III International Workshop

CLASSICAL AND QUANTUM INTEGRABLE SYSTEMS

Edited by L.G.Mardoyan
G.S.Pogosyan
A.N.Sissakian

Dubna 1998

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PREFACE

The 3d International Workshop on Classical and Quantum Integrable Systems was organized by the Joint Institute for Nuclear Research and Yerevan State University and was held in Yerevan (Armenia) from June 29 to July 04, 1998.

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- Classical and Quantum Integrable Systems with N Degrees of Freedom
- Dynamical Symmetries and Superintegrability
- Supersymmetry Aspects of Integrable Systems
- Nonlinear Phenomena and Integrability
- Geometrical Methods and Integrability
- Quantum Groups, Quadratic Algebras, Other Types of Symmetries
- Separation of Variables, Special Functions
- Contractions of the Classical and Quantum Groups
- Integrability in Classical and Quantum Gravity

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PROBLEM OF ROTATION IN ALTERNATIVE THEORIES OF GRAVITATION

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Abstract

From astrophysical observations it is possible to conclude, that the rotation of cosmic objects is the general property of celestial bodies. The problem of rotation, as one of unsolved problem of astrophysics, represents purely theoretical interest. The solution of this problem is would given possibility to find the external solutions of axial-symmetric fields in the frames of the different theories of gravitation and to calculate the internal and integral characteristics of rotating configurations, to define a role of rotation for stability and energetic reserves of celestial bodies.

In this report the Kaluza - Klein (five dimensional theory of gravitation), Einstein and Rosen (bimetric theory of gravitation) equations in the case of axial-symmetric fields are obtained. The rotation is considered as perturbation to a spherical problem. [1]-[2] All unknown functions are expanded by non dimensional parameter $\beta = \frac{\Omega^2}{8\pi G\rho_c}$, where G is the gravitational constant, ρ_c is the matter density in the center of the star. β in order of value is coinciding with relation of rotation energy to gravitational energy, which for the configurations is much less than unit. A successful choice of parameter of the perturbation theory allows us to be limited by linear approximation. In this approximation in the frame of Kaluza Klein's, Rosen's and Einstein's theories the analytical external solutions of field equations are founded.

The main theses of the bimetric theory can be explained in the terms of coordinateless differential forms [3]. The problem of a self-consistent choice of the ground metric of the curved space-time manifold is discussed [4]. It is proved that the choice of the flat space-time metric is closely connected with the choice of the frame of references and in this theory the inertial and gravitational effects can be separated. Curved space-time with axial symmetry is considered. Solved the problem of the slowly stationary rotating star configurations with the use of perturbation methods as well as in Einstein theory. The exact solution of the external gravitational field is found and defined all the physical characteristics of the rotating star such as the active masses and corresponding to them quadrupole momentum, the momentum of inertia and their dependences on the angular velocity of the rotation. Investigations of the inner structure of the star configurations for several astrophysical models has been done. In particular, an analytic solution of the problem of rotation in second order perturbation theory is found for star with the hypothetical equation of state $\rho = \alpha P$, where $\alpha = \sqrt{12} - 3$ [5].

1 Introduction

From astrophysical observations it is possible to conclude that the rotation of the cosmic objects is general property of celestial bodies. The problem of rotation, as one of unsolved problems of astrophysics, represents purely theoretical interest. It is necessary to offer a technique for its solution, as within the frameworks the Newton [6],[7] as in Einstein [2] and alternative gravitational theories. The solution of this problem is would given possibility to estimate the angular velocity of rotation for calculation of the internal and external characteristics of rotating configurations.

The problem of rotation of configurations is so complicated that the exact solution of this problem is impossible even within the framework of Newton theory. The account of effects in general theory of gravitation complicates the problem much more. It is enough to tell that there are not exact general axial symmetric external solutions for the Einstein and others equations. Therefore does not surprised that the methods of solution of the problem of rotation arising in the last decade are approximate.

In the most successful method of solution the rotation is considered as a perturbation to a static problem. All unknown functions are expanded with respect to dimensionless parameter $\beta = \Omega^2/8\pi G\rho_c \ll 1$. In particular the metric coefficients will be represented in the form

$$g_{\mu\nu} = g^{(0)}_{\mu\nu}\left(1 + \sum_n \beta^n \triangle g^{(n)}_{\mu\nu}\right),$$

Calculations show that the obtained results are satisfactory in the whole range of possible angular velocities, starting form zero up to limiting value $\Omega_{\max} = (\frac{GM}{R_e^3})^{1/2}$, where M is the mass of star, R_e is its equatorial radius, determined from the condition of absence of matter flow from the equator.

Let us consider the metric of gravitational field, created by stationary rotation of mass distribution. It is obvious, that in this case the gravitational field as the mass distribution will be axial symmetric as well,

$$g_{ik} = g_{ik}\left(r,\, \theta,\, \Omega\right).$$

In the common case angular velocity of rotation Ω depends on r and θ. Here we use the following definitions for the coordinates $x^0 = t, x^1 = r, x^2 = \theta, x^3 = \varphi$.

Let us make the transformation of coordinates $x^{0'} = -x^0, x^{1'} = x^{'}, x^{2'} = x^2, x^{3'} = -x^3$. By this transformation the sign of angular momentum wouldn't change. Hence the components of metric tensor g_{ik}do not change too, due to the invariance of interval for g_{ik} immediately follows, that

$$g_{01} = g_{13} = g_{23} = g_{20} = 0.$$

Therefore, in the common case of axial symmetry the 4-dimensional interval in the Einstein theory we write in the form

$$ds^2 = g^{00}(dx^0)^2 + g^{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + 2g_{12}dx^1dx^2 + 2g_{03}dx^0dx^3.$$

The future simplification of this expression we can get by suitable transformation of coordinates r and θ. We require, that in new coordinates $g'_{12} = 0$ and $g'_{33} = g'_{22}\sin^2\theta$.

In new designations for 4-dimensional interval we can write

$$\begin{aligned} ds^2 &= \left(e^\nu - e^\mu\omega^2 r^2 \sin^2\theta\right) dt^2 - e^\lambda dr^2 - r^2 e^\mu \left(d\theta^2 + \sin^2\theta d\varphi^2\right) + \\ &+ 2\omega r^2 e^\mu \sin^2\theta d\varphi dt, \end{aligned}$$

where λ, μ, ω and ν is the functions of r, θ and Ω.

The metric will be invariant corresponding to the time inversion $t \to -t$, when the angular velocity is changed the sign, hence all the components of metric tensor, besides ω, will be even functions of Ω.

2 Axial-symmetric fields in Kaluza-Klein theory

In the 5-dimensional Kaluza-Klein theory, wherein Einstein space-time is extended by the introduction of one extra dimension, and in the expression ds^2 we must add $-e^\eta d\psi^2$ [8]-[12]. The extra-coordinate is ψ. The function η is a function of r, θ, and even function of Ω. We assume, that all functions does not depend on extra coordinate ψ.

5-dimensional Kaluza-Klein metric is

$$\begin{aligned} ds^2 &= \left(e^\nu - e^\mu\omega^2 r^2 \sin^2\theta\right) dt^2 - e^\lambda dr^2 - r^2 e^\mu \left(d\theta^2 + \sin^2\theta d\varphi^2\right) + \\ &+ 2\omega r^2 e^\mu \sin^2\theta d\varphi dt - e^\eta d\psi^2. \end{aligned}$$

For nonzero components of Einstein tensor we get the following expressions:(equations in the vacuum)

$$G_0^0 = \frac{e^{-\mu}}{r^2} - e^{-\lambda}\left[\frac{1}{r^2} + \frac{\eta_1}{r} + \frac{\eta_1^2}{4} - \frac{\lambda_1}{r} - \frac{\eta_1\lambda_1}{4} + \frac{3\mu_1}{r} - \frac{\mu_1\lambda_1}{2} + \frac{3}{2}\mu_1^2 + \frac{\eta_{11}}{2} + \mu_{11}\right]$$

$$G_1^1 = \frac{e^{-\mu}}{r^2} - e^{-\lambda}\left[\frac{1}{r^2} + \frac{\eta_1}{r}\frac{\mu_1}{r} + \frac{\mu_1\eta_1}{2} + \frac{\mu_1^2}{4} + \frac{\nu_1}{r} + \frac{\eta_1\nu_1}{4} + \frac{\mu_1\nu_1}{2}\right]$$

$$\begin{aligned} G_3^3 &= G_2^2 = -e^{-\lambda}\left[\frac{\mu_{11} + \eta_{11} + \nu_{11}}{2} + \frac{\eta_1}{2r} + \frac{\eta_1^2}{4} - \frac{\lambda_1}{r} - \frac{\lambda_1\eta_1}{4} + \frac{\eta_1\mu_1}{4} + \frac{\eta_1\lambda_1}{4} + \right. \\ &\left. + \frac{\mu_1^2}{4} + \frac{\nu_1}{2r} + \frac{\eta_1\nu_1}{4} - \frac{\lambda_1\nu_1}{2} + \frac{\mu_1\nu_1}{2} + \frac{\nu_1^2}{2}\right] \end{aligned}$$

$$\begin{aligned} G_4^4 &= \frac{e^{-\mu}}{r^2} + e^{-\lambda}\left[\frac{\lambda_1}{r} - \frac{1}{r^2} - \frac{3\mu_1}{r} - \frac{\nu_1}{r} + \frac{\mu_1\lambda_1}{2} - \frac{3}{2}\mu_1^2 + \frac{\lambda_1\nu_1}{4} - \frac{\mu_1\nu_1}{2} - \right. \\ &\left. - \frac{\nu_1^2}{2} - \mu_{11} - \frac{\nu_{11}}{2}\right] \end{aligned}$$

To compare the results with the results of the Einstein theory one has to put $\eta = 0$.

2.1 Static case

For static field we have identity

$$\nu + \lambda + \eta = 0$$

Solutions in the vacuum in units $G = c = 1$ are the following

$$e^{\lambda_0} = A^{-(a+b)}; \quad e^{\nu_0} = A^a; \quad e^{\mu_0} = A^{1-(a+b)}; \quad e^{\eta_0} = A^b; \quad A = 1 - \frac{2M}{r},$$

where $a = const, b = const$,are arbitrary constants fulfilling the equation $a^2 + b^2 + ab = 1$. Transition to Einstein theory yields $a = 1, b = 0$ (Schwarzschild metric).

2.2 First order approximation by Ω

In the static case $\Omega = 0$ the components of metric tensor depend only on r. Since the diagonal components of metric tensor are even functions of Ω, in first order of approximation by Ω the functions λ, μ, ν, η are remained independent on angles. However in this approximation the metric contains a non diagonal component g_{03}, since ω is linear function of Ω. Therefore in this case we must add the equation for non-diagonal element of metric

$$e^{\mu-\lambda}\left[\omega_{11} + \omega_1\left(\frac{\eta_1 - \lambda_1 - \nu_1}{2} + 2\mu_1 + \frac{4}{r}\right)\right] + +\frac{1}{r^2}\left(\omega_{22} + \omega_2\, ctg(\theta)\right) = 0. \tag{1}$$

to the system of equations for static case.

Using the external spherically symmetric solutions

$$e^{\mu_0 - \lambda_0} = (1 - \frac{2M}{r})$$

and the definition $\omega = \sqrt{\beta}q$ the equation (1) can be rewritten

$$\left(1 - \frac{2M}{r}\right)\left[q_{11} + q_1\left(\frac{4}{r} + (2 - 2a - b)\frac{2M/r^2}{1 - \frac{2M}{r}}\right)\right] + +\frac{1}{r^2}\left(q_{22} + 3q_2\, ctg\theta\right) = 0.$$

The solution for q we seek in the form

$$q = \sum_{l=0}^{\infty} Q_l(r) P_l^{(1)}(\gamma),$$

where $\gamma = \cos\theta$, $P_l^{(1)}(\gamma)$–associated Legandre Polynomial, which are satisfying the equation

$$\frac{d^2 P_l^{(1)}}{d\theta^2} + 3\, ctg\theta \frac{dP_l^{(1)}}{d\theta} + (l-1)(l+2)P_l^{(1)} = 0.$$

Finally for the functions $Q_l(r)$ we get

$$\left(1-\frac{2M}{r}\right)\left[Q_l''+Q_l'\left(\frac{4}{r}+(2-2a-b)\frac{2M/r^2}{1-\frac{2M}{r}}\right)\right]+\frac{1}{r^2}(l-1)(l+2)Q_l(r)=0 \quad (2)$$

Introducing the new variable $x=2M/r$, from the equation (2) we have

$$x^2(1-x)\frac{d^2Q_l}{dx^2}-x\left[2-x(2-c)\right]\frac{dQ_l}{dx}-(l-1)(l+2)Q_l=0.$$

The solution of this equation we seek in the form

$$Q_l=x^{l+2}F_l(x)$$

and for $F_l(x)$ we get

$$x(1-x)\frac{d^2F_l}{dx^2}-\left[2l+2-x(2l+2+c)\right]\frac{dF_l}{dx}-(c+l-1)(l+2)F_l=0.$$

Solving this equation with the requirement, that $F_l(x)$ has to go to zero in infinity ($r\to\infty$), when $x\to 0$, we have

$$F_l(x)=B_lF(l+2,l+c-1,2l+2,x).$$

Here B_l are constants, $c=2-2a-b$ and $F(\alpha,\beta,x)-$ influence hypergeometric function. Thus

$$Q_l(x)=B_lF(l+2,l+c-1,2l+2,x).$$

The constants B_l have to be defined from continuity conditions of $\omega(r,\gamma)$ and its first derivative by r on the boundary of the configuration. Since in this approximation the shape of the configuration is sphere, we require the continuity on the surface of static configuration. Then easy to see that values of B_l with $l>1$ are equal to zero. So in the first order of approximation the function q depends only on r and

$$Q_1(x)=Bx^3F(3,c,4,x),$$

where

$$\begin{aligned} F(3,c,4,x) &= \frac{3}{x^3(1-c)(2-c)(3-c)}\times \\ &\quad \times\left[2-(1-x)^{1-c}(2+2(1-c)x+(1-c)(2-c)x^2\right] \end{aligned}$$

Function $Q_1(x)$ can be performed with the expression

$$Q_1(x) = \frac{3B}{(1-c)(2-c)(3-c)}[2-(1-x)^{1-c}(2+2(1-c)x+(\ -c)(2-c)x^2].$$

To compare this result we the similar one in Einstein theory one must take $a=1, b=0, c=0$ and $Q_1(x)=Bx^3$.

2.3 Second order approximation by Ω.

Here in after conveniently to use the following combinations of Einstein equations

$$G_0^0 - G_1^1 = 0; \quad G_2^2 + G_3^3 = 0; \qquad G_1^2 = 0; \qquad G_4^1 = 0 \tag{3}$$

The expansion of unknown function of small parameter β, can be presented as [1]

$$e^\lambda = e^{\lambda_0}(1+\beta f), \quad e^\nu = e^{\nu_0}(1+\beta\Phi), \quad e^\mu = (1+\beta u),$$

$$e^\eta = e^{\eta_0}(1+\beta v); \quad f = \sum_{l=0}^{\infty} f_l P_l(\cos\theta).$$

In the conside ing case $l = 0, 2$.

One of the equations from the system of equations (3) defines the additional condition

$$v_1 + u_1 + \Phi_1 - \frac{v}{2}\left[(1-a-2b) + \frac{2}{r}\right] - \frac{\Phi}{2}\left[(1-2a-b)\frac{A_1}{A} + \frac{2}{r}\right] - \frac{f}{2}\left(\frac{A_1}{A} + \frac{2}{r}\right) = K(r),$$

between the unknown functions, where $K(r)$ is an arbitrary function from r. This function we assume to be equal to zero.

From the field equations yields that there is a relation

$$v_l + \Phi_l + f_l = \varphi_l.$$

φ_l obey the definite condition

$$\left[-\frac{l(l+1)}{2r^2 A} + \frac{1}{r}\left(\frac{A_1}{A} + \frac{1}{r}\right)\right]\varphi_l = \frac{2A^{l-2a-b}}{3} r^2 q_1^2\left(\delta_{0l} - \delta_{2l}\right).$$

Using this equation we can integrate the system of field equations, since the result has a huge formula they are not written down.

This approximation allow us to find the mass, shape and quadrupole moment of rotating star.

3 Choice of the frame of reference

3.1 Einstein theory of gravitation

For the spherically symmetric distribution of matter

$$ds^2 = a(r,t)dt^2 + b(r,t)dr^2 + c(r,t)(d\theta^2 + \sin^2\theta d\varphi^2) + h(r,t)drdt.$$

The arbitrariness in choice of the frame of reference allows to transformations

$$r = r(r', t'), \qquad t = t(r', t'),$$

since the number of equations and unknowns are not equal

$$N_{equations} = 4 \neq N_{unknowns} = 6.$$

Using such transformations the one can write the metric in Schwarzschild form

$$ds^2 = e^{\nu(r,t)}c^2dt^2 - e^{\lambda(r,t)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

(scale transformations does not change the frame of reference), or in conformal Euclidean form

$$ds^2 = e^{\alpha(r,t)}c^2dt^2 - e^{\beta(r,t)}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)].$$

The transformation into free-falling frame gives

$$ds^2 = c^2dt^2 - e^{\beta(R,t)}dR^2 - r^2(R,t)(d\theta^2 + \sin^2\theta d\varphi^2).$$

The specification of the Einstein theory is that the frame of reference can be defined only from the solution of the problem.

3.2 Bimetric theory of gravitation

The flat metric γ_{ik} in bimetric theories [13],[14] allows us to separate the pure gravitational effects from effects of non-inertially of the frame of reference. The problem of the choice of the frame of reference can be illustrated in some special cases for spherically symmetric distribution of masses.

The metric of the flat space-time in spherical coordinates is

$$d\sigma^2 = c^2dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

Thus the metric of the curved space-time has to be

$$ds^2 = e^{2\Phi(r,t)}c^2dt^2 - e^{2\Psi(r,t)}dr^2 - r^2e^{2\chi(r,t)}(d\theta^2 + \sin^2\theta d\varphi^2) - 2f(r,t)drdt.$$

In arbitrary non inertial frames the flat metric is

$$d\sigma^2 = e^{2a(r,t)}c^2dt^2 - e^{2b(r,t)}dr^2 - r^2e^{2c(r,t)}(d\theta^2 + \sin^2\theta d\varphi^2) - 2v(r,t)drdt.$$

Calculations of the unknowns are the following: 8 components of the metric tensor, 2 components of $4-$velocity u^0 u^1, pressure $-P$, density $-\rho$. At all 12 unknowns.

The equations and connection are:

Equation of state (1 equation) $P = P(\rho)$, normalization of 4-velocity (1 equation) $u \cdot u = 1$. Field equations (4 equations)

$$N^i_k - \frac{1}{2}\delta^i_k N = -\frac{8\pi G}{c^2} k T^i_k;\, k = \sqrt{\frac{\det g}{\det \gamma}},$$

equations of hydrodynamics (2 equations)

$$T^i_{k;i} = 0\ ,\ k = 0, 1,$$

which are not follow from the field equations. Requirements on γ_{ik}, to be the metric of the flat space (2 equations): $P^i_{klm} = 0$, (r, t) –transformation (2 equations). At all 12 conditions.

So the system of the equations are complete.

3.2.1 Metric in isotropic coordinates

$$ds^2 = c^2 e^{2\Phi(r)} dt^2 - e^{2\Psi(r)} dr^2 - r^2 e^{2\chi(r)} (d\theta^2 + \sin^2\theta d\varphi^2),$$

$$d\sigma^2 = c^2 dt^2 - dr^2 - r^2 (d\theta^2 + \sin^2\theta d\varphi^2).$$

The external solutions of the field equations are

$$\Phi(r) = -\frac{GM}{c^2 r}, \qquad \Psi(r) = \chi(r) = -\frac{GM'}{c^2 r},$$

where M and M' are the constants of integrations and connected with the matter distribution via

$$M = 4\pi \int_0^R k(\rho + 3P) r^2 dr, \qquad M' = 4\pi \int_0^R k(\rho - P) r^2 dr.$$

3.2.2 The Metric of Schwarzschild

$$\begin{aligned} ds^2 &= c^2 e^{2A(R)} dt^2 - e^{2B(R)} dR^2 - R^2 (d\theta^2 + \sin^2\theta d\varphi^2), \\ d\sigma^2 &= c^2 dt^2 - e^{2\lambda(R)} dR^2 - R^2 e^{2\mu(R)} (d\theta^2 + \sin^2\theta d\varphi^2). \end{aligned}$$

From the conditions of the flatness we have

$$e^\lambda = \frac{d}{dR}\left(R\, e^\mu\right).$$

The solutions can be done in the parametric form

$$A = \frac{c_1}{r}, \quad B = -\ln\left(1 + \frac{c_2}{r}\right), \quad \lambda = \frac{c_2}{r} - \ln\left(1 + \frac{c_2}{r}\right), \quad \mu = \frac{c_2}{r},$$

where

$$R = r e^{-c_2/r}, \qquad c_1 = -\frac{GM}{c^2}, \qquad c_2 = -\frac{GM'}{c^2}$$

and $k = e^{A-3\mu}$.

3.2.3 Synchronous System of the Frame of reference $g_{00} = 1, g_{0\alpha} = 0$

a)Spherical symmetric case

The solution is obtained far from the star $r_g/r << 1$

$$
\begin{aligned}
ds^2 &= c^2 d\tau^2 - \frac{r_g}{r} dR^2 - r^2 (d\theta^2 + \sin^2\theta d\varphi^2), \\
d\sigma^2 &= c^2 \left(1 + \frac{r_g}{r}\right) d\tau^2 - \frac{r_g}{r} dR^2 - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right).
\end{aligned}
$$

where

$$
r = r_g \left(\frac{3\,(R - c\tau)}{2r_g}\right)^{2/3}.
$$

$r_g = 2GM/c^2$ is gravitational radius.

b)Axially-Symmetric distribution of Masses

The solution far from the star and in the frame of freely falling body and drugging under the forces of gravity is

$$
\begin{aligned}
d\sigma^2 &= c^2 \left(1 + \frac{r_g}{r} - \frac{\omega^2 r^2 \sin^2\theta}{c^2}\right) d\tau^2 - \frac{r_g}{r} dR^2 - r^2 (d\theta^2 + \sin^2\theta d\varphi^2) - \\
&\quad -2\frac{\omega^2 r^2 \sin^2\theta}{c^2} dR c d\tau + 2\frac{\omega r^2 \sin^2\theta}{c} d\varphi c d\tau - 2\frac{\omega r^2 \sin^2\theta}{c} dR d\varphi + \\
&\quad 6\beta b_1 \sin\theta \cos\theta d\theta c d\tau - 6\beta b \sin\theta \cos\theta dR d\theta.
\end{aligned}
$$

where

$$
\begin{aligned}
r &= r_g \left(\frac{3}{2}\frac{R - c\tau}{r_g}\right)^{2/3} \left[1 + c_0 \beta \left(\frac{3}{2}\frac{R - c\tau}{r_g}\right)^{-4/3} P_2(\cos\theta)\right]^{2/3}. \\
\omega &= \frac{c_2}{r^3}, \qquad b = \frac{c_0}{\sqrt{r}}, \qquad b_1 = \frac{c_1}{\sqrt{r}}
\end{aligned}
$$

β proportional to the Ω^2 square of the angular velocity of the rotating star. c_0, c_1, c_2- are constants of integration.

The third term in γ_{00} corresponding to centrifugal forces.

4 Problem of Rotation in the Bimetric Theory of Gravitation

Metric tensors of space-time manifold in frame bimetric theory of gravitation (BTG) are

$$
ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad d\sigma^2 = \gamma_{\mu\nu} dx^\mu dx^\nu. \tag{4}
$$

The tetrad and vector basis's for the flat space-time metric can be written in the form

$$\omega^0 = cdt, \quad \mathbf{e}_0 = \frac{1}{c}\frac{\partial}{\partial t}, \quad \omega^1 = dr, \quad \mathbf{e}_1 = \frac{\partial}{\partial r},$$

$$\omega^2 = rd\theta, \quad \mathbf{e}_2 = \frac{1}{r}\frac{\partial}{\partial\theta}, \quad \omega^3 = r\sin\theta d\varphi, \quad \mathbf{e}_3 = \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}.$$

In this basis the metric's (4) of axially-symmetric manifold can be written as [15]

$$\begin{aligned} d\sigma^2 &= \eta_{\alpha\beta}\omega^\alpha\omega^\beta, \quad \eta_{\alpha\beta} = diag(1,-1,-1,-1), \\ ds^2 &= e^{2\Phi}(\omega^0)^2 - e^{2\Psi}(\omega^1)^2 - e^{2\mu}(\omega^2 + \nu r\omega^1)^2 - e^{2\alpha}(\omega^3 + \omega r\sin\theta\omega^0)^2 \end{aligned}$$

So tetrad basis $\{\theta^\alpha\}$ and vector basis $\{\xi^\alpha\}$ of the curved space-time will be expressed in the following form

$$\theta^0 = \omega^0, \quad \xi_0 = \mathbf{e}_0 - r\omega\sin\theta\omega^0, \quad \theta^1 = \omega^1, \xi_1 = \mathbf{e}_1 - r\nu\mathbf{e}_2,$$

$$\theta^2 = \omega^2 + \nu r\omega^1, \quad \xi_2 = \mathbf{e}_2, \quad \theta^3 = \omega^3 + \omega r\sin\theta\omega^0, \quad \xi_3 = \mathbf{e}_3$$

The equations of gravitational field in BTG can be written in non-coordinate form

$$\mathbf{N} = -8\pi k\left(\mathbf{T} - \frac{1}{2}\mathbf{g}Sp\mathbf{T}\right),$$

where $\mathbf{N}$ will be expressed by the metric tensor of curved space $\mathbf{g}$ as

$$\mathbf{N} = \frac{1}{2}\nabla(\widetilde{g}\cdot\nabla\mathbf{g}).$$

$\mathbf{T}$ is the energy-momentum tensor of matter and non-gravitational fields and

$$Sp\mathbf{T} = g^{\mu\nu}T_{\mu\nu},$$

∇ is the operator of the covariant differential by $\gamma_{\mu\nu}$, $\widetilde{g} = g^{\mu\nu}\mathbf{e}_\mu\mathbf{e}_\nu$-projection to basis vectors of gravitation metric tensor $\mathbf{g}$.

The whole and exact expression for Rosen tensor $\mathbf{N}$ in the case of the axial symmetry is [16]:

$$\mathbf{N} = \left[-\Delta\Phi + \frac{1}{2}\left[(\omega + \omega_1 r)^2 + (\omega_2 + \omega ctg\theta)^2\right]\sin^2\theta e^{2\alpha-2\Phi}\right]\xi_0\omega^0 +$$

$$\left[(\Phi_1 - \alpha_1)(\omega + \omega_1 r) + \frac{\Phi_2 - \alpha_2}{r}(\omega_2 + \omega ctg\theta) - \frac{1}{2}(\omega_{11}r + 4\omega_1) + \frac{2\omega}{r} + \right.$$

$$\left. + \frac{1}{r}\left(\omega_{22} + 3\omega_2 ctg\theta + \omega\left(ctg^2\theta - 1\right)\right)\right]\sin\theta e_2\omega^0 + \left[-\Delta\alpha - \frac{1}{2}\left[(\omega + \omega_1 r)^2 + \right.\right.$$

$$(\omega_2 + \omega ctg\theta)^2] \sin^2\theta e^{2\alpha-2\Phi} + \frac{1}{2}\left(\frac{1}{r^2} - \nu^2 ctg^2\theta\right) e^{2\alpha-2\Psi} + \frac{ctg^2\theta}{2r^2} e^{2\alpha-2\mu} e_3\theta^3\Bigg] +$$

$$-\frac{1}{2}\left[\frac{2\nu ctg\theta}{r^2} - \left(\nu + \frac{ctg\theta}{r}\right)^2 e^{2\mu-2\Phi}\right] r\omega \sin\theta\xi_0\omega^3 - \frac{1}{2}\left[\frac{2\nu ctg\theta}{r} - \frac{1}{r^2} e^{2\Psi-2\alpha} -\right.$$

$$\left.-\left(\nu + \frac{ctg\theta}{r}\right)^2 e^{2\mu-2\alpha}\right] e_3\omega^3 + \frac{1}{2}\left[\omega_{11} r + 2\omega_1\left(2\alpha_1 - 2\Phi_1 + \frac{2}{r}\right)(\omega + \omega_1 r)\right.$$

$$\left.+\frac{1}{r}\left\{\omega_{22} + 2\omega_2 ctg\theta - \omega + (2\alpha_2 - 2\Phi_2 + ctg\theta)(\omega_2 + \omega ctg\theta)\right\}\right] e^{2\alpha-2\Phi} \sin\theta\xi_0\theta^3 +$$

$$+\frac{1}{2}\left[-2\triangle\Psi - \left(\nu_2 - \frac{1}{r}\right)\left(\nu_2 - \frac{1}{r} - \nu^2 r\right) e^{2\mu-2\Psi} - (\nu + \nu_1 r)^2 e^{2\mu-2\alpha} + \frac{2}{r^2} +\right.$$

$$\left.\frac{1}{r}\left(\nu ctg\theta - \frac{1}{r}\right) e^{2\alpha-2\Psi}\right]\xi_1\omega^1 + \frac{1}{2}\left[2(\Psi_1 - \mu_1)(\nu + \nu_1 r) + 2\frac{\Psi_2 - \mu_2}{r}\left(\nu_2 - \frac{1}{r}\right)\right.$$

$$-\nu\left(\nu_2 - \frac{1}{r}\right) + \frac{ctg\theta}{r^2}\left(1 - e^{2\alpha-2\mu}\right) + 2\frac{\Psi_2 - \mu_2}{r} e^{2\Psi-2\mu} - \omega^2 e^{2\Psi-2\Phi} \sin\theta\cos\theta -$$

$$\left.\nu_{11} r - 4\nu_1 - \frac{1}{r}(\nu_{22} + \nu_2 ctg\theta) - \frac{e^{2\Psi}}{r^2}\left(e^{-2\mu} - e^{2\alpha}\right) ctg\theta - \nu\left(2\nu_2 + \nu ctg\theta - \frac{1}{r}\right)\right]$$

$$\times e_2\omega^1 + \frac{1}{2}\left[-\frac{2\Psi_2\nu}{r} + \frac{1}{r}\left(\nu_2 - \frac{2}{r}\right) + \frac{e^{2\Psi}}{r^2}\left(e^{-2\mu} + e^{-2\alpha}\right) + \frac{1}{r}\left(\nu_2 + \nu ctg\theta + \nu^2 r\right)\right.$$

$$\left.-\omega^2 \sin^2\theta e^{2\Psi-2\Phi}\right] e_1\omega^1 + +\frac{1}{2}\left[-\frac{2\Psi_2}{r} + \frac{\nu}{r} - \nu\left(\nu_2 - \frac{1}{r} - \nu^2 r\right) e^{2\mu-2\Psi} + \frac{ctg\theta}{r} \times\right.$$

$$\left.\left(\nu ctg\theta - \frac{1}{r}\right) e^{2\alpha-2\Psi}\right]\xi_1\omega^2 + \frac{1}{2}\left[-2\triangle\mu + \left(\nu + \frac{ctg\theta}{r}\right)\frac{ctg\theta}{r} e^{2\mu-2\alpha} + \left(\nu_2 - \frac{1}{r}\right)\right.$$

$$\times\left(\nu_2 - \frac{1}{r} - \nu^2 r\right) e^{2\mu-2\Psi} - (\nu + \nu_1 r)^2 e^{2\mu-2\Psi}\frac{1}{r}(\nu_2 + \nu ctg\theta) - \left(\nu + \frac{ctg\theta}{r}\right) \times$$

$$\left.\times r\omega^2 \sin\theta\cos\theta e^{2\mu-2\Phi}\right] e_2\theta^2 + \frac{1}{2}\left[-\frac{2\mu_2\nu}{r} + \frac{1}{r}\left(\nu_2 - \frac{1}{r}\right) + \frac{1}{r^2}\left(1 - e^{2\Psi-2\mu}\right) -\right.$$

$$\left.\nu^2 - \frac{2\nu_2}{r} - \frac{ctg^2\theta}{r^2} e^{2\alpha-2\mu}\right] e_2\omega^2 + \frac{1}{2}\left[\frac{2\mu_2}{r^2} - -\nu\left(\nu_2 - \frac{1}{r} - \nu^2 r\right) e^{2\mu-2\Psi} + \frac{\nu}{r} +\right.$$

$$\left(\nu + \frac{ctg\theta}{r}\right)\frac{1}{r}e^{2\mu-2\alpha} - \left(\nu - \frac{ctg\theta}{r}\right) r\omega^2 \sin\theta e^{2\mu-2\Phi}\Bigg] e_1\theta^2 - \frac{ctg\theta}{2r^2}e_1\omega^1 -$$

$$-\frac{1}{2}\left[\nu_{11}r + 2\nu_1 + \left(2\mu_1 - 2\Psi_1 - \frac{2}{r}\right)(\nu + \nu_1 r) - \frac{1}{r}\left[\nu_{22} - 2\nu_2\nu r + \right.\right.$$

$$\left.\left.(2\mu_2 - 2\Psi_2 + ctg\theta)\left(\nu_2 - \frac{1}{r} - \nu^2 r\right)\right]\right] e^{2\mu-2\Psi}\xi_1\theta^2$$

Right side o' the field equations in case of stationary-rotating source is:

$$\mathbf{T} - \frac{1}{2}\mathbf{g}Sp\mathbf{T} = \frac{1}{2}\left[\left(\rho c^2 + P + u\right)\frac{\partial}{\partial t}dt - \left(\rho c^2 - P + u\right)\frac{\partial}{\partial\varphi}d\varphi - \left(\rho c^2 - P\right)\frac{\partial}{\partial r}dr\right.$$

$$\left. - \left(\rho c^2 - P\right)\frac{\partial}{\partial\theta}d\theta - u\frac{\partial}{\partial t}d\varphi - \left(\frac{\rho c^2 + P}{Q} - \omega\right)u\Omega\frac{\partial}{\partial\varphi}dt\right],$$

where

$$u = -\frac{2Q}{1 - \frac{(\omega+\Omega)Q}{\rho c^2 + P}}, \qquad Q = \left(\rho c^2 + P\right) r^2 \sin^2\theta\,(\omega + \Omega)\, e^{2\alpha-2\Phi}.$$

To complete the description of configuration we must add the equation of hydrodynamic equilibrium, i.e.

$$-\frac{dP}{\rho c^2 + P} = -d\left[\Phi + \frac{1}{2}\ln\left(1 - r^2(\omega + \Omega)^2\sin^2\theta e^{2\alpha-2\Phi}\right)\right],$$

to the system of field equations.

4.1 First order of approximation

The complete system of equations for metric coefficients for the rotating spherical star is

$$\Phi_{11} + \frac{2}{r}\Phi_1 = 4\pi k\,(\rho + 3P)\,, \quad \Psi_{11} + \frac{2}{r}\Psi_1 = \frac{2}{r^2}\sinh(2\Psi - 2\mu) - 4\pi k\,(\rho - P)\,,$$

$$\mu_{11} + \frac{2}{r}\mu_1 = -\frac{1}{r^2}\sinh(2\Psi - 2\mu) - 4\pi k\,(\rho - P)\,, \quad P_1 + (\rho + P)\Phi_1 = 0,$$

$$q_{11} + \left[\frac{4}{r} - \frac{2(M + M')}{r^2}\right]q_1 = \frac{2\left(M + M'\right)}{r^3} + 16\pi e^{\Phi_0 + 3\Psi_0}\,(\rho + 3P)\,q -$$

$$-\frac{1}{r^2}\left(q_{22} + 3q_2 ctg\theta\right) + 16\pi(\rho + P)e^{\Phi_0+3\Psi_0}\sqrt{8\pi\rho_c}.$$

The external solution of this equation (region out of star $P = \rho = 0$) in the units $G = c = 1$ is

$$\Phi = \Phi_0 = -\frac{M}{r}, \quad \Psi = \Psi_0 = \frac{M'}{r}, \quad \omega = \sqrt{3}q = \sqrt{3}\frac{c_1}{r^3}F\left(2,4,-\frac{2(M+M')}{r}\right)$$

where $F(\alpha,\beta,x)$-is influence hypergeometric function, c_1, M, M'-constants of integration

$$M = 4\pi\int_0^R (\rho+3P)\,e^{\Phi_0+3\Psi_0}r^2dr, \quad M' = 4\pi\int_0^R (\rho-P)\,e^{\Phi_0+3\Psi_0}r^2dr.$$

4.2 Second order of approximation

The system of the field equations for the second order corrections is:

$$\begin{aligned}
\Delta_0 X &= -8\pi e^{\Phi_0+3\Psi_0}(\rho_0-3P_0)\left[X+\frac{\rho^0-3P^0}{\rho_0-3P_0}\right],\\
\Delta_2 Y &= \frac{1}{2}(W(r)-V(r))+4\pi e^{\Phi_0+3\Psi_0}K(r)(\rho_0+P_0),\\
\Delta_2 A &= -8\pi e^{\Phi_0+3\Psi_0}(\rho_0-3P_0)\left[A+\frac{\rho^2-3P^2}{\rho_0-3P_0}\right], \qquad (5)\\
V(r) &= \Delta_2 F-\frac{4F}{r^2}-\frac{4\beta}{r^2}-\frac{8\chi^2}{r^2}, \quad \Delta_4\Gamma = 0,\\
\Delta_2 B-\frac{6B}{r^2}+\frac{18F}{r^2} &= 2V(r)-\frac{1}{2}W(r)-4\pi e^{\Phi_0+3\Psi_0}K(r)(\rho_0+P_0), \qquad (6)
\end{aligned}$$

where

$$\begin{aligned}
\Gamma &= B-2F+2\chi^2, \quad X = \varphi^0+f^0+2u^0, \quad Y = f^0-u^0,\\
A &= \varphi^2+f^2+2u^2, \quad B = f^2-u^2, \quad F = \varsigma^2 r
\end{aligned}$$

and

$$W(r) = \frac{1}{3}q^2r^2e^{2\Psi_0-2\Phi_0}\left(\frac{q_1}{q}+\frac{1}{r}\right)^2, \qquad V(r) = \frac{1}{3}q^2e^{2\Psi_0-2\Phi_0}.$$

Solution of the system (5) out of matter has a form

$$X = b_0y; \quad \varphi^0 = a_0y+\xi_0(y); \quad Y = \alpha_0y^3-\frac{1}{2}\varphi^0(y); \quad \Gamma = 0;$$

$$A = b_2y^3; \quad \varphi^2 = a_2y^3+\varphi^0(y); \quad B+6\chi = b_1y+\xi_0(y),$$

$$\xi_0(y) = 1+\frac{2\,c_1^2m^2}{y\;48}\left[e^{2y}\left(1-\frac{2}{y}+\frac{1}{2y^2}\right)+1+\frac{2}{y}-\frac{1}{y^2}+\frac{e^{-2y}}{2y^2}\right]$$

$$\varphi^0(y) = -\frac{c_1^2m^2}{48}\left[e^{2y}\left(1-\frac{2}{y}+\frac{5}{4y^2}\right)+1-\frac{1}{y}-\frac{1}{y^2}-\frac{e^{-2y}}{4y^2}\right],$$

where $y = m/2r$

The constants of integration are connected with inertial momentum, two masses and two quadrupole momentum of the star and defined by conditions of continuity of the external and internal solutions.

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INTEGRABLE DEFORMATIONS OF HAMILTONIAN SYSTEMS AND q-SYMMETRIES

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Abstract

The complete integrability of the hyperbolic Gaudin Hamiltonian and other related integrable systems is shown to be easily derived by taking into account their $sl(2,\mathbb{R})$ coalgebra symmetry. By using the properties induced by such a coalgebra structure, it can be proven that the introduction of any quantum deformation of the $sl(2,\mathbb{R})$ algebra will provide an integrable deformation for such systems. In particular, the Gaudin Hamiltonian arising from the non-standard quantum deformation of the $sl(2,\mathbb{R})$ Poisson algebra is presented, including the explicit expressions for its integrals of motion. A completely integrable system of nonlinearly coupled oscillators derived from this deformation is also introduced.

1 Introduction: on q-Poisson coalgebras

Let us consider the $sl(2)$ Poisson-Lie algebra

$$\{J_3, J_\pm\} = \pm 2\, J_\pm, \qquad \{J_-, J_+\} = \alpha\, J_3. \tag{1.1}$$

(where α is a real positive parameter and the usual $\alpha = 1$ case can be easily get by performing the change of basis $J_\pm \to J_\pm/\sqrt{\alpha}$). The Casimir function for this algebra reads

$$C(J_3, J_\pm) = \frac{\alpha}{4} J_3^2 - J_+ J_-. \tag{1.2}$$

The Poisson algebra (1.1) is endowed with a coalgebra structure [1, 2] by means of the "primitive" coproduct $\Delta : sl(2) \to sl(2) \otimes sl(2)$ defined as follows:

$$\Delta(J_3) = J_3 \otimes 1 + 1 \otimes J_3, \qquad \Delta(J_\pm) = J_\pm \otimes 1 + 1 \otimes J_\pm. \tag{1.3}$$

Compatibility between (1.3) and (1.1) means that Δ is a Poisson map: the three functions defined through (1.3) close also a Poisson $sl(2)$ algebra. The map (1.3) can be introduced in a similar manner for any Lie algebra. The coalgebra structure, together with the known relevance of Lie algebras in the context of Liouville integrability [3, 4], are the keystones of the universal construction of integrable systems introduced in [5] that we shall summarize in the next Section.

From that point of view, the search for q-Poisson algebras can be seen as the problem of finding deformations of a given algebra endowed with compatible deformations of the coproduct (1.3) preserving it as a Poisson map. In fact, two relevant and distinct structures deforming $sl(2,\mathbb{R})$ appeared in quantum group literature during last years (see, for instance, [6]), and can be realized as q-Poisson algebras as follows:

a) The *"standard"* deformation [7, 8] $sl_q(2)$, which is given by the following deformed Poisson brackets ($q = e^z$):

$$\{J_3, J_+\} = 2\,J_+, \quad \{J_3, J_-\} = -2\,J_-, \quad \{J_-, J_+\} = \frac{\sinh zJ_3}{z}, \tag{1.4}$$

which are compatible with the deformed coproduct

$$\begin{aligned}
\Delta(J_3) &= J_3 \otimes 1 + 1 \otimes J_3, \\
\Delta(J_+) &= J_+ \otimes e^{\frac{z}{2}J_3} + e^{-\frac{z}{2}J_3} \otimes J_+, \\
\Delta(J_-) &= J_- \otimes e^{\frac{z}{2}J_3} + e^{-\frac{z}{2}J_3} \otimes J_-,
\end{aligned} \tag{1.5}$$

in the sense that that (1.5) is a Poisson algebra homomorphism with respect to (1.4). The function

$$\mathcal{C}_z^{(s)}(J_3, J_\pm) = \left(\frac{\sinh(\frac{z}{2}J_3)}{z}\right)^2 - J_+\,J_-, \tag{1.6}$$

is the deformed Casimir for this Poisson coalgebra.

b) The *"non-standard"* quantum $sl(2,\mathbb{R})$ deformation [9, 10] is defined by:

$$\begin{aligned}
\Delta(J_-) &= J_- \otimes 1 + 1 \otimes J_-, \\
\Delta(J_+) &= J_+ \otimes e^{zJ_-} + e^{-zJ_-} \otimes J_+, \\
\Delta(J_3) &= J_3 \otimes e^{zJ_-} + e^{-zJ_-} \otimes J_3,
\end{aligned} \tag{1.7}$$

$$\{J_3, J_+\} = 2J_+ \cosh zJ_-, \qquad \{J_3, J_-\} = -2\frac{\sinh zJ_-}{z}, \qquad \{J_-, J_+\} = 4J_3. \tag{1.8}$$

The deformed Casimir reads

$$\mathcal{C}_z^{(n)}(J_3, J_\pm) = J_3^2 - \frac{\sinh zJ_-}{z}J_+. \tag{1.9}$$

The aim of this contribution is to present an application of this latter q-Poisson algebra in the construction of an integrable deformation of the Gaudin Hamiltonian. This deformation can be also interpreted in the context of a chain of nonlinearly coupled oscillators or, equivalently, in relation with a certain integrable perturbation of the motion of a particle under any central potential in the N-dimensional Euclidean space. Through these examples will show an intrinsic connection between quantum deformations and nonlinear interactions depending on the momenta. Results concerning the standard deformation of the Gaudin-Calogero system [11] can be found in [5, 12].

2 The formalism

By following [5], we can state the following result: any coalgebra (A, Δ) with Casimir element C can be considered as the generating symmetry of a large family of integrable systems. We shall consider here classical mechanical systems only and, consequently, we shall make use of Poisson realizations D of the algebra A of the form $D : A \to C^\infty(q,p)$. However, we recall that the formalism is also directly applicable to quantum mechanical systems. The constructive procedure is as follows.

Let (A, Δ) be a (Poisson) coalgebra with generators X_i $(i = 1, \dots, l)$ and Casimir function $C(X_1, \dots, X_l)$. This means that the coproduct $\Delta : A \to A \otimes A$ is a Poisson map. Let us consider the N-th coproduct map $\Delta^{(N)}$

$$\Delta^{(N)} : A \to A \otimes A \otimes \dots^{N)} \otimes A. \tag{2.10}$$

which is obtained (see [5]) by applying recursively the two-coproduct $\Delta^{(2)} \equiv \Delta$ in the form

$$\Delta^{(N)} := (id \otimes id \otimes \dots^{N-2)} \otimes id \otimes \Delta^{(2)}) \circ \Delta^{(N-1)}. \tag{2.11}$$

By taking into account that the m-th coproduct $(m \leq N)$ of the Casimir $\Delta^{(m)}(C)$ can be embedded into the tensor product of N copies of A as

$$\Delta^{(m)} : A \to \{A \otimes A \otimes \dots^{m)} \otimes A\} \otimes \{1 \otimes 1 \otimes \dots^{N-m)} \otimes 1\}, \tag{2.12}$$

it can be shown that,

$$\left\{\Delta^{(m)}(C), \Delta^{(N)}(X_i)\right\}_{A\otimes A\otimes \dots^{N)}\otimes A} = 0, \qquad i = 1, \dots, l, \qquad 1 \leq m \leq N. \tag{2.13}$$

With this in mind it can be proven [5] that, if $\mathcal{H}$ is an *arbitrary* (smooth) function of the generators of A, the N-particle Hamiltonian defined on $A \otimes A \otimes \dots^{N)} \otimes A$ as the N-th coproduct of $\mathcal{H}$

$$H^{(N)} := \Delta^{(N)}(\mathcal{H}(X_1, \dots, X_l)) = \mathcal{H}(\Delta^{(N)}(X_1), \dots, \Delta^{(N)}(X_l)), \tag{2.14}$$

fulfils

$$\left\{C^{(m)}, H^{(N)}\right\}_{A\otimes A\otimes \dots^{N)}\otimes A} = 0, \qquad 1 \leq m \leq N, \tag{2.15}$$

where the N functions $C^{(m)}$ $(m = 1, \dots, N)$ are defined through the m-th coproducts of the Casimir C

$$C^{(m)} := \Delta^{(m)}(C(X_1, \dots, X_l)) = C(\Delta^{(m)}(X_1), \dots, \Delta^{(m)}(X_l)), \tag{2.16}$$

and all the integrals of motion $C^{(m)}$ are in involution

$$\left\{C^{(m)}, C^{(n)}\right\} = 0, \qquad \forall m, n = 1, \dots, N. \tag{2.17}$$

Therefore, provided a non-trivial realization of A on a one-particle phase space is given, the N-particle Hamiltonian $H^{(N)}$ will be a function of N canonical pairs

(q_i, p_i) and is, by construction, completely integrable with respect to the ordinary Poisson bracket

$$\{f, g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right). \tag{2.18}$$

Moreover, its constants of motion will be given by the $C^{(m)}$ functions, all of them functionally independent since each of them depends on the first m pairs (q_i, p_i) of canonical coordinates. Note that with such one-particle realizations the first Casimir $C^{(1)}$ will be a number, and we are left with $N-1$ constants of motion with respect to $H^{(N)}$.

Let us stress now that the previous construction is valid for quantum algebras with no extra assumptions. Quantum algebras are also (deformed) coalgebras (A_z, Δ_z), and any function of the generators of a given quantum algebra with Casimir element C_z will provide, under a chosen deformed representation, a completely integrable Hamiltonian. Therefore, the obtention of quantum algebras is a direct method to get integrable deformations of those Hamiltonian systems with underlying coalgebra symmetry.

3 Oscillator chains from $sl(2, \mathbb{R})$ coalgebras

It is well-known that $sl(2, \mathbb{R})$ can be considered as a dynamical algebra for the one-dimensional harmonic oscillator. As we shall see in the sequel, the $sl(2, \mathbb{R})$ coalgebra will give us the complete set of integrals of the motion of a chain of N independent harmonic oscillators with the same frecuency ω. When the non-standard quantum deformation is considered, a new integrable oscillator chain with long range interactions depending on the momenta is obtained.

3.1 The non-deformed case

Let us consider the $sl(2, \mathbb{R})$ coalgebra (1.1) and (1.2) with $\alpha = 4$. A one-particle phase space realization of this Poisson algebra with vanishing Casimir is given by the functions:

$$f_-^{(1)} = D(J_-) = q_1^2, \qquad f_+^{(1)} = D(J_+) = p_1^2, \qquad f_3^{(1)} = D(J_3) = q_1 p_1. \tag{3.1}$$

From it, the harmonic oscillator Hamiltonian is recovered if the following linear function of the generators of $sl(2, \mathbb{R})$

$$\mathcal{H} = J_+ + \omega^2 J_-, \tag{3.2}$$

is represented through (3.1):

$$H^{(1)} = D(\mathcal{H}) = p_1^2 + \omega^2 q_1^2. \tag{3.3}$$

Now, the representation $(D \otimes D)$ when applied onto the prim tive coproduct (1.3), leads to the following two-particle phase space realization of $sl(2,\mathbb{R})$

$$f_-^{(2)} = q_1^2 + q_2^2, \qquad f_+^{(2)} = p_1^2 + p_2^2, \qquad f_3^{(2)} = q_1 p_1 + q_2 p_2. \tag{3.4}$$

that, in turn, gives rise to the uncoupled oscillator Hamiltonian:

$$H^{(2)} = (D \otimes D)(\Delta^{(2)}(\mathcal{H})) = f_+^{(2)} + \omega^2 f_-^{(2)} = p_1^2 + p_2^2 + \omega^2(q_1^2 + q_2^2). \tag{3.5}$$

Note that now the frecuency of both oscillators is the same. The phase space realization of the coproduct of the Casimir will give us the corresponding integral of motion

$$C^{(2)} = (D \otimes D)(\Delta^{(2)}(C)) = -(q_1 p_2 - q_2 p_1)^2, \tag{3.6}$$

that turns out to be the square of the angular momentum, as expec ed.

The construction of the m-dimensional functions from the m-th coproduct is straightforwardly obtained by the induction (2.11):

$$f_-^{(m)} = \sum_{i=1}^{m} q_i^2, \qquad f_+^{(m)} = \sum_{i=1}^{m} p_i^2, \qquad f_3^{(m)} = \sum_{i=1}^{m} q_i p_i. \tag{3.7}$$

From it, the uncoupled chain of N harmonic oscillators (all of them with the same frecuency) is obtained as the representation of the same dynamic al Hamiltonian (3.2):

$$H^{(N)} = f_+^{(N)} + \omega^2 f_-^{(N)} = \sum_{i=1}^{N} (p_i^2 + \omega^2 q_i^2) \tag{3.8}$$

together with the integrals of motion ($m = 2, \ldots, N$), that are deduced from the m-th coproducts of the Casimir and are shown to be

$$C^{(m)} = -\sum_{i<j}^{m} (q_i p_j - q_j p_i)^2. \tag{3.9}$$

Note that the integrals $C^{(m)}$ given by the $sl(2,\mathbb{R})$ coalgebra are just the quadratic Casimirs of the $so(m)$ algebras with $m = 2, \ldots, N$. It is also well known that the Hamiltonian (3.8) is $so(N)$ invariant, since it can be interpreted as the one for a particle moving on the N-dimensional Euclidean space under the potential $\omega^2 r^2$.

3.2 An integrable deformation from $U_z(sl(2,\mathbb{R}))$

Now, we introduce a one-particle deformed phase space realization of (1.8):

$$f_-^{(1)} = D_z(J_-) = q_1^2,$$

$$f_+^{(1)} = D_z(J_+) = \frac{\sinh zq_1^2}{zq_1^2} p_1^2, \qquad (3.10)$$
$$f_3^{(1)} = D_z(J_3) = \frac{\sinh zq_1^2}{zq_1} p_1.$$

This realization is also characterized by a vanishing deformed Casimir function (1.9)

Let us consider again the dynamical generator $\mathcal{H} = J_+ + \omega^2 J_-$. Under (3.10), we obtain the Hamiltonian

$$H_z^{(1)} = D_z(\mathcal{H}) = \frac{\sinh zq_1^2}{zq_1^2} p_1^2 + \omega^2 q_1^2. \qquad (3.11)$$

The corresponding two-particle phase space realization of $U_z(sl(2,\mathbb{R}))$ is obtained from both the coproduct (1.7) and (3.10):

$$f_-^{(2)} = q_1^2 + q_2^2,$$
$$f_+^{(2)} = \frac{\sinh zq_1^2}{zq_1^2} p_1^2 e^{zq_2^2} + \frac{\sinh zq_2^2}{zq_2^2} p_2^2 e^{-zq_1^2},$$
$$f_3^{(2)} = \frac{\sinh zq_1^2}{zq_1} p_1 e^{zq_2^2} + \frac{\sinh zq_2^2}{zq_2} p_2 e^{-zq_1^2}. \qquad (3.12)$$

The associated two-particle Hamiltonian is

$$H_z^{(2)} = \frac{\sinh zq_1^2}{zq_1^2} p_1^2 e^{zq_2^2} + \frac{\sinh zq_2^2}{zq_2^2} p_2^2 e^{-zq_1^2} + \omega^2(q_1^2 + q_2^2), \qquad (3.13)$$

and the deformed coproduct for the deformed Casimir (1.9) reads

$$C_z^{(2)} = -\frac{\sinh zq_1^2 \sinh zq_2^2}{z^2 q_1^2 q_2^2} (q_1 p_2 - q_2 p_1)^2 e^{-zq_1^2} e^{zq_2^2}. \qquad (3.14)$$

If we rewrite (3.13) in the form

$$H_z^{(2)} = H^{(2)} + p_1^2 \left(\frac{\sinh zq_1^2}{zq_1^2} e^{zq_2^2} - 1 \right) + p_2^2 \left(\frac{\sinh zq_2^2}{zq_2^2} e^{-zq_1^2} - 1 \right)$$
$$= H^{(2)} + z\,(p_1^2\, q_2^2 - p_2^2\, q_1^2) + o[z^2], \qquad (3.15)$$

the nature of the interaction introduced by the non-standard deformation with respect to (3.5) can be appreciated. Note that the series expansion (3.15) can be meaningful when the deformation parameter z is small, and it should be explored in order to analyse the dynamics.

The N-dimensional generalization for this system can be derived from the m-th order coproducts (2.11) induced from (1.7):

$$f_-^{(m)} = \sum_{i=1}^{m} q_i^2,$$

$$J_+^{(m)} = \sum_{i=1}^{m} \frac{\sinh zq_i^2}{zq_i^2} p_i^2 e^{zK_i^{(m)}(q^2)}, \qquad (3.16)$$

$$J_3^{(m)} = \sum_{i=1}^{m} \frac{\sinh zq_i^2}{zq_i} p_i e^{zK_i^{(m)}(q^2)}.$$

where the K-functions are defined as:

$$K_i^{(m)}(x) = -\sum_{k=1}^{i-1} x_k + \sum_{l=i+1}^{m} x_l. \qquad (3.17)$$

$$\begin{aligned} K_{ij}^{(m)}(x) &= K_i^{(m)}(x) + K_j^{(m)}(x) \\ &= -(x_i - x_j) - 2\sum_{k=1}^{i-1} x_k + 2\sum_{l=j+1}^{m} x_l. \qquad i < j. \end{aligned} \qquad (3.18)$$

Therefore, the N-dimensional Hamiltonian is just $H_z^{(N)} = J_+^{(N)} + \omega^2 J_-^{(N)}$ and the following constants of motion are deduced:

$$C^{(m)} = -\sum_{i<j}^{m} \frac{\sinh zq_i^2 \sinh zq_j^2}{z^2 q_i^2 q_j^2} (q_i p_j - q_j p_i)^2 e^{zK_{ij}^{(m)}(q^2)} \qquad (3.19)$$

Throughout all the computations leading to (3.19), the following property becomes useful:

$$\frac{\sinh(z\sum_{i=1}^{m} x_i)}{z} = \sum_{i=1}^{m} \frac{\sinh zx_i}{z} e^{zK_i^{(m)}(x)}. \qquad (3.20)$$

Note that under the limit $z \to 0$ we recover all the expressions presented in the previous paragraph.

3.3 Anharmonic chains and their deformation

We can now consider a more general dynamical Hamiltonian $\mathcal{H}$ of the form

$$\mathcal{H} = J_+ + \mathcal{F}(J_-), \qquad (3.21)$$

where $\mathcal{F}(J_-)$ is an arbitrary smooth function of J_-. The formalism summarized in Section 2 ensures that the corresponding system arising from the N-th coproduct of (3.21) is completely integrable, with $\mathcal{H}$ being *any* function of the coalgebra generators. Explicitly, this means that any N-particle Hamiltonian of the form

$$H^{(N)} = J_+^{(N)} + \mathcal{F}(J_-^{(N)}) = \sum_{i=1}^{N} p_i^2 + \mathcal{F}\left(\sum_{i=1}^{N} q_i^2\right), \qquad (3.22)$$

is completely integrable, and (3.9) are its constants of motion. Obviously, the integrability (in fact, superintegrability) of (3.22) is a well-known result, since (3.22) is just the Hamiltonian describing the motion of a particle in a N-dimensional Euclidean space under the action of a central potential. In terms of oscillator chains, the linear case $\mathcal{F}(J_-) = \omega^2 J_-$ leads to the previous harmonic case, and the quadratic one $\mathcal{F}(J_-) = J_-^2$ would give us an interacting chain of quartic oscillators. Further definitions of the function $\mathcal{F}$ would give us many other anharmonic chains, all of them sharing the same dynamical symmetry and the same integrals of the motion.

Let us now stress that, by using the non-standard quantum $sl(2,\mathbb{R})$ algebra, it is clear that a realization of (3.21) in terms of (3.16) gives us:

$$
\begin{aligned}
H_z^{(N)} &= \sum_{i=1}^{N} \frac{\sinh z q_i^2}{z q_i^2} p_i^2 e^{z K_i^{(N)}(q^2)} + \mathcal{F}\left(\sum_{i=1}^{N} q_i^2\right) \\
&= \sum_{i=1}^{N} p_i^2 + \mathcal{F}\left(\sum_{i=1}^{N} q_i^2\right) + \sum_{i=1}^{N} p_i^2 \left(\frac{\sinh z q_i^2}{z q_i^2} e^{z K_i^{(N)}(q^2)} - 1\right),
\end{aligned}
\tag{3.23}
$$

which is an integrable deformation of (3.22) with (3.19) being again the associated integrals. This result can be interpreted as a perturbation of the original anharmonic chain through long-range interacting terms depending on the momenta.

4 The Gaudin Hamiltonian and $sl(2,\mathbb{R})$ coalgebras

If we substitute the canonical realizations used until now in terms of angular momentum realizations of the same abstract $sl(2,\mathbb{R})$ Poisson coalgebra, the very same construction will lead us to a long-range interacting "classical spin chain" of the Gaudin type on which the quantum deformation can be easily implemented.

In particular, let us consider the classical angular momentum realization S

$$
g_3^{(1)} = S(J_3) = \sigma_3^1, \qquad g_+^{(1)} = S(J_+) = \sigma_+^1, \qquad g_-^{(1)} = S(J_-) = \sigma_-^1, \tag{4.1}
$$

where the variables σ_i^1 fulfil

$$
\{\sigma_3, \sigma_+\} = 2\sigma_+, \qquad \{\sigma_3, \sigma_-\} = -2\sigma_-, \qquad \{\sigma_-, \sigma_+\} = 4\sigma_3, \tag{4.2}
$$

and are constrained by a given constant value of the Casimir function (1.2) in the form $c_1 = (\sigma_3^1)^2 - \sigma_-^1 \sigma_+^1$.

As usual, m different copies of (4.1) (that, in principle, could have different values c_i of the Casimir) can be distinguished with the aid of a superscript σ_l^i. Then, the m-th coproduct provides the following realization of the non-deformed $sl(2,\mathbb{R})$ Poisson coalgebra :

$$
g_l^{(m)} = (S \otimes \dots^{m)} \otimes S)(\Delta^{(m)}(\sigma_l)) = \sum_{i=1}^{m} \sigma_l^i, \qquad l = +,-,3. \tag{4.3}
$$

Now, we can apply the usual construction and take $\mathcal{H}$ from (3.2). As a consequence, the uncoupled oscilator chain (3.8) is equivalent to

$$H^{(N)} = g_+^{(N)} + \omega^2 g_-^{(N)} = \sum_{i=1}^{m} (\sigma_+^i + \omega^2 \sigma_-^i), \tag{4.4}$$

and the Casimirs $C^{(m)}$ read ($m = 2, \dots, N$):

$$C^{(m)} = (g_3^{(m)})^2 - g_-^{(m)} g_+^{(m)} = \sum_{i=1}^{m} c_i + \sum_{i<j}^{m} (\sigma_3^i \sigma_3^j - \sigma_-^i \sigma_+^j - \sigma_-^j \sigma_+^i). \tag{4.5}$$

Note that these Casimirs are just Gaudin Hamiltonians of the hyperbolic type [13, 14] (in fact, we shall consider $C^{(N)}$ as the one defining a general Gaudin magnet).

The implementation of a non-standard deformation in the Gaudin system is now straightforward. The deformed angular momentum realization corresponding to the non-standard deformation $U_z(sl(2,\mathbb{R}))$ is:

$$\begin{aligned} g_-^{(1)} &= S_z(J_-) = \sigma_-^1, \\ g_+^{(1)} &= S_z(J_+) = \frac{\sinh z\sigma_-^1}{z\sigma_-^1} \sigma_+^1, \\ g_3^{(1)} &= S_z(J_3) = \frac{\sinh z\sigma_-^1}{z\sigma_-^1} \sigma_3^1, \end{aligned} \tag{4.6}$$

where the classical coordinates σ_l^1 are defined on the cone $c_1 = (\sigma_3^1)^2 - \sigma_-^1 \sigma_+^1 = 0$ (therefore, we are considering the zero realization).

