical objects: neutron stars, quasars, pulsars, almost homogeneous Cosmic Microwave Background radiation with temperature about three kelvins, filling the entire visible Universe, and much more. Astrophysicists found a redshift of spectral lines of atoms that emit photons at distant space objects obeying the Hubble law: The farther the object – the greater the redshift. Modern researchers, as once brave explorers of past centuries, have realized that they can already reach the limits of the visible Universe, those distances that a light beam flies over the lifetime of the Universe. Astrophysicists can see space objects remote from us at distances of the

order of the size of the Universe, and thus can determine the dependence of large redshift of the distances comparable to the size of the visible part of the Universe. New data for large values of the redshift suggest that our Universe is filled with basically not massive dust of distant galaxies, but invisible and mysterious substance of completely different nature, with a different equation of state, called dark energy. The results of measuring of the distribution of chemical elements in the Universe indicate the predominance of the photons in the era of the primary chemical synthesis of elements and negligibly small contribution of the visible baryonic matter (about 3 %) to the cosmic evolution. On the other hand, the speed of rotation of stars in spiral galaxies and the speed of rotation of galaxies in all giant superclusters, according to Newtonian mechanics, shows that apart from the baryonic matter of which we are composed dark matter is present in galaxies, the mass of which is ten times larger the mass of the visible baryonic matter. As a result of these recent discoveries, the following most pressing *questions of the Universe* face the Standard cosmology:

1. How was our Universe created?
2. What had been in the Universe before its appearance?
3. What is the Universe made?
4. What is the nature of the dark energy and the dark matter?
5. Why did background radiation flash in the early Universe?
6. How was the matter created?

7. How can the baryon asymmetry of the Universe in which one baryon corresponds to billion photons be explained?
8. Is it possible to build a physical theory that would not only answer all these questions, but could predict the evolution of matter in the Universe, just as Newton's celestial mechanics was able not only to explain to his contemporaries problems of the Universe current at that time, but also to calculate the movement of the planets and predict the existence of new planets really discovered later?

Modern physicists have to answer these questions and to *explain* them on the basis of the first principles. According to Wigner, there are three levels of "*explanation*":

1. New empirical phenomena and mechanisms of the type of mechanism of inflation.
2. New laws of dynamics.
3. New additional symmetry principles of theories of gravitation and elementary particles.

Recall that in the Standard cosmology to explain the data on the Supernova there is a mechanism of inflationary expansion of the Universe, just the one that was proposed and developed by modern physicists to solve the problems of the Standard cosmology. However, the initial energy density of the inflation of the Universe differs 10^{57} times from the present energy density of the inflation. This huge difference

has not yet found a convincing explanation at the level of new laws of dynamics in the framework of the *dynamic model* of inflation with a single scalar field (inflanton). On the other hand, the mechanism of *primary inflation* absolutizes the current values of the constants of the Newton and Hubble parameter measured by contemporary observers, just as Ptolemy's system absolutizes the position and velocity of the Earth observer in celestial mechanics.

To explain the data on the Supernova, the authors of this book have preferred the third level, choosing as new principles of symmetry the groups of affine and conformal transformations and the corresponding conformal Dirac's variational principle.

15.2 General discussion of results

15.2.1 Results of the work

Let us briefly list the results and conclusions of this monograph. For the formulation of the problem of classification of data on physical measurements and astrophysical observations we trace the evolution of ideas and mathematical methods of theoretical physics in the last five centuries of its development from Copernicus' principle of relativity to Einstein's principles of relativity, Poincaré's group and gauge theories. Irreducible unitary representations of Poincaré's group underlie the classification of quantum relativistic particles and quantum field theory, which describes the creation of particles, their decays and interactions.

The book is devoted to the construction of the quantum wave function of the Universe as a joint *unitary irreducible representation of affine and conformal groups* with the inclusion of the Standard Model of electroweak interactions and QCD. This arrangement consists of the following stages.

1. Derivation of the theory of gravitation as joint nonlinear realization of *affine and conformal groups* (Ogievetsky and Borisov) in the tangent Fock space in terms of Cartan's forms.
2. Choice of conformal measurement standards (Dirac), which allow us to separate the cosmic evolution of the devices of observation from the evolution of cosmic objects.
3. Choice of the reference frame of the Universe (Markov and Yukawa).
4. Solution of constraints in the event space (Dirac).
5. Primary and secondary quantization with the postulate of the vacuum (Fock).
6. The definition of the initial data according to the uncertainty principle (Blokhintsev).
7. Diagonalization of the operators of creation and annihilation of particles and the Universe (Bogoliubov).

It is shown that such a construction leads to the following results.

1. The postulate of the vacuum gives the arrow of the proper time interval and its beginning as a quantum anomaly.

2. The source of violation of conformal symmetry and the origin of the masses of elementary particles is the only normal ordering of field operators leading to Casimir's energies and condensates.
3. Planck's quantum of action leads to a hierarchy of cosmological scales for the matter fields, in accordance with their conformal weights (n). These scales include the current value of Hubble's parameter ($n = 0$), temperature of the Cosmic Microwave Background radiation ($n = 2$), the mass scale of the electroweak bosons ($n = 3$), and the Planck mass ($n = 4$).
4. Observational data on the Supernova (1998 – 2013), recalculated for Dirac's conformal long space-time intervals testify that our Universe is cold and almost empty.
5. The wave function of the quantum Universe as a joint irreducible unitary representation of affine and conformal symmetry groups is factorized by the wave function of the empty Universe and the S-matrix used in high energy physics to describe the processes of creation and interaction of particles.
6. The resulting S-matrix corresponds to the quantization of gauge fields in the reduced phase space of the field variables. Thus, the Hamiltonian formulation is a basis for unification of the theory of the gravitational field with the Standard Model of elementary particles, in which both the theories are considered at the quantum level in a certain frame of reference after solving of constraint equations.

7. The physical consequences of reduced quantization for QED, QCD, SM and the theory of gravity were considered. The solution of the Gaussian constraint in QED coincides with Dirac's approach to the quantization of electrodynamics in 1927. It is shown that the quantization of massive vector field is consistent with the principles of axiomatic approach to quantum field theory, in particular, with the postulate of the existence of the vacuum.
8. By solving the Gaussian constraint for the non-Abelian theory, in particular QCD, the corresponding generating functional was obtained in terms of gauge-invariant observables of color fields including their bound states. The method of generating functional is compared with the standard Faddeev – Popov path integral one. The difference of the modified QCD from the standard Faddeev – Popov approach consists in that the initial data of gauge-invariant observables of color fields are topologically degenerated. Destructive interference of these phase factors of the topological degeneration leads to zero amplitudes of creation of all color states. Therefore, this destructive interference can be interpreted as a purely kinematic confinement of colored particles and conditions. As a result of the kinematic confinement, there arises a quark-hadron duality, widely used in high-energy QCD to describe the deep inelastic interactions, where free quarks and gluons are treated as free partons with the standard propagators.
9. The normal ordering of field operators is a source of the sponta-

neous symmetry breaking and the appearance of dimensional parameters such as local two-particle correlation functions (Casimir's condensates) of quarks and gluons. The reduced QCD gives a possibility to obtain relations of these condensates with the parameters of low-energy interactions of quarks and mesons. In particular, we obtain the Gell-Mann – Oakes – Renner relation.

10. In the Standard Model of electroweak interactions the topological degeneration of non-Abelian fields is removed by their interactions with the Higgs field. After the normal ordering of the electroweak bosons and fermions in SM there appear quantum anomalies as two-particle correlations (*id est* the Casimir condensates). Assuming the universality of the ratio of the Casimir condensates of fields to their masses in power of their conformal weight, we estimated the mass spectrum of these bosons. The obtained mass of the Higgs boson $\sim 130 \pm 15$ GeV is in agreement with recent experimental data.
11. The affine gravitons (in terms of linear Cartan's forms) are described by the free action in the approximation of zero Newtonian interaction. We consider some arguments testified that the affine gravitons can play a role of the dark matter in spiral galaxies.
12. It is shown that in the empty Universe there is a possibility of the intensive vacuum creation of only gravitons and Higgs particles (and the corresponding longitudinal components of the vector bosons). One of the main results is the creation from the vac-

uum of an order of 10^{88} of the Higgs boson particles and longitudinal components of the vector fields, decays of which form all the matter content of the Universe, in accordance with current observational facts. The values of the baryon density, the ratio of the number of photons to the number of baryons, and the CMB temperature are in agreement with observations.

13. The data on the Supernovae, primordial nucleosynthesis and cosmological particle creation, re-calculated to units of relative standard length, can be described by a single regime of vacuum dominance of the Casimir energy.
14. On the basis of the developed Hamiltonian method the theory of cosmological perturbations is formulated. The Schwarzschild-type solutions are obtained with Jeans-like spatial oscillations.
15. Cosmic evolution of the masses leads to the capture of cosmic objects by a central field and provides a mechanism of formation of galaxies and clusters of galaxies; in this case there is a class of essentially ellipsoidal trajectories of galaxies. The reality of existence of this type of trajectories is confirmed by recent observations of the anisotropy of the Hubble flow velocities in the Local Group of galaxies by I.D. Karachentsev's group. Finally, the chaos of freely moving particles is organized by a cosmic motion in the observational structures of matter.
16. According to the cosmological modified Newtonian dynamics, the square of Newton's velocity of the galaxy in the COMA-type

superclusters is replaced by the sum of squares of two velocities (Hubble and Newton ones). Therefore, violation of the virial theorem found in clusters of galaxies and interpreted as evidence for the existence of dark matter, in the Conformal cosmology, can be explained by the Hubble velocities of these galaxies, without the dark matter halo, as was predicted by Einstein and Strauss in 1945 yet.

Thus, on the basis of the principles of the quantum Universe, formulated before 1974, it is possible to explain different properties of the world.

15.2.2 Discussion

One can send, imaginatively, an observer to the beginning of the Universe, just as Copernicus put his observer, imaginatively, on the Sun. Our observer knows that there was the beginning, and at the beginning there was an empty Universe. The observer finds that in this Universe there is only the dilaton zero mode and Casimir's energy. A method of measuring the dilaton is the redshift. There are systems of reference, co-moving to an empty element of the space, where the initial data of the dilaton are constants of motion. These constants of motion define its position (the present value of Newton's constant) and speed (the present value of the Hubble parameter). The principles of symmetry, of which the action of Dirac is derived, and the postulate of vacuum lead to a hierarchy of cosmological scales and the wave function of the Universe. The wave function of the Universe allows us to unify mod-

ern observational data on the Supernova with the latest experimental value of the Higgs particle mass.

The level of mathematical description of the nature, claimed in modern physics, formulates a lot of other questions. How can the affine group extend to include all fields of the Standard Model in the Goldstone fields? Why does the world have these symmetries, and not others? Why is there such a precise fit of the initial data and the dimensionless coupling constants under the anthropic principle?

Maybe in the future someone will find the answers to these questions in harmony of the very principles of symmetry, like Copernicus and Kepler found their answers to the questions of the Universe in harmony of epicycles, and Einstein, Weyl and other researchers in the 20th century found their answers in harmony of the laws of nature.

Steven Weinberg fatefully wrote in the Introduction of his book¹ about the future of the Standard Cosmological model: *“Can we really be sure of the standard model? Will new discoveries overthrow it and replace the present standard model with some other cosmogony, or even revive the steady-state model? Perhaps. I cannot deny a feeling of unreality in writing about the thirst three minutes as if we really know what we are talking about”*.

In this book, we gave theoretical and observational arguments in favour of Steven Weinberg’s predictions. New present-day discoveries, in particular, the recent LHC experimental data on a small value of the Higgs particle mass confirm the conformal symmetry principle on

¹Weinberg, S.: *The First Three Minutes: A Modern View of the Origin of the Universe*. Basic Books, New York (1977).

which the Standard Model of elementary particles is based. The origin of elementary particle masses is explained by the quantum anomaly of the vacuum postulate, not by the phenomenological Higgs potential. The same quantum anomaly in the form of the vacuum Casimir energy explained the present-day Supernovae observational data without the lambda-term in the framework of the Conformal cosmology.

Appendix A

Reduced Abelian field theory

A.1 Reduced QED

A.1.1 Action and frame of reference

Let us recall the Dirac approach to QED [1, 2, 3]. The theory is given by the well-known action

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} [i\rlap{\not{D}} - m] \psi + A_\mu j^\mu \right), \quad (\text{A.1})$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is tension, A_μ is a vector potential, ψ is the Dirac electron-positron bispinor field, and $j_\mu = e\bar{\psi}\gamma_\mu\psi$ is the charge current, $\rlap{\not{D}} \equiv \partial^\mu\gamma_\mu$. This action is invariant with respect to the collection of gauge transformations

$$A_\mu^\lambda = A_\mu + \partial_\mu\lambda, \quad \psi^\lambda = e^{+ie\lambda}\psi. \quad (\text{A.2})$$

The variational principle used for the action (A.1) gives the Euler – Lagrange equations of motion known as the Maxwell equations

$$\partial_\nu F^{\mu\nu} + j^\mu = 0, \quad (\text{A.3})$$

Physical solutions of the Maxwell equations are obtained in a fixed *inertial reference frame* distinguished by a unit time-like vector n_μ . This vector splits the gauge field A_μ into the time-like $A_0 = A_\mu n_\mu$ and space-like

$$A_\nu^\perp = A_\nu - n_\nu(A_\mu n_\mu)$$

components. Now we rewrite the Maxwell equations in terms of components

$$\Delta A_0 - \partial_0 \partial_k A_k = j_0, \quad (\text{A.4})$$

$$\square A_k - \partial_k [\partial_0 A_0 - \partial_i A_i] = -j_k. \quad (\text{A.5})$$

The field component A_0 cannot be a *degree of freedom* because its canonical conjugate momentum vanishes. The Gauss constraints (A.4) have the solution

$$A_0 + \partial_0 \Lambda = -\frac{1}{4\pi} \int d^3 y \frac{j_0(x_0, y_k)}{|\mathbf{x} - \mathbf{y}|}, \quad (\text{A.6})$$

where

$$\Lambda = -\frac{1}{\Delta} \partial_k A_k = \frac{1}{4\pi} \int d^3 y \frac{\partial_k A_k}{|\mathbf{x} - \mathbf{y}|} \quad (\text{A.7})$$

is a longitudinal component. The result (A.6) is treated as the *Coulomb potential field* leading to the *static* interaction.

