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Пертурбативные и непертурбативные исследования в теории поля в переменных светового фронта и операторные решения некоторых двумерных моделей

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С электронной версией диссертации можно ознакомиться на официальном сайте Объединенного института ядерных исследований в информационно-телекоммуникационной сети «Интернет» по адресу: *http://dissertations.jinr.ru*. С печатной версией диссертации можно ознакомиться в Научно-технической библиотеке ОИЯИ (г. Дубна, Московская область, ул. Жолио-Кюри, д. 6).

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General characterization of the work

Topicality of the research subject. Relativistic quantum field theory (QFT) is the conceptual and mathematical language of elementary particle physics. It has a few specific realizations like operator formalism or path integral approach, and methods starting from phenomenological or semi-phenomenological models through perturbation theory, numerical lattice simulations to highly abstract concepts like supersymmetry or string theory. All these particular schemes have been most often formulated in terms of the ”natural” space-time variables $x^\mu = (t, x, y, z)$. It has been known since the Dirac’s work on the forms of relativistic Hamiltonian dynamics [1] that in addition to this ”instant-form” or ”equal-time” form (called space-like (SL) here because the quantization rules are prescribed on a space-like initial surface) there is also a ”point form” (related to a Lorentz invariant quantization hypersurface) and the front form. The latter introduces the time variable $x^+ = t + z$ and the ”longitudinal” variable $x^- = t - z$, and correspondingly the space-time parametrization is $x^\mu = (x^+, x^-, x, y)$. Quantization rules are then prescribed on the hypersurface $x^+ = 0$, called also a ”null-plane” or ”light-front”. This ”innocent” change of variables actually changes dramatically the properties of the corresponding field theory from the structure of field equations and character of field variables (dynamical vs. constrained) up to the properties of the Fock space including the vacuum state. Specifically, purely kinematical arguments suggest, without any reference to dynamics, that quite generally the lowest-energy eigenstate of a full interacting Hamiltonian coincides with the vacuum of the free, non-interacting Hamiltonian. The latter property is based on the positivity of the light-front (LF) momentum $p^+ = p^0 + p^3$, conjugate to x^-. This means that contrary to the usual SL theory, where the only quantity with positively-definite spectrum is the energy, in the LF case there are two such quantities (operators) – p^+ and the LF energy $p^- = p^0 - p^3$. The spectral condition $p^+ \geq 0$ of the kinematical operator P^+ can be used to define the vacuum state as well as creation and annihilation operators for arbitrary x^+ even in the interacting case (in Heisenberg picture) [2]. Consequently, a consistent Fock expansion with momentum-dependent coefficients (wave functions) having the probabilistic interpretation similar to the non-relativistic quantum mechanics, can be applied to bound-state problem. All the above properties are unique and open a very promising avenue for a fresh formulation of QFT.

The development of the front form of dynamics after Dirac’s paper was somewhat non-uniform. It reappeared as the ”infinite-momentum frame” (IMF) approach independently in the context of the current algebra and perturbation theory in the mid of 1960’s (Fubini and Furlan [3], and Weinberg [4], respectively). The celebrated Feynman’s parton model was also formulated in the IMF [5]. Between 1969 and 1973 fundaments of the LF QFT were laid down in the work of Chang and Ma [6], Susskind [7], Leutwyler, Klauder and Streit [2], Rohrlich and collaborators [8], Kogut and Soper [9], Yan and collaborators [10], Leutwyler and Stern [11], and others. Later in 1970’s, important contributions were done by Maskawa and Yamawaki [12], Karmanov [13], and Brodsky and Lepage [14]. A more systematic development of the LF methods (both theoretical as well as more phenomenologically-oriented) was triggered by the work of Pauli and Brodsky in 1985 [15]. The important progress was then achieved for example within the LF renormalization-group approach [16, 17] and using the LF Tamm-Dancoff methods [18–20].

Although the light-front field theory emerged as a relatively independent direction of the theoretical research a few decades ago, there are still quite a few open questions and even controversies between individual research groups. The research activity is not as homogeneous as for example in the lattice computation community. It is however still very clear that the conceptual and technical advantages of the front form of dynamics are significant and promising. It is highly desirable to clarify the controversial points in the LF scheme, to further develop its conceptual and mathematical language, to improve our understanding of its basic properties like the vacuum structure and spontaneous symmetry breaking. Another important part of the LF studies should be the application of the scheme to the models and phenomena already understood in the SL form of the theory, in order to further test the LF formulation of QFT and to demonstrate its advantages. Solutions and approaches to particular aspects and problems of the light-front theory, included in the present thesis, such as the LF perturbation theory results, vacuum aspects, exactly solvable models, among others, has the ambition to contribute to such a development. The presented results are a part of the current endeavour of the international light-cone community to make progress in this promising research area.

State of progress of the research subject. There are a few groups worldwide which perform active research in the topics of the present thesis, both perturbative and non-perturbative. The Hamiltonian LF perturbation theory (LFPT - the ”old-fashioned”, time-ordered perturbation theory derived within the LF quantization) was developed by Kogut and Soper [9], Karmanov [13], and by Brodsky and Lepage [14] in 1970’ies and has since been used in numerous studies. However, quite often LF theorists [21] start from the covariant Feynman amplitudes of the process under study, and rewrite the Feynman integrals in terms of the LF variables recovering the LFPT amplitudes by contour integration in the complex k^- plane. This sometimes serves as a shortcut but on the other hand brings certain ambiguities or controversial results. The problem is related to the less convergent integrand in k^- variable than the corresponding SL form of the integrand which depends on the k^0 variable. Our approach assumes that the cleanest answers are obtained in the genuine LF Hamiltonian perturbation theory, although some mechanisms may look differently than in the LF Feynman approach. An example is given by the LF ”zero-modes” (ZMs), that is values of the integration variable $k^+ = 0$, which indeed often give non-negligible contributions in the first method. However, they actually do not exist in the genuine LF quantization of say the free massive LF scalar field and therefore also not in the LFPT. Their effect is replaced by the genuine LF non-ZM mechanisms, as we show in the first part of the thesis with the examples of the one-loop self-energy and scattering amplitudes in both the continuum and finite-volume treatments. In the latter case, the continuum limit reproduces the covariant amplitudes as the ZM corrections vanish. The genuine LFPT is also shown to predict correctly the LF vacuum amplitudes (bubbles) for scalar models, in agreement with the previously calculated, but not widely recognized, covariant Feynman diagrams evaluated in terms of the LF variables. Contrary to standard wisdom, non-vanishing of LF vacuum bubbles is not due to the Fourier mode carrying $k^+ = 0$, but due to the region of small k^+ values.

In the case of the spontaneous symmetry breaking (SSB) in the LF version, it is widely believed that this phenomenon occurs due to the constraint LF zero modes. This approach was formulated by Pinsky, van de Sande, Hiller and collaborators [22], Tsujimaru and Yamawaki [23], and others. In our work, we have made a step forward in developing a semiclassical picture of the broken phase of the self-interacting scalar theories, analogous to the usual SL textbook treatments.

The subject of exactly solvable 2-dimensional models has a long history starting more than 60 years ago in classical papers by Thirring [24], Schwinger [25], Schroer [26], Thirring and Wess [27, 28] and Fedebush [29]. We argue that despite numerous refinements and improvements, some important ingredients have not been fully incorporated in the solutions. This pertains first of all to the vacuum aspects, which are best studied in the Hamiltonian framework that however was not used in the previous SL studies. In particular, our diagonalization of the Thirring-model Hamiltonian by a Bogoliubov transformation lead to a physical vacuum state in terms of coherent states of composite scalar-field Fock operators, bilinear in the original fermion creation and annihilation operators. In the LF case, quantization of massless fields in D=1+1 has been a puzzle for a few decades. For the massless LF fermions (and for the related operator solution of the Schwinger model), it was argued that additional fermionic degrees of freedom have to be introduced "by hand" on the second initial surface $x^- = 0$ [30–32]. In our approach, the massless LF fields depend on just one variable x^+ or x^-. They are correctly obtained as the massless limit of the corresponding massive fields. This leads not only to the independent operator solutions of the solvable Thirring and Thirring-Wess models (not obtained in the LF approach before), but also generates a genuine form of the LF bosonization of fermion fields as well as correctly predicts correlation functions of the conformal field theory and the quantum Virasoro algebra. These issues have not been studied by LF methods before.

Numerical diagonalizations of the LF Hamiltonians for the relativistic models (within the discretized light-cone quantization, DLCQ) started by the Yukawa-model study in $D = 1 + 1$ by Pauli and Brodsky [15], and has since been applied to a variety of theories including gauge models and some realistic 4-dimensional models. Our contribution is the application of the numerical DLCQ method to the topological solutions (bound-states) in the broken phase. The mass of the quantum kink state was found for the first time together with some of its other properties.

Goals and purposes of the thesis. The main goal of the present thesis is to contribute to the development of the light-front form of field theory, conceptually and methodologically, at the level of perturbation theory as well as using non-perturbative approaches (operator methods, Hamiltonian formulation). Our LF perturbation-theory calculations both in the continuum and finite-volume versions (DLCQ) yield the correct self-energy and one-loop scattering amplitudes in a very simple manner, without a need to combine propagators and to introduce Feynman parameters, unlike the covariant perturbation theory. The efficiency of the LFPT in the finite volume as compared to the finite-volume "near-light cone" and "infinite-momentum" (IMF) treatments is shown along with the demonstration that the LF theory cannot be obtained as the "light-like limit" of the "near-LC" formulation. Another goal was to reconcile a contradiction between the vacuum amplitudes (bubbles) obtained using the LF evaluation of the covariant Feynman vacuum diagrams and their apparent vanishing in the genuine LF perturbation theory.

In the non-perturbative area, our goal was to show that the "triviality" of the LF vacuum does not exclude the possibility to describe the broken phase of the scalar models semiclassically, in a manner known from the SL theory. Degenerate vacua can be represented in terms of coherent states of scalar-field modes, in the finite-interval treatment with antiperiodic boundary condition. Similarly, vacuum degeneracy in a four-dimensional model with fermions follows from the presence of dynamical zero modes in the axial-vector charge operator. In the case of a gauge theory, the residual large gauge symmetry realized quantum-mechanically was shown to generate an infinite set of degenerate vacua summed to a gauge-invariant theta vacuum within the massive Schwinger model quantized in a finite volume in the light-cone gauge.

The purpose of our study of two-dimensional solvable models in the usual SL form was to generalize the known operator solutions of the Thirring and Thirring-Wess models by the inclusion of previously overlooked vacuum structure. Another goal was to get a complete solution of these models suitable for comparison with their LF versions, where the vacuum structure does not play a role and its effects are presumably incorporated in the operator part of the solution.

Our analysis of the massless LF fields (scalar and fermion) in D=$1+1$ had the goal to fill the gap and find a consistent quantization scheme for these specific fields that "live" on characteristic surfaces $x^{\pm}=0$. Application of the constructed quantization scheme in the area of solvable models and even in conformal-symmetry aspects demonstrated its consistency and efficacy: the LF operator solutions of the Thirring and Thirring-Wess models could have been found based on this quantization. The connection to the conformal field theory was also established, including the quantum version of the Virasoro algebra.

The purpose of the numerical DLCQ study of the $\phi^4(1+1)$ theory was to find quantum bound states in its broken phase. The obtained results, extrapolated to the continuum limit, confirmed our expectation and yielded the mass and other characteristics (number densities, shape of the form factor) of these quantum states, corresponding to classical topological solutions of the model.

Scientific novelty. The results of the thesis have the following novel aspects and qualities:

- A systematic comparison among the perturbative one-loop amplitudes of the self-interacting scalar models in the finite-volume treatment of the LF, near-LC and infinite-momentum schemes was performed for the first time. It clearly demonstrated advantages of the LF formulation as well as impossibility to define the LF form of QFT as a limit of the near-LC form, at least in the compactified treatment. The reason for the singularities emerging in that limit was revealed. The forward-scattering limit was correctly obtained without presence of the scalar zero mode contrary to claims in literature.

- The vacuum bubbles were shown to be non-zero in the LF perturbation theory, contrary to the prevailing opinion in the LF community. They are obtained as the limit of vanishing incoming momentum of the corresponding self-energy diagrams. Their values match those from the usual Feynman diagrams as well as from the LF evaluation of the Feynman amplitudes.

- A semiclassical picture of the spontaneous symmetry breaking was developed based on a unitary operator that shifts the scalar field to a state minimizing the LF energy. The vacuum is described by a coherent state of scalar-field modes. Such a direct approach has not been applied before, the prevailing opinion being that it is the form of the solution of the constraint equation that indicates the broken phase (a constant term in the solution). A direct construction of an infinite set of degenerate vacua for the sigma model with fermions was not given before either. It is based on the dynamical fermionic zero modes which have $p^+=0$ but can carry positive as well as negative values of the perpendicular momentum $p_\perp$ and thus may combine into vanishing $p_\perp$ in bilinear charge operators.

- The explicit construction of a degenerate set of vacua with internal structure was also derived within the massive Schwinger model in the light-cone gauge quantized in a finite-volume. The Fock-form gauge zero mode and its conjugate momentum was used for the first time in light-front studies in order to implement residual (large) gauge transformations in terms of the corresponding unitary operator. Previously, a non-relativistic quantum-mechanical treatment of the gauge zero mode effects was applied in literature. A novel feature in our approach is also the fermionic component of the vacuum structure obtained in the Fock representation.

- The correct full interacting vector current and non-trivial vacuum structure of the Thirring model were obtained based on an operator solution of the corresponding field equations. These features were missed in the previous studies. A similar solution of the Thirring-Wess model in terms of the fields present in the starting Lagrangian (i.e., without using auxiliary fields via Ansaetze) was obtained for the first time including the correct value of the axial anomaly.

- A consistent quantization of massless fields in D=1+1 was derived in the LF field theory. The massless LF scalar and fermion fields were obtained as massless limits of the corresponding two-dimensional massive fields. The previous attempts required to quantize the massless fermion field at two initial surfaces leading to a few unwanted consequences, while the massless LF quantum scalar field was not described before at all. The formulated quantization scheme opened the door to independent LF operator solutions of the Thirring, Thirring-Wess and Schwinger models. Similarly, it led to an independent LF form of bosonization of fermion fields in two space-time dimensions and to establishing a connection to conformal field theory.

- Detailed properties of quantum topological objects (kinks) in the two-dimensional ϕ^4 model were obtained by a numerical diagonalization of the corresponding LF Hamiltonian. Similar predictions only exist within the numerical lattice simulations but are restricted to the computation of the kink mass. The DLCQ approach based on the extracted information on the LF wave function, led to the insights into the ”parton” composition of the bound state and also to its relationship to the classical topological solution (Fourier transform of the form factor).

Theoretical and practical significance. Our research is purely theoretical, belongs to the fundamental science without direct practical applications. Its potential significance consists in extending and deepening of the methods of relativistic field theory and of the knowlegde related to the structure of the microworld. From the general point of view, our results might contribute to the opinion that the LF version of field theory is very well adapted to the description of relativistic processes, perhaps better than the conventional framework. More specifically, one of the goals of our studies has been to extend the ”canonical” picture of the LF form of relativistic dynamics by some concepts of the conventional SL form of QFT. This concerns for example the idea to extend the notion of the ”trivial” LF vacuum by transforming it to a more complex state (by means of certain zero-mode type of unitary operators, e.g.), which would however still be tractable analytically, in contrast to the SL theory, where the vacuum is typically a state with very complicated Fock structure, which can be obtained only from (unrealistic) non-perturbative calculations. Another related example is the existence of vacuum bubbles in the LF perturbation theory, which naively seem to vanish. Such ”non-conventional” ideas could be generalized to more realistic theories in future.

Methodology and the research methods. The general scheme utilized in the thesis is the quantum field theory expressed in terms of the light-front variables, in the continuum as well as finite-volume versions, and mostly in the Hamiltonian formalism. We applied methods of canonical quantization formulated in terms of LF space-time and field variables. The infrared regularization is achieved by enclosing the system in a finite box in x^- (or x^-, x and y in the 4-dimensional theories) and by imposing (anti)periodic boundary conditions in these variables. This leads to a denumerable (discrete) momentum basis, which allows one to carefully study the zero-mode aspects, and also facilitates numerical analyses (Hamiltonian matrix diagonalizations using the DLCQ method). In the perturbative approach, the Hamiltonian ("old-fashioned" or "time-ordered") LF perturbation theory is utilized because of its simplicity and transparency. In our non-perturbative studies, the operator and Hamiltonian methods in the light-front version are applied. In particular, unitary operators implementing symmetry properties at the quantum level, are an important ingredient. In the case of solvable models, solutions of field equations are obtained by solving the corresponding differential equations and incorporating quantum properties into these solutions, that is non-commutativity of the operators representing fields and observables. The products of the operators at the same space-time points are ill-defined and the point-splitting regularization (that is, interpreting the local product as a limit of the operators with arguments shifted by infinitesimal ϵ) is the most natural definition of such products, in particular fermionic currents. In many problems, the LF Fock representation turns out to be very efficient, leading to clear-cut solutions, for example in the case of quantum "anomalies" in the Thirring model and the LF Virasoro algebra (the commutation relations between the generators of the conformal transformations in the quantum case). Technique of Bogoliubov transformations has also been used for the diagonalization of the SL Hamiltonians. In our DLCQ study, the numerical diagonalization of the Hamiltonian matrix in the discrete basis is the main method.

The main results of the thesis submitted for the defense

1. The one-loop perturbative light-front (LF) amplitudes calculated in a finite volume (DLCQ method) have been shown to (i) match the covariant Feynman amplitudes in the continuum limit, but (ii) are not obtained as LF limits of amplitudes calculated in the conventional ("instant form" or "space-like" (SL)) theory compactified "close" to the light front. The latter method cannot be used to define DLCQ, as is sometimes assumed. The forward scattering amplitude in two-dimensional theory is correctly predicted without the zero mode which is actually not present in the LF massive scalar field as a dynamical degree of freedom.

2. The vacuum amplitudes (bubbles) in the LF perturbation theory are contrary to simple kinematical arguments non-zero and match the values known from usual as well as light-front evaluation of covariant Feynman amplitudes. This has been shown for self-interacting scalar models in continuum and also finite-volume treatments by considering the limit of vanishing incoming momentum of the associated self-energy diagrams.

3. A few mechanisms have been identified that extend the Fock vacuum (which is conventionally considered to be the full vacuum even in *interacting* LF theories) to truly physical vacua with non-trivial structure. The mechanisms are: coherent states of the scalar field that minimize the energy in the broken phase, unitary operators which implement residual large gauge symmetry

in the massive LF Schwinger model, and fermionic zero-mode terms in the axial charge that generate a family of degenerate vacua in the light-front sigma model with fermions.

4. A generalized operator solution of the Thirring model in the usual (SL) formulation has been found. The new solution incorporates a truly interacting fermion current as well as the true vacuum structure obtained by means of a Bogoliubov transformation diagonalizing the (bosonized) Hamiltonian of the model. A similar operator solution of the Thirring-Wess model was found for the first time in terms of the original fields present in the Lagrangian.

5. A notorious problem of quantization of massless LF fields in two space-time dimensions has been solved. In contrast to previous attempts, no initialization on two light-like surfaces is needed, the massless quantum fields emerge as the limit of the corresponding massive fields, with correct form of the two-point functions.

6. Starting from the above quantization scheme, the LF version of fermion-field bosonization has been derived. Also a connection of the massless LF fields in two dimensions to conformal field theory (CFT) has been established. Two-point functions of the energy-momentum tensor of CFT and the quantum Virasoro algebra were obtained in Fock representation. Based on the novel quantization, non-perturbative *light-front* operator solutions of the Thirring and Thirring-Wess models were found for the first time.

7. Physical properties of a quantum kink as a bound state in the broken phase of $\phi^4(1+1)$ theory, were determined. The mass, number densities and the Fourier transform of the kink's form factor were found *ab initio* using the non-perturbative diagonalization of the LF Hamiltonian in a finite discrete basis (the DLCQ method). No similar results were reported in the LF literature before, predictions from lattice simulations are less detailed.

Publications. The present thesis is based on 13 main articles, published in the peer-reviewed scientific journals, which are indexed in the data bases Web of Science and Scopus. The journals include Physics Letters B (6 publications), Physical Review D (3 publications), Few-Body Systems (3 publications). The full list is given below. The additional author's publications can be found in Bibliography.

Credibility and approbation of the results. Results of the presented thesis belong to the theoretical research dealing largely with the conceptual and model issues of the light-front form of the relativistic QFT where the direct comparison with experimental data is not available. Credibility of the results has been secured by confronting them with the results of competing research groups, by publishing them in high-quality scientific journals where the process of acceptance often included critical dialogues with referees, by presenting the results at international conferences and their discussion with renowned experts in the field. Our methodological guiding line has always been to choose and apply theoretical methods of the highest available clarity and simplicity, eliminating thereby potential ambiguities. The author of the thesis believes that the LF field theory posesses a remarkable deal of transparency and simplifying features as compared to the conventional SL field theory.