It is easy to check that the m-th order of the coproduct (1.7) in the above representation leads to the following functions

$$\begin{aligned} g_-^{(m)} &= \sum_{i=1}^{m} \sigma_-^i, \\ g_+^{(m)} &= \sum_{i=1}^{m} \frac{\sinh z\sigma_-^i}{z\sigma_-^i} \sigma_+^i e^{zK_i^{(m)}(\sigma_-)}, \\ g_3^{(m)} &= \sum_{i=1}^{m} \frac{\sinh z\sigma_-^i}{z\sigma_-^i} \sigma_3^i e^{zK_i^{(m)}(\sigma_-)}, \end{aligned} \tag{4.7}$$

that define the non-standard deformation of (4.4). Therefore, the following "non-standard Gaudin Hamiltonians" are obtained

$$C_z^{(m)} = (g_3^{(m)})^2 - \frac{\sinh z g_-^{(m)}}{z} g_+^{(m)}$$

$$= \sum_{i=1}^{m} \left(\frac{\sinh z\sigma_-^i}{z\sigma_-^i} \right)^2 e^{2zK_i^{(m)}(\sigma_-)} \left\{ (\sigma_3^i)^2 - \sigma_-^i \sigma_+^i \right\}$$

$$+ \sum_{i<j}^{m} \frac{\sinh z\sigma_-^i \sinh z\sigma_-^j}{z^2 \sigma_-^i \sigma_-^j} e^{zK_{ij}^{(m)}(\sigma_-)} (\sigma_3^i \sigma_3^j - \sigma_-^i \sigma_+^j - \sigma_+^i \sigma_-^j). \quad (4.8)$$

Since we are working in the zero representation with $(\sigma_3^i)^2 - \sigma_-^i \sigma_+^i = 0$, the expression (4.8) can be simplified

$$C_z^{(m)} = \sum_{i<j}^{m} \frac{\sinh z\sigma_-^i \sinh z\sigma_-^j}{z^2 \sigma_-^i \sigma_-^j} e^{zK_{ij}^{(m)}(\sigma_-)} (\sigma_3^i \sigma_3^j - \sigma_-^i \sigma_+^j - \sigma_+^i \sigma_-^j). \quad (4.9)$$

These integrals are the angular momentum counterparts to (3.19). Note that, as it was found for the standard case in [5], the deformation can be interpreted as the introduction of a variable range exchange [15] in the model (compare (4.9) with (4.5)).

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ON NORMALIZATION OF A CLASS OF POLYNOMIAL HAMILTONIANS

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Abstract

In this paper, the normalization of a class of polynomial Hamiltonians based on the symbolic computing is disscussed from both the ordinary and the inverse direction-points of view. The truncated three-particle Toda linear chain(3-TLC) and the regularized system of a planar hydrogen atom with the linear Stark effect (HLSE) are taken as examples to demonstrate the symbolic-computational approach to the ordinary and the inverse problems of the normalization.

1 Introduction.

It is widely known that the Birkhoff-Gustavson normal form (BGNF) expansion works effectively to study a behavior of nonlinear dynamical systems; the Hénon-Heiles system and Toda linear chain (TLC) are often taken as typical examples [1] to describe such an efficiency.

Since a major part of the BGNF expansion is made on the polynomial algebra [2], the BGNF expansion fits very well the symbolic computing on computers: For example, the symbolic computing program named GITA realizes the algebraic procedure of converting power-series Hamiltonians into their BGNF [3] on REDUCE 3.X $(X \geq 3)$[1].

One of the aim of this paper is to demonstrate how the normalization into BGNF works around integrable systems. One might think it unnecessary to normalize the Hamiltonian systems which are already integrable. However, the normalization of integrable systems is worth discussing especially in the case that the integrable systems admit the trajectories tending to singularities; as a typical example, the

[1]The authors are trying to realize the same procedure on Maple V

truncated three-particle Toda linear chain (3-TLC) is taken in order to demonstrate how the normalization of integrable systems works.

The 'inverse problem' of the BGNF expansion has been posed recently with the aim of applying, to certain systems of physical interests, quantum studies on the bifurcations in the BGNF systems [4, 5]. The inverse problem having been proposed reads as follows: '*Identify a class of dynamical systems which are reduced to the same BGNF up to a certain order*'. The inverse problem can be solved also by a symbolic computing; for example, by the symbolic computing program named GITA^{-1} proposed by the authors.

Another aim of this paper is to demonstrate how the inverse problem of normalization is solved. As an example, GITA^{-1} is applied to the regularized system of planar hydrogen atom with the linear Stark effect (HLSE) [6]. It is shown that a class of Liouville-type system share the same BGNF with the regularized system of HLSE.

The contents of this paper are organized as follows. In Sec. 2, the structures of GITA and GITA^{-1} for the general n-degree-of-freedom case is presented very briefly. In Sec. 3, the direct problem of 3-TLC is discussed to show the normalization is effective not only for non-integrable systems but also for integrable ones. In Sec. 4, the inverse problem of HLSE is discussed to show the way to identify a class of Hamiltonian systems which share certain BGNF Hamiltonian in common.

2 A review of GITA and GITA^{-1}

Let $H_{IN}(q,p)$ and $H_{OUT}(q,p)$ be the *input* and the *output* Hamiltonians, which are expressed as

$$H_\lambda(q,p) = \sum_{h=2}^{\infty} H_\lambda^{(h)}(q,p) \quad \text{with} \quad H_\lambda^{(2)}(q,p) = \frac{1}{2}\sum_{k=1}^{n}(p_k^2 + q_k^2), \tag{1}$$

where $H_\lambda^{(h)}$ $(\lambda = IN, OUT,\ h = 3, 4, \cdots)$ is a degree-h homogeneous polynomial in (q,p) expressed as

$$H_\lambda^{(h)}(q,p) = \sum_{|\alpha|+|\beta|=h} c_\lambda^{(h)}(\alpha,\beta) q^\alpha p^\beta \quad \text{with} \quad \begin{array}{l} q^\alpha p^\beta = q_1^{\alpha_1}\cdots q_n^{\alpha_n} p_1^{\beta_1}\cdots p_n^{\beta_n}, \\ |\alpha| = \sum_{k=1}^{n} \alpha_k,\ |\beta| = \sum_{k=1}^{n} \beta_k. \end{array} \tag{2}$$

Let $G_{IN}(\xi,\eta)$ and $G_{OUT}(\xi,\eta)$ be the *input* and the *output* BGNF Hamiltonian,

$$G_\lambda(\xi,\eta) = \sum_{j=1}^{\infty} G_\lambda^{(2j)}(\xi,\eta) \quad \text{with} \quad G_\lambda^{(2)}(\xi,\eta) = \frac{1}{2}\sum_{k=1}^{n}(\eta_k^2 + \xi_k^2), \tag{3}$$

where $G_\lambda^{(2j)}$ $(\lambda = IN, OUT,\ j = 2, 3, \cdots)$ is a degree-$2j$ homogeneous polynomial in (ξ, η) expressed as

$$G_\lambda^{(2j)}(\xi,\eta) = \sum_{|\alpha|+|\beta|=2j} \gamma_\lambda^{(2j)}(\alpha,\beta)\xi^\alpha\eta^\beta \quad \text{with} \quad \{G_\lambda^{(2)}, G_\lambda^{(2j)}\} = 0. \tag{4}$$

In eq. (4), α and β are multi-indices used in the same way as in eq. (2), and $\{\cdot,\cdot\}$ is the canonical Poisson bracket associated with the position variables ξ and the momentum ones η. The coefficients $\gamma_\lambda^{(2j)}(\alpha,\beta)$, $\quad (j = 2, 3, \cdots)$ are finded by solving the key BGNF equation

$$H(q, \frac{\partial W}{\partial \eta}) = G(\frac{\partial W}{\partial q}, \eta). \tag{5}$$

Here $W(q,\eta)$ is the generating function of the form, $W(q,\eta) = (q_1\eta_1 + q_2\eta_2) + \sum_{k=3} W^{(k)}(q\ \eta)$ ($W^{(k)}$: k-th order homogeneous polynomial in (q,η)), which should be identified together with $G(\xi,\eta)$. We will not, however, get the identification of $W(q,\eta)$ into detail here [2, 3].

Let us denote, by P_ℓ, the space of degree-ℓ homogeneous polynomials in $2n$-variables with real-coefficients, which can be identified with the vector space $\mathbf{R}^{N(n,\ell)}$, where $N(n,\ell)$ indicates the number of degree-ℓ monomials in $2n$-variables allowed to exist. Then, denoting such a correspondence by $\iota_\ell : P_\ell \to \mathbf{R}^{N(n,\ell)}$, we associate the vectors, $\vec{c}_\lambda^{(h)}$ and $\vec{\gamma}_\lambda^{(2j)}$, with H_λ and G_λ by

$$\vec{c}_\lambda^{(h)} = \iota_h(H_\lambda^{(h)}) \in \mathbf{R}^{N(n,h)} \quad \text{and} \quad \vec{\gamma}_\lambda^{(2j)} = \iota_{2j}(G_\lambda^{(2j)}) \in \mathbf{R}^{N(n,2j)}, \tag{6}$$

respectively. Further, using ι_ℓ, we express the differential operator,

$$D = \sum_{k=1}^{n} \{p_k(\partial/\partial q_k) - q_k(\partial/\partial p_k)\},$$

restricted on P_ℓ by the matrix $M^{(\ell)}$ acting on $\mathbf{R}^{N(n,\ell)}$; $\iota_\ell \circ D = M^{(\ell)} \circ \iota_\ell$.

After the preparation above, the h-th order part of Eq. (5) is put into the series of algebraic equations,

$$\vec{\gamma}_\lambda^{(h)} = M^{(h)}\{\vec{c}_\lambda^{(h)} + \Phi^{(h)}(\vec{c}_\lambda^{(h-1)}, \cdot, \cdot, \vec{c}_\lambda^{(2)})\} \quad (j = 3, 4, \cdots), \tag{7}$$

for $\vec{\gamma}_\lambda^{(h)}$ $(\lambda = IN, OUT)$ [2, 6], which are the very equations solved by GITA. Note that $\vec{\gamma}_\lambda^{(2j+1)}$ $(\lambda = IN, OUT)$ turn out to vanish [6].

We are now in a position to present what GITA^{-1} computes: Let us recall the inverse problem posed in Sec. 1, which is put in the following: '*For a given H_{IN}, identify all the possible (or a part of) H_{OUT}subject to $G_{IN} = G_{OUT}$ up to a certain order*'. Since $G_{IN} = G_{OUT}$ can read $\vec{\gamma}_{IN}^{(h)} = \vec{\gamma}_{OUT}^{(h)}$ $(h = 2, 3, \cdots)$, GITA^{-1} solves the series of equations,

$$M^{(h)}\vec{c}_{OUT}^{(h)} = \vec{\gamma}_{IN}^{(h)} - M^{(h)}\Phi^{(h)}(\vec{c}_{OUT}^{(h-1)}, \cdot, \cdot, \vec{c}_{OUT}^{(2)})\} \quad (j = 3, 4, \cdots), \tag{8}$$

for $\vec{c}^{(h)}_{OUT}$, where $\vec{\gamma}^{(h)}_{IN}$ are determined beforehand from H_{IN} through GITA (*i.e.* (1)-(7)). In the succeeding sections, we demonstrate how GITA and GITA^{-1} work on REDUCE 3.X ($X \geq 3$) in the direct problem of 3-TLC and in the inverse problem of HLSE.

3 The truncated three-particle Toda linear chain

Let us consider like an example of an integrable system: the three identical particles on the line governing by the Toda Hamiltonian [7]. The original Toda Hamiltonian can be reduces to the two-dimensional one

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{4}\{\exp\ (\xi) + \exp\ (\eta) + \exp\ (\zeta)\}, \tag{9a}$$

$$\xi = \sqrt{2}q_1 + \sqrt{\frac{2}{3}}q_2, \quad \eta = -\sqrt{2}q_1 + \sqrt{\frac{2}{3}}q_2, \quad \zeta = -2\sqrt{\frac{2}{3}}q_2, \quad \xi + \eta + \zeta = 0. \tag{9b}$$

It is easy to verify that the Hamiltonian system (9) possesses additional integral of motion in the form

$$I = \frac{4}{3}p_1(p_1^2 - 3p_2^2) + (p_1 - \sqrt{3}p_2)\exp\ (\xi) + (p_1 + \sqrt{3}p_2)\exp\ (\eta) - 2p_1\exp\ (\zeta). \tag{10}$$

Note that ansatzt $q_1 = \sqrt{6}x$, $q_2 = \sqrt{6}y$, $p_1 = \sqrt{6}p_x$, $p_2 = \sqrt{6}p_y$ and $H \to 6H$ brings the expressions (9) and (10) to the same ones as it is in the book [8].

As the Toda Hamiltonian has the C_{3v} symmetry its power expansion is determined fully through the two invariant functions $f = q_1^2 + q_2^2$ and $g = q_1^2 q_2 - \frac{1}{3}q_2^3$. Below the first power terms of the Taylor series for Hamiltonian (9) are written

$$H - \frac{3}{4} = \frac{1}{2}(p_1^2+p_2^2)+\frac{1}{2}(q_1^2+q_2^2)+\frac{1}{\sqrt{6}}(q_1^2q_2-\frac{1}{3}q_2^3)+\frac{1}{12}(q_1^2+q_2^2)^2+\frac{\sqrt{6}}{36}(q_1^2-q_2^2)(q_1^2q_2-\frac{1}{3}q_2^3)$$
$$+\frac{1}{180}[(q_1^2 + q_2^2)^3 + 2(q_1^2q_2 - \frac{1}{3}q_2^3)^2] + \ldots \tag{11a}$$

or

$$H - \frac{3}{4} = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}f + \frac{1}{\sqrt{6}}g + \frac{1}{12}f^2 + \frac{\sqrt{6}}{36}fg + \frac{1}{180}(f^3 + 2g^2)$$
$$+\frac{1}{90\sqrt{6}}f^2g + \frac{1}{15120}f(3f^3 + 16g^2) + \ldots \tag{11b}$$

Thus, it is seen that any truncated Toda's polynomial series generates a generalized Henon-Heiles Hamiltonian. Here a surprising situation arises: while the full Toda Hamiltonian (9) is integrated, its power expansion truncated in any finite degree presents a nonintegrable system 3-TCL.

In some manner this phenomenon may be explained by the behavior of the negative Gaussian curvature(NGC) domain on the respective potential energy surface PES of the Hamiltonian (11). Indeed, the NGC domain on the PES emerges if the highest degree (n_{max}) in the truncated series (9) is an odd number and, on the contrary, if the highest degree is an even number then such NGC domain does not appear at all. The emergence of the NGC domain is linked with the saddle points on the PES. Moreover, if we take into account more and more terms in the expansion (11) then the NGC domain moves upwards on the PES and in the case of the infinite series the NGC region vanishes.

Below we have constructed the Birkhoff-Gustavson normal form and the approximated integrals of motion for some truncated Toda's Hamiltonians in order to understand a dependence of the structure of phase space on the including of highest degree polynomial Hamiltonian (11). As an example, we present now the normal forms in the sixth $s_{max} = 6$ approach all but which obtain for the different highest degree of Toda's series from the value $n_{max} = 3$ (Henon-Heiles's Hamiltonian) to the $n_{max} = 5$. These Birkhoff-Gustavson normal forms are expressed below in the action-angle variables are obtained with help GITA procedure:

$$\xi_\nu = \sqrt{2I_\nu}\cos(\phi_\nu), \quad \eta_\nu = \sqrt{2I_\nu}\sin(\phi_\nu), \quad (\nu = 1,2). \tag{12}$$

$n_{max} = 3, \quad s_{max} = 6.$

$$G_3^{(6)} = (\frac{7}{1296}I_1^3 - \frac{7}{108}I_1^2 I_2 - \frac{7}{36}I_1^2 - \frac{7}{5184}I_1 I_2^2 + \frac{7}{432}I_2^3 - \frac{7}{144}I_2^2)cos(2\phi_2)$$
$$-\frac{155}{3888}I_1^3 - \frac{7}{108}I_1^2 I_2 + \frac{7}{5184}I_1 I_2^2 + \frac{35}{1296}I_2^3 - \frac{7}{144}I_2^2 - \frac{1}{12}I_1^2 + 2I_1 \tag{13a}$$

$n_{max} = 4, \quad s_{max} = 6.$

$$G_4^{(6)} = (\frac{22}{81}I_1^3 + \frac{1}{18}I_1^2 I_2 - \frac{1}{9}I_1^2 - \frac{11}{162}I_1 I_2^2 - \frac{1}{72}I_2^3 + \frac{1}{36}I_2^2)cos(2\phi_2)$$
$$-\frac{2}{243}I_1^3 + \frac{1}{18}I_1^2 I_2 + \frac{11}{162}I_1 I_2^2 - \frac{5}{216}I_2^3 - \frac{1}{36}I_2^2 + \frac{1}{3}I_1^2 + 2I_1 \tag{14a}$$

$n_{max} = 5, \quad s_{max} = 6.$

$$G_5^{(6)} = \frac{4}{81}I_1^3 - \frac{1}{108}I_1^2 I_2 - \frac{1}{9}I_1^2 - \frac{1}{81}I_1 I_2^2 + \frac{1}{432}I_2^3 + \frac{1}{36}I_2^2)cos(2\phi_2)$$
$$-\frac{32}{243}I_1^3 - \frac{1}{108}I_1^2 I_2 + \frac{1}{81}I_1 I_2^2 + \frac{5}{1296}I_2^3 - \frac{1}{36}I_2^2 + \frac{1}{3}I_1^2 + 2I_1 \tag{15a}$$

The second integrals in the corresponding approximation are also obtained by GITA procedure up to terms of degree s_{max} as quadratic form [3]

$$I^{(2)}(\xi[s_{max}], \eta[s_{max}]) = \Gamma(\xi[s_{max}], \eta[s_{max}]) - \sum_\nu \frac{1}{2}\omega_\nu(\xi_\nu^2[s_{max}] + \eta_\nu^2[s_{max}]), \; \nu = 1,2.$$

To obtain the integral in the original coordinates, GITA express the final variables $(\xi,\eta) = (\xi[s_{max}],\eta[s_{max}])$ through variables $(q = \xi[2], p = \eta[2])$ making $(s_{max} - 2)$ coordinate transformations $\nu = 1,2;\ s = 3,4,5,\dots,s_{max}$ in accordance with eqs.(5)and definition of the generation function via coefficiens $W^{(s)}(\xi_\nu[s-1],\eta_\nu[s])$

$$\xi_\nu[s] = \frac{\partial W^{(s)}}{\partial \eta_\nu[s]}(\xi_\nu[s-1],\eta_\nu[s]), \quad \eta_\nu[s-1] = \frac{\partial W^{(s)}}{\partial \xi_\nu[s-1]}(\xi_\nu[s-1],\eta_\nu[s])$$

As an example, we present them in the explicit form : $n_{max} = 3, \quad s_{max} = 6.$

$$\begin{aligned}
I^{(2)} = &-\frac{385}{124416}p_2^6 + \frac{3311}{41472}p_2^4p_1^2 - \frac{385}{41472}p_2^4q_2^2 - \frac{1589}{41472}p_2^4q_1^2 - \frac{5}{288}p_2^4 + \frac{1225}{5184}p_2^3p_1q_2q_1 \\
&+\frac{7}{18\sqrt{6}}p_2^3p_1q_1 - \frac{2849}{41472}p_2^2p_1^4 + \frac{287}{6912}p_2^2p_1^2q_2^2 - \frac{7}{18\sqrt{6}}p_2^2p_1^2q_2 \\
&-\frac{35}{256}p_2^2p_1^2q_1^2 - \frac{5}{144}p_2^2p_1^2 - \frac{385}{41472}p_2^2q_2^4 + \frac{5}{216\sqrt{6}}p_2^2q_2^3 \\
&+\frac{245}{2304}p_2^2q_2^2q_1^2 - \frac{5}{144}p_2^2q_2^2 + \frac{23}{72\sqrt{6}}p_2^2q_2q_1^2 + \frac{763}{41472}p_2^2q_1^4 \\
&+\frac{1}{16}p_2^2q_1^2 - \frac{7}{5184}p_2p_1^3q_2q_1 - \frac{7}{54\sqrt{6}}p_2p_1^3q_1 + \frac{217}{5184}p_2p_1q_2^3q_1 \\
&-\frac{7}{36\sqrt{6}}p_2p_1q_2^2q_1 - \frac{791}{5184}p_2p_1q_2q_1^3 - \frac{7}{36}p_2p_1q_2q_1 \\
&-\frac{7}{36\sqrt{6}}p_2p_1q_1^3 + \frac{847}{124416}p_1^6 - \frac{2821}{41472}p_1^4q_2^2 + \frac{7}{54\sqrt{6}}p_1^4q_2 \\
&+\frac{847}{41472}p_1^4q_1^2 - \frac{5}{288}p_1^4 + \frac{1771}{41472}p_1^2q_2^4 - \frac{37}{216\sqrt{6}}p_1^2q_2^3 - \frac{91}{2304}p_1^2q_2^2q_1^2 \\
&+\frac{1}{16}p_1^2q_2^2 + \frac{1}{8\sqrt{6}}p_1^2q_2q_1^2 + \frac{623}{41472}p_1^2q_1^4 - \frac{5}{144}p_1^2q_1^2 - \frac{545}{124416}q_2^6 \\
&+\frac{5}{216\sqrt{6}}q_2^5 + \frac{943}{41472}q_2^4q_1^2 - \frac{5}{288}q_2^4 - \frac{5}{108\sqrt{6}}q_2^3q_1^2 - \frac{1537}{41472}q_2^2q_1^4 - \frac{5}{288}q_1^4 \\
&-\frac{5}{144}q_2^2q_1^2 - \frac{5}{72\sqrt{6}}q_2q_1^4 - \frac{49}{124416}q_1^6
\end{aligned} \tag{13b}$$

$n_{max} = 4, \quad s_{max} = 6.$

$$\begin{aligned}
I^{(2)} = &\frac{35}{7776}p_2^6 + \frac{77}{2592}p_2^4p_1^2 + \frac{35}{2592}p_2^4q_2^2 - \frac{143}{2592}p_2^4q_1^2 + \frac{1}{72}p_2^4 + \frac{55}{324}p_2^3p_1q_2q_1 \\
&+\frac{2}{9\sqrt{6}}p_2^3p_1q_1 + \frac{7}{2592}p_2^2p_1^4 - \frac{11}{432}p_2^2p_1^2q_2^2 - \frac{2}{9\sqrt{6}}p_2^2p_1^2q_2
\end{aligned}$$

$$
\begin{aligned}
&-\frac{25}{432}p_2^2p_1^2q_1^2+\frac{1}{36}p_2^2p_1^2+\frac{47}{2592}p_2^2q_2^4-\frac{1}{54\sqrt{6}}p_2^2q_2^3\\
&+\frac{1}{48}p_2^2q_2^2q_1^2+\frac{1}{36}p_2^2q_2^2+\frac{5}{18\sqrt{6}}p_2^2q_2q_1^2-\frac{17}{2592}p_2^2q_1^4\\
&+\frac{1}{12}p_2^2q_1^2+\frac{41}{324}p_2p_1^3q_2q_1-\frac{2}{27\sqrt{6}}p_2p_1^3q_1+\frac{19}{324}p_2p_1q_2^3q_1\\
&-\frac{1}{9\sqrt{6}}p_2p_1q_2^2q_1+\frac{13}{324}p_2p_1q_2q_1^3-\frac{1}{9}p_2p_1q_2q_1\\
&-\frac{1}{9\sqrt{6}}p_2p_1q_1^3+\frac{49}{7776}p_1^6-\frac{157}{2592}p_1^4q_2^2+\frac{2}{27\sqrt{6}}p_1^4q_2\\
&+\frac{49}{2592}p_1^4q_1^2+\frac{1}{72}p_1^4-\frac{11}{2592}p_1^2q_2^4-\frac{7}{54\sqrt{6}}p_1^2q_2^3+\frac{1}{144}p_1^2q_2^2q_1^2\\
&+\frac{1}{12}p_1^2q_2^2+\frac{1}{6\sqrt{6}}p_1^2q_2q_1^2+\frac{53}{2592}p_1^2q_1^4+\frac{1}{36}p_1^2q_1^2+\frac{79}{7776}q_2^6\\
&-\frac{1}{54\sqrt{6}}q_2^5+\frac{1}{2592}q_2^4q_1^2+\frac{1}{72}q_2^4+\frac{1}{27\sqrt{6}}q_2^3q_1^2+\frac{131}{2592}q_2^2q_1^4+\frac{1}{72}q_1^4\\
&+\frac{1}{36}q_2^2q_1^2+\frac{1}{18\sqrt{6}}q_2q_1^4+\frac{53}{7776}q_1^6
\end{aligned}
\tag{14b}
$$

$n_{max}=5, \quad s_{max}=6.$

$$
\begin{aligned}
I^{(2)}=&-\frac{35}{15552}p_2^6+\frac{175}{5184}p_2^4p_1^2-\frac{35}{5184}p_2^4q_2^2-\frac{205}{5184}p_2^4q_1^2+\frac{1}{72}p_2^4+\frac{95}{648}p_2^3p_1q_2q_1\\
&+\frac{2}{9\sqrt{6}}p_2^3p_1q_1-\frac{175}{5184}p_2^2p_1^4-\frac{5}{846}p_2^2p_1^2q_2^2-\frac{2}{9\sqrt{6}}p_2^2p_1^2q_2\\
&-\frac{25}{288}p_2^2p_1^2q_1^2+\frac{1}{36}p_2^2p_1^2-\frac{11}{5184}p_2^2q_2^4-\frac{1}{54\sqrt{6}}p_2^2q_2^3\\
&+\frac{35}{864}p_2^2q_2^2q_1^2+\frac{1}{36}p_2^2q_2^2+\frac{5}{18\sqrt{6}}p_2^2q_2q_1^2+\frac{5}{5184}p_2^2q_1^4\\
&+\frac{1}{12}p_2^2q_1^2+\frac{25}{648}p_2p_1^3q_2q_1-\frac{2}{27\sqrt{6}}p_2p_1^3q_1+\frac{23}{648}p_2p_1q_2^3q_1\\
&-\frac{1}{9\sqrt{6}}p_2p_1q_2^2q_1-\frac{31}{648}p_2p_1q_2q_1^3-\frac{1}{9}p_2p_1q_2q_1\\
&-\frac{1}{9\sqrt{6}}p_2p_1q_1^3+\frac{35}{15552}p_1^6-\frac{275}{5184}p_1^4q_2^2+\frac{2}{27\sqrt{6}}p_1^4q_2\\
&+\frac{35}{5184}p_1^4q_1^2+\frac{1}{72}p_1^4+\frac{59}{5184}p_1^2q_2^4-\frac{7}{54\sqrt{6}}p_1^2q_2^3-\frac{19}{864}p_1^2q_2^2q_1^2
\end{aligned}
$$

$$+\frac{1}{12}p_1^2q_2^2+\frac{1}{6\sqrt{6}}p_1^2q_2q_1^2+\frac{43}{5184}p_1^2q_1^4+\frac{1}{36}p_1^2q_1^2+\frac{53}{15552}q_2^6$$
$$-\frac{1}{54\sqrt{6}}q_2^5+\frac{23}{5184}q_2^4q_1^2+\frac{1}{72}q_2^4+\frac{1}{27\sqrt{6}}q_2^3q_1^2+\frac{73}{5184}q_2^2q_1^4+\frac{1}{72}q_1^4$$
$$+\frac{1}{36}q_2^2q_1^2+\frac{1}{18\sqrt{6}}q_2q_1^4+\frac{43}{15552}q_1^6 \qquad (15b)$$

The above integrals (13b-15b) and corresponding Poincare sections are presented on Fig. 1-3. One can observe how the sequence of these sections step by step tends to the limited section of the exact Toda integral (10) which is shown on Fig.4. Note, the approximate integrals of motion will be described well theoretically regular phase trajectories similar to the other generalization of the Henon-Heiles dynamical system [9]. On this way one may expect to find additional criteria of a true choice of BGNF structure connected with exact integrals.

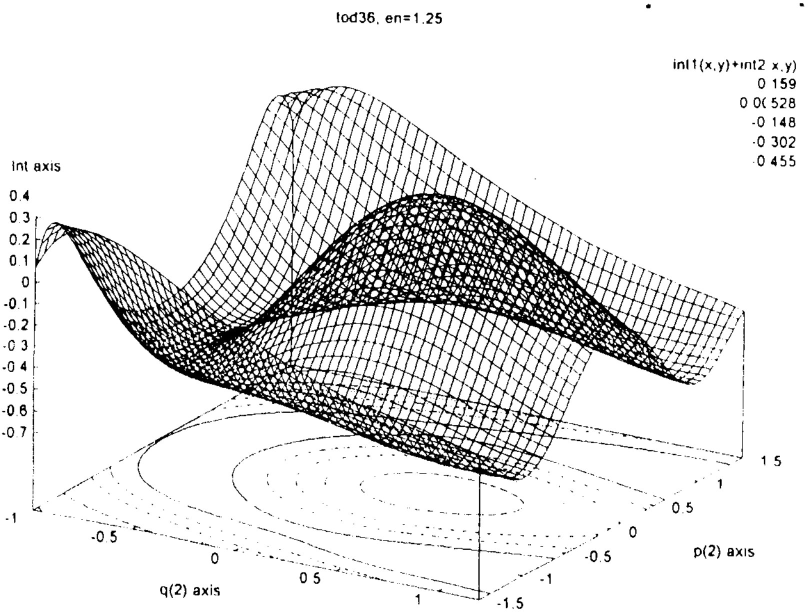

Fig.1 Approximate integral of motion 3-TLC(13b) $n_{max} = 3$, $s_{max} = 6$

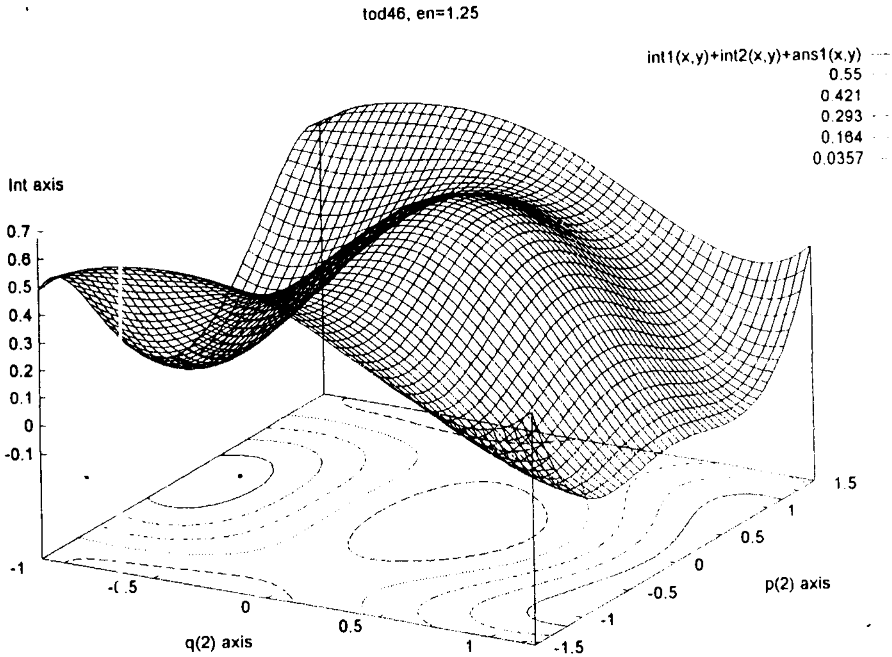

Fig.2 Approximate integral of motion 3-TLC(14b) $n_{max} = 4, s_{max} = 6$

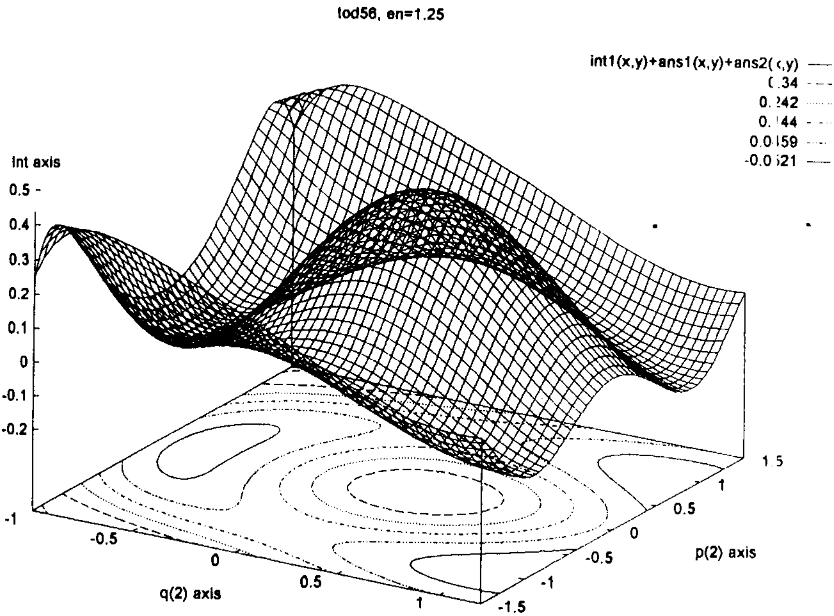

Fig.3 Approximate integral of motion 3-TLC(15b) $n_{max} = 5, s_{max} = 6$

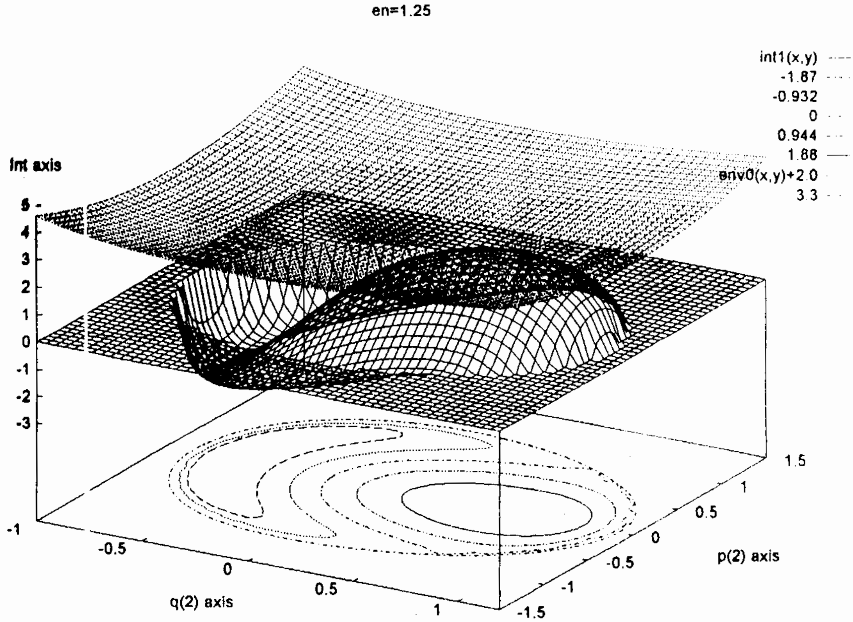

Fig.4 Exact integral of motion 3-TLC (11)

4 GITA^{-1} and the inverse problem of HLSE

GITA^{-1} consists of a core part and a subsidary part. The core part is derived from GITA [3] and the subsidiary part contains the procedures characteristic of GITA^{-1}, both of which are put together in a single file. The procedure list and the input data (the input Hamiltonian) are loaded at the beginning of running GITA^{-1}.

As an example of program fulfillment, we take the inverse problem of HLSE. Let the input Hamiltonian H_{IN} be

$$H_{IN}(q,p) = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + \frac{8}{3}\varepsilon(q_1^4 - q_2^4), \tag{16}$$

the Hamiltonian of regularized system of HLSE [6]. The input BGNF Hamiltonian, G_{IN}, for H_{IN} is calculated, up to the fourth order, to be

$$G_{IN}^{(4)}(\xi,\eta) = \varepsilon(\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2)(\xi_1^2 + \eta_1^2 - \xi_2^2 - \eta_2^2), \tag{17}$$

where ε is a parameter. The vector $\vec{\gamma}_{IN}^{(4)}$ in eq. (8) is hence fixed by the coefficients, $\gamma_{IN}^{(4)}(\alpha_1, \alpha_2, \beta_1, \beta_2)$, of $G_{IN}^{(4)}$ through (5). It is worth noting that all these preliminary calculations can be made by GITA.

The GITA^{-1} starts with generating the field P_ℓ to describe the output Hamiltonian $H_{OUT}^{(4)}$ all of whose coefficients $c_{OUT}^{(4)}(\alpha, \beta)$ are unidentified. Next, eqs. (1)-(7) with $\lambda = OUT$ are proceeded in GITA^{-1} to calculate $G_{OUT}^{(4)}$, all of whose coefficients, $\gamma_{OUT}^{(4)}(\alpha, \beta)$, expressed in terms of $c_{OUT}^{(4)}(\alpha, \beta)$ unidentified. On equating G_{IN} with G_{OUT} in GITA^{-1} , eq. (8) takes the following form:

$$
\begin{aligned}
3c_{OUT}^{(4)}(0,4,0,0) + c_{OUT}^{(4)}(0,2,0,2) + 3c_{OUT}^{(4)}(0,0,0,4) &= -8\varepsilon, \\
3c_{OUT}^{(4)}(4,0,0,0) + c_{OUT}^{(4)}(2,0,2,0) + 3c_{OUT}^{(4)}(0,0,4,0) &= 8\varepsilon,
\end{aligned}
\tag{18}
$$

where the coefficients, $c_{OUT}^{(4)}(k_1, k_2, \ell_1, \ell_2)$, not listed in (18) are set to zero.

Applying the subroutine SOLVE in REDUCE to eq. (18), we have

$$
\begin{aligned}
&c_{OUT}^{(4)}(0,0,4,0) = u(4), && c_{OUT}^{(4)}(0,0,0,4) = u(2), \\
&c_{OUT}^{(4)}(0,4,0,0) = (-3u(2) - u(1) - 8\varepsilon)/3, && c_{OUT}^{(4)}(0,2,0,2) = u(1), \\
&c_{OUT}^{(4)}(4,0,0,0) = (-3u(4) - u(3) + 8\varepsilon)/3, && c_{OUT}^{(4)}(2,0,2,0) = u(3),
\end{aligned}
\tag{19}
$$

as the solution of eq. (18), where $u(i)$ $(i = 1,2,3,4)$ denote the unidentified constants 'arbcomplex(i)' introduced automatically in SOLVE.

Finally, we identify the output Hamiltonian to be

$$
\begin{aligned}
H_{OUT}^{(4)}(q,p) &= u(4)p_1^4 + u(2)p_2^4 + u(3)q_1^2p_1^2 + u(1)q_2^2p_2^2 \\
&\quad + \frac{1}{3}\{-3u(4) - u(3) + 8\varepsilon\}\, q_1^4 + \frac{1}{3}\{-3u(2) - u(1) - 8\varepsilon\}\, q_2^4,
\end{aligned}
\tag{20}
$$

which admits G_{IN} given by eq. (17) as the BGNF. Note that if all the $u(i)$ vanish in eq. (20), H_{OUT} becomes identical with H_{IN} given by (16).

On setting $u(2) = u(4) = 0$ in eq. (20), H_{OUT} becomes

$$
\begin{aligned}
H_{OUT}(q,p) &= \frac{1 + 2u(3)q_1^2}{2}p_1^2 + \frac{1 + 2u(1)q_2^2}{2}p_2^2 \\
&\quad + q_1^2 + \frac{1}{3}\{-u(3) + 8\varepsilon\}\, q_1^4 + q_2^2 - \frac{1}{3}\{u(1) + 8\varepsilon\}\, q_2^4.
\end{aligned}
\tag{21}
$$

Surprisingly, H_{OUT} turns out to be a Hamiltonian admitting the separation of variables, which provides an integrable system accordingly. Although it seems to be incidental that we encounter such integrable systems after GITA^{-1} , one may expect to find a class of integrable systems whose Hamiltonians reduce to a given BGNF Hamiltonian.

5 Concluding Remarks

We would like to make a few remarks on the peculiarities in a realization of GITA^{-1}. (i) In generating the field P_ℓ and eq. (8), we have used REDUCE to implement the combinatorial algorithms and the list processing. (ii) As is seen from the algorithm (1)-(8) presented in Sec. 2, GITA^{-1} is proceeded by tracing back the procedures of GITA in principle. Hence we may expect that GITA and GITA^{-1} are put together to provide a unified symbolic computing program for various calculation around the BGNF expansion and integrable models in future.

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SOLUTIONS OF q-DEFORMED EQUATIONS WITH QUANTUM CONFORMAL SYMMETRY AND NONZERO SPIN

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Abstract

We consider the construction of explicit solutions of a hierarchy of q-deformed equations which are (conditionally) quantum conformal invariant. We give two types of solutions - polynomial solutions and solutions in terms of q-deformations of the plane wave. We use two q-deformations of the plane wave as a formal power series in the noncommutative coordinates of q-Minkowski space-time and four-momenta. One q-plane wave was proposed earlier by the first named author and B.S. Kostadinov, the other is new. The difference between the two is that they are written in conjugated bases. These q-plane waves are used here for the construction of solutions of the massless Dirac equation - one is used for the neutrino, the other for the antineutrino. It is also interesting that the neutrino solutions are deformed only through the q-pane wave, while the prefactor is classical. Thus, we can speak of a definite left-right asymmetry of the quantum conformal deformation of the neutrino-antineutrino system.

1 Introduction

One of the purposes of quantum deformations is to provide an alternative of the regularization procedures of quantum field theory. Applied to Minkowski space-time the quantum deformations approach is also an alternative to Connes' noncommutative geometry [1]. The first step in such an approach is to construct a noncommutative quantum deformation of Minkowski space-time. There are several possible such deformations, cf. [2, 3, 4, 5, 6, 7, 8]. We shall follow the deformation of [5] which is different from the others, the most important aspect being that it is related to a deformation of the conformal group.

The first problem to tackle in a noncommutative deformed setting is to analyze the behavior of the wave equations analogues. Thus, we start here with the study of a hierarchy of deformed equations derived in [9] with the use of quantum conformal symmetry. The hierarchy is parametrized by a natural number a. In fact, the case $a = 1$ corresponds to the q-d'Alembert equation, while for each $a > 1$ there are two couples of equations involving fields of conjugated Lorentz representations of dimension a. For instance, the case $a = 2$ coresponds to the massless Dirac equation, one couple of equations describing the neutrino, the other couple of equations describing the antineutrino, while the case $a = 3$ corresponds to the Maxwell equations.

In [9] the solution spaces were given only via their transformation properties under $U_q(sl(4))$ and quantum conformal symmetry. However, no explicit solutions, which are important for the applications, were given. This was started in [10] with the construction of solutions of the q-d'Alembert equation. Two classes of solutions were given: polynomial ones and a deformation of the plane wave. It is a formal power series in the noncommutative coordinates of q-Minkowski space-time and four-momenta. This q-plane wave has analogous properties to the classical one. In particular it has the properties of q-Lorentz covariance, and it satisfies the q-d'Alembert equation on the q-Lorentz covariant momentum cone. On the other hand, this q-plane wave is not an exponent or q-exponent, cf. [11]. Thus, it differs conceptually from the classical plane wave and may serve as a regularization of the latter. In the same sense it differs from the q-plane wave in the paper [12], which is not surprising, since there is used different q-Minkowski space-time (from [2, 3, 4] and different q-d'Alembert equation both based only on a (different) q-Lorentz algebra, and not on q conformal (or $U_q(sl(4))$) symmetry as in our case.

In the present paper we give polynomial solutions for all equations of the hierarchy. We give also another q-deformation of the usual plane wave written in a conjugated basis with respect to the first q-plane wave derived in [10]. We give solutions of the massless Dirac equation involving the two conjugated q-plane waves - one for the neutrino, the other for the antineutrino. It is also interesting that the neutrino solutions are deformed only through the q-pane wave, while the prefactor is classical. Thus, we can speak of a definite left-right asymmetry of the quantum conformal deformation of the neutrino-antineutrino system.

2 Preliminaries

First we introduce new Minkowski variables:

$$x_\pm \equiv x_0 \pm x_3 , \qquad v \equiv x_1 - ix_2 , \qquad \bar{v} \equiv x_1 + ix_2 , \tag{1}$$

which, (unlike the x_μ), have definite group-theoretical interpretation as part of a coset of the conformal group $SU(2,2)$ (or of $SL(4)$ with the appropriate conjugation)

[5]. The d'Alembert equation in terms of these variables is:

$$\Box\ \varphi = (\partial_- \partial_+ \ - \ \partial_v\ \partial_{\bar v})\ \varphi \ = \ 0\ , \tag{2}$$

while the Minkowski length is: $\mathcal{L} \ = \ x_- x_+ - v\bar v \ = \ x_0^2 - \vec x^2$.

In the q-deformed case we use the noncommutative q-Minkowski space-time of [5] which is given by the following commutation relations:

$$\begin{aligned} x_\pm v \ &= \ q^{\pm 1} v x_\pm \ , \quad x_\pm \bar v \ = \ q^{\pm 1} \bar v x_\pm \ , \\ x_+ x_- - x_- x_+ \ &= \ \lambda v \bar v \ , \quad \bar v v \ = \ v \bar v \ , \qquad (\lambda \equiv q - q^{-1})\ , \end{aligned} \tag{3}$$

with the deformation parameter being a phase: $|q| = 1$. The q-Minkowski length is:

$$\mathcal{L}_q \ = \ x_- x_+ - q^{-1} v \bar v\ . \tag{4}$$

It commutes with the q-Minkowski coordinates and has the correct classical limit $\mathcal{L}_{q=1} \ = \ \mathcal{L}$. Relations (3) are preserved by the anti-linear anti-involution ω acting as :

$$\omega(x_\pm) \ = \ x_\pm\ , \quad \omega(v) \ = \ \bar v\ , \quad \omega(q) \ = \ \bar q \ = \ q^{-1}\ , \quad (\omega(\lambda) \ = \ -\lambda)\ , \tag{5}$$

¿from which follows also that $\omega(\mathcal{L}_q) \ = \ \mathcal{L}_q$.

The solution space (for q-d'Alambert) consists of formal power series in the q-Minkowski coordinates:

$$\varphi \ = \ \sum_{j,n,\ell,m\in \mathbb{Z}_+} \mu_{jn\ell m}\ \varphi_{jn\ell m}\ , \qquad \varphi_{jn\ell m} \ = \ \hat\varphi_{jn\ell m},\ \tilde\varphi_{jn\ell m} \tag{6a}$$

$$\hat\varphi_{jn\ell m} \ = \ v^j\ x_-^n\ x_+^\ell\ \bar v^m \tag{6b}$$

$$\tilde\varphi_{jn\ell m} \ = \ \bar v^m\ x_+^\ell\ x_-^n\ v^j \ = \ \omega(\hat\varphi_{jn\ell m}) \tag{6c}$$

The solution spaces (6) are representation spaces of the quantum algebra $U_q(sl(4))$. The latter is defined, (cf. [13]), as the associative algebra over $\mathbb{C}$ with unit element $1_\mathcal{U}$, 'Chevalley' generators $k_i^\pm$, $X_i^\pm$, $i \ = \ 1,2,3$, and nontrivial relations ($[x]_q \equiv (q^x - q^{-x})/\lambda$) :

$$k_i k_j \ = \ k_j k_i\ , \quad k_i k_i^{-1} \ = \ k_i^{-1} k_i \ = \ 1_\mathcal{U}\ , \quad k_i X_j^\pm \ = \ q^{\pm c_{ij}} X_j^\pm k_i\ , \tag{7a}$$

$$[X_i^+, X_j^-] \ = \ \delta_{ij} \left(k_i^2 - k_i^{-2}\right) / \lambda\ , \tag{7b}$$

$$\left(X_i^\pm\right)^2 X_j^\pm \ - \ [2]_q X_i^\pm X_j^\pm X_i^\pm \ + \ X_j^\pm \left(X_i^\pm\right)^2 \ = \ 0\ , \quad |i-j| = 1\ . \tag{7c}$$

Further we suppose that q is not a nontrivial root of unity. The action of $U_q(sl(4))$ on $\hat\varphi_{jn\ell m}$ was given in [14], (cf. also [10]). Using those formulae we can find also the action on $\tilde\varphi_{jn\ell m}$:

$$\pi(k_1)\ \tilde\varphi_{jn\ell m} \ = \ q^{(j-n+\ell-m)/2}\ \tilde\varphi_{jn\ell m}\ , \tag{8a}$$

$$\pi(k_3)\ \tilde\varphi_{jn\ell m} \ = \ q^{(-j-n+\ell+m)/2}\ \tilde\varphi_{jn\ell m}\ , \tag{8b}$$

$$\pi(X_1^+)\ \tilde\varphi_{jn\ell m} \;=\; q^{-2+(-j+n+\ell-m)/2}\ [n]_q\ \tilde\varphi_{j+1,n-1,\ell m}\ + \\ +\ q^{-2+(-j+n-\ell+m)/2}\ [m]_q\ \tilde\varphi_{jn,\ell+1,m-1}\ , \tag{9a}$$

$$\pi(X_3^+)\ \tilde\varphi_{jn\ell m} \;=\; -\ q^{(-j-n+\ell+m)/2}\ [j]_q\ \tilde\varphi_{j-1,n,\ell+1,m}\ - \\ -\ q^{(-3j-n+3\ell+m)/2}\ [n]_q\ \tilde\varphi_{j,n-1,\ell,m+1}\ , \tag{9b}$$

$$\pi(X_1^-)\ \tilde\varphi_{jn\ell m} \;=\; q^{1+(j-n+\ell-m)/2}\ [j]_q\ \tilde\varphi_{j-1,n+1,\ell m}\ + \\ +\ q^{1+(-j+n+\ell-m)/2}\ [\ell]_q\ \tilde\varphi_{jn,\ell-1,m+1}\ , \tag{10a}$$

$$\pi(X_3^-)\ \tilde\varphi_{jn\ell m} \;=\; -\ q^{1+(j+3n-\ell-3m)/2}\ [\ell]_q\ \tilde\varphi_{j+1,n,\ell-1,m}\ - \\ -\ q^{1+(j+n-\ell-m)/2}\ [m]_q\ \tilde\varphi_{j,n+1,\ell,m-1}\ . \tag{10b}$$

We have written only the action of $X_j^\pm, k_j$ for $j = 1,3$ since we shall use only those. (Note that the representation formulae in [14] are for general holomorphic representations of $U_q(sl(4))$ characterized by three integers (r_1, r_2, r_3), (which here are $(0,-1,0)$), and the functions depend in general on two additional variables $z, \bar z$.)

Next we need the q-d'Alembert equation:

$$\Box_q\ \varphi \;\doteq\; \textstyle\sum_{j,n,\ell,m\in\mathbb{Z}_+}\ \mu_{jn\ell m}\ \Box_q\ \varphi_{jn\ell m} \;=\; 0\ , \tag{11a}$$

$$\Box_q\ \hat\varphi_{jn\ell m} \;=\; q^{1+n+2m+2j+\ell}[n]_q[\ell]_q\ \hat\varphi_{j,n-1,\ell-1,m}\ - \\ -\ q^{n+j+\ell+m}[j]_q[m]_q\ \hat\varphi_{j-1,n,\ell,m-1} \tag{11b}$$

$$\Box_q\ \tilde\varphi_{jn\ell m} \;=\; q^{n+\ell}[n]_q[\ell]_q\ \tilde\varphi_{j,n-1,\ell-1,m}\ - \\ -\ q^{n+j+\ell+m+1}[j]_q[m]_q\ \tilde\varphi_{j-1,n,\ell,m-1} \tag{11c}$$

(for (11b) cf. [9, 10]).

Remark: Equation (11) may be rewritten in a form closer to the $q=1$ in (2) by introducing q-difference operators. For this we first define the operators:

$$\hat M_\kappa^\pm\ \varphi \;=\; \textstyle\sum_{j,n,\ell,m\in\mathbb{Z}_+}\ \mu_{jn\ell m}\ \hat M_\kappa^\pm\ \varphi_{jn\ell m}\ , \qquad \kappa = \pm, v, \bar v\ , \tag{12a}$$

$$T_\kappa^\pm\ \varphi \;=\; \textstyle\sum_{j,n,\ell,m\in\mathbb{Z}_+}\ \mu_{jn\ell m}\ T_\kappa^\pm\ \varphi_{jn\ell m}\ , \qquad \kappa = \pm, v, \bar v\ , \tag{12b}$$

and $\hat M_+^\pm$, $\hat M_-^\pm$, $\hat M_v^\pm$, $\hat M_{\bar v}^\pm$, resp., acts on $\varphi_{jn\ell m}$ by changing by ± 1 the value of j, n, ℓ, m, resp., while $T_+^\pm$, $T_-^\pm$, $T_v^\pm$, $T_{\bar v}^\pm$, resp., acts on $\varphi_{jn\ell m}$ by multiplication by $q^{\pm j}, q^{\pm n}, q^{\pm \ell}, q^{\pm m}$, resp. Using the above we define the q-difference operators as follows:

$$\hat{\mathcal{D}}_\kappa\ \varphi \;=\; \frac{1}{\lambda}\ \hat M_\kappa^{-1}\ \left(T_\kappa - T_\kappa^{-1}\right)\ \varphi\ . \tag{13}$$

Using (12) and (13) then (11b,c) may be rewritten as:

$$\left(q\,\hat{\mathcal{D}}_-\,\hat{\mathcal{D}}_+\,T_v\,T_{\bar v}\;-\;\hat{\mathcal{D}}_v\,\hat{\mathcal{D}}_{\bar v}\right)\,T_v\,T_-\,T_+\,T_{\bar v}\,\hat\varphi \;=\; 0 \qquad (14a)$$

$$\left(\hat{\mathcal{D}}_-\,\hat{\mathcal{D}}_+\;-\;q\,\hat{\mathcal{D}}_v\,\hat{\mathcal{D}}_{\bar v}\,T_v\,T_{\bar v}\right)\,T_-\,T_+\;\tilde\varphi \;=\; 0 \qquad (14b)$$

Note that the operators in (12), (13), (14) for different variables commute, i.e., using these one is technically passing to commuting variables. Note that keeping the normal ordering it is straightforward to interchange commuting and noncommuting variables.

3 q-Plane Waves

We want to q-deform the plane wave. Clearly, the most general q-deformation is:

$$(\exp(k\cdot x))_q \;=\; \sum_{s=0}^{\infty}\frac{1}{[s]_q!}\,f_s\;, \qquad (15)$$

where f_s is a homogeneous polynomial of degree s in both sets of variables, i.e., q-momenta $(k_v, k_-, k_+, k_{\bar v})$ and q-Minkowski coordinates $(v, x_-, x_+, \bar v)$, such that $(f_s)|_{q=1} = (k\cdot x)^s$. Thus, we set $f_0 = 1$. One may expect that f_s for $s > 1$ would be equal or at least proportional to $(f_1)^s$, but the outcome would be that this is not the case. In order to proceed systematically we have to impose the conditions of q-Lorentz covariance and the q-d'Alembert equation.

The complexification of the q-Lorentz subalgebra of the q-conformal algebra is generated by $k_j^\pm$, $X_j^\pm$, $j = 1,3$. Using (9a,c), (10a,c) it is easy to check that:

$$\pi(X_j^\pm)\,\mathcal{L}_q \;=\; 0\;, \quad\Longrightarrow\quad \pi(X_j^\pm)\,(\mathcal{L}_q)^s \;=\; 0\;, \qquad j \;=\; 1,3. \qquad (16)$$

Since $(k\cdot x)^s$ is a scalar as $(\mathcal{L}_q)^s$, then also the q-deformations f_s should be scalars, and thus also should obey (16). In order to implement this we suppose that the momentum components are also non-commutative obeying the same rules (3) as the q-Minkowski coordinates, and that they commute with the coordinates. Also the ordering of the momentum basis will be the same for the coordinates. Taking all this into account one can see that a natural expression for f_s is:

$$f_s \;=\; \sum_{a,b,n\,\in\mathbb{Z}_+}\beta^s_{a,b,n}\,\frac{(-1)^{s-a-b}}{\Gamma_q(a-n+1)\,\Gamma_q(b-n+1)\,\Gamma_q(s-a-b+n+1)\,[n]_q!}\;\times$$
$$\times\; k_v^{s-a-b+n}\,k_-^{b-n}\,k_+^{a-n}\,k_{\bar v}^{n}\,v^n\,x_-^{a-n}\,x_+^{b-n}\,\bar v^{s-a-b+n}\;, \qquad (17)$$

where we have introduced some factors that are obvious from the correspondence with the case $q = 1$. (The expression in (17) does not involve terms that would vanish for $q = 1$. Actually, we shall see that such expressions would lead to noncovariant momenta light-cone.) In order to implement q-Lorentz covariance we impose the conditions:

$$\pi(X_j^\pm)\, f_s \;=\; 0, \qquad j \;=\; 1,3\;. \tag{18}$$

For this calculation we suppose that the q-Lorentz action on the noncommutative momenta is given by (8a,b), (9a,b), (10a,b). This was done in [10] and the following result for β was found:

$$\beta^s_{a,b,n} \;=\; q^{n(s-2a-2b+2n)\;+\;a(s-a-1)\;+\;b(-s+a+b+1)}\;\beta^s_{0,0,0}\;, \tag{19}$$

where

$$\left(\beta^s_{0,0,0}\right)^{-1} \;=\; \sum_{p=0}^{s} \frac{q^{(s-p)(p-1)+p}}{[p]_q!\;[s-p]_q!}\;. \tag{20}$$

The functions f_s obey the q-d'Alambert equation if the momenta are on the q-Lorentz covariant q-light cone:

$$\mathcal{L}^k_q \;=\; k_-k_+ \;-\; q^{-1}\,k_v k_{\bar v} \;=\; 0\;. \tag{21}$$

We turn now to the conjugated case. The q-plane wave now is:

$$\widetilde{\exp}_q(k\cdot x) \;=\; \sum_{s=0}^{\infty} \frac{1}{[s]_q!}\,\tilde f_s$$

$$\tilde f_s \;=\; \sum_{a,b,n} \tilde\beta^s_{a,b,n}\,\frac{(-1)^{s-a-b}}{\Gamma_q(a-n+1)\;\Gamma_q(b-n+1)\;\Gamma_q(s-a-b+n+1)\;[n]_q!}\;\times$$
$$\times k_{\bar v}^n k_+^{a-n} k_-^{b-n} k_v^{s-a-b+n}\;\; \bar v^{s-a-b+n}\, x_+^{b-n}\, x_-^{a-n}\, v^n \tag{22}$$

For q-Lorentz covariance we impose the conditions:

$$\pi(X_j^\pm)\,\tilde f_s \;=\; 0, \qquad j \;=\; 1,3\;. \tag{23}$$

We use the commutation relations for the momenta components as for as the q-Minkowski coordinates, and that momenta commute with the coordinates. We also have to use the twisted derivation rule [14] :

$$\pi(X_j^\pm)\,\psi\cdot\psi' \;=\; \pi(X_j^\pm)\,\psi\;\cdot\;\pi(k_j^{-1})\,\psi' \;+\; \pi(k_j)\,\psi\;\cdot\;\pi(X_j^\pm)\,\psi'\;, \tag{24}$$
$$\psi \;=\; k_{\bar v}^n k_+^{a-n} k_-^{b-n} k_v^{s-a-b+n}\;, \qquad \psi' \;=\; \bar v^{s-a-b+n}\, x_+^{b-n}\, x_-^{a-n}\, v^n\;.$$

The four conditions (23) bring eight relations between the coefficients $\tilde\beta$, however only three are independent, namely, the relations:

$$\tilde\beta^s_{a,b,n} \;=\; q^{2n+2a-b-2-s}\;\tilde\beta^s_{a-1,b,n}\;, \tag{25a}$$

$$\tilde\beta^s_{a,b,n} \;=\; q^{2n-a-2b+2+s}\;\tilde\beta^s_{a,b-1,n}\;, \tag{25b}$$

$$\tilde\beta^s_{a,b,n} \;=\; q^{2a+2b-4n+2-s}\;\tilde\beta^s_{a,b,n-1}\;, \tag{25c}$$

solving which we find the following solution:

$$\tilde{\beta}^s_{a,b,n} = q^{n(2a+2b-2n-s) + a(a-s-1) + b(s-a-b+1)} \tilde{\beta}^s_{0,0,0} \tag{26}$$

i.e., for each $s \geq 1$ only one constant remains to be fixed. Next we impose the q-d'Alembert equation:

$$\Box_q \tilde{f}_s = 0, \tag{27}$$

which holds trivially for $s = 0, 1$. For $s \geq 2$ we substitute (22) to obtain that (27) holds if the momentum operators are on the q-Lorentz covariant q-light cone (cf. (21)).

Now it remains only to fix the coefficient $\beta^s_{0,0,0}$. We note that for q=1 it holds:

$$(k \cdot x)|_{k \to x} = (x \cdot x) = \mathcal{L} , \tag{28}$$

and thus we shall impose the conditions:

$$(\tilde{f}_s)|_{k \to x} = (\mathcal{L}_q)^s . \tag{29}$$

A tedious calculation shows that:

$$\left(\tilde{\beta}^s_{0,0,0}\right)^{-1} = \sum_{p=0}^{s} \frac{q^{(p-s)(p-1)+p}}{[p]_q! \, [s-p]_q!} . \tag{30}$$

Note that $\left(\tilde{\beta}^s_{0,0,0}\right)^{-1} |_{q=1} = 2^s/s!$, as expected.