A.1.2 Elimination of time component

Dirac [1] proposed to eliminate the time component by substituting the manifest resolution of the Gauss constraints given by (A.6) into the initial action (A.1). This substitution – known as the reduction procedure – allows us to eliminate nonphysical pure gauge degrees of freedom. After this step the action (A.1) takes the form

$$S = \int d^4x \left(\frac{1}{2} (\partial_\mu A_k^\text{T})^2 + \bar{\psi} [i\rlap{\not{\partial}} - m] \psi - j_0 \partial_0 \Lambda - A_k^\text{T} j_k + \frac{1}{2} j_0 \frac{1}{\Delta} j_0 \right), \quad (\text{A.8})$$

where

$$A_k^\text{T} = \left(\delta_{ij} - \frac{\partial_i \partial_j}{\Delta} \right) A_j. \quad (\text{A.9})$$

This substitution leaves the longitudinal component Λ given by Eq. (A.7) without any kinetic term.

There are two possibilities. The first one is to treat Λ as the Lagrange factor that leads to the conservation law (A.3). In this approach, the longitudinal component is treated as an independent variable. This treatment violates gauge invariance because this component is gauge-variant and it cannot be measurable. Moreover, the time derivative of the longitudinal component in Eq. (A.6) looks like a physical source of the Coulomb potential. By these reasons we will not consider this approach in this paper.

In the second possibility, a measurable potential stress is identified with the gauge-invariant quantity (A.6)

$$A_0^\text{R} = A_0 - \frac{\partial_0 \partial_k}{\Delta} A_k, \quad (\text{A.10})$$

This approach is consistent with the principle of gauge invariance that identifies observables with gauge-invariant quantities. Therefore, ac-

According to the gauge-invariance, the longitudinal component should be eliminated from the set of degrees of freedom of QED, too.

A.1.3 Elimination of longitudinal component

This elimination is fulfilled by the choice of the “radiation variables” as gauge invariant functionals of the initial fields, *id est* “dressed fields” [1]

$$A_\mu^{\text{R}} = A_\mu + \partial_\mu \Lambda, \quad \psi^{\text{R}} = e^{ie\Lambda} \psi, \quad (\text{A.11})$$

In this case, the linear term $\partial_k A_k$ disappears in the Gauss law (A.4)

$$\Delta A_0^{\text{R}} = j_0^{\text{R}} \equiv e\bar{\psi}^{\text{R}} \gamma_0 \psi^{\text{R}}. \quad (\text{A.12})$$

The source of the gauge-invariant *potential field* A_0^{R} can be only an electric current j_0^{R} whereas the spatial components of the vector field A_k^{R} coincide with the transversal one

$$\partial_k A_k^{\text{R}} = \partial_k A_k^{\text{T}} \equiv 0. \quad (\text{A.13})$$

In this manner the frame-fixing $A_\mu = (A_0, A_k)$ is compatible with understanding of A_0 as a classical field, and the use of the Dirac dressed fields (A.11) of the Gauss constraints (A.4) leads to understanding of the variables (A.11) as gauge-invariant functionals of the initial fields.

A.1.4 Static interaction

Substitution of the manifest resolution of the Gauss constraints (A.4) into the initial action (A.1) calculated on constraints leads to that the

initial action can be expressed in terms of the gauge-invariant radiation variables (A.11) [1, 3]

$$S = \int d^4x \left(\frac{1}{2} (\partial_\mu A_k^R)^2 + \bar{\psi}^R [i\not{\partial} - m] \psi^R - A_k^R j_k^R + \frac{1}{2} j_0^R \frac{1}{\Delta} j_0^R \right). \quad (\text{A.14})$$

The Hamiltonian, which corresponds to this action, has the form

$$\begin{aligned} \mathcal{H} = & \frac{(\Pi_k^R)^2 + (\partial_j A_k^R)^2}{2} + p_\psi^R \gamma_0 [i\gamma_k \partial_k + m] \psi^R + \\ & + A_k^R j_k^R - \frac{1}{2} j_0^R \frac{1}{\Delta} j_0^R, \end{aligned} \quad (\text{A.15})$$

where Π_k^R, p_ψ^R are the canonical conjugate momentum fields of the theory calculated in a standard way. Hence, the vacuum can be defined as a state with minimal energy obtained as the value of the Hamiltonian for the equations of motion. Relativistic covariant transformations of the gauge-invariant fields are proved at the level of the fundamental operator quantization in the form of the Poincaré algebra generators [4]. The status of the theorem of equivalence between the Dirac radiation variables and the Lorentz gauge formulation is considered in [5, 6, 7, 8].

A.1.5 Comparison of radiation variables with the Lorentz gauge ones

The static interaction and the corresponding bound states are lost in any frame free formulation including the Lorentz gauge one. The action (A.8) transforms into

$$S = \int d^4x \left(-\frac{1}{2} (\partial_\mu A_\nu^L)^2 + \bar{\psi}^L [i\not{\partial} - m] \psi^L + A_\mu^L j^{L\mu} \right), \quad (\text{A.16})$$

where

$$A_\mu^L = A_\mu + \partial_\mu \Lambda^L, \quad \psi^L = e^{ie\Lambda^L} \psi, \quad \Lambda^L = -\frac{1}{\square} \partial^\mu A_\mu^L \quad (\text{A.17})$$

are the manifest gauge-invariant functionals satisfying the equations of motion

$$\square A_\mu^L = -j_\mu^L, \quad (\text{A.18})$$

with the current

$$j_\mu^L = -e\bar{\psi}^L \gamma_\mu \psi^L$$

and the gauge constraints

$$\partial_\mu A^{L\mu} \equiv 0. \quad (\text{A.19})$$

Really, instead of the potential, satisfying the Gauss constraints

$$\Delta A_0^R = j_0^R,$$

and two transverse variables in QED in terms of the radiation variables (A.11) we have here three independent dynamic variables, one of which A_0^L satisfies the equation

$$\square A_0^L = -j_0, \quad (\text{A.20})$$

and gives a negative contribution to the energy.

We can see that there are two distinctions of the ‘‘Lorentz gauge formulation’’ from the radiation variables. The first is the loss of Coulomb poles (*id est* static interactions). The second is the treatment of the time component A_0 as an independent variable with the negative contribution to the energy; therefore, in this case, the vacuum as the

state with the minimal energy is absent. In other words, one can say that the static interaction is the consequence of the vacuum postulate, too. The inequivalence between the radiation variables and the Lorentz ones does not mean violation of the gauge invariance, because both the variables can be defined as the gauge-invariant functionals of the initial gauge fields (A.11) and (A.17).

In order to demonstrate the inequivalence between the radiation variables and the Lorentz ones, let us consider the electron-positron scattering amplitude

$$T^R = \langle e^+, e^- | \hat{S} | e^+, e^- \rangle.$$

One can see that the Feynman rules in the radiation gauge give the amplitude in terms of the current $j_\nu = \bar{e}\gamma_\nu e$

$$\begin{aligned} T^R &= \frac{j_0^2}{\mathbf{q}^2} + \left(\delta_{ik} - \frac{q_i q_k}{\mathbf{q}^2} \right) \frac{j_i j_k}{q^2 + \nu\varepsilon} \\ &\equiv \frac{-j^2}{q^2 + \nu\varepsilon} + \frac{(q_0 j_0)^2 - (\mathbf{q} \cdot \mathbf{j})^2}{\mathbf{q}^2 [q^2 + \nu\varepsilon]}. \end{aligned} \quad (\text{A.21})$$

This amplitude coincides with the Lorentz gauge one

$$T^L = -\frac{1}{q^2 + \nu\varepsilon} \left[j^2 - \frac{(q_0 j_0 - \mathbf{q} \cdot \mathbf{j})^2}{q^2 + \nu\varepsilon} \right] \quad (\text{A.22})$$

when the box terms in Eq. (A.21) can be eliminated. Thus, the Faddeev equivalence theorem [6, 7] is valid if the currents are conserved

$$q_0 j_0 - \mathbf{q} \cdot \mathbf{j} = qj = 0, \quad (\text{A.23})$$

However, for the action with the external sources the currents are not conserved. Instead of the classical conservation laws we have the Ward

– Takahashi identities for Green functions, where the currents are not conserved

$$q_0 j_0 - \mathbf{q} \cdot \mathbf{j} \neq 0. \quad (\text{A.24})$$

In particular, the Lorentz gauge perturbation theory (where the propagator has only the light cone singularity $q_\mu q^\mu = 0$) cannot describe instantaneous Coulomb atoms; this perturbation theory contains only the Wick – Cutkosky bound states whose spectrum is not observed in the Nature.

Thus, we can give a response to the question: What are new physical results that follow from the Hamiltonian approach to QED in comparison with the frame-free Lorentz gauge formulation? In the framework of the perturbation theory, the Hamiltonian presentation of QED contains the static Coulomb interaction (A.21) forming instantaneous bound states observed in the Nature, whereas all frame free formulations lose this static interaction together with instantaneous bound states in the lowest order of perturbation theory on retarded interactions called the radiation correction. Nobody has proved that the sum of these retarded radiation corrections with the light-cone singularity propagators (A.22) can restore the Coulomb interaction that was removed from propagators (A.21) by hand on the level of the action.

A.2 Reduced vector boson theory

A.2.1 Lagrangian and reference frame

The classical Lagrangian of massive QED is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2V_\mu^2 + \bar{\psi}(i\rlap{\not{\partial}} - m)\psi + V_\mu j^\mu, \quad (\text{A.25})$$

In a fixed reference frame this Lagrangian takes the form

$$\begin{aligned} \mathcal{L} = & \frac{(\dot{V}_k - \partial_k V_0)^2 - (\partial_j V_k^T)^2 + M^2(V_0^2 - V_k^2)}{2} + \\ & + \bar{\psi}(i\rlap{\not{\partial}} - m)\psi + V_0 j_0 - V_k j_k, \end{aligned} \quad (\text{A.26})$$

where $\dot{V} = \partial_0 V$ and V_k^T is the transverse component defined by the action of the projection operator given in Eq. (A.9). In contrast to QED this action is not invariant with respect to gauge transformations. Nevertheless, from the Hamiltonian viewpoint the massive theory has the same problem as QED. The time component of the massive boson has a vanishing canonical momentum.

A.2.2 Elimination of time component

In [8], one supposed to eliminate the time component from the set of degrees of freedom like the Dirac approach to QED, *id est*, using the action principle. In the massive case it produces the equation of motion

$$(\Delta - M^2)V_0 = \partial_i \dot{V}_i + j_0. \quad (\text{A.27})$$

which is understood as constraints and has the solution

$$V_0 = \left(\frac{1}{\Delta - M^2} \partial_i V_i \right) + \frac{1}{\Delta - M^2} j_0. \quad (\text{A.28})$$

In order to eliminate the time component, let us insert (A.28) into the Lagrangian (A.26) [1, 8]

$$\begin{aligned}
\mathcal{L} = & \frac{1}{2} \left[(\dot{V}_k^{\text{T}})^2 + V_k^{\text{T}} (\Delta - M^2) V_k^{\text{T}} + j_0 \frac{1}{\Delta - M^2} j_0 \right] + \\
& + \bar{\psi} (\not{\partial} - m) \psi - V_k^{\text{T}} j_k + \\
& + \frac{1}{2} \left[\dot{V}_k^{\parallel} M^2 \frac{1}{\Delta - M^2} \dot{V}_k^{\parallel} - M^2 (V_k^{\parallel})^2 \right] - V_k^{\parallel} j_k + \\
& + j_0 \frac{1}{\Delta - M^2} \partial_k \dot{V}_k^{\parallel},
\end{aligned} \tag{A.29}$$

where we decomposed the vector field

$$V_k = V_k^{\text{T}} + V_k^{\parallel}$$

by means of the projection operator by analogy with (A.9). The last two terms are the contributions of the longitudinal component only. This Lagrangian contains the longitudinal component which is the dynamical variable described by the bilinear term. Now we propose the following transformation:

$$\begin{aligned}
& \bar{\psi} (\not{\partial} - m) \psi - V_k^{\parallel} j_k + j_0 \frac{1}{\Delta - M^2} \partial_k \dot{V}_k^{\parallel} = \\
& = \bar{\psi}^{\text{R}} (\not{\partial} - m) \psi^{\text{R}} - V_k^{\text{R}\parallel} j_k,
\end{aligned} \tag{A.30}$$

where

$$V_k^{\text{R}\parallel} = V_k^{\parallel} - \partial_k \frac{1}{\Delta - M^2} \partial_i V_i = -M^2 \frac{1}{\Delta - M^2} V_k^{\parallel}, \tag{A.31}$$

$$\psi^{\text{R}} = \exp \left\{ -i e \frac{1}{\Delta - M^2} \partial_i V_i \right\} \psi \tag{A.32}$$

are the radiation-type variables. It removes the linear term $\partial_i \dot{V}_i$ in the Gauss law (A.27). If the mass $M \neq 0$, one can pass from the initial variables V_k^{\parallel} to the radiation ones $V_k^{\text{R}\parallel}$ by the change

$$V_k^{\parallel} = \hat{Z} V_k^{\text{R}\parallel}, \quad \hat{Z} = \frac{M^2 - \Delta}{M^2} \quad (\text{A.33})$$

Now the Lagrangian (A.29) goes into

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \left[(\dot{V}_k^{\text{T}})^2 + V_k^{\text{T}}(\Delta - M^2)V_k^{\text{T}} + j_0 \frac{1}{\Delta - M^2} j_0 \right] + \bar{\psi}^{\text{R}}(\not{\partial} - m)\psi^{\text{R}} + \\ & + \frac{1}{2} \left[\dot{V}_k^{\text{R}\parallel} \hat{Z} \dot{V}_k^{\text{R}\parallel} + V_k^{\text{R}\parallel}(\Delta - M^2)\hat{Z} V_k^{\text{R}\parallel} \right] - V_k^{\text{T}} j_k - V_k^{\text{R}\parallel} j_k. \end{aligned} \quad (\text{A.34})$$

The Hamiltonian corresponding to this Lagrangian can be constructed in the standard canonical way. Using the rules of the Legendre transformation and canonical conjugate momenta

$$\Pi_{V_k^{\text{T}}}, \quad \Pi_{V_k^{\text{R}\parallel}}, \quad \Pi_{\psi^{\text{R}}}$$

we obtain

$$\begin{aligned} \mathcal{H} = & \frac{1}{2} \left[\Pi_{V_k^{\text{T}}}^2 + V_k^{\text{T}}(M^2 - \Delta)V_k^{\text{T}} + j_0 \frac{1}{M^2 - \Delta} j_0 \right] - \\ & - \Pi_{\psi^{\text{R}}} \gamma_0 (\not{\partial} + m) \psi^{\text{R}} \quad (\text{A.35}) \\ & + \frac{1}{2} \left[\Pi_{V_k^{\text{R}\parallel}} \hat{Z}^{-1} \Pi_{V_k^{\text{R}\parallel}} + V_k^{\text{R}\parallel} (M^2 - \Delta) \hat{Z} V_k^{\text{R}\parallel} \right] + \\ & + V_k^{\text{T}} j_k + V_k^{\text{R}\parallel} j_k. \end{aligned}$$

One can be convinced [8] that the corresponding quantum system has a vacuum as a state with minimal energy and correct relativistic transformation properties.