The results included in the thesis have been presented personally by the author at the following international conferences:

- Lightcone 2002: Structure of Hadrons and Nuclei on the Light Cone, LANL, Los Alamos, USA, August 2002
- Light Cone Workshop Hadrons and Beyond, University of Durham, United Kingdom, August 2003
- Workshop on Light-Cone QCD and Nonperturbative Hadron Physics, Cairns, Adelaide University, Australia, July 2005
- Light-Cone QCD and Nonperturbative Hadron Physics, University of Minnesota, Minneapolis, USA, May 2006
- Light-Cone 2009: Relativistic Hadronic and Particle Physics, ITA, Sao Jose dos Campos, Brazil, July 2009
- Light Cone 2010: Relativistic Hadronic and Particle Physics, Universidad de Valencia, Spain, June 2010
- LIGHTCONE 2011: Application of Ligh-Cone coordinates to Highly Relativistic Systems, Southern Methodist University, Dallas, USA, May 2011
- Light-Cone: Relativistic Hadronic and Particle Physics, University of Delhi, India, December 2012
- Light Cone Meeting, North Carolina State Unversity, Raleigh, USA, May 2014
- Light Cone 2015, INFN Frascati, Italy, September 2015
- Group 31, 31st International Colloquium on Group Theoretical Methods in Physics, Rio de Janeiro, Brazil, June 2016
- LIGHT CONE 2016, Universidade de Lisboa, Lisbon, Portugal, September 2016
- The XVIIth International Conference on Symmetry Methods in Physics, Yerevan, Armenia, July 2017
- Light Cone 2017, Frontiers in Light Front Hadron Physics: Theory and Experiment, University of Mumbai, Bombay, India, September 2017
- Light Cone 2018, Jefferson Laboratory, Newport News, USA, May 2018
- LIGHT CONE 2019 - QCD on the light cone: from hadrons to heavy ions, Ecole Polytechnique, Palaiseau, France, September 2019

Main publications on which the thesis is based

1. A. Harindranath, L. Martinovic, J. P. Vary, ”Compactification near and on the light front”, Physical Review D 62 (2000) 105015

2. L. Martinovic and J.P. Vary, ”Fermionic zero modes and spontaneous symmetry breaking on the light front”, Physical Review D 64 (2001) 105016

3. L. Martinovic, ”Large gauge transformations and light front vacuum structure”, Physics Letters B 509 (2001) 355-364

4. A. Harindranath, L. Martinovic, J. P. Vary, ”Perturbative S-matrix in discretized light cone quantization of two-dimensional phi**4 theory”, Physics Letters B 536 (2002) 250-258

5. D. Chakrabarti, A. Harindranath, L. Martinovic, J. P. Vary, ”Kinks in discrete light cone quantization”, Physics Letter B 582 (2004) 196-202

6. D. Chakrabarti, A. Harindranath, L. Martinovic, G. B. Pivovarov, J. P. Vary, ”A initio results for the broken phase of scalar ligh front field theory”, Physics Letters B 617 (2005) 92

7. L. Martinovic, ”Spontaneous symmetry breaking in light front field theory”, Physical Review D 78 (2008) 105009

8. L. Martinovic, P. Grangé, ”Hamiltonian formulation of exactly solvable models and their physical vacuum states”, Physics Letters B 724 (2013) 310-315

9. L. Martinovic, P. Grangé, ”Solvable models with massless light-front fermions”, Few Body Systems 56 (2015) 607-613

10. L. Martinovic, P. Grangé, ”Two-dimensional massless light front fields and solvable models”, Few Body Systems 57 (2016) 565-571

11. L. Martinovic, ”Two-dimensional massless light-front fields and conformal field theory”, In: Physical and Mathematical Aspects of Symmetries, Springer (2017)

12. L. Martinovic, ”Quantum field theory in two dimensions: Light-front versus space-like solutions”, Few Body Systems 58 (2017) 146-162

13. L. Martinovic, A. Dorokhov, ”Vacuum loops in light-front field theory”, Phys. Lett. B 811 (2020) 135929

Personal contribution of the author. The thesis is based on the work done at the Institute of Physics, Bratislava, Slovakia, at the Department of Physics and Astronomy, Iowa State University, Ames, USA, at the LPTA Laboratory Université Montpellier II, France and at the Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, Russia. I am the only author of 4 publications, 5 have one co-author, 2 have two co-authors. My contribution in publications with co-authors was essential or dominant and all the results of the thesis have been obtained by my active personal participation.

Structure and the size of the thesis. The thesis consists of the Introduction, 5 chapters, Bibliography and 3 Appendices. The total size is 173 pages, from which 148 is the main text. The number of figures is 15. The bibliography consists of 153 items.

CONTENTS OF THE THESIS

In the **INTRODUCTION**, the topicality of the studied themes is explained along with a broader context including a brief overview of the development of the light-front field theory since the seminal work of P. A. M. Dirac in 1949. Also the state of progress of the particular studied aspects is characterized and the main goals of the thesis are defined. Then we point out the scientific novelty of the results of our research together with its theoretical and practical significance, as well as applicability for additional studies. After that, a discussion of our general methodology and specific metods follows. Here we emphasize efficacy and clarity of the light-front operator methods and the Hamiltonian framework. We list seven main results of our thesis, submitted for defense, and justify the degree of credibility and approbation of the obtained results, in particular by the list of international scientific conferences and workshops, where the author delivered talks with the research topics of the thesis. The personal contribution of the author to the journal publications is also evaluated.

Chapter 1. Perturbation theory results. The advantages of the LF scheme manifest themselves already at the level of perturbation theory. Its version best adapted to the front form of the dynamics is the "old-fashioned" Hamiltonian PT, which is not manifestly covariant but simplifies the calculations. It does not involve integration over energy variable, replaces the Feynman propagators by energy denominators and from the very beginning introduces the LF momentum fractions which coincide with Feynman parameters x_i used in the covariant Feynman procedure. In the latter method, a few preliminary steps (like combination of denominators) are necessary. In this chapter, we applied the LFPT to one-loop self energy and scattering diagrams for self-interacting scalar models and compared its results to those of the infinite-momentum frame and near-LC, and in the continuum as well as finite-volume treatments. Then we studied the role of the scalar zero mode in the forward-scattering regime in the finite volume one-loop calculations for increased resolution parameter K (the continuum limit). Finally, we demonstrated that the LF vacuum amplitudes (bubbles) are non-zero in LFPT contrary to straightforward arguments based on LF momentum conservation and its positivity. The results agree with both the covariant Feynman calculations as well as with the Feynman integrals expressed in terms of the LF variables and evaluated by a careful regularization.

Problems related to compactification near and on the light front have become interesting also in the context of string theory [33]. Since zero modes pose a major challenge in the nonperturbative context, attempts have been made to perform the quantization on a space like surface [34] close to the light front (a parameter η characterizes the "closeness"). By taking $\eta \to 0$ one is supposed to reach the light front surface (sometimes also called "light-like" (LL)). However, this limiting procedure need not be smooth since a light front surface cannot be reached from a space-like surface by a *finite* Lorentz transformation. On the other hand, S-matrix elements should be independent of η for *any* value of η. Thus any η dependence in an S-matrix element signals breakdown of Lorentz invariance as in the results of Ref. [33].

The infinite momentum frame (IMF) [3, 4] is a concept that allows one to simulate perturbative LF theory calculations in an equal time framework by taking the external total longitudinal momentum to infinity. In scalar field theory, in the discretized version, one can ask whether one can simulate DLCQ perturbation theory by considering the IMF starting from the equal time formulation.

To clarify these issues, we have performed and compared perturbative calculations for scalar field

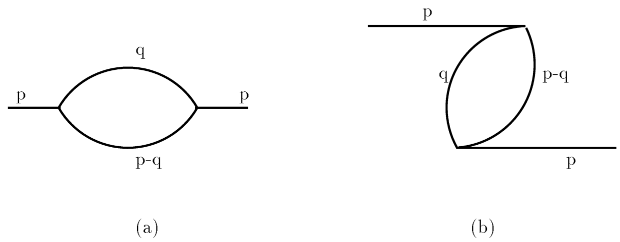

Figure 1: Two time-ordered one-loopdiagrams of the boson self-energy in the Hamiltonian perturbation theory for the $\lambda\phi^3$ model. Only the first diagram contributes in the LF case.

theory in the continuum and discretized versions of three formulations, namely, light front quantization, infinite momentum limit of equal time quantization and space-like quantization parameterized by η. For simplicity, we consider here mainly the self-energy diagram in ϕ^3 theory. The same overall picture emerges also in the case of scattering diagram in ϕ^4 theory.

First we compare results of the light front perturbation theory with those of the covariant PT in the continuum formulation. We consider the one loop self-energy diagram in ϕ^3 theory. Note that in this case there is only one time ordered diagram (the first diagram of Fig.1) in the light front case. Using the rules of light front old fashioned perturbation theory [9], we have

$$\Sigma(p^2) = \frac{\lambda^2}{2}\int_0^{p^+} \frac{dq^+ d^2q^\perp}{2(2\pi)^3}\,\frac{1}{q^+(p^+-q^+)}\,\frac{1}{p^- - \frac{(q^\perp)^2+m^2}{q^+} - \frac{(p^\perp-q^\perp)^2+m^2}{p^+-q^+} + i\epsilon}. \tag{1}$$

Introducing $y = q^+/p^+$, we get

$$\Sigma(p^2) = \frac{1}{2}\frac{\lambda^2}{2(2\pi)^3}\int_0^1 dy d^2q^\perp \frac{1}{y(1-y)p^2 - (q^\perp)^2 - m^2 + i\epsilon},\;\; p^2 = p^+p^- - (p^\perp)^2. \tag{2}$$

Next we derive this result starting from the Feynman diagram. The corresponding amplitude is

$$-i\Sigma(p^2) = \frac{1}{2}\frac{(-i\lambda)^2}{(2\pi)^4}\int d^4k\,\frac{i}{k^2-m^2+i\epsilon}\,\frac{i}{(p+k)^2-m^2+i\epsilon}. \tag{3}$$

Using $d^4k = \frac{1}{2}dk^+dk^-d^2k^\perp$, we have

$$\begin{aligned}\Sigma(p^2) \quad &= -\frac{i}{2}\frac{(-i\lambda)^2}{(2\pi)^4}\frac{1}{2}\int_{-\infty}^{+\infty} dk^+ \int d^2k^\perp \int_{-\infty}^{+\infty} dk^- \frac{1}{k^+(p^++k^+)} \times \\ &\times \frac{1}{k^- - \frac{(k^\perp)^2+m^2}{k^+} + i\frac{\epsilon}{k^+}}\frac{1}{p^- + k^- - \frac{(p^\perp+k^\perp)^2+m^2}{p^++k^+} + i\frac{\epsilon}{p^++k^+}}.\end{aligned} \tag{4}$$

Let us now perform the k^- integration. Let $p^+ > 0$. For $k^+ > 0$, $p^+ + k^+ > 0$, both poles are in the lower half of the complex k^- plane. We can close the contour in the upper half plane and the integral is zero. For $k^+ < 0$, if $p^+ < -k^+$, $p^+ + k^+ < 0$, both poles are in the upper half plane. We can close the contour in the lower half plane and the integral again is zero. For $p^+ > 0$, we get a non-vanishing contribution when $k^+ < 0$ and $k^+ > -p^+$:

$$\Sigma(p^2) \quad = \frac{1}{2}\frac{(-i\lambda)^2}{2(2\pi)^3}\int_{-p^+}^{0}\frac{dk^+d^2k^\perp}{k^+(p^++k^+)} \times$$
$$\times\frac{1}{p^- + \frac{(k^\perp)^2+m^2}{k^+} - \frac{(p^\perp-k^\perp)^2+m^2}{p^++k^+} - i\frac{\epsilon}{k^+} + i\frac{\epsilon}{p^++k^+}} \tag{5}$$

or

$$\Sigma(p^2) = \frac{\lambda^2}{2}\int_0^{p^+}\frac{dq^+d^2q^\perp}{2(2\pi)^3}\,\frac{1}{q^+(p^+-q^+)}\,\frac{1}{p^- - \frac{(q^\perp)^2+m^2}{q^+} - \frac{(p^\perp-q^\perp)^2+m^2}{p^+-q^+} + i\epsilon}. \tag{6}$$

We recover the expression (Eq. (1)) from old fashioned perturbation theory with energy denominator and integration over three momentum.

The same amplitudes can be calculated in the DLCQ framework. The mode expansion for the normal-mode field $\phi_n(\underline{x})$ is

$$\phi_n(\underline{x}) = \frac{1}{\sqrt{V}}\sum_{\underline{k}}\frac{1}{\sqrt{k^+}}\left[a_{\underline{k}}e^{-i\underline{kx}} + a^\dagger_{\underline{k}}e^{i\underline{kx}}\right]. \tag{7}$$

The corresponding DLCQ Hamiltonian, obtained from the energy-momentum tensor, is

$$P^- = \int_V d^3\underline{x}\left[m^2\phi^2 + (\partial_\perp\phi)^2 + \frac{\lambda}{3}\phi^3\right]. \tag{8}$$

It contains ZM terms, which have to be expressed by means of the normal-mode field $\phi_n(\underline{x})$. To do so we need to obtain the lowest-order solution of the ZM constraint

$$(m^2 - \partial_\perp^2)\phi_0 = -\frac{\lambda}{2}\int_{-L}^{L}\frac{dx^-}{2L}(\phi_0^2 + \phi_n^2). \tag{9}$$

In the Fock representation, the solution is

$$\phi_0(x^\perp) = -\frac{\lambda}{V}\sum_{\underline{k}_1,\underline{k}_2}\frac{\delta_{k_1^+,k_2^+}}{\sqrt{k_1^+k_2^+}}\frac{e^{-i(k_1^\perp-k_2^\perp)x^\perp}}{m^2+(k_1^\perp-k_2^\perp)^2}a^\dagger_{\underline{k}_1}a_{\underline{k}_2} - \frac{\lambda}{2m^2}\frac{1}{V}\sum_{\underline{k}_1}\frac{1}{k_1^+}, \tag{10}$$

The interacting Hamiltonian P^-_{int} contains a term, corresponding to the usual one of the continuum formulation, plus the ZM term, calculated to $O(\lambda^2)$:

$$P^-_{int} = P_{NM} + P^{-(2)}_{ZM}, \quad P_{NM} = \frac{\lambda}{3}\int_V d^3\underline{x}\phi_n^3, \tag{11}$$

$$P^{-(2)}_{ZM} = \int_V d^3\underline{x}\left[\phi_0(m^2-\partial_\perp^2)\phi_0 + \frac{\lambda}{3}(\phi_0\phi_n^2 + \phi_n\phi_0\phi_n + \phi_n^2\phi_0)\right] \tag{12}$$

The $O(\lambda^2)$ self-energy amplitude, corresponding to the first term in (11), is calculated by the Hamiltonian (time-ordered) perturbation theory formula

$$T_{fi} = \sum_n\frac{\langle\underline{p}'|P_{NM}|n\rangle\langle n|P_{NM}|\underline{p}\rangle}{p^- - p_n^-}, \tag{13}$$

where $|\underline{p}\rangle \equiv a^\dagger_{\underline{p}}|0\rangle$ and the summation runs over the two-particle intermediate states $|n\rangle \equiv 2^{-\frac{1}{2}} a^\dagger_{\underline{i}_1} a^\dagger_{\underline{i}_2}|0\rangle$. Σ corresponds to the invariant amplitude M_{fi} which differs by a kinematical factor from T_{fi}.

Its contribution to the boson self-energy in the first order perturbation theory is

$$\tilde{T}_{fi} = -\frac{1}{6}\frac{\delta_{\underline{p}',\underline{p}}}{\sqrt{p^+ V p'^+ V}}\frac{\lambda^2}{p^+}\sum_{q^\perp}\frac{1}{m^2+(p^\perp-q^\perp)^2}. \tag{14}$$

The corresponding M-amplitude vanishes in the continuum limit due to the extra L^{-1} factor:

$$\tilde{\Sigma}(p^+,p^\perp) = -\frac{1}{6}\frac{\lambda^2}{(2\pi)^2}\frac{1}{L}\frac{1}{p^+}\int d^2q^\perp\frac{1}{m^2+(p^\perp-q^\perp)^2}. \tag{15}$$

In this way, DLCQ calculation yields the correct covariant result for the one-loop self-energy in $\lambda\phi^3$ theory in the infinite-volume limit.

In order to calculate the one-loop scattering amplitude in DLCQ perturbation theory for ϕ^4 (3+1) model, we again need to derive the LF Hamiltonian with $O(\lambda^2)$ ZM effective interactions. Following the with the interaction $(4!)^{-1}\lambda\phi^4$, we find

$$P^-_{int} = \frac{2\lambda}{4!}\int_V d^3\underline{x}\;\phi_n^4(\underline{x}) \quad + \quad P^{-(2)}_{ZM},\;\; P^{-(2)}_{ZM} = \int_V d^3\underline{x}\left[\phi_0(m^2-\partial_\perp^2)\phi_0 + \frac{2\lambda}{4!}\;4\phi_0\phi_n^3\right]. \tag{16}$$

with the solution of the ZM constraint

$$\phi_0 = -\frac{\lambda}{3!}\frac{1}{m^2-\partial_\perp^2}\int_{-L}^{L}\frac{dx^-}{2L}\phi_n^3. \tag{17}$$

The invariant amplitude $\tilde{M}$ again vanishes for $L\to\infty$:

$$\tilde{M}_{fi}(\underline{p}_3-\underline{p}_1) = -\frac{\lambda^2}{8(2\pi)^3}\frac{1}{p_3^+-p_1^+}\frac{1}{L}\int d^2q^\perp\frac{1}{m^2+(q^\perp+p_1^\perp-p_3^\perp)^2} + (1\leftrightarrow 3). \tag{18}$$

In the continuum version of the **near-LF approach**, for the ϕ^3 self-energy, we have,

$$\begin{aligned}\Sigma(p^2) \;&= \frac{1}{2}\lambda^2\int_{-\infty}^{+\infty}\frac{dq_- d^2q^\perp}{(2\pi)^3}\Big(\frac{1}{\frac{1}{\eta^2}(E_{on}(p)-E_{on}(q)-E_{on}(p-q))+i\epsilon}\\ &\quad - \frac{1}{\frac{1}{\eta^2}(E_{on}(p)+E_{on}(q)+E_{on}(p-q))-i\epsilon}\Big) = \Sigma_I(p^2)+\Sigma_{II}(p^2).\end{aligned} \tag{19}$$

where $E_{on}(k) = \sqrt{(k_-)^2+\eta^2(m^2+(k^\perp)^2)}$. The two contributions correspond to two different time orderings (Figs.1) in old fashioned perturbation theory. By considering the $\eta\to 0$ limit of these expressions, one obtains the LF answer as the other contributions scale as η^2 and thus vanish.

The same diagram in the discretized form reads

$$\begin{aligned}\Sigma_I(p^2) = \frac{1}{2}\lambda^2\frac{1}{2L}\sum_n\int\frac{d^2q^\perp}{(2\pi)^2}\frac{1}{2\sqrt{(\frac{n\pi}{L})^2+\eta^2((q^\perp)^2+m^2)}}\times\\ \times\frac{1}{2\sqrt{(\frac{(j-n)\pi}{L})^2+\eta^2((p^\perp-q^\perp)^2+m^2)}}\frac{1}{\frac{1}{\eta^2}(E_i-E_I)+i\epsilon}\end{aligned} \tag{20}$$

where the energy of the intermediate (I) state is given by

$$E_I = \sqrt{(\frac{n\pi}{L})^2 + \eta^2((q^\perp)^2 + m^2)} + \sqrt{(\frac{(j-n)\pi}{L})^2 + \eta^2((p^\perp - q^\perp)^2 + m^2)}$$

and similarly for the initial state (E_i). The discretized longitudinal momenta are $q_- = n\pi L^{-1}, \;\; p_- = j\pi L^{-1}, \;\; n, j = 0, \pm 1, \pm 2, \ldots$ For $j, n \neq 0$, as $\eta \to 0$, we get the result independent of η and L.

For $n > j$, the amplitude vanishes as $\eta^2 L^2$ for fixed L. For $n = j = 0$, the amplitude diverges as $\frac{1}{\eta L}$. For Fig.1b, we have

$$\Sigma_{II}(p^2) = -\frac{1}{2}\lambda^2 \frac{1}{2L} \sum_n \int \frac{d^2 q^\perp}{(2\pi)^2} \frac{1}{2\sqrt{(\frac{n\pi}{L})^2 + \eta^2((q^\perp)^2 + m^2)}} \times$$
$$\times \frac{1}{2\sqrt{(\frac{(j-n)\pi}{L})^2 + \eta^2((p^\perp - q^\perp)^2 + m^2)}} \frac{1}{\frac{1}{\eta^2}(E_i + E_I) - i\epsilon}. \tag{21}$$

For $j, n \neq 0$, as $\eta \to 0$, the amplitude vanishes as $\eta^2 L^2$. For $n = j = 0$, the amplitude diverges as $\frac{1}{L\eta}$. It is not difficult to understand the origin of this divergence: since there is no dynamical scalar zero mode on the light front, the sum over intermediate states cannot include this mode. On the other hand, for arbitrarily small but non-zero η (space-like quantization) there is a dynamical zero mode in the sum over intermediate states. By requiring this state to exist in the limit we are not approaching the light front theory but some peculiar (divergent) regime of the space-like theory. The LF theory has its own mechanisms (constraints for zero modes) to replace this "missing" dynamical mode.

The same diagram in the **infinite momentum frame** reads

$$\Sigma(p^2) = \frac{\lambda^2}{2} \int_{-\infty}^{+\infty} \frac{d^3 q}{(2\pi)^3} \frac{1}{2E_q} \frac{1}{2E_{p-q}} \Big(\frac{1}{E_p - E_q - E_{p-q} + i\epsilon} + \frac{1}{E_p + E_q + E_{p-q} - i\epsilon} \Big) \tag{22}$$
$$= \Sigma_I(p^2) + \Sigma_{II}(p^2), \quad E_p = \sqrt{p^2 + (p^\perp)^2 + m^2}. \tag{23}$$

p is the third component of the three-vector $\mathbf{p}$. The two contributions correspond to two different time orderings in old fashioned perturbation theory. To facilitate the infinite momentum limit, we parametrize the internal momenta as follows: $\mathbf{q} = (xp, q^\perp)$, $\mathbf{p} - \mathbf{q} = ((1-x)p, p^\perp - q^\perp)$.

$\Sigma_I(p^2)$ in the $p \to \infty$ limit is

$$\Sigma_I(p^2) = \frac{1}{2} \frac{\lambda^2}{(2\pi)^3} \int_{-\infty}^{+\infty} dx d^2 q^\perp \; \frac{1}{2x} \; \frac{1}{2 \mid 1-x \mid p} \times \tag{24}$$
$$\times \frac{1}{p(1- \mid x \mid - \mid 1-x \mid) + \frac{m^2 + (p^\perp)^2}{2p} - \frac{m^2 + (q^\perp)^2}{2|x|p} - \frac{m^2 + (p^\perp - q^\perp)^2}{2|1-x|p}}.$$

Now we have to distinguish various regions. For $x \geq 0$, $1 - x \geq 0$, we get

$$\Sigma_I(p^2) = \frac{1}{2} \frac{\lambda^2}{(2\pi)^3} \int_0^1 dx \frac{1}{\frac{m^2 + (p^\perp)^2}{2} - \frac{m^2 + (q^\perp)^2}{2x} - \frac{m^2 + (p^\perp - q^\perp)^2}{2(1-x)}} \tag{25}$$

which agrees with the light front answer. Contributions from the other two refions are suppressed as p^{-2}. One also finds that in the infinite momentum limit ($p \to \infty$), the "backward going diagram" $\Sigma_{II}(p^2)$ vanishes as $\frac{1}{p^2}$ in accordance with Weinberg's results [4].

