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Proceedings of XVI Max Born Symposium
Supersymmetries and Quantum Symmetries

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Международное рабочее совещание «Суперсимметрии и квантовые симметрии 2001» (SQS'01) проходило с 21 по 25 сентября в Карпаче (Польша) в рамках XVI Симпозиума имени Макса Борна. Проведение рабочих совещаний по суперсимметрии было инициировано Виктором Исааковичем Огиевецким в 1993 г. в ОИЯИ. С 1997 г. эти совещания проходят каждые два года в память о выдающемся ученом В. И. Огиевецком. Симпозиумы имени Макса Борна организуются Институтом теоретической физики Вроцлавского университета с 1991 г. как продолжение семинаров по теоретической физике «Лейпциг–Вроцлав».

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Foreword

The International Workshop *"Supersymmetries and Quantum Symmetries 2001"* (SQS'01) was held 21-25 September of 2001 in Karpacz, Poland, in the framework of the Max Born Symposia organized by Institute for Theoretical Physics of the University of Wroclaw. The SQS workshops were founded by Victor I. Ogievetsky and, since 1993, were organized by JINR in Dubna. Since 1997 they are biennial and are intended to commemorate the contribution to theoretical physics of V.I. Ogievetsky who passed away in 1996. Max Born Symposia started in 1991 as a continuation of Seminars in Theoretical Physics "Leipzig-Wroclaw". The present meeting is sixteenth in this series.

Like the previous SQS workshops, SQS'01 was devoted to recent developments in supersymmetric theories including string theory and to mathematical aspects and applications of quantum groups. Supersymmetries, introduced in seventies as a tool to describe dynamics of fundamental interactions, imply the existence of anticommuting group parameters; quantum groups further generalize classical symmetries, replacing the notion of Lie groups and Lie algebras by noncommutative Hopf algebras. Since their start, the SQS workshops were aimed at providing a joint discussion of these two basic tools of the contemporary theoretical physics. A considerable attention at this meeting was paid to noncommutative geometries which underly quantum groups and have deep implications in string theory.

The workshop offered an excellent opportunity for research contacts between the world-known theorists and, in particular, supplied a good example of international scientific collaboration between a research group in Poland and Bogoliubov Laboratory of Theoretical Physics in Dubna. It was organized and financially supported by both Institutions: Institute for Theoretical Physics, University of Wroclaw, and Bogoliubov Laboratory of Theoretical Physics, JINR (Dubna). We would like also to acknowledge a support from the Bogoliubov - Infeld Foundation and Polish Ministry of Education and Sport. Based on the success of this meeting, we are convinced that the tradition of such joint research ventures should be continued in the future.

Evgeny Ivanov
Jerzy Lukierski
The Directors of International Workshop "SQS'01" organized as XVI Max Born Symposium

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Part I

Branes, AdS/CFT and Higher Spins

Massless Higher Spin Fields in the AdS Background and BRST Constructions for Nonlinear Algebras

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Abstract

The detailed description of the method of the construction of the nilpotent BRST charges for nonlinear algebras of constraints appearing in the description of the massless higher spin fields on the AdS_D background is presented. It is shown that the corresponding BRST charge is not uniquely defined, but this ambiguity has no impact on the physical content of the theory.

BRST – BFV method [1] is a powerful tool for the quantization of the wide class of the physical systems, invariant with respect to the gauge symmetry groups. The physical content of the theory is singled out by the BRST quantization condition

$$Q|Phys\rangle = 0 \tag{1}$$

for the nilpotent $Q^2 = 0$ BRST charge. Obviously due to the nilpotence property of the BRST charge the BRST quantization condition is gauge invariant i.e., the physical states are defined up to the transformation $|Phys\rangle' = |Phys\rangle + Q|\Lambda\rangle$ where the last term is a "spurious" state with the zero norm. Therefore to find the physical spectrum of the theory, one has to solve a cohomology problem of the corresponding BRST charge.

On the other hand the BRST method can be applied for the construction of the field theory, which corresponds to some first quantized system. Namely if one succeeds to construct the nilpotent BRST charge for the given system of constraints, then one can straightforwardly write the corresponding field theoretical Lagrangian of the form $L \sim \langle\Phi|Q|\Phi\rangle$ which gives the condition (1) as the equation of motion. At the same time the Lagrangian is gauge invariant under the transformations

$$\delta|\Phi\rangle = Q|\Lambda\rangle, \tag{2}$$

The later property is crucial for eliminating nonphysical degrees of freedom from the spectrum. The famous example of such kind of construction is BRST string field theory.

The method of construction of the nilpotent BRST charge is well known when the corresponding constraints form the closed linear Lie algebra. The situation becomes more

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complicated when one deals with the second – class constraints and (or) the algebra of constraints is nonlinear. In the present paper we analyze this situation and show, that at least for the system of constraints describing irreducible massless higher spin fields the corresponding BRST charge is not unique. Rather it can contain a number of free parameters which, nevertheless does not affect the physical content of the corresponding field theory. These parameters, leading to the noncanonical form of the BRST charge, can be present even in the case of closed linear algebra of constraints. The particular choice of these parameters ensures the possibility to simplify the expression for the BRST charge, and correspondingly the procedure of construction of the field theoretical Lagrangian becomes simpler as well.

Below we concentrate on the construction of the nilpotent BRST charges and corresponding field theoretical Lagrangian for the theory describing the higher spin fields in the flat space and in AdS spaces. In particular, we investigate an arbitrariness in the BRST charge, which is described by some free parameters, and show that the physical content of the corresponding field theory is independent of the values of these parameters.

First we briefly describe the system of the free higher massless spin fields propagating through the D – dimensional flat Minkowski space time in the framework of the BRST approach [2] and then generalize the method for the interaction with AdS_D background [3].

For the free massless higher spin fields the corresponding system of constraints imposed on the physical states includes the mass – shell constraint $L_0 = p_\mu^2$, transversality condition $L_1 = p_\mu a^\mu$, and tracelessness condition $L_2 = \frac{1}{2} a_\mu a^\mu$, where

$$p_\mu = \partial_\mu, \quad \left[a_\alpha, a_\beta^+\right] = \eta_{\alpha\beta}, \qquad \eta_{\alpha\beta} = diag(-1, 1, 1, ..., 1), \tag{3}$$

These constraints are written down in terms of creation and annihilation operators a_μ^+, a^μ. The general vector in the Fock space generated by these operators has the form

$$|\Phi\rangle = \sum_n \Phi_{\alpha_1 \alpha_2 \cdots \alpha_n}(x) a^{\alpha_1 +} a^{\alpha_2 +} \cdots a^{\alpha_n +} |0\rangle. \tag{4}$$

Obviously, the high spin fields $\Phi_{\alpha_1 \alpha_2 \cdots \alpha_n}(x)$ are automatically symmetrical with respect to the permutation of their indices and therefore representations of the corresponding flat space little group $O(d-2)$ are characterized by the Young tableaux with one row. Note, that more complicated Young tableaux need for their description more then one set of creation and annihilation operators.

The operators L_0, L_1, L_2 along with their conjugates L_1^+ and L_2^+ obey the following commutation relations.

$$\begin{aligned} &[L_1^+, L_1] = -L_0, \quad [L_1^+, L_2] = -L_1, \quad [L_2^+, L_1] = -L_1^+, \\ &[G_0, L_1] = -L_1, \quad [L_1^+, G_0] = -L_1^+, \end{aligned} \tag{5}$$

and

$$[G_0, L_2] = -2L_2, \quad [L_2^+, G_0] = -2L_2^+, \quad [L_2^+, L_2] = -G_0. \tag{6}$$

where we have denoted $G_0 \equiv a^{+\mu} a_\mu + \frac{D}{2}$.

The construction of the nilpotent BRST charge for this system of constraints is not straightforward since as it can be seen from (6) the constraints $L_2^\pm$ are of the second class. It is a consequence of the fact that the operator G_0 is strictly positive and therefore can not be considered as an additional constraint.

Let us first introduce the anticommuting ghost variables $\eta_0, \eta_1^+, \eta_1, \eta_2^+, \eta_2, \eta_G$ which correspond to the operators $L_0, L_1, L_1^+, L_2, L_2^+$ and G_0 and have the ghost number equal to 1, as well as the corresponding momenta $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_1^+, \mathcal{P}_2, \mathcal{P}_2^+, \mathcal{P}_G$ with the ghost number -1 obeying the anticommutation relations

$$\{\eta_0, \mathcal{P}_0\} = \{\eta_2, \mathcal{P}_2^+\} = \{\eta_2^+, \mathcal{P}_2\} = \{\eta_2, \mathcal{P}_2^+\} = \{\eta_2^+, \mathcal{P}_2\} = \{\eta_G, \mathcal{P}_G\} = 1. \tag{7}$$

Then we build the auxiliary representations of the generators of the algebra (6) in terms of the additional creation and annihilation operators b^+ and b, $[b, b^+] = 1$ with the help of the following construction using the Verma module. We introduce the vector in the space of the Verma module $|n\rangle_V = (L_2^+)^n|0\rangle_V$, $n \in N$, $L_2|0\rangle_V = 0$ and the corresponding vector in the Fock space

$$|n\rangle = (b^+)^n|0\rangle, \quad b|0\rangle = 0.$$

Mapping the vector in the space of Verma module into the vector in the Fock space one obtains the representations of the generators of algebra $SO(2,1)$ in terms of the variables b^+ and b

$$L^+_{2.aux} = b^+, \quad G_{0.aux} = 2b^+b + h, \quad L_{2.aux} = b^+bb + bh \tag{8}$$

where h is the highest weight of the Verma module $G_{0.aux}|0\rangle = h|0\rangle$. Then defining the operators $\tilde{L}_2^\pm = L_2^\pm + L^\pm_{2.aux}, \tilde{G}_0 = G_0 + G_{0.aux}$, we construct the corresponding "bare" nilpotent BRST charge for the total system of constraints (5)–(6)

$$\begin{aligned}\tilde{Q} \;&= \eta_0 L_0 + \eta_1 L_1^+ + \eta_1^+ L_1 + \eta_2 \tilde{L}_2^+ + \eta_2^+ \tilde{L}_2 + \\ &+\eta_G(-\tilde{G}_0 + 3 - \eta_1^+\mathcal{P}_1 - \mathcal{P}_1^+\eta_1 - 2\mathcal{P}_2^+\eta_2 - 2\eta_2^+\mathcal{P}_2) \\ &+\eta_2^+\eta_2\mathcal{P}_G - \eta_1^+\eta_1\mathcal{P} + \eta_1^+\mathcal{P}_1^+\eta_2 - \eta_2^+\eta_1\mathcal{P}_1.\end{aligned} \tag{9}$$

In order to avoid the condition of the type $G_0|\Phi\rangle = 0$ one has to get rid of the variables η_G and $\mathcal{P}_G$ in the BRST charge (9), keeping its nilpotence property at the same time. This can be done by replacing the parameter h by the expression

$$\pi = -(G_0 + 2b^+b - 3 + \eta_1^+\mathcal{P}_1 + \mathcal{P}_1^+\eta_1 + 2\mathcal{P}_2^+\eta_2 + 2\eta_2^+\mathcal{P}_2) \tag{10}$$

and then simply omit the dependence on $\mathcal{P}_G$ in the BRST charge (see [4] – [5] for the general treatment).

The "reduced" BRST charge thus takes the form

$$\begin{aligned}Q \;&= \eta_0 L_0 + \eta_2(L_2^+ + b^+) + \eta_1 L_1^+ + \eta_1^+ L_1 + \eta_2^+(L_2 - G_0 b + b - b^+bb) \\ &-\eta_1^+\eta_1\mathcal{P} + \eta_1^+\mathcal{P}_1^+\eta_2 - \eta_2^+\eta_1\mathcal{P}_1 + \eta_1^+\eta_2^+\mathcal{P}_1 b - \eta_2^+\mathcal{P}_1^+\eta_1 b - 2\eta_2^+\mathcal{P}_2^+\eta_2 b\end{aligned} \tag{11}$$

Finally to restore the hermiticity property of the BRST charge, which is lost due to the nonhermitian form of auxiliary representations (8) one has to define the scalar product in the Fock space as $\langle\Phi_1|K|\Phi_2\rangle$ were K is a nondegenerate kernel operator

$$K = Z^+Z, \quad Z = \sum_n \frac{1}{n!}(L_2^+)^n|0\rangle_V\langle 0|(b)^n \tag{12}$$

satisfying the property $KQ = Q^+K$.

Let us define the ghost vacuum in the way that the operators $\eta_0, \eta_1^+, \eta_2^+, \mathcal{P}_1^+$ and $\mathcal{P}_2^+$ are creation operators. Then the BRST invariant Lagrangian for our system will be.

$$L = \int \eta_0 \langle\chi|KQ|\chi\rangle \tag{13}$$

The state vector $|\chi\rangle$ along with the oscillators a_μ^+ and b^+ depends on the ghost creation operators as well and therefore the Lagrangian (13) contains some auxiliary fields. However one can show that after making use of the BRST gauge invariance (2) and the equations of motion with respect to the auxiliary fields, one is left with the only physical field with no ghost and b^+ dependence. This field is double traceless $L_2L_2|\Phi\rangle = 0$ and the corresponding Lagrangian for this field has the form [6]

$$L = \langle\Phi|L_0 - L_1^+L_1 - 2L_2^+L_0L_2 + L_2^+L_1L_1 + L_1^+L_1^+L_2 - L_2^+L_1^+L_1L_2|\Phi\rangle \tag{14}$$

and the Lagrangian (14), is invariant under the gauge transformations

$$\delta|\Phi\rangle = L_1^+|\lambda_1\rangle \tag{15}$$

with the traceless parameter of gauge transformations $L_2|\lambda_1\rangle = 0$.

Let us note that, the "bare" nilpotent BRST charge (9) has been constructed in the standard fashion adopted for the system of the first – class constraints i.e., the terms nonlinear in ghost variables have the form given in [1]. However there exists an ambiguity in the expression of the BRST charge for this system as we shall see below.

Now we generalize this procedure for the description of the higher massless integer spin fields interacting with D – dimensional Anti - de - Sitter background. The main feature we face in this case is the nonlinear character of the corresponding algebra of constraints [7]. In the AdS space the momentum operator p_μ (here we denote Einstein indexes as $\mu, \nu, ...$ and the tangent space indexes as $\alpha, \beta, ...$) gets modified as

$$p_\mu = \partial_\mu + \omega_\mu{}^{\alpha\beta}a_\alpha^+a_\beta. \tag{16}$$

Using this equation one cane easily check that the operators $L_1^\pm = a^{\mu\pm}p_\mu$ are mutually hermitian conjugated with respect to the integration measure $d^Dx\sqrt{-g}$. After introducing the covariant d'Alambertian

$$L_0 = g^{\mu\nu}(p_\mu p_\nu - \Gamma_{\mu\nu}^\lambda p_\lambda) \tag{17}$$

one obtains along with (6) the following commutation relations

$$\begin{aligned}
&[L_1^+, L_1] = -\tilde{L}_0, \quad [L_1^+, L_2] = -L_1, \quad [L_2^+, L_1] = -L_1^+,\\
&[\tilde{L}_0, L_1] = -2rL_1 + 4rG_0L_1 - 8rL_1^+L_2,\\
&[L_1^+, \tilde{L}_0] = -2rL_1^+ + 4rL_1^+G_0 - 8rL_2^+L_1,\\
&[G_0, L_1] = -L_1, \quad [L_1^+, G_0] = -L_1^+,
\end{aligned} \tag{18}$$

where

$$\tilde{L}_0 \equiv L_0 + r(-D + \frac{D^2}{4}) + 4rL_2^+L_2 - rG_0G_0 + 2rG_0, \tag{19}$$

and the parameter r being related to the inverse radius of the AdS_D space λ via $r = \lambda^2$.

As it can be seen from the relations (18) and (6) the operators obey the nonlinear algebra being analogous to the finite $W_3^{(2)}$ algebra [8]. Again our goal is to construct the nilpotent BRST charge for this system, where the operator G_0 is excluded from the total set of constraints.

As a first step we construct nilpotent BRST charge for the operators $L_0, L_1^\pm, L_2^\pm$ and G_0 satisfying the nonlinear algebra (6), (18). However in contrast to the case of the flat space – time i.e., when the constraints form the linear algebra there is no standard prescription for the construction of nilpotent BRST charge for the system of operators forming a nonlinear algebra. For this end we add step by step all possible higher order terms in the ghosts with arbitrary coefficients to the BRST charge (9) and require its nilpotence. Simultaneously we modify the operator at the linear term in ghost variable η_0, instead of being simply L_0 it becomes the nonlinear combination $\tilde{L}_0$ of L_0, G_0 and $L_2^\pm$ (19).

This procedure leads to the family of "bare" nilpotent BRST charges, which turn out to depend on three free parameters k_1, k_2, k_3, namely i.e, the expression of terms which contain higher degrees in the ghost variables turn out not to be uniquely defined. The possible solution of the problem (not necessarily the most general one) has the form

$$\tilde{Q}^1 = \tilde{Q}^1_0 + k_1\tilde{Q}^1_{k_1} + k_2\tilde{Q}^1_{k_2} + k_3\tilde{Q}^1_{k_3}, \tag{20}$$

where

$$\begin{aligned}
\tilde{Q}^1_0 &= \eta_0(\tilde{L}_0 + 6r) + \eta_1 L_1^+ + \eta_2 L_2^+ + \eta_1^+ L_1 + \eta_2^+ L_2 - \eta_G(G_0 - 3) \\
&+2r\eta_0\eta_1^+\mathcal{P}_1 - 8r\eta_0\eta_2^+\mathcal{P}_2 + 2r\eta_0\mathcal{P}_1^+\eta_1 \\
&-8r\eta_0\mathcal{P}_2^+\eta_2 - \eta_1^+\eta_1\mathcal{P}_0 + \eta_1^+\eta_G\mathcal{P}_1 + \eta_1^+\mathcal{P}_1^+\eta_2 - 4r\eta_1^+\mathcal{P}_G\eta_1 \\
&-\eta_2^+\eta_1\mathcal{P}_1 + 2\eta_2^+\eta_G\mathcal{P}_2 - \eta_2^+\mathcal{P}_G\eta_2 + \mathcal{P}_1^+\eta_G\eta_1 + 2\mathcal{P}_2^+\eta_G\eta_2 - 8r\eta_0\eta_1\mathcal{P}_1 L_2^+ \\
&-4r\eta_0\eta_1^+\mathcal{P}_1 G_0 + 8r\eta_0\eta_1^+\mathcal{P}_1^+ L_2 - 4r\eta_0\mathcal{P}_1^+\eta_1 G_0 - 12r\eta_0\eta_1^+\mathcal{P}_1^+\eta_1\mathcal{P}_1 \qquad (21) \\
\tilde{Q}^1_{k_1} &= -2\eta_0(G_0 - 3) - 6\eta_0\eta_2^+\mathcal{P}_2 - 6\eta_0\mathcal{P}_2^+\eta_2 - 3\eta_1^+\mathcal{P}_G\eta_1 - 2\eta_0\eta_1\mathcal{P}_1 L_2^+ \\
&-\eta_0\eta_1^+\mathcal{P}_1 G_0 - 2\eta_0\eta_1^+\mathcal{P}_2 L_1^+ + 2\eta_0\eta_1^+\mathcal{P}_1^+ L_2 - \eta_0\eta_1^+\mathcal{P}_G L_1 \\
&-\eta_0\mathcal{P}_1^+\eta_1 G_0 - 2\eta_0\mathcal{P}_2^+\eta_1 L_1 - \eta_0\mathcal{P}_G\eta_1 L_1^+ - 6\eta_0\eta_1^+\mathcal{P}_1^+\eta_1\mathcal{P}_1, \qquad (22) \\
\tilde{Q}^1_{k_2} &= -\eta_0(G_0 - 3) - \eta_0\eta_1^+\mathcal{P}_1 - 2\eta_0\eta_2^+\mathcal{P}_2 - \eta_0\mathcal{P}_1^+\eta_1 - 2\eta_0\mathcal{P}_2^+\eta_2 - \eta_1^+\mathcal{P}_G\eta_1 \qquad (23) \\
\tilde{Q}^1_{k_3} &= \eta_0\mathcal{P}_2^+\eta_2 + \eta_0\eta_2^+\mathcal{P}_2 + \eta_0\eta_1\mathcal{P}_1 L_2^+ + \eta_0\eta_1^+\mathcal{P}_2 L_1^+ - \eta_0\eta_1^+\mathcal{P}_1^+ L_2 + \eta_0\mathcal{P}_2^+\eta_1 L_1 \\
&+\eta_0\eta_1^+\eta_2^+\mathcal{P}_1\mathcal{P}_2 + 2\eta_0\eta_1^+\mathcal{P}_1^+\eta_1\mathcal{P}_1 + \eta_0\eta_1^+\mathcal{P}_1^+\mathcal{P}_G\eta_2 - \eta_0\eta_1^+\mathcal{P}_2^+\eta_2\mathcal{P}_1 \\
&-\eta_0\eta_2^+\mathcal{P}_1^+\eta_1\mathcal{P}_2 - \eta_0\eta_2^+\mathcal{P}_G\eta_1\mathcal{P}_1 + \eta_0\mathcal{P}_1^+\mathcal{P}_2^+\eta_1\eta_2 \qquad (24)
\end{aligned}$$

All these operators are nilpotent and mutually anticommuting so that their sum is nilpotent as well. Let us note that the particular choice of the parameters $k_1 = 0, k_2 = 0, k_3 = 8$ leads to the BRST charge constructed in [3], which includes terms up to the fifth order in ghosts. The simplest expression including only third powers of ghosts corresponds to the choice $k_1 = -2, k_2 = 4, k_3 = 0$:

$$\begin{aligned}
\tilde{Q}^1 &= \eta_0(\tilde{L}_0 + 6r) + \eta_1 L_1^+ + \eta_2 L_2^+ + \eta_1^+ L_1 + \eta_2^+ L_2 \\
&-\eta_G(G_0 - 3) - 2r\eta_0\eta_1^+\mathcal{P}_1 - 4r\eta_0\eta_2^+\mathcal{P}_2 \\
&-2r\eta_0\mathcal{P}_1^+\eta_1 - 4r\eta_0\mathcal{P}_2^+\eta_2 - \eta_1^+\eta_1\mathcal{P}_0 + \eta_1^+\eta_G\mathcal{P}_1 + \eta_1^+\mathcal{P}_1^+\eta_2 - 2r\eta_1^+\mathcal{P}_G\eta_1 \\
&-\eta_2^+\eta_1\mathcal{P}_1 + 2\eta_2^+\eta_G\mathcal{P}_2 - \eta_2^+\mathcal{P}_G\eta_2 + \mathcal{P}_1^+\eta_G\eta_1 + 2\mathcal{P}_2^+\eta_G\eta_2 - 4r\eta_0\eta_1\mathcal{P}_1 L_2^+ \\
&-2r\eta_0\eta_1^+\mathcal{P}_1 G_0 + 4r\eta_0\eta_1^+\mathcal{P}_2 L_1^+ + 4r\eta_0\eta_1^+\mathcal{P}_1^+ L_2 + 2r\eta_0\eta_1^+\mathcal{P}_G L_1 \\
&-2r\eta_0\mathcal{P}_1^+\eta_1 G_0 + 4r\eta_0\mathcal{P}_2^+\eta_1 L_1 + 2r\eta_0\mathcal{P}_G\eta_1 L_1^+.
\end{aligned} \tag{25}$$

To get rid of the nonphysical constraint G_0 we again take the auxiliary representations for the operators $L_2^\pm$ and G_0 in the form (8) and construct the "bare" nilpotent BRST charge adding step by step all possible higher order terms in the ghosts. The result is similar to (20), namely

$$\tilde{Q} = \tilde{Q}_0 + k_1\tilde{Q}_{k_1} + k_2\tilde{Q}_{k_2} + k_3\tilde{Q}_{k_3}, \tag{26}$$

but now

$$\begin{aligned}
\tilde{Q}_0 &= \eta_0(\tilde{L}_0 + 6r - 4rG_{0.aux}) + \eta_1 L_1^+ + \eta_1^+ L_1 \\
&+\eta_2(L_{2.aux}^+ + L_2^+) + \eta_2^+(L_{2.aux} + L_2) - \eta_G(G_{0.aux} + G_0 - 3) \\
&+2r\eta_0\eta_1^+\mathcal{P}_1 - 8r\eta_0\eta_2^+\mathcal{P}_2 + 2r\eta_0\mathcal{P}_1^+\eta_1 \\
&-8r\eta_0\mathcal{P}_2^+\eta_2 - \eta_1^+\eta_1\mathcal{P}_0 + \eta_1^+\eta_G\mathcal{P}_1 + \eta_1^+\mathcal{P}_1^+\eta_2 - 4r\eta_1^+\mathcal{P}_G\eta_1 - \eta_2^+\eta_1\mathcal{P}_1 \\
&+2\eta_2^+\eta_G\mathcal{P}_2 - \eta_2^+\mathcal{P}_G\eta_2 + \mathcal{P}_1^+\eta_G\eta_1 + 2\mathcal{P}_2^+\eta_G\eta_2 - 8r\eta_0\eta_1\mathcal{P}_1 L_2^+ \\
&-4r\eta_0\eta_1^+\mathcal{P}_1 G_0 + 8r\eta_0\eta_1^+\mathcal{P}_1^+ L_2 - 4r\eta_0\mathcal{P}_1^+\eta_1 G_0 - 12r\eta_0\eta_1^+\mathcal{P}_1^+\eta_1\mathcal{P}_1, \qquad (27)\\
\tilde{Q}_{k_1} &= -2\eta_0(G_{0.aux} + G_0 - 3) - 6\eta_0\eta_2^+\mathcal{P}_2 - 6\eta_0\mathcal{P}_2^+\eta_2 - 3\eta_1^+\mathcal{P}_G\eta_1 \\
&-2\eta_0\eta_1\mathcal{P}_1(L_{2.aux}^+ + L_2^+) + 2\eta_0\eta_1^+\mathcal{P}_1^+(L_{2.aux} + L_2) - \eta_0\eta_1^+\mathcal{P}_1(G_{0.aux} + G_0) \\
&-2\eta_0\eta_1^+\mathcal{P}_2 L_1^+ - \eta_0\eta_1^+\mathcal{P}_G L_1 - \eta_0\mathcal{P}_1^+\eta_1(G_{0.aux} + G_0) \\
&-2\eta_0\mathcal{P}_2^+\eta_1 L_1 - \eta_0\mathcal{P}_G\eta_1 L_1^+ - 6\eta_0\eta_1^+\mathcal{P}_1^+\eta_1\mathcal{P}_1, \qquad (28)\\
\tilde{Q}_{k_2} &= -\eta_0(G_{0.aux} + G_0 - 3) - \eta_0\eta_1^+\mathcal{P}_1 - 2\eta_0\eta_2^+\mathcal{P}_2 \\
&-\eta_0\mathcal{P}_1^+\eta_1 - 2\eta_0\mathcal{P}_2^+\eta_2 - \eta_1^+\mathcal{P}_G\eta_1, \qquad (29)\\
\tilde{Q}_{k_3} &= \eta_0\eta_2^+\mathcal{P}_2 + \eta_0\mathcal{P}_2^+\eta_2 + \eta_0\eta_1\mathcal{P}_1(L_{2.aux}^+ + L_2^+) - \eta_0\eta_1^+\mathcal{P}_1^+(L_{2.aux} + L_2) \\
&+\eta_0\eta_1^+\mathcal{P}_2 L_1^+ + \eta_0\mathcal{P}_2^+\eta_1 L_1 + \eta_0\eta_1^+\eta_2^+\mathcal{P}_1\mathcal{P}_2 + 2\eta_0\eta_1^+\mathcal{P}_1^+\eta_1\mathcal{P}_1 \\
&+\eta_0\eta_1^+\mathcal{P}_1^+\mathcal{P}_G\eta_2 - \eta_0\eta_1^+\mathcal{P}_2^+\eta_2\mathcal{P}_1 - \eta_0\eta_2^+\mathcal{P}_1^+\eta_1\mathcal{P}_2 \\
&-\eta_0\eta_2^+\mathcal{P}_G\eta_1\mathcal{P}_1 + \eta_0\mathcal{P}_1^+\mathcal{P}_2^+\eta_1\eta_2. \qquad (30)
\end{aligned}$$

Taking the free parameters as in (25) again leads to the simplest form of the BRST charge

$$\begin{aligned}
\tilde{Q} &= \eta_0(\tilde{L}_0 + 6r - 4rG_{0.aux}) + \eta_1 L_1^+ + \eta_1^+ L_1 \\
&+\eta_2(L_{2.aux}^+ + L_2^+) + \eta_2^+(L_{2.aux} + \eta_2^+ L_2) - \eta_G(G_{0.aux} + G_0 - 3) \\
&-2r\eta_0\eta_1^+\mathcal{P}_1 - 4r\eta_0\eta_2^+\mathcal{P}_2 - 2r\eta_0\mathcal{P}_1^+\eta_1 - 4r\eta_0\mathcal{P}_2^+\eta_2 - \eta_1^+\eta_1\mathcal{P}_0 \\
&+\eta_1^+\eta_G\mathcal{P}_1 + \eta_1^+\mathcal{P}_1^+\eta_2 - 2r\eta_1^+\mathcal{P}_G\eta_1 - \eta_2^+\eta_1\mathcal{P}_1 \\
&+2\eta_2^+\eta_G\mathcal{P}_2 - \eta_2^+\mathcal{P}_G\eta_2 + \mathcal{P}_1^+\eta_G\eta_1 + 2\mathcal{P}_2^+\eta_G\eta_2 + 4r\eta_0\eta_1\mathcal{P}_1 L_{2.aux}^+ \\
&-4r\eta_0\eta_1\mathcal{P}_1 L_2^+ + 2r\eta_0\eta_1^+\mathcal{P}_1 G_{0.aux} - 2r\eta_0\eta_1^+\mathcal{P}_1 G_0 + 4r\eta_0\eta_1^+\mathcal{P}_2 L_1^+ \\
&-4r\eta_0\eta_1^+\mathcal{P}_1^+ L_{2.aux} + 4r\eta_0\eta_1^+\mathcal{P}_1^+ L_2 + 2r\eta_0\eta_1^+\mathcal{P}_G L_1 + 2r\eta_0\mathcal{P}_1^+\eta_1 G_{0.aux} \\
&-2r\eta_0\mathcal{P}_1^+\eta_1 G_0 + 4r\eta_0\mathcal{P}_2^+\eta_1 L_1 + 2r\eta_0\mathcal{P}_G\eta_1 L_1^+. \qquad (31)
\end{aligned}$$

The further procedure goes on in the complete analogy with the case of the flat space – time background. Namely we obtain the reduced BRST charges Q from (26) after the transformation (10) and than construct the Lagrangian (13) being invariant under the gauge transformation (2). Though the BRST charge which corresponds to the system (18), (6) is not unique, one can show following the lines of [3] on the base of the explicit calculations that all these BRST charges after making the partial BRST gauge fixing in the Lagrangian (13) lead to the unique final form of the Lagrangian, which contains only one double traceless physical field $|\Phi\rangle$ $(L_2L_2|\Phi\rangle = 0)$

$$\begin{aligned}
\mathcal{L} &= \langle\Phi|\tilde{L}_0 - L_1^+L_1 - 2L_2^+\tilde{L}_0L_2 + L_2^+L_1L_1 + L_1^+L_1^+L_2 - L_2^+L_1^+L_1L_2 \\
&-r(6 - 4G_0 + 10L_2^+L_2 - 4L_2^+G_0L_2)|\Phi\rangle. \qquad (32)
\end{aligned}$$

The Lagrangian is invariant with respect to the gauge transformation (15) with the traceless parameter $|\lambda\rangle$ and describes massless irreducible higher spins in AdS space which correspond to the Young tableaux with one row in complete correspondence with [3]. The same result concerning non-uniqueness of the BRST charge will take place if we apply above construction to the system of constraints (5) – (6) describing the propagation of higher massless integer spin fields in the Minkowski space as well.

1 Conclusions

In this paper we have shown that the BRST charge which corresponds to the system of constraints describing higher massless integer spin fields is not unique. Rather there exists a family of charges even for the case of the flat space background. As a result the form and the number of terms which contain the third and higher degrees of the ghost variables may be different. However the final form of the corresponding gauge invariant Lagrangian and of the gauge transformations are such, that the physical spectrum of the theory does not depend on the particular choice of free parameters entering into the BRST charge. That means that in this particular case the cohomologies of the all family BRST charges given by the equation (26) both for the case of the flat and AdS_D backgrounds are the same. Moreover we conjecture that generically the analogous ambiguity if it is observed for other gauge systems should have no impact on the physical spectrum for the different particular choices of the free parameters. The number of free parameters can be different as well. Let us note also that the nilpotence of the BRST charge for the case of AdS_D background requires the modification of the mass – shell operator, which leads the constraint, giving the correct unitary massless representations of AdS_D group.

It seems interesting to apply this procedure to the more complicated representations of the AdS_D group, when the corresponding representations of the flat space time little group is described for two and more rows. It is interesting also to construct the auxiliary representations of the whole nonlinear algebra (18), (6) in terms of Verma module, as well as to study the possibility for the construction of finite dimensional representations for such kind of algebras.
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On the Fock Space Realizations of Nonlinear Algebras Describing the High Spin Fields in AdS Spaces

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Abstract

The method of construction of Fock space realizations of Lie algebras is generalized for nonlinear algebras. We consider as an example the nonlinear algebra of constraints which describe the totally symmetric fields with higher spins in the AdS space-time.

1 Introduction

It is well known that many physical systems are invariant with respect to nonlinear symmetries. As the most known examples one can mention the Kepler motion of the planets or the hydrogen atom. Though the general classification of nonlinear algebras does not exists, the examples of such algebras, as well as the description of physical systems in which they act, one can find in the literature. Some references and rather detailed investigation of the so called finite W-algebras see in [1].

The very interesting class of nonlinear symmetries arises in the description of the higher spin fields in the AdS space-time. They depend on the type of the corresponding Young tableaux[2] and become more complicated when the number N of raws is growing. The simplest of these symmetries is connected with the Young tableaux with $N = 1$ and describes the totally symmetric fields of higher ranks which correspond to unitary representations in the AdS space-time of arbitrary dimension.

Starting from a given physical system with known nonlinear symmetry one can construct the explicit expressions for the generators of the symmetry transformations, usually in terms of the harmonic oscillators connected with the physical observables. In the case at hand the generators are constructed in terms of a space-time derivatives and N additional harmonic oscillators, which describe spin degrees of freedom and can be viewed as some internal distances in the classical models of composite particles like a discrete string[3],[4]. No doubt that this representation of the algebra generators is not unique. The question arises, how one can construct other representations of nonlinear algebras? Moreover, the knowledge of such new realizations of the algebras can help in the search of physical systems having these algebras as a symmetries. It can also help in the investigations of mathematical properties of these algebras.

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In the present paper we propose the regular method of construction of realizations for nonlinear algebras. We begin with the description how the method works in the case of linear algebras, [5],[6], and after that generalize it for the simplest case ($N = 1$) of totally symmetric fields describing the higher spins in the AdS space-time. For this goal we construct the Verma module for this nonlinear algebra. The representation of the algebra generators is constructed in terms of some auxiliary harmonic oscillators as a result of establishing the one-to-one correspondence of the Verma module vectors with the vectors of the Fock space generated by these oscillators. In conclusions we mention some interesting mathematical problems connected with the construction and some possible developments.

2 Construction of the Fock space realizations in the Lie algebra case

In this section we describe the method, based on the construction given in [5],[6]. Let $\hat{H}^i$, $(i = 1, ..., k)$ and $\hat{E}^\alpha$ be the Cartan generators and root vectors of the algebra with the following commutation relations

$$\left[\hat{H}^i, \hat{E}^\alpha\right] = \alpha(i)\hat{E}^\alpha, \tag{2.1}$$

$$\left[\hat{E}^\alpha, \hat{E}^{-\alpha}\right] = \sum \alpha^i \hat{H}^i, \tag{2.2}$$

$$\left[\hat{E}^\alpha, \hat{E}^\beta\right] = N^{\alpha\beta}\hat{E}^{\alpha+\beta}. \tag{2.3}$$

Roots $\alpha(i)$ and parameters α^i, $N^{\alpha\beta}$ are structure constants of the algebra in the Cartan – Weyl basis.

Consider the highest weight representation of the algebra under consideration with the highest weight vector $|0\rangle_V$, annihilated by the positive roots

$$E^\alpha|0\rangle_V = 0 \tag{2.4}$$

and being the proper vector of the Cartan generators

$$H^i|0\rangle_V = h^i|0\rangle_V. \tag{2.5}$$

The representation which is given by (2.4) and (2.5) in the mathematical literature is called the Verma module [7]. Following the Poincare – Birkhoff – Witt theorem, the basis space of this representation is given by vectors

$$|n_1, n_2, \dots, n_r\rangle_V = (E^{-\alpha_1})^{n_1}(E^{-\alpha_2})^{n_2}\dots(E^{-\alpha_r})^{n_r}|0\rangle_V \tag{2.6}$$

where $\alpha_1, \alpha_2, \dots \alpha_r$ is some ordering of positive roots and $n_i \in N$.

Using the commutation relations of the algebra and the formula

$$AB^n = \sum_{k=0}^{n} \binom{n}{k} B^{n-k}[[\dots[A, B], B]\dots]$$

one can calculate the action of all generators on arbitrary vector of the Verma module. In [5] it was shown that, making use of the map

$$|n_1, n_2, \dots, n_r\rangle_V \longleftrightarrow |n_1, n_2, \dots, n_r\rangle \tag{2.7}$$

where $|n_1, n_2, \ldots, n_r\rangle$ are base vectors of the Fock space

$$|n_1, n_2, \ldots, n_r\rangle = (b_1^+)^{n_1}(b_2^+)^{n_2}\ldots(b_r^+)^{n_r}|0\rangle. \tag{2.8}$$

generated by creation and annihilation operators b_i^+, b_i $i = 1, 2, \ldots, r$ with the standard commutation relations

$$\left[b_i, b_j^+\right] = \delta_{ij}, \tag{2.9}$$

the generators of the algebra in the Fock space can be written as polynomials in creation operators.

As an explicit example of the construction given above let us consider the representations of the algebra, which can be used for the description of the higher spin fields in the flat space-time. This algebra is linear and its generators $L_1^+, L_2^+, L_1, L_2, G_0, L_0$, which single out the irreducible representation of the Poincare group have the following commutation relations

$$\begin{array}{lll}
[L_1^+, L_1] = -L_0, & [L_1^+, L_2] = -L_1, & [L_2^+, L_1] = -L_1^+, \\
[L_0, L_1] = 0, & [L_0, L_2] = 0, & [L_1^+, L_2^+] = 0, \\
[L_1^+, L_0] = 0, & [L_1, L_0] = 0, & [L_1, L_2] = 0, \\
[G_0, L_1] = -L_1, & [L_1^+, G_0] = -L_1^+, & [L_0, G_0] = 0,
\end{array} \tag{2.10}$$

with the $so(2,1)$ subalgebra

$$[G_0, L_2] = -2L_2, \quad [L_2^+, G_0] = -2L_2^+, \quad [L_2^+, L_2] = -G_0. \tag{2.11}$$

The defining relations for the highest vector of the Verma module are:

$$L_1|0\rangle_V = 0, L_2|0\rangle_V = 0, G_0|0\rangle_V = h|0\rangle_V, L_0|0\rangle_V = e|0\rangle_V.$$

As the base of the representation space one takes the following vectors

$$|n_1, n_2\rangle_V = (L_1^+)^{n_1}(L_2^+)^{n_2}|0\rangle_V \tag{2.12}$$

and after the simple calculations one obtains

$$\begin{aligned}
L_1^+|n_1, n_2\rangle_V &= |n_1 + 1, n_2\rangle_V \\
L_2^+|n_1, n_2\rangle_V &= |n_1, n_2 + 1\rangle_V \\
G_0|n_1, n_2\rangle_V &= (n_1 + 2n_2 + h)|n_1, n_2\rangle_V \\
L_0|n_1, n_2\rangle_V &= e|n_1, n_2\rangle_V \\
L_1|n_1, n_2\rangle_V &= n_2|n_1 + 1, n_2 - 1\rangle_V + en_1|n_1 - 1, n_2\rangle_V \\
L_2|n_1, n_2\rangle_V &= n_2(h + n_1 + n_2 - 1)|n_1, n_2 - 1\rangle_V + e\frac{n_1(n_1 - 1)}{2}|n_1 - 2, n_2\rangle_V.
\end{aligned} \tag{2.13}$$

Consider the auxiliary Fock space generated by the annihilation and creation operators b_1, b_2, b_1^+, b_2^+. If one takes the vectors in the Fock space

$$|n_1, n_2\rangle = (b_1^+)^{n_1}(b_2^+)^{n_2}|0\rangle \tag{2.14}$$

in one-to-one correspondence with the vectors (2.12) in the Verma module, it can be seen that the action of the following operators in the Fock space

$$\begin{aligned}
L_1^+ &= b_1^+ \\
L_2^+ &= b_2^+ \\
G_0 &= b_1^+ b_1 + 2b_2^+ b_2 + h \\
L_0 &= e, \\
L_1 &= b_1^+ b_2 + e b_1, \\
L_2 &= (b_1^+ b_1 + b_2^+ b_2 + h) b_2.
\end{aligned} \tag{2.15}$$

is identical to the expressions (2.13) for the Verma module. So, (2.15) gives realization of the algebra in the Fock space.

3 Construction of the Fock space realization in non-linear algebra case

In the AdS space-time the algebra (2.10) is modified by terms proportional to the parameter r which is the square of the inverse radius of this space. When $r \neq 0$ the algebra is nonlinear and have the following nonzero commutation relations [2]

$$\begin{aligned}
&[L_1^+, L_1] = -L_0, \quad [L_1^+, L_2] = -L_1, \quad [L_2^+, L_1] = -L_1^+, \\
&[L_0, L_1] = -2rL_1 + 4rG_0L_1 - 8rL_1^+L_2, \\
&[L_1^+, L_0] = -2rL_1^+ + 4rL_1^+G_0 - 8rL_2^+L_1, \\
&[G_0, L_1] = -L_1, \quad [L_1^+, G_0] = -L_1^+,
\end{aligned} \tag{3.1}$$

with the $so(2,1)$ subalgebra (2.11). The structure of this nonlinear algebra is simplified if one introduces new generator C instead of the generator L_0: $L_0 = C + 2K$, where K is the Casimir operator of the subalgebra $so(2,1)$

$$K = 4L_2^+L_2 - G_0G_0 + 2G_0, \tag{3.2}$$

The generator C commute with all other generators and is the central charge of the algebra.

It means that we are still able to define the Verma modules as the highest weight representation of this algebra under consideration with the highest weight vector $|0\rangle_V$, annihilated by L_1, L_2

$$L_1|0\rangle_V = 0, \quad L_2|0\rangle_V = 0, \tag{3.3}$$

and being the proper vector of the generators G_0, C and, correspondingly, of G_0, L_0

$$G_0|0\rangle_V = h|0\rangle_V, \quad L_0|0\rangle_V = e|0\rangle_V. \tag{3.4}$$

The basis of the representation space is given by vectors

$$(L_1^+)^{n_1}(L_2^+)^{n_2}|0\rangle_V = |n_1, n_2\rangle_V\,.$$

For further calculations we will define the operators K_1, K_2 with the help of the following relations

$$
\begin{aligned}
&[K, L_1^+] = L_1^+ - 2L_1^+ G_0 + 4L_2^+ L_1 = K_1 \\
&[L_0, L_1^+] = 2rK_1 \\
&[K_1, L_1^+] = 4L_2^+ L_0 - 2(L_1^+)^2 = K_2 \\
&[K_2, L_1^+] = 8rL_2^+ K_1 \\
&[K_1, L_2^+] = 0, \quad [K_2, L_2^+] = 0
\end{aligned}
\tag{3.5}
$$

It is easy to see that

$$
\begin{aligned}
K_1|0) &= (1-2h)|1,0) \\
K_2|0) &= 4e|0,1) - 2|2,0)\,.
\end{aligned}
$$

To obtain the explicit form of the representation we need the commutation rules with $(L_1^+)^n, L_2^+)^n$

$$
\begin{aligned}
G_0(L_1^+)^n &= (L_1^+)^n G_0 + n(L_1^+)^n \\
G_0(L_2^+)^n &= (L_2^+)^n G_0 + 2n(L_2^+)^n \\
L_0(L_1^+)^n &= (L_1^+)^n L_0 + \sum_{k=1} \frac{(8r)^k}{4} \binom{n}{2k-1} (L_1^+)^{n-2k+1} (L_2^+)^{k-1} K_1 + \\
&+ \sum_{k=1} \frac{(8r)^k}{4} \binom{n}{2k} (L_1^+)^{n-2k} (L_2^+)^{k-1} K_2 \\
L_1(L_1^+)^n &= (L_1^+)^n L_1 + n(L_1^+)^{n-1} L_0 + \sum_{k=1} \frac{(8r)^k}{4} \binom{n}{2k} (L_1^+)^{n-2k} (L_2^+)^{k-1} K_1 + \\
&+ \sum_{k=1} \frac{(8r)^k}{4} \binom{n}{2k+1} (L_1^+)^{n-2k-1} (L_2^+)^{k-1} K_2 \\
L_1(L_2^+)^n &= (L_2^+)^n L_1 + nL_1^+ (L_2^+)^{n-1} \\
L_2(L_1^+)^n &= (L_1^+)^n L_2 + n(L_1^+)^{n-1} L_1 + \binom{n}{2} (L_1^+)^{n-2} L_0 + \\
&+ \sum_{k=1} \frac{(8r)^k}{4} \binom{n}{2k+1} (L_1^+)^{n-2k-1} (L_2^+)^{k-1} K_1 + \\
&+ \sum_{k=1} \frac{(8r)^k}{4} \binom{n}{2k+2} (L_1^+)^{n-2k-2} (L_2^+)^{k-1} K_2 \\
L_2(L_2^+)^n &= (L_2^+)^n L_2 + n(L_2^+)^{n-1} G_0 + n(n-1)(L_2^+)^{n-1}
\end{aligned}
\tag{3.6}
$$

With the help of these formulas we obtain explicitly the Verma module representation

$$
\begin{aligned}
L_1^+|n_1, n_2\rangle_V &= |n_1+1, n_2\rangle_V \\
L_2^+|n_1, n_2\rangle_V &= |n_1, n_2+1\rangle_V \\
G_0|n_1, n_2\rangle_V &= (n_1 + 2n_2 + h)|n_1, n_2\rangle_V \\
L_0|n_1, n_2\rangle_V &= e|n_1, n_2\rangle_V + (1-2h) \sum_{k=1} \frac{(8r)^k}{4} \binom{n_1}{2k-1} |n_1 - 2k + 2, n_2 + k - 1\rangle_V +
\end{aligned}
$$

$$
\begin{aligned}
& + \sum_{k=1} \frac{(8r)^k}{4}\binom{n_1}{2k}\Big(4e|n_1-2k,n_2+k\rangle_V - 2|n_1-2k+2,n_2+k-1\rangle_V\Big) \\
L_1|n_1,n_2\rangle_V &= n_2|n_1+1,n_2-1\rangle_V + en_1|n_1-1,n_2\rangle_V + \\
& + \frac{(1-2h)}{4}\sum_{k=1}(8r)^k\binom{n_1}{2k}|n_1-2k+1,n_2+k-1\rangle_V + \\
& + e\sum_{k=1}(8r)^k\binom{n_1}{2k+1}|n_1-2k-1,n_2+k\rangle_V - \\
& - \frac{1}{2}\sum_{k=1}(8r)^k\binom{n_1}{2k+1}|n_1-2k+1,n_2+k-1\rangle_V \\
L_2|n_1,n_2\rangle_V &= n_2(h+n_1+n_2-1)|n_1,n_2-1\rangle_V + e\frac{n_1(n_1-1)}{2}|n_1-2,n_2\rangle_V + \\
& + \frac{(1-2h)}{4}\sum_{k=1}(8r)^k\binom{n_1}{2k+1}|n_1-2k,n_2+k-1\rangle_V + \\
& + e\sum_{k=1}(8r)^k\binom{n_1}{2k+2}|n_1-2k-2,n_2+k\rangle_V - \\
& - \frac{1}{2}\sum_{k=1}(8r)^k\binom{n_1}{2k+2}|n_1-2k,n_2+k-1\rangle_V
\end{aligned}
\tag{3.7}
$$

Now again if we use the operators b_1, b_2, b_1^+, b_2^+ we obtain the Fock space realization of the algebra (3.1)

$$
\begin{aligned}
L_1^+ &= b_1^+ \\
L_2^+ &= b_2^+ \\
G_0 &= b_1^+b_1 + 2b_2^+b_2 + h \\
L_0 &= \sum_{k=0}(8r)^k\left(\frac{e}{(2k)!} - \frac{2r(2h-1)}{(2k+1)!}b_1^+b_1 - \frac{4r}{(2k+2)!}(b_1^+)^2b_1^2\right)b_1^{2k}(b_2^+)^k \\
L_1 &= b_1^+b_2 + \sum_{k=0}(8r)^k\left(\frac{e}{(2k+1)!} - \frac{2r(2h-1)}{(2k+2)!}b_1^+b_1 - \frac{4r}{(2k+3)!}(b_1^+)^2b_1^2\right)b_1^{2k+1}(b_2^+)^k \\
L_2 &= (b_1^+b_1 + b_2^+b_2 + h)b_2 + \\
&+ \sum_{k=0}(8r)^k\left(\frac{e}{(2k+2)!} - \frac{2r(2h-1)}{(2k+3)!}b_1^+b_1 - \frac{4r}{(2k+4)!}(b_1^+)^2b_1^2\right)b_1^{2k+2}(b_2^+)^k
\end{aligned}
\tag{3.8}
$$

The infinite sums in this expressions are rather simple. For example, first of them can be written in terms of the formal variable $x = \sqrt{8rb_1^2b_2}$ as

$$
e\sum_{k=0}\frac{(8r)^k}{(2k)!}b_1^{2k}(b_2^+)^k = e\cosh x.
$$

All the other sums can be also written in terms of the $\cosh x$ and $\sinh x$.

4 Conclusions

In this paper we have generalized the method of constructions of Fock space representations for nonlinear algebras, developed for linear algebras in [5],[6]. The method uses the notion of the Verma module for an algebra under consideration. Taking as an example the very important physically case of higher spin fields in the AdS space-time we constructed the Verma module, as well as the Fock space representation for the generators of the corresponding nonlinear algebra.

From the mathematical point of view the construction of the Verma module gives the possibility to analyze its structure and search for the singular vectors in it. The general procedure will then help to construct the finite dimensional representations of considered nonlinear algebra. It would be interesting to carry out investigations along this line, as well as make the generalization on the case of more complicated nonlinear algebras connected with irreducible unitary representations in the AdS space-time of arbitrary dimensions, described by the Young tableaux with more then one raw.

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Curved BPS Domain Walls in Four and Five Dimensions

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Abstract

We review our recent work on curved (non Ricci flat) supersymmetric domain wall solutions in $\mathcal{N}=2$ gauged supergravity theories in four and five dimensions.

1 Introduction

According to the domain wall/QFT correspondence [1], the renomalization group flow in quantum field theories may be described in terms of domain wall solutions in dual gauged supergravity theories. So far, this correspondence has mainly been studied in the context of flat supersymmetric domain walls. It is, however, natural to ask whether there also exist curved (i.e. non Ricci flat) supersymmetric domain wall solutions. Here we review our recent work on curved BPS domain wall solutions in $\mathcal{N}=2$ gauged supergravity theories in four and five dimensions [2, 3]. In five dimensions, curved BPS domain wall solutions should provide a gravitational description of RG flow in dual four-dimensional field theories on backgrounds of constant curvature [4].

2 $\mathcal{N}=2$ gauged supergravity

Domain walls are codimension one solutions that separate the spacetime into regions corresponding to different vacua. In the simplest case, a domain wall is supported by a gauge potential that couples to its world volume. The field strength of this gauge potential is dual to a cosmological constant. In a more general setting with non-trivial couplings to scalar fields, this cosmological constant appears as an extremum of the potential term in the Lagrangian. The resulting solution then describes a flow towards an extremum, and if the potential possesses several extrema, the solution may interpolate between them. In the following we will be interested in constructing domain wall solutions in $\mathcal{N}=2$ gauged supergravity theories in four and five dimensions.

The $\mathcal{N}=2$ gauged supergravity theories that we consider are based on abelian vector multiplets (labeled by an index $I=0,\ldots,n_V$) and n_H hypermultiplets coupled to the $\mathcal{N}=2$ supergravity fields. Potentials that are allowed by $\mathcal{N}=2$ supersymmetry are

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then obtained by performing a gauging of (some of) the various global symmetries. There are two different types of gaugings, namely (i) one can either gauge some of isometries of the moduli space of ungauged $\mathcal{N} = 2$ supergravity or (ii) one can gauge (part of) the $SU(2)$ R-symmetry, which only acts on the fermions. In $\mathcal{N} = 2$ supergravity, at the two-derivative level, the moduli space is a direct product $\mathcal{M} = \mathcal{M}_V \times \mathcal{M}_H$, where $\mathcal{M}_V$ and $\mathcal{M}_H$ are parameterized by the scalars belonging to the vector multiplets and to the hypermultiplets, respectively. The hypermultiplet scalars parametrize a quaternionic Kähler geometry. The vector multiplet scalars, on the other hand, parametrize a special Kähler geometry in four dimensions, whereas in five dimensions they parametrize a very special Kähler geometry.

3 Four-dimensional curved BPS domain wall solutions

The four-dimensional $\mathcal{N} = 2$ gauged supergravity theories we will consider are obtained by performing a gauging either of a $U(1)$ subgroup of the $SU(2)$ R-symmetry or of a particular Abelian Killing symmetry of the universal hypermultiplet moduli space. We refer to [5]-[7] for a detailed description of the gauging. The former results in models with a potential term for the vector scalars only, whereas the latter results in a potential term for the vector scalars and for the dilaton field entering the universal hypermultiplet. In the context of string theory, the latter models can be obtained by Calabi-Yau threefold compactifications of type II string theory in the presence of internal H-fluxes [8]-[12]. Non-vanishing H-fluxes do allow for supersymmetric anti-de-Sitter groundstates with non-vanishing cosmological constant Λ_4. This implies that the associated four-dimensional field equations possess solutions describing non-flat gravitational backgrounds, such as domain walls.

In [2] we constructed four-dimensional domain wall solutions of the gauged supergravity theories described above. We showed that these walls are, in general curved, i.e. there is in general also a three-dimensional cosmological constant Λ_3 on the domain wall. This is due to the presence of a U(1)-Kähler connection in four-dimensional $\mathcal{N} = 2$ supergravity. We also showed that the behaviour of the scalar fields belonging to the vector multiplets is governed by a set of equations already known from the study of BPS black hole solutions in ungauged $\mathcal{N} = 2$ supergravity theories, namely the so-called attractor equations [13, 14].

The potentials appearing in the four-dimensional gauged supergravity theories described above are based on the following symplectic invariant superpotential,

$$W = \mathrm{e}^{s\phi}\left(\alpha_I L^I - \beta^I M_I\right) \quad , \quad s = 0, 2\,, \tag{1}$$

where the symplectic vector (L^I, M_I) $(I = 0, \ldots, n_V)$ denotes the sections which depend on the physical vector scalar fields, and ϕ is the dilaton field entering the universal hypermultiplet. The entries of the constant symplectic vector (α_I, β^I) correspond to the electric/magnetic $U(1)^{n_V+1}$ charges of the universal hypermultiplet and emerge in type IIA/B string compactifications as internal H-fluxes on Calabi-Yau threefolds with $h^{1,1}(h^{2,1}) = n_V$ for type IIA (type IIB) [9, 10]. The dependence of this superpotential on the vector scalars is, on the other hand, the direct analogue of the central charge Z of the $\mathcal{N} = 2$ supersymmetry algebra that plays a key role in the context of BPS black holes in ungauged $\mathcal{N} = 2$ supergravity. In the black hole context, the (α_I, β^I) are just the electric/magnetic $U(1)^{n_V+1}$ charges of the black holes.

We are interested in the construction of curved BPS domain wall solutions. These are solutions which are uncharged, which are supported by non-constant scalar fields and which have residual supersymmetry. We take the associated spacetime metric to be given by

$$2 = \mathrm{e}^{2U(z)} \hat{g}_{mn} dx^m dx^n + \mathrm{e}^{-2pU(z)} dz^2 \tag{2}$$

with $\hat{g}_{mn} = \hat{g}_{mn}(x^m)$. Here we denote spacetime indices by $x^\mu = (x^m, z), x^m = (t, x, y)$, and the corresponding tangent space indices by $a = (0, 1, 2, 3)$. The constant p is introduced for later convenience. We assume Lorentz invariance in the three-dimensional subspace $a = (0, 1, 2)$. The metric $\hat{g}_{mn}$ is thus a three-dimensional constant curvature metric, i.e. $\hat{R}_{mn} = -4l^{-2}\hat{g}_{mn}$. For the solution to be supersymmetric, we must take l to be imaginary [2], which corresponds to a three-dimensional anti-de Sitter spacetime. Since we take the solutions to be uncharged, we set the gauge fields to zero.

Supersymmetric domain wall solutions will, in general, only preserve part of $\mathcal{N} = 2$ supersymmetry. A condition on the supersymmetry parameters that is consistent with local Lorentz invariance in the subspace $a = (0, 1, 2)$ and that gives rise to BPS domain wall solutions is given by [2]

$$\epsilon_i = \bar{h}\, \sigma^1_{ij}\, \gamma^3\, \epsilon^j \ , \tag{3}$$

where $h = h(z)$ denotes a phase factor with chiral weight $c = -1$. The BPS nature of the solution then requires that [2]

$$2\mathrm{e}^{pU} \partial_z U = h\bar{W} + \bar{h}W \tag{4}$$

as well as

$$h\mathcal{D}_z \bar{h} = \tfrac{1}{2}\mathrm{e}^{-pU}\left(h\bar{W} - \bar{h}W\right) = \tfrac{2i}{|l|}\mathrm{e}^{-(1+p)U}. \tag{5}$$

The former describes the BPS flow equation for the warp factor U. The latter relates the imaginary part of $\bar{h}W$ to the U(1)-Kähler connection alluded to earlier. A non-vanishing U(1)-Kähler connection thus provides a three-dimensional cosmological constant $\Lambda_3 = |l|^{-2}$ on the domain wall, leading to a curved anti-de-Sitter like domain wall solution. On the black hole side, the same object corresponds to the angular momentum of stationary but in general non-static solutions [15]-[17].

A convenient form of the BPS flow equation for the vector scalars can be obtained in the following way [2]. We introduce the chiral invariant variables [2]

$$Y^I = \mathrm{e}^U\, \bar{h}\, L^I \quad , \quad F_I(Y) = \mathrm{e}^U\, \bar{h}\, M_I \quad . \tag{6}$$

Inserting (6) into $i(\bar{L}^I M_I - L^I \bar{M}_I) = 1$ yields

$$e^{2U} = i\Big[\bar{Y}^I F_I(Y) - Y^I \bar{F}_I(\bar{Y})\Big] \ . \tag{7}$$

The BPS flow equation for the scalar fields Y^I is then given by [2]

$$\partial_z \begin{pmatrix} Y^I - \bar{Y}^I \\ F_I - \bar{F}_I \end{pmatrix} = -i\, e^{s\phi + (1-p)U} \begin{pmatrix} \beta^I \\ \alpha_I \end{pmatrix} . \tag{8}$$

If we now set

$$e^{s\phi} = e^{(p-1)U} \ , \tag{9}$$

then we may integrate (8),

$$\begin{pmatrix} Y^I - \bar{Y}^I \\ F_I - \bar{F}_I \end{pmatrix} = i \begin{pmatrix} h^I - \beta^I z \\ h_I - \alpha_I z \end{pmatrix} = i \begin{pmatrix} H^I \\ H_I \end{pmatrix}, \tag{10}$$

where the (H^I, H_I) denote harmonic functions. These equations, which determine the behaviour of the physical vector scalar fields $z^A = Y^A/Y^0$, are also known from the study of BPS black hole solutions in ungauged $\mathcal{N} = 2$ supergravity theories, where they are called attractor equations [13, 14]. Thus we see that the BPS flow equations (8) are equivalent to the set of algebraic equations (10). A solution to the attractor equations (10) expresses the complex scalar fields Y^I in terms of the harmonic functions. Via (7) then also the warp factor U gets expressed in terms of harmonic functions.

Let us now return to (9). For the case $s = 0$ (corresponding to an Abelian gauging of the $SU(2)$ R-symmetry), (9) holds for the convenient choice $p = 1$. For the case $s = 2$ (corresponding to the Abelian gauging of a particular isometry of the universal hyper moduli space) we will show below that (9) solves the BPS equation for the dilaton (the non-constant hyper scalar field) provided that $p = -3$.

Next, consider inserting (10) into (5). This yields

$$|l|^{-1} = -\tfrac{1}{4}\, e^{s\phi} \left(\alpha_I h^I - \beta^I h_I\right) \ . \tag{11}$$

Since l has to be constant, we conclude that it is not possible to have a non-vanishing three-dimensional cosmological constant $\Lambda_3 = |l|^{-2}$ for the case $s = 2$. That is, for $s = 2$ the domain wall solutions have to be flat ($\alpha_I h^I - \beta^I h_I = 0$), whereas for $s = 0$ they may be curved. This implies that in the latter case there is no restriction on the integration constants (h^I, h_I), whereas in the former case there are such restrictions.

Let us perform two consistency checks on the solution given above. Let us first return to (4). Using (8) as well as (9), equation (4) may be rewritten as

$$\partial_z e^{2U} = \Big(\alpha_I (Y^I + \bar{Y}^I) - \beta^I (F_I + \bar{F}_I)\Big) = i\partial_z\Big(\bar{Y}^I F_I - Y^I \bar{F}_I\Big) \ , \tag{12}$$

which is in perfect agreement with (7). Next, let us consider $h\mathcal{D}_z\bar{h} = h\partial_z\bar{h} - i\mathcal{A}_z$, where $\mathcal{A}_z = \mathcal{A}_z^Y - \frac{i}{2}\partial_z \log\frac{\bar{h}}{h}$, $\mathcal{A}_z^Y = \frac{1}{2}e^{-2U}(Y^I - \bar{Y}^I)\overleftrightarrow{\partial_z}(F_I - \bar{F}_I)$. Using (10) it follows that $h\mathcal{D}_z\bar{h} = -i\mathcal{A}_z^Y = -\frac{i}{2}e^{-2U}(\alpha_I h^I - \beta^I h_I)$ which, upon using (11), precisely yields (5).

Finally, the BPS flow equation for the dilaton (the non-constant hyper scalar field) is given by [2]

$$e^{pU}\partial_z e^{-2\phi} = 4\,\bar{h}\,(\alpha_I L^I - \beta^I M_I) = 4\,e^{-2\phi}\,\bar{h}W\ . \tag{13}$$

We infer from (13) that $\bar{h}W = h\bar{W}$, and hence we obtain from (5) that $|l|^{-1} = \infty$. Using (4) it follows that (13) is solved by

$$e^{-2(\phi-\phi_0)} = e^{4U}\ . \tag{14}$$

This is consistent with (9) provided that $p = -3$.

Let us briefly summarize some of the properties of the domain wall solutions we have constructed. For both cases ($s = 0, 2$) we find that the behaviour of the vector scalar fields is determined by a set of attractor equations given in (10). For the case $s = 0$, the hyper scalars are all constant and the domain wall solutions are, in general, curved, with the three-dimensional cosmological constant $\Lambda_3 = |l|^{-2}$ determined by (11). The associated spacetime metric is given by (2) with $p = 1$. For the case $s = 2$, on the other hand, we find that the dilaton field exhibits the run-away behaviour (14). The domain wall solutions must be flat, and the associated spacetime metric is given by (2) with $p = -3$.

4 Five-dimensional curved BPS domain wall solutions

The five-dimensional $\mathcal{N} = 2$ gauged supergravity theories that we consider are in the class constructed in [18], describing the general coupling of n_V abelian vector multiplets and of n_H hypermultiplets to supergravity. The scalar fields ϕ^x ($x = 1, \ldots, n_V$) of the vector multiplets parametrize a very special manifold. The hypermultiplet scalars q^X, on the other hand, parametrize a quaternionic Kähler geometry determined by $4n_H$-beins $f_X^{iA}(q^X)$, with the $SU(2)$ index $i = 1, 2$ and the $Sp(2n_H)$ index $A = 1, \ldots, 2n_H$, raised and lowered by the symplectic metrics ε_{ij} and C_{AB}. We refer to [18] for more details.

The scalar potential in such theories is given by [19]

$$\mathcal{V} = -6\,W^2 + \frac{9}{2}g^{\Lambda\Sigma}\partial_\Lambda W\partial_\Sigma W + \frac{9}{2}W^2(\partial_x Q^s)(\partial^x Q^s)\ . \tag{15}$$

Here $g_{\Lambda\Sigma}$ denotes the metric of the complete scalar manifold, which is positive definite, involving the scalars of both the vector and the hypermultiplets. $W(\phi, q)$ and $Q^s(\phi, q)$ denote the norm and the $SU(2)$ phases, respectively, which appear in the decomposition of the triplet of Killing prepotentials P^s, i.e. $P^s = \sqrt{\frac{3}{2}}WQ^s$ with $Q^sQ^s = 1$. Note that in (15), the derivatives acting on the $SU(2)$ phases Q^s are only computed with respect to the scalars of the vector multiplets.

It is convenient to rewrite (15) as follows [4]

$$\mathcal{V} = -6\,W^2 + \frac{9}{2}\,\Gamma^{-2}\,g^{xy}\partial_x W \partial_y W \;+ \frac{9}{2}\,g^{XY}\partial_X W \partial_Y W \tag{16}$$

where

$$\Gamma^{-2}(\phi, q) = 1 + W^2\,\frac{g^{xy}(\partial_x Q^s)(\partial_y Q^s)}{g^{xy}\partial_x W \partial_y W}\;. \tag{17}$$

We are interested in the construction of curved BPS domain wall solutions. These are solutions which are uncharged, which are supported by non-constant scalar fields and which have residual supersymmetry. We take the associated spacetime metric to be given by

$$ds^2 = \mathrm{e}^{2U(r)}\hat{g}_{mn}dx^m dx^n + dr^2\;, \tag{18}$$

where $x^\mu = (x^m, r)$ (with $x^m = (t, x, y, z)$) and $\hat{g}_{mn} = \hat{g}_{mn}(x^m)$. We denote the corresponding tangent space indices by $a = (0, 1, 2, 3, 5)$. The metric $\hat{g}_{mn}$ is taken to be a four-dimensional constant curvature metric, i.e. $\hat{R}_{mn} = -12 l^{-2}\,\hat{g}_{mn}$, with the four-dimensional cosmological constant proportional to l^{-2}. For the solution to be supersymmetric, we must take l to be imaginary [3], which corresponds to a four-dimensional anti-de Sitter spacetime. We allow for a non-trivial dependence of the scalar fields on the coordinate r, and we write $\phi' = \partial\phi/\partial r$, $q' = \partial q/\partial r$ as well as $U' = \partial U/\partial r$.

We observe that in contrast to what happens in four dimensions (c.f. (11)) the cosmological constant on the domain wall is now an independent quantity, not related to any of the objects appearing in very special geometry.

For solutions with spacetime line element (18), the appropriate projector condition on the supersymmetry transformation parameters ϵ_i leading to residual supersymmetry is given by [3]

$$i\gamma_5\epsilon_i = A(r)\;Q_i{}^j\epsilon_j + B(r)\;M_i{}^j\epsilon_j\;. \tag{19}$$

Here $Q = iQ^s\sigma^s$ and $M = iM^s\sigma^s$ denote $SU(2)$-valued matrices satisfying $Q_i{}^j Q_j{}^k = -\delta_i{}^k, M_i{}^j M_j{}^k = -\delta_i{}^k$ (i.e. $Q^sQ^s = 1, M^sM^s = 1$). Q and M are orthogonal in $SU(2)$ space, i.e. $Q^sM^s = 0$. The consistency of (19) then yields

$$A^2(r) + B^2(r) = 1\;. \tag{20}$$

A curved BPS domain wall solution is supported by a non-trivial warp factor U as well as by non-constant scalar fields. The curved BPS flow equation for the warp factor U is given by [3]

$$U' = \pm\gamma g W\;, \tag{21}$$

where

$$\gamma(r) = \sqrt{1 - \frac{4e^{-2U}}{|l|^2 g^2 W^2}} \,. \tag{22}$$

The BPS flow equation for the vector scalars, on the other hand, is given by

$$\phi^{x\,\prime} = \mp 3g\, \gamma^{-1}\, g^{xy}\partial_y W \,. \tag{23}$$

The flow equations for the warp factor U and for the vector scalars ϕ were also derived in [20] in the context of non-supersymmetric five-dimensional gravity theories.

The flow equation for the hyper scalar fields is more complicated than the one for the vector scalars (23) and will be discussed in [4].

In the case that a given gauged supergravity model contains vector multiplets, a solution to the BPS flow equations (21) and (23) will also have to satisfy the following consistency condition due to supersymmetry [3, 4]

$$\Gamma(\phi, q) = \gamma(r) \,. \tag{24}$$

If a gauged supergravity model contains only hypermultiplets, then this consistency condition is absent.

It can be shown that the various integrability conditions associated to $\delta\psi_{\mu i} = 0$ are all satisfied [4].

It is not guaranteed that a solution to the BPS flow equations will also automatically solve the equations of motion for the scalar fields [21, 4]. Thus, in order to construct a curved BPS domain wall solution one should proceed as follows. First one constructs a solution to the BPS flow equations. It may be that in order to achieve this one has to expand γ in powers of $|l|^{-2}$ and to solve the flow equations iteratively as a power series in $|l|^{-2}$. Then one checks whether the consistency condition (24) is met by this solution (this only applies to models which include vector multiplets). If so, then one finally checks the scalar equations of motion. If they are satisfied, one has managed to construct a curved BPS domain wall solution.

The RG flow interpretation of curved BPS domain wall solutions will be discussed in [4].

As an application of our results, it would be interesting to construct a curved version of the flow discussed in [19], which is an $\mathcal{N} = 2$ version of the flow discussed by Freedman, Gubser, Pilch and Warner [22]. This is a flow in $\mathcal{N} = 4$ super-Yang-Mills theory broken to an $\mathcal{N} = 1$ theory by the addition of a mass term for one of the three adjoint chiral superfields [23].

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Non-Abelian Born-Infeld Revisited

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Abstract

We discuss the non-abelian Born-Infeld action, including fermions, as a series in α'. We review recent work establishing the complete result to α'^2, and its impact on our earlier attempts to derive the Born-Infeld action using κ-symmetry.

1 Introduction

The Born-Infeld action for the Maxwell field was originally introduced to obtain a finite electrostatic energy for the electron. This is achieved by constructing a nonlinear theory for the Maxwell field with an upper bound on the allowed electric field. With just this requirement the theory is far from unique. In his review [1] Born indicates that he prefers the following action:

$$\mathcal{L}_{\mathrm{BI}} = -\sqrt{-\det(\eta_{\mu\nu} + F_{\mu\nu})}\,. \tag{1}$$

His reasons for this choice are that the combination of metric and fieldstrength suggests a geometric interpretation, and that this particular example satisfies the requirement of electric-magnetic duality: the equations of motion are symmetric under the exchange of electric and magnetic fields.

These two properties are partly resposible for the fact that the Born-Infeld action reappeared in string theory, as the effective action for the open (super)string. Since then a lot of work has been devoted to studying the properties of (1). Calculations of scattering amplitudes in the bosonic string [2, 3] and in the superstring [4] revealed the terms of order F^4 in the expansion of (1) and their fermionic partners. The same result was obtained by extending the super-Maxwell action in $d = 10$ with F^4 terms and requiring supersymmetry [5].

The Born-Infeld action is also the effective theory for the vector field on the worldvolume of a D-brane. For $p < 9$ this requires a generalization of (1) with worldvolume scalar fields describing the transverse modes of the D-brane. For $p = 9$, the spacetime filling brane, the ordinary Born-Infeld theory is recovered. In [6] the complete result for $p = 9$ was presented.

In the case of n parallel branes the gauge invariance is extended to $U(1)^n$. Open strings stretching between the branes give rise to massive vector fields, which however become massless when all branes overlap. In that case there are n^2 massless vector fields on the common worldvolume, and the gauge invariance is enhanced to $U(n)$ [7]. The worldvolume theory for a system of n overlapping D9-branes is called the non-abelian Born-Infeld theory, and coincides with the effective theory for open superstrings with $U(n)$ Chan-Paton factors. For $p < 9$ transverse scalars, also in the adjoint representation of $U(n)$, appear.

The construction of the non-abelian Born-Infeld theory has turned out to be a difficult problem [8]. However, a lot of progress has been made this year and in the following sections we will give an overview of recent work. In Section 2 we will expand the Born-Infeld theory in α', and discuss symmetric traces. In Section 3 we will show that there is no need for α' terms. In Section 4 we discuss the recent results of [9] on α'^2, and our string theory calculation [10] at this order. We compare with a κ-symmetry [11] approach in Section 5, and discuss higher orders in α' in Section 6.

2 The α' expansion and symmetric traces

In this paper we will discuss the Born-Infeld theory as a nonlinear deformation of the supersymmetric Yang-Mills theory in $d = 10$. We define fields such that the dimension of the fieldstrength dim $F = 2$, and the adjoint Majorana-Weyl fermion χ has dim $\chi = 3/2$. The action then takes on the form

$$\mathcal{L} = \frac{1}{g^2}\Big(F^2 + \alpha' F^3 + \alpha'^2 F^4 + \alpha'^3 F^5 + \ldots\Big)\,, \tag{2}$$

where in each order in α' we should add the fermionic partners, keeping in mind that dim $\alpha'^2\bar{\chi}\partial\chi = 0$. The Yang-Mills coupling constant g plays no role in the following and can be set equal to one.

The terms at α'^0 are

$$\mathcal{L}_0 = -\tfrac{1}{4}F^{\mu\nu a}F_{\mu\nu}{}^a + \tfrac{1}{2}\bar{\chi}^a \not{D}\chi^a\,, \tag{3}$$

where the indices $a, b, \ldots$ indicate the $U(n)$ adjoint representation. The action (3) is invariant under the following supersymmetry transformations:

$$\delta_0 A_\mu{}^a = \bar{\epsilon}\gamma_\mu\chi^a\,, \qquad \delta_0\chi^a = \tfrac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu}{}^a\,. \tag{4}$$

Besides the linear supersymmetry the action has an obvious nonlinear sypersymmetry, related to the fact that a D-brane in type II theories breaks half of the maximal $d = 10$ supersymmetry. This takes on the form

$$\delta_0\chi^a = \eta^a\,, \tag{5}$$

where η is constant. The property of being constant is not preserved under Yang-Mills transformations, therefore we need to restrict η by the condition

$$f^{abc}\eta^c = 0\,. \tag{6}$$

This means that η commutes with all $U(n)$ generators, and is therefore in the $U(1)$ factor of $U(n)$.

The question is then how to extend (3) to higher orders in α' in a supersymmetric way. A first result on the α'^2 contribution was given in [5]. There it was shown that a supersymmetric extension can be obtained if the terms at order α'^2 are written as a symmetric trace, that is, the terms F^4 should be written as

$$\mathrm{tr}\, T_{(a}T_bT_cT_{d)}F^aF^bF^cF^d \equiv \mathrm{Str}\, F^4\,, \tag{7}$$

(where the parentheses indicate symmetrization) and similarly for the terms bilinear in the fermions (quartic fermions were not considered in [5]). With this symmetric trace the α'^2 terms are a straightforward generalization of the abelian result [5].