4 Polynomial solutions for nonzero spin

As we mentioned for $a \in \mathbb{N} + 1$ there are two couples of equations involving fields of conjugated Lorentz representations of dimension a. One of the couples is [9]:

$$\{q^a[a-1-N_z]_q \hat{\mathcal{D}}_v T_- - \hat{\mathcal{D}}_- \hat{\mathcal{D}}_z\} \, T_z T_v T_+ \, \hat{\varphi} = 0 \tag{31}$$

$$\{q^a[a-1-N_z]_q \hat{\mathcal{D}}_+ T_- T_{\bar{v}} T_v - \hat{\mathcal{D}}_{\bar{v}} \hat{\mathcal{D}}_z\} \, T_z T_v T_+ \, \hat{\varphi} = 0 \tag{32}$$

The solutions are polynomials in z of degree $a-1$ and formal power series in the q-Minkowski coordinates:

$$\hat{\varphi} = \sum_{i,j,n,l,m \in \mathbb{Z}_+} p_{ijnlm} \, z^i v^j x_-^n x_+^l \bar{v}^m \tag{33}$$

Substituting (33) in (31) and (32) we obtain two recurrence relations:

$$p_{i+1,j,n+1,lm} = q^{a+n} \frac{[a-i-1]_q [j+1]_q}{[i+1]_q [n+1]_q} \, p_{i,j+1,nlm} \tag{34}$$

$$p_{i+1,jnl,m+1} \;=\; q^{a+j+n+m}\,\frac{[a-i-1]_q[l+1]_q}{[i+1]_q[m+1]_q}\;p_{ijn,l+1,m} \tag{35}$$

Solving (34) and (35) we obtain, respectively:

$$p_{ijnlm} \;=\; q^{i(a+n-1)-\frac{i(i-1)}{2}}\,\frac{[a-1]_q![j+i]_q![n-i]_q!}{[a-i-1]_q![i]_q![j]_q![n]_q!}\;p_{0,j+i,n-i,lm} \tag{36}$$

$$p_{ijnlm} \;=\; q^{i(a+j+n+m-1)}\,\frac{[a-1]_q![l+i]_q![m-i]_q!}{[a-i-1]_q![i]_q![l]_q![m]_q!}\;p_{0jn,l+i,m-i} \tag{37}$$

Combining (36) and (37) we obtain:

$$\begin{aligned} p_{nmdbc} \;&=\; q^{n(a+d+c+\frac{n}{2}-\frac{1}{2})+m(c+m+n)}\,\frac{[a-1]_q![b]_q![d]_q![c]_q!}{[a-n-1]_q![n]_q![m]_q![d-m]_q!}\times \\ &\times\frac{1}{[b-n-m]_q![c+n+m]_q!}\;p_{00dbc} \end{aligned} \tag{38}$$

Accordingly, the solution of (31) and (32) is given using (33) as follows (since p_{00dbc} are constants):

$$\begin{aligned} \hat{\varphi}_{dbc} \;&=\; \sum_{m=0}^{\min(d,b)}\sum_{n=0}^{a-1} q^{n(a+d+c+\frac{n}{2}-\frac{1}{2})+m(c+m+n)}\,\frac{[a-1]_q![b]_q![d]_q![c]_q!}{[a-n-1]_q![n]_q![m]_q![d-m]_q!}\times \\ &\times\frac{1}{[b-n-m]_q![c+n+m]_q!}\;z^n v^m x_-^{d-m} x_+^{b-n-m}\bar{v}^{c+n+m} \end{aligned} \tag{39}$$

Formula (39) is valid also for noncommutative coordinates. For commuting coordinates it may be written in more compact form as follows:

$$\hat{\varphi}_{dbc} \;=\; x_-^d x_+^b \bar{v}^c\; F_D^q\left(-b, 1-a, -d, c+1; q^{a+d+c-\frac{1}{2}}\frac{z\bar{v}}{x_+}, q^c\frac{v\bar{v}}{x_+x_-}\right) \tag{40}$$

where F_D^q is a q-deformation of a double hypergeometric function:

$$F_D^q[a,b_1,b_2,c,z_1,z_2] = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{\frac{n^2}{2}+m^2+nm}(a)^q_{n+m}(b_1)^q_n(b_2)^q_m}{(c)^q_{n+m}[n]_q![m]_q!}\;z_1^n z_2^m \tag{41}$$

which for $q=1$ is given by [15], (f-la 5.7.1.6). For $a=1$ these solutions coincide with the results of [10].

We pass now to the other couple of equations [9]:

$$([a-N_{\bar{z}}-1]_q\hat{\mathcal{D}}_+T_{\bar{v}} - q^a\hat{\mathcal{D}}_v\hat{\mathcal{D}}_{\bar{z}}T_-)\;T_-T_{\bar{v}}\;\hat{\varphi}' \;=\; 0 \tag{42}$$

$$([a-N_{\bar{z}}-1]_q\hat{\mathcal{D}}_{\bar{v}} - q^a\hat{\mathcal{D}}_-\hat{\mathcal{D}}_{\bar{z}}T_v^2T_-)\;T_{\bar{v}}\;\hat{\varphi}' \;=\; 0 \tag{43}$$

The solutions are polynomials in $\bar{z}$ of degree $a-1$ and formal power series in the q-Minkowski coordinates:

$$\hat{\varphi}' = \sum_{i,j,n,l,m \in \mathbb{Z}_+} p_{ijnlm}\ v^m x_-^n x_+^l \bar{v}^j \bar{z}^i \tag{44}$$

Substituting (44) in (42) and (43) we obtain two recurrence relations:

$$p_{i+1,jnl,m+1} = q^{-a+j-n-m} \frac{[a-i-1]_q [l+1]_q}{[i+1]_q [m+1]_q}\ p_{ijn,l+1,m} \tag{45}$$

$$p_{i+1,j,n+1,l,m} = q^{-a-n-2m} \frac{[a-i-1]_q [j+1]_q}{[i+1]_q [n+1]_q}\ p_{i,j+1,nlm} \tag{46}$$

The solution of (45) and (46) is:

$$\begin{aligned} p_{nmd'b'c'} &= q^{-n(a+d'+c'+\frac{n}{2}-\frac{1}{2})+m(c'+m+n)} \frac{[a-1]_q![b']_q![d']_q![c']_q!}{[a-n-1]_q![n]_q![m]_q![d'-m]_q!} \times \\ &\times \frac{1}{[b'-n-m]_q![c'+n+m]_q!}\ p_{00d'b'c'} \end{aligned} \tag{47}$$

Accordingly, the solution of (42) and (43) is given by:

$$\begin{aligned} \hat{\varphi}'_{d'b'c'} &= \sum_{m=0}^{min(d',b')} \sum_{n=0}^{a-1} q^{-n(a+d'+c'+\frac{n}{2}-\frac{1}{2})+m(c'+m+n)} \frac{[a-1]_q![b']_q![d']_q![c']_q!}{[a-n-1]_q![n]_q![m]_q![d'-m]_q!} \times \\ &\times \frac{1}{[b'-n-m]_q![c'+n+m]_q!}\ v^{c'+m+n} x_-^{d'-m} x_+^{b'-n-m} \bar{v}^m \bar{z}^n = \\ &= x_-^{d'} x_+^{b'} \bar{v}^{c'}\ F'^q_D \left(-b', 1-a, -d', c'+1; q^{-(a+d+c-\frac{1}{2})} \frac{z\bar{v}}{x_+}, q^{c'} \frac{v\bar{v}}{x_+ x_-} \right) \end{aligned} \tag{48}$$

where another deformation of the same double hypergeometric function as above is used:

$$F'^q_D[a', b'_1, b'_2, c', z_1, z_2] = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q^{\frac{-n^2}{2}+m^2+nm} (a')^q_{n+m} (b'_1)^q_n (b'_2)^q_m}{(c')^q_{n+m} [n]_q! [m]_q!}\ z_1^n z_2^m \tag{49}$$

The first formula in (48) is valid also for noncommutative coordinates. For $a=1$ these solutions coincide with (39), (40), and all reduce to the result of [10].

5 Solutions of the massless q-Dirac equation in terms of q-plane waves

As it was shown in [9] if a function satisfies (31) and (32) or (42) and (43) then it satisfies also the q-d'Alambert equation. Thus, it is justified to look for solutions in terms of q-plane waves.

Here we shall restrict to the case $a = 2$. In this case the relevant equations are:

$$([1 - N_z]_q \hat{\mathcal{D}}_v - q\hat{\mathcal{D}}_- \hat{\mathcal{D}}_z T_-)\ T_v\ \hat{\varphi} = 0 \tag{50}$$

$$([1 - N_z]_q \hat{\mathcal{D}}_+ T_{\bar{v}}^{-1} - q\hat{\mathcal{D}}_{\bar{v}} \hat{\mathcal{D}}_z T_- T_v)\ T_-\ \hat{\varphi} = 0 \tag{51}$$

$$([1 - N_{\bar{z}}]_q \hat{\mathcal{D}}_+ T_{\bar{v}} T_v^{-1} - q^2 \hat{\mathcal{D}}_v \hat{\mathcal{D}}_{\bar{z}} T_-)\ T_- T_{\bar{v}}\ \hat{\varphi}' \ = \ 0 \tag{52}$$

$$([1 - N_{\bar{z}}]_q \hat{\mathcal{D}}_{\bar{v}} - q^2 \hat{\mathcal{D}}_- \hat{\mathcal{D}}_{\bar{z}} T_v^2)\ T_{\bar{v}}\ \hat{\varphi}' \ = \ 0 \tag{53}$$

where (50), (51) are written in the conjugated basis, while (52), (53) are obtained by setting $a = 2$ in equations (42), (43). The functions $\hat{\varphi}$, $\hat{\varphi}'$ are polynomials of first degree in z, $\bar{z}$, respectively, and can be wriiten as:

$$\hat{\varphi} = \hat{\varphi}_0 + z\hat{\varphi}_1\ , \qquad \hat{\varphi}' = \hat{\varphi}'_0 + \bar{z}\hat{\varphi}'_1 \tag{54}$$

We note that the above equations are a q-deformation of the massless Dirac equation. Indeed, for $q = 1$ they can be rewritten in the two-component form of the massless Dirac equation. It is well known that the latter splits into independent equations for the neutrino $\Phi_{(-)}$ and the antineutrino $\Phi_{(+)}$:

$$(\partial_{x_0} \pm (\sigma_1 \partial_{x_1} + \sigma_2 \partial_{x_2} + \sigma_3 \partial_{x_3}))\ \Phi_{(\pm)}(x) \ = \ 0 \tag{55}$$

where σ_k are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{56}$$

It is easy to see that $\Phi_{(\pm)}$ are expressed through our functions (for $q = 1$) as:

$$\Phi_{(+)} \ = \ \tfrac{1}{2} \begin{pmatrix} \hat{\varphi}_0 \\ -\hat{\varphi}_1 \end{pmatrix}\ , \qquad \Phi_{(-)} \ = \ \tfrac{1}{2} \begin{pmatrix} \hat{\varphi}'_1 \\ \hat{\varphi}'_0 \end{pmatrix}\ . \tag{57}$$

Thus our field $\hat{\varphi}$ corresponds to the antineutrino, while $\hat{\varphi}'$ corresponds to the neutrino.

We start first with the q-deformation of the neutrino equations (52) and (53). We shall look for solutions in terms of the q-plane wave (17):

$$\hat{\varphi}' \ = \ \sum_{s=0}^{\infty} \frac{1}{[s]_q!}\ \psi'_s \tag{58}$$

where ψ'_s are the analogs of f_s, so we shall solve:

$$([1 - N_{\bar{z}}]_q \hat{\mathcal{D}}_+ T_{\bar{v}} T_v^{-1} - q^2 \hat{\mathcal{D}}_v \hat{\mathcal{D}}_{\bar{z}} T_-)\ T_- T_{\bar{v}}\ \psi'_s \ = \ 0 \tag{59}$$

$$([1 - N_{\bar{z}}]_q \hat{\mathcal{D}}_{\bar{v}} - q^2 \hat{\mathcal{D}}_- \hat{\mathcal{D}}_{\bar{z}} T_v^2)\ T_{\bar{v}}\ \psi'_s \ = \ 0 \tag{60}$$

Furthermore we shall make the following Ansatz:

$$\psi'_s \ = \ (\alpha k_+ + \beta k_- + \gamma k_v + \delta k_{\bar{v}} + \bar{z}(\alpha' k_+ + \beta' k_- + \gamma' k_v + \delta' k_{\bar{v}}))\ f_s \tag{61}$$

where $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta'$ are constants to be determined. We substitute (61) in (59) and (60) for commutative Minkowski coordinates and noncommutative momenta on the q-light cone. Solving we find that:

$$\beta = 0; \quad \gamma = 0; \quad \alpha' = 0; \quad \delta' = 0; \quad \beta' = -\delta; \quad \gamma' = -\alpha \tag{62}$$

Thus, the general solution of (59) and (60) is:

$$\psi_s'^{\alpha,\delta} = (\alpha k_+ + \delta k_{\bar{v}} - (\delta k_- + \alpha k_v)\bar{z}) \; f_s \tag{63}$$

and so the two independent solutions are given in terms of the q-plane wave:

$$\psi'^{(1)} = (k_+ - k_v \bar{z}) \; \exp_q(k \cdot x) \tag{64a}$$
$$\psi'^{(2)} = (k_{\bar{v}} - k_- \bar{z}) \; \exp_q(k \cdot x) \tag{64b}$$

Let us stress that the prefactors do not depend on q, i.e., they coincide with the classical ones (which, of course, are obtained by a much shorter calculation).

Now we pass to the antineutrino field:

$$\hat{\varphi} = \sum_{s=0}^{\infty} \frac{1}{[s]_q!} \psi_s \tag{65}$$

where we shall solve the q-deformed equations:

$$([1 - N_z]_q \hat{\mathcal{D}}_v - q\hat{\mathcal{D}}_- \hat{\mathcal{D}}_z T_-) \; T_v \; \psi_s = 0 \tag{66}$$

$$([1 - N_z]_q \hat{\mathcal{D}}_+ T_{\bar{v}}^{-1} - q\hat{\mathcal{D}}_{\bar{v}} \hat{\mathcal{D}}_z T_- T_v) \; T_- \; \psi_s = 0 \tag{67}$$

Analogously to the above we shall write:

$$\psi_s = (\alpha k_+ + \beta k_- + \gamma k_v + \delta k_{\bar{v}} + z(\alpha' k_+ + \beta' k_- + \gamma' k_v + \delta' k_v)) \; \tilde{f}_s \tag{68}$$

where $\tilde{f}_s$ is from the conjugated q-plane wave. [If we use the other basis the prefactors will depend on s and we would not be able to express the solutions in terms of a q-plane wave.] Now the general solution of (66) and (67) is:

$$\psi_s^{\alpha,\gamma} = (\alpha k_+ + \gamma k_v - qz(\alpha k_{\bar{v}} + \gamma k_-)) \; \tilde{f}_s \tag{69}$$

and so the two independent solutions of (50) and (51) are:

$$\tilde{\psi}^{(1)} = (k_+ - qk_{\bar{v}} z) \; \widetilde{\exp}_q(k \cdot x) \tag{70a}$$
$$\tilde{\psi}^{(2)} = (k_v - qk_- z) \; \widetilde{\exp}_q(k \cdot x) \tag{70b}$$

Note that unlike the neutrino case, the antineutrino prefactors are q-deformed.

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QUANTUM CHAOS AND EXCEPTIONAL POINTS

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Abstract

The intricate connection between the distribution of exceptional points and particular fluctuation properties of level spacings is discussed. A procedure is given to determine the distribution of the exceptional points in the λ-plane for a Hamiltonian of the form $H_0 + \lambda H_1$.

1 Introduction

Quantum chaos has attracted great attention during the past few years. This is remarkable since, in contrast to classical chaos, a proper definition is not available as of yet. However, there is overwhelming experimental evidence, that, on a quantum mechanical level, nature shows characteristic features when systems are investigated which have a classical chaotic counterpart [1]. Even more striking is the fact that some of such features are still awaiting a satisfactory explanation although, as is the case in ionisation of highly excited Rydberg states in hydrogen [2], the dynamical situation is supposed to be fully understood. It appears, that the semiclassical limit ($\hbar \to 0$) of a system which is classically chaotic, is not understood. The formal root of this difficulty lies in the non-uniform nature of the limit

$$\lim_{\hbar \to 0} e^{iHt/\hbar}. \tag{1}$$

It is obvious that energy and time scales are crucial for the limit attained. The fascinating aspect for the physicist seems to be the necessity of having more detailed experimental information to tackle this formal problem. Whether or not we have to re-think our concepts of quantum mechanics remains to be seen; it appears to be certain that we get at least a more thorough understanding of quantum mechanics from such studies.

In this paper we attempt to address the questions: which are the common properties of the *quantum mechanical operators* that give rise to spectral properties that are ascribed to quantum chaos. If a matrix representation of a Hamiltonian which originates from a classically chaotic analogy is given, which is the mathematical mechanism that yields the special features of the spectrum within the particular range of the parameter where classical chaos is discerned. While cases with a classical analogy have no intrinsic statistical element, a further puzzling question is: why is it that a statistical approach without physical input reproduces such good agreement of the statistical properties of the spectrum.

We believe that the common root to the answer of these questions lies in what is called the exceptional points [3] of an operator. Most physical problems in quantum mechanics can be formulated by the Hamiltonian $H_0 + \lambda H_1$ where the parameter λ can play the role of a perturbation parameter, or it may serve to effect a phase transition, or it may under variation steer the system from an ordered into a chaotic regime. The exceptional points of the full operator are the points λ for which two eigenvalues coalesce. Here we exclude genuine degeneracies of the selfadjoint problem, in other words, the eigenvalues coincide for no real λ. The exceptional points occur in the complex λ -plane. Note that the operator is not selfadjoint for complex λ -values.

The definition of exceptional points is general and applies in particular to operators in an infinite dimensional space, also when the spectrum of the operator has a continuum part. In the present work we restrict ourselves to finite dimensional matrices H_0 and H_1 as in this case the role of the exceptional points and the associated Riemann sheet structure is thoroughly understood [4, 5]. We do not believe that restriction to matrices has a major impact on our conclusions since virtually all the practical work even in connection with quantum chaos is done in a finite dimensional matrix space.

The physical significance of the exceptional points is due to their relation with avoided level crossing for real λ-values. The spectrum $E_k(\lambda)$, $k = 1, \ldots, N$ has branch point singularities at the exceptional points, in fact, two of the N levels are connected via a square root branch point. If this happens near to the real λ-axis, a level repulsion will occur for the two levels for real λ-values. Globally, all the exceptional point singularities determine the shape of the whole spectrum. There is a nice analogy to the more widely known connection between the pole singularities of the scattering function and the shape of the cross section: in a similar way as the positions of the poles including their statistical properties determine the measurable cross section, the exceptional points determine the shape of the spectrum and in particular the occurrences of avoided level crossings. The distribution of the exceptional points will therefore determine the fluctuation properties of level spacings.

The positions of the exceptional points are fixed in the complex λ-plane and are determined solely by H_0 and H_1. For large matrices it is prohibitive to determine the positions of the exceptional points. However, it is possible to determine the distribution reasonably well from the knowledge of the two operators. A high density of exceptional points is a sufficient prerequisite for the occurrence of quantum chaos. In the following section we recapitulate the basics about exceptional points for matrices to render the paper self-contained.

2 Exceptional points and unperturbed lines

Avoided level crossing is always associated with exceptional points [4, 6, 7] if it occurs for the levels $E_k(\lambda)$ of the Hamiltonian $H_0 + \lambda H_1$. The exceptional points are square root branch point singularities in the complex λ-plane. We give an elementary example for illustration and briefly list the essential aspects with regard to exceptional points.

Consider a two dimensional matrix problem where H_0 is diagonal with eigenvalues ε_1 and ε_2, while H_1 is represented in the form

$$H_1 = U \cdot D \cdot U^{-1}. \tag{2}$$

Here, the diagonal matrix D contains the eigenvalues ω_1 and ω_2 of the matrix H_1 and U is the rotation

$$U = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}. \tag{3}$$

The eigenvalues of the problem $H_0 + \lambda H_1$ are

$$E_{1,2}(\lambda) = \frac{\varepsilon_1 + \varepsilon_2 + \lambda(\omega_1 + \omega_2)}{2} \pm R \tag{4}$$

where

$$R = \left\{ \left(\frac{\varepsilon_1 - \varepsilon_2}{2}\right)^2 + \left(\frac{\lambda(\omega_1 - \omega_2)}{2}\right)^2 + \frac{1}{2}\lambda(\varepsilon_1 - \varepsilon_2)(\omega_1 - \omega_2)\cos 2\varphi \right\}^{1/2}. \tag{5}$$

Clearly, when $\varphi = 0$ the spectrum is given by the two lines

$$E_k^0(\lambda) = \varepsilon_k + \lambda\omega_k \qquad k = 1, 2 \tag{6}$$

which intersect at the point of degeneracy $\lambda = -(\varepsilon_1 - \varepsilon_2)/(\omega_1 - \omega_2)$. When the coupling between the two levels is turned on by switching on φ the degeneracy is lifted and avoided level crossing occurs. Now the two levels coalesce in the complex λ-plane where R vanishes which happens at the complex conjugate points

$$\lambda_c = -\frac{\varepsilon_1 - \varepsilon_2}{\omega_1 - \omega_2}\exp(\pm 2i\varphi). \tag{7}$$

At these points, the two levels $E_k(\lambda)$ are connected by a square root branch point, in fact the two levels are the values of one analytic function on two different Riemann sheets.

These considerations carry over to an N-dimensional problem [4]. The diagonal matrix H_0 contains the elements ε_k and D the elements ω_k, $k = 1, \ldots, N$; the matrix U is now an N-dimensional rotation which can be parameterised by $N(N-1)/2$ angles. (In the quoted paper a parameterisation was chosen so that U is unity when all angles are zero.) The exceptional points are determined by the simultaneous solution of the equations

$$
\begin{aligned}
\det(E - H_0 - \lambda H_1) &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}E}\det(E - H_0 - \lambda H_1) &= 0.
\end{aligned}
\tag{8}
$$

There are generically $N(N-1)$ solutions which occur in complex conjugate pairs in the λ-plane. At those points the N levels $E_k(\lambda)$ are connected in pairs by square root branch points when they are analytically continued into the complex λ-plane. Since the positions of the singularities determine the shape of the spectrum, and in particular the fluctuation properties, a closer analysis is indicated. As is exemplified in the next section, the crucial condition for the occurrence of level statistics ascribed to quantum chaos is a high density of exceptional points in the complex plane within a small window of real λ-values.

To get an idea about the density of exceptional points we introduce the concept of *unperturbed lines*. Clearly, when U is the unit matrix (all angles are zero), the spectrum of $H_0 + \lambda H_1 = H_0 + \lambda D$ is given by the lines $\varepsilon_k + \lambda\omega_k$ with $k = 1, \dots, N$. The $N(N-1)/2$ intersection points of the N lines depend on the relative order of the numbers ε_k and ω_k. If both sequences are in ascending order, all intersections occur at negative λ-values; conversely, if one sequence is ascending and the other descending all intersections occur at positive λ-values. In general, the order which is appropriate for the actual problem, is expected to lie between the two extremes. To find out the appropriate order we are guided by the asymptotic behaviour of the levels $E_k(\lambda)$ of the full problem. For large values of λ the leading terms are given by

$$E_k(\lambda) = \lambda\omega_k + \alpha_k + \dots \tag{9}$$

where the dots stand for first and higher order terms in $1/\lambda$. Neglecting these terms, Eq.9 yields just the unperturbed lines with the appropriate association of slopes ω_k and intercepts α_k. From perturbation theory we find the latter to be the diagonal elements of the 'backwards' rotated H_0, *viz.*

$$\alpha_k = (U^{-1} \cdot H_0 \cdot U)_{k,k} \,. \tag{10}$$

The significance of the procedure lies in the easy availability of the parameters of the unperturbed lines from the knowledge of H_0 and H_1 alone. The exceptional points of the full problem are expected to lie near to the intersection points of the unperturbed lines (for the two dimensional problem the real parts of the exceptional points coincide with the points of intersection). Hence, knowledge of H_0 and H_1 alone enables us to predict for which values of λ strong fluctuations of the energy levels are expected to occur and where they certainly do not occur.

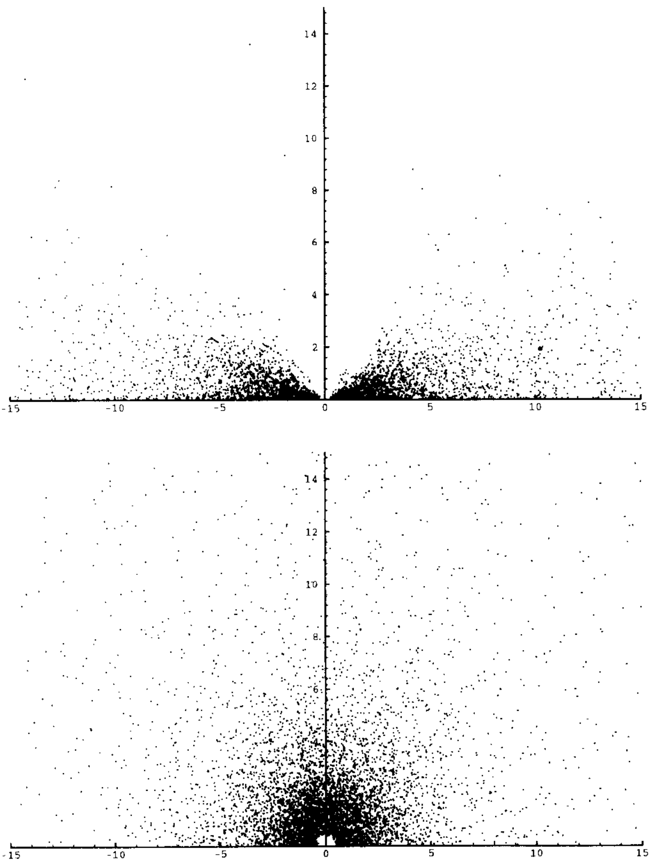

Fig.1. A sample of exceptional points in the upper λ-plane for $\varphi_0 \approx \varphi_{\text{crit}}/2$ (top) and $\varphi_0 \geq \varphi_{\text{crit}}$ (bottom). An ensemble of 140 cases of 10×10 matrices is displayed. The apparent hole around the origin is due to the finite sa nple size and is insignificant.

3 Distribution in the complex plane

While the intersection points of the unperturbed lines give a fair indication about the range of the values of λ where quantum chaos occurs, they do not give an indication about the distribution in the complex plane. To make progress we consider the N-dimensional matrix model where H_0 is diagonal with elements ε_i which we assume to be different. The form of H_1 is again chosen as in Eq.(2) where the elements of D are ω_i and the orthogonal matrix U is a random orthogonal matrix. For $U \equiv 1$ the levels of the full problem $H = H_0 + \lambda H_1$ are given by $E_i = \varepsilon_i + \lambda\omega_i$. The intersection points of these straight lines have a well defined distribution if the ε_i or the ω_i or both are uniformly distributed in a certain (large) interval. The distribution is given by $P(x) = c/x^2$ where c is a constant and x is the unfolded energy variable. We now switch on the random angles from which the random orthogonal matrix U is constructed by chosing them from the interval $[-\varphi_0, \varphi_0]$. As soon as φ_0 is nonzero, each intersection point will become a pair of exceptional points. For very small values of φ_0 the exceptional points lie near to the real axis from which they emerge. With increasing φ_0 the exceptional points move further into the complex λ-plane. In Fig.1 examples of the pattern are given for φ_0 about $\varphi_{\text{crit}}/2$ and for $\varphi_0 \geq \varphi_{\text{crit}}$. An empirical value for φ_{crit} is $\pi/(2\sqrt{N})$; the distribution does not change for $\varphi_0 > \varphi_{\text{crit}}$. It turns out that the distribution in the plane is again given by $P(r) = c/r^2$ where r is the distance from the origin. It is remarkable that it depends only on the distance from the origin; in other words, it is independent of a direction, that is of an angle against the real axis, if $\varphi_0 \geq \varphi_{\text{crit}}$.

The relation to quantum chaos lies in the corresponding nearest neighbour distributions (NND). We obtain a pure Poisson distribution for $\varphi_0 = 0$ reflecting an integrable case. A genuine Wigner distribution emerges if φ_0 has reached its critical value; it remains a Wigner distribution for all values of $\varphi_0 > \varphi_{\text{crit}}$. If $\varphi_0 < \varphi_{\text{crit}}$ the NND is between the two, it can be fitted by a Brody distribution.

4 Summary and discussion

The connection between fluctuations in the level spacings and the distribution of the exceptional points of the Hamiltonian $H_0 + \lambda H_1$ has been discussed. A constructive procedure to obtain a distribution of the real parts, and thus of regions of λ where quantum chaos is to be expected, has been given in terms of the unperturbed lines. The usefulness of the approach has been demonstrated by a number of examples given in the literature, where it has been found that quantum chaos is the more likely to occur the higher the density of exceptional points [8].

We mention that it was argued in a previous paper [7] that transitional regions of finite Fermi systems which undergo a phase transition are generically expected to show patterns ascribed to quantum chaos. This aspect has been further pursued and confirmed recently [9]. Exceptional points also explain why traditional approximation methods like a mean field approach are bound to fail in these regions. The reasoning is based on the augmented occurrence of exceptional points in the transitional region. The singularities limit the radius of convergence of a perturbative expansion.

With the approach adopted in this paper one might try to give a characterisation of quantum chaos: quantum chaos is quantum mechanics under special conditions. If avoided level crossing (or tunnelling) occurs on a large scale then quantum chaos is likely to occur. The high sensitivity [7] under generic perturbation of spectrum and state vectors is explained in terms of the high density of exceptional points. However, in contrast to the classical case, the transition from order to chaos is smooth, although it can be dramatic.

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THE FAMILIES OF ORTHOGONAL, UNITARY AND QUATERNIONIC UNITARY CAYLEY–KLEIN ALGEBRAS AND THEIR CENTRAL EXTENSIONS

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Abstract

The families of quasi-simple or Cayley–Klein algebras associated to anti-hermitian matrices over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are described in a unified framework. These three families include simple and non-simple real Lie algebras which can be obtained by contracting the pseudo-orthogonal algebras $so(p,q)$ of the Cartan series B_l and D_l, the special pseudo-unitary algebras $su(p,q)$ in the series A_l, and the quaternionic pseudo-unitary algebras $sq(p,q)$ in the series C_l. This approach allows to study many properties for all these Lie algebras simultaneously. In particular their non-trivial central extensions are completely determined in arbitrary dimension.

1 The three main families of CK algebras

The aim of this contribution is twofold. First, we present the structure of the three main series of Cayley–Klein (CK) algebras (orthogonal, unitary and quaternionic unitary) as associated to antihermitian matrices over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ [1, 2]. These families include real simple Lie algebras in the four Cartan series as well as many non-simple Lie algebras which can be obtained from the simple ones by a sequence of contractions [3, 4]. The CK approach is quite useful in the study of many structures associated to the algebras in a given CK family, such as their symmetric homogeneous spaces [5], Casimir invariants [6], quantum deformations [7], etc. This formalism allows a clear view of the behaviour of such properties under contraction.

And second, as an example of this approach, we focus on the complete and general results concerning the non-trivial central extensions [8, 9, 10] for *all the algebras* in the three main CK families.

We consider N real coefficients $\omega = (\omega_1, \omega_2, \ldots, \omega_N)$ and two-index coefficients ω_{ab} defined in terms of the former by

$$\omega_{ab} = \omega_{a+1}\omega_{a+2}\cdots\omega_b \qquad a,b = 0,1,\ldots,N \quad a<b \qquad \omega_{aa} := 1. \tag{1.1}$$

Let $\mathbb{K}$ denote an associative division algebra; this can be either the reals $\mathbb{R}$, complex $\mathbb{C}$ or quaternions $\mathbb{H}$. The space $\mathbb{K}^{N+1}$ can be endowed with a hermitian (sesqui)linear form $\langle\,.\,|\,.\,\rangle_\omega : \mathbb{K}^{N+1} \times \mathbb{K}^{N+1} \to \mathbb{K}$ defined by

$$\langle\, a|\, b\rangle_\omega := \bar{a}^0 b^0 + \bar{a}^1\omega_1 b^1 + \bar{a}^2\omega_1\omega_2 b^2 + \ldots + \bar{a}^N\omega_1\cdots\omega_N b^N = \sum_{i=0}^{N} \bar{a}^i\omega_{0i} b^i \tag{1.2}$$

where $a,\ b \in \mathbb{K}^{N+1}$ and $\bar{a}^i$ means the conjugation in $\mathbb{K}$ of the component a^i. The underlying metric is provided by the matrix

$$\mathcal{I}_\omega = \mathrm{diag}\,(1,\,\omega_{01},\,\omega_{02},\ldots,\,\omega_{0N}) = \mathrm{diag}\,(1,\,\omega_1,\,\omega_1\omega_2,\ldots,\,\omega_1\cdots\omega_N). \tag{1.3}$$

The antihermitian CK family over $\mathbb{K}$ is defined as the family of Lie algebras of the groups of linear isometries of this hermitian metric over the space $\mathbb{K}^{N+1}$. Thus the isometry condition for an element U of the corresponding Lie group

$$\langle U\, a|U\, b\rangle_\omega = \langle\, a|\, b\rangle_\omega \qquad \forall\ a,\ b \in \mathbb{K}^{N+1}, \tag{1.4}$$

leads to the following relations for the group element U and generator X:

$$U^\dagger\mathcal{I}_\omega U = \mathcal{I}_\omega \qquad X^\dagger\mathcal{I}_\omega + \mathcal{I}_\omega X = 0. \tag{1.5}$$

For $\mathbb{K} = \mathbb{R},\ \mathbb{H}$ these matrices span a simple Lie algebra. For $\mathbb{K} = \mathbb{C}$ the algebra so determined is not simple, but becomes simple if we add the unimodularity condition:

$$\det(U) = 1 \qquad \mathrm{trace}(X) = 0, \tag{1.6}$$

which is known as the *special* condition for the Lie algebra/group Hence from condition (1.5) (and also (1.6) when $\mathbb{K} = \mathbb{C}$) we obtain $\mathcal{I}_\omega$-antihermitian $(N+1)\times(N+1)$ matrices over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ which generate the three families of quasi-simple or CK algebras displayed in the following table [1, 2]:

$\mathbb{K}$	CK family	Generators	Dimension
$\mathbb{R}$	Orthogonal $so_\omega(N+1)$	$J_{ab} = -\omega_{ab}e_{ab} + e_{ba}$	$\frac{1}{2}N(N+1)$
$\mathbb{C}$	Complex special unitary $su_\omega(N+1)$	$J_{ab} = -\omega_{ab}e_{ab} + e_{ba}$ $M_{ab} = i(\omega_{ab}e_{ab} + e_{ba})$ $B_l = i(e_{l-1,l-1} - e_{ll})$	$(N+1)^2 - 1$
$\mathbb{H}$	Quaternionic unitary $sq_\omega(N+1)$	$J_{ab} = -\omega_{ab}e_{ab} + e_{ba}$ $M^\alpha_{ab} = i_\alpha(\omega_{ab}e_{ab} + e_{ba})$ $E^\alpha_a = i_\alpha e_{aa}$	$2(N+1)^2 + (N+1)$

Hereafter it is assumed that $a, b = 0, 1, \ldots, N$ and $a < b$; $l = 1, \ldots, N$; i is the complex unit; $\alpha = 1, 2, 3$ and $i_1 = i$, $i_2 = j$, $i_3 = k$ are the usual quaternionic units; and e_{ab} is the $(N+1)\times(N+1)$ matrix with a single 1 entry in row a, column b.

When all the coefficients $\omega_a \neq 0$ these CK families include real simple algebras in the four Cartan series:

• The orthogonal CK family $so_\omega(N+1)$ embraces the pseudo-orthogonal algebras $so(p,q)$ with $p+q=N+1$ in the Cartan series $B_{\frac{N}{2}}$ for even N, and $D_{\frac{N+1}{2}}$ for odd N.
• The special unitary CK family $su_\omega(N+1)$ comprises the special pseudo-unitary algebras $su(p,q)$ with $p+q=N+1$ in the Cartan series A_N.
• The quaternionic unitary CK family $sq_\omega(N+1)$ includes the quaternionic pseudo-unitary algebras $sq(p,q)$ with $p+q=N+1$ (the usual name for these algebras is $sp(p,q)$) in the Cartan series C_{N+1}.

In all cases, p and q are the number of positive and negative terms in the matrix $\mathcal{I}_\omega$ (1.3), so when all $\omega_a > 0$ we recover the compact real forms $so(N+1)$, $su(N+1)$ and $sq(N+1)$. If we set one or several coefficients ω_a equal to zero, we obtain a non-simple Lie algebra. This process is equivalent to perform the limit $\omega_a \to 0$ which corresponds to carry out an Inönü–Wigner contraction [11]. Therefore each CK family includes simple as well as non-simple members and the latter appear by means of contractions. These non-simple Lie algebras preserve properties related to simplicity, and are also called quasi-simple [12]. For instance, it has been shown in [6] that all the algebras in the orthogonal CK family $so_\omega(N+1)$ share the same number of functionally independent Casimir operators no matter how contracted is the Lie algebra. This not longer true if we go beyond the CK family as the abelian algebra clearly shows: all its generators are invariants.

These families of quasi-simple algebras can be obtained by applying the graded contraction theory [13, 14] to any simple member in each family (e.g., $so_\omega(N+1)$ is provided by $\mathbb{Z}_2^{\otimes N}$ graded contractions starting from $so(N+1)$ [3, 4]). An alternative approach to these algebras can be found in [15].

In the sequel we write down the Lie brackets for these three families of Lie algebras. We remark that the values of each *real* coefficient ω_a can be reduced to $+1$, -1 or 0 (by means of a rescaling), so that for a given N each CK family contains 3^N Lie algebras (some of them are isomorphic as abstract Lie algebras).

The commutation relations of the *orthogonal CK family* $so_\omega(N+1)$ can be obtained from the matrix generators J_{ab} and read [1, 8]:

$$[J_{ab},J_{ac}]=\omega_{ab}J_{bc} \quad [J_{ab},J_{bc}]=-J_{ac} \quad [J_{ac},J_{bc}]=\omega_{bc}J_{ab} \quad [J_{ab},J_{de}]=0. \tag{1.7}$$

Hereafter whenever three indices a,b,c appear they are always assumed to verify $a<b<c$; whenever four indices a,b,d,e appear, $a<b$, $d<e$ and all of them are assumed to be different; and there is no any implied sum over repeated indices.

The Lie brackets of the *special unitary CK family* $su_\omega(N+1)$ are given by [1, 9]:

$$\begin{array}{lll}
[J_{ab},J_{ac}]=\omega_{ab}J_{bc} & [J_{ab},J_{bc}]=-J_{ac} & [J_{ac},J_{bc}]=\omega_{bc}J_{ab} \\
[M_{ab},M_{ac}]=\omega_{ab}J_{bc} & [M_{ab},M_{bc}]=J_{ac} & [M_{ac},M_{bc}]=\omega_{bc}J_{ab} \\
[J_{ab},M_{ac}]=\omega_{ab}M_{bc} & [J_{ab},M_{bc}]=-M_{ac} & [J_{ac},M_{bc}]=-\omega_{bc}M_{ab} \\
[M_{ab},J_{ac}]=-\omega_{ab}M_{bc} & [M_{ab},J_{bc}]=-M_{ac} & [M_{ac},J_{bc}]=\omega_{bc}M_{ab} \\
[J_{ab},J_{de}]=0 & [M_{ab},M_{de}]=0 & [J_{ab},M_{de}]=0
\end{array}$$
$$[J_{ab},B_l]=(\delta_{a,l-1}-\delta_{b,l-1}+\delta_{bl}-\delta_{al})M_{ab}$$
$$[M_{ab},B_l]=-(\delta_{a,l-1}-\delta_{b,l-1}+\delta_{bl}-\delta_{al})J_{ab} \tag{1.8}$$

$$[J_{ab}, M_{ab}] = -2\omega_{ab} \sum_{s=a+1}^{b} B_s \qquad [B_k, B_l] = 0. \tag{1.9}$$

If we discard the condition (1.6) and consequently, further to the generators of $su_\omega(N+1)$ we add the matrix $I = i\sum_{a=0}^{N} e_{aa}$, we obtain the *unitary* CK family $u_\omega(N+1)$ of dimension $(N+1)^2$ and with commutators given by (1.8), (1.9) and

$$[J_{ab}, I] = 0 \qquad [M_{ab}, I] = 0 \qquad [B_l, I] = 0. \tag{1.10}$$

When all $\omega_a \neq 0$ we find the pseudo-unitary algebras $u(p,q)$ which are not simple (they are in the semisimple series $A_N \oplus D_1$) but their pattern is similar to that of $su(p,q)$ with respect to $\mathcal{I}_\omega$.

The commutation rules of the *quaternionic unitary CK family* $sq_\omega(N+1)$ turn out to be [1, 10]:

$$\begin{array}{lll}
[J_{ab}, J_{ac}] = \omega_{ab} J_{bc} & [J_{ab}, J_{bc}] = -J_{ac} & [J_{ac}, J_{bc}] = \omega_{bc} J_{ab} \\
[M^\alpha_{ab}, M^\alpha_{ac}] = \omega_{ab} J_{bc} & [M^\alpha_{ab}, M^\alpha_{bc}] = J_{ac} & [M^\alpha_{ac}, M^\alpha_{bc}] = \omega_{bc} J_{ab} \\
[J_{ab}, M^\alpha_{ac}] = \omega_{ab} M^\alpha_{bc} & [J_{ab}, M^\alpha_{bc}] = -M^\alpha_{ac} & [J_{ac}, M^\alpha_{bc}] = -\omega_{bc} M^\alpha_{ab} \\
[M^\alpha_{ab}, J_{ac}] = -\omega_{ab} M^\alpha_{bc} & [M^\alpha_{ab}, J_{bc}] = -M^\alpha_{ac} & [M^\alpha_{ac}, J_{bc}] = \omega_{bc} M^\alpha_{ab} \\
[J_{ab}, J_{de}] = 0 & [M^\alpha_{ab}, M^\alpha_{de}] = 0 & [J_{ab}, M^\alpha_{de}] = 0 \\
[J_{ab}, E^\alpha_d] = (\delta_{ad} - \delta_{bd}) M^\alpha_{ab} & [M^\alpha_{ab}, E^\alpha_d] = -(\delta_{ad} - \delta_{bd}) J_{ab} & \\
[J_{ab}, M^\alpha_{ab}] = 2\omega_{ab}(E^\alpha_b - E^\alpha_a) & [E^\alpha_a, E^\alpha_b] = 0 &
\end{array} \tag{1.11}$$

$$\begin{array}{lll}
[M^\alpha_{ab}, M^\beta_{ac}] = \omega_{ab}\varepsilon_{\alpha\beta\gamma} M^\gamma_{bc} & [M^\alpha_{ab}, M^\beta_{bc}] = \varepsilon_{\alpha\beta\gamma} M^\gamma_{ac} & [M^\alpha_{ac}, M^\beta_{bc}] = \omega_{bc}\varepsilon_{\alpha\beta\gamma} M^\gamma_{ab} \\
[M^\alpha_{ab}, M^\beta_{de}] = 0 & [M^\alpha_{ab}, M^\beta_{ab}] = 2\omega_{ab}\varepsilon_{\alpha\beta\gamma}(E^\gamma_a + E^\gamma_b) & \\
[M^\alpha_{ab}, E^\beta_d] = (\delta_{ad} + \delta_{bd})\varepsilon_{\alpha\beta\gamma} M^\gamma_{ab} & [E^\alpha_a, E^\beta_b] = 2\delta_{ab}\varepsilon_{\alpha\beta\gamma} E^\gamma_a &
\end{array}$$

where $\varepsilon_{\alpha\beta\gamma}$ is the completely antisymmetric unit tensor with $\varepsilon_{123} = 1$, and whenever three quaternionic indices α, β, γ appear, they are assumed to be different.

Finally, we would like to remark that in addition to these three main 'signature' families of CK algebras, whose simple members $so(p,q)$, $su(p,q)$, $sq(p,q)$ can be realised as antihermitian matrices over either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, there are other CK families [1, 2]. In the C_{N+1} Cartan series, the remaining real Lie algebra is the symplectic $sp(2(N+1), \mathbb{R})$, which can be interpreted in terms of CK families either as the single simple member of its own CK family $sp_\omega(2(N+1), \mathbb{R})$, or alternatively, as the unitary family $u_\omega((N+1), \mathbb{H}')$ over the algebra of the split quaternions $\mathbb{H}'$ (a pseudo-orthogonal variant of quaternions, where i_1, i_2, i_3 still anticommute, but their squares are $i_1^2 = -1, i_2^2 = 1, i_3^2 = 1$; this is not a division algebra). Likewise there are other CK families associated to the remaining real simple Lie algebras, namely: $su^*(2(N+1)) \approx sl(N+1, \mathbb{H})$ in the Cartan series A_{2N+1}, $sl(N+1, \mathbb{R}) \approx su(N+1, \mathbb{C}')$ also in A_N, and $so^*(2N)$ in D_N.

2 Central extensions

Let $\mathcal{G}$ an r-dimensional Lie algebra with generators $\{X_1, \ldots, X_r\}$ and structure constants C^k_{ij}. A generic central extension $\overline{\mathcal{G}}$ of $\mathcal{G}$ by the one-dimensional algebra generated by Ξ is a Lie algebra with $(r+1)$ generators $\{X_1, \ldots, X_r, \Xi\}$ and commutation relations:

$$[X_i, X_j] = \sum_{k=1}^{r} C_{ij}^k X_k + \xi_{ij}\Xi \qquad [\Xi, X_i] = 0. \tag{2.1}$$

Hence we consider an initial extension coefficient ξ_{ij} associated to *each* Lie bracket $[X_i, X_j]$. As $\overline{\mathcal{G}}$ must be a Lie algebra, the extension coefficients ξ_{ij} must be antisymmetric in their indices, $\xi_{ji} = -\xi_{ij}$, and must fulfil the following conditions coming from the Jacobi identities for the generators X_i, X_j, X_l in $\overline{\mathcal{G}}$:

$$\sum_{k=1}^{r} \left(C_{ij}^k \xi_{kl} + C_{jl}^k \xi_{ki} + C_{li}^k \xi_{kj} \right) = 0. \tag{2.2}$$

Therefore as a first step in the study of the central extensions of a given Lie algebra $\mathcal{G}$ we have to solve the set of linear equations (2.2) involving all initially possible extension coefficients ξ_{ij} (2.1). Afterwards we have to find out which of these coefficients are trivial, that is, which can be removed from (2.1) by means of a change of basis in $\overline{\mathcal{G}}$. Explicitly, if for a set of arbitrary real numbers μ_k we perform a change of generators

$$X_k \to X_k' = X_k + \mu_k \Xi, \tag{2.3}$$

then the commutators for the generators $\{X_1', \ldots, X_r', \Xi\}$ are given by (2.1) with a new set of extension coefficients

$$\xi_{ij}' = \xi_{ij} - \sum_{k=1}^{r} C_{ij}^k \mu_k, \tag{2.4}$$

where $\delta\mu(X_i, X_j) = \sum_{k=1}^{r} C_{ij}^k \mu_k$ is the two-coboundary generated by μ. Extension coefficients differing by a two-coboundary correspond to equivalent extensions, and those extension coefficients which are a two-coboundary, $\xi_{ij} = -\sum_{k=1}^{r} C_{ij}^k \mu_k$, correspond to trivial extensions. The classes of equivalence of non-trivial two-cocycles determine the second cohomology group of the Lie algebra, $H^2(\mathcal{G}, \mathbb{R})$.

The procedure we have just described can be carried out for each family of CK algebras in a global way, so that this unified approach avoids at once and for all the need of a case-by-case study for any given algebra in the family. In general, the extension coefficients arising for each CK family can be casted into three types according to their behaviour under contraction (when some ω_a vanish):

• *Type I extension coefficients:* they correspond to central extensions which are trivial for all the CK algebras belonging to a given family simultaneously, no matter of how many coefficients ω_a are equal to zero. Therefore they can be removed at once by means of redefinitions as (2.3).

• *Type II extension coefficients:* they give rise to non-trivial extensions when some $\omega_a = 0$, and to trivial ones otherwise. Therefore these extensions become non-trivial through contractions and are called pseudoextensions [16, 17].

• *Type III extension coefficients:* they must fulfil some additional conditions in such manner that whenever they are non-zero they are always non-trivial. Therefore these extensions cannot appear through the pseudoextension process.

In the sequel we present the complete results on the problem of finding all non-trivial central extensions for the three main families of CK algebras. We will directly discard type I extensions which are trivial for all CK algebras in each family.

2.1 Central extensions of the orthogonal CK algebras

The non-zero commutators of any central extension $\overline{so}_\omega(N+1)$ of the orthogonal CK algebra $so_\omega(N+1)$ are given [8] by:

$$\begin{array}{lll} [J_{ab}, J_{bc}] = -J_{ac} & & \\ [J_{ab}, J_{a\,b+1}] = \omega_{ab} J_{b\,b+1} + \omega_{a\,b-1}\alpha^F_{b\,b+1}\Xi & [J_{ab}, J_{ac}] = \omega_{ab} J_{bc} & \text{for } c > b+1 \\ [J_{ac}, J_{a+1\,c}] = \omega_{a+1\,c} J_{a\,a+1} + \omega_{a+2\,c}\alpha^L_{a\,a+1}\Xi & [J_{ac}, J_{bc}] = \omega_{bc} J_{ab} & \text{for } b > a+1 \\ [J_{a\,a+1}, J_{c\,c+1}] = \beta_{a+1\,c+1}\Xi & [J_{a\,a+2}, J_{a+1\,a+3}] = -\omega_{a+2}\beta_{a+1\,a+3}\Xi. & \end{array} \tag{2.5}$$

The extension is completely characterized by the following extension coefficients:

• Two type II coefficients, α^L_{01}, $\alpha^F_{N-1\,N}$. The extension α^L_{01} (resp. $\alpha^F_{N-1\,N}$) is non-trivial if $\omega_2 = 0$ (resp. $\omega_{N-1} = 0$) and is trivial otherwise.

• $(N-2)$ type II *pairs*, $\alpha^F_{a+1\,a+2}$, $\alpha^L_{a+1\,a+2}$ $(a = 0, 1, \ldots, N-3)$, fulfilling

$$\omega_{a+3}\alpha^F_{a+1\,a+2} = \omega_{a+1}\alpha^L_{a+1\,a+2}. \tag{2.6}$$

The two extensions $\alpha^F_{a+1\,a+2}$, $\alpha^L_{a+1\,a+2}$ are both non-trivial when $\omega_{a+1} = 0$ and $\omega_{a+3} = 0$, and both are simultaneously trivial otherwise.

• $(N-1)(N-2)/2$ type III coefficients $\beta_{b+1\,d+1}$ $(b = 0, 1, \ldots, N-3$ and $d = b+2, \ldots, N-1)$ which must satisfy the following additional relations:

$$\begin{array}{llll} \text{If} \quad d = b+2: & \omega\beta_{b+1\,b+3} = 0 & \text{for} & \omega = \omega_b, \omega_{b+1}\omega_{b+2}, \omega_{b+2}\omega_{b+3}, \omega_{b+4} \\ \text{If} \quad d > b+2: & \omega\beta_{b+1\,d+1} = 0 & \text{for} & \omega = \omega_b, \omega_{b+2}, \omega_d, \omega_{d+2} \end{array} \tag{2.7}$$

where it is understood that conditions as $\omega_0\beta = 0$ or $\omega_{N+1}\beta = 0$ are not present. Whenever the extension $\beta_{b+1\,d+1}$ is non-zero it is always non-trivial.

The pseudoextension character of type II coefficients $\alpha^F_{a+1\,a+2}$, $\alpha^L_{a+1\,a+2}$ means that if $\omega_{a+1} \neq 0$ and $\omega_{a+3} \neq 0$ both can be simultaneously removed by applying a redefinition of the generator $J_{a+1\,a+2}$:

$$J_{a+1\,a+2} \to J'_{a+1\,a+2} = J_{a+1\,a+2} + \frac{\alpha^F_{a+1\,a+2}}{\omega_{a+1}}\Xi = J_{a+1\,a+2} + \frac{\alpha^L_{a+1\,a+2}}{\omega_{a+3}}\Xi. \tag{2.8}$$

The equality is guaranteed by the condition (2.6). Thus in this case both extensions are trivial. If we perform, for instance, the contraction $\omega_{a+1} \to 0$, the relation (2.6) implies that $\omega_{a+3}\alpha^F_{a+1\,a+2} = 0$, so if $\omega_{a+3} \neq 0$ the extension $\alpha^F_{a+1\,a+2}$ vanishes, while $\alpha^L_{a+1\,a+2}$ can be eliminated by using (2.8). Therefore both extensions become non-trivial only through the two contractions $\omega_{a+1} = 0$ and $\omega_{a+3} = 0$. The same happens for the remaining type II extensions α^L_{01} and $\alpha^F_{N-1\,N}$ with respect to ω_2 and ω_{N-1}, respectively. We remark that type III extensions do not appear under such process.

Let us illustrate now these results with some interesting algebras [8]:

(1) $so(p,q)$ with $p+q=N+1$ (all $\omega_a \neq 0$).
All type III coefficients vanish due to (2.7), and all type II are trivial as they can be removed by means of redefinitions (2.8). Therefore, as it is well known (Whitehead's lemma) these algebras have no non-trivial extensions: $\dim(H^2(so(p,q),\mathbb{R}))=0$.

(2) $iso(p,q)$ with $p+q=N$ ($\omega_1=0$ and the others are non-zero).
These algebras include the Euclidean $iso(N)$ and Poincaré $iso(N-1,1)$ ones. The case $N=2$ is special since $\omega_1=\omega_{N-1}=0$, so that there is single non-trivial extension $\alpha^F_{N-1\,N}=\alpha^F_{12}$ and $\dim(H^2(iso(p,q),\mathbb{R}))=1$; this extension corresponds to a uniform and constant force field in the $1+1$ Minkowskian free kinematics for the Poincaré algebra $iso(1,1)$. However this extension is no longer possible if $N>2$ ($0=\omega_1\neq\omega_{N-1}$) and there are no non-trivial type II extensions; furthermore it can be checked that all type III ones vanish so that $\dim(H^2(iso(p,q),\mathbb{R}))=0$.

(3) $iiso(p,q)$ with $p+q=N-1$ ($\omega_1=\omega_2=0$ and the others are non-zero).
We have to distinguish three cases according to the values of N:

N	Non-trivial extensions	$\dim(H^2(iiso(p,q),\mathbb{R}))$
2	$\alpha^L_{01},\alpha^F_{12}$	2
3	$\alpha^L_{01},\alpha^F_{23},\beta_{13}$	3
>3	α^L_{01}	1

The physical role of these extensions for the Galilean algebra $iiso(N-1)$ is as follows: the only non-trivial coefficient which appears for any N, α^L_{01}, is the mass; α^F_{12} is a constant force for $iiso(1)$; α^F_{23} is a sort of non-relativistic remainder of the Thomas precession for $iiso(2)$ [18] and β_{13} has no physical meaning and disappears once we move from the Galilean algebra to the corresponding Lie group.

(4) $ii\ldots iso(1)$ (all $\omega_a=0$).
This is the most contracted CK algebra: the orthogonal flag algebra. All the conditions (2.6) and (2.7) are fulfilled and all possible extensions are non-trivial, that is, there are $2(N-1)$ type II and $(N-1)(N-2)/2$ type III non-trivial extensions.

2.2 Central extensions of the unitary CK algebras

The commutation relations of any central extension $\overline{su}_\omega(N+1)$ of the special unitary CK algebra $su_\omega(N+1)$ can be written [9] as the commutation relations (1.8) together with:

$$[J_{ab},M_{ab}]=-2\omega_{ab}\sum_{s=a+1}^{b}B_s+\sum_{s=a+1}^{b}\omega_{a\,s-1}\omega_{sb}\,\alpha_s\Xi \qquad [B_k,B_l]=\beta_{kl}\Xi \quad k<l \quad (2.9)$$

which will replace those in (1.9). The possible extension coefficients are

- N type II coefficients α_k ($k=1,\ldots,N$). Each of them gives rise to a non-trivial extension if $\omega_k=0$ and to a trivial one otherwise.

• $N(N-1)/2$ type III coefficients β_{kl} ($k < l$ and $k, l = 1, \ldots, N$), sa isfying

$$\omega_k \beta_{kl} = 0 \qquad \omega_l \beta_{kl} = 0. \tag{2.10}$$

Thus, β_{kl} vanishes when at least one of the parameters ω_k, ω_l is different from zero. When β_{kl} is non-zero it is always non-trivial.

On the other hand, the Lie brackets of any central extension $\overline{u}_\omega(N+1)$ of the unitary CK algebra $u_\omega(N+1)$ are given [9] by (1.8), (2.9) and:

$$[J_{ab}, I] = 0 \qquad [M_{ab}, I] = 0 \qquad [B_k, I] = \gamma_k \Xi \tag{2.11}$$

which will replace those in (1.10). The extension is completely characterized by the above extension coefficients α_k and β_{kl} together with:
• N type III coefficients γ_k ($k = 1, \ldots, N$) satisfying

$$\omega_k \gamma_k = 0. \tag{2.12}$$

When γ_k is non-zero, the extension that it determines is non-trivial.

Unlike the orthogonal CK family we can give now the following closed expressions for the dimension of the second cohomology group of these two unitary CK families which establish the number of non-trivial extension coefficients according to the number n of coefficients ω_k which are equal to zero for a given unitary CK algebra:

$$\dim\left(H^2(su_\omega(N+1), \mathbb{R})\right) = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2} \tag{2.13}$$

$$\dim\left(H^2(u_\omega(N+1), \mathbb{R})\right) = n + \frac{n(n-1)}{2} + n = \frac{n(n+3)}{2}. \tag{2.14}$$

The first term n in the sum of (2.13) and (2.14) corresponds to the central extensions α_k, the second term $\frac{n(n-1)}{2}$ to the β_{kl} and the third term n in (2.14) to the γ_k.

As far as the special unitary CK algebras is concerned, we find that the second cohomology group is trivial for the simple $su(p,q)$ algebras with all $\omega_a \neq 0$ ($n = 0$). The inhomogeneous $iu(p,q)$ algebras with $\omega_1 = 0$ and all other constants $\omega_a \neq 0$ ($n = 1$) have, in any dimension, a single non-trivial extension: α_1. The algebras with $\omega_1 = \omega_2 = 0$ and the remaining $\omega_a \neq 0$ ($n = 2$) have always three non-trivial extensions: α_1, α_2 and β_{12}. The special unitary flag algebra with all $\omega_a = 0$ ($n = N$) has the maximum number of non-trivial extensions: $N(N+1)/2$. A similar discussion can be performed for the CK family $u_\omega(N+1)$.

2.3 Central extensions of the quaternionic unitary CK algebras

The algebraic structure of the CK family $sq_\omega(N+1)$ seems to be much more complicated than those in the orthogonal and unitary CK families. However, the solution to the central extension problem comes as a surprise and is quite simple to state

[10]: all central extensions of any Lie algebra belonging to the quaternionic unitary CK family $sq_{\omega}(N+1)$ are always trivial for any N and for any values of the set of constants $\omega_1, \omega_2, \ldots, \omega_N$:

$$\dim\left(H^2(sq_{\omega}(N+1), \mathbb{R})\right) = 0. \tag{2.15}$$

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INÖNÜ - WIGNER CONTRACTIONS FOR INTERBASES EXPANSIONS ON 2 AND 3 - DIMENSIONAL SPHERES

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Abstract

Basis function for representations of rotation groups, corresponding to different chains of subgroups, are realized by different hyperspherical functions. They are related amongst each other by interbases expansion coefficients. The asymptotic limits of these interbases expansions are obtained when the corresponding rotation groups O(3) and O(4) are contracted to the Euclidean groups E(2) and E(3).

1 Introduction

A recent article [1] was devoted to the relation between the separation of variables in the Laplace-Beltrami operator on an n-sphere S_n and the Laplace operator on the Euclidean space E_n. The connection between the two spaces is studied using the theory of Lie algebra and Lie group contractions [2]. The two groups related by the contraction procedure are the isometry groups of the two spaces, i.e. the rotation group O(n+1) and the Euclidean group E(n), respectively.

Let us consider the Laplace-Beltrami equation

$$\Delta_{LB}\Psi(u_1, u_2 ...., u_n) = -\lambda\Psi(u_1, u_2, ..., u_n) \tag{1}$$

on some homogeneous space M. We denote the isometry group of M and its Lie algebra G and L, respectively. Separated eigenfunction of Δ_{LB} can be characterized as the common eigenfunctions of a complete set of commuting operators Y_a, $a = 1, 2, .., n$

$$Y_a\Psi = -\lambda_a\Psi, \qquad [Y_a, Y_b] = 0, \qquad \Psi = f_1(u_1)f_2(u_2)....f_n(u_n). \tag{2}$$

The set of operators $\{Y_a, a = 1, 2, .., n\}$ includes the Laplace-Beltrami operator and consists of second order operators in the enveloping algebra of L. The simplest types of coordinates are obtained if all operators Y_a in the set are Casimir operators of subalgebras of L. Different types of coordinate systems and different types of eigenfunctions Ψ correspond to different types of subalgebra chains $L \supset L_1 \supset L_2 ....$ The corresponding coordinates are called subgroup type coordinates.

Smorodinsky, Vilenkin and collaborators introduced a graphical method, the "method of trees", for characterizing different subgroup type coordinates. The method is explained in the original articles [3, 4] and subsequent book [5]. The relation between tree diagrams and subgroup (or subalgebra) diagrams is presented in ref. [1].

The contractions considered in [1] were "analytical contractions". The contraction parameter is the radius of the sphere R. It is introduced into the generators of the Lie algebra L, hence also into the operators $\{Y_a\}$ and thus into eigenfunction Ψ and eigenvalues λ_a. The effect of taking the limit $R \to \infty$, when S_n goes to E_n, was studied for coordinates, operators, eigenfunctions and eigenvalues.

In physical application one makes use of concrete basis functions of representations and one needs other objects as well. Thus, in nuclear physics, or any many-body theory more generally, it is often necessary to expand different bases in terms of each other. The different bases are related by unitary transformations. The matrix elements of transformations between different bases for the group O(n+1), i.e. between different types of hyperspherical (or polyspherical) functions, are called "T-coefficients" and have been calculated explicitly [6].

The purpose of this article is to study the contractions of the interbases expansions and thus of the T - coefficients, in the limit $R \to \infty$.