In the finite-volume or discretized calculations, we again restrict the longitudinal coordinate to a finite interval. Specifically, we set $-L < x^3 < +L$. The longitudinal momenta $q^3 \to q^3_n = \frac{\pi}{L}n$, $n = 0, \pm 1, \pm 2, \ldots$. The field operator at $t = 0$ becomes

$$\phi(x) = \sum_n \int d^2q^\perp \frac{1}{\sqrt{4L\omega_n}} \Big[a_n(q^\perp) e^{i\frac{n\pi x^3}{L} + iq^\perp\cdot x^\perp} + a^\dagger_n(q^\perp) e^{-i\frac{n\pi x^3}{L} - iq^\perp \cdot x^\perp}\Big]. \tag{26}$$

Let us consider the ϕ^3 self energy. For the external momentum we set $p = (\frac{j\pi}{L}, p^\perp)$. We obtain

$$\Sigma(p^2) = \frac{1}{2}\lambda^2 \frac{1}{2L} \sum_n \int \frac{d^2q^\perp}{(2\pi)^2} \frac{1}{2\sqrt{(\frac{n\pi}{L})^2 + M(q^\perp)}} \frac{1}{2\sqrt{(\frac{(j-n)\pi}{L})^2 + M(p^\perp - q^\perp)}}$$
$$\Big(\frac{1}{\sqrt{(\frac{j\pi}{L})^2 + M(p^\perp)} - \sqrt{(\frac{n\pi}{L})^2 + M(q^\perp)} - \sqrt{(\frac{(j-n)\pi}{L})^2 + M(p^\perp - q^\perp)}}$$
$$- \frac{1}{\sqrt{(\frac{j\pi}{L})^2 + M(p^\perp)} + \sqrt{(\frac{n\pi}{L})^2 + M(q^\perp)} + \sqrt{(\frac{(j-n)\pi}{L})^2 + M(p^\perp - q^\perp)}} \Big), \tag{27}$$

where $M(p^\perp) \equiv (p^\perp)^2 + m^2$, etc. If we take the continuum limit, then $\frac{1}{2L}\sum_n \to \frac{dq}{2\pi}$ and we obtain the result of the previous subsection. Then, taking the infinite momentum limit, the second contribution drops out and we get the light front answer from the first contribution alone.

The same results are obtained with regard to the relationship between the LF, near-LF and infinite-momentum frame one-loop perturbative scattering amplitudes [35] (finite-volume vs. continuum formulations). The conclusion is that the $\eta \to 0$ limit does not lead us to the light front theory but to a peculiar non-covariant regime of the discretized space-like theory.

On the other hand, the continuum version of the near light front old fashioned perturbation theory reproduces the light front answers (which agree with the covariant ones). But since this formulation has no particular advantages there is no real reason to use it in practical calculations.

The conclusion for the LF theory is that the "light-like compactification" is feasible and DLCQ is a consistent scheme. The discretized formulation of the quantum theory on the LL surface does exist as a straightforward LF field theory, but *not* as a limit of a space-like compactification.

Forward scattering amplitude in a finite volume. We consider the scattering amplitude at one loop level in ϕ^4 theory. p_1, p_2 are the initial momenta and p_3, p_4 are the final momenta. Let us denote $s = (p_1+p_2)^2 = (p_1+p_2)^+(p_1+p_2)^- = (p_1^+ + p_2^+)^2 m^2/p_1^+p_3^+$ and $t = (p_1-p_3)^2 = (p_1-p_3)^+(p_1-p_3)^- = -(p_1^+ - p_3^+)^2 m^2/p_1^+p_3^+$. According to the rules of light front perturbation theory, we have to consider two cases separately. For $p_1^+ > p_3^+$, which defines one possible x^+ ordering, the amplitude (see Fig.2a) is

$$\begin{aligned} M_{fi} &= \frac{1}{2}\frac{\lambda^2}{4\pi}\,\theta(p_1^+ - p_3^+) \int_0^{p_1^+ - p_3^+} dq_1^+ \,\frac{1}{q_1^+}\,\frac{1}{p_1^+ - p_3^+ - q_1^+} \times \\ &\qquad \times \frac{1}{p_1^- + p_2^- - p_3^- - p_2^- - q_1^- - (p_1 - p_3 - q_1)^-} \\ &= \frac{1}{2}\frac{\lambda^2}{4\pi m^2}\frac{p_1^+p_3^+}{p_1^+ + p_3^+}\frac{\theta(p_1^+ - p_3^+)}{p_1^+ - p_3^+}\int_0^{p_1^+ - p_3^+} dq_1^+ \Big[\frac{1}{q_1^+ - p_1^+} - \frac{1}{q_1^+ + p_3^+}\Big], \end{aligned} \tag{28}$$

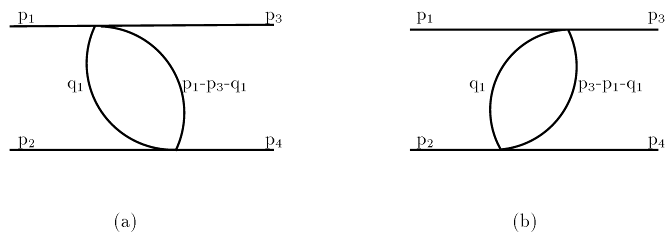

Figure 2: ϕ^4 scattering diagrams in time-ordered LF perturbation theory

We are interested in the forward scattering amplitude, i.e., in $\mid p_1^+ - p_3^+ \mid \to 0$ limit. In this limit q_1^+ is very small compared to both p_1^+ and p_3^+ and it is legitimate to expand the integrands. We get,

$$\frac{1}{q_1^+ - p_1^+} - \frac{1}{q_1^+ + p_3^+} \approx -\frac{p_1^+ + p_3^+}{p_1^+ p_3^+}, \quad \frac{1}{q_1^+ - p_3^+} - \frac{1}{q_1^+ + p_1^+} \approx -\frac{p_1^+ + p_3^+}{p_1^+ p_3^+}. \tag{29}$$

Thus, in the forward scattering limit, we get $M_{fi} = -\lambda^2/(8\pi m^2)$.

Using the second-order formula with $|p\rangle \to |p_1^+, p_2^+\rangle$, $|p'\rangle \to |p_3^+, p_4^+\rangle$ and with four-particle intermediate states, one finds for the second-order normal mode scattering amplitude the expression

$$T_{fi} = \frac{\delta_{p_4^+ + p_3^+, p_2^+ + p_1^+} \theta(p_3^+ - p_1^+)}{(2L)^2 \sqrt{p_4^+ p_3^+ p_2^+ p_1^+}} \frac{\lambda^2}{4} \sum_{q_1^+} \frac{1}{q_1^+ (p_3^+ - p_1^+ - q_1^+)} \frac{1}{p_3^- - p_1^- - q_1^- - (p_3 - p_1 - q_1)^-}$$

plus another term with $1 \leftrightarrow 3$. In DLCQ, we have,

$$t = (p_1^+ - p_3^+)(p_1^- - p_3^-) = -m^2 \frac{(p_1^+ - p_3^+)^2}{p_1^+ p_3^+} = -m^2 \frac{(n_1 - n_3)^2}{n_1 n_3}, \tag{30}$$

independent of L. For convenience, we set $m^2 = 1.0$ and without loss of generality take $p_1^+ > p_3^+$. The scattering amplitude (up to the irrelevant factor $\frac{\lambda^2}{8\pi}$) is

$$M(t) = \frac{n_1 n_3}{n_1 + n_3} \frac{1}{n_1 - n_3} \sum_{n=1}^{n_1 - n_3} \Big[\frac{1}{n - n_1} - \frac{1}{n + n_3} \Big]. \tag{31}$$

For the case of antiperiodic boundary conditions, the scattering amplitude in the t-channel is

$$M(t) = 2 \frac{n_1 n_3}{n_1 + n_3} \frac{1}{n_1 - n_3} \sum_{n=1}^{n_1 - n_3 - 1} \Big[\frac{1}{n - n_1} - \frac{1}{n + n_3} \Big]. \tag{32}$$

The results for the forward-scattering amplitude are plotted in the few figures below.

The conclusion is that the continuum limit of DLCQ produces the correct covariant limit for the one-loop scattering amplitude including processes with $p^+ = 0$ exchange in the t-channel.

Vacuum diagrams in LF perturbation theory. A simple kinematical argument (positivity and conservation of the LF momentum) suggests that vacuum amplitudes are forbidden in the LF field theory as the corresponding vertices would violate momentum conservation. On the other hand, when the covariant Feynman perturbative amplitudes are rewritten in terms of the LF variables and the k^- integration is done in a mathematically consistent way (using appropriate regularization), the

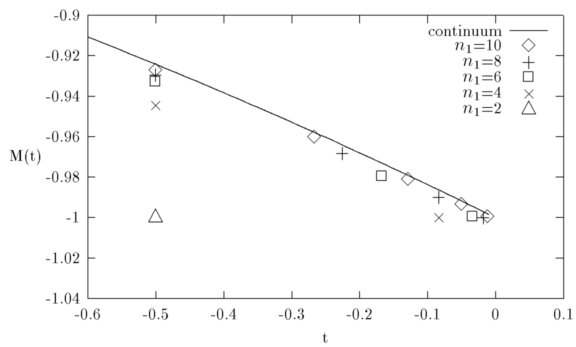

Figure 3: The behaviour of the amplitude $M(t)$ for $n_1 = 2, 4, 6, 8, 10$ for small values of t.

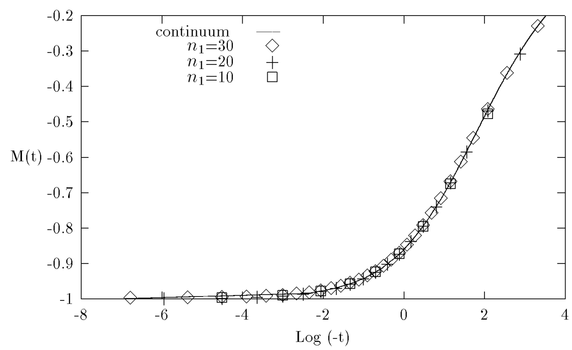

Figure 4: The amplitude $M(t)$ plotted as a function of $log(-t)$ in DLCQ for $n_1 = 10, 20, 30$ and compared with the continuum result.

LF vacuum amplitudes are found to be non-vanishing and in agreement with the standard evaluation of the Feynman integrals. This contradicts the abovementioned direct LF evaluation and leads to questions concerning the equivalence [36] between the usual SL and the LF schemes, and even a failure of the genuine LF theory.

We show here that the self-energy diagrams, when represented in the LF perturbation theory, reduce for external momentum $p = 0$ to nonzero limits equal to the values of vacuum bubbles known from the conventional Feynman diagrams.

When the LFPT rules are applied to vacuum diagrams, the result is ill-defined [10], as the delta function requires all three momenta to be equal to zero:

$$\tilde{V} \sim \int_0^\infty \frac{dk_1^+}{k_1^+} \int_0^\infty \frac{dk_2^+}{k_2^+} \int_0^\infty \frac{dk_3^+}{k_3^+} \frac{\delta(k_1^+ + k_2^+ + k_3^+)}{(-\mu^2)[\frac{1}{k_1^+} + \frac{1}{k_2^+} + \frac{1}{k_3^+}]}. \tag{33}$$

To resolve this inconsistency, one has to start from the graph with nonvanishing external momentum p and to compute the $p = 0$ limit in the corresponding integrals. In this way, the expression

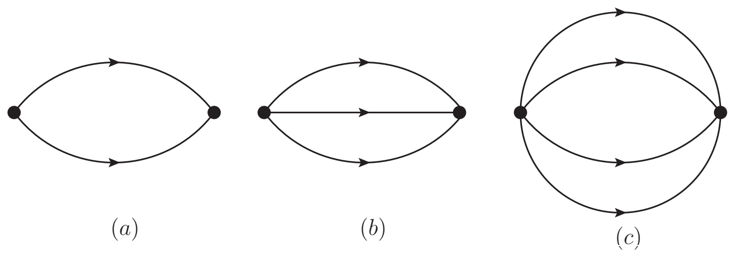

Figure 5: Vacuum diagrams (a) $\Sigma_3(0) \equiv V_2(\mu)$ (equal to the Green function of $\phi^2(x)\phi^2(0)$ composite operator in free theory [36]), (b) $\Sigma_4(0) \equiv V_3(\mu)$ for the ϕ^3 model and (c) for ϕ^4 model (not discussed in the main text). The arrows indicate the momentum flow.

(33) is first replaced by the self-energy

$$\Sigma_4(p^2) = \tilde{N}_4\lambda^2 \int\limits_0^{p^+} \frac{dk_1^+}{k_1^+} \int\limits_0^{p^+-k_1^+} \frac{dk_2^+}{k_2^+(p^+ - k_1^+ - k_2^+)} \frac{1}{p^- - \frac{\mu^2}{k_1^+} - \frac{\mu^2}{k_2^+} - \frac{\mu^2}{p^+-k_1^+-k_2^+} + i\epsilon}. \quad (34)$$

$\tilde{N}_4^{-1} = 3!(4\pi)^2$. Introducing the dimensionless variables $x = \frac{k_1^+}{p^+}$, $y = \frac{k_2^+}{p^+}$, $\Sigma_4(p)$ becomes

$$\Sigma_4(p^2) = \int\limits_0^1 \frac{dx}{x} \int\limits_0^{1-x} \frac{dy}{y(1-x-y)} \frac{\tilde{N}_4\lambda^2}{p^2 - \frac{\mu^2}{x} - \frac{\mu^2}{y} - \frac{\mu^2}{1-x-y}}. \quad (35)$$

Now we can set $p = 0$. The integral over the variable y can be performed explicitly, yielding

$$F(x) = \frac{1}{\mu^2} \frac{4x}{\sqrt{3x^2 - 2x - 1}} \arctan \frac{1-x}{\sqrt{3x^2 - 2x - 1}}. \quad (36)$$

The numerical computation of the second integral gives

$$\Sigma_4(0) = \lambda^2 \tilde{N}_4 \int\limits_0^1 \frac{dx}{x} F(x) \equiv V_3(\mu) = -\frac{\lambda^2}{\mu^2}\tilde{N}_4 C, \ \ C = 2.343908\ldots, \quad (37)$$

in the complete agreement with the space-like computations [37].

This result can be understood in detail in a simpler case of the one-loop self-energy diagram in the $\lambda\phi^3$ theory (see Fig.5a) with two external lines attached). The Feynman amplitude is

$$-i\Sigma_3(p^2) = \frac{1}{2}\frac{(-i\lambda)^2}{(2\pi)^2} \int d^2k\, G(k)G(p-k), \quad (38)$$

The vacuum bubble ($p = 0; \lambda = 1$) rewritten in terms of the LF variables,

$$V_2(\mu) = \frac{i}{16\pi^2} \int\limits_{-\infty}^{+\infty} dk^+ \int\limits_{-\infty}^{+\infty} dk^- \frac{1}{(k^+k^- - \mu^2 + i\epsilon)^2}. \quad (39)$$

is correctly evaluated with a cutoff Λ [10], leading to the correct form($\tilde{N}_3 = -i/16\pi^2$)

$$V_2(\mu) = \frac{\tilde{N}_3}{\mu^2} \int\limits_{-\infty}^{+\infty} dk^+ \Big[\frac{1}{k^+ + i\epsilon} - \frac{1}{k^+ - i\epsilon}\Big] = \frac{\tilde{N}_3}{\mu^2}(-2\pi i) \int\limits_{-\infty}^{+\infty} dk^+ \delta(k^+) = -\frac{1}{8\pi}\frac{1}{\mu^2}. \quad (40)$$

Table 1: Smooth approach of the one-loop self-energy amplitude $\Sigma_3(p)$ of the $\lambda\phi^3$ model from Eq.(45) to its (rescaled) value $S = 1$ at $p = 0$.

p^2/μ^2	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	0
S	1.50902	1.09320	1.00667	0.99890	0.99813	0.99805

This diagram sheds light upon the mechanism at work in the genuine LF case. The LFPT formula for the self-energy Σ_3 (38)

$$\Sigma_3(p) = \frac{1}{8\pi}\int_0^{p^+} \frac{dk^+}{k^+(p^+ - k^+)} \frac{1}{p^- - \frac{\mu^2}{k^+} - \frac{\mu^2}{p^+ - k^+} + i\epsilon} \to \frac{1}{8\pi}\int_0^1 \frac{dx}{p^2 x(1-x) - \mu^2 + i\epsilon} \tag{41}$$

upon introducing $x = k^+/p^+$. For $p = 0$ one indeed easily reproduces $V_2(\mu)$ of (40) (see Fig. 9(a)). Working directly with the first form of (41) and taking $p^+ = p^- = \eta$ for simplicity, we get

$$\Sigma_3(\eta) = -\frac{1}{4\pi}\big(G(\eta) - G(0)\big), \quad G(k) = \frac{\arctan\left(\frac{2k-\eta}{\sqrt{4\mu^2 - k^2}}\right)}{\eta\sqrt{4\mu^2 - \eta^2}}. \tag{42}$$

The expansion for infinitesimal η gives

$$\Sigma_3(\eta) = -\frac{1}{8\pi}\frac{1}{\mu^2}\Big[1 + \frac{\eta^2}{4\mu^2} + \mathcal{O}(\eta^4)\Big]. \tag{43}$$

The correct result is recovered for $\eta = 0$ due to compensation between the η^{-1} factor in theintegrand and the interval length η. Starting with $\eta = 0$ yields ill-defined expression.

The DLCQ analog of the $\Sigma_3(p)$ amplitude is ($\mathcal{N}_3$ being the normalization constant)

$$\Sigma_3(p) = \mathcal{N}_3 \sum_{k^+}^{p^+} \frac{1}{k^+(p^+ - k^+)} \frac{1}{\left[p^- - \frac{\mu^2}{k^+} - \frac{\mu^2}{p^+ - k^+}\right]} \ , p^+ = \frac{2\pi}{L}K, \ k^+ \equiv k_n^+ = \frac{2\pi}{L}n, \tag{44}$$

$n = 1, 2, \ldots, K-1$. For $p = 0$ and periodic BC, $\Sigma_3(0) = V_2(\mu)$ becomes

$$V_2(\mu) = -\frac{1}{8\pi\mu^2}S, \ \ S = \sum_{n=1}^{K-1} \frac{1}{n(K-n)} \frac{1}{\left[\frac{1}{n} + \frac{1}{K-n}\right]} = \frac{K-1}{K}. \tag{45}$$

For $K \to \infty$, S obviously converges to the continuum value 1. The same result is obtained for the antiperiodic BC [38]. In Table 1, the smooth approach of the self-energy to the vacuum-loop value as $p \to 0$ is shown. The vacuum bubble with three internal lines can be computed in the same way.

Chapter summary: We calculated one-loop self-energy and scattering diagrams of scalar self-interacting models in D=3+1 using continuum and finite-volume versions of three approaches: the genuine LF theory, the "near-LF" theory extrapolated to the light front, and the infinite-momentum frame approach. The covariant results were obtained in the continuum limit of the finite-volume DLCQ calculations. In the infinite-momentum frame method, only the LF diagrams gave non-zero contributions. The amplitudes obtained in the compactified near-LF formulation were found to have singular light-front limits, the reason for this behaviour was clarified as the difference in the character of scalar-field zero modes in the two formulations. The DLCQ formulation thus cannot be obtained as the "light-like" limit of the SL quantization. In two dimensional ϕ^4 theory, the perturbative DLCQ amplitudes were also found to match the covariant scattering amplitudes including in the forward-scattering limit, and even without the (non-existent) LF scalar zero mode. A similar situation was discovered also in the case of vacuum bubbles: while in the evaluation of the covariant vacuum Feynman diagrams in terms of the LF variables for the self-interacting scalar models the non-zero values of the vacuum amplitudes are due to the sharp $k^+ = 0$ contribution (k^+ is the loop variable), the situation in the genuine LF theory is somewhat different. The Fourier mode carrying $k^+ = 0$ is not present in the LF perturbation theory but the correct results are nevertheless obtained if one uses a regularized approach with non-zero incoming momentum p, which adequatly treats the integration region close to $k^+ = 0$. Passing to the relative LF variables (which coincide with the Feynman parameters) maps this picture to the $p = 0$ limit of the corresponding self-energy Feynman diagrams.

Chapter 2. Spontaneous symmetry breaking and LF vacuum. Spontaneous symmetry breaking (SSB) is a fundamental non-perturbative property of certain quantum field-theoretical models. It occurs when the Hamiltonian of a theory is symmetric under a group of transformations while the ground state is non-invariant. This implies that the vacuum is not unique since its non-invariance means that it is transformed to another vacuum state. Hence there must be a family of ground states which all correspond to the same energy, i.e. they are degenerate.

This overall picture of the broken phase is well understood in the conventional field theory. It is generally believed however that it is not possible in the LF theory because the Fock vacuum, i.e. the state without particles, is the "final" physical vacuum, since it is an eigenstate of the complete Hamiltonian, not only its free part. Hence there is just one unique vacuum, and no degenerate set, necessary for the standard picture of SSB, can be constructed. The detailed reason is that charges, i.e. the symmetry generators in a scalar theory always annihilate the LF Fock vacuum because due to positivity of the momentum p^+ they cannot contain terms composed of purely creation operators [39,40] if there are no dynamical zero modes, i.e. independent degrees of freedom carrying $p^+ = 0$, in the theory. Without such terms it is not possible to transform the LF Fock vacuum into a more complex object and one cannot construct multiple vacua.

We have succeeded in developing a simple LF picture of the symmetry breaking. It is based on a concept which is close to the scheme known from the conventional field theory. We gave tree potential mechanisms in three models: the 2-dimensional scalar ϕ^4 theory in the broken phase, in the $O(2)$-symmetric sigma model with fermions and in the massive Schwinger model in the light-cone gauge with a residual ("large") gauge symmetry present in the finite volume treatment.