The discussion about non-abelian Born-Infeld has been dominated by the possibility that the complete result might be symmetric trace (see [12] for a careful discussion of this issue). The full result at order α'^n must be of the form

$$S = S_1 + S_2 + S_3 , \tag{8}$$

with

$$\begin{aligned} S_1 &\sim \operatorname{Str} F^{n+2} , \\ S_2 &\sim F^n[F,F] , \\ S_3 &\sim F^n\{D,D\}F , \end{aligned} \tag{9}$$

that is, all terms which are not written as a symmetric trace must contain commutators of fields (or of derivatives of fields). In case of higher derivative terms, as in S_3, commutators of derivatives need not appear since they can be rewritten as terms in S_2. Of course more derivatives than indicated in S_3 may appear. It was known that for the terms given in [5] at order α'^2 $S_2 = S_3 = 0$. But is was also known from earlier results from string theory [13] that S_3 terms are needed at order α'^3 and that they are not in the form of a symmetric trace.

Therefore the conjecture that the full answer is a symmetric trace cannot be true, a statement that will derive further support from results to be discussed later. However, it is still possible that the symmetric trace part of S will turn out to be a useful approximation to the full result.

3 Contributions at order α'

There are three possible terms in the action at order α':

$$\mathcal{L}_1 = b_1 f^{abc} F_{\mu\nu}{}^a F^{\nu\rho\, b} F_\rho{}^{\mu\, c} + (b_2 f^{abc} + b_3 d^{abc}) F^{\mu\nu\, a}\, \bar{\chi}^b \gamma_\mu D_\nu \chi^c . \tag{10}$$

Here $f^{abc} = \operatorname{tr} T^a[T^b, T^c]$ and $d^{abc} = \operatorname{tr} T^a\{T^b, T^c\}$.

The terms containing fermions can be eliminated from $\mathcal{L}_0 + \mathcal{L}_1$ by a field redefinition. Note that by a change ΔA and $\Delta\chi$ the order α'^0 action changes as

$$\mathcal{L}_0 \to \mathcal{L}_0 - \Delta A_\mu{}^a D_\nu F^{\mu\nu\, a} + \Delta\bar{\chi}^a \not{D} \chi^a . \tag{11}$$

This implies that any term in the effective action that is proportional to the order α'^0 equations of motion can be eliminated by a field redefinition. The fermionic term in (10) proportional to b_2 can be rewritten as

$$-\tfrac{1}{2} b_2 f^{abc} D_\nu F^{\mu\nu\, a}\, \bar{\chi}^b \gamma_\mu \chi^c , \tag{12}$$

while for the b_3-term we find

$$+\tfrac{1}{2} b_3 d^{abc} F^{\mu\nu\, a}\, \bar{\chi}^b \gamma_{\mu\nu} \not{D} \chi^c . \tag{13}$$

Therefore both terms can be eliminated.

This is in agreement with string amplitude calculations which give a vanishing on-shell three-point function. String theory therefore tells us that the bosonic term in (10) should be absent. However, this term cannot be redefined away. Therefore the coefficient b_1 should be set equal to zero. We will now show that supersymmetry also forces us to do this.

The idea of order by order in α' supersymmetry is that the transformation of the action

$$\mathcal{L}_n = \sum_k^n \mathcal{L}_k \tag{14}$$

by the transformation rules of order α'^l, $0 \leq l < n$, should be compensated for by a new order α'^n transformation rule of the fields. This is possible only if the transformation of $\mathcal{L}_n$ is proportional to order α'^0 equations of motion.

The variation of $\mathcal{L}_1$, after the fermionic terms have been redefined away, takes on the form:

$$\begin{aligned} \delta_0 \mathcal{L}_1 &= 6b_1 f^{abc} D_\mu \delta_0 A_\nu{}^a F^{\nu\rho\, b} F_\rho{}^{\mu\, c} \\ &= 3b_1 f^{abc} \bar{\epsilon}\gamma_\nu \chi^a D^\nu F^{\rho\mu\, b} F_{\rho\mu}{}^c - 6b_1 f^{abc} \bar{\epsilon}\gamma_\nu \chi^a F^{\nu\rho\, b} D_\mu F_\rho{}^{\mu\, c}\,, \end{aligned} \tag{15}$$

We see that last term in the variation is proportional to the equation of motion, but the other one is not. Thus the b_1-term cannot be supersymmetrized, and we must set $b_1 = 0$.

In conclusion, at order α' two fermionic terms in the action are allowed. They do not contribute to the on-shell three-point function, and can be redefined away. Although one is not forced to do this redefinition, we will from now on assume that there is no order α' contribution to the action.

4 The order α'^2 terms

The full order α'^2 action including quartic fermions was recently constructed with superspace methods [9]. The result is, in our notation,

$$\begin{aligned} \mathcal{L}_2 = \; & S^{abcd}[\tfrac{1}{8} F_{\mu\nu}{}^a F_{\lambda\rho}{}^b F^{\mu\lambda\, c} F^{\nu\rho\, d} - \tfrac{1}{32} F_{\mu\nu}{}^a F^{\mu\nu\, b} F_{\lambda\rho}{}^c F^{\lambda\rho\, d} - \tfrac{1}{4} F_{\mu\nu}{}^a F^{\mu\rho\, b} \bar{\chi}^c \gamma^\nu D_\rho \chi^d \\ & - \tfrac{1}{16} F^{\mu\nu\, a} D_\mu F^{\lambda\rho\, b} \bar{\chi}^c \gamma_{\nu\lambda\rho} \chi^d + \tfrac{1}{24} \bar{\chi}^a \gamma^\mu D^\nu \chi^b \bar{\chi}^c \gamma_\mu D_\nu \chi^d \\ & + f^{def}(- \tfrac{7}{192} F^{\mu\nu\, a} \bar{\chi}^b \gamma_{\mu\nu\rho} \chi^c \bar{\chi}^e \gamma^\rho \chi^f + \tfrac{1}{1152} F^{\mu\nu\, a} \bar{\chi}^b \gamma^{\lambda\rho\sigma} \chi^c \bar{\chi}^e \gamma_{\mu\nu\lambda\rho\sigma} \chi^f)]\,. \end{aligned} \tag{16}$$

One sees that the result is a symmetric trace, except for two four-fermion terms. The method of [9] guarantees supersymmetry, but of course this can also be checked explicitly. Imposing supersymmetry, using the method indicated in Section 3, also gives the supersymmetry transformation rules at α'^2. This was done by [14], with the aim of extending the α'^2 supersymmetry to to the context of the non-commutative Yang-Mills theory, as well as in [5], for the transformation rules linear in χ. Let us present the complete result here. The calculation requires the use of the commutator of two covariant derivatives

$$[D_\mu, D_\nu]\chi^a = f^{abc} F_{\mu\nu}{}^b \chi^c\,, \tag{17}$$

as well as the $U(n)$ identity

$$f^{ab(c} S^{def)a} = 0\,. \tag{18}$$

The transformation rules for linear supersymmetry are then

$$
\begin{aligned}
\delta_2 A_\mu{}^a &= S^{abcd}\big[-\tfrac{3}{16} F^{\nu\rho\, b} F_{\nu\rho}{}^c \bar\epsilon\, \gamma_\mu \chi^d + \tfrac{1}{2} F_{\mu\rho}{}^b F^{\nu\rho\, c} \bar\epsilon\, \gamma_\nu \chi^d - \tfrac{1}{8} F_{\mu\nu}{}^b F_{\lambda\rho}{}^c \bar\epsilon\, \gamma^{\nu\lambda\rho} \chi^d \\
&\quad + \tfrac{1}{32} F^{\nu\lambda\, b} F^{\rho\sigma\, c} \bar\epsilon\, \gamma_{\mu\nu\lambda\rho\sigma} \chi^d + \tfrac{1}{96} \bar\epsilon\, \gamma^{\nu\rho} \not{D} \chi^b \bar\chi^c \gamma_{\mu\nu\rho} \chi^d + \tfrac{1}{288} \bar\epsilon\, \gamma_{\mu\nu\rho\sigma} \not{D} \chi^b \bar\chi^c \gamma^{\nu\rho\sigma} \chi^d \\
&\quad - \tfrac{1}{48} \bar\epsilon\, \gamma_\nu D_\rho \chi^b \bar\chi^c \gamma_\mu{}^{\nu\rho} \chi^d - \tfrac{1}{96} \bar\epsilon\, \gamma_{\mu\nu\rho} D_\sigma \chi^b \bar\chi^c \gamma^{\nu\rho\sigma} \chi^d + \tfrac{1}{288} \bar\epsilon\, \gamma_{\nu\rho\sigma} D_\mu \chi^b \bar\chi^c \gamma^{\nu\rho\sigma} \chi^d \big] \quad (19) \\
\delta_2 \bar\chi^a &= S^{abcd}\big[\tfrac{1}{8} F^{\mu\nu\, b} F_{\mu\lambda}{}^c F_{\nu\rho}{}^d \bar\epsilon\, \gamma^{\lambda\rho} - \tfrac{1}{32} F^{\mu\nu\, b} F_{\mu\nu}{}^c F_{\lambda\rho}{}^d \bar\epsilon\, \gamma^{\lambda\rho} \\
&\quad - \tfrac{1}{192} F^{\mu\nu\, b} F^{\lambda\rho\, c} F^{\sigma\tau\, d} \bar\epsilon\, \gamma_{\mu\nu\lambda\rho\sigma\tau} - \tfrac{5}{192} F^{\mu\nu\, b} \bar\epsilon\, \gamma^\lambda \chi^c \bar\chi^d \overleftarrow{\not{D}} \gamma_{\mu\nu\lambda} \\
&\quad + \tfrac{1}{96} F^{\mu\nu\, b} \bar\epsilon\, \gamma_{\mu\lambda\rho} \chi^c \bar\chi^d \overleftarrow{\not{D}} \gamma_\nu{}^{\lambda\rho} - \tfrac{1}{1152} F^{\mu\nu\, b} \bar\epsilon\, \gamma_{\mu\nu\lambda\rho\sigma} \chi^c \bar\chi^d \overleftarrow{\not{D}} \gamma^{\lambda\rho\sigma} \\
&\quad - \tfrac{19}{32} F^{\mu\nu\, b} \bar\epsilon\, \gamma_\mu \chi^c D_\nu \bar\chi^d - \tfrac{1}{96} F^{\mu\nu\, b} \bar\epsilon\, \gamma^\rho \chi^c D_\mu \bar\chi^d \gamma_{\nu\rho} \\
&\quad - \tfrac{5}{96} F^{\mu\nu\, b} \bar\epsilon\, \gamma^\rho \chi^c D_\rho \bar\chi^d \gamma_{\mu\nu} + \tfrac{3}{32} F^{\mu\nu\, b} \bar\epsilon\, \gamma_{\mu\nu}{}^\rho \chi^c D_\rho \bar\chi^d \\
&\quad - \tfrac{1}{24} F^{\mu\nu\, b} \bar\epsilon\, \gamma_\mu{}^{\rho\sigma} \chi^c D_\rho \bar\chi^d \gamma_{\nu\sigma} + \tfrac{13}{192} F^{\mu\nu\, b} \bar\epsilon\, \gamma_\mu{}^{\rho\sigma} \chi^c D_\nu \bar\chi^d \gamma_{\rho\sigma} \\
&\quad - \tfrac{1}{64} F^{\mu\nu\, b} \bar\epsilon\, \gamma^{\rho\sigma\tau} \chi^c D_\rho \bar\chi^d \gamma_{\mu\nu\sigma\tau} - \tfrac{1}{192} F^{\mu\nu\, b} \bar\epsilon\, \gamma^{\rho\sigma\tau} \chi^c D_\mu \bar\chi^d \gamma_{\nu\rho\sigma\tau} \\
&\quad - \tfrac{1}{192} F^{\mu\nu\, b} \bar\epsilon\, \gamma_{\mu\nu}{}^{\rho\sigma\tau} \chi^c D_\rho \bar\chi^d \gamma_{\sigma\tau} + \tfrac{1}{768} F^{\mu\nu\, b} \bar\epsilon\, \gamma_\mu{}^{\lambda\rho\sigma\tau} \chi^c D_\nu \bar\chi^d \gamma_{\lambda\rho\sigma\tau} \\
&\quad + \tfrac{1}{192} D_\mu F_{\nu\lambda}{}^b \bar\chi^c \gamma^{\mu\rho\sigma} \chi^d \bar\epsilon\, \gamma^{\nu\lambda}{}_{\rho\sigma} - \tfrac{5}{96} D_\mu F_{\nu\lambda}{}^b \bar\chi^c \gamma^{\nu\lambda\rho} \chi^d \bar\epsilon\, \gamma^\mu{}_\rho \big] . \quad (20)
\end{aligned}
$$

The modifications at α'^2 to the transformation rules for nonlinear supersymmetry were presented in [9].

Note that there is an ambiguity in determining the transformation rules. Certain terms in the variation are proportional to the product of the fermionic equation of motion and the equation of motion for the vector field. In that case one may cancel this contribution by either changing the fermionic or the bosonic transformation rule. We have always put such terms in the transformation rules for the bosons.

In a recent paper [10] we have calculated the open string four-point function. This allows the calculation of terms quartic in the basic fields in the effective action. This calculation also gives the order α'^2 action except the last terms in (16) which are a product of five fields. Let us briefly recall the basics of such a string calculation [15, 16]. The four-point function is given by the following formula:

$$
\begin{aligned}
A_4 &= -8ig^2\, K(1,2,3,4)\, (T_{1\,abcd} G(s,u) + \\
&\quad + T_{2\,abcd} G(s,t) + T_{3\,abcd} G(t,u)) ,
\end{aligned} \tag{21}
$$

where g is the Yang-Mills coupling constant and s, t and u are the standard Mandelstam variables satisfying $s + t + u = 0$. K contains the wave-functions of the external fermion lines, the T_i contain the traces over $U(n)$ generators, and satisfy in particular the relation

$$
(T_1 + T_2 + T_3)_{abcd} = S_{abcd} . \tag{22}
$$

G is the Veneziano amplitude

$$
\begin{aligned}
G(s,t) &= \frac{1}{st} \frac{\Gamma(1-\alpha' s)\Gamma(1-\alpha' t)}{\Gamma(1-\alpha'(s+t))} \\
&= \frac{1}{st} - \frac{\pi^2 \alpha'^2}{6} - \zeta(3) u \alpha'^3 + \ldots
\end{aligned} \tag{23}
$$

From this it is already clear that the four-point function at order α'^2 contains a symmetric trace over the Yang-Mills indices. For higher orders in α' this will not occur.

This does not immediately imply that in the effective action which reproduces at tree-level this string four-point function at order α'^2 will be a symmetric trace. If the effective action contains either or both of the fermionic α' contributions discussed in the previous section, these contribute to the four-point function through reducible diagrams, and the α'^2 contribution will then not be a symmetric trace. It is only when the α' terms are all absent that the full α'^2 four-point function is produced by the irreducible diagrams from the α'^2 effective action. Full details are given in [10].

With this proviso our result for the effective action at order α'^2 agrees completely with the result of [9].

Clearly the four-point function also has contributions at higher orders in α'. The terms in the effective action at order α'^3 were recently constructed from this part of the four-point function [17].

5 The fate of κ-symmetry

In a recent paper [11] we proposed a method to obtain the non-abelian Born-Infeld action using κ-symmetry. In such a construction one starts with an $N=2$ supersymmetric theory on the worldvolume. The fields are (in the abelian case) the embedding coordinates $X^\mu(\sigma)$ of the brane, a 32-components target space spinor $\theta(\sigma)$, and the worldvolume vector $A_i(\sigma)$. The symmetries acting on these fields are a global $N=2$ supersymmetry (parameter ϵ), a local $N=2$ supersymmetry with parameter $\kappa(\sigma)$, abelian gauge transformations and worldvolume reparametrisations with parameters $\xi^i(\sigma)$. On the fermions θ the fermionic symmetries act as

$$\delta\theta = \alpha'^{-1}(-\epsilon + (1+\Gamma)\kappa)\,, \tag{24}$$

where Γ is a field-dependent object satisfying $\Gamma^2=1$. This implies that by gauge fixing κ-symmetry half of the fermions can be gauged away, leaving only the $N=1$ fermion χ. The worldvolume reparametrisations can be gauge fixed by identifying the embedding coordinates with the worldvolume coordinates σ (the static gauge). After this gauge fixing two global supersymmetries remain, one realized linearly (the parameter ϵ from previous sections), the other realized nonlinearly.

In the abelian case, κ-symmetry has been very useful in obtaining the effective action for single Dp-branes in flat [6] and curved backgrounds [18]. In particular, the complete result for the abelian $D9$-brane was given in [6].

In our approach for the non-abelian case we start with a non-abelian κ-symmetry. The idea is that we now have n^2 type II fermions on the worldvolume, and to the reduce all of these to 16-component type I fermions we need n^2 local fermionic symmetries, i.e., θ should transform as

$$\delta\theta^a = \alpha'^{-1}(-\epsilon^a + (\delta^{ab}+\Gamma^{ab})\kappa^b)\,, \tag{25}$$

again with $\Gamma^2=1$. The parameter ϵ^a is constant, and therefore has to satisfy $f^{abc}\epsilon^c=0$. Early on in our analysis we discovered that our approach could only work in the static gauge, so there are no embedding coordinates X^μ.

We obtained a κ-symmetric action of the following schematic form:

$$\mathcal{L} = F^2 + \bar\theta\partial\theta(1+\alpha' F + \alpha'^2 F^2) \tag{26}$$

Invariance under κ-symmetry requires

$$\delta A = \bar\kappa(1+\alpha' F + \ldots)\gamma\theta \tag{27}$$

We checked the κ-invariance to order α' in the variation. This gave us only the terms at order α'^2 with fermions (26). It turned out that these were not in the form of a symmetric trace, which is in contradiction with the results of the previous section. There are now three possibilities:

- Note that (26) also contains terms of order α'. We know from the previous sections that these can be redefined away, and it might be that this redefinition modifies the α'^2 terms to a symmetric trace.
- It might be that supersymmetry does not fix the action uniquely, and that our κ-symmetric action leads a different supersymmetric invariant (which then would have no obvious relation to string theory).
- κ-symmetry, based on our assumptions, cannot be extended to order α'^2.

We have checked that the first two possibilities are not realized, so we conclude that at least one of the assumptions used in deriving (26) is false. Let us briefly review the main assumptions:

- κ-symmetry is non-abelian, each type I fermion is doubled. This still seems the most reasonable starting point to us. A proposal with only one κ-symmetry in a non-abelian Born-Infeld action has been proposed by [19].
- For n overlapping branes we use only one set of worldvolume coordinates and embedding coordinates (before going to the static gauge). In principle one could think of a more ambitious approach, which starts with as many coordinate systems and embedding coordinates as there are branes. The difficulty here is to impose the enlarged (non-abelian?) reparametrization invariance.

Presently it is not clear how to proceed with κ-symmetry in the non-abelian context. Because the nonlinear supersymmetry is still present at order α'^2 it is neverthless tempting to believe that the results of the previous section are the gauged fixed version of formulation with a larger, local symmetry.

6 Higher orders in α'

For a long time the only information on orders α'^k, $k > 2$ came from [13]. Recently more details about the case α'^3 have been obtained, while some partial results for order α'^4 are also known.

Let us start with α'^3. In [20] the one-loop bosonic five-point function in supersymmetric $N = 4$ $d = 4$ Yang-Mills theory is evaluated. The corresponding contribution to the effective action must be supersymmetric. If supersymmetry uniquely determines this term in the effective action, then it should be the same as the corresponding contribution to the non-abelian Born-Infeld theory.

In [21] the bosonic terms at α'^3 have been determined by requiring that a particular BPS solution, related to branes intersecting at angles [22], of the α'^0 field equations, remains a solution of the deformation of the Yang-Mills theory with nonlinear terms. It has been shown that in the abelian case this method gives the complete Born-Infeld theory to all orders in α' [23].

There are two types of terms in the result of [20, 21], those with F^5 and the terms of type $F^2(DF)^2$. In ten dimensions there six independent F^5 terms, while there are only four in $d = 4$. The $d = 4$ result therefore leads to a two-parameter solution in $d = 10$. For the terms with derivatives there is no such ambiguity. Indeed, all authors ([13, 20, 21, 17] agree on the the $F^2(DF)^2$ terms. For the F^5 terms the free parameters in the ten-dimensional version of [20] cannot be chosen in such a way that it agreement with [21] is obtained. The ordering of the Yang-Mills fields in the [13] result is not sufficiently specified to make a comparison with the new results possible.

So at this stage there are two candidates for the α'^3 terms in the open superstring effective action. One way to settle this problem would be to use the Noether procedure to supersymmetrize the α'^3 terms. Such a calculation is presently under way [24]. One starts with an Ansatz containing all bosonic terms and the terms bilinear in the fermions, and requires supersymmetry (four-fermion terms will not be considered at this stage). This procedure is simplified somewhat by the following observation. The supersymmetry transformation rules obtain corrections for each order of α'. In the absence of order α' terms in the action the first modifications occur at order α'^2. At order α'^3 the action and transformation rules of order α'^2 then play no role. This implies also that the α'^3 terms will be multiplied by an arbitrary constant, which can only be determined by a string theory calculation (and turns out to be proportional to $\zeta(3)$ [13]). Thus, as far as supersymmetry is considered, there are two invariants at this stage, one containing $\mathcal{L}_0$, $\mathcal{L}_2$ and all other terms at higher orders implied by supersymmetry, and another one starting with $\mathcal{L}_0$ and $\mathcal{L}_3$. More invariants are likely to appear at still higher orders in α'. String theory will lead to a particular combination of these invariants.

At order α'^4 some higher derivative terms are known from [17], and partial results on the bosonic terms were presented in [25].

Although we still seem to be far from a complete result, a lot of information has become available in the past year, and more is likely to appear next year. Hopefully the new results will reveal sufficient structure to recognize the form of the complete answer.

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Bihamiltonian Approach to the Closed String Model in the Background Fields

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Abstract

The closed string model in the background gravity field and the antisymmetric B-field is considered as the bihamiltonian system in assumption,that string model is the integrable model for particular kind of the background fields. It is shown, that bihamiltonity is origin of two types of the T-duality of the closed string models. The dual nonlocal Poisson brackets, depending of the background fields and of their derivatives, are obtained. The integrability condition is formulated as the compatibility of the bihamiltonity condition and the Jacobi identity of the dual Poisson bracket. It is shown, that the dual brackets and dual hamiltonians can be obtained from the canonical (PB) and from the initial hamiltonian by imposing of the second kind constraints on the initial dynamical system, on the closed string model in the constant background fields, as example. The closed string model in the constant background fields is considered without constraints, with the second kind constraints and with first kind constraints as the B-chiral string. The two particles discrete closed string model is considered as two relativistic particle system to show the difference between the Gupta-Bleuler method of the quantization with the first kind constraints and the quantization of the Dirac bracket with the second kind constraints.

1 Introduction

The bihamiltonian approach [1, 2, 3] to the integrable systems was initiated by Magri [4] for the investigation of the integrability of the KdV equation. A finite dimensional dynamical system with 2N degrees of freedom x^a, $a = 1...2N$ is integrable, if it is described by the set of the n integrals of motion $F_1, ..F_n$ in involution under some Poisson bracket (PB)

$$\{F_i, F_k\}_{PB} = 0 \tag{1}$$

The dynamical system is completely solvable, if $n = N$. Any of the integral of motion (or any linear combination of them) can be considered as the hamiltonian

$$H_k := F_k \tag{2}$$

The bihamiltonity condition has following form

$$\dot{x}^a = \frac{dx^a}{dt} = \{x^a, H_1\}_1 = ... = \{x^a, H_N\}_N \tag{3}$$

The hierarchy of new (PB) is arose in this connection.

$$\{,\}_1, \{,\}_2, ..\{,\}_N \tag{4}$$

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The hierarchy of new dynamical systems is arose under the new time coordinates t_k.

$$\frac{dx^a}{dt_{n+k}} = \{x^a, H_n\}_{k+1} \tag{5}$$

The new equations of motion are describe the new dynamical systems, which are dual to the original system, with the dual set of the integrals of motion. The dual set of the integrals of motion can be obtained from the original it by the mirror transformations and by the contraction of the integrals of motion algebra. The contraction of the integral of motion algebra means, that the dynamical system is belong to the orbits of corresponding generators and is describe the invariant subspace. The set of the commuting integrals of motion is belong to Cartan subalgebra of this algebra. Consequently, duality is property of the integrable models. KdV equation is one of the most interesting examples of the infinite dimensional integrable mechanical systems with soliton solutions. We are considered the dynamical systems with constraints. In this case, first kind constraints are the generators of gauge the transformations and they are integrals of motion. First kind constrains $F_k(x^a) \approx 0$, $k = 1, 2...$ form the algebra of constraints under some (PB).

$$\{F_i, F_k\}_{PB} = C^l_{ik} F_l \approx 0 \, . \tag{6}$$

The structure functions C^l_{ik} may be functions of the phase space coordinates in general case. The second kind constraints $f_k(x^a) \approx 0$ are the representations of the first kind constraints algebra. The second kind constraints is defined by the condition $\{f_i, f_k\} = C_{ik} \neq 0$. The reversible matrix C_{ik} is not constraint and it is function of phase space coordinates also. The second kind constraints take part in deformation of the $\{,\}_{PB}$ to the Dirac bracket $\{,\}_D$. As rule, such deformation leads to the nonlinear and to the nonlocal brackets. First kind constraints are imposed upon the vector states under the quantization: $F_k|\Psi>= 0$. The same spectrum of the excitations and of the wave functions are obtained under the Gupta- Bleuler method of the quantization. One-half of the second kind constraints can be considered as first kind constraints and they must be imposed upon vector states in Gupta- Bleuler method of quantization. The (PB) is not deformed to the Dirac bracket in this connection. The bihamiltonity condition leads to the dual (PB), which are nonlinear and nonlocal brackets as rule. We suppose, that the dual brackets can be obtained from the initial canonical bracket under the imposition of the second kind constraints. We make this conclusion from the consideration of two dynamical models, closed string model in the constant background fields and two particles discrete closed string model, as examples. The Gupta- Bleuler method of the quantization may be more preferable in some case, if the dual bracket is nonlocal. We have applied [5] bihamiltonity approach to the investigation of the integrability of the closed string model in the arbitrary background gravity field and antisymmetric B-field. The bihamiltonity condition and the Jacobi identities for the dual brackets have considered as the integrability condition for a closed string model. They led to some restrictions on the background fields. The local dual (PB) of the similar type have considered [6] in the application to the hamiltonian hydrodynamical models. The (PB) of the hydrodynamical type for the phase coordinate functions $u^i(x,t)$ is defined by formula

$$\{u^i(x), u^k(y)\} = g^{ik}(u(x))\partial_x\delta(x-y) + b^{ik}_l(u(x))u^l_x\delta(x-y) \tag{7}$$

There $g^{ik}(u), b^{ik}_l(u)$ are the arbitrary functions of the phase space coordinates and $u_x = \partial_x u$. The Jacobi identity is satisfied under the following conditions:

1. Tensor g^{ik} is symmetric tensor and it is define some metric on the phase space.
2. $b_l^{ik}(u) = -g^{ij}\Gamma_{jl}^k(u)$ and connection Γ_{jl}^k is consistent to metric g^{ik} and it has zero curvature and zero torsion.
Therefore, there are such local coordinates, that $g^{ik}=$ const, $b_{jl}^k = 0$. This (PB) was used for description of the hamiltonian system of the hydrodynamical type. That is systems with functionals of the hydrodynamical type. The density of this functionals does not depend of the derivatives $u_x^k, u_{xx}^k, ..$ and hamiltonian is functional of the hydrodynamical type also.
In opposite this models, the functionals of the closed string model is depended of the derivatives of the string coordinates. As result, we need to introduce additional nonlocal term with the step function $\epsilon(x-y) = 2\partial_x^{-1}\delta(x-y)$, which is the origin of the difficulty of the Jacobi identity proof.
The plan of the paper is following. In the second section we are considered closed string model in the arbitrary background gravity field and antisymmetric B-field as the bihamiltonian system. We suppose, that this model is integrable model for some configurations of the background fields. The bihamiltonity condition and the Jacobi identities for the dual (PB) must be result to the integrability condition, which is restrict the possible configurations of the background fields. The well known examples of the integrable gravity models with the gravity metric tensor, which is depended of one or of two variables only. In this paper we are assumed the metric dependence of the arbitrary number of the variables for generality and we did not analyzed the particular cases of the metric dependence. In the third section we are considered three examples of the closed string model in the constant background fields: without constraints, with the second kind constraints and the B-chiral string with the first kind constraints. In the four section we are considered two particles discrete closed string model to show the difference between the (PB) structure under the Gupta-Bleuler method of the quantization and under the quantization of the system with the second kind constraints.

2 Closed string in the background fields

The closed string in the background gravity field and the antisymmetric B-field is described by first kind constraints

$$h_1 = \frac{1}{2}g^{ab}(x)[p_a - \alpha B_{ac}(x)x'^c][p_b - \alpha B_{bd}(x)x'^d] + \frac{1}{2}g_{ab}(x)x'^a x'^b \approx 0, \; h_2 = p_a x'^a \approx 0 \tag{8}$$

where $a, b = 0, 1, ...D-1$, $x^a(\sigma), p_a(\sigma)$ are the periodical functions on σ with the period on π , α -arbitrary parameter. The original (PB) are the symplectic (PB)

$$\{x^a(\sigma), p_b(\sigma')\}_1 = \delta_b^a\delta(\sigma - \sigma') \qquad \{x^a, x^b\}_1 = \{p_a, p_b\}_1 = 0 \tag{9}$$

The hamiltonian equations of motion of the closed string, in the arbitrary background gravity field and antisymmetric B-field under the hamiltonian $H_1 = \int\limits_0^\pi h_1 d\sigma$ and (PB) $\{,\}_1$, are

$$\begin{aligned}
\dot{x}^a &= g^{ab}[p_b - \alpha B_{bc}x^{'c}] \qquad (10)\\
\dot{p}_a &= \alpha B_{ab}g^{bc}p_c^{'} + [g_{ab} - \alpha^2 B_{ac}g^{cd}B_{db}]x^{''b} - \\
&-\frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}p_b p_c - \alpha\frac{\partial}{\partial x^a}(B_{bd}g^{dc})x^{'b}p_c + \alpha\frac{\partial}{\partial x^b}(B_{ad}g^{dc})x^{'b}p_c - \\
&-\frac{1}{2}\frac{\partial}{\partial x^a}[g_{bc} - \alpha^2 B_{bd}g^{de}B_{eb}]x^{'b}x^{'c} + \frac{\partial}{\partial x^b}[g_{ac} - \alpha^2 B_{ad}g^{de}B_{ec}]x^{'b}x^{'c}
\end{aligned}$$

The dual (PB) are obtained from the bihamiltonity condition

$$\dot{x}^a = \{x^a, \int_0^\pi h_1 d\sigma^{'}\}_1 = \{x^a, \int_0^\pi h_2 d\sigma^{'}\}_2 \qquad (11)$$

$$\dot{p}_a = \{p_a, \int_0^\pi h_1 d\sigma^{'}\}_1 = \{p_a, \int_0^\pi h_2 d\sigma^{'}\}_2$$

and they have following form

$$\begin{aligned}
&\{A(\sigma), B(\sigma^{'})\}_2 = \qquad (12)\\
&\frac{\partial A}{\partial x^a}\frac{\partial B}{\partial x^b}[[\omega^{ab}(\sigma) + \omega^{ab}(\sigma^{'})]\epsilon(\sigma^{'} - \sigma) + \\
&[\Phi^{ab}(\sigma) + \Phi^{ab}(\sigma^{'})]\frac{\partial}{\partial \sigma^{'}}\delta(\sigma^{'} - \sigma) + \\
&[\Omega^{ab}(\sigma) + \Omega^{ab}(\sigma^{'})]\delta(\sigma^{'} - \sigma)] + \\
&\frac{\partial A}{\partial p_a}\frac{\partial B}{\partial p_b}[[\omega_{ab}(\sigma) + \omega_{ab}(\sigma^{'})]\epsilon(\sigma^{'} - \sigma) + \\
&[\Phi_{ab}(\sigma) + \Phi_{ab}(\sigma^{'})]\frac{\partial}{\partial \sigma^{'}}\delta(\sigma^{'} - \sigma) + \\
&[\Omega_{ab}(\sigma) + \Omega_{ab}(\sigma^{'})]\delta(\sigma^{'} - \sigma)] + \\
&[\frac{\partial A}{\partial x^a}\frac{\partial B}{\partial p_b} + \frac{\partial A}{\partial p_b}\frac{\partial B}{\partial x^a}][[\omega_b^a(\sigma) + \omega_b^a(\sigma^{'})]\epsilon(\sigma^{'} - \sigma) \\
&+[\Phi_b^a(\sigma) + \Phi_b^a(\sigma^{'})]\frac{\partial}{\partial \sigma^{'}}\delta(\sigma^{'} - \sigma)] + \\
&[\frac{\partial A}{\partial x^a}\frac{\partial B}{\partial p_b} - \frac{\partial A}{\partial p_b}\frac{\partial B}{\partial x^a}][\Omega_b^a(\sigma) + \Omega_b^a(\sigma^{'})]\delta(\sigma^{'} - \sigma)
\end{aligned}$$

The arbitrary functions $A, B, \omega, \Phi, \Omega$ are the functions of the $x^a(\sigma), p_a(\sigma)$. The functions ω^{ab}, ω_{ab}, Φ^{ab}, Φ_{ab} are the symmetric functions on a, b and Ω^{ab}, Ω_{ab} are the antisymmetric functions to satisfy the condition $\{A, B\}_2 = -\{B, A\}_2$. The equations of motion under the hamiltonian $H_2 = \int_0^\pi h_2(\sigma^{'})d\sigma^{'}$ and (PB) $\{,\}_2$ are

$$\begin{aligned}
\dot{x}^a &= -2\omega_b^a x^b + 4\omega^{ab}p_b + 2\Phi^{ab}p_b^{''} - 2\Phi_b^a x^{''b} + 2\Omega_b^a x^{'b} - 2\Omega^{ab}p_b^{'} + \qquad (13)\\
&\int_0^\pi d\sigma^{'}[\omega_b^a x^{'a} + \frac{\partial \omega^{ac}}{\partial x^b}x^{'b}p_c + \frac{\partial \omega^{ac}}{\partial p_b}p_b^{'}p_c]\epsilon(\sigma^{'} - \sigma) \\
&+(\frac{\partial \Phi^{ac}}{\partial x^b}x^{'b} + \frac{\partial \Phi^{ac}}{\partial p_b}p_b^{'})p_c^{'} - (\frac{\partial \Phi_c^a}{\partial x^b}x^{'b} + \frac{\partial \Phi_c^a}{\partial p_b}p_b^{'})x^{'c}
\end{aligned}$$

$$\dot{p}_a = -2\omega_{ab}x^b - 2\Phi_{ab}x^{''b} + 2\Omega_{ab}x^{'b} + +4\omega_a^b p_b + 2\Phi_a^b p_b^{''} + 2\Omega_a^b p_b^{'} + \tag{14}$$
$$\int_0^\pi d\sigma^{'}(\omega_{ab}x^{'b} - \frac{\partial^2\omega^{cd}}{\partial x^a \partial x^b}x^{'b}p_c p_d - \frac{\partial \omega^{bc}}{\partial x^a}p_b p_c^{'})\epsilon(\sigma^{'} - \sigma)$$
$$-(\frac{\partial\Phi_{ac}}{\partial x^b}x^{'b} + \frac{\partial\Phi_a^c}{\partial p_b}p_b^{'})x^{'c} + (\frac{\partial\Phi_a^c}{\partial x^b}x^{'b} + \frac{\partial\Phi_a^c}{\partial p_b}p_b^{'})p_c^{'}$$

The bihamiltonity condition (11) is led to the two constraints

$$-2\omega_b^a x^b + 4\omega^{ab}p_b + 2\Phi^{ab}p_b^{''} - -2\Phi_b^a x^{''b} + 2\Omega_b^a x^{'b} - 2\Omega^{ab}p_b^{'} + \tag{15}$$
$$\int_0^\pi d\sigma^{'}[\omega_b^a x^{'a} + \frac{\partial\omega^{ac}}{\partial x^b}x^{'b}p_c + \frac{\partial\omega^{ac}}{\partial p_b}p_b^{'}p_c]\epsilon(\sigma^{'} - \sigma)$$
$$+(\frac{\partial\Phi^{ac}}{\partial x^b}x^{'b} + \frac{\partial\Phi^{ac}}{\partial p_b}p_b^{'})p_c^{'} -$$
$$-(\frac{\partial\Phi_c^a}{\partial x^b}x^{'b} + \frac{\partial\Phi_c^a}{\partial p_b}p_b^{'})x^{'c} = g^{ab}p_b - \alpha g^{ab}B_{bc}x^{'c}$$

$$-2\omega_{ab}x^b - 2\Phi_{ab}x^{''b} + 2\Omega_{ab}x^{'b} + \tag{16}$$
$$+4\omega_a^b p_b + 2\Phi_a^b p_b^{''} + 2\Omega_a^b p_b^{'} + \int_0^\pi d\sigma^{'}(\omega_{ab}x^{'b} -$$
$$-\frac{\partial^2\omega^{cd}}{\partial x^a\partial x^b}x^{'b}p_c p_d - \frac{\partial\omega^{bc}}{\partial x^a}p_b p_c^{'})\epsilon(\sigma^{'} - \sigma)$$
$$-(\frac{\partial\Phi_{ac}}{\partial x^b}x^{'b} + \frac{\partial\Phi_a^c}{\partial p_b}p_b^{'})x^{'c} +$$
$$+(\frac{\partial\Phi_a^c}{\partial x^b}x^{'b} + \frac{\partial\Phi_a^c}{\partial p_b}p_b^{'})p_c^{'} =$$

$$= \alpha B_{ab}g^{bc}p_c^{'} + [g_{ab} - \alpha^2 B_{ad}g^{dc}B_{cb}]x^{'b} -$$
$$-\frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}p_b p_c - \alpha\frac{\partial}{\partial x^a}(B_{bd}g^{dc})x^{'b}p_c +$$
$$+\alpha\frac{\partial}{\partial x^b}(B_{ad}g^{dc})x^{'b}p_c -$$
$$-\frac{1}{2}\frac{\partial}{\partial x^a}[g_{bc} - \alpha^2 B_{bd}g^{de}B_{eb}]x^{'b}x^{'c} +$$
$$\frac{\partial}{\partial x^b}[g_{ac} - \alpha^2 B_{ad}g^{de}B_{ec}]x^{'b}x^{'c}$$

In really, there is the list of the constraints depending on the possible choice of the unknown functions ω, Ω, Φ. In the general case, there are as the first kind constraints as the second kind constraints too. Also, it is possible to solve the constraints equations as the equations for the definition of the functions ω, Ω, Φ. We are considered last possibility and we obtained the following consistent solution of the bihamiltonity condition.

$$\Phi^{ab} = 0, \Omega^{ab} = 0, \Phi_b^a = 0, \omega^{ab} = Cg^{ab} \tag{17}$$
$$\omega_{ab} = \frac{C}{2}\frac{\partial^2 g^{cd}}{\partial x^a\partial x^b}p_c p_d,\ \omega_b^a = -C\frac{\partial g^{ac}}{\partial x^b}p_c,$$

$$\Phi_{ab} = -\frac{1}{2}[g_{ab} - \alpha^2 B_{ac} g^{cd} B_{db}],$$

$$\Omega_{ab} = \frac{1}{2}(\frac{\partial\Phi_{bc}}{\partial x^a} - \frac{\partial\Phi_{ac}}{\partial x^b})x'^c -$$

$$-(\frac{\partial\Omega^c_b}{\partial x^a} - \frac{\partial\Omega^c_a}{\partial x^b})p_c, \frac{\partial\omega^{ab}}{\partial p_c} = 0,$$

$$2\Omega^a_b = -\alpha g^{ac} B_{ca}, \frac{\partial g^{ab}}{\partial x^c} x^c = n g^{ab}, C = \frac{1}{2(n+2)}$$

The metric tensor $g^{ab}(x)$ is the homogeneous function of x^a order n and C is arbitrary constant. In the difference of the (PB) of the hydrodynamical type, we are needed to introduce the separate (PB) for the coordinates of the Minkowski space and for the momenta because, the gravity field is not depend of the momenta. Although, this difference is vanished under the such constraint as $f(x^a, p_a) \approx 0$. One can see, that the main term in the (PB) with the metric tensor is the term with the step function $\epsilon(\sigma - \sigma')$. The functions ω^b_a, Ωab are proportional to the connection and to the torsion. The function ω_{ab} is proportional to the curvature and the product of the connections. The dual (PB) for the phase space coordinates are

$$\{x^a(\sigma), x^b(\sigma')\}_2 = [\omega^{ab}(\sigma) + \omega^{ab}(\sigma')]\epsilon(\sigma' - \sigma)$$

$$\begin{aligned}\{p_a(\sigma), p_b(\sigma')\}_2 = [\omega_{ab}(\sigma) + \omega_{ab}(\sigma')]\epsilon(\sigma' - \sigma) + \\ [\Phi_{ab}(\sigma) + \Phi_{ab}(\sigma')]\frac{\partial}{\partial\sigma'}\delta(\sigma' - \sigma) + \\ [\Omega_{ab}(\sigma) + \Omega_{ab}(\sigma')]\delta(\sigma' - \sigma)\end{aligned} \tag{18}$$

$$\begin{aligned}\{x^a(\sigma), p_b(\sigma')\}_2 = [\omega^a_b(\sigma) + \omega^a_b(\sigma')]\epsilon(\sigma' - \sigma) \\ +[\Omega^a_b(\sigma) + \Omega^a_b(\sigma')]\delta(\sigma' - \sigma)\end{aligned}$$

$$\begin{aligned}\{p_a(\sigma), x^b(\sigma')\}_2 = [\omega^a_b(\sigma) + \omega^a_b(\sigma')]\epsilon(\sigma' - \sigma) \\ +[\Omega^a_b(\sigma) + \Omega^a_b(\sigma')]\delta(\sigma' - \sigma)\end{aligned}$$

The functions $\omega^{ab}(x), \omega_{ab}(x), \Phi_{ab}(x), \Omega_{ab}(x)$, $\omega^a_b(x), \Omega^a_b(x)$ is defined in (17). It is rather easy to prove the Jacobi identities for the local part of the dual (PB) $\{,\}_2$. It does not understand, how to prove the Jacobi identities for the nonlocal part of it. The principal term of the Jacobi identities with the step functions only is the term with the structure function $\omega^{ab}(x)$.

$$[\frac{\partial g^{ab}(\sigma)}{\partial x^d}[g^{dc}(\sigma) + g^{dc}(\sigma'')] - \tag{19}$$

$$-\frac{\partial g^{ac}(\sigma)}{\partial x^d}[g^{db}(\sigma) + g^{db}(\sigma')]]\epsilon(\sigma' - \sigma)\epsilon(\sigma'' - \sigma) +$$

$$+[\frac{\partial g^{cb}(\sigma)}{\partial x^d}[g^{da}(\sigma') + g^{da}(\sigma)] -$$

$$-\frac{\partial g^{ab}(\sigma)}{\partial x^d}[g^{dc}(\sigma') + g^{dc}(\sigma'')]]\epsilon(\sigma - \sigma')\epsilon(\sigma'' - \sigma') +$$

$$[\frac{\partial g^{ac}(\sigma'')}{\partial x^d}[g^{db}(\sigma'') + g^{db}(\sigma')] - \frac{\partial g^{cb}(\sigma'')}{\partial x^d}[g^{da}(\sigma'') + g^{da}(\sigma)]]\epsilon(\sigma - \sigma'')\epsilon(\sigma' - \sigma'') = 0$$

It is possible to reduce this condition to the unique equation

$$[\frac{\partial g^{ab}(\sigma)}{\partial x^d}[g^{dc}(\sigma) + g^{dc}(\sigma'')] - \frac{\partial g^{ac}(\sigma)}{\partial x^d}[g^{db}(\sigma) + g^{db}(\sigma')]]\epsilon(\sigma' - \sigma)\epsilon(\sigma'' - \sigma) = 0 \tag{20}$$

One of the possible way of the solution of this problem is the consideration of the metric tensor on the phase space to count the contribution of the structure functions Ω and Φ to this expression. The second possible way is the consideration of the second kind constraints from the list of the constraints (15),(17), instead of the solution (17). It is necessary to introduce this constraints to the initial model with the hamiltonian H_1, the (PB) $\{,\}_1$ and to obtain the bihamiltonity condition in this case. As we will see later on the some examples, the second kind constraints of the $f(x^a, p_a) \approx 0$ type can lead to the nonlocal Dirac bracket with the step function from the one side, and they can introduce the dependence of the metric tensor of the momenta on the solutions of this constraints from the other side. At present, this problems are under the consideration.

3 Constant background fields

The bihamiltonity condition (15),(17) is reduced to the following constraints on the phase space

$$\begin{aligned} &-2\omega^a_b x^b + 4\omega^{ab} p_b + 2\Phi^{ab} p''_b - 2\Phi^a_b x''^b + \\ &+2\Omega^a_b x'^b = g^{ab} p_b - \alpha g^{ab} B_{bc} x'^c,\ \Omega^{ab} = 0 \\ &-4\omega_{ab} x^b - 2\Phi_{ab} x''^b + 2\Omega_{ab} x'^b + 4\omega^b_a p_b + \\ &+2\Phi^b_a p''_b + 2\Omega^b_a p'_b = \alpha B_{ab} g^{bc} p'_c + \\ &+[g_{ab} - \alpha^2 B_{ac} g^{cd} B_{db}] x''^b \end{aligned} \tag{21}$$

There is the unique solution without constraints

$$\omega^{ab} = \frac{1}{4} g^{ab},\ 2\Omega^a_b = -\alpha g^{ac} B_{cb} = \alpha B_{bc} g^{ca},$$
$$-2\Phi_{ab} = g_{ab} - \alpha^2 B_{ac} g^{cd} B_{db} \tag{22}$$

The rest structure functions are equal zero. In this section we are supplemented the bihamiltonity condition (11) by the mirror transformations of the integrals of motion.

$$\dot{x}^a = \{x^a, \int_0^\pi h_1 d\sigma'\}_1 = \{x^a, \int_0^\pi \pm h_2 d\sigma'\}_{\pm 2} \tag{23}$$

The dual (PB) are

$$\begin{aligned} \{x^a(\sigma), x^b(\sigma')\}_{\pm 2} &= \pm\frac{1}{2} g^{ab} \epsilon(\sigma' - \sigma) \\ \{x^a(\sigma), p_b(\sigma')\}_{\pm 2} &= \mp\alpha g^{ac} B_{cb} \delta(\sigma' - \sigma) \\ \{p_a(\sigma), p_b(\sigma')\}_{\pm 2} &= \\ &= \mp[g_{ab} - \alpha^2 B_{ac} g^{cd} B_{db}] \frac{\partial}{\partial\sigma'} \delta(\sigma' - \sigma) \end{aligned} \tag{24}$$

The dual dynamical system

$$\dot{x}^a = \{x^a, \pm H_2\}_1 = \{x^a, H_1\}_{\pm 2} \tag{25}$$

is the left(right) chiral string

$$\dot{x}^a = \pm x'^a, \ \dot{p}_a = \pm p'_a \tag{26}$$

In the terms of the Virasoro operators

$$L_k = \frac{1}{4\pi}\int_0^\pi (h_1 + h_2)e^{ik\sigma}d\sigma \tag{27}$$
$$\bar{L}_k = \frac{1}{4\pi}\int_0^\pi (h_1 - h_2)e^{ik\sigma}d\sigma$$

the first kind constraints form the $Vir \oplus Vir$ algebra under the (PB) $\{,\}_1$.

$$\begin{aligned} \{L_n, L_m\}_1 &= -i(n-m)L_{n+m} \\ \{\bar{L}_n, \bar{L}_m\}_1 &= -i(n-m)\bar{L}_{n+m} \\ \{L_n, \bar{L}_m\}_1 &= 0 \end{aligned} \tag{28}$$

The dual set of the integrals of motion is obtained from initial it by the mirror transformations

$$H_1 \to \pm H_2, L_0 \to \pm L_o, \bar{L}_0 \to \mp \bar{L}_0, \tau \to \sigma \tag{29}$$

and by the contraction of the first kind constraints algebra $L_n = 0$, or $\bar{L}_n = 0$, $n \neq 0$.

3.1 Second kind constraints

Another way to obtain the dual brackets is the imposition of the second kind constraints on the initial dynamical system, by such manner, that $F_i = F_k$ for $i \neq k, i, k = 1, 2, ...$ on the constraints surface $f(x^a, p_a) = 0$. Let us consider the closed string model with $B_{ab} = 0$ for simplicity.

$$\begin{aligned} h_1 &= \frac{1}{2}g^{ab}p_a p_b + \frac{1}{2}g_{ab}x'^a x'^b, \\ h_2 &= p_a x'^a \end{aligned} \tag{30}$$

The constraints $f_a^{(-)}(x,p) = p_a - g_{ab}x'^b \approx 0$ or $f_a^{(+)} = p_a + g_{ab}x'^b \approx 0$(do not simultaneously) are the second kind constraints.

$$\begin{aligned} \{f_a^{(\pm)}(\sigma), f_b^{(\pm)}(\sigma')\}_1 &= C_{ab}^{(\pm)}(\sigma - \sigma') = \\ &= \pm 2g_{ab}\frac{\partial}{\sigma'}\delta(\sigma' - \sigma) \end{aligned} \tag{31}$$

The inverse matrix $(C^{(\pm)})^{-1}$ has following form

$$C^{(\pm)ab}(\sigma - \sigma') = \pm\frac{1}{4}g^{ab}\epsilon(\sigma' - \sigma) \tag{32}$$

There is only one set of the constraints, because consistency condition

$$\{f^{(\pm)}(\sigma), H_1\}_1 = f^{'(\pm)}(\sigma) \approx 0, ...$$
$$\{f^{(\pm)(n)}(\sigma), H_1\}_1 = f^{(\pm)(n+1)}(\sigma) \approx 0 \tag{33}$$

is not produce the new sets of constraints. By using the standard definition of the Dirac bracket, we are obtained following Dirac brackets for the phase space coordinates.

$$\{x^a(\sigma), x^b(\sigma')\}_D = \pm\frac{1}{4}g^{ab}\epsilon(\sigma' - \sigma), \tag{34}$$
$$\{p_a(\sigma), p_b(\sigma')\}_D = \mp\frac{1}{2}g_{ab}\frac{\partial}{\sigma'}\delta(\sigma' - \sigma),$$
$$\{x^a(\sigma), p_b(\sigma')\}_D = \frac{1}{2}\delta^a_b\delta(\sigma' - \sigma)$$

The equations of motion under the hamiltonians $H_1 = h_1, H_2 = h_2$ and Dirac bracket

$$\dot{x}^a = \{x^a, H_1\}_D = \{x^a, H_2\}_D = g^{ab}p_b = \pm x'^a \tag{35}$$
$$\dot{p}_a = \{p_a, H_1\}_1 = \{p_a, H_2\}_D = g_{ab}x'^b = \pm p'_a$$

are coincide on the constraints surface. The dual brackets $\{,\}_{\pm 2}$ are coincide with the Dirac brackets also. The contraction of the algebra of the first kind constraints means that the integrals of motion $H_1 = H_2$ are coincide on the constraints surface too.

3.2 B-chiral string

Let us consider the following constraint from the list (24)

$$\varphi_a = p_a + \beta B_{ab}x'^b \approx 0 \tag{36}$$

The consistency condition

$$\{\varphi_a, H_1\}_1 = (\alpha + \beta)B_{ab}g^{bc}\varphi'_c +$$
$$+[g_{ab} - (\alpha + \beta)^2 B_{ac}g^{cd}B_{db}]x''^b \approx 0 \tag{37}$$

show, that under the additional condition on the B-field

$$g_{ab} = (\alpha + \beta)^2 B_{ac}g^{cd}B_{db}, \tag{38}$$

the constraints $\varphi_a \approx 0$ are first kind constraints. The motion equations are

$$\dot{x}^a = -(\alpha + \beta)g^{ab}B_{bc}x'^c \tag{39}$$
$$\dot{p}_a = -(\alpha + \beta)B_{ab}g^{bc}p'_c, \; \ddot{x}^a = x''^a$$

This model is the bihamiltonian model under (PB) (26) also. The B-chiral string model is dual to the chiral model also.

4 Two particles discrete string

In this section we are considered two particles discrete closed string model, as two relativistic particles model, to show the difference between the Dirac brackets quantization and the Gupta-Bleuler quantization methods. Two and three pieces discrete string in Gupta-Bleuler method of quantization was considered in the paper [7] and it is described by the following constraints

$$\begin{aligned} h = \frac{1}{4}(p^2 + q^2) + \omega_0^2 r^2 \approx 0, \\ f_1 = pq \approx 0, f_3 = qr \approx 0, \\ f_2 = pr \approx 0, f_4 = \frac{1}{4}q^2 - \omega_0^2 r^2 \end{aligned} \tag{40}$$

This model is the two relativistic particles system with the oscillator interaction. The constraints (40) are two particles discrete analog of the Virasoro constraints and p, q, r are the collective variables $p^a = p_2^a + p_1^a, q^a = p_2^a - p_1^a$, $r^a = x_2^a - x_1^a$. Under the hamiltonian $H = h$ and the canonical (PB) $\{r^a, q^b\} = 2\eta^{ab}$, the constraints f_i are the second kind constraints.

$$\begin{aligned} \{f_1, f_2\} = 2p^2 \neq 0, \\ \{f_3, f_4\} = q^2 + 4\omega_0^2 r^2 \neq 0 \end{aligned} \tag{41}$$

The string coordinates are satisfied to following Dirac brackets.

$$\begin{aligned} \{r_a, r_b\}_D = \frac{r_b q_a - r_a q_b}{q^2 + \omega_0^2 r^2}, \{q_a, q_b\}_D = 4\omega_0 \{r_a, r_b\}, \\ \{r_a, q_b\}_d = 2(\eta_{ab} - \frac{p_a p_b}{p^2} - -\frac{q_a q_b + 4\omega_0^2 r_a r_b}{q^2 + 4\omega_0^2 r^2}) \end{aligned} \tag{42}$$

In the terms of the amplitudes $a_a^{(+)}, a_a$ of the equations of motion solutions

$$\begin{aligned} r_a = a_a e^{i\omega_0 \tau} + a_a^{(+)} e^{-i\omega_0 \tau}, \\ q_a = 2i\omega_0 (a_a e^{i\omega_0 \tau} - a_a^{(+)} e^{-i\omega_0 \tau}) \end{aligned} \tag{43}$$

they have form

$$\begin{aligned} \{r_a, r_b\}_D = \frac{i(a_b^{(+)} a_a - a_a^{(+)} a_b)}{2\omega_0 a_k^{(+)} a_k}, \\ \{q_a, q_b\}_D = 2\omega_0 \{r_a, r_b\}, \\ \{r_a, q_b\}_D = 2(\eta_{ab} - \frac{p_a p_b}{p^2} - \frac{a_a^{(+)} a_b + a_b^{(+)} a_a}{2a_k^{(+)} a_k}) \end{aligned} \tag{44}$$

This Dirac brackets is possible to solve in the terms of the variables $a, a^{(+)}$.

$$\{a_a, a_b^{(+)}\}_D = \frac{i}{2\omega_0}(\eta_{ab} - \frac{p_a p_b}{p^2} - \frac{i a_a^{(+)} a_b}{2\omega_0 a_k^{(+)} a_k}) \tag{45}$$

The hamiltonian and the linear combinations of the constraints $f_1, .f_4$ have following form.

$$H = \frac{1}{4}p^2 + 4\omega_0^2 a_k^{(+)} a_k, \tag{46}$$

$$f_1 = p_k a^k, f_2 = p_k a_k^{(+)}, f_3 = a_k a^k, f_4 = a_k^{(+)} a^{(+)k}$$

Under the quantization $[,] \to i\{,\}_D$, $a_k, a_k^{(+)} \to$ operators $a_k, a_k^{(+)}$, the commutation relation is

$$[a_k, a_l^{(+)}] = -\frac{1}{2\omega_0}(\eta_{kl} - \frac{p_k p_l}{p^2}) + \\ + \frac{1}{2\omega_0}(a^{(+)i} a_i)^{-1} a_k^{(+)} a_l \tag{47}$$

Last term of the commutation relation has transposed Lorentz indices k, l. The wave function

$$|\Psi_n(p)> = a_{k_1}^{(+)}, a_{k_2}^{(+)} ... a_{k_n}^{(+)} \Psi_{k_1 k_2 ... k_n}(p)|0> \tag{48}$$

of the physical states must to be the own function of the hamiltonian H on the constraints surface. Let us consider the two particles excited state, for example.

$$H|\Psi_2(p)> = (\frac{1}{4}p^2 - 4\omega_0^2)\Psi_{k_1 k_2}(p) a_{k_1}^{(+)} a_{k_2}^{(+)} |0> \tag{49}$$

+ terms, which are proportional to the expressions

$$p^{k_1}\Psi_{k_1 k_2}(p), \eta^{k_1 k_2}\Psi_{k_1 k_2}(p), a_l^{(+)} a^{(+)l} \Psi_{k_1 k_2}(p)$$

Consequently, we must to impose additional conditions on the wave function $p^k \Psi_{kl} = 0$, $\eta^{kl}\Psi_{kl} = 0$ to satisfy the request about the own function. Last term $a_i^{(+)} a^{(+)i} \Psi_{kl}$ is vanished on the constraints surface. In contrast to the nonlinear and to the nonlocal Dirac bracket (45), we have the canonical (PB) and two first kind constraints $p_k a^k \approx 0$,$a_k a^k \approx 0$ in the Gupta-Bleuler method of the quantization. The first kind constraints H, f_1, f_3 are imposed on the vector states and they are led to the following equations on the wave function.

$$(p^2 - 16\omega_0^2)\Psi_{k_1 k_2 .. k_n}(p) = 0, \\ p^k \Psi_{k k_2 .. k_n}(p) = 0, \eta^{kl}\Psi_{kl .. k_n}(p) = 0 \tag{50}$$

5 Relativistic particle in the constant electromagnetic field

The relativistic particle in the constant background electromagnetic field is described by the Hamiltonian

$$H = \frac{1}{2}[(p_a + i\beta B_{ab} x_b)^2 + m^2] \tag{51}$$

The electromagnetic field is $A_a(x) = -2F_{ab}x_b$=$-B_{ab}x_b$. The simplest constraint from (21) is

$$\varphi_a = p_a + i\alpha B_{ab} x_b \approx 0 \tag{52}$$

The consistency condition

$$\{\varphi_a, H\} = i\alpha(\alpha+\beta)B_{ab}\varphi_b \tag{53}$$

shows that there is a unique set of constraints if $\alpha + \beta = 0$. They are the second class constraints $\{\varphi_a, \varphi_b\} = 2i\alpha B_{ab}$. There is the following algebra of the phase space coordinates under the Dirac bracket

$$\{x_a, x_b\}_D = \frac{-i}{2\alpha}(B^{-1})_{ab}, \{x_a, p_b\}_D = \frac{1}{2}\eta_{ab}$$
$$\{p_a, p_b\}_D = \frac{-i\alpha}{2}B_{ab} \tag{54}$$

The motion equation under the Dirac bracket

$$\dot{x}_a + 2i\alpha B_{ab}x_b = 0,\ \dot{p}_a + 2i\alpha B_{ab}x_b = 0 \tag{55}$$

has the solution $x_a(\tau) = \{e^{-2i\alpha B\tau}\}_{ab}x_b(0)$. The quantization of the Dirac bracket results in the following commutation relations.

$$[x_a, x_b] = \frac{1}{2\alpha}(B^{-1})_{ab}, [p_a, p_b] = \frac{\alpha}{2}B_{ab}$$
$$[x_a, p_b] = \frac{i}{2}\eta_{ab} \tag{56}$$

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Towards Higher-N Superextensions of Born-Infeld Theory

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Abstract

We give a brief account of supersymmetric Born-Infeld theories with extended supersymmetry, including those with partially broken supersymmetry. Some latest developments in this area are presented. One of them is $N = 3$ supersymmetric Born-Infeld theory which admits a natural off-shell formulation in $N = 3$ harmonic superspace.

1 Introduction

The main source of current interest in superextensions of the Born-Infeld (or Dirac-Born-Infeld) action is the fact that these super BI actions constitute an essential part of the worldvolume actions of Dp-branes (see [1] and refs. therein). It is tempting to formulate super BI actions in a manifestly supersymmetric way using an off-shell superfield approach. In this approach the worldvolume SUSY is linearly realized, as opposed to the approach proceeding from a gauge-fixed form of the Green-Schwarz Dp-actions [2], in which all SUSYs are nonlinear and non-manifest. As was shown in [3], the $N = 1, d = 4$ super BI theory first formulated in [4] in terms of superfields can be re-interpreted as a theory of partial spontaneous breaking of global $N = 2$ SUSY (PBGS) down to $N = 1$ SUSY, with vector $N = 1$ multiplet as the relevant Goldstone multiplet. Thus the PBGS approach provides a powerful way of deriving super BI actions with extra hidden nonlinearly realized supersymmetries.

This contribution is a brief survey of super BI theories of this kind known to date. A short description of the recently constructed $N = 3$ supersymmetric BI theory [5] is also given.

2 Super BI Theories as PBGS Systems

2.1 $N = 2 \rightarrow N = 1, d = 3$ *BI theory.* The basic object of this simplest super BI theory [6] is the $N = 1, d = 3$ spinor gauge superfield strength $\mu^a(x^{bc}, \theta^d)$, $a, b, c, d = 1, 2$, subjected to the constraint

$$D_a \mu^a = 0 \ , \tag{1}$$

where $D_a = \partial/\partial\theta^a + 1/2\,\theta^b\partial_{ab}$. A nonlinear extension of the free $N = 1, d = 3$ Maxwell action, with the $d = 3$ BI action as the bosonic core, reads

$$\begin{aligned} S &\sim \int d^3x d^2\theta\, w \ , \\ w &\equiv \frac{1}{2}\frac{\mu^2}{1+\sqrt{1-D^2\mu^2}} \ . \end{aligned} \tag{2}$$

Besides being manifestly $N = 1$ supersymmetric, this action possesses the hidden nonlinear SUSY

$$\delta_\eta \mu_a = \eta_a \left(1 - D^2 w\right) + \eta^b \partial_{ab} w \ . \tag{3}$$

The superfield μ^a is the Goldstone superfield associated with the spontaneous breaking of the $N = 2, d = 3$ SUSY,

$$\{Q_a, Q_b\} = \{S_a, S_b\} = P_{ab} \ , \quad \{Q_a, S_b\} = 0 \ , \tag{4}$$

down to the $N = 1$ one generated by Q_a, P_{ab}. The Goldstone fermion ψ^a associated with the spontaneously broken generators S_a in the standard nonlinear realizations approach (with making use of the exponential parametrization of the Poincaré supergroup elements) is related to μ^a by

$$\psi^a = \frac{\mu^a}{1 - D^2 w} \ . \tag{5}$$

In components, (2) gives rise to a static-gauge form of the space-filling D2-brane action.

Now let us briefly outline how the super BI action (2) can be deduced. The derivation follows a generic method [7, 8] which is also applicable to other PBGS cases [9, 10].

The starting point is the appropriate linear realization of the considered PBGS pattern. It is obtained by embedding the $N = 1, d = 3$ Maxwell superfield strength μ_a into a linear $N = 2, d = 3$ multiplet. The latter should have such a transformation law under the S-supersymmetry that μ_a transforms with an inhomogeneous term $\sim \eta_a$ and so can be interpreted as the Goldstone fermion of linear realization.

The appropriate $N = 2, d = 3$ supermultiplet was proposed in [11] as a deformation of the $N = 2, d = 3$ Maxwell multiplet. The superfield constraints defining this deformed multiplet can be taken in the form [10]

$$\begin{aligned} &(\text{a}) \quad \left[(D)^2 - (D^\zeta)^2\right] W = -2i \ , \\ &(\text{b}) \quad D^a D^\zeta_a W = 0 \ , \end{aligned} \tag{6}$$

where $W(x, \theta, \zeta)$ is a real $N = 2, d = 3$ superfield (ζ_a is an extra Grassmann coordinate). The standard S-supersymmetry transformation law of W,

$$\delta_\eta W = -\eta^a \left(\frac{\partial}{\partial \zeta^a} - \frac{1}{2}\zeta^b \partial_{ab}\right) W \ , \tag{7}$$

implies the following transformation laws for the irreducible $N = 1$ superfield components of $W(x, \theta, \zeta)$, $\mu_a \equiv -iD^\zeta_a W|_{\zeta=0}$ and $w \equiv i/2\, W|_{\zeta=0}$,

$$\begin{aligned} &(\text{a}) \quad \delta_\eta \mu_a = \eta_a \left(1 - D^2 w\right) + \eta^b \partial_{ab} w \ , \\ &(\text{b}) \quad \delta_\eta w = \frac{1}{2}\, \eta^a \mu_a \ . \end{aligned} \tag{8}$$

It is easy to check that eq. (8a) is consistent with the Bianchi identity (1) (which is none other than eq. (6b)).

The additional homogeneously transforming $N = 1$ superfield $w(x, \theta)$ can be expressed in terms of the Goldstone-Maxwell one μ_a by enforcing nonlinear constraints the precise form of which is dictated by the generic method of refs. [7]-[10] applied to the given system.

As the first step, one defines superfields $\tilde{\mu}_a$ and $\tilde{w}$ as finite S-supersymmetry transforms of μ_a and w, with the transformation parameter η^a being replaced by $-\psi^a(x,\theta)$:

$$
\begin{aligned}
\tilde{\mu}_a &= \mu_a - \psi_a\left(1 - D^2 w\right) - \psi^b \partial_{ab} w - \frac{1}{4}\psi^2 \partial_{ab}\mu^b , \\
\tilde{w} &= w - \frac{1}{2}\psi^a \mu_a + \frac{1}{4}\psi^2\left(1 - D^2 w\right) .
\end{aligned} \tag{9}
$$

These quantities homogeneously transform with respect to the whole $N=2, d=3$ SUSY. Therefore, one can covariantly equate them to zero,

$$
\tilde{\mu}_a = \tilde{w} = 0 . \tag{10}
$$

From these covariant constraints one derives the equivalence relation between ψ^a and μ^a (5), as well as the relation

$$
w = \frac{1}{4}\frac{\mu^2}{1 - D^2 w} . \tag{11}
$$

These are precisely the equations postulated in [6] (up to a rescaling of w). They can be used to express w in terms of either ψ^a or μ^a

$$
w = \frac{1}{4}\frac{\psi^2}{1 + \frac{1}{4}D^2\psi^2} = \frac{1}{2}\frac{\mu^2}{1 + \sqrt{1 - D^2\mu^2}} . \tag{12}
$$

This composite superfield is just the Goldstone superfield Lagrangian density in the action (2).

The same superfield D2-brane action can be written in a manifestly $N=2$ supersymmetric form as an integral over the whole $N=2$ superspace, with either W^2 or $N=2, d=3$ Fayet-Iliopoulos term as the Lagrangian densities.

2.2 $N=2 \to N=1, d=4$ BI theory. This case [3] corresponds to the partial breaking of $N=2, d=4$ SUSY,

$$
\begin{aligned}
&\left\{Q_\alpha, \bar{Q}_{\dot\alpha}\right\} = \left\{S_\alpha, \bar{S}_{\dot\alpha}\right\} = 2P_{\alpha\dot\alpha} , \\
&\left\{Q_\alpha, S_\beta\right\} = 0 ,
\end{aligned} \tag{13}
$$

down to the $N=1, d=4 \propto (Q, \bar{Q}, P)$, with a vector $N=1, d=4$ multiplet as the Goldstone one. In terms of the $N=1, d=4$ Maxwell superfield strength $W^\alpha(z), \bar{W}^{\dot\alpha}(z)$ $(z = (x^{\alpha\dot\alpha}, \theta^\alpha, \bar\theta^{\dot\alpha}))$,

$$
\bar{D}^{\dot\alpha} W^\beta = 0 , \; D^\alpha W_\alpha + \bar{D}_{\dot\alpha}\bar{W}^{\dot\alpha} = 0 , \tag{14}
$$

the Goldstone superfield action ($N=1, d=4$ BI action) reads

$$
\begin{aligned}
&S \sim \int d^4x_L d^2\theta \, \phi , \\
&\phi = W^2 + \frac{1}{2}\bar{D}^2 \frac{W^2\bar{W}^2}{1 - \frac{1}{2}A + \sqrt{1 - A + \frac{1}{4}B^2}} , \\
&A \equiv \frac{1}{2}\left(D^2W^2 + \bar{D}^2\bar{W}^2\right) , \\
&B \equiv \frac{1}{2}\left(D^2W^2 - \bar{D}^2\bar{W}^2\right) .
\end{aligned} \tag{15}
$$

The equivalence relation between the canonical nonlinear realization Goldstone spinor superfields $\psi^\alpha, \bar{\psi}^{\dot\alpha}$ and $W^\alpha, \bar{W}^{\dot\alpha}$ is [10]

$$W^\alpha = \psi^\alpha\left(1 - \frac{1}{4}\bar{D}^2\bar{\phi}\right) + \dots , \tag{16}$$

where ... stand for terms with x-derivatives. The nonlinear SUSY acts as

$$\delta_\eta W_\alpha = \eta_\alpha\left(1 - \frac{1}{4}\bar{D}^2\bar{\phi}\right) - i\bar{\eta}^{\dot\alpha}\partial_{\alpha\dot\alpha}\phi \ . \tag{17}$$

In components, (15) describes the space-filling D3-superbrane in a static gauge.

The action (15) can be deduced [10] from the appropriate linear realization of the PBGS pattern $N=2 \to N=1, d=4$, following the procedure similar to that applied in the $N=2 \to N=1, d=3$ case above. The starting point is $N=2, d=4$ Goldstone-Maxwell multiplet in the $N=2$ superspace $z = (x^{\alpha\dot\alpha}, \theta_i^\alpha, \bar{\theta}^{\dot\alpha i})$. It is defined by the following deformation [12] of the standard $N=2$ Maxwell superfield strength constraints

$$\begin{aligned} &(\mathrm{a})\ \ D^{ik}\mathcal{W} - \bar{D}^{ik}\bar{\mathcal{W}} = iM^{(ik)} \ , \\ &(\mathrm{b})\ \ D^i_\alpha\bar{\mathcal{W}} = \bar{D}_{\dot\alpha i}\mathcal{W} = 0 \ . \end{aligned} \tag{18}$$

Here,

$$D^i_\alpha = \frac{\partial}{\partial\theta^\alpha_i} - i\bar{\theta}^{\dot\alpha i}\partial_{\alpha\dot\alpha} \ , \ \bar{D}_{\dot\alpha i} = -\frac{\partial}{\partial\bar{\theta}^{\dot\alpha i}} + i\theta^\alpha_i\partial_{\alpha\dot\alpha} \ ,$$
$$D^{ik} = D^{\alpha i}D^k_\alpha \ , \ \bar{D}^{ik} = \bar{D}^i_{\dot\alpha}\bar{D}^{\dot\alpha k} \ ,$$

and $M^{ik} = M^{ki}$ is a triplet of constants which explicitly break the automorphism $SU(2)_A$ of $N=2$ supersymmetry down to $U(1)_A$ and satisfy the pseudo-reality condition

$$\overline{(M^{ik})} = \epsilon_{in}\epsilon_{km}M^{nm} \ .$$

Now we pass to the $N=1$ superfield notation by relabelling the Grassmann coordinates and spinor derivatives as

$$\theta^\alpha_1 \equiv \theta^\alpha \ , \ \theta^\alpha_2 \equiv \zeta^\alpha, \ D^1_\alpha \equiv D_\alpha, \ D^2_\alpha \equiv D^\zeta_\alpha \ .$$

In order to have the off-shell S-supersymmetry (acting as ζ-supertranslations) spontaneously broken while the Q-supersymmetry unbroken, we choose the following frame with respect to the explicitly broken $SU(2)_A$

$$M^{12} = 0 \ , \quad M^{11} = M^{22} = m \ , \tag{19}$$

where m is a real constant. Like in the case of D2-brane it is fixed up to rescaling of W. A convenient choice is

$$m = -2 \ .$$

It will be also convenient to choose a basis in $N=2$ superspace where the chirality with respect to the variable ζ^α is manifest

$$\bar{D}^\zeta_{\dot\alpha} = -\frac{\partial}{\partial\bar{\zeta}^{\dot\alpha}} \ , \quad D^\zeta_\alpha = \frac{\partial}{\partial\zeta^\alpha} - 2i\bar{\zeta}^{\dot\alpha}\partial_{\alpha\dot\alpha} \ . \tag{20}$$

In this basis, constraints (18) imply the following structure of the superfield $\mathcal{W}(x,\theta,\zeta)$

$$\mathcal{W} = -\frac{i}{2}\phi + i\zeta^\alpha W_\alpha - i\frac{1}{2}\zeta^2\left(1 - \frac{1}{4}\bar{D}^2\bar{\phi}\right),$$

where ϕ and W_α are chiral $N=1$ superfields

$$\bar{D}_{\dot\alpha}\phi = \bar{D}_{\dot\alpha}W_\alpha = 0\ , \tag{21}$$

and the fermionic superfield W_α obeys the $N=1$ Maxwell superfield strength constraint (14).

The S-supersymmetry transformation of the $N=2$ superfield $\mathcal{W}$

$$\delta_\eta \mathcal{W} = -\left[\eta^\alpha \frac{\partial}{\partial\zeta^\alpha} + \bar{\eta}^{\dot\alpha}\left(\frac{\partial}{\partial\bar{\zeta}^{\dot\alpha}} + 2i\zeta^\alpha\partial_{\alpha\dot\alpha}\right)\right]\mathcal{W}$$

implies the following ones for its $N=1$ superfield components ϕ and W_α

$$\begin{aligned} &\delta_\eta\phi = 2(\eta W)\ , \quad \delta_\eta\bar{\phi} = 2(\bar{W}\bar{\eta})\ , \\ &\delta_\eta W_\alpha = \eta_\alpha\left(1 - \frac{1}{4}\bar{D}^2\bar{\phi}\right) - i\bar{\eta}^{\dot\alpha}\partial_{\alpha\dot\alpha}\phi\ , \\ &\delta_\eta \bar{W}_{\dot\alpha} = \overline{(\delta_\eta W_\alpha)}\ . \end{aligned} \tag{22}$$

The superfield W_α shows up an inhomogeneous shift $\sim \eta_\alpha$ in its transformation, so it is the Goldstone fermion of the linear realization of the considered $N=2 \to N=1,\ d=4$ PBGS pattern (Goldstone-Maxwell $N=1$ superfield).

After this one can start the algorithmic procedure of passing to the relevant nonlinear realization. It goes in full analogy with the $N=2 \to N=1, d=3$ case. Firstly one constructs finite η-transformations of the superfields ϕ and W_α proceeding from the infinitesimal ones (22). As the next step, one replaces, in the transformed superfields, the transformation parameters by the original nonlinear realization Goldstone fermions, $\eta_\alpha \to -\psi_\alpha$, $\bar{\eta}_{\dot\alpha} \to -\bar{\psi}_{\dot\alpha}$. At last, one imposes the covariant constraints on the so defined superfields

$$\tilde{\phi} = \tilde{W}_\alpha = 0\ , \tag{23}$$

and obtains the appropriate relations between ϕ, W_α and ψ_α. After some algebra, one ends up with the simple relations

$$\phi = \frac{W^2}{1 - \frac{1}{4}\bar{D}^2\bar{\phi}}\ , \quad \bar{\phi} = \frac{\bar{W}^2}{1 - \frac{1}{4}D^2\phi}\ , \tag{24}$$

which, up to a rescaling of ϕ, are just those postulated in [3] and derived from the nilpotency condition in [13]. Their solution is the expression for ϕ in (15).

Note that the Goldstone superfield Lagrangian density for the $N=2 \to N=1$ PBGS in (15) can be equivalently rewritten in $N=2, d=4$ superspace in terms of the constrained $N=2$ gauge superfield strength $\mathcal{W}, \bar{\mathcal{W}}$ as the Fayet-Iliopoulos term or as the kinetic term of $N=2$ Goldstone-Maxwell multiplet.