2 Contractions of interbases expansions for O(3)

In the case of the S_2 - sphere only two trees exist (Fig.1). They are topologically equivalent and correspond to equivalent subgroup chains O(3)$\supset$O(2). However, in the 3-body problem, one chain privileges particles (1,2), the other particles (2,3). The spherical functions $Y_{lm}(\theta_1, \theta_2)$ corresponding to these trees are connected by the transformation

$$Y_{lm_1}\left(\frac{\pi}{2} - \theta_1', \theta_2'\right) = \sum_{m_2=-l}^{l} D^l_{m_2,m_1}(\frac{\pi}{2}, \frac{\pi}{2}, 0)\, Y_{lm_2}(\theta_1, \theta_2) \qquad (3)$$

where $D^l_{m_2,m_1}(\alpha, \beta, \gamma) = e^{-im_2\alpha} d^l_{m_2,m_1}(\beta) e^{-im_1\gamma}$ is the Wigner function [7] and the angles in both sides of the expansions are connected by the relations

$$\begin{aligned} u_0 &= R\cos\theta_1 &&= R\cos\theta_1'\cos\theta_2' \\ u_1 &= R\sin\theta_1\cos\theta_2 &&= R\cos\theta_1'\sin\theta_2' \\ u_2 &= R\sin\theta_1\sin\theta_2 &&= R\sin\theta_1' \end{aligned}$$

We use an integral representation for the function $d^l_{m_2,m_1}(\pi/2)$

$$d^l_{m_2,m_1}(\frac{\pi}{2}) = (-1)^{\frac{l-m_1}{2}} \frac{2^l}{\pi} \left\{ \frac{(l+m_2)!(l-m_2)!}{(l+m_1)!(l-m_1)!} \right\}^{1/2}$$

$$\int_0^\pi (\sin\alpha)^{l-m_1} (\cos\alpha)^{l+m_1} e^{2im_2\alpha} d\alpha,$$

and the formulas [8]

$$\cos(2n\alpha) = T_n(\cos 2\alpha), \qquad \sin(2n\alpha) = \sin 2\alpha \cdot U_{n-1}(\cos 2\alpha),$$

where $T_l(x)$ and $U_l(x)$ are Tchebyshev polynomials of the first and second kind. After integrating over α, we obtain a representation of the Wigner D functions in terms of the hypergeometrical function ${}_3F_2$ (of argument 1):

$$D^l_{m_2,m_1}(\frac{\pi}{2}, \frac{\pi}{2}, 0) = \frac{(-1)^{\frac{l+m_2-m_1}{2}}}{\sqrt{\pi} J!} \sqrt{(l+m_2)!(l-m_2)!} \tag{4}$$

$$\begin{cases} \left\{ \dfrac{\Gamma(\frac{l+m_1+1}{2})\Gamma(\frac{l-m_1+1}{2})}{\Gamma(\frac{l+m_1+2}{2})\Gamma(\frac{l-m_1+2}{2})} \right\}^{\frac{1}{2}} {}_3F_2 \left(\begin{matrix} -m_2, m_2, \frac{l+m_1+1}{2} \\ \frac{1}{2}, l+1 \end{matrix} \middle| 1 \right) & (l-m_1) - \text{even} \\ \frac{2il}{(l+1)} \left\{ \dfrac{\Gamma(\frac{l+m_1+2}{2})\Gamma(\frac{l-m_1+2}{2})}{\Gamma(\frac{l+m_1+1}{2})\Gamma(\frac{l-m_1+1}{2})} \right\}^{\frac{1}{2}} {}_3F_2 \left(\begin{matrix} 1-m_2, m_2+1, \frac{l+m_1+2}{2} \\ \frac{3}{2}, l+2 \end{matrix} \middle| 1 \right) & (l-m_1) - \text{odd} \end{cases}$$

Consider now the contraction limit $R \to \infty$ in the expansion (3). For large R we put

$$l \sim kR, \qquad m_1 \sim k_1 R, \qquad \theta_1 \sim \frac{r}{R}, \qquad \theta_1' \sim \frac{y}{R}, \qquad \theta_2' \sim \frac{x}{R},$$

where $k^2 = k_1^2 + k_2^2$, and have

$$\lim_{R\to\infty} \frac{1}{\sqrt{R}} Y_{lm_2}(\theta_1, \theta_2) = (-1)^{\frac{m_2+|m_2|}{2}} \sqrt{k} J_{|m_2|}(kr) \frac{e^{im_2\theta_2}}{\sqrt{2\pi}},$$

$$\lim_{R\to\infty} (-1)^{-\frac{l-|m_1|}{2}} Y_{lm_1}\left(\frac{\pi}{2} - \theta_1', \theta_2'\right) = \sqrt{\frac{k}{k_2}} \frac{e^{ik_1 x}}{\pi} \begin{cases} \cos k_2 y, \ (l-|m_1|) - \text{even}, \\ -i\sin k_2 y, \ (l-|m_1|) - \text{odd}, \end{cases}$$

Using known asymptotic formulas for the ${}_3F_2$ functions and Γ - functions in eq. (4) we obtain:

$$\lim_{R\to\infty}(-1)^{-\frac{l-|m_1|}{2}}\sqrt{R}\,D^l_{m_2,m_1}\left(\frac{\pi}{2},\frac{\pi}{2},0\right) = (-1)^{\frac{m_2}{2}}\sqrt{\frac{2}{\pi k}}$$

$$\times\begin{cases} \left(\frac{k^2}{k_2}\right)^{\frac{1}{4}}\ {}_2F_1\left(-m_2,m_2;\ \frac{1}{2};\ \frac{k+k_1}{2k}\right) & (l-m_1)-\text{even},\\ \\ -im_2\ \left(\frac{k_2}{k^2}\right)^{\frac{1}{4}}\ {}_2F_1\left(-m_2+1,m_2+1;\ \frac{3}{2};\ \frac{k+k_1}{2k}\right) & (l-m_1)-\text{odd}. \end{cases}$$

$$= (-1)^{\frac{3m_2}{2}}\sqrt{\frac{2}{\pi k_2}}\begin{cases} \cos m_2\phi, & (l-m_1)-\text{even},\\ \\ i\sin m_2\phi, & (l-m_1)-\text{odd}, \end{cases} \tag{5}$$

Multiplying the interbases expansion (3) by the factor $(-1)^{-\frac{l-|m_1|}{2}}$ and taking the contraction limit $R\to\infty$ we obtain ($\theta\equiv\theta_2$, $m\equiv m_2$)

$$e^{ik_1x}\begin{Bmatrix}\cos k_2y\\ \sin k_2y\end{Bmatrix} = \sum_{m=-\infty}^{\infty}(i)^{|m|}\begin{Bmatrix}\cos m\varphi\\ -\sin m\varphi\end{Bmatrix} J_{|m|}(kr)\,e^{im\theta} \tag{6}$$

or in exponential form

$$e^{ikr\cos(\theta-\varphi)} = \sum_{m=-\infty}^{\infty}(i)^m\,J_m(kr)\,e^{im(\theta-\varphi)} \tag{7}$$

The inverse expansion is

$$J_m(kr)e^{im\theta} = \frac{(-i)^m}{2\pi}\int_0^{2\pi} e^{im\varphi-ikr\cos(\theta-\varphi)}\,d\varphi. \tag{8}$$

For $\theta=0$ the two last formulas are equivalent to well known formulas in the theory of Bessel functions [8], namely expansions of plane waves in terms of cylindrical ones and vice versa.

3 Contractions on the interbases expansions for O(4)

Let us consider now the interbases expansions for the hyperspherical functions on the three-dimensional sphere S_3. There are only three (see Fig.2.) elementary notrivial transformations between trees for S_3.

1. First we consider the expansion between trees on Fig.2(a) [6]

$$\Psi_{Jnm}(\theta_1, \theta_2, \theta_3) = \sum_{l=|m|}^{J} (i)^{l-|m|} (-1)^{\frac{J-|m|-n}{2}} C^{l,|m|}_{\frac{J}{2},\frac{|m|+n}{2};\frac{J}{2},\frac{|m|-n}{2}} \Psi_{Jlm}(\theta_1', \theta_2', \theta_3), \quad (9)$$

where

$$\begin{aligned} u_0 &= R\cos\theta_1\cos\theta_2 = R\cos\theta_1' \\ u_1 &= R\cos\theta_1\sin\theta_2 = R\sin\theta_1'\cos\theta_2' \\ u_2 &= R\sin\theta_1\cos\theta_3 = R\sin\theta_1'\sin\theta_2'\cos\theta_3 \\ u_3 &= R\sin\theta_1\sin\theta_3 = R\sin\theta_1'\sin\theta_2'\sin\theta_3, \end{aligned}$$

$C^{l,\gamma}_{a,\alpha;b,\beta}$ - are the Clebsch-Gordan coefficients for the SU(2) group and the corresponding hyperspherical functions have the form:

$$\Psi_{Jnm}(\theta_1,\theta_2,\theta_3) = \frac{\sqrt{2J+2}}{2\pi}\sqrt{\frac{(\frac{J+|m|+|n|}{2})!(\frac{J-|m|-|n|}{2})!}{(\frac{J+|m|-|n|}{2})!(\frac{J-|m|+|n|}{2})!}}\, e^{in\theta_2} e^{im\theta_3}$$

$$\times (\sin\theta_1)^{|m|}(\cos\theta_1)^{|n|}\, P^{(|m|,|n|)}_{\frac{J-|m|-|n|}{2}}(\cos 2\theta_1), \quad (10)$$

$$\Psi_{Jlm}(\theta_1',\theta_2',\theta_3) = \frac{\sqrt{(2J+1)(J+l+1)!(J-l)!}}{2^{l+1}\Gamma(J+\frac{3}{2})}$$

$$\times(\sin\theta_1')^l\, P^{(l+\frac{1}{2},l+\frac{1}{2})}_{J-l}(\cos\theta_1')\, Y_{lm}(\theta_2',\theta_3), \quad (11)$$

where $P^{(\alpha,\beta)}_n(x)$ are Jacobi polynomials. In the contraction limit $R \to \infty$ and

$$\theta_1' \to \frac{r}{R} \qquad \theta_1 \to \frac{\rho}{R}, \qquad \theta_2 \to \frac{x_1}{R}, \qquad J \sim kR, \qquad n \sim k_1 R,$$

where $r = \sqrt{x_1^2+\rho^2} = \sqrt{x_1^2+x_2^2+x_3^2}$, $k = \sqrt{k_1^2+p^2} = \sqrt{k_1^2+k_2^2+k_3^2}$, we have [1]

$$\lim_{R\to\infty} \frac{1}{\sqrt{R}} \Psi_{Jnm}(\theta_1,\theta_2,\theta_3) = \Phi_{kk_1m}(x_1,\rho,\theta_3) = \sqrt{\frac{k}{\pi}}\, J_{|m|}(p\rho)e^{ik_1x_1}\,\frac{e^{im\theta_3}}{\sqrt{2\pi}}, \quad (12)$$

and

$$\lim_{R\to\infty} \frac{1}{R}\Psi_{Jlm}(\theta_1',\theta_2',\theta_3) = \Phi_{klm}(r,\theta_2',\theta_3) = \sqrt{\frac{k}{r}}\, J_{l+\frac{1}{2}}(kr)\, Y_{lm}(\theta_2',\theta_3). \quad (13)$$

Taking the Clebsch-Gordan coefficients in the form

$$C^{l,|m|}_{\frac{J}{2},\frac{|m|+n}{2};\frac{J}{2},\frac{|m|-n}{2}} = (-1)^{\frac{J-|m|-n}{2}} \frac{(J)!}{(|m|)!} \sqrt{\frac{(2l+1)(l+|m|)}{(J-l)!(J+l+1)!(l-|m|)!}}$$

$$\times \sqrt{\frac{(\frac{J+|m|-|n|}{2})!(\frac{J-|m|+|n|}{2})!}{(\frac{J+|m|+|n|}{2})!(\frac{J-|m|-|n|}{2})!}}\ {}_3F_2 \left\{ \begin{matrix} -\frac{J-n-|m|}{2}, -l, l+1, \\ -J, |m|+1. \end{matrix} \Big| 1 \right\} \tag{14}$$

in the contraction limit $R \to \infty$, we get

$$\lim_{R\to\infty} \sqrt{R}(-1)^{\frac{J-|m|-n}{2}} C^{l,|m|}_{\frac{J}{2},\frac{|m|+|n|}{2};\frac{J}{2},\frac{|m|-|n|}{2}} = W^l_{k|m|}(\cos\phi) = \sqrt{\frac{(2l+1)(l+|m|)!}{k(l-|m|)!}}$$

$$\frac{(\sin\phi)^{|m|}}{2^{|m|}|m|!}{}_2F_1\left(-l+|m|, l+|m|+1; |m|+1; \frac{1-\cos\phi}{2}\right) = \sqrt{\frac{2}{k}}\ \mathcal{P}^{|m|}_l(\cos\phi), \tag{15}$$

where

$$\mathcal{P}^{|m|}_l(x) = \sqrt{\frac{(2l+1)(l-|m|)!}{2(l+|m|)!}}\ P^{|m|}_l(x)$$

are the orthonormalized Legendre polynomials and $\cos\phi = p/k$. Thus the interbases expansion (9) transforms to the expansion between the cylindrical and spherical bases for the Helmholtz equation

$$\Phi_{kk_1m}(x_1,\rho,\theta_3) = \sum_{l=|m|}^{\infty} W^l_{k|m|}(\cos\phi)\,\Phi_{klm}(r,\theta'_2,\theta_3), \tag{16}$$

We use the formula

$$\int_0^\pi W^l_{k|m|}(\cos\phi)\,W^{*l'}_{k|m|}(\cos\phi)\,\sin\phi d\phi = 2\,\delta_{l,l'} \tag{17}$$

to obtain the inverse expansion

$$\Phi_{klm}(r,\theta'_2,\theta_3) = \frac{1}{2}\int_0^\pi W^{*l}_{k,|m|}(\cos\phi)\Phi_{kk_1m}(x_1,\rho,\theta_3)\sin\phi\, d\phi. \tag{18}$$

Putting the exact form of the functions (12)-(13) and interbases coefficients (15) into the expansions (16) and (18), we obtain

$$\frac{1}{\sqrt{2\pi}}\,e^{ikr\cos\phi\cos\theta'_2}\,J_{|m|}(p\rho) = \sum_{l=|m|}^{\infty}(i)^{l+m}\frac{1}{\sqrt{kr}}\,J_{l+\frac{1}{2}}(kr)\,\mathcal{P}^{|m|}_l(\cos\phi)\,\mathcal{P}^{|m|}_l(\cos\theta'_2) \tag{19}$$

$$\frac{1}{\sqrt{kr}} J_{l+\frac{1}{2}}(kr)\mathcal{P}_l^{|m|}(\cos\theta_2') = \frac{(-i)^{l+m}}{\sqrt{2\pi}} \int_0^\pi e^{ikr\cos\phi\cos\theta_2'} J_{|m|}(p\rho)\mathcal{P}_l^{|m|}(\cos\phi)\sin\phi\, d\phi$$

The last two expansions coincide with well known formulas in the theory of the Bessel functions [8].

2. The second expansion is:

$$\Psi_{Jnm}(\theta_1,\theta_2,\theta_3) = \sum_{l=|m|}^{J} T^l_{Jnm}\Psi_{Jlm}(\theta_1',\theta_2',\theta_3). \tag{20}$$

where

$$\begin{aligned}
u_0 &= R\cos\theta_1\cos\theta_2 &&= R\cos\theta_1' \\
u_1 &= R\cos\theta_1\sin\theta_2\cos\theta_3 &&= R\sin\theta_1'\cos\theta_2'\cos\theta_3 \\
u_2 &= R\cos\theta_1\sin\theta_2\sin\theta_3 &&= R\sin\theta_1'\cos\theta_2'\sin\theta_3 \\
u_3 &= R\sin\theta_1 &&= R\sin\theta_1'\sin\theta_2',
\end{aligned}$$

[see Fig.2(b)]. The hyperspherical wavefunctions corresponding to these two trees are

$$\Psi_{Jnm}(\theta_1,\theta_2,\theta_3) = \frac{\sqrt{(2J+1)(J+n+1)!(J-n)!}}{2^{n+1}\Gamma(J+\frac{3}{2})}$$

$$\times(\cos\theta_1)^n\, P_{J-n}^{(n+\frac{1}{2},n+\frac{1}{2})}(\sin\theta_1)\, Y_{nm}(\theta_2,\theta_3),$$

$$\Psi_{Jlm}(\theta_1',\theta_2',\theta_3) = \frac{\sqrt{(2J+1)(J+l+1)!(J-l)!}}{2^{l+1}\Gamma(J+\frac{3}{2})}$$

$$\times(\sin\theta_1')^l\, P_{J-l}^{(l+\frac{1}{2},l+\frac{1}{2})}(\cos\theta_1')\, Y_{lm}(\frac{\pi}{2}-\theta_2',\theta_3).$$

The interbases coefficients T^l_{Jnm} have the following form [6]

$$T^l_{jnm} = \left[\frac{1+(-1)^{J-n+l-m}}{2}\right](-1)^{\frac{J-n+l-m}{2}}\frac{2^{l+n-2m}}{|m|!}\frac{\Gamma(\frac{J-n-l+|m|}{2}+1)}{\Gamma(\frac{J+n+l-|m|}{2}+1)}$$

$$\sqrt{\frac{(2l+1)(2n+1)(n+|m|)!(l+|m|)!(J-l)!(J-n)!}{(n-|m|)!(l-|m|)!(J+n+1)!(J+l+1)!}}$$

$${}_4F_3\left\{\begin{matrix} -\frac{n-|m|}{2}, -\frac{n-|m|-1}{2}, -\frac{l-|m|}{2}, -\frac{l-|m|-1}{2} \\ |m|+1, -\frac{J+n+l-|m|}{2}, \frac{J-n-l+|m|}{2}+1 \end{matrix}\middle|\, 1\right\}.$$

In the contraction limit $R \to \infty$ and

$$\theta_1 \sim \frac{x_3}{R}, \qquad \theta_2 \sim \frac{\rho}{R}, \qquad \theta_1' \sim \frac{r}{R}, \qquad J \sim kR, \quad n \sim pR,$$

where $r = \sqrt{\rho^2 + x_3^2}$ and $k = \sqrt{p^2 + k_3^2}$, we obtain [1]

$$\begin{aligned}\lim_{R\to\infty} \frac{(-1)^{-\frac{J-n}{2}}}{\sqrt{R}} \Psi_{Jnm}(\theta_1, \theta_2, \theta_3) &= \Phi_{kpm}(\rho, x_3, \theta_3) \\ &= \sqrt{k}\, J_m(p\rho)\, \frac{e^{im\theta_3}}{\pi} \begin{cases} \cos k_3 x_3, & J-n - \text{even} \\ i \sin k_3 x_3, & J-n - \text{odd} \end{cases}\end{aligned}$$

$$\lim_{R\to\infty} \frac{1}{R} \Psi_{Jlm}(\theta_1', \theta_2', \theta_3) = \Phi_{klm}(r, \theta_2', \theta_3) = \sqrt{\frac{k}{r}}\, J_{l+\frac{1}{2}}(kr)\, Y_{lm}(\frac{\pi}{2} - \theta_2', \theta_3).$$

For the contractions of interbases coefficients T^l_{Jnm} we get

$$\begin{aligned}\lim_{R\to\infty} (-1)^{-\frac{J-n}{2}} \sqrt{R}\, T^l_{Jnm} &= W^l_{k|m|}(\cos\phi) = \frac{(-1)^{\frac{l-m}{2}}}{|m|!} \sqrt{\frac{(2l+1)(l+|m|)!}{2k(l-|m|)!}} \\ &(\cot\phi)^{|m|+\frac{1}{2}} (\sin\phi)^l\, {}_2F_1\left(-\frac{l-|m|}{2}, -\frac{l-|m|-1}{2}; |m|+1; -\cot^2\phi\right) \\ &= (-1)^{\frac{l+|m|}{2}} \sqrt{\frac{2}{k}} (\cot\phi)^{\frac{1}{2}}\, \mathcal{P}_l^{|m|}(\sin\phi), \qquad (21)\end{aligned}$$

where $\cos\phi = p/k$. The interbasis expansion in (20) transforms to the expansion between the cylindrical and spherical bases for the Helmholtz equation

$$\frac{J_m(p\rho)}{\sqrt{\pi}} \left\{ \begin{array}{c} \cos k_3 x_3 \\ \\ i \sin k_3 x_3 \end{array} \right\} = \sum_l \frac{(-1)^{\frac{l-|m|}{2}}}{\sqrt{kr}} J_{l+\frac{1}{2}}(kr) (\cot\phi)^{\frac{1}{2}} \mathcal{P}_l^{|m|}(\sin\phi) \mathcal{P}_l^{|m|}(\sin\theta_2') \quad (22)$$

where the top line on the left hand side corresponds to a summation over $l = |m|, |m|+2, |m|+4, \ldots$ and the bottom one to a summation over $l = |m|+1, |m|+3, \ldots$ on the right hand side. The E(3) expansion (22) is related to the expansion (19) by the substitution $k_1 = k\cos\phi \to k_3$, $x_1 = r\cos\theta_2' \to x_3$, $\phi \to \pi/2 - \phi$ and $\theta_2' \to \pi/2 - \theta_2'$.

3. The third expansion is:

$$\Psi_{Jlm}(\theta_1,\theta_2,\theta_3) = \sum_{n=-(J-|m|)}^{(J-|m|)} (-i)^{l-|m|}\, C^{l,|m|}_{\frac{J}{2},\frac{|m|-n}{2};\frac{J}{2},\frac{|m|+n}{2}}\, \Psi_{Jmn}(\theta_1',\theta_2',\theta_3) \tag{23}$$

where n has the same parity as $J-|m|$ and

$$\begin{aligned} u_0 &= R\cos\theta_1\cos\theta_2\cos\theta_3 = R\cos\theta_1'\cos\theta_3' \\ u_1 &= R\cos\theta_1\cos\theta_2\sin\theta_3 = R\cos\theta_1'\sin\theta_3' \\ u_2 &= R\cos\theta_1\sin\theta_2 \qquad\quad = R\sin\theta_1'\cos\theta_2' \\ u_3 &= R\sin\theta_1 \qquad\qquad\quad = R\sin\theta_1'\sin\theta_2', \end{aligned}$$

[see Fig.2(c)] The corresponding hyperspherical function is:

$$\Psi_{Jlm}(\theta_1,\theta_2,\theta_3) = \frac{\sqrt{(2J+1)(J+l+1)!(J-l)!}}{2^{l+1}\Gamma(J+\frac{3}{2})}$$
$$(\cos\theta_1)^l\, P_{J-l}^{(l+\frac{1}{2},l+\frac{1}{2})}(\sin\theta_1)\, Y_{lm}\left(\frac{\pi}{2}-\theta_2,\theta_3\right),$$

and the wave function $\Psi_{Jmn}(\theta_1',\theta_2',\theta_3')$ is given by (10) (with n replaced by m).

The contraction in this case (see Fig.2(c) and eq. (24) below) will involve 3 quantum numbers J, l and m. Eq. (14)expressinf Clebsch-Gordan coefficients in terms of the ${}_3F_2$ function is not convenient for taking this limit. Instead, we use the following integral representation [6]

$$C^{l,|m|}_{\frac{J}{2},\frac{|m|-n}{2};\frac{J}{2},\frac{|m|+n}{2}} = (-i)^{l-|m|}(-1)^{\frac{J-|m|+n}{2}} \left\{\frac{(l+|m|)!(\frac{J-|m|-n}{2})!(\frac{J-|m|+n}{2})!}{(\frac{J+|m|-n}{2})!(\frac{J+|m|+n}{2})!(l-|m|)!}\right\}^{1/2}$$
$$\frac{\sqrt{(2l+1)(J-l)!(J+l+1)!}}{2^{l+|m|+2}\Gamma(J+3/2)}\,\frac{1}{\sqrt{\pi}}\int_0^{2\pi}(\sin\phi)^{l-|m|}\,P_{J-l}^{(l+\frac{1}{2},l+\frac{1}{2})}(\cos\phi)e^{in\phi}d\phi$$

and the formulas [8]

$$P_n^{(\alpha,\alpha)}(\cos\phi) = \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)n!}$$
$$\times\begin{cases} {}_2F_1(-\frac{n}{2},\frac{n+1}{2}+\alpha;\alpha+1;\sin^2\phi) & n-\text{even}, \\ \cos\phi\,{}_2F_1(-\frac{n-1}{2},\frac{n}{2}+\alpha+1;\alpha+1;\sin^2\phi) & n-\text{odd}. \end{cases}$$

After integrating over ϕ, we obtain a representation of the Clebsch-Gordan coefficients in terms of the hypergeometrical function ${}_4F_3$ (of argument 1):

$$C^{l,|m|}_{\frac{J}{2},\frac{|m|-n}{2};\frac{J}{2},\frac{|m|+n}{2}} = (i)^{l-|m|}(-i)^{\frac{J-|m|+2n}{2}} \frac{\sqrt{2l+1}}{2^{2l}} \sqrt{\frac{(J+l+1)!(\frac{J-|m|-n}{2})!(\frac{J+|m|-n}{2})!}{(J-l)!(\frac{J-|m|+n}{2})!(\frac{J+|m|+n}{2})!}}$$

$$\begin{cases} \dfrac{\sqrt{(l-|m|)!(l+|m|)!}}{\Gamma(\frac{l+|m|-n+2}{2})\Gamma(\frac{l-|m|-n+2}{2})} \dfrac{\Gamma(\frac{J-l+1}{2})}{\Gamma(\frac{J+l+3}{2})} {}_4F_3 \left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2}, \frac{J+l+2}{2}, \frac{l-J}{2} \\ \frac{1}{2}, \frac{l-|m|-n+2}{2}, \frac{l+|m|-n+2}{2} \end{matrix} \middle| 1 \right) & (J-l) - \text{even} \\ \dfrac{-in\sqrt{(l-|m|)!(l+|m|)!}}{\Gamma(\frac{l+|m|-n+3}{2})\Gamma(\frac{l-|m|-n+5}{2})} \dfrac{\Gamma(\frac{J-l}{2})}{\Gamma(\frac{J+l+2}{2})} {}_4F_3 \left(\begin{matrix} \frac{1-n}{2}, \frac{2-n}{2}, \frac{J+l+3}{2}, \frac{l-J+1}{2} \\ \frac{3}{2}, \frac{l-|m|-n+3}{2}, \frac{l+|m|-n+3}{2} \end{matrix} \middle| 1 \right) & (J-l) - \text{odd} \end{cases}$$

(To our knowledge, this expression is new). In the contraction limit $R \to \infty$ and

$$\theta_1 \sim \frac{x_3}{R}, \quad \theta_2 \sim \frac{x_2}{R}, \quad \theta_3 \sim \frac{x_1}{R}, \quad \theta_1' \sim \frac{\rho}{R}, \quad J \sim kR, \quad l \sim pR, \quad m \sim k_1 R. \tag{24}$$

we get

$$\lim_{R\to\infty} (-1)^{-\frac{J-|m|}{2}} \Psi_{Jlm}(\theta_1,\theta_2,\theta_3) = \sqrt{\frac{2kp}{\pi k_2 k_3}} \frac{e^{ik_1x_1}}{\pi}$$

$$\begin{cases} \cos k_2x_2 \cos k_3x_3 & (J-|m|) - \text{even}, \quad (l-|m|) - \text{even} \\ -i\sin k_2x_2 \cos k_3x_3 & (J-|m|) - \text{odd}, \quad (l-|m|) - \text{even}, \\ -i\cos k_2x_2 \sin k_3x_3 & (J-|m|) - \text{even}, \quad (l-|m|) - \text{odd}, \\ -\sin k_2x_2 \sin k_3x_3 & (J-|m|) - \text{odd}, \quad (l-|m|) - \text{odd}. \end{cases} \tag{25}$$

$$\lim_{R\to\infty} (-i)^{l-|m|} 2(-1)^{-\frac{J-|m|}{2}} \sqrt{R}\, C^{l,|m|}_{\frac{J}{2},\frac{|m|-n}{2};\frac{J}{2},\frac{|m|+n}{2}} = \sqrt{\frac{8p}{(k^2-k_1^2)\pi}} (\sin 2\phi)^{-\frac{1}{2}}$$

$$\times \begin{cases} \cos n\phi, & (J-|m|) - \text{even}, \quad (l-|m|) - \text{even}, \\ -i\sin n\phi, & (J-|m|) - \text{odd}, \quad (l-|m|) - \text{even}, \\ -i\sin n\phi, & (J-|m|) - \text{even}, \quad (l-|m|) - \text{odd}, \\ -\cos n\phi, & (J-|m|) - \text{odd}, \quad (l-|m|) - \text{odd}, \end{cases} \tag{26}$$

where $\cos\phi = (p^2 - k_1^2)/(k^2 - k_1^2)$ and $k^2 = p^2 + k_3^2 = k_1^2 + k_2^2 + k_3^2$. Substituting the formulas (12), (25) and (26) into the expansion (23) we have

$$\begin{Bmatrix} \cos k_2x_2 \cos k_3x_3 \\ \sin k_2x_2 \cos k_3x_3 \\ \cos k_2x_2 \sin k_3x_3 \\ \sin k_2x_2 \sin k_3x_3 \end{Bmatrix} = \sum_{n=-\infty}^{\infty} \begin{Bmatrix} \cos n\phi \\ \sin n\phi \\ \sin n\phi \\ \cos n\phi \end{Bmatrix} J_{|n|}(q\rho) e^{in\theta_2'}$$

where

$$\tan\theta_2' := \frac{x_3}{x_2}, \qquad q^2 = k_2^2 + k_3^2, \qquad \rho^2 = x_2^2 + x_3^2, \qquad \cos^2\phi = \frac{k_2^2}{k_2^2 + k_3^2}$$

Thus the interbasis expansion (23) transforms to the expansion between Cartesian and cylindrical bases for the Helmholtz equation on E_2.

4 Conclusions

In this article we rectricted ourselves to subgroup type coordinates only and moreover to the lowest dimensional spheres S_2 and S_3. Two earlier articles were devoted to contractions of separated basis functions that correspond to nonsubgroup type coordinates, in particular elliptic coordinates on S_2 and on the hyperboloid H_2 [9, 10]. It would also be possible to obtain interbases expansions for other types of bases, though so far this has not been done.

We mention that though the two different trees on S_2 are topologically equivalent, the contraction to E_2 destroys the symmetry between them. Indeed, one system of coordinates on S_2 goes into polar coordinates on E_2, the other into Cartesian ones. Correspondingly, contracted interbasis expansions give relations between plane and cylindrical waves namely eq. (7) and (8). These expansion formulas are of course well known.

Five types of trees exist for S_3. Only two of them correspond to mutually nonisomorphic subgroup chains. After the contraction, we obtain expansions between functions separated in spherical, cylindrical and Cartesian coordinates, respectively. We have presented only 3 of the 10 possible interbasis expansions and their contractions. The others are either obtained by composing the 3 elementary ones, or correspond to transitions within an O(3) subgroup. The transition between spherical and Cartesian basis in E(3) is obtained by composing the Cartesian to cylindrical and cylindrical to spherical ones.

An article on constractions of interbases expansions on S_n for n arbitrary, is in progress.

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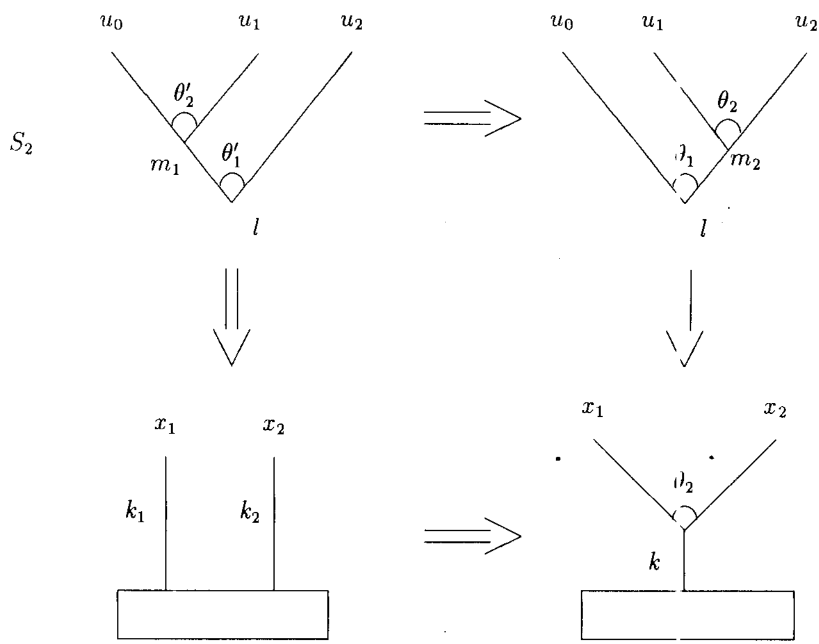

Figure 1. Interbasis expansions contracted from $O(3)$ to $E(2)$.

Figure 2. Elementary interbasis expansions contracted from $O(4)$ to $E(3)$.

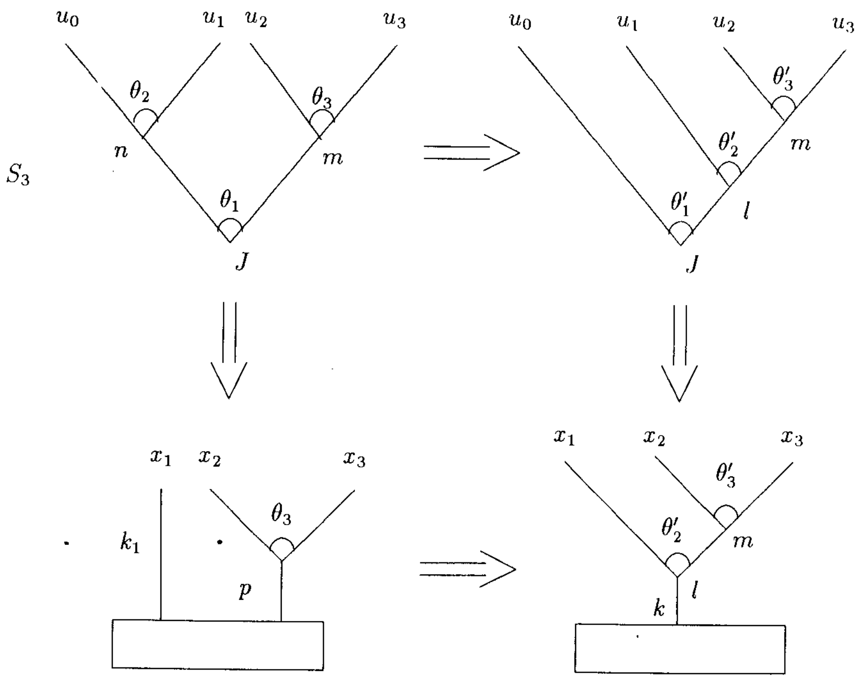

Figure 2a.

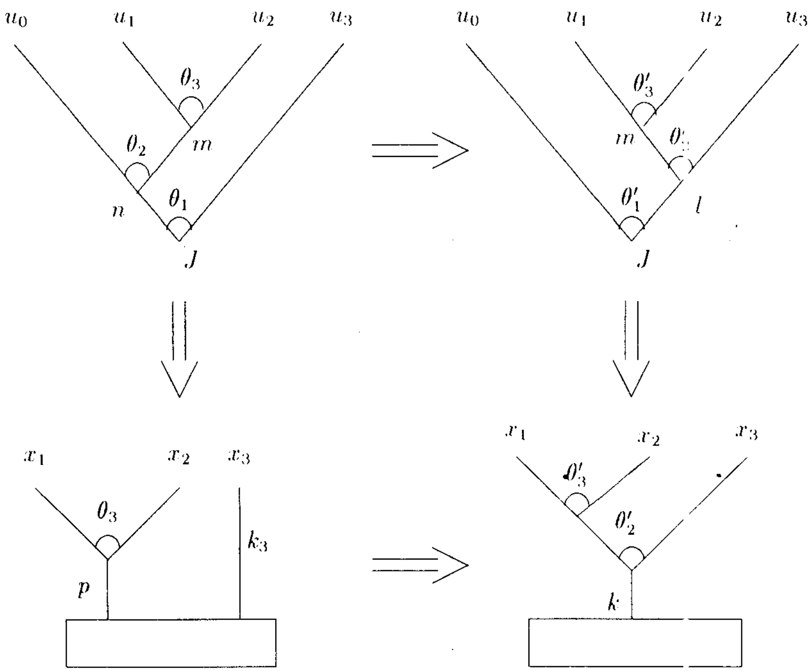
u_0
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Figure 2b.

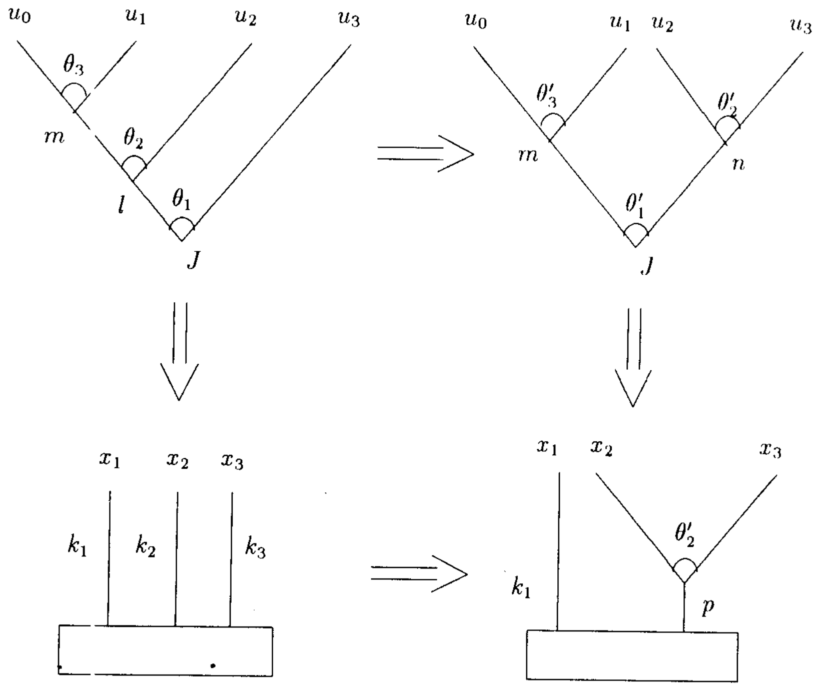

Figure 2c.

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THE CONSTRUCTION OF DISCRETE TIME INTEGRABLE SYSTEMS

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In the following *CT* means *continuous time*, *DT* means *discrete time*, *M* means *mechanics* and $\underset{c}{=}$ refers to any equality holding modulo equations of motion.

There is no space here to discuss our motivation for constructing discrete time mechanics. This may be found in [1] and references therein. Suffice it to say that continuous time and space have long been regarded with suspicion and that theorists have considered various alternatives, usually proposing that at short distances ($\simeq$ large momenta) interesting physics may occur. Our interest here is in the consequences of taking temporal discretisation literally, not regarding it as an approximation to continuous time, but as a fundamental aspect of a novel form of mechanics. We will review the principles behind our *DT* mechanics [1], which differs from the approach of various other workers in that only our time is discretised.

Turning to the subject area of this Workshop, Arnold's definition of Liouville integrability states that for an integrable *CT* system with r degrees of freedom there exist r sufficiently regular, functionally independent functions $F^i(\mathbf{p},\mathbf{q})$ which are in involution, i.e. their pair-wise Poisson brackets vanish, i.e. $\{F^i, F^j\}_{PB} = 0$. One of the F^i is the Hamiltonian of the system. These Poisson brackets are local in time in *CTM*, and they (or Dirac brackets in the case of second class constraints) correspond to equal time commutators in canonical quantisation. We discuss this later in the context of temporal non-locality, which is an inherent feature of *DTM*.

In the field of integrable systems (a tiny subset of all dynamical systems) temporal discretisation has been discussed by several authors, but no general principles have been stated. This leads to the questions: *can we find different discretisations of the same CT system?* and: *is there a 'best' or 'proper' way to discretise?* The answers are *yes* and *possibly*. Our work has concentrated on the second question.

In our *DT* dynamics, systems are viewed at a sequence of discrete times $t_M < t_{M+1} < \ldots < t_{N-1} < t_N$ called *nodes*. *Links* are the intervals between the nodes. Dynamical variables come in two sorts: *node variables* such as $\psi_n^\alpha \equiv \psi^\alpha(t_n)$, defined on the nodes only and *link variables* ϕ_n^β defined on the links connecting the nodes. We distinguish two sorts of *DTM*. *Stroboscopic mechanics* (*SCM*) is just *CTM* viewed at discrete times (see below). In this case all trajectories are continuous in time and it is meaningful to discuss intermediate times $t_i < t < t_{i+1}$ if we so choose. The other sort is genuine *DTM*, where the *CT* limit need not exist for certain trajectories and in principle it is meaningless to discuss intermediate times.

Another issue concerns the intervals $T_n \equiv t_{n+1} - t_n$. These may all be fixed independently of the dynamics (*rigid DTM*) or they may be determined by the dynamics (*non-rigid DTM*). Examples of rigid *DTM* are the discrete Toda lattice discussed below and our own work on *DTM* [1]. Examples of non-rigid *DTM* are T.D. Lee's *DTM* (below) and Moser-Veselov elliptic billiards (below).

An $\mathbf{r}^{th}$-order function is any term or expression of the form $f(\mathbf{q}_n, \mathbf{q}_{n+1}, \ldots, \mathbf{q}_{n+r})$. We formulate our equations of motion using a first-order *system function* (*DT* 'Lagrangian') $\Lambda^n \equiv \Lambda\left(\psi_n^\alpha, \psi_{n+1}^\alpha, \phi_n^\beta\right)$. A number of discretisations of integrable systems can be based on such functions. Note that an asymmetry between node and link variables occurs at this level in our approach, and that the system function Λ^n is much more like a Hamilton's principal function than a Lagrangian, being non-local in time. Starting with the action sum $A_D = \sum_{n=M}^{N-1} \Lambda^n$, the equations of motion follow from Hamilton's principle extended to *DT*, $\delta A_D \underset{c}{=} 0$, giving

$$\frac{\partial}{\partial \psi_n^\alpha}\left\{\Lambda^n + \Lambda^{n-1}\right\} \underset{c}{=} 0; \quad M < n < N, \quad \frac{\partial}{\partial \phi_n^\beta}\Lambda^n \underset{c}{=} 0, \quad M \leq n < N. \tag{1}$$

In the case of non-rigid *DTM* we have the additional equations

$$\frac{\partial}{\partial T_n}\Lambda^n \underset{c}{=} 0, \qquad M \leq n < N. \tag{2}$$

The following are examples of discretised integrable systems for which the system function approach works.

The Toda lattice

The *CT model* discussed by *Symes* [2] is given by the Lagrangian

$$L = \tfrac{1}{2}\sum_a (\dot{x}^a)^2 - \sum_\alpha e^{x^{a+1} - x^a}. \tag{3}$$

In 1995 *Suris* [3] gave two different discretisations based on system functions. The first has system function

$$\Lambda = \sum \frac{e^{Ux-x} - 1 + x - Ux}{T} - T\sum e^{x - USx} \tag{4}$$

with equation of motion

$$\frac{Ux - 2x + \bar{U}x}{T^2} \underset{c}{=} \frac{1}{T^2}\ln\left\{\frac{1 - T^2\exp\left(\bar{U}x - S\bar{U}x\right)}{1 - T^2\exp\left(SUx - Ux\right)}\right\} \tag{5}$$

whereas the other has system function

$$\Lambda = \sum \frac{\left(Ux - x\right)^2}{2T} - \frac{1}{T}\sum\int_0^{x - \bar{S}Ux} \ln\left\{1 + T^2 e^\eta\right\} d\eta \tag{6}$$

with equation of motion

$$\frac{Ux - 2x + \bar{U}x}{T^2} \underset{c}{=} \frac{1}{T^2}\ln\left\{\frac{1 + T^2\exp\left(SUx - x\right)}{1 + T^2\exp\left(x - \bar{U}Sx\right)}\right\}. \tag{7}$$

The operators U and S are the temporal and spatial step operators respectively with $\bar{U}$, $\bar{S}$ their respective inverses, and we use the notation $x_n \to x$, $x_{n+1} \to Ux$, $x_{n-1} \to \bar{U}x$ and suppress the temporal index n. This example shows that temporal discretisation is not a unique process, unless some new principles are employed.

The Calogero-Moser model

A discretisation was given by *Nijhoff & Pang* [4] with the system function

$$\Lambda^n = \sum_{a,b} \ln |x^a_{n+1} - x^b_n| - \sum_{a \neq b} \ln |x^a_n - x^b_n|. \tag{8}$$

DT Noether Theorem

The DT analogue of Noether's theorem is readily found. If the system function is invariant to the infinitesimal transformation $\psi^\alpha_n \to \psi^\alpha_n + \delta\psi^\alpha_n$, $\phi^\beta_n \to \phi^\beta_n + \delta\phi^\beta_n$ then the first order quantity

$$C^n \equiv \sum_\alpha \delta\psi^\alpha_n \partial\Lambda^n / \partial\psi^\alpha_n \tag{9}$$

is an invariant of the motion. Generally, continuous symmetries found in CTM will occur in DTM with the exception of translation in time. Therefore, we expect that, for a given integrable non-relativistic CT system with r invariants, our discretisation approach should generate at least $r-1$ DT analogues. It is the non-existence of a Hamiltonian which makes the discretisation of integrable systems interesting.

There are two possible ways around this problem. First, Logan suggested [5] that if an infinitesimal transformation gave a change in the system function of the form $\delta\Lambda^n = v_{n+1} - v_n$ then we could construct an associated invariant (called a *Logan invariant* by us) possibly analogous to the CT Hamiltonian. Unfortunately, finding such a transformation may be near impossible for highly non-linear systems. Second, Lee's variant of non-rigid DTM gives us a method of generating an invariant which, from Dirac's reparametrisation approach discussed below, is a possible analogue of the Hamiltonian.

To illustrate the problem with the Hamiltonian, consider the rigid, regular DT system function

$$\Lambda = \frac{(Ux - x)^2}{2T} - \tfrac{1}{2}T\,(UV + V)\,, \quad V = V(x). \tag{10}$$

The equation of motion, which is explicit in Ux, is

$$\frac{Ux - 2x + \bar{U}x}{T^2} \underset{c}{=} -\partial_x V. \tag{11}$$

The analogue of canonical momentum may be defined via $p \underset{c}{=} -\partial_x \Lambda$, $Up \underset{c}{=} \partial_{Ux}\Lambda$ and now we define the "Hamiltonian" $\tilde{H}\,(Up, x) \equiv Up\,(Ux - x) - \Lambda$. Then we find

$$\frac{\partial \tilde{H}}{\partial Up} \underset{c}{=} Ux - x, \quad \frac{\partial \tilde{H}}{\partial x} \underset{c}{=} p - Up, \tag{12}$$

from which we deduce that the two-form $dx \wedge dp$ is conserved, viz $U(dx \wedge dp) \underset{c}{=} dx \wedge dp$. Unfortunately, the $\tilde{H}$ is not conserved in general, a fact rediscovered by many authors. The origin of this problem (if it be regarded as a problem) is two-fold; first, there is no continuous time translation in genuine DT mechanics, so the concept of a generator of such a transformation is meaningless, and second, the step operator U is not a derivation, i.e., it does not obey the Leibnitz rule of differentiation of products.

Energy and Dirac's reparametrisation

Before we discuss Lee's *DTM,* consider Dirac's approach [6] to energy. Given a CT Lagrangian $L = L(\mathbf{q}, \dot{\mathbf{q}}, t)$ Dirac rewrote the action integral in the generally covariant form $A_{if} = \int_0^1 d\lambda\, \tilde{L}$, where $t \rightarrow t(\lambda)$, $t' \equiv dt/d\lambda > 0$, $t_i = t(0)$, $t_f = t(1)$, $\tilde{L} \equiv t' L(\tilde{\mathbf{q}}, \tilde{\mathbf{q}}'/t', t)$ with $\tilde{\mathbf{q}}(\lambda) \equiv \mathbf{q}(t)$. With t treated as a dynamical variable and Hamilton's action principle extended to variations of t, then in addition to the usual Euler-Lagrange equations of motion for the q^i we find the equation of motion

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \dot{\mathbf{q}} - L\right\} \underset{c}{=} -\frac{\partial L}{\partial t}, \tag{13}$$

which is the usual rate of change of energy equation. When there is no explicit time dependence in L we have conservation of energy. We should ask, is there some analogue of this approach in *DTM*? The answer is *yes*.

In rigid *DTM* there are two related fundamental questions connected with energy; i) is there a DT analogue of the Hamiltonian, and ii) if so, how to construct it? In *CTM* H is the generator of continuous translations in time and we construct H directly from L using the rule $H \equiv \mathbf{p} \cdot \dot{\mathbf{q}} - L$. In *DTM* we simply don't have this. Yet if *DTM* is in some sense a discretisation of *CTM*, why should we not have a DT analogue of H?

Ad-hoc discretisations may or may not have invariants corresponding to the CT Hamiltonian. Integrable rigid discretisations such as the DT Toda lattice discussed above were constructed with a Poisson bracket approach in mind, and because of their very special construction have a conserved 'Hamiltonian' (or rather Hamiltonians, because it is clear there is no uniqueness in the discretisation). In general however, there will be no guarantee of this.

A systematic way forward is to discretise via the non-rigid approach of T.D. Lee, which generates a DT analogue of the Hamiltonian automatically, which for autonomous systems will be a dynamical invariant. However, the Poisson structure may not then be so evident.

Before we discuss Lee's *DTM*, we discuss the concept of non-locality in time in *CTM*, and the possibility of constructing non-local in time brackets.

Stroboscopic mechanics

If a system is integrable in CT mechanics it will certainly be integrable in *SCM*. This proves that we can always find examples of integrable DT systems. However, not all DT systems are *SCM* systems, so it's a good idea to understand how Poisson brackets (which are local in time) can be compatible with *SCM* (which is non-local in time).

When we consider **non-local in time** (*NLT*) quantities in *CTM,* it is surprising but true that such objects occur naturally, although some may be not so familiar. A good example is the action integral $A[\mathbf{q}] \equiv \int_{t_i}^{t_f} dt\, L$ itself. Another example is Hamilton's principal function (*HPF*), defined as the action integral evaluated on a true or classical trajectory. We now consider the construction of *NLT* brackets in *CTM.*

Peierl's Poisson brackets (*PPB*)

Poisson brackets occur in Arnold's statement of integrability in *CTM.* The *PPB* [7] allows us to define an *NLT* bracket which reduces in the limit of equal times to the conventional *PB.* Moreover, it is formulated in configuration space and not in phase space, so it is an inherently useful object to use in *SCM.*

Consider a configuration space $\mathbf{q} \equiv \{q^1, q^2, \ldots, q^n\}$. Define the functional derivatives

$$\frac{\delta q^j(t_2)}{\delta q^i(t_1)} = \delta_{ij}\delta(t_1 - t_2). \tag{14}$$

Given a first order Lagrangian $L = L(\mathbf{q}, \dot{\mathbf{q}}, t)$ the action integral $A[\mathbf{q}]$ is just the integral $A[\mathbf{q}] \equiv \int_{t_i}^{t_f dt} L|_q$ over a particular trajectory $\mathbf{q}$. Then we define

$$A_i \equiv \frac{\delta}{\delta q^i(t_i)} A[\mathbf{q}], \quad A_{ij} \equiv \frac{\delta^2}{\delta q^i(t_i)\,\delta q^j(t_j)} A[\mathbf{q}], \tag{15}$$

etc. using deWitt's compact notation [8]. The second order Euler-Lagrange equations of motion are derived from the principle $S_i \equiv A_i[\mathbf{q}_c] \underset{c}{=} 0$, where the $\mathbf{q}_c$ is the true or classical trajectory.

The *retarded Green's function* $G_{ij}^{(-)}(t_1, t_2)$ satisfies the equation

$$\int dt \sum_j S_{ij}(t_1, t)\, G_{jk}^{(-)}(t, t_2) = -\delta_{ij}\delta(t_1 - t_2). \tag{16}$$

and the condition $G_{ij}^{(-)}(t_1, t_2) = 0, t_1 < t_2$. The *advanced Green's function* $G_{ij}^{(+)}$ $(t_1 - t_2)$ is defined by $G_{ij}^{(+)}(t_1, t_2) \equiv G_{ji}^{(-)}(t_2, t_1)$ and then the *Peierls propagator kernel* $G_{ij}(t_1 - t_2)$ is defined by

$$G_{ij}(t_1, t_2) \equiv G_{ij}^{(+)}(t_1, t_2) - G_{ij}^{(-)}(t_1, t_2). \tag{17}$$

Then the *PPB* $\{f, g\}_{PPB}$ of two functions f, g on configuration space is defined as

$$\{f, g\}_{PPB} \equiv \int dt_1 dt_2 \frac{\delta f}{\delta q^i(t_1)} G_{ij}(t_1, t_2) \frac{\delta g}{\delta q^j(t_2)}. \tag{18}$$

In a more condensed notation [8] we may write this in the form as $\{f, g\}_{PPB} = \delta_1 f\, G_{12}\, \delta_2 g$, where the repeated index implies appropriate integration and summation. The *PPB* provides a natural way to construct brackets which are *NLT.* We find for example

$$\left\{q^i(t_1), q^j(t_2)\right\}_{PPB} = G^{ij}(t_1 - t_2). \tag{19}$$

This is an important result, because it tells us that in SCM there is a concept of a NLT bracket between canonical coordinates.

For functions f, g of the q^i and their derivatives evaluated at definite times, we find

$$\lim_{t_2 \to t_1} \{f(t_1), g(t_2)\}_{PPB} = \{f, g\}_{PB}, \tag{20}$$

where the Poisson bracket is evaluated at equal time t_1. In [8] deWitt discusses the relationship between group invariants, the dynamical equation of motion and the Jacobi identity in some detail. He also shows that the equation of motion can be cast in the NLT form

$$\dot{q}^i(t_1) \underset{c}{=} \left\{q^i(t_1), H(t_2)\right\}_{PPB}, \tag{21}$$

and proves that the Jacobi identity using the Peierls' bracket is satisfied for any three of the group invariants. This is the link between CTM Liouville integrability and SCM, which in turn gives a link to genuine DTM.

example: the CT harmonic oscillator

Given the Lagrangian

$$L = \tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}m\omega^2 x^2, \tag{22}$$

then the Peierls propagator kernel is just $G(t) = -\sin(\omega t)/m\omega$. Hence we find the NLT bracket

$$\{x(t_1), x(t_2)\}_{PPB} = -\frac{1}{m\omega}\sin[\omega(t_1 - t_2)]. \tag{23}$$

From the Lagrangian, we define the conjugate momentum $p \equiv \partial L/\partial\dot{x} = m\dot{x}$ and so we find

$$\{x(t_1), p(t_2)\}_{PPB} = \cos[\omega(t_1 - t_2)], \tag{24}$$

which reduces in the CT limit to the standard PB $\{x, p\}_{PB} = 1$ Now a general solution of the equation of motion is

$$x_1 \underset{c}{=} -\Delta(1-2)\overleftrightarrow{\partial_2} x_2 = \partial_2\Delta(1-2)x_2 - \Delta(1-2)\dot{x}_2 \tag{25}$$

where Δ is the Schwinger function given by $\Delta(t) = mG(t)$, with

$$\lim_{2 \to 1} \partial_2\Delta_{1-2} = 1, \qquad \lim_{2 \to 1} \Delta_{12} = 0. \tag{26}$$

A canonical quantisation of this system gives the rule

$$[\hat{x}(t_1), \hat{x}(t_2)] = -\frac{i\hbar}{m\omega}\sin[\omega(t_1 - t_2)]. \tag{27}$$

Now consider a DT analogue of this system. Experience shows [9] that the appropriate system function corresponding to (22) is of the form

$$\Lambda^n \equiv \tfrac{1}{2}\alpha\left(x_{n+1}^2 + x_n^2\right) - \beta x_{n+1}x_n, \quad \beta \neq 0, \tag{28}$$

where α and β are real constants, with equation of motion

$$x_{n+1} \underset{c}{=} 2\eta x_n - x_{n-1}, \qquad \eta \equiv \alpha/\beta. \tag{29}$$

For quantisation, we may use the Schwinger action principle extended to DT [9], i.e. we take

$$\delta\langle\phi, N|\psi, M\rangle = i\hbar\langle\phi, N|\delta\hat{A}_{NM}|\psi, M\rangle, \tag{30}$$

and then we find

$$[\hat{x}_{n+1}, \hat{x}_n] = -ih/\beta, \tag{31}$$

which is the DT analogue of (27). For an example from field theory, we find the free scalar field commutator [10]

$$[\hat{\varphi}_{n+1}(\mathbf{x}), \hat{\varphi}_n(\mathbf{y})] = \frac{-6i}{4\pi T|\mathbf{x}-\mathbf{y}|}e^{-\sqrt{\mu^2+6/T^2}|\mathbf{x}-\mathbf{y}|}. \tag{32}$$

T. D. Lee's DT mechanics

In 1983 T.D. Lee [11] considered the discretisation of

$$A_C \equiv \int_{t_i}^{t_f} L\,dt, \qquad L = \tfrac{m}{2}\dot{\mathbf{r}}^2 - V(\mathbf{r}): \tag{33}$$

in the form

$$A_C \to A_D \equiv \sum_{n=M}^{N-1}\Lambda^n + u\left\{\sum_{n=M}^{N-1}T_n - T_{MN}\right\} \tag{34}$$

where

$$\Lambda^n \equiv \tfrac{1}{2}\frac{m(\mathbf{r}_{n+1}-\mathbf{r}_n)^2}{T_n} - \tfrac{1}{2}T_n[V(\mathbf{r}_{n+1}) + V(\mathbf{r}_n)]. \tag{35}$$

Here the T_n are considered dynamical degrees of freedom and u is a Lagrange multiplier. Then the equations of motion are

$$m\frac{\mathbf{r}_{n+1}-\mathbf{r}_n}{T_n} + m\frac{\mathbf{r}_{n-1}-\mathbf{r}_n}{T_{n-1}} \underset{c}{=} -\tfrac{1}{2}(T_n + T_{n-1})\frac{\partial}{\partial\mathbf{r}_n}V(\mathbf{r}_n), \tag{36}$$

$$\tfrac{1}{2}\frac{m(\mathbf{r}_{n+1}-\mathbf{r}_n)^2}{T_n^2} + \tfrac{1}{2}[V(\mathbf{r}_{n+1}) + V(\mathbf{r}_n)] \underset{c}{=} u, \qquad \sum_{n=M}^{N-1}T_n \underset{c}{=} T_{MN}. \tag{37}$$

By inspection, equations (37) correspond to Dirac's energy equation (13) for an autonomous system. For a rigid regular case we would have $T_n = T$, giving only the equations

$$m\frac{\mathbf{r}_{n+1} - 2\mathbf{r}_n + \mathbf{r}_{n-1}}{T^2} \underset{c}{=} -\frac{\partial}{\partial\mathbf{r}_n}V(\mathbf{r}_n), \tag{38}$$

with no obvious energy invariant.

An interesting point is that for the analogue of a path integral quantisation of a non-rigid model we might need to consider the number of links to be variable, from one to infinity. We would not naturally expect this to be the case for rigid *DTM*.

Moser-Veselov Elliptic billiards

We now discuss a simple system [12] for which Lee's approach works. Consider a configuration space $\mathbf{x} \equiv (x^1, x^2, \ldots, x^n)$ with action sum $A[\mathbf{x}] \equiv \sum_{n=M}^{N-} |\mathbf{x}_{n+1} - \mathbf{x}_n|$ and constraint $\mathbf{x}_n^T B \mathbf{x}_n = 1$, where B is a symmetric positive definite matrix. Then the equations of motion are

$$\mathbf{x}_{n+1} - \mathbf{x}_n \underset{c}{=} \mu_n \mathbf{y}_n, \qquad \mathbf{y}_n - \mathbf{y}_{n-1} \underset{c}{=} \nu_n B \mathbf{x}_n \tag{39}$$

$$\mathbf{y}_n \equiv \frac{\mathbf{x}_{n+1} - \mathbf{x}_n}{|\mathbf{x}_{n+1} - \mathbf{x}_n|}, \quad \mu_n = -2\frac{\mathbf{x}_n^T B \mathbf{y}_n}{\mathbf{y}_n^T B \mathbf{y}_n}, \quad \nu_n = -2\frac{\mathbf{y}_{n-1} B \mathbf{x}_n}{\mathbf{x}_n^T B^2 \mathbf{x}_n}. \tag{40}$$

This can be rewritten as an example of Lee's *DTM* as follows. Between the events $(\mathbf{x}_n, t_n)$ and $(\mathbf{x}_{n+1}, t_{n+1})$ the *CT HPF* S^n is based on free particle motion and given by $S^n = m(\mathbf{x}_{n+1} - \mathbf{x}_n)^2 / 2T_n$. Then the action sum is

$$A_{MN} = \sum_{n=M}^{N-1} S^n + \sum_{n=M}^{N} \tfrac{1}{2}\lambda\left(\mathbf{x}_n^T B \mathbf{x}_n - 1\right) + u\left\{\sum_{n=M}^{N-1} T_n - T_{MN}\right\}. \tag{41}$$

and hence the equations of motion are

$$\frac{m(\mathbf{x}_{n+1} - \mathbf{x}_n)}{T_n} - \frac{m(\mathbf{x}_n - \mathbf{x}_n)}{T_{n-1}} \underset{c}{=} \lambda_n B \mathbf{x}_n, \quad M < n < N \tag{42}$$

$$\frac{m(\mathbf{x}_{n+1} - \mathbf{x}_n)^2}{2T_n^2} \underset{c}{=} =u, \quad M \le n < N \tag{43}$$

$$\sum_{n=M}^{N-1} T_n \underset{c}{=} T_{MN}, \quad \mathbf{x}_n^T B \mathbf{x}_n \underset{c}{=} 1, \quad M \le n \le N. \tag{44}$$

Then we find $\mu_n = |\mathbf{x}_{n+1} - \mathbf{x}_n|$, $\mu^2 = 2T_n^2 u/m, \lambda_n = \mu_n m \nu_n / T_n$, giving complete agreement between the Moser-Veselov equations (40)and our results (44). This suggests that at least for some discretised integrable systems, there may be value in pursuing Lee's approach.

Virtual paths

Even if we use Lee's approach, the construction of a system function remains a problem. We have proposed an approach based on the concept of *virtual path* [9]. The basic idea is that if *DT* is fundamental, then there is a level below which there is no meaning to time. So, take a given link $[t_n, t_{n+1}]$, and consider the variables associated with it. For a node variable ψ, construct a suitable *virtual path* running from ψ_n to ψ_{n+1}. For example, for point particles the virtual path from n to $n+1$ might be $\psi_\lambda \equiv \lambda\psi_{n+1} + \bar{\lambda}\psi_n$, $\bar{\lambda} \equiv 1 - \lambda$. For link variables, the virtual path may just the variable itself. The choice of virtual path is really part of our modelling of the system and different virtual paths will give different system functions. Then our virtual path prescription is

$$L\left(\dot{\psi},\psi\right) \rightarrow \Lambda^n = T_n\langle L\left(D_\lambda\psi_\lambda,\psi_\lambda\right)\rangle, \tag{45}$$

where $\langle f_\lambda\rangle \equiv \int_0^1 d\lambda f_\lambda$ and $D_\lambda \equiv \frac{\partial_\lambda}{T_n}$. For example, the standard Lagrangian $L = \frac{1}{2}m\dot{x}^2 - V(x)$ goes over to the system function

$$\Lambda^n = \frac{m\left(x_{n+1}-x_n\right)^2}{T_n} - T_n\int_0^1 d\lambda V\left(\lambda x_{n+1}+\bar{\lambda}x_n\right). \tag{46}$$

The point here is that the integration over the virtual path parameter λ leaves us with a system function depending only on the node and link variables. The choice of virtual path will influence the details of this function, however. For example, gauge invariance can be readily incorporated in this approach, but the virtual paths become non-linear in this case.

Summary

Our proposed strategy for finding integrable discretisations would be the following: 1) find an integrable CT model to be discretised; 2) rewrite the Lagrangian as a non-rigid system function using appropriate virtual paths; 3) find all the continuous symmetries and hence determine the DT Noether invariants; 4) vary with respect to the T_n's (using Lagrange multipliers) to get Lee's analogue of the conserved energy; 5) investigate the symplectic structure; 6) quantise via the DT Schwinger action principle.

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QUANTUM AND CLASSICAL STOCHASTIC DYNAMICS: EXACTLY SOLVABLE MODELS BY SUPERSYMMETRIC METHODS

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Abstract

A supersymmetric method for the construction of so-called conditionally exactly solvable quantum systems is reviewed and extended to classical stochastic dynamical systems characterized by a Fokker-Planck equation with drift. A class of drift-potentials on the real line as well as on the half line is constructed for which the associated Fokker-Planck equation can be solved exactly. Explicit drift potentials, which describe mono-, bi-, meta-or unstable systems, are constructed and their decay rates and modes are given in closed form.