Semiclassical picture of SSB in 2-dimensional scalar models. We start from the Lagrangian of $\phi^4(1+1)$ model expressed in terms of the LF variables

$$\mathcal{L}_{lf} = 2\partial_+\phi\partial_-\phi + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4, \quad L \leq x^- \leq L. \tag{46}$$

We choose $\phi(-L) = -\phi(L)$ at our finite interval. The basic commutator

$$\left[\phi(0,x^-), \Pi_\phi(0,y^-)\right] = i\delta_a(x^- - y^-), \;\; \Pi_\phi = 2\partial_-\phi, \;\; \delta_a(x^-) = 1/2\partial_-\epsilon_a(x^-), \tag{47}$$

where $\delta_a(x^-)$ is the antiperiodic delta function, permits one to define a unitary operator

$$U(b) = \exp\big[-ib\int\limits_{-L}^{+L}\frac{dx^-}{2}\Pi_\phi(x^-)\big] = e^{-4ib\phi(L)} \tag{48}$$

which translates the field $\phi(x^-)$ by a constant b. Minimizing the expectation value of the LF energy between the transformed (shifted) vacuum states $|b\rangle = U(b)|0\rangle$, we get the semiclassical relations $\langle b|P^-|b\rangle = Lb^2(\frac{\lambda}{2}b^2 - \mu^2)$ which has a non-trivial minimum for $b^2 = \frac{\mu^2}{\lambda} \equiv v^2$. The LF energy density is lower in the new vacuum $|v\rangle$:

$$\langle v|\frac{P^-}{2L}|v\rangle = -\frac{\mu^4}{4\lambda} \;\; < \;\; \langle 0|\frac{P^-}{2L}|0\rangle = 0. \tag{49}$$

The vacuum expectation value (VEV) of the scalar field in this state coincides with the position of the minimum of the classical potential:

$$\langle v|\phi(x^-)|v\rangle = \langle 0|U^{-1}(v)\phi(x^-)U(v)|0\rangle = \frac{\mu}{\sqrt{\lambda}}\epsilon_\Lambda(x^- - L) = \frac{\mu}{\sqrt{\lambda}}. \tag{50}$$

In the Fock representation, one has a coherent state representing the physical vacuum of the model in the semi-quantum approximation:

$$|v\rangle = \exp\big\{v\sum_{n=1/2}^{\Lambda}\tilde{c}_n\big(a_n^\dagger - a_n\big)\big\}|0\rangle = \mathcal{N}\exp\big\{v\sum_{n=1/2}^{\Lambda}\tilde{c}_n a_n^\dagger\big\}|0\rangle, \tag{51}$$

with $\tilde{c}_n = 4(-1)^{n-1/2}/\sqrt{\pi n}$. The other usual properties of the broken phase are then found in this simple model as well as in the three-dimensional model with complex scalar fields, where one ca derive the Goldstone theorem by means of the usual arguments known in the SL theory. In the case of periodic boundary conditions, one can solve the zero-mode constraint perturbatively and obtain two Hamiltonians, each corresponding to one of the two semiclassical minima of the potential energy.

Fermion zero modes and LF symmetry breaking. We demonstrate that dynamical fermion zero modes provide a suitable mechanism for a description of vacuum degeneracy in a simple non-gauge field theory with fermions. Charges, which are the generators of global symmetries of the given system, contain a ZM part and consequently transform the trivial vacuum into a continuous set of degenerate vacuum states. This leads to a SSB in the usual sense [41–45] with non-zero vacuum expectation values of certain operators and a massless NG state in the spectrum of states. To simplify our discussion of SSB in the LF field theory, we will consider a version of the $O(2)$-symmetric sigma model with fermions [23, 46] specified by the Lagrangian density

$$\mathcal{L} = \overline{\psi}\big(\frac{i}{2}\gamma^\mu \overleftrightarrow{\partial_\mu} - m\big)\psi + \frac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\pi\partial^\mu\pi) - \frac{1}{2}\mu^2(\sigma^2 + \pi^2) - g\overline{\psi}(\sigma + i\gamma^5\pi)\psi, \tag{52}$$

The Lagrangian (52) is invariant under the global $U(1)$ transformation $\psi \to \exp(-i\alpha)\psi$ and for $m = 0$ also under the axial transformation

$$\psi \to \exp(-i\beta\gamma^5)\psi, \;\; \psi^\dagger \to \psi^\dagger\exp(i\beta\gamma^5), \tag{53}$$

$$\sigma \to \sigma\cos 2\beta - \pi\sin 2\beta, \;\; \pi \to \sigma\sin 2\beta + \pi\cos 2\beta. \tag{54}$$

Rewriting the above Lagrangian in terms of the LF variables and performing the Legendre transformation, one finds for the LF Hamiltonian

$$\begin{aligned} P^- &= \int_V d^3\underline{x}\,\Big[(\partial_k\sigma)^2 + (\partial_k\pi)^2 + \mu^2(\sigma^2+\pi^2) + \psi_+^\dagger\big(m\gamma^0 - i\alpha^k\partial_k\big)\psi_- + \\ &+ \; g\psi_+^\dagger\gamma^0(\sigma + i\gamma^5\pi)\psi_- + h.c.\Big], \end{aligned} \quad (55)$$

where $d^3\underline{x} \equiv \frac{1}{2}dx^- d^2x_\perp$. The Dirac matrices are defined as $\gamma^\pm = \gamma^0 \pm \gamma^3$, $\alpha^k = \gamma^0\gamma^k$, the LF projection operators as $\Lambda_\pm = \frac{1}{2}\gamma^0\gamma^\pm$ and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. $\Lambda_\pm$ separate the fermi field into the independent component $\psi_+ = \Lambda_+\psi$ and the dependent one $\psi_- = \Lambda_-\psi$.

The infrared-regularized formulation is achieved by enclosing the system into a three-dimensional box. The fields are decomposed into the zero-mode (subscript 0) and normal-mode (NM, subscript n) parts. One finds that $\psi_-, \psi_-^\dagger, \sigma_0, \pi_0$ are non-dynamical fields with vanishing conjugate momenta, while $\psi_+, \psi_+^\dagger, \sigma_n, \pi_n$ are dynamical. To quantize the model, we assume at $x^+ = 0$

$$\{\psi_{+i}(\underline{x}), \psi_{+j}^\dagger(\underline{y})\} = \frac{1}{2}\delta_{ij}\delta^3(\underline{x}-\underline{y}),\; i,j = 1,4, \quad (56)$$

$$\big[\sigma_n(\underline{x}), \partial_-\sigma_n(\underline{y})\big] = \big[\pi_n(\underline{x}), \partial_-\pi_n(\underline{y})\big] = \frac{i}{2}\delta_n^3(\underline{x}-\underline{y}). \quad (57)$$

$\delta^3(\underline{x}) = \delta_0^3 + \delta_n^3(\underline{x})$ is the periodic delta function with $\delta_0^3 = 1/V$ being its global zero-mode part. The anticommutator (56) can be derived using the expansion

$$\psi_+(\underline{x}) = \sum_{\substack{p^+,p^\perp \\ s=\pm\frac{1}{2}}} \frac{u(s)}{\sqrt{V}}\big(b(\underline{p},s)e^{-i\underline{px}} + d^\dagger(\underline{p},-s)e^{i\underline{px}}\big), \quad (58)$$

$$\{b(\underline{p},s), b^\dagger(\underline{p}',s')\} = \{d(\underline{p},s), d^\dagger(\underline{p}',s')\} = \delta_{s,s'}\delta_{\underline{p},\underline{p}'}. \quad (59)$$

In the chiral representation, the spinors are $u^\dagger(s=\frac{1}{2}) = (1\,0\,0\,0), u^\dagger(s=-\frac{1}{2}) = (0\,0\,0\,1)$, where s is the LF helicity. The summations in Eq.(59) run over discrete momenta $p^+ = 2\pi L^{-1}n,\; n = 0, 1\ldots, N \to \infty,\; p^i = \pi L_\perp^{-1}n^i,\; n^i = 0, \pm1, \ldots, \pm N_\perp \to \infty$.

The non-dynamical fields satisfy the constraints

$$2i\partial_-\psi_- = \big[m\gamma^0 - i\alpha^k\partial_k + g\gamma^0\left(\sigma + \gamma^5\pi\right)\big]\,\psi_+, \quad (60)$$

$$(\partial_\perp^2 - \mu^2)\sigma_0 = g\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2L}(\psi_+^\dagger\gamma^0\psi_- + H.c.),\;\; (\partial_\perp^2 - \mu^2)\pi_0 = g\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2L}(i\psi_+^\dagger\gamma^0\gamma^5\psi_- + H.c.).$$

We shall set $m = 0$ henceforth to maintain the axial-vector symmetry of the Hamiltonian. The scalar fields are kept massive [23] to avoid infrared problems. Using the solution ψ_{-n} of the constraint (60), the interacting part of P^- takes the form ($\underline{y} = (y^-, x_\perp)$)

$$\begin{aligned} P^-_{int} &= \int_V d^3\underline{x}\,\Big[\mu^2(\sigma_0^2 + \pi_0^2) + (\partial_k\sigma_0)^2 + (\partial_k\pi_0)^2\Big] + \\ &+ ig\int_V d^3\underline{x}\;\psi_+^\dagger(\underline{x})\Sigma^\dagger(\underline{x})\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2L}\frac{1}{2}\epsilon_n(x^- - y^-)\Big[i\gamma^k\partial_k\psi_{+n}(y^-, x_\perp) + h.c. - g\Sigma(\underline{y})\psi_+(\underline{y})\Big], \end{aligned} \quad (61)$$

P^-_{int} and the solution ψ_{-n} are not closed expressions due to the presence of σ_0, π_0, which in turn are given by their own constraints (60), depending on ψ_{-n}. However, this is not an obstacle for determining the symmetry properties of the Hamiltonian, which are of primary importance in the present approach. The symmetries are implemented by the unitary operators $U(\alpha) = \exp(-i\alpha Q)$, $V(\beta) = \exp(-i\beta Q^5)$:

$$\psi_+(\underline{x}) \rightarrow e^{-i\alpha}\psi_+(\underline{x}) = U(\alpha)\psi_+(\underline{x})U^\dagger(\alpha), \;\; \psi_+(\underline{x}) \rightarrow e^{-i\beta\gamma^5}\psi_+(\underline{x}) = V(\beta)\psi_+(\underline{x})V^\dagger(\beta). \quad (62)$$

While the NM parts of the charge operators Q and Q^5

$$Q = \int_V d^3\underline{x} j^+(\underline{x}) = 2\int_V d^3\underline{x}\psi_+^\dagger\psi_+, \;\; Q^5 = 2\int_V d^3\underline{x}\Big[\psi_+^\dagger\gamma^5\psi_+ + 2\Big(\sigma_n\partial_-\pi_n - \pi_n\partial_-\sigma_n\Big)\Big] \quad (63)$$

are diagonal in creation and annihilation operators, the ZM parts contain also off-diagonal terms:

$$Q_0 = \sum_{p_\perp, s}\Big[b_0^\dagger(p_\perp, s)b_0(p_\perp, s) - d_0^\dagger(p_\perp, s)d_0(p_\perp, s) + \Big(b_0^\dagger(p_\perp, s)d_0^\dagger(-p_\perp, -s) + H.c.\Big)\Big] \quad (64)$$

$$Q_0^5 = \sum_{p_\perp, s} 2s\Big[b_0^\dagger(p_\perp, s)b_0(p_\perp, s) + d_0^\dagger(p_\perp, s)d_0(p_\perp, s) + \Big(b_0^\dagger(p_\perp, s)d_0^\dagger(-p_\perp, -s) + H.c.\Big)\Big].$$

The commuting ZM charges Q_0, Q_0^5 *do not* annihilate the LF vacuum $|0\rangle$ defined by $b(\underline{p}, s)|0\rangle = d(\underline{p}, s)|0\rangle = 0$. However, their vacuum expectation values are zero as they have to be. In this way, the vacuum of the model transforms under $U(\alpha), V(\beta)$ as $|0\rangle \rightarrow |\alpha\rangle = \exp(i\alpha Q_0)|0\rangle$, $|0\rangle \rightarrow |\beta\rangle = \exp(i\beta Q_0^5)|0\rangle$, where

$$|\alpha\rangle = \exp\left(i\alpha\sum_{p_\perp, s}\Big[b_0^\dagger(p_\perp, s)d_0^\dagger(-p_\perp, -s) + H.c.\Big]\right)|0\rangle, \quad (65)$$

$$|\beta\rangle = \exp\left(i\beta\sum_{s} 2s\Big[b_0^\dagger(p_\perp, s)d_0^\dagger(-p_\perp, -s) + h.c.\Big]\right)|0\rangle. \quad (66)$$

The vacua contain ZM fermion-antifermion pairs with opposite helicities. Thus, the global symmetry of the Hamiltonian (62) leads to an infinite set of translationally invariant states $|\alpha, \beta\rangle = U(\alpha)V(\beta)|0\rangle$ ($P^+|\alpha, \beta\rangle = P^\perp|\alpha, \beta\rangle = 0$), labeled by two real parameters. Since $U(\alpha), V(\beta)$ commute with P^-, the vacua are degenerate in the LF energy. The Fock space can be built from any of them since they are unitarily equivalent (in a finite volume). The Goldstone theorem can be derived in the next step as we have all the ingredients for the usual proof of the theorem [43–45, 47].

Vacuum structure of the LF massive Schwinger model. Schwinger model, that is $QED(1+1)$ with massless fermion field, has for decades served as a prototype of gauge theories in the usual SL form of the theory because the model incorporates in a simplified form certain non-perturbative and vacuum-related properties expected to be realized im more realistic theories (such as vacuum degeneracy, theta angle, chiral symmetry breaking, the $U(1)$ problem). Adding the mass term for fermions results in an essentially more complicated dynamics which unlike its massless version is not solvable exactly at the level of field equations.

The Lagrangian density of the two-dimensional QED without any gauge-fixing term is

$$\mathcal{L} = \frac{i}{2}\overline{\psi}(x)\gamma^\mu \overset{\leftrightarrow}{\partial_\mu} \psi(x) - m\overline{\psi}(x)\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - ej_\mu(x)A^\mu(x), \tag{67}$$

where m is the mass of the Fermi field $\psi(x)$ and e is the coupling constant, i.e. the electric charge. The gauge potential $A^\mu(x)$ as well as the vector current $j^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi(x)$ has two components. The electromagnetic tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The γ-matrices in the SL formalism are expressed as before in terms of Pauli matrices, $\gamma^0 = \sigma^1, \gamma^1 = i\sigma^2$. The conjugate Fermi field is $\psi = \psi^\dagger\gamma^0$.

In terms of LF space and time variables $x^\mu = x^\pm$, the Lagrangian (67) has the form

$$\mathcal{L}_{LF} = i\psi_2^\dagger \overset{\leftrightarrow}{\partial_+} \psi_2 + i\psi_1^\dagger \overset{\leftrightarrow}{\partial_-} \psi_1 + \frac{1}{2}(\partial_+A^+ - \partial_-A^-)^2 - m(\psi_2^\dagger\psi_1 + \psi_1^\dagger\psi_2) - \frac{e}{2}j^+A^- - \frac{e}{2}j^-A^+.$$

The corresponding field equations read

$$2i\partial_+\psi_2 = m\psi_1 + eA^-\psi_2, \quad \partial_+(\partial_+A^+ - \partial_-A^-) = -\frac{e}{2}j^-, \tag{68}$$

$$2i\partial_-\psi_1 = m\psi_2 + eA^+\psi_1, \quad \partial_-(\partial_+A^+ - \partial_-A^-) = \frac{e}{2}j^+. \tag{69}$$

We will consider the theory on a finite interval $-L \le x^- \le L$ with (anti)periodic fields: $\psi(-L) = -\psi(L), A^\pm(-L) = A^\pm(L)$. This implies that the gauge field can be decomposed into the x^--independent zero-mode (ZM) part $A_0^\pm$ and the normal mode (NM) part $A_n^\pm(x^-)$. The classical Lagrangian (68) is then invariant under the gauge transformations (GT)

$$\psi_2(x) \to e^{-ie\Lambda(x)}\psi_2(x), \quad A^\pm(x) \to A^\pm(x) - 2\partial_\mp\Lambda(x). \tag{70}$$

We study vacuum properties of the LF massive Schwinger model in the finite-volume version of the light cone gauge which in the continuum theory is defined by the condition $A^+(x) = 0$. For periodic gauge field, the gauge transformation (70) imply that the zero mode A_0^+ cannot be removed by any choice of the gauge function. Hence it is a physical, gauge-invariant, dynamical degree of freedom.

The gauge ZM and dynamical fermion field satisfy the (anti)commutation relations

$$\left[A_0^+(x^+), \Pi_{A_0^+}(x^+)\right] = \frac{i}{L}, \quad \{\psi_2(x^-,x^+), {\psi_2}^\dagger(y^-,x^+)\} = \frac{1}{2}\delta_a(x^- - y^-), \tag{71}$$

where $\Pi_{A_0^+} = \partial_+A_0^+$ is the momentum conjugate to A^+. The Fermi field is expanded at $x^+ = 0$ as

$$\psi_2(x^-) = \frac{1}{\sqrt{2L}}\sum_{n=\frac{1}{2}}^{\infty}\left(b_n e^{-\frac{i}{2}p_n^+x^-} + d_n^\dagger e^{\frac{i}{2}p_n^+x^-}\right), \; \{b_n, b_{n'}^\dagger\} = \{d_n, d_{n'}^\dagger\} = \delta_{n,n'}, \tag{72}$$

where $p_n^+ = \frac{2\pi}{L}n, n = \frac{1}{2}, \frac{3}{2}, \ldots$ With $A_n^+ = 0,\; A_0^- = 0$, the ψ_1-constraint and the Gauss' law are

$$2i\partial_-\psi_1 = m\psi_2 + eA_0^+\psi_1, \quad -\partial_-^2A_n^- = \frac{e}{2}j^+ \tag{73}$$

and can be inverted to find $\psi_1(x)$ and $A_n^-(x)$. The Hamiltonian in terms of independent fields is

$$P^- = L\Pi_{A_0^+}^2 - \frac{e^2}{4}\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}\int_{-L}^{+L}\frac{\mathrm{d}y^-}{2L}j^+(x^-)\mathcal{G}_2(x^- - y^-)j^+(y^-) +$$

$$+m_f^2\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}\int_{-L}^{+L}\frac{\mathrm{d}y^-}{2L}\left[{\psi_2}^\dagger(x^-)\mathcal{G}_a(x^- - y^-; A_0^+)\psi_2(y^-) + h.c.\right]. \tag{74}$$

$\mathcal{G}_2$ is the periodic Green's function $\sim (p+)^{-2}$ corresponding to the operator ∂_-^2 in the Gauss' law and the antiperiodic $\mathcal{G}_a$ given in terms of the sign function ϵ_a is

$$\mathcal{G}_a(x^- - y^-; A_0^+) = \frac{1}{4i}\exp\Big(-\frac{ie}{2}(x^- - y^-)A_0^+\Big)\Big[\epsilon_a(x^- - y^-) + i\epsilon_a(L - y^-)\tan(\frac{eL}{2}A_0^+)\Big],$$

The Gauss' law in the ZM sector means electric neutrality of the physical states, $Q|phys\rangle = 0$.

The LF Hamiltonian (74) exhibits a residual symmetry which is not explicitly present in the continuum formulation. It corresponds to transformations with non-trivial topological properties.

For the considered $U(1)$ theory, the corresponding gauge function has the form $\Lambda_\nu = \frac{\pi}{L}\nu x^-$ and defines a winding number ν:

$$\Lambda_\nu(L) - \Lambda_\nu(-L) = 2\pi\nu, \ \ \nu \in Z. \tag{75}$$

Thus, the residual gauge symmetry of the Hamiltonian (74) is

$$A_0^+ \to A_0^+ + \frac{2\pi}{eL}\nu, \ \ \psi_+(x^-) \to e^{-i\frac{\pi}{L}\nu x^-}\psi_+(x^-). \tag{76}$$

Let us discuss the ZM part of the symmetry first. At the quantum level, it is convenient to work with the rescaled ZM operators $\hat{\zeta}$ and $\hat{\pi}_0$:

$$A_0^+ = \frac{2\pi}{eL}\hat{\zeta}, \ \ \Pi_{A_0^+} = \frac{e}{2\pi}\hat{\pi}_0, \ \ \Big[\hat{\zeta}, \hat{\pi}_0\Big] = i. \tag{77}$$

The box length dropped out from the ZM commutator. The shift transformation of A_0^+ is implemented by the unitary operator $\hat{Z}_1$, $\hat{\zeta} \to \hat{Z}_1\hat{\zeta}\hat{Z}_1^\dagger = \hat{\zeta} + 1, \ \ \hat{Z}_1 = \exp(i\hat{\pi}_0)$. The transformation of the ZM operator $\hat{\zeta}$ is accompanied by the transformation of the vacuum state. The latter is defined by $a_0\Psi_0 = 0$, where $a_0(a_0^\dagger)$ is the usual annihilation (creation) operator of a boson quantum:

$$a_0 = \frac{1}{\sqrt{2}}\Big(\hat{\zeta} + i\hat{\pi}_0\Big), \ \ a_0^\dagger = \frac{1}{\sqrt{2}}\Big(\hat{\zeta} - i\hat{\pi}_0\Big), \ \ \Big[a_0, a_0^\dagger\Big] = 1. \tag{78}$$

Implementation of the large gauge transformations for the dynamical fermion field $\psi_2(x)$ is based on the commutator

$$\big[\psi_2(x^-), j^+(y^-)\big] = \psi_2(y^-)\delta_a(x^- - y^-) \tag{79}$$

which follows from the basic anticommutation relation (71). The unitary operators $\hat{F}_\nu = (\hat{F}_1)^\nu$ that implement the phase transformation (76) are

$$\psi_2(x^-) \to \hat{F}_\nu\psi_2(x^-)\hat{F}_\nu^\dagger, \quad \hat{F}_\nu = \exp\left[i\frac{\pi}{L}\nu\int_{-L}^{+L}\frac{\mathrm{d}x^-}{2}x^- j^+(x^-)\right]. \tag{80}$$

The Hilbert space transforms correspondingly. But since physical states carry $Q = 0$ and the operators $b_k^\dagger d_l^\dagger$, which create these states, are invariant, it is only the vacuum state that changes to

$$|0\rangle \to \hat{F}_\nu|0\rangle = \exp\left[\nu\sum_{m=1}^{\infty}\frac{(-1)^m}{m}(A_m^\dagger - A_m)\right]|0\rangle \equiv |\nu; f\rangle. \tag{81}$$

The boson Fock operators $A_m, A_m^\dagger$

$$A_m^\dagger = \sum_{k=\frac{1}{2}}^{m-\frac{1}{2}} b_k^\dagger d_{m-k}^\dagger + \sum_{k=\frac{1}{2}}^{\infty}\Big[b_{m+k}^\dagger b_k - d_{m+k}^\dagger d_k\Big], \tag{82}$$

satisfy $\left[A_m, A^\dagger_{m'}\right] = m\delta_{m,m'}$. They emerge naturally after taking a Fourier transform of the $j^+(x^-)$ current expressed in terms of fermion modes. This yields

$$j^+(x^-) = \frac{1}{L}\sum_{m=1}^{\infty}\left[A_m e^{-\frac{i}{2}p^+_m x^-} + A^\dagger_m e^{\frac{i}{2}p^+_m x^-}\right] \tag{83}$$

as well as the exponential operator in Eq.(81). The states $|\nu; f\rangle$ are not invariant under $\hat{F}_1$: $|\nu; f\rangle \rightarrow |\nu+1; f\rangle$. The gauge-invariant *theta vacuum* is obtained as [47]

$$|\theta\rangle = \sum_{\nu=-\infty}^{+\infty} e^{-i\nu\theta}\left(\hat{T}_1\right)^\nu |0\rangle, \quad \hat{T}_1|\theta\rangle = e^{i\theta}|\theta\rangle, \quad \hat{T}_1 = \hat{Z}_1\hat{F}_1. \tag{84}$$

Thus $|\theta\rangle$ is an eigenstate of $\hat{T}_1$ with the eigenvalue $\exp(i\theta)$ - it is invariant up to a phase [47,48].