2.3 $N=4 \to N=2, d=4$ BI theory. The $N=2, d=4$ super BI action can be constructed in terms of the $N=2$ Maxwell superfield strength $\mathcal{W}(z), \bar{\mathcal{W}}(z)$, $z = (x^{\alpha\dot\alpha}, \theta^\alpha_i, \bar{\theta}^{\dot\alpha i})$, defined by the constraints

$$\bar{D}^i_{\dot\alpha}\mathcal{W} = D^\alpha_i\bar{\mathcal{W}} = 0\ , \quad D^{ik}\mathcal{W} = \bar{D}^{ik}\bar{\mathcal{W}}\ , \tag{25}$$

$(i, k = 1, 2)$, which is the $M^{ik} = 0$ version of the constraints (18). The simplest $N = 2$ BI action was proposed in [14]. It goes into the previously given $N = 1$ BI action after the appropriate reduction, but it does not possess a second nonlinearly realized $N = 2$ SUSY. So it cannot be interpreted as the Goldstone superfield action for the PBGS option $N = 4 \to N = 2$ in $d = 4$.

A recursive procedure of restoring the correct action was developed in [15, 16]. The point of departure is the following $N = 4, d = 4$ superalgebra

$$
\begin{aligned}
&\left\{Q^i_\alpha, \bar{Q}_{\dot{\alpha}j}\right\} = \left\{S^i_\alpha, \bar{S}_{\dot{\alpha}j}\right\} = 2\delta^i_j P_{\alpha\dot{\alpha}} \,, \\
&\left\{Q^i_\alpha, S^j_\beta\right\} = 2\varepsilon^{ij}\varepsilon_{\alpha\beta} Z \,.
\end{aligned} \tag{26}
$$

The basic Goldstone superfield supporting the 1/2 breaking of this $N = 4$ SUSY is a scalar one $W, \bar{W}$ associated with the complex central charge generator $Z, \bar{Z}$. It obeys a nonlinear version of the constraints (25) and is related to $\mathcal{W}, \bar{\mathcal{W}}$ by a complicated equivalence redefinition. Up to the sixth order, the $N = 4 \to N = 2$ Goldstone-Maxwell superfield action and transformations of the central charge and second hidden SUSY (with the generators $S, \bar{S}$) read

$$
\delta\mathcal{W} = f - \frac{1}{2}\bar{D}^4(fA) + \frac{1}{4}\Box(\bar{f}\bar{A}) + \frac{1}{4i}\bar{D}^{i\dot{\alpha}}\bar{f}D^\alpha_i\,\partial_{\alpha\dot{\alpha}}\bar{A} \,, \tag{27}
$$

$$
\begin{aligned}
A &= \bar{\mathcal{W}}^2\left(1 + \frac{1}{2}D^4\mathcal{W}^2\right), \; D^4 = \frac{1}{48}D^{ik}D_{ik} \,, \\
f &= c + 2i\,\eta^{i\alpha}\theta_{i\alpha} \,,
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
S^{(6)}_{bi} &\sim \int d\zeta_L \mathcal{W}^2 + \text{c.c.} + \frac{1}{2}\int dz\{\, \mathcal{W}^2\bar{\mathcal{W}}^2\left[2 + \left(D^4\mathcal{W}^2 + \bar{D}^4\bar{\mathcal{W}}^2\right)\right] \\
&\quad - \frac{1}{9}\mathcal{W}^3\Box\bar{\mathcal{W}}^3 + O(W^8)\,\} \,,
\end{aligned} \tag{29}
$$

where dz and $d\zeta_L$ are measures of integration over the whole $N = 2$ superspace and its chiral subspace. In [16] the action was restored up to 10th order using some algorithmic iteration procedure. The full action contains the standard BI action and some disguised form of the Nambu-Goto action for two physical scalar fields. It is a gauge-fixed action of D3-superbrane in $D = 6$.

Let us say a few words on the aforementioned recursive procedure of restoring the full superfield action and its relation to the approach based on the linear realizations of PBGS exemplified in the previous two subsections.

As was proposed in [16], the linear Goldstone-Maxwell multiplet relevant to the case at hand is an infinite-dimensional linear multiplet of $N = 4$ superalgebra (26), with a non-trivial realization of the central charges $Z, \bar{Z}$. To be more precise, one embeds the $N = 2$ Maxwell superfield strength $\mathcal{W}$ into an infinite-dimensional $N = 4$ multiplet

$$
\mathcal{W},\ \bar{\mathcal{W}},\ \mathcal{A}_n,\ \bar{\mathcal{A}}_n, \qquad (n = 0, 1, 2, \ldots) \,, \tag{30}
$$

where $\mathcal{A}_n$ are chiral (otherwise unconstrained) $N = 2$ superfields,

$$
\bar{D}_{\dot{\alpha}i}\mathcal{A}_n = 0 \,, \; D^i_\alpha\bar{\mathcal{A}}_n = 0 \,. \tag{31}
$$

The following transformations

$$
\begin{aligned}
\delta\mathcal{W} &= f - \frac{1}{2}\bar{D}^4(f\bar{\mathcal{A}}_0) + \frac{1}{4}\Box(\bar{f}\mathcal{A}_0)\frac{1}{4i}\bar{D}^{i\dot{\alpha}}\bar{f}D^{\alpha}_i\partial_{\alpha\dot{\alpha}}\mathcal{A}_0 \,, \\
\delta\mathcal{A}_0 &= 2f\mathcal{W} + \frac{1}{4}\bar{f}\Box\mathcal{A}_1 + \frac{1}{4i}\bar{D}^{i\dot{\alpha}}\bar{f}D^{\alpha}_i\partial_{\alpha\dot{\alpha}}\mathcal{A}_1, \\
\delta\mathcal{A}_1 &= 2f\mathcal{A}_0 + \frac{1}{4}\bar{f}\Box\mathcal{A}_2 + \frac{1}{4i}\bar{D}^{i\dot{\alpha}}\bar{f}D^{\alpha}_i\partial_{\alpha\dot{\alpha}}\mathcal{A}_2
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
&\ldots\ldots\ldots \\
\delta\mathcal{A}_n &= 2f\mathcal{A}_{n-1} + \frac{1}{4}\bar{f}\Box\mathcal{A}_{n+1} + \frac{1}{4i}\bar{D}^{i\dot{\alpha}}\bar{f}D^{\alpha}_i\partial_{\alpha\dot{\alpha}}\mathcal{A}_n, \quad (n \geq 1)
\end{aligned} \tag{33}
$$

where the function f was defined in (28), close off shell both among themselves and with those of the manifest $N=2$ supersymmetry just according to the superalgebra (26). The central charge $(c,\bar{c})$ transformations non-trivially act on this infinite tower of $N=2$ superfields.

A good candidate for the chiral $N=2$ Lagrangian density is the superfield $\mathcal{A}_0$. Indeed, the "action"

$$
S = \int d^4x d^4\theta \mathcal{A}_0 + \int d^4x d^4\bar{\theta}\bar{\mathcal{A}}_0 \tag{34}
$$

is invariant with respect to the transformation (33) up to surface terms.

It remains to define covariant constraints which would express $\mathcal{A}_0$, $\bar{\mathcal{A}}_0$ in terms of $\mathcal{W}$, $\bar{\mathcal{W}}$, with preserving the linear representation structure (32), (33). Because of an infinite number of $N=2$ superfields $\mathcal{A}_n$, there should exist an infinite set of constraints expressing these superfields through the basic Goldstone ones $\mathcal{W}$, $\bar{\mathcal{W}}$. The procedure of deducing this set of constraints was described in [16]. The first two constraints read

$$
\begin{aligned}
\phi_0 &= \mathcal{A}_0\left(1 - \frac{1}{2}\bar{D}^4\bar{\mathcal{A}}_0\right) - \mathcal{W}^2 \\
&\quad - \sum_{k=1}\frac{(-1)^k}{2\cdot 8^k}\mathcal{A}_k\Box^k\bar{D}^4\bar{\mathcal{A}}_k = 0\,, \\
\phi_1 &= \Box\mathcal{A}_1 + 2\left(\mathcal{A}_0\Box\mathcal{W} - \mathcal{W}\Box\mathcal{A}_0\right) \\
&\quad - \sum_{k=0}\frac{(-1)^k}{2\cdot 8^k}\Big(\Box\mathcal{A}_{k+1}\Box^k\bar{D}^4\bar{\mathcal{A}}_k \\
&\quad - \mathcal{A}_{k+1}\Box^{k+1}\bar{D}^4\bar{\mathcal{A}}_k\Big) = 0\,,
\end{aligned} \tag{35}
$$

and so on.

At present we do not know how to explicitly solve this set of constraints and to find a closed expression for the Lagrangian densities $\mathcal{A}_0$, $\bar{\mathcal{A}}_0$. We can only recover the general solution by iterations. E.g.,

$$
\begin{aligned}
&\mathcal{A}_0 = \mathcal{W}^2 + \mathcal{A}_0^{(4)} + \mathcal{A}_0^{(6)} + \mathcal{A}_0^{(8)} + \ldots\,, \\
&\mathcal{A}_0^{(4)} = \frac{1}{2}\mathcal{W}^2\bar{D}^4\bar{\mathcal{W}}^2\,, \ \ldots\,.
\end{aligned} \tag{36}
$$

The action, up to the 10th order in $\mathcal{W},\bar{\mathcal{W}}$, was found in [16]. It turned out to coincide, at least up to the 8th order, with the action deduced in [17] from the requirements of self-duality and invariance under nonlinear shifts of $\mathcal{W},\bar{\mathcal{W}}$ (the $c,\bar{c}$ transformations in our notation). This is an indication that the full $N=4 \to N=2$ BI action is also self-dual like its $N=2 \to N=1$ prototype [3].

2.4 $N=8 \to N=4, d=4$ BI theory. No manifestly $N=4$ supersymmetric off-shell actions are known for $N=4, d=4$ Maxwell theory, so no such actions can be defined for the BI deformations of the latter. The best what one can hope to gain is $N=4$ BI actions with the $N=1$, $N=2$ [1] or at most $N=3$ [5] manifest off-shell SUSYs. However, it is still possible to derive the superfield equations of motion of $N=4$ BI theory, in a manifestly $N=4$ supersymmetric form and with one extra nonlinearly realized $N=4$ SUSY, within the nonlinear realizations formalism applied to the following $N=8, D=4$ superalgebra [15]:

$$
\begin{aligned}
&\left\{Q^i_\alpha, \bar{Q}_{\dot{\alpha}j}\right\} = \left\{S^i_\alpha, \bar{S}_{\dot{\alpha}j}\right\} = 2\delta^i_j P_{\alpha\dot{\alpha}}\,, \\
&\left\{Q^i_\alpha, S^j_\beta\right\} = \varepsilon_{\alpha\beta} Z^{ij} \qquad (37)\\
&\bar{Z}_{ij} = \left(Z^{ij}\right)^* = \frac{1}{2}\varepsilon_{ijkl} Z^{kl}\,. \qquad (38)
\end{aligned}
$$

The basic Goldstone superfield supporting partial breakdown of this SUSY down to $N=4, d=4$ SUSY $\propto (Q,P)$ is an $N=4$ superfield W_{ik} , $\bar{W}^{ij} = \frac{1}{2}\varepsilon^{ijkl}W_{kl}$, associated with the generator Z^{ik}. One also introduces spinor Goldstone superfields $\psi^\alpha_i, \bar{\psi}^{\dot{\alpha}i}$ associated with S^i_α and $\bar{S}_{\dot{\alpha}j}$. The covariant superfield equations of the $N=8 \to N=4$ super BI theory read

$$
\begin{aligned}
&(a) \quad \psi_{\alpha i} + \frac{2i}{3}\mathcal{D}^j_\alpha W_{ij} = 0, \\
&(b) \quad \mathcal{D}^k_\alpha W_{ij} - \frac{1}{3}\left[\delta^k_i \mathcal{D}^m_\alpha W_{mj} - (i \leftrightarrow j)\right] = 0 \qquad (39)
\end{aligned}
$$

(plus their c.c.). Here $\mathcal{D}^j_\alpha, \bar{\mathcal{D}}^{\dot{\alpha}}_j$ are the appropriate covariantization of the flat $N=4$ spinor derivatives. They nonlinearly depend on $\psi^\alpha_i, \bar{\psi}^{\dot{\alpha}}_i$ which can be covariantly expressed through derivatives of W_{kl} by eq. (39a). Eq. (39b) is a covariantization of the standard superfield constraints of the on-shell $N=4$ Maxwell theory [18], and it contains the dynamical equations for the component fields. One of them is a disguised form of the BI equations. The full set of equations is a manifestly worldvolume supersymmetric form of the equations of the gauge-fixed D3-superbrane in $D=10$, with 6 physical bosonic fields of $N=4$ Goldstone-Maxwell multiplet being transverse brane coordinates.

As was already mentioned, one cannot hope to construct an off-shell $N=4$ supersymmetric superfield action for this super BI system, since no such action exists even for the ordinary $N=4$ gauge theory. However, for the first non-trivial term in the BI action, the quartic term $\sim F^{\alpha\beta}F_{\alpha\beta}\bar{F}^{\dot{\alpha}\dot{\beta}}\bar{F}_{\dot{\alpha}\dot{\beta}}$, $N=4$ completions with $N=1$ and $N=2$ off-shell supersymmetries are known. These were given, respectively, in terms of $N=1$ superfields [1] and $N=2$ projective superfields [19]. Here we present this completion in terms of off-shell $N=2$ harmonic superfields [20].

In the harmonic superspace (HSS) description [21, 22], $N=4$ vector multiplet is represented by the standard gauge $N=2$ superfield strength $\mathcal{W}(z), \bar{\mathcal{W}}(z)$ and the analytic hypermultiplet superfield $q^{+a}(\zeta, u)$. Here $\{\zeta, u\} \equiv \{x^{\alpha\dot{\beta}}, \theta^{+\alpha}, \bar{\theta}^{+\dot{\alpha}}, u^{\pm i}\}$ are co-ordinates of an analytic subspace of $N=2$ harmonic superspace $\{z, u^{\pm i}\}$ and $u^{\pm i}, u^{+i}u^-_i = 1$, are harmonic variables parametrizing some internal 2-sphere S^2. The indices a and i are doublet indices of two mutually commuting $SU(2)$ groups, $a = 1,2; i = 1,2$. Further details can be found in [21, 22]. The free $N=4$ Maxwell theory action is given by

$$S_{free}^{N=4} = \frac{1}{8}\int d\zeta_L \mathcal{W}^2 + \text{c.c.} - \frac{1}{2}\int d\zeta^{(-4)} q^{+a} D^{++} q_a^+ \,, \tag{40}$$

where D^{++} is the analyticity-preserving harmonic derivative and $d\zeta^{(-4)}$ is the measure of integration over the analytic superspace. Besides being manifestly $N=2$ supersymmetric, the action (40) is invariant under one more $N=2$ SUSY which forms $N=4, d=4$ SUSY together with the manifest $N=2$ one. For our purposes it suffices to know only the on-shell form of transformations of this hidden SUSY:

$$\begin{aligned} &\delta\mathcal{W} = \frac{1}{2}\bar{\epsilon}^{\dot{\alpha}a}\,\bar{D}^-_{\dot{\alpha}} q_a^+ \,, \quad \delta\bar{\mathcal{W}} = \frac{1}{2}\epsilon^{\alpha a}\, D^-_{\alpha} q_a^+ \,, \\ &\delta q_a^{\pm} = \frac{1}{4}\left(\epsilon_a^{\beta} D^{\pm}_{\beta}\mathcal{W} + \bar{\epsilon}_a^{\dot{\alpha}}\bar{D}^{\pm}_{\dot{\alpha}}\bar{\mathcal{W}}\right) , \end{aligned} \tag{41}$$

where $\epsilon^{\alpha a}, \bar{\epsilon}^{\dot{\alpha}a}$ are the Grassmann transformation parameters, $D^{\pm}_{\alpha}, \bar{D}^{\pm}_{\dot{\alpha}}$ are harmonic projections of the flat $N=2$ spinor derivatives and $q^{-a} \equiv D^{--}q^{+a}$, D^{--} being the second harmonic derivative. The quartic superfield term which is invariant under the transformations (41) and yields the correct quartic term $\sim F^2\bar{F}^2$ in the component BI action (with the correct relative coefficient w.r.t. the free Maxwell action) is uniquely restored, up to terms vanishing on the free mass shell,

$$\begin{aligned} &S_4^{N=4} = \frac{1}{32}\int dz du\,\{\,\mathcal{W}^2\bar{\mathcal{W}}^2 - 4(q^+\cdot q^-) \\ &\times[\mathcal{W}\bar{\mathcal{W}} - \frac{1}{3}(q^+\cdot q^-)]\,\}, \\ &(q^+\cdot q^-) \equiv q^{+a}q_a^- \,. \end{aligned} \tag{42}$$

Here du is the measure of integration over harmonics ($\int du \cdot 1 = 1$). The problem of $N=4$-completing of higher-order terms of the $N=2$ BI action (even of its simplest version [14]) is technically very complicated because the form of the hidden on-shell $N=4$ transformations (41) is modified from order to order in superfields.

It is remarkable that in $N=3$ HSS [23] one can construct an off-shell $N=3$ superextension of the *full* BI action [5].

3 N=3 Born-Infeld Theory

3.1 Introduction. Since the $N>2$ super BI actions are extensions of the corresponding super Maxwell actions, the necessary condition of the existence of some off-shell super BI action is the existence of such an action for the relevant free super Maxwell theory. The maximally supersymmetric off-shell formulation of $N=4$ gauge theory is that with manifest $N=3$ supersymmetry. It was given in [23] in the framework of $N=3$ harmonic superspace (HSS).

In [5], starting from this formulation, we have constructed an $N=3$ superextension of the full BI action. As distinct from the previously known $N=1$ and $N=2$ super BI actions, the construction of the $N=3$ BI action is by no means a straightforward order-by-order supersymmetrization of the bosonic BI action. The main novel feature stems from the crucial property that the Grassmann-analytic gauge potentials of $N=3$ gauge theory in $N=3$ HSS [23] contain, besides the physical fields including the standard gauge potential A_m, also an infinite number of the auxiliary fields. Among them there is an independent bispinor field $H_{\alpha\beta}=H_{\beta\alpha}$. The correct bilinear Maxwell term in the component action arises only after elimination of this field by its algebraic equation of motion. The $N=3$ gauge superfield strength contains the combination $V_{\alpha\beta}=\frac{1}{4}[H_{\alpha\beta}+F_{\alpha\beta}(A)]$ of the auxiliary field and the gauge field strength.

The auxiliary component $V_{\alpha\beta}$ can be interpreted as a Legendre-type transform variable for the gauge field strength $F_{\alpha\beta}(A)$. It turns out that this specific Legendre transform of the standard bosonic BI action is determined by a real function E of the single variable $a=V^2\bar{V}^2$ where $V^2=V^{\alpha\beta}V_{\alpha\beta}$. The problem of $N=3$ supersymmetrization of the BI action is then reduced to the construction of self-interaction superfield terms of the order 4k in the auxiliary field $V_{\alpha\beta}$. All these terms can be constructed as the appropriate powers of the off-shell $N=3$ superfield strengths and their spinor derivatives in the framework of an analytic subspace of $N=3$ HSS. A generic function $E(a)$ exhausts the complete set of the $SO(2)$ self-dual nonlinear extensions of the Maxwell action, the BI one being a special representative of them. All such actions can be $N=3$ supersymmetrized off shell.

3.2 Elements of $N=3$ harmonic formalism. The fundamental objects of the abelian $N=3$ gauge theory are three harmonic gauge potentials living as unconstrained superfields on the (4+6|8)-dimensional analytic subspace $H(4{+}6|8)=\{\zeta,u\}$ of $N=3$ HSS

$$\begin{aligned}&V^1_2(\zeta,u)\ , \quad V^1_3(\zeta,u)\ , \quad V^2_3\ ,\\ &V^1_2=-(\widetilde{V^2_3})\ , \qquad V^1_3=(\widetilde{V^1_3})\ .\end{aligned} \tag{43}$$

The definition of the generalized conjugation $\sim$ preserving $N=3$ Grassmann harmonic analyticity and the precise content of the analytic coordinate set $\{\zeta,u\}$ can be found in [23, 5]. The potentials undergo abelian gauge transformations with a real analytic parameter $\lambda(\zeta,u)$:

$$\delta V^1_2=iD^1_2\lambda\ , \quad \delta V^1_3=iD^1_3\lambda\ , \quad \delta V^2_3=iD^2_3\lambda\ . \tag{44}$$

The potential V^1_3 can be consistently expressed in terms of the two remaining ones by imposing the conventional constraint

$$\hat{V}^1_3\equiv D^1_2V^2_3-D^2_3V^1_2\ . \tag{45}$$

The free $N=3$ gauge theory action has the following form:

$$S_2(V^1_2,V^2_3)=-\frac{1}{4f^2}\int d\zeta({}^{33}_{11})du[\,V^2_3D^1_3V^1_2+\frac{1}{2}(D^1_2V^2_3-D^2_3V^1_2)^2\,]\ , \tag{46}$$

where the analytic superspace integration measure $d\zeta({}^{33}_{11})du=d^4x_Ad^8\theta_A({}^{33}_{11})du$ is defined in [23, 5] and we have introduced the coupling constant f of dimension -2, so that $[V^1_2]=-2$ and the gauge field strength is dimensionless. Besides an infinite number of gauge components accounted for by the gauge freedom (44), the gauge potentials possess an infinite number of the auxiliary field components. The latter disappear only on the mass shell defined by the free equations of motion following from (46):

$$\begin{aligned}&F^{11}_{23}=D^1_3V^1_2-D^1_2\hat{V}^1_3=0\ ,\\ &F^{12}_{33}=D^2_3\hat{V}^1_3-D^1_3V^2_3=0\ .\end{aligned} \tag{47}$$

For our further purposes it will be important to know the full structure of the bosonic $SU(3)$ singlet sector in the component expansion of the off-shell analytic potentials V_2^1 and V_3^2 in the WZ gauge. A simple analysis yields

$$v_2^1 = \theta_2^\alpha \bar\theta^{1\dot\beta} A_{\alpha\dot\beta} + i(\theta_2)^2 \bar\theta^{1(\dot\alpha}\bar\theta^{2\dot\beta)}\bar H_{\dot\alpha\dot\beta} + i(\theta_2)^2(\bar\theta^1\bar\theta^2)C \;, \tag{48}$$

$$v_3^2 = \theta_3^\alpha \bar\theta^{2\dot\beta} A_{\alpha\dot\beta} - i\theta_2^{(\alpha}\theta_3^{\beta)}(\bar\theta^2)^2 H_{\alpha\beta} - i(\theta_2\theta_3)(\bar\theta^2)^2 C, \tag{49}$$

where $H_{\alpha\beta} = H_{\beta\alpha}$, $\bar C = C$, and the spinor representation for the gauge field strength was used

$$F_{\alpha\beta}(A) \equiv \partial^{\dot\beta}_{(\alpha} A_{\beta)\dot\beta} \;,\; \bar F_{\dot\alpha\dot\beta}(A) \equiv \partial^{\beta}_{(\dot\alpha} A_{\beta\dot\beta)} \;,$$
$$\partial^{\dot\beta}_{\alpha}\bar F_{\dot\beta\dot\alpha} - \partial^{\beta}_{\dot\alpha} F_{\beta\alpha} = 0 \;. \tag{50}$$

We observe that the auxiliary dimensionless symmetric tensor and scalar fields $H_{\alpha\beta}$ and C are present in the off-shell $SU(3)$ singlet sector in parallel with the gauge potential $A_{\alpha\dot\beta}$ and its covariant field strength. The fields $H_{\alpha\beta}, \bar H_{\dot\alpha\dot\beta}$ play a crucial role in constructing $N=3$ supersymmetric BI action.

The gauge fields part of the off-shell super $N=3$ Maxwell component Lagrangian corresponding to (46) is

$$L_2(F,H,C) = \frac{1}{16f^2}[\,H^2 + \bar H^2 - 6\,(\bar H\bar F + HF) + F^2 + \bar F^2 + 8C^2\,] \;. \tag{51}$$

Eliminating the auxiliary fields $H_{\alpha\beta}, \bar H_{\dot\alpha\dot\beta}, C$ by their algebraic equations of motion

$$H_{\alpha\beta} = 3\,F_{\alpha\beta} \;,\; \bar H_{\dot\alpha\dot\beta} = 3\,\bar F_{\dot\alpha\dot\beta} \;,\; C = 0 \;, \tag{52}$$

we arrive at the standard Maxwell action

$$L_2(F) = -\frac{1}{2f^2}(F^2 + \bar F^2) = -\frac{1}{4f^2}\mathcal{F}^{mn}\mathcal{F}_{mn} \;, \tag{53}$$

where $\mathcal{F}_{mn} = \partial_m A_n - \partial_n A_m$.

The basic building-blocks of the $N=3$ BI action are the analytic superfield strengths. Like in the $N=2$ gauge theory in $N=2$ HSS [24], one firstly defines the non-analytic abelian connections V_1^2, V_2^3 via the harmonic zero-curvature equations

$$D_2^1 V_1^2 - D_1^2 V_2^1 = 0 \;, \quad D_3^2 V_2^3 - D_2^3 V_3^2 = 0 \;, \tag{54}$$

where $V_2^3 = -\widetilde{V_1^2}$, $\delta V_2^3 = iD_2^3\lambda$, $\delta V_1^2 = iD_1^2\lambda$ and the explicit form of the harmonic derivatives is given in [5]. Then the mutually conjugated Grassmann-analytic off-shell superfield strengths of the $N=3$ Maxwell theory are constructed as follows [25]:

$$W_{23} = \frac{1}{4}(\bar D_3)^2 V_2^3, \; \bar W^{12} = -\frac{1}{4}(D^1)^2 V_1^2. \tag{55}$$

These off-shell superfield strengths satisfy the following Grassmann analyticity conditions:

$$\overline{D}_{2\dot\alpha} W_{23} = \overline{D}_{3\dot\alpha} W_{23} = D^1_\alpha W_{23} = 0 \;,$$
$$D^1_\alpha \bar W^{12} = D^2_\alpha \bar W^{12} = \overline{D}_{3\dot\alpha}\bar W^{12} = 0 \tag{56}$$

and harmonic differential conditions

$$D_3^2 W_{23} = 0 \quad , \; D_2^1 \bar{W}^{12} = 0 \; . \tag{57}$$

We shall need the full off-shell $SU(3)$ singlet component structure of $W_{23}, \bar{W}^{12}$. The explicit expressions for relevant parts of the latter read

$$w_{23} = i\theta_2^\alpha \theta_3^\beta V_{\alpha\beta} - (\theta_2)^2 \theta_3^\alpha \bar{\theta}^{2\dot{\beta}} \partial_{\dot{\beta}}^\beta V_{\alpha\beta} \; , \tag{58}$$

$$\bar{w}^{12} = i\bar{\theta}^{1\dot{\alpha}} \bar{\theta}^{2\dot{\beta}} \bar{V}_{\dot{\alpha}\dot{\beta}} + (\bar{\theta}^2)^2 \theta_2^\alpha \bar{\theta}^{1\dot{\beta}} \partial_\alpha^{\dot{\alpha}} V_{\dot{\alpha}\dot{\beta}} \; , \tag{59}$$

where

$$V_{\alpha\beta} = \frac{1}{4}(H_{\alpha\beta} + F_{\alpha\beta}) \; , \; \bar{V}_{\dot{\alpha}\dot{\beta}} = \overline{(V_{\alpha\beta})} \; .$$

One can directly check that $w_{23}, \bar{w}^{12}$ on their own obey the off-shell conditions (56) and (57).

The free Maxwell Lagrangian (51) (with $C = 0$), being rewritten through the newly introduced auxiliary fields $V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}}$, reads

$$L_2(F, H, 0) \equiv B_2(F, V) = \frac{1}{f^2}[V^2 + \bar{V}^2 - 2(VF + \bar{V}\bar{F}) + \frac{1}{2}(F^2 + \bar{F}^2)] \; . \tag{60}$$

The algebraic equations of motion for $V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}}$ giving rise to the standard Lagrangian (53) are simply

$$V_{\alpha\beta} = F_{\alpha\beta} \; , \quad \bar{V}_{\dot{\alpha}\dot{\beta}} = \bar{F}_{\dot{\alpha}\dot{\beta}} \; . \tag{61}$$

From the above discussion one infers two important properties of the off-shell description of $N = 3$ gauge theory in $N = 3$ HSS having no direct analogs in the $N = 1$ and $N = 2$ cases. First, the free Maxwell component Lagrangian appears in the unusual forms (51) or (60), while its standard form is recovered only after eliminating the auxiliary fields $V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}}$ by their linear algebraic equations of motion (61). Secondly, the off-shell superfield strengths contain just these tensor auxiliary fields, but not the ordinary gauge field strengths $F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}$.

These surprising features suggest a non-standard approach to constructing nonlinear and non-polynomial superextensions of the off-shell $N = 3$ Maxwell theory. One should modify (51) by proper terms which are nonlinear (and/or non-polynomial) in the auxiliary fields $V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}}$, such that nonlinearities in $F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}$ inherent in the BI action are regained as the result of eliminating these auxiliary fields by their *nonlinear* equations of motion. Then one can hope to $N = 3$ supersymmetrize the terms nonlinear in $V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}}$ with the help of the above superfield strengths $W_{23}, \bar{W}^{12}$ which contain just these auxiliary fields.

3.3 $N = 3$ BI action. Without entering into details, the modification of the free Maxwell Lagrangian (60), such that it becomes the correct bosonic BI Lagrangian,

$$L_{BI}(F, \bar{F}) = \frac{1}{f^2}\left[1 - \sqrt{-\det(\eta_{mn} + \mathcal{F}_{mn})}\right]$$

$$\equiv \frac{1}{f^2}[1 - Q(\varphi, \bar{\varphi})] \; , Q(\varphi, \bar{\varphi}) = \sqrt{1 + X}, \tag{62}$$

$$\varphi = F^2 \; , \; \bar{\varphi} = \bar{F}^2 \; ,$$

$$X(\varphi, \bar{\varphi}) \equiv (\varphi + \bar{\varphi}) + (1/4)(\varphi - \bar{\varphi})^2 \; , \tag{63}$$

after elimination of the auxiliary fields $V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}}$ by their algebraic equations of motion, is as follows [5]

$$
\begin{aligned}
B(F,V) &= B_2(F,V) + \frac{1}{f^2}E(V^2,\bar{V}^2) \\
&= \frac{1}{f^2}[\,\nu + \bar{\nu} - 2(VF + \bar{V}\bar{F}) + \frac{1}{2}(\varphi + \bar{\varphi}) + E(\nu\bar{\nu})]\,. \qquad (64)
\end{aligned}
$$

Here $\nu \equiv V^2, \bar{\nu} \equiv \bar{V}^2$ and $E(\nu\bar{\nu})$ is a real function of the single argument $\nu\bar{\nu} \equiv a$ defined by the following equations

$$
E(a) = 2[2t^2(a) + 3t(a) + 1]\ ,\ E(0) = 0, \qquad (65)
$$

$$
t^4(a) + t^3(a) - \frac{1}{4}a = 0\ ,\ t(0) = -1\ . \qquad (66)
$$

Note that the generic choice of the function $E(a)$ corresponds to a wide class of self-dual (by Legendre transformation) extensions of Maxwell action, the BI one being merely a particular case of these nonlinear actions. [1]

The problem of constructing a manifestly $N = 3$ supersymmetric superfield action which would yield, in the bosonic sector, the F, V form (64) of the BI action amounts to setting up a collection of superfield monomials which extend the appropriate terms in the power expansion of the function $E(V^2\bar{V}^2)$ defined in (65), (66).

Given the function $E(V^2\bar{V}^2)$, we introduce the new function $\hat{E}(V^2\bar{V}^2)$ by

$$
E(V^2\bar{V}^2) = \frac{1}{2}V^2\bar{V}^2\hat{E}(V^2\bar{V}^2)\ , \qquad (67)
$$

with $\hat{E}(a) = 1 - a/4 + O(a^2)$. Then the whole sequence of higher order terms in the $N = 3$ generalization of the BI-action can be written as a closed expression in the analytic superspace,

$$
S_E = \frac{1}{32f^2}\int dud\zeta(^{33}_{11})(W_{23})^2(\bar{W}^{12})^2\hat{E}(A)\ . \qquad (68)
$$

Here A is the following real analytic superfield:

$$
A = \frac{1}{2^{11}}(D^1)^2(\bar{D}_3)^2[D^{2\alpha}W_{12}D^2_\alpha W_{12}\,\bar{D}_{2\dot{\alpha}}\bar{W}^{23}\bar{D}^{\dot{\alpha}}_2\bar{W}^{23}] = V^2\bar{V}^2 + \dots\ , \qquad (69)
$$

$$
\begin{aligned}
&W_{12} = D^3_1 W_{23} = -4i\,\theta^{(\alpha}_1\theta^{\beta)}_2 V_{\alpha\beta} + \dots\ , \\
&\bar{W}^{23} = -D^3_1\bar{W}^{12}\ . \qquad (70)
\end{aligned}
$$

Thus we have obtained an $N = 3$ generalization of the Born-Infeld action using the off-shell Grassmann-analytic potentials V^1_2 and V^2_3

$$
S^{N=3}_{BI} = S_2 + S_E\ . \qquad (71)
$$

The substitution of generic function $\hat{E}(A) = 1 + O(A)$ into (68) yields $N = 3$ superextensions of a wide class of the self-dual nonlinear deformations of Maxwell theory.

Finally, we notice that the above $N = 3$ BI action is a minimal $N = 3$ extension of the bosonic BI action. It still remains to examine whether it admits a treatment

[1] See [26] for a detailed discussion of the self-duality issues in this new setting.

as a $N = 3$ Goldstone-Maxwell superfield action describing an off-shell PBGS option $N = 6 \to N = 3, d = 4$ which could amount on shell to the option $N = 8 \to N = 4, d = 4$. By analogy to the situation with $N = 2$ BI action [14, 15, 16], one could expect that for such an interpretation to be possible the above action should be modified by some extra terms with extra x-derivatives on them.

4 Concluding Remarks

In conclusion, it is worth mentioning a few important unsolved problems in supersymmetric BI theories.

- Constructing a BI deformation of the off-shell $N = (1,0), d = 6$ super Maxwell action. Such super BI action should amount to a static gauge of the space-filling D5-superbrane action in a flat Minkowski background.
- Adapting the nonlinear realizations formalism to the case of non-abelian BI actions. This would seemingly require introducing the notion of generalized supersymmetry algebras incorporating the non-abelian gauge group structure.
- Constructing superconformally invariant versions of $N = 2$ and $N = 3$ super BI actions reviewed in this article. Such modifications could play an important role in the context of the AdS/CFT correspondence [27]-[30], providing the PBGS form of the effective worldvolume actions of D3-superbranes on superbackgrounds with the $\mathrm{AdS}_n \times S^m$-type bosonic body ($\mathrm{AdS}_5 \times S^1$ in the $N = 2$ case and $\mathrm{AdS}_5 \times S^5$ in the $N = 3$ and $N = 4$ cases).

Note that an $N = 4$ completion of the F^4 term [31] of the $N = 2$ superconformal BI action in $N = 2$ HSS was found in a recent paper [32]. It radically differs from its non-conformal counterpart (42):

$$S_4^{conf} \sim \int dzdu \,\{ \ln \frac{\mathcal{W}}{\Lambda} \ln \frac{\bar{\mathcal{W}}}{\Lambda} + (X-1)\frac{\ln(1-X)}{X} + [\,\mathrm{Li}_2(X) - 1\,]\,\}\,. \tag{72}$$

Here

$$X = -2\,\frac{q^+ \cdot q^-}{\bar{\mathcal{W}}\mathcal{W}}\,, \tag{73}$$

and

$$\mathrm{Li}_2(X) = -\int_0^X \frac{\ln(1-t)}{t}\,dt = \sum_{n=1}^{\infty} \frac{1}{n^2} X^n$$

is Euler dilogarithm (Λ is an arbitrary scale). The action (72) is invariant under both $N = 2$ superconformal group and $N = 4$ supersymmetry (41), hence it is invariant under the whole $N = 4$ superconformal group.

As distinct from (42), the contributions to the component F^4 term come from both the pure $\mathcal{W}, \bar{\mathcal{W}}$ and mixed $\mathcal{W}, \bar{\mathcal{W}}, q^{\pm}$ pieces of (72). It reads

$$\sim \frac{F^2 \bar{F}^2}{(|\varphi|^2 + f^{ia} f_{ia})^2}\,, \tag{74}$$

where $\varphi(x), \bar{\varphi}(x)$ ($< \varphi > \neq 0$) and $f^{ia}(x)$ are physical bosonic fields of the vector $N=2$ multiplet and hypermultiplet. Together they form a 6-dimensional multiplet of the R-symmetry group $SU(4)$ of $N=4$ SUSY. The $SU(4)$ invariant square of these fields in the denominator of (74) ensures the scale and conformal invariance of the corresponding x-space action and can be identified, from the $\mathrm{AdS}_5 \times S^5$ D3-superbrane perspective [27], with the fifth (radial) co-ordinate of AdS_5.

It would be extremely interesting to find a superconformally invariant version (if existing) of the $N=3$ BI action (71). Such an action is expected to be unique and to provide a manifestly off-shell $N=3$ supersymmetric superfield form of the abelian D3-superbrane action on $\mathrm{AdS}_5 \times S^5$. It is very important to know this hypothetical maximally worldvolume supersymmetric form of the D3-brane action both for further clarifying the AdS/CFT correspondence and, as a closely related goal, for exploring the precise structure of the quantum low-energy effective action in $N=4$ SYM theory.

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Superbranes in AdS background*

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Abstract

Proceeding from the 1/2 partial breaking of global $N = 1$ AdS_3 and AdS_4 supersymmetries in the nonlinear realizations approach, we derive the worldvolume superfield actions of AdS_3 superstring and AdS_4 supermembrane. Their bosonic cores are the Maldacena-type conformally invariant actions of the AdS_3 string and AdS_4 membrane.

1 Introduction

Superbranes partially break the translational invariance and space-time supersymmetry and so they can be considered as notable examples of systems with partial breaking of global supersymmetry (PBGS). In the approach with PBGS as the guiding principle, the manifestly worldvolume supersymmetric superbrane actions emerge as the Goldstone superfield actions associated with nonlinear realizations of some global space-time supersymmetry spontaneously broken down to a smaller supersymmetry. The Goldstone superfields carry the superbrane physical worldvolume multiplets and are identified with coordinates of some superspace of the initial supersymmetry.

Until now, only branes on a flat background were treated in the PBGS framework (see, e.g., [1, 2, 3]). To incorporate general curved backgrounds, one should generalize the PBGS approach to the local supersymmetry by properly turning on supergravity, both on the super worldvolume and in the target superspace. As one of the approaches to this difficult problem, it is natural to start by considering PBGS branes on some particular curved superbackgrounds. Here we elaborate on the simplest examples of this kind, the type I superstring on the AdS_3 superspace and $N = 1$ AdS_4 supermembrane.

The AdS (super)groups are realized as (super)conformal groups on the boundary (super)spaces whose bosonic dimension is smaller by 1 compared to the bulk AdS (super)spaces. Thus, the PBGS approach to AdS superbranes of co-dimension 1 should deal with proper nonlinear realizations of superconformal groups. Whereas nonlinear realizations of superconformal symmetries in superspace were already considered in literature [4], it turns out that relevant to the PBGS point of view is a rather special type of them. In both considered cases, the underlying AdS supersymmetries are realized as Goldstone superfield-modified superconformal transformations of the worldvolume superspaces.

2 Type I superstring on AdS_3

Our starting point will be a realization of the corresponding PBGS pattern on the superspace coordinates. We are going to find a realization of the $N = 1$ AdS_3 supergroup, such that only its $N = (1, 0), d = 2$ Poincaré subsupergroup is realized linearly.

*This talk is based on the results obtained in the collaboration with F.Delduc and E.Ivanov

The $N=1$ AdS_3 superalgebra in the obvious notation reads:

$$
\begin{aligned}
&[M_{ab}, M_{cd}] = \epsilon_{ac} M_{bd} + \epsilon_{ad} M_{bc} + \epsilon_{bc} M_{ad} + \epsilon_{bd} M_{ac} \equiv (M) \ , \\
&[M_{ab}, R_{cd}] = (R) \ , \quad [R_{ab}, R_{cd}] = m^2 (M) \ , \{Q_a, Q_b\} = 2R_{ab} + 2m M_{ab} \ , \\
&[M_{ab}, Q_c] = \epsilon_{ac} Q_b + \epsilon_{bc} Q_a \ , \quad [R_{ab}, Q_c] = m\left(\epsilon_{ac} Q_b + \epsilon_{bc} Q_a\right) \ .
\end{aligned}
\tag{2.1}
$$

One of the possible bases in this superalgebra is the conformal basis, in which it is recognized as the minimal $N=1, d=2$ superconformal algebra (a direct sum of $N=1, d=1$ superconformal algebra $osp(1|2)$ and conformal $d=1$ algebra $sl_2 \sim so(1,2)$):

$$
sl_2 \propto \left\{ \begin{array}{l} L_{-1} = \frac{1}{4m}\left(R_{22} - m M_{22}\right) \\ L_0 = \frac{1}{4}\left(M_{12} - \frac{1}{m} R_{12}\right) \\ L_1 = \frac{1}{4m}\left(R_{11} - m M_{11}\right) \end{array} \right\}, \ osp(1|2) \propto \left\{ \begin{array}{l} \hat{L}_{-1} = \frac{1}{4m}\left(R_{22} + m M_{22}\right) \\ \hat{L}_0 = \frac{1}{4}\left(M_{12} + \frac{1}{m} R_{12}\right) \\ \hat{L}_1 = \frac{1}{4m}\left(R_{11} + m M_{11}\right) \\ G_{-1/2} = \frac{1}{2\sqrt{m}} Q_2 \\ G_{1/2} = \frac{1}{2\sqrt{m}} Q_1 \end{array} \right\}
$$

Two mutually commuting sl_2 subalgebras constitute the finite-dimensional $d=2$ conformal and AdS_3 algebra $so(2,2)$. From the $d=2$ point of view, the generators M_{12} and R_{12} are, respectively, the $SO(1,1)$ Lorentz and dilatation generators.

For our purposes more appropriate is another basis which is a superextension of the so-called "solvable subgroup basis" of the AdS_3 group $SO(2,2)$ [5]:

$$
\begin{aligned}
&P_{++} = R_{22} + m M_{22}, \ P_{--} = R_{11} - m M_{11}, \ K_{++} = M_{22}, \ K_{--} = M_{11}, \\
&U = M_{12}, \ Z = R_{12}, \ Q_+ = Q_2, \ Q_- = Q_1 \ .
\end{aligned}
\tag{2.2}
$$

The main peculiarities of this basis are as follows:

- $[P_{++}, P_{--}] = 0$, therefore we may introduce the flat coordinates $x^{\pm\pm}$ as parameters associate with $P_{\pm\pm}$;
- The generators $\{U, K_{\pm\pm}\}$ form $so(1,2)$ and $K_{\pm\pm}$ span the coset $SO(1,2)/SO(1,1)$;
- The generator P_{--} commutes with $Q_\pm$ while P_{++} commutes only with Q_+. Thus in this basis the $N=(1,0), d=2$ Poincarè superalgebra $\propto (Q_+, P_{\pm\pm})$ is explicit and this allows us to deal with standard $N=(1,0)$ superfields as the worldvolume ones;
- In the flat limit $m \to 0$ the algebra (2.1) becomes the central-charge extended $N=(1,1), d=3$ super Poincarè algebra (amounting to $N=1, D=3$ one);
- The $N=(1,0), d=2$ Poincaré supertranslations subalgebra together with the dilatations generator Z form the maximal solvable subalgebra [5] of $N=1$ AdS_3 superalgebra.

Now we wish to construct a nonlinear realization of this $N=1$ AdS_3 supergroup with unbroken $N=(1,0), d=2$ Poincaré supersymmetry. Our basic object is a superfield $\Phi(x^\pm, \theta^\pm)$ which we associate with the generator Z in the following coset element:

$$
g = e^{x^{\pm} P_{\pm}} e^{\theta^+ Q_+} e^{\theta^- Q_-} e^{\Phi Z}
\tag{2.3}
$$

The left action of the full supergroup on this element amounts to the following transformations of the coordinates:

$$
\begin{aligned}
&\delta x^{++} = a^{++} - \varepsilon^+\theta^+ - 4m\varepsilon^-\theta^+x^{++} + 2m\alpha x^{++}, \\
&\delta x^{--} = a^{--} - \varepsilon^-\theta^- e^{4m\Phi} + 2m\alpha x^{--}, \\
&\delta\theta^+ = \varepsilon^+ + 4m\varepsilon^- x^{++} + m\alpha\theta^+, \ \delta\theta^- = \varepsilon^- + 4m\varepsilon^-\theta^+\theta^- - m\alpha\theta^-, \\
&\delta\Phi = -2\varepsilon^-\theta^+ + \alpha,
\end{aligned} \tag{2.4}
$$

where $a^{\pm\pm}, \varepsilon^{\pm}, \alpha$ are the group parameters related to $P_{\pm\pm}, Q_{\pm}$ and Z, respectively. One observes that the spinor derivative of Φ with respect to θ^+ is shifted by ε^-, so this derivative is a Goldstone fermion and the nonlinear ε^- supersymmetry is actually spontaneously broken in this realization, despite the fact that its generator Q_- picks up the coordinate θ^- as the coset parameter. This realization is an analog of the "quasi-linear" realization in the case of the $D = 3$ PBGS superstring in the flat background [6], and it goes into the latter in the limit $m \to 0$.

From the "active" transformation properties of the Goldstone superfield Φ with respect to the broken supersymmetry Q_-:

$$
\begin{aligned}
\hat{\delta}\Phi &= -2\varepsilon^-\theta^+ - \varepsilon^-\left(1 + 4m\theta^+\theta^-\right)\frac{\partial}{\partial\theta^-}\Phi - 4m\varepsilon^- x^{++}\left(D_+\Phi - 2\theta^+\partial_{++}\Phi\right) \\
&\quad + \varepsilon^-\theta^- e^{4m\Phi}\partial_{--}\Phi,
\end{aligned} \tag{2.5}
$$

where $D_+ = \partial/\partial\theta^+ + \theta^+\partial_{++}$, one may easily find the transformation properties of the $N = (1,0)$ components of $\Phi(x^{\pm\pm}, \theta^{\pm}) \equiv \phi(x^{\pm\pm}, \theta^+) + \theta^-\psi_-(x^{\pm\pm}, \theta^+)$:

$$
\begin{aligned}
&\hat{\delta}\phi = -2\varepsilon^-\theta^+ - \varepsilon^-\psi_- - 4m\varepsilon^- x^{++}\left(D_+\phi - 2\theta^+\partial_{++}\phi\right), \\
&\hat{\delta}\psi_- = -4m\varepsilon^-\theta^+\psi_- - 4m\varepsilon^- x^{++}\left(D_+\psi_- - 2\theta^+\partial_{++}\psi_-\right) - \varepsilon^- e^{4m\phi}\partial_{--}\phi.
\end{aligned} \tag{2.6}
$$

One sees that $\psi_+ \equiv D_+\phi$ is the inhomogeneously transforming Goldstone fermion for the broken Q_- supersymmetry, while ψ_- is an extra homogeneously transforming $N = (1,0)$ superfield. Using it, one can construct an invariant of both supersymmetries:

$$
S \sim \int d^2x d\theta^+ e^{-4m\phi}\psi_- \,. \tag{2.7}
$$

To make this invariant a meaningful action, we are led to covariantly express the superfield ψ_- in terms of our basic Goldstone superfield ϕ. To find the proper constraints we shall follow the general strategy of [7, 8] which includes the following steps:

- One finds the finite supersymmetric transformations of $\psi_+ \equiv D_+\phi$ and ψ_-. In these transformations, one replaces the constant parameter ε^- by an arbitrary, for the moment, spinor $N = (1,0)$ superfield ξ^- taken with the sign minus;

- Finally, one shows that it is possible to define the transformation properties of the superfield ξ^- with respect to both supersymmetries in such a way that putting the transformed superfields $D_+\phi$ and ψ_- with the above replacements equal to zero is an invariant procedure.

Carrying out this procedure, we get the following system of equations:

$$\psi_+ + 2\xi^- + \xi^- D_+\psi_- - 4m\xi^- x^{++}\left(\partial_{++}\phi - 2\theta^+\partial_{++}\psi_+\right) = 0,$$
$$\psi_- - 4m\xi^- x^{++}\left(D_+\psi_- - 2\theta^+\partial_{++}\psi_-\right) - \xi^- e^{4m\phi}\partial_{--}\phi = 0. \tag{2.8}$$

These equations can be easily solved for ψ_- and ξ^-:

$$\psi_- = -\frac{e^{4m\phi}D_+\phi\partial_{--}\phi}{1+\sqrt{1-e^{4m\phi}\partial_{++}\phi\partial_{--}\phi}}\,,\quad \xi_- = -\frac{\psi_+}{2+D_+\psi_-}\,. \tag{2.9}$$

Thus, the action (2.7) for the type I superstring in super-AdS$_3$ background is

$$S = \int d^2x d\theta^+ \frac{D_+\phi\partial_{--}\phi}{1+\sqrt{1-e^{4m\phi}\partial_{++}\phi\partial_{--}\phi}}\,. \tag{2.10}$$

By construction it is invariant with respect to both supersymmetries and, therefore, with respect to the super-AdS$_3$ symmetry defined by the superalgebra (2.1).

It is instructive to explicitly write the bosonic part of the action (2.10):

$$S = \int d^2x e^{-4m\varphi}\left(1-\sqrt{1-e^{4m\varphi}\partial_{++}\varphi\partial_{--}\varphi}\right)\,, \tag{2.11}$$

where $\varphi \equiv \phi|_{\theta^+=0}$. To see that the action (2.11) describes a string embedded into the AdS$_3$ background, let us introduce $U = e^{-2m\phi}$ and rescale $x^\pm = \frac{1}{2m}\tilde{x}^\pm$. Now one can rewrite (2.11), up to some overall constant factor, in the form

$$S = \int d^2\tilde{x}\, U^2 \left(1-\sqrt{1-\frac{(\tilde{\partial}_{++}U\tilde{\partial}_{--}U)}{U^4}}\right)\,. \tag{2.12}$$

Thus S is recognized as $d=2$ analog of the Maldacena scale-invariant brane action on AdS$_5$ [9] (actually, of the scalar fields piece of the full D3-brane action).

3 AdS$_4$ supermembrane

Our starting point in this case will be the $N=1$ AdS$_4$ superalgebra $osp(1|4)$ in the $d=3$ spinor $SL(2,R)$ notation, once again in the basis which is a natural generalization of (2.2) and is most convenient for our purposes

$$\begin{aligned}
&[M_{ab},M_{cd}] = \varepsilon_{ac}M_{bd}+\varepsilon_{ad}M_{bc}+\varepsilon_{bc}M_{ad}+\varepsilon_{bd}M_{ac} \equiv (M)_{ab,cd} = -(M)_{cd,ab}\,,\\
&[K_{ab},K_{cd}] = -(M)_{ab,cd}\,,\ [M_{ab},K_{cd}] = (K)_{ab,cd}\,,\ [M_{ab},P_{cd}] = (P)_{ab,cd}\,,\\
&[K_{ab},D] = -2P_{ab}+2mK_{ab}\,,\ [P_{ab},D] = -2mP_{ab}\,,\ [P_{ab},P_{cd}] = 0\,,\\
&[K_{ab},P_{cd}] = -2\left(\varepsilon_{ac}\varepsilon_{bd}+\varepsilon_{bc}\varepsilon_{ad}\right)D - m\,(M)_{ab,cd}\,,\\
&\{Q_a,Q_b\} = 2P_{ab}\,,\ \{S_a,S_b\} = 2P_{ab}-4m\,K_{ab}\,,\ \{Q_a,S_b\} = 2\varepsilon_{ab}D - 2mM_{ab}\,,\\
&[M_{ab},Q_c] = \varepsilon_{ac}Q_b+\varepsilon_{bc}Q_a \equiv (Q)_{ab,c}\,,\ [M_{ab},S_c] = (S)_{ab,c}\,,\\
&[K_{ab},Q_c] = (S)_{ab,c}\,,\ [K_{ab},S_c] = -(Q)_{ab,c}\,,\\
&[P_{ab},Q_c] = 0\,,\ [P_{ab},S_c] = -2m\,(Q)_{ab,c}\,,\ [D,Q_a] = m\,Q_a\,,\ [D,S_a] = -m\,S_a\,.
\end{aligned} \tag{3.1}$$

Here, $a, b = 1, 2$ and the contraction parameter m is proportional to the inverse AdS$_4$ radius.

The $SO(1,2)$ generators M_{ab} together with K_{ab} form the algebra of $D = 4$ Lorentz group $SO(1,3)$. The advantage of the basis (3.1) is that it manifests the $N = 1, d = 3$ super Poincaré subalgebra of $osp(1|4)$ and still yields the $N = 1, D = 4$ super Poincaré algebra (in the $d = 3$ notation) in the contraction limit $m = 0$. The $N = 1, d = 3$ Poincaré supertranslations subalgebra $\propto (Q_a, P_{ab})$ together with the generator D form the maximal solvable subalgebra of $osp(1|4)$.

We wish to construct a $OSp(1|4)$ extension of the AdS$_4$ membrane action [9, 10]

$$S = \int d^3x\, e^{-6mq} \left(1 - \sqrt{1 - \frac{1}{2} e^{4mq} \partial^{ab} q \partial_{ab} q}\right) , \tag{3.2}$$

such that it possesses a manifest $N = 1, d = 3$ supersymmetry extending the manifest $d = 3$ Poincaré worldvolume invariance of (3.2), and becomes the PBGS action of the flat $N = 1, D = 4$ supermembrane [11] in the contraction limit $m = 0$. Just like the latter action is the Goldstone superfield action for the 1/2 breaking of the $N = 1, D = 4$ Poincaré supersymmetry down to the $N = 1, d = 3$ one, the action we are seeking for is expected to be a Goldstone superfield action describing the 1/2 spontaneous breaking of the $OSp(1|4)$ supersymmetry down to its $N = 1, d = 3$ super Poincaré subgroup.

We shall construct such an action along the same lines as in Sect. 2. As a first step we need to define the appropriate analog of the "intermediate" nonlinear realization (2.4).

As a natural extension of the coset element (2.3) we choose the following one:

$$g = e^{x^{ab} P_{ab}} e^{\theta^a Q_a} e^{\psi^a S_a} e^{u(z) D} e^{\Lambda^{ab}(z) K_{ab}} . \tag{3.3}$$

Here, the parameters $z \equiv \left(x^{ab}, \theta^a, \psi^a\right)$ are $N = 2, d = 3$ superspace coordinates, while $u = u(z)$ and $\Lambda^{ab}(z)$ are Goldstone superfields given on this superspace. The subspace spanned by the coordinate set $\zeta \equiv \left(x^{ab}, \theta^a\right)$ is the standard flat $N = 1, d = 3$ superspace in which $N = 1, d = 3$ Poincaré supertranslations $\propto (Q_a, P_{ab})$ are realized in a standard way:

$$\delta x^{ab} = a^{ab} - \frac{1}{2}\left(\epsilon^a \theta^b + \epsilon^b \theta^a\right) , \quad \delta\theta^a = \epsilon^a . \tag{3.4}$$

These transformation laws are obtained by acting on (3.3) from the left by the group element $g_0 = e^{a^{ab} P_{ab}} e^{\epsilon^a Q_a}$.

The rest of the $OSp(1|4)$ transformations except for the $SO(1,2)$ rotations is nonlinearly realized on the coset coordinates, mixing the $N = 2$ superspace coordinates with the Goldstone superfield $u(z)$. Acting on (3.3) from the left by different supergroup elements, it is easy to find the explicit form of these transformations.

Covariant derivatives of the Goldstone superfield $u(z)$ can be constructed by the supercoset element (3.3) following the generic guidelines of the nonlinear realizations method. Of actual need for us will be spinor covariant derivatives. Without entering into details, these are as follows

$$\tilde{\nabla}^Q_a u = \frac{1}{\sqrt{1 - 2\lambda^2}} \left(\nabla^Q_a - 2\lambda^b_a \nabla^S_b\right) u , \ \tilde{\nabla}^S_a u = \frac{1}{\sqrt{1 - 2\lambda^2}} \left(\nabla^S_a + 2\lambda^b_a \nabla^Q_b\right) u , \tag{3.5}$$

$$\nabla^Q_a u = e^{mu} \left(D_a - m\psi^2 \frac{\partial}{\partial\psi^a}\right) u - 2e^{mu}\psi_a \equiv \hat{\nabla}^Q_a u - 2e^{mu}\psi_a ,$$

$$\nabla^S_a u = e^{-mu}\left[\frac{\partial}{\partial\psi^a} + e^{4mu}\left(\psi^b\partial_{ab} + 3m\psi^2 D_a\right)\right] u\ . \tag{3.6}$$

Here

$$D_a = \frac{\partial}{\partial\theta^a} + \theta^b\partial_{ab}\ , \quad \{D_a, D_b\} = 2\partial_{ab}\ , \tag{3.7}$$

is the standard covariant spinor derivative of $N=1, d=3$ Poincaré supersymmetry. The Goldstone superfield λ^{ab} is related to Λ^{ab} as

$$\lambda^{ab} \equiv \frac{\tanh\sqrt{2\Lambda^2}}{\sqrt{2\Lambda^2}}\Lambda^{ab}\ ,\ \lambda^2 = \lambda^{ab}\lambda_{ab}\ . \tag{3.8}$$

What we have at this stage, is a nonlinear realization of the $N=1$ AdS_4 supergroup on the $N=2, d=3$ Goldstone superfield $u(x,\theta,\psi)$. The first component in the θ,ψ expansion of u can be regarded as the Goldstone dilaton field. The spinor derivative $D_a u$ is shifted by η_a under the S -supersymmetry, suggesting that we actually face the 1/2 spontaneous breaking of the AdS_4 supersymmetry, with $D_a u|_{\psi=0}$ as the corresponding Goldstone fermionic $N=1$ superfield. However, u contains quite a few extra component fields having no immediate Goldstone interpretation.

To construct the minimal Goldstone multiplet, we shall generalize the method which was applied in [6] to $d=2$ PBGS systems and, in [12], to the case of flat-space $N=1, D=4$ supermembrane. Following the reasonings of [12] and keeping in mind that the minimal scalar multiplets of $N=1$ AdS_4 supergroup are represented by chiral $N=1, D=4$ or $N=2, d=3$ superfields, we shall regard the Goldstone superfield $u(z)$ to be *complex* and subject it to the covariant chirality constraint

$$\left(\nabla^Q_a - i\nabla^S_a\right) u = 0\ . \tag{3.9}$$

This condition is equivalent to the similar one in terms of the full covariant derivatives (3.5) in view of the relation

$$\nabla^Q_a \pm i\nabla^S_a = \frac{\delta^b_a \mp i\lambda^b_a}{\sqrt{1-2\lambda^2}}\left(\tilde\nabla^Q_b \pm i\tilde\nabla^S_b\right)\ , \tag{3.10}$$

and so it is covariant with respect to the whole $OSp(1|4)$. Using the explicit form of the covariant derivatives (3.6), it is straightforward to check that (3.9) is self-consistent, in the sense that the appropriate integrability condition is satisfied

$$\begin{aligned}&\{\hat\nabla^Q_a - i\nabla^S_a, \hat\nabla^Q_b - i\nabla^S_b\}u - 2\left(\hat\nabla^Q_a - i\nabla^S_a\right)e^{mu}\psi_b - 2\left(\hat\nabla^Q_b - i\nabla^S_b\right)e^{mu}\psi_a\\ &= T^c_{ab}\left(\nabla^Q_c - i\nabla^S_c\right)u\ .\end{aligned} \tag{3.11}$$

The constraint (3.9) can be solved by expanding u in powers of ψ^a. It is easy to check that (3.9) expresses all terms in this expansion in terms of $u_0 \equiv u|_{\psi^a=0}$ and D_a derivatives of u_0. In particular,

$$\frac{\partial u}{\partial\psi^a}|_{\psi=0} = -ie^{2mu}D_a u|_{\psi=0}\ . \tag{3.12}$$

Thus the complex $N=1, d=3$ superfield

$$u_0(x,\theta) \equiv q(x,\theta) + i\Phi(x,\theta)\ , \quad q^\dagger = -q\ ,\ \Phi^\dagger = -\Phi\ , \tag{3.13}$$

incorporates the full irreducible field content of the $N = 2, d = 3$ Goldstone chiral superfield $u(x, \theta, \psi)$. Its S-supersymmetry transformation reads

$$\delta u_0 = Lu_0 + 2\eta^a\theta_a + ie^{2mu}\eta^a D_a u_0 \ , \tag{3.14}$$

where

$$L \equiv -m\left(\theta^2\eta^a + 4x^{ab}\eta_b\right)\frac{\partial}{\partial\theta^a} + 4m\eta_c\theta^b x^{ac}\partial_{ab} \ . \tag{3.15}$$

For the imaginary and real parts of u_0 eq. (3.14) implies the following transformation rules

$$\begin{aligned} \delta q &= Lq - e^{2mq}\eta^a\left[\sin(2m\Phi)D_a q + \cos(2m\Phi)D_a\Phi\right] + 2\eta^a\theta_a \ , \\ \delta\Phi &= L\Phi + e^{2mq}\eta^a\left[\cos(2m\Phi)D_a q - \sin(2m\Phi)D_a\Phi\right] \ . \end{aligned} \tag{3.16}$$

The nonlinear realization we have at this step is still non-minimal in the following sense. Besides the $N = 1$ superfield $q(x, \theta)$ which contains all Goldstone fields required by the 1/2 breaking of $OSp(1|4)$ down to its $N = 1, d = 3$ Poincaré subgroup ($q|_{\theta=0}$ for the dilatations, $(D_a q)|_{\theta=0}$ for the broken S-transformations and $\partial_{ab}q|_{\theta=0}$ for the broken $SO(1,3)/SO(1,2)$ transformations), there is an extra non-Goldstone $N = 1, d = 3$ superfield $\Phi(x, \theta)$. The last step is to eliminate the latter in terms of q and its derivatives by imposing some nonlinear covariant constraint on $u_0(x, \theta)$, analogous to the constraints imposed in the flat case [11]. The precise form of such a constraint can be found by applying the general procedure of refs. [7, 8, 6] to the given case. We skip details and present the final form of the constraint

$$\Phi = \frac{e^{2mq}D^a q D_a q}{4 + e^{2mq}D^2\Phi} \ . \tag{3.17}$$

It can be directly checked to be covariant with respect to (3.16).

From our superfield u_0 we can construct the following two simplest invariants (up to normalization factors)

$$S_1 = \frac{1}{2}\int d^3x d^2\theta\left(e^{-4mu_0} + e^{4mu_0^\dagger}\right) = \int d^3x d^2\theta e^{-4mq}\cos(4m\Phi) \ , \tag{3.18}$$

and

$$S_2 = -\frac{1}{2im}\int d^3x d^2\theta\left(e^{-4mu_0} - e^{4mu_0^\dagger}\right) = \frac{1}{m}\int d^3x d^2\theta e^{-4mq}\sin(4m\Phi) \ . \tag{3.19}$$

After solving the constraint (3.17)

$$\Phi = \frac{e^{2mq}D^a q D_a q}{2 + \sqrt{4 + e^{4mq}D^2(D^b q D_b q)}} \ , \tag{3.20}$$

in the Lagrangians in S_1 and S_2 only lowest terms in Φ survive in view of the nilpotency of Φ in (3.20). So the actions take the form

$$S_1 \sim \int d^3x d^2\theta e^{-4mq} \ , \tag{3.21}$$

$$S_2 \sim \int d^3x d^2\theta \frac{e^{-2mq} D^a q D_a q}{2+\sqrt{4+e^{4mq} D^2 (D^b q D_b q)}}\,. \tag{3.22}$$

Relevant for our purposes is just S_2, because it contains the kinetic term of the Goldstone superfield $q(\zeta)$. Its bosonic part, with the fermions omitted and the auxiliary field $B = D^2 q|_{\theta=0}$ eliminated by its equation of motion ($B = 0$ in the vanishing fermions limit), is recognized as the AdS$_4$ membrane action (3.2).

We come to the conclusion that the Goldstone superfield action (3.22) is the natural superextension of the conformally-invariant AdS$_4$ membrane action. Besides being manifestly invariant under $N = 1, d = 3$ Poincaré supersymmetry, it is invariant under the nonlinearly realized part of $N = 1$ AdS$_4$ supersymmetry $OSp(1|4)$ which acts on the $N = 1, d = 3$ superworldvolume as the Goldstone superfield-modified $d = 3$ superconformal transformations (eqs. (3.16) with (3.20) standing for Φ). Thus it is a PBGS superfield form of the worldvolume action of $N = 1$ AdS$_4$ supermembrane. In the limit $m \to 0$, it reproduces the known PBGS action of $N = 1, D = 4$ supermembrane in the flat Minkowski background [11].

4 Conclusions

Proceeding from a 1/2 partial breaking of the $N = 1$ AdS$_3$ and AdS$_4$ supersymmetries in the nonlinear realizations description, we have constructed the worldvolume superfield actions for AdS$_3$ superstring and AdS$_4$ supermembrane. Their main characteristic feature is that the spontaneously broken parts of AdS supersymmetries are realized as a Goldstone-superfield modified $d = 2$ and $d = 3$ superconformal symmetry. Like in the case of flat supermembrane [11] and other PBGS systems, the superfield Lagrangian densities of the AdS$_3$ superstring and AdS$_4$ supermembrane PBGS actions are not of a tensor form, they are shifted by a full derivative under the broken parts of the AdS supersymmetry transformations. In this sense, they resemble WZNW or Chern-Simons terms.

Our consideration here can be regarded as a first step towards constructing analogous worldvolume superfield actions for more interesting examples of branes on the superbackgrounds with the AdS$_n \times S^m$ bosonic part, including the appropriate Dp-branes. It also remains to be examined how such actions are related to the more familiar Green-Schwarz type ones.

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On Superbranes with non–Abelian Worldvolume Fields

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Abstract

We discuss symmetry properties of actions which describe dynamics of multiple brane configurations and analyze a possible way of making them supersymmetric. Supersymmetric and κ–invariant ND0-brane models are proposed.

1 Introduction

Branes attract great deal of attention of the theoretical community because they play or may play an important role in the construction and implications of the unified theory of fundamental interactions based on string theory and M–theory. Branes are stable BPS configurations of string and M–theory, and they ensure and reflect the existence of various string/M–theory dualities. Branes are "on the boundary" of the AdS/CFT correspondence, and they provide us with a natural geometrical mechanism of breaking (super)symmetry spontaneously. Various brane world and cosmology scenari have been proposed, and possibilities of getting the Standard Model at intersections of multiple brane configurations have been analyzed. So there are many motivations for studying branes.

The main purpose of this paper is to discuss symmetry properties of actions which describe dynamics of multiple brane configurations and analyze a possible way of making them supersymmetric. This topic is intrinsically related to the problem of the construction of a relevant non–Abelian Dirac–Born–Infeld theory which would reproduce the superstring theory effective action.

To better understand peculiarities about multiple brane actions it would be instructive to first recall properties of worldvolume actions for single superbranes, which for different types of superbranes have been constructed by different authors and which all have a generic structure. We shall concentrate on Dirichlet branes [1, 2, 3].

2 Single (Dirichlet) Branes

The Dirichlet branes are hypersurfaces on which open strings can end. They propagate in space–time and their dynamics is described by worldvolume actions whose structure and properties are known in detail for the single branes. We shall consider the general structure of the single brane action with the example of a D3–brane propagating in type IIB D=10 (flat) superspace [1].

The action, which is an integral over four-dimensional D3-brane worldvolume, has the following form[1]

$$S = -\int d^4\xi\sqrt{-det[\Pi_a^M\Pi_b^N\eta_{MN} + F_{ab} + B_{ab}]}$$

$$+\int_{\mathcal{M}_4}[C_{(4)}(x,\theta) + F_{(2)}\wedge C_{(2)}(x,\theta)], \tag{1}$$

where ξ^a (a=0,1,2,3) are worldvolume coordinates,

$$\Pi_a^M = \partial_a x^M + i\theta^I\gamma^M\partial_a\theta^I,$$

x^M $(M = 0,1,\cdots,9)$ and θ_α^I $(I = 1,2; \quad \alpha = 1,\cdots,16)$ are respectively bosonic and fermionic (Majorana-Weil) coordinates of type IIB D=10 superspace;
$F_{ab} = \partial_a A_b(\xi) - \partial_b A_a(\xi)$ is the field strength of an Abelian worldvolume gauge field associated with the end of the string, and
$B_{(2)}(x,\theta)$, $C_{(2)}(x,\theta)$ and $C_{(4)}(x,\theta)$ are the worldvolume pull backs of two- and four-form fields of type IIB supergravity whose functional dependence on x, θ and their worldvolume derivatives is exactly known when type IIB superspace is flat (see [1] for details).

By construction the action (1) is invariant under target space Lorentz transformations and type IIB D=10 supersymmetry

$$\delta\theta^I = \epsilon^I, \quad \delta x^M = -i\epsilon^I\gamma^M\theta^I, \quad A_a(\xi) \neq 0. \tag{2}$$

Note that the worldvolume vector field transforms nontrivially under space-time supersymmetry. The exact form of its variation the reader may find in [1].

The action is invariant under worldvolume reparametrizations (diffeomorphisms) $\xi^a \rightarrow f^a($ which allow one to impose the static gauge by splitting the D3-brane space-time coordinates along and transverse to the brane and identifying the longitudinal coordinates with the worldvolume coordinates

$$x^M = (x^a(\xi), x^i(\xi)), \quad a = 0,1,2,3; \quad i = 1,\cdots,6$$

$$x^a = \xi^a\,. \tag{3}$$

In addition the action is also invariant under fermionic κ-symmetry with a spinorial parameter $\kappa_\beta^J(\xi)$

$$\delta\theta_\alpha^I = P_{\alpha J}{}^{\beta I}\kappa_\beta^J, \quad \delta x^M = i\delta\theta\gamma^M\theta, \quad \delta A_a \neq 0. \tag{4}$$

which because of the projector $P_{\alpha J}{}^{\beta I}$ allows one to eliminate one of θ^I, for instance to put $\theta_\alpha^1 = 0$.

Because of the local worldvolume diffeomorphisms and κ-symmetry (which one can gauge fix as discussed above) the physical degrees of freedom of the D3-brane form an $N = 4$, $D = 4$ vector supermultiplet

$$A_a(\xi), \quad x^i(\xi), \quad \psi_\alpha(\xi) = \theta_\alpha^2\,. \tag{5}$$

which on the mass shell contains eight bosonic and eight fermionic degrees of freedom.

[1]We put the D-brane tension to unity.

The D3–brane action in the static gauge [2] is the component action for $N = 4$ supersymmetric Dirac–Born–Infeld theory

$$S = \int d^4\xi \sqrt{-det(\eta_{ab} + \partial_a x^i \partial_b x^i + F_{ab} + \mathcal{O})}, \tag{6}$$

where $\mathcal{O}(\psi)$ stands for the fermionic terms

$$\mathcal{O}(\psi) = 2i\psi(\gamma_a + \partial_a x^i \gamma_i)\partial_b\psi + \psi\gamma^M \partial_a\psi\psi\gamma_M\partial_b\psi\,. \tag{7}$$

This action possesses additional $N = 4$ supersymmetry which is nonlinearly realized on the fields (5) and is spontaneously broken (see e.g. [2] for the discussion of this point).

We have thus demonstrated that the form and the symmetry properties of the world-volume actions for the single superbranes are exactly known and well understood from various points of view.

We now turn to the consideration of multiple brane configurations.

3 Multiple D–Branes and the Appearance of Non–Abelian Fields

3.1 The ND–brane action

If we have a stack of N D–branes, strings can end on each of them and/or can link different branes together. As a result [4], if the distances between the D–branes tend to zero, in the limit when they coincide there occurs the enhancement of $U(1)^N$ Abelian symmetries of individual branes up to a $U(N)$ group, and corresponding non–Abelian massless gauge fields $A_a(\xi)$ in the adjoint representation of $U(N)$ appear on the worldvolume of the coincident branes. In the supersymmetric case the vector fields $A_a(\xi)$ have non–Abelian scalar and spinor superpartners as in the case of the single D–brane (5). An interesting effect which takes place in systems of N D–branes coupled to Ramond–Ramond fields first observed by Myers [8] is that, for instance, N D0–branes blow up into a higher dimensional abelian D–brane configuration. This has also been demonstrated to happen with N D1–branes growing into a D5–brane [9].

Now the problem arises how to construct the action for the N D–branes. An "honest" but hardly realizable way would be to start with the sum of the actions (1) for non–interacting single Dp–branes and to add terms which describe interactions of the D–branes via strings

$$S = \int d^{p+1}\xi_1 L(\xi_1) + \cdots + \int d^{p+1}\xi_N L(\xi_N) + \sum_{n\neq m=1}^{N} S_{int}(\xi_n, \xi_m)\,. \tag{8}$$

It is not quite clear how to determine the detailed structure and symmetries of the interaction terms $S_{int}(\xi_n, \xi_m)$, but in the absence of these terms the action (8) is invariant under N copies of worldvolume diffeomorphisms, N target space supersymmetries and N κ–symmetries. It has been assumed (see for example a discussion in [5]) that the interaction terms $S_{int}(\xi_n, \xi_m)$ might have a structure such that all these symmetries become non–Abelian in the limit where the branes coincide, and in [5] a possibility of constructing a supersymmetric action for N space–filling D9–branes in type IIB D=10 superspace with a non-Abelian κ-symmetry has been considered up to quadratic terms in the variation of fields, but then the conclusion has been made [6] that such a construction would fail beyond the quadratic approximation. At least in the case of N space-filling branes, also the possibility of the existence of U(N) valued worldvolume diffeomorphisms seems not natural since all the branes occupy one and the same space parametrized by a single set of coordinates. So we shall take another point of view on the structure and symmetries of coincident branes. Since the N D9-branes are space filling they necessarily coincide, therefore there is actually only a single brane carrying non-Abelian vector and spinor fields on its worldvolume and the action of this brane should be invariant under conventional "Abelian" diffeomorphisms, target-space supersymmetry and (possibly) κ-symmetry. Then, since actions for lower dimensional systems of coincident Dp-branes can be obtained from the N D9-brane action by T-duality the same reasoning may be applied for the systems of these branes as well. Namely, let us consider them as a qualitatively new *single* brane configuration which is created when N Dp-branes stack up together. We shall call it the N(onabelian) Dp-brane, because it propagates non-Abelian scalar, vector and spinor fields on its worldvolume [11].

The bosonic action for N coincident Dp-branes has been obtained by T-dualization of the ND9-brane action in [7, 8] and by T-dualization of ND0-brane action in [10]. The action has the following form

$$S = S_{WZ} - \int d^{p+1}\xi \, \mathrm{Tr}\Big\{ e^{-\phi} \times\sqrt{-det\,(P\,[E_{ab} + E_{ai}(Q^{-1} - \delta)^{ij}E_{jb}] + F_{ab})}\sqrt{det(Q^{i}{}_{j})}\Big\}\,, \tag{9}$$

where ϕ is the dilaton field. $F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b]$ is the field strength of a worldvolume $U(N)$ gauge field $A_a(\xi)$. In addition, the action (9) depends on $D-p-1$ scalar fields $\Phi^i(\xi)$ and on their covariant derivatives $D_a\Phi^i = \partial_a\Phi^i + i[A_a, \Phi^i]$ taking values in the space of the adjoint representation of $U(N)$ (i.e. they are $N \times N$ Hermitian matrices).

P denotes the pull back onto the worldvolume of the matrix

$$E_{MN} = G_{MN} + B_{MN} \tag{10}$$

composed of the target-space metric G_{MN} and an NS-NS field B_{MN}. The explicit form of the pullback of E_{MN} (10) is

$$P[E]_{ab} = E_{ab} + E_{ai}\, D_b\Phi^i + D_a\Phi^i\, E_{ib} + D_a\Phi^i D_b\Phi^j\, E_{ij}. \tag{11}$$

The matrix $Q^i{}_j$ in (9) has the form

$$Q^i{}_j \equiv \delta^i{}_j + i\,[\Phi^i, \Phi^k]\, E_{kj}\,. \tag{12}$$

The diagonal elements of the $N \times N$ matrix Φ^i are associated with transverse space coordinates of the N Dp-branes when they are separated.