1 Introduction.

Many physical phenomena of nature are characterized by some basic differential equations. For example, quantum-mechanical phenomena are described by Schrödinger's equation, which dictates the dynamics of some quantum system represented by a Hamilton operator. One is therefore primarily interested in finding all eigenvalues and eigenstates of such Hamiltonians. As a consequence, finding a large class of, in this sense, analytically exactly solvable quantum systems is an important goal and this search has already been initiated by Schrödinger using the so-called factorization method [1, 2]. This factorization method is basically equivalent to Darboux's method [3] already invented 1882 and applied to Sturm-Liouville problems. Recently, these methods have attracted new attention [4], in particular in connection with supersymmetric (SUSY) quantum mechanics [5]. For a recent approach and further references see, for example, [6, 7, 8].

Somehow unnoticed by those working on the quantum mechanical problem the same basic ideas have been utilized in constructing exactly solvable classical stochastic systems described by a Fokker-Planck equation [9, 10, 11]. In fact, the Fokker-Planck equation can be put into the form of an imaginary-time Schrödinger equation exhibiting SUSY. Therefore, exactly solvable SUSY quantum Hamiltonians also provide exactly solvable Fokker-Planck-type systems. Whereas until now these methods have been applied to mono- and bistable systems [9, 10, 11], they correspond to a situation with unbroken SUSY, we present here also examples with broken SUSY giving rise to meta- or unstable drift potentials.

In the next two sections we briefly review the basic concepts of SUSY quantum mechanics and its close connection with the Fokker-Planck equation [5 12]. Section 4 reviews our recent approach [6, 7, 8] towards a construction of the most general class of SUSY partners of a given quantum mechanical Hamiltonian. This approach is applied to the Fokker-Planck equation and allows to construct new drift potentials with known decay rates and modes. Two examples are discussed in some detail. The first one is related to the linear harmonic oscillator having unbroken SUSY and leads to bistable double-well or monostable single-well drift potentials already found by Hongler and Zheng [9]. The second example is the class related to the radial harmonic oscillator with broken SUSY. Here the drift potential is either metastable or unstable.

2 Supersymmetric quantum mechanics.

To begin with, let us briefly review the basics of SUSY quantum mechanics or, to be more precise, Witten's model of SUSY quantum mechanics [5]. This model consists of a pair of standard Schrödinger Hamiltonians (units are such that $\hbar = m = 1$)

$$H_\pm = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\pm(x)\,, \qquad V_\pm(x) = \frac{1}{2}\Big(W^2(x) \pm W'(x)\Big)\,. \tag{1}$$

Evidently, this pair is completely characterized by the so-called SUSY potential W, which is assumed to be real-valued and an at least once differentiable function inside the configuration space. This configuration space will be either the real line $\mathbb{R}$ or the positive half line $\mathbb{R}^+$. Thus the above Hamiltonians act on a suitable linear space of square-integrable functions over $\mathbb{R}$ and $\mathbb{R}^+$, respectively. With the help of the supercharge operators

$$A = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x} + W(x)\right)\,, \quad A^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x} + W(x)\right) \tag{2}$$

the above pair of SUSY-partner Hamiltonians factorizes, $H_+ = AA^\dagger \geq 0$, $H_- = A^\dagger A \geq 0$, and obviously obeys the intertwining relations $AH_- = H_+A$ and $H_-A^\dagger = A^\dagger H_+$. As a consequence H_+ and H_- are essentially isospectral. In other words, their strictly positive energy eigenvalues coincide and the corresponding eigenstates are related via the supercharges. However, there may exist an additional vanishing eigenvalue for one of these Hamiltonians. In this case SUSY is said to be unbroken and by standard convention [5] this additional eigenvalue is assumed to belong to H_-. If such a state does not exist SUSY is said to be broken. To summarize, for unbroken SUSY we have

$$\begin{aligned} &E_0^- = 0\,, \qquad E_{n+1}^- = E_n^+ > 0\,, \qquad \psi_0^-(x) = \psi_0^-(0)\exp\left\{-\int_0^x \mathrm{d}z\, W(z)\right\}\,, \\ &\psi_{n+1}^- = (E_n^+)^{-1/2}A^\dagger\psi_n^+\,, \qquad \psi_n^+ = (E_{n+1}^-)^{-1/2}A\psi_{n+1}^-\,, \end{aligned} \tag{3}$$

whereas for broken SUSY these relations read

$$E_n^- = E_n^+ > 0\ , \qquad \psi_n^- = (E_n^+)^{-1/2} A^\dagger \psi_n^+\ , \qquad \psi_n^+ = (E_n^-)^{-1/2} A \psi_n^-\ . \tag{4}$$

Here we have denoted the eigenfunctions and eigenvalues of $H_\pm$ by $\psi_n^\pm$ and $E_n^\pm$, respectively. That is,

$$H_\pm \psi_n^\pm = E_n^\pm \psi_n^\pm\ , \qquad n = 0, 1, 2, \ldots, \tag{5}$$

and we also note that throughout this paper we will consider, without loss of generality, quantum systems with a purely discrete spectrum.

We conclude this section by noting that for broken as well as unbroken SUSY one can obtain the complete spectral information of one Hamiltonian, say H_-, if the eigenvalues and eigenstates of the corresponding partner, here H_+, and the SUSY potential W are known.

3 Classical stochastic dynamics.

As mentioned in the Introduction we are also interested in classical systems with a stochastic dynamics governed by the Fokker-Planck equation

$$\frac{\partial}{\partial t}\, m_t(x, x_0) = \frac{1}{2}\, \frac{\partial^2}{\partial x^2}\, m_t(x, x_0) + \frac{\partial}{\partial x}\Big(U'(x) m_t(x, x_0)\Big)\ . \tag{6}$$

Here $m_t(x, x_0)$ denotes the transition-probability density of a macroscopic degree of freedom to be found at time t at position x if it initially has been at x_0, i.e. $m_0(x, x_0) = \delta(x - x_0)$. This degree of freedom is subjected to an external force characterized by the real-valued drift potential U and to a stochastic random force (white noise) resulting in the diffusive term on the right-hand side of (6) with diffusion constant set equal to 1/2. Making the ansatz [12]

$$m_t(x, x_0) = \mathrm{e}^{-U(x)} K_t(x, x_0) \tag{7}$$

leads to

$$-\frac{\partial}{\partial t}\, K_t(x, x_0) = \left[-\frac{1}{2}\, \frac{\partial^2}{\partial x^2} + \frac{1}{2}\, U'^2(x) - \frac{1}{2}\, U''(x)\right] K_t(x, x_0)\ , \tag{8}$$

which may be interpreted as an imaginary-time Schrödinger equation for the SUSY Hamiltonian H_- with a SUSY potential given by the first derivative of the drift potential, $W := U'$. Hence, the desired transition-probability density is given via the Euclidean propagator for H_-:

$$m_t(x, x_0) = \exp\{U(x_0) - U(x)\} \langle x| \exp\{-tH_-\} |x_0\rangle\ . \tag{9}$$

It is obvious that the decay modes and decay rates of this classical stochastic dynamical system are related to the eigenfunctions and eigenvalues of H_-. To be more explicit, let us consider the cases of unbroken and broken SUSY separately.

For unbroken SUSY the ground-state energy of H_- vanishes and as a consequence there exists a stationary, i.e. time-independent, probability distribution. Noting that $\psi_0^-(x) = \psi_0^-(x_0)\exp\{U(x_0) - U(x)\}$ the transition-probability density can be put into the form

$$m_t(x, x_0) = \left[\psi_0^-(x)\right]^2 + \exp\{U(x_0) - U(x)\} \sum_{n=1}^{\infty} e^{-tE_n^-} \psi_n^-(x)\psi_n^-(x_0) , \tag{10}$$

which clearly shows that the strictly positive eigenvalues of H_- and the associated eigenfunctions characterize the decay rates and decay modes of the system. In addition the ground-state wavefunction represents the stationary distribution. That is, a stable system is characterized by an unbroken SUSY.

In the case of broken SUSY, by definition, H_- does have strictly positive eigenvalues only and cannot lead to a stationary distribution. In other words, broken SUSY is related to unstable systems. Here the transition probability density has the form

$$m_t(x, x_0) = \exp\{U(x_0) - U(x)\} \sum_{n=0}^{\infty} e^{-tE_n^-} \psi_n^-(x)\psi_n^-(x_0) . \tag{11}$$

Finally, we mention that in both cases we may invert the drift potential, $U \to -U$, which amounts in replacing H_- by H_+. Hence, due to the presence of SUSY the inverted drift potential has the same decay rates as the original one. Only in the case of unbroken SUSY a stable system becomes unstable upon inversion. This is expressed in the fact that the vanishing eigenvalue $E_0^- = 0$ is missing in the spectrum of H_+. The decay modes of the original and the inverted drift potential are also clearly related by the SUSY transformations (3) and (4), respectively.

4 Designing exactly solvable models.

From the discussion in the previous two sections it is clear that once the spectral data, that is, the eigenvalues and eigenfunctions, of H_+ are known we also know these data for the corresponding superpartner H_- and in turn the decay rates and modes for the Fokker-Planck equation with drift potential $U(x) = \int_{x_0}^{x} dz\, W(z)$. In order to construct new exactly solvable systems we will search for the most general class of superpartners associated with a given exactly solvable quantum Hamiltonian H_+. Such a construction method has recently been given [6, 7, 8] and we briefly review it here.

In order to find the most general class of SUSY partners of a given SUSY Hamiltonian we make the following ansatz

$$W(x) = \Phi(x) + \frac{u'(x)}{u(x)} , \tag{12}$$

where Φ is a known SUSY potential belonging to some shape-invariant (i.e. exactly solvable) potential. See, for example, Table 5.1. in ref. [5]. If we now assume that u

is a solution of

$$u''(x) + 2\,\Phi(x)\,u'(x) - b\,u(x) = 0\ , \qquad b \in \mathbb{R}\ , \tag{13}$$

we find that

$$V_+(x) = \frac{1}{2}\,\Phi^2(x) + \frac{1}{2}\,\Phi'(x) + \frac{b}{2}\ . \tag{14}$$

By construction V_+ is up to the additive constant $b/2$ shape-invariant and, therefore, the eigenvalues and eigenfunctions of H_+ are exactly known. The corresponding partner potentia can be put into the form

$$V_-(x) = \frac{1}{2}\,\Phi^2(x) - \frac{1}{2}\,\Phi'(x) + \frac{u'(x)}{u(x)}\left(2\,\Phi(x) + \frac{u'(x)}{u(x)}\right) - \frac{b}{2}\ , \tag{15}$$

which, in genera , is not shape-invariant and thus a *new* exactly solvable quantum mechanical potential. Similarly we may construct also a *new* drift potential

$$U(x) = \int_{x_0}^{x} \mathrm{d}z\,\Phi(z) + \log u(x) \tag{16}$$

for which the associated decay rates and modes are known exactly.

Let us note here that we cannot take an arbitrary solution of (13) for the construction of new quantum-mechanical and drift potentials. In order to circumvent domain questions of the operators involved we will take only those solutions of (13) into account which are strictly positive. Hence, no singularities inside the configuration space will appear in V_- and U. For this reason the new Schrödinger potentials V_- obtained in this way have been called conditionally exactly solvable [6]. In the following two subsections we will present two examples with unbroken and broken SUSY, respectively. For further details and examples see [8].

4.1 A system with unbroken SUSY.

The first example we are going to present is characterized by a linear SUSY potential, $\Phi(x) = x$, with configuration space given by the real line, $x \in \mathbb{R}$. This SUSY potential gives rise to the harmonic-oscillator potential

$$V_+(x) = \tfrac{1}{2}\,(x^2 + b + 1) \tag{17}$$

and the corresponding eigenvalues and eigenfunctions of H_+ are

$$E_n^+ = n + b/2 + 1\ , \qquad \psi_n^+(x) = \left[\sqrt{\pi}\,2^n n!\right]^{-1/2} H_n(x)\exp\{-x^2/2\}\ . \tag{18}$$

Here H_n denotes a Hermite polynomial of order n. In this case the general solution of (13) can be expressed in terms of confluent hypergeometric functions,

$$u(x) = {}_1F_1\left(-\frac{b}{4}, \frac{1}{2}, -x^2\right) + \beta\,x\,{}_1F_1\left(\frac{2-b}{4}, \frac{3}{2}, -x^2\right) \tag{19}$$

and is strictly positive for $b > -2$ and $|\beta| < 2\Gamma(\frac{b}{4}+1)/\Gamma(\frac{b+2}{4})$, cf. ref. [7, 8]. The corresponding partner potential is given by

$$V_-(x) = \frac{1}{2}\,x^2 - \frac{b+1}{2} + \frac{u'(x)}{u(x)}\left[2\,x + \frac{u'(x)}{u(x)}\right] \tag{20}$$

and plots of it for various values of the parameters b and β can be found in Figure 1 of [7] and Figures 1 and 2 of [8]. Here we only note that for large $|x|$ this potential becomes asymptotically that of a harmonic oscillator. For values of b close to the lower bound -2 this potential exhibits two double-wells near the origin. Whereas for larger values of b these double wells merge to a single well at the origin. For $\beta = 0$ the potential V_- is symmetric about $x = 0$. This symmetry is broken for non-vanishing β. Noting that SUSY is unbroken for all allowed values of the parameters, the spectral properties of the corresponding Hamiltonian H_- are easily obtained from (3):

$$E_0^- = 0, \qquad E_{n+1}^- = E_n^+ = n + b/2 + 1, \qquad \psi_0^-(x) = \frac{\psi_0^-(0)}{u(x)}\,\exp\{-x^2/2\},$$

$$\psi_{n+1}^-(x) = \frac{\exp\{-x^2/2\}}{\left[\sqrt{\pi}\,2^{n+1}n!(n+b/2+1)\right]^{1/2}}\left(H_{n+1}(x) + H_n(x)\,\frac{u'(x)}{u(x)}\right). \tag{21}$$

These results also allow a complete study of the Fokker-Planck equation for the drift potential

$$U(x) = \frac{1}{2}\,x^2 + \log u(x)\,. \tag{22}$$

In fact, this drift potential is stable and has a stationary distribution given by $[\psi_0^-(x)]^2$. The decay rates and decay modes are given explicitly by E_{n+1}^- and ψ_{n+1}^- in (21). Plots of this drift potential for various values of the parameters can be found in Figures 1 of [9]. Here we again briefly mention that for b close to its lower limit U has the shape of a bistable (double-well) potential being symmetric about the origin for $\beta = 0$. For larger values of b this drift potential develops also a stable single-well shape.

To conclude this subsection let us mention that the results presented here are not new. They have first been derived by Hongler and Zheng [9] in 1982, which was even before the work of Mielnik [4] who was searching for more general factorizations of the harmonic oscillator Hamiltonian. His results can be viewed as special cases of those presented here. The advantage of the present approach is that it can be applied not only to the harmonic oscillator-like systems but to all exactly solvable ones [8]. In particular, there exist also shape-invariant quantum systems which allow for a broken SUSY [5]. That is, one may be able to design drift potentials which are metastable and can be solved exactly. To our knowledge, such exactly solvable systems are not available so far. Except, of course, the rather trivial case of a piecewise linear drift potential. In the next subsection we are going to present an exactly solvable metastable drift potential, which is related to the radial harmonic oscillator Hamiltonian.

4.2 A system with broken SUSY.

In this subsection we will consider the case where the SUSY potential is given by

$$\Phi(x) = x + \frac{\gamma}{x}, \qquad \gamma > 0, \tag{23}$$

where the condition put on the parameter γ leads to a SUSY potential Φ which characterizes a shape-invariant SUSY pair of radial harmonic oscillator-like Hamiltonians with broken SUSY. Due to the condition (13) even the SUSY potential (12) with above Φ gives rise to the radial harmonic oscillator potential

$$V_+(x) = \frac{x^2}{2} + \frac{\gamma(\gamma-1)}{2x^2} + \gamma + \frac{b+1}{2}, \tag{24}$$

which is exactly solvable and leads to the following spectral properties of H_+:

$$E_n^+ = 2n + 2\gamma + 1 + \frac{b}{2}, \quad \psi_n^+(x) = \left[\frac{2\,n!}{\Gamma(n+\gamma+1/2)}\right]^{1/2} x^\gamma\, e^{-x^2/2}\, L_n^{(\gamma-1/2)}(x^2)\,. \tag{25}$$

Here $L_n^{(\nu)}$ denotes an associated Laguerre polynomial of degree n and index ν. As in the previous case the general solution of (13) can be expressed in terms of confluent hypergeometric functions. Here, however, we are only interested in cases with broken SUSY and with this constraint the most general solution reads

$$u(x) = {}_1F_1(-\frac{b}{4}, \gamma + \frac{1}{2}, -x^2) = e^{-x^2}{}_1F_1(\gamma + \frac{b+2}{4}, \gamma + \frac{1}{2}, x^2)\,. \tag{26}$$

This solution will be strictly positive if $b > -4\gamma - 2$. The corresponding SUSY partner potential reads

$$V_-(x) = \frac{x^2}{2} + \frac{\gamma(\gamma+1)}{2x^2} + \gamma - \frac{b+1}{2} + \frac{u'(x)}{u(x)}\left(2x + \frac{2\gamma}{x} + \frac{u'(x)}{u(x)}\right) \tag{27}$$

and is plotted, for example, as Figure 5 in ref. [8]. The eigenvalues of the Schrödinger Hamiltonian H_- for this potential are identically to those of H_+ given in (25). The eigenfunctions can also be obtained from those in (25) via the SUSY transformation (4) and read

$$\psi_n^-(x) = \left[\frac{2\,n!}{(n+\gamma+\frac{1}{2}+\frac{b}{4})\Gamma(n+\gamma+1/2)}\right]^{1/2} x^{\gamma+1}\, e^{-x^2/2}$$

$$\left(L_n^{(\gamma+1/2)}(x^2) + \frac{u'(x)}{2\,x\,u(x)}\right). \tag{28}$$

Again, as in the previous case we can construct a drift potential with decay rates and modes given by the eigenvalues and eigenfunctions of H_-. This drift potential is explicitly given by

$$U(x) = \frac{1}{2}x^2 + \gamma \log x + \log u(x) =$$

$$= -\frac{1}{2}x^2 + \gamma \log x + \log \left[{}_1F_1(\gamma + \frac{b+2}{4}, \gamma + \frac{1}{2}, x^2) \right] \tag{29}$$

and because of broken SUSY does not have a stationary distribution. To be more explicit, for small $x > 0$ it has a logarithmic "hole" at the origin, i.e. $U(x) \approx -\gamma |\log x|$ for $x \ll 1$, whereas for large $x \to \infty$ it becomes asymptotically that of a harmonic oscillator, i.e. $U(x) \approx x^2/2$ for $x \gg 1$. In addition to that, for values of b close to its lower limit $-4\gamma - 2$ this drift potential exhibits a local minimum. In other words, it is metastable. In fact, the smallest decay rate is given by $E_0^- = 2\gamma + 1 + b/2 \searrow 0$ for $b \searrow -4\gamma - 2$. For larger values of b this local minimum disappears and the drift potential (29) becomes unstable. A typical plot of this potential is given in Figure 1 where we have shown (29) for $\gamma = 1$ and $b \in (-6, -5)$. Note that in Figure 1 we have plotted $U(x)$ versus $\exp\{-x\}$.

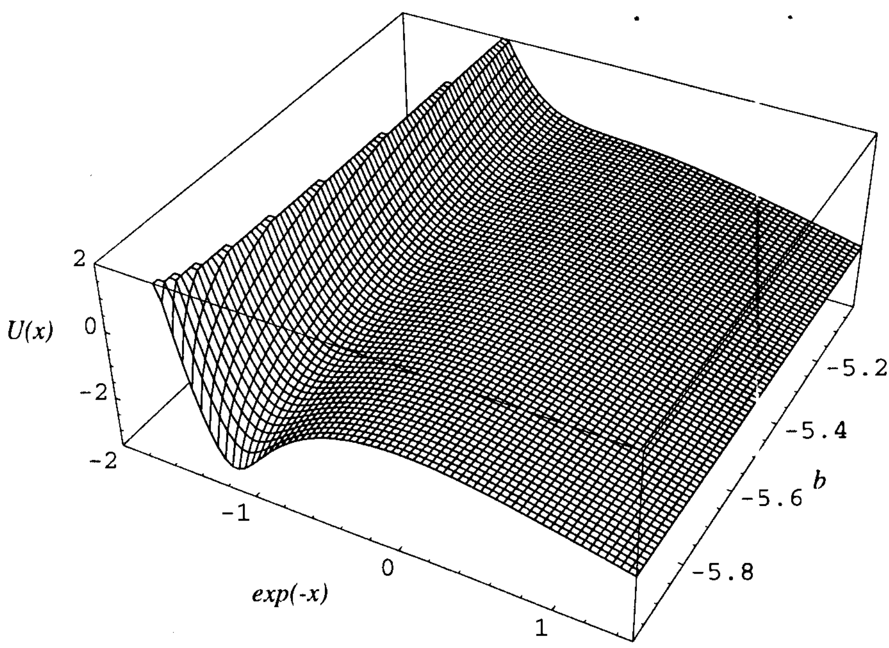

Figure 1: A family of drift potentials (29) for $\gamma = 1$ showing a transition from metastability to instability with increasing b. The corresponding decay rates and decay modes are known in closed form. Note that we have plotted $U(x)$ versus $\exp\{-x\}$.

5 Final remarks.

In this paper we have extended a recent approach to the construction of drift potentials for which the associated Fokker-Planck equation can be solved exactly. As starting point we have chosen a SUSY potential Φ which generates a pair of shape-invariant quantum mechanical potentials leading to exactly solvable quantum Hamiltonians. This SUSY potential can be perturbed (by adding u'/u) in such a way that one of these Hamiltonians, H_+, remains in the class of shape-invariant exactly solvable Hamilton operators. The partner Hamiltonian H_-, however, is a new one and due to SUSY its spectral properties can be obtained from H_+ in a straightforward way. Using the close relations between SUSY quantum mechanics and the Fokker-Planck equation we have used these results to find also new drift potentials with exactly known decay rates and modes. In this paper we have constructed stable as well as unstable drift potentials associated with unbroken and broken SUSY, respectively. The examples for unbroken SUSY are actually not new and have already been studied by Hongler and Zheng [9]. However, the present results for broken SUSY leading, in particular, to metastable drift potentials are new. To our knowledge, these are the first (on the positive half line) analytical metastable drift potentials for which the decay rates and modes can be given in closed analytical form. Besides some practical applications these metastable potentials can also serve as a testing ground for approximation methods. Recall that for metastable drift potentials the fluctuation operator for the classical "bounce" solution has a negative eigenvalue and, therefore, leads to serious singularities within a saddle-point approximation [13]. The standard treatment in such unstable situations goes back to Langer [14] and is based on some analytical continuation techniques. These formal treatments can now be applied to and (for the first time) tested with the examples presented in Section 4.2.

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SUPERINTEGRABILITY AND SPECIAL FUNCTIONS IN MATHEMATICAL PHYSICS

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Abstract

In this talk we outline the basic ideas relating to the notion of superintegrability. The fundamental notion we introduce is that of simultaneous separability of the Schrödinger equation in more than one coordinate system. The energy observable is degenerate for potentials of this type and the corresponding observables that arise from the simultaneous separability close quadratically under repeated commutation. We give examples of these systems and indicate how superintegrability can be of use, particularly in relation to bound states.

1 Introduction

It has long been known that there are potentials for which a given Hamiltonian system in classical mechanics admits a solution via separation of variables in more than one coordinate system [1,2]. The methodical search for such potentials was initiated by Smorodinsky, Winternitz et.al. [3,4] in two and three dimensions. A subset of such systems is called *maximal* in dimension N if there exist 2N-1 functionally independent integrals of motion. In this talk we show how to find the finite solutions (bound state eigenfunctions) for those quantum mechanical systems in two dimensions which admit separation of variables in at least two coordinate systems. We do this for the corresponding systems defined in two dimensional Euclidean space and indicate how these ideas work for quantum mechanical systems defined on the two dimensional sphere [5].

2 The Euclidean plane and the two dimensional sphere

2.1 To illustrate the basic idea we can consider the example of the Schrödinger equation with potential

$$V(x,y) = \frac{1}{2}\left(\omega^2(x^2+y^2) + \frac{k_1^2-\frac{1}{4}}{x^2} + \frac{k_2^2-\frac{1}{4}}{y^2}\right),$$

i.e.,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Psi - \left(\omega^2(x^2+y^2) + \frac{k_1^2-\frac{1}{4}}{x^2} + \frac{k_2^2-\frac{1}{4}}{y^2}\right)\Psi = -2E\Psi.$$

This equation can be solved by the separation of variables ansatz $\Psi = \psi_1(u)\psi_2(v)$ for a suitable choice of coordinates $x = x(u,v)$, $y = y(u,v)$. In fact this equation separates in three coordinate systems: *Cartesian* coordinates (x,y); *Polar* coordinates $x = r\cos\theta$, $y = r\sin\theta$, and *Elliptical* coordinates

$$x^2 = c^2\frac{(u_1-e_1)(u_2-e_1)}{(e_1-e_2)}, \qquad y^2 = c^2\frac{(u_1-e_2)(u_2-e_2)}{(e_2-e_1)}.$$

The bound state energies are given by

$$E_n = \omega(2n+2+k_1+k_2)$$

for integer n. The wave functions for each of these coordinate systems are:
1. *Cartesian* coordinates

$$\Psi_{n_1,n_2}(x,y) = 2\omega^{\frac{1}{2}(k_1+k_2+2)}\sqrt{\frac{n_1!n_2!}{\Gamma(n_1+k_1+1)\Gamma(n_2+k_2+1)}}\,x^{(k_1+\frac{1}{2})}y^{(k_2+\frac{1}{2})}$$

$$e^{-\frac{\omega}{2}(x^2+y^2)}L_{n_1}^{k_1}(\omega x^2)L_{n_2}^{k_2}(\omega y^2)$$

where $n = n_1 + n_2$,
2. *polar* coordinates

$$\Psi(r,\theta) = \Phi_q^{(k_1,k_2)}(\theta)\omega^{\frac{1}{2}(2q+k_1+k_2+1)}\sqrt{\frac{2m!}{\Gamma(m+2q+k_1+k_2+1)}}$$

$$e^{(-\omega r^2/2)}r^{(2q+k_1+k_2+1)}L_m^{2q+k_1+k_2+1}(\omega r^2)$$

where $n = m + q$ and

$$\Phi_q^{(k_1,k_2)}(\theta) = \sqrt{2(2q + k_1 + k_2 + 1)\frac{q!\Gamma(k_1 + k_2 + q + 1)}{\Gamma(k_2 + q + 1)\Gamma(k_1 + q + 1)}}$$

$$\times(\cos\theta)^{k_1+(1/2)}(\sin\theta)^{k_2+(1/2)}P_q^{(k_1,k_2)}(\cos 2\theta).$$

3. *elliptical* coordinates

$$\Psi = e^{-\omega(x^2+y^2)}x^{k_1+\frac{1}{2}}y^{k_2+\frac{1}{2}}\prod_{m=1}^{n}\left(\frac{x^2}{\theta_m - e_1} + \frac{y^2}{\theta_m - e_2} - c^2\right)$$

where use has been made of the identity

$$\frac{x^2}{\theta - e_1} + \frac{y^2}{\theta - e_2} - c^2 = -c^2\frac{(u_1 - \theta)(u_2 - \theta)}{(\theta - e_1)(\theta - e_2)}.$$

The zeros θ_j satisfy the relations

$$\frac{k_1 + 1}{\theta_m - e_1} + \frac{k_2 + 1}{\theta_m - e_2} + \sum_{j\neq m}\frac{2}{(\theta_m - \theta_j)} - \omega = 0.$$

Because of the separability properties of the Schrödinger equation in these coordinate systems there are second order symmetries. These are second order partial differential operators L which are such that if Ψ is a solution then so is $L\Psi$. A basis for such operators is

$$L_1 = \partial_x^2 + \frac{(\frac{1}{4} - k_1^2)}{x^2} - \omega^2 x^2, \qquad L_2 = \partial_y^2 + \frac{(\frac{1}{4} - k_2^2)}{y^2} - \omega^2 y^2$$

$$M = (x\partial_y - y\partial_x)^2 + (\frac{1}{4} - k_1^2)\frac{y^2}{x^2} + (\frac{1}{4} - k_2^2)\frac{x^2}{y^2} - \frac{1}{2}.$$

The separable eigenfunctions already given are eigenfunctions of the symmetry operators L_1, M and $M + e_2 L_1 + e_1 L_2$ with eigenvalues

$$\lambda_c = -\omega(2n_1 + k_1 + 1), \qquad \lambda_p = (2q + k_1 + k_2 + 1)^2 + (1 + k_1^2 + k_2^2),$$

$$\lambda_e = 2(\ - k_1)(1 - k_2) - 2e_2\omega(k_1 + 1) - 2e_1\omega(k_2 + 1) - \omega^2 e_1 e_2 -$$

$$4\sum_{m=1}^{q}[e_2\frac{k_1 + 1}{\theta_m - e_1} + e_1\frac{k_2 + 1}{\theta_m - e_2}].$$

The algebra constructed by repeated commutators is

$$[L_1, M] = [M, L_2] = R, \qquad [L_i, R] = -4\{L_i, L_j\} + 16\omega^2 M, \qquad i \neq j,$$

$$[M, R] = 4\{L_1, M\} - 4\{L_2, M\} + 8(1-k_2^2)L_1 - 8(1-k_1^2)L_2,$$

$$R^2 = \frac{8}{3}\{M, L_1, L_2\} + \frac{64}{3}\{L_1, L_2\} + 16\omega^2 M^2 - 16(1-k_2^2)L_1^2$$

$$-16(1-k_1^2)L_2^2 - \frac{128}{3}\omega^2 M - 64\omega^2(1-k_1^2)(1-k_2^2).$$

These relations are quadratic.

2.2 In two real dimensions there are four potentials that have the multiseparation property. The second is

$$V(x,y) = \omega^2(4x^2+y^2) - \frac{(\frac{1}{4}-k_2^2)}{y^2}.$$

The corresponding Schrödinger equation is separable in two coordinate systems: *Cartesian* coordinates and *parabolic* coordinates

$$x = \frac{1}{2}(\xi^2-\eta^2), y = \xi\eta.$$

2.3 The third potential is

$$V(x,y) = -\frac{\alpha}{\sqrt{x^2+y^2}} + \frac{1}{4\sqrt{x^2+y^2}}\left(\frac{(k_1^2-\frac{1}{4})}{\sqrt{x^2+y^2}+x} + \frac{(k_2^2-\frac{1}{4})}{\sqrt{x^2+y^2}-x}\right).$$

The corresponding Schrödinger equation is separable in two coordinate systems: *polar*, *parabolic* and *modified elliptic* coordinates, where

$$\xi^2 = c^2\frac{(u_1-e_1)(u_2-e_1)}{(e_1-e_2)}, \qquad \eta^2 = c^2\frac{(u_1-e_2)(u_2-e_2)}{(e_2-e_1)}.$$

This last coordinate system can be written as

$$x = \sqrt{\frac{(U_1-E_1)(U_2-E_1)}{4(E_1-E_2)}}, \qquad y = \sqrt{\frac{(U_1-E_2)(U_2-E_2)}{4(E_2-E_1)}} - 2\sqrt{E_1-E_2},$$

where $E_1 = -e_1e_2$, $E_2 = -\frac{1}{4}(e_1+e_2)^2$ and $U_j = u_j^2 - u_j(e_1+e_2)$.

2.4 The fourth potential is

$$V(x,y) = -\frac{\alpha}{\sqrt{x^2+y^2}} + \frac{B_1}{4}\frac{\sqrt{\sqrt{x^2+y^2}+x}}{\sqrt{x^2+y^2}} + \frac{B_2}{4}\frac{\sqrt{\sqrt{x^2+y^2}-x}}{\sqrt{x^2+y^2}}.$$

Separation occurs here in *parabolic* and *parabolic coordinates of the second type*

$$x = \mu\nu, \quad y = \frac{1}{2}(\mu^2 - \nu^2).$$

As an illustration of the utility of the notion of a quadratic algebra consider the last potential given. A basis for the quadratic algebra consists of L_1, L_2 and H with defining relations

$$[R, L_1] = -4L_2H + B_1B_2, \qquad [R, L_2] = 4L_1H + \frac{1}{2}(B_1^2 - B_2^2)$$

$$R^2 = 4L_1^2H + 4L_2^2H - 16\alpha^2H + (B_2^2 - B_1^2)L_1 - 2B_1B_2L_2 - 2\alpha^2(B_1^2 + B_2^2)$$

with $R = [L_1, L_2]$. If we look for eigenfunctions of the operators L_1, L_2 respectively, we have

$$L_1\varphi_m = \lambda_m\varphi_m, \qquad L_2\psi_n = \rho_n\psi_n.$$

If we write

$$L_1\psi_n = \sum_\tau C_{n\tau}\psi_\tau$$

then the quadratic algebra relations imply

$$[(\rho_n - \rho_\tau)^2 + 8E]C_{n\tau} = -[\frac{1}{2}(B_1^2 - B_2^2) - 16\alpha E]\delta_{n\tau}$$

$$\sum_\tau C_{n\tau}C_{\tau\sigma}(2\rho_\tau - \rho_n - \rho_\sigma) = (8E\rho_n + B_1B_2 + 16\alpha E)\delta_{n\sigma}.$$

These relations in turn imply that

$$C_{nn} = -\frac{\frac{1}{2}(B_1^2 - B_2^2) + 16\alpha E}{8E}$$

and $C_{nn+1} = C^*_{n+1n}$ are the only nonzero coefficents. Indeed they can essentially be determined by the relation

$$4\sqrt{-2E}\,(|C_{n,n+1}|^2 - |C_{n-1,n}|^2) = 8E\rho_n + B_1B_2 + 16\alpha E$$

where the eigenvalues λ_m and ρ_n are given by

$$\lambda_m = 2\alpha - \frac{B_1^2}{8E} - (2m+1)\sqrt{-2E}, \quad \rho_n = 2\alpha - \frac{(B_1+B_2)^2}{16E} - (2n+1)\sqrt{-2E}$$

and the quantisation condition for E is

$$4\alpha - \frac{B_1^2 + B_2^2}{8E} = -(q+2)\sqrt{-2E}$$

for integer q.

These ideas work also for separable coordinates on the two dimensional sphere. Two important examples are

2.5 The potential

$$V = \frac{1}{2}\left[\frac{k_1^2 - \frac{1}{4}}{s_1^2} + \frac{k_2^2 - \frac{1}{4}}{s_2^2} + \frac{k_3^2 - \frac{1}{4}}{s_3^2}\right]$$

where as usual $s_1^2 + s_2^2 + s_3^2 = 1$. The corresponding Schrödinger equation has the form

$$\left[(s_1\frac{\partial}{\partial s_2} - s_2\frac{\partial}{\partial s_1})^2 + (s_1\frac{\partial}{\partial s_3} - s_3\frac{\partial}{\partial s_1})^2 + (s_3\frac{\partial}{\partial s_2} - s_2\frac{\partial}{\partial s_3})^2\right]\Psi +$$
$$\left[\frac{(k_1^2 - \frac{1}{4})}{s_1^2} + \frac{(k_2^2 - \frac{1}{4})}{s_2^2} + \frac{(k_3^2 - \frac{1}{4})}{s_3^2}\right]\Psi = -2E\Psi.$$

This equation admits solution via separation of variables in two coordinate systems: *spherical* coordinates

$$s_1 = \sin\theta\cos\varphi, \quad s_2 = \sin\theta\sin\varphi, \quad s_3 = \cos\theta$$

and *elliptical* coordinates

$$s_i^2 = \frac{(u_1 - e_i)(u_2 - e_i)}{(e_1 - e_j)(e_i - e_j)}, \qquad i,j,k = 1,2,3 \qquad i \neq j \neq k.$$

Indeed a basis for the symmetries of Schrödinger's equation with this potential is

$$L_{ij} = (s_i\partial_{s_j} - s_j\partial_{s_i})^2 + (\frac{1}{4} - k_i^2)\frac{s_j^2}{s_i^2} + (\frac{1}{4} - k_j^2)\frac{s_i^2}{s_j^2} - \frac{1}{2}, \quad i \neq j.$$

These symmetries satisfy the quadratic algebra relations

$$[L_{ij}, R] = 4\{L_{ij}, L_{jk}\} - 4\{L_{ij}, L_{ik}\} + 8(1 - k_i^2)L_{jk} - 8(1 - k_j)^2 L_{ik},$$

$$R^2 = -\frac{4}{3}\{L_{ij}, L_{ik}, L_{jk}\} + \frac{64}{3}\{L_{ij}, L_{ik}\} + \frac{64}{3}\{L_{ij}, L_{jk}\} + \frac{64}{3}\{L_{ik}, L_{jk}\} -$$
$$16(1-k_k^2)L_{ij}^2 - 16(1-k_j^2)L_{ik}^2 - 16(1-k_i^2)L_{jk}^2 +$$
$$\frac{128}{3}(1-k_i^2)L_{jk} + \frac{128}{3}(1-k_j^2)L_{ik} + \frac{128}{3}(1-k_k^2)L_{ij}, \quad i \neq j \neq k.$$

The eigenfunctions with bound state energy eigenvalues

$$E_p = \frac{1}{2}(2p+2+k_1+k_2+k_3)^2 - \frac{1}{8}$$

in these coordinate systems are: *polar*

$$\Psi = (\sin\theta)^{-1}\Phi_n^{(k_2,k_1)}(\varphi)\Phi_m^{(2n+k_1+k_2,k_3)}(\theta)$$

where $p = m + n$, and *elliptical*

$$\Psi = (\prod_{\ell=1}^{3} s_\ell^{(k_\ell+\frac{1}{2})})\prod_{j=1}^{q}\left(\frac{s_1^2}{\theta_j - e_1} + \frac{s_2^2}{\theta_j - e_2} + \frac{s_3^2}{\theta_j - e_3}\right)$$

where

$$\frac{k_1+1}{\theta_m - e_1} + \frac{k_2+1}{\theta_m - e_2} + \frac{k_3+1}{\theta_m - e_3} + \sum_{j\neq m}\frac{2}{(\theta_m - \theta_j)} = 0$$

and $q = p$. Here we have made use of the identity

$$\frac{s_1^2}{\theta_j - e_1} + \frac{s_2^2}{\theta_j - e_2} + \frac{s_3^2}{\theta_j - e_3} = \frac{\Pi_{\ell=1}^{2}(u_\ell - \theta_j)}{\Pi_{m=1}^{3}(\theta_j - e_m)}.$$

The separable eigenfunctions already given are eigenfunctions of the symmetry operators L_{12} and $e_3L_{12} + e_2L_{13} + e_1L_{23}$:

$$\lambda_S = (2n + k_1 + k_2 + 1)^2,$$

$$\lambda_E = -2[k_1(e_2+e_3) + k_2(e_1+e_3) + k_3(e_2+e_1) + e_3k_1k_2 + e_1k_2k_3 + e_2k_1k_3] -$$
$$\frac{3}{2}(e_1+e_2+e_3) - 4e_2e_3(k_1+1)\sum_{m=1}^{q}\frac{1}{\theta_m - e_1} -$$
$$4e_2e_1(k_3+1)\sum_{m=1}^{q}\frac{1}{\theta_m - e_3} - 4e_1e_3(k_2+1)\sum_{m=1}^{q}\frac{1}{\theta_m - e_2}.$$

2.6 A second multiseparable potential on the sphere is

$$V = -\frac{\alpha s_3}{\sqrt{s_1^2+s_2^2}} + \frac{1}{4\sqrt{s_1^2+s_2^2}}\left[\frac{(k_1^2-\frac{1}{4})}{\sqrt{s_1^2+s_2^2}+s_1} + \frac{(k_2^2-\frac{1}{4})}{\sqrt{s_1^2+s_2^2}-s_1}\right].$$

Schrödinger's equation for this potential has the form

$$\left[(s_1\frac{\partial}{\partial s_2}-s_2\frac{\partial}{\partial s_1})^2+(s_1\frac{\partial}{\partial s_3}-s_3\frac{\partial}{\partial s_1})^2+(s_3\frac{\partial}{\partial s_2}-s_2\frac{\partial}{\partial s_3})^2\right]\Psi+$$

$$\left(-\frac{\alpha s_3}{\sqrt{s_1^2+s_2^2}} + \frac{1}{4\sqrt{s_1^2+s_2^2}}\left[\frac{(k_1^2-\frac{1}{4})}{\sqrt{s_1^2+s_2^2}+s_1} + \frac{(k_2^2-\frac{1}{4})}{\sqrt{s_1^2+s_2^2}-s_1}\right]\right)\Psi = -2E\Psi.$$

This equation admits solution via separation of variables in two coordinates systems: *spherical* and *elliptical coordinates of modified type*

$$s_1' = \cos f\, s_1 + \sin f\, s_3, \qquad s_2' = s_2, \qquad s_3' = -\sin f\; s_1 + \cos f\; s_3$$

where

$$s_i^2 = \frac{(y_1-e_i)(y_2-e_i)}{(e_1-e_j)(e_i-e_j)}, \qquad i,j,k = 1,2,3 \quad i\neq j\neq k,$$

and

$$\sin f = \sqrt{\frac{(e_2-e_1)}{(e_3-e_1)}}, \qquad e_1 = e_2+\frac{1}{4}(E_+ + E_-)^2, \qquad e_3 = e_2+\frac{1}{4}(E_+ - E_-)^2,$$

$$y_j = e_2 + \frac{1}{4}(E_+^2+E_-^2)+\frac{1}{4}E_+E_-(Z_j+\frac{1}{Z_j}), \quad j=1,2.$$

Indeed if we use the variables

$$U_3^2 = Z_1Z_2, \quad U_1^2 = \frac{(Z_1+\Omega_-)(Z_2+\Omega_-)}{(\Omega_-^2-1)}, \quad U_2^2 = \frac{(Z_1+\Omega_+)(Z_2+\Omega_+)}{(\Omega_+^2-1)}$$

where

$$\Omega_+ = \frac{E_+}{E_-}, \Omega_- = \frac{E_-}{E_+},$$

then, putting $k_3 = \sqrt{2(E-i\alpha)+1/4}$ and $\hat{E} = i\alpha + E$ and multiplying the Schrödinger equation by $(Z_1Z_2)^{-1}-1$, we see that the resulting equation has the form

$$H\Psi = (U_1\frac{\partial}{\partial U_2}-U_2\frac{\partial}{\partial U_1})^2+(U_1\frac{\partial}{\partial U_3}-U_3\frac{\partial}{\partial U_1})^2+(U_3\frac{\partial}{\partial U_2}-U_2\frac{\partial}{\partial U_3})^2)\Psi+$$

$$\left[\frac{(\frac{1}{4}-k_1^2)}{U_1^2}+\frac{(\frac{1}{4}-k_2^2)}{U_2^2}+\frac{(\frac{1}{4}-k_3^2)}{U_3^2}\right]\Psi+2\hat{E}\Psi=0$$

which is essentially the same form as for the first potential. The bound state quantisation condition has the form

$$E=\frac{1}{2}\left[2q+2+k_1+k_2+\sqrt{\frac{1}{4}+2(E-i\alpha)}\right]^2-\frac{1}{8}.$$

Outlook for this subject

Some natural questions arise from the outline of superintegrability given above.

[1]. What in general do the quadratic algebra relations imply?

[2]. How does one characterize the quadratic algebras?

[3]. Can we make systemmatic statements about superintegrability of Schrödinger's equation with a given potential in n dimensions?

[4]. What are the complete sets of superintegrable potentials in complex spaces?

These are all questions for which a solution seems possible. In addition to these considerations are the relations between the special functions that appear in the solutions of these problems. This is a topic that is currently also being investigated.

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GENERALIZED DIFFERENTIALS GEODESICS OF HIGHER ORDER

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Abstract

We present a Z_3-graded generalization of the exterior calculus, elaborated and published in a recent series of articles. A new type of differentials d^2x^k is introduced, and binary and ternary relations that must be satisfied by the differentials dx^k and d^2x^m ensuring that $d^3f(x^k) = 0$ are displayed. We then discuss how the notions of connection, torsion and curvature can be extended to this new differential calculus. A possibility of using a third order generalization of geodesic equation in the description of the two-body problem is also evoked.

1 Introduction.

It is well known that when we try to solve a linear differential equation of second order, the first integral can be found usually without much pain. Nevertheless, as a rule the first derivative of the function to be found is a complicated and highly nonlinear expression. In the simplest case of a pendulum of mass m, the second order linear equation

$$ml\,\frac{d^2\theta}{dt^2} + mg\,sin\theta = 0$$

leads to the first integral of energy $E = \frac{ml^2}{2}\,(\frac{d\theta}{dt})^2 - \frac{1}{2}\,mgl\,cos\theta = Const$, yielding the well-known elliptic integral defining implicitly the function $\theta(t)$:

$$\frac{d\theta}{dt} = \sqrt{\frac{2E}{ml^2} + \frac{g}{l}cos\theta} \quad \Longrightarrow \quad t - t_0 = \int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\frac{2E}{ml^2} + \frac{g}{l}cos\,\theta}} =$$

which can not be evaluated analytically. The analytical solution can be found in the linear approximation, when one can replace $sin\theta$ by θ, and under the integral, $cos\theta$ by $1 - \frac{\theta^2}{2}$; then the integral yields the function $Arcsin\,\theta$.

We would like to draw the attention to the corollary of this fact, namely, that an extra *differentiation* of a non-linear equation may lead to a linear problem that can be treated more easily.

The use of differential equations of order higher than two is also inspired by our recent investigations concerning the generalizations of the exterior differential calculus, in which the operator d is no more nilpotent in the usual sense, i.e. $d^2 f \neq 0$, but one imposes $d^N = 0$ instead ([1], [3], [6], [2], [4], [5]). Here we shall explore essentially the case of $N = 3$, although the generalization for other values of N is straightforward. In what follows, we shall try to show how these differentials of higher order naturally lead to covariant equations of higher order generalizing the usual geodesic equation, which may be used for simplifying certain non-linear problems.

The usual exterior differential calculus is based on the identification of the differentials of local coordinates, dx^k, $k = 1, 2, \ldots D$, (D = the dimension of the manifold) with the Grassmann algebra of *exterior forms* spanned by all the skew-symmetric tensor products of differentials, e.g.

$$dx^i \wedge dx^k = \frac{1}{2}\left(dx^i \otimes dx^k - dx^k \otimes dx^i\right) \qquad \text{etc.,} \tag{1}$$

supplemented with the anti-Leibniz rule for the exterior differentiation:

$$d\,(\omega \wedge \phi) = (d\omega) \wedge \phi + (-1)^{deg(\omega)}\, \omega \wedge d\phi \tag{2}$$

where deg(ω) is the degree of the exterior form ω.

It is easy to prove now that $d^2 = 0$ when acting on any p-form. In particular, on a function of coordinates $f(x^k)$ one gets, taking into account the rule (2) and recalling that $\partial_k f$ is of degree 0 :

$$df = \partial_k f\, dx^k\,; \quad d^2 f = \partial^2_{ik} f\, dx^i \wedge dx^k + \partial_k f\, d^2 x^k = 0, \tag{3}$$

which is consistent with the assumption $d^2 x^k = 0$ on the basic differentials (which are just other functions on the manifold), and the fact that *symmetric* quantity $\partial_{ik} f$ is contracted with the *anti-symmetric quantity* $dx^i \wedge dx^k$.

In spite of the fact that in the exterior calculus $d^2 x^k = 0$, this formal expression appears in the notation for the second derivative of a function. The expression $\frac{d^2 x^k}{dt^2}$ can be interpreted as a short-hand notation of an iterated action of the derivation $\frac{d}{dt}$, but it can be also given an autonomous meaning as the ratio between the *second differential* $d^2 x^k$ and the *square* of the differential dt, i.e.

$$\frac{d^2 x^k}{dt^2} = \frac{d}{dt}\frac{dx^k}{dt} = \frac{d^2 x^k}{(dt)^2}$$

Clearly, $d^2 x^k$ can not coexist with the exterior product $dx^k \wedge dx^m$, but it can be related with the symmetric part of the tensor product $dx^k \otimes dx^m$. Let us suppose indeed that $d^2 x^k \neq 0$, but $d^3 x^k = 0$, and in general, $d^3 f = 0$ instead. Continuing to apply the rules (2) and (3), but replacing -1 with the primitive third root of unity, $q = e^{\frac{2\pi i}{3}}$, we have then: (for simplicity, we have suppressed the tensor product sign $\otimes$)

$$d^3 f = \partial^3_{mik} f\, dx^m dx^i dx^m + \partial^2_{ik} f\, d^2 x^i dx^k + q\, \partial^2_{ik} f\, dx^i d^2 x^k + \partial^2_{ik} f\, dx^i d^2 x^k + \partial_k f\, d^3 x^k$$

After a simple re-arrangement and re-labeling of indices, and using the fact that $d^3 = 0$, we get

$$d^3 f = \partial^3_{ikm} f \, dx^i dx^k dx^m + \partial^2_{ik} f \left[(1+q) dx^i d^2 x^k + d^2 x^k dx^i \right] = 0 \tag{4}$$

As it can be seen, there are two types of forms of degree 3: ternary products of the first-order differentials, or binary products of first- and second-order differentials. Because these combinations are linearly independent, the corresponding expressions must vanish separately. Because $1 + q = -q^2$, the last term will vanish identically if we set

$$dx^i d^2 x^k = q \, d^2 x^k dx^i \qquad \text{and} \qquad d^2 x^k dx^i = q^2 \, dx^i d^2 x^k. \tag{5}$$

Now, assuming that the *associative* algebra spanned by the first- and second-order differentials is Z_3-graded, dx^k being of degree one, and $d^2 x^m$ of degree two, the grades adding up *modulo* 3 under the associative multiplication, we see that the second differential $d^2 x^k$ in the above formula can be replaced by a product $dx^k dx^m$, thus suggesting the following ternary relations:

$$dx^i dx^k dx^m = q \, dx^k dx^m dx^i = q^2 \, dx^m dx^i dx^k \tag{6}$$

which make the terms containing the third derivatives vanish, too, because

$$\partial^3_{ikm} f = \frac{1}{3} (\partial^3_{ikm} f + \partial^3_{kmi} f + \partial^3_{mik} f) , \qquad \text{and} \qquad 1 + q + q^2 = 0.$$

The associativity together with the above ternary relation imposes that all the fourth-order products must vanish. The proof is straightforward: for a product of any four 1-differentials one has:

$$(dx^i dx^k dx^l) dx^m = (q dx^k dx^l dx^i) dx^m = q dx^k (dx^l dx^i dx^m) =$$
$$q^2 dx^k (dx^i dx^m dx^l) = (q^2 dx^k dx^i dx^m) dx^l = (q^3 dx^i dx^m dx^k) dx^l =$$
$$= q^3 dx^i (dx^m dx^k dx^l) = q^4 dx^i (dx^k dx^l dx^m) = q dx^i dx^k dx^l dx^m = 0 \tag{7}$$

By extension, we shall require that all the products of differentials of order four and higher must vanish: thus, we shall admit that

$$dx^i dx^k d^2 x^m = 0, \quad d^2 x^k d^2 x^m = 0, \qquad \text{etc.}$$

There is another important difference between the associative algebra of this type and the usual Z_2-graded Grassmann algebra, namely, the functions over the manifold form a right module over this algebra. This means that we consider only the expressions in which a function stands always on the left, multiplied from the right by a product of differentials. In contrast, these expressions can be multiplied by functions on the left; but we can not multiply one such expression by another (no internal multiplication can be defined in a module).

To give an illustration how this Z_3-graded differential calculus works, let us compute the differentials of a 1-form $A = A_k(x)\, dx^k$

$$dA = \partial_i A_k\, dx^i dx^k + A_k\, d^2 x^k;$$

$$d^2 A = \partial_m \partial_i A_k\, dx^m dx^i dx^k + \partial_i A_k\, d^2 x^i dx^k + q\, \partial_m A_k\, dx^m d^2 x^k + \partial_m A_k\, dx^m d^2 x^k \quad (8)$$

Taking into account the commutation relations and the fact that

$$\partial^2_{ik} A_m\, dx^i dx^k dx^m = \frac{1}{3}\left(\partial^2_{ik} A_m + q \partial^2_{mi} A_k + \partial^2_{km} A_i\right) dx^i dx^k dx^m,$$

everything can be expressed as

$$d^2 A = F_{ik}\, d^2 x^i dx^k - \frac{1}{2}\left(\partial_i F_{km} + \partial_k F_{im}\right) + \frac{i\sqrt{3}}{2}\left(\partial_i F_{km} - \partial_k F_{im}\right) \quad (9)$$

which is a manifestly covariant expression, depending only on the Maxwell tensor and its derivatives.

2 Covariant basis of Z_3-graded differentials.

If we suppose that the formal expression $d^2 f$ does not vanish identically as it is the case in the usual Z_2-graded exterior calculus of forms, then we must abandon the antisymmetry of the product of 1-forms in this algebra. The vanishing of the expression $d^2 f$ could be given an intrinsic sense in any local coordinate system because one supposes that simultaneously $d^2 = 0$, so it applied to any second differential of a local coordinate, be it dx^k or $d^2 y^{k'}$, and parallelly, taking into account the symmetry of partial second derivatives,

$$\frac{\partial^2 f}{\partial x^k \partial x^m} = \frac{\partial^2 f}{\partial x^m \partial x^k},$$

it had to be postulated that the product of 1-forms must be antisymmetric:

$$dx^k\, dx^m = -dx^m\, dx^k$$

Under a change of local coordinates, $x^k \to y^{m'}(x^k)$, and $x^k = x^k(y^{m'})$, the basis of 1-forms transformed as a covariant tensor, i.e. $dx^k = \frac{\partial x^k}{\partial y^{m'}}\, dy^{m'}$. However, had we abandoned the postulate $d^2 = 0$ and the antisymmetry of the product of 1-forms, the *second differentials* of the coordinates, which are for the time being purely formal expressions, would not transform as covariant tensors because of the non-homogeneous term:

$$d^2 x^k = d\left(\frac{\partial x^k}{\partial y^{m'}}\, dy^{m'}\right) = \frac{\partial^2 x^k}{\partial y^{l'} \partial y^{m'}}\, dy^{l'} dy^{m'} + \frac{\partial x^k}{\partial y^{m'}}\, d^2 y^{m'} \quad (10)$$

Introducing *connection coefficients* Γ^k_{lm} we shall define the *covariant second differentials* D^2x^k as

$$D^2x^k = d^2x^k + \Gamma^k_{lm}\, dx^l\, dx^m \tag{11}$$

(in order to make our notation homogeneous, from now on we sha l also note the first differentials - which are naturally covariant - with capital D, i.e. we shall identify $Dx^k = dx^k$.). Note that the above equation can be still interpreted in terms of Grassmann algebra of exterior forms: if we still impose $d^2 = 0$ and the antisymmetry of the exterior product, the covariant 2-form D^2x^k is equal to the torsion 2-form. The vanishing of D^2x^k is then equivalent to the condition of null torsion, which is valid in all coordinate systems.

Let us suppose now that the differentials dx^k and d^2x^m satisfy the relations imposed by the condition $d^3 = 0$ derived before, i.e., with $q = e^{\frac{2\pi i}{3}}$:

$$dx^k\, dx^l\, dx^m = q\, dx^l\, dx^m\, dx^k \qquad \text{and} \qquad dx^k\, d^2x^m = q\, d^2x^m\, dx^k$$

It is obvious that these relations remain valid if we replace the orcinary first and second differentials by their *covariant* counterparts:

$$Dx^k\, Dx^l\, Dx^m = q\, Dx^l\, Dx^m\, Dx^k \qquad \text{and} \qquad Dx^k\, D^2x^m = q\, D^2x^m\, Dx^k \tag{12}$$

which span a covariant basis of the same Z_3-graded algebra, which has the property of transforming covariantly under the change of a basis.

Now, although D^2x^k represents a tensorial quantity, its covariant differential $D(D^2x^k) = D^3x^k$ can not be computed by simple iteration, i.e. by applying the same formula as for the covariant differential of Dx^k. As a matter of fact, D^3x^k has to be a tensorial quantity of third degree, which in covariant basis should contain both $Dx^k\, D^2x^m$ and $Dx^kDx^lDx^m$. That is why we should write:

$$D^3x^k = D(D^2x^k) = d(D^2x^k) + B^k_{lm}\, Dx^lD^2x^m + C^k_{lmn}\, Dx^l\, Dx^m\, Dx^n \tag{13}$$

with new coefficients of two kinds, whose transformation properties under coordinate change are yet to be derived, and which will assure the tensorial character of D^3x^k. Acting with the operator d on D^2x^k yields the explicit result:

$$\begin{aligned} D^3x^k \;&=\; d^3x^k + \partial_n\Gamma^k_{lm}\, dx^ndx^ldx^m + \Gamma^k_{lm}\, d^2x^ldx^m + q\,\Gamma^k_{lm}\, dx^l\, d^2x^m + \\ &\quad + B^k_{lm}\, Dx^lD^2x^m + C^k_{lmn}\, Dx^l\, Dx^m\, Dx^n \end{aligned} \tag{14}$$

Now, using the fact that $d^2x^l = D^2x^l - \Gamma^l_{rm}\, dx^r\, dx^m$, we can express D^3x^k by means of covariant quantities only:

$$\begin{aligned} D^3x^k \;&=\; d^3x^k + \Big[\, B^k_{lm} + q^2\,\Gamma^k_{ml} + q\,\Gamma^k_{lm}\,\Big]\, Dx^lD^2x^m + \\ &\quad + \Big[\, C^k_{lmn} + \partial_l\Gamma^k_{mn} - \Gamma^r_{lm}\Gamma^k_{rn} - q\,\Gamma^r_{mn}\Gamma^k_{lr}\,\Big]\, Dx^lDx^mDx^m \end{aligned}$$

$$= \quad d^3x^k + \tilde{B}^k_{lm}\, Dx^l D^2 x^l + \tilde{C}^k_{lmn}\, Dx^l Dx^m Dx^n \tag{15}$$

Now we shall proceed by analogy with the Z_2-graded case. As we have seen, the condition $d^3 = 0$ implies also the ternary and binary q-commutation relations with $q = e^{\frac{2\pi i}{3}}$. This means that in the final expression we remain with

$$D^3x^k = \tilde{B}^k_{lm}\, Dx^l D^2 x^m + \tilde{C}^k_{lmn}\, Dx^l Dx^m Dx^n,$$

It is obvious that if we want to impose the tensorial behavior on D^3x^k, then both coefficients $\tilde{B}^l_{l n}$ and $\tilde{C}^k_{lmn}$ must transform as tensors given the manifestly tensorial character of the products of differentials they are contracted with. In contrast, the coefficients B^k_{ln} and C^k_{lmn} have obviously non-tensorial character, which is compensated by the connection coefficients and their derivatives entering the definitions of $\tilde{B}^k_{lm}$ and $\tilde{C}^{l}_{lmn}$. In order to get the transformation rules for the coefficients B^k_{lm} and C^k_{lmn} we use the observation that the formula (13) implicitly determines how the covariant differential D is acting on the second order differentials. Indeed the left-hand side of (13) can be written in the form

$$\begin{aligned} D^3x^k &= D(D^2x^k) = D(d^2x^k + \Gamma^k_{rs}\, Dx^r Dx^s) \\ &= D(d^2x^k) + \frac{\partial \Gamma^k_{rs}}{\partial x^l}\, Dx^l Dx^r Dx^s + (q\, \Gamma^k_{rs} + q^2\, \Gamma^k_{sr})\, dx^r D^2 x^s. \end{aligned} \tag{16}$$

Before we express $D(d^2x^k)$ in terms of the coefficients B^i_{lm} and C^i_{lmn} let us introduce the following notations. Any 3-index quantity $\mathcal{R}_{lmn}$ can be split into *three* parts belonging to three irreducible representations of the cyclic group Z_3:

$$\mathcal{R}_{lmn} = \mathcal{R}_{(lmn)} + \mathcal{R}_{\{lmn\}} + \mathcal{R}_{(lmn]},$$

defined as follows:

$$\begin{aligned} \mathcal{R}_{(lmn)} &= \frac{1}{3}(\,\mathcal{R}_{lmn} + \mathcal{R}_{nlm} + \mathcal{R}_{mnl}); \\ \mathcal{R}_{\{lmn\}} &= \frac{1}{3}(\,\mathcal{R}_{lmn} + q^2\, \mathcal{R}_{nlm} + q\, \mathcal{R}_{mnl}); \\ \mathcal{R}_{[lmn]} &= \frac{1}{3}(\,\mathcal{R}_{lmn} + q\, \mathcal{R}_{nlm} + q^2\, \mathcal{R}_{mnl}). \end{aligned} \tag{17}$$

Now we can express $D(d^2x^k)$ in terms of coefficients B^i_{lm} and C^i_{lmn} as follows

$$D(d^2x^k) = B^i_{lm}\, Dx^l D^2 x^m + (C^i_{lms} - \Gamma^i_{r[s}\Gamma^r_{lm]} - \Gamma^r_{[ms}\Gamma^i_{l]r})\, Dx^l Dx^m Dx^s. \tag{18}$$

Differentiating covariantly both sides of (10) one obtains the relation

$$D(d^2x^k) = \frac{\partial x^k}{\partial y^{k'}}\, D(d^2 y^{k'}) - \frac{\partial^2 x^k}{\partial y^{k'} \partial y^{l'}} \Gamma^{l'}_{r's'} Dx^{k'} Dx^{r'} Dx^{s'}. \tag{19}$$

In order to give the transformation rules a more compact form we shall use the following notations

$$U^i_{j'} = \frac{\partial x^i}{\partial y^{j'}}, \qquad \partial_j U^{i'}_k = \frac{\partial^2 y^{i'}}{\partial x^k \partial x^j}.$$

Then replacing $D(d^2x^k)$ and $D(d^2y^{k'})$ in the above formula by their expressions in terms of the coefficients B^i_{lm} and C^i_{lms} according to (18) and collecting together similar terms we get the transformation rules

$$B^i_{lm} = B^{i'}_{l'm'}\, U^i_{i'} U^{l'}_l U^{m'}_m$$

$$C^i_{lms} = U^i_{i'} U^{l'}_l U^{m'}_m U^{n'}_n\, C^{i'}_{l'm's'} + U^i_{i'} U^{n'}_{[n} \partial_m U^{\tau'}_{l]}\, \Gamma^{i'}_{\tau' n'} + U^i_{s'} \partial_r U^{s'}_{[n} U^{l'}_{l} U^{m'}_m U^{\tau}_{\tau'}\, \Gamma^{\tau'}_{l'm'} +$$
$$U^i_{s'} \partial_\tau U^{s'}_{[n} \partial_l U^{t'}_{m]} U^{\tau}_{t'} + U^i_{i'} U^{l'}_{[n} \partial_l U^{r'}_{m]}\, \Gamma^{i'}_{l'r'} + U^i_{s'} \partial_\tau U^{s'}_{[n} U^{m'}_l U^{n'}_{m]} U^{\tau}_{\tau'}\, \Gamma^{\tau'}_{m'n'} + U^i_{s'} U^{\tau}_{t'} \partial_r U^{s'}_{[n} \partial_l U^{t'}_{m]}$$

3 The Z_3-graded analogue of torsion tensor.

As in the case of torsion in ordinary exterior calculus, the tensorial character is displayed only by the irreducible part of these coefficients displaying the corresponding symmetry.