A.2.3 Quantization

We start the quantization procedure from the canonical quantization by using the following equal time canonical commutation relations (ETCCRs):

$$\left[\hat{\Pi}_{V_k^T}, \hat{V}_k^T \right] = i\delta_{ij}^T \delta^3(\mathbf{x} - \mathbf{y}), \quad (\text{A.36})$$

$$\left[\hat{\Pi}_{V_k^{\text{R}\parallel}}, \hat{V}_k^{\text{R}\parallel} \right] = i\delta_{ij}^{\parallel} \delta^3(\mathbf{x} - \mathbf{y}). \quad (\text{A.37})$$

The Fock space of the theory is built by the ETCCRs

$$\left[a_{(\lambda)}^-(\pm k), a_{(\lambda')}^+(\pm k') \right] = \delta^3(\mathbf{k} - \mathbf{k}') \delta_{(\lambda)(\lambda')}; \quad (\text{A.38})$$

$$\{ b_{\alpha}^-(\pm k), b_{\alpha'}^+(\pm k') \} = \delta^3(\mathbf{k} - \mathbf{k}') \delta_{\alpha\alpha'}; \quad (\text{A.39})$$

$$\{ c_{\alpha}^-(\pm k), c_{\alpha'}^+(\pm k') \} = \delta^3(\mathbf{k} - \mathbf{k}') \delta_{\alpha\alpha'}. \quad (\text{A.40})$$

with the vacuum state $|0\rangle$ defined by the relations

$$a_{(\lambda)}^-|0\rangle = b_{\alpha}^-|0\rangle = c_{\alpha}^-|0\rangle = 0. \quad (\text{A.41})$$

The field operators have the Fourier decompositions in the plane wave basis

$$\begin{aligned} V_j(x) &= \int [d\mathbf{k}]_v \epsilon_j^{(\lambda)} \left[a_{(\lambda)}^+(\omega, \mathbf{k}) e^{-i\omega t + i\mathbf{k}\mathbf{x}} + a_{(\lambda)}^-(\omega, -\mathbf{k}) e^{i\omega t - i\mathbf{k}\mathbf{x}} \right] \\ \psi(x) &= \sqrt{2m_s} \int [d\mathbf{k}]_s \left[b_{\alpha}^+(k) u_{\alpha} e^{-i\omega t + i\mathbf{k}\mathbf{x}} + c_{\alpha}^-(k) \nu_{\alpha} e^{i\omega t - i\mathbf{k}\mathbf{x}} \right] \\ \psi^+(x) &= \sqrt{2m_s} \int [d\mathbf{k}]_s \left[b_{\alpha}^-(k) u_{\alpha}^+ e^{i\omega t - i\mathbf{k}\mathbf{x}} + c_{\alpha}^+(-k) \nu_{\alpha}^+ e^{-i\omega t + i\mathbf{k}\mathbf{x}} \right] \end{aligned}$$

with the integral measure

$$[d\mathbf{k}]_{v,s} = \frac{1}{(2\pi)^{3/2}} \frac{d^3\mathbf{k}}{\sqrt{2\omega_{v,s}(\mathbf{k})}}$$

and the frequency of oscillations

$$\omega_{v,s}(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m_{v,s}^2}.$$

One can define the vacuum expectation values of the instantaneous products of the field operators

$$V_i(t, \vec{x})V_j(t, \vec{y}) =: V_i(t, \vec{x})V_j(t, \vec{y}) : + \langle V_i(t, \vec{x})V_j(t, \vec{y}) \rangle, \quad (\text{A.42})$$

$$\bar{\psi}_\alpha(t, \vec{x})\psi_\beta(t, \vec{y}) =: \bar{\psi}_\alpha(t, \vec{x})\psi_\beta(t, \vec{y}) : + \langle \bar{\psi}_\alpha(t, \vec{x})\psi_\beta(t, \vec{y}) \rangle, \quad (\text{A.43})$$

where

$$\langle V_i(t, \vec{x})V_j(t, \vec{y}) \rangle = \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{2\omega_v(\mathbf{k})} \sum_{(\lambda)} \epsilon_i^{(\lambda)} \epsilon_j^{(\lambda)} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}, \quad (\text{A.44})$$

$$\langle \bar{\psi}_\alpha(t, \vec{x})\psi_\beta(t, \vec{y}) \rangle = \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{2\omega_s(\mathbf{k})} (\mathbf{k}\vec{\gamma} + m)_{\alpha\beta} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} \quad (\text{A.45})$$

are the Pauli – Jordan functions.

A.2.4 Propagators and condensates

The vector field in the Lagrangian (A.34) is given by the formula

$$V_i^R = \left[\delta_{ij}^T + \hat{Z}^{-1} \delta_{ij}^{\parallel} \right] V_j = V_i^T + \hat{Z}^{-1} V_i^{\parallel}. \quad (\text{A.46})$$

Hence, the propagator of the massive vector field in radiative variables is

$$D_{ij}^R(x-y) = \langle 0|TV_i^R(x)V_j^R(y)|0\rangle = \quad (\text{A.47})$$

$$= -i \int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq \cdot (x-y)}}{q^2 - M^2 + i\epsilon} \left(\delta_{ij} - \frac{q_i q_j}{\mathbf{q}^2 + M^2} \right).$$

Together with the instantaneous interaction described by the current-current term in the Lagrangian (A.34) this propagator leads to the amplitude

$$T^R = D_{\mu\nu}^R(q) \tilde{j}^\mu \tilde{j}^\nu = \frac{\tilde{j}_0^2}{\mathbf{q}^2 + M^2} + \left(\delta_{ij} - \frac{q_i q_j}{\mathbf{q}^2 + M^2} \right) \frac{\tilde{j}_i \tilde{j}_j}{q^2 - M^2 + i\epsilon} \tag{A.48}$$

of the current-current interaction which differs from the acceptable one

$$T^L = \tilde{j}^\mu D_{\mu\nu}^L(q) \tilde{j}^\nu = -\tilde{j}^\mu \frac{g_{\mu\nu} - \frac{q_\mu q_\nu}{M^2}}{q^2 - M^2 + i\epsilon} \tilde{j}^\nu. \tag{A.49}$$

The amplitude given by Eq. (A.48) is the generalization of the radiation amplitude in QED. As it was shown in [8], the Lorentz transformations of classical radiation variables coincide with the quantum ones and they both (quantum and classical) correspond to the transition to another Lorentz frame of reference distinguished by another time-axis, where the relativistic covariant propagator takes the form

$$D_{\mu\nu}^R(q|n) = \frac{-g_{\mu\nu}}{q^2 - M^2 + i\epsilon} + \frac{n_\mu n_\nu (qn)^2 - [q_\mu - n_\mu(qn)][q_\nu - n_\nu(qn)]}{(q^2 - M^2 + i\epsilon)(M^2 + |q_\mu - n_\mu(qn)|^2)}, \tag{A.50}$$

where n_μ is determined by the external states. Remember that the conventional local field massive vector propagator takes the form (A.49)

$$D_{\mu\nu}^L(q) = -\frac{g_{\mu\nu} - \frac{q_\mu q_\nu}{M^2}}{q^2 - M^2 + i\epsilon}. \tag{A.51}$$

In contrast to this conventional massive vector propagator the radiation-type propagator (A.50) is regular in the limit $M \rightarrow 0$ and is well behaved for large momenta, whereas the propagator (A.51) is singular. The radiation amplitude (A.48) can be rewritten in the alternative form

$$T^{\text{R}} = -\frac{1}{q^2 - M^2 + i\epsilon} \left[\tilde{j}_\nu^2 + \frac{(\tilde{j}_i q_i)^2 - (\tilde{j}_0 q_0)^2}{q^2 + M^2} \right], \quad (\text{A.52})$$

for comparison with the conventional amplitude defined by the propagator (A.51). One can find that for a massive vector field coupled to a conserved current ($q_\mu \tilde{j}^\mu = 0$) the collective current-current interactions mediated by the radiation propagator (A.50) and by the conventional propagator (A.51) coincide

$$\tilde{j}^\mu D_{\mu\nu}^{\text{R}} \tilde{j}^\nu = \tilde{j}^\mu D_{\mu\nu}^{\text{L}} \tilde{j}^\nu = T^{\text{L}}. \quad (\text{A.53})$$

If the current is not conserved

$$\tilde{j}_0 q_0 \neq \tilde{j}_k q_k,$$

the collective radiation field variables with the propagator (A.50) are inequivalent to the initial local variables with the propagator (A.51) and the amplitude (A.48). The amplitude (A.53) in the Feynman gauge is

$$T^{\text{L}} = -\frac{j^2}{q^2 - M^2 + i\epsilon}, \quad (\text{A.54})$$

and corresponds to the Lagrangian

$$\mathcal{L}_F = \frac{1}{2}(\partial_\mu V_\mu)^2 - j_\mu V_\mu + \frac{1}{2}M^2 V_\mu^2 \quad (\text{A.55})$$

In this theory the time component has a negative contribution to the energy. According to this, a correctly defined vacuum state could not exist. Nevertheless, the vacuum expectation value

$$\langle V_\mu(x)V_\mu(x) \rangle$$

coincides with the values for two propagators (A.50) and (A.51) because in both these propagators the longitudinal part does not give a contribution if one treats them as derivatives of constant like

$$\langle \partial V_\mu(x)V_\mu(x) \rangle = \partial \langle V_\mu(x)V_\mu(x) \rangle = 0.$$

In this case, we have

$$\langle V_\mu(x)V_\mu(x) \rangle = -\frac{2}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{\omega_v(\mathbf{k})}, \quad (\text{A.56})$$

$$\langle \bar{\psi}_\alpha(x)\psi_\alpha(x) \rangle = -\frac{m_s}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{\omega_s(\mathbf{k})}, \quad (\text{A.57})$$

where m_s , M_v are masses of the spinor and vector fields.

Bibliography

- [1] Dirac, P.A.M.: *The quantum theory of the emission and absorption of radiation*. Proc. Roy. Soc. London. **A 114**, 243 (1927).
Dirac, P.A.M.: *Gauge-invariant formulation of quantum electrodynamics*. Can. J. Phys. **33**, 650 (1955)
- [2] Heisenberg, W., Pauli, W.: *Zur quantendynamik der wellenfelder*. Z. Phys. **56**, 1 (1929).
Heisenberg, W., Pauli, W.: *Zur quantentheorie der wellenfelder. II*. Z. Phys. **59**, 168 (1930)
- [3] Polubarinov, I.V.: *Equations of quantum electrodynamics*. Phys. Part. & Nuclei. **34**, 741 (2003).
- [4] Schwinger, J.: *Non-abelian gauge fields. Relativistic invariance*. Phys. Rev. **127**, 324 (1962).
- [5] Pervushin, V.N.: *Dirac variables in gauge theories*. Phys. Part. & Nuclei. **34**, 679 (2003)
- [6] Faddeev, L.D., Popov, V.N.: *Feynman diagrams for the Yang – Mills field*. Phys. Lett. **B 25**, 29 (1967)

-
- [7] Faddeev, L.D.: *Feynman's integral for singular Lagrangians*.
Theor. and Math. Phys. **1**, 3 (1969)
- [8] Pavel, H.P., Pervushin, V.N.: *Reduced phase-space quantization
of massive vector theory*. Int. J. Mod. Phys. **A 14**, 2885 (1999).
[arXiv: hep-th/9706220]

Appendix B

Quantum field theory for bound states

B.1 Ladder approximation

The generating functional of quantum field theory for bound states can be presented by means of the relativistic generalization of the Hubbard – Stratonovich (HS) transformation [1, 2]. The Hubbard – Stratonovich transformation is an exact mathematical transformation

$$\exp[-ax^2/2] = [2\pi a]^{-1/2} \int_{-\infty}^{+\infty} dy \exp[-ixy - y^2/(2a)]. \quad (\text{B.1})$$

The basic idea of the HS transformation is to reformulate a system of particles interacting through two-body potentials (10.18) in the theory (10.19) into a system of independent particles interacting with a bilocal auxiliary field $\mathcal{M}^{\text{ab}}(x, y)$. The HS transformation was invented by the Russian physicist Ruslan L. Stratonovich and popularized by the

British physicist John Hubbard.

$$Z_\psi = \int d\psi d\bar{\psi} e^{\iota W_{\text{instant}}[\psi\bar{\psi}] + \iota S[J^*, \eta^*, \bar{\eta}^*]} = \quad (\text{B.2})$$

$$= \int d\psi d\bar{\psi} e^{-\frac{1}{2}(\psi\bar{\psi}, \mathcal{K}\psi\bar{\psi}) - (\psi\bar{\psi}, G_0^{-1}) + \iota S[J^*, \eta^*]} = \quad (\text{B.3})$$

$$= \int \left[\prod_{x,y,a,b} d\mathcal{M}^{\text{ab}}(x, y) \right] \exp\{\iota W_{\text{eff}}[\mathcal{M}] + \iota(\eta\bar{\eta}, G_{\mathcal{M}})\}. \quad (\text{B.4})$$

The effective action in Eq. (B.4) can be decomposed in the form

$$W_{\text{eff}}[\mathcal{M}] = -\frac{1}{2}N_c(\mathcal{M}, \mathcal{K}^{-1}\mathcal{M}) + \iota N_c \text{Tr} \ln(1 + \Phi), \quad (\text{B.5})$$

$$\text{Tr} \ln(1 + \Phi) = \sum_{n=1}^{\infty} \frac{1}{n} \Phi^n. \quad (\text{B.6})$$

Here $\Phi \equiv G_0\mathcal{M}$, Φ^2 , Φ^3 , *etc.* mean the following expressions

$$\begin{aligned} \Phi(x, y) &\equiv G_0\mathcal{M} = \int d^4z G_0(x, z)\mathcal{M}(z, y), \\ \Phi^2 &= \int d^4x d^4y \Phi(x, y)\Phi(y, x), \\ \Phi^3 &= \int d^4x d^4y d^4z \Phi(x, y)\Phi(y, z)\Phi(z, x), \text{ etc.} \end{aligned} \quad (\text{B.7})$$

The first step to the semi-classical quantization of this construction [1] is the determination of its minimum of the effective action

$$N_c^{-1} \frac{\delta W_{\text{eff}}(\mathcal{M})}{\delta \mathcal{M}} \equiv -\mathcal{K}^{-1}\mathcal{M} + \frac{\iota}{G_0^{-1} - \mathcal{M}} = 0. \quad (\text{B.8})$$

This equation is known as the Schwinger – Dyson one. We denote the corresponding classical solution for the bilocal field by $\Sigma(x - y)$. It depends only on the difference $x - y$ at $A^* = 0$ because of translation invariance of vacuum solutions.