The term S_{WZ} in (9) is a Wess-Zumino term which describes the coupling of the NDp-brane system to lower as well as to higher-rank RR fields (compare with (1)), the coupling to the latter is possible because of the non-commutativity of the adjoint scalar fields Φ^i [8, 10]. It has been shown [12] that S_{WZ} is invariant under gauge transformations of the antisymmetric background fields, as in the case of the single branes.

Finally, it should be noted that all the background fields in the non–Abelian action (9) are assumed to be the same functionals of the adjoint scalars Φ^i as they are in the Abelian theory of a single Dp–brane [13, 14]. Further details on the structure of the action (9) may be found in [8].

3.2 Problems

The NDp–brane action has several important problems. One of them is the "trace" problem already paused in [15] which is similar to answering the question "What is the exact form of the non–Abelian Dirac–Born–Infeld action which is the effective action of superstring theory?" . This problem has been intensively discussed in the literature and will not be touched here. We shall simply adopt that the trace in eq. (9) is symmetrized [15], which means that all non–Abelian quantities inside the trace can be regarded as commuting ones, which drastically simplifies the analysis of the action.

The action (9) is constructed in the static gauge, it is therefore not worldvolume diffeomorphism invariant and it is not invariant under a full space–time symmetry of the target space (except for the space–filling branes). For instance in $D = 10$ the Lorentz group $SO(1,9)$ is broken down to $SO(1,p) \times SO(9-p)$.

Another problem is how to supersymmetrize multiple brane actions and to make them invariant under κ–symmetry, or in other words what is the basic symmetry principle for introducing fermionic modes into the construction?

We now plunge into the consideration of these problems.

3.3 Restoration of worldvolume reparametrization invariance

To make the action (9) worldvolume diffeomorphism invariant we shall follow the point of view, introduced above, that eq. (9) is essentially the action for N *coincident* D–branes, i.e. for a *single* NDp–brane, because the integration is taken over a single set of worldvolume coordinates and not over N worldvolumes as in the schematic multiple brane action (8).

We should identify the transverse coordinates of the single NDp–brane in space–time. A natural choice is the trace of the scalar matrix $\Phi^i(\xi)$ which is associated with the center of mass of the N brane configuration

$$x^i(\xi) = \frac{1}{N} Tr \Phi^i(\xi). \tag{13}$$

The full set of ND–brane coordinates in space–time is then $x^M = (\xi^a,\, x^i(\xi))$, which upon removing the static gauge $x^a = \xi^a$ becomes

$$x^M = (x^a(\xi),\, x^i(\xi))\,. \tag{14}$$

We have thus extracted the $U(1)$ scalars $x^i(\xi)$ from the $U(N)$ scalars and identified them with the transverse target space coordinates of the ND–brane. The traceless scalars

$$\varphi^i(\xi) = \Phi^i - x^i \cdot \mathrm{I} \quad \subset \quad SU(N)$$

which take values in the space of the adjoint representation of $SU(N)$ and the $U(N)$ vector fields $A_a(\xi)$ are considered to be *pure* worldvolume fields living on the ND–brane. One should therefore forget about the space–time origin of $\varphi^i(\xi)$.

3.4 The reparametrization invariant action for an ND–brane of codimension one

To simplify further analysis let us consider ND–branes of codimension one [11]. These are branes whose worldvolume has one dimension less than the dimension of target space, i.e. $p+1=D-1$. In this case there is only one $U(N)$–valued scalar and one coordinate $x^\perp$ transverse to the brane

$$\Phi^\perp = x^\perp \cdot \mathbf{I} + \varphi, \tag{15}$$

hence the matrix Q in (12) is equal to one, the ND–brane action simplifies and in the flat background has the following form (where we have introduced the ND–brane coordinates x^M and restored reparametrization invariance as has been described in the previous subsection)

$$S = \int \mathrm{Tr}\left[\sum C^{(n+1)}\, e^B\right] e^F$$
$$- \int d^{p+1}\xi \, \mathrm{Tr}\left[-det\left(\partial_a x^M \partial_b x^N \eta_{MN} + F_{ab} + D_a\varphi D_b\varphi + 2\partial_{(a} x^\perp D_{b)}\varphi\right)\right]^{\frac{1}{2}}, \tag{16}$$

where $C^{(n+1)}$ are Ramond–Ramond (n+1)–form potentials with $n=p,p-2,p-4,\cdots$.

As one can see the last term in the determinant mixes the transverse ND–brane coordinate with the non–Abelian $SU(N)$ scalar. Thus the action is not Lorentz invariant in target space though we have restored the worldvolume reparametrization invariance. At present it is not clear yet whether breaking of Lorentz invariance is an intrinsic property of coincident brane configurations, or whether it is still present in a somewhat hidden non–linearly realized way, or this is just a shortcoming of existing ND–brane actions[2].

If we drop the last "bad" term from the action (16) it becomes Lorentz invariant. Thus there may exist ND–brane configurations which preserve (manifest) Lorentz invariance (i.e. are Lorentz covariant) and their study may be of interest on its own. Below we shall study both Lorentz non–covariant and Lorentz covariant models.

Though the ND–brane systems based on the action (16) are not Lorentz covariant, they are invariant under target–space (Poincare) translations

$$x^M \quad \to \quad x^M + a^M\,. \tag{17}$$

This is a necessary condition for the possibility of the supersymmetrization of the ND–brane systems which we shall carry out below for a simple model of an ND0–brane in type IIA D=2 superspace.

4 ND0–branes in type IIA D=2 superspace

4.1 Bosonic ND0–brane action

Since there is no place for dynamical vector fields on the worldline (parametrized by τ), the ND0–brane action contains only (derivatives of) ND0–brane coordinates $x^M(\tau)$ (M=0,1) and an SU(N) traceless scalar matrix $\varphi(\tau)$

$$S = -m \int d\tau \, \mathrm{Tr}\sqrt{-\dot{x}^M \dot{x}_M - \dot{\varphi}^2 - 2\dot{x}^1\dot{\varphi}}, \tag{18}$$

[2]There are other examples of dynamical systems with broken Lorentz invariance, these are superparticles which preserve 1/4 supersymmetry [19] and not half as the conventional ones.

where m is the mass of the single D0–brane (the mass of the N D0–branes being, naturally, Nm).

The action is worldvolume reparametrization invariant hence there must be a first class constraint on the dynamical variables associated with this local symmetry. If $\dot{\phi}$ was zero the constraint would have the form of the standard mass shell condition for a particle of mass Nm

$$p_M p^M + (Nm)^2 = 0\,, \tag{19}$$

where $p_M = (p_0, p_1)$ is the particle canonical momentum conjugate to x^M.

When $\dot{\varphi} \neq 0$ the first class constraint becomes [11]

$$p_M p^M + M^2(p_\varphi, p_1) = 0\,, \tag{20}$$

where

$$M^2 = \left[\left(Tr\sqrt{(\frac{1}{N}p_1 + p_\varphi)^2 + m^2} \right)^2 - p_1^2 \right] \tag{21}$$

is the effective mass of the system which depends on the momentum p_φ conjugate to the non–Abelian scalar $\varphi(\tau)$ and on the spatial momentum p_1[3], the dependence on the latter reflecting Lorentz non–invariance of the ND0–brane.

For further consideration it is convenient to present the ND0–brane action in the first–order form

$$S = \int d\tau \mathrm{Tr} \left\{ \frac{1}{N} p_M \dot{x}^M + p_\varphi \dot{\varphi} - \frac{e(\tau)}{2N}(p_M p^M + M^2(p_\varphi, p_1)) \right\}, \tag{22}$$

where $e(\tau)$ is the Lagrange multiplier for the mass shell condition.

4.2 Supersymmetrization

We shall now generalize the action (22) to be supersymmetric in target type IIA superspace whose Grassmann directions are parametrized by a two–component Majorana spinor θ^α ($\alpha = 1, 2$) or a pair of (one–component) Majorana-Weil spinors $\theta^\pm = \frac{1}{2}(1 \pm \gamma^3)\theta$, where $\gamma^3 = \gamma^0\gamma^1$ and $\gamma^M = \gamma^0, \gamma^1$ are $D = 2$ Dirac matrices[4].

To guess the form of the supersymmetric ND0–brane action let us recall the structure of the standard action for a type IIA D=2 superparticle of a mass Nm (see e.g. [16])

$$S = \int d\tau \left\{ p_M(\dot{x}^M + i\theta\gamma^M\dot{\theta}) - \frac{e(\tau)}{2}[p_M p^M + (Nm)^2] + iNm\,\theta\gamma^2\dot{\theta} \right\}, \tag{23}$$

where the last term in (23) is the Wess–Zumino or the Chern–Simons term. The action (23) is invariant under the $D = 2$ Lorentz rotations, Poincare translations, global supersymmetry

$$\delta_\epsilon \theta^\alpha = \epsilon^\alpha, \quad \delta_\epsilon x^M = -i\epsilon\gamma^M\theta, \qquad \delta_\epsilon P_M = 0 = \delta_\epsilon e \tag{24}$$

[3]The exact expressions of p_M and p_φ in terms of $\dot{x}^M$ and $\dot{\varphi}$ are given in [11].

[4]Note that the superspace under consideration is a direct low dimensional analog of type IIA $D = 10$ superspace.

and local fermionic κ–symmetry

$$\delta_\kappa \theta = (p_M \gamma^M + Nm\gamma^2)\kappa, \quad \delta_\kappa x^M = i\delta\theta\gamma^M\theta, \quad \delta_\kappa e = 4i\kappa^\alpha\dot{\theta}_\alpha, \quad \delta_\kappa p_M = 0, \tag{25}$$

where $\kappa^\alpha(\tau)$ is a Grassmann–odd Majorana spinor parameter of the κ–symmetry.

We are now in a position to write down the supersymmetric generalization of the ND0–brane action (22). To this end in the action (23) we should replace the mass Nm with the effective mass (21) and add the kinetic term for the non–Abelian $SU(N)$ scalar field φ. We shall also add a kinetic term for a fermionic superpartner of $\varphi(\tau)$ the worldvolume Grassmann–odd field $\psi(\tau)$. The super–ND0–brane action is [11]

$$\begin{aligned} S \;=\; & \int d\tau \mathrm{Tr}\left\{\frac{1}{N}p_M(\dot{x}^M + i\theta\gamma^M\dot{\theta}) \right. & (26)\\ & \left. -\frac{e(\tau)}{2N}\left[p_M p^M + M^2(p_\varphi, p_1)\right] + i\frac{M(p_\varphi,p_1)}{N}\theta\gamma^2\dot{\theta} + p_\varphi\dot{\varphi} - i\psi\dot{\psi}\right\}. \end{aligned}$$

The action (26) is invariant under the following κ–symmetry variations of fields

$$\delta_\kappa\theta = \left(p_M\gamma^M + M(p_\varphi,p_1)\gamma^2\right)\kappa(\tau),$$

$$\delta_\kappa x^M = i\delta\theta\gamma^M\theta \,+\, i\delta\theta\gamma^2\theta\,\frac{\partial M(p_\varphi,p_1)}{\partial p_1}\,\delta_1^M\,, \tag{27}$$

$$\delta_\kappa e = 4i\kappa^\alpha\dot{\theta}_\alpha,$$

$$\delta_\kappa\varphi = i\delta\theta\gamma^2\theta\left[\frac{\partial M(p_\varphi,p_1)}{N\partial p_\varphi}\right]_{trless}, \;\delta_\kappa\psi = 0\,. \tag{28}$$

In (27) $\delta_1^M = (0,1)$ is a unit space–like vector, which reflects Lorentz non–invariance of the model.

The action (26) is invariant under the following global $N=2$ supersymmetry transformations of the fields

$$\delta_\epsilon\theta^\alpha = \epsilon^\alpha, \quad \delta_\epsilon x^M = -i\epsilon\gamma^M\theta - i\epsilon\gamma^2\theta\frac{\partial M(p_\varphi,p_1)}{\partial p_1}\delta_1^M, \tag{29}$$

$$\delta_\epsilon\varphi = -i\epsilon\gamma^2\theta\left[\frac{\partial M(p_\varphi,p_1)}{N\partial p_\varphi}\right]_{trless}, \;\delta_\epsilon\psi = 0\,. \tag{30}$$

Note that since $M(p_\varphi,p_1)$, is a function of p_φ and p_1 the global supersymmetry transformation of the spatial coordinate x^1 gets modified (as under κ–symmetry (27)), which is the price for the model to be non–invariant under $D=2$ Lorentz transformations.

The modified global supersymmetry transformations (29), (30) of x^1 and φ close on generalized global bosonic 'translations' of x^1 and φ, therefore the supersymmetry algebra of the transformations (29)–(30) generated by the Poisson brackets of the Noether supercharges derived from the action (26) has the following form (26)

$$\{Q_\alpha, Q_\beta\} = 2ip_M\gamma^M_{\alpha\beta} + 2iM(p_\varphi,p_1)\,\gamma^2_{\alpha\beta}\,, \tag{31}$$

where

$$Q_\alpha = \pi_\alpha + ip_M(\gamma^M\theta)_\alpha + iM(p_\varphi,p_1)\,(\gamma^2\theta)_\alpha\,, \tag{32}$$

and π_α is the momentum conjugate to θ^α.

We observe that the superalgebra (31) has the "central charge" term proportional to the effective mass $M(p_\varphi, p_1)$ which arises because the $SU(N)$ adjoint scalars and x^1 nontrivially transform under supersymmetry. Such a "central extension" of the superalgebra is akin to the superalgebras associated with the single superbranes having the Wess–Zumino terms in their actions [17] and gauge fields on their worldvolumes which vary under target–space supersymmetry transformations [18].

The ND0–brane action (26) is also invariant under the following rigid worldvolume linear supersymmetry transformations of φ and ψ with a $U(N)$ adjoint fermionic parameter α

$$\delta\psi = [\alpha p_\varphi]_{trless}, \qquad \delta\varphi = 2i[\alpha\psi]_{trless}. \tag{33}$$

4.3 The Lorentz–covariant super–ND0–brane system

We now consider the Lorentz-covariant counterpart of the above model. For this we assume the effective mass (21) to be independent of the spatial momentum p_1, which is amount to dropping the term $\dot{x}^1\dot{\varphi}$ in the action (18)

$$M(p_\varphi) = Tr\sqrt{p_\varphi^2 + m^2\,\mathbf{I}}\,. \tag{34}$$

The supersymmetric and κ invariant action for the Lorentz invariant ND0–brane is

$$S = \int d\tau \mathrm{Tr}\left\{\frac{1}{N}p_M(\dot{x}^M + i\theta\gamma^M\dot{\theta})\right. \tag{35}$$

$$\left. -\frac{e(\tau)}{2N}\left(p_M p^M + M^2(p_\varphi)\right) + i\frac{M(p_\varphi)}{N}\theta\gamma^2\dot{\theta} + p_\varphi\dot{\varphi} - i\psi\dot{\psi}\right\}.$$

The variation properties of the fields with respect to the symmetries of the action (35) are the same as written in equations (24), (27)–(33), except that now the spatial coordinate x^1 transforms in the standard way under the κ–symmetry, $D = 2$ supersymmetry and translations.

By integrating over the momenta p_M and p_φ the Lorentz–covariant action (35) can be rewritten in the second order form[5]

$$S = -m\int d\tau Tr\left[-i\psi\dot{\psi} + \sqrt{\left(\sqrt{-(\dot{x}^M + i\theta\gamma^M\dot{\theta})^2} + i\theta\gamma^2\dot{\theta}\right)^2 - \dot{\phi}^2}\,\right]. \tag{36}$$

5 Interactions of Worldline Fermions and Generalization to Higher Dimensions

5.1 Fermion interactions

In the actions (26) and (35) the $SU(N)$ fermions have only kinetic terms, which can be generalized to include interactions with bosonic fields of the ND0–brane. A generic structure of the interaction term is

$$S_{int} = -\int d\tau\, Tr\, iV(p_\varphi, \varphi, x^M, p_M)\,\psi\dot{\psi}\,. \tag{37}$$

[5]For the non–covariant model we have not found a simple second–order form of the action.

where $V(p_\varphi, \varphi, x^M, p_M)$ is an $N \times N$ matrix which depends on the momenta and the non–Abelian scalar.

It would be of interest to find the most general form of V which is compatible with the target–space and worldline supersymmetry. An example of such an interaction term is the one in which V is a function of p_1 and p_φ only

$$S_{int} = -\int d\tau\, Tr\, iV(p_\varphi, p_1)\,\psi\dot{\psi}\,. \tag{38}$$

If in the action (26) we replace the fermion kinetic term with eq. (38) the resulting action will be supersymmetric, κ–invariant and will remain invariant under the worldline supersymmetry (33) provided the variations of φ and x^1 get modified as

$$\delta\varphi = 2i\left[\alpha\psi V(1 + \frac{1}{2}\frac{\partial lnV}{\partial p_\varphi}p_\varphi)\right]_{trless}\,. \tag{39}$$

$$\delta x^1 = iTr[\alpha\psi\frac{\partial V}{\partial p_1}p_\varphi]\,.$$

In the case of the Lorentz–covariant model of the previous section we should choose $V(p_\varphi)$ to be independent of the ND0–brane spatial momentum p_1, then x^1 will not transform under (39).

5.2 A non–Abelian super–D0–brane in higher dimensions

The models based on the actions (26) and (35) can be generalized to higher space–time dimensions D, to contain a higher number of non–Abelian bosonic fields φ^i (which we assume to be equal to the number of transverse spatial dimensions $D-1$, as in the case of the N coincident D0–branes) and non–Abelian fermionic fields which can take values in the vector or a spinor representation of the group $SO(D-1)$.

Below we shall give an example of the action for a non-Abelian D0–brane in type IIA $D = 10$ superspace which possesses all the supersymmetries of the ND0–brane model in $D = 2$, provided that the variations of the fields are modified accordingly,

$$\begin{aligned} S = \int d\tau \mathrm{Tr} \Big\{ & \frac{1}{N} p_M(\dot{x}^M + i\theta\gamma^M\dot{\theta}) \\ & - \frac{e(\tau)}{2N}\left[p_M p^M + M^2(p_{\varphi^i}, p_i)\right] + i\frac{M(p_{\varphi^i}, p_i)}{N}\theta\gamma^{11}\dot{\theta} \\ & + p_{\varphi^i}\dot{\varphi}^i - i\sum_{n=1}^{9} V_n(p_\varphi, p_j)(\psi^i\dot{\psi}^i)^n \Big\}, \end{aligned} \tag{40}$$

where $M = 0, 1, \cdots, 9$, $i = 1, \cdots, 9$ and

$$M^2(p_{\varphi^i}, p_i) = \left[\left(Tr\sqrt{(\frac{1}{N}p_i + p_{\varphi^i})^2 + m^2}\right)^2 - p_i^2\right].$$

As above, in (40) we have assumed the trace to be symmetric for the Grassmann–even matrices and antisymmetric for the Grassmann–odd matrices, and the last term is a polinomial in $\psi^i\dot{\psi}^i$ with the coefficients V_n depending on the momenta.

The Lorentz–covariant $D = 10$ ND0–brane model is obtained from (40) by requiring that M and V_n do not depend on p_i.

With an appropriate choice of V_n the action (40) may be regarded as a supersymmetric generalization of the bosonic action for N coincident D0–branes (9) in the approximation where the commutator of the scalar fields $[\phi^i, \phi^j]$ (12) is dropped out.

6 Conclusion

In this contribution we have raised a question about the Lorentz invariance of the actions for N coincident Dp–branes and have found that this problem requires further analysis and deeper insight. We have demonstrated that models of ND0–branes allow for supersymmetric extensions invariant under the *'abelian'* κ–symmetry and worldline supersymmetries if one interprets these models as single D0–branes carrying non–Abelian bosonic and fermionic fields on their worldlines.

As a generalization of these results it would be of interest to construct a supersymmetric version of the ND0–brane model with interactions of non–Abelian fields involving the commutator of the $SU(N)$ scalars (12), and to get supersymmetric extensions of NDp–branes of higher dimension and codimension.

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Domain Walls and Spaces of Special Holonomy

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Abstract

We review the relations between a family of domain-wall solutions to M-theory and gravitational instantons with special holonomy. When oxidized into the maximal-dimension parent supergravity, the transverse spaces of these domain walls become cohomogeneity-one spaces with generalized Heisenberg symmetries and a homothetic conformal symmetry. These metrics may also be obtained as scaling limits of generalized Eguchi-Hanson metrics, or, with appropriate discrete identifications, from generalized Atiyah-Hitchin metrics, thus providing field-theoretic realizations of string-theory orientifolds.

1 Introduction

Sets of magnetically charged p-branes with a compact transverse space necessarily include both positive and negative tension branes. This is because, although one allows the Bianchi identities for the field strength $F_{[n]}$ supporting the brane to be violated by magnetic source terms, the resulting $dF_{[n]}$ should still have a vanishing integral over a compact space. This "cohomology condition" is at the origin of the pairing of positive and negative tension orbifold planes in the Hořava-Witten construction [1] and in the dimensionally reduced interpretation of this construction in terms of fixed 3-branes in $D = 5$ supergravity [3]. Codimension-one BPS solutions are characterized by a linear harmonic function, and the equality of positive and negative charges in such solutions is a consequence of the necessity for this function to be continuous in the compact transverse space, thus requiring equality of 'upwards' and 'downwards' kinks. The same basic pattern is found in the more phenomenological approach of Randall and Sundrum [2] for branes crafted from patched sections of $D = 5$ anti de Sitter space.

For purely gravitational solutions, we shall review below how a simple sum-rule argument establishes the impossibility of resolving all the singularities in domain-wall spacetimes of this type with a compact transverse space [4]. A domain wall (*i.e.* a $(D-2)$-brane) must be supported by the combined effects of gravity and a scalar field with a potential. Descending from higher dimensions, this may be viewed as the result of a dimensional reduction in which the supporting form field $\mathcal{F}_{nm}$ has been reduced to a constant 0-form in the "internal" dimensions. The source-free Einstein equations in this case may be organized so as to have a total derivative on one side and a sum of non-negative terms on the other. Integration of the total derivative over a compact transverse space then yields zero, forcing the vanishing of the derivative of the scalar field that would otherwise have to support the domain wall, thus ruling out a non-trivial brane spacetime. Of course, if a source is present the situation is different, and non-trivial spacetimes can in that case be found. But with a source present, the spacetime has a singularity.

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Although it is not possible to find a solution with all singularities resolved on a compact transverse space, it is still of interest to look for resolved domain-wall solutions with noncompact transverse spaces. Such a solution could be viewed as "half" of a Hořava-Witten configuration. It is of particular interest to see whether the negative tension objects can be resolved, as these are the ones for which no acceptable delta-function source exists in string theory. Moreover, experience shows that the possibilities for resolving singularities may be greater if one dimensionally oxidizes a solution into the highest-dimensional parent supergravity, since dimensional reduction can introduce singularities in otherwise non-singular spacetimes [5].

The tension T of a brane solution can be read off from the change in the extrinsic curvature $K_{\mu\nu}$, since by the Israel matching condition for extrinsic curvatures on + and − sides of a domain wall, one has

$$\Delta K_{\mu\nu} = K^{(+)}_{\mu\nu} - K^{(-)}_{\mu\nu} = -\frac{8\pi G}{3} T \tag{1}$$

where in coordinates locally adapted to the brane geometry the extrinsic curvature is given by $K_{\mu\nu} = \frac{1}{2} n^{\hat\lambda}\partial_{\hat\lambda} g_{\mu\nu}$, where $n^{\hat\lambda}$ is the normal to the domain-wall worldvolume. As we shall shortly be considering both D and $D-1$ dimensional contexts, we adopt the convention that "hatted" indices refer to the full embedding spacetime containing the brane, while "unhatted" indices refer only to the brane worldvolume coordinates x^μ. In a BPS domain-wall spacetime characterized by a linear harmonic function $H = c + my$ of the transverse coordinate y, with metric

$$ds^2 = H^{\frac{4}{\Delta(D-2)}} dx^\mu dx_\mu + H^{\frac{4(D-1)}{\Delta(D-2)}} dy^2 \ , \tag{2}$$

the extrinsic curvature is proportional to the slope parameter m. Thus, in a standard domain wall setup where one reflects the harmonic function, *e.g.* at $y = 0$, so $H = c + m|y|$, the tension (1) is proportional to $-m$. Accordingly, for a negative tension domain wall, the warp factor $e^{2A(y)} = H^{\frac{4}{\Delta(D-2)}}$ of the embedding D-dimensional spacetime *grows* as one moves away from the brane, and conversely, the warp factor for a positive tension brane *decreases* as one moves away from the brane. In the following, we will review what is known about the possibility of resolving the singularities at such branes.

2 Difficulties with Singularity Resolutions

Quite a lot of effort has been devoted to the question of resolving domain-wall singularities. This has been done by analysis of the possible renormalization group flows for the supporting scalar fields [6, 7] in the embedding spacetime. Another approach is *via* sum rules derived from the Einstein equations [4]. For example, one can start from the action for gravity coupled to a nonlinear sigma model for a set of scalars ϕ^I, $I = 1, \ldots, N$ with a potential $V(\phi)$ and possible sources J_α:

$$I_{g,\phi} = \int d^{d+1}x\sqrt{-g}\left(\frac{1}{2\kappa^2}R - G_{IJ}(\phi)\partial_M\phi^I\partial_N\phi^J - V(\phi) - \sum_\alpha J_\alpha(\phi)\delta(y - y_\alpha)\right) . \tag{3}$$

For a generalized domain-wall spacetime, one takes the metric ansatz

$$ds^2 = e^{2A(y)} g_{\mu\nu}(x) dx^\mu dx^\nu + e^{2B(y)} \ , \tag{4}$$

leading to Einstein equations that can be combined to give on the left-hand side a total derivative of the warp factor,

$$A'' = \frac{\kappa^2}{12}(T^\mu_\mu - 4T^5_5) - \frac{1}{12}e^{-2A}R_{\text{brane}} \,, \tag{5}$$

where R_{brane} is the Ricci scalar for the brane worldvolume metric $g_{\mu\nu}$. Substituting for $T_{\mu\nu}$ and T_{55} and integrating $\oint dy$ over a compact transverse space, the total derivative A'' drops out and so one has the sum rule

$$\oint dy \left(g_{IJ}\phi^{I\prime}\phi^{J\prime} + \sum_\alpha J_\alpha(\phi)\delta(y-y_\alpha) + \frac{1}{4\kappa^2}e^{-2A}R_{\text{brane}} \right) = 0 \,. \tag{6}$$

Thus, in order to resolve the singularities of a set of domain walls with a compact transverse space and with vanishing worldvolume Ricci scalar R_{brane}, the absence of sources J_α would imply $\oint dy\,(g_{IJ}\phi^{I\prime}\phi^{J\prime}) = 0$ and for positive definite sigma model metric g_{IJ} this would require constant scalar fields, $\phi^{I\prime} = 0\ \forall_I$. But nontrivial scalars are needed to support a domain-wall spacetime, so this rules out the possibility of resolving all the domain-wall singularities in a compact transverse space.

If one cannot resolve a full Hořava-Witten pair of domain walls, then one can try the next best thing, *i.e.* to resolve just half of it, removing the second brane to infinity and looking for a nonsingular solution for just one brane. Here one encounters a further difficulty: "double-ended" spacetimes with growing warp factor as one recedes from the brane cannot be resolved. The best chance of resolving such a spacetime would be after oxidizing it from D dimensions up into a solution of its maximal-dimension parent supergravity, *e.g.* into $\hat{D} = 11$, where singularities in dilatonic scalars are not a problem because there are no such scalars. After this dimensional oxidation, one is not dealing anymore with a brane of codimension one, but instead the $(\hat{D} - D + 1)$ dimensional transverse space now becomes a space of cohomogeneity one, with the $y = $ const slices representing homogeneous spaces on which the $(\hat{D} - D)$ dimensional oxidation has been performed. A negative tension domain wall in D dimensions with $H = 1 + m|y|$, $m > 0$, would lift after such dimensional oxidation to a spacetime with a "saddle," reducing to a minimum cross-sectional volume in the extra $d = (\hat{D} - D)$ dimensions and then re-expanding. But for solutions to the purely gravitational sector of $D = 11$ supergravity, there cannot be smooth solutions of this sort, as a result of the Cheeger-Gromoll theorem [8], or as can be seen by the following simple argument [9]. First transform the transverse y coordinate into $\tilde{y}$, the proper distance from the cross section of minimum volume, and consider the metric ansatz $ds^2 = d\tilde{y}^2 + g_{ij}(x,\tilde{y})dx^i dx^j$, where $i, j = 1, \dots d = (\hat{D} - D)$. Let $V(x,y) = \sqrt{\det g_{ij}}$ and consider $\Theta = V'/V$, which is the expansion rate of geodesic congruence in this spacetime. This quantity is subject to the Raychaudhuri inequality

$$\frac{d\Theta}{d\tilde{y}} \leq -\frac{1}{d}\Theta^2 - \Sigma_{ij}\Sigma^{ij} \,, \tag{7}$$

where $\Sigma_{ij} = \frac{\partial g_{ij}}{\partial \tilde{y}} - \frac{1}{d}g^{rs}\frac{\partial g_{rs}}{\partial \tilde{y}}g_{ij}$. Thus, if Θ is negative at some value of $\tilde{y}$, it remains negative always, so a nonsingular "saddle" spacetime is not possible, at least in this purely gravitational setting.

Thus, we now come to consider "single-sided" domain walls, whose BPS metric can be taken to be derived from the harmonic function $H = my$. The "single-sided" space is obtained from the double-sided space by making a Z_2 identification between the two halves.

A priori, the single-sided case may look even less promising for singularity resolution than the double-sided one, since one is trading in a fairly mild delta-function singularity in the double-sided metric with $H = c + m|y|$ for a more serious single-sided singularity in the curvature where $H = 0$. However, one has to remember that problems with singularities can be improved after dimensional oxidation, so one needs to keep an open mind at this point.

3 Homothetic Heisenberg Brane Metrics

Let us now consider a specific class of domain wall solutions [9] which oxidize up to a very suggestive class of metrics in $\hat{D} = 11$. As a basic example, consider a domain wall in $D = 8$, supported purely by fields derived from the $\hat{D} = 11$ metric. In order to generate the scalar potential needed to support this domain wall in $D = 8$, the dimensional reduction down from $\hat{D} = 11$ needs to have a constant flux turned on in the compact 9 & 10 directions: $\mathcal{F}^1_{9\,10} = m$, where $\mathcal{F}^1$ is the field strength of the Kaluza-Klein vector $\mathcal{A}^1$ arising in the initial $\hat{D} = 11 \to 10$ reduction. The resulting 6-brane metric in $D = 8$ is

$$ds_8^2 = H^{\frac{1}{6}} dx^\mu dx_\mu + H^{\frac{7}{6}} dy^2 \ . \tag{8}$$

As we have noted above, the best chance for a resolution of the singularity at the surface where $H = 0$ is to be found after oxidation on coordinates (z_1, z_2, z_3) back up to $\hat{D} = 11$, yielding the metric

$$ds_{11}^2 = dx^\mu dx_\mu + ds_4^2 \tag{9}$$

where the transverse metric ds^4 is

$$ds_4^2 = H^{-1}(dz_1 + m z_3 dz_2)^2 + H(dy^2 + dz_2^2 + dz_3^2) \ . \tag{10}$$

The metric (10) has some striking properties [8]. First of all, it is Kähler. To see this, let us adopt an orthonormal basis of vierbein 1-forms

$$\begin{aligned} e^0 &= H^{\frac{1}{2}} dy & e^1 &= H^{-\frac{1}{2}}(dz_1 + m z_3 dz_2) \\ e^2 &= H^{\frac{1}{2}} dz_2 & e^3 &= H^{\frac{1}{2}} dz_3 \ . \end{aligned} \tag{11}$$

Using these, we can give the Kähler form $J = e^0 \wedge e^1 - e^2 \wedge e^3 = \frac{1}{2} J_{ij} dw^i \wedge dw^j$. where $w^i = (y, z_1, z_2, z_3)$. Using J_i^j, one can make holomorphic $P_i{}^j = \frac{1}{2}(\delta_i{}^j - \mathrm{i} J_i{}^j)$ and antiholomorphic $Q_i{}^j = \frac{1}{2}(\delta_i{}^j + \mathrm{i} J_i{}^j)$ projectors. The Darboux form for the metric (10) is achieved by solving the differential equations $Q_i{}^j \partial_j \zeta^\mu = 0$ for complex coordinates ζ^μ, $\mu = 1, 2$. A sample solution to these equations is

$$\begin{aligned} \zeta^1 &= z_3 + \mathrm{i} z_2 \\ \zeta_2 &= y + \mathrm{i} z_1 - \tfrac{1}{4} m(z_2^2 + z_3^2) + \frac{\mathrm{i}}{2} m z_2 z_3 \ . \end{aligned} \tag{12}$$

Changing over to these complex coordinates, the metric becomes

$$ds_4^2 = 2 g_{\mu\bar{\nu}} d\zeta^\mu d\bar{\zeta}^{\bar{\nu}} = H |d\zeta^1|^2 + H^{-1} |d\zeta^2 + \tfrac{1}{2} m \bar{\zeta}^1 d\zeta^1|^2 \ , \tag{13}$$

where now $H = [1 + m(\zeta^2 + \bar{\zeta}^2) + \frac{1}{2}|\zeta^1|^2]^{\frac{1}{2}}$. The Kähler structure is then made explicit by writing $g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} K$ with $K = 2H^3/(3m^2)$.

Constructing the curvature for the vierbeins (11) shows it to be self-dual in the $d=4$ Euclidean-signature dimensions w^i, and to have SU(2) holonomy. The suspicion arises thus that the metric (10) may have something to do either with the gravitational instanton K3 or with the Eguchi-Hanson metric. We will confirm these associations by considering the symmetries of the metric (10).

For simplicity, we now let the slope parameter be $m=1$, and rewrite the metric (10) using the proper distance $\tilde{y}=\frac{2}{3}y^{3/2}$ as the radial coordinate. The metric (10) becomes

$$ds^2 = d\tilde{y}^2 + (\frac{3\tilde{y}}{2})^{-\frac{2}{3}}\sigma_3^2 + (\frac{3\tilde{y}}{2})^{\frac{2}{3}}(\sigma_1^2+\sigma_2^2) \ , \tag{14}$$

where $\sigma_1 = dz_2$, $\sigma_2 = dz_3$, $\sigma_3 = dz_1 + z_3 dz_2$. The σ_i are 1-forms satisfying the algebra

$$d\sigma_1 = 0 \qquad d\sigma_2 = 0 \qquad d\sigma_3 = -\sigma_1 \wedge \sigma_2 \ . \tag{15}$$

They are left-invariant with respect to the *Heisenberg group* of 3×3 upper-triangular matrices

$$g = \begin{pmatrix} 1 & k_2 & k_1 \\ 0 & 1 & k_3 \\ 0 & 0 & 1 \end{pmatrix} \tag{16}$$

with corresponding Killing vectors

$$R_1 = \frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_1} \ , \qquad R_2 = \frac{\partial}{\partial z_3} \ , \qquad R_3 = \frac{\partial}{\partial z_1} \tag{17}$$

satisfying the Heisenberg algebra

$$[R_1,R_3] = [R_2,R_3] = 0 \ , \qquad [R_1,R_2] = -R_3 \ . \tag{18}$$

In addition to the Heisenberg symmetry, the metric (10) is characterized by a rate of growth in volume between $\tilde{y}_0$ and $\tilde{y}$ that goes like $\tilde{y}^{\frac{4}{3}}$.

The Heisenberg symmetry and the rate of volume growth for the metric (10) led Gibbons and Rychenkova [8] to identify it with a singular (BKTY) limit of the K3 metric identified by Tian & Yau and Bando & Kobayashi [10]. In this singular limit, K3 is rendered noncompact: the parts of the manifold that do not correspond to the metric (10) are "blown away" to infinity. The Heisenberg (or "Nil") symmetry appears only in this singular limit, since the compact K3 has no Killing vectors. Accordingly, we shall call such limits "Heisenberg limits." From the present example, we learn that domain-wall solutions with singularities can sometimes be resolved into nonsingular spaces after taking a single-sided interpretation of the original space and then dimensionally oxidizing into the highest-dimension parent supergravity theory. In this construction, the original single transverse coordinate y becomes firstly the radial coordinate for a space of cohomogeneity one, in which the $y=$ const slices are homogeneous subspaces. In the subsequent resolution into a smooth space like K3, this cohomogeneity-one structure may be lost.

Another characteristic property of the metric (10), which will generalize to other examples of this sort, is its homothetic scaling symmetry. The transformation

$$z_1 \to \lambda^2 z_1 \ , \qquad z_2 \to \lambda z_2 \ , \qquad z_3 \to \lambda z_3 \ , \qquad y \to \lambda y \tag{19}$$

causes the metric to scale by a constant factor, $ds_4^2 \to \lambda^3 ds_4^2$. A conformal symmetry of this sort, with a constant scaling factor for the metric, is known as a "homothety." The corresponding Killing vector is

$$D = y\frac{\partial}{\partial y} + 2z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3} \; ; \tag{20}$$

the existence of this conformal symmetry derives from the fact that the Heisenberg algebra is itself scale invariant under

$$\sigma_1 \to \lambda\sigma_1 \; , \qquad \sigma_2 \to \lambda\sigma_2 \; , \qquad \sigma_3 \to \lambda^2\sigma_3 \; . \tag{21}$$

Note in particular that the "slope" parameter m is not changed in this transformation, so, unlike apparently similar integration constants for higher codimension branes, m does not fix a dimensional quantity that determines how the geometry changes as one moves away from the $H = 0$ surface; indeed, if it did, there could be no such scaling symmetry. The scaling symmetry is obtained, however, only for the single-sided metric with $H = my$ that we are considering here; the double-sided case with $H = \text{const} + m|y|$ breaks the scaling symmetry at the point of reflection.

4 Relation to the Eguchi-Hanson metric

The resolution discussed above for the metric (10) is highly implicit, owing to the fact that the explicit metric on K3 is unknown. At the expense of retaining the singularity, however, one may find a rather more explicit relation to a self-dual metric that has frequently been discussed in connection with approximations to K3, namely the Eguchi-Hanson metric [11]. The Eguchi-Hanson metric is

$$ds^2_{\rm EH} = \left(1 + \frac{\tilde{Q}}{\tilde{r}^4}\right)^{-1} d\tilde{r}^2 + \tfrac{1}{4}\tilde{r}^2 \left(1 + \frac{\tilde{Q}}{\tilde{r}^4}\right)\tilde{\sigma}_3^2 + \tfrac{1}{4}\tilde{r}^2 9\sigma_1^2 + \sigma_2^2) \; , \tag{22}$$

where the $\tilde{\sigma}_i$ are left-invariant 1-forms with respect to SU(2), satisfying the algebra

$$d\tilde{\sigma}_1 = -\tilde{\sigma}_2 \wedge \tilde{\sigma}_3 \; , \qquad d\tilde{\sigma}_2 = -\tilde{\sigma}_3 \wedge \tilde{\sigma}_1 \; , \qquad d\tilde{\sigma}_3 = -\tilde{\sigma}_1 \wedge \tilde{\sigma}_2 \; . \tag{23}$$

If one now performs an Inönü-Wigner contraction,

$$\tilde{\sigma}_1 = \lambda\sigma_2 \; , \qquad \tilde{\sigma}_2 = \lambda\sigma_2 \; , \qquad \tilde{\sigma}_3 = \lambda^2\sigma_3 \; , \tag{24}$$

then in the limit $\lambda \to 0$, one obtains the previous differential algebra (15) for $\sigma_1, \sigma_2, \sigma_3$.

In order to take this limit in the metric, one needs first to rescale the radial coordinate and the charge, $\tilde{r} = \lambda^{-1}r$, $\tilde{Q} = \lambda^{-6}Q$, and also needs to make a coordinate transformation $y = \frac{1}{4}mr^2$. Then, upon making the identification $Q = \frac{16}{m^4}$, one obtains the internal sector (10) of the $D = 8$ domain wall metric, with $\sigma_1 = mdz_2$, $\sigma_2 = mdz_3$, $\sigma_3 = m(dz_1 + mz_3dz_2)$. Thus the scaling by λ blows up the EH region $\tilde{r} \approx 0$ into the region described by the domain-wall metric (10).

Note that one is dealing here with a singular version of the Eguchi-Hanson metric with $\tilde{Q} > 0$ (the non-singular metric has $\tilde{Q} < 0$, and is complete for $\tilde{r} \geq |\tilde{Q}|^{\frac{1}{4}}$), so this construction does not actually resolve the domain-wall singularity.

Within the unscaled EH metric, the far region where $\tilde{r} \to \infty$ asymptotically tends to a Ricci-flat cone over RP^3. This region has a scaling symmetry $\tilde{r} \to \tilde{\lambda}\tilde{r}$, $ds^2_{\mathrm{EH}} \to \tilde{\lambda} ds^2_{\mathrm{EH}}$, generated by a homothetic conformal Killing vector $E = \tilde{r}\frac{\partial}{\partial \tilde{r}}$. In fact, one also has in this region $\nabla_\mu E_\nu = g_{\mu\nu}$ and since $\nabla_{[\mu}E_{\nu]} = 0$, it follows that E is a gradient (this is also clear from its explicit form). So E is also hypersurface orthogonal.

In the middle region of the EH metric, $0 << \tilde{r} << \infty$, the E scaling symmetry is lost. This accords with the fact that the slices at fixed $\tilde{r}$ are orbits of SU(2), which does not have a scaling symmetry.

The region near the EH singularity $\tilde{r} \approx 0$ blows up, as we have seen, into the metric (10) which has the Heisenberg group symmetry and also a homothetic conformal Killing vector $D = y\frac{\partial}{\partial y} + 2z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3}$. This conformal Killing vector is not, however, proportional to a gradient, so it is not hypersurface-orthogonal.

5 Higher Dimensions and Orientifolds

Starting from a series of gravitational domain-wall solutions in lower dimensions, with less preserved supersymmetry than in the above $D = 8 \leftrightarrow d = 4$ example, one can similarly oxidize up to $\hat{D} = 11$ and so obtain from their transverse spaces a series of cohomogeneity-one manifolds of special holonomy with varying amounts of preserved supersymmetry.

For example, we can consider a domain wall in $D = 6$ for the theory obtained by turning on fluxes $\mathcal{F}_{9,10}$ and $\mathcal{F}_{7,8}$ for the gravitational Kaluza-Klein vector $\mathcal{A}_m$ arising in the original $D = 11 \to D = 10$ reduction (in the notation of Ref. [12], these fluxes correspond to the zero-form field strengths $\mathcal{F}^1_{(0)\,23}$ and $\mathcal{F}^1_{(0)\,45}$). In $D = 6$, one then obtains a single-sided domain wall metric depending on a linear harmonic function $H = my$ in the codimension-one transverse space,

$$ds^2 = H^{\frac{1}{2}} dx^\mu dx_\mu + H^{\frac{5}{2}} dy^2 \qquad \mu = 0, 1, \ldots, 4\,. \tag{25}$$

Oxidizing this metric up to $\hat{D} = 11$, one obtains a metric $d\hat{s}^2 = dx^\mu dx_\mu + ds_6^2$, from which one may extract the $d = 6$ cohomogeneity-one part

$$ds_6^2 = H^{-2}[dz_1 + m(z_3 dz_2 + z_5 dz_4)]^2 + H^2 dy^2 + H(dz_2^2 + \cdots + dz_5^2)\,. \tag{26}$$

Defining an orthonormal basis for this metric by

$$\begin{aligned} e^0 &= H dy & e^1 &= H^{-1}[dz_1 + m(z_3 dz_2 + z_5 dz_4)] \\ e^2 &= H^{\frac{1}{2}} dz_2 & e^3 &= H^{\frac{1}{2}} dz_3 \\ e^4 &= H^{12} dz_4 & e^5 &= H^{\frac{1}{2}} dz_5\,, \end{aligned} \tag{27}$$

one finds a Kähler form $J = e^0\wedge e^1 - e^2\wedge e^3 - e^4\wedge e^5$. The metric (26) has SU(3) holonomy, corresponding to the $\frac{1}{4}$ supersymmetry preserved by the $D = 6$ domain wall.

By a generalization of the arguments in the $d = 4$ case presented above, the $d = 6$ metric (26) may also be obtained as a Heisenberg limit of a smooth gravitational instanton, in this case a Calabi-Yau 3-fold [8]. Now the volume grows as $\tilde{y}^{\frac{3}{2}}$ as one recedes from the singularity, and, after taking a scaling limit analogous to the BKTY one, the metric develops a generalized Heisenberg symmetry with respect to which which the homogeneous $\tilde{y} =$ const slices are group orbits having the structure of a T^1 bundle over T^4 [9]. For

$H = my$ one finds again a homothetic conformal Killing vector $D = y\frac{\partial}{\partial y} + 3z_1\frac{\partial}{\partial z_1} + \frac{3}{2}z_2(\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3} + z_4\frac{\partial}{\partial z_4} + z_5\frac{\partial}{\partial z_5})$. And, similarly to the $d = 4$ example, the $d = 6$ metric (26) can also be obtained from a $d = 6$ generalization of the Eguchi-Hanson metric [9].

Higher-dimensional cohomogeneity-one gravitational domain walls of special holonomy may also be obtained as Heisenberg limits of corresponding-dimensional gravitational instantons. For example, one has a $\frac{1}{4}$ supersymmetric $D = 5 \leftrightarrow d = 7$ metric with G_2 holonomy,

$$\begin{aligned} ds_7^2 &= H^4dy^2 + H^{-2}[dz_1 + m(z_4dz_3 + z_6dz_5)]^2 + H^{-2}[dz_2 + m(z_5dz_3 - z_6dz_4)]^2 \\ &\quad + H^2(dz_3^2 + \cdots + dz_5^2)\ , \end{aligned} \tag{28}$$

or a $\frac{1}{16}$ supersymmetric $D = 4 \leftrightarrow d = 8$ metric with Spin(7) holonomy,

$$\begin{aligned} ds_8^2 &= H^6dy^2 + H^{-2}[dz_1 + m(z_5dz_4 + z_7dz_6)]^2 + H^{-2}[dz_2 + m(z_6dz_4 - z_7dz_5)]^2 \\ &\quad + H^{-2}[dz_3 + m(z_7dz_4 + z_6dz_5)]^2 + H^3(dz_4^2 + \cdots + dz_7^2)\ . \end{aligned} \tag{29}$$

The relations between domain walls and gravitational instantons that we have reviewed here all have negative tension, as can be seen from their growth in volume as one recedes from the singularity and in analogy with the matching condition (1). Moreover, we have seen that there can be Heisenberg limits to domain wall spacetimes from various previously known spacetimes: we have seen implicit limits to such domain walls from nonsingular K3 and Calabi-Yau spaces and also more explicit limits (but without resolving the singularities) from generalized Eguchi-Hanson spaces.

Now, in string theory, one does not have acceptable source actions for negative tension objects, since the corresponding brane waves would have negative kinetic energy. Thus it is fortunate that the cases we have investigated can actually be resolved into portions of nonsingular gravitational instanton spaces, which do not require such sources. The negative tension objects that string theory does contain, however, are orientifold planes. These avoid having unacceptable negative energy brane waves since they are fixed in position by virtue of their $\mathbb{Z}_2$ identifications: the orientifold plane is located at the $\mathbb{Z}_2$ fixed point. Returning to the effective supergravity theory, one is, accordingly, interested in finding spacetimes corresponding to orientifold planes. One such orientifold plane is related to our basic 6-brane example (9,10). In this case, the orientifold spacetime corresponds to the product of $d = 7$ Minkowski space times the Atiyah-Hitchin (AH) metric [13], which can be written in the form

$$ds_{\rm AH}^2 = dt^2 + a^2(t)\sigma_1^2 + b^2(t)\sigma_2^2 + c^2(t)\sigma_3^2\ . \tag{30}$$

At large t, one has $a(t) \to b(t)$ and the AH metric tends to the Taub-NUT metric, which in appropriate coordinates is

$$ds_{\rm TN}^2 = 4(1 + \frac{2M}{r})^{-1}(d\psi + \cos\theta d\phi) + (1 + \frac{2M}{r})(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)) \tag{31}$$

with, however, $M < 0$ and which is here taken subject to the $\mathbb{Z}_2$ identification $(\psi, \theta, \phi) \leftrightarrow (\psi - \pi, \pi - \theta, \phi + \pi)$. Taking a Heisenberg limit of this metric, one reobtains our conical spacetime (10), but subject now to the $\mathbb{Z}_2$ identification $(z_1, z_2, z_3) \leftrightarrow (-z_1, -z_2, z_3)$. Accordingly, the AH metric is described as being "asymptotically locally conical" (ALC).

In string theory, the macroscopic curvature of a supergravity solution is interpreted as arising from the accumulation of charge $\leftrightarrow$ tension for a stack of microscopic extended objects. Accordingly, one may interpret our basic 6-brane solution (9) also as the Heisenberg limit of the metric for a stack of orientifold planes, subject also now to the above $\mathbb{Z}_2$ identification. The ADM mass M of this spacetime is negative, corresponding to the negative tension. However, there are no negative energy brane waves on such a background because the $\mathbb{Z}_2$ identification eliminates translational zero modes. This ALC solution manages to avoid being inconsistent with the positive mass theorem because its structure at infinity is not simply-connected, owing to the $\mathbb{Z}_2$ identification, and this interferes with the solution of the Dirac equation subject to boundary conditions at infinity, which is an essential element of the positive mass theorem proof.

Higher-dimensional examples of ALC metrics can be found using the recent explicit constructions of metrics with Spin(7) holonomy [14] and G_2 holonomy [15].

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Part II

Noncommutativity, Integrability and Quantum Groups

Noncommutative Geometry, Noncommutative Field Theories and String Field Theories*

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Abstract

In this lecture notes we review applications of noncommutative geometry to construction of noncommutative field theories and string field theories. We consider noncommutative quantum field theories emphasizing an issue of their renormalizability and the UV/IR mixing. We describe main ingredients of cubic string field theories and methods which are used to study the cubic open string field theory around the tachyon vacuum.

1 Introduction

It is well known that the gauge invariance principle is a basic principle in construction of modern field theories. Both general relativity and Yang-Mills theory use the gauge invariance. Another example of gauge invariant theory is string field theory [1]. To formulate this theory it is useful to employ notions of noncommutative differential geometry.

Noncommutative geometry has appeared in physics in the works of the founders of quantum mechanics. Heisenberg and Dirac have argued that the phase space of quantum mechanics must be noncommutative and it should be described by quantum algebra. After works of von Neumann and more recently by Connes mathematical and physical investigations in noncommutative geometry became very intensive.

Noncommutative geometry uses a generalization of the known duality between a space and the algebra of functions on it [2, 3]. One can take $\mathcal{A}$ to be an abstract noncommutative associative algebra $\mathcal{A}$ with a multiplication $\star$ and interpret $\mathcal{A}$ as an "algebra of functions" on a (nonexisting) noncommutative space. One can introduce many geometrical notions in this setting. A basic set consists of an external derivative Q (or an algebra $\mathcal{G}$ of derivatives) and an integral, which satisfy ordinary axioms. In Section 3 we will present several examples of noncommutative spaces.

Let us sketch motivations/applications of models of noncommutative differential geometry (NCDG) in physics:

- Noncommutative space-time;
- Large N reduced models (quenched Yang-Mills theory);
- D-brane physics, (M)atrix theory;
- Description of the behavior of models in external fields;
- Noncommutative Solitons and Instantons;

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- Smoothing of singularities;
- String Field Theory (SFT).

Noncommutative Space-Time

- Uncertainty principle in gravity

There is an absolute limitation on length measurements in quantum gravity

$$\ell \geq \ell_{Pl}. \tag{1}$$

The reason is the following. If we want to locate a particle we need to have energy greater than the Planck mass $m_{Pl} = (\hbar c G^{-1})^{\frac{1}{2}}$ (G is Newton's constant). The corresponding gravitational field will have a horizon at $r = 2Gm_{Pl}c^{-2} = 2\ell_{Pl}$, shielding whatever happens inside the Schwarzschild radius and therefore the smallest "quanta" of space and time exists. It is tempting to relate the uncertainty δx in the length measurement (1) to the noncommutativity of coordinates

$$[x^\mu, x^\nu] = i\hbar\theta^{\mu\nu}. \tag{2}$$

The idea of quantization of space-time using noncommutative coordinates x_μ is an old one [4]-[7]. Snyder [5] has proposed to use the noncommutative space to improve the UV behavior in quantum field theory. An extension of the Snyder's idea was considered in [8] where the field theory with the momentum in the de Sitter space was considered. A proposal for a general (pseudo)Riemannian geometry in the momentum space was discussed in [9].

Deriving relation (1) from (2) by analogy with the derivation of the Heisenberg uncertainty inequality one gets the problem because (1) has to hold even for one-dimensional space. In [10] it was suggested that (1) indicates that at the Planck scale one should use non-Archimedean geometry. For recent discussion of space-time noncommutativity see [11]-[15]. The space-time stringy uncertainty has been discussed in [12, 16].

- UV divergences in QFT

One of earlier motivations to consider field theories on noncommutative space-time (noncommutative field theories) was a hope that it would be possible to avoid quantum field theory divergences [11, 12, 3, 18, 14, 17]. Now a commonly accepted belief is that a theory on a noncommutative space is renormalizable if (and only if) the corresponding commutative theory is renormalizable. Results on the one-loop renormalizability of noncommutative gauge theory [19]-[21] as well as the two-loop renormalizability of the noncommutative scalar ϕ^4_4 theory [22] support this belief. However the renormalizability does not guarantee that the theory is well-defined. There is a mixing of the UV and the IR divergences [23, 22]. For further developments see [24] and refs. therein.

D-branes and M(atrix) theory

Connes, Douglas and Schwartz [25] have shown that supersymmetric gauge theory on a noncommutative torus is naturally related to a compactification of the M(atrix) theory. M(atrix) theory is a maximally supersymmetric quantum mechanics with the action:

$$\int dt \, \mathrm{Tr} \sum_{i=1}^{9} (D_t X)^2 - \sum_{i<j} [X^i, X^j]^2 + \chi^\dagger (D_t + \Gamma_i X^i) \chi. \tag{3}$$

Here $D_t = \partial/\partial t + iA_0$, and variation over A_0 yields a Gauss law. This action is also known as a regularized form of the supermembrane action. Equations of motion for the action (3) admit the following static solutions

$$[X^\mu, X^\nu] = i\hbar\theta^{\mu\nu} \tag{4}$$

that are nothing but commutation relations (2) defining noncommutative space-time.

Compactification on torus

$$U_i^{-1} X^j U_i = X^j + \delta_i^j 2\pi R_i.$$

leads to the relations

$$U_i U_j = e^{i\theta_{ij}} U_j U_i,$$

which define the algebra of functions on a noncommutative torus with the noncommutativity parameter θ, (see [26] for reviews and further developments).

One of the natural questions is whether noncommutative gauge theory can be used to describe more general compactifications in string theory. In order to have progress in this direction one has to develop the theory of noncommutative quantum gauge fields not only on the torus, but also on more general manifolds [27]-[29]. A framework for a noncommutative gauge theory on Poisson manifolds by using recently developed deformation quantization [30, 31] has been discussed in [17].

Models in external fields

- Dipole in a magnetic field

It is well known that the position operators in quantum mechanics in the presence of a magnetic field does not commute. A model consisting of a pair of opposite charges moving in a strong magnetic field can be described as a particle moving on the noncommutative 2-plane [32].

- Quantum Hall effect

The model describing electrons in a magnetic field projected to the lowest Landau level can be naturally treated as a noncommutative field theory. Thus noncommutative field theory is relevant to the theory of the quantum Hall effect [32, 33]. It has been proposed that a good description of the fractional quantum Hall effect can be obtained using noncommutative Chern-Simons theory [34, 35].

- Strings in a magnetic field

Noncommutative geometry appears in string theory with a nonzero B-field [36] (see [24] for refs. on earlier considerations). The action is

$$\int_\Sigma \frac{d^2z}{4\pi\alpha'}\left(g_{ij}\partial_a x^i \partial^a x^j - 2\pi i\alpha' B_{ij}\epsilon^{ab}\partial_a x^i \partial_b x^j\right). \tag{5}$$

Seiberg and Witten have found a limit in which the entire string dynamics is described by a supersymmetric gauge theory on a noncommutative space and shown an equivalence between ordinary gauge fields and noncommutative gauge fields.

Noncommutative Solitons and Instantons

In commutative scalar field theories there is Derrick's theorem, which prohibits the existence of finite energy solitons in two and more spatial dimensions. The proof is based on a simple scaling argument according to which no finite size minimum can exist, since upon shrinking all length scales both kinetic and potential energies decrease. This argument does not work in the presence of a length scale $\sqrt{\theta}$. It has been shown in [37], that for sufficiently large θ stable solitons exist in the noncommutative scalar field theory. For example, for a cubic potential solitons are defined as solutions of the following equation

$$\phi = \phi \star \phi. \tag{6}$$

While in a commutative theory this equation would admit only constant solutions, in a noncommutative theory there are countable many solutions, see [38] for more details.

As for noncommutative instantons it was found that the ADHM construction admits a generalization to the noncommutative case [39]. Noncommutative instantons in the limit $\theta \to 0$ are pure $U(1)$ gauge configurations with a singularity at the origin. For nonzero θ instantons are nonsingular objects with a size of order $\sqrt{\theta}$.

Smoothing of singularities

Solutions of a local field theory usually can be extended to the solutions of noncommutative theory. As a rule in such extensions any singularities disappear due to the position-space uncertainty. In particular, the noncommutative rank 1 theory has nonsingular instanton, monopole and vortex solutions.

Solitons in noncommutative theories can be stable even when their commutative counterparts are not, so noncommutativity provides a natural mechanism of stabilization for objects of the size $\sqrt{\theta}$. This behavior is naturally related to nonlocality. In other words, noncommutativity inevitably leads to nonlocality.

String Field Theory

One of the main motivations to construct string field theory (SFT) — an off-shell formulation of a string theory — was a hope to study non-perturbative phenomena in string theory [1, 40]. The Witten SFT action for open bosonic string is

$$S = \frac{1}{2}\int \Phi \star Q\Phi + \frac{1}{3}\int \Phi \star \Phi \star \Phi. \tag{7}$$

Open bosonic string has a tachyon that leads to a classical instability of the perturbative vacuum. In the early works by Kostelecky and Samuel [41] it was proposed to use SFT to describe condensation of the tachyon to a stable vacuum. By using the level truncation scheme they have shown that the tachyon potential in open bosonic string has a nontrivial minimum. Further [42], the level truncation method has been applied to study an effective potential of auxiliary fields in the cubic superstring field theory (SSFT) [43, 44]. It was found that some of the low-lying auxiliary scalar fields acquire non-zero vacuum expectation values that was considered as an indication of supersymmetry breaking in this vacuum.

Later on, Sen has proposed [45] to interpret the tachyon condensation as a decay of unstable D-brane. In the framework of this interpretation the vacuum energy of the open bosonic string in the Kostelecky and Samuel vacuum cancels the tension of the unstable space-filling 25-brane. This cancellation has been checked with high accuracy [46](see [47] for review).

The cubic open string field theory around the tachyon vacuum, or the vacuum string field theory (VSFT) was proposed in [49] and it is investigated now very intensively (see refs. in [24]). This VSFT has the same form as (7) but with a differential operator $\mathcal{Q}$, which is related with shifted charged

$$Q_{sh} = Q + \{\Phi_{vac}, \ \cdot \ \}.$$

by field redefinitions

$$\mathcal{Q} = RQ_{sh}R^{-1}.$$

The triviality of the cohomology of the new BRST operator $\mathcal{Q}$ around the tachyon vacuum is expected [45]. This means that this theory should not contain physical open string excitations. The evidence in favor of this conjecture was also found by demonstrating the absence of kinetic term for the tachyon field in nonperturbative vacuum [50]. VSFT is conjectured to describe the appearance of closed strings after the open string tachyon condensation. Moreover, it was suggested [49] that after some (maybe singular) field redefinition Q_{new} can be written as a pure ghost operator $\mathcal{Q}$. If this conjecture is true it is instructive to search solutions of VSFT equation of motion

$$\mathcal{Q}\Phi + \Phi \star \Phi = 0 \tag{8}$$

in the factorized form $\Phi = \Phi_{matter} \otimes \Phi_{ghost}$, thus obtaining a projector-like equation for matter sector:

$$\Phi_{matter} = \Phi_{matter} \star \ \Phi_{matter}. \tag{9}$$

An equation similar to (9) has appeared in construction of solitonic solutions in noncommutative field theories in the large coupling limit [37].

As an example of the projector-like solution, there is a state called sliver, $|\Xi\rangle$. This projector was constructed in the oscillator formalism by Kostelecky and Potting [48]. In fact a wedge state as a candidate to solve (9) was appeared first in the CFT framework [52]. Then it was at first identified with the sliver numerically [51] and later on by direct calculations [53]. The sliver state is special in a sense that it can be defined in an arbitrary boundary CFT [52].

The projector-like form of eq. (9) gave a new impulse for the development of the half-string formalism [1],[54],[55], which drastically simplifies Witten's $\star$-product. To construct projections more systematically, it is useful to find a matrix representation of open string fields. To see matrix representation in a more transparent way it is fruitful to use the so-called "comma" form of the overlap vertex [54],[56],[53]. The Bogoliubov transformation that makes the sliver to be a new vacuum requires the 3-string vertex to take a comma form [48]. The Bogoliubov transformation relating the sliver and initial Fock vacuum cannot be defined as an operator in the Fock space [61]. This is not surprising since we deal with infinite number of degrees of freedom. One can use the dressing transformation technique to give to the sliver a meaning of a state in the Hilbert space. Just due to the infinite dimensional nature of the star product appear associativity anomaly and twist anomaly.

Several tools are used to study VSFT:

1. the functional approach, where string fields are discussed in terms of left-right modes functionals, is developed in [55],

2. the operator approach, where explicit Fock-space operators for left-right parts of the string are explored, is presented in [54].

In [49] a simple linear form for $\mathcal{Q} = c_o + \sum f_n(c_n + (-1)^n c_n^\dagger)$ was proposed. Under this assumption the ghost part of the equation admits an analytical investigation [59, 60, 61, 62, 63]. Since an analytic tachyon vacuum solution is unknown, it is impossible to derive this form of VSFT action [49] from the initial Witten action (7). However, one can check if the proposed VSFT action passes some tests to be acceptable. In particular, in [50] a classical solutions of VSFT that reproduces the perturbative open string vacuum is constructed (see [60, 63]). Several attempts of getting closed strings from VSFT have been recently performed [64, 62].

This consideration concerns only bosonic cubic SFT. To study the same questions in the superstring case it is worth to work with a formalism which deals with equations of motions in the simple form (8) without any modifications. String fields in 0 picture [43], [44] give us a suitable formalism.

2 Axioms of Noncommutative Differential Geometry

Let $\mathcal{A}$ be an abstract associative algebra with the multiplication $\star$

$$\star: \ \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \tag{10}$$

One can interpret $\mathcal{A}$ as an "algebra of functions" on a (nonexisting) noncommutative space.

Let us introduce some basic geometrical notions on this algebra $\mathcal{A}$: external derivative Q (or an algebra $\mathcal{G}$ of derivatives)

$$Q : \mathcal{A} \to \mathcal{A} \tag{11}$$

and integral (linear map)

$$\int : \mathcal{A} \to \mathbb{C}. \tag{12}$$

Noncommutative space is the set

$$\left(\mathcal{A}, Q, \int\right), \tag{13}$$

which satisfies the following axioms:

- associativity

$$A \star (B \star C) = (A \star B) \star C, \quad A, B, C \in \mathcal{A}; \tag{14}$$

- nilpotency

$$Q^2 = 0; \tag{15}$$

- Leibnitz rule

$$Q(A \star B) = (QA) \star B) + A \star (QB), \quad A, B \in \mathcal{A}; \tag{16}$$

- cyclicity property

$$\int (A \star B) = \int (B \star A); \tag{17}$$

- "integration by part"

$$\int (QA) = 0. \tag{18}$$

In the most applications the algebra $\mathcal{A}$ is also equipped with a natural $\mathbb{Z}_2$-grading: $|A \star B| = |A| + |B|$. For example, in Section ?? $\mathbb{Z}_2$-grading is usual boson/ferimon super-grading. One has to modify Leibnitz rule (16) and cyclicity property (17) to be in agreement with $\mathbb{Z}_2$-grading:

$$Q(A \star B) = (QA) \star B + (-1)^{|A|} A \star (QB), \tag{19}$$

$A, B \in \mathcal{A}$;

$$\int (A \star B) = (-1)^{|A||B|} \int (B \star A), \quad A, B \in \mathcal{A} \tag{20}$$

where $(-1)^{|A||B|} = -1$ if both A and B are odd elements and equal 1 otherwise.

If $\mathcal{A}$ admits a Lie algebra of derivatives one can develop a noncommutative bundles theory generalizing the standard theory. Let $\mathcal{G}$ be a Lie algebra of derivatives on $\mathcal{A}$ and $\alpha_1, \ldots, \alpha_d$ be generators of $\mathcal{G}$. Then one can generalize the notion of non-commutative space (13) to the set

$$\left(\mathcal{A}, \mathcal{G}, \int\right), \tag{21a}$$

which satisfies axioms (14) and (17) as well as

$$\alpha_i(A \star B) = (\alpha_i A) \star B + A \star (\alpha_i B), \tag{21b}$$

where $A, B \in \mathcal{A}$, $\alpha_i \in \mathcal{G}$; and

$$\int \alpha_i \ A = 0. \tag{21c}$$

If V is a projective module over $\mathcal{A}$ (i.e. a "vector bundle over $\mathcal{A}$") one defines a connection in V as the set of linear operators $\nabla_1, \dots, \nabla_d$ on V satisfying

$$\nabla_i(a\phi) = a\nabla_i(\phi) + \alpha_i(a)\phi \tag{22}$$

for all $a \in \mathcal{A}$ and $\phi \in V$, $i = 1, \dots, d$. The curvature of connection ∇_i is

$$F_{ij} = \nabla_i\nabla_j - \nabla_j\nabla_i - f_{ij}^k\nabla_k \tag{23}$$

belongs to the algebra of endomorphisms of the $\mathcal{A}$-module V.

A profit from such kinds of constructions is that we can immediately write an action that is invariant under gauge transformations. If we deal with the set of axioms (15)-(19) we can define the gauge transformation as

$$\delta A = Q\Lambda + A \star \Lambda - \Lambda \star A, \quad A, \Lambda \in \mathcal{A}. \tag{24}$$

The gauge invariant action is [1]

$$S[A] = \frac{1}{2}\int A \star QA + \frac{1}{3}\int A \star A \star A. \tag{25}$$

This is a generalization of the Chern-Simons action.

In the case of axioms (14), (17), (22), (21b) and (21c) one has a generalization of the Yang-Mills action

$$S = -\frac{1}{4}\int (F_{ij}F^{ij}) \tag{26}$$

with

$$F_{ij} = \alpha_i A_j - \alpha_j A_i + A_i \star A_j - A_j \star A_i. \tag{27}$$

The action is invariant under the following gauge transformations

$$\delta(A_i) = \alpha_i(\omega) + A_i \star \omega - \omega \star A_i, \tag{28}$$

where $\omega \in \mathcal{A}$ is a gauge parameter.

There are several examples of realization of axioms (2.2) and (2.3). We will discuss the following realizations of axioms (21):

- The d-dimensional noncommutative torus $\mathbb{T}_\theta^d$.
- The d-dimensional noncommutative Euclidian space $\mathbb{R}_\theta$.

String field theory (SFT) presents an example of axioms (2.2). There is also a realization of axioms (2.2) on matrices.

3 Examples.

3.1 Noncommutative torus $\mathbb{T}^d_\theta$.

The d-dimensional noncommutative torus is defined by its algebra $\mathcal{A}_\theta$ with generators $U_1, \ldots, U_d$ satisfying the relations

$$U_i U_j = e^{2i\pi\theta_{ij}} U_j U_i$$

where $i, j = 1, \ldots, d$ and $\theta = (\theta_{ij})$ is a real antisymmetric matrix. The algebra $\mathcal{A}_\theta$ is equipped with an antilinear involution $*$ obeying $U_i^* = U_i^{-1}$ (i.e. $\mathcal{A}_\theta$ is a *-algebra). An element of $\mathcal{A}_\theta$ is a power series

$$f = \sum f(p_1, ..., p_d) U_1^{p_1} ... U_d^{p_d}$$

where $p = (p_1, ..., p_d) \in Z^d$ and the sequence of complex coefficients $f(p_1, ..., p_d)$ decreases faster than any power of $|p| = |p_1| + ... + |p_d|$ when $|p| \to \infty$. The function $f(p)$ is called the symbol of the element f. We denote by U^p the product $U_1^{p_1} ... U_d^{p_d}$. Then one has $U^p U^k = e^{2i\pi\varphi(p,k)} U^{p+k}$, where $\varphi(p, k) = \sum \varphi_{ij} p_i p_j$ and φ_{ij} is a matrix obtained from θ after deleting all its elements below the diagonal. To simplify the product rule we replace U^p by $e^{i\pi\varphi(p,q)} U^p$ so that we have

$$U^p U^k = e^{i\pi\theta(p,k)} U^{p+k}.$$

If f and g are two elements of $\mathcal{A}_\theta$,

$$f = \sum_p f(p) U^p, \quad g = \sum_k g(k) U^k$$

then the product

$$fg = \sum_{p,k} f(p) g(k) U^p U^k =$$

$$= \sum_{p,k} f(p) g(k) e^{2i\pi\theta(p,k)} U^{p+k} = \sum_q (f \star g)(q) U^q,$$

where the star-product $(f \star g)(q)$ of symbols $f(p)$ and $g(k)$ is

$$(f \star g)(q) = \sum_p f(p) g(q-p) e^{i\pi\theta(p,q-p)}.$$

The differential calculus on the noncommutative torus is introduced by means of the derivations ∂_j defined as

$$\partial_j U^p = i p_j U^p, \quad j = 1, \ldots, d.$$

They satisfy the Leibnitz rule $\partial_j(fg) = \partial_j f \cdot g + f \cdot \partial_j g$ for any $f, g \in \mathcal{A}_\theta$.

The integral of $f = \sum f(p) U^p$ is defined as $\int f = f(0)$, which is in correspondence with the commutative case. The integral has the property of being the trace on the algebra $\mathcal{A}_\theta$, i.e. $\int fg = \int gf$ for any $f, g \in \mathcal{A}_\theta$. Moreover one has

$$\int \partial_j f \cdot g = -\int \partial_j g \cdot f.$$

The gauge field A_i on the noncommutative torus is defined as

$$A_i = \sum_{p \in Z^d} A_i(p) U^p, \quad i = 1, \dots, d.$$

Here $A_i(p)$ is a sequence of $N \times N$ complex matrices indexed by a space-time index. It corresponds to the Fourier representation of the ordinary gauge theory on commutative torus. The gauge field is antihermitian, $A_i^* = -A_i$, or $A_i(p)^* = -A_i(-p)$. A_i is an element of a matrix algebra with coefficients in $\mathcal{A}_\theta$ and its curvature is defined as

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j],$$

where ∂_i is a derivative on $\mathcal{A}_\theta$ defined above. The Yang-Mills action is

$$S = -\frac{1}{4} \int \mathrm{tr}(F_{ij} F^{ij}).$$

This action is invariant under gauge transformations

$$A_i \to \Omega A_i \Omega^{-1} + \Omega \partial_i \Omega^{-1}, \quad F_{ij} \to \Omega F_{ij} \Omega^{-1},$$

where Ω is a unitary element of the algebra of matrices over $\mathcal{A}_\theta$.

3.2 Noncommutative flat space $\mathbb{R}_\theta$.

The main property of the noncommutative flat space R^d_θ space is the non-zero commutator of the coordinates

$$[x_i, x_j] = i\theta_{ij}$$

where θ is called the parameter of noncommutativity. We begin with the formulation of the Moyal product as one of the important realization of such multiplication law.

Let us consider one-dimensional quantum mechanics in the Hilbert space of square integrable functions on the real line $L^2(\mathbf{R})$ with ordinary canonical operators of position $\hat{q}$ and momentum $\hat{p}$,

$$[\hat{p}, \hat{q}] = -i\hbar,$$

acting as $\hat{q}\psi(x) = x\psi(x)$, $\hat{p}\psi(x) = -i\hbar d\psi(x)/dx$. If a function of two real variables $f(q,p)$ is given in terms of its Fourier transform

$$f(q,p) = \int e^{i(rq+sp)} \tilde{f}(r,s) dr ds,$$

then one can associate with it an operator $\hat{f}$ in $L^2(\mathbf{R})$ by the following formula

$$\hat{f} = \int e^{i(r\hat{q}+s\hat{p})} \tilde{f}(r,s) dr ds.$$

This procedure is called the Weyl quantization and the function $f(q,p)$ is called the symbol of the operator $\hat{f}$ [30, 31]. One has the correspondence

$$\hat{f} \longleftrightarrow f = f(q,p).$$

If $\hat{f}^*$ is the Hermitian adjoint to $\hat{f}$ then its symbol is $f^* = f^*(q,p)$

$$\hat{f}^* \longleftrightarrow \hat{f}^*(q,p) = \bar{f}(q,p).$$

If two operators $\hat{f}_1$ and $\hat{f}_2$ are given with symbols $f_1(q,p)$ and $f_2(q,p)$ then the symbol of product $\hat{f}_1\hat{f}_2$ is given by the Moyal product

$$f_1 \star f_2 = (f_1 \star f_2)(p,q)$$

as [30, 31]

$$(f_1 \star f_2)(p,q) =$$
$$= e^{i\hbar L}(f_1(q_1,p_1)f_2(q_2,p_2))|_{q_1=q_2=q,\;\; p_1=p_2=p},$$

where

$$L = \frac{1}{2}(\frac{\partial^2}{\partial q_1 \partial p_2} - \frac{\partial^2}{\partial q_2 \partial p_1}).$$

4 Noncommutative φ^4-model.

The simplest nontrivial model which can demonstrate the most important properties of the noncommutative field theories is a non-commutative φ^4-model [18].

The theory is defined by the action

$$S = S_0 + S_{int} \tag{29}$$

$$= \int d^dx\,[\frac{1}{2}(\partial_\mu\varphi)^2 + \frac{1}{2}m^2\varphi^2 + \frac{g}{4!}(\varphi\star\varphi\star\varphi\star\varphi)(x)],$$

where $\star$ is a Moyal product

$$(f\star g)(x) = e^{i\xi\theta^{\mu\nu}\partial_\mu\otimes\partial_\nu}f(x)\otimes g(x),$$

ξ is a deformation parameter, $\theta^{\mu\nu}$ is a non-degenerate skew-symmetric real constant matrix, $\theta^2 = -1$, d is even. In this section we deal with four dimensional Euclidean space. Also, we introduce the convenient notation

$$p_1 \wedge p_2 \equiv \xi p_1 \theta p_2.$$

Let us rewrite the interaction term in the Fourier representation

$$S_{int} = \frac{g}{4!(2\pi)^d}\int dp_1 dp_2 dp_3 dp_4\, e^{-ip_1\wedge p_2 - ip_3\wedge p_4} \times \tag{30}$$

$$\varphi(p_1)\varphi(p_2)\varphi(p_3)\varphi(p_4)\delta(p_1+p_2+p_3+p_4).$$

There are the following distinguished properties of the deformed theory as compared with standard local φ^4_d model:

- There are non-local phase factors in the vertex.
- These factors provide regularization for some loop integrals but not for all [19, 20, 21, 17, 32].

- To have renormalizability the sum of divergences in each order of perturbation theory must have a phase factor already present in the action.

To single out phase factors it is convenient to use the 't Hooft double-line graphs and a notion of planar graphs. For planar graphs the phase factors do not affect the Feynman integrations at all (see [18] for details). In particular the planar graphs have exactly the same divergences as in the commutative theory [32]. There are no superficial divergences in nonplanar graphs since they are regulated by the phase factors [19, 20, 21, 17, 32]. Moreover, oscillating phases regulate also divergent subgraphs, unless they are not divergent planar subgraphs.

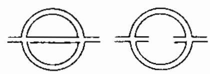

Figure 1: Planar graph with a nonplanar subgraph.