Here is what we do mean by this. As in the usual case the connection coefficients Γ^k_{lm} could be split into two parts, the torsion (antisymmetric part) and the symmetric part,

$$\Gamma^k_{lm} = \frac{1}{2}\left[\Gamma^k_{lm} + \Gamma^k_{ml}\right] + \frac{1}{2}\left[\Gamma^k_{lm} - \Gamma^k_{ml}\right] = \Gamma^k_{(lm)} + S^k_{lm}.$$

so the four-index symbols C^k_{lmn} as we have mentioned earlier can be split into *three* parts belonging to three representations of the cyclic group Z_3 :

$$C^k_{lmn} = C^k_{(lmn)} + C^k_{\{lmn\}} + C^k_{(lmn]}.$$

Therefore, only the part $\tilde{C}^k_{[lmn]}$ should appear in the final expression:

$$D^3 x^k = \tilde{B}^k_{lm}\, Dx^l D^2 x^m + \tilde{C}^k_{[lmn]}\, Dx^l Dx^m Dx^n =$$

$$= \left(B^k_{lm} + q^2\,\Gamma^k_{ml} + q\,\Gamma^k_{lm}\right) Dx^l D^2 x^m + \tilde{C}^k_{[lmn]}\, Dx^l Dx^m Dx^n \tag{20}$$

$$\text{with} \qquad \tilde{C}^k_{lmn} = C^k_{lmn} + \partial_l \Gamma^k_{mn} - \Gamma^r_{lm}\Gamma^k_{rn} - q\,\Gamma^r_{mn}\Gamma^k_{lr}$$

Because the coefficients in both terms on the right-hand side are tensors, we can start to investigate their intrinsic properties. Among these, the condition of reality should be applied to both coefficients separately. Starting with the first coefficient, $\tilde{B}^k_{lm}$, and recalling that $q = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ and $q^2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$, we have explicitly

$$\tilde{B}^k_{lm} = B^k_{lm} - \frac{1}{2}\left(\Gamma^k_{ml} + \Gamma^k_{lm}\right) + \frac{i\sqrt{3}}{2}\left(\Gamma^k_{lm} - \Gamma^k_{ml}\right)$$

The imaginary part is the *torsion tensor* of the connection Γ^k_{lm}, so the reality of the coefficient $\tilde{B}^k_{lm}$ is equivalent with the vanishing of the torsion, leaving only the *symmetric* part of Γ^k_{lm}. So, from now on, we can write

$$\tilde{B}^k_{lm} = B^k_{lm} - \Gamma^k_{lm}, \quad \text{with} \quad \Gamma^k_{lm} = \Gamma^k_{ml}.$$

This means that the B^k_{lm} transform as connection coefficients, so that the difference $B^k_{lm} - \Gamma^k_{lm}$ is a tensor. As a corollary, the vanishing of D^3x^k implies that $B^k_{lm} = \Gamma^k_{lm}$ and $\Gamma^k_{lm} = \Gamma^k_{ml}$.

The symmetry of the connection coefficients makes it possible to identify the tensor appearing in the coefficients $\tilde{C}^k_{lmn}$. As a matter of fact, only the part $\tilde{C}^k_{[lmn]}$ is relevant here, the other two irreducible parts' contribution vanishing when contracted with the covariant expression $Dx^l Dx^m Dx^n$. The part of $\tilde{C}^k_{lmn}$ containing the connection coefficients and their derivatives should be also Z_3-anti-symmetrized, yielding

$$\frac{1}{3}\left(C^k_{lmn} + qC^k_{nlm} + q^2C^k_{mnl} + \partial_l\Gamma^k_{mn} + q\partial_n\Gamma^k_{lm} + q^2\partial_m\right) Dx^l Dx^m Dx^n \Gamma^k_{nl}$$

$$-\frac{1}{3}\left(\Gamma^r_{lm}\Gamma^k_{rn} - q\Gamma^r_{nl}\Gamma^k_{rm} - q^2\Gamma^r_{mn}\Gamma^k_{rl} - q\Gamma^r_{mn} - q^2\Gamma^r_{lm}\Gamma^k_{nr} - \Gamma^r_{nl}\Gamma^k_{rm}\right) Dx^l Dx^m Dx^n$$

It is not difficult, taking into account the symmetries, to identify the final result in terms of the Riemann tensor: $\tilde{C}^k_{lmn}\, Dx^l Dx^m Dx^n$ is equal to

$$\left(C^k_{[lmn]} + \frac{1}{3}\,[\,R^k_{nlm} + R^k_{mln}\,] + \frac{q}{3}\,[R^k_{mnl} + R^k_{lnm}\,] + \frac{q^2}{3}\,[R^k_{lmn} + R^k_{nml}\,]\right) Dx^l Dx^m Dx^n$$

Taking into account that

$$C^k_{[lmn]} = \frac{1}{3}\left(C^k_{[lmn]} + q\, C^k_{[nlm]} + q^2\, C^k_{[mnl]}\right)$$

and assuming that the coefficients $C^k_{[lmn]}$ are real, the vanishing of the above expression leads to the equality

$$C^k_{[lmn]} = -\,[\,R^k_{nlm} \,+\, R^k_{mln}\,] \tag{21}$$

The analogy with the usual exterior differential calculus is now obvious. In the usual case the condition $D^2x^k = 0$ implied the vanishing of torsion,

$$S^k_{lm} = \frac{1}{2}[\Gamma^k_{lm} - \Gamma^k_{ml}] = 0,$$

whereas now, in our Z_3-graded case, the similar condition $D^3x^k = 0$ implies not only the vanishing of torsion, but also determines entirely the coefficients B^k_{lm} (equal to $\Gamma^k_{(ml)}$), and partly the coefficients C^k_{lmn} namely, their q-skew-symmetric part $C^k_{[lmn]}$

(equal then to the expression $-[R^k_{nlm} + R^k_{mln}]$). By analogy, one can impose similar conditions on the "conjugate" q^2-skewsymmetric part $C^k_{\{mnl\}}$, def ning it e.g. as $C^k_{\{mnl\}} = C^k_{[lnm]}$. However, the *totally symmetric* part $C^k_{(lmn)}$, which is not a tensor, is still undefined, because its contribution cancels automatically when contracted with the 3-form $Dx^l Dx^m Dx^n$.

We also believe that thee exists a tensorial quantity that can be obtained with the combinations of a certain number of derivatives of $C_{(lmn)}$, maybe up to the third order, just like the Riemann tensor is produced out of the Christoffel symbols. Until now we haven't found the proper expression.

4 Generalized geodesics of third order

The symmetric part of C^k_{lmn} together with the coefficients B^k_{lm} may be used for the definition of a new kind of parallel transport and generalized geodesic curves. One can define, independently of usual covariant derivative of a vector along a parametrized curve $x^k(\lambda)$ determined by connection coefficients Γ^k_{lm}.

$$\frac{DY^k}{D\lambda} = \frac{dY^k}{d\lambda} + \Gamma^k_{lm}\frac{dx^l}{d\lambda}Y^m = 0, \tag{22}$$

a second-order covariant derivative which is not an iteration of the first one:

$$\frac{\mathcal{D}^2Y^k}{\mathcal{D}\lambda^2} = \frac{d^2Y^k}{d\lambda^2} + E^k_{lm}\frac{dx^k}{d\lambda}\frac{DY^m}{D\lambda} + F^k_{lm}\frac{D^2x^l}{D\lambda^2}Y^m + G^k_{lmn}\frac{dx^l}{d\lambda}\frac{dx^m}{d\lambda}Y^n \tag{23}$$

where we use a different notation, $\frac{\mathcal{D}}{\mathcal{D}\lambda}$ in order to stress that we don't consider here a simple iteration of the usual covariant differentiation $\frac{D}{D\lambda}$. The coefficients E^k_{lm} and F^k_{lm} are not identical a priori; all we need to know about the transformation properties of E^k_{lm}, F^k_{lm} and G^k_{lmn} is that the resulting quantity $\frac{\mathcal{D}^2Y^k}{\mathcal{D}\lambda^2}$ transforms as a vector under a coordinate change.

If we replace the vector field $Y^k(x^m(\lambda))$ by the vector $\frac{dx^k}{d\lambda}$ tangent to the curve, we obtain a third-order generalization of the geodesic equation:

$$\frac{\mathcal{D}^3x^k}{\mathcal{D}\lambda^3} = \frac{d^3x^k}{d\lambda^3} + [E^k_{lm} + F^k_{ml}]\frac{dx^l}{d\lambda}\frac{D^2x^m}{D\lambda^2} + G^k_{lmn}\frac{dx^l}{d\lambda}\frac{dx^m}{d\lambda}\frac{dx^n}{d\lambda} = 0 \tag{24}$$

Now the $\frac{dx^k}{d\lambda}$ etc. are commutative entities, so that $G^k_{lmn} = G^k_{(lmn)}$. Had we iterated the usual covariant derivative in the above equation, then the coefficients $E^k_{lm} + F^k_{lm}$ and $C^k_{(lmn)}$ would be completely determined from the connection coefficients Γ^k_{lm} and their derivatives; however, we can introduce more general coefficients having the required transformation properties and independent of Γ^k_{lm}. This is reminiscent of a similar situation one level below, when the Christoffel connection is totally determined by the metric, but a larger class of affine connections exist which are independent of metric.

The generalized geodesic equation of third order (24) defines a larger class of curves that the usual geodesics and may be of interest in probing certain geometrical objects. For example, in the flat Euclidean space the solutions of the equation

$$\frac{d^3\mathbf{r}}{d\lambda^3} = 0, \qquad \text{which are} \qquad \mathbf{r}(\lambda) = \mathbf{a}\frac{\lambda^2}{2} + \mathbf{b}\,\lambda + \mathbf{c}, \tag{25}$$

where $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three arbitrary vectors easily identified with the initial conditions at $\lambda = 0$:

$$\mathbf{c} = \mathbf{r}\mid_{\lambda=0}, \quad \mathbf{b} = \frac{d\mathbf{r}}{d\lambda}\mid_{\lambda=0}, \quad \mathbf{a} = \frac{d^2\mathbf{r}}{d\lambda^2}\mid_{\lambda=0},$$

include not only the straight lines, but also all possible hyperbolae. If we want to include the closed trajectories, we must project the solutions of similar equation in a four-dimensional Euclidean space onto its arbitrary three-dimensional subspace.

Thus, all trajectories which are solutions of the Kepler problem, i.e. the set of all conics in a 3-dimensional Euclidean space, can be put in a one-to-one correspondence with the full set of solutions of a homogeneous *third order* differential equation $\frac{d^3\mathbf{r}}{d\lambda^3} = 0$ in a four-dimensional Euclidean space.

After identifying the constants that fix this correspondence, the time dependence (i.e. the law of motion $\mathbf{r}(t)$) will be encoded in the functional dependence $\lambda = \lambda(t)$, which we can establish only implicitly, in the form of an integral (usually of elliptic type).

If the equation (25) is replaced by its covariant generalization (24), then the influence of various perturbations, including the effects of a post-Newtonian approximation of General Relativity, may be encoded in the coefficients E^k_{lm}, F^k_{lm} and G^k_{lmn}. We believe that this approach can lead to some interesting developments in the research of approximate solutions of the two-body problem in General Relativity, including eventually some modifications or deviations due to certain supplementary effects.

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NEUTRON STARS IN MULTIDIMENSIONAL EINSHTEIN THEORY OF GRAVITATION. DIFFERENT VERSIONS OF THE EQUATION OF STATE

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Abstract

The radius, mass and other parameters of static spherically symmetric neutron stars are calculated in multidimensional space-time for different versions of the equation of state of superdense matter. A model with one Ricci flat internal space of any dimension is used.

1 Introduction

The hypothesized existence of additional space dimensions seems to adequately reflect the dynamics of interactions over small distances ($r \ll 10^{-16}$cm) typical for grand unified theories. This is corroborated by the achievements of string theory [1] and supergravitation theory [2, 3] (also see [4, 5] and the references therein on the theory of multidimensional space-time). The main efforts made in the framework of Einstein theory of multidimensional space-time have been directed to fitting the predictions of multidimensional cosmological models (the universe arising from some multidimensional formation) to observed data. At the same time, it is fairly important that the effects of additional dimensions may be manifested also in island type gravitating systems.

The present work is a continuation of [6]. We determine the radius, mass and other parameters of static spherically symmetric neutron stars in the space-time of an arbitrary dimension, for different versions of the equation of state of stellar matter.

2 Field equations

In the field theory limit of the superstring theory, the gravitation is described with sufficient accuracy by the Einstein equations

$$R_{AB} - \frac{1}{2}g_{AB}R = 8\pi æ T_{AB}, \tag{1}$$

in which R_{AB} is the Ricci tensor of multidimensional space-time with the line element of the form

$$ds^2 = g_{AB}dx^A dx^B, \qquad A, B = 0, 1, 2, ...k + 3 \tag{2}$$

and a signature $\left(+\underbrace{-...-}_{k+3}\right)$ for the metric tensor g_{AB}. In (1) T_{AB} is the multidimensional energy-momentum tensor and $æ$ is the multidimensional analog of the gravitational constant (the speed of light c=1). One is to remember that the boson and supersymmetric string models of grand unified theories have a meaning only in the space-time of dimensionality D=k+4=26 and 10, respectively [1,3]. We confine ourselves to the space-time model

$$R^D = T \times R^3_{(0)} \times R^k_{(1)}, \qquad D = 1 + 3 + k = 10 \text{ or } 26. \tag{3}$$

in which the inner space $R^k_{(1)}$ formed by the additional dimensions is the Ricci-flat:

$$R_{(1)ab} = 0, \qquad a, b = 4, 5, ..., k + 3 \tag{4}$$

and compact.

For a static, spherically symmetric celestial body the line element can be chosen in the form

$$ds^2 = ds^2_0 - g_{(1)ab}e^{\sigma}dx^a dx^b, \tag{5}$$

where

$$ds^2_0 = e^{\nu}dt^2 - e^{\lambda}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)], \tag{6}$$

is the line element of the four-dimensional space-time $R^4 = T \times R^3_{(0)}$ in isotropic spherical coordinates r, θ and φ, ν, λ, σ are functions of the coordinate r. What concerns the $g_{(1)ab}$, it is a metric tensor of the inner space-time $R^k_{(1)}$ and depends on the coordinate x^a in the subspace $R^k_{(1)}$.

The energy-momentum tensor in the multidimensional space-time may be represented as

$$T^A_B = diag(\varepsilon, -p, -p, -p, \underbrace{-p_1, ..., -p_1}_{k}) \tag{7}$$

Here ε and p are the energy density and pressure of the superdense matter of the neutron star, and p_1 is the effective pressure corresponding to the additional dimensions.

The solution of the field equations outside the neutron star is as follows

$$e^{\nu} = \left(\frac{1-v}{1+v}\right)^{\beta}, \quad e^{\lambda} = \left(1-v^2\right)^2 \left(\frac{1+v}{1-v}\right)^{\beta(1+\alpha k)}, \quad e^{\sigma} = \left(\frac{1-v}{1+v}\right)^{\alpha\beta}, \tag{8}$$

where $v = r_g/2\beta r$,

$$r_g = 2GM \tag{9}$$

is the gravitational radius of the neutron star, M is its gravitating mass and α, β are connected by a relation

$$\beta^2 = \frac{8}{1+\alpha^2 k + (1+\alpha k)^2}. \tag{10}$$

The integration constant α is determined by an integration over the distribution of matter inside the star [7]:

$$\alpha = \frac{2\bar{b}}{2+k(1-\bar{b})}, \qquad b = \frac{3p-\varepsilon-2p_1}{\varepsilon+3p}, \tag{11}$$

where

$$\bar{b} = \frac{\int b(\varepsilon+3p)\sqrt{|g|}d^{k+3}x}{\int(\varepsilon+3p)\sqrt{|g|}d^{k+3}x}. \tag{12}$$

The functions $\nu(r)$, $\lambda(r)$, and $\sigma(r)$ are determined by means of the field equations. One can write three of them as:

$$\begin{aligned} R_0^0 &= 8\pi æ\left(T_0^0 - \frac{T}{k+2}\right), \\ R_0^0 + R_3^3 + R_a^a &= 8\pi æ\left(T_1^1 + T_2^2\right), \\ R_b^a &= 8\pi æ\left(T_b^a - \delta_b^a \frac{T}{k+2}\right), \end{aligned} \tag{13}$$

where δ_b^a is the Kronecker symbol. We shall also use the component A=1 of the matter response equation

$$T_{A;B}^{B} = 0. \tag{14}$$

So we have four equations for six variables:

$$\nu(r), \quad \lambda(r), \quad \sigma(r), \quad \varepsilon(r), \quad p(r), \quad p_1(r) \tag{15}$$

and we need two others.

As it is known [8,9], the pressure p of superdense degenerate matter is a function of its energy density ε.

In Table I we give information concerning 4 versions of the equation of state $p = p(\varepsilon)$ that we have used in our numerical calculations.

Table I

Verson	References	Equation of state
1	[9]	Grigorian & Sahakian
2	[10]	Pandhriapande 1971 (neutron), BBP[a], BPS[b] and FMT[c]
3	[11]	PS[d],AØ[e]-5, BBP[a]−2 and BBP[a]-1
4	[11]	PS[d],WFF[f]-3, BBP[a]−2, BBS[b]-1, and HZD[g]

[a]Baym, Bete & Pethik 1971
[b]Baym, Bete & Sutherland 1971
[c]Feynman, Metropolis & Teller 1949
[d]Pandharipande & Smith 1975
[e]Arnsten & Østgard 1984
[f]Wiringa, Fiks & Fabrocini1988
[g]Haensel, Zdunik & Dobaczewski 1989

We have no information concerning the dependence of effective pressure p_1 on ε and p. We have used a simple linear approximation

$$p_1 = -\varepsilon/2 + ap, \tag{16}$$

where a is some adjustable parameter. Such an approximation is in accordance with known post-Newtonian effects of the relativistic gravity [6].

Having Eqns. (16) and fixing a and k (dimension of the inner space), we can solve Eqns. (13) and (14) for different values of the pressure $p(0)$ at the center of the neutron star.

3 Results of numerical calculations

In Fig.1 we give the mass M of superdense spherically symmetric configurations (in solar units) in space-time with D=10 and a=-1, 1.5, and 4 as a function of the logarithm to the base ten of central pressure $p(0)$ (in SGS units). The neutron stars correspond to points to the left of the first maximum of each curve. For a=1.5 the results of multidimensional theory coincide with General Relativity predictions. It is clear that the maximum mass of the neutron stars depends on the value of a. Presented in this figure are results of calculation for one version of the equation of state of superdense mater. Results of calculations for another versions of the equation of state are presented in Figs.2-4. From these data it follows that the maximum mass of neutron stars depends essentially on the value of the parameter a: $M_{max} = M_{max}(a)$ for all versions of the equation of state used.

The dependence of M_{max} on a is given in Fig.5 for equations of state under consideration. It seems that the maximum mass has the greatest value at $3 \leq a \leq 4$ that exceeds the corresponding greatest General Relativity mass ($M_{max} = 2.14 M_{\odot}$). The M_{max} falls off monotonically with distance from its greatest value and this conclusion is valid for all versions of the equation of state used in our calculation. Such a dependence of M_{max} on a enables us to determine the allowable values of a by comparing our results with observed data [12] (the dashed line in the figure). Such a comparison leads to the strict limits: $1 \leq a \leq 7$.

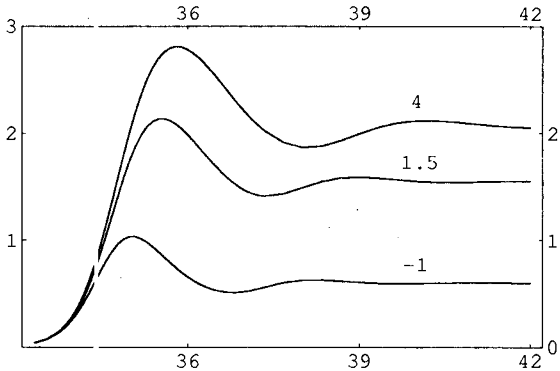


Figure 1:

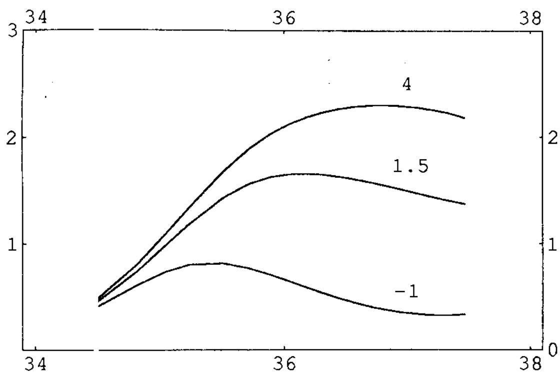
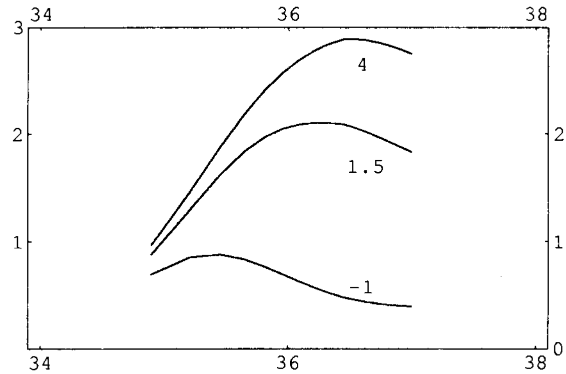


Figure 2:



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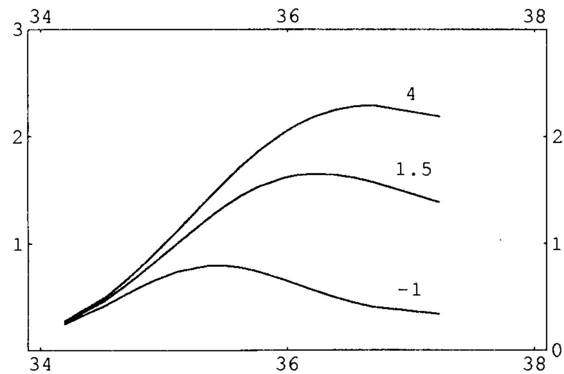

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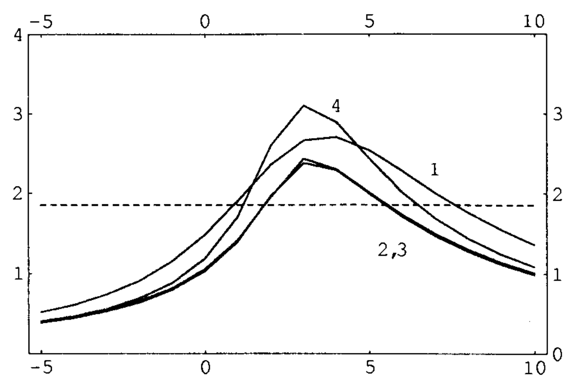

Figure 5:

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SEPARABLE FOUR-DIMENSIONAL HARMONIC OSCILLATORS AND REPRESENTATIONS OF THE POINCARÉ GROUP.

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Abstract

It is possible to construct representations of the Lorentz group using four-dimensional harmonic oscillators. This allows us to construct three-dimensional wave functions with the usual rotational symmetry for space-like coordinates and one-dimensional wave function for time-like coordinate. It is then possible to construct a representation of the Poincaré group for a massive particles having the $O(3)$ internal space-time symmetry in its rest frame. This oscillator can also be separated into two transverse components and the two-dimensional world of the longitudinal and time-like coordinates. The transverse components remain unchanged under Lorentz boosts, while it is possible to construct the squeeze representation of the $O(1,1)$ group in the space of the longitudinal and time-like coordinates. While the squeeze representation forms the basic language for squeezed states of light, it can be combined with the transverse components to form the representation of the Poincarè group for relativistic extended particles.

1 Introduction.

Harmonic oscillators played the essential role in the development of quantum mechanics. From the mathematical point of view, the present form of quantum mechanics did not advance too far from the oscillator framework. Thus, the first relativistic wave function has to be the oscillator wave function. With this point in mind, Dirac in 1945 wrote down a normalizable four-dimensional wave function and attempted to construct representations of the Lorentz group [1], and started a history of the oscillators which can be Lorentz-boosted [2, 3].

Let us consider the Minkowskian space consisting of the three-dimensional space of (x, y, z) and one time variable t. Then the quantity $(x^2 + y^2 + z^2 - t^2)$ remains invariant under Lorentz transformations. On the other hand, $(x^2 + y^2 + z^2 + t^2)$ is not invariant. Thus, the exponential form

$$\exp\left(x^2 + y^2 + z^2 - t^2\right) \tag{1}$$

is a Lorentz-invariant quantity, while the form

$$\exp\left(x^2 + y^2 + z^2 + t^2\right) \tag{2}$$

is not. The Gaussian function of Eq.(2) is localized in the t variable. It was Dirac who wrote down this normalizable Gaussian form in 1945 [1]. In 1963, the author of this report was fortunate enough to hear directly from Prof. Dirac that the physics of special relativity is much richer than writing down Lorentz-invariant quantities. This could mean that we should study more systematically the above normalizable form under the influence of Lorentz boosts, and this study has led to the observation that Lorentz boosts are squeeze transformations [4, 5].

The exponential form of Eq.(1) is Lorentz-invariant but cannot be normalized. This aspect was noted by Feynman *et al.* in their 1971 paper [6]. In their paper, Feynman *et al.* tell us not to use Feynman diagrams but to use oscillator wave functions when we approach relativistic bound states. In this report, we shall use much of the formalism given in Ref. [6], but not their Lorentz-invariant wave functions which are not normalizable.

In this report, we discuss the oscillator representations of the $O(3,1)$ and explain why this representation is adequate for internal space-time symmetry of relativistic extended hadrons. This group has $O(1,1)$ as a subgroup which describes Lorentz boosts along a given direction. It is shown that Lorentz boosts are squeeze transformations. It is shown also that the infinite-dimensional unitary representation of this squeeze group constitutes the mathematical basis for squeezed states of light.

In Sec. 2, we discuss the three-parameter subgroups of the six-parameter Lorentz group which leaves the four-momentum of the particle invariant. These groups govern the internal space-time symmetries of relativistic particles, and they are called Wigner's little groups [7]. In Sec. 3, it is shown that the covariant harmonic oscillators constitute a representation of the Poincaré group for relativistic extended particles. If we add Lorentz boosts to the oscillator formalism, the symmetry group becomes non-compact, and its unitary representations become infinite-dimensional. In Sec. 4, we study an infinite-dimensional unitary representation for the harmonic oscillator formalism.

2 Subgroups of the Lorentz Group.

The Poincaré group is the group of inhomogeneous Lorentz transformations, namely Lorentz transformations followed by space-time translations. This group is known as the fundamental space-time symmetry group for relativistic particles. This ten-parameter group has many different representations. For space-time symmetries, we study first Wigner's little groups. The little group is a maximal subgroup of the six-parameter Lorentz group whose transformations leave the four-momentum of a given particle invariant. The little group therefore governs the internal space-time symmetry of the particles. Massive particles in general are known as spin degrees of freedom. Massless particles in general have the helicity and gauge degrees of freedom.

Let J_i be the three generators of the rotation group and K_i be the three boost generators. They then satisfy the commutation relations

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \qquad [J_i, K_j] = i\epsilon_{ijk}K_k, \qquad [K_i, K_j] = -i\epsilon_{ijk}J_k. \tag{3}$$

The three-dimensional rotation group is a subgroup of the Lorentz group. If a particle is at rest, we can rotate it without changing the four-momentum. Thus, the little group for massive particles is the three-parameter rotation group. If the particle is boosted, it gains a momentum along the boosted direction. If it is boosted along the z direction, the boost operator becomes

$$B_3(\eta) = \exp\left(-i\eta K_3\right). \tag{4}$$

The little group is then generated by

$$J_i' = B_3(\eta)J_3\left(B_3(\eta)\right)^{-1}. \tag{5}$$

These boosted $O(3)$ generators satisfy the same set of commutation relations as the set for $O(3)$.

Table I. Covariance of the energy-momentum relation, and covariance of the internal space-time symmetry groups. The quark model and the parton model are two different manifestations of the same covariant entity.

Massive, Slow	COVARIANCE	Massless, Fast
$E = p^2/2m$	Einstein's $E = mc^2$	$E = cp$
S_3 S_1, S_2	Wigner's Little Group	S_3 Gauge Trans.
Quarks	Covariant Harmonic Oscillators	Partons

If the parameter η becomes infinite, the particle becomes like a massless particle. If we go through the contraction procedure spelled out by Inonu and Wigner in 1953

[8], the $O(3)$-like little group becomes contracted to the $E(2)$-like little group for massless particles generated by J_3, N_1, and N_2 [9, 10],where

$$N_1 = K_1 - J_2, \qquad N_2 = K_2 + J_1. \tag{6}$$

These N generators are known to generate gauge transformations for massless particles [11, 12]. Gauge transformations for spin-1 photons are well known. As for massless spin-1/2 particles, neutrino polarizations are due to gauge invariance.

The transition from massive to massless particles is illustrated in the second row of Table I. In Sec. 3, we shall discuss how a massive particle with space-time extension be boosted from its rest frame to an infinite-momentum frame. This aspect is illustrated in the third row of Table I.

3 Covariant Harmonic Oscillators.

If we construct a representation of the Lorentz group using normalizable harmonic oscillator wave functions, the result is the covariant harmonic oscillator formalism [3]. The formalism constitutes a representation of Wigner's O(3)-like little group for a massive particle with internal space-time structure. This oscillator formalism has been shown to be effective in explaining the basic phenomenological features of relativistic extended hadrons observed in high-energy laboratories. In particular, the formalism shows that the quark model and Feynman's parton picture [13] are two different manifestations of one relativistic entity [3, 14].

The essential feature of the covariant harmonic oscillator formalism is that Lorentz boosts are squeeze transformations [4]. In the light-cone coordinate system, the boost transformation expands one coordinate while contracting the other so as to preserve the product of these two coordinate remains constant. We shall show that the parton picture emerges from this squeeze effect.

The covariant harmonic oscillator formalism has been discussed exhaustively in the literature, and it is not necessary to give another full-fledged treatment in the present paper. We shall discuss here one of the most puzzling problems in high-energy physics, namely whether quarks are partons. It is now a well-accepted view that hadrons are bound states of quarks. This view is correct if the hadron is at rest or nearly at rest. On the other hand, it appears as a collection of partons when it moves with a speed very close to that of light. This is called Feynman's parton picture [13].

Let us consider a bound state of two particles. For convenience, we shall call the bound state the hadron, and call its constituents quarks. Then there is a Bohr-like radius measuring the space-like separation between the quarks. There is also a time-like separation between the quarks, and this variable becomes mixed with the longitudinal spatial separation as the hadron moves with a relativistic speed. There are no quantum excitations along the time-like direction. On the other hand, there is the time-energy uncertainty relation which allows quantum transitions. It is possible to accommodate these aspect within the framework of the present form of quantum mechanics. The uncertainty relation between the time and energy variables is the c-number relation, which does not allow excitations along the time-like coordinate. We shall see that the covariant harmonic oscillator formalism accommodates this narrow window in the present form of quantum mechanics.

Let us consider a hadron consisting of two quarks. If the space-time position of two quarks are specified by x_a and x_b respectively, the system can be described by the variables

$$X = (x_a + x_b)/2, \qquad x = (x_a - x_b)/2\sqrt{2}. \tag{7}$$

The four-vector X specifies where the hadron is located in space and time, while the variable x measures the space-time separation between the quarks. In the convention of Feynman *et al.* [6], the internal motion of the quarks bound by a harmonic oscillator potential of unit strength can be described by the Lorentz-invariant equation

$$\frac{1}{2}\left\{x_\mu^2 - \frac{\partial^2}{\partial x_\mu^2}\right\}\psi(x) = \lambda\psi(x). \tag{8}$$

We use here the space-favored metric: $x^\mu = (x, y, z, t)$.

It is possible to construct a representation of the Poincaré group from the solutions of the above differential equation [3]. If the hadron is at rest, the solution should take the form

$$\psi(x,y,z,t) = \phi(x,y,z)\left(\frac{1}{\pi}\right)^{1/4}\exp\left(-t^2/2\right), \tag{9}$$

where $\phi(x,y,z)$ is the wave function for the three-dimensional oscillator. If we use the spherical coordinate system, this wave function will carry appropriate angular momentum quantum numbers. Indeed, the above wave function constitutes a representation of Wigner's $O(3)$-like little group for a massive particle [3]. There are no time-like excitations, and this is consistent with our observation of the real world. It was Dirac who noted first this space-time asymmetry in quantum mechanics [15]. However, this asymmetry is quite consistent with the $O(3)$ symmetry of the little group for hadrons.

Since the three-dimensional oscillator differential equation is separable in both the spherical and Cartesian coordinate systems, the spherical form of $\phi(x,y,z)$ consists of Hermite polynomials of x, y, and z. If the Lorentz boost is made along the z direction, the x and y coordinates are not affected, and can be dropped from the wave function. The wave function of interest can be written as

$$\psi^n(z,t) = \left(\frac{1}{\pi}\right)^{1/4}\exp\left(-t^2/2\right)\phi_n(z), \tag{10}$$

with

$$\phi_n(z) = \left(\frac{1}{\pi n! 2^n}\right)^{1/2} H_n(z)\exp(-z^2/2), \tag{11}$$

where $\psi^n(z)$ is for the n-th excited oscillator state. The full wave function $\psi^n(z,t)$ is

$$\psi_0^n(z,t) = \left(\frac{1}{\pi n! 2^n}\right)^{1/2} H_n(z) \exp\left\{-\frac{1}{2}\left(z^2 + t^2\right)\right\}. \tag{12}$$

The subscript 0 means that the wave function is for the hadron at rest. The above expression is not Lorentz-invariant, and its localization undergoes a Lorentz squeeze as the hadron moves along the z direction [3].

It is convenient to use the light-cone variables to describe Lorentz boosts. The light-cone coordinate variables are

$$u = (z+t)/\sqrt{2}, \qquad v = (z-t)/\sqrt{2}. \tag{13}$$

In terms of these variables, the Lorentz boost along the z direction,

$$\begin{pmatrix} z' \\ t' \end{pmatrix} = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix}, \tag{14}$$

takes the simple form

$$u' = e^{\eta} u, \qquad v' = e^{-\eta} v, \tag{15}$$

where η is the boost parameter and is $\tanh^{-1}(v/c)$.

The wave function of Eq.(12) can be written as

$$\psi_o^n(z,t) = \psi_0^n(z,t) = \left(\frac{1}{\pi n! 2^n}\right)^{1/2} H_n\left((u+v)/\sqrt{2}\right) \exp\left\{-\frac{1}{2}(u^2+v^2)\right\}. \tag{16}$$

If the system is boosted, the wave function becomes

$$\psi_\eta^n(z,t) = \left(\frac{1}{\pi n! 2^n}\right)^{1/2} H_n\left((e^{-\eta}u + e^{\eta}v)/\sqrt{2}\right) \times \exp\left\{-\frac{1}{2}\left(e^{-2\eta}u^2 + e^{2\eta}v^2\right)\right\}. \tag{17}$$

In both Eqs. (16) and (17), the localization property of the wave function in the uv plane is determined by the Gaussian factor, and it is sufficient to study the ground state only for the essential feature of the boundary condition. The wave functions in Eq.(16) and Eq.(17) then respectively become

$$\psi^0(z,t) = \left(\frac{1}{\pi}\right)^{1/2} \exp\left\{-\frac{1}{2}(u^2+v^2)\right\}. \tag{18}$$

If the system is boosted, the wave function becomes

$$\psi_\eta(z,t) = \left(\frac{1}{\pi}\right)^{1/2} \exp\left\{-\frac{1}{2}\left(e^{-2\eta}u^2 + e^{2\eta}v^2\right)\right\}. \tag{19}$$

We note here that the transition from Eq.(18) to Eq.(19) is a squeeze transformation. The wave function of Eq.(18) is distributed within a circular region in the uv plane, and thus in the zt plane. On the other hand, the wave function of Eq.(19) is distributed in an elliptic region.

4 Unitary Infinite-dimensional Representation.

Let us go back to Eq.(16) and Eq.(17). We are now interested in writing them in terms of the one-dimensional oscillator wave functions given in Eq.(11). After some standard calculations [3], we can write the squeezed wave function as

$$\psi_\eta^n(z,t) = \left(\frac{1}{\cosh\eta}\right)^{n+1} \sum_k \left(\frac{(n+k)!}{n!k!}\right)^{1/2} (\tanh\eta)^k \phi_{n+k}(z)\phi_n(t). \tag{20}$$

If the parameter η becomes zero, this form becomes the rest-frame wave function of Eq.(16).

It is sometimes more convenient to use the parameter β defined as

$$\beta = \tanh\eta. \tag{21}$$

This parameter is the speed of the hadron divided by the speed of light. In terms of this parameter, the expression of Eq.(20) can be written as

$$\psi_\eta^n(z,t) = \left(1-\beta^2\right)^{(n+1)/2} \sum_k \left(\frac{(n+k)!}{n!k!}\right)^{1/2} \beta^k \phi_{n+k}(x)\phi_n(t). \tag{22}$$

If we take the integral

$$\int |\psi_\eta^n(z,t)|^2 dz dt = \left(1-\beta^2\right)^{n+1} \sum_k \left(\frac{(n+k)!}{n!k!}\right) \beta^{2k}. \tag{23}$$

The sum in the above expression is the same as the binomial expansion of

$$\left(1-\beta^2\right)^{-(n+1)}.$$

Thus, the right hand side of Eq.(23) is 1. The power series expansion of Eq.(20) reflects a well-known but hard-to-prove mathematical theorem that unitary representations of non-compact groups are infinite-dimensional. The Lorentz group is a non-compact group.

If $n = 0$, the above form becomes simplified to

$$\psi_\eta(z,t) = \left(1-\beta^2\right)^{1/2} \sum_k \beta^k \phi_k(z)\phi_k(t). \tag{24}$$

This is the power series expansion of Eq.(19). This relatively simple form is very useful in many other branches of physics.

It is well known that the mathematics of the Fock space in quantum field theory is that of harmonic oscillators. Among them, the coherent-state representation occupies a prominent place because it is the basic language for laser optics. Recently, two photon coherent states have been observed and the photon distribution is exactly like that of the wave function given in Eq.(20). These coherent states are commonly called squeezed states. It is very difficult to see why the word "squeeze" has to be associated with the power series expansion given in Eq.(20) or Eq.(22). It is however quite clear from the expression of Eq.(19) that Gaussian distribution is squeezed. Thus, the above representation tells us how squeezed states are squeezed.

Next, let us briefly discuss the role of this infinite dimensional representation in understanding the density matrix. For simplicity, we shall work with the squeezed ground-state wave function. From the wave function of Eq.(24), we can construct the pure-state density matrix

$$\rho_\eta(z,t;z',t') = \psi_\eta(z,t)\psi_\eta(z',t'), \tag{25}$$

which satisfies the condition $\rho^2 = \rho$:

$$\rho_\eta(z,t;z',t') = \int \rho_\eta(z,t;z'',t'')\rho_\eta(z'',t'';z',t')dz''dt''. \tag{26}$$

However, there are at present no measurement theories which accommodate the time-separation variable t. Thus, we can take the trace of the ρ matrix with respect to the t variable. Then the resulting density matrix is

$$\rho_\eta(z,z') = \int \psi_\eta(z,t)\left\{\psi_\eta(z',t)\right\}^* dt = \left(1-\beta^2\right)\sum_k \beta^{2k}\phi_k(z)\phi_k^*(z'). \tag{27}$$

It is of course possible to compute the above integral using the analytical expression given in Eq.(19). The result is

$$\rho_\eta(z,z') = \left(\frac{1}{\pi\cosh 2\eta}\right)^{1/2} \exp\left\{-\frac{1}{4}[(z+z')^2/\cosh 2\eta + (z-z')^2\cosh 2\eta]\right\}. \tag{28}$$

This form of the density matrix satisfies the trace condition

$$\int \rho(z,z)dz = 1. \tag{29}$$

The trace of this density matrix is one, but the trace of ρ^2 is less than one, as

$$Tr\left(\rho^2\right) = \int \rho_\eta^n(z,z')\rho_\eta^n(z',z)dz'dz = \left(1-\beta^2\right)^2\sum_k \beta^{4k}, \tag{30}$$

which is less than one and is $1/\left(1+\beta^2\right)$. This is due to the fact that we do not know how to deal with the time-like separation in the present formulation of quantum mechanics. Our knowledge is less than complete.

The standard way to measure this ignorance is to calculate the entropy defined as [16, 17]

$$S = -Tr\left(\rho\ln(\rho)\right). \tag{31}$$

If we pretend to know the distribution along the time-like direction and use the pure-state density matrix given in Eq.(25), then the entropy is zero. However, if we do not know how to deal with the distribution along t, then we should use the density matrix of Eq.(27) to calculate the entropy, and the result is [18]

$$S = 2[(\cosh\eta)^2 \ln(\cosh\eta) - (\sinh\eta)^2 \ln(\sinh\eta)]. \tag{32}$$

In terms of the velocity parameter, this expression can be written as

$$S = \frac{1}{1-\beta^2}\ln\frac{1}{1-\beta^2} - \frac{\beta}{1-\beta^2}\ln\frac{\beta}{1-\beta^2}. \tag{33}$$

From this we can derive the hadronic temperature [19].

In this report, the time-separation variable t played the role of an unmeasurable variable. The use of an unmeasurable variable as a "shadow" coordinate is not new in physics and is of current interest [20]. Feynman's book on statistical mechanics contains the following paragraph [21].

When we solve a quantum-mechanical problem, what we really do is divide the universe into two parts - the system in which we are interested and the rest of the universe. We then usually act as if the system in which we are interested comprised the entire universe. To motivate the use of density matrices, let us see what happens when we include the part of the universe outside the system.

In the present paper, we have identified Feynman's rest of the universe as the time-separation coordinate in a relativistic two-body problem. Our ignorance about this coordinate leads to a density matrix for a non-pure state, and consequently to an increase of entropy.

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DEFORMED QUANTUM RELATIVISTIC PHASE SPACES–AN OVERVIEW*

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Abstract

We describe three ways of modifying the relativistic Heisenberg algebra - first one not linked with quantum symmetries, second and third related with the formalism of quantum groups. The third way is based on the identification of generalized deformed phase space with the semidirect product of two dual Hopf algebras describing quantum group of motions and the corresponding quantum Lie algebra. As an example the κ-deformation of relativistic Heisenberg algebra is given, determined by κ-deformed D=4 Poincaré symmetries.

1 Introduction

The "naive" canonical quantization scheme, based on the Heisenberg relations ($i = 1, \ldots N$)

$$[\widehat{x}_i, \widehat{p}_j] = i\hbar\delta_{ij} \tag{1.1a}$$

$$[\widehat{x}_i, \widehat{x}_j] = [\widehat{p}_i, \widehat{p}_j] = 0 \tag{1.1b}$$

has been applied to quantum mechanics ($N = 3$; in general $N = 3n$ finite) as well as to quantum field theory ($N = \infty$). The modification of canonical quantization scheme in the presence of interactions in quantum field theory is well known, and the difficulty with existence of canonical ET limits causes the appearance of infinite wave function renormalization. It appear however, that also quantum-mechanical system in the presence of quantized gravitational forces can not be quantized canonically. The quantized gravitational field modifies the classical space time by replacing classical Riemannian geometry by at present not yet well understood quantum Riemannian geometry. One of expected properties of such a quantized geometry is the noncommutativity of space coordinates ($[x_i, x_j] \neq 0$) [1-4]. Further if we supplement the fourth Minkowski coordinate $x_0 = ct$ and add $p_0 = \frac{E}{c}$, one gets the following relativistic extension of the relations (1.1a-b)

$$[\widehat{x}_\mu, \widehat{p}_\nu] = i\hbar\eta_{\mu\nu} \tag{1.2a}$$

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$$[\widehat{x}_\mu, \widehat{x}_\nu] = [\widehat{p}_\mu, \widehat{p}_\nu] = 0 \tag{1.2b}$$

Again the relations (1.2a-b) will not be valid for arbitrary small distances, because taking into consideration the quantum gravity effects $[x_\mu, x_\nu] \neq 0$ [1-4]†

In this lecture I would like to review briefly different ways of obtaining deformations of relativistic phase space algebra (1.2a-b). The distinction between different ways depends on the role played by quantum deformations of respective symmetry groups. We shall distinguish the following three classes of deformations of the Heisenberg algebra (1.2a-b):

i) The deformed relations remain covariant under classical symmetry group. Such a deformation we shall call as described in the framework of classical symmetries (see Sect. 2).

ii) The quantum phase space coordinates are the corepresentations of quantum symmetry group and the deformed relations are covariant under the coaction of quantum symmetries. Such formalism is based on differential calculus on noncommutative spaces, with the derivatives representing momenta (see Sect. 3). In particular the deformations of the relations (1.2a-b) is obtained if we consider quantum phase space as described by the vector corepresentation of quantum Lorentz group (deformed Minkowski space) supplemented with noncommutative vector fields (deformed fourmomenta).

iii) The generalized quantum relativistic phase space is described by the semidirect product of two dual Hopf algebras, describing respectively the quantum Poincaré group and quantum Poincaré algebra. It appears that if we perform the contraction by putting trivial Lorentz group generators ($\Lambda_\mu{}^\nu = \delta_\mu{}^\nu$) one obtains the eight-dimensional deformed relativistic phase space subalgebra, span by the translation generators of quantum Poincaré group and fourmomentum generators of quantum Poincaré algebra. In particular we shall present in Sect. 4 such an example derived from deformed κ–Poincaré symmetries. In Sect. 5 we shall present some final remarks.

2 Deformations of Relativistic Heisenberg Algebra in the Framework of Classical Relativistic Symmetries

The first well-known proposal of noncommutativity for four relativistic space-time coordinates is due to Snyder ([5]; see also [6]). He assumed that (we put $\hbar = 1$):

$$[\widehat{x}_\mu, \widehat{x}_\nu] = i\ell^2\, M_{\mu\nu}\,, \tag{2.1}$$

†This conditions is weaker than $[\widehat{x}_i, \widehat{x}_j] \neq 0$ because one can have in relativistic case $[\widehat{x}_i, \widehat{x}_j] = 0$ but $[\widehat{x}_0, \widehat{x}_i^-] \neq 0$ (see also Sect. 4).

where $M_{\mu\nu}$ are Lorentz algebra generators, and

$$[M_{\mu\nu}, \widehat{x}_\rho] = i\left(\eta_{\nu\rho}\widehat{x}_\mu - \eta_{\mu\nu}\widehat{x}_\rho\right) \tag{2.2}$$

In order to describe the deformation of relativistic phase space (1.2a-b) we employ the commuting fourmomenta ($[p_\mu, p_\nu] = 0$). We get

$$\begin{aligned} M_{\mu\nu} &= i\left(p_\mu \frac{\partial}{\partial p^\nu} - p_\nu \frac{\partial}{\partial p^\mu}\right) \\ \widehat{x}_\mu &= i\left(1+\ell^2 p^2\right)^{\frac{1}{2}} \frac{\partial}{\partial p^\mu}, \end{aligned} \tag{2.3}$$

and

$$[\widehat{x}_\mu, \widehat{p}_\nu] = i\left(1+\ell^2 p^2\right)^{\frac{1}{2}} \eta_{\mu\nu}. \tag{2.4}$$

It is clear from the relations (2.1-4) that the deformed formalism is covariant under the <u>classical</u> Lorentz symmetries. Moreover, the relations (2.1-2) indicate that in the momentum space one can introduce fifth momentum coordinate p_5, satisfying the five - dimensional mass-shell condition $p_5^2 - p^2 = \frac{1}{\ell^2}$ or $p_5 = \frac{1}{l}(1+\ell^2p^2)^{\frac{1}{2}}$. It appears that the formalism has build in classical O(3,2) anti-de-Sitter symmetry with the deformation parameter ℓ equal to the inverse de-Sitter radius in five-dimensional momentum space. This property further was developed by Kadyshevsky [6] and his collaborators [7,8] as a way of introducing regularization in local relativistic field theory.

Second model which we would like to quote here has been recently proposed by Dopplicher, Fredenhagen and Roberts (DFR) [1]. Starting from the analysis of the uncertainty relations in the presence of quantum gravity effects (see also [2-4]) they proposed the following model of noncommutative space-time coordinate:

$$[\widehat{x}_\mu, \widehat{x}_\nu] = i\,\ell_p^2\,\Sigma_{\mu\nu}, \tag{2.5}$$

where $\Sigma_{\mu\nu} = -\Sigma_{\nu\mu}$ is the two-tensor, commuting with the coordinates $\widehat{x}_\mu$. It appears that the relation (2.5) can be supplemented by the <u>classical</u> relations $[p_\mu, p_\nu] = 0$, $[\widehat{x}_\mu, p_u] = i\eta_{\mu\nu}$. The DFR phase space relations are covariant under the classical Lorentz symmetries, provided that the Lorentz generators $M_{u\nu}$ contain additional term rotating the tensor components $\Sigma_{\mu\nu}$

$$[M_{\mu\nu}, \Sigma_{\rho\tau}] = i\left(\eta_{\nu\rho}\,\Sigma_{\mu\tau} - \eta_{\mu\rho}\,\Sigma_{\nu\tau} + \eta_{\mu\tau}\,\Sigma_{\nu\rho} - \eta_{\nu\tau}\,\Sigma_{\mu\rho}\right). \tag{2.6}$$

Moreover, because $\Sigma_{\mu\nu}$ are x_μ-independent translation-invariant operator, we see that the DFR formalism is invariant under the classical D=4 Poincaré transformations.

The relation (2.5) can be written for any D, but only in D=2 one can use standard Poincaré generators, because the central two-tensor $\Sigma_{\mu\nu}$ will be described by a scalar:

$$\Sigma_{\mu\nu}^{\mathrm{D=2}} = \eta_{\mu\nu}\cdot\Sigma. \tag{2.7}$$

Interestingly, if we consider D=2 Euclidean space, and corresponding (2+1)-dimensional Galilei group, the relation (2.5) describes the second central extension of classical (2+1)–dimensional algebra [9,10], with the nonrelativistic coordinates described by boost generators.

The DFR deformations is very mild, and its dynamical consequences can be calculated explicitly (for D=4 see [1,11]; for Euclidean D=2 case see [12]).

It should be pointed out that both deformations presented in this section introduce besides the phase space generators $Y_A = (\widehat{x}_\mu, \widehat{p}_\mu)$ also additional generators U_r (Lorentz generators $M_{\mu\nu}$ or tensorial central charges $\Sigma_{\mu\nu}$), which form together consistent associative algebra:

$$
\begin{aligned}
[Y_A, Y_B] &= \Omega_{AB}(Y,U)\,, \\
[Y_A, U_r] &= \Omega_{Ar}(Y,U)\,, \\
[U_r, U_s] &= \Omega_{rs}(Y,U)\,.
\end{aligned}
\tag{2.8}
$$

The generalized quantum symplectic structure ($\Omega_{AB} = -\Omega_{BA}, \Omega_{Ar}, \Omega_{rs} = -\Omega_{sr}$) satisfy Jacobi identities. In Snyder case the algebra (2.8) is the anti-de-Sitter algebra in five-momentum space constrained by the mass-shell condition $p_5^2 - p^2 = \frac{1}{\ell^2}$.

In general case discussing the geometric and algebraic nature of deformed phase space algebra (2.8) one should ask three questions:

i) Under which symmetries the relations (2.8) are covariant? In the following two paragraphs we shall consider the examples of deformations covariant under quantum symmetries, described by noncommutative and noncocommutative Hopf algebra playing the role of symplectomorphism group.

ii) which reality conditions for the generators Y_A, U_r can be consistently imposed?

iii) which are the Hilbert space (or C*-algebra) realizations of the deformed quantum phase space algebra (2.8)?

In this brief review we shall discuss the deformations in purely algebraic framework, and we shall be mainly concerned with the answer to the first question.

3 Deformations of Heisenberg Algebra Covariant under the Quantum Group Transformations

For canonical case, describing relativistic quantum mechanics (see (1.2a-b) the basic set of relations (2.8) is reduced to the first one with $\Omega_{AB} = \begin{pmatrix} 0 & \eta_{\mu\nu} \\ -\eta_{\mu\nu} & 0 \end{pmatrix}$, and the symplectomorphism group is described by Sp(8;R). The symplectomorphism algebra sp(8;R) contains the diagonal subalgebra O(3,1) describing the classical Lorentz transformations in space - time and four-momentum sectors.

First step in considering deformed Heisenberg algebra (1.2) is to consider deformed Lorentz symmetry. At present there are well-known all possible quantum Lorentz symmetries, in the form of real $\star$-Hopf algebra [13.14]. The problem which has not been yet solved is the embeddings of all these quantum Lorentz groups into the possible quantum symplectomorphism groups described by quantum deformations of Sp(8;R).

At this point we would like to observe, that Heisenberg algebra (1.1) can be also presented as an algebra of creation and annihilation operators $a_i = \frac{1}{\sqrt{2}}(\widehat{x}_i + i\widehat{p}_i)$, $a_i^+ = \frac{1}{\sqrt{2}}(\widehat{x}_i - i\widehat{p}_i)$ satisfying well-known relation (we put $h = 1$)

$$
\begin{aligned}
&[a_i^+, a_j] = \delta_{ij} \\
&[a_i, a_j] = [a_i^+, a_j^+] = 0
\end{aligned}
\tag{3.1}
$$

If $i = 1, \ldots, N$ the symplectomorphism groups is Sp(2N), but there is a subgroups $U(N) \subset Sp(2N)$ of holomorphic transformations $a'_i = U_i{}^j a_j$ where $U_i^j \in U(N)$. The problem of deformation of relations (3.1) consistent with well-known Drinfeld-Jumbo deformation $U_q(N)$ has been solved by Woronowicz and Pusz ([15]; see also [16]). The set of q-deformed commutation relations looks as follows ($[A,B]_q \equiv AB - qBA$):

$$
\begin{aligned}
&[a_i, a_j^+]_q = 0 && i > j \\
&[a'_i, a_j^+]_{q^2} = 1 + \left(q^2 - 1\right)\sum_{k=0}^{i-1} a_k^+ a_k && i = j \\
&[a_i, a_j]_q = [a_i^+, a_j^+]_{q^{-1}} = 0 && i < j
\end{aligned}
\tag{3.2}
$$

It appears that the n-dimensional modules $(a_1, \ldots, a_N)$ and $(a_1^+, \ldots, a_N^+)$ form the corepresentation of quantum holomorphic symplectomorphism group $U_q(N)^\ddagger$. This construction has been further generalized to the supersymmetric set of bosonic and fermionic creation and annihilation operators [19] and to the case of quantum covariance group described by a quantum R-matrix [16,20].

In order to describe the quantum deformation of the relativistic phase space (1.2a-b) one should perform the following steps:

i) Specify the quantum deformation $O(3,1)^{(q)}$ of Lorentz group O(3 1), which only in particular cases can be described in terms of quantum R-matrices for the spinorial Lorentz group SL(2;c).

ii) Determine the algebra of quantum Minkowski space $\mathcal{M}_4^{(q)}$, with four generators of algebra describing the fundamental representation of $O^{(q)}(3,1)$.

‡For the definition of $U_q(N)$ see [17,18].

(iii) Define the quantum fourmomenta p_μ by noncommutative derivatives (noncommutative vector fields) acting on the functions on $\mathcal{M}_4^{(q)}$.

It appears that depending on the type of deformations, we need to consider besides the generators $Y_A = (\widehat{x}_\mu, \widehat{p}_\mu)$ also additional generators U_r (see (2.8)). We shall mention here the following two cases:

i) Drinfeld-Jimbo deformation of Lorentz group.

Such a case was elaborated in detail by Munich group [21-23]. The relations (2.8) are realized if we supplement 7 additional generators $U_r = (M_{\mu\nu}, \Lambda)$, where $M_{\mu\nu}$ are the Lorentz generators, and Λ describe the scaling generator. An interesting property of this scheme is the classical nature of time coordinate (i.e. $\widehat{x}_0$ can be chosen an central element of $\mathcal{M}_4^{(q)}$).

ii) Podleś class of deformations of Poincaré group, providing 8-dimensional quantum phase space.

It is known [24,25] that the Drinfeld-Jimbo deformation of Lorentz group can not be extended to quantum Poincaré group. Only the deformations of Lorentz group with quantum R-matrix satisfying $R^2 = 1$ permits the extension to quantum Poincaré group $\mathcal{P}_4^{(q)}$ [26]. The translation sector of $\mathcal{P}_4^{(q)}$ describes the generators $\widehat{x}_\mu$ of $\mathcal{M}_4^{(q)}$:

$$(R-1)_{\rho\tau}{}^{\mu\nu}\left(\widehat{x}^\rho \widehat{x}^\tau - Z^{\rho\tau}{}_\nu \widehat{x}^\nu + T^{\rho\tau}\right) = 0 \tag{3.3}$$

where R is quantum O(3,1) matrix satisfying $R^2 = 1$ and $Z_{\mu\nu}{}^\rho$, $C_{n\nu}$ are numerical dimensionless coefficients, satisfying suitably set of constraints (see [26]). The fourdimensional differential calculus, with four independent basic one-forms and corresponding four dual derivatives, are obtained if the parameters in eq. (3.3) satisfy additional conditions [27]. In such a case the space-time noncommutativity (3.4) can be supplemented by the following remaining relations of deformed algebra (1.2a-b)

$$p_u \widehat{x}^\nu - R^{\nu\rho}{}_{\mu\tau} \widehat{x}^\tau p_\rho = \delta_\mu{}^\nu + Z^{\nu\rho}{}_\mu p_\rho \tag{3.4a}$$

$$p_u p_\nu - R^{\tau\rho}{}_{\nu\mu} p_\rho p_\tau = 0 \tag{3.4b}$$

where p_ν can be identified with the noncommutative derivatives.

4 Deformed Phase Space as Heisenberg Double of Quantum Space–Time Symmetries

In previous paragraphs we described deformed relativistic phase space as a deformed Heisenberg algebra covariant under classical (Sect. 2) or quantum (Sect. 3) relativistic symmetries. In such an approach to "deformed physics" the basic object is a quantum phase algebra, and the quantum symmetries in form of respective Hopf algebras describe its covariance properties. It should be stressed that the assumptions of covariance of deformed phase space algebra (deformed Heisenberg algebra) under classical or quantum symmetries restricts substantially all possible class of deformations.

In this Section we shall consider the scheme in which primary object is the quantum symmetry. The deformed relativistic phase space will be obtained directly from a dual pair of Hopf algebras, describing quantum Poincaré group $\mathcal{P}_4^{(q)}$ and quantum Poincaré algebra $P_4^{(q)}$. The realizations describing the noncommutativity of coordinates and momenta will be obtained by introducing the semidirect product $\mathcal{P}_4^{(q)} \bowtie P_4^{(q)}$, or so–called Heisenberg double [28,29].

If we have two dual Hopf algebras H, $\widetilde{H}$, with duality generated by inner non-degenerate product $< \widetilde{a}, a >$ ($a \in H; \widetilde{a} \in \widetilde{H}$), the Heisenberg double $\mathcal{H}(H)$ is the algebra $H \otimes \widetilde{H}$ with the cross multiplication given by the relations:

$$\widetilde{a} \circ a = a_{(1)} < \widetilde{a}_{(1)}, a_{(2)} > \widetilde{a}_{(2)} \tag{4.1}$$

where we use the notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$ etc. In particular one can apply this definition to the multiplication of two Abelian Hopf algebras describing translation sector of classical Poincaré group

$$\left(\widetilde{H} = \{x^\mu\};\ [x^\mu, x^\nu] = 0, \qquad \Delta(x^k) = x^k \otimes 1 + 1 \otimes x^k\right)$$

and fourmomentum sector of classical Poincaré algebra ($H = \{p_k; [p^k, p^v] = 0$, $\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k$). Introducing the Planck constant $\hbar$ into the duality relation

$$< x^\mu, p_{\ \nu} >= i\hbar\delta^\mu_{\ \nu} \tag{4.2a}$$

one obtains from (4.1)

$$x^\mu \cdot p_\nu = p_\nu < 1, 1 > x^\mu + 1 \cdot < x^\mu, p_\nu > \cdot 1 \tag{4.2b}$$

Because $< 1, 1 >= 1$, substituting (4.2a) we obtain from (4.2a) the classical relativistic Heisenberg algebra (1.2a-b). Such a scheme can be generalized to the case of any quantum - deformed relativistic symmetries. In this lecture I shell provide the example of so–called κ–deformations of D=4 relativistic symmetries.

We choose the following realization of the κ-Poincaré algebra in bicrossproduct basis [2,8] ($\mu, \nu, \lambda, \rho = 0, 1, 2, 3; i, j = 1, 2, 3$ and $g^{\mu\nu} = g_{\mu\nu} = (-1, 1, 1, 1)$)

-algebra sector

$$\begin{aligned}
&[P_\mu, P_\nu] = 0 \\
&[M_{\mu\nu}, M_{\lambda\sigma}] = i\hbar(g_{\mu\sigma}M_{\nu\lambda} + g_{\nu\lambda}M_{\mu\sigma} - g_{\mu\lambda}M_{\nu\sigma} - g_{\mu\sigma}M_{\mu\lambda} \\
&[M_{ij}, P_\mu] = -i\hbar(g_{i\mu}P_j - g_{j\mu}P_i) \\
&[M_{i0}, P_0] = i\hbar P_i
\end{aligned} \tag{4.3}$$

$$[M_{i0}, P_j] = i\delta_{ij}\left(\hbar^2\kappa \sinh\left(\frac{P_0}{\hbar\kappa}\right) e^{-\frac{P_0}{\hbar\kappa}} + \frac{1}{2\kappa}\vec{P}^2\right) - \frac{i}{\kappa}P_iP_j$$

-coalgebra sector

$$\Delta(M_{ij}) = M_{ij} \otimes I + I \otimes M_{ij}$$

$$\Delta(M_{k0}) = M_{k0} \otimes e^{-\frac{P_0}{\hbar\kappa}} + I \otimes M_{k0} + \frac{1}{\hbar\kappa} M_{kl} \otimes P_l \tag{4.4}$$

$$\Delta(P_0) = P_0 \otimes I + I \otimes P_0$$

$$\Delta(P_k) = P_k \otimes e^{-\frac{P_0}{\hbar\kappa}} + I \otimes P_k$$

where $\hbar$ denotes Planck's constant and κ deformation parameter. The formulas for the antipode and counit are omitted because they are not essential for this construction.