The next step is the expansion of the effective action around the point of minimum $\mathcal{M} = \Sigma + \mathcal{M}'$,

$$W_{eff}(\Sigma + \mathcal{M}') = W_{eff}^{(2)} + W_{int}; \quad (\text{B.9})$$

$$W_{eff}^{(2)}(\mathcal{M}') = W_Q(\Sigma) + N_c \left[-\frac{1}{2} \mathcal{M}' \mathcal{K}^{-1} \mathcal{M}' + \frac{i}{2} (G_\Sigma \mathcal{M}')^2 \right], \quad (\text{B.10})$$

$$W_{int} = \sum_{n=3}^{\infty} W^{(n)} = i N_c \sum_{n=3}^{\infty} \frac{1}{n} (G_\Sigma \mathcal{M}')^n, \quad (\text{B.11})$$

$$G_\Sigma = (G_0^{-1} - \Sigma)^{-1}. \quad (\text{B.12})$$

The bilocal function $\mathcal{M}'(x, y)$ in terms of the Jacobi – type variables

$$z = x - y, \quad X = \frac{x + y}{2}$$

can be decomposed over the complete set of orthonormalized solutions Γ of the classical equation

$$\frac{\delta^2 W_{eff}(\Sigma + \mathcal{M}')}{\delta \mathcal{M}'^2} \cdot \Gamma = 0. \quad (\text{B.13})$$

This series takes the form:

$$\begin{aligned} \mathcal{M}'(x, y) &= \mathcal{M}'(z|X) = \\ &= \sum_H \int \frac{d^3 \mathcal{P}}{(2\pi)^3 \sqrt{2\omega_H}} \int \frac{d^4 q e^{iq \cdot z}}{(2\pi)^4} \times \\ &\times \left[e^{i\mathcal{P} \cdot X} \Gamma_H(q^\perp | \mathcal{P}) a_H^+(\mathcal{P}) + e^{-i\mathcal{P} \cdot X} \bar{\Gamma}_H(q^\perp | \mathcal{P}) a_H^-(\mathcal{P}) \right], \end{aligned} \quad (\text{B.14})$$

with a set of quantum numbers (H) including masses

$$M_H = \sqrt{\mathcal{P}_\mu^2}$$

and energies

$$\omega_H = \sqrt{\mathcal{P}^2 + M_H^2}.$$

The bound state creation and annihilation operators obey the commutation relations

$$\left[a_{H'}^-(\mathcal{P}'), a_H^+(\mathcal{P}) \right] = \delta_{H'H} \delta^3(\mathcal{P}' - \mathcal{P}) . \quad (\text{B.15})$$

The corresponding Green function takes the form

$$\begin{aligned} \mathcal{G}(q^\perp, p^\perp | \mathcal{P}) &= \quad (\text{B.16}) \\ &= \sum_H \left(\frac{\Gamma_H(q^\perp | \mathcal{P}) \bar{\Gamma}_H(p^\perp | -\mathcal{P})}{(\mathcal{P}_0 - \omega_H - i\varepsilon) 2\omega_H} - \frac{\bar{\Gamma}_H(p^\perp | \mathcal{P}) \Gamma_H(p^\perp | -\mathcal{P})}{(\mathcal{P}_0 - \omega_H - i\varepsilon) 2\omega_H} \right) . \end{aligned}$$

To normalize vertex functions Γ , we can use the "free" part of the effective action (B.10) for the quantum bilocal meson \mathcal{M}' with the commutation relations (B.15). The substitution of the off-shell

$$\sqrt{\mathcal{P}^2} \neq M_H$$

decomposition (9.13) into the "free" part of effective action defines the reverse Green function of the bilocal field $\mathcal{G}(\mathcal{P}_0)$

$$W_{eff}^{(0)}[\mathcal{M}] = 2\pi \delta(\mathcal{P}_0 - \mathcal{P}'_0) \sum_H \int \frac{d^3\mathcal{P}}{\sqrt{2\omega_H}} a_H^+(\mathcal{P}) a_H^-(\mathcal{P}) \mathcal{G}_H^{-1}(\mathcal{P}_0) \quad (\text{B.17})$$

where $\mathcal{G}_H^{-1}(\mathcal{P}_0)$ is the reverse Green function which can be represented as a difference of two terms

$$\mathcal{G}_H^{-1}(\mathcal{P}_0) = I(\sqrt{\mathcal{P}^2}) - I(M_H^{\text{ab}}(\omega)) \quad (\text{B.18})$$

where $M_H^{ab}(\omega)$ is the eigenvalue of the equation for small fluctuations (B.11) and

$$I(\sqrt{\mathcal{P}^2}) = iN_c \int \frac{d^4q}{(2\pi)^4} \times \text{tr} \left[G_{\Sigma_b} \left(q - \frac{\mathcal{P}}{2} \right) \bar{\Gamma}_{ab}^H(q^\perp | - \mathcal{P}) G_{\Sigma_a} \left(q + \frac{\mathcal{P}}{2} \right) \Gamma_{ab}^H(q^\perp | \mathcal{P}) \right],$$

where

$$\underline{G}_\Sigma(q) = \frac{1}{\not{q} - \underline{\Sigma}(q^\perp)}, \quad \underline{\Sigma}(q) = \int d^4x \Sigma(x) e^{iqx} \quad (\text{B.19})$$

is the fermion Green function. The normalization condition is defined by the formula

$$2\omega = \frac{\partial \mathcal{G}^{-1}(\mathcal{P}_0)}{\partial \mathcal{P}_0} \Big|_{\mathcal{P}_0 = \omega(\mathcal{P}_1)} = \frac{dM(\mathcal{P}_0)}{d\mathcal{P}_0} \frac{dI(M)}{dM} \Big|_{\mathcal{P}_0 = \omega}. \quad (\text{B.20})$$

Finally, we get that solutions of equation (B.13) satisfy the normalization condition [3]

$$iN_c \frac{d}{d\mathcal{P}_0} \int \frac{d^4q}{(2\pi)^4} \text{tr} \left[\underline{G}_\Sigma \left(q - \frac{\mathcal{P}}{2} \right) \bar{\Gamma}_H(q^\perp | - \mathcal{P}) \underline{G}_\Sigma \left(q + \frac{\mathcal{P}}{2} \right) \Gamma_H(q^\perp | \mathcal{P}) \right] = 2\omega_H. \quad (\text{B.21})$$

The achievement of the relativistic covariant constraint-shell quantization of gauge theories is the description of both the spectrum of bound states and their S-matrix elements.

It is convenient to write the relativistic-invariant matrix elements for the action (B.9) in terms of the field operator

$$\Phi'(x, y) = \int d^4x_1 G_\Sigma(x - x_1) \mathcal{M}'(x_1, y) = \Phi'(z|X).$$

Using the decomposition over the bound state quantum numbers (H)

$$\Phi'(z|X) = \sum_H \int \frac{d^3\mathcal{P}}{(2\pi)^{3/2}\sqrt{2\omega_H}} \int \frac{d^4q}{(2\pi)^4} \times \quad (\text{B.22})$$

$$(e^{i\mathcal{P}\cdot X} \Phi_H(q^\perp|\mathcal{P}) a_H^+(\mathcal{P}) + e^{-i\mathcal{P}\cdot X} \bar{\Phi}_H(q^\perp|-\mathcal{P}) a_H^-(\mathcal{P})) ,$$

where

$$\Phi_{H(\text{ab})}(q^\perp|\mathcal{P}) = G_{\Sigma a}(q + \mathcal{P}/2) \Gamma_{H(\text{ab})}(q^\perp|\mathcal{P}) , \quad (\text{B.23})$$

we can write the matrix elements $W^{(n)}$ (B.10) for the interaction between the vacuum and the n -bound state [4]

$$\langle H_1\mathcal{P}_1, \dots, H_n\mathcal{P}_n | iW^{(n)} | 0 \rangle =$$

$$= -i(2\pi)^4 \delta^4 \left(\sum_{i=1}^n \mathcal{P}_i \right) \prod_{j=1}^n \left[\frac{1}{(2\pi)^3 2\omega_j} \right]^{1/2} M^{(n)}(\mathcal{P}_1, \dots, \mathcal{P}_n), \quad (\text{B.24})$$

$$M^{(n)} = \int \frac{i d^4q}{(2\pi)^4 n} \sum_{\{i_k\}} \Phi_{H_{i_1}}^{\text{a}_1, \text{a}_2}(q|\mathcal{P}_{i_1}) \times$$

$$\Phi_{H_{i_2}}^{\text{a}_2, \text{a}_3} \left(q - \frac{\mathcal{P}_{i_1} + \mathcal{P}_{i_2}}{2} | \mathcal{P}_{i_2} \right) \Phi_{H_{i_3}}^{\text{a}_3, \text{a}_4} \left(q - \frac{2\mathcal{P}_{i_2} + \mathcal{P}_{i_1} + \mathcal{P}_{i_3}}{2} | \mathcal{P}_{i_3} \right) \times$$

$$\dots \Phi_{H_{i_n}}^{\text{a}_n, \text{a}_1} \left(q - \frac{2(\mathcal{P}_{i_2} + \dots + \mathcal{P}_{i_{n-1}}) + \mathcal{P}_{i_1} + \mathcal{P}_{i_n}}{2} | \mathcal{P}_{i_n} \right), \quad (\text{B.25})$$

where $\{i_k\}$ denotes permutations over i_k .

Expressions (B.16), and (B.25) represent Feynman rules for the construction of a quantum field theory with the action (B.10) in terms of bilocal fields.

B.2 Bethe – Salpeter Equations

Equations for the spectrum of the bound states (B.13) can be rewritten in the form of the Bethe – Salpeter (BS) one [5, 6]

$$\Gamma = i\mathcal{K}(x, y) \int d^4 z_1 d^4 z_2 G_\Sigma(x - z_1) \Gamma(z_1, z_2) G_\Sigma(z_2 - y) . \quad (\text{B.26})$$

In the momentum space with

$$\underline{\Gamma}(q|\mathcal{P}) = \int d^4 x d^4 y e^{i\frac{x+y}{2}\mathcal{P}} e^{i(x-y)q} \Gamma(x, y)$$

the Coulomb type kernel we obtain the following equation for the vertex function ($\underline{\Gamma}$):

$$\begin{aligned} \underline{\Gamma}(k, \mathcal{P}) &= \quad \quad \quad (\text{B.27}) \\ &= i \int \frac{d^4 q}{(2\pi)^4} \underline{V}(k^\perp - q^\perp) \not\ell \left[\underline{G}_\Sigma \left(q + \frac{\mathcal{P}}{2} \right) \Gamma(q|\mathcal{P}) \underline{G}_\Sigma \left(q - \frac{\mathcal{P}}{2} \right) \right] \not\ell \end{aligned}$$

where $\underline{V}(k^\perp)$ means the Fourier transform of the potential,

$$k_\mu^\perp = k_\mu - \ell_\mu(k \cdot \ell)$$

is the relative momentum transversal with respect to ℓ_μ , and \mathcal{P}_μ is the total momentum.

The quantity $\underline{\Gamma}$ depends only on the transversal momentum

$$\underline{\Gamma}(k|\mathcal{P}) = \underline{\Gamma}(k^\perp|\mathcal{P}),$$

because of the instantaneous form of the potential $\underline{V}(k^\perp)$ in any frame. The Bethe – Salpeter equation (B.26) for potential independent of the longitudinal momentum allows to make integration over it which at

the rest frame is equal to q_0^1 . We consider the Bethe – Salpeter equation (B.27) after integration over the longitudinal momentum. The vertex function takes the form

$$\Gamma_{ab}(k^\perp|\mathcal{P}) = \int \frac{d^3q^\perp}{(2\pi)^3} \underline{V}(k^\perp - q^\perp) \not{\ell} \Psi_{ab}(q^\perp) \not{\ell}, \quad (\text{B.28})$$

where the bound state wave function Ψ_{ab} is given by

$$\begin{aligned} \Psi_{ab}(q^\perp) &= \quad (\text{B.29}) \\ &= \not{\ell} \left[\frac{\bar{\Lambda}_{(+)\text{a}}(q^\perp \Gamma_{ab}(q^\perp|\mathcal{P}) \Lambda_{(-)\text{b}}(q^\perp)}{E_T - \sqrt{\mathcal{P}^2} + i\epsilon} + \frac{\bar{\Lambda}_{(-)\text{a}}(q^\perp \Gamma_{ab}(q^\perp|\mathcal{P}) \Lambda_{(+)\text{b}}(q^\perp)}{E_T + \sqrt{\mathcal{P}^2} - i\epsilon} \right] \not{\ell}. \end{aligned}$$

Here the sum of one-particle energies of the two particles (a) and (b)

$$E_T = E_a + E_b$$

defined by (10.27) and the notation (10.26)

$$\bar{\Lambda}_{(\pm)}(q^\perp) = S^{-1}(q^\perp) \Lambda_{(\pm)}(0) S(q^\perp) = \Lambda_{(\pm)}(-q^\perp). \quad (\text{B.30})$$

has been introduced.