Chapter summary: We have presented three potential mechanism capable to extend the "trivial" LF vacuum to a ground state with certain structure which is relatively simple and contrary to the conventional SL form of the theory is not of a dynamical origin. The shift operator in the case of self-interacting scalar models allowed one to find the semiclassical (approximate) ground state corresponding to the minimum of the LF energy in the broken phase. The off-diagonal terms present in the zero-mode part of the axial-vector charge in the LF sigma model with fermions generate an infinite family of the degenerate vacuum states labeled by a continuous parameter reflecting the symmetry of the corresponding LF Hamiltonian. The existence of associated massless Nambu-Goldstone boson can be then demonstrated along the lines used in the conventional SL form of the theory. In a simple gauge theory setting, a residual ("large") gauge symmetry of the massive LF Schwinger model quantized in the light-cone gauge at a finite interval in x^- was shown, when realized on a quantum level, to transform the LF "bare" vacuum to a more complex state with internal Fock structure. A gauge-invariant theta vacuum was then derived. All these results will possibly require further refinements but indicate promising ways to extend the notion of the LF vacuum as a structureless "empty" state towards a more complex ground state which may be necessary for the LF theory to correctly and completely describe physics related to vacuum structure in the usual SL formulation, like spontaneous or dynamical symmetry breaking and formation of vacuum condensates.

Chapter 3. Hamiltonian treatment of solvable models in the conventional theory. Two-dimensional models have played a major role in the development of QFT. Two best known examples arte the Thirring [24] and Schwinger [25] models. Their main virtue is that because of their (mathematical) simplicity, they can be solved at the level of field equations. Since their birth more than half a century ago, these non-perturbative operator solutions have yielded valuable insights into the structure of QFT. Of course, not all the discovered mechanisms and insights can be taken to more realistic models as they are peculiar to the two-dimensional world. On the other hand, many general properties and concepts are "visible" already in one space dimension. It seems however that not all of the intrinsic properties of the solvable models have been understood to the full depth. The Hamiltonian treatment has only rarely been used. Although the present thesis is devoted mostly to LF field theory, the well-established space-like solutions intended at the beginning of our efforts to be used mainly for comparison, turned out to be suitable for certain modifications and improvements.

The gradient-coupling model: SL vs LF solution. This is the simplest relativistic model whose dynamics can be solved completely, without any approximations. We have chosen this model to illustrate the operator methods later used in a more complicated context and to compare the two forms (SL vs LF) of the relativistic field theory at an elementary level.

The classical Lagrangian density of the model in the covariant form reads

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^\nu \overset{\leftrightarrow}{\partial_\nu} \Psi - m\overline{\Psi}\Psi + \frac{1}{2}\partial_\nu\phi\partial^\nu\phi - \frac{1}{2}\mu^2\phi^2 - g\partial_\nu\phi J^\nu, \;\; J^\nu = \overline{\Psi}\gamma^\nu\Psi. \tag{85}$$

It describes a coupled system of the fermion and scalar field interacting via the gradient coupling. We shall work in the representation where the SL γ-matrices are given by $\gamma^0 = \sigma^2, \gamma^1 = i\sigma^1$ with σ^1, σ^2 being the Pauli matrices. J^ν is the interacting current of massive fermions. For $\mu = 0$, the model was studied by B. Schroer [26], for axial-vector current interaction it is known as the Rothe-Stamatescu model ($m = 0, \mu \neq 0$) [49]. The corresponding Euler-Lagrange equations

$$i\gamma^\mu\partial_\mu\Psi = m\Psi + g\partial_\mu\phi\gamma^\mu\Psi, \;\; \partial_\mu\partial^\mu\phi + \mu^2\phi = g\partial_\mu J^\mu. \tag{86}$$

are solved by

$$\Psi(x) = e^{ig\phi(x)}\psi(x), \;\; i\gamma^\mu\partial_\mu\psi(x) = m\psi(x). \tag{87}$$

irrespectively of the nature of the scalar field (free or interacting). However, for the conserved current J^μ, the scalar field is free. It is quantized by the Fock commutator as $\left[a(k^1), a^\dagger(l^1)\right] = \delta(k^1 - l^1)$, where $\hat{k}.x \equiv \omega(k^1)t - k^1x^1, \omega(k^1) = \sqrt{k_1^2 + \mu^2}$, is

$$\phi(x) = \int_{-\infty}^{+\infty} \frac{dk^1}{\sqrt{4\pi\omega(k^1)}}\left[a(k^1)e^{-i\hat{k}\cdot x} + a^\dagger(k^1)e^{i\hat{k}\cdot x}\right] \equiv \phi^{(+)}(x) + \phi^{(-)}(x). \tag{88}$$

The free massive fermion field is expanded and quantized as

$$\psi(x) = \int_{-\infty}^{+\infty} \frac{dp^1}{\sqrt{4\pi E(p^1)}}\left[u(p^1)b(p^1)e^{-i\hat{p}\cdot x} + v(p^1)d^\dagger(p^1)e^{i\hat{p}\cdot x}\right], \;\; E(p^1) = \sqrt{p_1^2 + m^2},$$
$$u^\dagger(p^1) = (\sqrt{p^+}, \sqrt{p^-}), v(p^1)^\dagger = (\sqrt{p^+}, -\sqrt{p^-}), \;\; p^\pm = E(p^1) \pm p^1,$$
$$\{b(p^1), b^\dagger(q^1\} = \{d(p^1), d^\dagger(q^1)\} = \delta(p^1 - q^1). \tag{89}$$

At quantum level, the solution (87) has to be regularized by normal ordering:

$$\Psi(x) = Z^{1/2}(\epsilon)e^{-ig\phi^{(-)}(x)}e^{-ig\phi^{(+)}(x)}\psi(x), \tag{90}$$

$D^{(+)}(z)$ is the corresponding two-point correlation function. Application of the point-splitting regularization to the interacting current yields:

$$J^\mu(x) = s\lim_{\epsilon\to 0}\frac{1}{2}\big\{Z(\epsilon)\overline{\psi}(x + \frac{\epsilon}{2})e^{ig\phi^{(-)}(x+\frac{\epsilon}{2})}e^{ig\phi^{(+)}(x+\frac{\epsilon}{2})}\gamma^\mu e^{-ig\phi^{(-)}(x-\frac{\epsilon}{2})}$$
$$\times e^{-ig\phi^{(+)}(x-\frac{\epsilon}{2})}\psi(x - \frac{\epsilon}{2}) + H.c.\big\} =: \overline{\psi}(x)\gamma^\mu\psi(x) : + \frac{g}{2\pi}\partial^\mu\phi(x). \tag{91}$$

There is no need to subtract the vacuum expectation value (VEV) part by hand (as is conventionally done) if one defines it as a hermitian sum of two terms.

The quantum vector current has received a correction (”anomaly”) $\partial_\mu J^\mu(x) = g(2\pi)^{-1}\partial_\mu\partial^\mu\phi(x)$

i.e. its divergence is non-zero [50]. The only effect of this quantum term is a finite mass "renormalization". Similarly, one finds that for $m = 0$ the quantum axial-vector current is conserved:

$$J_5^\mu(x) =: \overline{\psi}(x)\gamma^\mu\gamma^5\psi(x) : -\frac{g}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi(x), \;\; \partial_\mu J_5^\mu(x) = 0. \tag{92}$$

In Fock representation, the full quantum Hamiltonian acquires the form

$$H = H_{0B} + H_{0F} + H_{int}, \;\; H_{0B} = \int\limits_{-\infty}^{+\infty} \mathrm{d}p^1 \omega(p^1) a^\dagger(p^1) a(p^1), \tag{93}$$

$$H_{0F} = \int\limits_{-\infty}^{+\infty} \mathrm{d}p^1 E(p^1)\Big[b^\dagger(p^1)b(p^1) + d^\dagger(p^1)d(p^1)\Big],$$

$$H_{int} = \frac{g}{2\sqrt{\pi}}\int\limits_{-\infty}^{+\infty} \mathrm{d}k^1 \big[c^\dagger(k^1)a(k^1) + a^\dagger(k^1)c(k^1) + a^\dagger(k^1)c^\dagger(k^1) + a(k^1)c(k^1)\big],$$

where the composite boson operators, satisfying $[c(k^1), c^\dagger(l^1)] = \delta(k^1 - l^1)$ correspond to the vector current in the bosonized form (see below),

$$j^\mu(x) = -\frac{i}{\sqrt{2}\pi}\int \frac{dk^1}{\sqrt{2|k^1|}} k^\mu \big\{c(k^1)e^{-i\hat{k}\cdot x} - c^\dagger(k^1)e^{i\hat{k}\cdot x}\big\}. \tag{94}$$

Here we have already switched to the case of massless fermions for simplicity, so $E(p^1) = |p^1|$.

The explicit Fock form of the massless fermion field and its vector current is

$$\psi(x) = \int \frac{dp^1}{2\pi}\Big\{b(p^1)u(p^1)e^{-i\hat{p}\cdot x} + d^\dagger(p^1)v(p^1)e^{i\hat{p}\cdot x}\Big\}, \; \hat{p}^\mu = (|p^1|, p^1), \tag{95}$$

where the spinors u, v are the massless limits of the massive spinors. The operators of the bosonized current (94) are found by Fourier-transforming the expression for the bilinear fermionic current,

$$c(k^1) = \frac{i}{\sqrt{k^0}}\int dp^1 \big\{\theta\big(p^1k^1\big)\big)\big[b^\dagger(p^1)b(p^1 + k^1) - d^\dagger(p^1)d(p^1 + k^1)\big] +$$
$$+\epsilon(p^1)\theta\big(p^1(p^1 - k^1)\big)d(k^1 - p^1)b(p^1)\big\}. \tag{96}$$

The Hamiltonian H_{int} in (94) is non-diagonal and the Fock vacuum $|0\rangle$ is not an eigenstate of the full Hamiltonian H. This is a typical situation in the SL theory - the true vacuum state is not known, it should be found in complicated dynamical calculations. In our simple model, a construction of the vacuum state is possible by a Bogoliubov transformation, which is designed to bring H to a diagonal form (see the Thirring model below). The true ground state of H is then found as

$$|\Omega\rangle = N \exp\big[\int\limits_{-\infty}^{+\infty} \mathrm{d}k^1 \gamma(g) c^\dagger(-k^1) a^\dagger(k^1)\big]|0\rangle. \tag{97}$$

A difficulty arises however, when one compares the present results with the picture emerging from the LF analysis. The covariant Lagrangian in terms of LF space-time and field variables reads:

$$\mathcal{L}_{lf} = i\Psi_2{}^\dagger \overleftrightarrow{\partial}_+ \Psi_2 + i\Psi_1{}^\dagger \overleftrightarrow{\partial}_- \Psi_1 - m(\Psi_1{}^\dagger\Psi_2 + \Psi_2{}^\dagger\Psi_1) +$$
$$+ 2\partial_+\phi\partial_-\Phi - \frac{1}{2}\mu^2\phi^2 - g\partial_+\phi J^+ - g\partial_-\phi J^-, \tag{98}$$

The Euler-Lagrange equations in the component form read

$$2i\partial_+\Psi_2 = m\Psi_1 + 2g\partial_+\phi\Psi_2, \quad 2i\partial_-\Psi_1 = m\Psi_2 + 2g\partial_-\phi\Psi_1. \tag{99}$$

As in the SL case, these equations can be easily solved in terms of the free fields as

$$\Psi(x^+, x^-) = e^{ig\phi(x^+,x^-)}\psi(x^+, x^-), \tag{100}$$

where the free fields obey the corresponding equations

$$\big(4\partial_+\partial_- + \mu^2\big)\phi(x^+, x^-) = 0, \;\; \big(i\gamma^+\partial_+ + i\gamma^-\partial_- - m\big)\psi(x^+, x^-) = 0. \tag{101}$$

The second equation in (99) is a constraint. It relates the Ψ_1 field component to the Ψ_2 and ϕ fields in a non-dynamical manner. As a result, the interaction-free Hamiltonian is obtained:

$$P^- = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x^-}{2} T^{+-}(x^-) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x^-}{2} \Big[m\big({\psi_1}^\dagger\psi_2 + {\psi_2}^\dagger\psi_1\big) + \mu^2\phi^2\Big] \tag{102}$$

in a clear contradiction to the SL Hamiltonian, which does contain an interaction term.

The contradiction is removed when one realizes that the true field variables are the free fields in this solvable model. The solutions are composite operators made up from free fields. The Lagrangian has to be re-expressed in terms of them first (analogously to inserting a constraint into Lagrangian). As the result, one arrives at a free-field Lagrangian and Hamiltonian

$$H = \int_{-\infty}^{+\infty} \mathrm{d}x^1 \big[-i\psi^\dagger\alpha^1\partial_1\psi + m\psi^\dagger\gamma^0\psi + \frac{1}{2}\Pi_\phi^2 + \frac{1}{2}(\partial_1\phi)^2 + \frac{1}{2}\mu^2\phi^2\big], \tag{103}$$

also in the SL case. However, the correct Heisenberg equations are generated with this Hamiltonian. The physical vacuum of the model obviously coincides with the Fock vacuum. The only trace of the interacting theory is the non-canonical form of the anticommutation relation of the interacting fermion field, the quantum correction in the currents (91,92), and the form of the correlation functions. The latter are expressed in terms of the correlation functions of free fields,

$$\begin{aligned}\langle vac|\Psi_\alpha(x)\overline{\Psi}_\beta(y)|vac\rangle &= \langle 0| : e^{-ig\phi(x)} : \psi_\alpha(x)\overline{\psi}_\beta(y) : e^{ig\phi(y)} : |0\rangle = \\ &= e^{g^2 D^{(+)}(x-y)} S^{(+)}_{\alpha\beta}(x-y),\end{aligned} \tag{104}$$

where, with N_0, J_0, K_0 being the Bessel functions,

$$\begin{aligned}&D^{(+)}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle, \\ &D^{(+)}(z) = -\frac{1}{4}\theta(z^2)\Big[N_0(\mu\sqrt{z^2}) + i\mathrm{sgn}(z^0)J_0(\mu\sqrt{z^2})\Big] + \frac{1}{2\pi}\theta(-z^2)K_0(\mu\sqrt{-z^2}).\end{aligned} \tag{105}$$

The fermionic two-point function is

$$S^{(+)}_{\alpha\beta}(x-y) = \langle 0|\psi_\alpha(x)\overline{\psi}_\beta(y)|0\rangle, \; S^{(+)}_{\alpha\beta}(z) = \Big(i\gamma^\mu\partial_\mu + m\Big)_{\alpha\beta} D^{(+)}(z). \tag{106}$$

Repeating the same procedure for the LF case, one finds the the same result as before. The both forms of the theory now fully agree.

The Thirring model: new aspects. Thirring model played an important role in history of QFT. A. Wightman's Cargése lectures [51] and B. Klaiber's Boulder lectures [52] reflect the stages of the gradual understanding of the model. In the latter paper, the infra-red regularized operator solution, compatible with the axioms of QFT, was written down and the corresponding n-point correlation functions were constructed. It may therefore seem that all physical aspects of the model have been clarified. This however is not quite true: in a few papers by Faber and Ivanov [53], an existence of a broken phase was argued, based on Nambu – Jona-Lasinio BCS-like Ansatz for the ground state, e.g.. A similar conclusion was done by Fujita et al. [54] using the Bethe Ansatz solution. A systematic Hamiltonian study based on the model's solvability was given only relatively recently [50].

The Lagrangian density of the Thirring model is

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^{\mu} \overleftrightarrow{\partial_{\mu}} \Psi - \frac{1}{2}gJ_{\mu}J^{\mu}, \;\; J^{\mu} = \overline{\Psi}\gamma^{\mu}\Psi. \tag{107}$$

Field equations and current conservation

$$i\gamma^{\mu}\partial_{\mu}\Psi(x) = gJ^{\mu}(x)\gamma_{\mu}\Psi(x), \;\; \partial_{\mu}J^{\mu}(x) = 0. \tag{108}$$

The general solution is

$$\Psi(x) = e^{-i(g/\sqrt{\pi})\left(\alpha j(x) - \beta\gamma^5\tilde{j}(x)\right)}\psi(x), \;\; \gamma^{\mu}\partial_{\mu}\psi(x) = 0 \tag{109}$$

with $\alpha + \beta = 1$. "Potentials" $j(x)$ and $\tilde{j}(x)$ are connected to the free vector current by $\partial_{\mu}j(x) = -\sqrt{\pi}j_{\mu}(x),\;\partial_{\mu}\tilde{j}(x) = \sqrt{\pi}\epsilon_{\mu\nu}j^{\nu}(x)$. Presence of the free potential - as has been done in the Klaiber's work [52] - corresponds to replacing $J^{\mu}(x)$ by $j^{\mu}(x)$ in the field equation. This restrictive assumption does not lead to the most general solution and actually is not necessary. A more general solution can be obtained as follows [50]. Consider the $\beta = 0$ case for simplicity:

$$\Psi(x) = e^{i(g/\sqrt{\pi})J(x)}\psi(x), \;\; \partial_{\mu}J(x) = -\sqrt{\pi}J_{\mu}(x) \tag{110}$$

with the unknown potential $J(x)$ of the interacting current $J^{\mu}(x)$. Regularizing (110) by normal-ordering the exponential like we did in the gradient-coupling model, and calculating the corresponding current using the point-split product of the above $\Psi^{\dagger}$ and Ψ, we find an "anomalous" term

$$J^{\mu}(x) =: \overline{\psi}(x)\gamma^{\mu}\psi(x) : + \frac{g}{2\pi}J^{\mu}(x) \Rightarrow J^{\mu}(x) = G(g)j^{\mu}(x), \; G(g) = (1 - \frac{g}{2\pi})^{-1}, \tag{111}$$

which is just a quantum correction to the classical form of the current. Thus the interacting current is equal to the rescaled (by a coupling-constant dependent factor) free current. It is again convenient to work with the bosonized current with the composite operators $c(k^1), c^{\dagger}(k^1)$ defined in (96). The "potential" $j(x)$ is then nothing but a free massless scalar field:

$$j(x) = \frac{1}{\sqrt{2\pi}}\int \frac{dk^1}{\sqrt{2|k^1|}}\left\{c(k^1)e^{-i\hat{k}\cdot x} + c^{\dagger}(k^1)e^{i\hat{k}\cdot x}\right\}. \tag{112}$$

The problem with the two-dimensional massless scalar field however is that it strictly speaking does not exist since its two-point functions is infrared divergent:

$$D_0^{(+)}(x-y) = \langle 0|j(x)j(y)|0\rangle = \frac{1}{4\pi}\int_{-\infty}^{+\infty} \frac{dk^1}{|k^1|}e^{-i\hat{k}\cdot x} = \infty. \tag{113}$$

This divergence may be simply regularized by introducing an infrared cutoff λ.

True ground state of the massless Thirring model. The correct Hamiltonian $H = H_0 + H_{int}$ is given in terms of the interacting current (111):

$$H = \int_{-\infty}^{+\infty} \mathrm{d}x^1 \Big[-i\psi^\dagger \alpha^1 \partial_1 \psi - \frac{1}{2} g \big(J^0 J^0 - J^1 J^1 \big) \Big] \tag{114}$$

In the Fock representation, the free Hamiltonian is given by (94) with $E(p^1) \to |p^1|$. The interacting Hamiltonian greatly simplifies in terms of the operators $c(k^1), c^\dagger(k^1)$:

$$H_g = G^2(g) \frac{g}{\pi} \int_{-\infty}^{+\infty} dk^1 |k^1| \Big[c^\dagger(k^1) c^\dagger(-k^1) + c(k^1) c(-k^1) \Big]. \tag{115}$$

Obviously $H = H_0 + H_g$ is not diagonal and $|0\rangle$ is not its eigenstate. The construction of the physical vacuum state can be done using the method of [55]. One considers the unitary operator U,

$$U = e^{iS}, \;\; S = -\frac{i}{2} \int_{-\infty}^{+\infty} \mathrm{d}p^1 \gamma(p^1) \big[c^\dagger(p^1) c^\dagger(-p^1) - c(p^1) c(-p^1) \big]. \tag{116}$$

It is useful to form new free and interacting Hamiltonians

$$\hat{H}_0 = H_0 - T, \quad \hat{H}_g = H_g + T, \;\; T = \int_{-\infty}^{+\infty} \mathrm{d}k^1 |k^1| c^\dagger(k^1) c(k^1). \tag{117}$$

$\hat{H}_g$ transforms non-trivially due to

$$\big[S, c(k^1) \big] = i\gamma(k^1) c^\dagger(-k^1), \quad \big[S, c^\dagger(k^1) \big] = i\gamma(k^1) c(-k^1), \;\; \gamma(-k^1) = \gamma(k^1). \tag{118}$$

A diagonal form of the Hamiltonian is obtained if $\gamma(k^1)$ takes on the value $\gamma_d = \frac{1}{2} \mathrm{arctanh}(2G(g)\frac{g}{\pi})$. Thus we have achieved

$$e^{iS} \big(\hat{H}_0 + \hat{H}_g \big) e^{-iS} |0\rangle = 0 \tag{119}$$

and $|\Omega\rangle = e^{-iS}|0\rangle$ is the new vacuum state. Explicitly,

$$|\Omega\rangle = \exp \Big[-\frac{1}{2} \gamma_d \int_{-\infty}^{+\infty} \mathrm{d}p^1 \big[c^\dagger(p^1) c^\dagger(-p^1) - c(p^1) c(-p^1) \big] \Big] |0\rangle. \tag{120}$$

It corresponds to a coherent state of pairs of composite bosons (bilinear in fermion Fock operators) with zero total momentum:

$$P^1 |\Omega\rangle = 0, \;\; P^1 = \int_{-\infty}^{+\infty} \mathrm{d}p^1 p^1 \big[b^\dagger(p^1) b(p^1) + d^\dagger(p^1) d(p^1) \big]. \tag{121}$$

The vacuum $|\Omega\rangle$ is invariant under $U(1)$ and $U_A(1)$ transformations. The vacuum state $|\Omega\rangle$ thus corresponds to the symmetric phase. In other words, there is no chiral symmetry breaking. This contradicts the results [53]. However, the true vacuum should be an eigenstate of the full Hamiltonian. $|\Omega\rangle$ is such a state while the BCS Ansatz of the construction [53] is not.