So, at first sight it seems that the proof of renormalizability is rather trivial. One has divergences only in planar graphs, so one can use the fact that planar theory is renormalizable if its scalar counterpart does [65]. In other words, one can expect that it is enough to take the planar approximation of an ordinary theory, find divergences within this approximation and write counterterms in the noncommutative theory as divergent parts of planar graphs multiplied on the phase factors. However, these arguments work only for superficial divergences. The situation is more subtle for divergent subgraphs. The reason is that a planar graph can contain nonplanar subgraphs (see simple example on Fig.1) and these divergences should be also removed. Therefore, renormalizability of the theory (29) is not obvious. This gave us [22] a reason to consider explicitly the two-loop renormalization of the theory (29)

In explicit calculations we will use single-line graphs and symmetric vertices. After the symmetrization of (30) we get (up to numerical factor)

$$\int \prod_{i=1}^{4} dp_i \, \varphi(p_i) \, \delta(p_1 + p_2 + p_3 + p_4) \times \tag{31}$$

$$[\cos(p_1 \wedge p_2) \cos(p_3 \wedge p_4) + \cos(p_1 \wedge p_3) \cos(p_2 \wedge p_4) +$$

$$\cos(p_1 \wedge p_4) \cos(p_2 \wedge p_3)]$$

and the vertex is a sum of three terms (see Fig.2). Here we would like to say a few words

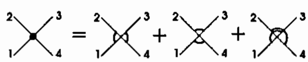

Figure 2: The symmetric vertex.

about a general technique of calculations in the presence of non-local phase factors in the vertex. In our calculations we always denote external momenta by p_i and loop momentum by k. In order to perform an integration over loop momenta the α-representation and

Feynman parametrization are used[66]. Using α-representation we get the Gauss type integral over internal momentum

$$\int e^{-\omega A\omega + l\omega} d^D\omega = \frac{\pi^{D/2}}{\sqrt{\det A}} e^{\frac{1}{4} l A^{-1} l}. \tag{32}$$

Namely, an expression for an arbitrary graph is a product of n propagators with exponential factor coming from trigonometric structure of the vertex. We put the parameter α_i in correspondence to the i-th propagator.

The most general expression for a one loop graph can be written schematically as

$$L_1 = \int d^d k \exp\left[-(ak^2 + lk + M^2) + ib\theta k\right],$$

where a depends on α_i, l and M depend on α_i and p_j, b depends on p_j. Using (32) the integration over k is evident and the result is

$$L_1 = \frac{\pi^{d/2}}{a^{d/2}} e^{-M^2 + \frac{1}{4a}(l^2 - b^2 + 2il\theta b)}$$

The next step is an integration over parameters α_i. There are two formulae we work with

$$\int_0^\infty \alpha^{\nu-1} e^{-\mu\alpha} d\alpha = \frac{1}{\mu^\nu}\Gamma(\nu), \tag{33}$$

$$\int_0^\infty \alpha^{\nu-1} e^{-\gamma\alpha - \frac{\beta}{\alpha}} d\alpha = 2\left(\frac{\beta}{\gamma}\right)^{\nu/2} K_\nu(2\sqrt{\beta\gamma}). \tag{34}$$

Note that function $\Gamma(\nu)$ has simple poles at points $\nu = 0, -1, -2, \ldots$, which correspond to UV divergencies (the momentum space ultraviolet divergencies become a small α divergence). Meanwhile the RHS of (34) for $\beta \neq 0$ has no singularities at points $\nu = 0, -1, -2, \ldots$, since the factor $e^{-\frac{\beta}{\alpha}}$ produces a regularization for small α. It is easy to show that the integral

$$\int L_1 d\alpha_1 \ldots d\alpha_n$$

also is well defined if $b \neq 0$.

Now we are going to compute explicitly one-loop counterterms using dimensional regularization $d = 4 - 2\epsilon$. We will also present the explicit form of finite part for two point and four point 1PI functions $\Gamma^{(2)}$ and $\Gamma^{(4)}$ in the one loop approximation. We use the standard notations for perturbation expansion of 1PI-functions, $\Gamma^{(i)} = \sum_n g^n \Gamma_n^{(i)}$, $\Gamma^{(2)} = \Gamma^{(2)}_{f.p.} + \Delta\Gamma^{(2)}$ and $\Gamma^{(4)} = \Gamma^{(4)}_{f.p.} + \Delta\Gamma^{(4)}$.

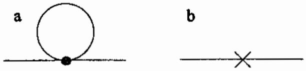

Figure 3: $\Gamma_1^{(2)} - \Delta\Gamma_1^{(2)}$.

The only graph 3a contributes to $\Gamma_1^{(2)}$ and

$$\Gamma_1^{(2)} = -\frac{g(\mu^2)^\epsilon}{6(2\pi)^d}\int dk \frac{2+\cos 2p\wedge k}{k^2+m^2}.$$

Here we have introduced a new parameter μ of dimension of mass to leave the action dimensionless in $4-2\epsilon$ dimensions.

We present this expression as a sum of planar and nonplanar parts $\Gamma_1^{(2)} = \Gamma_{1,pl}^{(2)} + \Gamma_{1,npl}^{(2)}$, where

$$\Gamma_{1,pl}^{(2)} = -\frac{g(\mu^2)^\epsilon}{6(2\pi)^d}\int dk \frac{2}{k^2+m^2},$$

$$\Gamma_{1,npl}^{(2)} = -\frac{g(\mu^2)^\epsilon}{6(2\pi)^d}\int dk \frac{\cos 2p\wedge k}{k^2+m^2}.$$

We see that only the planar part has a divergence

$$\Gamma_{1,pl}^{(2)} = \frac{g}{32\pi^2}\frac{2}{3}m^2\left(\frac{1}{\epsilon}+\psi(2)-\ln\frac{m^2}{4\pi\mu^2}+O(\epsilon)\right) \qquad (35)$$

and this divergent part is subtracted by the counterterm 3b.

The nonplanar part can be calculated explicitly using α-representation

$$\Gamma_{1,npl}^{(2)} = -\frac{g}{32\pi^2}\frac{2}{3}\sqrt{\frac{m^2}{\xi^2p^2}}K_1(2m\xi|p|) \qquad (36)$$

and as has been mentioned above has no divergencies. Note that this representation takes place only for $p\neq 0$.

$\Gamma_2^{(4)}$ is a sum of s-, t- and u-channel graphs, $\Gamma_2^{(4)} = \Gamma_{2,s}^{(4)} + \Gamma_{2,t}^{(4)} + \Gamma_{2,u}^{(4)}$. The explicit form of $\Gamma_{2,s}^{(4)}$ is

$$\Gamma_{2,s}^{(4)} = \frac{g^2(\mu^2)^{2\epsilon}}{18(2\pi)^d}\int \frac{\mathcal{P}_{4a}(\{p\},k)}{(k^2+m^2)((P+k)^2+m^2)}dk$$

where $P = p_1+p_2$,

$$\mathcal{P}_{4a}(\{p\},k) =$$
$$[2\cos(p_1\wedge p_2)\cos(k\wedge P)+\cos(p_1\wedge p_2+(p_1-p_2)\wedge k)]\times$$
$$[2\cos(p_3\wedge p_4)\cos(k\wedge P)+\cos(p_3\wedge p_4+(p_4-p_3)\wedge k)].$$

The trigonometric polynomial $\mathcal{P}_{4a}$ can be rewritten in the form

$$2\cos(p_1\wedge p_2)\cos(p_3\wedge p_4)+\sum_j{}' c_j e^{i\Phi^j(p)+ib^j(p)\wedge k} \qquad (37)$$

where the sum $\sum'$ goes over all j for which linear functions $b^j(p)$ are nonzero for almost all $\{p\}$. This representation is useful to separate the terms which do not depend on internal momentum and, therefore, produce divergencies. Other terms have phase factors which contain k and provide a finite answer. Thus, the only first term in (37) contributes to a pole part (see Fig. 4a) and we have

$$\Gamma_{2,s}^{(4)} = \frac{g^2}{32\pi^2}\frac{2}{9}[\cos(p_1\wedge p_2)\cos(p_3\wedge p_4)$$

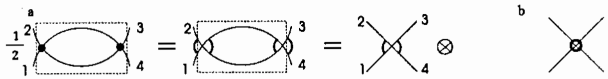

Figure 4: a) $\Delta\Gamma^{(4)}_{2,s}$, b) Cross denotes the 4-vertex counterterm.

$$\left(\frac{1}{\epsilon} + \psi(1) - \int_0^1 dx \ln\frac{m^2 + x(1-x)P^2}{4\pi\mu^2}\right) +$$

$$+ \sum_j{}' c_j e^{i\Phi^j(p)} \int_0^1 dx K_0(X_j) e^{ixb^j \wedge P}\Bigg].$$

where $X_j = \sqrt{\xi^2 b^{j2}(m^2 + P^2 x(1-x))}$. This representation is well defined only if all b^j are nonzero, i.e. for non exceptional momenta. It is the matter of a simple algebra to sum up the divergent parts of $\Gamma^{(4)}_{2,s}$, $\Gamma^{(4)}_{2,t}$ and $\Gamma^{(4)}_{2,u}$ to obtain a graph with the symmetric vertex.

Therefore, at the one-loop we have

$$\Delta\Gamma^{(2)}_{1l} = \frac{g}{48\pi^2}\frac{m^2}{\epsilon},$$

$$\Delta\Gamma^{(4)}_{1l} = \frac{g^2}{16\pi^2}\frac{1}{9\epsilon}[\cos(p_1 \wedge p_2)\cos(p_3 \wedge p_4) +$$
$$\cos(p_1 \wedge p_3)\cos(p_2 \wedge p_4) + \cos(p_1 \wedge p_4)\cos(p_2 \wedge p_3)].$$

It is evident that our renormalized 1PI functions do not have the limit $\xi \to 0$. Due to this there is a nontrivial mixing of UV and IR divergencies (see [23, 22] for details). It turns out that a type of the UV divergency of the integral corresponding to the Feynman graph in a local model

$$J_{UV}(p) = \int f(k,p)dk$$

is the same as a type of the IR behavior of the integral corresponding to the Feynman graph in a noncommutative model

$$J_{IR}(\xi p) = \int e^{i\xi k\theta p} f(k,p)dk.$$

For example, the logarithmically divergent integral $J_{UV} \sim \log\Lambda$ corresponds to a logarithmic singularity $\log(\xi|\theta p|)$ in J_{IR}, the quadratically divergent integral J_{UV} corresponds to a quadratic singularity $(\xi|\theta p|)^{-2}$ in J_{IR}, and so on.

Indeed, the UV behavior of the Feynman graph is defined by its index of divergence ω. For $\omega > 0$ one has the asymptotic UV behaviour as $\Lambda \to \infty$

$$J^{as}_{UV} = \int_0^\Lambda k^{\omega-1} dk = \frac{1}{\omega}\Lambda^\omega. \tag{38}$$

The IR behavior is described by integral

$$J^{as}_{IR} = \int_0^\infty k^{\omega-1} e^{i\xi k} dk \tag{39}$$

as $\xi \to 0$. This integral is the Fourier transform of the distribution $k_+^{\omega-1}$

$$J_{IR}^{as} = ie^{i\frac{\pi}{2}(\omega-1)}\frac{\Gamma(\omega)}{(\xi+i0)^{\omega}}, \tag{40}$$

i.e. for $\omega > 0$ one has the correspondence with UV behaviour (38).

To see logarithmic singularities on ξ let us note that

$$\int_m^{\Lambda} k^{-1}dk \sim C\int_{-\Lambda}^{\Lambda}\frac{d^2k}{k^2+m^2} \sim \log\Lambda \tag{41}$$

and the corresponding non local integral is

$$J_{IR}^{as,0}(\xi p) = \int \frac{e^{i\xi p\wedge k}}{k^2+m^2}d^2k. \tag{42}$$

The latter integral can be calculated explicitly and the result is

$$J_{IR}^{as,0} = 2\pi K_0(m\xi|p|). \tag{43}$$

The modified Bessel function $K_0(z)$ has an expansion

$$K_0(z)(\xi p) = -\ln(z) + \ln(2) - C + O(z^2)$$

where C is Euler constant, i.e. $J_{IR}^{as,0}(\xi p)$ has the logarithmic dependence on ξ.

IR poles appear in the corrections to propagators and can produce IR divergencies in multi-loop graphs even in the massive theories. One has not to worry about logarithms, since in the origin the logarithm is an integrable function. Let us consider a example. One can calculate explicitly the finite part of the tadpole graph in the noncommutative φ^4 theory. The answer is given by the sum of (36) and (35). The behavior of (36) in the limit $p^2 \to 0$ (the same as the limit $\xi \to 0$) is the following

$$\Gamma^{(2)}_{1,f.p.}\underset{p^2\to 0}{\sim}\frac{c}{(\xi|p|)^2}.$$

This result can be easily obtained from the series expansion of the modified Bessel function $K_1(z)$

$$K_1(z) = 1/z + \frac{1}{2}z\ln(z) + (\frac{1}{2}C - \frac{1}{4} - \frac{1}{2}\ln(2))z + O(z^3)$$

where C is Euler constant. Caused by this asymptotic there are problems with an IR behavior of graphs with tadpoles. They produce divergence in the IR region if the number of insertions is three or more. Thus, theories with a real scalar field have problems with infrared behavior [23, 22] originated in multi one-loop insertions.

5 Cubic String Field Theory

5.1 Witten's realization of axioms on space of string functionals.

Witten [1] presented realization of the set (13-20) in the case when $\mathcal{A}$ is taken to be the space of string fields

$$\mathcal{F} = \{\Psi[X(\sigma); c(\sigma), b(\sigma)]\} \tag{44}$$

which can be described as functionals of the matter $X(\sigma)$, ghost $c(\sigma)$ and antighost fields $b(\sigma)$ corresponding to an open string in 26 dimensions with $0 \leqslant \sigma \leqslant \pi$. For this string field theory Q_B is the usual open string BRST charge of the form

$$Q_B = \int_0^\pi d\sigma\, c(\sigma) \left(T_B(\sigma) + \frac{1}{2} T_{bc}(\sigma) \right) . \tag{45}$$

T_B and T_{bc} are stress tensors for the matter field and ghosts.

The star product $\star$ is defined by gluing the right half of one string to the left half of the other using a delta function interaction. The star product factorizes into separate matter and ghost parts. For the matter fields the star product is given by

$$(\Psi \star \Phi) [X(\sigma)] \equiv \int \prod_{\frac{\pi}{2} \leq \sigma \leq \pi} dX'(\sigma)\, dX''(\pi - \sigma)$$

$$\prod_{\frac{\pi}{2} \leq \sigma \leq \pi} \delta[X'(\sigma) - X''(\pi - \sigma)]\ \Psi[X'(\sigma)] \Phi[X''(\sigma)],$$

$$X(\sigma) = X'(\sigma) \quad \text{for} \quad 0 \leq \sigma \leq \frac{\pi}{2};$$

$$X(\sigma) = X''(\sigma) \quad \text{for} \quad \frac{\pi}{2} \leq \sigma \leq \pi . \tag{46}$$

The integral over a string field factorizes into matter and ghost parts, and in the matter sector is given by

$$\int \Psi = \int \prod_{0 \leq \sigma \leq \pi} dX'(\sigma) \times \tag{47}$$

$$\prod_{0 \leq \sigma \leq \frac{\pi}{2}} \delta[X'(\sigma) - X'(\pi - \sigma)]\ \Psi[X'(\sigma)].$$

Performing a Fourier mode expansion of the 26 matter fields X^μ through

$$X^\mu(\sigma) = x_0^\mu + \sqrt{2\alpha'} \sum_{n=1}^{\infty} x_n^\mu \cos(n\sigma) \tag{48}$$

one can consider string field $\Phi[X(\sigma)]$ (we consider there only the X-part of string field for simplicity and we omit the space-time index μ) as depending on the set $\{x_n\}$, $\Phi = \Phi[\{x_n\}]$. Then one can regard $\Phi[\{x_n\}]$ a wave function in the coordinate representation of a system with an infinite number of degrees of freedom.

5.2 String functionals as matrices or symbols for operators.

One can consider string functionals as vector states in the coordinate representation

$$\Psi[X(\sigma)] = \langle X(\sigma) | \Psi \rangle . \tag{49}$$

Our goal is to obtain the mapping from string fields $|\Psi\rangle$ to matrices

$$|\Psi\rangle \Longleftrightarrow \Psi_{\mathbf{n},\mathbf{m}} \tag{50}$$

so that

$$|\Psi\rangle \star |\Phi\rangle \Longleftrightarrow \sum_{\mathbf{k}} \Psi_{\mathbf{n,k}} \Psi_{\mathbf{k,m}}, \tag{51}$$

$$\int |\Psi\rangle \Longleftrightarrow Tr\Psi = \sum_{\mathbf{k}} \Psi_{\mathbf{k,k}}. \tag{52}$$

Also we want to find a map from string fields to operators

$$|\Psi\rangle \Longleftrightarrow \hat{\Psi} \tag{53}$$

with

$$|\Psi\rangle \star |\Phi\rangle \Longleftrightarrow \hat{\Psi}\hat{\Phi}. \tag{54}$$

In other words we look for a map between string functionals and operators

$$\Psi[X(\sigma)] \Longleftrightarrow \hat{\Psi}, \tag{55}$$

such that

$$\Psi[X(\sigma)] \star \Phi[X(\sigma)] \Longleftrightarrow \hat{\Psi}\hat{\Phi}. \tag{56}$$

There is an analogy between this map and the relation between operators and their symbols [58]. Using this analogy one can interpret the Witten multiplication as a Moyal product of two functionals.

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Basic Twisting Factors and Factorization Properties of Twists

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Abstract

All the explicitly known twisting deformations for universal enveloping algebras of simple Lie algebras g are factorizable. They can be performed as a sequence of successive elementary twistings. Only three types of nontrivial elementary twists (*basic twisting factors*) are found in these decompositions. The factorization property together with the general structure of the basic factors can be used to find new twist deformations. To illustrate these possibilities a new form is obtained for the twisting element with the parabolic carrier subalgebra.

1 Introduction

Let g be a simple Lie algebra over C with the root system $\Lambda(\mathrm{g})$. We study quantum deformations of the universal enveloping algebra $U(\mathrm{g})$ considered as a Hopf algebra with *primitive* coproducts for the generators $g \in \mathrm{g}$, $\Delta(g) = g \otimes 1 + 1 \otimes g$. The deformations of $U(\mathrm{g})$ governed by the coboundary Lie-Poisson structures play the significant role having nontrivial applications in mathematical physics. The coboundary Lie-Poisson structure is fixed by the r-matrix, the solution of the classical Yang-Baxter equation (CYBE).

The set of nonskewsymmetric solutions $r(\mathrm{g})$ of CYBE was described by Belavin and Drinfeld [1].

For the special case of the Cremmer-Gervais quantization $U_{CG}(gl(N))$ Kulish and Mudrov [2] have found the twist connecting the Hopf algebra $U_{CG}(gl(N))$ with the standard quantization $U_q(gl(N))$ [3],[4]. Etingof, Schedler and Schiffman [5] had proved that such twists can be constructed for any r-matrix from the Belavin-Drinfeld list and an arbitrary simple Lie algebra g. This solves the quantization problem for Lie-Poisson structures described by nonskewsymmetric r-matrices. A compact expression for such twists was found by Isaev and Ogievetsky [6].

For skewsymmetric r-matrices Drinfeld [7] had reduced the quantization problem to the solution of the *twist equations*:

$$\begin{aligned} &\mathcal{F}_{12}(\Delta \otimes id)\mathcal{F} = \mathcal{F}_{23}(id \otimes \Delta)\mathcal{F}, \\ &(\epsilon \otimes id)\, \mathcal{F} = (id \otimes \epsilon)\, \mathcal{F} = 1 \otimes 1. \end{aligned} \tag{1}$$

The solutions of (1) are called the *twisting elements.* Given a Hopf algebra $\mathcal{A}(m, \Delta, S, \eta, \epsilon)$ and a twisting element $\mathcal{F} \in \mathcal{A} \otimes \mathcal{A}$ we have a map $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{A}_{\mathcal{F}}$, where the Hopf algebra $\mathcal{A}_{\mathcal{F}}(m, \Delta_{\mathcal{F}}, S_{\mathcal{F}}, \eta, \epsilon)$ has the same multiplication, unit and counit (as in $\mathcal{A}$), the twisted

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coproduct $\Delta_{\mathcal{F}} = \mathcal{F}\Delta\mathcal{F}^{-1}$ and the twisted antipode $S_{\mathcal{F}}$. There is an important subclass of twists that satisfy the *factorized twist equations* [15],

$$\begin{aligned}(\Delta \otimes id)\mathcal{F} &= \mathcal{F}_{13}\mathcal{F}_{23},\\ (id \otimes \Delta_F)\mathcal{F} &= \mathcal{F}_{12}\mathcal{F}_{13}.\end{aligned} \tag{2}$$

When $\mathcal{A}$ is a universal enveloping algebra $U(\mathrm{g})$ the minimal subalgebra $\mathbf{L} \subset \mathrm{g}$ on which $\mathcal{F} \in U(\mathbf{L}) \otimes U(\mathbf{L})$ is defined is called the *carrier subalgebra* of the twist.

The knowledge of the twisting element is important in applications. When $\mathcal{A}$ is quasitriangular with the $\mathcal{R}$-matrix $\mathcal{R}$ then such is $\mathcal{A}_{\mathcal{F}}$ with $\mathcal{R}_{\mathcal{F}} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}$. For any skewsymmetric r-matrix the solution of the equations (1) can be constructed [7] by means of Campbell-Hausdorff series using some special extension of the initial universal enveloping algebra. However, these constructions are obviously inappropriate for an efficient usage of quantum $\mathcal{R}$-matrices.

The classification scheme for skewsymmetric classical r-matrices was developed by Stolin [8].

There are different classes of multiparametric skew r-matrices that are quantized explicitly [9],[10],[11]. For them the solutions of the twist equations are found in terms of elementary functions on $\mathbf{L}$ thus providing the possibility to construct explicitly the deformed algebras and the corresponding $\mathcal{R}$-matrices. Most of these sets of twisting elements have important algebraic properties induced by the structure of the root system $\Lambda(\mathrm{g})$ [11]. In this report the factorization property of twists will be studied. It can be shortly formulated as the possibility to factorize the twisting element $\mathcal{F}$ into such "two-dimensional" parts $\mathcal{F}_i$

$$\mathcal{F} = \mathcal{F}_k \ldots \mathcal{F}_2\mathcal{F}_1, \tag{3}$$

that every $\mathcal{F}_i$ is a twist for the Hopf algebra deformed by the previous factors. Thus the decomposition (3) presents $\mathcal{F}$ as a sequence of elementary twists. Trivial factors (coboundaries) do not contribute to the r-matrix. Nontrivial $\mathcal{F}_i$'s can be characterized as "two-dimensional" as far as they are in one-to-one correspondence with the summands of the classical skewsymmetric r-matrix. For nontrivial factors the function $\ln(\mathcal{F}_i)$ is a tensor product of two elements $f_1, f_2 \in \mathcal{A}$.

We call the nontrivial twists $\mathcal{F}_i$ the *basic twisting factors*. It will be demonstrated that all the explicitly known twists can be presented in the factorized form (3), moreover all of them can be composed of only three types of basic twisting factors (and some trivial twists).

2 Basic Twisting Factors

Consider a quasitriangular Hopf algebra $\mathcal{A}(m, \Delta, S, \eta, \epsilon, \mathcal{R}; \xi_1, \ldots, \xi_{k-1})$ which is the deformation of the universal enveloping algebra $U(\mathrm{g})$ with the deformation parameters $\xi_1, \ldots, \xi_{k-1}$. The deformation can be coordinated so that there exists a smooth curve with the parameter ξ and the expansion

$$\mathcal{R}_{\mathcal{A}}(\xi) = 1 \otimes 1 + \xi r_{\mathcal{A}} + o(\xi). \tag{4}$$

This testifies that the quantization was performed in the direction pointed by the classical r-matrix $r_{\mathcal{A}}$. From now on we shall denote by r the skew part of $r_{\mathcal{A}}$. Notice that we have a multiparametric family $\mathcal{A}(\xi_1, \ldots, \xi_{k-1})$ of Hopf algebras. Different boundaries of the variety $\mathcal{A}(\xi_1, \ldots, \xi_{k-1})$ can correspond to different classical limits and we can have different r-matrices for $\mathcal{A}$.

The *basic twisting factor* (BTF) $\Phi = \Phi(\xi_k)$ is an element of $\mathcal{A} \otimes \mathcal{A}$ such that

1. it is a parametric solution of the twist equations (1) for $\mathcal{A}(\xi_1, \ldots, \xi_{k-1})$

2. its logarithm $\ln(\Phi)$ has the form

$$\ln(\Phi) = f_1 \otimes f_2, \qquad f_1, f_2 \in \mathcal{A} \tag{5}$$

3. in the variety parametrized by $\xi_1, \ldots, \xi_{k-1}, \xi_k$ there exists a smooth curve $C(\psi)$ (where ψ is the parameter of C) such that in the neighborhood of its origin the antisymmetrized logarithm of the twisting element $(id - \tau)\ln(\Phi)$ has the form:

$$\begin{aligned} &(id - \tau)\ln(\Phi) = \psi g_1 \wedge g_2 + o(\psi), \\ &g_1, g_2 \in \mathbf{g} \end{aligned} \tag{6}$$

and

$$r_{\mathcal{A}_\Phi} = r_{\mathcal{A}} + g_1 \wedge g_2 \tag{7}$$

is a solution of CYBE for $\mathbf{g}$.

According to the definition the minimal carrier subalgebra of BTF is two-dimensional. As far as there are two inequivalent two-dimensional Lie algebras we must have two basic twisting factors. In fact we have at least three inequivalent BTF's. The reason is that the dimension-of-carrier arguments are valid only when the twisting element is applied to the undeformed universal enveloping algebra. For an arbitrary Hopf algebra the notion of the carrier changes and is not more correlated with the classical r-matrix, its relations with the Frobenius subalgebras in $\mathbf{g}$ are much more complicated. Still there is at least one type of twists that has all the properties of BTF though it cannot has a carrier in a Lie algebra (see point 3 below).

The restriction of a BTF to the curve $C(\psi)$,

$$\Phi(\xi_1, \ldots \xi_k)|_{C(\psi)} = \Phi(\psi), \tag{8}$$

is called *the canonical form* of the BTF. Notice that the parameter ξ_k is not necessarily independent of those that are present in the Hopf algebra $\mathcal{A}(\xi_1, \ldots, \xi_{k-1})$. As it follows from the definition ξ_k must provide the possibility to determine the canonical parameter ψ and establish the connection with the classical r-matrix. In particular this means that to introduce ξ_k (or ψ) we can use only the automorphisms of the carrier that do not change its structure constants.

Below the basic twisting factors are presented in their canonical form.

1. The first BTF was obtained in 1990 by Reshetikhin [15]. It was formulated for an Abelian Hopf algebra $\mathcal{AB}$ of arbitrary dimension. Let $\{l_i; i = 1, \ldots, k\}$ be the primitive elements of $\mathcal{AB}$. The following expression is a solution of (1):

$$\mathcal{F}_{AB} = \exp(\sum_{i,j} \eta_{i,j} l_i \otimes l_j), \quad i, j = 1, \ldots, k.$$

Only the antisymmetric part of the matrix η contribute to the r-matrix. So we shall consider only antisymmetric η's. The twist $\mathcal{F}_{AB}$ can be obviously factorized. For any pair $l, l' \in \{l_i | i = 1, \ldots, k\}$ there exists a solution of (1) and the corresponding η plays the role of canonical parameter. We call these solutions the *Reshetikhin basic twisting factors* (RF's):

$$\Phi_R = \exp(\psi l \otimes l'). \tag{9}$$

The costructure of the carrier is not changed by such twists though the universal $\mathcal{R}$-matrix is nontrivial,

$$\mathcal{R}_R = (\Phi_J)_{21}\Phi_J^{-1} = \exp(\psi l' \wedge l). \tag{10}$$

The corresponding classical r-matrix is

$$r_R = l' \wedge l. \tag{11}$$

The factor RF induces deformations in the costructure of $\mathbf{g}$ when the carrier $\mathbf{L}$ is nontrivially injected in $\mathbf{g}$ and $\mathbf{g}$ is nonabelian.

2. In 1992 Ogievetsky had found the so called Jordanian BTF (JF) [16]. Its carrier is the two-dimensional Borel algebra $\mathbf{B}$,

$$[H, E] = E. \tag{12}$$

For $U(\mathbf{B})$ the following element is the solution of (1):

$$\Phi_J = e^{H \otimes \sigma(\psi)} \tag{13}$$

with

$$\sigma(\psi) = \ln(1 + \psi E). \tag{14}$$

This basic factor belongs to the class of twists that satisfy the factorized equations (2). As far as (for BTF's) we have the relation (5) to fulfill the conditions (2) it is sufficient to have primitive element f_1 in $\mathcal{A}$ and check that f_2 will be primitive in $\mathcal{A}_F$. It is easy to prove that the basic factor Φ_J has such a property. The result of the deformation $\Phi_J : U(\mathbf{B}) \longrightarrow U_J(\mathbf{B})$ is the Hopf algebra $U_J(\mathbf{B})$ with the coproducts:

$$\begin{array}{l} \Delta_J(H) = H \otimes e^{-\sigma(\psi)} + 1 \otimes H, \\ \Delta_J(E) = E \otimes e^{\sigma(\psi)} + 1 \otimes E. \end{array} \tag{15}$$

The universal element and its classical counterpart are as follows:

$$\begin{array}{l} \mathcal{R}_J = (\Phi_J)_{21}\Phi_J^{-1} = e^{\sigma(\psi) \otimes H} e^{-H \otimes \sigma(\psi)}, \\ r_J = E \wedge H \end{array} \tag{16}$$

.

3. The third BTF was first considered as a part of the so called *extended Jordanian twist* (EJ) [13]. It is easy to see that it can be treated as an independent twisting factor. Consider the universal enveloping $U(\mathbf{H})$ of the Hiesenberg algebra $\mathbf{H}$:

$$[A, B] = E, \quad [A, E] = [B, E] = 0. \tag{17}$$

It can be supplied with the nontrivial costructure defined by the expressions

$$\begin{array}{l}\Delta_J(A) = A \otimes e^{\alpha\sigma(\xi)} + 1 \otimes A, \\ \Delta_J(B) = B \otimes e^{\beta\sigma(\xi)} + 1 \otimes B, \\ \Delta_J(E) = E \otimes e^{\sigma(\xi)} + 1 \otimes E, \\ \qquad \alpha + \beta = 1\end{array} \tag{18}$$

(where $\sigma(\xi) = \ln(1 + \xi E)$). Thus we get the Hopf algebra $\mathcal{H}_J(\xi) = U_J(\mathbf{H})$. For this algebra the following elements are the solutions of the twist equations (1):

$$\begin{array}{l}\Phi_E = \exp(A \otimes Be^{-\beta\sigma(\xi)}), \\ \Phi_{E'} = \exp(-B \otimes Ae^{-\alpha\sigma(\xi)}).\end{array} \tag{19}$$

These BTF's are called the *extending factors* (EF's) or simply the *extensions*. They induce the deformations of the costructure in $U_J(\mathbf{H})$, for example $\Phi_E : \mathcal{H}_J(\xi) \longrightarrow \mathcal{H}_E(\xi)$, where the coproducts of the algebra $\mathcal{H}_E(\xi)$ are fixed by the relations

$$\begin{array}{l}\Delta_E(A) = A \otimes e^{-\beta\sigma(\xi)} + 1 \otimes A, \\ \Delta_E(B) = B \otimes e^{\beta\sigma(\xi)} + e^{\sigma(\xi)} \otimes B, \\ \Delta_E(E) = E \otimes e^{\sigma(\xi)} + 1 \otimes E.\end{array} \tag{20}$$

Here the parameter ψ coincides with ξ in $\mathcal{H}_J(\xi)$. The reason is that the EF factors are the discrete twists and can only borrow the parameter from $\mathcal{A}(\xi) = \mathcal{H}_J(\xi)$. So far we cannot check the third point of the definition because $\mathcal{H}_J(\xi)$ is not quasitriangular. To connect EF with the r-matrix it is necessary to find the quasitriangular Hopf algebra containing $\mathcal{H}_J(\xi)$. Such algebras exist, the minimal one is the universal enveloping algebra $U_J(\mathbf{L}(\alpha, \beta))$ with the four-dimensional carrier $\mathbf{L}(\alpha, \beta)$:

$$\begin{array}{l}[H, E] = E, \quad [H, A] = \alpha A, \\ [H, B] = \beta B, \quad [A, B] = E, \\ [E, A] = [E, B] = 0, \\ \alpha + \beta = 1.\end{array} \tag{21}$$

Notice that the parameters α and β that appear in (18) are to coincide with the corresponding eigenvalues of the operator $\mathrm{ad}(H)$. The costructure of $U_J(\mathbf{L}(\alpha, \beta))$ is defined by (20) and the coproduct

$$\Delta_J(H) = H \otimes e^{-\sigma(\xi)} + 1 \otimes H.$$

The Hopf algebra $U_J(\mathbf{L}(\alpha, \beta))$ can be obtained by applying to $U(\mathbf{L}(\alpha, \beta))$ the Jordanian factor JF, $\Phi_J : U(L) \longrightarrow U_J(L)$. Here we have the $\mathcal{R}$-matrix $\mathcal{R}_J = e^{\sigma(\xi)\otimes H}e^{-H\otimes\sigma(\xi)}$. Let us put $\xi = \xi_1$ and further apply $\Phi_E(\xi_2)$ to $U_J(\mathbf{L}(\alpha, \beta))$. The necessary condition for the parameters is $\xi_2 = \xi_1 = \psi$. Thus we have ascertained that the expressions (19) are the EF's in the canonical form with the deformation parameter $\psi = \xi$. Two factors JF and EF can be applied to $U(L(\alpha, \beta))$ successively forming the extended Jordanian twist [13]: $\mathcal{F}_{EJ} = \Phi_E \Phi_J$. The twisted algebra $U_{EJ}(L(\alpha, \beta))$ is triangular with the universal element

$$\mathcal{R}_{EJ} = e^{\psi Be^{-\beta\sigma(\psi)}\otimes A}e^{\sigma(\psi)\otimes H} \times e^{-H\otimes\sigma(\psi)}e^{-\psi A\otimes Be^{-\beta\sigma(\psi)}} \tag{22}$$

and the r-matrix

$$r_{EJ} = E \wedge H + B \wedge A \tag{23}$$

that contains the term corresponding to the EF factor. (All the constructions above can be equally performed for $\Phi_{E'}$ – the second type of EF (see (19))). Despite the close relations between EJ and EF it is important that the factor EF can be considered as an independent BTF.

3 Compositions

In this section we shall demonstrate how the basic factors are composed, the known twists are factorized into sequences of BTF's and trivial twists and also how the canonical expressions for BTF's can be deformed in such sequences.

1. In the root system $\Lambda(\mathbf{g})$ of a simple Lie algebra $\mathbf{g}$ choose any long root as the initial root λ_0 and consider the set π of its constituent roots:

$$\begin{aligned} \pi = \;& \{\lambda', \lambda'' \mid \lambda' + \lambda'' = \lambda_0; \\ & \lambda' + \lambda_0, \lambda'' + \lambda_0 \notin \Lambda(g)\} \\ \pi = \;& \pi' \cup \pi''; \qquad \pi' = \{\lambda'\}, \pi'' = \{\lambda''\}. \end{aligned}$$

Consider the subset $\Lambda_{\lambda_0}^{\perp}$ of roots orthogonal to λ_0 and the corresponding subalgebra $\mathbf{g}_2 \equiv \mathbf{g}_{\lambda_0}^{\perp} \subset \mathbf{g} \equiv \mathbf{g}_1$. (For $sl(N)$ we have $\mathbf{g}_2 = sl(N-2)$.) It was shown in [9], [11] that for any classical Lie algebra it is possible to compose chains of twists corresponding to the sequence of injections $\mathbf{g}_p \subset \ldots \subset \mathbf{g}_2 \subset \mathbf{g}_1$:

$$\begin{aligned} &\mathcal{F}_{\mathcal{B}_{1\prec p}}(\xi_1, \ldots, \xi_p) = \prod_{\lambda' \in \pi'_p} e^{\xi_p E_{\lambda'} \otimes E_{\lambda_0^p - \lambda'} e^{-\frac{1}{2}\sigma_{\lambda_0^p}(\xi_p)}} \times e^{\{H_{\lambda_0^p} \otimes \sigma_{\lambda_0^p}(\xi_p)\}} \times \mathcal{F}_{s_{p-1}} \times \\ &\prod_{\lambda' \in \pi'_{p-1}} e^{\xi_{p-1} E_{\lambda'} \otimes E_{\lambda_0^{p-1} - \lambda'} e^{-\frac{1}{2}\sigma_{\lambda_0^{p-1}}(\xi_{p-1})}} \times e^{H_{\lambda_0^{p-1}} \otimes \sigma_{\lambda_0^{p-1}}(\xi_{p-1})} \times \ldots \\ &\mathcal{F}_{s_1} \times \prod_{\lambda' \in \pi'_1} e^{\xi_1 E_{\lambda'} \otimes E_{\lambda_0^1 - \lambda'} e^{-\frac{1}{2}\sigma_{\lambda_0^1}(\xi_1)}} \times e^{\{H_{\lambda_0^1} \otimes \sigma_{\lambda_0^1}(\xi_1)}. \end{aligned} \tag{24}$$

All the exponents in these chains are the canonical BTF's of two types: extension and Jordanian. Each JF has its proper parameter and each EF depends only on the parameter borrowed from only one preceding Jordanian factor. Each BTF has the corresponding term in the r-matrix

$$r_{\mathcal{B}_{1\prec p}}(\xi_1, \ldots, \xi_p) = \sum_{i=1}^{p} \sum_{\lambda' \in \pi'_i} \xi_i E_{\lambda'} \otimes E_{\lambda_0^i - \lambda'} + \sum_{i=1}^{p} \xi_i H_{\lambda_0^i} \otimes E_{\lambda_0^i}. \tag{25}$$

The chain in the form (24) contains also the trivial factors $\mathcal{F}_{s_i}$ different for different series of simple algebras (for the series A_n they degenerate into tensor units, for details of their construction see [11]).

2. In [14] the quantum Jordanian twist was constructed. It is an analogue of the Jordanian twist for standard quantization of Lie algebras. In this case $\mathcal{A}(\xi_1) = U_q(\mathbf{g})$ and $\xi_1 = h = \ln q$. The quantum Jordanian twisting element has the following form:

$$\mathcal{F}_{q\mathcal{J}}(h, \xi_2) = e^{H \otimes \omega(\xi_2)} e^{-hH \otimes H} \tag{26}$$

with

$$\omega = \ln\left((\xi_2 E + 1)\, e^{hH}\right). \tag{27}$$

The first factor of $\mathcal{F}_{q\mathcal{J}}$ is trivial while the second is the JF. When $\xi_1 = h = 0$ the factor $e^{H\otimes\omega}$ obtains its canonical form.

3. The twists that enable the transitions from the standard quantization to the other types classified in the Belavin-Drinfeld list (see [2], [5], [6]) can be expressed in terms of BTF's. From the r-matrices' point of view the purpose of that twisting is to convert one type of quantization into the other, but not to enlarge the carrier subalgebra. The twisting elements have the same parameters as the initial quantum algebra $\mathcal{A}(q) = U_q(\mathrm{g})$. They can be scaled to obtain the ordinary BTF form. For example, the twisting element that transforms the standard $U_q(sl(3))$ into the Cremmer-Gervais quantization looks like

$$\begin{aligned} \mathcal{F}_{CG}(h) &= e_{q^2}^{-\mu E_1\otimes F_2} e^{-h\eta_{i,j} H_i\otimes H_j}, \\ \mu &= q - q^{-1}. \end{aligned} \tag{28}$$

As far as $\eta_{i,j}$ is arbitrary and μ can be multiplied by an arbitrary scaling factor one can cancel the dependence on h. Then for $h = 0$ we get the canonical forms of two Reshetikhin factors and the trivial twist factor depending on the symmetric part of $\eta_{i,j}$.

In the cases presented above the deformation of the twisting factors was induced by the standard quantization of the universal enveloping algebra. One can find also the deformed EF's in the chains of twists. In particular this happens when the compensating trivial twists are cancelled in $\mathcal{F}_{\mathcal{B}_{1\prec p}}(\xi_1, \ldots, \xi_p)$. Such cancellation leads to the deformation of the remaining factors of the chain. The reason is that all the factors $\mathcal{F}_{s_i}$ are cohomologically trivial only for the costructures deformed by the previous links. But they could be nontrivial when the links that follow the particular $\mathcal{F}_{s_i}$ are applied. One can drag all $\mathcal{F}_{s_i}$'s to the very end of the chain. This highly cumbersome transformation will deform the exponential factors though they will remain BTF's and the factorization property will still be valid. It can be checked that the product $\prod_i \mathcal{F}_{s_i}$ is a solution of (1) for the coproducts transformed by the remaining part of the chain. This means that the sequence $(\prod_i \mathcal{F}_{s_i})^{-1}\mathcal{F}_{\mathcal{B}_{1\prec p}}(\xi_1, \ldots, \xi_p)$ is also a twist. In general it is composed by the deformed BTF's (except the factors of the first link of the chain that are always in the canonical form). To describe such effects the notion of the *deformed carrier space* was introduced (see [17]).

4 Quantizing the Parabolic r-Matrix

In this section we shall consider the situation when the deformation of the basic factor is produced by the nonstandard quantization and at the same time this BTF cannot recover its canonical form with the help of trivial twists (like $\mathcal{F}_{s_i}$ in (24)).

Let $\mathrm{g} = sl(3)$ and

$$r_{\wp} = H_{23}^{\perp} \wedge E_{13} + E_{12} \wedge E_{23} + H_{13}^{\perp} \wedge E_{32}. \tag{29}$$

with

$$\begin{aligned}
H_{13}^{\perp} &= \frac{1}{3}E_{11} - \frac{2}{3}E_{22} + \frac{1}{3}E_{33}, \\
H_{23}^{\perp} &= \frac{2}{3}E_{11} - \frac{1}{3}E_{22} - \frac{1}{3}E_{33}, \\
H_{23} &= E_{22} - E_{33} = H_{23}^{\perp} - 2H_{13}^{\perp}
\end{aligned}$$

(E_{ij} are the ordinary matrix units). The element $r_{\wp} \in \mathbf{g} \wedge \mathbf{g}$ is the parabolic r-matrix defined on the carrier $\wp$ generated by the Borel subalgebra $\mathbf{B}(sl(3))$ and the element E_{32}, i.e. on the parabolic subalgebra of $sl(3)$.

According to our main statement the twisting element $\mathcal{F}_{\wp}$ (that will quantize $r_{\wp}$) can be presented as a product of basic twisting factors. The form (29) of the r-matrix suggests that two of them correspond to the extended Jordanian twist. It follows immediately that the third factor is also a JF whose structure might be deformed by the previous twists. We start the composition of $\mathcal{F}_{\wp}$ by the JF and EF factors on the carrier subalgebra $\mathbf{L}$ generated by E_{12}, E_{13}, E_{23} and H_{23}:

$$\mathcal{F}_{EJ} = \Phi_{E'}\Phi_J = e^{(-\psi E_{23} \otimes E_{12} e^{\sigma_{13}(\psi)})} e^{(H_{23} \otimes \sigma_{13}(\psi))} \tag{30}$$

with $\sigma_{13} = \ln(1 + \psi E_{13})$. The Cartan element is chosen here to be H_{23} in order to have the simplest possible form for the coproducts of the last two generators that we need: $H_{13}^{\perp}$ and E_{32};

$$\begin{aligned}
\Delta_{EJ}(H_{13}^{\perp}) &= H_{13}^{\perp} \otimes 1 + 1 \otimes H_{13}^{\perp}, \\
\Delta_{EJ}(E_{32}) &= E_{32} \otimes e^{-2\sigma_{13}(\xi_1)} + 1 \otimes E_{32}.
\end{aligned} \tag{31}$$

Now for the deformed algebra $\mathcal{A}(\xi_1) = U_{EJ}(\wp; \xi_1)$ and we are to construct the (deformed) JF for the Borel subalgebra generated by $H_{13}^{\perp}$ and E_{32},

$$\Phi_J(\xi_{1,2}) = e^{H_{13}^{\perp} \otimes \omega(\xi_{1,2})}. \tag{32}$$

Here the element $f_1 = H_{13}^{\perp}$ is primitive, thus only the function $f_2 = \omega(\xi_{1,2})$ is supposed to be deformed (different from the canonical $\sigma_{32}(\xi_2)$).

As we have seen in the Example 2 in the deformed case the Borel subalgebra is to be considered in the form

$$[H_{13}^{\perp}, e^{\omega(\xi_{1,2})} - \gamma(\xi_{1,2})] = e^{\omega(\xi_{1,2})} - \gamma(\xi_{1,2}), \tag{33}$$

where the element $\gamma(\xi_{1,2}) \in U_{EJ}(\wp; \xi_1)$ commutes with $H_{13}^{\perp}$,

$$[H_{13}^{\perp}, \gamma(\xi_{1,2})] = 0, \tag{34}$$

and is group-like,

$$\Delta_{EJ}(\gamma(\xi_{1,2})) = \gamma(\xi_{1,2}) \otimes \gamma(\xi_{1,2}). \tag{35}$$

To find the solution of the factorized equations (2) we must construct $\omega(\xi_{1,2})$ that will be primitive in $\mathcal{A}_J(\xi_{1,2}) = U_{JEJ}(\wp; \xi_{1,2})$, or in other terms, to impose the condition

$$e^{\mathrm{ad}(H_{13}^{\perp} \otimes \omega)} \circ \Delta_{EJ}(e^{\omega}) = e^{\mathrm{ad}(H_{13}^{\perp} \otimes \omega)} \circ \Delta_{EJ}(e^{\omega} - \gamma) + e^{\mathrm{ad}(H_{13}^{\perp} \otimes \omega)} \circ (\gamma \otimes \gamma) = e^{\omega} \otimes e^{\omega}. \tag{36}$$

If $(e^{\omega} - \gamma)$ is quasiprimitive this condition can be simplified:

$$\Delta_{EJ}(e^{\omega}) = (e^{\omega} - \gamma) \otimes 1 + \gamma \otimes e^{\omega}. \tag{37}$$

Our deformations are smooth and in the limit $\xi_1 \longrightarrow 0$ the basic factor $\Phi_J(\xi_{1,2})$ must have the canonical form $\Phi_J(0, \xi_2) = e^{H_{13}^{\perp} \otimes \sigma_{32}(\xi_2)}$. So we can consider the deformed factor in terms of the expansions,

$$\begin{array}{l} \gamma(\xi_{1,2}) = 1 + \sum_{i=1} \gamma_i(\xi_2)\xi_1^i, \\ e^{\omega(\xi_{1,2})} - \gamma(\xi_{1,2}) = \xi_2 E_{32} + \sum_{i=1} \eta_i(\xi_2)\xi_1^i, \end{array} \tag{38}$$

The corresponding power series for the left-hand side in (37) can be found using the expression (30). Now we can construct the deforming functions γ_i and η_i. It can be proved that in our case the solutions contain only the first and the second order of the deformation parameter ξ_1:

$$e^{\omega(\xi_{1,2})} - \gamma(\xi_1) = \xi_2 E_{32} + 2\xi_1\xi_2 E_{32}E_{13} + \xi_1^2\xi_2 E_{32}E_{13}^2, \gamma(\xi_1) = 1 + 2\xi_1 E_{13} + \xi_1^2 E_{13}^2. \tag{39}$$

This result can be presented in a compact form

$$e^{\omega(\xi_{1,2})} = e^{\sigma_{32}(\xi_2)} e^{2\sigma_{13}(\xi_1)}. \tag{40}$$

Thus we have obtained for the basic Jordanian factor $\Phi_J(\xi_2)$ the deformed expression

$$\Phi_J(\xi_{1,2}) = \exp(H_{13}^{\perp} \otimes \ln(e^{\sigma_{32}(\xi_2)} e^{2\sigma_{13}(\xi_1)})) \tag{41}$$

that satisfy the factorized twist equations for Δ_{EJ}. We can apply this JF to the previously deformed algebra $\Phi_J(\xi_{1,2}) : U_{EJ}(\wp; \xi_1) \longrightarrow U_{JEJ}(\wp; \xi_{1,2})$. The composition of twist deformations
$\Phi_J(\xi_{1,2})\mathcal{F}_{EJ}(\xi_1) : U(\wp) \longrightarrow U_{JEJ}(\wp; \xi_{1,2})$ is the parabolic twist factorized into a sequence of BTF's:

$$\mathcal{F}_{\wp'}(\xi_{1,2}) = \Phi_J(\xi_{1,2})\Phi_{E'}(\xi_1)\Phi_J(\xi_1) = e^{H_{13}^{\perp} \otimes \ln(e^{\sigma_{32}(\xi_2)} e^{2\sigma_{13}(\xi_1)})} e^{-\xi_1 E_{23} \otimes E_{12} e^{\sigma_{13}(\xi_1)}} \times e^{H_{23} \otimes \sigma_{13}(\xi_1)}. \tag{42}$$

This solution of twist equations corresponds to the r-matrix $r_{\wp}$ (29). The parabolic twist was first constructed in [12]. There also the factorization properties were exploited and the generalized Verma relations were used to obtain the twisting element in the form

$$\mathcal{F}_{\wp}(\xi_{1,2}) = e^{H_{13}^{\perp} \otimes \ln(e^{\sigma_{13}(\xi_1)} e^{\sigma_{32}(\xi_2)} e^{\sigma_{13}(\xi_1)})} \times e^{-\xi_1 E_{23} \otimes E_{12} e^{\sigma_{13}(\xi_1)}} e^{H_{23} \otimes \sigma_{13}(\xi_1)}. \tag{43}$$

In the calculations presented above the factorization property was combined with the perturbation approach. As a result we obtained the new deformed Jordanian factor. Notice that the parabolic twists $\mathcal{F}_{\wp}(\xi_{1,2})$ are $\mathcal{F}_{\wp'}(\xi_{1,2})$ inequivalent.

5 Conclusions

It is not evident that one can consider all the twists (for the universal enveloping algebras of simple algebras) to be factorizable into the basic factors. At the same time we have seen that these properties are valid in quite different situations including quantum algebras and carrier subalgebras with generators from positive and negatives sectors of $\Lambda(\mathbf{g})$. Moreover the factorization ansatz can be used to construct new twists. Its application simplifies considerably the solution of twist equations reducing it to the problem of finding the suitable deformations for functions $f_{1,2}$ in (5).

There could be more than three basic twisting factors. In the Lie algebras (whose root system contains long series of roots) a pair of generators can have relations different from those typical to the three known BTF's (Abelian, Borel or Heisenberg).

The factorization property means that any twisted "symmetry" can be constructed step by step, deforming each time only one pair of variables from the classical phase space. It would be important to relate this effect to the complete solvability of the corresponding quantum systems.

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Generalized Grasmann Algebras and Extended Supersymmetric Models

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Abstract

It is shown that the fermionic Heisenberg-Weyl algebra with $2N = D$ fermionic generators is equivalent to the generalized Grassmann algebra with two fractional generators. The 2,3 and 4 dimensional Heisenberg - Weyl algebra is explicitly given in terms of the fractional generators. These algebras are used for the formulation of the $N = 2,3,4$ extended supersymmetry. As an example we reformulate the Lax approach of the supersymmetric Korteweg - de Vries equation in terms of the generators of the generalized Grassmann algebra.

1. Introduction

Generalized Grassmann algebras have been introduced firstly in [1] in the framework of the 2D conformal field theories. Later these algebras were rediscovered in the contexts of quantum groups [2] and fractional generalizations of supersymmetric quantum mechanics [3]. Next, these algebras were used for the generalization of the supersymmetry to the fractional supersymmetry [4], [5], [6], [7].

Interestingly the generalized Grassmann algebras (or their intimate analogs) have found applications in the construction of the finite dimensional (cyclic) representations of the quantum groups, covariant quantum algebras and zero-mode algebras in WZNW models (see e.g. [2], [8], [9], [6], [10]). The generalized Grassmann algebras should be also useful for the construction of finite dimensional representations of the quadratic diffusion algebras [11] which are employed in the theory of one-dimensional stochastic processes with exclusion. The fractional extensions of the Virasoro algebra [4] utilizes generalized Grassmann algebras as well. Recently generalized Grassmann algebras appeared in the systems on linear lattices with periodic boundary conditions [12] (see also [9]). In this paper we find the irreducible representations of the Clifford algebras by realizing them in terms of the generalized Grassmann algebras.

The main idea of the introduction of the generalized Grassmann algebras is to replace the usual fermionic condition $\theta^2 = 0$ by the more general nilpotence condition $\theta^{p+1} = 0$. The p number in the physical literature is usually called the order of parastatistics. Due to the big interest to such generalization it is tempting to try to use these generalized algebras to the supersymmetric solitonic equations.

In this paper we propose the method of the reformulations, in terms of the generalized Grassmann variables, the usual extended supersymmetric models. We concentrate our attention to the supersymmetric Korteweg - de Vries (SKdV) equation only. However our approach could be used to any $N = 1,2,3,4$ supersymmetric model as well.

We explicitly show that the fermionic Heisenberg-Weyl algebra with $2N = D$ fermionic generators is equivalent to the generalized Grassmann algebra with two fractional generators θ, ∂ where the order of parastatistics p is $2^N - 1$. We explicitly construct the 2,3,4-

dimensional Heisenberg-Weyl algebras in terms of the generalized Grassmann algebras with $p+1=4,8,16$. It appears that this Heisenberg-Weyl algebra is closely related to the Clifford algebras in Euclidean spaces of the dimensions $D=4,6,8$. One can use this observation to formulate the $N=2,3,4$ extended supersymmetry. All of this allowed us to formulate the $N=2$ SKdV equations in terms of the generalized Grassmann algebras for $p+1=4$.

The paper is organized as follows: In Section 2 we recall the basic facts about generalized Grassmann algebras. In Section 3 the Weyl construction for the Heisenberg-Weyl and Clifford algebras is discussed. In Section 4 we construct fermionic $N=2,3,4$ Heisenberg-Weyl algebras in terms of the generators of the Z_4, Z_8 and Z_{16} generalized Grassmann algebras. In Section 5 we discuss the example of the $N=1,2$ SKdV equations rewritten in terms of Z_2 and Z_4 graded algebra respectively. The Section 6 contains conclusions.

2. Generalized Grassmann Algebra

Consider a deformed oscillator algebra with two generators θ and ∂ satisfying[1], [2], [5]

$$\partial\,\theta - q\,\theta\,\partial = I\,, \tag{2.1}$$

$$\theta^{p+1} = \partial^{p+1} = 0 \quad \Rightarrow \quad q^{p+1} = 1\,, \tag{2.2}$$

where p is an integer. Note that the complete basis of this algebra is given by the $(p+1)^2$ elements $(\theta^n\,\partial^m)$ $(0 \le n, m \le p)$ and this algebra is isomorphic to the algebra of matrices $Mat(p+1)$ [2], [17]. The grading in this algebra is

$$deg\,(\theta^n\,\partial^m) = n - m\,. \tag{2.3}$$

It is convenient to introduce the grading operator [2]

$$\omega = \partial\,\theta - \theta\,\partial \quad \Rightarrow \quad \omega^{p+1} = 1\,, \tag{2.4}$$

which defines the automorphisms

$$\omega\,\theta\,\omega^{-1} = q\,\theta\,, \quad \omega\,\partial\,\omega^{-1} = q^{-1}\,\partial\,. \tag{2.5}$$

Using (2.1) and (2.4) we obtain useful relations

$$\theta\,\partial = \frac{\omega - 1}{q-1}\,, \quad \partial\,\theta = \frac{q\,\omega - 1}{q-1}\,. \tag{2.6}$$

The algebra only defined by (2.1), but not (2.2), has been firstly considered in [13]. This algebra is closely connected to the algebra of Macfarlane - Biedenharn quantum oscillators [14]. For the nontrivial case, when q is root of unity, this connection has been discussed in [9].

A characteristic equation $\omega^{p+1}=1$ for the grading operator (2.4) can be rewritten in the form

$$(\omega-1)\,(\omega-q)\ldots(\omega-q^p) = \prod_{n=0}^{p}\,(\omega-q^n) = 0\,, \tag{2.7}$$

and one can introduce the projection operators

$$P_k = \prod_{n=0,n\neq k}^{p} \frac{(\omega-q^n)}{(q^k-q^n)}\,, \quad (k=0,\ldots,p)\,. \tag{2.8}$$

These projectors satisfy the following identities

$$\begin{array}{c} \sum_{k=0}^{p} P_k = 1\,, \quad P_n\,P_k = \delta_{nk}\,P_k, \\ \omega\,P_k = P_k\,\omega = q^k\,P_k, \\ P_0\,\theta = 0\,, \quad \theta\,P_{k-1} = P_k\,\theta\,, \quad \theta\,P_p = 0, \\ \partial\,P_0 = 0\,, \quad P_{k-1}\,\partial = \partial\,P_k\,, \quad P_p\,\partial = 0. \end{array} \tag{2.9}$$

In view of the nilpotence conditions (2.2) one can deduce

$$\begin{array}{ccc} \theta^{p+1-n}\,\partial^{p+1-n}\,\partial^n = 0 & \Rightarrow & \prod_{k=0}^{p-n}(\omega - q^k)\,\partial^n = 0. \\ \theta^n\,\theta^{p+1-n}\,\partial^{p+1-n} = 0 & \Rightarrow & \theta^n\,\prod_{k=0}^{p-n}(\omega - q^k) = 0. \end{array} \tag{2.10}$$

It means that

$$\theta^n\,P_m = 0 = P_m\,\partial^n \quad (p - n < m \le p). \tag{2.11}$$

A generic function u, with θ as an argument, can be expanded in power series as

$$u(\theta) = u_0 + \theta\,u_1 + \ldots + \theta^p\,u_p. \tag{2.12}$$

where u_i are in general noncommutative coefficients. The left action of θ on $u(\theta)$ is given by

$$\theta\,u(\theta) = \theta\,u_0 + \theta^2\,u_1 + \ldots + \theta^p\,u_{p-1}. \tag{2.13}$$

To define the left action of the operator ∂ on the function $u(\theta)$ (2.12) we find, using the relation (2.1), that

$$\partial\,\theta^n = (1 + q + q^2 + ... + q^{n-1})\,\theta^{n-1} + q^n\,\theta^n\,\partial = (n)_q\,\theta^{n-1} + q^n\,\theta^n\,\partial, \tag{2.14}$$

where the q-number $(n)_q$ is

$$(n)_q = \frac{(1 - q^n)}{1 - q}. \tag{2.15}$$

It is clear that ∂ can be considered as a generalization of the ordinary derivative and using (2.14) we obtain

$$\partial\,(u(\theta)) = u_1 + (2)_q\,\theta\,u_2 + \ldots + (p)_q\,\theta^{p-1}\,u_p. \tag{2.16}$$

The equation $q^{p+1} = 1$ is a consequence of the nilpotence condition $\theta^{p+1} = 0$ (2.2) and relation (2.14) for $n = p + 1$. If in addition we require that $\partial(\theta^n) \neq 0$ for $n < p + 1$, then q should be a primitive root of unity ($q^n \neq 1$ for all $n < p + 1$) and one can choose $q = \exp(2\pi i/(p + 1))$. In this case, generators θ and ∂ (2.1), (2.2) are reduced to the ordinary fermionic ($p = 1$) and bosonic ($p \to \infty$) creation and annihilation operators.

Equations (2.13) and (2.16) give a matrix representation for θ and ∂. Indeed, arbitrary function $u(\theta)$ (2.12) can be realized as $(p + 1)$ dimensional vector

$$u = \begin{pmatrix} u_0 \\ \vdots \\ u_p \end{pmatrix}, \quad u_k = < k|u > . \tag{2.17}$$

To reproduce the actions of θ and ∂ on the function $u(\theta)$ (2.13) and (2.16) we need the following matrix representation of θ and ∂

$$\theta_{km} = \begin{pmatrix} 0 & . & . & 0 \\ 1 & 0 & . & . \\ . & . & 0 & . \\ 0 & . & 1 & 0 \end{pmatrix} =< k|\theta|m >,$$

$$\partial_{km} = \begin{pmatrix} 0 & 1 & 0 & . & 0 \\ . & 0 & (2)_q & 0 & . \\ . & . & . & . & . \\ . & . & . & 0 & (p)_q \\ 0 & . & . & . & 0 \end{pmatrix} =< k|\partial|m > . \tag{2.18}$$

The corresponding matrix representations of ω (2.4) and projectors (2.8) are

$$\omega_{km} = \begin{pmatrix} 1 & 0 & . & 0 \\ 0 & q & . & . \\ . & . & . & . \\ 0 & . & 0 & q^p \end{pmatrix} =< k|\omega|m >,$$

$$(P_n)_{km} = \delta_{kn}\,\delta_{km}, \tag{2.19}$$

and one can obtain the relations

$$\begin{aligned} \omega^T = \omega\,, \quad \partial^T = (\tfrac{\omega-1}{q-1})\,\theta, \\ \theta^T = \partial\,\left(\textstyle\sum_{k=1}^{p} \frac{1}{(k)_q}\,P_k\right), \end{aligned} \tag{2.20}$$

where T is a transposition of the matrices. The last equation in (2.20) has been obtain with the help of projectors (2.8).

In eqs. (2.17) - (2.19) we have introduced the ladder of $p+1$ states $|m>$ $(m = 0, \ldots, p)$ defined by: $\partial|0>= 0$, $\theta|m>= |m+1>$, $\partial|m>= (m)_q\,|m-1>$. The dual states $<k|$ satisfy the orthogonality condition $<k|m>= \delta_{km}$. The matrix θ_{km} has the form $\delta_{k,m+1}$ and in general $(\theta^n)_{km} = \delta_{k,m+n}$, while for the matrix ∂ we have $(\partial^n)_{km} \sim \delta_{k+n,m}$. The form of the matrices (θ^n), (∂^n) and ω corresponds to our choice of Z_{p+1} grading (2.3) which is naturally called in [2] as "along diagonal" grading. We use the matrix representations (2.18), (2.19) in our calculations below.

Note that the algebra (2.1), with additional relations (2.2), represents a special case of the more general algebra

$$\mathcal{D}\,\Theta - q\,\Theta\,\mathcal{D} = I\,, \tag{2.21}$$

$$\Theta^{p+1} = c_1\,, \quad \mathcal{D}^{p+1} = c_2 \;\Leftrightarrow\; q^{p+1} = 1\,, \tag{2.22}$$

where operators c_i are central elements, $deg(\Theta) = 1$ and $deg(\mathcal{D}) = -1$. For all fixed values of c_i these algebras are isomorphic to the algebra of matrices $Mat(p+1)$ and, therefore, the generators $\Theta, \mathcal{D}$ can be expressed in terms of the elements θ, ∂ of the generalized Grassmann algebra (2.1), (2.2). In the case $c_i = 0$ we have $\Theta = \theta$, $\mathcal{D} = \partial$. Let $c_1 \neq 0$ (the case $c_2 \neq 0$ is considered below) and operators Θ and $\mathcal{D}$ are

$$\Theta = \theta + \frac{1}{(p)_q!}\,\partial^p\,c_1\,, \quad \mathcal{D} = \Theta^{-1}\,\frac{z\,\omega - 1}{q-1}\,, \tag{2.23}$$

where z is a function of central elements c_i which is fixed below, $(n)_q! := (1)_q \cdot (2)_q \cdots (n)_q$. To find the expression for $\mathcal{D}$ we have used (2.21) and the identity

$$\mathcal{D}\Theta - \Theta\mathcal{D} = z\,\omega\,.$$

Note that the operator Θ in the matrix representation has the familiar form of the cyclic matrix:

$$\Theta_{km} = \begin{pmatrix} 0 & . & 0 & c_1 \\ 1 & 0 & . & 0 \\ . & . & 0 & . \\ 0 & . & 1 & 0 \end{pmatrix}.$$

Operators Θ, $\mathcal{D}$ (2.23) automatically satisfy conditions (2.21), (2.22) except eq. $\mathcal{D}^{p+1} = c_2$ which connects the parameter z with the central elements c_i:

$$c_1\,c_2\,(1-q)^{p+1} = \prod_{k=0}^{p} (1 - z\,q^{-k}\,\omega) = 1 - z^{p+1}, \tag{2.24}$$

where we have used that $\omega^{p+1} = 1 = q^{p+1}$.

The case $c_2 \neq 0$ can be considered analogously and one can take

$$\mathcal{D} = \partial + \frac{1}{(p)_q!}\,\theta^p\,c_2\,, \quad \Theta = \frac{z\,\omega - 1}{q-1}\,\mathcal{D}^{-1}, \tag{2.25}$$

instead of (2.23) with the same relation (2.24) on the elements c_i and z. This relation is rewritten in the form of Fermat curve $y^{p+1} + z^{p+1} = 1$ where $y = (c_1\,c_2)^{1/(p+1)}\,(1-q)$. At the end of this Section we note that the differential calculus on the generalized Grassmann algebras [2], [17] is closely related to the differential calculus on the finite groups Z_n [15].

3. The Heisenberg-Weyl and Clifford Algebras

The N-dimensional Heisenberg-Weyl algebra is generated by $2N$ operators θ_i and ∂_i ($i = 1,2,..,N$) satisfying the following commutation relation

$$\begin{aligned} \theta_\mu\theta_\nu + \theta_\nu\theta_\mu &= 0, \\ \partial_\mu\partial_\nu + \partial_\nu\partial_\mu &= 0, \\ \partial_\mu\theta_\nu + \theta_\nu\partial_\mu &= \delta_{\mu\nu}. \end{aligned} \tag{3.26}$$

It is straightforward to check that

$$\gamma_\mu = \partial_\mu + \theta_\mu\,, \quad \gamma_{N+\mu} = i(\partial_\mu - \theta_\mu), \tag{3.27}$$

(where $\mu = 1,2,..,N$) is a realization of $2N$ - dimensional Euclidean Clifford algebra $[\gamma_A, \gamma_B]_+ = 2\delta_{AB}$. For our purposes it is convenient to recall the Weyl construction [16] of the generators θ_μ, ∂_μ of the algebra (3.26)

$$\theta_\mu = \underbrace{\hat\omega \otimes \hat\omega \otimes .... \otimes \hat\omega}_{(\mu-1)times} \otimes \hat\theta \otimes \underbrace{I \otimes .... \otimes I}_{(N-\mu)times}, \tag{3.28}$$

$$\partial_\mu = \underbrace{\hat\omega \otimes \hat\omega \otimes .... \otimes \hat\omega}_{(\mu-1)times} \otimes \hat\partial \otimes \underbrace{I \otimes ...... \otimes I}_{(N-\mu)times}, \tag{3.29}$$

via the generators $\hat{\theta}$ and $\hat{\partial}$ of the 1-dimensional fermionic Heisenberg - Weyl algebra (2.1), (2.2) (for $p = 1$)

$$\hat{\theta}\,\hat{\partial} + \hat{\partial}\,\hat{\theta} = 1\,, \quad \hat{\theta}^2 = 0 = \hat{\partial}^2\,, \quad \hat{\omega} = [\hat{\partial},\,\hat{\theta}]. \tag{3.30}$$

Using relations (3.28) and (3.29) one can construct the irreducible representation of the HW algebra (3.26) and even dimensional Clifford algebras using the representation (2.18), (2.19) of the generalized Grassmann algebra for $p = 1$:

$$\begin{gathered}\hat{\theta} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ \hat{\partial} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ \hat{\omega} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{gathered} \tag{3.31}$$

Recall that the tensorial product (in particular the tensorial products in (3.28), (3.29)) of two matrices A and B ($n \otimes n$ and $m \otimes m$ respectively) is defined by the following $n \cdot m \otimes n \cdot m$ matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & . & . & a_{1n}B \\ . & . & . & . \\ . & . & . & . \\ a_{n1}B & . & . & a_{nn}B \end{pmatrix}. \tag{3.32}$$

The matrices (3.28), (3.29) are $2^N \times 2^N$ dimensional matrices. Our task is to embed the $2N$ dimensional Heisenberg-Weyl algebra (3.26) into the generalized Grassmann algebra (2.1), (2.2) for some fixed p. It is clear (by considering the dimensions of the matrix representations (2.18), (2.19) and (3.28), (3.29)) that the Heisenberg-Weyl algebra can be realized in terms of the q-operators only if $p + 1 = 2^N$ and, thus, p must be odd.

Remark 1. Matrix representations of the generators θ_μ and ∂_μ (3.28), (3.29)), (3.31) satisfy conditions

$$\theta_\mu^T = \partial_\mu\,, \quad \partial_\mu^T = \theta_\mu\,, \quad \theta_\mu^* = \theta_\mu\,, \quad \partial_\mu^* = \partial_\mu \tag{3.33}$$

where T is a transposition and $*$ is a complex conjugation of the matrices. It means that eqs. (3.27) define the real representation of the Clifford algebras.

Remark 2. The Weyl construction (3.28), (3.29) can be generalized for the case when the algebra $\{\hat{\theta}, \hat{\partial}\}$ is taken to be Z_{p+1} algebra (2.1), (2.2) for arbitrary p. In this way one can construct multidimensional generalized Grassmann algebras (see [17], [9] for details) and the algebras of covariant quantum oscillators [2].

4. $N = 2, 3, 4$ Heisenberg-Weyl Algebras via $Z_{4,8,16}$ Generalized Grassmann Algebras

4.1 $N = 2$ Heisenberg-Weyl algebra

To understand what we mean with embedding of algebras, let us consider the case $N = 2$ (or $p = 3$). We will consider only primitive roots q. It means that for $q^4 = 1$, we have to put

$q^2 = -1$. The Weyl representation (3.28), (3.29) gives

$$\theta_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \theta_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{4.34}$$

$$\partial_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \partial_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.35}$$

Note that the non zero elements of θ_1 and ∂_1 are in the sites $(k, k-2)$ and $(k-2, k)$, while the non zero elements of θ_2 and ∂_2 are in the sites $(k, k-1)$ and $(k-1, k)$. It means that the generators θ_i and ∂_i can be expressed (according with the along diagonal grading (2.3)) in terms of the generators of the generalized Grassmann algebra as

$$\begin{aligned} \theta_1 &= t_0\theta^2 + t_1\theta^3\partial, \\ \theta_2 &= t_0'\theta + t_1'\theta^2\partial + t_2'\theta^3\partial^2. \end{aligned} \tag{4.36}$$

$$\begin{aligned} \partial_1 &= d_0\partial^2 + d_1\theta\partial^3, \\ \partial_2 &= d_0'\partial + d_1'\theta\partial^2 + d_2'\theta^2\partial^3. \end{aligned} \tag{4.37}$$

Our aim is to determine the coefficients t_i, t_i' and d_i, d_i' in (4.36) and (4.37) in such a way that generators θ_μ, ∂_ν obey the defining relations (3.26). According with the representation (2.19) we find that

$$\theta_1 = \theta^2\,, \quad \theta_2 = \theta\,(P_0 - P_2) = (P_1 - P_3)\,\theta. \tag{4.38}$$

Since the generators ∂_μ are related to the generators θ_μ by the transpositions (3.33) and using (2.20) one can deduce from (4.38) the expressions

$$\begin{aligned} \partial_1 &= \tfrac{1}{(2)_q}\,\partial^2 \left(P_2 + \tfrac{1}{(3)_q}\,P_3\right), \\ \partial_2 &= \partial \left(P_2 - \tfrac{1}{(3)_q}\,P_3\right). \end{aligned} \tag{4.39}$$

Since q is a primitive root of the equation $q^4 = 1$ the following identities hold:

$$\begin{gathered} q^2 = -1\,, \quad (2)_q = 1 + q\,, \quad (3)_q = 1 + q + q^2 = q \\ \Downarrow \\ (2)_q\,(3)_q = (1+q)\,q = q - 1, \end{gathered} \tag{4.40}$$

and using relations (2.1), (2.6) one can deduce another representation for the generators of the $N = 2$ Heisenberg-Weyl algebra

$$\theta_1 = \theta^2\,, \quad \theta_2 = \theta - \theta^2\,\partial + q\,\theta^3\,\partial^2,$$

$$\partial_1 = \frac{1}{2}(1-q)\,\partial^2 - \theta\partial^3\,, \quad \partial_2 = \partial - \theta\partial^2 + (1+q)\,\theta^2\partial^3.$$

One can express these generators in terms of the operator ω using (4.38), (4.39) and (2.10)

$$\partial_1 = (\tfrac{\omega-1-q}{1-q})\,\partial^2,$$

$$\theta_2 = \theta\,(\tfrac{\omega-q}{1-q})^2\,, \quad \partial_2 = \omega\,(\tfrac{\omega-q}{1-q})\,\partial.$$

4.2 $N = 3$ Heisenberg-Weyl algebra

Using Weyl construction (3.28), (3.29) one can obtain the following matrix representations for the generators θ_μ $(\mu = 1, 2, 3)$:

$$\begin{aligned} \theta_1 &= \textstyle\sum_{m=0}^{3} e_{m+4,m}, \\ \theta_2 &= \textstyle\sum_{m=0}^{1} (e_{m+3,m} - e_{m+6,m+3}), \\ \theta_3 &= e_{1,0} - e_{3,2} - e_{5,4} + e_{7,6}, \end{aligned} \tag{4.41}$$

where the matrix units e_{ij} have been introduced

$$(e_{ij})_{km} = \delta_{ik}\,\delta_{jm}\,, \quad e_{ij}\,e_{kl} = \delta_{jk}\,e_{il}.\,. \tag{4.42}$$

According with the representation (2.19) we find that

$$\begin{aligned} &\theta_1 = \theta^4\,, \quad \theta_2 = \theta^2\,(P_0 + P_1 - P_4 - P_5) = \\ &\quad (P_2 + P_3 - P_6 - P_7)\,\theta^2, \\ &\theta_3 = \theta\,(P_0 - P_2 - P_4 + P_6) = \\ &\quad (P_1 - P_3 - P_5 + P_7)\,\theta. \end{aligned} \tag{4.43}$$

Applying the transpositions (3.33) and using (2.20) one can deduce from (4.43) the expressions for generators ∂_μ:

$$\begin{aligned} \partial_1 &= \partial^4(\textstyle\sum_{k=4}^{7} \frac{1}{(k-3)_q(k-2)_q(k-1)_q(k)_q})P_k, \\ \partial_2 &= \partial^2(\textstyle\sum_{k=2,3} - \sum_{k=6,7})\frac{(k-2)_q!}{(k)_q!}P_k, \\ \partial_3 &= \partial(\textstyle\sum_{k=1,7} - \sum_{k=3,5})\frac{1}{(k)_q}P_k. \end{aligned} \tag{4.44}$$

4.3 $N = 4$ Heisenberg-Weyl algebra

Using Weyl construction (3.28), (3.29) one can obtain the following matrix representations for the generators θ_μ $(\mu = 1, 2, 3)$:

$$\begin{aligned} \theta_1 &= \textstyle\sum_{m=0}^{7} e_{m+8,m}, \\ \theta_2 &= (\textstyle\sum_{m=0}^{3} - \sum_{m=8}^{11})\,e_{m+4,m}, \\ \theta_3 &= (\textstyle\sum_{m=0,1,12,13} - \sum_{m=4,5,8,9})\,e_{m+2,m}, \\ \theta_4 &= (\textstyle\sum_{m=0,6,10,12} - \sum_{m=2,4,8,14})\,e_{m+1,m}. \end{aligned} \tag{4.45}$$

According with the representation (2.19) we find that

$$\begin{aligned} \theta_1 &= \theta^8\,, \\ \theta_2 &= (\textstyle\sum_{m=4}^{7} - \sum_{m=12}^{15})\,P_m\theta^4\,, \\ \theta_3 &= (\textstyle\sum_{m=2,3,14,15} - \sum_{m=6,7,10,11})\,P_m\theta^2\,, \\ \theta_4 &= (\textstyle\sum_{m=1,7,11,13} - \sum_{m=3,5,9,15})\,P_m\theta. \end{aligned} \tag{4.46}$$

Applying the transpositions (3.33) and using (2.20) one can deduce from (4.43) the expressions for generators ∂_μ:

$$\partial_1 = \partial^8 \left(\sum_{k=8}^{15} \tfrac{(k-6)_q!}{(k)_q!} P_k\right),$$
$$\partial_2 = \partial^4 \left(\sum_{k=4}^{7} - \sum_{k=12}^{15}\right) \tfrac{(k-4)_q!}{(k)_q!} P_k,$$
$$\partial_3 = \partial^2 \left(\sum_{k=2,3,14,15} - \sum_{k=6,7,10,11}\right) \tfrac{(k-2)_q!}{(k)_q!} P_k,$$
$$\partial_4 = \partial \left(\sum_{k=1,7,11,13} - \sum_{k=3,5,9,15}\right) \tfrac{1}{(k)_q} P_k.$$

5. Example: $N = 2$ SKDV Equations in Terms of Z_4 Generalized Grassmann Algebra

We start with the discussion of the $N = 1$ super-extension of the KdV equation

$$\frac{\partial \Phi}{\partial t} = -\partial_x\Big((\partial_x^2 \Phi) + 3\Phi(\mathcal{D}\Phi)\Big). \tag{5.47}$$

It is known (see e.g.[20]) that the Lax operator L with the Lax pair representation in this case can be taken in the form

$$L = \partial_x^2 + \Phi\,\mathcal{D} \quad ; \quad \frac{\partial L}{\partial t} = \left[L, L^{\frac{3}{2}}_{\geq 0}\right], \tag{5.48}$$

where ≥ 0 denotes the projection onto the purely (super)differential part of the operator, $\partial_x \equiv \frac{\partial}{\partial x}$ and we have used the odd superfield $\Phi(x,\theta) = \psi(x) + \hat{\theta}\, u(x)$ and covariant superderivative $\mathcal{D} = \hat{\partial} + \hat{\theta}\, \partial_x$.