Acting with the operators (B.30) on equation (B.28) one gets the equations for the wave function ψ in an arbitrary moving reference frame

$$\begin{aligned} (E_T(k^\perp) \mp \sqrt{\mathcal{P}^2}) \Lambda_{(\pm)\text{a}}^{(\ell)}(k^\perp) \Psi_{ab}(k^\perp) \Lambda_{(\mp)\text{b}}^{(\ell)}(-k^\perp) &= \quad (\text{B.31}) \\ = \Lambda_{(\pm)\text{a}}^{(\ell)}(k^\perp) \int \frac{d^3q^\perp}{(2\pi)^3} \underline{V}(k^\perp - q^\perp) \Psi_{ab}(q^\perp) \Lambda_{(\mp)\text{b}}^{(\ell)}(-k^\perp). \end{aligned}$$

¹This integral has poles of the product of two Green functions of the parton-quarks (or leptons in QED)

$$\frac{i}{2\pi} \int dq^0 \frac{1}{(q_0 - a - i\epsilon)(q_0 + b + i\epsilon)} = \frac{1}{a + b}.$$

All these equations (B.28) and (B.31) have been derived without any assumption about the smallness of the relative momentum $|k^\perp|$ and for an arbitrary total momentum

$$\mathcal{P}_\mu = \left(\sqrt{M_A^2 + \mathcal{P}^2}, \quad \mathcal{P} \neq 0 \right). \quad (\text{B.32})$$

We expand the function Ψ on the projection operators

$$\Psi = \Psi_+ + \Psi_-, \quad \Psi_\pm = \Lambda_\pm^{(\ell)} \Psi \Lambda_\mp^{(\ell)}. \quad (\text{B.33})$$

According to Eq. (B.29), Ψ satisfies the identities

$$\Lambda_+^{(\ell)} \Psi \Lambda_+^{(\ell)} = \Lambda_-^{(\ell)} \Psi \Lambda_-^{(\ell)} \equiv 0, \quad (\text{B.34})$$

which permit the determination of unambiguous expansion of Ψ in terms of the Lorentz structures:

$$\Psi_{\mathbf{a},\mathbf{b}\pm} = S_{\mathbf{a}}^{-1} \left(\gamma_5 L_{\mathbf{a},\mathbf{b}\pm}(q^\perp) + (\gamma_\mu - \ell_\mu \ell) N_{\mathbf{a},\mathbf{b}\pm}^\mu \right) \Lambda_\mp^{(\ell)}(0) S_{\mathbf{b}}^{-1}, \quad (\text{B.35})$$

where

$$L_\pm = L_1 \pm L_2, \quad N_\pm = N_1 \pm N_2.$$

In the rest frame $\ell_\mu = (1, 0, 0, 0)$ we get

$$N^\mu = (0, N^i); \quad N^i(q) = \sum_{a=1,2} N_a(q) e_a^i(q) + \Sigma(q) \hat{q}^i.$$

The wave functions L, N^α, Σ satisfy the following equations.

1. Pseudoscalar particles.

$$M_L \overset{0}{L}_2(\mathbf{p}) = E \overset{0}{L}_1(\mathbf{p}) - \int \frac{d^3q}{(2\pi)^3} V(\mathbf{p} - \mathbf{q}) (c_p^- c_q^- - \xi \mathbf{s}_p^- \mathbf{s}_q^-) \overset{0}{L}_1(\mathbf{q});$$

$$M_L \overset{0}{L}_1(\mathbf{p}) = E \overset{0}{L}_2(\mathbf{p}) - \int \frac{d^3q}{(2\pi)^3} V(\mathbf{p}-\mathbf{q})(\mathbf{c}_p^+ \mathbf{c}_q^+ - \xi \mathbf{s}_p^+ \mathbf{s}_q^+) \overset{0}{L}_2(\mathbf{q}) .$$

Here, in all equations, we use the following definitions

$$E(\mathbf{p}) = E_a(\mathbf{p}) + E_b(\mathbf{p}) , \quad (\text{B.36})$$

$$\mathbf{c}_p^\pm = \cos[v_a(\mathbf{p}) \pm v_b(\mathbf{p})] , \quad (\text{B.37})$$

$$\mathbf{s}_p^\pm = \sin[v_a(\mathbf{p}) \pm v_b(\mathbf{p})] , \quad (\text{B.38})$$

$$\xi = \hat{p}_i \cdot \hat{q}_i , \quad (\text{B.39})$$

where E_a, E_b are one-particle energies and v_a, v_b are the Foldy – Wouthuysen angles of particles (a,b) given by Eqs. (10.27) and (10.28).

2. Vector particles.

$$M_N \overset{0}{N}_2^\alpha = E \overset{0}{N}_1^{\alpha-} - \int \frac{d^3q}{(2\pi)^3} V(\mathbf{p}-\mathbf{q}) \{ (\mathbf{c}_p^- \mathbf{c}_q^- \underline{\delta}^{\alpha\beta} + \mathbf{s}_p^- \mathbf{s}_q^- (\underline{\delta}^{\alpha\beta} \xi - \eta^\alpha \underline{\eta}^\beta)) \overset{0}{N}_1^\beta + (\eta^\alpha \mathbf{c}_p^- \mathbf{c}_q^+) \overset{0}{\Sigma}_1 \} ;$$

$$M_N \overset{0}{N}_1^\alpha = E \overset{0}{N}_2^{\alpha-} - \int \frac{d^3q}{(2\pi)^3} V(\mathbf{p}-\mathbf{q}) \{ (\mathbf{c}_p^+ \mathbf{c}_q^+ \underline{\delta}^{\alpha\beta} + \mathbf{s}_p^+ \mathbf{s}_q^+ (\underline{\delta}^{\alpha\beta} \xi - \eta^\alpha \underline{\eta}^\beta)) \overset{0}{N}_2^\beta + (\eta^\alpha \mathbf{c}_p^+ \mathbf{c}_q^-) \overset{0}{\Sigma}_2 \} .$$

$$\eta^\alpha = \hat{q}_i \hat{e}_i^\alpha(p),$$

$$\underline{\eta}^\alpha = \hat{p}_i \hat{e}_i^\alpha(q),$$

$$\underline{\delta}^{\alpha\beta} = \hat{e}_i^\alpha(q) \hat{e}_i^\beta(p).$$

3. Scalar particles.

$$\begin{aligned}
& M_{\Sigma} \overset{0}{\Sigma}_2 = E \overset{0}{\Sigma}_1 - \\
& - \int \frac{d^3q}{(2\pi)^3} V(\mathbf{p} - \mathbf{q}) \{ (\xi \mathbf{c}_p^+ \mathbf{c}_q^+ + \mathbf{s}_p^+ \mathbf{s}_q^+) \overset{0}{\Sigma}_1 + (\underline{\eta}^\beta \mathbf{c}_p^- \mathbf{c}_q^+) \overset{0}{N}_1^\beta \} ; \\
& M_{\Sigma} \overset{0}{\Sigma}_1 = E \overset{0}{\Sigma}_2 - \\
& - \int \frac{d^3q}{(2\pi)^3} V(\mathbf{p} - \mathbf{q}) \{ (\xi \mathbf{c}_p^- \mathbf{c}_q^- + \mathbf{s}_p^- \mathbf{s}_q^-) \overset{0}{\Sigma}_2 + (\underline{\eta}^\beta \mathbf{c}_p^+ \mathbf{c}_q^-) \overset{0}{N}_2^\beta \} .
\end{aligned}$$

The normalization of these solutions is uniquely determined by equation (B.21)

$$\frac{2N_c}{M_L} \int \frac{d^3q}{(2\pi)^3} \{ L_1(\mathbf{q}) L_2^*(\mathbf{q}) + L_2(\mathbf{q}) L_1^*(\mathbf{q}) \} = 1 , \quad (\text{B.40})$$

$$\frac{2N_c}{M_N} \int \frac{d^3q}{(2\pi)^3} \{ N_1^\mu(\mathbf{q}) N_2^{\mu*}(\mathbf{q}) + N_2^\mu(\mathbf{q}) N_1^{\mu*}(\mathbf{q}) \} = 1 , \quad (\text{B.41})$$

$$\frac{2N_c}{M_{\Sigma}} \int \frac{d^3q}{(2\pi)^3} \{ \Sigma_1(\mathbf{q}) \Sigma_2^*(\mathbf{q}) + \Sigma_2(\mathbf{q}) \Sigma_1^*(\mathbf{q}) \} = 1 . \quad (\text{B.42})$$

If the atom is at rest ($\mathcal{P}_\mu = (M_A, 0, 0, 0)$) equation (B.31) coincides with the Salpeter equation [6]. If one assumes that the current mass m^0 is much larger than the relative momentum, then the coupled equations (B.28) and (B.31) turn into the Schrödinger equation. In the rest frame ($\mathcal{P}_0 = M_A$) equation (10.27) for a large mass ($m^0/|q^\perp| \rightarrow \infty$) describes a nonrelativistic particle

$$\begin{aligned}
E_a(\mathbf{k}) &= \sqrt{(m_a^0)^2 + \mathbf{k}^2} \simeq m_a^0 + \frac{1}{2} \frac{\mathbf{k}^2}{m_a^0}, \\
\tan 2v &= \frac{k}{m^0} \rightarrow 0; \quad S(\mathbf{k}) \simeq 1; \quad \Lambda_{(\pm)} \simeq \frac{1 \pm \gamma_0}{2}.
\end{aligned}$$

Then, in equation (B.31) only the state with positive energy remains

$$\Psi_P^{\alpha\beta} \simeq \Psi_{(+)}^{\alpha\beta} = [\Lambda_{(+)} \gamma_5]^{\alpha\beta} \sqrt{4\mu} \psi_{\text{Sch}}, \quad \Lambda_{(-)} \Psi_P^{\alpha\beta} \Lambda_{(+)} \simeq 0, \quad (\text{B.43})$$

where

$$\mu \equiv \frac{m_a \cdot m_b}{(m_a + m_b)}.$$

And finally the Schrödinger equation results in

$$\begin{aligned} \left[\frac{1}{2\mu} \mathbf{k}^2 + (m_a^0 + m_b^0 - M_A) \right] \psi_{\text{Sch}}(\mathbf{k}) &= \\ &= \int \frac{d^3 q}{(2\pi)^3} V(\mathbf{k} - \mathbf{q}) \psi_{\text{Sch}}(\mathbf{q}), \end{aligned} \quad (\text{B.44})$$

with the normalization

$$\frac{1}{(2\pi)^3} \int d^3 q |\psi_{\text{Sch}}|^2 = 1.$$

For an arbitrary total momentum \mathcal{P}_μ (B.32) equation (B.44) takes the form

$$\begin{aligned} \left[-\frac{1}{2\mu} (k_\nu^\perp)^{-2} + (m_a^0 + m_b^0 - \sqrt{\mathcal{P}^2}) \right] \psi_{\text{Sch}}(k^\perp) &= \\ &= \int \frac{d^3 q^\perp}{(2\pi)^3} V(k^\perp - q^\perp) \psi_{\text{Sch}}(q^\perp), \end{aligned} \quad (\text{B.45})$$

where

$$k_\mu^\perp = k_\mu - \frac{\mathcal{P}\mathbf{k}}{M_H^2} \mathcal{P}_\mu,$$

and describes a relativistic atom with nonrelativistic relative momentum $|k^\perp| \ll m_{a,b}^0$. In the framework of such a derivation of the Schrödinger equation it is sufficient to define the total coordinate as $X = (x + y)/2$, independently of the magnitude of the masses of the two particles forming an atom.

In particular, the Coulomb interaction leads to a positronium at

rest with the bilocal wave function (B.43)

$$\Psi_P^{\alpha\beta}(\mathbf{z}) = \left(\frac{1 + \gamma_0}{2} \gamma_5 \right)^{\alpha\beta} \underline{\psi}_{\text{Sch}}(\mathbf{z}) \sqrt{\frac{m_e}{2}}, \quad (\text{B.46})$$

$$\underline{\psi}_{\text{Sch}}(\mathbf{z}) = \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{z}} \psi_{\text{Sch}}(\mathbf{z}); \quad (\text{B.47})$$

where $\underline{\psi}_{\text{Sch}}(\mathbf{z})$ is the Schrödinger normalizable wave function of the relative motion

$$\left(-\frac{1}{m_e} \frac{d^2}{d\mathbf{z}^2} - \frac{\alpha}{|\mathbf{z}|} \right) \underline{\psi}_{\text{Sch}}(\mathbf{z}) = \epsilon \underline{\psi}_{\text{Sch}}(\mathbf{z}) \quad (\text{B.48})$$

with the normalization

$$\int d^3 z \|\underline{\psi}_{\text{Sch}}(\mathbf{z})\|^2 = 1$$

where $M_P = (2m_e - \epsilon)$ is the mass of a positronium, $(1 + \gamma_0)/2$ is the projection operator on the state with positive energies of an electron and positron.