The two-point function is defined as

$$C_2(x-y) = \langle vac|\Psi(x)\overline{\Psi}(y)|vac\rangle. \tag{122}$$

In the usual treatment [52, 56], $|vac\rangle$ is replaced by $|0\rangle$. In that case, commuting the fermion operators through the exponentials and the exponentials themselves, one arrives at

$$C_2(x-y) = e^{\frac{g^2}{\pi}D_0^{(+)}(x-y)}e^{-2g\left[D_0^{(+)}(y-x)+\gamma^5\tilde{D}_0^{(+)}(y-x)\right]}\langle 0|\psi(x)\overline{\psi}(y)|0\rangle. \tag{123}$$

Here, $\mu = e^{\gamma_E}\lambda$ and $x^2 = x^+x^- = (x^0+x^1)(x^0-x^1)$. $D_0^{(+)}(x)$ and $\tilde{D}_0^{(+)}(x)$ are two sorts of massless two-point functions [52]. Calculation with $|\Omega\rangle$ is more complicated and leads to:

$$\langle\Omega|\Psi(x)\overline{\Psi}(y)|\Omega\rangle = F(x-y;\kappa)C_2(x-y). \tag{124}$$

The function $F(z;\kappa) \to 1$ if $\kappa \to 0$. Its explicit form is under study.

The Thirring-Wess model: new operator solution. The dynamics of the Thirring-Wess model is defined by the Lagrangian

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^\mu \overset{\leftrightarrow}{\partial_\mu} \Psi - \frac{1}{4}\tilde{G}_{\mu\nu}\tilde{G}^{\mu\nu} + \mu_0^2\tilde{B}_\mu\tilde{B}^\mu - eJ_\mu\tilde{B}^\mu, \quad \tilde{G}_{\mu\nu} = \partial_\mu\tilde{B}_\nu - \partial_\nu\tilde{B}_\mu. \tag{125}$$

The model was proposed by Brown [28] and independently by Thirring and Wess [27]. The first work puts an unnecesssary emphasis on manifestly "gauge-invariant" definitions of several operators. The second paper uses a few Ansaetze to solve the field equations. Here we will show that the field equations can be solved directly in terms of the fields present in the original Lagrangian (125).

The set of coupled field equations (massless Dirac plus Proca) has the form

$$i\gamma^\mu\partial_\mu\Psi(x) = e\gamma^\mu\tilde{B}_\mu(x)\Psi(x), \;\; \partial_\mu\tilde{G}^{\mu\nu} + \mu_0^2\tilde{B}^\nu = eJ^\nu, \tag{126}$$

Taking ∂_ν of the Proca equation yields $\partial_\mu\tilde{B}^\mu = 0$. With this condition, the Dirac equation is solved in terms of $\tilde{B}^0(x)$ and the free fermion field $\psi(x)\gamma^\mu\partial_\mu\psi = 0$ as ($\epsilon(x)$ is the sign function):

$$\Psi(x) = \exp\Big\{-\frac{ie}{2}\gamma^5\int_{-\infty}^{+\infty}\mathrm{d}y^1\epsilon(x^1-y^1)\tilde{B}^0(y^1,t)\Big\}\psi(x). \tag{127}$$

As before, the product of two fermion operators has to be regularized by the point-splitting, $x^\mu \pm \frac{\epsilon^\mu}{2}$, and the interacting currents are given as the limits $\epsilon \to 0$ of the non-local products,

$$J^\mu_{(5)}(x) = \frac{1}{2}\Big[\Psi^\dagger(x+\frac{\epsilon}{2})\gamma^0\gamma^\mu(\gamma^5)\Psi(x-\frac{\epsilon}{2}) + H.c.\Big].$$

The free massless fermion field is quantized according to (95). The free massive vector field $B^0(x)$, which will also be needed, is expanded as

$$B^0(x) = \frac{1}{\sqrt{2\pi}}\int\frac{dk^1}{\sqrt{2E(k^1)}}\frac{k^1}{\mu_0}\big[a(k^1)e^{-i\hat{k}\cdot x} + a^\dagger(k^1)e^{i\hat{k}\cdot x}\big], \tag{128}$$

while the second component of B^μ is determined from the condition $\partial_0B^0 + \partial_1B^1 = 0$. We find, by means of the split free current and using the definition of the symmetric limit:

$$J^\mu(x) = j^\mu(x) + \frac{e}{\pi}\tilde{B}^\mu(x), \;\; J^\mu_5(x) = j^\mu_5(x) + \frac{e}{\pi}\epsilon^{\mu\nu}\tilde{B}_\nu(x), \;\; a(x) \equiv \partial_\mu J^\mu_5 = \frac{e}{2\pi}\epsilon_{\mu\nu}\tilde{G}^{\mu\nu}(x).$$

The essential point in this derivation is that the expression in the exponential contains a term of order $O(\epsilon)$ which cancels a singularity in the free-field contraction, generating in this way a finite term.

The quantum vector current is conserved, while the divergence of the axial current $a(x)$ is non-zero. The anomalous term surprisingly coincides with the usual result in the Schwinger model - which is a gauge theory - although no exponential of the integral over the gauge field (necessary in the Schwinger model to maintain gauge invariance, see the next section) was inserted. A similar result was obtained relatively recently in a much more complicated way in [57].

The Proca equations are solved by $\tilde{B}^{\nu}(x) = B^{\nu}(x) + \frac{e}{\mu^2}j^{\nu}(x)$. Then the Lagrangian and subsequently also the Hamiltonian can be expressed in terms of the independent field variables - the free massless fermion field, its vector current, and the free $B^{\mu}(x)$ with the rescaled mass μ as

$$H = H_0 + H_{int}, \quad H_{int} = -e\frac{\mu_0^2}{\mu^2}\int_{-\infty}^{+\infty} \mathrm{d}x^1 \left[2\left(B^0 j^0 - B^1 j^1\right) + e\frac{\mu_0^2}{\mu^2}\left(j_0^2 - j_1^2\right)\right]. \tag{129}$$

The Fock form of the Hamiltonian (129), corresponding free fields (95) and (128) is non-diagonal: it contains terms like $a^{\dagger}(k_1)c^{\dagger}(k_2)$ that do not annihilate the Fock vacuum $|0\rangle$. Diagonalization of the Hamiltonian has again to be performed by a suitable Bogoliubov transformation. This, together with the calculation of the non-perturbative correlation functions of the model from the interacting $\Psi(x)$ and $\tilde{B}^{\mu}(x)$ solutions, requires further study.

The operator solution of the Schwinger model: problematic aspects. Our analysis of this prototype gauge theory starts from the modified Lagrangian

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^{\mu} \overleftrightarrow{\partial_{\mu}} \Psi - \frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - eJ_{\mu}\tilde{A}^{\mu} - G(x)\partial_{\mu}\tilde{A}^{\mu} + \frac{1}{2}(1-\gamma)G^2(x), \tag{130}$$

Here $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ and the last two terms with the auxiliary field $G(x)$ [58, 59] replace the usual term $-\frac{\lambda}{2}\left(\partial_{\mu}\tilde{A}^{\mu}\right)^2$, which is conventionally added to impose the Lorentz gauge.

The gauge-fixing term in the Lagrangian guarantees restriction to an arbitrary covariant gauge (parameter γ) in which neither the condition $\partial_{\mu}A^{\mu}(x) = 0$ nor the Maxwell equations can be satisfied at the operator level. The redundant gauge degrees of freedom of this gauge are expressed in terms of ghost fields having zero norm and carrying vanishing momentum and energy [60].

The gauge freedom in the above Lagrangian has been restricted (fixed) only partially, it is still invariant with respect to the gauge transformations $A^{\mu}(x) \to A^{\mu}(x) - \partial_{\mu}\Lambda(x)$, parametrized by the gauge function obeying $\partial_{\mu}\partial^{\mu}\Lambda(x) = 0 \Rightarrow \partial_0^2\Lambda = \partial_1^2\Lambda$. The field equations following from (130) are

$$\partial_{\mu}F^{\mu\nu}(x) = eJ^{\nu}(x) - \partial^{\nu}G(x), \;\; \partial_{\mu}A^{\mu}(x) = (1-\gamma)G(x), \;\; \partial_{\mu}\partial^{\mu}G(x) = 0, \tag{131}$$

$$i\gamma^{\mu}\partial_{\mu}\Psi(x) = e\gamma^{\mu}A_{\mu}(x)\Psi(x) \tag{132}$$

To have the original theory, we must impose a condition $G^{(+)}(x)|phys\rangle = 0$ on physical states. It may be viewed as a generalization of the Gupta-Bleuler condition [61] $\partial_{\mu}A^{(+)\mu}|phys\rangle = 0$. Choosing $\gamma = 1$, the gauge condition is satisfied at the operator level. The solution of the Dirac equation is analogous to the Thirring-Wess model case (here however A^μ field is unphysical):

$$\Psi(x) = \exp\Big\{-\frac{ie}{2}\gamma^5 \int\limits_{-\infty}^{+\infty} \mathrm{d}y^1 \epsilon(x^1 - y^1) A^0(y^1, t)\Big\}\psi(x), \quad \gamma^\mu\partial_\mu\psi = 0. \tag{133}$$

Then the vector and axial-vector currents are calculated from (133) via point-splitting. They are invariant under the residual gauge transformations only if a modified exponential of the gauge field is added, leading to the interacting currents equal to the free ones, with the axial anomaly restricted to the zero-mode sector [62, 63].

Here one should mention for the sake of comparison briefly the seminal solution of the model due to Lowenstein and Swieca [48]. It is based on the Ansatz for the gauge field (where $\tilde{\partial_\mu} = \epsilon_{\mu\nu}\partial^\nu$)

$$A_\mu = -\frac{\sqrt{\pi}}{e}\big(\tilde{\partial}_\mu\Sigma + \partial_\mu\tilde{\eta}\big), \quad \Rightarrow \Psi(x) =: e^{i\sqrt{\pi}\gamma^5\big(\Sigma(x)+\eta(x)\big)} : \psi(x). \tag{134}$$

In the $\partial_\mu A^\mu = 0$ gauge, one finds $\partial_\mu\partial^\mu\tilde{\eta} = 0$ and $F_{\mu\nu} = \frac{\sqrt{\pi}}{e}\epsilon_{\mu\nu}\partial_\rho\partial^\rho\Sigma$, $\partial_\mu J^\mu_5 = \frac{e}{2\pi}e_{\mu\nu}F^{\mu\nu}$. As a consequence of the anomaly, the gauge field is found to obey massive equation (the Schwinger mechanism). Our finding however is that the anomaly calculation is actually based on a gauge-non-invariant axial-vector current. Also, the Hamiltonian of the model, which has not been derived in [48], is non-diagonal and consequently the Fock vacuum $|0\rangle$ is not its eigenstate, i.e. it is not the true ground state of the model (for subsequent topological considerations).

Chapter summary: Solvable models have been important "play grounds" for developing our understanding of non-perturbative structure of quantum field theory. Here we have argued that some aspects of their operator solutions were not understood completely, mainly because the Hamitonian formulation was rarely used. We have shown that the SL and LF versions of the derivative-coupling model are mutually consistent if the correct identification of the true degrees of freedom is made. Although the spectrum of states of the model consists of free particles, non-trivial quantum effects are present due to the necessity to correctly define products of fermion field operators (the currents), leading to "anomalous" terms. The same is true for the more complex Thirring model, where the physical vacuum has been found by a Bogoliubov transformation that diagonalizes the Hamiltonian, which became quadratic in effective boson operators bilinear in the original fermion field. We have given a new operator solution of the Thirring-Wess model without a need to introduce auxiliary fields (via Ansaetze), which leads to the axial-vector anomaly in a very simple way. We have also pointed out certain drawbacks of the fundamental operator solution of the Schwinger model, related to the incorrect treatment of the residual gauge freedom in the covariant (Landau) gauge.

Chapter 4. Solvable models and two-dimensional massless LF fields. The seemingly very simple task of quantization the massless LF fields in D=1+1 turned out to be a real puzzle for a few decades. We solved this problem starting from the corresponding massive fields. The consistency of the new quantization scheme was proved by the independent LF bosonization of massless LF fermion fields, by the correct relationship to the conformal field theory and also by LF operator solutions of the solvable models, discussed in the previous chapter in the usual SL formulation.

Massless LF fields in two dimension. A very important and interesting question is the relation between the usual "equal-time" (SL) theory and the light-front form. This issue was extensively studied recently [64] with the conclusion that the two schemes can simply be related to each other.

Our experience rather suggests principal differences between the LF and SL quantizations. Could it be that the LF version conceptually as well as technically simplifies the structure of QFT while still maintaining potential for reliable predictions? The area of 2D solvable relativistic models represents a very suitable theoretical environment to study these questions [65, 66].

The two-dimensional massless LF fields, being the essential elements for exact operator solutions of the models, have not been understood and correctly quantized for a few decades. A simple and natural way of quantizing them has been given relatively recently in [66, 67].

Our quantization of the massless **LF scalar field** starts from the free massive field. Its covariant Lagrangian density and the corresponding field equation takes in terms of the LF variables the form

$$\mathcal{L} = 2\partial_+\phi\partial_-\phi - \frac{1}{2}\mu^2\phi^2, \quad \big(4\partial_+\partial_- + \mu^2\big)\phi(x) = 0. \tag{135}$$

The quantum solution of the field equation (135) is

$$\phi(x) = \int_0^{+\infty} \frac{dk^+}{\sqrt{4\pi k^+}}\big[a(k^+)e^{-\frac{i}{2}k^+x^- - \frac{i}{2}\frac{\mu^2}{k^+}x^+} + a^\dagger(k^+)e^{\frac{i}{2}k^+x^- + \frac{i}{2}\frac{\mu^2}{k^+}x^+}\big], \tag{136}$$

with the creation and annihilation operators obeying $\big[a(k^+), a^\dagger(l^+)\big] = \delta(k^+ - l^+)$, $a(k^+)|0\rangle = 0$. From (136) we get the conjugate momentum $\pi(x) = 2\partial_-\phi(x)$ and the field time derivative $\theta(x) = 2\partial_+\phi(x)$. In addition to the correlation function $D_0^{(+)}(z) = \langle 0|\phi(x)\phi(y)|0\rangle$, we shall also need

$$D_1^{(+)}(z) = \langle 0|\phi(x)\pi(y)|0\rangle, \;\; D_2^{(+)}(z) = \langle 0|\phi(x)\theta(y)|0\rangle, \tag{137}$$

$$D_i^{(+)}(z) = i\int_0^{\infty} \frac{dk^+}{4\pi} f_i(k^+)e^{-\frac{i}{2}k^+(z^- - i\epsilon^-) - \frac{i}{2}\frac{\mu^2}{k^+}(z^+ - i\epsilon^+)}. \tag{138}$$

Here $f_0(k^+) = -\frac{i}{k^+},\; f_1(k^+) = 1,\; f_2(k^+) = \frac{\mu^2}{k^{+2}}$. The small imaginary parts in the exponential are necessary for the existence of the integrals. The latter are evaluated in terms of the (modified) Bessel functions $J_\nu(z), N_\nu(z), K_\nu(z), \nu = 0, 1$. Explicitly,

$$\begin{aligned} D_1^{(+)}(z) = &-\theta\big(z^2\big)\frac{\mu}{4}\sqrt{\frac{z^+}{z^-}}i\Big[J_1\big(\mu\sqrt{z^2}\big) - i\,\mathrm{sgn}(z^+)N_1\big(\mu\sqrt{z^2}\big)\Big] \\ &- \theta\big(-z^2\big)\mathrm{sgn}(z^+)\frac{\mu}{4\pi}\sqrt{-\frac{z^+}{z^-}}K_1\big(\mu\sqrt{-z^2}\big), \quad D_2^{(+)}(z) = D_1^{(+)}(z^+ \leftrightarrow z^-). \end{aligned} \tag{139}$$

Both $D_1^{(+)}$ and $D_2^{(+)}$ have a non-vanishing massless limit (due to $K_1(z) \sim 1/z$ for small z):

$$D_1^{(+)}(x - y; \mu^2 = 0) \;\; = \;\; \frac{1}{2\pi}\frac{1}{(x^- - y^- - i\epsilon^-)}, \tag{140}$$

and similarly for $D_2^{(+)}(x - y; \mu^2 = 0$ with $x^- - y^- \to x^+ - y^+$. These results suggest that there must exist massless analogs of $\phi(x), \pi(x), \theta(x)$ reproducing (140).

Indeed, from the LF massless Klein-Gordon equation $\partial_+\partial_-\tilde{\phi}(x) = 0$, one expects

$$\tilde{\phi}(x) = \tilde{\phi}(x^+) + \tilde{\phi}(x^-). \tag{141}$$

as a form of the general solution. The mass dependence resides only in the plane-wave factor [2], therefore the massless limit of the massive solution (136) yields $\tilde{\phi}(x^-)$:

$$\tilde{\phi}(x^-) = \int_0^{+\infty} dk^+ \left[\tilde{a}(k^+)e^{-\frac{i}{2}k^+x^-} + \tilde{a}^\dagger(k^+)e^{\frac{i}{2}k^+x^-}\right]. \tag{142}$$

The second piece $\tilde{\phi}(x^+)$ can also be recovered from (136) after the change of variables $k^+ = \frac{\mu^2}{k^-}$. Then in the massless limit one finds the commutation relations

$$\left[\tilde{a}(k^-), \tilde{a}^\dagger(l^-)\right] = \delta(k^- - l^-), \;\; \left[\tilde{a}(k^+), \tilde{a}^\dagger(l^-)\right] = 0. \tag{143}$$

and the massless field itself

$$\tilde{\phi}(x^+) = \int_0^{+\infty} dk^- \left[\tilde{a}(k^-)e^{-\frac{i}{2}k^-x^+} + \tilde{a}^\dagger(k^-)e^{\frac{i}{2}k^-x^+}\right], \tag{144}$$

and similarly for $\theta(x^+)$ and $\pi(x^-)$. The basic commutators, following from (144) and (143), are

$$\left[\tilde{\phi}(x^-), \tilde{\phi}(y^-)\right] = -\frac{i}{4}\epsilon(x^- - y^-), \;\; \left[\tilde{\phi}(x^+), \tilde{\phi}(y^+)\right] = -\frac{i}{4}\epsilon(x^+ - y^+). \tag{145}$$

Also, one readily verifies that the two-point functions calculated from the massless fields coincides with the massless limits (140) of the massive functions. The massless momentum operators are

$$P^+ = \int_0^{+\infty} dk^+ k^+ \tilde{a}^\dagger(k^+)\tilde{a}(k^+), \quad P^- = \int_0^{+\infty} dk^+ k^- \tilde{a}^\dagger(k^-)\tilde{a}(k^-) \tag{146}$$

The same procedure can be applied to the **light front fermion field**. The field equation (2D Dirac equation) $i\gamma^\mu\partial_\mu\psi(x) = m\psi(x)$ decomposes into a dynamical and a constraint equation:

$$2i\partial_+\psi_2(x) = m\psi_1(x), \;\; 2i\partial_-\psi_1(x) = m\psi_2(x) \Rightarrow \partial_+\psi_2 = 0, \; \partial_-\psi_1 = 0. \tag{147}$$

The resultant representation is

$$\tilde{\psi}_2(x^-) = \frac{1}{\sqrt{4\pi}} \int_0^{+\infty} dp^+ \left[\tilde{b}(p^+)e^{-\frac{i}{2}p^+x^-} + \tilde{d}^\dagger(p^+)e^{\frac{i}{2}p^+x^-}\right], \tag{148}$$

$$\tilde{\psi}_1(x^+) = \frac{1}{\sqrt{4\pi}} \int_0^{+\infty} dp^- \left[\tilde{b}(p^-)e^{-\frac{i}{2}p^-x^+} - \tilde{d}^\dagger(p^-)e^{\frac{i}{2}p^-x^+}\right], \tag{149}$$

$$\{\tilde{b}(p^-), \tilde{b}^\dagger(q^-)\} = \{\tilde{d}(p^-), \tilde{d}^\dagger(q^-)\} = \delta(p^- - q^-), \tag{150}$$

$$\{\tilde{b}(p^-), \tilde{b}^\dagger(q^+)\} = \{\tilde{d}(p^-), \tilde{d}^\dagger(q^+)\} = 0. \tag{151}$$

The expected form of the field anticommutators is then obtained:

$$\left\{\tilde{\psi_1}(x^+), \tilde{\psi_1}^\dagger(y^+)\right\} = \delta(x^+ - y^+), \quad \left\{\tilde{\psi_2}(x^-), \tilde{\psi_2}^\dagger(y^-)\right\} = \delta(x^- - y^-). \tag{152}$$

The two-point function calculated from the massless $\tilde{\psi}_1(x^+)$ coincides with the massless limit of the massive 2-point function. From the field expansions (148), (149), the bilinear operators (the currents $(:\psi_1^\dagger\psi_1:, :\psi_2^\dagger\psi_2:)$ and the scalar densities $\psi_2^\dagger\psi_1 \pm \psi_1^\dagger\psi_2$) are easily constructed.

Thus, a consistent framework for massless LF fermion quantum field has been established. No new variables had to be introduced, the massive fields contain the necessary information.

Bosonization is a remarkable property of two-dimensional field theory: fermion fields can be represented in terms of boson variables [68, 69]. It turns out that its LF version is particularly simple. It is based on the natural decomposition of the massless $\phi(x)$ and $\psi(x)$ fields, $\phi(x) = \phi(x^+) + \phi(x^-)$, $\psi^T(x) = (\psi_1(x^+), \psi_2(x^-))$.