The operators $\hat{\theta}$, $\hat{\partial}$ are the same as in (3.30), (3.31). Since the fermionic field $\psi(x)$ anticommutes with $\hat{\theta}$ and $\hat{\partial}$ (3.31) we should represent it in the form

$$\psi(x) = \hat{\omega}\, \hat{\psi}(x) = \begin{pmatrix} \hat{\psi} & 0 \\ 0 & -\hat{\psi} \end{pmatrix}, \tag{5.49}$$

which guaranties the anticommutation of ψ with $\hat{\theta}$ and $\hat{\partial}$. Substitution of the matrix representations (3.31) and (5.49) in (5.48) gives the matrix representation for the $N = 1$ SKdV Lax operator

$$L = \partial_x^2 + \begin{pmatrix} \hat{\psi} & 0 \\ u & -\hat{\psi} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \partial_x & 0 \end{pmatrix} = \partial_x^2 + \begin{pmatrix} 0 & \hat{\psi} \\ -\hat{\psi}\partial_x & u \end{pmatrix}. \tag{5.50}$$

Here the field $\hat{\psi}(x)$ should be considered as a fermionic field while $u(x)$ is a bosonic Virasoro field. Thus, we represent the Lax operator L for super-KdV equation in two different but absolutely equivalent forms. The first formula (5.48) gives the representation of the L-operator in terms of Z_2 graded algebra (2.1), (2.2), while the second formula (5.50) gives corresponding graded matrix realization.

Now the discussion of the $N = 2$ SKdV Lax operators is in order. It is well known that there are three different completely integrable $N = 2$ supersymmetric extensions of the Korteweg-de Vries equations [18] [19]. All these extensions could be written as

$$\frac{\partial \Phi}{\partial t} = \partial_x \Big(-(\partial_x^2 \Phi) + 3\Phi(\mathcal{D}_1\mathcal{D}_2\Phi) + \frac{1}{2}(\alpha - 1)(\mathcal{D}_1\mathcal{D}_2\Phi^2) + \alpha\Phi^3\Big), \tag{5.51}$$

where $\alpha = 4, -2, 1$ and $\Phi(x, \theta_1, \theta_2) = w(x) + \theta_1\, \psi_1(x) + \theta_2\, \psi_2(x) + \theta_2\, \theta_1\, u(x)$. We would like to show that the Lax operators for these generalizations could be written in terms of the Z_4 generalized Grassmann algebra. Let us notice that the Lax operators for the $\alpha = 1$ case is

$$L_1 = \partial_x - \partial_x^{-1}\, \mathcal{D}_1\, \mathcal{D}_2 \cdot \Phi(x, \theta_1, \theta_2)\ , \tag{5.52}$$

where the superderivatives $\mathcal{D}_i$ are

$$\mathcal{D}_1 = \partial_1 + \theta_1\, \partial_x\ , \quad \mathcal{D}_2 = \partial_2 + \theta_2\, \partial_x . \tag{5.53}$$

The supersymmetric KdV equation is obtained from the Lax pair representation as

$$\frac{\partial L_1}{\partial t} = \left[L_1, (L_1^3)_{\geq 0}\right] \tag{5.54}$$

The Lax operators for the $\alpha = 4, -2$ are described in terms of the previous operator L_1 as

$$L_{-2} = L_1^\dagger\, L_1 \quad , \quad L_4 = (L_1^\dagger\, L_1)_{\geq 0}\ , \tag{5.55}$$

where $\dagger$ denotes the hermitian conjugation. From that reason we consider therefore the case $\alpha = 1$ only.

The component $u(x)$ is a Virasoro field and variables θ_i and ψ_j anticommute with each other. It means that we can represent ψ_i in the form

$$\psi_i = (\hat{\omega} \otimes \hat{\omega}) \cdot \hat{\psi}_i = \begin{pmatrix} \hat{\psi}_i & 0 & 0 & 0 \\ 0 & -\hat{\psi}_i & 0 & 0 \\ 0 & 0 & -\hat{\psi}_i & 0 \\ 0 & 0 & 0 & \hat{\psi}_i \end{pmatrix}$$

$$[\theta_i,\, \hat{\psi}_j] = 0 = \{\hat{\psi}_i,\, \hat{\psi}_j\}_+ .$$

In matrix notations we have

$$\begin{aligned}
\mathcal{D}_1 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \partial_x & 0 & 0 & 0 \\ 0 & \partial_x & 0 & 0 \end{pmatrix}, \\
\mathcal{D}_2 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ \partial_x & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -\partial_x & 0 \end{pmatrix}, \\
\mathcal{D}_1\, \mathcal{D}_2 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -\partial_x & 0 \\ 0 & \partial_x & 0 & 0 \\ \partial_x^2 & 0 & 0 & 0 \end{pmatrix}, \\
\Phi &= \begin{pmatrix} w & 0 & 0 & 0 \\ \hat{\psi}_2 & w & 0 & 0 \\ \hat{\psi}_1 & 0 & w & 0 \\ -u & -\hat{\psi}_1 & \hat{\psi}_2 & w \end{pmatrix}.
\end{aligned} \tag{5.56}$$

Finally we write the Lax operator (5.52) in the matrix form

$$L = \partial_x \mathbf{1} - \begin{pmatrix} 0 & 0 & 0 & -\partial_x^{-1} \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \partial_x & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} w & 0 & 0 & 0 \\ \hat{\psi}_2 & w & 0 & 0 \\ \hat{\psi}_1 & 0 & w & 0 \\ -u & -\hat{\psi}_1 & \hat{\psi}_2 & w \end{pmatrix}. \tag{5.57}$$

The superfield Φ and covariant derivatives $\mathcal{D}_i$ in terms of Z_4 variables are represented in the form

$$\Phi = w(x) + \theta\,(P_0 + P_2)\,\hat{\psi}_2(x) + \theta^2\,(P_0 - P_1)\,\hat{\psi}_1(x) - \theta^3\,u(x). \tag{5.58}$$

$$\begin{gathered} \mathcal{D}_1 = \tfrac{1}{(2)_q}\,\partial_\theta^2\left(P_2 + \tfrac{1}{(3)_q}\,P_3\right) + \theta^2\,\partial_x \\ \mathcal{D}_2 = \partial_\theta\left(P_1 - \tfrac{1}{(3)_q}\,P_3\right) + \theta\,(P_0 - P_2)\,\partial_x, \end{gathered} \tag{5.59}$$

and using these relations we obtain

$$\partial_x^{-1}\,\mathcal{D}_1\,\mathcal{D}_2 = \frac{(q+1)}{2}\,\partial_\theta^3\,\partial_x^{-1} + \frac{(q-1)}{2}\,\partial_\theta\,P_2 + \theta\,P_1 + \theta^3\,\partial_x. \tag{5.60}$$

Let us introduce the Z_4 covariant derivative

$$\begin{gathered} \mathcal{D} = \partial_\theta + \tfrac{1}{(3)_q!}\,\theta^3\,\partial_x = \partial_\theta - \tfrac{1}{2}\,(q+1)\,\theta^3\,\partial_x \\ \mathcal{D}^4 = \partial_x, \end{gathered} \tag{5.61}$$

where the operators θ and ∂_θ generate the Z_4 algebra (2.1), (2.2) for $p = 3$ and parameter q satisfy $q^2 = -1$. Now we express all ∂_θ^k via operator θ and Z_4- covariant derivative $\mathcal{D}$:

$$\partial_\theta = \mathcal{D} + \tfrac{1}{2}\,(q+1)\,\theta^3\,\partial_x\ ,$$

$$\partial_\theta^2 = \mathcal{D}^2 + \left(q\,\mathcal{D}\,\theta^3 + \tfrac{(q+1)}{2}\,\theta^2\right)\,\partial_x\ ,$$

$$\partial_\theta^3 = \mathcal{D}^3 + \left(\tfrac{(q-1)}{2}\,\mathcal{D}^2\,\theta^3 + \tfrac{(q-1)}{2}\,\mathcal{D}\,\theta^2 + q\,\theta\right)\,\partial_x\ .$$

Using this relation the differential operator (5.60) is rewritten as

$$\left\{\frac{q+1}{2}\,\partial_x^{-1}\,\mathcal{D}^3 - \frac{1}{2}\,\mathcal{D}^2\,\theta^3 + \mathcal{D}\Big(\frac{(q-1)}{2}\,P_2 - \frac{1}{2}\,\theta^2\Big) + \theta\left(P_1 + \frac{(q-1)}{2}\right) + \theta^3\,\partial_x\right\}\ .$$

Finally we obtain the expression for the Lax operator L (5.57) as a pseudodifferential operator with respect to the fractional covariant derivative $\mathcal{D}$:

$$L = \partial_x + \mathcal{D}^{-1}\cdot u_{-1} + u_0 + \mathcal{D}\cdot u_1 + \mathcal{D}^2\cdot u_2 + \partial_x\cdot u_4\ , \tag{5.62}$$

where the fractional superfields $u_k(x,\theta)$ are defined in terms of the unique fractional prepotential Φ (5.58).

$$u_{-1} = \frac{1}{q-1}\,\Phi\ , \quad u_0 = \theta\left(\frac{1}{q+1} - P_1\right)\,\Phi\ ,$$

$$u_1 = \left(\frac{1-q}{2}\,P_2 + \frac{1}{2}\,\theta^2\right)\,\Phi\ , \quad 2\,u_2 = -u_4 = \theta^3\,\Phi\ .$$

6. Conclusion

In this parer we have shown that the fermionic Heisenberg-Weyl algebra with $2N = D$ fermionic generators is equivalent to the generalized Grassmann algebra with two fractional generators of the order of the parastatistic: $p = 2^N - 1$. The 2,3 and 4 dimensional Heisenberg - Weyl algebras (they are related to the 4,6 and 8 dimensional Clifford algebras) were explicitly given. The $N = 2, 3, 4$ extended supersymmetries can be described in terms of these algebras. We reformulate the Lax approach for the supersymmetric Korteweg - de Vries equation in terms of the generators of the generalized Grassmann algebra. Note that the theory of the pseudodifferential operators on the $(1|1)$ superline [20] can be directly generalized to the theory of pseudodifferential operators on the fractional superline (x, θ) (we plan to return to these problems in one of the forthcoming papers). On the other hand, in the similar manner, one can construct the Z_{p+1}- graded extensions of the KdV and Kadomtsev-Petviashvily hierarchies. These extensions should be probably related to the graded matrix generalizations of the KdV and KP equations.

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On the Construction of a Noncommutative Action for the $D = 11$ Supermembrane

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Abstract

We formulate an action for the $D = 11$ Supermembrane over the Weyl algebra bundle in terms of a noncommutative geometry. The action reduces on a Darboux chart to the $D = 11$ Supermembrane action. We propose an extension of the Seiberg-Witten map obtaining a geometrical construction of the action depending only on the class of admissible symplectic connections. We briefly review the noncommutative approach over the Weyl algebra bundle.

1 Introduction

The formulation of superstrings in 10 dimensions , supermembranes and super5 branes in 11 dimensions in terms of noncommutative geometry has been recently analized from different points of view, for a recent review see [1]. Most of the work has been performed in the context of the interaction of open strings with constant antisymmetric background fields which are assumed to be much larger that the background metric. The precise limits in which the noncommutative geometry arises were discussed in [2].

The approach we followed in [3] and [4] was to emphasize the residual gauge symmetry of the $D = 11$ supermembrane in the light cone gauge , the area preserving diffeomorphisms. In two dimensions they are the same as the symplectomorphisms on the world volume . The idea was then to formulate a new action for the supermembrane in terms of a noncommutative symplectic geometry. The symplectic two-form arises naturally from the existence of a non trivial central charge on the supersymmetric algebra. In order to formulate the action in terms of a noncommutative geometry the fields describing the supermembrane action were lifted to Yang-Mills connections and associated gauge fields over the Weyl algebra bundle, where a general framework for the symplectic formulation of noncommutative geometry was introduced by Fedosov in [5]. One of the results in [4] was the construction of a geometrical action over the Weyl bundle which reduces to the well known action of the $D = 11$ supermembrane in the light cone gauge when the geometrical objects are projected down from the Weyl algebra bundle to fields over the world volume and a formal parameter in the expansion of the elements of the Weyl algebra tends to zero.

In this paper we first briefly review Fedosov's general framework and the introduction of Yang Mills connections over the Weyl Bundle. We then propose an extension of the action in [4] which becomes independent of the symplectic connections introduced by Fedosov.

To do so we introduce a map similar to the one used by Seiberg and Witten in [2] but on the Weyl algebra bundle. This new map has the advantage that it is globally defined over the world volume, it is not a local map as the Seiberg-Witten one. This is an important advantage to the construction we followed in [4] with the usual formulations. The current approach considers constant symplectic structure, constructed from the constant background antisymmetric field, and it is then valid only over a Darboux chart. The price paid in order to have a global cobnstruction is the introduction of a symplectic connection over the world volume that, in general, is not unique. There is a class of admissible symplectic connections which may be introduced. One would like then to have an action which depends only on the class of symplectic connections. This is the problem we address in this paper.

2 Yang Mills connections over the Weyl bundle

In [5] a formal Weyl algebra W_x corresponding to a symplectic space $T_x\Sigma$, where Σ is a symplectic manifold (Σ, ω) of dimension $2n$ and $T_x\Sigma$ is its tangent space at x, is an algebra whose elements are given by a formal series

$$a(y, h) = \sum_{k,p\geq 0} h^k a_{k,\mu_1\ldots\mu_p} y^{\mu_1} \ldots y^{\mu_p} \tag{1}$$

where h is a formal parameter, $\mu_0 = 0$ and μ_p runs from 1 to $2n$ when $p \neq 0$. To order terms in the summation, we give the following degrees to variables: deg $y^\mu = 1$, deg $h = 2$ and we order by increasing degrees $2k + p$. The Weyl product of elements $a, b \in W_x$ is defined as

$$a \circ b = \sum_{k=0}^{\infty} (-\frac{ih}{2})^k \frac{1}{k!} \omega^{\mu_1\nu_1} \ldots \omega^{\mu_k\nu_k} \cdot \frac{\partial^k a}{\partial y^{\mu_1} \ldots \partial y^{\mu_k}} \frac{\partial^k b}{\partial y^{\nu_1} \ldots \partial y^{\nu_k}} \tag{2}$$

This product is associative and independent of the basis in $T_x\Sigma$.The union of W_x defines the Weyl algebra bundle W. We will consider q-forms on Σ with values in W. A torsion free symplectic connection preserving the symplectic structure is introduced on the tangent bundle of Σ. This connection is then lifted to the Weyl bundle W by considering its action directly on the coefficients of the expansion 1, we denote it $\mathcal{D}_S$

$$\mathcal{D}_S a = dx^\rho \wedge D_\rho a \tag{3}$$

where D_ρ denotes the symplectic connection on Σ. General covariant derivatives $\mathcal{D}$ on the bundle may be considered with one-form connections γ globally defined on Σ and with values in W,

$$\mathcal{D}a = \mathcal{D}_S a + \frac{i}{h}[\gamma, a] \tag{4}$$

The two-form Ω

$$\Omega = R + \mathcal{D}_S\gamma + \frac{i}{2h}[\gamma, \gamma] \tag{5}$$

is the Weyl curvature of the connection $\mathcal{D}$. R is the curvature of the connection $\mathcal{D}_S$. The Weyl curvature satisfies the Bianchi identity

$$\mathcal{D}\Omega = \mathcal{D}_S\Omega + \frac{i}{h}[\gamma, \Omega] = 0 \tag{6}$$

moreover, for any section $a \in W \otimes \Lambda$

$$\mathcal{D}^2 a = \frac{i}{h}[\Omega, a] \tag{7}$$

In general, transitions on the bundle $T\Sigma$ will induce transitions on the algebra W.The infinitesimal gauge transformations on elements of the algebra are expressed as automorphisms given by

$$a \to a + [a, \lambda] \tag{8}$$

with 'infinitesimal' $\lambda \in W$.
The corresponding gauge transformations for the connections $\mathcal{D}$ are

$$\mathcal{D} \to \mathcal{D} + \mathcal{D}\lambda. \tag{9}$$

implying

$$\mathcal{D}a \to \mathcal{D}a + [\mathcal{D}a, \lambda]. \tag{10}$$

Abelian connections $\mathcal{D}_A$ are connections $\mathcal{D}$ with Weyl curvature Ω being a central form of the algebra. Let us denote it Ω_A. It then satisfies

$$[\Omega_A, a] = 0 \tag{11}$$

for any section $a \in W$. Associated with $\mathcal{D}_A$ there is a subalgebra of W, denoted W_A, defined by

$$W_A = \Big\{a \in W : \mathcal{D}_A a = 0\Big\}. \tag{12}$$

There is a one to one correspondence between the C^∞ functions $a_0(x)$ over Σ and the elements of W_A. In fact, given $a \in W_A$, the projection is defined as

$$\sigma a := a(x, y = 0, h) = a_0(x), \tag{13}$$

and given $a_0(x)$ there is a unique element $a \in W_A$ with such projection. If a and $b \in W_A$, its Weyl product is projected to the globally defined $\star$-product

$$\sigma(a \circ b) = a_0 \star b_0 \tag{14}$$

In the particular case when $\omega_{\mu\nu}$ is constant and the symplectic connection is zero, the formula agrees with the Moyal product. In [4] we constructed the Yang Mills connection over the Weyl algebra bundle. Let Σ be a symplectic manifold with a symplectic two-form $\omega_{\mu\nu}dx^\mu \wedge dx^\nu$. In this section, we assume ω to be an arbitrary non-degenerate closed two-form over Σ. A set of multi-beins is defined by

$$\omega_{\mu\nu} = \varepsilon^i_\mu \varepsilon^j_\nu \epsilon_{ij}, \tag{15}$$

where ϵ_{ij} is the canonical symplectic tensor. Because of Darboux theorem, locally we always have

$$\varepsilon^i_\mu = \partial_\mu g^i. \tag{16}$$

We may consider an atlas where on each chart we have (16) . The transitions on g^i between different charts preserve the symplectic structure (15). The multi-bein ε^i_μ will then have transitions over Σ, otherwise, one would have a set of $2n$ non-singular vector fields globally defined over Σ, but this is not true in general.

Let us discuss the transitions on intersection of charts in more detail. Consider two open sets U and $\hat{U}$, $U \bigcap \hat{U} = \emptyset$ in which

$$\varepsilon^i_\mu = \partial_\mu g^i, \text{ and } \hat{\varepsilon}^{\mathrm{i}}_\mu = \partial_\mu \hat{\mathrm{g}}^{\mathrm{i}} \tag{17}$$

respectively. In $U \cap \hat{U}$ we then have

$$\omega_{\mu\nu} = \varepsilon^i_\mu \varepsilon^j_\nu \epsilon_{ij} = \hat{\varepsilon}^i_\mu \hat{\varepsilon}^j_\nu \epsilon_{ij}, \tag{18}$$

from which we obtain

$$\hat{\varepsilon}^i_\mu = S^i_j \varepsilon^j_\mu \text{ where } S^i_j = \epsilon^{ik} \hat{\varepsilon}^\mu_k \varepsilon^l_\mu \epsilon_{lj}, \tag{19}$$

we define the inverse of ϵ by $\epsilon^{ij}\epsilon_{jk} = \delta^i_k$. One may verify that S preserves the canonical symplectic tensor and hence $S \in Sp(2n)$. Consequently in order to have a global construction over Σ, one must begin by introducing a symplectic $Sp(2n)$ connection on the tangent bundle. We first consider the following symplectic connection over Σ,

$$\Theta^\lambda_{\mu\nu} \equiv \zeta_{\mu\nu\rho}\omega^{\rho\lambda} + \frac{1}{3}\frac{\partial \omega_{\nu\rho}}{\partial x^\mu}\omega^{\rho\lambda} + \frac{1}{3}\frac{\partial \omega_{\mu\rho}}{\partial x^\nu}\omega^{\rho\lambda} \tag{20}$$

where $\zeta_{\mu\nu\rho}$ is a totally symmetric tensor. This is the most general expression for a connection satisfying

$$(\frac{\partial}{\partial x^\mu} + \Theta_\mu)\omega_{\rho\lambda} = 0 \tag{21}$$

here $\Theta^\lambda_{\mu\nu}$ is expressed in terms of $\omega_{\mu\nu}$ and its derivatives. It is invariant under the $Sp(2n)$ transition of the multi-bein. We now consider the following torsion free connection on the tangent space

$$\Gamma^j_{\mu i} = \varepsilon^\nu_i (\frac{\partial \varepsilon^j_\nu}{\partial x^\mu} - \Theta^\lambda_{\mu\nu}\varepsilon^j_\lambda) \tag{22}$$

it transforms as a $Sp(2n)$ connection under $Sp(2n)$ transformations on the tangent space. In fact,

$$\hat{\Gamma}^j_{\mu i} = (S^{-1})^l_i \Gamma^k_{\mu l} S^j_k - (S^{-1})^k_i \frac{\partial S^j_k}{\partial x^\mu} \tag{23}$$

this connection is symplectic on the tangent space. We may construct from it the most general symplectic connection on the tangent space in the following way. Let us denote

$$\breve{D}_\mu \equiv \partial_\mu + \Gamma_\mu, \tag{24}$$

a symplectic connection must satisfy

$$(\breve{D}_\mu + \Delta\Gamma_\mu)\epsilon_{ij} = 0 \tag{25}$$

this equation has the general solution

$$\Delta\Gamma^j_{\mu i} = \frac{1}{3}(\breve{D}_\mu \epsilon_{il})\epsilon^{lj} + \frac{1}{3}\varepsilon^k_\mu \varepsilon^\nu_i (\breve{D}_\nu \epsilon_{kl})\epsilon^{lj} + \varepsilon^k_\mu \tilde{\zeta}_{(ilk)}\epsilon^{lj} \tag{26}$$

$\Delta\Gamma^j_{\mu i}$ is acovariant vector on the world volume and a tensor under $Sp(2n)$ transformations. Since the connection (24) is symplectic the first two terms of the right hand side member in (26) are zero. We may finally construct our symplectic connection D , when acting on mixed indices vectors V^i_ν it yields

$$D_\mu V^i_\nu = \frac{\partial V^i_\nu}{\partial x^\mu} + (\Gamma_\mu + \Delta\Gamma_\mu)^i_l V^l_\nu - \Theta^\lambda_{\mu\nu} V^i_\lambda, \tag{27}$$

it satisfies

$$D_\mu \omega_{\rho\lambda} = 0, \quad D_\mu \epsilon_{ij} = 0 \tag{28}$$

and it has the right transformation law on the world volume and in the tangent space.

The final form of the Yang Mills connection is

$$\mathcal{D} = \frac{i}{h}[G_i e^i, \bullet] + \frac{i}{h}[\gamma, \bullet] \tag{29}$$

where G_i obeys the following equations

$$\mathcal{D}_A G_i = 0, \ \sigma G_i = \epsilon_{ij} g^j(x) \tag{30}$$

where $g^j(x)$ is defined in (16) and γ_i also obeys

$$\mathcal{D}_A \gamma_i = 0. \tag{31}$$

The curvature of the connection $\mathcal{D}$ is then given by

$$\Omega = \frac{i}{2h}[G, G] + \frac{i}{h}[G, \gamma] + \frac{i}{2h}[\gamma, \gamma], \tag{32}$$

it satisfies the Bianchi identity

$$\mathcal{D}\Omega = 0 \tag{33}$$

this property follows from the Jacobi identity for the bracket. The first term in (32) reduces in the flat limit to

$$\frac{i}{2h}[G, G] = -\frac{1}{2} e^i \wedge e^j \epsilon_{ij} = -\omega \tag{34}$$

The projection of Ω has in general the expression

$$\begin{aligned} \sigma\Omega \ &= \ -\omega + \mathcal{F} - \frac{h^2}{96}\left(R_{jkli}(D_{\hat{j}} D_{\hat{k}} D_{\hat{l}} \mathcal{A}_m)\right) - \left((\frac{1}{4} R_{\hat{j}\hat{k}\hat{l}p} \epsilon^{pq} D_q \mathcal{A}_m)\right) \epsilon^{j\hat{j}} \epsilon^{k\hat{k}} \epsilon^{l\hat{l}} e^i \wedge e^m \\ &\quad - \frac{h^2}{96 \cdot 8} R_{jkli} R_{\hat{j}\hat{k}\hat{l}m} \epsilon^{j\hat{j}} \epsilon^{k\hat{k}} \epsilon^{l\hat{l}} e^i \wedge e^m + O(h^3).. \end{aligned} \tag{35}$$

where the curvature is constructed from the $Sp(2n)$ symplectic connection (27), the remaining terms are higher order in h and depend also on the derivatives of the curvature. The curvature $\mathcal{F}$ is the Yang Mills field strength

$$\mathcal{F} = \frac{1}{2} e^i \wedge e^j (D_i \mathcal{A}_j - D_j \mathcal{A}_i + \frac{i}{h}\{\mathcal{A}_i, \mathcal{A}_j\}_{star}) \tag{36}$$

constructed now with the $Sp(2n)$ covariant symplectic derivative introduced in (27), notice that the *star* bracket in (36) is the global generalization of the Moyal bracket over the whole symplectic manifold obtained in [5]. We notice that, because of (16) and (19), the first covariant symplectic derivative of g^i is a simple derivative. We will assume the same transformation law under $Sp(2n)$ for $\mathcal{A}$.

3 Supermembrane action and the Seiberg - Witten map

The starting point in the construction of the noncommutative action for the supermembrane in [4] was to consider a nontrivial central charge of the supersymmetric algebra.

This condition defines in a natural way a nondegenerate closed two-form ω over the spatial world volume, which may be taken to be a Riemann surface. This closed two-form is invariant under the area preserving diffeomorphisms which is the residual gauge symmetry of the supermembrane in the light cone gauge. We may then write the following Hamiltonian density on the Weyl algebra , see [4] for more details,

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}(\acute{P}^M \circ *\acute{P}^M) + \frac{1}{2}(\acute{\Pi} \circ *\acute{\Pi}) - \frac{1}{2h^2}([G_i e^i + \gamma, X^M] \circ *[G_i e^i + \gamma, X^M]) \\ &\quad - \frac{1}{4h^2}([X^M, X^N] \circ *[X^M, X^N]) + \frac{1}{2}(\Omega \circ *\Omega) \end{aligned} \tag{37}$$

where the Hodge $*$ is constructed using an induced Riemannian metric.

We may inmediately project out the center components of this Hamiltonian yielding

$$\begin{aligned}\sigma\mathcal{H} &= \frac{1}{2}(\acute{P}^M)^2\omega + \frac{1}{2}(\acute{\Pi} \wedge *\acute{\Pi}) - (D_{\mathcal{A}}X^M \wedge *D_{\mathcal{A}}X^M) \\ &\quad - \frac{1}{4h^2}\{X^M, X^N\}^2_{star}\omega + \frac{1}{2}(\mathcal{F} - \omega)(*\mathcal{F} - *\omega) + \\ &\quad + \text{more terms}\end{aligned} \tag{38}$$

where $D_{\mathcal{A}} = D_S X^M + \{\mathcal{A}, X^M\}_{star}$ corresponds to the first terms in a projection of the gauge covariant derivative,

$$\begin{aligned}\sigma\mathcal{D}_i X^M &= \frac{i}{h}\{G_i, X^M\}_{star} + \frac{i}{h}\{\mathcal{A}_i, X^M\}_{star} = D_{S\,i}X^M + \frac{i}{h}\{\mathcal{A}_i, X^M\}_{star} \\ &\quad + \text{curvature terms} \ldots\end{aligned} \tag{39}$$

If we consider the terms of $\sigma\mathcal{H}$ independent on the formal parameter h we obtain exactly the $D = 11$ supermembrane Hamiltonian in the light cone gauge. Let us know consider a generalization of the Seiberg-Witten map to the Weyl bundle, we will then use this map in order to write an action for the supermembrane depending only on the class of symplectic connections introduced on the previous section. the construction will be based on the Hamiltonian (37).

The Seiberg-Witten map [2] is a map between gauge equivalent classes of noncommutative gauge fields and commutative ones. That is,

$$\mathcal{A}(x) \overset{S-Wmap}{\Leftrightarrow} A(x) \tag{40}$$

such that

$$\Delta_g \mathcal{A} = d\lambda + \{\mathcal{A}, \lambda\} \Leftrightarrow \Delta_g A(x) = d\tilde{\lambda} \tag{41}$$

where λ and $\tilde{\lambda}$ are infinitesimal gauge parameters. This map was explicitly constructed for a constant background antisymmetric field. It may be generalized to the Weyl algebra bundle for arbitrary symplectic structures in the following way. We may associate, in a unique way, to each Yang Mills connection over the Weyl algebra bundle an abelian connection $\mathcal{D}_A$ whose curvature is of the form

$$\Omega_A = F + h\Omega_1 + h^2\Omega_2 + \ldots \tag{42}$$

where F $\Omega_1, \Omega_2, \ldots$ are closed two-forms which are gauge invariant. Moreover, this map associates to each non-commutative gauge equivalent class, corresponding to $\mathcal{D}$, the gauge equivalent class of D_A:

$$\mathcal{D}\lambda = \partial\lambda + [\gamma, \lambda] \to d(\lambda + s)$$

where $s = s(\gamma, \lambda)$. If we define the projection

$$\sigma\gamma = \mathcal{A}(x, h), \tag{43}$$

then

$$\mathcal{A}(x, h) = A(x) + hA_1 + h^2 A_2 + \ldots \tag{44}$$

It turns out that

$$\begin{aligned} F &= F(A) && (45)\\ \Omega_1 &= \Omega_1(A, A_1) && (46)\\ \Omega_2 &= \Omega_2(A, A_1, A_2). && (47) \end{aligned}$$

and so on. If we impose now the conditions

$$\Omega_i = 0, i = 1, \ldots, \infty \tag{48}$$

we exactly recover the Seiberg-Witten map over any Darboux chart. The procedure provides then a global extension of the Seiberg=Witten map.

There is also a geometrical construction based on a Poisson bracket, instead of the Weyl one, which is relevant in our discussion. We consider the vector bundle $\mathcal{P}$ constructed with the same geometrical objects defined in section 2 but instead of constructing a Weyl bracket from the noncommutative product, we introduce a Poisson bracket.

All the analysis of section 2 follows in a similar manner, in particular the construction of Yang-Mills connections and the abelian ones. An action may be constructed from such geometry by replacing the Weyl brackets in (37) by the Poisson ones. It turns out that for such construction the action is independent of the formal parameter h. Moreover its projection over the central components gives rise to the $D = 11$ Supermembrane action in the light cone gauge [3]. We may now consider in this geometry the construction of a Seiberg-Witten map, furthermore, we may construct a map between gauge equivalent classes of noncommutative Yang-Mills connections over the Weyl algebra bundle and the gauge equivalent classes of Yang-Mills connections over $\mathcal{P}$. It is then possible to rewrite the action of the $D = 11$ Supermembrane in terms of the noncommutative Yang-Mills connections and gauge fields X^M, thec first relevant terms of such action agree with (37). The action depends only on the class of admissible symplectic connections, two elements of the class differ by a totally symmetric symbol. We will discuss the explicit construction of this action in a forthcoming paper.

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Gauge Invariance for Non-Commutative Space-Time

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Abstract

The realization of the two-dimensional Poincare algebra in terms of the non-commutative differential calculus on the commutative algebra of functions is considered. The corresponding algebra of functions is commutative and is defined by the spectrum of unitary irreducible representations of the two-dimensional De Sitter group. The Gauge invariance principle consistent with this quantum geometry is considered

1 Introduction

The suggestions to consider the non-commutative space-time at small distances are as old as quantum theory itself. [1] Today it is commonly accepted that the most probable scale for these "small" distances is that of Grand Unified Theories, i.e., the scale close to the Planck one

$$l_0 = \sqrt{\frac{\hbar G_N}{c^3}} \simeq 10^{-33} cm \qquad (1)$$

One of the most convincing arguments for this is that at this scale the curvature radius of space-time is of the order of the De Broglie wave length of a test particle.

Recently the possibility of non-commutative space to emerge in the framework of the string theory has been indicated first in [6]. More recently Yang-Mills theories on non-commutative spaces have emerged in the context of M-theory compactified on a torus in the presence of constant background three-form field, or as a low-energy limit of open strings in a background B-field describing the fluctuations of the D-brane world volume. (See the review article [7] and references therein.)

It is worth mentioning a series of recent papers [5]-[8] where it has been shown that the curved momentum space and the corresponding Snyder-like quantum space naturally arise when considering the 2+1 model of gravity interacting with the scalar field. The canonical momenta belong to the hyperboloid in three-dimensional space, Lobachevsky space.

One of the first NG models is connected with the modification of the momentum space and goes back to H.Snyder (see review article [1] for further references, and also

[1]There is a terminological unexactness here: non-commutative space-time is often referred to as the quantized space. But the space in quantum theory is quantized "by definition", simply because the position is a measurable quantity and as any other physical quantity is presented by the operators $\widehat{x}_\mu$. In standard quantum theory different components of the position operator mutually commute and can be simultaneously diagonalized, i.e., represented by numbers. By this reason the coordinates are referred to as "non-quantized". Hence, when introducing noncommutativity in geometry it is better to speak about the non-commutative Geometry (NG).

[2] - [4]). In this approach, usual commuting position operators were changed by the non-commutative quantities of a concrete form, more precisely by the boost generators of the De Sitter or Anti De Sitter momentum space. The geometries of momentum and configurational spaces are closely connected. As quantum mechanical operators of energy and momentum are the derivation operators in space-time, it is clear that transfer to the non-commutative geometry requires the modification of the geometry of momentum space. And vise versa. It is not obvious which is prime. Choosing the momentum space with geometry different from the standard Minkowski one we obtain different (in general non-commutative) geometry of space-time.

The change of the geometry of the momentum space leads to the modification of the procedure of extension of the S-matrix off the mass shell, i.e., to a different dynamical description. Establishing the geometry of the momentum space off the mass shell is in fact an additional axiom of quantum field theory (QFT). Actually, in the standard QFT, the axiom that the geometry of the momentum space off the mass shell is the pseudo-euclidean (Minkowski) is accepted as an evident fact, without saying. We can think that some background interaction exists which modifies the geometry of the momentum space [1]. In consequence of the change of the geometry of the p-space the space-time becomes quantum (non-commutative). We stress that the physical meaning of the geometry and topology of the momentum space has no clear physical interpretation as yet. The space-time groups considered in QFT as covariance groups are the isometry groups of space-time.

The explicit character of Snyder's approach to space-time quantization has a remarkable consequence: we can define the common spectrum of a (complete) set of operators constructed from $\hat{x}_\mu$ and some other operators from the centrum of the universal enveloping algebra of the De Sitter Lie algebra and consider them as points ξ_μ of the new quantum (and non-commutative in general) space-time. It can be shown that a formulation of the generalized causality condition and QFT in terms of the points of this new numerical quantum space-time is as comprehensive procedure as it is in the usual QFT with the Minkowskian space-time. In this approach, the structure of the singular field theoretic functions is entirely reconstructed as compared to the standard QFT, and the corresponding perturbation theory is free of ultraviolet divergences [4].

2 Non-commutative configurational space

Let us consider the two-dimensional momentum space of constant curvature modelled by the two-dimensional surface embedded into the three-dimensional pseudo-Euclidean space

$$p_L p^L = p_\mu p^\mu - p_2^2 = -1 \quad L = 0, 1, 2, \quad \mu = 0, 1 \qquad (2)$$

The "plane waves" in this case, i.e., the kernels of the Fourier transform connecting the p-space and the new configurational space and at the same time the state vectors describing the free motion of the particle are the matrix elements of the unitary irreducible representations of the isometry group of the space (2) $\langle \xi \mid p \rangle$. Or in other words, the surface (2) is the uniform space of this group. [2]. To describe these matrix elements, we use the hyper-spherical coordinates

[2]Evidently, dimensional physical constants including the fundamental length l_0 must enter expression (2). We use the unit system $\hbar = c = l$. All relations of the theory must go over into the standard ones when l_0 can be considered as a small quantity. This will be called the correspondence limit.

$$p^0 = \sinh\zeta \quad p^1 = \cosh\zeta \sin\omega \quad p^2 = \cosh\zeta\cos\omega$$
$$-\infty < \zeta < \infty \qquad -\pi < \omega < \pi \tag{3}$$

In the correspondence limit

$$p^2 \approx 1, \quad p^0 \approx p^1 \approx 0, \qquad \zeta \to 0, \quad \omega \to 0 \tag{4}$$

the generalized plane waves $\langle \xi \mid p\rangle$ have the form

$$\langle \xi \mid p\rangle = \langle \sigma, n \mid \zeta, \omega\rangle = \frac{2^{\sigma+1}\kappa(\sigma, n)}{\sqrt{2\pi}\cosh\zeta} e^{in\omega} P_{n-\frac{1}{2}}^{-(\sigma+\frac{1}{2})}\left(\xi^0 \tanh\zeta\right) \tag{5}$$

where $P_\nu^\mu(z)$ are the associated Legendre functions, ξ^0 is the sign of the discrete time n, and

$$\kappa(\sigma, n) = \Gamma\left(\frac{\sigma+n+2}{2}\right)\Gamma\left(\frac{\sigma-n+2}{2}\right) \tag{6}$$

Let us list some important properties of the generalized plane waves

1. Behavior at the origin

$$\langle \xi \mid 0\rangle = 1 \tag{7}$$

2. For principal series of the irreps

$$\sigma = i\Lambda - \frac{1}{2},\ 0 \le \Lambda < \infty, \text{ or } \sigma = k = 0, 1, 2, \ldots, \quad n = 0, \pm 1, \pm 2, \ldots \tag{8}$$

We call the set $\xi = (\sigma, n)$ a point of quantum space.

3. Orthogonality and completeness

$$\frac{1}{(2\pi)^2}\int d\Omega_p \langle \xi \mid p\rangle\langle p \mid \xi'\rangle = \mu^{-1}(\sigma)\delta_{\xi^0\xi'^0}\delta_{nn'}\delta(\Lambda - \Lambda')$$

$$\frac{1}{(2\pi)^2}\int d\Omega_\xi \langle p \mid \xi\rangle\langle \xi \mid p'\rangle = \delta(p(-)p') = |p^2|\delta(\vec{p} - \vec{p}') \tag{9}$$

$$d\Omega_p = \frac{dp^0 dp^1}{|p^2|},$$

$$\mu(\sigma, n) = \frac{\left(\sigma+\frac{1}{2}\right)\cot\pi\left(\sigma+\frac{1}{2}\right)}{2}\frac{\Gamma\left(\frac{\sigma+n+2}{2}\right)\Gamma\left(\frac{-\sigma+n+1}{2}\right)}{\Gamma\left(\frac{\sigma+n+1}{2}\right)\Gamma\left(\frac{-\sigma+n}{2}\right)}$$

The volume element $d\Omega_\xi$ symbolizes the integration over the continuous part of the proper time σ and summation over its discrete part as well as the summation over the discrete time n (cf. [4]).

4. The correspondence limit

$$\zeta \simeq p^0, \quad \omega \simeq p^1, \quad p^2 \simeq 1$$
$$n \simeq x^1, \quad \sqrt{\sigma^2 + n^2} \simeq |x^0| \tag{10}$$
$$\langle \xi \mid p\rangle \to e^{ip_\mu x^\mu}$$

Relations (10) suggest our identification of σ and n as quantum analogs of the two-dimensional space-time coordinates (interval and time).

3 Holomorphic realization

Let us consider the holomorphic realization of the plane waves (5). We start with the continuous part of the spectrum of the interval σ. Introducing the new variables

$$z = i\Lambda + n = \sigma + n + \frac{1}{2}, \quad \text{and} \quad \overline{z} = -i\Lambda + n = -\sigma + n - \frac{1}{2} \tag{11}$$

and using the connection between associated Legendre functions and hypergeometric functions [9] we come to the following holomorphic representation for the plane wave

$$\langle \xi \mid p\rangle = \langle \sigma, n \mid \zeta, \omega\rangle = \langle z, \overline{z} \mid \zeta, \omega\rangle =$$

$$= \frac{2^{\frac{z-\overline{z}+1}{2}} e^{i\omega\frac{z+\overline{z}}{2}}}{\sqrt{2\pi}\cosh\zeta}\kappa(z, \overline{z}) P_{\frac{z+\overline{z}-1}{2}}^{-\left(\frac{z-\overline{z}}{2}\right)}\left(\xi^0\tanh\zeta\right) = (\cosh\zeta)^{-\frac{z-\overline{z}+1}{2}} e^{i\omega\frac{z+\overline{z}}{2}}.$$
$$\cdot F\left(\frac{z+\frac{1}{2}}{2}, \frac{-\overline{z}+\frac{1}{2}}{2}, \frac{1}{2}; \tanh^2\zeta\right) -$$
$$-2\xi^0\tanh\zeta \cdot \zeta(z, \overline{z}) F\left(\frac{z+\frac{3}{2}}{2}, \frac{-\overline{z}+\frac{3}{2}}{2}, \frac{3}{2}; \tanh^2\zeta\right) \tag{12}$$

where

$$\zeta(z, \overline{z}) = \frac{\Gamma\left(\frac{z+\frac{3}{2}}{2}\right)\Gamma\left(\frac{-\overline{z}+\frac{3}{2}}{2}\right)}{\Gamma\left(\frac{z+\frac{1}{2}}{2}\right)\Gamma\left(\frac{-\overline{z}+\frac{1}{2}}{2}\right)} \tag{13}$$

As the hypergeometric function is the entire function of its first two parameters we can easily prove that the plane wave $\langle z, \overline{z} \mid \zeta, \omega\rangle$ is an analytic function in its first argument z and anti-analytic in its second argument $\overline{z}$. This function obeys as well as $\zeta(z, \overline{z})$ the symmetry condition

$$\overline{f}(z, \overline{z}) = f(\overline{z}, z) \tag{14}$$

At the same time it is well known fact that hypergeometric functions don't obey any differential relation in its parameters, but obey the recurrence relations [9]. This makes them the subjects to the non-commutative differential calculus. The holomorphic representation leads to the simplest form of such a calculus. We introduce this calculus starting with basic relations

$$[z,\, dz] = dz,\ [\overline{z},\, d\overline{z}] = d\overline{z},\ [\overline{z},\, dz] = 0,\ [z,\, d\overline{z}] = 0,\ [z,\, \overline{z}] = 0 \tag{15}$$

in which in contrast with the standard (commutative) calculus the coordinates z, and $\overline{z}$ commute between themselves but don't commute with corresponding differentials. It can be shown that the comprehensive differential calculus based on the relations (15) exists [1]. We can introduce the generalized interior derivatives right $\overrightarrow{\partial}$ and left $\overleftarrow{\partial}$ of the function $f\,(z,\, \overline{z})$ as

$$d_z f = [dz,\, f] = \overrightarrow{\partial_z} f\, dz = dz\, \overleftarrow{\partial_z} f \tag{16}$$

and similar formulae for the non-commutative differentiation in $\overline{z}$. The Leibnitz rule is fulfilled for the exterior differentiations (16)

$$d_z(fg) = (d_z f)g + f(d_z g) \tag{17}$$

and in modified form for the interior derivatives

$$\overrightarrow{\partial_z}(fg) = \left(\overrightarrow{\partial_z} f\right) g + f\left(\overrightarrow{\partial_z} g\right) + \left(\overrightarrow{\partial_z} f\right)\left(\overrightarrow{\partial_z} g\right)$$

$$\tag{18}$$

$$\overleftarrow{\partial_z}(fg) = \left(\overleftarrow{\partial_z} f\right) g + f\left(\overleftarrow{\partial_z} g\right) + \left(\overleftarrow{\partial_z} f\right)\left(\overleftarrow{\partial_z} g\right)$$

We refer the reader for the detailed theory of non-commutative differential forms on the commutative algebra of functions to [1, 7, 15] and deliver here the only information necessary for introducing the physical operators. Using right and left non-commutative Hodge operators $\overrightarrow{*}$ and $\overleftarrow{*}$ ([1]) we define the symmetrized interior derivatives in the form

$$\partial^s_z = \frac{1}{2}(\overrightarrow{*} + \overleftarrow{*})\, dz \qquad \partial^c_z = \frac{1}{2}(\overrightarrow{*} - \overleftarrow{*})\, dz + 1 \tag{19}$$

It can be proved that non-commutative interior derivatives in our case can be expressed as the finite-difference derivatives [1, 7, 15].

$$\overrightarrow{\partial} f = \left(e^{\frac{\partial}{\partial z}} - 1\right) f(z,\overline{z}), \quad \overleftarrow{\partial} f = \left(1 - e^{-\frac{\partial}{\partial z}}\right) f(z,\overline{z}) \tag{20}$$

momentum operators

$$\begin{aligned} p^+ &= \frac{1}{(z-\overline{z})}\left\{\left(z+\frac{1}{2}\right) e^{2\frac{\partial}{\partial z}} - \left(\overline{z}+\frac{1}{2}\right) e^{2\frac{\partial}{\partial \overline{z}}}\right\} \\ p^- &= \frac{1}{(z-\overline{z})}\left\{\left(z-\frac{1}{2}\right) e^{-2\frac{\partial}{\partial z}} - \left(\overline{z}-\frac{1}{2}\right) e^{-2\frac{\partial}{\partial \overline{z}}}\right\} \\ p^0 &= -\xi^0 \frac{4}{(z-\overline{z})}\, \zeta\,(z,\overline{z})\, \sinh\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\right) \\ p^+ &= p^2 + ip^1 \qquad p^- = p^2 - ip^1 \end{aligned} \tag{21}$$

Operators p^L mutually commute. Their common eigen-functions are the plane waves (5) with eigen-values (3). Momentum operators obey the hermiticity condition

$$\mu^{-1}\left(p^+\right)^\dagger \mu = p^- \qquad \mu^{-1}\left(p^0\right)^\dagger \mu = p^0 \tag{22}$$

Now we write down the three generators

$$M^{12} = \frac{z+\overline{z}}{2}, \quad M^{10} = 2\,\xi^0\, \zeta\,(z,\overline{z})\, \partial^{(s)}_{\frac{z+\overline{z}}{2}}, \quad M^{20} = 2i\,\xi^0\, \zeta\,(z,\overline{z})\, \partial^{(c)}_{\frac{z+\overline{z}}{2}} \tag{23}$$

which are as well as (21) the non-commutative differential operators and complete the set (21) up to the Lie algebra of the inhomogeneous 2 - dimensional De Sitter group.

For $\partial^{(s)}_{\frac{z\pm\overline{z}}{2}}$ and $\partial^{(c)}_{\frac{z\pm\overline{z}}{2}}$ the following relations

$$\partial^{(s)}_{\frac{z\pm\overline{z}}{2}} = \partial^{(s)}_z \partial^{(c)}_{\overline{z}} \pm \partial^{(c)}_z \partial^{(s)}_{\overline{z}} \qquad \partial^{(c)}_{\frac{z\pm\overline{z}}{2}} = \partial^{(c)}_z \partial^{(c)}_{\overline{z}} \pm \partial^{(s)}_z \partial^{(s)}_{\overline{z}} \tag{24}$$

are fulfilled

Let us consider the gauge transformation localized in non-commutative space-time

$$\psi'(\xi) = \Omega(\xi)\, \psi(\xi) \qquad \Omega(\xi)^\dagger = \Omega(\xi)^{-1} \tag{25}$$

Unlike the usual theory the gauge transformation entangles the components of momenta:

$$\Omega^{-1}(\xi)\, P^L \Omega(\xi) = {C^L}_K(\xi)\, P^K \tag{26}$$

For the sake of simplicity we write down the ${C^L}_K(z,\, \overline{z})$-matrix separately for the cases when it depends on only one of the variables $\frac{z-\overline{z}}{2}$ or $\frac{z+\overline{z}}{2}$. For $\Omega = \Omega(\frac{z-\overline{z}}{2})$ we have

$${C^L}_K\left(\frac{z-\overline{z}}{2}\right) = \left[\left(\partial^{(c)}_{\frac{z-\overline{z}}{2}}\, \Omega\left(\frac{z-\overline{z}}{2}\right)\right) - \frac{2i}{z-\overline{z}}\left(\partial^{(s)}_{\frac{z-\overline{z}}{2}}\, \Omega\left(\frac{z-\overline{z}}{2}\right)\right)\hat{\Sigma}\right]^L{}_K \tag{27}$$

where

$$\hat{\Sigma} = \begin{pmatrix} \frac{i}{2} & -M^{10} & -M^{20} \\ -M^{10} & \frac{i}{2} & M^{12} \\ -M^{20} & -M^{12} & \frac{i}{2} \end{pmatrix} \tag{28}$$

For the case $\Omega = \Omega(\frac{z+\overline{z}}{2}) = e^{i\,\lambda(\frac{z+\overline{z}}{2})}, \quad \lambda = -i\,\ln\Omega$

$${C^L}_K\left(\frac{z+\overline{z}}{2}\right) = e^{i\left(\partial^{(c)}_{\frac{z+\overline{z}}{2}}\lambda(\frac{z+\overline{z}}{2})\right)} \begin{pmatrix} e^{i\left(\left(\partial^c_{\frac{z+\overline{z}}{2}} - 1\right)\lambda\right)} & 0 & 0 \\ 0 & \cos\left(\partial^s_{\frac{z+\overline{z}}{2}}\lambda\right) & -\sin\left(\partial^s_{\frac{z+\overline{z}}{2}}\lambda\right) \\ 0 & \sin\left(\partial^s_{\frac{z+\overline{z}}{2}}\lambda\right) & \cos\left(\partial^s_{\frac{z+\overline{z}}{2}}\lambda\right) \end{pmatrix} \tag{29}$$

Also the following the following relations are true

$$\Omega^{-1}(z,\, \overline{z})\, P_L \Omega(z,\, \overline{z}) = P_K {C^\dagger}_L^K(z,\, \overline{z}) \qquad C^\dagger = C^{-1} \tag{30}$$

It is easily seen that in a consequence of (26) and (30) the De Sitter condition (2) is invariant in respect to the gauge transformations (25). It can be easily shown [4] that to make the theory gauge invariant we must introduce the complex De Sitter vector $\hat{A}^L$ of electromagnetic field and require that it transforms similarly to (26) and (30):

$$\Omega^{-1}(z, \bar{z})\, \hat{A}^L\, \Omega(z, \bar{z}) = C^L{}_K(z, \bar{z})\, \hat{A}^K$$

$$\Omega^{-1}(z, \bar{z})\, \hat{A}^\dagger_L\, \Omega(z, \bar{z}) = \hat{A}^\dagger_K C^\dagger_L{}^K(z, \bar{z}) \tag{31}$$

It follows from (31) that the components of electromagnetic field do not commute with the gauge function and in a consequence do not commute between themselves. Introducing the covariant derivatives

$$\hat{D}^L = -i\left(p^L - \hat{A}^L\right) \tag{32}$$

we obtain the De Sitter invariant equations for the matter fields. The tensor of electromagnetic field is given as

$$\hat{F}^{KL} = \left[\hat{D}^L, \hat{D}^K\right] = \left[\hat{p}^L, \hat{A}^K\right] - \left[\hat{p}^K, \hat{A}^L\right] - \left[\hat{A}^L, \hat{A}^K\right] \tag{33}$$

The action of the electromagnetic field is

$$S = Tr \int \hat{F}^{KL} \hat{F}_{KL}\, d\Omega_\xi \tag{34}$$

and the non-commutative analog of the D'Alembert equation takes the form

$$\left[\hat{D}_K, \hat{F}^{KL}\right] = 0 \tag{35}$$

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Integrable Tops and non-Commutative Torus

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Abstract

We consider the hydrodynamics of the ideal fluid on a 2-torus and its Moyal deformations. The both type of equations have the form of the Euler-Arnold tops. The Laplace operator plays the role of the inertia-tensor. It is known that 2-d hydrodynamics is non-integrable. After replacing of the Laplace operator by a distinguish pseudo-differential operator the deformed system becomes integrable. It is an infinite rank Hitchin system over an elliptic curve with transition functions from the group of the non-commutative torus. In the classical limit we obtain an integrable analog of the hydrodynamics on a torus with the inertia-tensor operator $\bar{\partial}^2$ instead of the conventional Laplace operator $\partial\bar{\partial}$.

1 Introduction

The Euler-Arnold tops (EAT) are Hamiltonian systems defined on the coadjoint orbits of groups [1]. Particular examples of such systems are the Euler top related to SO(3), its SO(N) generalization [2] and the hydrodynamic of the ideal incompressible fluid on a space M. The corresponding group of the latter system is SDiff(M). We consider here the case $\dim(M) = 2$ and then restrict M to be a torus T^2.

EAT are completely determined by their Hamiltonians, since the Poisson structure is fixed to be related to the Kirillov-Kostant form on the coadjoint orbits. The Hamiltonians are determined by the inertia-tensor operator $\mathfrak{J}$ mapping the Lie algebra $\mathfrak{g}$ to the Lie coalgebra $\mathfrak{g}^*$. Special choices of $\mathfrak{J}$ lead to completely integrable systems (see review [3]). In the case of the 2d hydrodynamics $\mathfrak{J}$ is the Laplace operator and it turns out that the theory is non-integrable [4]. The goal of this paper are integrable models related to SDiff. Some integrable models related to SDiff were considered in [5, 6, 7, 8].

These type of models can be described as the classical limit of integrable models when the commutators in the Lax equations are replaced by the Poisson brackets. This approach was proposed in Ref.[9], where the dispersionless KP hierarchy was constructed and later developed in numerous publications (see review [10]).

Here we use the same strategy defining an integrable system on the non-commutative torus (NCT) and then taking the classical limit to SDiff(T^2). The starting point in our construction is the integrable GL($N,\mathbb{C}$) EAT introduced in Ref.[11, 12]. Their inertia-tensor operators depend on the module τ, $Im\tau > 0$ of an elliptic curve. This curve is the basic spectral curve in the Hitchin description of the model. We consider here a special limit $N \to \infty$ of GL($N,\mathbb{C}$) that leads to the group G_θ of NCT, where θ is the Planck constant.[1] The group G_θ is defined as the set of invertible elements of the NCT algebra $\mathcal{A}_\theta$. It can be embedded in GL(∞) and in this way G_θ can be described as a special

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[1] For Manakov's top $\lim N \to \infty$ was considered in Ref.[13].

limit of GL($N,\mathbb{C}$). We define EAT related to G_θ depending on a parameter τ. Then, we construct the Lax operator with the spectral parameter on an elliptic curve with the same parameter τ.

In the classical limit $\theta \to 0$, $G_\theta \to$SDiff(T^2) and the inertia-tensor operator $\mathfrak{J}$ takes the form $\bar{\partial}^2$. The conservation laws survive in this limit while commutators in the Lax hierarchy become the Poisson brackets. It turns out that the classical limit is essentially the same as the rational limit of the basic elliptic curve, such that the product of the Planck constant θ and the half periods of the basic curve are constant. In this way in the classical limit NCT becomes dual to the basic elliptic curve.

2 The Lie algebra of the non-commutative torus

Here we reproduce some basic results about NCT and related to it Lie algebra Sin_θ [14].

1. *Non-commutative torus.*

Quantum torus $\mathcal{A}_\theta$ is the unital algebra with two generators (U_1, U_2) that satisfy the relation

$$U_1U_2 = \mathbf{e}(\theta)U_2U_1, \ \mathbf{e}(\theta) = e^{2\pi i\theta}, \ \theta \in [0,1)\,. \tag{2.1}$$

Elements of $\mathcal{A}_\theta$ are the double sums

$$\mathcal{A}_\theta = \{x = \sum_{m,n\in\mathbb{Z}} a_{mn}U_1^mU_2^n, \ a_{mn} \in \mathbb{C}\}\,,$$

where a_{mn} are rapidly decreasing on the lattice $\mathbb{Z}^2$

$$\sup_{m,n\in\mathbb{Z}}(1+m^2+n^2)^k|a_{mn}|^2 < \infty, \text{ for } k \in \mathbb{N}. \tag{2.2}$$

The trace functional $\mathrm{tr}(x)$ on $\mathcal{A}_\theta$ is defined as $\mathrm{tr}(x) = a_{00}$. It is positive definite and satisfies the evident identities

$$\mathrm{tr}(x^*x) = \sum_{m,n\in\mathbb{Z}} |a_{mn}|^2 \geq 0\,,$$

$$\mathrm{tr}(1) = 1, \ \ \mathrm{tr}(xy) = \mathrm{tr}(yx\,,$$

where the star involution acts as $U_a^* = U_a^{-1}$, $a_{mn}^* = \bar{a}_{mn}$. The relation with the commutative algebra of smooth functions on the two dimensional torus

$$T^2 = \{\mathbb{R}^2/\mathbb{Z}\oplus\mathbb{Z}\} \sim \{0 < x \leq 1, 0 < y \leq 1\}. \tag{2.3}$$

comes from the identification

$$U_1 \to \mathbf{e}(x), \ \ U_2 \to \mathbf{e}(y), \tag{2.4}$$

and the multiplication on T^2 becomes the Moyal multiplication:

$$(f * g)(x,y) := fg+ \tag{2.5}$$

$$\sum_{n=1}^{\infty} \frac{(\pi\theta)^n}{n!}\varepsilon_{r_1s_1}\ldots\varepsilon_{r_ns_n}(\partial^n_{r_1\ldots r_n}f)(\partial^n_{s_1\ldots s_n}g).$$

Another representation with the operators, that acts on the space of functions on $\mathbb{R}/\mathbb{Z}$, has the form

$$U_1 \to \exp(i\varphi),\ U_2 \to \mathbf{e}(\theta\partial_\varphi). \tag{2.6}$$

Finally, we can identify U_1, U_2 with matrices in $\mathrm{GL}(\infty)$

$$U_1 \to Q,\ U_2 \to \Lambda, \tag{2.7}$$

where Q and Λ are defined as

$$Q = \mathrm{diag}(\mathbf{e}(j\theta)),\ \Lambda = E_{j,j+1},\ j \in \mathbb{Z}.$$

The trace functional in the Moyal identification (2.4) is the integral

$$\mathrm{tr} f = \int_{T^2} f dx dy = f_{00}. \tag{2.8}$$

2. Sin-algebra

Define the following quadratic combinations of the generators

$$T_{mn} = \frac{i}{2\pi\theta}\mathbf{e}\left(\frac{mn}{2}\theta\right) U_1^m U_2^n. \tag{2.9}$$

Their commutator has the form of the sin-algebra

$$[T_{mn}, T_{m'n'}] = \frac{1}{\pi\theta}\sin\pi\theta(mn' - m'n)T_{m+m',n+n'}. \tag{2.10}$$

We denote Sin_θ the Lie algebra of the series

$$\psi = \sum_{mn}\psi_{mn}T_{mn} \tag{2.11}$$

with rapidly decreasing coefficients (2.2) and the commutator (2.10).

In the Moyal representation (2.5) the commutator has the form

$$[f(x,y), g(x,y)] = \{f,g\}^*,$$

where

$$\{f,g\}^* = \frac{1}{\theta}(f*g - g*f). \tag{2.12}$$

The algebra Sin_θ has a central extension $\widehat{Sin}_\theta$ that comes from the one-cocycles of $\mathcal{A}_\theta$. The corresponding additional term in (2.10) has the form

$$(am + bn)\delta_{m,-m'}\delta_{n,-n'}. \tag{2.13}$$

Let θ be a rational number $\theta = p/N$, where $p, N \in \mathbb{N}$ with $\gcd(p,N) = 1$. Then Sin_θ has an ideal

$$I_N = \{T_{mn} - T_{m+Nl,n+Nk},\ k,l \in \mathbb{Z}\}, \tag{2.14}$$

and the factor $Sin_{p/N}/I_N$ for odd N is isomorphic to $\mathrm{GL}(N,\mathbb{C})$. Similarly, we can consider the ideal

$$\hat{I}_N = \{T_{mn} - T_{m+Nl,n},\ l \in \mathbb{Z}\}, \tag{2.15}$$

in $\widehat{Sin}_\theta$. The factor $\widehat{Sin}_\theta/\hat{I}_N$ is isomorphic to the central extended loop algebra $\hat{L}(\mathrm{gl}(N,\mathbb{C}))$.

3 2d-hydrodynamics on $\mathcal{A}_\theta$

1. 2-d hydrodynamics

Let $\mathbf{v} = (V_x, V_y)$ be the velocity vector of the ideal incompressible fluid on a compact manifold M ($\dim(M) = 2$, $\mathrm{div}\mathbf{v} = 0$). [2] The Euler equation for 2d hydrodynamics takes the form [1]

$$\partial_t \mathrm{curl}\mathbf{v} = \mathrm{curl}[\mathbf{v}, \mathrm{curl}\mathbf{v}], \tag{3.1}$$

where $\mathrm{curl}\mathbf{v} = \partial_x V_y - \partial_y V_x$ is the vorticity of the vector field $\mathbf{v}$.

Let $\psi(x,y)$ be the stream function. It is the Hamiltonian generating the vector field $\mathbf{v}$

$$i_{\mathbf{v}} dx \wedge dy = d\psi.$$

In other words $V_x = -\partial_y\psi,\ V_y = \partial_x\psi$. There is a isomorphisms of the Lie algebra of vector fields $\mathrm{div}\mathbf{v} = 0$ on M and the Poisson algebra of the stream functions $\mathfrak{g} = \{\psi\}$ on M defined up to constants

$$\{\psi_{\mathbf{v}_1}, \psi_{\mathbf{v}_2}\} = \psi_{[\mathbf{v}_1,\mathbf{v}_2]}$$

Let $\mathfrak{g}^*$ be the dual space of distributions on M. The vorticity $\mathcal{S} = \mathrm{curl}\mathbf{v}$ of the vector field $\mathbf{v}$

$$\mathcal{S} = -\Delta\psi$$

can be considered as an element from $\mathfrak{g}^*$. The Euler equation (3.1) for $\mathcal{S}$ takes the form

$$\partial_t\mathcal{S} = \{\mathcal{S}, \psi\},\ \text{ or }\ \partial_t\mathcal{S} = \{\mathcal{S}, \Delta^{-1}\mathcal{S}\}. \tag{3.2}$$

We can look on (3.2) as the Euler-Arnold equation for the rigid top related to the Lie algebra $\mathfrak{g}$, where the Laplace operator is the map

$$\Delta : \mathfrak{g} \to \mathfrak{g}^*$$

that plays the role of the inertia-tensor. The phase space of the system is a coadjoint orbit of the group of the canonical transformations $\mathrm{SDiff}(M) = \exp\{\psi, \cdot\}$. The equation (3.2) takes the form

$$\partial_t\mathcal{S} = \mathrm{ad}^*_{\nabla H}\mathcal{S}, \tag{3.3}$$

where $\nabla H = \frac{\delta H}{\delta \mathcal{S}} = \psi$ is the variation of the Hamiltonian

$$H = \frac{1}{2}\int_M \mathcal{S}\Delta^{-1}\mathcal{S} = \int_M \psi\Delta\psi. \tag{3.4}$$

There is infinite set of Casimirs defining the coadjoint orbits

$$C_k = \int_M \mathcal{S}^k. \tag{3.5}$$

Certainly, these integrals cannot provide the integrability of the system [4].

Consider a particular case, when M is a two-dimensional torus (2.3). In terms of the Fourier modes s_{mn} of the vorticity

$$\mathcal{S} = \sum_{mn} s_{mn}\mathbf{e}(mx + ny)$$

[2] For simplicity we assume that the measure on M is $dx \wedge dy$, though all expressions can be written in a covariant way.

the Hamiltonian (3.4) takes the form

$$H = -\frac{1}{8\pi^2}\sum_{mn}\frac{1}{m^2+n^2}s_{mn}s_{-m,-n}\,, \tag{3.6}$$

and we come to the equation

$$\partial_t s_{mn} = -\frac{1}{8\pi^2}\sum_{j,k}\frac{jn-km}{j^2+k^2}s_{jk}s_{m-j,n-k}\,. \tag{3.7}$$

2. 2d hydrodynamics on non-commutative torus.

We can consider the similar construction by replacing the Poisson brackets by the Moyal brackets (2.12) [15, 16]. Introduce the vorticity $\mathcal{S}$ as an element of Sin^*_θ

$$\mathcal{S} = \sum_{mn} s_{mn}T_{mn}. \tag{3.8}$$

The equation (3.2) takes the form

$$\partial_t\mathcal{S} = \{\mathcal{S}, \Delta^{-1}\mathcal{S}\}^*\,,$$

or for the Fourier modes

$$\partial_t s_{mn} = -\frac{1}{8\pi^3\theta}\sum_{j,k}\frac{\sin\pi\theta(jn-km)}{j^2+k^2}s_{jk}s_{m-j,n-k}\,. \tag{3.9}$$

This system is the Euler-Arnold top on the group G_θ of invertible elements of $\mathcal{A}_\theta$ and the coadjoint orbits are defined by the same Casimirs (3.5) as for SDiff(T^2).

4 Elliptic rotator on $\mathcal{A}_\theta$

Let $\wp$ be the Weierstrass function depending on the modular parameter τ, Im$\tau > 0$.

$$\wp(u;\tau) = \frac{1}{u^2} + \sum_{j,k}{}' \left(\frac{1}{(j+k\tau+u)^2} - \frac{1}{(j+k\tau)^2}\right). \tag{4.1}$$

We replace the inverse inertia-tensor Δ^{-1} of the hydrodynamics on the pseudodifferential operator $\mathfrak{J}^{-1} = J:\ \mathcal{S}\to\psi$ acting in a diagonal way on the Fourier coefficients (3.8):

$$J:\ s_{mn}\to\wp\begin{bmatrix} m\\ n\end{bmatrix}s_{mn} = \psi_{mn}. \tag{4.2}$$

Here

$$\wp\begin{bmatrix} m\\ n\end{bmatrix} = \wp\left((m+n\tau)\theta;\tau\right).$$

We consider the Euler-Arnold top with the inertia-tensor defined by J^{-1} (4.2). It has the Hamiltonian

$$H_\theta = \frac{\theta^2}{2}\int_{T^2}\mathcal{S}J(\mathcal{S}) = \frac{\theta^2}{2}\int_{T^2}\psi J^{-1}(\psi) = \frac{\theta^2}{2}\sum_{mn}\wp\begin{bmatrix} m\\ n\end{bmatrix}s_{mn}s_{-m,-n}, \tag{4.3}$$

where the integral is the trace functional (2.8). The equation of motion in the form of the Moyal brackets has the standard form

$$\partial_t\mathcal{S} = \theta^2\{\mathcal{S}, J(\mathcal{S})\}^*\,, \tag{4.4}$$

or for the Fourier components

$$\partial_t s_{mn} = \frac{\theta}{\pi}\sum_{j,k}s_{jk}s_{m-j,n-k}\wp\begin{bmatrix} j\\ k\end{bmatrix}\sin\pi\theta(jn-km)\,. \tag{4.5}$$

Consider the classical limit $\theta\to 0$ of this system when the Moyal brackets in (4.4) pass to the standard Poisson brackets. In this case we come to the top on the group SDiff(T^2). Since

$$\lim_{\theta\to 0}\theta^2\wp\begin{bmatrix} m\\ n\end{bmatrix} = \frac{1}{(m+n\tau)^2}$$

the Hamiltonian (4.3) takes the form

$$H = \frac{1}{2}\int_{T^2}\psi(\bar\partial)^2\psi\,, \tag{4.6}$$

where $\bar\partial = \frac{1}{\rho}(\tau\partial_x - \partial_y)$, $\rho = \tau - \bar\tau$. The operator $\bar\partial^2$ plays the role of the inertia-tensor. It replaces the Laplace operator $\Delta\sim\partial\bar\partial$ for $\tau = i$ $(\partial = \frac{1}{\rho}(-\bar\tau\partial_x + \partial_y))$ of the standard hydrodynamics.

5 Integrability of elliptic rotator on $\mathcal{A}_\theta$

1. The Lax pair.

We will prove here that the Hamiltonian system of the elliptic rotator (4.4), (4.5) have an infinite set of involutive integrals of motion in addition to the Casimirs (3.5). It will follow from the Lax form

$$\partial_t L = [L, M] \tag{5.1}$$

of the equations (4.4), (4.5). To define the Lax operator we introduce the basic spectral curve

$$E_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})\,, \tag{5.2}$$

with the already defined modular parameter τ. The Lax operator $L(z)$ is $(1,0)$-form on the spectral parameter $z\in E_\tau$ taking value in the coalgebra Sin^*_θ.

Introduce the following functions

$$\phi(u,z) = \frac{\vartheta(u+z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}\,, \tag{5.3}$$

where $\vartheta(u)$ is the odd theta-function on E_τ,

$$\vartheta(z|\tau) = \mathbf{e}\left(\frac{\tau}{8}\right)\sum_{n\in\mathbf{Z}}(-1)^n e^{\pi i(n(n+1)\tau+2nz)}\,, \tag{5.4}$$

and

$$\varphi\begin{bmatrix} m\\ n\end{bmatrix}(z) = \mathbf{e}(-n\theta z)\phi(-(m+n\tau)\theta, z)\,, \tag{5.5}$$

$$f\begin{bmatrix} m \\ n \end{bmatrix}(z) = \mathbf{e}(-n\theta z)\partial_u\phi(u,z)|_{u=-(m+n\tau)\theta}\,. \tag{5.6}$$

Then the Lax operator takes the form

$$L = \sum_{mn} s_{mn}\varphi\begin{bmatrix} m \\ n \end{bmatrix}(z)T_{mn}\,, \tag{5.7}$$

and

$$M = \sum_{mn} s_{mn}f\begin{bmatrix} m \\ n \end{bmatrix}(z)T_{mn}\,. \tag{5.8}$$

The equivalence of (5.1) and (4.5) follows from the Calogero functional equation

$$\varphi\begin{bmatrix} m \\ j \end{bmatrix} f\begin{bmatrix} j \\ n \end{bmatrix} - \varphi\begin{bmatrix} j \\ n \end{bmatrix} f\begin{bmatrix} m \\ j \end{bmatrix} = \left(\wp\begin{bmatrix} m \\ j \end{bmatrix} - \wp\begin{bmatrix} j \\ n \end{bmatrix}\right)\varphi\begin{bmatrix} m \\ n \end{bmatrix},$$

and the identity

$$\varphi\begin{bmatrix} m \\ n \end{bmatrix}(z)\varphi\begin{bmatrix} -m \\ -n \end{bmatrix}(z) = \wp(z) - \wp\begin{bmatrix} m \\ n \end{bmatrix}.$$

2. *Hitchin description.*

It turns out that the elliptic rotator is the Hitchin system [17, 18] on E_τ. It was proved in [12] for the group $\mathrm{GL}(N,\mathbb{C})$. Here we replace $\mathrm{GL}(N,\mathbb{C})$ by the infinite-dimensional group G_θ. It means that we consider the infinite rank vector bundle P over E_τ with the structure group G_θ. The Hitchin systems of infinite rank were introduced in [12, 19]. Here we reproduce the properties of the concrete Lax equation as the Hitchin system. The Lax operator plays the role of the Higgs field. It satisfies the Hitchin equation

$$\bar\partial L = 0, \quad Res\, L|_{z=0} = \mathcal{S},$$

with the quasi-periodicity conditions

$$L(z+1) = QL(z)Q^{-1},$$

$$L(z+\tau) = \Lambda L(z)\Lambda^{-1}\,.$$

It is easy to see that (5.7) does satisfy these conditions.

3. *Integrals of motion.*

In the Hitchin construction the integrals are defined by means of the $(-j,1)$-differentials $\mu_j \in \Omega^{(-j,1)}(E_\tau)$. Let us choose the representatives from $\Omega^{(-j,1)}(E_\tau)$ that form a basis in the cohomology space $H^1(E_\tau,\Gamma^j)$, $(\dim H^1 = j)$

$$\mu_j = (\mu_{0,j}\partial_z^{j-1}\otimes d\bar z, \mu_{2,j}\partial_z^{j-1}\otimes d\bar z \ldots \mu_{j,j}\partial_z^{j}\otimes d\bar z)\,.$$

Let $\chi(z,\bar z)$ be a characteristic function of a small neighborhood of $z=0$. We can choose $\mu_{s,j}$ in the form

$$\mu_{0,j} \sim \theta^j\bar\partial(\bar z - z)(1-\chi(z,\bar z))\,,^{3}$$
$$\mu_{s,j} = c_{s,j}z^{s-1}\bar\partial\chi(z,\bar z) \text{ for } j>1,\ j\ge s>1, \tag{5.9}$$
$$c_{s,j} \sim \theta^{j-s}\,.$$

[3] $\mu_{0,2}$ is the Beltrami differential

The integrals

$$I_{s,j} = \frac{1}{j}\int_{E_\tau}\int_{T^2}(L^j)\mu_{s,j} \tag{5.10}$$

are well defined and generate the infinite set of conservation laws. Here we integrate over the basic spectral curve E_τ (5.2) and NCT $\mathcal{A}_\theta$. The conservation laws can be extracted from the expansion of the elliptic function

$$\frac{1}{j}\int_{T^2} L^j(z) = I_{0,j} + \sum_{r=2}^{j} I_{r,j}\wp^{(r-2)}(z),\ (j=2,\ldots)\,.$$

In particular,

$$\int_{T^2} L^2(z) = I_{0,2} + \wp(z)\int_{T^2}\mathcal{S}^2,\ H = I_{0,2}\,.$$

Note that

$$I_{j,j} \sim C_j = \int_{T^2}\mathcal{S}^j$$

are the Casimirs (3.5).