Bibliography

- [1] Ebert, D., Pervushin, V.N.: *Dynamical breakdown of chiral symmetry and abnormal perturbation expansions*. Problems of Gauge Theories. Barbashov, B.M., Nesterenko, V.V. (Eds.) JINR D-2-2004-66, Dubna, 62 (2004).
[<http://thsun1.jinr.ru/pervush/doc/bnp-v5e.pdf>]
- [2] Pervushin, V.N., Reinhardt, H., Ebert, D.: *Continual integral in collective variables and its application to nuclear and hadron physics*. Part. & Nuclei. **10**, 1114 (1979)
- [3] Nakanishi, N.: *A general survey of the theory of the Bethe–Salpeter equation*. Suppl. Prog. Theor. Phys. **43**, 1 (1969)
- [4] Kalinovsky, Yu.L., Kallies, W., Kuranov, B.N., Pervushin, V.N., and Sarikov, N.A.: *Bilocal meson lagrangians and potential model*. Phys. Part. & Nuclei. **49**, 1709 (1989)
- [5] Salpeter, E.E.: *Mass corrections to the fine structure of hydrogen-like atoms*. Phys. Rev. **87**, 328 (1952)
- [6] Salpeter, E.E., Bethe, H.A.: *A relativistic equation for bound-state problem*. Phys. Rev. **84**, 1232 (1951)

Appendix C

Abel – Plana formula

Only a few series in mathematics can be calculated in an exact form. Therefore, it is very important to express the sums of series in terms of contour integrals. One of the popular methods is based on the following theorem [1].

Let a function $f(z)$ be holomorphic in the strip $a < \Re z < b$ and satisfy the inequality

$$|f(x + iy)| \leq M e^{a|y|}, \quad a < 2\pi. \quad (\text{C.1})$$

Then at $k \geq a + 1, n \leq b - 1, n > k$, and for any $0 < \theta < 1$

$$\begin{aligned} \sum_{s=k}^n f(s) &= \int_{k+\theta-1}^{n+\theta} f(x) dx + \\ &+ \frac{1}{2i} \int_0^{\theta+i\infty} [f(n+z) - f(k-1+z)](\cot \pi z + i) dz + \\ &+ \frac{1}{2i} \int_0^{\theta-i\infty} [f(k-1+z) - f(n+z)](\cot \pi z - i) dz. \end{aligned} \quad (\text{C.2})$$

Proof.

We denote by C_h a rectangle

$$k - 1 + \theta < \Re z < n + \theta, \quad |\Im z| < h,$$

which in view of the conditions on k and n is in the band $a < \Re z < b$, and by J an integral of $f(z) \cot \pi z$ along C_h . According to the residue theorem, we have for $z = s$:

$$J = 2\pi i \sum_k^n \operatorname{res} f(z) \cot \pi z = 2i \sum_k^n f(s).$$

We denote by C_h^+ the upper half of C_h , and by C_h^- the lower half C_h , and the direction of C_h^+ and C_h^- we assume to be the direction of a point $z = k - 1 + \theta$ to a point $z = n + \theta$. Then we have

$$J = \int_{C_h^-} f(z) \cot \pi z dz - \int_{C_h^+} f(z) \cot \pi z dz$$

and

$$\begin{aligned} J &= \int_{C_h^-} f(z)(\cot \pi z - i) dz + i \int_{C_h^-} f(z) dz - \\ &\quad - \int_{C_h^+} f(z)(\cot \pi z + i) dz + i \int_{C_h^+} f(z) dz. \end{aligned}$$

The integral of $f(z)$ depends only on the ends of the loop, so the integrals of $f(z)$ along C_h^+ and C_h^- can be replaced by the integral over

the interval $(k - 1 + \theta, n + \theta)$. Hence,

$$J = 2i \int_{k-1+\theta}^{n+\theta} f(x) dx + \int_{C_h^-} f(z)(\cot \pi z - i) dz - \int_{C_h^+} f(z)(\cot \pi z + i) dz.$$

Further,

$$\begin{aligned} & \int_{C_h^+} f(z)(\cot \pi z + i) dz = \\ &= \int_0^{\theta+ih} [f(k-1+z) - f(n+z)](\cot \pi z + i) dz + \\ &+ \int_{h-1+\theta+ih}^{n+\theta+ih} f(z)(\cot \pi z + i) dz. \end{aligned}$$

Since for $k - 1 + \theta \leq x \leq n + \theta$

$$\begin{aligned} & \left| \int_{k-1+\theta+ih}^{n+\theta+ih} f(z)(\cot \pi z + i) dz \right| \leq \\ & \leq (n - k + 1) |f(x + ih)| |\cot \pi(x + ih) + i|. \end{aligned}$$

Since

$$|\cot \pi(x + ih) + i| < \frac{2}{e^{2\pi h} - 1}$$

at $h > 0$, then, in force (C.1), we have at $h \rightarrow +\infty$:

$$\left| \int_{k-1+\theta+ih}^{n+\theta+ih} f(z)(\cot \pi z + i) dz \right| \leq (n - k + 1) M e^{ah} \frac{2}{e^{2\pi h} - 1} \rightarrow 0$$

and

$$\begin{aligned} \int_{C_h^+} (\cot \pi z + \imath) dz &= \\ &= \int_{\theta}^{\theta+\imath\infty} [f(k-1+z) - f(n+z)](\cot \pi z + \imath) dz. \end{aligned}$$

Analogously, for the integral C_h^- . Comparing the two expressions derived for J , we arrive at the formula (C.2).

Corollary.

Let a function $f(z)$ be holomorphic in the half-plane $\Re z > 0$ and satisfy the inequality

$$|f(x + iy)| < \varepsilon(x)e^{a|y|}, \quad 0 < a < 2\pi,$$

where $\varepsilon(x) \rightarrow 0$ under $x \rightarrow +\infty$. Then for any $0 < \theta < 1$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\sum_1^n f(s) - \int_{\theta}^{n+\theta} f(x) dx \right) &= \tag{C.3} \\ &= \frac{1}{2\imath} \int_{\theta}^{\theta-\imath\infty} f(z)(\cot \pi z - \imath) dz - \frac{1}{2\imath} \int_{\theta}^{\theta+\imath\infty} f(z)(\cot \pi z + \imath) dz. \end{aligned}$$

This formula is called the *Abel – Plana formula*. It results from (C.2), so that

$$\left| \int_{\theta}^{\theta \pm \imath\infty} f(n+z)(\cot \pi z \mp \imath) dz \right| \leq \varepsilon(n+\theta) \int_0^{\infty} M e^{-(2\pi-a)y} dy \rightarrow 0$$

as $n \rightarrow \infty$.

In quantum field theory an infinite number of degrees of freedom leads to zero vacuum fluctuations that give a divergent contribution

to the physical values. To calculate the quantum values of the energy-momentum tensor the Abel – Plana formula (C.3) reduces to the form [2]:

$$\sum_{n=0}^{\infty} F(n) = \int_0^{\infty} F(x) dx + \frac{1}{2}F(0) + i \int_0^{\infty} \frac{F(it) - F(-it)}{\exp(2\pi t) - 1} dt. \quad (\text{C.4})$$

The first term on the right-hand side is the energy-momentum tensor of unlimited space. Regularization is reduced to its subtraction. For the regularized sum of the divergent series (C.4) we obtain the formula [3]:

$$\text{reg} \sum_{n=0}^{\infty} F(n) = \frac{1}{2}F(0) + i \int_0^{\infty} \frac{F(it) - F(-it)}{\exp(2\pi t) - 1} dt. \quad (\text{C.5})$$

A modification of the derivation of formula (C.4) allows one to get a regularization analogous to (C.5) for divergent series in which the sum is over a half-integer values of the argument [3]:

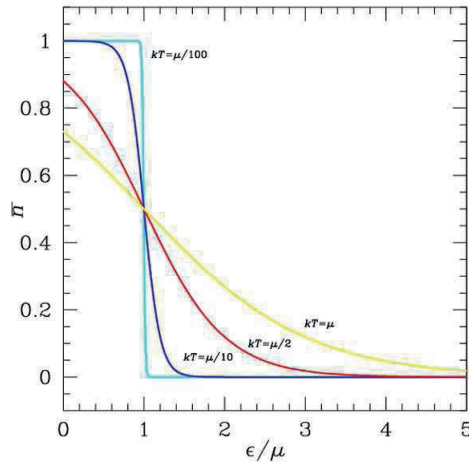
$$\text{reg} \sum_{n=0}^{\infty} F\left(n + \frac{1}{2}\right) = -i \int_0^{\infty} \frac{F(it) - F(-it)}{\exp(2\pi t) + 1} dt. \quad (\text{C.6})$$

This formula is used for carrying out calculations with a fermion field.

The Casimir effect consists of a polarization of the vacuum of quantized fields which arises as a result of a change in the spectrum of vacuum oscillations. Calculations of the effect for manifolds of various configurations and for fields with various spins with using the Abel–Plana formulae (C.5), and (C.6) for the regularization are presented in paper [4]. The distribution functions for bosons and fermions over energy

$$f_{\mp}(\epsilon) = \frac{1}{\exp(\epsilon - \mu)/k_B T \mp 1}$$

are similar to the expressions under the integral signs in (C.5) and (C.6). See, for example, a family of curves $f_+(\epsilon)$ in Fig.16.1.



A family of curves $f_+(\epsilon)$ describes the Fermi–Dirac distribution of identical fermions over single-particle energy states, where no more than one fermion can occupy a state. Using the Fermi–Dirac distribution, one can find the distribution of identical fermions over energy, where more than one fermion can have the same energy.

Bibliography

- [1] Evgrafov, M.A.: *Analytic Functions*. Dover Publications. (1978)
- [2] Bateman, H., Erdélyi, A.: *High Transcendental Functions*. McGraw–Hill Book Co. (1953)
- [3] Grib, A.A., Mamaev, S.G., Mostepanenko, V.M.: *Vacuum Quantum Effects in Strong Fields*. Friedmann Lab. Publ. (1994)
- [4] Mostepanenko, V.M., Trunov, N.N.: *The Casimir effect and its applications*. *Physics–Uspekhi*. **31**, 965 (1988)

Appendix D

Functional Cartan forms

D.1 Dynamical model with high derivatives

Albert Einstein in fifth edition of his book “The Meaning of Relativity” added his paper “Relativistic theory of the non-symmetric field”, written with collaboration with B. Kaufman [1]. A system of differential equations of motion does not determine the field completely. There still remain certain free data. The smaller the number of free data, the “stronger” is the system. Einstein introduced a notion of “strength” of a system of field equations.

How can we determine the degree of freedom of the functions? This problem is studied by Whittaker under considering spherical harmonics [2]. How can we set the initial conditions for systems of differential

equations if there are gauge degrees of freedom or identities? The problem of identifying the dynamical variables is associated with the formulation of the Cauchy problem [3]. There is an interesting problem without dynamical degrees of freedom [4].

Let us consider here an instructive example of a system with constraints: a string theory whose Lagrangian is the n th power of the Gauss curvature of a space-time ($n \in \mathbb{N}, n > 1$) [5]. Insofar the Hilbert functional of gravitation in (1+1)-dimensional space-time gives the Gauss–Bonnet topological invariant, we take as a Lagrangian the Gauss curvature in the n th power. The theory keeps its covariance. Although the calculations are cumbersome, the problem can be solved fully. It turns out to be a useful example of using the generalized Maurer–Cartan forms.

We analyze the dynamics of a space–time metric taken in the ADM form:

$$(g_{\mu\nu}) = \begin{pmatrix} \alpha^2 + \beta^2 & \gamma\beta \\ \gamma\beta & \gamma^2 \end{pmatrix}, \quad \sqrt{g} = \alpha\gamma, \quad (\text{D.1})$$

where the metric functions $\alpha(t, x)$ and $\beta(t, x)$ have the meaning of Lagrange multipliers. The Gauss curvature can be expressed by [6]

$$R = -\frac{1}{2\alpha^3\gamma^2} \det \begin{pmatrix} \alpha & \beta & \gamma \\ \dot{\alpha} & \dot{\beta} & \dot{\gamma} \\ \alpha' & \beta' & \gamma' \end{pmatrix} - \frac{1}{2\alpha\gamma} \left(\left[\frac{(\gamma^2) \cdot - (\gamma\beta)'}{\alpha\gamma} \right] \cdot - \left[\frac{(\gamma\beta) \cdot - (\alpha^2 + \beta^2)'}{\alpha\gamma} \right] \right). \quad (\text{D.2})$$

The functional of action is in the form

$$S = \frac{1}{2} \int_{t,x} R^n \alpha \gamma = \tag{D.3}$$

$$= \frac{1}{2} \int_{t,x} (\alpha \gamma)^{1-n} \left[\left(\frac{\beta' - \dot{\gamma}}{\alpha} \right)' + \left(\frac{\beta \dot{\gamma}}{\alpha \gamma} \right)' - \left(\frac{(\alpha^2 + \beta^2)'}{2\alpha \gamma} \right)' \right]^n,$$

where $\int_{t,x}$ denotes integration over the space-time. Varying S by the metric one gets the Euler—Lagrange equations:

$$\begin{aligned} \frac{\partial L}{\partial \alpha} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{\alpha}} - \frac{\partial}{\partial x} \frac{\partial L}{\partial \alpha'} + \frac{\partial^2}{\partial x^2} \frac{\partial L}{\partial \alpha''} &= 0, \\ \frac{\partial L}{\partial \beta} - \frac{\partial}{\partial x} \frac{\partial L}{\partial \alpha \beta'} + \frac{\partial^2}{\partial t \partial x} \frac{\partial L}{\partial \beta'} + \frac{\partial^2}{\partial x^2} \frac{\partial L}{\partial \beta''} &= 0, \\ \frac{\partial L}{\partial \gamma} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{\gamma}} - \frac{\partial}{\partial x} \frac{\partial L}{\partial \gamma'} + \frac{\partial^2}{\partial t^2} \frac{\partial L}{\partial \ddot{\gamma}} + \frac{\partial^2}{\partial t \partial x} \frac{\partial L}{\partial \dot{\gamma}'} &= 0, \end{aligned} \tag{D.4}$$

where L is the density of the Lagrange function.