We start with $\psi_2(x^-)$. Assume that it can be represented as

$$\varphi_2(x^-) = C : e^{i\alpha\phi(x^-)} : \ = Ce^{i\alpha\phi^{(-)}(x^-)}e^{i\alpha\phi^{(+)}(x^-)}. \tag{153}$$

It is now sufficient to adjust the constants C and α in such a way, that two φ_2 with different arguments anticommute and $\varphi_2(x^-)$, $\varphi_2^\dagger(y^-)$ satisfy the anticommutation relation (152). The required form is

$$\hat{\varphi}_2(x^-) = Ce^{i2\sqrt{\pi}\phi^{(-)}(x^-)}e^{i2\sqrt{\pi}\phi^{(+)}(x^-)}, \ \ \hat{\varphi}_1(x^+) = Ce^{i2\sqrt{\pi}\phi^{(-)}(x^+)}e^{i2\sqrt{\pi}\phi^{(+)}(x^+)}, \tag{154}$$

where $C = \sqrt{\frac{\lambda e^{\gamma_E}}{4\pi}}$. Using the point-splitting regularization, the bosonized vector current is

$$j^+(x^-) = \frac{2}{\sqrt{\pi}}\partial_-\phi(x^-), \ \ j^-(x^+) = \frac{2}{\sqrt{\pi}}\partial_+\phi(x^+). \tag{155}$$

It correctly reproduces the Schwinger term in both current-current commutators:

$$\left[j^+(x^-), j^+(y^-)\right] = \frac{i}{\pi}\partial_x\delta(x^- - y^-), \ \ \left[j^-(x^+), j^-(y^+)\right] = \frac{i}{\pi}\partial_x\delta(x^+ - y^+). \tag{156}$$

Similarly, for the scalar densities one obtains in a straightforward way

$$\overline{\psi}(x)\psi(x) = \frac{\lambda e^{\gamma_E}}{4\pi}\cos\left(2\sqrt{\pi}\phi(x)\right), \ \ \overline{\psi}(x)\gamma^5\psi(x) = i\frac{\lambda e^{\gamma_E}}{4\pi}\sin\left(2\sqrt{\pi}\phi(x)\right). \tag{157}$$

Thus the LF version of bosonization yields the results known from the SL theory.

Relation to conformal field theory. Additional properties of the massless LF scalar field has been worked out for the sake of comparison with the holomorphic conformal formulation [70].

The LF Hamiltonian density $T^{+-}(x)$ of the free massless scalar field $\phi(x) = \phi(x^+) + \phi(x^-)$ vanishes, as required by the conformal symmetry. The other components of the energy-momentum tensor are nonvanishing:

$$T^{++}(x^-) = 4 : \partial_-\phi(x^-)\partial_-\phi(x^-) :, \quad T^{--}(x^+) = 4 : \partial_+\phi(x^+)\partial_+\phi(x^+) : . \tag{158}$$

Here

$$\partial_+\phi(x) = \frac{1}{2i}\int_0^{+\infty}\frac{dk^-\sqrt{k^-}}{\sqrt{4\pi}}\left[a(k^-)e^{-\frac{i}{2}k^-x^+} - a^\dagger(k^-)e^{\frac{i}{2}k^-x^+}\right] \tag{159}$$

and analogously for $\partial_-\phi(x)$ (with $k^- \to k^+$). The LF Hamiltonian (146) can also be obtained as the x^+-integral of the density $T^{--}(x^+)$, analogously to P^+ which is the x^--integral of $T^{++}(x^-)$. Compute now a few additional correlation functions

$$\langle 0|\theta(x^+)\theta(y^+)|0\rangle = \frac{1}{\pi}\frac{1}{(x^+ - y^+ - i\delta^+)^2}, \ \ \langle 0|\pi(x^-)\pi(y^-)|0\rangle = \frac{1}{\pi}\frac{1}{(x^- - y^- - i\delta^-)^2}. \tag{160}$$

as well as those between the components of the energy-momentum tensor

$$\langle 0|T^{--}(x^+)T^{--}(y^+)|0\rangle = \frac{2}{\pi^2}\frac{1}{(x^+ - y^+ - i\delta^+)^4}, \tag{161}$$

and analogously for $T^{++}(x^-)$. In the holomorphic formulation of the CFT in 2D [70, 71], the Laurent expansion in the variables

$$z = e^{\frac{2\pi}{L}\zeta},\ \ \bar{z} = e^{\frac{2\pi}{L}\bar{\zeta}},\ \ \text{where}\ \ \zeta = \tau - ix,\ \ \bar{\zeta} = \tau + ix, \tag{162}$$

is used. It is based on radial quantization that uses the euclidean time τ, with $t \to -i\tau$. We need to reformulate our results in the form of infinite series to conform with the discrete picture of [70].

Thus, we consider the massive field in a finite box of lenght $2L$ in x^- or $2T$ in x^+ with periodic boundary conditions $\phi(x^+, x^- = -L) = \phi(x^+, x^- = L)$, $\phi(x^+ = -T, x^-) = \phi(x^+ = T, x^-)$ and perform the massless limit as we did in the continuum-theory case:

$$\phi(x^-) = \sum_{n=1,2,...}\frac{1}{\sqrt{2Lk_n^+}}\left[a_n e^{-\frac{i}{2}k_n^+ x^-} + a_n^\dagger e^{\frac{i}{2}k_n^+ x^-}\right], \tag{163}$$

$$\phi(x^+) = \sum_{n=1,2,...}\frac{1}{\sqrt{2Lk_n^-}}\left[\bar{a}_n e^{-\frac{i}{2}k_n^- x^+} + \bar{a}_n^\dagger e^{\frac{i}{2}k_n^- x^+}\right], \tag{164}$$

$$\left[a_m, a_n^\dagger\right] = \left[\bar{a}_m, \bar{a}_n^\dagger\right] = \delta_{m,n},\ \left[\bar{a}_m, a_n^\dagger\right] = 0. \tag{165}$$

Since $\mu = 0$, ϕ_0 can be non-zero. It reduces however to a constant. Its conjugate momentum thus vanishes. The energy-momentum tensor in the discrete representation reads, with $K = -\frac{\pi}{L^2}$:

$$T^{++}(x^-) = K\sum_{m,n}\epsilon(m)\epsilon(n)\sqrt{|m||n|} : a_m a_n : e^{-i\frac{\pi}{L}(n+m)x^-}, \tag{166}$$

$$T^{--}(x^+) = K\sum_{m,n}\epsilon(m)\epsilon(n)\sqrt{|m||n|} : \bar{a}_m \bar{a}_n : e^{-i\frac{\pi}{L}(n+m)x^+}. \tag{167}$$

They can be transformed to a ”Virasoro form” by simply taking a Fourier transform. Indeed, assume that $T^{++}(x^-)$ can be represented as

$$T^{++}(x^-) = \frac{1}{4L^2}\sum_{l=0,\pm1,...} L_l e^{-i\frac{\pi}{L}lx^-}. \tag{168}$$

The operator coefficients L_l are then obtained by inverting this relation:

$$L_l = 2L\int_{-L}^{+L} dx^- e^{i\frac{\pi}{L}lx^-} T^{++}(x^-),\ \ L_0 = 4LP^+. \tag{169}$$

Inserting $T^{++}(x^-)$ in the Fock form (166) into (169) gives

$$L_n = -4\pi\sum_{k=\pm1,...}\epsilon(k)\epsilon(n-k)\sqrt{|k||n-k|}a_k a_{n-k}. \tag{170}$$

A straightforward calculation based on the commutators (165) yields the LF version of the Virasoro algebra, including the c-number term, not present at the classical level (the ”central extension”),

$$\left[L_n, L_m\right] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \tag{171}$$

where c is the "central charge". The completely paralel treatment can be given for the component $T^{--}(x^+)$, leading to the algebra (171) with $L_n \to \bar{L}_n$. Obviously, one has $\big[L_n, \bar{L}_m\big] = 0$.

The full agreement with the CFT result is obtained. Similar results are obtained for fermions.

All the LF results can be easily transformed to the conformal (holomorphic or antiholomorphic) form by switching to the euclidean time and defining the variables ζ and $\bar{\zeta}$ (162). We get

$$\langle 0|\pi(\zeta)\pi(\zeta')|0\rangle = -\frac{1}{(\zeta-\zeta')^2},\;\; \langle 0|T(\zeta)T(\zeta')|0\rangle = \frac{c}{2}\frac{1}{(\zeta-\zeta')^4},\; c=1. \tag{172}$$

The above field expansions now read

$$\phi(\zeta) = \frac{1}{\sqrt{4\pi}}\sum_{n=\pm1,\pm2,\ldots}\frac{1}{\sqrt{|n|}}a_n z^n,\;\; [a_m, a_n] = \delta_{m+n,0}, \tag{173}$$

$$\phi(\bar{\zeta}) = \frac{1}{\sqrt{4\pi}}\sum_{n=\pm1,\pm2,\ldots}\frac{1}{\sqrt{|n|}}\bar{a}_n \bar{z}^n,\;\; [\bar{a}_m, \bar{a}_n] = \delta_{m+n,0}. \tag{174}$$

It is analogous to the transition to the conformal field in the conventional treatment.

Solution of the LF Thirring model. The consistent quantization of the massless LF fermion field enables one to find independent exact LF operator solutions of a few two-dimensional models. The Thirring model is the most natural starting example. Its Lagrangian and field equation is

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^\mu \overset{\leftrightarrow}{\partial_\mu} \Psi + \frac{1}{2}g J_\mu J^\mu,\quad i\gamma^\mu\partial_\mu\Psi = -g\gamma_\mu J^\mu,\quad J^\mu(x) = \overline{\Psi}(x)\gamma^\mu\Psi(x). \tag{175}$$

In the LF theory, these become

$$\mathcal{L} = i\Psi_2^\dagger \overset{\leftrightarrow}{\partial_+} \Psi_2 + i\Psi_1^\dagger \overset{\leftrightarrow}{\partial_-} \Psi + gJ^+J^-, \tag{176}$$

$$2i\partial_+\Psi_2 = -gJ^-\Psi_2,\;\; 2i\partial_-\Psi_1 = -gJ^+\Psi_1,\;\; J^+ = 2\Psi_2^\dagger\Psi_2,\;\; J^- = 2\Psi_1^\dagger\Psi_1. \tag{177}$$

Defining the potentials $J(x)$, $\tilde{J}(x)$ and $J(x^\pm)$ by

$$\begin{aligned} &J(x) = J(x^+) + J(x^-),\;\; \tilde{J}(x) = -J(x^+ + J(x^-)),\\ &J^+(x^-) = \frac{2}{\sqrt{\pi}}\partial_- J(x) = \frac{2}{\sqrt{\pi}}\partial_-\tilde{J}(x),\;\; J^-(x^+) = \frac{2}{\sqrt{\pi}}\partial_+ J(x) = -\frac{2}{\sqrt{\pi}}\partial_+\tilde{J}(x), \end{aligned} \tag{178}$$

the most general solution of Eqs.(177) is

$$\Psi(x) = e^{i\frac{g}{\sqrt{\pi}}\big(\alpha J(x) - \beta\gamma^5\tilde{J}(x)\big)}\psi(x),\quad \alpha+\beta = 1,\; J^\mu(x) = G(g)j^\mu(x). \tag{179}$$

The factor $G(g) = (1-\frac{g}{2\pi})^{-1}$ results from calculating the interacting current from the solution (179) as the point-split product [65]: the free current gets renormalized by interaction in quantum theory. This aspect was omitted in the original solution [52]. The quantum (normal-ordered) solution is

$$\Psi_1(x) = e^{i\lambda G\big((\alpha-\beta)j^{(-)}(x^+)+j^{(-)}(x^-)\big)}\psi_1(x^+)e^{i\lambda G\big((\alpha-\beta)j^{(+)}(x^+)+j^{(+)}(x^-)\big)}, \tag{180}$$

$$\Psi_2(x) = e^{i\lambda G\big(j^{(-)}(x^+)+(\alpha-\beta)j^{(-)}(x^-)\big)}\psi_2(x^-)e^{i\lambda G\big(j^{(+)}(x^+)+(\alpha-\beta)j^{(+)}(x^-)\big)},\;\; \lambda \equiv \frac{g}{\sqrt{\pi}}. \tag{181}$$

The expansion of the $\psi(x)$ field is given in (149). Bosonization by Fourier transformation gives

$$j^+(x^-) = \frac{1}{i\sqrt{\pi}}\int_0^\infty \frac{dk^+\,k^+}{\sqrt{4\pi k^+}}\big[c(k^+)e^{-\frac{i}{2}k^+x^-} - c^\dagger(k^+)e^{\frac{i}{2}k^+x^-}\big],\;\big[c(k^+), c^\dagger(l^+)\big] = \delta(k^+ - l^+),$$

and analogously for $j^-(x^+)$, where $b(k^+), d(k^+), c(k^+) \to b(k^-), d(k^-), c(k^-)$ in

$$\hat{c}(k^+) = \int_0^\infty ds^+ \big[b^\dagger(s^+)b(s^+ + k^+) - d^\dagger(s^+)d(s^+ + k^+) + d(p^+)b(k^+ - p^+)\theta(k^+ - s^+)\big]. \quad (182)$$

The corresponding potentials (”integrated currents”) $j(x^+)$ and $j(x^-)$ are just the two components of the massless scalar field discussed in the previous paragraphs. The interacting potential $J(x) = G(g)j(x)$. The Hamiltonian of the model acquires a very simple form

$$P^- = \int_{-\infty}^{+\infty} \frac{dx^-}{2} T^{+-}(x) = g \int_{-\infty}^{+\infty} \frac{dx^-}{2} J^+(x^-)J^-(0) = gG^2(g)Q\tilde{Q}, \;\; Q = \hat{c}(0), \;\; \tilde{Q} = j^-(0). \quad (183)$$

The 2-point functions can be calculated from the solutions (181) using the operator identities $Ae^B = e^\lambda e^B A$, if $[A, B] = \lambda A$, λ being a c-number, and $e^A e^B = e^{[A,B]}e^B e^A$, if $[A, B]$ is a c-number. For example, we get, with $z = x - y$,

$$\langle 0|\Psi_1(x)\Psi_1{}^\dagger(y)|0\rangle = \frac{-\mu(2\pi)^{-1}}{(i\mu z^+ + \delta)^{1+\frac{\frac{g^2}{\pi}G^2(\alpha-\beta)^2 - 2gG(\alpha-\beta)}{4\pi}}} \frac{1}{(i\mu z^- + \delta)^{\frac{g^2}{4\pi^2}}}. \quad (184)$$

Its Fourier transform has no pole indicating absence of asymptotic states [52, 56].

Solution of the LF Thirring-Wess model. The Lagrangian of the model is

$$\begin{aligned} \mathcal{L} &= \frac{i}{2}\overline{\Psi}\gamma^\mu \overset{\leftrightarrow}{\partial_\mu} \Psi - \frac{1}{4}\tilde{G}_{\mu\nu}\tilde{G}^{\mu\nu} + \frac{1}{2}\mu_0^2 \tilde{B}_\mu \tilde{B}^\mu - e\tilde{B}_\mu J^\mu, \\ \tilde{G}_{\mu\nu} &= \partial_\mu \tilde{B}_\nu - \partial_\nu \tilde{B}_\mu, \;\; J^\mu(x) = \overline{\Psi}(x)\gamma^\mu\Psi(x), \end{aligned} \quad (185)$$

where $\Psi(x)$ ($\tilde{B}^\mu(x)$) is the interacting massless fermion (massive vector) field. The solvability of the theory implies that one can find an operator solution of the coupled system of the Dirac and Proca equations, which in the LF form read

$$2i\partial_+\Psi_2 = e\tilde{B}^-\Psi_2, \quad 4\partial_+\partial_-\tilde{B}^+ + \mu_0^2\tilde{B}^+ = eJ^+, \quad (186)$$

$$2i\partial_-\Psi_1 = e\tilde{B}^+\Psi_1, \quad 4\partial_+\partial_-\tilde{B}^- + \mu_0^2\tilde{B}^- = eJ^-. \quad (187)$$

The classical solution of the Dirac equation involves the free massless LF fermion field components:

$$\begin{aligned} \Psi_1(x) &= \exp\Big\{ -\frac{ie}{2}\int_{-\infty}^{+\infty} \frac{\mathrm{d}y^-}{2}\frac{1}{2}\epsilon(x^- - y^-)\tilde{B}^+(x^+, y^-)\Big\}\psi_1(x^+), \\ \Psi_2(x) &= \exp\Big\{ +\frac{ie}{2}\int_{-\infty}^{+\infty} \frac{\mathrm{d}y^-}{2}\frac{1}{2}\epsilon(x^- - y^-)\tilde{B}^+(x^+, y^-)\Big\}\psi_2(x^-). \end{aligned} \quad (188)$$

The interacting quantum current appearing in the Proca equation is found by means of the point-splitting procedure as before:

$$J^+(x) = 2 : \psi_2^\dagger(x^-)\psi_2(x^-) : -\frac{e}{2\pi}\tilde{B}^+(x^+, x^-), \quad (189)$$

$$J^-(x) = 2 : \psi_1^\dagger(x^+)\psi_1(x^+) : -\frac{e}{2\pi}\tilde{B}^-(x^+, x^-). \tag{190}$$

It is conserved due to the condition $\partial_\mu \tilde{B}^\mu = 0$. On the other hand, the axial-vector current $J^\mu(x) = \overline{\Psi}(x)\gamma^\mu\gamma^5\Psi(x)$ is "anomalous",

$$\partial_\mu J_5^\mu = -\frac{e}{2\pi}\partial_+\tilde{B}^+ = \frac{e}{4\pi}\epsilon_{\mu\nu}\tilde{G}^{\mu\nu}. \tag{191}$$

When (190) is inserted to the Proca equation, the quantum correction renormalizes the bare mass μ_0 as $\mu_0^2 + \frac{g^2}{2\pi} \equiv \mu^2$:

$$(4\partial_+\partial_- + \mu^2)\tilde{B}^\nu = ej^\nu \Rightarrow \tilde{B}^\nu = B^\nu + \frac{e}{\mu^2}j^\nu, \quad (4\partial_+\partial_- + \mu^2)B^\nu = 0. \tag{192}$$

The interacting vector-meson field is thus given entirely in terms of the free fields. The solutions (188) is then on the quantum level regularized by normal ordering. The free current $j^\mu(x)$ built from the components (148), (149) can be bosonized as in the Thirring model case. The free LF vector-meson field, quantized by $[a(k^+), a^\dagger(l^+)] = \delta(k^+ - l^+)$, is expanded as

$$B^+(x) = \int_0^{+\infty} \frac{dk^+}{\sqrt{4\pi k^+}}\frac{k^+}{\mu}\big[a(k^+)e^{-i\hat{k}\hat{x}} + a^\dagger(k^+)e^{i\hat{k}\hat{x}}\big]. \tag{193}$$

$$\big[B^+(x^-), \Pi_{B^+}(y^-)\big] = i\delta(x^- - y^-), \quad \Pi_{B^+} = \partial_+B^+ - \partial_-B^- = 2\partial_+B^+. \tag{194}$$

The component $B^-(x)$ is determined from $\partial_+B^+ + \partial_-B^- = 0$. The field expansions (194), (182) can be used to obtain the Fock form of the corresponding LF Hamiltonian

$$P^- = \frac{1}{2}\int_{-\infty}^{+\infty} dx^-\Big[-\mu_0^2B^+B^- - \mu_0^2\frac{e}{\mu^2}\big(B^+j^- + B^-j^+\big) - \mu_0^2\frac{e^2}{\mu^4}j^+j^-\Big]. \tag{195}$$

This, along with the nonperturbative computation of the correlation functions of the model, is the subject of subsequent studies.

Operator solution of the LF Schwinger model: preliminary results. With the new quantization scheme for massless LF 2D fields, a new strategy towards the Schwinger model in LF field theory was formulated. Previous attempts did not produce a consistent physical picture of the model [30–32, 72, 73]. Our approach is based on the original fermionic form of the Lagrangian, restricted to the covariant (Feynman) gauge. A simple operator solution of the corresponding Dirac equation is found in terms of the free massless LF fermion field and the LF gauge field, i.e. without the Ansatz method. The correct axial anomaly is again obtained as a consequence of the point-splitting regularization of the fermion current.

We start from the same Lagrangian in the covariant gauge, as we did in the SL treatment. In terms of the LF variables, the Lagrangian in the Feynman gauge ($\gamma = 0$, $G(x)$ being the auxuliary, gauge-fixing field)) takes the form

$$\mathcal{L} = i\Psi_2^\dagger \overset{\leftrightarrow}{\partial}_+ \Psi_2 + i\Psi_1^\dagger \overset{\leftrightarrow}{\partial}_- \Psi_1 + \frac{1}{2}(\partial_+A^+ - \partial_-A^-)^2 - \frac{e}{2}J^-A^+ - \frac{e}{2}J^+A^-$$
$$-G(\partial_+A^+ + \partial_-A^-) + \frac{1}{2}G^2, \quad J^\mu = (J^+, J^-) = (\Psi_2^\dagger\Psi_2, \Psi_1^\dagger\Psi_1). \tag{196}$$

The corresponding field equations are

$$2i\partial_+\Psi_2 = eA^-\Psi_2, \quad \partial_+(\partial_+A^+ - \partial_-A^-) + \frac{e}{2}J^- = \partial_+G, \tag{197}$$

$$2i\partial_-\Psi_1 = eA^+\Psi_1, \quad \partial_-(\partial_-A^- - \partial_+A^+) + \frac{e}{2}J^+ = \partial_-G. \tag{198}$$

The field $G(x)$, satisfying $(\partial_+A^+ + \partial_-A^-) = G$, is composed of positive and negative-frequency parts $G^{(+)}$ and $G^{(-)}$. The physical subspace is given by $G^{(+)}|phys\rangle = 0$ [58]. The field variables in the Dirac eq. are neatly separated in the solution [74]

$$\Psi_2(x) = e^{-\frac{ie}{2}\int_{-\infty}^{+\infty} dy^+ \frac{1}{2}\epsilon(x^+-y^+)A^-(y^+,x^-)}\psi_2(x^-), \tag{199}$$

$$\Psi_1(x) = e^{-\frac{ie}{2}\int_{-\infty}^{+\infty} dy^- \frac{1}{2}\epsilon(x^--y^-)A^+(x^+,y^-)}\psi_1(x^+). \tag{200}$$

The residual gauge symmetry of the Lagrangian (196) is parametrized by the gauge function $\Lambda(x)$:

$$A^+ \to A^+ - 2\partial_-\Lambda, \quad A^- \to A^- - 2\partial_+\Lambda, \;\; \partial_+\partial_-\Lambda(x) = 0 \Rightarrow \Lambda(x) = \Lambda(x^+) + \Lambda(x^-). \tag{201}$$

Note that the transformation law for the interacting fermion field (200) is determined by the transformation for $A^\pm$ together with the laws for the free fermion field $\psi_2 \to \exp\{ie\Lambda(x^-)\}\psi_2, \psi_1 \to \exp\{ie\Lambda(x^+)\}\psi_1$, which are symmetry transformations for the free massless Lagrangian $\mathcal{L}_{0f} = [i\psi_2^\dagger\partial_+\psi_2 + i\Psi_1^\dagger\partial_-\psi_1] + H.c.$ Quantum currents are again constructed from the operator solution as regularized (point-split) products:

$$\begin{aligned} J^+(x) &= \lim_{\epsilon\to 0}\left[\Psi_2^\dagger(x+\frac{\epsilon}{2})\Psi_2(x-\frac{\epsilon}{2}) + H.c.\right] \\ &= j^+(x^-) + \frac{e}{2\pi}\int_{-\infty}^{+\infty} dy^+\frac{1}{2}\epsilon(x^+-y^+)\partial_-A^-(y^+,x^-), \end{aligned} \tag{202}$$

$$\begin{aligned} J^-(x) &= \lim_{\epsilon\to 0}\left[\Psi_1^\dagger(x+\frac{\epsilon}{2})\Psi_1(x-\frac{\epsilon}{2}) + H.c.\right] \\ &= j^-(x^+) + \frac{e}{2\pi}\int_{-\infty}^{+\infty} dy^-\frac{1}{2}\epsilon(x^--y^-)\partial_+A^+(x^+,y^-). \end{aligned} \tag{203}$$

where $j^+(x^-) = 2 : {\psi_2}^\dagger(x^-)\psi_2(x^-) :,\; j^-(x^+) = 2 : {\psi_1}^\dagger(x^+)\psi_1(x^+) :$. One immediately finds

$$\partial_\mu J^\mu(x) = \partial_+J^+ + \partial_-J^- = \frac{e}{2\pi}(\partial_+A^+ + \partial_-A^-) \equiv \frac{e}{2\pi}G, \tag{204}$$

$$\partial_\mu J_5^\mu(x) = \partial_+J^+ - \partial_-J^- = -\frac{e}{2\pi}(\partial_+A^+ - \partial_-A^-). \tag{205}$$

The first relation expresses the conservation of the vector current in the physical subspace, while the second represents the axial-vector ”anomaly”.