Using the following duality relations, extending (4.2)

$$< x^\mu, P_\nu >= i\hbar\delta^\mu_\nu \qquad < \Lambda^\mu{}_\nu, M_{\alpha\beta} >= i\hbar\left(\delta^\mu_\alpha g_{\nu\beta} - \delta^\mu_\beta g_{\nu\alpha}\right) \tag{4.5}$$

we obtain the commutation relations defining κ–Poincaré group [3,8] in the form

- algebra sector

$$[x^\mu, x^\nu] = \tfrac{i}{\kappa}\left(\delta^\mu_0 x^\nu - \delta^\nu_0 x^\mu\right)$$

$$[\Lambda^\mu_\nu, x^\lambda] = -\tfrac{i}{\kappa}\left(\left(\Lambda^\mu_0 - \delta^\mu_0\right)\Lambda^\lambda_\nu + \left(\Lambda^0_\nu - \delta^0_\nu\right)g^{\mu\lambda}\right) \tag{4.6}$$

$$[\Lambda^\mu_\nu, \Lambda^0_\mu] = 0$$

-coalgebra sector

$$\Delta(x^\mu) = \Lambda^\mu{}_\alpha \otimes x^\alpha + x^\mu \otimes I$$

$$\Delta(\Lambda^\mu{}_\nu) = \Lambda^\mu{}_\alpha \otimes \Lambda^\alpha{}_\nu \tag{4.7}$$

The commutation relations (4.3) and (4.6) one can supplemented by the following relations obtained from (4.1), (4.4) and (4.7)

- cross relations:

$$[P_k, x_l] = i\hbar\delta_{kl} \qquad [P_0, x_0] = i\hbar$$

$$[P_k, x_0] = -\tfrac{i}{\kappa}P_k \qquad [P_0, x_l] = 0$$

$$[P_\mu, \Lambda^\alpha_\beta] = 0 \tag{4.8}$$

$$[M_{\alpha\beta}, \Lambda^\mu{}_\nu] = i\hbar\left(\delta^\mu_\beta\Lambda_{\alpha\nu} - \delta^\mu_\alpha\Lambda_{\beta\nu}\right)$$

$$[M_{\alpha\beta}, x^\mu] = i\hbar\left(\delta^\mu_\beta x_\alpha - \delta^\mu_\alpha x_\beta\right) + \tfrac{i}{\kappa}\left(\delta^0_\beta M_\alpha{}^\mu - \delta^0_\alpha M_\beta{}^\mu\right)$$

where

$$M_\alpha{}^\mu = g^{\mu\rho} M_{\alpha\rho} \qquad M^\mu{}_\alpha = g^{\mu\rho} M_{\rho\alpha}$$

The relations (4.3), (4.6) and (4.8) give us the Heisenberg double $\mathcal{H}(\mathcal{P}_k)$ of κ-Poincaré algebra firstly presented in [32]. In order to define the κ-deformed phase space we should consider subalgebra of $\mathcal{H}(\mathcal{P}_k)$ given by the following commutation relations:

$$\begin{aligned}
&[x_k, x_l] = 0 && [P_\mu, P_\nu] = 0 \\
&[x_0, x_k] = \tfrac{i}{\kappa} x_k && \\
&[x_k, P_l] = i\hbar\delta_{kl} && [x_k, P_0] = 0 \\
&[x_0, P_l] = \tfrac{i}{\kappa} P_l && [x_0, P_0] = -i\hbar
\end{aligned} \tag{4.9}$$

For $\kappa \to \infty$ we get the standard nondeformed phase space satisfying the Heisenberg commutation relations (1.2a-b).

It should be stressed that the κ-deformed phase space (4.9) is not a Hopf algebra, however, separately in the translation sector (generators x_μ) and fourmomentum sector (generators P_μ) one can read-off from the relations (4.4) and (4.7) (in eq. (4.7) after contraction $\Lambda_\beta^\beta \to \delta_\alpha^4$) the coalgebraic structure. We obtain that in κ-deformed phase space one can add two coordinates and two momenta as follows:

$$x_\mu^{(1+2)} = x_\mu^{(1)} + x_\mu^{(2)} \tag{4.10a}$$

$$P_0^{(1+2)} = P_0^{(1)} + P_0^{(2)} \tag{4.10b}$$

$$\vec{P}^{(1+2)} = \vec{P}^{(1)} e^{-\frac{P_0^{(2)}}{\hbar\kappa}} + \vec{P}^{(2)} \tag{4.11c}$$

We see that due to the κ-deformation there is modified the addition law of two three-momenta.

Analogous Heisenberg double construction can be performed for any quantum Poincaré group listed by Podleś and Woronowicz [26]. It should be mentioned however, that before solving such a task we should obtain the complete list of quantum Poincaré algebras, dual to quantum groups from [26] - the list which at present is yet not known.

5 Final Remarks

The considerations of deformed quantum phase spaces goes along the following three lines:

- i) Mathematical consequences of deforming space time symmetries. In such a way one obtains the set of possible algebraic schemes, which we outlined in this talk.

- ii) Investigation of dynamical consequences of modifying Einstein gravity scheme (e.g. superstring theory) on Heisenberg uncertainty relations (see e.g. [32,33])

- iii) Considering different "gedenken-experiments" for the measurement of distances and time intervals in quantum theory, which include into analysis the quantum property of the observer (reference system) (see e.g. [34-36]). It appears that the "classical" measuring devices, with infinite mass as well as definite position and velocity are the idealizations in the description of measuring process which is not realistic. It appears that introducing "realistic" measurement theory in quantum mechanics we are bound to modify the standard Heisenberg uncertainty relations, with Planck length $l_p = \frac{\hbar}{m_p c} \simeq 10^{-33}$ cm playing a crucial role in the modifications.

We see that the future modified models of quantum mechanics should have on one side solid mathematical and dynamical bases, on the other should be consistent with "realistic" measurement theory. A modest effort in such a direction has been presented in [36], where there are given arguments based on "realistic" measurement theory for the κ-deformation of relativistic phase space, presented in Sect. 4. At present, however, the choice of the deformation which is the most physical one is an open question.

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HIDDEN SYMMETRY OF THE YANG–COULOMB MONOPOLE

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Abstract

Bound system composed of the Yang monopole coupled to a particle of the isospin by the $SU(2)$ and Coulomb interaction is considered. The generalized Runge–Lenz vector and the $SO(6)$ group of hidden symmetry are established. It is also shown that the group of hidden symmetry make it possible to calculate the spectrum of system by a pure algebraic method.

During the last few years attempts was made to find the electromagnetic duality phenomenon [1] in the framework of quantum mechanics [2-11]. One of the systems arising in these searches is the Yang–Coulomb monopole (YCM), a unique example of an integrable non–Abelian system.

The YCM is defined as a five–dimensional bound system composed of the Yang monopole [8] and a particle of the isospin. Both the monopole and particle are assumed also having electric charges of the opposite signs. Thus, the monopole-particle coupling is realized not only by the $SU(2)$ gauge field but also by the Coulomb interaction. At large distances the Coulomb structure becomes immaterial and YCM seems to be pure Yang monopole.

In this article, we analyze the symmetry properties of the YCM. We establish the $SO(6)$ group of hidden symmetry of YCM and use this symmetry for calculation of the energy spectrum of YCM by an algebraic method.

Let us introduce the space $\Omega = \mathbb{R}^5(x_j) \otimes S^3$ where $\mathbb{R}^5(x_j)$ and S^3 have the meaning of the configuration and the gauge space of YCM. We keep the following notation: $j = 0, 1, 2, 3, 4$; $a = 1, 2, 3$; $\hat{T}_a$ are the $SU(2)$ gauge group generators; $\vec{A}^a$ is the triplet of Yang monopole's gauge potentials; σ^a are the Pauli matrices. The YCM is governed by the Hamiltonian

$$\hat{H} = \frac{1}{2m}\hat{\pi}^2 + \frac{\hbar^2}{2mr^2}\hat{T}^2 - \frac{e^2}{r}$$

where $r = (x_j x_j)^{1/2}$, $\hat{T}^2 = \hat{T}_a\hat{T}_a$, $\hat{\pi}^2 = \hat{\pi}_j\hat{\pi}_j$,

$$\hat{\pi}_j = -i\hbar\frac{\partial}{\partial x_j} - \hbar A_j^a\hat{T}_a$$

$$A_i^a = \frac{2i}{r(r+x_0)} \tau_{ij}^a x_j$$

and τ^a are the 5×5 matrices having the following explicit form

$$\tau^1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i\sigma^1 \\ 0 & i\sigma^1 & 0 \end{pmatrix}, \quad \tau^2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i\sigma^3 \\ 0 & -i\sigma^3 & 0 \end{pmatrix}, \quad \tau^3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{pmatrix}$$

and satisfying $[\tau^a, \tau^b] = i\epsilon_{abc}\tau^c$.

Each term of the $\vec{A}^a$–triplet coincides with the gauge potential of the five–dimensional Dirac monopole with a unit topological charge and the line of singularity extended along the axis $x_0 \leq 0$.

It is obvious that $\tau_{ij}^a = -\tau_{ji}^a$ and, therefore, A_j^a are orthogonal to x_j. Moreover, it follows from

$$4\tau_{ij}^a \tau_{jk}^b = \delta_{ab}\left(\delta_{ik} - \delta_{i0}\delta_{k0}\right) + 2i\epsilon_{abc}\tau_{ik}^c$$

that the vectors A_j^a are also orthogonal to each other

$$A_j^a A_j^b = \frac{r - x_0}{r^2(r+x_0)}\delta_{ab}$$

The following fundamental commutation relations are valid:

$$[\hat{\pi}_i, x_k] = -i\hbar\delta_{ik}, \quad [\hat{\pi}_i, \hat{\pi}_k] = i\hbar^2 F_{ik}^a \hat{T}_a$$

where

$$F_{ik}^a = \partial_i A_k^a - \partial_k A_i^a + \epsilon_{abc} A_i^b A_k^c$$

is the triplet of gauge fields. In a more explicit form

$$F_{ik}^{a(+)} = \frac{1}{r^2}\left[\left(x_k + r\delta_{k0}\right) A_i^{a(+)} - \left(x_i + r\delta_{i0}\right) A_k^{a(+)} - 2i\tau_{ik}^a\right]$$

The last expression follows from the formula:

$$\epsilon_{abc}\tau_{ij}^b \tau_{km}^c = \frac{i}{2}\Big[\left(\delta_{i0}\delta_{k0} - \delta_{ik}\right)\tau_{jm}^a - \left(\delta_{i0}\delta_{m0} - \delta_{im}\right)\tau_{jk}^a +$$

$$+ \left(\delta_{j0}\delta_{m0} - \delta_{jm}\right)\tau_{ik}^a - \left(\delta_{j0}\delta_{k0} - \delta_{jk}\right)\tau_{im}^a\Big]$$

The straighforward computation gives

$$F_{ij}^{a(+)} F_{jk}^{b(+)} = \frac{1}{r^6}\left(x_i x_k - r^2\delta_{ik}\right)\delta_{ab} + \frac{1}{r^2}\epsilon_{abc} F_{ik}^{c(+)}$$

Since YCM is a non–Abelian extension of the five–dimensional Coulomb problem, it is natural to try to construct an analog of the Runge–Lenz vector for YCM by passing from $\mathbb{R}^3$ to $\mathbb{R}^5(x_j)$ and taking account of the gauge field. The first step has been made many years ago [12]

$$\hat{M}_k = \frac{1}{2\sqrt{m}}\left(\hat{p}_i\hat{l}_{ik} + \hat{l}_{ik}\hat{p}_i + \frac{2me^2}{\hbar}\frac{x_k}{r}\right)$$

where $\hat{p}_i = -\hbar\partial/\partial x_i$, $\hbar\hat{l}_{ik} = x_i\hat{p}_k - x_k\hat{p}_i$. The second step can be realized by the substitution [8]: $\hat{p}_i \to \hat{\pi}_i$, $\hat{l}_{ik} \to \hat{L}_{ik}$ where

$$\hat{L}_{ik} = \frac{1}{\hbar}\left(x_i\hat{\pi}_k - x_k\hat{\pi}_i\right) - r^2 F^a_{ik}\hat{T}_a$$

It is possible to verify that

$$[\hat{L}_{ik}, x_j] = i\delta_{ij}x_k - i\delta_{kj}x_i, \quad [\hat{L}_{ik}, \hat{\pi}_j] = i\delta_{ij}\hat{\pi}_k - i\delta_{kj}\hat{\pi}_i$$

$$[\hat{L}_{ij}, \hat{L}_{mn}] = i\delta_{im}\hat{L}_{jn} - i\delta_{jm}\hat{L}_{in} - i\delta_{in}\hat{L}_{jm} + i\delta_{jn}\hat{L}_{im},$$

i.e., $\hat{L}_{ik}$ indeed are the generators of the group $SO(5)$ and $[\hat{H}, \hat{L}_{ik}] = 0$.

A long manipulation exercise yields $[\hat{H}, \hat{M}_k] = 0$ which means that in fact $\hat{M}_k$ is the analog of the Runge–Lenz vector for YCM. It can also be shown that

$$[\hat{L}_{ij}, \hat{M}_k] = i\delta_{ik}\hat{M}_j - i\delta_{jk}\hat{M}_i, \quad [\hat{M}_i, \hat{M}_k] = -2i\hat{H}\hat{L}_{ik}$$

These commutation rules generalize relations known from the theory of the Coulomb problem [13].

Finally, let us introduce the 6×6 matrix

$$\hat{D} = \begin{pmatrix} \hat{L}_{ij} & -\hat{M}'_j \\ \hat{M}'_j & 0 \end{pmatrix}$$

where $\hat{M}'_i = \left(-2\hat{H}\right)^{-1/2}\hat{M}_i$. The components $\hat{D}_{\mu\nu}$ (here $\mu, \nu = 0, 1, 2, 3, 4, 5$) satisfy the commutation relations

$$[\hat{D}_{\mu\nu}, \hat{D}_{\lambda\rho}] = i\delta_{\mu\lambda}\hat{D}_{\nu\rho} - i\delta_{\nu\lambda}\hat{D}_{\mu\rho} - i\delta_{\mu\rho}\hat{D}_{\nu\lambda} + i\delta_{\nu\rho}\hat{D}_{\mu\lambda},$$

i.e., $\hat{D}_{\mu\nu}$ are the generators of the group $SO(6)$. Since $[\hat{H}, \hat{D}_{\mu\nu}] = 0$, one concludes that YCM is provided by the $SO(6)$ group of hidden symmetry.

The Casimir operators for $SO(6)$ are [14]

$$\begin{aligned}
\hat{C}_2 &= \frac{1}{2}\hat{D}_{\mu\nu}\hat{D}_{\mu\nu} \\
\hat{C}_3 &= \epsilon_{\mu\nu\rho\sigma\tau\lambda}\hat{D}_{\mu\nu}\hat{D}_{\rho\sigma}\hat{D}_{\tau\lambda} \\
\hat{C}_4 &= \frac{1}{2}\hat{D}_{\mu\nu}\hat{D}_{\nu\rho}\hat{D}_{\rho\tau}\hat{D}_{\tau\mu}
\end{aligned}$$

According to [15], the eigenvalues of these operators can be taken as

$$
\begin{aligned}
C_2 &= \mu_1(\mu_1+4)+\mu_2(\mu_2+2)+\mu_3^2 \\
C_3 &= 48(\mu_1+2)(\mu_2+1)\mu_3 \\
C_4 &= \mu_1^2(\mu_1+4)^2+6\mu_1(\mu_1+4)+\mu_2^2(\mu_2+2)^2+\mu_3^4-2\mu_3^2
\end{aligned}
$$

where μ_1, μ_2 and μ_3 are the positive integer or half–integer numbers and $\mu_1 \geq \mu_2 \geq \mu_3$.

Direct calculations lead to the representation

$$
\begin{aligned}
\hat{C}_2 &= -\frac{e^4 m}{2\hbar^2 \hat{H}} + 2\hat{T}^2 - 4 \qquad (1) \\
\hat{C}_3 &= 48\left(-\frac{me^4}{2\hbar^2 \hat{H}}\right)^{1/2} \hat{T}^2 \\
\hat{C}_4 &= \hat{C}_2^2 + 6\hat{C}_2 - 4\hat{C}_2\hat{T}^2 - 12\hat{T}^2 + 6\hat{T}^4
\end{aligned}
$$

From the last equation we can obtain another expression for the eigenvalue C_4

$$C_4 = \left[C_2 - 2T(T+1)\right]^2 + 6\left[C_2 - 2T(T+1)\right] + 2T^2(T+1)^2$$

and conclude that

$$C_2 - 2T(T+1) = \mu_1(\mu_1+4) \qquad (2)$$

$$\mu_2^2(\mu_2+2)^2 + \mu_3^4 - 2\mu_3^2 = 2T^2(T+1)^2 \qquad (3)$$

The energy levels of YCM can be derived from (1) and (2)

$$\epsilon_{\mu_1}^T = -\frac{me^4}{2\hbar^2(\mu_1+2)^2} \qquad (4)$$

The substitution of the eigenvalues of $\hat{H}$ and $\hat{T}^2$ in the equation for $\hat{C}_3$ gives one more formula for C_3

$$C_3 = 48(\mu_1+2)T(T+1)$$

Now, we have two expressions for C_3 and the comparison leads to the relation

$$T(T+1) = (\mu_2+1)\mu_3 \qquad (5)$$

Comparing this with (2), we have the equation

$$\left(\mu_2^2 - \mu_3^2\right)\left[(\mu_2+2)^2 - \mu_3^2\right] = 0$$

Since $\mu_3 \leq \mu_2$, one concludes that $\mu_3 = \mu_2$. Then, from (5) it follows that $\mu_2 = T$ and, therefore, μ_1 in (4) takes only values $\mu_1 = T, T+1, T+2, ...,$ – the result known from our previous paper [11].

Let us complete the present consideration of YCM by three remarks:
(a) It is known [16] that the five–dimensional Coulomb problem is provided with the hidden $SO(6)$ symmetry. From this position, our paper states that the inclusion of the $SU(2)$ gauge field doesn't break the hidden symmetry of the initial system. The same phenomenon takes place for the Abelian particle–charge system [17].
(b) The elegant approach presented above to solve the YCM eigenvalue problem has recently been used for the same purpose in connection with the so–called $SU(2)$ Kepler problem [18].
(c) The eigenfunctions of YCM and the degeneracy of the energy levels has been considered in our recent publication [11].

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GENERALIZATION OF SUSY N=2 KDV EQUATION

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Abstract

We describe three different extended (N=2) supersymmetric Lax operator of the Korteweg de Vries Equation. These Lax operators are next generalized to more complicated form, including additionals fields. These generalized Lax operators create a hudge class of new integrable models for which the hamiltonian structure is considered.

1 Introduction.

Integrable Hamiltonian systems occupy an important place in diverse branches of theoretical physics as exactly solvable models of fundamental physical phenomena ranging from nonlinear hydrodynamics to string theory [1,2,3]. The general Kadomtsev - Petviashvilli (KP) system is 1+1 dimensional integrable model containing an infinite number of fields. In the Sato approach, the KP hierarchy is described by the isospectral deformations of the eigenvalue problem $L\Psi = \lambda\psi$ for the pseudodifferential operator $L = \partial + U_2\partial^{-1} + U_3\partial^{-2}$ which is given by

$$L_{t_n} = [B_n, L]\,, \tag{1}$$

where $n = 2, 3, ...$ and B_n is the differential part of the microdifferential operator L^n. If we require that L satisfies the additional condition that $L^n = B_n, n \geq 2$ then the hierarchy of equation given by Eq. (1) are reduced to the hierarchy of (1+1) dimensional integrable systems called the n-reduced KP hierarchy. For example, the Korteweg - de Vries equation and the Boussinesq equation belong to the two-reduced and three-reduced KP hierarchies, respectively.

On the other side, a new type of reduction have been proposed recently in a series of articles, which leads many (2+1) dimensional integrable systems to (1+1) dimensional integrable systems [4-5]. For example, by assuming that L satisfies the constraints

$$L^n = B_n + q\partial^{-1}r, \tag{2}$$

we can obtain the so called k-constrained KP hierarchy. Interestingly, the one - constrained KP hierarchy coincide with the AKNS hierarchy, and the two - constrained KP hierarchy coincides with the Yajima - Oikawa hierarchy. The k-constrained KP hierarchy was shown to possess Lax pairs, recursion operatorsand the bi-Hamiltonian structures.

In this paper we would like to discuss how this situations looks like in the supersymmetric case.

The idea to use the extended supersymmetry (SUSY) for the generalization of the soliton equations appeared almost in parallel to the usage of the SUSY in the quantum field theory [6]. The main idea of SUSY is to treat bosons and fermions operators equally. The first results, concerned the construction of classical field theories with fermionic and bosonic fields depending on time and one space variable, can be found in [7-9]. In many cases, the addition of fermion fields does not guarantee that the final theory becomes the SUSY invariant. Therefore this method was named the fermionic extension in order to distinguish it from the fully SUSY way.

We have at the moment many different procedures [10-26] of the supersymmetrization of the soliton equations. From the soliton point of view we can distinguish two different recipes. In the first we add to the theory new anticommuting Grassmann valued functions only while in the second case we also add the new commuting functions. In many cases, the addition of fermion fields does not guarentee that the final theory becomes supersymmetric invariant.

From this classification one can infer that the second approach is more important then the first one. We extend in this case, our system of equation, to the new larger system of equations. This may be viewed as a bonus, but this extended case is in no way more fundamental than the non-extended one.

In this paper we would like to consider the supersymmetric Korteweg de Vries equation and its different genetralization, using second receipt. The interest to such generalization is motivated by observation that special type of such generalization describes the Lax operator for SUSY KdV equation in higher supersymmetric dimensions. Moreover it gives rise to hudge classess of new integrable equations.

The paper is organized as follows. In the first chapter we describe the classical Lax's approach to the Korteweg de Vries equation. The superymmetrization of this equation is described in the next section while last section contains the classifications of different generalizations of these supersymmetric equation. All calculations presented in this paper have been obtained by extensive applications of the symbolic computation language REDUCE and special package written by author SUSY2 [27].

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2 Lax operator for the classical KdV Equation

It is possible to write down the famus Korteweg de Vries equation

$$u_t = -u_{xxx} + 6uu_x, \tag{3}$$

using two different Lax operators

$$L_1 = \partial^2 + u, \tag{4}$$

$$L_2 = \partial + \partial^{-1}u. \tag{5}$$

Both these operators give rise to the standart or to the nonstandard Lax pair representation of the KdV equation. For the first case we have

$$\frac{\partial L_1}{\partial t} = \left[L_1, \left(L_{1,+}^{\frac{3}{2}}\right)\right], \tag{6}$$

while for the second case

$$\frac{\partial L_2}{\partial t} = \left[L_2, \left(L_{2,\geq 1}^{3}\right)\right]. \tag{7}$$

Here $+$ denotes the projection onto the differential part of the pseudodifferential operators while $\geq$ the projection onto the purely differential part.

Tyhis KdV equation is hamiltonian equation which usally is written down as

$$u_t = (-\partial^3 + 2\partial u + 2u\partial)\frac{\delta}{\delta u}\int dx u^2. \tag{8}$$

The Lax formulation is very usefull in investigations of the (bi) hamiltonian property of the given system of equations. The knowledge of Lax operator allows us to construct conserved currents for the given model.

3 The extended supersymmetric KdV equation

It appeared that there are three different integrable extension of the extended $N = 2$ supesymmetric KdV equation. These cases are contained in the following one-parameter family of super-Hamiltonian evolution equations [12]

$$\begin{aligned}
\Phi_t &= -\Phi_{xxx} + 3(\Phi D_1 D_2 \Phi)_x + \frac{1}{2}(a-1)(D_1 D_2 \Phi^2)_x + 3a\Phi^2\Phi_x \\
&= \left(D_1 D_2 \partial + 2\partial\Phi + 2\Phi\partial - D_1 \Phi D_1 - D_2 \Phi D_2\right) \\
&\quad \frac{\delta}{\delta\Phi}\int dx d\theta_1 d\theta_2 \frac{1}{2}\left(\Phi D_1 D_2 \Phi + \frac{a}{3}\Phi^3\right),
\end{aligned} \tag{9}$$

where $\Phi = \Phi(x, \theta_1, \theta_2)$ ia a commuting superfunction of space and two Grassmannian variables θ_1 and θ_2, D_i is the usual supersymmetric derivative $D_i = \partial_{\theta_i} + \theta_i \partial$.

Here the parameter a enumerates our extensions and take the values $a = -2, 1, 4$ for which this system consitute the integrable model [12,24]. The supersymmetric Lax operators takes the following form [12,17,24]

$$\begin{aligned}
a = -2: \qquad & L = \partial^2 + D_1 \Phi D_2 - D_2 \Phi D_1, \\
& L = \partial^2 + \Phi\partial - (D\Phi)D,
\end{aligned} \tag{10}$$

$$a = 4: \qquad L = -(D_1 D_2 + \Phi)^2,$$
$$L = \partial - \Phi - \bar{D}\partial^{-1}(D\Phi), \tag{11}$$

$$a = 1: \qquad L = \partial - D_1 D_2 \partial^{-1}\Phi,$$
$$L = \partial - \left[D, \bar{D}\right]\partial^{-1}\Phi. \tag{12}$$

Here $(D\Phi)$ denotes the value of the action of D onto Φ and $D = \partial_{\theta_1} - \frac{1}{2}\theta_2\partial, \bar{D} = \partial_{\theta_2} - \frac{1}{2}\theta_1\partial$. For the first three Lax operators we have the usual Lax pair

$$\frac{\partial L}{\partial t} = 4\left[L, (L_+^{\frac{3}{2}})\right], \tag{13}$$

while for the last three operators we have the nonstandard Lax pair

$$\frac{\partial L}{\partial t} = \left[L, (L^3)_{\geq 1}\right]. \tag{14}$$

where $(+)$ denotes the projection onto the super-differential part of the operator and (≥ 1) the projection onto the strictly super - differential part of the operator. All these Lax pairs give us the super-hamiltonina formulations of the supersymmetric KdV equation whose Hamiltonian operator is connected with the supersymmetric extension of the Virasoro algebra.

4 Susy generalization of KdV equation

At the moment we have two different approaches in this direction. In the first we include additional chiral fields to the supersymmetric Lax operator while in the second we include ordinary superfunctions to the Lax operator. Let us consider these cases indenpendently.

In the first approach it is possible to generalize Lax operator for $a = -2, 4$ only. Explicitely we have the following generalized Lax operator:

$$a = -2: \qquad L = \partial^2 + \Phi\partial - (D\Phi)\bar{D} - \sum_{i=1}^{M}(F_i G_i + F_i \bar{D}\partial^{-1}(DG_i)), \tag{15}$$

$$a := 4 \qquad L = \partial - \Phi - \bar{D}\partial^{-1}(D\Phi) - \sum_{i=1}^{M}(F_i G_i + F_i \bar{D}\partial^{-1}(DG_i)), \tag{16}$$

where M is fixed number while F_i, G_i are chiral fields such that $(DF_i) = (\bar{D}G_i) = 0$. These generalization have been considered in details in [19] where the hamiltonian structure have been considered also. Moreover in it was shown that the Lax operator constitute also the Lax operator for the extended $N = 4$ supersymmetric KdV equation. The Lax operators presented here create the new hierarchy of supersymmetric integrable models. Explicitely the second flow for the generalized $a = 4$ Lax operator has the following form:

$$\Phi_t = -\left[D, \bar{D}\right]\Phi_x - 4\Phi\Phi_x + \sum_{i=1}^{M} \bar{D}F_i DG_i, \tag{17}$$

$$F_{it} = F_{ixx} + 4D(\Phi\bar{D}F_i), \tag{18}$$

$$G_{it} = -G_{ixx} + 4\bar{D}(\Phi DG_i). \tag{19}$$

For the second approach where we include the usual supersymmetric field we have the generalization for $a = 4, 1$ case only.

For the $a = 1$ case it is possible to consider the following supersymmetric Lax operator [21]

$$L = \partial - \left[D, \bar{D}\right]\partial^{-1}\Phi - \sum_{i=1}^{M} \left[D, \bar{D}\right]\partial^{-1}G_i\left[D, \bar{D}\right]\partial^{-1}G_i. \tag{20}$$

where G_i are usual superfermionic fields. The third flow in this hierarchy generate the following system of equations for $M = 1$

$$\begin{aligned}\Phi_t \quad &= \Big[- \Phi_{xx} - \Phi^3 + 3\Phi\left[D, \bar{D}\right]\Phi + 3G_x\left[D, \bar{D}\right]G - 12\Phi DG\bar{D}G \\ &\qquad + 6\bar{D}\Phi GDG + 6D\Phi G\bar{D}G\Big]_x,\end{aligned} \tag{21}$$

$$\begin{aligned}G_t \quad &= -G_{xxx} + 6\Phi\left(\bar{D}\Phi DG + D\Phi\bar{D}G\right) + 3\left(\Phi\left[D, \bar{D}\right]G\right)_x \\ &\qquad - 3\Phi^2 G_x - 6G_x DG\bar{D}G - 6\left(GDG\bar{D}G\right)_x.\end{aligned} \tag{22}$$

One can check that the equations (21-22) are covariant with respect to the following transformations of an extra hidden supersymmetry:

$$\delta\Phi = \varepsilon G_x, \qquad \delta G = \varepsilon\Phi, \tag{23}$$

where ε is a Grassmann parameter. This additional supersymmetry together with explicit $N = 2$ ones form $N = 3$ supersymmetry. This system of equation is just the $N = 3$ supersymmetric KdV equation which can be obtained from the Hamiltonian [17] .

$$\begin{aligned}H = -3\int dx d\theta_1 d\theta_2 \Big(&\Phi\left[D, \bar{D}\right]\Phi + \frac{1}{3}\Phi^3 - G\left[D, \bar{D}\right]G_x \\ &+ 2\Phi GG_x + 8\Phi DG\bar{D}G\Big)\end{aligned} \tag{24}$$

$$\Phi_t = \{H, \Phi\}, \qquad G_t = \{H, G\}. \tag{25}$$

if we use the $N=3$superconformal algebra Poisson brackets, which read in terms of $N=2$ supercu:rents $\Phi(Z)$ and $G(Z)$ as follows:

$$
\begin{aligned}
\{\Phi(Z_1),\Phi(z_2)\} &= \Big(\frac{1}{2}[D,\bar{D}]\partial+\partial\Phi+\Phi\partial+\hbar D\Phi D+D\Phi\bar{D}\Big)\Delta(Z_1-Z_2),\\
\{\Phi(Z_1 0,G(Z_2)\} &= \Big(\partial G+\frac{1}{2}G\partial-\bar{D}GD-DG\bar{D}\Big)\Delta(Z_1-Z_2),\\
\{G(Z_1),G(Z_2)\} &= \frac{1}{2}\Big(\Phi+\big[D,\bar{D}\big]\Big)\Delta(Z_1-Z_2).
\end{aligned}
\tag{26}
$$

Here $\Delta(Z_1-Z_2)=(\theta_1-\theta_2)(\bar{\theta}_1-\bar{\theta}_2)\delta(z_1-z_2)$ is $n+2$ supersymmetric delta function and all operat rs in r.h.s. are evaluated in the second point. The Lax operator considered previously is a particular case of the hudge family of Lax operators of the form:

$$
\begin{aligned}
L \;=\; & \partial + \big[D,\bar{D}\big]\partial^{-1}+\\
& \sum_{i=1}^{M}\Big(\big[D,\bar{D}\big]\partial^{-1}H_i\partial^{-1}\big[D,\bar{D}\big]K_i-\big[D,\bar{D}\big]\partial^{-1}K_i\partial^{-1}\big[D,\bar{D}\big]\partial^{-1}H_i\Big)+\\
& \sum_{j=1}^{S}\Big(\big[D,\bar{D}\big]\partial^{-1}F_j\partial^{-1}\big[D,\bar{D}\big]G_j+\big[D,\bar{D}\big]\partial^{-1}G_j\partial^{-1}\big[D,\bar{D}\big]\partial^{-1}F_j\Big),
\end{aligned}
\tag{27}
$$

where H_i,K_i are the superbosonic functions while G_j,F_j are the superfermionic functions ,M and S are fixed numbers. This generalization give us a hudge class of integrable generalizations of the solitons equation. In particular the following Lax operator

$$L=\partial+\partial^{-1}\big[D,\bar{D}\big]\phi+H-\partial^{-1}\big[D,\bar{D}\big]H\big[D,\bar{D}\big] \tag{28}$$

belongs to this class and its third flow is

$$
\begin{aligned}
\Phi_t &= \Big[-\Phi_{xx}-3\big[D,\bar{D}\big]\Phi\Phi-\Phi^3\\
&\qquad +12DH\bar{D}H-6\bar{D}\Phi DH-6d\Phi\bar{D}H\Big]_x,
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
H_t &= \Big[-6DH_x\bar{H}+6DH\bar{H}_x+6\bar{D}\Phi DH\Phi-3H_x^3\\
&\qquad 6D\Phi\bar{D}H\Phi+H_{xxx}+3\big(\big[D,\bar{D}\big]H\big)^2+3\big(\Phi\big[D,\bar{D}\big]H\big)_x\Big].
\end{aligned}
\tag{30}
$$

This equation reasembles the Hirota-Satsuma equation in the bosonic sector.

For the $a=4$ case let us consider the following generalizations,

$$
\begin{aligned}
L_b &= D_1D_2+\Phi+H\partial^{-1}D_1D_2K-K\partial D_1DH &(31)\\
L_f &= D_1D_2+\Phi+G\partial^{-1}D_1D_2F+F\partial^{-1}D_1D_2G, &(32)
\end{aligned}
$$

where H, K are the superbosons for L_b while G, F. Now it is rather technical problem to construct the Lax pair system which generate a rich hierarchy of integrable equation. This construction has been considered in more details in [25] where the hamiltonian structure has been discussed also. To finish this consideration let us consider the particular case of the L_f operator in the form

$$L = D_1 D_2 + \Phi + 2H\partial^{-1} D_1 D_2 H. \tag{33}$$

The second flow give rise to the following system of equations

$$\Phi_t = \Big(- D_1 D_2 \Phi - 2\Phi^2 + 4HH_x - 2(D_1 H)^2 - 2(D_2 H)^2 \Big)_x /2, \tag{34}$$

$$H_t = \Big(- 2D_1 D_2 H_x - 4H_x \Phi - (D_1 \Phi) D_2 H) - (D_2 \Phi D_1 H) - H\Phi_x \Big)/4. \tag{35}$$

The bosonic sector of this system of equations, in which $\Phi = \phi_0 + \theta_1\theta_2\phi_1$ and $H = \theta_1 h_0 + \theta_2 h_1$ where Φ_0, Φ_1, h_0, h_1 are usual classical fields reads:

$$\phi_0 = \Big(- \phi_1 - 2\phi_0^2 - 2h_0^2 - 2h_1^2 \Big)_x /2 \tag{36}$$

$$\phi_1 = \Big(\phi_{0xx} - 4\phi_1\phi_0 + 8h_0 h_{1x} - 8h_1 h_{0x} \Big)_x /2 \tag{37}$$

$$h_{0t} = \Big(- 2h_{1xx} - 2\phi_{0x} h_0 - 4\phi_0 h_{0x} - \phi_1 h_1 \Big)/2 \tag{38}$$

$$h_{1t} = \Big(2h_{0xx} - 2\phi_{0x} h_1 - 4\phi_0 h_{1x} + \phi_1 h_0 \Big)/2 \tag{39}$$

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HIGHER LOOP STRING COSMOLOGY AS A DYNAMICAL SYSTEM

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1 Introduction

The cosmological consequences of string theory and their possible manifestations are important both for observational verification of theory and for the resolution of a number of problems in modern cosmology. Increasingly growing number of works in this direction (see [1], [2], [3] and references therein) has resulted to the formation of a new area of cosmological investigations - String Cosmology. Recently, various cosmological models with dilaton and antisymmetric tensor field have been investigated to leading order in string tension as well as with higher-curvature corrections. In most cases the universe evolves toward strong coupling region where quantum loop corrections become important and the tree-level effective action can not reliably describe the dynamics. This motivates the consideration of models with loop corrections. Another motivation is related to the fact, that the string loop effects may naturally generate different nonmonotonic coupling functions in the effective action and consequently a relaxation mechanism (Damour - Polyakov mechanism) [4] by which various moduli fields are attracted towards their present vacuum expectation values. In particular, in [5] it was shown that an inflationary era within Damour - Polyakov mechanism could solve the Polonyi moduli problem. Note that loop corrections together with the higher curvature terms are natural sources providing the necessary conditions for graceful exit in pre big-bang models of cosmological inflation [6], [7].

In the present paper we investigate cosmological models with variable dilaton in the framework of low-energy effective string gravity with higher genus corrections. In section 2 the structure of corresponding effective action is discussed and cosmological field equations are derived. In section 3 the exactly solvable cases are considered. These include pure gravi-dilaton anisotropic models with Ricci-flat subspaces, models with massless scalars and Kalb-Ramond field and radiation-dominated models. For the general case of the barotropic perfect fluid as a nongravitational source the set of cosmological equations can be presented in the form of the third-order autonomous dynamical system. The corresponding phase-space analysis is presented in section 4.

2 String effective action and cosmological model

At lowest order in string tension the string effective action can be written down as [9], [4]

$$S=\int d^Dx\sqrt{\left|\widetilde{G}\right|}\left[-\widetilde{F}_R(\varphi)\widetilde{R}-4\widetilde{F}_\varphi(\varphi)\partial_M\varphi\widetilde{\partial}^M\varphi+\widetilde{L}_m\left(\varphi,\widetilde{G}_{MN},\psi\right)\right], \qquad (1)$$

where φ is the dilaton field, $\widetilde{R}$ denotes curvature scalar of D - dimensional metric $\widetilde{G}_{MN}$, $\widetilde{L}_m$ - is the Lagrangian density of other fields, collectively noted by a symbol ψ and, in general, depending on the metric and dilaton. Hereafter the symbol $\sim$ above the letters specifies the quantities in string conformal frame, the metric of which is the metric of corresponding σ- model. Note that the string coupling is given by the vacuum expectation value of the dilaton field: $g_s=\langle e^{2\varphi}\rangle$.

The functions $\widetilde{F}_K(\varphi), K=R,\varphi,m$ (F_m is dilaton coupling function to the various types of matter fields) receive string perturbative as well as nonperturbative corrections. For the heterotic string we can write the following expansion

$$\widetilde{F}_K(\varphi)=e^{-2\varphi}\left(1+\sum_{l=1}^{\infty}\widetilde{Z}_K^{(l)}e^{2l\varphi}\right) \qquad (2)$$

where dimensionless coefficient $\widetilde{Z}_K^{(l)}$ - represents l - loop contribution, and the parameter of loop expansion is $e^{2\varphi}$: each loop of the string diagrams gives the contribution $\sim e^{2\varphi}$. At $\varphi\ll-1$ the system is in the weak coupling region and is described by tree approximation of the string diagrams. Note that for the type-I and type-II superstrings the kinetic terms of the R-R fields do not contain dilaton dependence at tree-level.

The action (1) is written in the string conformal frame. In various cases it is convenient to consider Einstein (E-) conformal frame, where the term with Ricci scalar has the standard canonical form. E- and string frames are related by conformal transformation of D - dimensional metric:

$$G^{MN}=\Omega^2(\varphi)\widetilde{G}^{MN},\quad \Omega(\varphi)=\Omega_E(\varphi)\equiv\widetilde{F}_R^{1/(1-n)} \qquad (3)$$

Up to divergent terms the E-frame action takes the form

$$S=\int d^Dx\sqrt{|G|}\left[-R-4F_\varphi(\varphi)\partial_M\varphi\partial^M\varphi+L_m(\varphi,G_{MN},\psi)\right], \qquad (4)$$

Here we assumed, that $\widetilde{F}_R>0$. If this function is negative valued, the conformal factor can be chosen similarly (3) with the absolute value. Now in the corresponding action the Ricci scalar will enter with positive sign, and the corresponding gravitational constant will be negative. The function at the kinetic term of dilaton field and matter Lagrangian are related to the corresponding string frame functions by

$$4F_\varphi(\varphi)=-\frac{n}{n-1}\left(\frac{\widetilde{F}_R'}{\widetilde{F}_R}\right)^2+4\frac{\widetilde{F}_\varphi}{\widetilde{F}_R},\quad L_m=\Omega_E^D\widetilde{L}_m\left(\varphi,\Omega_E^2G_{MN},\psi\right) \qquad (5)$$

For $F_\varphi < 0$ by introducing a new scalar field ϕ as

$$d\phi = 2\sqrt{-F_\varphi}d\varphi \tag{6}$$

the kinetic term can be written in the canonical form. Below we shall consider just this case. At tree-level $F_\varphi = -1/(n-1)$ and the new scalar field is proportional to the dilaton: $\phi = 2\varphi/\sqrt{n-1}$.

Another important conformal frame of scalar - tensor theories is the so-called Jordan frame, in which the nongravitational part of the action does not contain the scalar field. In this frame the laws of evolution of the nongravitat onal fields take the same form as in multidimensional GR.

Here we shall consider D - dimensional homogeneous cosmological model with spacetime structure $R \otimes M^1 \otimes ... \otimes M^P$ and with the metric

$$G_{MN} = diag\left(N^2(t), ..., -R_i^2(t)\, g_{lm}^{(i)}, ...\right), \tag{7}$$

where M_i is a maximally symmetric n_i dimensional space , $\sum n_i = n$, $g_{lm}^{(i)}$ is the metric in this space, $R_i(t)$ and N are corresponding scale factors anc lapse function. From the field equations it follows, that for the metric (7) the energy - momentum tensor is diagonal and has anisotropic perfect fluid form with the energy density ε, and the effective pressure p_i in the subspace M^i. The cosmological field equations are simplest in terms of the Einstein metric. By introducing new scalar field ϕ in accordance with (6), the system of cosmological equations can be written as

$$\dot{H}_i + \left(\sum_{i=1}^{p} n_i H_i - \dot{N}/N\right) H_i = N^2 b_i \varepsilon - N^2 k_i \frac{n_i - 1}{R_i^2} \tag{8}$$

$$\begin{aligned}
\ddot{\phi} + \left(\sum_{i=1}^{p} n_i H_i - \dot{N}/N\right)\dot{\phi} &= \frac{1}{2} N^2 \alpha \varepsilon \\
N^2\varepsilon + \dot{\phi}^2 - \sum k_i n_i N^2 \frac{n_i - 1}{R_i^2} &= \sum_{i,l=1}^{p} a_{il} H_i H_l
\end{aligned}$$

where the overdots denote time derivatives , $k_i = -1, 0, 1$ for subspaces with negative, zero and positive curvatures, correspondingly,

$$\begin{aligned}
H_i &= \dot{R}_i/R_i, \quad b_i = \frac{1}{2}\left[a_i + \frac{1}{n-1}\left(1 - \sum_{i=1}^{p} n_i a_i\right)\right], \quad a_i = p_i/\varepsilon, \qquad (9) \\
\alpha &= \frac{1}{\varepsilon\sqrt{|G|}} \frac{\delta L\sqrt{|G|}}{\delta\phi}, \quad a_{il} = n_i n_l - n_l \delta_{il}
\end{aligned}$$

It can be seen that in the case of sources for which the Jordan frame can be realized, we have

$$\alpha = (1 - na)\, d\ln\left(\Omega_J/\Omega_E\right)/d\phi \tag{10}$$

where Ω_J corresponds to the Jordan frame conformal factor. In (8) the choice $N = 1$ corresponds to the E-frame comoving time. From the second equation of (8) it follows, that for the solutions with constant dilaton $\varphi = \varphi_0$, the values φ_0 have to be roots of the equation $\alpha(\varphi) = 0$. The dilaton can be fixed both by dilaton potential, or by nontrivial dependence of Lagrangian of the other fields on dilaton The second way is known as Damour - Polyakov mechanism [4]. The solutions with constant dilaton coincide with corresponding solutions of multidimensional GR.

In the cosmological context the continuity equation takes the form

$$\dot{\varepsilon}/\varepsilon + \sum_{i=1}^{p} n_i(1+a_i)H_i + \alpha\,\dot{\phi} = 0, \tag{11}$$

where the last term in LHS appears due to direct interaction of the dilaton field with nongravitational sector. Note that in the Jordan frame such an interaction is absent and $\alpha = 0$.

In the case of the equation of state with constant a_i and for the function α, depending only on dilaton field , the integration of equation (11) yields

$$\varepsilon = const \cdot \exp\left(-\int \alpha(\varphi)\, d\varphi\right) \prod_{i=1}^{p} R_i^{-n_i(1+a_i)} \tag{12}$$

As an example of an additional source we shall consider system of massless scalar fields ψ_i with the E-frame Lagrangian density

$$L_m = \sum_i F_{\psi_i}(\varphi)\partial_M\psi_i\partial^M\psi_i \tag{13}$$

Within the framework of homogeneous cosmological models assuming that $\psi_i = \psi_i(t)$ we obtain for corresponding energy density and pressure

$$\varepsilon = p_i = \sum_i F_{\psi_i}\dot{\psi}_i^2/N^2, \quad a_i = 1, \quad i = 1, ..., p, \tag{14}$$

By using the equation of motion for the field ψ_i the expressions for the energy density and function α take the form

$$\varepsilon = V^{-2}\sum_i C_i^2 F_{\psi_i}^{-1}, \quad \alpha = \frac{1}{V^2\varepsilon}\sum C_i^2 F'_{\psi_i}/F_{\psi_i}^2, \quad V = \prod_{i=1}^{p} R_i^{n_i} \tag{15}$$

where C_i are integration constants.

Another important example of an additional source is the antisymmetric Kalb - Ramond field B_{MN} with strength H_{MNP} and Lagrangian density $L_m = \frac{1}{2}F_H(\varphi)$ $H_{MNP}H^{MNP}$. In $D = 4$ the corresponding equations of motion are solved by the ansatz

$$H^{MNP} = \frac{1}{\sqrt{|G|}}F_H^{-1}\varepsilon^{MNPQ}\partial_Q h, \tag{16}$$

where ε^{MNPQ} is completely antisymmetric 4-tensor, h is the pseudoscalar axion field. From the Bianci identity $\partial_{[Q}H_{MNP]} = 0$ the equation of motion for the field h follows:

$$\Box h - \partial^M h \partial_M F_H / F_H = 0 \tag{17}$$

The corresponding Lagrangian density has the form

$$L_m = F_h \partial^M h \partial_M h, \quad F_h = 1/(2F_H) \tag{18}$$

Thus in this case the Kalb - Ramond field is equivalent to the pseudoscalar field h and the corresponding formulas are a special case of the previous example. However it is necessary to note, that if for usual scalar fields in the string frame at tree-level $\widetilde{F}_{\psi_i} = e^{-2\varphi}$, for the heterotic string axion, as it follows from (18), $\widetilde{F}_h = e^{2\varphi}/2$.

3 Exactly solvable cases

In this section we shall consider cases when the set of cosmological equations (8) is exactly integrable in terms of integrals containing dilaton coupling functions.

3.1 Pure gravi - dilaton solutions

We start with the simple case of Ricci - flat subspaces ($k_i = 0$) and zero nongravitational sources ($L_m = 0$). In terms of the E-frame comoving time the solutions to cosmological equations can be written as

$$H_i = \frac{H_{i0}}{t - t_0}, \quad R_i = R_{i0} \left| t - t_0 \right|^{H_{i0}}, \quad \sum_{i=1}^{p} n_i H_{i0} = 1 \tag{19}$$

$$\phi_{10} \ln |t - t_0| = 2 \int \sqrt{-F_\varphi} d\varphi, \quad \phi_{10}^2 + \sum_{i=1}^{p} n_i H_{i0}^2 = 1 \tag{20}$$

where $H_{i0}, R_{i0}, \phi_0, t_0$- are integration constants. The last relation is a consequence of the constraint equation. Then $\phi_{10} = 0$ from (19),(20) one obtains the multidimensional generalization of the Kasner solution of the D - dimensional GR. Note that unlike the case of Kasner solution, the solution (19),(20) has an isotropic limit when one obtains $H_{10} = 1/n, \quad \phi_{10} = \pm\sqrt{1 - 1/n}$.

The corresponding solution in the string frame can be found by transformation

$$\widetilde{R}_i = \widetilde{F}_R^{1/(1-n)} R_i, \quad dt_s = \widetilde{F}_R^{1/(1-n)} dt \tag{21}$$

and has the form

$$\widetilde{R}_i = R_{i0} \widetilde{F}_R^{1/(1-n)} \exp\left(\frac{2H_{i0}}{\phi_{10}} \int \sqrt{-F_\varphi} d\varphi \right), \tag{22}$$

$$t_s = \pm \frac{2}{\phi_{10}} \int \sqrt{-F_\varphi} \widetilde{F}_R^{1/(1-n)} \exp\left(\frac{2}{\phi_{10}} \int \sqrt{-F_\varphi} d\varphi \right) d\varphi \tag{23}$$

where t_s is string frame synchronous time coordinate. For given coupling functions we can obtain the function $\varphi(t_s)$ by inversing equation (23) and then derive the scale factors from (22). At tree level from (22) and (23) one obtains the solution previously derived in [10], [11].

3.2 Solutions with modulus and Kalb - Ramond fields

In the equations (8) as an additional source we shall consider a set of scalar fields (moduli fields) and Kalb - Ramond field. As it has been mentioned already, if we accept the ansatz (16), the Kalb - Ramond field is reduced to the scalar field with Lagrangian (18). In particular, in cosmological context we shall assume

$$\varepsilon = p_i = L_m = \sum_i{}' F_{\psi_i} \dot{\psi}_i^2 / N^2 \equiv \sum_i F_{\psi_i} \dot{\psi}_i^2 / N^2 + F_h \dot{h}^2 / N^2 \tag{24}$$

and prime specifies the fact that the corresponding sum includes also the contribution of the axion field. It follows from (24), that for a source under consideration all coefficients a_i are equal to 1, and therefore all coefficients b_i are equal to zero. For these values and for Ricci - flat subspaces the solution of the first equation (8) for scale factors in synchronous system of coordinates is still determined by the relation (19). The energy density is determined by (see (15))

$$\varepsilon = \frac{1}{(t-t_0)^2} \sum_i{}' \frac{c_i^2}{F_{\psi_i}}. \tag{25}$$

The substitution of this expression into the constraint equation and integration of the obtained equation leads to the following result

$$\ln|t-t_0| = \pm \int A(\varphi) d\varphi, \quad A(\varphi) = 2\sqrt{-F_\varphi} \left[1 - \sum_{i=1}^{p} n_i H_{i0}^2 - \sum_i{}' c_i^2 / F_{\psi_i} \right]^{-1/2} \tag{26}$$

Here the functions F_{ψ_i} are related to the corresponding string frame functions by the relation $F_{\psi_i} = \widetilde{F}_{\psi_i} / \widetilde{F}_R$. Further for given function $\varphi(t)$ one finds

$$\psi_i = c_i \int \frac{A(\varphi) d\varphi}{F_{\psi_i}(\varphi)}. \tag{27}$$

For the given coupling functions in (1) the formulas (19), (26), (27) determine the evolution of the corresponding cosmological model in the E-frame. The string frame solutions can be found by transformations (21) and have the form

$$\widetilde{R}_i = R_{i0} \widetilde{F}_R^{1/(1-n)} \exp\left(\pm H_{i0} \int A(\varphi) d\varphi \right), \tag{28}$$

$$t_s = \pm sgn(t - t_0) \int A(\varphi) \widetilde{F}_R^{1/(1-n)} \exp\left(\pm \int A(\varphi) d\varphi \right) a\varphi \tag{29}$$

Note that unlike the case of the E-frame, the solutions for string frame scale factors do not coincide with corresponding vacuum solutions. At tree-level and for $n = 3$ from (28) and (29) one obtains the solutions alresdy derived in [8] for the case of isotropic models with one modulus.

3.3 Isotropic spatially-curved models

We shall now consider the isotropic models with curved space. For a set of scalar fields and Kalb - Ramond field the equation for scale factor coincides with the corresponding vacuum equation and is solved most simply in terms of conformal time η, corresponding to the gauge $N = R$:

$$R = R_m \left| \frac{1}{\sqrt{k}} \sin\left[\sqrt{k}(n-1)\eta \right] \right|^{1/(n-1)} \tag{30}$$

In view of this from the constraint equation we find for energy density

$$N^2 \varepsilon + \left(\frac{d\phi}{d\eta} \right)^2 = \frac{kn(n-1)}{\sin^2\left[\sqrt{k}(n-1)\eta \right]}. \tag{31}$$

In what follows we shall assume $0 \leq (n-1)\eta \leq \pi$ for $k = 1$ and $0 \leq (n-1)\eta < \infty$ for $k = 0, -1$. The solutions in the other regions can be obtained from here. By integrating the constraint equation for dilaton field we find the following relation

$$\pm \ln z = 2(n-1) \int \sqrt{-F_\varphi} \left[n(n-1) - \sum_i{}' c_i^2 / F_{\psi_i} \right]^{-1/2} d\varphi \tag{32}$$

where we have introduced a new variable z according to

$$z = \left| \frac{1}{\sqrt{k}} \tan\left[\sqrt{k}(n-1)\frac{\eta}{2} \right] \right| \tag{33}$$

For given functions $\widetilde{F}_K(\varphi)$ in the action (1) this formula together with (30) determines the dynamics of corresponding cosmological model. For the function $\varphi(\eta)$, determined from (32), the dependence of the other fields on time can be derived from

$$\psi_i = \frac{c_i}{n-1} \int F_{\psi_i}^{-1}(\varphi(z)) \frac{dz}{z}. \tag{34}$$

In particular, for $\psi_i = h$ one obtains the equation for the axion field. For the given coupling functions, the expressions (30), (32), (34) determine the evolution of the isotropic cosmological models with $k = 0, \pm 1$ in terms of conformal time coordinate η, which is connected to E - frame synchronous time coordinate by relation

$$t - t_0 = \pm \int R d\eta = \pm \frac{2R_0}{n-1} \int_0^z \frac{z^{1/(n-1)} dz}{(1+kz^2)^{n/(n-1)}}, \tag{35}$$

where the integral in the right hand side can be expressed through hypergeometric function. The formula (30) for the scale factor can be written down as

$$R = R_0 \left(\frac{z}{1+kz^2} \right)^{1/(n-1)}. \tag{36}$$

The corresponding solutions in string frame can be found by conformal transformation and have the form

$$\widetilde{R} = R_0 \left(\frac{z \widetilde{F}_R^{-1}}{1+kz^2} \right)^{1/(n-1)}, \quad t_s = \pm \frac{2R_0}{n-1} \int \left(\frac{z \widetilde{F}_R^{-1}}{(1+kz^2)^n} \right)^{1/(n-1)} dz$$

These relations determine string frame cosmological solutions, provided we specify the dilaton coupling functions. For open models ($k = -1$) the variable z varies within limits $0 \leq z \leq 1$, thus, as it follows from (35), time coordinate is not limited. In the case of closed models $0 \leq z < \infty$, and t varies in finite limits:

$$|t - t_0| \leq t_{em} \equiv \frac{R_0}{n-1} B \left(\frac{n}{2(n-1)}, \frac{n}{2(n-1)} \right), \tag{37}$$

where $B(x, y)$ - is Euler β-function. For the models with a flat space from (36) we have $R \sim |t - t_0|^{1/n}$, which is the isotropic limit of the solution (19). In the limit $t \to t_0$ the contribution of curvature terms is negligible and the behavior of open and closed models is the same as in a flat case. For an open model and $t \to \infty$ from (36) and (35) we have $R \sim t$.

3.4 Radiation-dominated solutions

For the radiation $a = 1/n$ and therefore according to (10) $\alpha = 0$. Integrating scalar field equation one has

$$2 \int \sqrt{-F_\varphi(\varphi)} d\varphi = const \pm \sqrt{\frac{n}{n-1}} \ln \left(1 + \frac{2}{u} + 2H\sqrt{u_0} \right), u = \left(\frac{R}{R_0} \right)^{n-1}. \tag{38}$$

where u_0, R_0 are integration constants. For the models with $k = 1$ the possible values of the function u are limited: $u \leq u_m$, where the maximum value can be expressed via constant u_0 by means of relation $u_m = \left(1 + \sqrt{1 + 4u_0}\right)/2u_0$.

The scale factor dependence on the conformal time is determined from

$$2u_0 u = \left(k + \sqrt{|4u_0 + k|}\right) \times \begin{array}{ll} \sin\left[\sqrt{k}(n-1)(\eta - \eta_0)\right]/\sqrt{k}, & u_0 > -k/4 \\ \cosh\left[(n-1)(\eta - \eta_0)\right], & k = -1, u_0 < 1/4 \\ \sqrt{u_0}\left[(n-1)^2(\eta - \eta_0)^2/(4u_0) - 1\right], & k = 0 \end{array} \tag{39}$$

(when $u_0 = 1/4$, $k = -1$ one has the solution $u = -2 + 2\exp\left[(n-1)(\eta - \eta_0)\right]$). The E-frame synchronous time coordinate t can be expressed via u as

$$t = \frac{\sqrt{u_0}R_0}{n-1}\int \frac{u^{1/(n-1)}du}{\sqrt{1 + u - ku_0u^2}}. \tag{40}$$

The function $\phi(\eta)$ can be found either by inserting (39) into (38) or more simply by direct integration of $\phi = const \cdot \int d\eta/u$ with the help of (39).

The scale factor and comoving time coordinate for the corresponding radiation-dominated solution in the string frame can be found by conformal transformation and have the form

$$\widetilde{R} = R_0\left[u/\widetilde{F}_R(\varphi)\right]^{1/(n-1)}, \quad \widetilde{t} = \pm\frac{\sqrt{u_0}}{n-1}\int \frac{\widetilde{R}(u)du}{\sqrt{1 + u - u_0u^2}} \tag{41}$$

For the given coupling functions $\widetilde{F}_\varphi(\varphi)$ and $\widetilde{F}_R(\varphi)$ in the action (1) we can obtain the function $\varphi(u)$ by inverting the equation (38) and then derive scale factor $\widetilde{R}(u)$ and commoving time $\widetilde{t}(u)$ from (41). Therefore the formulae (38) and (41) determine the string frame radiation-dominated solution in parametric form as a function of u, provided we specify the dilaton coupling functions. By choosing $\widehat{F}_\varphi = \widetilde{F}_R = e^{-2\varphi}$ one obtains the corresponding tree-level solutions.

4 Qualitative analysis in general case

In previous section we have considered some special cases for which cosmological field equations are exactly solvable in terms of integrals containing dilaton coupling functions. For general case of barotropic fluid with equation of state $p = a\varepsilon$, $-1 \leq a \leq 1$, this is not the case. In this section we shall consider the qualitative evolution of the isotropic models by using dynamical system approach.

By introducing a new independent variable τ and the function $h(\tau)$ in accordance with

$$d\tau = \sqrt{H^2 + k}d\eta, \qquad h = \frac{d\ln R}{d\tau} = \frac{H}{\sqrt{H^2 + k}} \tag{42}$$

and substituting $N^2\varepsilon$ from the constraint equation into the ones for scale factor and scalar field the equations of motion can be presented in the form of third-order autonomous dynamical system

$$\begin{aligned} \frac{d\phi}{d\tau} &= x, \qquad \frac{dx}{d\tau} = \left[n(n-1) - x^2\right]\left[\alpha(\phi)/2 - bhx\right], \\ \frac{dh}{d\tau} &= \left(1 - h^2\right)\left[(n-1)(nb-1) - bx^2\right], \quad b = \frac{1-a}{2(n-1)} \end{aligned} \tag{43}$$

This system is invariant under the transformations

$$\tau \to -\tau, \quad x \to -x, \quad h \to -h \tag{44}$$

relating expanding and contracting models, and

$$\phi \to -\phi, \quad x \to -x, \quad \alpha \to -\alpha \tag{45}$$

relating the models, for two, in general different, functions $\alpha(\phi)$ and $-\alpha(-\phi)$. The phase trajectories describing the models with nonnegative energy density lie in the region

$$n(n-1) - x^2 \geq 0. \tag{46}$$

Here the sign of equality corresponds to the pure gravi-dilaton models ($\varepsilon = 0$), which are considered in previous section. The solutions with $h = \pm 1$ correspond to the spatially-flat models. In this case the system (43) is reduced to the second order dynamical sistem.

The key aspect for qualitative analysis is to determine the critical points and their stability. At finite part of the phase space (ϕ, x, h) the critical points of the system (43) are the points

$$(\phi_0, 0, \pm 1), \quad \alpha(\phi_0) = 0 \tag{47}$$

where upper/lower sign corresponds to the expanding/contracting models. We shall consider the first of these cases. The results for the contracting models can be obtained from here by means of transformation (44). Note that the points (47) are GR solutions for $k = 0$. The constant dilaton solutions for $k = 1$ are presented by segments $\phi = \phi_0$, $x = 0$, $|h| < 1$, and for $k = -1$ by ray $\phi = \phi_0$, $x = 0$, $|h| > 1$.

Near the critical point (47) the equation for h drops from the other two ones and the corresponding eigenvalues are equal to

$$\lambda_{1,2} = -k_0 \pm \sqrt{k_0^2 + n(n-1)\alpha_0'/2}, \quad \lambda_3 = n(1+a) - 2 \tag{48}$$

where $\alpha_0' = (d\alpha/d\phi)_{\phi=\phi_0}$, $k_0 = n(1-a)/4$. The eigenvalues $\lambda_{1,2}$ define the nature of the critical point for the trajectories in the invariant subspace $h = 1$, corresponding to the spatially-flat models. The standard qualitative analysis shows that for spatially-flat models the point (47) is (1) saddle when $\alpha_0' > 0$, (2) center when $a = 1$, $\alpha_0' < 0$, (3) stable focus when $\alpha_0' < -\alpha_0^2/8$, $\alpha_0 = 2b\sqrt{n(n-1)}$, (4) improper node when $\alpha_0' = -\alpha_0^2/8$ and (5) node when $0 > \alpha_0' > -\alpha_0^2/8$. In the case of expanding models the existence for the function $\alpha(\phi)$ of zero $\phi = \phi_0$ with $\alpha'(\phi_0) < 0$ leads to the efficient mechanism for the dilaton stabilization and relaxation of string effective gravity to the GR. Indeed, for the relation of the dilaton variations at some time moments t_i and t_e we have

$$\frac{\phi(t_e) - \phi_0}{\phi(t_i) - \phi_0} \sim \exp\left\{-|Re\lambda_1|\left[\tau(t_e) - \tau(t_i)\right]\right\} = \left[\frac{R(t_i)}{R(t_e)}\right]^{|Re\lambda_1|} \tag{49}$$

In the cases 3) and 4), when critical point is focus or improper node one has $|Re\lambda_1| = n(1-a)/4$. In the inflationary stage $a = -1$ and the relation of the scale factors at the beginning and at the end of inflation is $\sim e^{-65}$. If initially the dilaton shift ~ 1, one receives that at the end of inflation the dilaton variations are strongly suppressed ($n = 3$) $\phi(t_e) - \phi_0 < 10^{-42}$. Assuming that for the various sources the zero of the function $\alpha(\phi)$ is the same (in particular, this is the case when the dilaton coupling functions are universal) the subsequent expansion stages (radiation- and matter-dominated) leads to the further suppression of these variations: $|\delta\phi| < 10^{-49}$ at present epoch [5]. Such variations are much more far from observational restrictions following from the tests of the equivalence principle. Thus this scheme leads to the natural suppression of the dilaton variations in the expanding universe and lies in the basis of dilaton stabilizing mechanism known as Damour-Polyakov mechanism. Note that this scheme will work also in the case 5), if the value of $|\gamma_0'|$ is not too small.