Consider, for example, the integrals, that have the third order in the field $\mathcal{S}$. Let

$$E_1(u) = \partial_u \log\vartheta(u)\,,$$

be the first Eisenstein function.[4] The second Eisenstein function is

$$E_2(u) = \wp(u) + 2\zeta(1/2)\,.$$

For the functions $\phi(u,z)$ (5.3) we have the following relation. If $u_1+u_2+u_3=0$ then

$$\begin{aligned}\phi(u_1,z)\phi(u_2,z)\phi(u_3,z) = &-\frac{1}{2}E_2'(z)+ \\ &E_2(z)\left(E_1(u_1)+E_1(u_2)+E_1(u_3)\right)+ \\ &E_2(u_3)\left(E_1(u_1)+E_1(u_2)+E_1(u_3)\right) - \frac{1}{2}E_2'(u_3)\,.\end{aligned} \tag{5.11}$$

This relation allows to calculate $\int_{T^2} L^3(z)$. Let

$$E_2\begin{bmatrix} m \\ n \end{bmatrix} = E_2((m+n\tau)\theta),$$

$$E_1\begin{bmatrix} m \\ n \end{bmatrix} = E_1((m+n\tau)\theta)\,.$$

Then in terms of the Fourier modes $\mathcal{S} = \{s_{mn}\}$ the integrals take the form

$$I_{1,3} = \frac{\theta^3}{3}\sum_{\sum m_j = \sum n_j = 0}\prod_{j=1}^{3} s_{m_j n_j}\left(E_2\begin{bmatrix} m_3 \\ n_3 \end{bmatrix}\sum_j E_1\begin{bmatrix} m_j \\ n_j \end{bmatrix} - \frac{1}{2}E_2'\begin{bmatrix} m_3 \\ n_3 \end{bmatrix}\right), \tag{5.12}$$

[4] It is related to the Weierstrass zeta-function as $E_1(u) = \zeta(u) - 2\zeta(1/2)u$.

$$I_{2,3} = -\frac{\theta}{6} \sum_{\sum m_j = \sum n_j = 0} \prod_{j=1}^{3} s_{m_j n_j} \sum_j E_1 \begin{bmatrix} m_j \\ n_j \end{bmatrix} . \tag{5.13}$$

The integrals (5.10) allow to write the hierarchy of the commuting flows

$$\partial_{s,j} \mathcal{S} = \{\nabla I_{s,j}, \mathcal{S}\}^* \quad (\partial_{s,j} = \partial_{t_{s,j}}) . \tag{5.14}$$

They have the Lax representation with L (5.7)

$$\partial_{s,j} L = [L, M_{s,j}] := \{L, M_{s,j}\}^* , \tag{5.15}$$

and $M_{s,j}$ is partly fixed by the equation

$$\bar{\partial} M_{s,j} = -L(z)^{j-1} \mu_{s,j} .$$

4. Classical limit.
In the limit $\theta \to 0$ the Lax hierarchy (5.15) is replaced by the classical equations

$$\partial_{s,j} L^{(-1)} = \{L^{(-1)}, M_{s,j}^{(0)}\} ,$$

where

$$L^{(-1)} = \frac{s_{mn}}{(m + n\tau)}$$

and $M_{s,j} = M_{s,j}^{(0)} + O(\theta)$. The integrals of motion in this limit survive. We already pointed the form of the Hamiltonian $H = I_{0,2}$ (4.6). The third order integrals take the forms

$$I_{1,3} = \frac{1}{3} \sum_{\sum m_j = \sum n_j = 0} \prod_{j=1}^{3} s_{m_j n_j} \left(\frac{1}{(m_3 + n_3\tau)^2} \sum_j \frac{1}{m_j + n_j\tau} - \frac{1}{2(m_3 + n_3\tau)^3} \right) ,$$

$$I_{2,3} = -\frac{1}{6} \sum_{\sum m_j = \sum n_j = 0} \prod_{j=1}^{3} s_{m_j n_j} \times \sum_j \frac{1}{m_j + n_j\tau} .$$

It turns out that the classical limit in some sense is the same as the rational limit of the basic spectral curve E_τ. Let ω_1, ω_2 be two half-periods of E_τ, $\tau = \omega_2/\omega_1$. The rational limit means that ω_1, $\omega_2 \to \infty$. In this case the Weierstrass function is degenerate as

$$\wp(u) \to \frac{1}{u^2} .$$

Consider the double limit $\theta \to 0$, ω_1, $\omega_2 \to \infty$ such that

$$\lim \omega_1 \theta = 1, \quad \lim \omega_2 \theta = \tau$$

It follows from the definition of the Weierstrass function $\wp(u; \omega_1, \omega_2)$ that in this limit

$$\wp \begin{bmatrix} m \\ n \end{bmatrix} \to \frac{1}{(m + n\tau)^2} .$$

Rescale now the Hamiltonian (4.3) by the dropping the θ^2 multiplier. The new Hamiltonian in the double limit takes the form (4.6).

6 Conclusion

There are three related subjects that are not covered here. The first will be elucidated in separated publications.

- The elliptic rotator on $GL(N, \mathbb{C})$ can be transformed to the elliptic Calogero-Moser system (ECM) by the symplectic Hecke correspondence [12]. For general orbits we obtain ECM system with spin. The spin degrees of freedom disappear for the most degenerated orbits. The similar approach can be developed for the elliptic rotator on $\mathcal{A}_\theta$. In other words, there should exists a symplectic Hecke correspondence between a special thermodynamical limit of ECM and the elliptic rotator on $\mathcal{A}_\theta$. The general thermodynamical limit of the ECM system leads to ECM system related to $GL(\infty)$, but here we consider the NCT subgroup and it means that the limit is special. For $\theta = p/N$ after the factorization over the ideal (2.14) we come back to the ECM system for $GL(N, \mathbb{C})$. I plan to describe ECM system for $\mathcal{A}_\theta$ and its symplectic (Hecke) transformation to the elliptic EAT for $\mathcal{A}_\theta$.
- The EAT can be considered on the central extended algebra $\widehat{Sin_\theta}$ (2.13). For the hydrodynamics ($\mathfrak{J} = \Delta$) this case was investigated in Ref. [15]. Incorporating of the central charge drastically changes the dynamics of EAT. In particular, the integrals (5.10) are no longer the conserved quantities, as well as the Hamiltonian (4.3). For ECM system related to $GL(N, \mathbb{C})$ this construction leads to the two-dimensional ECM integrable field theory [12, 19]. In the group theoretical terms it means that we pass from $GL(N, \mathbb{C})$ to the central extended loop group $\hat{L}(GL(N, \mathbb{C}))$.

The same symplectic transformation that maps ECM system to the elliptic EAT is working in the two-dimensional case. In particular, the two-dimensional ECM system for $N = 2$ is mapped to the Landau-Lifshitz equation [12]. In the limit $N \to \infty$ we obtain the symplectic map from the thermodynamical limit of the ECM field theory to the central extended elliptic rotator. The both cases can be described as the systems related to $\widehat{Sin_\theta}$. When $\theta = p/N$ after the factorizing over the ideal (2.15) we come back to the systems considered in [12, 19].

- Two different tori are incorporated in our construction - NCT $\mathcal{A}_\theta$ and the basic spectral curve E_τ. In the classical limit they become dual. It seems natural to replace E_τ on another NCT $\mathcal{A}_{\theta'}$. In a general setting it means a generalization of the Hitchin systems to the non-commutative case. One attempt in this direction was done in Ref.[20]. As we have seen, the theta-functions with characteristics are the building blocks for the Lax operators. The analogs of theta-functions on NCT are known [21, 22, 23]. We can hope that they will play the essential role in the construction of the Hitchin type systems in the non-commutative situation.

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Quantum Lobachevsky Spaces and Wave Functions of Relativistic Toda Lattice

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Abstract

A twist of quantum Lorentz group depending on two parameters and a one-parametric family of quantum Lobachevsky spaces are considered. For three collections of these parameters the Casimir operators are connected with the two-body relativistic open Toda lattice Hamiltonians. Their eigen-functions are the modified Bessel-Jackson functions of three types. The principal series of unitary irreducible representations of the quantum Lorentz groups is constructed. The matrix elements in the irreducible spaces are the Bessel-Macdonald-Jackson functions. They are the wave functions of the two-body relativistic open Toda lattice.

1 INTRODUCTION

In the classical Harmonic Analysis the zonal spherical functions play the role of the wave-functions of some integrable models [1]. In particular, the Bessel functions are the zonal spherical functions related to the group symmetries of the Euclidean spaces. On the other hand, they are the wave-functions of the rational Calogero-Moser model.

The modified Bessel functions and the Bessel-Macdonald functions are the Fourier transform on a horosphere of the eigen-functions of the Laplace-Beltrami operator in the horospheric coordinates on the Lobachevsky space. From the point of view of integrable systems the Laplace-Beltrami operator in horospheric coordinates gives rise to the simplest form of the open Toda Hamiltonian. B. Kostant generalized this connection and established the similar relations between an arbitrary open Toda lattice and the Whittaker model of irreducible representations of the splitted simple group G [2].

In [3] we applied the Kostant scheme to the quantum Lorentz group constructed by Podlez and Woronowicz [4]. It turns out that the corresponding integrable system is the open two-body relativistic Toda lattice. Its Hamiltonian is related to the Casimir operator of the quantum Lorentz group being written in an analog of the horospheric coordinates on the corresponding quantum Lobachevsky space. Up to the q-exponents multipliers the eigen-functions are the modified q-Bessel-Jackson and the q-Bessel-Jackson-Macdonald functions. Their properties were investigated in [5, 6, 7]. Recently, the integral representations for the wave functions of the N-body case based on the Whittaker model of $\mathcal{U}_q(\mathrm{SL}(N,\mathbb{R}))$ was presented in [8].

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In this work we investigate the two-body case in detail. Our analysis is based on the Whittaker model for the quantum Lorentz group $\mathcal{U}_q(\mathrm{SL}(2,\mathbb{C}))$. In this way we describe the group theoretical approach to the modified q-Bessel-Jackson and the q-Bessel-Jackson-Macdonald functions. For this purpose we construct unitary irreducible representations of a twisted family of quantum Lorentz groups. There are several special types of twists. For our purpose the most interesting will be the twist of Reshetikhin type [9]. As by product we obtain some results in the Harmonic Analysis on quantum Lobachevsky spaces. The quantum Lobachevsky spaces play the role of homogeneous spaces with respect to the twisted quantum Lorentz groups. Three members from the family are distinguished. For these cases the Casimir operators realized on the quantum Lobachevsky spaces lead to second order difference operators. Up to a conjugation they coincide with the quantum relativistic two-body open Toda Hamiltonians. Their eigen-functions are the three types of the modified Bessel-Jackson functions. We construct an analog of the principle series unitary irreducible representations. Then we consider some special matrix elements of group elements acting in an irreducible space of the class-one representations. They are the wave functions of the Toda lattice. These matrix elements have the form of the double integral. It follows from [6] that they are q-Bessel-Jackson-Macdonald functions.

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2 Classical Case

1. Lorentz group. The matrix Lorentz group is the group of complex unimodular matrices of second order

$$G = \mathrm{SL}(2,\mathbb{C}) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \ (\alpha\delta - \beta\gamma = 1).$$

Its Lie algebra has three generators

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ h = \frac{1}{2}\mathrm{diag}(1,-1), \ f = e^T$$

with commutators

$$[e,f] = 2h, \ [h,e] = e, \ [h,f] = -f.$$

They generate $\mathrm{sl}(2,\mathbb{R})$ over $\mathbb{R}$ and $\mathrm{sl}(2,\mathbb{C})$ over $\mathbb{C}$. As a real algebra $\mathrm{sl}(2,\mathbb{C})$ has six generators and one should consider the second copy of $\mathrm{sl}(2,\mathbb{R})$ generated by $(\bar{e},\bar{h},\bar{f})$.

2 Principal series. The principal series representations $\pi_{\nu,n}$ of G are induced by the characters of the Borel subgroup of G. It can be realized in the space of sections of a linear bundle $\mathcal{L}_{\nu,n}$ over $\mathbf{P}^1$

$$V_{\nu,n} = \Gamma(\mathcal{L}_{\nu,n}) \sim \tag{2.1}$$

$$\mathcal{A}^{(-\frac{r}{2},-\frac{\tilde{r}}{2})}(\mathbf{P}^1) \ (r = i\nu + n - 1, \ \tilde{r} = i\nu - n - 1).$$

The sections have the form $f(z,\bar{z})(dz)^{-\frac{r}{2}}(d\bar{z})^{-\frac{\tilde{r}}{2}}$, where $f(z,\bar{z})$ are smooth functions with the asymptotic

$$f(z,\bar{z})_{z\to\infty} \sim z^r\bar{z}^{\tilde{r}}. \tag{2.2}$$

The restriction of $\pi_{\nu,n}(g)$ on $V_{\nu,n}$ (2.1) has form

$$\pi_{\nu,n}(g)f(z,\bar{z}) = (\gamma z + \alpha)^{i\nu+n-1}(\bar{\gamma}\bar{z} + \bar{\alpha})^{i\nu-n-1}f(zg,\bar{z}\bar{g}). \tag{2.3}$$

Since $\bar{r}+\bar{\bar{r}}=-2$ there is a Hermitian form on $\mathcal{A}^{(-\frac{r}{2},-\frac{\bar{r}}{2})}(\mathbf{P}^1)$

$$<f_1|f_2>=\int_{\mathbf{P}^1} f_1(z,\bar{z})\overline{f_2(z,\bar{z})}dzd\bar{z}. \tag{2.4}$$

The representation $\pi_{\nu,n}$ (2.3) is realized in $V_{\nu,n}$ by the unitary operators because

$$<\pi_{\nu,n}(g)f_1|f_2>=<f_1|\pi_{\nu,n}(g^{-1})f_2>.$$

The infinitesimal version of this construction takes the form:

$$e\to T^+=\partial_z,\quad h\to T^3=-z\partial_z+\frac{r}{2},$$

$$f\to T^-=-z^2\partial_z+rz,\quad (r=n-1+i\nu) \tag{2.5}$$

$$\bar{e}\to\bar{T}^+=\partial_{\bar{z}},\ \bar{h}\to\bar{T}^3=-\bar{z}\partial_{\bar{z}}+\frac{\bar{r}}{2},$$

This action preserves the asymptotic (2.2).

There are two Casimir operators of $\mathrm{SL}(2,\mathbb{C})$

$$\Omega=h^2+h+fe,\quad \tilde{\Omega}=\bar{h}^2+\bar{h}+\bar{f}\bar{e}.$$

They become the scalar operators in $V_{\nu,n}$

$$\Omega|\Psi>=(\frac{1}{4}(i\nu+n)^2-\frac{1}{4})|\Psi>, \tag{2.6}$$

$$\tilde{\Omega}|\Psi>=(\frac{1}{4}(i\nu-n)^2-\frac{1}{4})|\Psi>,\quad |\Psi>\in V_{\nu,n}.$$

If $n=0$ then there exists a $\mathrm{SU}(2)$ invariant vector in $V_{\nu,0}$

$$(1+|z|^2)^{i\nu-1}.$$

3. Lobachevsky space. A subclass of these representations $r=i\nu-1$, $(n=0)$ (the representations of class one) are related to the decomposition of functions on the Lobachevsky space $\mathbf{L}=\mathrm{SU}(2)\backslash\mathrm{SL}(2,\mathbb{C})$. The later can be realized as the space of second order unimodular Hermitian positive definite matrices. The Iwasawa decomposition

$$g=kb,\ g\in\mathrm{SL}(2,\mathbb{C}),\ k\in\mathrm{SU}(2),\ b\in AN$$

allows to introduce the horospheric coordinates on $\mathbf{L}$:

$$x=b^\dagger b=\begin{pmatrix}\bar{h}h & \bar{h}hz\\ \bar{z}\bar{h}h & \bar{z}\bar{h}hz+(\bar{h}h)^{-1}\end{pmatrix}.$$

The triple $(H=\bar{h}h,z,\bar{z})$ is uniquely determined by x. It is called the horospheric coordinates of x.

Consider the space of smooth complex valued integrable functions $\mathcal{R}_{\mathbf{L}}=\{f(H,z,\bar{z})\}$ on $\mathbf{L}$:

$$\int|f(H,z,\bar{z})|^2HdHdzd\bar{z}<\infty. \tag{2.7}$$

The Lie operators generated by the right shifts act on $\mathcal{R}_{\mathbf{L}}$ as

$$e\to\mathcal{D}^+=\partial_z,h\to\mathcal{D}^3=\frac{1}{2}H\partial_H-z\partial_z,$$

$$f\to\mathcal{D}^-=Hz\partial_H-z^2\partial_z+H^{-2}\partial_{\bar{z}}. \tag{2.8}$$

The generators $\bar{e},\bar{h},\bar{f}$ give rise to the complex conjugate operators.

Consider a family $V_\nu,\ \nu\in\mathbb{R}$ of subspaces in $\mathcal{R}_{\mathbf{L}}$

$$V_\nu=\{f(\bar{z},aH,z)=a^{(i\nu-1)}f(\bar{z},H,z)\}.$$

Assume that e,h,f act on the subspace of holomorphic ($\bar{z}$ independent) functions and $\bar{e},\bar{h},\bar{f}$ act on the subspace of antiholomorphic functions. After comparison (2.8) with (2.5) one concludes that $V_\nu=V_{\nu,0}$. The invariant integral (2.7) coincides with the hermitian form (2.4) on the irreducible subspace.

The Casimir operators in the horospheric coordinates take the form

$$\Omega=\frac{1}{4}H^2\partial_H^2+\frac{3}{4}H\partial_H+H^{-2}\partial_{z\bar{z}}^2,\quad \tilde{\Omega}=\bar{\Omega}.$$

According to (2.6) they are scalar operators on V_ν

$$\Omega|_{V_\nu}=-\frac{1}{4}(\nu^2+1)\mathrm{Id}.$$

4. Whittaker functions. Let $U_{\nu,\mu}=\{f_\nu(\bar{z},H,z)\}$ be the space of smooth functions on $\mathbf{L}$ satisfying the following conditions:
(i) $\Omega f_\nu(\bar{z},H,z)=-\frac{1}{4}(\nu^2+1)f_\nu(\bar{z},H,z)$;
(ii) $\partial_z f(\bar{z},H,z)=i\mu f(\bar{z},H,z)$, $\partial_{\bar{z}}f(\bar{z},H,z)=i\mu f(\bar{z},H,z)$.

The last condition means that

$$f_\nu(\bar{z},H,z)=\exp i\mu(z+\bar{z})F_\nu(H).$$

Substituting this expression in the equation (i) one obtains

$$(\frac{1}{4}H^2\partial_H^2+\frac{3}{4}H\partial_H-H^{-2}\mu^2)F_\nu(H)=-\frac{1}{4}(\nu^2+1)F_\nu(H). \tag{2.9}$$

Two independent solutions of this equation are modified Bessel functions

$$H^{-1}I_{\pm i\nu}(2\mu H^{-1}).$$

The bounded solution is the Bessel-Macdonald function

$$H^{-1}K_{i\nu}(2\mu H^{-1})=H^{-1}\frac{\pi(I_{-i\nu}(2\mu H^{-1})-I_{i\nu}(2\mu H^{-1}))}{2i\sinh\nu}. \tag{2.10}$$

The bounded solution (2.10) can be represented as a special matrix element of the operator

$$g(T^3|\varphi)=\exp 2(T^3\otimes\varphi)\in\mathcal{U}_{\mathrm{SL}}(2,\mathbb{C})\otimes\mathcal{A}_{\mathbf{L}},H=e^\varphi,$$

which is a special group element defined in the tensor product of the universal enveloping algebra $\mathcal{U}(\mathrm{SL}(2,\mathbb{C}))$ and the group algebra $\mathcal{A}_{\mathbf{L}}$. It acts on the irreducible space V_ν. Define two vectors $\psi_L,\ \psi_R$ such that

$$T^+\psi_R=i\mu\psi_R,\quad \bar{T}^+\psi_R=i\mu\psi_R,\quad (T^3-\bar{T}^3)\psi_L=0.$$

It means that ψ_R is the eigen-vector of the nilpotent subgroup N, and ψ_L is the $SU(2)$-invariant vector. The state ψ_R is called *the Whittaker vector.* Then it follows from (2.5) that

$$\psi_R = \exp i\mu(z+\bar{z}), \quad \psi_L = \frac{1}{(1+|z|^2)^{1-i\nu}}.$$

In fact, ψ_R does not lie in V_ν, but rather it is a distribution over some subspace of V_ν. Nevertheless, the integral $< \psi_L|g(T^3|\varphi)|\psi_R >$ is well defined. This matrix element satisfies (2.9) and

$$< \psi_L|g(T^3|\varphi)|\psi_R >= \exp((i\nu+1)\varphi)\times \tag{2.11}$$
$$\times \int\int d\bar{z}dz \frac{\exp(i\mu(z+\bar{z}))}{(1+e^{2\varphi}|z|^2)^{1+i\nu}} \sim H^{-1}K_{i\nu}(2\mu H^{-1}).$$

In this formula the angular integration in the polar coordinates allows to rewrite it as

$$H^{-1}K_{i\nu}(2\mu H^{-1}) == \Gamma(1+i\nu)H^{-i\nu-1}\int_0^\infty \frac{J_0(2\mu\rho)\rho}{(H^{-2}+\rho^2)^{1+i\nu}}d\rho.$$

This construction can be generalized to an arbitrary splitted Lie group. The bounded solution is called *the Whittaker function.*

3 Quantum Lorentz group

1. General construction. Deformations of the group algebra $\mathcal{A}(\mathrm{SL}(2,\mathbb{C}))$ based on the Iwasawa and the Gauss decompositions were considered in [4] and [10]. Later three parameter deformations were found in [11].

We start from a pair of the standard $\mathcal{U}_q(\mathrm{SL}(2,\mathbb{R}))$ Hopf algebra. The first one is generated by A, B, C, D and the unit with the relations

$$AD = DA = 1,\ AB = qBA,\ BD = qDB,$$
$$AC = q^{-1}CA,\ CD = q^{-1}DC, \tag{3.1}$$
$$[B,C] = \frac{1}{q-q^{-1}}(A^2-D^2).$$

It is a Hopf algebra where the coproduct is defined as

$$\Delta(A) = A\otimes A,\ \Delta(D) = D\otimes D,$$
$$\Delta(B) = A\otimes B + B\otimes D,$$
$$\Delta(C) = A\otimes C + C\otimes D,$$

with the counit

$$\varepsilon\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix},$$

and the antipode

$$S\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \begin{pmatrix} D & -q^{-1}B \\ -qC & A\end{pmatrix}.$$

There is a copy of this algebra $\mathcal{U}_q^*(\mathrm{SL}_2)$ generated by A^*, B^*, C^*, D^* with the relations following from (3.1). The star generators commute with A, B, C, D. The coproduct for the

star generators is determined by $\Delta(X^*) = (\Delta(X))^*$ and the antipode is extracted from the rule $S\circ * \circ S \circ * = id$. The pair of two independent copies $\mathcal{U}_q(\mathrm{SL}(2,\mathbb{R}))$ and $\mathcal{U}_q^*(\mathrm{SL}(2,\mathbb{R}))$, which commute and cocommute, defines the untwisted version of the quantum Lorentz algebra $\mathcal{U}_q(\mathrm{SL}(2,\mathbb{C}))$. Note that this form differs from the Iwasawa [4] and the Gauss [10] constructions.

The $\mathcal{U}_q(\mathrm{SU}(2))$ subalgebra [13] is defined as

$$\mathcal{U}_q(SU_2) = \{B^* = C,\ \ C^* = B,\ \ A^* = A\}.$$

There are two Casimir elements in $\mathcal{U}_q^{(r,s)}(\mathrm{SL}(2,\mathbb{C}))$ which commute with any $u \in \mathcal{U}_q^{r,s}(\mathrm{SL}(2,\mathbb{C}))$.

$$\Omega_q := \frac{(q^{-1}+q)(A^2+A^{-2})-4}{2(q^{-1}-q)^2} + \frac{1}{2}(BC+CB),$$
$$\tilde{\Omega}_q = \Omega_q^*. \tag{3.2}$$

To go further we following V.Drinfeld [12] introduce depending on two parameters twist in the form [9]. Let $A = q^K$, $A^* = q^{K^*}$ and

$$\mathcal{F}(r,s) = q^{rK^*\otimes K + K\otimes sK^*} \in \mathcal{U}_q(\mathrm{SL}(2,\mathbb{R}))\otimes\mathcal{U}_q^*(\mathrm{SL}(2,\mathbb{R})).$$

The element $\mathcal{F}(r,s)$ is the so-called left twisting because it satisfies

$$(\mathcal{F}\otimes 1)(\Delta\otimes id)\mathcal{F} = (1\otimes\mathcal{F})(id\otimes\Delta)\mathcal{F},$$
$$(\varepsilon\otimes id)\mathcal{F} = 1 = (id\otimes\varepsilon)\mathcal{F}.$$

These properties allow to generate a new coproduct from the old one:

$$\Delta^{\mathcal{F}} = \mathcal{F}\Delta\mathcal{F}^{-1}.$$

In our case

$$\Delta^{\mathcal{F}}(A) = \Delta(A) = A\otimes A,$$
$$\Delta^{\mathcal{F}}(B) = (A^*)^{-r}A\otimes B + B\otimes D(A^*)^s, \tag{3.3}$$
$$\Delta^{\mathcal{F}}(C) = (A^*)^{r}A\otimes C + C\otimes D(A^*)^{-s}.$$

The antipode is changed in a consistent way with the new coproduct.

$$m(S^{\mathcal{F}}\otimes id)\Delta^{\mathcal{F}}(u) = m(id\otimes S^{\mathcal{F}})\Delta^{\mathcal{F}}(u) = \varepsilon(u),$$

where m is the multiplication. So we have

$$S^{\mathcal{F}}\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \begin{pmatrix} D & -q^{-1}(A^*)^{r-s}B \\ -q(A^*)^{s-r}C & A\end{pmatrix}.$$

In this way we obtain the twisted algebra $\mathcal{U}_q^{\mathcal{F}}(\mathrm{SL}(2,\mathbb{R}))$. The algebra $(\mathcal{U}_q^*)^{\mathcal{F}}(\mathrm{SL}(2,\mathbb{R}))$ with $\mathcal{F}$ depending on the same pair (r,s) can be constructed in the similar way.

$$\Delta^{\mathcal{F}}(A^*) = \Delta(A^*) = A^*\otimes A^*,$$
$$\Delta^{\mathcal{F}}(B^*) = A^{-r}A^*\otimes B^* + B^*\otimes D^*A^s, \tag{3.4}$$

$$\Delta^{\mathcal{F}}(C^*) = A^r A^* \otimes C^* + C^* \otimes D^* A^{-s}.$$

$$S^{\mathcal{F}}\begin{pmatrix} A^* & B^* \\ C^* & D^* \end{pmatrix} = \begin{pmatrix} D^* & -\bar{q}A^{r-s}B^* \\ -\bar{q}^{-1}A^{s-r}C^* & A^* \end{pmatrix}.$$

The pair $\mathcal{U}_q^{\mathcal{F}}(\mathrm{SL}(2,\mathbb{R}))$ and $(\mathcal{U}_q^*)^{\mathcal{F}}(\mathrm{SL}(2,\mathbb{R}))$ forms the family of commutative, noncocommutative quantum Lorentz algebras $\mathcal{U}_q^{(r,s)}(\mathrm{SL}(2,\mathbb{C}))$. In particular, for $r=1,\ s=1$ its dual object is just the Gauss form of the quantum Lorentz algebra [10].

2. Quantum Lobachevsky space. The quantum Lobachevsky space $\mathbf{L}_{\kappa,q}$ is the associative $*$-algebra over $\mathbb{C}$ with a unity and three generators

$$(z^*, H, z),\quad H^* = H,\ (z)^* = z^*,\ 1^* = 1.$$

The commutation relations depend on two parameters $\kappa, q \in \mathbb{C}$

$$zH = \kappa H z, \qquad z^*H = \bar{\kappa}^{-1}Hz^*, \tag{3.5}$$

$$zz^* = az^*z - bH^{-2},\quad a = \left(\frac{\kappa}{q}\right)^2,\quad b = \bar{\kappa}^{-1}\left(\frac{\kappa}{q}\right)^2\left(1 - q^2\frac{\bar{\kappa}}{\kappa}\right) \tag{3.6}$$

The both parameters should be tune in a such a way that the coefficients a, b become real. It is happened in two cases
(i) κ and q are real,
(ii) $|q| = 1$ and $\mathrm{Arg}(q) = \mathrm{Arg}(\kappa)$.

Proposition 3.1 *$\mathbf{L}_{\kappa,q}$ is a right $\mathcal{U}_q^{(0,s)}(\mathrm{SL}(2,\mathbb{C}))$-module.*

Proof.
Define the right action of $\mathcal{U}_q^{(0,s)}(\mathrm{SL}(2,\mathbb{C}))$ on $\mathbf{L}_{\kappa,q}$:

$$\begin{gathered}
z^*.A = z^*,\ H.A = q^{\frac{1}{2}}H,\ z.A = q^{-1}z,\\
z^*.A^* = \left(\frac{\kappa}{q}\right)^{\frac{2}{s}} z^*,\ H.A^* = \left(\frac{q}{\kappa}\right)^{\frac{1}{s}} H,\ z.A^* = z,\\
z^*.B = 0,\quad H.B = 0\ \ z.B = q^{-\frac{1}{2}}\\
z^*.C = q^{\frac{3}{2}}\kappa^{-1}H^{-2},\ H.C = Hz,\ z.C = -q^{\frac{1}{2}}z^2,\\
x.B^* = 0,\quad x.C^* = 0\ \text{ for any } x \in \mathbf{L}_{\kappa,q}.
\end{gathered} \tag{3.7}$$

Similarly, the left action of A^*, A, B^*, C^* takes the form

$$\begin{gathered}
A^*.z^* = \bar{q}^{-1}z^*,\ A^*.H = \bar{q}^{\frac{1}{2}}H,\ A^*.z = z,\\
A.z^* = z^*,\ A.H = \left(\frac{\bar{q}}{\bar{\kappa}}\right)^{\frac{1}{s}} H,\ A.z = \left(\frac{\bar{\kappa}}{\bar{q}}\right)^{\frac{2}{s}} z\\
B^*.z^* = \bar{q}^{-\frac{1}{2}},\ B^*.H = 0,\ B^*z = 0,\\
C^*.z^* = -\bar{q}^{\frac{1}{2}}(z^*)^2,\ C^*.H = z^*H,\\
C^*.z = \bar{q}^{\frac{3}{2}}\bar{\kappa}^{-1}H^{-2}.
\end{gathered} \tag{3.8}$$

The coproduct in $\mathcal{U}_q^{(0,s)}(\mathrm{SL}(2,\mathbb{C}))$ (3.3),(3.4) is compatible with the commutation rules in $\mathbf{L}_{\kappa,q}$ (3.5), (3.6). Similar, the $*$-structures in $\mathbf{L}_{\kappa,q}$ and in $\mathcal{U}_q^{(0,s)}(\mathrm{SL}(2,\mathbb{C}))$ are also compatible. It means that (3.7) and (3.8) are related as

$$(x.K)^* = K^*.x^*.$$

for $K = A, B, C$ and $x = z^*, H, z$. $\square$

To eliminate the ambiguities related to the noncommutativity of $\mathbf{L}_{\kappa,q}$ we consider only the ordered monomials putting z^* on the left side, z on the right side and keeping H in the middle of the monomials:

$$: f(z^*, H, z) := \sum_{m,k,n} c_{m,k,n} w(m,k,n),$$

$$w(m,k,n) = z^{*m}H^k z^n.$$

The actions (3.7), (3.8) on the ordered monomials $w(m,k,n)$ take the form

$$\begin{gathered}
w(m,k,n).A = q^{-n+\frac{k}{2}}w(m,k,n),\\
w(m,k,n).A^* = \left(\frac{q}{\kappa}\right)^{(-2m+k)/s} w(m,k,n),\\
w(m,k,n).B = q^{-n+\frac{k+1}{2}}\frac{1-q^{2n}}{1-q^2}w(m,k,n-1),\\
w(m,k,n).C =\\
= q^{n-\frac{3(k-1)}{2}}\kappa^{k-1}\frac{1-q^{2m}}{1-q^2}w(m-1,k-2,n)-\\
-q^{-n+\frac{k+3}{2}}\frac{1-q^{2n-2k}}{1-q^2}w(m,k,n+1),
\end{gathered} \tag{3.9}$$

$$\begin{gathered}
A^*.w(m,k,n) = \bar{q}^{-m+\frac{k}{2}}w(m,k,n),\\
A.w(m,k,n) = \left(\frac{\bar{q}}{\bar{\kappa}}\right)^{(-2n+k)/s} w(m,k,n),\\
B^*.w(m,k,n) = \bar{q}^{-m+\frac{k+1}{2}}\frac{1-\bar{q}^{2m}}{1-\bar{q}^2}w(m-1,k,n),\\
C^*.w(m,k,n) =\\
= \bar{q}^{m-\frac{3(k-1)}{2}}\bar{\kappa}^{k-1}\frac{1-\bar{q}^{2n}}{1-\bar{q}^2}w(m,k-2,n-1)-\\
-\bar{q}^{-m+\frac{k+3}{2}}\frac{1-\bar{q}^{2m-2k}}{1-\bar{q}^2}w(m+1,k,n)
\end{gathered} \tag{3.10}$$

and

$$\begin{gathered}
w(m,k,n).\Omega_q = q^{-k+1}\frac{(1-q^{k+1})^2}{(1-q^2)^2}w(m,k,n)+\\
+\left(\frac{\kappa}{q}\right)^{k-1}\frac{(1-q^{2m})(1-q^{2n})}{(1-q^2)^2}w(m-1,k-2,n-1).
\end{gathered}$$

4 Principle series of unitary representations

The principle series representations for quantum Lorentz groups were considered in [14, 15]. Here we present the formulae for principle series operators inspired by the action of operators on the quantum Lobachevsky spaces. In what follows we forget the coalgebraic structure and for this reason use the notation $\mathcal{U}_q(\mathrm{SL}(2,\mathbb{C}))$.

Consider the same space $V_{\nu,n}$ as in the classical case (2.1). But now we introduce another Hermitian form

$$< f|g >= q^{\frac{i\nu+n-1}{2}}\bar{q}^{\frac{i\nu-n+1}{2}}\int\int \overline{f(\bar{q}^{\frac{i\nu-n+1}{2}}\bar{z}, q^{\frac{i\nu+n-1}{2}}z)}g(\bar{q}^{\frac{i\nu-n+1}{2}}\bar{z}, q^{\frac{i\nu+n-1}{2}}z)d\bar{z}dz. \tag{4.1}$$

Define the action π_ν of $\mathcal{U}_q(\mathrm{SL}(2,\mathbb{C}))$ on $V_{\nu,n}$. We assume that $\mathcal{U}_q(\mathrm{SL}_2)$ acts only on the holomorphic part of $V_{\nu,n}$, while $\mathcal{U}_q^*(\mathrm{SL}_2)$ acts on the antiholomorphic part. Following our notations in Section 3 we write the holomorphic action from the right hand side, while the antiholomorphic action from the left hand side. The right action takes the form

$$\begin{aligned} f(z).\pi_\nu(A) &= q^{\frac{i\nu+n-1}{2}}f(q^{-1}z),\\ f(z).\pi_\nu(B) &= q^{\frac{i\nu+n}{2}}\frac{f(q^{-1}z)-f(qz)}{1-q^2}z^{-1},\\ f(z).\pi_\nu(C) &= \frac{z}{1-q^2}[-q^{\frac{i\nu+n+2}{2}}f(q^{-1}z)+q^{-\frac{3(i\nu+n)}{2}+3}f(qz)]. \end{aligned} \tag{4.2}$$

Similarly, for the left action we have

$$\begin{aligned} \pi_\nu(A^*).f(\bar{z}) &= \bar{q}^{\frac{1}{2}(i\nu-n-1)}f(\bar{q}^{-1}\bar{z}),\\ \pi_\nu(B^*).f(\bar{z}) &= \bar{q}^{\frac{i\nu-n}{2}}\frac{f(\bar{q}^{-1}\bar{z})-f(\bar{q}\bar{z})}{1-\bar{q}^2}\bar{z}^{-1},\\ \pi_\nu(C^*).f(\bar{z}) &= \frac{\bar{z}}{1-\bar{q}^2}[-\bar{q}^{\frac{i\nu-n+2}{2}}f(\bar{q}^{-1}\bar{z})+\bar{q}^{-\frac{3(i\nu-n)}{2}+3}f(\bar{q}\bar{z})]. \end{aligned} \tag{4.3}$$

Proposition 4.1 *(i) The operators π_ν (4.2), (4.3) are irreducible representations of $\mathcal{U}_q(\mathrm{SL}(2,\mathbb{C}))$;*
(ii) for $\nu \in \mathbb{R}$ they are unitarity with respect to the form (4.1):

$$< g|f.\pi_\nu(u) >=< \pi_\nu(S(u^*)).g|f >,$$

$$g, f \in V_{\nu,n}, \quad u \in \mathcal{U}_q(\mathrm{SL}(2,\mathbb{C})).$$

Proof. The direct calculations show that the operators $\pi_\nu(u)$ satisfy the same commutation relations as u (3.1). Moreover, the representation actions π_ν preserves the asymptotic (2.2).

Another way to see that π_ν satisfy the commutation relations for the case $n = 0$ is to consider the subspace $\tilde{V}_\nu$ of $\mathbf{L}_{\kappa,q}$ of the form

$$f(z^*, cH, z) = c^{i\nu-1}f(z^*, H, z).$$

In other words, the basis functions in $\tilde{V}_\nu$ are $w(m, i\nu-1, n) = (z^*)^m H^{i\nu-1}z^n$. Starting from (3.9), (3.10) we define the action of $\mathcal{U}_q^{(0,s)}(\mathrm{SL}(2,\mathbb{C}))$ on $\tilde{V}_\nu$ in a such way that the

right actions of $\pi_\nu(A,B,C)$ do not touch z^*-variable ($m = 0$ in (3.9)). Similarly, the left acting operators $\pi_\nu(A^*,B^*,C^*)$ do not touch z-variable ($n = 0$ in (3.10)). Then $\tilde{V}_\nu$ is invariant with respect to π_ν. Since the dependence on H is fixed one can put $H = 1$. Then we come to the representation (4.2) and (4.3), where V_ν is defined as $V_\nu = \tilde{V}_\nu|_{H=1}$.

Consider the action of the Casimirs (3.2) on V_ν. They become the scalar operators

$$\pi_\nu(\Omega) = ([-\frac{i\nu+n}{2}]_q)^2\mathrm{Id}, \quad \pi_\nu(\Omega^*) = ([-\frac{i\nu-n}{2}]_{\bar{q}})^2\mathrm{Id}.$$

In this sense the representations π_ν are irreducible.

The unitarity is proved by direct calculations. □

In the classical limit we come to the principle series representations of $\mathrm{SL}(2,\mathbb{C})$ since (2.5)

$$\lim_{q\to1}\pi_\nu(B) = T^+, \quad \lim_{q\to1}\pi_\nu(C) = T^+, \quad \lim_{q\to1}\pi_\nu(A) = (1+T^3),$$

and the Hermitian form (4.1) takes the classical form (2.4).

It follows from (4.2) that ν is defined up to a constant. Two representations (ν, n) and (ν', n) are equivalent if

$$\nu' = \nu + \frac{4\pi m}{\hbar}, \quad m \in \mathbb{Z}, \quad \hbar = \ln q,$$

and we assume that $\nu \in \mathbb{R}$, $0 \le \nu \le \frac{4\pi}{\hbar}$.

The case $n = 0$ corresponds to the class one representations $V_{\nu,0} = V_\nu$. These representations are related to the quantum Lobachevsky space. Moreover, as in the classical limit there exists a $\mathcal{U}_q(\mathrm{SU}(2))$ invariant state in V_ν. It will be found in Section 6.

5 The q^2-Bessel functions

We take here q and κ to be real, and put $\kappa = q^\delta$.

Consider the eigenvalue problem for the Casimir (3.2) acting on $\mathbf{L}_{\kappa,q}$

$$f_\nu^\delta(z^*, H, z).\Omega_q^\nu = ([\frac{i\nu}{2}]_q)^2 f_\nu^\delta(z^*, H, z).$$

Thereby, $f_\nu^{(\delta)}(z^*, H, z) \in V_\nu$. Assume

$$\Omega_q = \Omega_q - q^{-i\nu+2}\frac{(1-q^{i\nu})^2}{(1-q^2)^2}.$$

We seek solutions of the equation $f_\nu^{(\delta)}(z^*, H, z).\Omega_q^\nu = 0$, where $f_\nu^{(\delta)}(z^*, H, z)$ takes the form

$$f^{(\delta)}(z^*, H, z) = e_{q^2}(i\mu(1-q^2)z^*)F_\nu^{(\delta)}(H)e_{q^2}(i\mu(1-q^2)z),$$

and e_{q^2} is the q^2-exponential $e_{q^2}(x) = \frac{1}{(x,q^2)_\infty}$.

It follows from (3.9) and properties of q^2-exponential that $F_\nu^{(\delta)}(H)$ is the solution of the difference equation

$$\frac{qF_\nu^{(\delta)}(qH) - (q^{i\nu}+q^{-i\nu})F_\nu^{(\delta)}(H) + q^{-1}F_\nu^{(\delta)}(q^{-1}H)}{(1-q^2)^2} - \mu^2 q^{-\delta-1}H^{-2}F_\nu^{(\delta)}(q^{\delta-1}H) = 0. \tag{5.1}$$

Substitution $F_\nu^{(\delta)}(H) = H^{-1}\mathcal{F}_\nu^{(\delta)}(H^{-1})$ and $H^{-1} = x$ leads to the difference equation for $\mathcal{F}_\nu^{(\delta)}(x)$

$$\mathcal{F}_\nu^{(\delta)}(q^{-1}x) - (q^{i\nu} + q^{-i\nu})\mathcal{F}_\nu^{(\delta)}(x) + \mathcal{F}_\nu^{(\delta)}(qx) - \\ -\mu^2 q^{-2\delta}(1-q^2)^2 x^2 \mathcal{F}_\nu^{(\delta)}(q^{1-\delta}x) = 0 \tag{5.2}$$

Define the second order difference operator

$$\Delta_q\psi(x) = \frac{\psi(qx) - 2\psi(x) + \psi(q^{-1}x)}{(q-q^{-1})^2}.$$

Then (5.2) takes the form

$$\Delta_q \mathcal{F}_\nu^{(\delta)}(x) - \mu^2 q^{-2\delta+2} x^2 \mathcal{F}_\nu^{(\delta)}(q^{1-\delta}x) = ([\frac{i\nu}{2}]_q)^2 \mathcal{F}_\nu^{(\delta)}(x).$$

It is just the discrete analog of the quantum Liouville equation corresponding to the two-body case of the quantum open relativistic Toda lattices [?] with depending on the parameter δ Hamiltonians

$$\hat{H} = -\Delta_q + \mu^2 q^{-2\delta+2} x^2 (T_q)^{1-\delta},$$

where T_q is the shift operator $T_q\psi(x) = \psi(qx)$. The wave-functions of these systems are among the solutions of (5.2). It follows from (5.2) that the second ordered difference equations arise only for three values $\delta = 0, 1, 2$. The other cases lead to higher ordered difference equations. In what follows we consider only these three cases.

The solutions of (5.2) are the modified q^2-Bessel functions [5]

$$J_{i\nu}^{(j)}(2i\mu(1-q^2)q^{-\frac{1}{2}\delta}x; q^2) = \frac{q^{-\frac{1}{2}i\delta\nu}}{\Gamma_{q^2}(i\nu+1)} \times$$

$$\times \sum_{k=0}^{\infty} \frac{(-1)^k q^{(2-\delta)k(k+i\nu)-k\delta}(1-q^2)^{2k}(i\mu x)^{i\nu+2k}}{(q^2,q^2)_k (q^{2i\nu+2},q^2)_k}.$$

Here

$$j = \begin{cases} 1 & \text{for} \quad \delta = 2 \\ 2 & \text{for} \quad \delta = 0 \\ 3 & \text{for} \quad \delta = 1. \end{cases} \tag{5.3}$$

Three values of j correspond to the known functions [5, 6, 7, 16]:

$$I_{i\nu}^{(j)}(2\mu(1-q^2)q^{-\frac{1}{2}\delta}x; q^2) = e^{\frac{1}{2}\nu\pi} J_{i\nu}^{(j)}(2i\mu(1-q^2)q^{-\frac{1}{2}\delta}x; q^2) -$$

$$- \begin{cases} \text{modified } q^2\text{–Bessel function of type } 1, j = 1 \\ \text{modified } q^2\text{–Hahn} - \text{Exton function}, j = 3 \\ \text{modified } q^2\text{–Bessel function of type } 2, j = 2 \end{cases}$$

Obviously $I_{-i\nu}^{(j)}(2\mu(1-q^2)q^{-\frac{1}{2}\delta}x; q^2)$ is the solution of (5.2) also. The corresponding Wronskians are not vanishing. Thus, the modified q^2-Bessel functions $I_{i\nu}^{(j)}$, $I_{-i\nu}^{(j)}$ form the fundamental system. Though they are unbounded functions, and our goal to find their linear combination providing the bounded solution as in the classical case (2.11).

6 The q-Whittaker functions

In this Section we assume as before that $\kappa = q^\delta$ and q are real.

Consider an analog of the group element defined on $\mathcal{U}_q(\mathrm{SL}(2,\mathbb{C})) \bigotimes \mathbf{L}_{\kappa,q}$. [1]

$$g(A|H) = e^{\mathcal{K}\otimes 2\phi}, \; A = e^{\hbar\mathcal{K}}, \; H = e^\phi, \; \hbar = \ln q.$$

The action of $\pi_\nu(g(A|H))$ in V_ν (see (4.2))

$$\pi_\nu(g(A|H)) : f(\bar{z}, z) \to H^{i\nu-1} f(\bar{z}, H^{-2}z).$$

We rewrite the Hermitian form (4.1) as

$$< f|g >= q^{i\nu+n-1} \int\int \overline{f(\bar{z},z)} g(q^{i\nu-n+1}\bar{z}, q^{i\nu+n-1}z) d\bar{z}dz. \tag{6.1}$$

By means of (6.1) we define the matrix element of $\pi_\nu(g(A,H))$ in V_ν

$$F_\nu^{(\delta)}(H) =< \psi_L|\psi_R.g(A|H) >=< S(g(A^*|H)).\psi_L|\psi_R >, \tag{6.2}$$

We drop here and in what follows the notion of the representation π_ν.

Let $\psi_L, \ \psi_R \in V_\nu$ satisfy the following conditions

$$C^*.\psi_L(\bar{z},z) = \psi_L(\bar{z},z).B$$

$$B^*.\psi_R.B = -\bar{\mu}\mu q^{1-\delta}\psi_R.A^{2(\delta-1)}.$$

In other words, from (4.2), (4.3)

$$B^*.\psi_R(\bar{z},z).B = -\bar{\mu}\mu q^{(i\nu-2)(\delta-1)}\psi_R(\bar{z}, q^{2-2\delta}z).$$

Thus ψ_R can be considered as the *quantum Whittaker vector* for $\mathcal{U}_q(\mathrm{SL}(2,\mathbb{C}))$.

Proposition 6.1 *$F_\nu^{(\delta)}(H)$ satisfies (5.1)*

Proposition 6.2 *The matrix element $F_\nu^{(\delta)}(H) =< \psi_L|g(A,H)|\psi_R >$ (6.2) has the form of the double integral*

$$F_\nu^{(\delta)}(H) = q^{i\nu-1}H^{i\nu-1} \int\int \frac{(-q^{2i\nu+4}\bar{z}z, q^2)_\infty}{(-q^2\bar{z}z, q^2)_\infty} \xi_1(q^{i\nu+1}z^*)\xi_2(q^{i\nu-1}H^{-2}z) dz dz^*, \tag{6.3}$$

where

$$\xi_1(\bar{z}) = {}_1\Phi_1(0; -q; q, -i\mu(1-q^2)q^{1-i\nu}\bar{z}),$$

$$\xi_2(z) = \begin{cases} {}_0\Phi_2(-; 0, -q; q, -i\mu(1-q^2)q^{3-i\nu}z), \delta = 0 \\ {}_1\Phi_1(0; -q; q, -i\mu(1-q^2)qz), \delta = 1 \\ {}_3\Phi_1(0,0,0; -q; q, -i\mu(1-q^2)q^{i\nu-1}z), \delta = 2, \end{cases}$$

and ${}_r\Phi_s$ is the basic hypergeometric series.

[1]The group elements g are defined in the tensor product $\mathcal{U}_q \otimes \mathcal{A}_q$ and satisfy the relation $\Delta g \otimes id = g_{12}g_{23}$ [17, 18].

It is easily to show that the (6.3) is the q-analog of the (2.11). If we rewrite the integral (6.3) in the polar coordinates $z = \rho e^{i\phi}$

$$F_\nu^{(\delta)}(H) = 2q^{i\nu-1}H^{i\nu-1}\int_0^\infty \frac{(-q^{2i\nu+4}\rho^2, q^2)_\infty}{-q^2\rho^2, q^2)_\infty}\rho d\rho \times$$

$$\times \int_{-\pi}^{\pi} \xi_1(q^{i\nu+1}\rho e^{-i\phi})\xi_2(q^{i\nu-1}H^{-2}\rho e^{i\phi})d\phi,$$

and represent the functions ξ_1 and ξ_2 as series, we can calculate the inner integral. So

$$F_\nu^{(\delta)}(H) = 4\pi q^{i\nu-1}H^{i\nu-1}\int_0^\infty \frac{(-q^{2i\nu+4}\rho^2, q^2)_\infty}{(-q^2\rho^2, q^2)_\infty} \times$$

$$\times J_0^{(j)}(2\mu(1-q^2)q^{(i\nu-1)\frac{\delta}{2}+1}H^{-1}\rho; q^2)\rho d\rho.$$

Setting $q^{1+\frac{i\nu\delta}{2}}\rho = r$ we obtain finally

$$F_\nu^{(\delta)}(H) = 4\pi q^{i\nu(\delta+1)}H^{i\nu-1} \times$$

$$\times \int_0^{\infty e^{\frac{i\nu\delta}{2}\ln q}} \frac{(-q^{2i\nu+2-i\nu\delta}r^2, q^2)_\infty}{(-q^{-i\nu\delta}r^2, q^2)_\infty} \times$$

$$\times J_0^{(j)}(2\mu(1-q^2)q^{-\frac{\delta}{2}}H^{-1}r; q^2)rdr =$$

$$= -\frac{8\pi q^{-\nu^2+\frac{3}{2}i\nu\delta}\ln q}{(1-q^2)\Gamma_{q^2}(i\nu+1)A_{i\nu}^{|1-\delta|}}\mu^{i\nu} \times$$

$$\times H^{-1}K_{i\nu}^{(j)}(2\mu(1-q^2)q^{-\frac{\delta}{2}}H^{-1}; q^2),$$

where

$$A_{i\nu} = \sqrt{\frac{I_{i\nu}^{(2)}(2; q^2)}{I_{-i\nu}^{(2)}(2; q^2)}},$$

and $j = 1, 2, 3$ are connected with $\delta = 2, 0, 1$ by relations (5.3) (see [19]). The curve of integration for $\delta = 2$ ($j = 1$) should satisfy the condition $\nu \ln q \neq (k + \frac{1}{2})\pi$.

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Area Preserving Diffeomorphisms and the Dynamics of Nonrelativistic Charged Point Particles on a Plane

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Abstract

We deform the interaction between nonrelativistic point particles on a plane and a Chern-Simons field to obtain an action invariant with respect to time-dependent area-preserving diffeomorphisms. The deformed and undeformed Lagrangians are connected by a point transformation leading to a classical Seiberg-Witten map between the corresponding gauge fields. The Schroedinger equation derived by means of Moyal-Weyl quantization from the effective two-particle interaction exhibits
– a singular metric, leading to a splitting of the plane into an interior (bag-) and an exterior region,
– a singular potential (quantum correction) with singularities located at the origin and at the edge of the bag.
We list some properties of the solutions of the radial Schroedinger equation.

1 Introduction

Two-dimensional incompressible fluids, in particular quantum-hall fluids (QHF's) are well known to be invariant with respect to time-independent area-preserving diffeomorphisms (cp.[1]). In a particle picture a QHF is usually described by neglecting the kinetic energy compared to the magnetic field term leading to noncommutative geometry and a reduced phase space (cp.[2]). In the present paper we generalize this picture by allowing

– the particles to move in full phase space,

- the area-preserving diffeomorphisms to become time-dependent.

To do this we consider a deformation of the interaction of nonrelativistic charged point particles on a plane coupled to a Chern-Simons (CS) field such that

– the deformed action is invariant with respect to time-dependent area-preserving diffeomorphisms $\nu_{2,t}$,

– the deformed and undeformed particle Lagrangians are connected by a point transformation leading to the classical analogue of a Seiberg-Witten (SW) map between the deformed and undeformed gauge fields.

There is a second reason for studying such a model. A theory, unifying translational- and $U(1)$-gauge invariance in 2d contains nine independent gauge fields: six dreibein components and three $U(1)$-gauge fields [3]. Our model shows that the restriction of the

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group of local translations to its subgroup $\nu_{2,t}$ reduces the number of independent gauge fields to three as the dreibeins are built from the deformed U(1)-gauge fields.
We begin by constructing the deformed Lagrangian, consider the equations of motion (EOM) and discuss the properties of the deformed gauge fields. After gauge fixing we solve the nonlinear Gauss-constraint for the two-particle problem and quantize it by means of the Moyal-Weyl prescription. We discuss the structure of the resulting radial Schroedinger equation and list some properties of its solutions. We conclude with some final remarks.

2 Deformed Lagrangian

Infinitesimal elements of $\nu_{2,t}$ are defined by†

$$\delta x_i = -\theta\epsilon_{ij}\partial_j\Lambda(\underline{x},t) \quad , \quad i,j=1,2 \tag{1}$$

at fixed time t, where Λ is an infinitesimal gauge function and θ is a finite deformation parameter. The corresponding change of a field $f(\underline{x},t)$ is defined by

$$\delta_0 f(\underline{x},t) := f'(\underline{x},t) - f(\underline{x},t) \tag{2}$$

or, if we include the coordinate change, we define

$$\delta f(\underline{x},t) := f'(\underline{x}',t) - f(\underline{x},t) \tag{3}$$

Now, we want to deform

$$L_{\text{part}} := \frac{1}{2}\dot{x}_i^2 + e(A_i(\underline{x},t)\dot{x}_i + A_0(\underline{x},t)) \tag{4}$$

in such a way that $\hat{L}_{part}$ becomes quasi-invariant with respect to (1) where $\hat{A}_\mu$ transform as‡

$$\delta\hat{A}_\mu(\underline{x},t) = \partial_\mu\Lambda(\underline{x},t) \tag{5}$$

or, equivalently

$$\delta_0\hat{A}_\mu(\underline{x},t) = \partial_\mu\Lambda(\underline{x},t) + \theta\epsilon_{ij}\partial_i\hat{A}_\mu\partial_j\Lambda \tag{6}$$

ie. the gauge transformations of the $\hat{A}_\mu$ fields mix local $U(1)$-transformations with space transformations (1) (cp. [5])§.

In order to deform the first term in (4) we define invariant coordinates η_i (cp. [8])

$$\eta_i(\underline{x},t) := x_i + \theta\epsilon_{ij}\hat{A}_j(\underline{x},t). \tag{7}$$

Obviously, we have

$$\delta\eta_i = 0\ . \tag{8}$$

Then, by means of the invariant velocity ξ_i

$$\xi_i := \frac{d}{dt}\eta_i(\underline{x},t), \tag{9}$$

†Such transformations are used also in [4].
‡Deformed quantities are marked by a hat.
§The idea of mixing coordinate transformations with gauge transformations has been developed in [6] and extended to noncommutative space in [7].

we define $\hat{E}_{kin}$

$$\hat{E}_{kin} := \frac{1}{2}\xi_i^2 . \tag{10}$$

The deformation of the interaction term in (4) is more involved. It is given by

$$e(A_i\dot{x}_i + A_0) \rightarrow e(\hat{A}_i\dot{x}_i + \hat{A}_0) + \frac{e\theta}{2}\epsilon_{ij}\hat{A}_i\frac{d}{dt}\hat{A}_j =: \hat{L}_{int}, \tag{11}$$

where an additional CS-term is needed because $\delta\dot{x}_i \neq 0$ and because we want $\hat{L}_{int}$ to be quasi-invariant:

$$\delta\hat{L}_{int} = e\frac{d}{dt}(\Lambda - \frac{\theta}{2}\epsilon_{ij}\hat{A}_i\partial_j\Lambda). \tag{12}$$

To do this it is advantageous to replace $\hat{L}_{part}$ by its first-order form

$$\begin{aligned}\hat{L}_{\rm part} &= \dot{x}_k(\xi_k + e\hat{A}_k + \theta\epsilon_{ij}(\xi_i + \frac{e}{2}\hat{A}_i)\partial_k\hat{A}_j) - \frac{1}{2}\xi_i\xi_i \\ &\quad +\theta\epsilon_{ij}(\partial_t\hat{A}_j)(\xi_i + \frac{e}{2}\hat{A}_i) + e\hat{A}_0.\end{aligned} \tag{13}$$

By varying the action $\hat{S}_{part}$ with respect to ξ_i and x_i we get (9) as a constraint

$$\xi_i = \dot{x}_i + \theta\epsilon_{ij}\frac{d}{dt}\hat{A}_j \tag{14}$$

and the invariant nonlinear Lorentz-force equation

$$\dot{\xi}_k = \frac{e}{1+\theta\hat{F}}(\epsilon_{kl}\xi_l\hat{F} + \hat{F}_{ko}) \tag{15}$$

with the invariant field strength $\hat{F}_{\mu\nu}$

$$\hat{F}_{\mu\nu} := \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu + \theta\epsilon_{ik}(\partial_i\hat{A}_\mu)\partial_k\hat{A}_\nu \tag{16}$$

and

$$\hat{F}_{ij} = \epsilon_{ij}\hat{F} . \tag{17}$$

We note that in the particular case of a constant external magnetic field B, ie. for $\hat{F} = B$, (14) and (15) are equivalent to the EOM given by Duval and Horvathy [9] for a particle which possesses a nonvanishing second central charge k of the planar Galilei group [10] with $ek = -\theta$.
Finally, the CS-interaction of the $\hat{A}_\mu$ field invariant with respect to the gauge transformations (5), is given by [5]

$$\hat{L}_{CS} = \frac{\kappa}{2}\int d^2x\epsilon^{\mu\nu\rho}\hat{A}_\mu(\partial_\nu\hat{A}_\rho + \frac{\theta}{3}\epsilon_{ik}(\partial_i\hat{A}_\nu)(\partial_k\hat{A}_\rho)). \tag{18}$$

3 Deformed gauge fields

Usually the invariant velocity ξ_i is defined in terms of nonrelativistic dreibein fields E_μ^ν $(\mu,\nu = 0,1,2)$¶ [8]

$$\xi_i = E_k^i\dot{x}_k + E_0^i . \tag{19}$$

¶The space indices can be taken equivalently as lower or upper indices.

Comparing (19) with (14) and considering time as fixed leads to dreibeins expressed in terms of gauge fields $\hat{A}_k$

$$E_\mu^a := \theta\epsilon_{ak}\partial_\mu\hat{A}_k + \delta_{a\mu} \tag{20}$$

$$E_0^0 = 1 \quad \text{and} \quad E_k^0 = 0 ,$$

which, due to (5), transform covariantly with respect to $\nu_{2,t}$. Note that in the case of arbitrary local translations as an invariance group, dreibeins and the $\hat{A}_\mu$ are independent of each other [3]. Only the restriction to the subgroup $\nu_{2,t}$ allows the relation (20).
From the transformation law (6) we infer that the $\hat{A}_\mu$ are the gauge fields of the classical limit of a noncommutative $U(1)$-gauge theory. This raises the question of a possible Seiberg-Witten (SW) map [9] between the deformed and undeformed gauge fields $\hat{A}_\mu$ and A_μ. For this we consider the point transformation

$$x_i \rightarrow \eta_i(\underline{x}, t) \tag{21}$$

and redefine the gauge fields

$$\hat{A}_\mu(\underline{x}, t) \rightarrow A_\mu(\underline{\eta}, t) \tag{22}$$

so that

$$\hat{L}_{\rm part}(\hat{A}_\mu(\underline{x}, t), \dot{x}_i) = L_{\rm part}(A_\mu(\underline{\eta}, t), \dot{\eta}_i) \tag{23}$$

(21)-(23) defines the classical analogue of an inverse SW-map between gauge fields on a commutative space.‖

This inverse SW-map may be given explicitly in terms of inverse dreibeins $\{e_\mu^\nu\}$

$$A_\nu(\underline{\eta}(\underline{x}, t), t) = \frac{1}{2}\hat{A}_\mu(\underline{x}, t)(\delta_\nu^\mu + e_\nu^\mu(\underline{x}, t)) \tag{24}$$

with, according to (20),

$$\begin{aligned} e_j^i &= \frac{\delta_{ij} - \theta\epsilon_{ik}\partial_k\hat{A}_j}{1+\theta\hat{F}}, \\ e_0^i &= -\theta\epsilon_{jk}\partial_t\hat{A}_k e_j^i, \\ e_0^0 &= 1 \quad \text{and} \quad e_i^0 = 0 . \end{aligned} \tag{25}$$

Solving (24) for $\hat{A}_\mu$ to leading order in θ we obtain

$$\hat{A}_\mu = A_\mu - \frac{\theta}{2}\epsilon_{ik}A_i(\partial_k A_\mu + F_{k\mu}) + 0(\theta^2) \tag{26}$$

in agreement with [12].

‖A different definition of such a classical SW-map has been given in [13].

4 Gauge fixing and residual symmetry

$\hat{A}_0$ is a Lagrange multiplier whose variation in the total action

$$\hat{S} = \hat{S}_{\text{part}} + \hat{S}_{\text{field}} \tag{27}$$

leads to the Gauss-constraint

$$\hat{F}(\underline{x}, t) = -\frac{1}{\kappa} \sum_{\alpha=1}^{N} e_\alpha \delta(\underline{x} - \underline{x}_\alpha). \tag{28}$$

Eq. (28) can be integrated once

$$\hat{A}_k + \frac{\theta}{2} \epsilon_{\ell S} \hat{A}_\ell \partial_k \hat{A}_S = -\frac{1}{2\pi\kappa} \sum_\alpha e_\alpha \partial_k \phi(\underline{x} - \underline{x}_\alpha) + \partial_k \lambda(\underline{x}, t) \tag{29}$$

where

$$\phi(\underline{x}) := \arctan \frac{x_2}{x_1} \tag{30}$$

is a singular gauge function which has to be regularized (cp. [11]) and λ is an arbitrary gauge function to be determined by fixing the asymptotic behaviour of $\hat{A}_\mu$. For that we follow closely the procedure described in [11]:

i) We decompose

$$\hat{A}_\mu = \tilde{A}_\mu + \hat{A}_\mu^{as} \tag{31}$$

with

$$\tilde{A}_\mu \overset{r\to\infty}{\to} 0(r^{-1}). \tag{32}$$

In order to fulfill (32) the $\tilde{A}_i$ should be chosen as solutions of $(29)_{\lambda=0}$.

ii) We require the asymptotically Euclidean metric leading to

$$\hat{A}_j^{as} = \frac{1}{\theta} \epsilon_{jk} a_k(t) \tag{33}$$

ie. the $\hat{A}_j^{as}$ transform covariantly with respect to translations, local in time (residual symmetry). This requirement fixes also $\hat{A}_0^{as}$. Thus our procedure fixes λ (gauge fixing).

iii) We redefine the Lagrangian in terms of the new variables $\{\tilde{A}_\mu, a_i(t)\}$ such that the solutions of the Euler-Lagrange equations minimize the new action.

5 The classical two-particle problem

We consider two identical particles of charge e. Applying the Legendre transformation to our Lagrangian and using the Gauss-constraint (28) we obtain

$$H = \frac{1}{2} \sum_{\alpha=1}^{2} \xi_{i,\alpha}^2 \tag{34}$$

with the constraint

$$\sum_{\alpha=1}^{2} \xi_{i,\alpha} = 0 \tag{35}$$

arising from the variation of $\dot{a}_i(t)$ in the redefined action. In order to express the $\xi_{i,\alpha}$ in terms of canonical variable $\{\underline{x}_\alpha, \underline{p}_\alpha\}$ we need the $\tilde{A}_k$ at the particle positions $\underline{x} = \underline{x}_{\substack{1\\2}}$ to be solutions of

$$\tilde{A}_k + \frac{\theta}{2} \epsilon_{\ell S} \tilde{A}_\ell \partial_k \tilde{A}_S = \frac{e}{2\pi\kappa} \epsilon_{k\ell} \sum_{\alpha=1}^{2} \left(\frac{(x - x_\alpha)_\ell}{|\underline{x} - \underline{x}_\alpha|^2} \right)_{reg} \tag{36}$$

We find in obvious notation

$$\tilde{A}_{\substack{1\\k,2}} = \pm \epsilon_{k\ell} (x_1 - x_2)_\ell \chi(|\underline{x}_1 - \underline{x}_2|) \tag{37}$$

with

$$\chi(r) := \frac{1}{\tilde{\theta}} \left(1 - \left(1 - \frac{\tilde{\theta}}{r^2} \right)^{1/2} \right) \tag{38}$$

where

$$\tilde{\theta} := \frac{e\theta}{\pi\kappa}. \tag{39}$$

With (37) and the position and momentum variables for the relative motion

$$\underline{x} := \underline{x}_1 - \underline{x}_2, \qquad \underline{p} = \frac{1}{2}(\underline{p}_1 - \underline{p}_2) \tag{40}$$

we obtain by means of a straightforward computation the two-particle Hamiltonian H in terms of canonical variables

$$H = \left(p_k - \frac{e^2}{2\pi\kappa} \epsilon_{k\ell} \frac{x_\ell}{r^2} \right) \left(p_{k'} - \frac{e^2}{2\pi\kappa} \epsilon_{k'\ell'} \frac{x_{\ell'}}{r^2} \right) g^{kk'} \tag{41}$$

with the inverse metric tensor $g^{kk'}$ given by

$$g^{kk'} := (1 - \tilde{\theta}/r^2)^{-1} \delta_{kk'} - \frac{\tilde{\theta}(2 - \tilde{\theta}/r^2)}{r^2 - \tilde{\theta}} \frac{x_k x_{k'}}{r^2}. \tag{42}$$

In plane spherical coordinates (41) reads

$$H = p_r^2 (1 - \tilde{\theta}/r^2) + \frac{(\ell + \frac{e^2}{2\pi\kappa})^2}{r^2 - \tilde{\theta}} \tag{43}$$

where ℓ is the canonical angular momentum

$$\ell := \epsilon_{ik} x_i p_k. \tag{44}$$

From (43), respectively (41,42), we conclude that H is singular at $r^2 = \tilde{\theta} =: r_0^2$ if $\tilde{\theta} > 0$ and so that $E \gtrless 0$ for $r \gtrless r_0$. Thus we conclude that we have no communication between the interior ($r < r_0$) and the exterior ($r > r_0$) space regions. We have a geometric bag determined by the singularity of our dynamically generated metric.

6 The two-particle Schrödinger equation

By applying the Moyal-Weyl quantization procedure (cp. [14] eq. (3.7)) to H given in terms of Cartesian variables (eq. (41)) we obtain the radial Schrödinger equation

$$\left(-\frac{\hbar^2}{r}\partial_r r(1-\tilde{\theta}/r^2)\partial_r + \frac{\overline{m}^2}{r^2-\tilde{\theta}} + V(r) - E\right)\varphi_{\overline{m}}(r) = 0 \tag{45}$$

with a singular potential $V(r)$

$$V(r) := -\frac{\hbar^2\tilde{\theta}}{2}\left(\frac{1}{(r^2-\tilde{\theta})^2} - \frac{1}{r^4}\right) \tag{46}$$

and a fractional (anyonic) angular momentum

$$\overline{m} := m + \frac{e^2}{2\pi\kappa}\,, \qquad m \in \mathbf{Z}\,. \tag{47}$$

Let us list some properties of the solutions of (45):

i) For $r \to r_0$ we obtain a behaviour which is typical for the singularity there (cp. [15])

$$\varphi_{\overline{m}}(r) \simeq \begin{cases} C(\tilde{\theta}-r^2)^{1/4}\exp\left(\frac{-(\tilde{\theta}/2)^{1/2}}{(\tilde{\theta}-r^2)^{1/2}}\right) & \text{if } r < r_0 \\ (r^2-\tilde{\theta})^{1/4}\left(A\cos\left(\frac{(\tilde{\theta}/2)^{1/2}}{(r^2-\tilde{\theta})^{1/2}}\right) + \right. & \\ \left. +B\sin\left(\frac{(\tilde{\theta}/2)^{1/2}}{(r^2-\tilde{\theta})^{1/2}}\right)\right) & \text{if } r > r_0 \end{cases} \tag{48}$$

With the required continuity of the partial radial current $j_{\overline{m}}$ at $r = r_0$ we infer from $(48)_{r<r_0}$ that

$$j_{\overline{m}}(r_0) = 0\,. \tag{49}$$

Therefore A/B in (48) must be a real number.

Due to (49) we have no communication between the interior (bag-) and the exterior region also in the quantum case. In particular, a scattering wave ariving from the exterior region is totally reflected at the edge of the bag. Thus the bag acts like a white hole.

ii) For $r \to 0$ we obtain

$$\varphi_{\overline{m}}/r) \simeq (r^2)^{S_\pm}\,, \qquad S_\pm := \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{2}}\right) \tag{50}$$

ie. all solutions are regular at the origin (cp. [15]).

iii) From (48) and (50) we infer that all solutions are square integrable within the bag region. As we have no additional boundary condition at hand which would determine a discrete spectrum, we expect the spectrum to be continuous. Such a situation is characteristic for singular potentials (cp. [15]).

7 Final remarks

We have shown that a deformed interaction between charged point particles and a CS-field, made invariant with respect to time-dependent area-preserving diffeomorphisms, leads to a two-particle Schroedinger equation of a highly singular nature. We have obtained some properties of its solutions but a complete discussion of its solution structure is still lacking. Work on the continuum generalization and the inclusion of an external magnetic field is in progress.

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Centrally Extended Residual Symmetries in the Presence of a Constant EM Background

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Abstract

In this talk I discuss the properties of the symmetry algebra (denominated "residual symmetry") of a QFT in the presence of a non-vanishing, constant, electro-magnetic background. It is shown that this problem can be formulated and solved on a purely Lie-algebraic ground (i.e. it is model-independent).
One of its simplest consequences consists in recovering the central extension in the commutator of the translations, associated with the abelian $U(1)$ charge.
For illustrative purposes the case of the $(2+1)$-dimensional Poincaré invariant QFTs coupled with a constant external EM background is discussed in detail. The deformed (and gauge-fixing dependent) surviving Poincaré generators are explicitly computed. In the generic case the residual symmetry algebra is isomorphic to $u(1) \oplus \mathcal{P}_c(2)$, where $\mathcal{P}_c(2)$ is the centrally extended 2-dimensional Poincaré algebra.
The connection with the Noncommutative Field Theories is briefly mentioned. A short discussion concerning the possible physical implications and the outline of the forthcoming research is given.

1 Introduction

In the present days the issues of Noncommutative Field Theory are vastly explored. Much of the activity on this topic was in consequence of the Seiberg and Witten's observation [1] that a non-commutative gauge theory may be equivalently described by a commuting gauge theory formulated in terms of ordinary (not star) products of a commuting vector potential A_μ, together with an explicit dependence on $\theta^{\alpha\beta}$, which is regarded as a constant background. The equivalence established by Seiberg and Witten motivated much of the following investigations on Field Theories formulated in terms of the Moyal star-product (for more information and references see, e.g., the now available book collecting the Proceedings of the Winter School held here in Karpacz few months ago [2]).

On the other hand, the Seiberg and Witten's observation can inspire a complementary and, so to speak, more "conservative" point of view, namely the investigation of QFT's in a given constant background within the standard commuting space-time framework. This is the viewpoint advocated here.

In this talk I will furnish an answer to the following question: which symmetries survive w.r.t the free case, for a system described by a QFT in a given constant background (which, in practical applications, can be regarded as reached adiabatically). This is why I have employed the notion of "residual symmetry" in the title, even if the term "residual" could be misleading. It appears to suggest that such symmetries form just a subalgebra of the original symmetry algebra. I will prove in the following that this is *not* the case.

My motivation in investigating such a problem came after the work of [3]. Based on some previous results, especially of Cangemi and Jackiw [4, 5], the analysis of [3]

concerned the very simple, almost trivial, model of a free massive complex boson in $1+1$ dimensions minimally coupled to an external gauge-field.

The main result of [3] consists in the proof that, in the presence of a constant electric field E, the Lie algebra of symmetries of the action is the centrally extended Poincaré algebra in $1+1$ dimension. Basically, what happens is that the originally internal $U(1)$ global charge for $E=0$, for $E \neq 0$ appears as a central charge in the commutator of the momenta. Despite the extreme simplicity of the model, the result of [3] is nevertheless noteworthy because it produces another mechanism to produce centrally extended symmetries, besides the known ones based on the arising of anomalies, either quantum [6], or even classically [7]. A systematic analysis of all these features is presented in [8].

On the other hand, the result of [3] concerns a very specific model. In this talk I will prove that the analysis of the residual symmetries and their central extensions can be performed by using solely Lie-algebraic methods, which are model-independent. Due to such a reformulation of the problem it is tempting to say that it is the property of the symmetries themselves which dictates how the residual symmetries are rearranged and the central extensions make their appearance. The generators of a surviving symmetry correspond to a deformation of the original generators.

In consequence of this purely Lie-algebraic investigation, it turns out that the results here obtained have a more general validity (they can be applied, e.g., also to interacting theories). Furthermore, the computations involved are technically simple, which make them easily applicable to more elaborated frameworks, such as standard and higher-dimensional field theories.

In this talk I will limit myself to treat and compute the residual symmetry algebra associated to a Poincaré-invariant theory in $(2+1)$ dimensions coupled with an abelian $U(1)$ external gauge-field, assumed in a constant electric and magnetic background. The choice of this example is just for illustrative purposes. The analysis of the residual symmetries originated by more complicated Lie and super-Lie algebra is currently under investigation (further comments will be furnished in the Conclusions). The mathematical techniques employed in such analysis, however, are straightforward generalizations of the techniques furnished here.

2 Statement of the Problem

For pedagogical reasons it is convenient to formulate and solve the problem by working out a specific example. It will be clear, however, that the problem and the method for its solution can be straightforwardly generalized to more complicated algebraic structures than the one here discussed. The case here treated concerns the computation of the residual symmetry for generic Poincaré-invariant field theories in $(2+1)$-dimension, coupled with an external constant EM background. The two-dimensional case of [3] is recovered by performing a dimensional reduction.

In the absence of the external electric and magnetic field, the action $\mathcal{S}$ is assumed to be invariant under a 7-parameter symmetry, namely the six generators of the $(2+1)$-Poincaré symmetry which, when acting on scalar fields (the following discussion however is valid no matter which is the spin of the fields) are represented by

$$\begin{aligned} P_\mu &= -i\partial_\mu, \\ M_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu), \end{aligned} \qquad (1)$$

(the metric is chosen to be $+--$). The remaining symmetry generator corresponds to the internal global $U(1)$ charge that will be denoted as Z.

It is further assumed that in the action $\mathcal{S}$ the dependence on the classical background field is expressed in terms of the covariant gauge-derivatives

$$D_\mu = \partial - ieA_\mu,$$

with e the electric charge.

In the presence of constant external electric and magnetic fields, the $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ field is constrained to satisfy

$$F^{0i} = E^i, \qquad F^{ij} = \epsilon^{ij} B, \tag{2}$$

where $\mu, \nu = 0,1,2$ and $i,j = 1,2$. The fields E^i and B are constant. Without loss of generality the x^1, x^2 spatial axis can be rotated so that $E^1 \equiv E$, $E^2 = 0$. Throughout the text this convention is respected.

In order to recover (2), the gauge field A_μ must depend at most linearly on the coordinates $x^0 \equiv t$, $x^1 \equiv x$ and $x^2 \equiv y$.

The gauge-transformation

$$A_\mu \mapsto A_\mu{}' = A_\mu + \frac{1}{e}\partial_\mu \alpha(x^\nu) \tag{3}$$

allows to conveniently choose for A_μ the gauge-fixing

$$\begin{aligned} A_0 &= 0, \\ A_i &= E_i t - \frac{B}{2}\epsilon_{ij} x^j. \end{aligned} \tag{4}$$

The above choice is a good gauge-fixing in the sense that it completely fixes the gauge (no gauge-freedom is left). It is easily proven that the discussion which follows is independent of the choice of the gauge-fixing. In particular, the residual symmetry is a true physical symmetry and its symmetry algebra is the same no matter which gauge-fixing has been chosen.