The gauge function $\Lambda(x^\pm)$ can have a non-trivial topology (polynomials in x^+ and x^-). The linear form is sufficiently general. The corresponding ”large” gauge transformations may be conveniently analyzed on a finite interval in both x^+, x^- with periodic boundary conditions.

Further aspects that have to be obtained in the present scheme include the complete Schwinger mechanism of vector-boson mass generation, chiral symmetry, vacuum degeneracy and fermion condensate. These issues require further study [74].

Chapter summary: In this chapter, we have studied various aspects of two-dimensional massless light-front fields. First, we have found a method how to quantize these fields unambiguously. Previous attempts did not yield satisfactory results. Consistency of the quantized massless LF fields was checked by the agreement of their two-point functions with the massless limit of the two-point functions of the corresponding massive fields, by the simple and correct derivation of the fermion-field bosonization, and by a correct and natural connection to the conformal field theory, including the quantum Virasoro algebra between the 2-dimensional Poincaré group generators. Finally, based on the new quantization scheme, simple and transparent operator solutions of the LF Thirring, Thirring-Wess and Schwinger model were constructed. The established properties of the models are in agreement with their space-like counterparts.

Chapter 5. Quantum kinks in a non-perturbative DLCQ study. In the Section 2, we presented LF perturbative calculations using the discretized (finite-interval) form of the self-interacting scalar models. This DLCQ method was initially applied by Pauli and Brodsky [15] to non-perturbative computations of the mass spectra and associated LF wave functions for the Yukawa model in two dimensions, because the method is suitable for building consistent Fock space expansions (many-particle Fock space). In contrast to the conventional SL field theory, this is a well defined step due to the fact that the LF Fock vacuum is an eigenstate of the full LF Hamiltonian, not just its free part. The hamiltonian matrix H_{ij} is then calculated from the (rescaled) Hamilton operator in Fock representation $H = \frac{2\pi}{L}P^-$ as $\langle i|H|j\rangle$. The states symbolically denoted by $|j\rangle$ are composed from j particles with the momenta $p_1^+, p_2^+ \dots p_j^+$ ($p_1^+ \equiv \frac{2\pi}{L}k_1$, etc.) in such a way that $p_1^+ + p_2^+ + \dots = P^+$ where P^+ is the total LF momentum of the system. The number of state grows rapidly with the resolution K defined as $P^+ = 2\pi L^{-1}K$, and the resultant large hamiltonian matrix is calculated and diagonalized numerically on a computer. In this last subsection, we consider the two-dimensional scalar model with quartic self-interaction in this nonperturbative setting.

In the variational approach, kinks can be well-approximated by coherent states. This appears to have two implications for the Fock space expansion in our discretized approach: one may need an infinite number of bosons to describe solitons, and, since the dimensionless total longitudinal momentum K automatically provides a cutoff on the number of bosons, convergence in K may be difficult to achieve for a kink-like state [75]. Our study shows how a nonperturbative evaluation of topological excitations and their observables is feasible in a finite Fock basis.

A popular nonperturbative numerical approach to field theory is the Euclidean lattice formulation. In the topologically non-trivial sector of the two-dimensional ϕ^4 theory, results available from lattice simulations are far limited to the determination of the kink mass [76]. The results for the configuration average of the kink profile are not smooth and are difficult to interpret, probably due to finite volume limitations. In this situation, the DLCQ approach represents a powerful method capable to generate predictions for physical observables nonperturbatively and from first principles. We will present results of DLCQ computations for the case of both antiperiodic and periodic boundary conditions and compare the results with the classical topological solution (kinks in this case) [77,78].

The LF Hamiltonian of the $\lambda\phi^4(1+1)$ model in the broken phase is

$$P^- = \int_{-L}^{+L} \frac{\mathrm{d}x^-}{2} \Big[-\mu^2\phi^2 + \frac{2\lambda}{4!}\phi^4 \Big] \equiv \frac{L}{2\pi} H \tag{206}$$

where L defines our compact domain $-L \le x^- \le +L$. The main goal in this section will is to compute the energy spectrum of H or equivalently of $M^2 = P^+P^- = KH$. In DLCQ with antiperiodic BC, the field expansion has the form

$$\phi(x^-) = \frac{1}{\sqrt{4\pi}} \sum_n \frac{1}{\sqrt{n}} \left[a_n e^{-i\frac{n\pi}{L}x^-} + a_n^\dagger e^{i\frac{n\pi}{L}x^-} \right], \;\; n = \frac{1}{2}, \frac{3}{2}, \ldots . \tag{207}$$

From the discrete Fock expansion of the (anti)periodic scalar field one obtains the Hamiltonian

$$\begin{aligned} H \;\; &= \;\; -\mu^2 \sum_n \frac{1}{n} a_n^\dagger a_n + \frac{\lambda}{4\pi} \sum_{k\le l, m\le n} \frac{1}{N_{kl}^2} \frac{1}{\sqrt{klmn}} a_k^\dagger a_l^\dagger a_n a_m \delta_{k+l,m+n} + \\ &\quad + \frac{\lambda}{4\pi} \sum_{k, l\le m\le n} \frac{1}{N_{lmn}^2} \frac{1}{\sqrt{klmn}} \left[a_k^\dagger a_l a_m a_n + a_n^\dagger a_m^\dagger a_l^\dagger a_k \right] \; \delta_{k,l+m+n} \end{aligned} \tag{208}$$

with $N_{kl} = 1$ for $k \ne l$ or $\sqrt{2}$ for $k = l$, and with $N_{lmn} = \sqrt{2!}, \; l = m \ne n, \; l \ne m = n$, $N_{lmn} = \sqrt{3!}, \; l = m = n$, and equal to 1 otherwise.

The final result of the DLCQ numerical computations are certain sets of numerical data that need to be analyzed and interpreted. A useful analytical guidance was provided by the "constrained variational approach" of [79]. The result of the constrained variational calculation, being semi-classical, is especially reliable in the weak coupling region and we can use its functional form to extract the kink mass from the numerical results of matrix diagonalization.

Let us first discuss the simpler unconstrained variational treatment for *antiperiodic boundary conditions*. Choose as a trial state the coherent state

$$\mid\alpha\rangle = \mathcal{N} \exp\Big(\sum_n \alpha_n a_n^\dagger \Big) \mid 0\rangle \Rightarrow \frac{\langle\alpha\mid\phi(x^-)\mid\alpha\rangle}{\langle\alpha\mid\alpha\rangle} = \frac{1}{\sqrt{4\pi}} f(x^-). \tag{209}$$

$\mathcal{N}$ is a normalization factor. Assuming the kink profile of $f(x^-)$, one gets $\alpha_{m-\frac{1}{2}} = \sqrt{\frac{3}{g}}\,\frac{i}{\pi}$, while the number density of bosons with momentum fraction $x = \frac{j}{K}$ is

$$\chi(x) = \frac{\langle\alpha\mid a_j^\dagger a_j \mid\alpha\rangle}{\langle\alpha\mid\alpha\rangle} = \alpha_j^2. \tag{210}$$

In the unconstrained variational calculation, the expectation value of the LF momentum operator is infinite for an infinite number of modes, because $f(x^-)$ is discontinous at $x^- = 0$, hence the space derivative in the momentum operator is infinite. To cure this deficiency for the present purpose, it is convenient to switch to a constrained variational calculation [79]. In the limit $\langle K\rangle \to \infty$ the expectation value of the Hamiltonian H in the generalized coherent states has the functional form

$$\frac{\langle\alpha\mid H\mid\alpha\rangle}{\langle\alpha\mid\alpha\rangle} = -\frac{6\pi\mu^4}{\lambda} + \frac{32\mu^6}{\lambda^2\langle K\rangle} \,. \tag{211}$$

Interpreting the state $|\alpha\rangle$ to be a kink state, we identify the first term as the vacuum energy density which is the classical vacuum energy density. The second term is identified as $\frac{M_{kink}^2}{\langle K\rangle}$. Then we get the classical kink mass $M_{kink} = \frac{4\sqrt{2}\mu^3}{\lambda}$.

An observable that yields considerable insight to the spatial structure of the topological object is the Fourier transform of its form factor, which is defined as the matrix element of the field operator in a physical state. We compute the Fourier transform of the form factor of the lowest state which, according to Goldstone and Jackiw [80], in the weak coupling (static) limit, represents the kink profile, i.e. a quantum counterpart of the classical solution. Let $\mid K\rangle$ and $\mid K'\rangle$ denote this state with momenta K and K'. The corresponding formula of the continuum theory reads

$$\int_{-\infty}^{+\infty} dq^{+}\exp\{-\frac{i}{2}q^{+}a\}\langle K' \mid \Phi(x^{-}) \mid K\rangle = \phi_c(x^{-}-a). \tag{212}$$

had to be adapted to the application within the DLCQ method.

The numerical results are as follows. For integer (half integer) values of K we have even (odd) number of particles. The dimensionality of the matrix in the even and odd sectors for different values of K is for example equal to $295,\ 61316,\ 813177$ for $K = 295,\ 39.5,\ 54.5$ and to $336,\ 67243,\ 880962$ for $K = 16,\ 40,\ 55$. All results presented here were obtained on small clusters of computers (≤ 15 processors) using the MFD code adapted to bosons [81] with the Lanczos diagonalization method. Since the Hamiltonian exhibits the $\phi \to -\phi$ symmetry, the even and odd

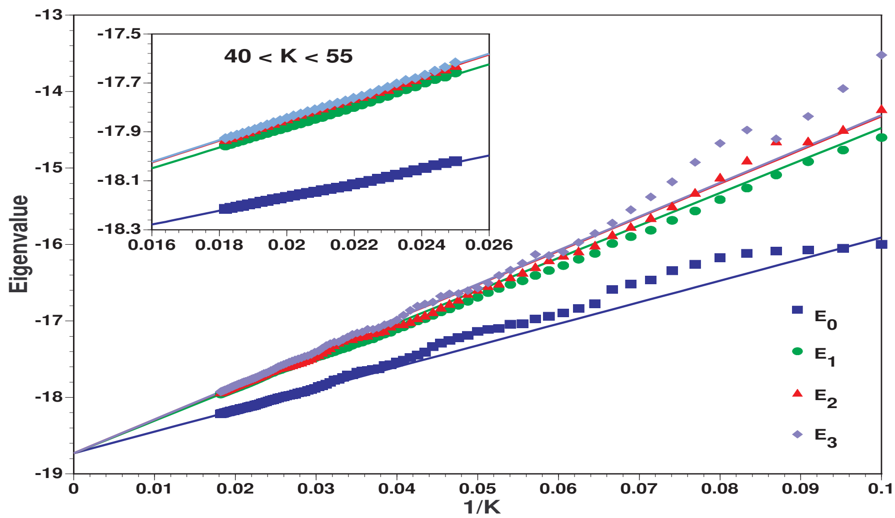

Figure 6: Lowest four eigenvalues for even and odd sectors as a function of $\frac{1}{K}$ for λ=1.0. The inset shows the details over the range $40 \leq K \leq 55$. The discrete points are the DLCQ eigenvalues while the straight lines are the linear fits to the $40 \leq K \leq 55$ data constrained to have the same intercept.

particle sectors of the theory are decoupled, i.e. matrix elements of the Hamiltonian between these two sectors vanish. A clear signal of SSB is the degeneracy of the spectrum in the even and odd particle sectors. Thus at finite K, we can compare the spectra for an integer K value (even particle sector) and its neighbouring half integer K value (odd particle sector) and look for degenerate states.

In Fig. 6 we show the lowest four energy eigenvalues in the broken symmetry phase for the even and odd particle sectors for λ=1.0 as a function of $\frac{1}{K}$. The points represent results at half integer increments in K from $K = 10$ to $K = 55$. The overall trend is towards smoother behavior at higher K. The smooth curves in Fig. 6 are linear fits to the eigenvalues in the range from $K = 40$ to $K = 55$ constrained to have the same intercept.

With guidance from the constrained variational calculation, see Eq. (211), we can extract the kink mass from the linear fit to the DLCQ data for the ground state eigenvalue. We fit the $\lambda = 1.0$ data in the range $40 \leq K \leq 55$ to a linear form $(C_1 + C_2/K)$.

Next we examine the behaviour of the number density $\chi(x)$ for the kink state. In Fig. 7 we show $\chi(x)$ for $K = 55$ and $K = 54.5$ for λ = 1. For this coupling the number densities for even and odd sectors are almost identical to each other indicative of degenerate states.

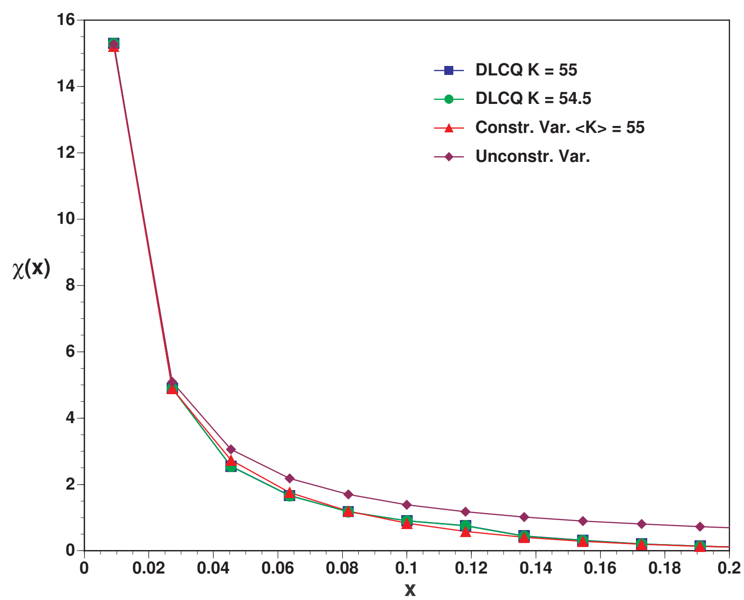

Figure 7: The number density $\chi(x)$ for even $(K = 55)$ and odd $(K = 54.5)$ sectors for $\lambda = 1$ compared with unconstrained and constrained $(\langle K \rangle = 55)$ variational results.

Following Goldstone and Jackiw, we calculated the Fourier transform of the form factor of the kink state at weak coupling. In Fig. 8(a) we show the profile for $\lambda = 1$ at three values of K. It is clear that at the profile is that of a kink which appears reasonably converged with increasing K.

Let us highlight the main results of the DLCQ analysis of quantum kinks for the case of *periodic boundary conditions*. The scalar field can be decomposed in this case as $\phi(x^-) = \phi_0 + \varphi(x^-)$, where ϕ_0 is the zero mode operator. Since it will be deliberately omitted in our Hamiltonian, $\varphi(x^-)$ is the normal mode operator (207) where now the index n runs over integers instead of half-integers.

We again diagonalize this Hamiltonian in the basis of all many-boson configurations at a fixed K where K is the sum of the values of the dimensionless momenta of all bosons in the configuration.

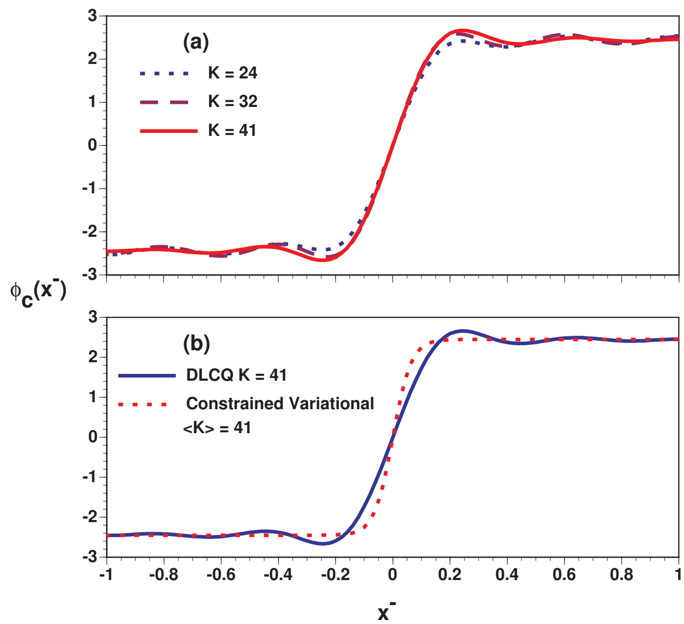

Figure 8: Fourier Transform of the kink form factor at λ=1; (a) results for $K = 24, 32$, and 41 each obtained with DLCQ eigenstates from 11 values of K centered on the designated K value; (b) comparison of DLCQ profile at K=41 with constrained variational result with $\langle K \rangle = 41$.

The Hamiltonian is symmetric under $\phi \to -\phi$ and thus, with PBC, the Hamiltonian matrix becomes block diagonal in even and odd particle number sectors. The dimensionality of the largest matrix we solve, $K = 60$, is equal to 483 338 (even sector) and 483 129 (odd sector).

Since we dropped the $P^+ = 0$ mode, degenerate vacuum states, characterized by a spatially uniform field expectation value, are not explicitly present in our formulation. However, one may expect degeneracy of the energy levels in the even and odd particle sectors at sufficiently high resolution, K. The lowest state is expected to be a kink-antikink configuration and should have a positive invariant mass (twice the mass of the single kink at weak coupling). One can extract this mass from finite K results analogously to the case of antiperiodic BC. Next, we compute the Fourier transform of the formfactor for the lowest state, using the discrete version of the formula (212). The Fourier transform represents the classical kink-antikink profile in the weak coupling limit and thus yields information about the spatial structure of the low lying states (see Fig. 9).

Chapter summary: The results of the present chapter can be summarized as follows. We have demonstrated the phenomenon of spontaneous symmetry breaking in a discretized light front approach without P^+ zero mode and calculated several nonperturbative physical quantities. The degeneracy of energy levels is both a signature of spontaneous symmetry breaking and essential for the existence of kinks. We have found that a finite Fock space yields features of the lowest excitation that are similar to those of a variational coherent state ansatz. We have extracted the quantum kink mass and the vacuum energy density for small λ by extrapolating our lowest eigenvalue to the continuum limit. At weak coupling, the mass of the quantum kink is closer to the classical value

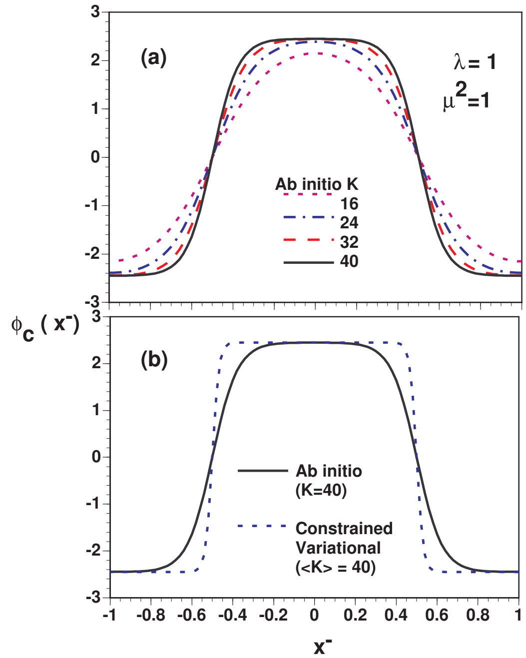

Figure 9: Fourier Transform of the kink-antikink form factor at $\lambda = 1.0$. Results are plotted in units of L. (a) Convergence with K. (b) Comparison of our result (Ab initio) (K=40) with constrained variational calculation ($\langle K \rangle = 40$).

than to the semi-classical mass. We have extracted the number density of elementary constituents of the lowest state and compared it with the coherent state prediction. We have also evaluated the Fourier transform of the lowest state form factor in a fully non-perturbative quantum approach and obtained a kink profile (antiperiodic BC) and a kink-antikink profile (periodic BC). These results can be interpreted as indicative of the viability of DLCQ for addressing non-trivial phenomena in quantum field theory.

There are three **Appendices** in the thesis.

In the **Appendix A** some details of the "near-light front" approach to field theory are presented.

In the **Appendix B**, the derivation and rules of the Hamiltonian (time-ordered) light-front perturbation theory are given.

In the **Appendix C**, some details of the constrained variational method, used for the extraction of kink parameters from the DLCQ data, are presented.

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