As it follows from (48) for the models $k = \pm 1$ the critical point is unstable when $1 + a > 2/n$. The corresponding GR solutions tend to the spatially flat solution (point (47) with the upper sign) as $t \to -\infty$. In the case $1 + a < 2/n$ the GR solutions with $k = \pm 1$ tend to the solution (47) in the limit $t \to +\infty$ (inflationary models). Then additionally $Re\lambda_{1,2} < 0$, the critical point is stable and the solutions of string cosmology with variable dilaton tend to the solution (47) too.

Now we turn to the investigation of the behavior of the phase trajectories at infinity of the phase-space. For this it is convenient to compactify the phase space. In the case of the closed models it is sufficient to use the mapping

$$e^{\phi} = \frac{y}{1-y}, \quad 0 \le y \le 1 \tag{50}$$

The corresponding dynamical system differs from (43) by the first of the equations and the phase space is the rectangular parallelepiped

$$0 \le y \le 1, \quad |x| \le \sqrt{n(n-1)}, \quad |h| \le 1 \tag{51}$$

The boundaries of this parallelepiped are invariant subspaces. The trajectories on the boundaries $h = 1, -1$ correspond to the spatially-flat expanding and contracting models, respectively, and the trajectories on the boundaries $x = \pm\sqrt{n(n-1)}$ represent the pure gravi-dilaton models.

For $h = 1$ (the case $h = -1$ can be obtained by transformation (4)) the critical points on the boundary of (51) are the points (y, x, h) with coordinates ($\alpha = \alpha(y)$)

$$\left(j, \pm\sqrt{n(n-1)}, 1\right), \quad j = 0, 1 \tag{52}$$

$$(j, \alpha(j)/2b, 1), \quad |\alpha(j)| \le \alpha_0 \tag{53}$$

The eigenvalues for these points are

$$\lambda_1 = \pm(-1)^j\sqrt{n(n-1)}, \quad \lambda_2 = n(1-a)(1 \mp \alpha(j)/\alpha_0), \quad \lambda_3 = 2(n-1) \tag{54}$$

for the point (52), and

$$\lambda_1 = (-1)^j \alpha(j)/2b, \quad \lambda_2 = -\frac{n}{2}(1-a)(1-\alpha^2(j)/\alpha_0^2 \tag{55}$$

$$\lambda_3 = 2n(n-1)b\left[\alpha^2(j)/\alpha_0^2 + 1/nb - 1\right] \tag{56}$$

for the points (53). Thus, for the models with $k = \pm 1$ the points (52) are always unstable. The points (53) are stable when

$$(-1)^j \alpha(j) < 0, \quad 1 - \frac{\alpha^2(j)}{\alpha_0^2} > \frac{2(n-1)}{n(1-a)} \tag{57}$$

Note that the second of these conditions is possible only if $a < 2/n - 1$. Now by taking into account (48) we conclude that for $k = \pm 1$ all critical points with $h = 1$ are unstable when $a > 2/n - 1$.

In the case $a < 2/n - 1$ there are critical points laying on the boundaries $y = 0, 1$ of parallelepiped (51):

$$\left(j, \pm\sqrt{(n-1)(n-1/b)}, h_1(\alpha(j))\right), \quad h_1 = \frac{\pm\alpha(j)}{\alpha_0\sqrt{1-1/nb}}, \quad j = 0, 1 \tag{58}$$

For these points the eigenvalues are determined by the relations

$$\lambda_1 = \pm(-1)^j \sqrt{(n-1)(n-1/b)} \tag{59}$$

$$\lambda_{2,3} = \frac{n-1}{2}h_1\left[-1 \pm \sqrt{1 - 8(nb-1)(1-h_1^{-2})}\right] \tag{60}$$

When $h_1^2 < 1$ the points (58) correspond to the closed models and are unstable, as the eigenvalues λ_2 and λ_3 have different signs. When $h_1^2 > 1$ the solutions corresponding to the points (58) describe the open models. For the case of upper sign the point with $j = 0$ is unstable, and the point with $j = 1$ is stable when $\alpha(y = 1) > 0$ and corresponds to the expanding models. Thus, the solution corresponding to the point (58) can be stable only for open expanding models: $\pm\alpha(j) > 0$, $\pm(-1)^j < 0$, $h_1^2 > 1$.

For the open models $|h| > 1$, to compactify phase plane in addition to (50) we have to map

$$z = 1/h, \quad |z| \leq 1 \tag{61}$$

By introducing the new independent variable $dT = hd\tau$ the corresponding dynamical system can be written as

$$\frac{dy}{dT} = xyz(1-y), \quad \frac{dx}{dT} = \left[n(n-1) - x^2\right]\left[\alpha z/2 - bx\right] \tag{62}$$

$$\frac{dz}{dT} = z\left(1 - z^2\right)\left[(n-1)(nb-1) - bx^2\right]$$

As above we shall consider the expanding models (the case of contracting models can be obtained by transformation $x \to -x$, $z \to -z$). The phase space of the system (62) is the parallelepiped

$$0 \leq y \leq 1, \quad |x| \leq \sqrt{n(n-1)}, \quad 0 \leq z \leq 1 \tag{63}$$

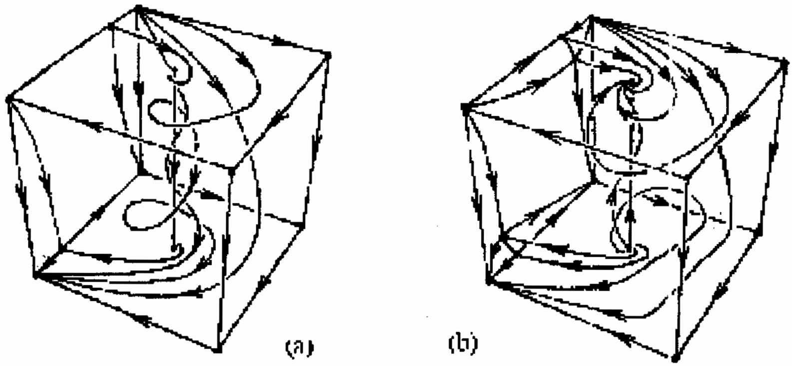

Fig. 1. Phase–space diagrams in the case of the models with positive spatial curvature compactified into space (y, x, h) for (a) $a > 2/n - 1$ and (b) $a < 2/n - 1$.

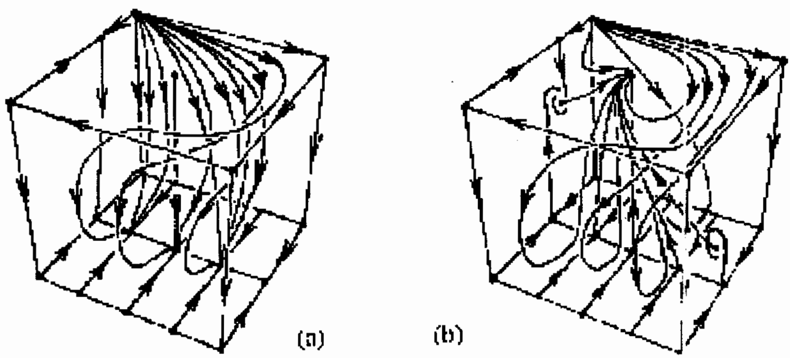

Fig. 2. Phase–space diagrams in the case of the models with negative spatial curvature compactified into space $(y, x, z = 1/h)$ for (a) $a > 2/n - 1$ and (b) $a < 2/n - 1$.

and the critical points lie on the boundaries $z = 1$ and $z = 0$. For the first of these cases $(h = 1)$ the critical points and their nature are the same as in the previous case and correspond to the spatially flat models. Thus we shall consider the second case. At $z = 0$ $(h = \infty)$ the solutions of the system (62) are the segments $z = 0$, $y = constant$. All points of the segments $(x = 0, z = 0)$ and

($x = \pm\sqrt{n(n-1)}, z = 0$) are critical points. The eigenvalues corresponding to the axis z are equal to

$$\lambda_3 = (n-1)(nb-1) \tag{64}$$

for the points $x = z = 0$, and

$$\lambda_3 = 1 - n \tag{65}$$

for the points $x = \pm\sqrt{n(n-1)}$, $z = 0$. In the second case the critical points are unstable. The points $x = z = 0$ are stable when $a > 2/n - 1$ and unstable otherwise. By taking into account that for the first of these cases the critical points for $z = 1$ are unstable, we conclude that all models with $k = -1$ finish evolution on the segment $x = 0$, $z = 0$. Near the corresponding critical point the trajectories of the dynamical system enter to this point lying in the plane (x, z). In this plane the corresponding critical point is an stable node with the eigenvalues

$$\lambda_2 = -nb(n-1), \quad \lambda_3 = (n-1)(nb-1) \tag{66}$$

In the case $a < 2/n - 1$ the final point of the evolution for the models with $k = -1$ are the points on the boundary $z = 1$, corresponding to the models with $k = 0$, and the points (58) under the conditions specified above (see the paragraph after the formula (59).

On the basis of the analysis carried out above it is possible to construct the phase portraits of the cosmological models for arbitrary function $\alpha(\phi)$, defined by dilaton coupling functions in the string effective Lagrangian.. As an illustration in Fig. 1 and 2 we present the phase portraits for the closed and open models in the case of the function $\alpha(\phi)$, satisfying to the boundary conditions $0 < \alpha(-\infty) < \alpha_0$, $\alpha(+\infty) < -\alpha_0$ and with a single zero. For the models with nonnegative energy density the phase space is the parallelepiped (51) in the case $k = 1$ (Fig. 1) and (63) in the case $k = -1$ (Fig. 2). The trajectories lying on the upper boundaries ($h = 1$) represent an invariant subspace and correspond to the spatially flat models. In Fig. 1 the bottom boundary also corresponds to the spatially flat models ($h = -1$, models of contraction). The vertical segments $\phi = \phi_0$, $x = 0$, $|h| < 1$ and $0 < z < 1$ represent the corresponding GR solutions. Let us consider the presented phase portraits in turn.

1. $k = 1$, $a > 2/n - 1$ (Fig. 1a). The function $h(\tau)$ is decreasing. The only stable critical point is $(0, -\sqrt{n(n-1)}, -1)$. All the models , except the ones corresponding the separatrixes of the saddle points and GR solutions, begin evolution in the weak coupling region from the point $(0, \sqrt{n(n-1)}, 1)$ at some finite time moment of the E-frame synchronous time coordinate, and finish the evolution at the another finite time moment in the weak coupling region in the state of contraction. Thus the pure gravi-dilaton solutions are early and late time attractors. The GR solution corresponding to the constant dilaton, is unstable.

2. $k = 1, a < 2/n - 1$ (Fig. 1b). Now there is a region of the phase space, $|x| < \sqrt{(n-1)(n-1/b)}$, with the inflationary segments of trajectories, $dh/d\tau > 0$. There are four qualitatively different classes of solutions with variable dilaton:

   (a) models which begin and finish the evolution in the weak coupling region and look like to those of Fig. 1a. However, now there are segments of trajectories with $dh/d\tau > 0$;

   (b) models beginning the evolution in the weak coupling region near the pure gravi-dilaton spatially flat expanding solution (the point $(0, \sqrt{n(n-1)}, 1)$). After the initial noninflationary expansion at some moment $dh/d\tau = 0$, the evolution enters to the inflationary era, $dh/d\tau > 0$, and the trajectories enter to the point (47) with $h = 1$ (the transitions from $dh/d\tau < 0$ to $dh/d\tau > 0$ can take place several times). All the models of this class tend to the corresponding GR solution, and the curvature effects quickly becomes inessential. In this scheme the dilaton variations becomes sufficiently damped to avoid the contradiction with the observations. This is just Damour-Polyakov mechanism of dilaton stabilization;

   (c) models starting the contraction from the point (47) with the lower sign, near the GR solution and finishing the evolution in the point $(0, -n^{1/2}\sqrt{n-1}, -1)$ near the pure gravi-dilaton solution. Note that in intermediate stages the contraction - expansion transitions can take place. As in the first case the cosmological evolution does not stabilize the dilaton;

   (d) trajectories starting from the point (47) with the lower sign and entering the point (47) with the lower sign. For these models during the evolution the initial contraction turns into the expansion, the curvature effects of the space quickly becomes inessential and the models tend to the corresponding spatially flat GR solution.

3. $k = -1, a > 2/n - 1$ (Fig. 2a). E-frame expanding models are presented. The phase portrait for contracting models can be obtained by transformation (44) (note that for the dynamical system (62) the variable T does not change the sign). The function $h(\tau)$ is increasing and the function $H(t)$ is decreasing.. All the models, except the special ones, begin evolution in the neighborhood of the spatially flat pure gravi-dilaton solution and finish it on the segment $x = 0$, $z = 0$ on the lower boundary.. The corresponding limiting value of the dilaton depends on the initial conditions.

4. $k = -1, a < 2/n - 1$ (Fig. 2b). After the initial noninflationary expansion in the vicinity of the pure gravi-dilaton solution the models enter into the inflationary era and tends to the GR solution. The inflation leads to the strong suppression of the dilaton variations.. The trajectories quickly tends to the GR flat solution. There are also a class of models beginning the evolution on the segment $x = 0$, $z = 0$ (separatrixes of the degenerate saddle points on

this segment). Thus, unlike to the case of the closed models (Fig. 1b), now he dilaton stabilization takes place for the all models. As for the considered example of the function $\alpha(y)$, we have assumed $\alpha(0) > 0$ and $\alpha(1) < 0$, then the critical points (58) can not be stable.

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A NOTE ON QUANTUM PITCHFORK/SADDLE-NODE BIFURKATIONS IN A PERTURBED OSCILLATOR WITH FOUR-PARAMETRES

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Abstract

In this paper, a quantum mechanical study is made for a 1:1 resonant perturbed oscillator in the Birkhoff-Gustavson normal form with *four* parameters, which exhibits both pitchfork and saddle-node bifurcations in its periodic trajectories in classical theory. On quantizing that oscillator, it is shown that two kinds degeneracies of energy levels are shown to take place as quantum counterparts of those bifurcations.

1 Introduction.

It has been an interesting question whether significant phenomena, such as chaos, bifurcations, ..., etc., exhibited in classical dynamical systems admit their quantum counterparts or not (see [N] and references therein). In a series of papers [U1-3] of the author, quantum counterparts of bifurcations were studied in certain 1:1 resonant perturbed oscillators in the Birkhoff-Gustavson normal form with *two/three* parameters.

This paper is a continuation of [U1-3], and aims to study quantum counterparts of pitchfork/saddlenode bifurcations of periodic trajectories in a 1:1 resonant perturbed oscillator with *four* parameters in the Birkhoff-Gustavson normal form. Through the torus quantization, two kinds of degeneracies of energy levels are shown to exist as quantum counterparts of the pitchfork/saddle-node bifurcation. The quantum study of the saddle-node bifurcation has been prepared already [U4], and an extended version of this note is in preparation [U5].

The perturbed oscillator to be dealt with in this paper is a Hamiltonian system $(\mathbf{R}^2 \times \mathbf{R}^2, dp \wedge dq, H_\alpha)$ defined as follows. Let $(\mathbf{R}^2 \times \mathbf{R}^2, dp \wedge dq)$ be the phase space endowed with the canonical symplectic 2-form, $dp \wedge dq = \sum_{j=1}^{2} dp_j \wedge dq_j$, where q and p are the Cartesian coordinates of $\mathbf{R}^2$'s. The H_α is the Hamiltonian given by

$$H_\alpha = J + \alpha_1 J L_2 + \alpha_2 J L_3 + \frac{\alpha_3}{2} L_2^2 + \frac{\alpha_4}{2} L_3^2 \tag{1}$$

where J, L_1, L_2, and L_3 are defined by

$$\begin{aligned} J &= \frac{1}{2}(p_1^2+q_1^2)+\frac{1}{2}(p_2^2+q_2^2), & L_1 &= q_1q_2+p_1p_2, \\ L_2 &= q_1p_2-q_2p_1, & L_3 &= \frac{1}{2}(p_1^2+q_1^2)-\frac{1}{2}(p_2^2+q_2^2). \end{aligned} \tag{2}$$

The $\alpha = (\alpha_j)_{j=1,\cdots,4}$ in H_α are the parameters taking their values in the region,

$$\mathcal{P} = \{\alpha = (\alpha_1, \cdots, \alpha_4) \in \mathbf{R}^4 \mid \alpha_j > 0\ (j = 1, \cdots, 4),\ a_3 > a_4\}. \tag{3}$$

The H_α is can be regarded as a Hamiltonian of a perturbed system of the 1:1 resonant harmonic oscillator, on account of $H_0 = J$.Moreover, H_α is in the Birkhoff-Gustavson normal form since it Poisson-commutes with J. The cases of $\alpha_1 = \alpha_4 = 0$ and of $\alpha_4 = 0$ were dealt with in [U1-3].

The contents of this paper are summarized as follows. In Sec. 2, it is shown that the perturbed oscillator exhibits both pitchfork and saddle-node bifurcations in its periodic trajectories,whose bifurcation sets are given by

$$\mathcal{B}^P = \{\alpha \in \mathcal{P} \mid \phi^P(\alpha) = 0\}, \quad \mathcal{B}^S = \{\alpha \in \mathcal{P} \mid \phi^S(\alpha) = 0\}, \tag{4}$$

respectively, where $\phi^P(\alpha)$ and $\phi^S(\alpha)$ are the functions defined by

$$\phi^P(\alpha) = 1 - \left(\frac{\alpha_1}{\alpha_3}\right)^2 - \left(\frac{\alpha_2}{\alpha_4}\right)^2, \quad \phi^S(\alpha) = (\alpha_4-\alpha_3)^{2/3} - (\alpha_1^{2/3}+\alpha_2^{2/3}). \tag{5}$$

With Section 3, the torus quntization of the perturbed oscillator starts. Since the perturbed oscillator admits J as a first integral, all the regular compact connected components of the level sets of H_α and J, provide a family of invariant tori [AM] required for the torus quantization. After the introduction of the torus quantization condition, a sufficient condition for degeneracies of energy levels is derived, so that two kinds of degeneracies of energy levels are suggested as quantum pitchfork/saddle-node bifurcations.

In Sec. 4, one of those kinds of degeneracies is shown to be a quantum pitchfork bifurcation: The set of the values of parameters,α, at which the degeneracies take place is bounded by the classical pitchfork bifurcation set $\mathcal{B}^P$.

In Sec. 5, the other kind of degeneracies is shown to be a quantum saddle-node bifurcation: The closure of the set of the values of parameters, α, at which the degeneracies take place is bounded by the classical saddle-node bifurcation set $\mathcal{B}^S$.

2 Bifurcations of Periodic Trajectories. Classical Theory.

Since J is a first integral of the perturbed oscillator, the $U(1)$ action,

$$p + iq \in \mathbf{R}^2 \times \mathbf{R}^2 \mapsto \exp(is)(p+iq) \in \mathbf{R}^2 \times \mathbf{R}^2 \quad (s \in [0, 2\pi]), \tag{6}$$

generated by J is admitted as a symmetry of the perturbed oscillator. Hence the perturbed oscillator can be reduced by the $U(1)$ action to a Hamiltonian system on the quotient manifold, $J^{-1}(\nu)/U(1)$, of the level manifold,

$$J^{-1}(\nu) = \{(q,p) \in \mathbf{R}^2 \times \mathbf{R}^2 \mid J(q,p) = \nu > 0\}, \tag{7}$$

of J. The $J^{-1}(\nu)/U(1)$ is shown to be diffeomorphic to S^2, the two-dimensional sphere with unit radius, by using the Hopf map,

$$\pi_\nu : (q,p) \in J^{-1}(\nu) \mapsto x = (1/\nu)\,(L_1(q,p),\, L_2(q,p),\, L_3(q,p)) \in S^2 \subset \mathbf{R}^3, \tag{8}$$

of $J^{-1}(\nu)$ to S^2. Through π_ν, the Hamiltonian equation for the perturbed oscillator is reduced to an 'Euler-like' equation [U2-3],

$$\frac{dx}{dt} = -\frac{2}{\nu}\, x \times \operatorname{grad} H^{\mathrm{red}}_{\alpha,\nu} \quad (x \in S^2 \subset \mathbf{R}^3,\ \times : \text{the vector product}) \tag{9}$$

on S^2, where $H^{\mathrm{red}}_{\alpha,\nu}$ is the reduced Hamiltonian defined on S^2 of the form,

$$H^{\mathrm{red}}_{\alpha,\nu} = \nu + \nu^2 \left\{ b_2 x_2 + b_3 x_3 + \frac{a_2}{2} x_2^2 + \frac{a_3}{2} x_3^2 \right\}, \tag{10}$$

which is induced from H_α through π_ν. The reduction procedure provides us with the following crucial fact [K,CR]:

Fact 1: *For every equilibrium point of Eq. (9), there exists on each $J^{-1}(\nu)$ the unique periodic trajectory which is projected to the equilibrium point through π_ν. Such equilibrium point and its associated periodic trajectory share the same stability signature.*

Owing to Fact 1, we look into the e.p.'s of Eq. (9) instead of the periodic trajectories of the perturbed oscillator.

Lemma 2: *All the possible equilibrium points of Eq. (9) are classified as in Table 1 according to the signatures of ϕ^S and ϕ^P defined by (5). In Table 1, 'e', 'h', and 'd' stand for e. p.'s with a definite Jacobi matrix, with an indefinite one, and with a degenerate one, and the roman digits indicate numbers of them.*

	$\phi^P(\alpha) < 0$	$\phi^P(\alpha) = 0$	$\phi^P(\alpha) > 0$
$\phi^S(\alpha) < 0$	2-e	1-e, 1-d	3-e, 1-h
$\phi^S(\alpha) = 0$	2-e, 1-d	1-e, 2-d	3-e, 1-h, 1-d
$\phi^S(\alpha) > 0$	3-e, 1-h	2-e, 1-h, 1-d	4-e, 2-h

Table 1: The classification of the equilibrium points of Eq. (10)

Each row of Table 1 implies that a pitchfork bifurcation [MR] takes place for the e.p.'s when α passes across the set $\mathcal{B}^P$ defined by (4) with (5). So does each column of Table 1 that a saddle-node bifurcation [MR] takes place when α passes across the set $\mathcal{B}^S$. Recalling Fact 1, we have the following:

Theorem 3: *On every $J^{-1}(\nu)$, the periodic trajectories of the perturbed oscillator allowed to exist by Fact 1 exhibit the the pitchfork (resp. saddle-node) bifurcations as the parameter $\alpha \in \mathcal{P}$ passes across $\mathcal{B}^P$ (resp. $\mathcal{B}^S$) defined by (4) with (5).*

3 Setting up the Torus Quantization.

A. Family of Invariant Tori Since J is a first integral, we are able to construct a family of invariant tori required for the quantization [AM] by collecting all the compact connected regular component of the level sets of J and H_α expressed as

$$M_\alpha(\nu, E) = \{(q,p) \in \mathbf{R}^2 \times \mathbf{R}^2 \mid J(q,p) = \nu,\ H_\alpha(q,p) = \nu + \nu^2 E\}. \tag{11}$$

We note that, according to the Arnold theorem, each of such components is diffeomorphic to the two dimensional torus, which is called the invariant torus [AM].We get a classification of the topology of $M_\alpha(\nu, E)$ [U5] in the following by using the local triviality property [Nk] of the principal $U(1)$ bundle $\pi_\nu : J^{-1}(\nu) \to S^2$.

Lemma 4: *If a level set, $M_\alpha(\nu, E)$, of H_α and J is regular,it is diffeomorphic to one of the following listed in Table 2, where $\coprod$ stands for the disjoint union operation.*

	$\phi^P(\alpha) < 0$	$\phi^P(\alpha) = 0$	$\phi^P(\alpha) > 0$
$\phi^S(\alpha) < 0$	T^2	T^2	$T^2, T^2 \coprod T^2$
$\phi^S(\alpha) = 0$	T^2	T^2	$T^2, T^2 \coprod T^2$
$\phi^S(\alpha) > 0$	$T^2, T^2 \coprod T^2$	$T^2, T^2 \coprod T^2$	$T^2, T^2 \coprod T^2$

Table 2: The classification of the regular $M_\alpha(\nu, E)$

B. Quantization Condition We are now in a position to introduce the quantization condition for the invariant tori:An invariant torus is quantizable if and only if the integral condition,

$$\frac{1}{2\pi\hbar} \oint_c p^T dq - \frac{1}{4}\mathcal{M}(c) \in \mathbf{Z} \text{ (the set of all the integers)}, \tag{12}$$

is valid for every (smooth) loop c in the torus, where $\hbar$ = (Plank's constant)/2π [AM]. The first term of (12) is the action integral and the second one a quarter of the Maslov index.The quantizable tori are said to represent 'quasi-classical' states of a given Hamiltonian system [W].

Owing to the Stokes' theorem, we have to estimate (12) only for a pair of topologically independent non-contractible loops in each invariant torus T^2:Since the local triviality [Nk] of the $U(1)$ bundle,$\pi_\nu : J^{-1}(\nu) \to S^2$ provides us with the diffeomorphism, $T^2 \simeq U(1) \times \pi_\nu(T^2)$, for T^2, one of the non-contractible loops, denoted by c^1, is the factor space coming from the $U(1)$ action (6):

$$c^1 = \left\{(q,p) \in T^2 \,|\, p + iq = \exp(is)(\tilde{p} + i\tilde{q}),\ s \in [0, 2\pi]\right\}, \tag{13}$$

where $(\tilde{q},\tilde{p}) \in T^2$ is fixed arbitrarily. The other loop, denoted by c^2, of T^2 is constructed by using a local cross section [Nk], $\sigma_\nu : x(\neq (0,0,-1)) \in S^2 \mapsto \sqrt{2\nu}(Q(x),P(x)) \in J^{-1}(\nu)$, of $\pi_\nu : J^{-1}(\nu) \to S^2$, where

$$Q(x) = \left(\sqrt{(1+x_3)/2},\ x_1/\sqrt{2(1+x_3)}\right), \quad P(x) = \left(0,\ x_2/\sqrt{2(1+x_3)}\right). \tag{14}$$

The c^2 for T^2 subject to $(0,0,-1)^T \notin \pi_\nu(T^2_{\alpha,\pm}(\nu,E))$ is defined by

$$c^2 = \sigma_\nu(\pi_\nu(T^2)) = \{(q,p) \in T^2 \mid (q,p) = \sqrt{2\nu}(Q(x),P(x)),\ x \in \pi_\nu(T^2)\}, \tag{15}$$

which is oriented oppositely to the solution of (9) along $\pi_\nu(c^2)$.

C. Sufficient Condition for Degeneracies Before estimating (12), we are to think of a sufficient condition for degeneracies of energy levels. Within the torus quantization, an energy level, say $\mathcal{E}$, admits is degenerate if and only if there are at least two quantizable invariant tori both on which H_α takes the common value $\mathcal{E}$. Assuming such quantizable tori denoted by T_j^2 $(j = 1,2)$ are included in the level sets, $M_\alpha(\mu_j, E_j)$, we have $\mathcal{E} = \nu_1 + \nu_1^2 E_1 = \nu_2 + \nu_2^2 E_2$. Hence, as a sufficient condition for this equation, we obtain $\nu_1 = \nu_2 = \exists\nu'$, and $E_1 = E_2 = \exists E'$, which implies $T_1^2, T_2^2 \subseteq M_\alpha(\nu', E')$. To summarize, we have the following:

Lemma 5: *A degeneracy of energy levels takes place if there exists at least one level set, $M_\alpha(\nu,E)$, homeomorphic to $T^2 \coprod T^2$ both of whose T^2 components are quantizable.*

We identify the families of the invariant tori made by collecting all the $M_\alpha(\nu,E) \simeq T^2 \coprod T^2$ which emerge due to $\phi^P(\alpha) > 0$ (resp. $\phi^S(\alpha) > 0$:

Lemma 6: *Let $\lambda_0 = 0$ and λ_k $(k = 1,2,3)$ be the solutions of the equation,*

$$\{\alpha_1/(\lambda-\alpha_3)\}^2 + \{\alpha_2/(\lambda-\alpha_4)\}^2 = 1, \tag{16}$$

subject to $\lambda_1 < \alpha_3 < \lambda_2 < \lambda_3 < \alpha_4$, where λ_2 and λ_3 are admissible only in the case of $\phi^S(\alpha) > 0$. With λ_k $(k = 0,1,2,3)$, let E_k be associated by

$$E_j = \frac{\alpha_1^2}{(\lambda_j-\alpha_3)} + \frac{\alpha_2^2}{(\lambda_j-\alpha_4)} + \frac{\alpha_1^2\alpha_3}{2(\lambda_j-\alpha_3)^2} + \frac{\alpha_2^2\alpha_4}{2(\lambda_j-\alpha_4)^2} \tag{17}$$

Then the family of invariant tori emerging due to $\phi^P(\alpha) > 0$ consists of all the $M_\alpha(\nu,E) \simeq T^2 \coprod T^2$ subject to $E \in (E_0, E_1)$, which will be denoted by $\mathcal{T}^P$. The family emerging due to $\phi^S(\alpha) > 0$ does of those subject to $E \in (E_2, E_3)$, which will be denoted by $\mathcal{T}^S$.

4 Quantum Pitchfork Bifurcation.

So far we have specified the family of invariant tori, $\mathcal{T}^P$, associated with the pitchfork bifurcation, we estimate the quantization condition (12) for the invariant tori belonging to $\mathcal{T}^P$, and then determine the values of α at which the degeneracies allowed to exist by Lemma 5 for $\mathcal{T}^P$

It is convenient to denote the invariant tori in $M_\alpha(\nu, E) \simeq T^2 \coprod T^2 \in \mathcal{T}^P$ by $T_P^2(\alpha, \pm; \nu, E)$, which stand for

$$\begin{aligned} T_P^2(\alpha, +; \nu, E) &= M_\alpha(\nu, E) \cap \{(q,p) \in \mathbf{R}^2 \times \mathbf{R}^2 | L_1(q,p) > 0\} \\ T_P^2(\alpha, -; \nu, E) &= M_\alpha(\nu, E) \cap \{(q,p) \in \mathbf{R}^2 \times \mathbf{R}^2 | L_1(q,p) < 0\}. \end{aligned} \tag{18}$$

Accordingly, we denote by $c_P^j(\alpha, \pm; \nu, E)$ $(j = 1, 2)$ the pair of non-contractible loops defined by (13) and (15) for $T_P^2(\alpha, \pm; \nu, E)$.

Equation (12) for $c_P^1(\alpha, \pm; \nu, E)$ is easily evaluated to give

$$\nu = \hbar(n+1) \qquad (n = 0, 1, \cdots), \tag{19}$$

which agrees with the usual quantized values of J. Since the canonical transformation, $(q_1, q_2, p_1, p_2) \mapsto (q_1, -q_2, p_1, -p_2)$, causes the interchange of $T_P^2(\alpha, \pm; \nu, E)$, the Maslov index and the action integral for $c_P^2(\alpha, \pm; \nu, E)$ are calculated to be

$$\mathcal{M}(c_P^2(\alpha, \pm; \nu, E)) = -2, \quad \frac{1}{2\pi\hbar} \oint_{c_P^2(\alpha,+;\nu,E)} p^T dq = \frac{1}{2\pi\hbar} \oint_{c_P^2(\alpha,-;\nu,E)} p^T dq, \tag{20}$$

so that we have the following.

Lemma 7 *If one of $T_P^2(\alpha, \pm; \nu, E)$ is quantizable, then so is the other.*

We continue estimating (12) to identify the values of α at which degeneracies of energy levels take place allowed to exist by Lemma 5 for $\mathcal{T}^P$. A key equation is

$$\frac{1}{2\pi\hbar} \oint_{c_P^2(\alpha,-;\nu,E)} p^T dq = -\frac{\nu}{\pi} A^P(\alpha; E), \tag{21}$$

where $A^P(\alpha; E)$ is the area enclosed to the left by $\tilde{c}_P^2(\alpha, -; E) \subset S^2$. Note that $A^P(\alpha; E)$ is well-defined even in the case of $E = E_1$ as the area enclosed to the left by a simple component of the separatorix given by $H_{\alpha,\nu}^{\text{red}} = \nu + \nu^2 E_1$. The properties of $A_n^P(\alpha; E)$ to be utilized are presented in Appendix.

Let α be fixed to satisfy $\phi^P(\alpha) > 0$ and ν be quantized as (19). The increasing and the continuity properties of $A^P(\alpha, E)$ (see Lemma 13 in Appendix) are put together with (21) to show that there exists at least a pair of quantizable tori, $T^2(\alpha, \pm; \hbar(n+1), E)$ if and only if

$$A^P(\alpha, E_1) > 2\pi/(n+1) \tag{22}$$

holds true. Since Eq. (22) is in fact valid for every sufficiently large n, we have the following recalling Lemma 7:

Lemma 8 *For every α belonging to*

$$\mathcal{D}^P = \{\alpha \in \mathcal{P} \,|\, \phi^P(\alpha) > 0\} \tag{23}$$

there exists at least a pair of quantizable tori, $T_P^2(\alpha, \pm; \hbar(n+1), E)$ with large n.

Since $\mathcal{T}^P$ cannot exist anymore, we have the following.

Theorem 9 *The degeneracies of energy levels associated with $\mathcal{T}^P$ is a quantum pitchfork bifurcation in the sense that the classical bifurcation set $\mathcal{B}^P$ bounds $\mathcal{D}^P$.*

5 Quantum Saddle-Node Bifurcation.

Now that we have found the quantum pitchfork bifurcation, we move on to find a quantum saddle-node bifurcation in the energy levels associated with $\mathcal{T}^S$. Let us denote the invariant tori in $\mathcal{T}^S$ by

$$T_S^2(\alpha,+;\nu,E) = M_\alpha(\nu,E) \cap \{(q,p)\in \mathbf{R}^2\times\mathbf{R}^2 | L_2(q,p) - \sqrt{2\nu}/(\lambda_2-\alpha_3) > 0\}$$
$$T_S^2(\alpha,-;\nu,E) = M_\alpha(\nu,E) \cap \{(q,p)\in \mathbf{R}^2\times\mathbf{R}^2 | L_2(q,p) - \sqrt{2\nu}/(\lambda_2-\alpha_3) < 0\} \quad (24)$$

Accordingly, we denote by $c_S^j(\alpha,\pm;\nu,E)$ $(j=1,2)$ the non-contractible loops defined by (13) and (15) for $T_S^2(\alpha,\pm;\nu,E)$. While we have (19) and the first equation of (20) again for these loops, the second equation of (20) does not hold true anymore because the pair of tori, $T_S^2(\alpha,\pm;\nu,E)$, does not admit the symmetry admissible for $T_P^2(\alpha,\pm;\nu,E)$. Hence, we have to think of another way to estimate estimate (12) for $c_S^2(\alpha,\pm;\nu,E)$: The way is to put the sufficient condition for the degeneracies given by Lemma 5 into the condition,

$$\oint_{c_S^2(\alpha,+;\nu,E)} pdq + \oint_{c_S^2(\alpha,-;\nu,E)} pdq \in 2\pi\mathbf{Z}, \tag{25}$$

with

$$\oint_{c_S^2(\alpha,+;\nu,E)} pdq + \pi \in 2\pi\mathbf{Z}. \tag{26}$$

The l. h. s. of (25) can be put in a complex integration of the form,

$$\begin{aligned}\text{l.h.s. of (25)} = \nu \int_{\Gamma(\alpha;E)} &\frac{\alpha_4}{[S(z)+2\alpha_2[R(z)]^{1/2}]^{1/2}} \times \left[\left(1-\frac{\alpha_2}{\alpha_4}+\frac{[R(z)]^{1/2}}{\alpha_4}\right)\right.\\ &\left.+\frac{z}{[R(z)]^{1/2}}\left(1-\frac{\alpha_4}{\alpha_4+\alpha_2-[R(z)]^{1/2}}\right)(\alpha_3 z+\alpha_1)\right]dz \end{aligned}\tag{27}$$

with

$$\begin{aligned} S(z) &= \alpha_4(\alpha_3-\alpha_4)z^2 + 2\alpha_4\alpha_2 z + (\alpha_4^2 - 2E\alpha_4 - 2\alpha_2^2) \\ R(z) &= -\alpha_3\alpha_4 z^2 - 2\alpha_4\alpha_1 z + 2E\alpha_4 + \alpha_2^2, \end{aligned}\tag{28}$$

The integration is taken along a non-contractible loop $\Gamma(\alpha;E)$ on the Riemann surface diffeomorphic to T^2 of $[S(z)+2\alpha_2[R(z)]^{1/2}]^{1/2}$. Under (19), a calculation of (27) with the Residue theorem [JS] results in

$$\alpha_1/\sqrt{\alpha_3(\alpha_3-\alpha_4)} = m/(n+1) \qquad (n=1,2,,\cdots,\ m=1,2,\cdots,n), \tag{29}$$

a necessary condition for the degeneracies associated with $\mathcal{T}_S$.

We are now in a position to estimate (26). A key equation is

$$\frac{1}{2\pi\hbar}\oint_{c_S^2(\alpha,-;\nu,E)} p^T dq = -\frac{\nu}{\pi}A^S(\alpha;E) \tag{30}$$

similar to (21), where $A^S(\alpha; E)$ is the area enclosed to the left by $\tilde{c}_S^2(\alpha, -; E) \subset S^2$. Note that $A^S(\alpha\ E)$ is well-defined also in the case of $E = E_3$ as the area enclosed to the left by simple component of the separatorix given by $H^{\mathrm{red}}_{\alpha,\nu} = \nu + \nu^2 E_3$. The properties of $A^S(\alpha; E)$ to be utilized are presented in Appendix.

In view of (19) and (29), our estimation will be made under the assumption,

$$\phi^S(\alpha) > 0 \quad \text{with} \quad (19) \quad \text{and} \quad (29). \tag{31}$$

Under (31), the decreasing and the continuity properties of $A^S(\alpha, E)$ (see Lemma 13 in Appendix) are put together with (30) to show that there exists at least one invariant torus if and only if

$$A^3(\alpha; E_3) > 2\pi/(n+1) \tag{32}$$

holds true. Under (31), $A(\alpha, E_3)$ will be denoted by $A_n^m(\alpha_2; \alpha_3, \alpha_4)$ henceforth, on account of $\alpha_1 = m\sqrt{\alpha_3 - \alpha_4}/(n+1)$. Note that $A_n^m(\alpha_2; \alpha_3, \alpha_4)$ satisfies (39) in Appendix, which will play a key role below.

We wish to describe α for which (32) holds true in more detail. Let α_3 and α_4 be fixed arbitrarily to be $\alpha_3 > \alpha_4$ (cf. (3)). Then under (31), α_2 can range over $(0, a_n^m(\alpha_3, \alpha_4))$, where $a_n^m(\alpha_3, \alpha_4)$ is defined by

$$a_n^m(\alpha_3, \alpha_4) = (\alpha_3 - \alpha_4)^{2/3} - \{m^2\alpha_3(\alpha_3 - \alpha_4)\}^{1/3}/\{(n+1)\}^{2/3} \tag{33}$$

which obviously satisfies (39) in Appendix. In view of (32), let us look into the solution of the equation

$$A^{\kappa m}_{\kappa(n+1)-1}(\alpha_2; \alpha_3, \alpha_4) = \frac{2\pi}{\kappa(n+1)} = \frac{2\pi}{\{\kappa(n+1)-1\}+1}, \tag{34}$$

for α_2 in $(0, a^{\kappa m}_{\kappa(n+1)-1}(\alpha_3, \alpha_4))$ under (31). If κ is chosen to be sufficiently large, the decreasing property of $A^{\kappa m}_{\kappa(n+1)-1}(\alpha_2; \alpha_3, \alpha_4)$ for α_2 (see Lemma 13) implies the existence of the unique solution of (34) which will be denoted by $\alpha^{\kappa}_{m,n}(\alpha_3, \alpha_4)$. Hence recalling the decreasing property of $A_n^m(\alpha_2; \alpha_3, \alpha_4)$ again, we have a sufficient condition for the degeneracies:

Lemma 10: *Under (31), let the integer κ be large enough for (34) to admit the unique solution $\alpha^{\kappa}_{m,n}(\alpha_3, \alpha_4)$.*

Since $\mathcal{D}$ can be shown to be dense in the subset of $\mathcal{P}$ determined by $\phi^S(\alpha) > 0$ (cf. [U3]), we have the following:

Theorem 12: *The degeneracies of energy levels associated with $\mathcal{T}^S$ is a quantum saddle-node bifurcation in the sense that the classical saddle-node bifurcation set $\mathcal{B}^S$ bounds the closure of $\mathcal{D}^S$.*

6 Concluding Remark.

We have shown that, in the four-parameter perturbed oscillator, the degeneracies of energy levels associated with the families of invariant tori $\mathcal{T}_P$ (resp. $\mathcal{T}_S$) are quantum pitchfork (resp. saddle-node) bifurcations.From the results above and those in [U1-3], the structures of the set of the values of parameters admitting the degeneracies associated with pitchfork/saddle-node bifurcations are conjectured as follows: For pitchfork bifurcation, the parameter-set might be identical with either of the sets partitioned by the classical bifurcation set. In contrast with this, for saddle-node bifurcation, the parameter-set might be dense in either of the sets partitioned by the classical bifurcation set. It will be checked in future studies on the remaining cases; the five/six-parameter cases.

Appendix

We present, very briefly, several properties of the areas, $A^P(\alpha, E)$ (resp. $A^S(\alpha, E)$), enclosed to the left by the loops, $\tilde{c}_P^2(\alpha, -; E)$ (resp. $\tilde{c}_S^2(\alpha, -; E)$, and thereby those of $A_n^m(\alpha_2; \alpha_3, \alpha_4) = A_S(\alpha, -; E_3)$. It is convenient to introduce the isometry χ of $S^2\backslash\{x_2 = 1\}$ to D, the open disk with the unit radius, which is defined by

$$(\xi, \eta) = \chi(x) = \left(x_1/\sqrt{2(1-x_2)},\, x_3/\sqrt{2(1-x_2)}\right). \tag{35}$$

Through χ, $A^P(\alpha, E)$ (resp. $A^S(\alpha, E)$) are mapped isometrically to the area on D enclosed to the left by the loops expressed as

$$\alpha_3\xi^8 + 2\alpha_3 C_1\xi^6 + (C_1^2 + 2\alpha_3 C_2)\xi^4 + (2C_1C_2 + \alpha_2^2)\xi^2 + \{C_2^2 + \alpha_2^2\eta^2(\eta^2 - 1)\} = 0 \tag{36}$$

with $E \in (E_0, E_1]$ and $\xi < 0$ (resp. with $E \in [E_2, E_3)$ and $\eta < \alpha_1/(\lambda_2 - \alpha_3)$), where C_1 and C_2 are given by

$$C_1 = (2\alpha_3 - \alpha_4)\eta\ 2 + (\alpha_1 - \alpha_3), \tag{37}$$

$$C_2 = (\alpha_3 - \alpha_4)\eta^4 - (\alpha_3 - \alpha_4 - \alpha_1)\eta^2 + \left(\frac{\alpha_3}{4} - \frac{\alpha_1}{2} - \frac{E}{2}\right). \tag{38}$$

Estimating (36), we have the following:

Lemma 13 *$A^P(\alpha, E)$ is continuous in both E and α, and increasing with E. $A^S(\alpha, E)$ is continuous in both E and α, and decreasing with E. Further, $A^P(\alpha, E)$ (resp. $A^S(\alpha, E)$) tend to zero as $E \downarrow E_0$ (resp. $E \uparrow E_3$).*

For $A_n^m(\alpha_2; \alpha_3, \alpha_4)$ identical with $A^S(\alpha, E_2)$ under (31) and $a_n^m(\alpha_3, \alpha_4)$, we have

$$A_n^m(\alpha_2; \alpha_3, \alpha_4) = A_{\kappa(n+1)-1}^{\kappa m}(\alpha_2; \alpha_3, \alpha_4) \quad a_n^m(\alpha_3, \alpha_4) = a_{\kappa(n+1)-1}^{\kappa m}(\alpha_3, \alpha_4), \tag{39}$$

($\kappa = 1, 2, \cdots$), on account of $m/(n+1) = \kappa m/\{(\kappa(n+1) - 1) + 1\}$.

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TRIANGULAR REPRESENTATION OF HAMILTONIANS AND GENERALIZED GAUSS HYPERGEOMETRIC SERIES

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Abstract

Within a (generalized) power-series method the $D+1$–term recurrences are interpreted as a linear algebraic Schrödinger equation with a triangularized Hamiltonian matrix. In partitioned notation, this equation may be read and solved as two-term recurrences for $D-$*dimensional* arrays of coefficients. For illustration, bound states in all the shape-invariant-like potentials are then shown obtainable in two steps. The first one is analytic: in terms of certain generalized $\Gamma-$functions, both (by definition, asymptotically correct) Jost solutions are constructed as $D-$*dimensional* matrix generalizations of the Gauss hypergeometric function $_2F_1$. In the second step we recommend a matching of the right and left Jost solution in the origin. This is shown significantly facilitated by a $\mathcal{PT}-$like double-parity symmetry of our particular illustrative models.

1 The generalized power series method.

Bound states are often constructed, in an orthonormalized basis $\{|n\rangle\}$, via a variational Ansatz

$$|\Psi\rangle \approx |\Psi_N\rangle = \sum_{n=0}^{N} |n\rangle\, d_n, \qquad N \to \infty. \tag{1}$$

This gives Hamiltonians H represented by hermitean matrices. Their diagonalization becomes a routine numerical procedure [1]. Still, the requirement of orthonormality $\langle m|n\rangle = \delta_{mn}$ is nontrivial and may even become counter-productive. For example, the well known exact eigenstates

$$\Psi^{\{PT\}}(x) = \frac{d_0 \tanh^p x}{\cosh^\kappa x}\; {}_2F_1\left(\frac{1}{2}(\kappa+1+p+A), \frac{1}{2}(2\kappa+p-A); 1+\kappa; \frac{1}{\cosh^2 x}\right)$$

of the one-dimensional Pöschl-Teller model $H^{\{PT\}} = p^2 - A(A+1)/\cosh^2 x$ (with energies $E = -\kappa^2$ and parities $(-1)^p$, $p = 0,\ 1$) may most easily be produced in a *non-orthogonal* basis $\{|n^{\{PT\}}\rangle\}$ [2].

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The use of an "optimal", Hamiltonian-adapted basis is characteristic, e.g., for the numerical method of Lanczos iterations [3]. It might be recalled here as a variational parallel and main methodical inspiration of our forthcoming construction. Unfortunately, the analogies between variational and non-variational methods are not too strong; the related mathematics must be analysed with due care. In our illustrative example, the insertion of the Ansatz (1) (with "the best" basis $\{|n^{\{PT\}}\rangle\}$) into the differential Schrödinger equation $(H^{\{PT\}} - E)|\Psi\rangle = 0$ only leads to a non-hermitean linear algebraic problem

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & \dots \\ b_0 & a_1 & 0 & 0 & \dots \\ 0 & b_1 & a_2 & 0 & \dots \\ & & \ddots & \ddots & \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \end{pmatrix} = 0. \tag{2}$$

Still, its "diagonalization" (i.e., explicit specification of all the coefficients d_n in terms of a few Γ-functions) is immediate: We have a norm $d_0 \neq 0$ and explicit $d_1 = -d_0 b_0 / a_1$ etc, with $a_1 \neq 0$ etc.

The "next-to-solvable" cases (to be considered here) induce, as a rule, more lower diagonals in eq. (2) [4]. One has to replace eq. (2) by another non-hermitean equation. We may only preserve its two-diagonal (or rather two-term recurrent) form via its D-dimensional partitioning,

$$\begin{pmatrix} A_0 & 0 & 0 & 0 & \dots \\ B_0 & A_1 & 0 & 0 & \dots \\ 0 & B_1 & A_2 & 0 & \dots \\ & & \ddots & \ddots & \end{pmatrix} \begin{pmatrix} D_0 \\ D_1 \\ D_2 \\ \vdots \end{pmatrix} = 0 \tag{3}$$

where we collect, say,

$$D_n := \begin{pmatrix} d_{m+1} \\ \dots \\ d_{m+D} \end{pmatrix}, \qquad m = nD - m_0, \qquad n = 0, 1, \dots$$

with $m_0 = 1$ or, better, $m_0 = D$. Thus, the solution of the latter non-hermitean, triangular-matrix problem is again recurrent and easy,

$$D_j = -A_j^{-1} B_j D_{j-1}, \qquad j = 1, 2, \dots . \tag{4}$$

This is the core of our present proposal. We intend to demonstrate that the degree and range of parallelism between equations (2) and (3) (and, first of all, between their respective "exact" and "next-to-exact" solutions and their interpretations) is quite surprising. This might prove unexpectedly helpful in multiple physical applications.

2 Asymmetric triangularized Hamiltonians.

Let us remind the reader that in one dimension the closed (a. k. a. shape invariant) Gauss hypergeometric bound-state solutions only exist for the Rosen-Morse well $V^{\{RM\}}(x) = (- - -A(A+1) + B \sinh 2x)/\cosh^2 x$ and for its "scarf" modification $V^{\{sc\}}(x) = [(2A+1)B \sinh x + B^2 - A(A+1)]/\cosh^2 x$ (cf. Table 4.1 in review [5] for more details). Both these forces degenerate to their above-mentioned PT predecessor in the limit $B \to 0$. Here we intend to show that their $D = 1$ hypergeometric solutions admit an immediate $D > 1$ generalization applicable, e.g., to the further elementary forces

$$V_s^{\{M\}}(f,x) = \frac{f}{\cosh^M x}, \qquad V_a^{\{N\}}(g,x) = \frac{g \cdot \sinh x}{\cosh^N x} \tag{5}$$

and their superpositions. For this purpose we shall introduce a parameter $a \geq 0$ and indices $p = 0,1$, $q = 0,1$ and $\kappa + 2n + p = \sigma(n,p) = \sigma$ and contemplate the following generalization of $\langle x|n^{\{PT\}}\rangle\}$ with a multiple index $\mu = \mu(n,p,q) = 4n + 2p + q \geq 1$,

$$\langle x|\Xi_\mu\rangle \equiv \xi_{n,p,q,a,\kappa}(x) = \frac{\sinh^{1-q} x}{\cosh^{\kappa+2n+p} x}\, e^{a \arctan(\sinh x)}.$$

After differentiation with respect to the coordinate x we may determine the action of the kinetic Hamiltonian $T = - - -\partial_x^2$,

$$T|\Xi_{\mu(n,p,0)}\rangle = -(\sigma-1)^2 |\Xi_{\mu(n,p,0)}\rangle + (2\sigma - 1)\, a\, |\Xi_{\mu(n,p,1)}\rangle +$$
$$+(\sigma^2 + \sigma - a^2) |\Xi_{\mu(n+1,p,0)}\rangle - (2\sigma+1)\, a\, |\Xi_{\mu(n+1,p,1)}\rangle,$$
$$T|\Xi_{\mu(n,p,1)}\rangle = -\sigma^2 |\Xi_{\mu(n,p,1)}\rangle + (2\sigma+1)\, a\, |\Xi_{\mu(n+1,p,0)}\rangle + (\sigma^2 + \sigma - a^2) |\Xi_{\mu(n+1,p,1)}\rangle.$$

The states with $p = 0, 1$ stay decoupled. At $a = 0$ (to be fixed from now on) the action of T also conserves the spacial parity q. We may choose the two different and asymptotically WKB-compatible initial kets $|0\rangle$ with $p = 1 - q = 0$ or 1, i.e., $\langle x|\Xi_{\mu(0,0,1)}\rangle \equiv \cosh^{-\kappa} x$ or $\langle x|\Xi_{\mu(0,1,0)}\rangle \equiv \sinh x \cdot \cosh^{-\kappa-1} x$. The repeated action of $H = T + V$ on $|0\rangle$ generates then the two Krylov-like (or Lanczos-like) bases. In this sense, our forces (5) split to the four different categories as listed in Table 1.

Table 1. Reduced bases $\{|n\rangle\} \equiv \{|\Xi_{k(n)}\rangle\}$ for potentials (5).

n	0	1	2	3	4	...	0	1	2	3	4	...
potential	k_n (first option)						k_n (second option)					
$V_s^{(2K)}(x)$	1	5	9	13	17	...	2	6	10	14	18	...
$V_s^{(2K+1)}(x)$	1	3	5	7	9	...	2	4	6	8	10	...
$V_a^{(2L)}(x)$	1	4	5	8	9	...	2	3	6	7	10	...
$V_a^{(2L+1)}(x)$	1	5	6	9	10	...	2	5	6	9	10	...

Recalling the first line of Table 1 for a more thorough discussion, we notice that the $p-$ and $q-$preserving Hamiltonians $H = T + V_s^{(2K)}(x)$ act irreducibly on the two alternative $p = 1-q = 0$ or 1 bases. At $K = 1$ this reproduces our above observation and replacement of the Pöschl-Teller differential equation $(H^{\{PT\}} - E)|\Psi\rangle = 0$ by the strictly two-diagonal algebraic equation $Q(p)\vec{d} = 0$ (3) with $D = 1$. At $K = 2$, the coupling f would move one step down and require $D = 2$. A neat illustration at $D = K = 3$ with $\alpha_j = -j\,(2\kappa + j)$ and $\beta_j = (\kappa + j)(\kappa + j + 1)$ reads

$$Q(0) = \left(\begin{array}{c|ccc|c} 0 & 0 & 0 & 0 & \dots \\ \hline \beta_0, & \alpha_2 & 0 & 0 & \dots \\ 0 & \beta_2, & \alpha_4 & 0 & \dots \\ f & 0 & \beta_4, & \alpha_6 & \\ \hline & \ddots & & \ddots & \ddots \end{array} \right), \qquad Q(1) = \left(\begin{array}{c|ccc|c} 0 & 0 & 0 & 0 & \dots \\ \hline \beta_1, & \alpha_2 & 0 & 0 & \dots \\ 0 & \beta_3, & \alpha_4 & 0 & \dots \\ f & 0 & \beta_5, & \alpha_6 & \\ \hline & \ddots & & \ddots & \ddots \end{array} \right)$$

etc. The "forgotten" second symmetric model $V_s^{(2K+1)}(x)$ still complies with the $q-$preserving twins $|\Xi_{2n+1}\rangle$ and $|\Xi_{2n+2}\rangle$, $n = 0, 1, \dots$. The mere three neighbouring diagonals appear in their lower triangular representants of Hamiltonains $Q = Q(q)$ at $K = 0$ so that a two-dimensional partitioning suffices in recipe (3). A three-dimensional partitioning is needed for $K = 1$ while $D = 2K + 1$ further on.

The second half of Table 1 describes the $q-$mixing bases and antisymmetric potentials $V_a^{(N)}(x)$. The action of antisymmetic models $V_a^{(2L)}(x) = g \cdot \sinh x \cdot \cosh^{-2L} x$ with $L \geq 1$ preserves the value of p in the basis,

$$V_a^{(2L)}(x)|\Xi_{\mu(n,p,q)}\rangle = (1-q)\,g \cdot |\Xi_{\mu(n+L-1,p,1-q)}\rangle + (-1)^{1-q} g \cdot |\Xi_{\mu(n+L,p,1-q)}\rangle.$$

With the same abbreviations as above, we get the explicit "antisymmetric scarf" (AS) $L = 1$ Hamiltonian at $p = 0$,

$$Q^{(AS)}(0) = \left(\begin{array}{c|cc|cc|cc|c} 0 & & & & & & & \\ \hline g & \alpha_1 & & & & & & \\ \beta_0 & g & \alpha_2 & & & & & \\ \hline 0 & \beta_2 & g & \alpha_3 & & & & \\ & -g & \beta_2 & g & \alpha_4 & & & \\ \hline & & 0 & \beta_4 & g & \alpha_5 & & \\ & & & -g & \beta_4 & g & \alpha_6 & \\ \hline & & & & \ddots & \ddots & \ddots & \ddots \end{array} \right) \qquad (6)$$

etc. The last row in Table 1 seems to be most puzzling. Firstly, the simplest $V_a^{(1)}(x) = g \cdot \tanh x$ has to be eliminated as not confining at all. Secondly, the reducibility of action of $H = p^2 + V_a^{(2L+1)}(x)$ is controlled by numbers $I \equiv 2p + q$ (modulo 4) which look mysterious. Elements with $I = 1$ and $I = 2$ never mix with their unphysical, $I = 0$ and $I = 3$ counterparts. Thirdly, both our initial choices of $|0\rangle = |\Xi_1\rangle$ and $|0\rangle = |\Xi_2\rangle$ generate the same output. The difference will only lie in the elements of Q.

3 Application: Bound states.

Expansions (1) will define Jost (i.e., asymptotically correct) solutions of Schrödinger equations, provided only that we prove their convergence at all coordinates. This type of convergence (different form the possible loss of normalizability) is virtually controlled by the mere asymptotically increasing kinetic matrix elements in Q. The changes in the potential may only play a marginal role.

Let us pick up the AS model with $L = 1$ and $D = 2$ for detailed study. In its Hamiltonian (6) the contribution of the coupling g is clearly separated from the growing and energy-dependent kinetic terms. This is a core of the proof. Technically, it recalls eq. (1),

$$|\Psi^{(AS)}(p)\rangle = \sum_{j=0}^{\infty} |\Xi_{\mu(j+1-p,p,0)}\rangle \cdot d^{(0)}_{2j+1-p}(p) + \sum_{j=0}^{\infty} |\Xi_{\mu(j,p,1)}\rangle \cdot a^{(1)}_{2j+p}(p). \qquad (7)$$

We visualise our linear algebraic Schrödinger equation (3) as two-dimensional recurrences (4) for doublets of coefficients $[D_{n+1-p}]_j = [D_{n+1-p}(p)]_j = d^{(0)}_{2n+j-p}(p)$ with

$$[-A_j(p)]^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\alpha_{2j+p} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix} \begin{pmatrix} 1/\alpha_{2j+p-1} & 0 \\ 0 & 1 \end{pmatrix}$$

and $j = 1,\ 2,\ p = 0,\ 1$ and $n = 0, 1, \ldots$. As long as we have $0 > \alpha_1 > \alpha_2 > \ldots$ at any $\kappa > 0$, our vectors of solutions are well defined and unique. Asymptotically, their upper and lower components decouple in a $p-$independent way,

$$[D_j(p)]_q \ \left[\equiv d^{(q)}_{2j+p+q-1}(p)\right] = \left[1 + \frac{1-4q}{2j} + \mathcal{O}\left(\frac{1}{j^2}\right)\right] [D_{j-1}(p)]_q, \qquad q = 0 \text{ or } 1. \qquad (8)$$

Off the origin, the discussion of convergence of both our infinite series (7) is short. As long as $\cosh x > 1$ for all the nonzero and real coordinates $x \neq 0$, the estimate (8) implies that our series (7) are absolutely convergent. For all the (complex) couplings g and energies $-\kappa^2$ this follows from an ordinary geometric criterion. The same conclusion may immediately be extended to complex coordinates $x + iy$ which lie out of the fairly narrow strip $|x| \leq \ln(1+\sqrt{2})$ or, better, to the exterior of a wiggly bounded domain $|\cosh(x+iy)| = \sqrt{\sinh^2 x + \cos^2 y} \leq 1$. A refined Raabe criterion is only needed (and implies an indeterminate behaviour of the type $0 \times \infty$) in the limit $x \to 0$. Strictly speaking, this entitles us to work on a punctured domain of $x \in (-\infty, 0) \bigcup (0, \infty)$. Generically (i.e., at unphysical energies) we may only preserve the analyticity in the origin at the cost of violation of the "second" asymptotic boundary condition (by each analytic continuations of our "one-sided" Jost solutions). Vice versa, we have to match the left and right Jost solution at $x = 0$, presumably, numerically.

At both $p = 0$ or $p = 1$, our Jost solutions (7) with coefficients $d_j = d^{(q)}_j(p)$ consist of the two separate summations, each with a well defined spacial parity $(-1)^q$,

$|\Psi^{(AS)}(p)\rangle = |\Psi^{(even)}(p)\rangle + |\Psi^{(odd)}(p)\rangle$. The first few coefficients $d_n^{(q)}(p) = d_n^{(q)}(p,g)$ are easily derived

$$d_0^{(1)}(0) = \ , \quad d_1^{(0)}(0) = -g/\alpha_1, \quad d_2^{(1)}(0) = -\beta_0/\alpha_2 + g^2/(\alpha_1\alpha_2), \quad \ldots,$$

$$d_0^{(0)}(1) = 1, \quad d_1^{(1)}(1) = -g/\alpha_1, \quad d_2^{(0)}(1) = -\beta_1/\alpha_2 + g^2/(\alpha_1\alpha_2), \quad \ldots .$$

and seen to exhibit an unexpected symmetry,

$$d_j^{(q)}(p,-g) = (-1)^{p+q+1}\, d_j^{(q)}(p,g).$$

A double parity operator may be introduced such that our generalized hypergeometric functions $\varphi^{(p)}(g,x,\kappa) = \langle x|\Psi^{(AS)}(p)\rangle$ (7) become its eigenstates,

$$\hat{P}\,\varphi^{(p)}(g,x,\kappa) = \varphi^{(p)}(-g,-x,\kappa) = (-1)^p\,\varphi^{(p)}(g,x,\kappa). \tag{9}$$

In a reversal of this argument, any bound state $\Psi^{(AS)}(x)$ may be assigned, say, a positive $\hat{P}-$parity. Summarizing, we may recommend its following $g-$independent normalization

$$\Psi^{(AS)}(x) = \mathcal{M}\,\varphi^{(0)}(g,x,\kappa) + g\cdot\mathcal{N}\,\varphi^{(1)}(g,x,\kappa), \tag{10}$$

Marginally, let us note that the $\hat{P}-$symmetry of our Hamiltonians is, at purely imaginary couplings, an important quantum-mechanical analogue of $\mathcal{PT}$ invariance in field theory [6]. Unfortunately, the "simplest possible" Rosen Morse model $V^{\{RM\}}(x)$ with imaginary $B = ih/2$ and parametrization $h = -\sigma^2\sin 2\tau \in (-\infty,\infty)$ fails to behave in the standard way: Its closed Gauss hypergeometric solutions

$$\Psi(x) = (\sqrt{y})^{\sigma\cos\tau - i\sigma\sin\tau}\left(\sqrt{1-y}\right)^{\sigma\cos\tau + i\sigma\sin\tau}\ {}_2F_1(a,b;c;y) \tag{11}$$

with $a = \sigma\cos\tau + A + 1$, $b = \sigma\cos\tau - A$, $c = 1 + \sigma\exp(-i\tau)$ and $y = (1+\tanh x)/2$ remain normalizable at all the real energies $E = -\sigma^2\cos 2\tau \in (-\infty,\infty)$.

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