The differential equations of the extremals (D.4) are very complicated. The matter is clearer in the Hamiltonian formulation since we deal with a nondegenerated theory with higher derivatives. So we use a slightly modified version of the Ostrogradski method [7]. It is relevant to introduce, along with generalized coordinates (α, β, γ) , the new variable

$$u \equiv \frac{\beta' - \dot{\gamma}}{\alpha}. \tag{D.5}$$

Then the action in the coordinates $(\alpha, \beta, \gamma, u)$ takes the form

$$S = \frac{1}{2} \int_{t,x} (\alpha \gamma)^{1-n} \left[\dot{u} - \left(\frac{\beta u + \alpha'}{\gamma} \right)' \right]^n. \tag{D.6}$$

Momentum densities are calculated, using the functional derivatives

$$\pi_u \equiv \frac{\delta S}{\delta \dot{u}} = \frac{\partial L}{\partial \dot{u}} = \frac{n}{2} \alpha \gamma^{1-n} \left[\dot{u} - \left(\frac{\beta u + \alpha'}{\gamma} \right)' \right]^{n-1},$$

$$\begin{aligned}\pi_\alpha &\equiv \frac{\delta S}{\delta \dot{\alpha}} = \frac{\partial L}{\partial \dot{\alpha}}, \\ \pi_\beta &\equiv \frac{\delta S}{\delta \dot{\beta}} = -\frac{\partial}{\partial x} \frac{\partial L}{\partial \dot{\beta}'}, \\ \pi_\gamma &\equiv \frac{\delta S}{\delta \dot{\gamma}} = \frac{\partial L}{\partial \dot{\gamma}} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \ddot{\gamma}} - \frac{\partial}{\partial x} \frac{\partial L}{\partial \dot{\gamma}'}. \end{aligned}$$

Taking into consideration (D.5), the Hamiltonian

$$H = \int_x (\pi_u \dot{u} + \pi_\alpha \dot{\alpha} + \pi_\beta \dot{\beta} + \pi_\gamma \dot{\gamma} - L[\alpha, \dot{\alpha}, \alpha', \alpha''; \beta, \beta', \beta'', \beta'''; \gamma, \dot{\gamma}, \gamma', \ddot{\gamma}, \gamma'])$$

becomes

$$H = \int_x (\pi_u \dot{u} + \pi_\alpha \dot{\alpha} + \pi_\beta \dot{\beta} + \pi_\gamma (\beta' - \alpha u) - L[\alpha, \alpha', \alpha''; \beta, \beta'; \gamma, \gamma'; u, \dot{u}, u']).$$

Ignoring boundary terms one has the following expression for the Hamiltonian:

$$\begin{aligned} H &= \\ &= \int_x \left(\pi_\alpha \dot{\alpha} + \pi_\beta \dot{\beta} + \alpha \left[(n-1) \left(\frac{2}{n^n} \right)^{1/(n-1)} \gamma \pi_u^{n/(n-1)} - u \pi_\gamma + \left(\frac{\pi'_u}{\gamma} \right)' \right] \right. \\ &\quad \left. + \beta \left[-u \left(\frac{\pi'_u}{\gamma} \right) - \pi'_\gamma \right] \right). \end{aligned}$$

Along with the equations of motion obtained by varying the Hamiltonian by variables $u(t, x)$, $\gamma(t, x)$, there are two differential constraints:

$$\begin{aligned} (n-1) \left(\frac{2}{n^n} \right)^{1/(n-1)} \gamma \pi_u^{n/(n-1)} - u \pi_\gamma + \left(\frac{\pi'_u}{\gamma} \right)' &= 0, \\ u \left(\frac{\pi'_u}{\gamma} \right) + \pi'_\gamma &= 0. \end{aligned}$$

The system can be integrated once and then takes the form

$$\frac{1}{2n-1} \left(\frac{2}{n} \right)^{n/(n-1)} \pi_u^{(2n-1)/(n-1)} + \left(\frac{\pi'_u}{\gamma} \right)^2 + \pi_\gamma^2 = c(t),$$

$$u\pi'_u + \gamma\pi'_\gamma = 0,$$

where $c(t)$ is an arbitrary function of time.

The Hamiltonian formulation is defined by the Poisson structure \hat{J} on the functional phase space. Its nonzero brackets are

$$\{\gamma(t, x), \pi_\gamma(t', x')\} = \delta(t - t')\delta(x - x'),$$

$$\{u(t, x), \pi_u(t', x')\} = \delta(t - t')\delta(x - x').$$

In geometrodynamics, U could be considered as the phase space of the Wheeler—De Witt space.

Then we construct on the basis of constraints the functionals

$$\begin{aligned} \Phi[\phi] &= \\ &= \int_{t,x} \left(\frac{1}{2n-1} \left(\frac{2}{n} \right)^{n/(n-1)} \pi_u^{(2n-1)/(n-1)} + \left(\frac{\pi'_u}{\gamma} \right)^2 + \pi_\gamma^2 - c(t) \right) \phi(t, x), \\ \Xi[\xi] &= \int_{t,x} (u\pi'_u + \gamma\pi'_\gamma)\chi(t, x) \end{aligned}$$

and calculate their Poisson bracket

$$\{\Phi, \Xi\} = \int_{t,x;t',x'} \frac{\delta\Phi}{\delta z} \hat{J} \frac{\delta\Phi}{\delta z}. \tag{D.7}$$

The result of the calculation is

$$\{\Phi[\phi], \Xi[\xi]\} = \Phi[(\phi\chi)'] + \int c(t)\phi(t, x).$$

So the differential constraints form a closed algebra (there are no other constraints in the theory) and they do not annihilate the Poisson bracket. We can express the variables π_γ and u from the constraints as

$$\pi_\gamma^2 = c(t) - \frac{1}{2n-1} \left(\frac{2}{n} \right)^{n/(n-1)} \pi_u^{(2n-1)/(n-1)} - \left(\frac{\pi'_u}{\gamma} \right)^2,$$

$$u = -\gamma \left(\frac{\pi'_\gamma}{\pi'_u} \right).$$

D.2 Variational De Rham complex

For the investigation of covariant theories, mathematical tools of the theory of variational complexes [8] that are generalization of the De Rham complexes of differential forms prove to be useful. The variational complexes are decomposed into two components. The first part is obtained by reformulation of the De Rham complex onto spaces of a set of differential functions set on $V \subset X \times D$, where X is a space of independent variables and U is a space of dependent variables. A differential r -form is given by

$$\omega^r = \sum_J P_J[u] dx^j, \quad (\text{D.8})$$

where P_J are differential functions and

$$dx^J = dx^{j_1} \wedge \dots \wedge dx^{j_r}, \quad 1 \leq j_1 < \dots < j_r \leq p$$

constitute the basis of a space of differential r -forms $\wedge_r T^*X$.

Since for relativistic theories a consequence of covariance of the description is that the Hamiltonian is zero, we will be interested here only in the second part of the variational complex. Let us suppose the Hamiltonian constraint to be resolved. Differential forms are active on “horizontal” variables X from M , and vertical forms are constructed analogously—they are active on “vertical” variables u and their derivatives. A vertical k -form is a finite sum

$$\hat{\omega}^k = \sum P_J^\alpha[u] du_{J_1}^{\alpha_1} \wedge \dots \wedge du_{J_k}^{\alpha_k}, \quad (\text{D.9})$$

where P_j^α are differential functions. Here independent variables are like parameters.

Insofar the vertical form $\hat{\omega}$ is built on a space of finite jets $M^{(n)}$, a vertical differential has properties of bilinearity, antidifferentiation, and closure like an ordinary differential. Here we use functional forms connected with the introduced vertical forms, as functionals connected with differential functions.

Let $\omega^k = \int_x \hat{\omega}^k$ be a functional k -form corresponding to a vertical k -form $\hat{\omega}^k$. A variational differential of a form ω^k is a functional $(k+1)$ -form corresponding to a vertical differential of a form $\hat{\omega}^k$. The basic properties are deduced from the properties of the vertical differential, so we get a variational complex. A variational differential defines an exact complex

$$0 \xrightarrow{\delta} \Lambda_*^0 \xrightarrow{\delta} \Lambda_*^1 \xrightarrow{\delta} \Lambda_*^2 \xrightarrow{\delta} \Lambda_*^3 \xrightarrow{\delta} \dots \quad (\text{D.10})$$

on spaces of functional forms on M .

Of particular interest in theoretical physics problems are functional forms: $\omega^0, \omega^1, \omega^2$. In the present problem, after the constraints are utilized, we get a functional 1-form as a generalization of a differential Cartan form for dynamical systems:

$$\omega^1 = \int_{t,x} \left[\pi_\gamma \left(t, \pi_u, \left(\frac{\pi'_u}{\gamma} \right) \right) d\gamma - u \left(t, \pi_u, \left(\frac{\pi'_u}{\gamma} \right), \left(\frac{\pi'_u}{\gamma} \right)' \right) d\pi_u \right] \quad (\text{D.11})$$

Equations of motion are obtained as a condition of the closedness of the 1-form: $\delta\omega^1 = 0$.

But, as we demonstrate below, there is a 0-form ω^0 :

$$\omega^0 = \int_{t,x} \hat{\omega}^0(t, \gamma, \pi_u) \tag{D.12}$$

so that $\delta\omega^0 = \omega^1$, *id est* ω^1 is not only a closed form, it is an exact one.

Acting by the operator of the variational differential δ on the form (D.12), we get

$$\delta\omega^0 = \int_{t,x} \left[\frac{\delta\omega^0}{\delta\gamma} d\gamma + \frac{\delta\omega^0}{\delta\pi_u} d\pi_u \right]. \tag{D.13}$$

From this we find conditions on $\hat{\omega}^0(t, \gamma, \pi_u)$:

$$\frac{\partial\hat{\omega}^0}{\partial\gamma} = \pi_\gamma \left(t, \pi_u, \left(\frac{\pi'_u}{\gamma} \right) \right), \tag{D.14}$$

$$\frac{\partial\hat{\omega}^0}{\partial\pi_u} = \frac{\partial}{\partial x} \left(\frac{\partial\hat{\omega}^0}{\partial\pi'_u} \right) = -u \left(t, \pi_u, \left(\frac{\pi'_u}{\gamma} \right), \left(\frac{\pi'_u}{\gamma} \right)' \right). \tag{D.15}$$

The system of differential equations (D.14), (D.15) can be solved analytically:

$$\begin{aligned} \hat{\omega}^0 = & \int_{t,x} \gamma \left[c(t) - \frac{1}{2n-1} \left(\frac{2}{n} \right)^{n/(n-1)} \pi_u^{(2n-1)/(n-1)} - \left(\frac{\pi'_u}{\gamma} \right)^2 \right]^{1/2} + \\ & + \int_{t,x} \pi'_u \arcsin \left[\frac{\pi'_u}{\gamma} \left(c(t) - \frac{1}{2n-1} \left(\frac{2}{n} \right)^{n/(n-1)} \pi_u^{(2n-1)/(n-1)} \right)^{-1/2} \right], \end{aligned}$$

where $\pi_u(\alpha, \dot{\alpha}, \alpha', \alpha''; \beta, \beta', \beta'', \dot{\beta}, \dot{\beta}', \dot{\beta}''; \gamma, \dot{\gamma}, \ddot{\gamma}, \dot{\gamma}')$ in the initial variables is

$$\pi_u = \frac{1}{\alpha\gamma} \left[\left(\frac{\beta' - \dot{\gamma}}{\alpha} \right)' + \left(\frac{2\beta\dot{\gamma} - (\alpha^2 + \beta^2)'}{2\alpha\gamma} \right)' \right]. \tag{D.16}$$

We get a generalized De Rham variational complex:

$$0 \xrightarrow{\delta} \Lambda_*^0 \xrightarrow{\delta} \Lambda_*^1 \xrightarrow{\delta} 0 \tag{D.17}$$

because the operator of the variational differential δ is nilpotent: $\delta^2 = 0$. So the generalized De Rham cohomology group is trivial. Translating into a physical language we conclude that the functional of action (D.3) does not define any dynamical problem.

Bibliography

- [1] Einstein, A.: *The Meaning of Relativity*. Fifth edition: Including *The Relativistic theory of the non-symmetric field*. Princeton (1955)
- [2] Whittaker, E.T., Watson, G.N.: *A Course of Modern Analysis*. Cambridge University Press (1963)
- [3] Burlankov, D.E.: *Local structure of functional spaces and dynamical variables of gauge-invariant fields*. Theor. Math. Phys. **39**, 293 (1979)
- [4] Burlankov, D.E., Pavlov, A.E.: *Variational forms and two-dimensional R^2 -gravity*. Int. J. Mod. Physics. **A 4**, 5177 (1989)
- [5] Pavlov, A.E.: *Two - dimensional R^n gravitation*. Int. J. Theor. Physics. **36**, 2107 (1997)
- [6] Pogorelov, A.B.: *Differential Geometry*. Nauka, Moscow (1969)
- [7] Dubrovin, B.A., Novikov, S.P., Fomenko, A.T.: *Modern Geometry: Methods and Applications*. Nauka, Moscow (1986)
- [8] Olver, P.: *Applications of Lie Groups to Differential Equations*. Springer, New York (1986)

Appendix E

Dynamics of the mixmaster model

E.1 Pseudo–Euclidean generalized Toda chain

The metric of the mixmaster model [1] is

$$ds^2 = N^2 dt^2 - e^{2\alpha} (e^{2\beta})_{ij} \omega^i \omega^j, \quad (\text{E.1})$$

where the differential forms

$$\begin{aligned} \omega^1 &= \sin \psi d\theta - \cos \psi \sin \theta d\phi, \\ \omega^2 &= \cos \psi d\theta + \sin \psi \sin \theta d\phi, \\ \omega^3 &= -(d\psi + \cos \theta d\phi) \end{aligned} \quad (\text{E.2})$$

are expressed through the Euler angles (ψ, θ, ϕ) on $SO(3)$ group. The structure constants of the corresponding algebra $so(3)$ appear in the

relations

$$d\omega^i = \frac{1}{2}\epsilon_{ijk}\omega^i \wedge \omega^j.$$

The symmetric traceless matrix $(\beta)_{ij}$ can be presented in the form

$$(\beta)_{ij} = \text{diag}(\beta_+ + \beta_- \sqrt{3}, \beta_+ - \beta_- \sqrt{3}, -2\beta_+),$$

where β_+, β_- are two field amplitudes as generalized coordinates.

The Misner's cosmological model does not belong to the completely integrable systems [2]. It is an example of the pseudo-Euclidean generalized Toda chains at a level of energy $H = 0$ [3]. The Hamiltonian has the form

$$H = \frac{1}{2}(-p_\alpha^2 + p_+^2 + p_-^2) + \exp(4\alpha)V(\beta_+, \beta_-), \quad (\text{E.3})$$

where the potential function $V(\beta_+, \beta_-)$ is an exponential polynomial:

$$V(\beta_+, \beta_-) = \exp(-8\beta_+) + \exp(4\beta_+ + 4\sqrt{3}\beta_-) + \exp(4\beta_+ - 4\sqrt{3}\beta_-) - \\ - 2\exp(4\beta_+) - 2\exp(-2\beta_+ + 2\sqrt{3}\beta_-) - 2\exp(-2\beta_+ - 2\sqrt{3}\beta_-).$$

The Hamiltonian of the generalized Toda chain has the form

$$H = \frac{1}{2} \langle \mathbf{p}, \mathbf{p} \rangle + \sum_{i=1}^N c_i v_i, \quad (\text{E.4})$$

where \langle, \rangle is a scalar product in the Minkowski space $\mathbb{R}^{1,n-1}$, c_i are some real coefficients, $v_i \equiv \exp(\mathbf{a}_i, \mathbf{q})$, $(,)$ is a scalar product in the Euclidean space \mathbb{R}^n , and \mathbf{a}_i are real vectors. For the considered mixmaster model: $n = 3, N = 6$. Pseudo-euclidity of a momentum space is a distinctive peculiarity of gravitational problems so they cannot be referred to as analytical dynamics problems, where the corresponding form quadratic in momenta is the kinetic energy.

E.2 Kovalevski exponents

On the other hand, the cosmological models can be considered as dynamical systems [4]. So it is possible to carry out strict methods of analysis traditionally used in the analytical mechanics, and adopt them to systems like (E.3). Let us apply the Painlevé test for calculation of Kovalevski exponents [5]. The term “Kovalevski exponents” was introduced in paper [6], thus marking an outstanding contribution of the Russian woman to the solution of the important problem of integration of rigid body rotation.

Expanding the $2n$ -dimensional phase space to the $2N$ -dimensional one by homeomorphism $(\mathbf{p}, \mathbf{q}) \mapsto (\mathbf{v}, \mathbf{u})$:

$$v_i \equiv \exp(\mathbf{a}_i, \mathbf{q}), \quad u_i \equiv \langle \mathbf{a}_i, \mathbf{p} \rangle, \quad i = 1, \dots, N, \quad (\text{E.5})$$

one gets a Hamiltonian system which equations of motion are the autonomous homogeneous differential equations with polynomial right side:

$$\dot{v}_i = u_i v_i, \quad \dot{u}_i = \sum_{j=1}^N M_{ij} v_j, \quad i = 1, \dots, N. \quad (\text{E.6})$$

The matrix \hat{M} is constructed of scalar products of vectors \mathbf{a}_i in the Minkowski space $\mathbb{R}^{1, n-1}$:

$$M_{ij} \equiv -c_j \langle \mathbf{a}_i, \mathbf{a}_j \rangle.$$

The system of equations (E.6) is quasi-homogeneous. The power of quasi-homogeneity on variables u_i is one, and on v_i is two. The property of integrability of a dynamical system appears in a character of singularities of solutions so only singular points represent particular

interest for investigation. The differential equations have the following partial meromorphic solutions:

$$u_i = \frac{U_i}{t}, \quad v_i = \frac{V_i}{t^2}, \quad i = 1, \dots, N, \quad (\text{E.7})$$

the coefficients U_i, V_i satisfy the system of algebraic equations

$$2V_i = U_i V_i, \quad -U_i = \sum_{j=1}^N M_{ij} V_j.$$

Now let us analyze the special types of solutions. Let $V_1 \neq 0$, the rest $V_2, V_3, \dots, V_N = 0$, then we get a solution: if $M_{11} \neq 0$, then

$$V_1 = \frac{2}{M_{11}}, \quad U_1 = -2, \quad U_2 = -\frac{2M_{21}}{V_{11}}, \dots, U_N = -\frac{2M_{N1}}{M_{11}}.$$

Analogously the last solutions will be obtained. If for some i : $V_i \neq 0$, and $V_j = 0$ for all $j \neq i$, then at $M_{ii} \neq 0$ we get

$$U_i = -2, \quad U_j = -2M_{ji}/M_{ii} \quad \text{for all } i \neq j.$$

It follows from the obtained solutions that the significant point of analysis is a nonequality of the corresponding diagonal element of the matrix \hat{M} to zero that it is possible in the case of isotropy of the vector \mathbf{a}_i . It is a principal distinctive feature of pseudo-Euclidean chains.

For investigating single-valuedness of the obtained solutions we use the Lyapunov method [5] based on studying the behavior of their variations:

$$\frac{d}{dt}(\delta u_i) = \sum_{j=1}^N M_{ij} \delta v_j,$$

$$\frac{d}{dt}(\delta v_i) = \frac{U_i \delta v_i}{t} + \frac{V_i \delta u_i}{t^2}, \quad i = 1, \dots, N.$$

We seek their solutions in the form of

$$\delta u_i = \xi_i t^{\rho-1}, \quad \delta v_i = \eta_i t^{\rho-2}, \quad i = 1, \dots, N.$$

Then for searching the coefficients ξ_i, η_i one gets a linear homogeneous system of equations with the parameter ρ :

$$(\rho - 2 - U_i)\eta_i = V_i \xi_i, \quad (\text{E.8})$$

$$(\rho - 1)\xi_i = \sum_{j=1}^N M_{ij}\eta_j, \quad i = 1, \dots, N. \quad (\text{E.9})$$

Values of the parameter ρ are called the *Kovalevski exponents*.

Let us consider solutions when $V_i \neq 0$. If $\eta_1 \neq 0$ and the rest $\eta_i = 0$, then from the first system of equations (E.8) one gets $\xi_1 = M_{11}\rho\eta_1/2$, a substitution of it into the second system (E.9) gives a condition of values of the parameter ρ :

$$\rho(\rho - 1) - 2 = 0,$$

id est $\rho_1 = -1, \rho_2 = 2$. The remaining equations (E.9) give us solutions $\xi_i = \xi_i(\eta_1, \rho)$.

Let $\eta_2 \neq 0$, then $\eta_3, \eta_4, \dots, \eta_N = 0$,

$$\rho = 2 - 2\frac{M_{21}}{M_{11}}, \quad \rho\eta_1 = 2\frac{\xi_1}{M_{11}}.$$

The second system gives functions $\xi_i = \xi_i(\eta_2), i = 1, 2, \dots, N$, and so on. As a result, having looked through all solutions of the first series for the case $V_1 \neq 0$, we obtain the formula for the spectrum ρ :

$$\rho = 2 - 2\frac{\langle \mathbf{a}_i, \mathbf{a}_1 \rangle}{\langle \mathbf{a}_1, \mathbf{a}_1 \rangle}, \dots, i = 2, 3, \dots, N.$$

As a final result, having considered the rest solutions, we obtain a formula for the Kovalevski exponents ρ that generalizes the Adler—van Moerbeke formula [7] for the case of indefinite spaces:

$$\rho = 2 - 2 \frac{\langle \mathbf{a}_i, \mathbf{a}_k \rangle}{\langle \mathbf{a}_k, \mathbf{a}_k \rangle}, \quad i \neq k, \quad \langle \mathbf{a}_k, \mathbf{a}_k \rangle \neq 0. \quad (\text{E.10})$$

The requirement $\rho \in \mathbb{Z}$ is a necessary condition for meromorphy of solutions on a complex plane of t . It should be noticed that while obtaining formula (E.10) no restrictions on a metric signature were imposed. It is correct not only for spaces of the Minkowski signature.

Now let us apply the elaborated method to analyze the integrability of the mixmaster model of the Universe, “root vectors” of which have the form:

$$\begin{aligned} \mathbf{a}_1(4, -8, 0), \quad \mathbf{a}_2(4, 4, 4\sqrt{3}), \quad \mathbf{a}_3(4, 4, -4\sqrt{3}), \\ \mathbf{a}_4(4, 4, 0), \quad \mathbf{a}_5(4, -2, 2\sqrt{3}), \quad \mathbf{a}_6(4, -2, -2\sqrt{3}). \end{aligned}$$

“Cartan matrix” composed of scalar products of the “root vectors” in the Minkowski space has the form:

$$\langle \mathbf{a}_i, \mathbf{a}_j \rangle = 48 \begin{pmatrix} 1 & -1 & -1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & -1 \\ -1 & -1 & 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & -1/2 & -1/2 \\ 0 & 0 & -1 & -1/2 & 0 & -1/2 \\ 0 & -1 & 0 & -1/2 & -1/2 & 0 \end{pmatrix}.$$

One gets three “root vectors” disposed out of a light cone (space-like vectors), the rest three are isotropic on the light cone. Using the

generalized Adler—van Moerbeke formula (E.10), taking account of zero norm of three vectors, we get integer $\rho_1 = 2, \rho_2 = 4$. As the Killing metric in the generalized Adler—van Moerbeke formula (E.10) is indefinite, it should be pointed out in the classification scheme of the noncompact Lie algebras for getting exact solutions of Toda lattices, as it was done in [3].

Due to isotropic character of three vectors, we transit from the Misner phase variables to some other [4]

$$(\alpha, \beta_+, \beta_-; p_\alpha, p_+, p_-) \mapsto (X, Y, Z; p_x, p_y, p_z).$$

Now the Hamiltonian has a more symmetric form:

$$\begin{aligned} X &= \frac{1}{12} \exp(2(\alpha + \beta_+ + \sqrt{3}\beta_-)), & Y &= \frac{1}{12} \exp(2(\alpha + \beta_+ - \sqrt{3}\beta_-)), \\ Z &= \frac{1}{12} \exp(2(\alpha - 2\beta_+)); \\ p_x &= \frac{1}{12}(2p_\alpha + p_+ + \sqrt{3}p_-), & p_y &= \frac{1}{12}(2p_\alpha + p_+ - \sqrt{3}p_-), & p_z &= \frac{1}{6}(p_\alpha - p_+). \end{aligned}$$

The equations of motion are represented as Hamiltonian equations on a direct sum of two-dimensional solvable Lie algebras

$$g(6) = g(2) \oplus g(2) \oplus g(2) :$$

$$\{X, p_x\} = X, \quad \{Y, p_y\} = Y, \quad \{Z, p_z\} = Z \quad (\text{E.11})$$

with the Hamiltonian H :

$$H = -\frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{4}(p_x + p_y + p_z)^2 - 2(X^2 + Y^2 + Z^2) + (X + Y + Z)^2.$$

The Hamiltonian has a form of a kinetic energy of a top:

$$H = \frac{1}{2} \sum_{i,j=1}^6 I_{ij} x^i x^j,$$

where the phase variables are enumerated as

$$x_1 = X, \quad x_2 = Y, \quad x_3 = Z; \quad x_4 = p_x, \quad x_5 = p_y, \quad x_6 = p_z,$$

and the energy tensor I_{ij} has a block type. So the mixmaster cosmological model can be considered as the Euler—Poincaré top on a Lie algebra (E.11). The Euler—Poincaré equations are generalization of the famous dynamical Euler equations describing a rotation of a rigid body with the corresponding algebra of rotations $so(3)$.

The partial meromorphic solution of the obtained system of differential equations is $x_i = C_i/t$. Then the problem is reduced to investigation of a spectrum of Kovalevski's matrix

$$K_{ij} = (c_{jk}^i I_{kl} + c_{jk}^l I_{ki}) C^l + \delta_{ij}, \quad (\text{E.12})$$

where c_{ij}^k are the structure constants of the algebra and C_i are solutions of an algebraic system:

$$C_i + c_{ij}^k I_{jl} C_k C_l = 0.$$

Calculations give an integer-valued spectrum of the matrix (E.12):

$$\rho = -1, 1, 1, 2, 2, 2,$$

which point to the regular character of behavior of the considered dynamical system.

Bibliography

- [1] Misner, Ch.: *Mixmaster Universe*. Phys. Rev. Lett. **22**, 1071 (1969)
- [2] Lichtenberg, A.J., Lieberman, M.A.: *Regular and Chaotic Dynamics*. Springer, New York (1992)
- [3] Bogoyavlenski, O.I.: *Methods in the Qualitative Theory of Dynamical Systems in Astrophysics and Gas Dynamics*. Springer (1985)
- [4] Pavlov, A.E.: *The mixmaster cosmological model as a pseudo-Euclidean generalized Toda chain*. [arXiv: gr-qc/9504034].

Pavlov, A.E.: *Mixmaster cosmological model as a top of Euler – Poincaré*. Abstracts of Int. Conf. “Geometrization of Physics - III”. Kazan State University. 69 (1997)
- [5] Kovalevskaya, S.V.: *Scientific Papers*. Nauka, Moscow (1948)
- [6] Yoshida, H.: *Necessary condition for the existence of algebraic first integrals*. Celestial mechanics. **31**, 363 (1983)

- [7] Adler, M., Moerbeke, P. van: *Kowalewski's asymptotic method, Kac—Moody Lie algebras and regularization*. Commun. Math. Phys. **83**, 83 (1982)

**More
Books!** 



yes
i want morebooks!

Buy your books fast and straightforward online - at one of world's fastest growing online book stores! Environmentally sound due to Print-on-Demand technologies.

Buy your books online at

www.get-morebooks.com

Kaufen Sie Ihre Bücher schnell und unkompliziert online – auf einer der am schnellsten wachsenden Buchhandelsplattformen weltweit! Dank Print-On-Demand umwelt- und ressourcenschonend produziert.

Bücher schneller online kaufen

www.morebooks.de



VDM Verlagsservice-
gesellschaft mbH

VDM Verlagsservicegesellschaft mbH

Heinrich-Böcking-Str. 6-8
D - 66121 Saarbrücken

Telefon: +49 681 3720 174
Telefax: +49 681 3720 1749

info@vdm-vsg.de
www.vdm-vsg.de