Due to (4), the action $\mathcal{S}$ explicitly depends on the x^μ coordinates entering A_μ. The simplest way to compute the symmetry property of an action such as $\mathcal{S}$ which explicitly depends on the coordinates consists in performing the following trick. At first A_μ is regarded on the same foot as the other fields entering $\mathcal{S}$ and assumed to transform as standard vector field under the global Poincaré transformations, namely

$$A_\mu{}'(x^{\rho\prime}) = \Lambda_\mu{}^\nu A_\nu(x^\rho) \tag{5}$$

for $x^{\mu\prime} = \Lambda^\mu{}_\nu + a^\mu$.

For a generic infinitesimal Poincaré transformation, however, the transformed A_μ gauge-field no longer respects the gauge-fixing condition (4). In the active transformation viewpoint only fields are entitled to transform, not the space-time coordinates themselves. A_μ plays the role of a fictitious field, inserted to take into account the dependence of the action $\mathcal{S}$ on the space-time coordinates caused by the non-trivial background. Therefore, the overall infinitesimal transformation δA_μ should be vanishing. This result can

be reached if an infinitesimal gauge transformation (3) $\delta_g(A_\mu)$ can be found in order to compensate for the infinitesimal Poincaré transformation $\delta_P(A_\mu)$, i.e.

$$\delta(A_\mu) = \delta_P(A_\mu) + \delta_g(A_\mu) = 0. \tag{6}$$

Only those Poincaré generators which admit a compensating gauge-transformation satisfying the (6) condition provide a symmetry of the $\mathcal{S}$ action (and therefore enter the residual symmetry algebra). This is a plain consequence of the original assumption of the Poincaré and manifest gauge invariance for the action $\mathcal{S}$ coupled to the gauge-field A_μ.

Notice however that the original Poincaré generators are deformed by the presence of extra-terms associated to the compensating gauge transformation. Let p denote a generator of (1) which "survives" as a symmetry in the presence of the external background. The effective generator of the residual symmetry is

$$\hat{p} = p + \ldots,$$

where $\ldots$ denotes the terms arising from the compensating gauge transformation associated to p. Such extra terms $\ldots$ are gauge-fixing dependent. The "residual symmetry generator" $\hat{p}$ can only be expressed in a gauge-dependent manner. However, two gauge-fixing choices are related by a gauge transformation $\mathbf{g}$. It can be easily seen that the same residual symmetry generator expressed in the new gauge-fixing and denoted as $\tilde{p}$, is related by an Adjoint transformation

$$\tilde{p} = \mathbf{g}\hat{p}\mathbf{g}^{-1} \tag{7}$$

to the previous one. Therefore the residual symmetry algebra does not dependent on the choice of the gauge fixing and is a truly physical characterization of the action $\mathcal{S}$.

The extra-terms $\ldots$ are necessarily linear in the space-time coordinates when associated with a translation generator, and bilinear when associated to a surviving Lorentz generator. They are the reason for the arising of the central term in the commutator of the deformed translation generators.

The scheme being clear, it is just a matter of straightforward computation to perform the analysis of the residual symmetries in different contexts and starting from different symmetry algebras. In the next section the results for the residual symmetry of the Poincaré invariance in $(2+1)$ dimensions are furnished.

3 The residual symmetry for the (2+1) Poincaré case.

The framework and the conventions have been illustrated in the previous section, here just the results are quoted. The residual symmetry algebra of the $(2+1)$-Poincaré theory involves, besides the global $U(1)$ generator Z, the three deformed translations and just one deformed Lorentz generator (the remaining two are broken, not providing a symmetry).

With the (4) gauge-fixing choice the deformed translations are explicitly given by

$$\begin{aligned} P_0 &= -i\partial_t - eEx, \\ P_1 &= -i\partial_x - \frac{e}{2}By, \\ P_2 &= -i\partial_y + \frac{e}{2}Bx. \end{aligned} \tag{8}$$

The deformed generator of the residual Lorentz symmetry is explicitly given, in the same gauge-fixing and for $E \neq 0$, by

$$
\begin{aligned}
M &= i(x\partial_t + t\partial_x) - i\frac{B}{E}(y\partial_x - x\partial_y) + \\
&\frac{e}{2}(Et^2 + Ex^2 - Bty).
\end{aligned} \tag{9}
$$

The residual symmetry algebra is given by the following commutation relations

$$
\begin{aligned}
[P_0, P_1] &= iEZ, \\
[P_0, P_2] &= 0, \\
[P_1, P_2] &= iBZ,
\end{aligned} \tag{10}
$$

together with

$$
\begin{aligned}
[M, P_0] &= -iP_1, \\
[M, P_1] &= -iP_0 - i\frac{B}{E}P_2, \\
[M, P_2] &= i\frac{B}{E}P_1.
\end{aligned} \tag{11}
$$

The $U(1)$ charge Z is no longer decoupled from the other symmetry generators, but it appears in (10) as a central charge.

The residual symmetry algebra of the (1+1) dimensions of ref. [3] is recovered from the P_0, P_1, M, Z subalgebra and corresponds to the centrally extended $2D$ Poincaré algebra thoroughly studied in [5]. This is, however, the residual symmetry algebra for any $(1+1)$-dimensional theory coupled with an external constant electric background, not only the symmetry algebra of the specific model studied in [3].

The 5-generator solvable, non-simple Lie algebra of residual symmetries admits a canonical presentation, obtained by a careful choice of the generators in its presentation.

At first it should be noticed that

$$
\tilde{Z} =_{def} BP_0 + EP_2 \tag{12}
$$

commutes with all the other $*$ generators

$$
[\tilde{Z}, *] = 0, \tag{13}
$$

so that the residual symmetry algebra is given by a direct sum of $u(1)$ and a 4-generator algebra. The latter algebra is isomorphic to the centrally extended two-dimensional Poincaré algebra. Such an algebra is of Minkowskian or Euclidean type according to whether $E > B$ or respectively $E < B$ (the case $E = B$ is degenerate). This point can be intuitively understood due to the predominance of the electric or magnetic effect (in the absence of the electric field the theory is manifestly rotational invariant, so that the Lorentz generator is associated with the Euclidean symmetry). We have explicitly, for $B > E$, that the algebra

$$
\begin{aligned}
[\overline{M}, S_1] &= iS_2, \\
[\overline{M}, S_2] &= -iS_1
\end{aligned} \tag{14}
$$

is reproduced by

$$
\begin{aligned}
\overline{M} &= EM\frac{1}{\sqrt{B^2 - E^2}}, \\
S_1 &= P_0 + \frac{B}{E}P_2, \\
S_2 &= \frac{\sqrt{B^2 - E^2}}{E}P_1,
\end{aligned} \tag{15}
$$

while for $E > B$ the algebra

$$
\begin{aligned}
[\overline{M}, T_1] &= iT_2, \\
[\overline{M}, T_2] &= iT_1
\end{aligned} \tag{16}
$$

is reproduced by

$$
\begin{aligned}
\overline{M} &= \frac{1}{\sqrt{E^2 - B^2}}M, \\
T_1 &= P_0 + \frac{B}{E}P_2, \\
T_2 &= -\sqrt{E^2 - B^2}P_1.
\end{aligned} \tag{17}
$$

In both cases the commutator between the translation generators S_1, S_2, and respectively T_1, T_2, develops the central term proportional to Z which can be conveniently normalized.

The residual symmetry algebra of the $(2+1)$ case for generic values of E and B (the $E = B$ case is degenerate) is therefore given by the direct sum

$$
u(1) \oplus \mathcal{P}_c(2). \tag{18}
$$

No new residual symmetry algebra is produced starting from the three-dimensional case (and in contrast with the 2-dimensional and ordinary Poincaré cases).

Three central charges are found, namely Z, $\tilde{Z}$ and the order two Casimir of the centrally extended Poincaré algebra (see [5]).

The introduction of a constant EM background implies, in some sort, that for the peculiar three-dimensional case the effective theory corresponds to a field theory in one dimension less (albeit a noncommutative one), with respect to the physical space-time dimension of the formulated problem. Notice that this is true for $(2+1)$, while it is not so in the case of $(1+1)$ dimensions since the unique Lorentz generator "survives" as a symmetry generator. In $D = 4$ the residual symmetry algebra admits seven generators (instead of five). Its explicit form will be reported elsewhere.

Some considerations are in order. The above residual symmetry algebra should be regarded on the same foot as the ordinary Poincaré invariance for theories formulated in a constant background. Issues such as the spin-statistics connection in the presence of a non-trivial background could be formulated in terms of the residual symmetry algebra. Furthermore it should be possible, e.g., to discuss the consequences of such symmetries at the level of Feynman diagrams. The notion of residual symmetry could be helpful in analyzing questions like the renormalization property for perturbative field theories in the presence of a given background. A vast range of concrete applications can be found for the realm of residual symmetries.

I conclude this section with two further remarks. The first one concerns the noncommutativity. The noncommutativity is implied by the presence, in the commutator of the deformed translation generators, of a central extension. Such deformed symmetry generators are associated with the corresponding Noether charges, which should develop the same central extensions. In reality we should further check that no extra anomaly is produced by the quantum Noether charges. This analysis, however, requires a detailed investigation of the concrete given model and is less likely being conducted in a model-independent way. To avoid such kind of problems, we can just assume working with *classical* field theories.

The non-commutativity of the momenta is a converse picture with respect to the noncommutativity of the spacetime coordinates which is usually discussed in the current literature. At least in some particular cases, the connection between the two pictures is well-established. In [9] such a connection is explained in detail for the case of a point-particle moving on a plane, in the presence of a strong, perpendicular to the plane, magnetic field and at the lowest Landau level. It should be noticed that this particular case can be recovered by specializing the residual symmetry algebra obtained in this paper and given by the formulas (10) and (11). Due to the rather complete discussion of [9] there is no need here to spend more words on that.

Finally, let me point out another reference, given by [10], in which the noncommutativity of a (2+1) theory (with just Galilean, instead of Poincaré invariance) was discussed.

4 Conclusions

In this talk I have discussed the issue of the residual symmetry in the presence of a constant electro-magnetic background. I worked out in detail the case of the originally $(2+1)$-invariant Poincaré theory and shown that its residual symmetry is isomorphic to the direct sum

$$u(1) \oplus \mathcal{P}_c(2), \tag{19}$$

the latter being the centrally extended two-dimensional Poincaré algebra. It should be noticed that the $u(1)$ generator *does not* correspond to the $U(1)$ global charge generator Z which, in the presence of a constant background, appears as a central charge in the commutator algebra of the deformed translation generators. Despite the fact that the explicit computations presented regarded a particular algebra, the method employed is quite general and can be applied straightforwardly to any initial symmetry algebra. Such symmetries include Poincaré invariance in any dimension and the conformal transformations, as well as their supersymmetric extensions. Issues related with the partial breaking of extended supersymmetries can be studied in the light of this method. All such cases are currently under investigation and the final results will be reported in a forthcoming paper [11].

The algebra of the residual symmetries, *being not* a subalgebra of the symmetry algebra in the absence of electro-magnetic background, is interesting in itself. It can define for instance some dynamical models. It is tempting for instance to use, for more general residual symmetries, the approach of [4] in which a gravitational theory based on the centrally extended two-dimensional Poincaré algebra is constructed. For what concerns the residual symmetry of the $(2+1)$-dimensional Poincaré invariance, the results of [4]

can be immediately borrowed, due to the (19) decomposition of such an algebra. In more complicated cases such as the Minkowskian $(3+1)$-dimensional case, decompositions like (19) no longer apply.

The present talk concerns only the residual symmetries arising from a constant background of an abelian $U(1)$ gauge-invariance. Generalizations can be done in two ways, i.e. considering the residual symmetries for constant background in the presence of higher rank antisymmetric tensor fields. Such backgrounds are relevant in the string/brane context [12].

Conversely, residual symmetries can be studied also for non-constant background. In the presence of a linear homogeneous electric background it is known that a phenomenon of particle production is produced [13]. It is quite tempting to investigate such a problem in the light of the residual symmetries of the model. Investigations in this direction are currently under consideration.

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Part III

Supersymmetry and Quantum Field Theory

On the Local Symmetries of Gravity and Supergravity Models

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Abstract

We present here a detailed analysis of the local symmetries of supergravity in an arbitrary dimension D, both in the component and superfield approaches, using a field–space democracy point of view. As an application, we discuss briefly how a complete description of the local gauge symmetries clarifies the properties of the supergravity–superbrane coupled system in the standard background superfield approximation for supergravity.

1 Introduction

A recent supersymmetric analysis of the supergravity–superbrane interaction [1, 2] in which supergravity is described by its group manifold action (not as a background), as well as other related models [3, 4, 5], makes desirable a reexamination of the rôle of local supersymmetry and, more generally, of the gauge symmetries in supergravity models.

We present here a detailed account of local symmetries, beginning in Sec.2 by the differential forms formulation of D–dimensional general relativity. We describe in Sec.3 the complete set of its spacetime gauge symmetries, including diffeomorphism invariance whose discussion, together with general coordinate transformations, becomes especially relevant when the interacting system of (super)gravity and (super–p–)brane is considered. This is so because in the super–p–brane action the spacetime variables play a dynamical rôle.

We explain how the presence of diffeomorphism invariance provides the possibility of presenting the general coordinate invariance in an equivalent form, its 'variational version', which does not act on the spacetime coordinates (see [6] for the $D = 4$ $N = 1$ superfield supergravity case). In Sec. 4 we consider the second Noether theorem for general relativity. Then, in Sec. 5, we describe the general structure of the action for the standard, component formulation of supergravity, its local symmetries and their associated Noether identities.

Sec. 6 is devoted to the superspace general coordinate symmetry and other gauge symmetries for the on–shell superfield formulation of supergravity, where supergravity is described by the set of constraints on the superspace torsion, which imply dynamical equations. We apply this knowledge in Sec.7 to uncover the relation between the local supersymmetry and the κ–symmetry of the supermembrane in a $D = 11$ superfield supergravity background.

2 D– dimensional General Relativity in differential form

D-dimensional gravity models can be formulated in terms of the moving frame or vielbein fields $e_\mu{}^a(x)$ (tetrad in $D=4$), which determine the vielbein one–forms on spacetime M^D,

$$e^a(x) = dx^\mu e_\mu{}^a(x) \ . \tag{1}$$

A change of frame is given by a matrix of the local $SO(1, D-1)$ group,

$$\begin{gathered} e^a(x) \to e^{a\prime}(x) = e^b(x)\Lambda_b{}^a(x) \, , \\ \Lambda_a{}^c \eta_{cd} \Lambda_b{}^d = \eta_{ab} \ = diag(1, -1, \ldots, -1) \, . \end{gathered} \tag{2}$$

This local Lorentz symmetry is the first gauge symmetry of gravity theory to be noted. Its infinitesimal form is

$$\begin{gathered} \delta_L x^\mu = 0 \, , \quad \delta_L e^a(x) = e^b(x) L_b{}^a(x) \, , \\ L^{ab}(x) = -L^{ba}(x) \, , \end{gathered} \tag{3}$$

where $\Lambda_b{}^a(x) = \delta_b{}^a + L_b{}^a(x) + \mathcal{O}(L^2)$. It is convenient to introduce the spin connection

$$w^{ab}(x) = dx^\mu w_\mu^{ab}(x) = -w^{ba}(x) \, , \tag{4}$$

with the transformation rule $w^{ab}(x) \to w^{ab\prime}(x) = (\Lambda^{-1}(x) d\Lambda(x))^{ab} + (\Lambda^{-1}(x) w(x) \Lambda(x))^{ab}$, or

$$\delta_L w^{ab}(x) = \mathcal{D}L^{ab} \equiv dL^{ab} - 2L^{[a|c} w_c{}^{|b]} \, . \tag{5}$$

The spin connection can be either expressed from the beginning through the vielbein field by imposing the covariant torsion constraint

$$T^a := \mathcal{D}e^a \equiv de^a - e^b \wedge w_b{}^a = 0 \tag{6}$$

(second order approach), or treated as an independent variable in the action principle (first order approach). In both cases the dynamics is determined by the *Einstein–Hilbert* (EH) action, which can be written in terms of differential forms on M^D as (see *e.g.* [7, 8] and refs. therein)

$$S_{D,G} = \int_{M^D} R^{ab} \wedge e_{ab}^{\wedge(D-2)} \, , \tag{7}$$

where $e_{a_1 \ldots a_q}^{\wedge(D-q)} \equiv \frac{1}{(D-q)!} \varepsilon_{a_1 \ldots a_q b_1 \ldots b_{D-q}} \, e^{b_1} \wedge \ldots \wedge e^{b_{D-q}}$ and R^{ab} is the Riemann curvature,

$$R^{ab} = dw^{ab} - w^{ac} \wedge w_c{}^b = \frac{1}{2} dx^\nu \wedge dx^\mu R_{\mu\nu}{}^{ab} \equiv \frac{1}{2} e^d \wedge e^c R_{cd}{}^{ab} \, . \tag{8}$$

The variation of the EH action (7) reads[1]

$$\delta S_{D,G} = -(-1)^D \int_{M^D} M_{(D-1)a} \wedge \delta e^a + \int_{M^D} e_{abc}^{\wedge(D-3)} \wedge T^c \wedge \delta w^{ab} \, , \tag{9}$$

[1]since $\delta e_{ab}^{\wedge(D-2)} = -(-1)^D e_{abc}^{\wedge(D-3)} \wedge \delta e^c$, $d(e_{ab}^{\wedge(D-2)}) = -(-1)^D e_{abc}^{\wedge(D-3)} \wedge de^c$ and $\mathcal{D}(e_{ab}^{\wedge(D-2)}) = -(-1)^D e_{abc}^{\wedge(D-3)} \wedge T^c$.

where

$$\begin{gathered} M_{(D-1)a} := R^{bc} \wedge e_{abc}^{\wedge(D-3)} \equiv dx_\mu^{\wedge(D-1)} M_a^\mu \, , \\ M_a^\mu \propto \left(R_{ca}{}^{cb} - \tfrac{1}{2}\delta_a{}^b R_{cd}{}^{cd}\right) e_b{}^\mu(x) \, , \end{gathered} \tag{10}$$

is the Einstein tensor as a $(D-1)$–form.

The δe^a variation produces the free Einstein equation

$$M_{(D-1)a} = 0 \ \Leftrightarrow \ R_{ca}{}^{cb} - \tfrac{1}{2}\delta_a{}^b R_{cd}{}^{cd} = 0 \, . \tag{11}$$

In the *first order approach*, eq. (9) also gives the torsion constraint (6)

$$\begin{aligned} \frac{\delta S_{D,G}}{\delta w_\mu^{ab}(x)} = 0 \ \Rightarrow \ & e_{abc}^{\wedge(D-3)} \wedge T^c = 0 \\ \Rightarrow \ & T^c \equiv \tfrac{1}{2} e^b \wedge e^a T_{ab}{}^c = 0 \, . \end{aligned} \tag{12}$$

In the *second order approach*, the torsion constraint (6) is imposed *ab initio*, so that

$$\delta S_{D,G}|_{T^a=0} = -(-1)^D \int_{M^D} M_{(D-1)a} \wedge \delta e^a \, . \tag{13}$$

Thus, varying the EH action one can ignore the dependence of the spin connection on the vielbein (see [12] and refs. therein).

3 Gauge symmetries of General Relativity

3.1 Diffeomorphism invariance

All the above formalism is written in terms of differential forms on spacetime. Clearly, these are invariant under an arbitrary change of local coordinates *i.e.*, they are M^D–diffeomorphisms invariant

$$x^\mu \mapsto x^{\mu\prime} = x^\mu + \delta_{diff} x^\mu \equiv x^\mu + b^\mu(x) \, , \tag{14}$$

$$e^a(x) \mapsto e^{\prime a}(x') = e^a(x) \, , \text{ etc.} \tag{15}$$

In field theory analyses, where *only* fields, such as $e_\mu^a(x)$, are considered as dynamical variables, this obvious invariance can be ignored in favour of general coordinate symmetry (see below). However, when the coupled system of (super)gravity and (super)brane is considered in the framework of an action principle (see [1, 2]) the set of dynamical variables includes, besides the fields $e_\mu^a(x)$ *etc.*, the local coordinate functions $\hat{x}^\mu(\xi)$ defined by the map $\hat{\phi} : W^{p+1} \mapsto M^D$, where W^{p+1} is the worldvolume with local coordinates $\xi^i = (\tau, \sigma^1, \ldots, \sigma^p)$. This suggests adopting a field–space democracy approach [10] where the fields ($e_\mu^a(x)$ *etc.*) and the spacetime coordinates x^μ are treated on equal footing.

3.2 General coordinate symmetry and its 'variational version'

Besides being local Lorentz and diffeomorphism invariant, the EH action (7) is invariant under general coordinate transformations as we discuss below.

To derive the equations of motion for a field theory from the variational principle, as *e.g.* eq. (11) for general relativity, one uses arbitrary variations of the fields only, *e.g.* $\delta' e^a_\mu(x)$, so that

$$\delta' x^\mu = 0 \,, \qquad \delta' e^a(x) = dx^\mu \delta' e^a_\mu(x) \,. \tag{16}$$

On the other hand, a general variation $\delta e^a(x)$ is

$$\delta e^a(x) \equiv \delta' e^a(x) + \delta_{\delta x} e^a(x) \,, \tag{17}$$

where $\delta_{\delta x}$ denotes the variation due to the change $x \to x' = x + \delta x$. The $\delta_{\delta x}$ variation is given by the Lie derivative $L_{\delta x} = d i_{\delta x} + i_{\delta x} d$. For instance,

$$\begin{aligned} \delta_{\delta x} e^a(x) &:= e^a(x+\delta x) - e^a(x) = L_{\delta x} e^a(x) = d i_{\delta x} e^a(x) + i_{\delta x} d e^a(x) = \\ &= \mathcal{D}(i_{\delta x} e^a(x)) + i_{\delta x} T^a(x) + e^b\, i_{\delta x} w_b{}^a \,, \end{aligned} \tag{18}$$

where

$$\begin{aligned} i_{\delta x} e^a = \delta x^\mu e^a_\mu(x) \,, \quad i_{\delta x} w^{ab} = \delta x^\mu w^{ab}_\mu(x) \,, \\ i_{\delta x} T^a(x) = e^b i_{\delta x} e^c T_{cb}{}^a(x) \,; \end{aligned} \tag{19}$$

the last term in (18), $e^b i_{\delta x} w_b{}^a = e^b\, \delta x^\mu w_{\mu b}{}^a(x)$, is the local Lorentz rotation induced by δx^μ.

In the above notation, diffeomorphism invariance ((14), (15)) can be formulated as a symmetry under the transformations

$$\delta_{diff} x^\mu = b^\mu(x) \,, \tag{20}$$

$$\delta_{diff} e^a(x) = \delta'_{diff} e^a(x) + L_b e^a(x) = 0 \,. \tag{21}$$

Thus, $\delta'_{diff} e^a_\mu(x)$ is defined by $-(L_b e^a)_\mu$, *i.e.*

$$\delta'_{diff} e^a(x) = -L_b e^a \,, \tag{22}$$

and $\delta_{diff} S_{D,G} = 0$ follows as an evident consequence of (13) and (21).

In contrast, general coordinate transformations or local translations are defined as arbitrary displacements of the spacetime points,

$$x^\mu \mapsto x^{\mu\prime} = x^\mu + \delta_{gc} x^\mu \equiv x^\mu + t^\mu(x) \tag{23}$$

(*cf.* (20)) which *induce* differential forms transformations as *e.g.*

$$e^a(x) \mapsto e^a(x') = e^a(x + \delta_{gc} x) \equiv e^a(x+t) \,, \tag{24}$$

i.e.,

$$\delta_{gc} e^a(x) = \delta_{\delta_x} e^a(x) \equiv \delta_t e^a(x) = L_t e^a(x) = \mathcal{D}(t^a(x)) + i_t T^a(x) + e^b\, i_t w_b{}^a \tag{25}$$

(as in (18)) where $t^a(x) = i_t e^a(x) \equiv t^\mu(x) e^a_\mu(x)$.

Consider a D-form $\mathcal{L}_D$ on M^D, involving the $\wedge$ product of forms, their exterior derivatives and, possibly, the $*$ Hodge operator. Then $\mathcal{L}_D(x) \mapsto \mathcal{L}_D(x') = \mathcal{L}_D(x+t)$, and

$$\delta_{gc} \mathcal{L}_D = L_b \mathcal{L}_D = (i_t d + d i_t)\mathcal{L}_D = d(i_t \mathcal{L}_D) \,, \tag{26}$$

since any D-form on M^D is closed. Thus, *any* $S = \int_{M^D} \mathcal{L}_D$ is general coordinate invariant. In particular, the EH action (7) possesses this invariance.

Thus, one may look at δ_{diff} and δ_{gc}, respectively, as passive and active forms of the same spacetime coordinates transformations, unaffecting any theory defined through an integral of a D-form $\mathcal{L}_D$ on M^D. This picture changes when the action of a p-brane, which is given by an integral $\int_{W^{p+1}} \hat{\mathcal{L}}_{p+1}$ on the $(p+1)$-dimensional worldvolume W^{p+1}, is considered together with $\int_{M^D} \mathcal{L}_D$. The coupled action $\int_{M^D} \mathcal{L}_D + \int_{W^{p+1}} \hat{\mathcal{L}}_{p+1}$ will still possess spacetime diffeomorphism invariance provided $\hat{\mathcal{L}}_{p+1}$ is formulated in terms of pull-backs of spacetime differential forms (so that eq. (14) implies $\hat{x}^\mu(\xi) \mapsto \hat{x}^{\mu\prime}(\xi) = \hat{x}^\mu(\xi) + b^\mu(\hat{x}(\xi))$), but it will not be spacetime general coordinate invariant, since such an invariance means equivalence between different spacetime points, and points on the brane are not equivalent to points outside it.

Let us go back to the pure gravity case. On account of diffeomorphism invariance, one can use equivalently (rather than $\delta_{gc}(t)$, eqs. (23)–(25)), $\delta_{gc}(t)$ followed by a diffeomorphism (eqs. (20), (22)) with $b^\mu(x) = -t^\mu(x)$, $\delta_{diff}(b = -t)$. As we are dealing with a local Lorentz invariant theory, we may also add a local Lorentz transformation with parameter $L^{ab} = -i_t w^{ab}$ and get

$$\tilde{\delta}_{gc}(t^a) := \delta_{gc}(t^\mu) + \delta_{diff}(b^\mu = -t^\mu) + \delta_{gc}(L^{ab} = -i_t w^{ab}) \,. \tag{27}$$

This $\tilde{\delta}_{gc}(t^a)$ will be called, following [6], the 'variational version' of the general coordinate transformation $\delta_{gc}(t^a)$. Indeed, it does not act on x^μ,

$$\tilde{\delta}_{gc} x^\mu := 0 \,, \tag{28}$$

so that, *e.g.* (eqs. (16), (25)),

$$\tilde{\delta}_{gc} e^a(x) := \mathcal{D}t^a + e^c t^b T_{bc}{}^a(x) = \tilde{\delta}'_{gc} e^a(x) \,. \tag{29}$$

Thus $\tilde{\delta}_{gc}$ provides the complete general coordinate variation of a differential form, *e.g.* $\delta_{gc} e^a(x) = \tilde{\delta}_{gc} e^a(x)$, as a result of the field variation $\tilde{\delta}'_{gc}$ only.

In the second order approach, where $T^a = 0$, the $\tilde{\delta}_{gc}$ transformations (29) simplify and acquire the characteristic form of a gauge field transformation,

$$\tilde{\delta}_{gc} e^a(x)|_{T^a=0} = \mathcal{D}t^a \,. \tag{30}$$

This provides the possibility of treating gravity as a gauge theory of local translations in their variational form $\tilde{\delta}_{gc}$ (see [9] for early discussions of gravity as a gauge theory).

Note, finally, that since the above D-form $\mathcal{L}_D$ is diffeomorphism invariant and $\delta_{gc} S = 0$ by (26), it follows directly from (27) that $\tilde{\delta}_{gc} S = 0$ also.

4 Second Noether theorem applied to General Relativity

The invariance of the EH action (7) under the variational version of the general coordinate transformations $\tilde{\delta}_{gc}(t^a)$,

$$\tilde{\delta}_{gc} S_{D,G} = (-1)^D \int_{M^D} \mathcal{D} M_{(D-1)a}\, t^a(x) = 0 \,, \tag{31}$$

follows from the fact that the Bianchi identity $\mathcal{D}R^{ab} \equiv 0$ and the torsion constraint (6) imply

$$\mathcal{D}M_{(D-1)a} \equiv 0 \tag{32}$$

(for simplicity, we use in this section the second order approach). This is the so-called *Noether identity* (NI) which reflects the presence of a gauge symmetry, here the symmetry under $\tilde{\delta}_{gc}$.

In general, the second Noether theorem states that any gauge symmetries, $\delta_{gauge}S = 0$, given by transformation rules that involve the derivatives of the local parameter up to k-th order, are in one–to–one correspondence with their associated NIs, *i.e.* with identically satisfied relations between the (*l.h.* sides of the) Lagrangian equations of motion and their derivatives up to k-th order.

To discuss also δ_{diff} and δ_{gc} in this framework, consider the second order approach to D-dimensional general relativity in a field–space democracy context [10], where coordinates and fields are treated on the same footing. Then, the dynamical variables are the vielbein field $e^a_\mu(x)$ and the spacetime coordinate x^μ. Their Lagrangian equations are

$$\mathcal{N}_\mu = 0\,, \quad \mathcal{N}_\mu := \tfrac{\delta S}{\delta x^\mu} \qquad \text{and} \tag{33}$$
$$M^\mu_a = 0\,, \quad M^\mu_a := -(-1)^D \tfrac{\delta S}{\delta e^a_\mu(x)}\,, \tag{34}$$

where M^μ_a defines the differential D-form $M_{(D-1)a}$, eq. (10).

To find an explicit expression for $\mathcal{N}_\mu$ one uses the splitting (17) of the general variation of the EH action (13)

$$\delta S_{D,G} = -(-1)^D \int_{M^D} M_{(D-1)a} \wedge (\delta' e^a(x) + \delta_{\delta x} e^a(x))\,. \tag{35}$$

The Einstein equation (34) now follows from the $\delta' e^a(x)$ variation, while the δx^μ variation entering $\delta_{\delta x} e^a(x) = \mathcal{D}(i_{\delta x} e^a(x)) + e^b\, i_{\delta x} w_b{}^a$ (eq. (18) with $T^a = 0$) results in eq. (33) with $\mathcal{N}_\mu$ defined by

$$d^D x \mathcal{N}_\mu = (-1)^D \mathcal{D}M_{(D-1)a}\, e^a_\mu - (-1)^D M_{(D-1)a} \wedge e^b w_{\mu b}{}^a\,. \tag{36}$$

The variational version of the general coordinate transformations $\tilde{\delta}_{gc}$, (28)–(30), as well as the local Lorentz transformations δ_L, (3), do not act on the spacetime coordinates. As a result, the NIs reflecting the invariance under $\tilde{\delta}_{gc}$ and δ_L only involve the *l.h.* side of eq. (10) (eqs. (34))

$$\tilde{\delta}_{gc} S_{D,G} = 0 \quad \Leftrightarrow \quad \mathcal{D}M_{(D-1)a} \equiv 0\,, \tag{37}$$
$$\tilde{\delta}_L S_{D,G} = 0 \quad \Leftrightarrow \quad M_{(D-1)[a} \wedge e_{b]} \equiv 0\,. \tag{38}$$

In contrast, for the general coordinate symmetry in its original form δ_{gc}, (23), (24), the basic transformations are arbitrary changes of the spacetime points, eq. (23). Thus,

$$\delta_{gc} S_{D,G} = 0 \qquad \Leftrightarrow \qquad \mathcal{N}_\mu := \tfrac{\delta S}{\delta x^\mu} \equiv 0\,. \tag{39}$$

Using eq. (36) together with (37) and (38) one finds that the identity (39) holds indeed.

It might look strange that two equivalent formulations of the same general coordinate symmetry, $\tilde{\delta}_{gc}$ and δ_{gc}, have different NIs. The reason is simple: a linear combination of

these NIs reproduces the NI for diffeomorphism invariance δ_{diff}, (20), (21) (or, equivalently (14), (15)). Indeed, the explicit form of $\mathcal{N}_\mu$ (36) actually provides us with such NI

$$\delta_{diff} S = 0 \;\Leftrightarrow\; d^D x \mathcal{N}_\mu - (-1)^D \times (\mathcal{D}M_{(D-1)a} e^a_\mu - M_{(D-1)a} \wedge e^b\, w_{\mu b}{}^a) \equiv 0\,. \tag{40}$$

As the two terms inside the brackets are identically equal to zero due to the NIs for $\tilde{\delta}_{gc}$ and δ_L, (eqs. (37) and (38)) the NI (40) implies (39) and *viceversa*. This translates the definition of $\tilde{\delta}_{gc}$ in eq. (27) to the language of the second Noether theorem.

5 D-dimensional supergravity

5.1 Local supersymmetry and supergravity

Supergravity (see *e.g.* [12] and refs. therein) is the gravity theory invariant under local supersymmetry δ_{ls}. This is a local symmetry involving a fermionic (Grassmann) spinor parameter $\epsilon^{\underline{\alpha}}(x)$, ($\underline{\alpha} = 1, \dots n$ is a D-dimensional Lorentz spinor index, $n = \dim(Spin(1, D-1))$. Hence $\delta_{ls}(\epsilon^{\underline{\alpha}})$ mixes the graviton field, *i.e.* the vielbein $e^a_\mu(x)$, with a fermionic field, the gravitino or Rarita–Schwinger field $\psi^{\underline{\alpha}}_\mu(x)$. Specifically,

$$\delta_{ls} e^a_\mu(x) = -2i\psi^{\underline{\alpha}}_\mu(x) \Gamma^a_{\underline{\alpha\beta}} \epsilon^{\underline{\beta}}(x)\,. \tag{41}$$

Using the fermionic one–form $e^{\underline{\alpha}}(x)$,

$$e^{\underline{\alpha}}(x) := dx^\mu \psi^{\underline{\alpha}}_\mu(x) = e^a \psi^{\underline{\alpha}}_a(x)\,, \tag{42}$$

eq. (41) reads

$$\delta_{ls} e^a(x) = -2i e^{\underline{\alpha}}(x) \Gamma^a_{\underline{\alpha\beta}} \epsilon^{\underline{\beta}}(x)\,, \tag{43}$$

The vector–spinor gravitino field has the proper index structure to be the gauge field for local supersymmetry. Thus it is natural to assume that

$$\delta_{ls} \psi^{\underline{\alpha}}_\mu(x) = \mathcal{D}_\mu \epsilon^{\underline{\alpha}}(x)\,, \tag{44}$$

or, equivalently

$$\delta_{ls} e^{\underline{\alpha}}(x) = \mathcal{D}\epsilon^{\underline{\alpha}}(x)\,. \tag{45}$$

The guess (44) or (45) is supported by the fact that the linearized form δ^0_{ls} of (45),

$$\delta^0_{ls} e^{\underline{\alpha}}(x) = d\epsilon^{\underline{\alpha}}(x)\,,\; \delta^0_{ls} \psi^{\underline{\alpha}}_\mu(x) = \partial_\mu \epsilon^{\underline{\alpha}}(x)\,, \tag{46}$$

is an evident symmetry of the free D-dimensional Rarita–Schwinger (RS) action on flat spacetime

$$S^{RS}_D = -\tfrac{2i}{3} \int_{R^D} de^{\underline{\alpha}} \wedge e^{\underline{\beta}} \wedge (dx)^{\wedge(D-3)}_{\mu\nu\rho} \Gamma^{\mu\nu\rho}_{\underline{\alpha\beta}}$$
$$\propto \int d^D x \psi^{\underline{\alpha}}_\mu \Gamma^{\mu\nu\rho}_{\underline{\alpha\beta}} \partial_\nu \psi^{\underline{\beta}}_\rho\,. \tag{47}$$

The first candidate for a locally supersymmetric action is the sum of the free EH action (7) and the RS action (47) 'covariantized' with respect to the local Lorentz transformations (3)

$$S_D^{2+3/2} = \int_{M^D} \mathcal{L}_D^2 + \int_{M^D} \mathcal{L}_D^{3/2} \,, \tag{48}$$

$$\mathcal{L}_D^2 = R^{ab} \wedge e_{ab}^{\wedge(D-2)} \,, \tag{49}$$

$$\mathcal{L}_D^{3/2} = -\tfrac{2i}{3} \mathcal{D}e^{\underline{\alpha}} \wedge e^{\underline{\beta}} \wedge e_{abc}^{\wedge(D-3)} \, \Gamma^{abc}_{\underline{\alpha\beta}} \,. \tag{50}$$

For $D=4$, $N=1$, $S_D^{2+3/2}$ is indeed locally supersymmetric under (43), (45) and provides the action for the simple $D=4$ supergravity [11] (see also [12, 13]).

$$S_{4,SG} = S_{D=4}^{2+3/2} \,. \tag{51}$$

5.2 General structure of the supergravity action and equations of motion

In higher dimensions (in particular, in $D=10,11$) the supergravity multiplet involves a set of antisymmetric tensor gauge fields $C_{\mu_1\ldots\mu_q}(x)$ described by differential forms

$$C_q \equiv \tfrac{1}{q!} dx^{\mu_q} \wedge \ldots \wedge dx^{\mu_1} C_{\mu_1\ldots\mu_q}(x) \,, \tag{52}$$

(C_3 for $D=11$; C_2, C_4 and B_2 in $D=10$ type IIB, C_1, C_3 and B_2 in $D=10$ type IIA, *etc.*) and, in $4<D<11$, scalar fields (*e.g.* dilaton ϕ in $D=10$ type IIA and IIB and axion C_0 in $D=10$ type IIB) and spinors. Thus, in general

$$S_{SG,D} = \int_{M^D} (\mathcal{L}_D^2 + \mathcal{L}_D^{3/2} + \mathcal{L}_D{}^{\leq 1}) \,; \tag{53}$$

$\mathcal{L}_D{}^{\leq 1}$ includes, in particular, the kinetic term for the q–form gauge fields

$$\propto \int d^Dx \, det(e_\mu^a) \, \mathcal{H}_{\mu_1\ldots\mu_{q+1}} \mathcal{H}^{\mu_1\ldots\mu_{q+1}} + \ldots \,, \tag{54}$$

where

$$\mathcal{H}_{q+1} := dC_q - c_1 e^{\underline{\alpha}} \wedge e^{\underline{\epsilon}} \wedge \bar{\Gamma}^{(q-1)}{}_{\underline{\alpha\epsilon}} \tag{55}$$

$$= \tfrac{1}{(q+1)!} dx^{\mu_{q+1}} \wedge dx^{\mu_1} \mathcal{H}_{\mu_1\ldots\mu_{q+1}}(x) \,, \tag{56}$$

$$\bar{\Gamma}^{(k)}{}_{\underline{\alpha\epsilon}} := \tfrac{1}{k!} e^{a_1} \wedge \ldots \wedge e^{a_k} \Gamma_{a_1\ldots a_p \underline{\alpha\epsilon}} \,, \tag{57}$$

is the generalized field strength of C_q.

These kinetic terms can be written in a first order form (which is suitable for discussing the relation with superspace approach, see [7, 8]) if one adds to every gauge q–form C_q an auxiliary antisymmetric tensor field $F_{a_1\ldots a_{q+1}}(x) = F_{[a_1\ldots a_{q+1}]}(x)$. These fields can be used to construct the $(q+1)$–forms and the $(D-q-1)$–forms

$$F_{q+1} \equiv \tfrac{1}{(q+1)!} e^{a_{q+1}} \wedge \ldots \wedge e^{a_1} F_{a_1\ldots a_{q+1}}(x) \,, \tag{58}$$

$$\mathcal{F}_{D-q-1} = e_{a_1\ldots a_{q+1}}^{\wedge(D-q-1)} F^{a_1\ldots a_{q+1}}(x) \,, \tag{59}$$

which allow us to write the kinetic term(s) (54) as

$$\mathcal{L}_D{}^{\leq 1} = c(\mathcal{H}_{q+1} - \tfrac{1}{2} F_{q+1}) \wedge \mathcal{F}_{D-q-1} + \ldots \,, \tag{60}$$

where the terms denoted by dots do not contain $F^{a_1\ldots a_{q+1}}(x)$.

Indeed, the variation of $F^{a_1\ldots a_{q+1}}(x)$ leads to the algebraic equation

$$\mathcal{H}_{q+1} - F_{q+1} = 0 \tag{61}$$

which identifies the auxiliary field $F^{a_1\ldots a_{q+1}}(x)$ with the generalized field strength of the tensor gauge field $C_{a_1\ldots a_q}(x)$,

$$F_{a_1\ldots a_{q+1}}(x) = (q+1)\nabla_{[a_1} C_{a_2\ldots a_{q+1}]} + \ldots =$$
$$= (q+1) e_{[a_1}^{\mu_1} \ldots e_{a_{q+1}]}^{\mu_{q+1}} \partial_{\mu_1} C_{\mu_2\ldots\mu_{q+1}} + \ldots \,,$$

where the dots denote the terms with torsion and fermions. Substituting eq. (61) into the Lagrangian form (60) one arrives at the standard, second order approach, representation for the kinetic term of the gauge field $C_{\mu_1\ldots\mu_q}(x)$, eq. (54). On the other hand, varying (53), (60), with respect to the gauge field(s) $C_{\mu_1\ldots\mu_q}(x)$ one finds

$$G_{(D-q)} \equiv d(e_{a_1\ldots a_{q+1}}^{\wedge(D-q-1)} F^{a_1\ldots a_{q+1}}) + \ldots = 0 \,, \tag{62}$$

which, after the use of eq. (61), becomes the dynamical gauge field equation.

For future reference we note that the equation $\delta S_{D,SG}/\delta w^{ab} = 0$ determines the 'improved' constraint on the spacetime torsion T^a (*cf.* (6)),

$$T^a + ie^{\underline{\alpha}} \wedge e^{\underline{\epsilon}} \Gamma^a_{\underline{\alpha\epsilon}} = 0 \,, \tag{63}$$

while $\delta e^{\underline{\alpha}}$ and δe^a provide the differential form expression for the RS and Einstein equations of supergravity

$$\Psi_{(D-1)\underline{\alpha}} := \tfrac{4i}{3} \mathcal{D}e^{\underline{\epsilon}} \wedge e_{abc}^{\wedge(D-3)} \Gamma^{abc}_{\underline{\epsilon\alpha}} + \ldots = 0 \,, \tag{64}$$

$$M_{(D-1)\,a} := R^{bc} \wedge e_{abc}^{\wedge(D-3)} + \ldots = 0 \,. \tag{65}$$

For simplicity, we will not consider here the cases where the supergravity multiplet involves scalar and spinor fields. Thus our basic examples are $D=3,4$ *and* 11 supergravity.

In the above notation a generic variation of the supergravity action reads

$$\begin{aligned} \delta S_{D,SG} = &-(-1)^D \int_{M^D} M_{(D-1)a} \wedge \delta e^a + (-1)^D \int_{M^D} \Psi_{(D-1)\underline{\alpha}} \wedge \delta e^{\underline{\alpha}} + \\ &+ \int_{M^D} (-1)^D e_{abc}^{\wedge(D-3)} \wedge (T^c + ie^{\underline{\gamma}} \wedge e^{\underline{\delta}} \Gamma^c_{\underline{\gamma\delta}}) \wedge \delta w^{ab} + \\ &+ c \int_{M^D} (\mathcal{H}_{q+1} - F_{q+1}) \wedge e_{a_1\ldots a_{q+1}}^{\wedge(D-q-1)} \delta F^{a_1\ldots a_{q+1}} + c(-1)^{Dp} \int_{M^D} G_{D-q} \wedge \delta C_q \,. \end{aligned} \tag{66}$$

5.3 Local supersymmetry, general coordinate symmetry and Noether identities

The above first order form of the supergravity action (see [8]), eq. (53) with (49), (50) and (60), is written in terms of differential forms on M^D (including the covariant zero–forms $F_{a_1\ldots a_{q+1}}(x)$) and thus is invariant under M^D–diffeomorphisms, defined by eqs. (20), (22) and the analogous ones for $e^{\underline{\alpha}}(x)$ *etc.*

The action of D-dimensional supergravity (53), being a generalization of the EH general relativity action, eq. (7), possesses local Lorentz invariance (3) and general coordinate invariance under δ_{gc} (eqs. (23), (25)) or, equivalently, under its variational version $\tilde{\delta}_{gc}$ (eq. (27)). Moreover, it is invariant under local supersymmetry transformations $\delta_{ls}(\epsilon^{\underline{\alpha}})$,

$$\begin{aligned} \delta_{ls}x^\mu &= 0\,, & (67)\\ \delta_{ls}e^a(x) &= -2ie^{\underline{\alpha}}(x)\Gamma^a_{\underline{\alpha\epsilon}}\epsilon^{\underline{\epsilon}}(x)\,, & (68)\\ \delta_{ls}e^{\underline{\alpha}}(x) &= \mathcal{D}\epsilon^{\underline{\alpha}}(x) + \epsilon^{\underline{\epsilon}}(x)\mathcal{M}_{1\underline{\epsilon}}{}^{\underline{\alpha}}(x)\,, & (69)\\ \delta_{ls}C_{p+1}(x) &= 2c_1 e^{\underline{\alpha}}\wedge\bar{\Gamma}^{(p)}{}_{\underline{\alpha\epsilon}}\,\epsilon^{\underline{\epsilon}}(x)\,, & (70)\\ \delta_{ls}w^{ab}(x) &= \mathcal{W}^{ab}_{1\underline{\epsilon}}(x)\epsilon^{\underline{\epsilon}}(x)\,, & (71)\\ \delta_{ls}F^{a_1\ldots a_{q+1}}(x) &= S^{a_1\ldots a_{q+1}}_{\underline{\epsilon}}\epsilon^{\underline{\epsilon}}(x)\,, & (72) \end{aligned}$$

where $S^{a_1\ldots a_{q+1}}_{\underline{\epsilon}}(x)$ and the one–forms $\mathcal{M}_{1\underline{\epsilon}}{}^{\underline{\alpha}}(x)$, $\mathcal{W}_{1\underline{\epsilon}}{}^{ab}(x)$ are constructed from the fields of the supergravity multiplet and the auxiliary fields $F_{a_1\ldots a_{p+2}}(x)$ (*cf.* (43), (45)).

Then, the experience of Secs. 3,4 allows one to conclude (actually without any further calculations) that the general coordinate symmetry in its variational form $\tilde{\delta}_{gc}$, eqs. (28), (29), and the local supersymmetry δ_{ls}, eqs. (67)–(72), are reflected by Noether identities relating the *l.h.* sides of the *field* equations only, namely

$$\begin{aligned} \mathcal{D}\Psi_{(D-1)\underline{\alpha}} - 2iM_{(D-1)a}\wedge e^{\underline{\epsilon}}\Gamma^a_{\underline{\alpha\epsilon}} + \ldots &\equiv 0\,, & (73)\\ \mathcal{D}M_{(D-1)a} - \ldots &\equiv 0\,, & (74) \end{aligned}$$

where the terms denoted by dots turn out to be proportional to the *l.h.* sides of eqs. (61)–(64), but *not* of the Einstein equation (65). For example, for $D=4$ $N=1$ supergravity the full NIs (73), (74) read

$$\begin{aligned} &\mathcal{D}\Psi_{3\underline{\alpha}} - 2iM_{3a}\wedge e^{\underline{\beta}}\Gamma^a_{\underline{\alpha\beta}} - \Gamma_{a\underline{\alpha\beta}}\,\mathcal{D}e^{\underline{\beta}}\wedge(T^a + ie^{\underline{\gamma}}\wedge e^{\underline{\delta}}\Gamma^a_{\underline{\gamma\delta}}) \equiv 0\,, \qquad (75)\\ &\mathcal{D}M_{3a} - \frac{1}{2}\epsilon_{abcd}R^{bc}\wedge(T^d + ie^{\underline{\alpha}}\wedge e^{\underline{\beta}}\Gamma^d_{\underline{\alpha\beta}}) \equiv 0\,. \end{aligned}$$

To check that $\delta_{ls}S_{D,SG}=0$ (or $\tilde{\delta}_{gc}S_{D,SG}=0$) implies (73) (or (74)) and *viceversa* it is sufficient to insert (67)–(72) (or (28), (29)) in the general expression for the supergravity action variation (66),

$$\begin{aligned} \delta_{ls}S_{D,SG} &= -(-1)^D\int_{M^D}(M_{(D-1)a}\wedge\delta_{ls}e^a - \Psi_{(D-1)\underline{\alpha}}\wedge\delta_{ls}e^{\underline{\alpha}} + \ldots) = \\ &= -(-1)^D\int_{M^D}(-2iM_{(D-1)a}\wedge e^{\underline{\beta}}\Gamma^a_{\underline{\alpha\epsilon}} + \mathcal{D}\Psi_{(D-1)\underline{\alpha}} + \ldots)\epsilon^{\underline{\alpha}} = 0. \end{aligned}$$

Then, one sees again (*cf.* Sec. 4) that the gauge invariance of the action and the Noether identities imply each other.

6 Supergravity in superspace

The local supersymmetry $\delta_{ls}(\epsilon^{\underline{\alpha}})$, eqs. (67)–(72), has a structure which resembles that of the variational copy of the general coordinate transformations, $\tilde{\delta}_{gc}(t^a)$ (eqs. (23), (25)), but with a fermionic parameter. The similarity can be recognized also from the structure of the Noether identities, (73), (74). This is one more reason for the existence of superspace $\Sigma^{(D|n)}$ (originally introduced [14] in connection with global supersymmetry)

with coordinates $Z^M = (x^\mu, \theta^\alpha)$, $\alpha = 1,\ldots,n$, where, *e.g.* $n = 2^{[D/2]}$ for $N=1$, $D\neq 10$ and $N=2$, $D=10$. The holonomic or coordinate basis for the cotangent superspace is provided by $dZ^M = (dx^\mu, d\theta^\alpha)$, while the general unholonomic basis (with $Spin(1,D-1)$ indices denoted by underlined greek letters) is defined by the supervielbein forms [2]

$$E^A(Z) = (E^a(Z), E^{\underline{\alpha}}(Z)) = dZ^M E_M^A(Z)\,. \qquad (76)$$

The differential geometry of spacetime can be extended to superspace [6]. In particular, introducing the spin connection superform

$$w^{ab} = dZ^M w^{ab}_M(Z)\,,\ w_{\underline{\alpha}}{}^{\underline{\beta}} \equiv \tfrac{1}{4}w^{ab}\Gamma_{ab\underline{\alpha}}{}^{\underline{\beta}}\,, \qquad (77)$$

one can define superspace torsions and curvature

$$\begin{aligned} T^a &:= \mathcal{D}E^a = dE^a - E^b\wedge w_b{}^a\\ &:= \tfrac{1}{2}E^A\wedge E^B T_{BC}{}^a\,, & (78)\\ T^{\underline{\alpha}} &:= \mathcal{D}E^{\underline{\alpha}} = dE^{\underline{\alpha}} - E^{\underline{\beta}}w_{\underline{\alpha}}{}^{\underline{\beta}}\\ &:= \tfrac{1}{2}E^A\wedge E^B T_{BC}{}^{\underline{\alpha}}\,, & (79)\\ R^{ab} &:= dw^{ab} - w^{ac}\wedge w_c{}^b\\ &:= \tfrac{1}{2}E^A\wedge E^B R_{BC}{}^{ab}\,, & (80) \end{aligned}$$

as well as, when the supergravity multiplet contains antisymmetric tensor gauge fields $C_q(x)$, the generalized field strengths

$$\begin{aligned} \mathcal{H}_{q+1} &:= dC_q - c_1E^{\underline{\alpha}}\wedge E^{\underline{\epsilon}}\wedge\bar{\Gamma}^{(q-1)}{}_{\underline{\alpha\epsilon}} & (81)\\ &= \tfrac{1}{(q+1)!}E^{A_{q+1}}\wedge E^{A_1}\mathcal{H}_{A_1\ldots A_{q+1}}(Z)\,, & (82)\\ \bar{\Gamma}^{(k)}{}_{\underline{\alpha\epsilon}} &:= \tfrac{1}{k!}E^{a_1}\wedge\ldots\wedge E^{a_k}\Gamma_{a_1\ldots a_p\underline{\alpha\epsilon}}\,, & (83) \end{aligned}$$

of the various gauge superforms

$$C_q := \tfrac{1}{q!}E^{A_{q+1}}\wedge\ldots\wedge E^{A_1}C_{A_1\ldots A_q}(Z)\,. \qquad (84)$$

6.1 Local supersymmetries of supergravity in superspace

The differential forms on $\Sigma^{(D|n)}$ are invariant under arbitrary changes of coordinates, *i.e.* under local superspace diffeomorphisms,

$$\begin{aligned} Z^M \mapsto Z^{M\prime} &= Z^M + b^M(Z)\,, & (85)\\ E^A(Z) \mapsto E^{A\prime}(Z') &= E^A(Z)\,,\ \text{etc.}\,, & (86) \end{aligned}$$

for which (*cf.* (20), (21))

$$\begin{aligned} \delta_{sdiff}Z^M &= b^M(Z)\,, & (87)\\ \delta_{sdiff}E^A &\equiv \delta'_{sdiff}E^A + L_bE^A = 0\,,\ \text{etc.}\,. & (88) \end{aligned}$$

The superspace local Lorentz transformation δ_L, with superfield parameter $L^{ab}(Z) = -L^{ba}(Z)$, is also a manifest symmetry of supergravity.

[2]Such superforms, but depending on a Goldstone fermion $\Theta^{\underline{\alpha}}(x)$ rather than on the superspace coordinate θ^β, were already used in [13].

The superspace general coordinate transformations are defined by an arbitrary change of the local superspace coordinates (as in (87)),

$$\delta_{sgc} Z^M = t^M(Z) \ , \tag{89}$$

but, in contrast with (88),

$$\delta_{sgc} E^A = L_t E^A = \mathcal{D} i_t E^A + i_t T^A + E^B i_t w_B{}^A \ , \tag{90}$$

$$i_t w_B{}^A = t^M w_{MB}{}^A \ , \ i_t E^A = t^M E_M^A \ , \quad \text{etc.}$$

$$w_B{}^A = diag(w_b{}^a, w_{\underline{\beta}}{}^{\underline{\alpha}}) \ . \tag{91}$$

Using the superdiffeomorphism invariance, the variational copy $\tilde{\delta}_{sgc}$ [6] of δ_{sgc} is defined by (*cf.* (27))

$$\tilde{\delta}_{sgc}(t^A) = \delta_{sgc}(t^M) + \delta_{sdiff}(b^M = -t^M) + \delta_L(L^{ab} = -i_t w^{ab}) \ ,$$

$$t^A(Z) = i_t E^A(Z) \equiv t^M(Z) E_M^A(Z) \ . \tag{92}$$

Again $\tilde{\delta}_{sgc}$ does not act on the superspace coordinates (x^μ, θ^α), but acts on superforms as the Lie derivative (*cf.* (29))

$$\tilde{\delta}_{sgc} Z^M := 0 \ , \tag{93}$$

$$\tilde{\delta}_{sgc} E^A(Z) = dZ^M \tilde{\delta}'_{sgc} E_M^A(Z) := \mathcal{D} t^A + E^C t^B T_{BC}{}^A(Z) \ , \tag{94}$$

$$\tilde{\delta}_{sgc} w^{ab}(Z) := E^D t^C R_{CD}{}^{ab}(Z) \ . \tag{95}$$

In particular, the fermionic part $\tilde{\delta}_{sgcf}(\epsilon^{\underline{\alpha}})$ of $\tilde{\delta}_{sgc}$, determined by the parameter $t^A(Z) = (0, \epsilon^{\underline{\alpha}}(Z))$, $\tilde{\delta}_{sgcf}(\epsilon^{\underline{\alpha}}(Z)) = \tilde{\delta}_{sgc}(0, \epsilon^{\underline{\alpha}}(Z))$, can be called superspace local supersymmetry. Its action is given by

$$\tilde{\delta}_{sgcf} Z^M := 0 \ , \tag{96}$$

$$\tilde{\delta}_{sgcf} E^a(Z) := E^C \epsilon^{\underline{\beta}} T_{\underline{\beta}C}{}^a(Z) \ , \tag{97}$$

$$\tilde{\delta}_{sgcf} E^{\underline{\alpha}}(Z) := \mathcal{D}\epsilon^{\underline{\alpha}} + E^C \epsilon^{\underline{\beta}} T_{\underline{\beta}C}{}^{\underline{\alpha}}(Z) \ , \tag{98}$$

$$\tilde{\delta}_{sgcf} w^{ab}(Z) := E^D \epsilon^{\underline{\gamma}} R_{\underline{\gamma}D}{}^{ab}(Z) \ , \tag{99}$$

and is similar, albeit not identical, to the straightforward extension of the local supersymmetry transformations δ_{ls} (eqs. (68)–(72)) to superspace. We will see below that the desired identification of $\tilde{\delta}_{sgcf}|_{\theta=0} = \tilde{\delta}_{sgcf}(\epsilon^{\underline{\alpha}}(x,0))$, with $\delta_{ls}(\epsilon^{\underline{\alpha}}(x))$ appears when the superspace constraints are taken into account.

6.2 Superspace constraints

The unrestricted supervielbein and spin connection contain a large amount of fields (mostly unwanted). The supergravity multiplets can be extracted from the supervielbein by imposing *covariant* constraints on the superspace torsions, curvature and the gauge superform field strengths. The main constraints have the form

$$T^a = iE^{\underline{\alpha}} \wedge E^{\underline{\beta}} \Gamma^a_{\underline{\alpha}\underline{\beta}} \ , \tag{100}$$

$$\mathcal{H}_{q+1} := dC_q - c_1 E^{\underline{\alpha}} \wedge E^{\underline{\epsilon}} \wedge \bar{\Gamma}^{(q-1)}{}_{\underline{\alpha}\underline{\epsilon}} = \frac{1}{(q+1)!} E^{a_{q+1}} \wedge \ldots \wedge E^{a_1} F_{a_1 \ldots a_{q+1}}(Z) \ , \tag{101}$$

and can be derived as a straightforward extension of the component, first order form supergravity eqs. (63), (61) to superspace. This fact is not accidental. It reflects the existence of the so-called group manifold or 'rheonomic' approach to supergravity [7, 8], which provides the bridge between the component and superfield formalism (see also Sec. 2 of [1] for a brief review).

6.3 Local supersymmetry of $(D = 11)$ supergravity constraints

After the constraints (100), (101) are taken into account, the fermionic general coordinate transformations $\tilde{\delta}_{sgcf}$ simplify. In particular,

$$\tilde{\delta}_{sgcf} Z^M = 0 \ , \tag{102}$$

$$\tilde{\delta}_{sgcf} E^a(Z) = -2iE^{\underline{\alpha}} \Gamma^a_{\underline{\alpha}\underline{\beta}} \epsilon^{\underline{\beta}} \ , \quad \text{etc.} \tag{103}$$

Now one can easily see that $\tilde{\delta}_{sgcf} E^a|_{\theta=0}$ becomes identical to $\tilde{\delta}_{ls} e^a$, $\tilde{\delta}_{sgcf} E^a|_{\theta=0} = \tilde{\delta}_{ls} e^a$, after the usual identification of the supergravity forms with the leading components of superforms, $(E^a, E^{\underline{\alpha}})|_{\theta=0, d\theta=0} = (e^a, e^{\underline{\alpha}})$, *etc.*, is made.

To be specific, let us consider $D = 11$ supergravity [15] $(a = 0, 1, \ldots, 10 \ , \ \underline{\alpha} = 1, \ldots, 32)$. Here the superspace constraints (100), (101),

$$T^a = -iE^{\underline{\gamma}} \wedge E^{\underline{\beta}} \Gamma^a_{\underline{\gamma}\underline{\beta}} \ , \tag{104}$$

$$\mathcal{H}_4 \equiv dC_3 - \tfrac{1}{2} E^{\underline{\alpha}} \wedge E^{\underline{\beta}} \wedge \bar{\Gamma}^{(2)}_{\underline{\alpha}\underline{\beta}} = \tfrac{1}{4!} E^{c_4} \wedge \ldots \wedge E^{c_1} \ F_{c_1 \ldots c_4} \ , \tag{105}$$

imply

$$T^{\underline{\alpha}} = \tfrac{1}{2} E^b \wedge E^c T_{cb}{}^{\underline{\alpha}} - \tfrac{1}{2} E^b \wedge E^{\underline{\beta}} T_{b\underline{\beta}}{}^{\underline{\alpha}} \ , \tag{106}$$

$$T_{b\underline{\beta}}{}^{\underline{\alpha}} = \tfrac{i}{9} (F_{bb_1b_2b_3} (\Gamma^{b_1b_2b_3})_{\underline{\beta}}{}^{\underline{\alpha}} + \tfrac{1}{8} F^{b_1b_2b_3b_4} (\Gamma_{bb_1b_2b_3b_4})_{\underline{\beta}}{}^{\underline{\alpha}}) \ , \tag{107}$$

$$R^{ab} = -2iE^{\underline{\alpha}} \wedge E^{\underline{\beta}} T_{\underline{\beta}}{}^{[a\underline{\gamma}} \Gamma^{b]}_{\underline{\gamma}\underline{\alpha}} + E^c \wedge E^{\underline{\alpha}} (iT^{ab\underline{\beta}} \Gamma_{c\underline{\beta}\underline{\alpha}} - 2iT_c{}^{[a|\underline{\beta}} \Gamma^{|b]}_{\underline{\beta}\underline{\alpha}}) +$$

$$+ \tfrac{1}{2} E^d \wedge E^c \ R_{cd}{}^{ab} \ . \tag{108}$$

Using (104)–(108), the superspace local supersymmetry (96)–(99) takes the form

$$\tilde{\delta}_{sgcf} Z^M = 0 \ , \tag{109}$$

$$\tilde{\delta}_{sgcf} E^a = -2iE^{\underline{\alpha}} \Gamma^a_{\underline{\alpha}\underline{\beta}} \epsilon^{\underline{\beta}}(Z) \ , \tag{110}$$

$$\tilde{\delta}_{sgcf} E^{\underline{\alpha}} = \mathcal{D}\epsilon^{\underline{\alpha}}(Z) + E^b T_{\underline{\beta}b}{}^{\underline{\alpha}} \ \epsilon^{\underline{\beta}}(Z) \ , \tag{111}$$

$$\tilde{\delta}_{sgcf} C_3 = E^{\underline{\alpha}} \wedge \bar{\Gamma}^{(2)}_{\underline{\alpha}\underline{\beta}} \ \epsilon^{\underline{\beta}}(Z) \ , \tag{112}$$

$$\tilde{\delta}_{sgcf} F_{a_1a_2a_3a_4} = 3! T_{[a_1a_2}{}^{\underline{\alpha}} \Gamma_{a_3a_4]\underline{\alpha}\underline{\beta}} \epsilon^{\underline{\beta}}(Z) \ , \tag{113}$$

$$\tilde{\delta}_{sgcf} w^{ab} = -4iE^{\underline{\alpha}} \ T_{(\underline{\alpha}}{}^{[a\,\underline{\gamma}} \ \Gamma^{b]}{}_{\underline{\beta})\underline{\gamma}} \ \epsilon^{\underline{\beta}}(Z) +$$

$$+ iE^c (T^{ab\,\underline{\alpha}} \Gamma_{c\underline{\alpha}\underline{\beta}} - 2T_c{}^{[a\,\underline{\alpha}} \Gamma^{b]}{}_{\underline{\alpha}\underline{\beta}}) \epsilon^{\underline{\beta}} \ . \tag{114}$$

Setting $\theta = 0$, $d\theta = 0$ in eqs. (110)–(114), one arrives at the *on-shell* version of the local supersymmetry transformations characteristic of the component supergravity action[3] *i.e.*,

[3]Alternatively, substituting $\tilde{\theta}(x)$ for θ in (110)–(114) one obtains the *on-shell* version of the local supersymmetry transformations characteristic of the group manifold or rheonomic action for D=11 supergravity [8].

the actual local supersymmetry transformation which leaves the action invariant differs from the pull-backs of (110)–(114) to M^D by terms which vanish on the mass shell [4].

The discussion of the previous section suggests the following observation. The same transformation rules for superfields (superforms), (110)–(114) appear for the original form of the fermionic general coordinate transformations with

$$\delta_{sgcf} Z^M = \epsilon^{\underline{\beta}}(Z) E_{\underline{\beta}}^M(Z) , \tag{115}$$

$$\Leftrightarrow \begin{cases} i_{\delta_{sgcf}} E^a = \delta_{sgcf} Z^M E_M^a = 0 , \\ i_{\delta_{sgcf}} E^{\underline{\alpha}} = \delta_{sgcf} Z^M E_M^{\underline{\alpha}} = \epsilon^{\underline{\alpha}}(Z) , \end{cases}$$

$$\delta_{sgcf} E^a = -2i E^{\underline{\alpha}} \Gamma^a_{\underline{\alpha}\underline{\beta}} \epsilon^{\underline{\beta}}(Z) , \tag{116}$$

$$\delta_{sgcf} E^{\underline{\alpha}} = \mathcal{D}\epsilon^{\underline{\alpha}}(Z) + E^b T_{\underline{\beta} b}{}^{\underline{\alpha}} \epsilon^{\underline{\beta}}(Z) , \tag{117}$$

$$\delta_{sgcf} C_3 = E^{\underline{\alpha}} \wedge \bar{\Gamma}^{(2)}_{\underline{\alpha}\underline{\beta}} \epsilon^{\underline{\beta}}(Z) , \tag{118}$$

$$\delta_{sgcf} F_{a_1 a_2 a_3 a_4} = 3! T_{[a_1 a_2}{}^{\underline{\alpha}} \Gamma_{a_3 a_4] \underline{\alpha}\underline{\beta}} \epsilon^{\underline{\beta}}(Z) , \tag{119}$$

$$\delta_{sgcf} w^{ab} = -4i E^{\underline{\alpha}} T_{(\underline{\alpha}}{}^{[a\,\underline{\gamma}} \Gamma^{b]}{}_{\underline{\beta})\underline{\gamma}} \epsilon^{\underline{\beta}}(Z) + i E^c (T^{ab\,\underline{\alpha}} \Gamma_{c\underline{\alpha}\underline{\beta}} - 2T_c{}^{[a\,\underline{\alpha}} \Gamma^{b]}{}_{\underline{\alpha}\underline{\beta}}) \epsilon^{\underline{\beta}} . \tag{120}$$

Thus the $D = 11$ superspace constraints are invariant under *both* δ_{sgcf} and $\tilde{\delta}_{sgcf}$. This reflects the superdiffeomorphism invariance (85)–(88) of forms.

7 Local supersymmetry and κ–symmetry of a superbrane in a supergravity background

To see why a full account of the local gauge symmetries in supergravity can be relevant, let us now consider the standard description of a super-p-brane moving in a supergravity background [5].

Consider, *e.g.* the supermembrane (M2-brane) in the $D = 11$ supergravity background defined by the constraints (104)–(105). Its action is [16]

$$S_{11,2} = \int_{W^3} \tfrac{1}{3!} * \hat{E}_a \wedge \hat{E}^a - \hat{C}_3(\hat{x}, \hat{\theta}) , \tag{121}$$

where

$$\hat{E}^a = d\hat{Z}^M(\xi) E_M^a(\hat{Z}) = d\xi^i \partial_i \hat{Z}^M E_M^a(\hat{Z}) , \tag{122}$$

$$\hat{C}_3 = \tfrac{1}{3!} d\hat{Z}^{M_3} \wedge \ldots \wedge d\hat{Z}^{M_1} C_{M_1 M_2 M_3}(\hat{Z}) , \tag{123}$$

are the pull-backs $\hat{\phi}^*(E^a)$, $\hat{\phi}^*(C_3)$ of the supervielbein and gauge field superforms on the $\Sigma^{(11|32)}$ superspace by the map

$$\hat{\phi} : W^3 \to \Sigma^{(11|32)} , \qquad \hat{\phi} : \xi^i \mapsto \hat{Z}^M(\xi) , \tag{124}$$

[4]Note that the restoration of such terms is an involved technical problem. However, the use of the second Noether theorem can simplify the proof of the local supersymmetry of the action, as it allows to work with equations of motion instead of the general variation.

[5]Such description could be regarded as the background field approximation to a fully dynamical description of supergravity—super-p-brane system based on a coupled action including both the supergravity and super-p-brane Lagrangians [1].

so that $\hat{Z}^M = \hat{Z}^M(\xi)$ *etc.* As supergravity is treated as a background, the set of dynamical variables includes only the local supercoordinate functions $\hat{Z}^M(\xi) = (\hat{x}^\mu(\xi), \hat{\theta}^\alpha(\xi))$, which define the worldvolume as a surface in superspace, $\{Z^M \in \Sigma^{(11|32)} \mid Z^M = \hat{Z}^M(\xi)\}$. Hence the basic variations, $\delta\hat{Z}^M(\xi)$, can be recognized as a counterpart of superspace general coordinate transformations (89), (90), and can be split into the bosonic and fermionic parts

$$i_\delta \hat{E}^a \equiv \delta\hat{Z}^M(\xi) \hat{E}_M^a(\hat{Z}(\xi)) ,$$
$$i_\delta \hat{E}^{\underline{\alpha}} \equiv \delta\hat{Z}^M(\xi) \hat{E}_M^{\underline{\alpha}}(\hat{Z}(\xi)) . \tag{125}$$

Taking into account the constraints, one finds that the fermionic variations of the supercoordinate functions, $\delta_f \hat{Z}^M(\xi)$, defined by (*cf.* (115))

$$i_{\delta_f} \hat{E}^a = 0 , \qquad i_{\delta_f} \hat{E}^{\underline{\alpha}} \neq 0 ,$$
$$\Leftrightarrow \quad \delta_f \hat{Z}^M(\xi) = i_{\delta_f} \hat{E}^{\underline{\alpha}}(\xi) E_{\underline{\alpha}}^M(\hat{Z}(\xi)) , \tag{126}$$

lead to

$$\delta_f \hat{E}^a = -2i \hat{E}^{\underline{\alpha}} \Gamma^a_{\underline{\alpha}\underline{\beta}} i_{\delta_f} \hat{E}^{\underline{\beta}} , \tag{127}$$

$$\delta_f \hat{C}_3 = \hat{E}^{\underline{\alpha}} \wedge \hat{\bar{\Gamma}}^{(2)}_{\underline{\alpha}\underline{\beta}} i_{\delta_f} \hat{E}^{\underline{\beta}} \tag{128}$$

(*cf.* (115), (116), (118), where the rôle of $\epsilon^{\underline{\alpha}}(Z)$ is played by $i_\delta \hat{E}^{\underline{\alpha}}$). Hence,

$$\begin{aligned} \delta_f S_{11,2} &= \int_{W^3} \tfrac{1}{2} * \hat{E}_a \wedge \delta_f \hat{E}^a - \delta_f \hat{C}_3 = \\ &= \int_{W^3} (-i * \hat{E}_a \wedge \hat{E}^{\underline{\alpha}} \Gamma^a_{\underline{\alpha}\underline{\epsilon}} - \hat{E}^{\underline{\alpha}} \wedge \hat{\bar{\Gamma}}^{(2)}_{\underline{\alpha}\underline{\epsilon}}) i_{\delta_f} \hat{E}^{\underline{\epsilon}} \\ &= -i \int_{W^3} * \hat{E}_a \wedge \hat{E}^{\underline{\alpha}} (\Gamma^a (I - \bar{\gamma}))_{\underline{\alpha}\underline{\beta}} i_{\delta_f} \hat{E}^{\underline{\beta}} , \end{aligned} \tag{129}$$

where

$$\bar{\gamma} \equiv \frac{i}{3! \sqrt{|g|}} \epsilon^{ijk} \hat{E}_i^a \hat{E}_j^b \hat{E}_k^c \Gamma_{abc} \tag{130}$$

is the well known matrix satisfying $\mathrm{tr}\bar{\gamma} = 0$, $\bar{\gamma}^2 = I$, that enters in the M2-brane κ-*symmetry* projector $\frac{1}{2}(1 + \bar{\gamma})$ [16]. Thus, for $i_{\delta_f} \hat{E}^{\underline{\alpha}} = i_{\delta_\kappa} \hat{E}^{\underline{\alpha}} := (1 - \bar{\gamma})^{\underline{\alpha}}{}_{\underline{\beta}} \kappa^{\underline{\beta}}(\xi)$ we find $\delta_\kappa S_{11,2} = 0$, which expresses the fundamental κ–symmetry of the supermembrane [16].

We see that when computing the fermionic variation δ_f (eqs. (126), (127), (128)) of the supermembrane action we actually perform a superspace fermionic general coordinate transformation δ_{sgcf}, eqs. (115)–(118), pulled back to W^3: $\phi^*(\delta_{sgcf}) = \delta_f$. The variation δ_f produces the superbrane equations of motion on W^3, $\hat{\Xi}_{\underline{\beta}} := * \hat{E}_a \wedge \hat{E}^{\underline{\alpha}} (\Gamma^a (I - \bar{\gamma}))_{\underline{\alpha}\underline{\beta}} = 0$. Thus, the whole variation δ_f is not a local symmetry of the dynamical system including the superbrane (otherwise, the brane dynamics would be trivial in the 'fermionic' directions). However, this fermionic equation becomes an identity when multiplied by $(1 + \bar{\gamma})$, *i.e.* $\hat{\Xi}_{\underline{\beta}}(1 + \bar{\gamma}) \equiv 0$. This is the Noether identity (Sec. 4, 5.3) for κ–symmetry. On the other hand, as $\phi^*(\delta_{sgcf}) = \delta_f$, this means that the breaking of δ_{sgcf} by the supermembrane is partial and that the part of δ_{sgcf} preserved on W^3 is given by the κ–symmetry transformations. Moreover, as the brane action possesses manifest local Lorentz and diffeomorphism invariances, we can use (27) to conclude that $\delta_f S_{11,2} \equiv \delta_{sgcf} S_{11,2}$ is equal to $\tilde{\delta}_{sgcf} S_{11,2}$.

Hence $\tilde{\delta}_{sgcf} S_{11,2} = 0$ for the superfield supersymmetry transformations (109)–(112) with the parameter restricted on W^3 to be of the form $\epsilon^{\underline{\alpha}}(\hat{Z}) = (1-\bar{\gamma})^{\underline{\alpha}}{}_{\beta}\kappa^{\underline{\beta}}(\xi)$, and we can state that κ–symmetry is just the part of the local supersymmetry which is preserved by the brane action[6].

8 Concluding remarks

The above considerations indicate that
i) The κ–symmetry of the superbrane in the superfield supergravity background is the part of the superfield local supersymmetry characteristic of the supergravity constraints which is not broken by the presence of the superbrane.
ii) In any complete Lagrangian description of the supergravity—superbrane coupled system which includes the standard superbrane action, the local supersymmetry will be partially broken. Any coupled action describing both supergravity and the superbrane will possess not more than 1/2 of the local supersymmetry characteristic of the 'free' supergravity action.
iii) As the superbrane action is written in terms of pull–backs of superspace differential forms and, possibly, the *worldvolume* Hodge star operator, the complete coupled action evidently possesses superdiffeomorphism symmetry δ_{sdiff}.
iv) As the superfield local supersymmetry can be equivalently considered as originated either from the superspace general coordinate transformations δ_{sgcf}, (115)–(120), or from their variational copy $\tilde{\delta}_{sgcf}$, (109)–(114), we conclude that the coupled system of *supergravity and bosonic p-brane* should possess 1/2 of the local supersymmetry characteristic of the free supergravity, if the bosonic p-brane appears to be the $\hat{\theta}(\xi) = 0$ 'limit' of a superbrane [2].

We hope to return to these points in forthcoming publications.

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Quantum T-dualisation up to Two Loops

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Abstract

We first consider the most general σ-model defined on a group manifold $(G_L \times G_R)/G_D$. Its metric, invariant under the action of G_L, includes appropriate parameters describing all the possible breakings of the group G_R. A direct computation of the Ricci tensor shows that the one-loop renormalisability of its T-dualised σ-model is strictly equivalent to the one-loop renormalisability of the original model for any symmetry breaking. Then, restricting ourselves to the simple example of the T-dualised $SU(2)$ σ-model, we show that, even if this model has been claimed to be non-renormalisable at the two-loop order, it is still possible to define a correct quantum theory - at least up to this order - by modifying its target space metric in a finite manner at the one-loop order.

1 Introduction

The subject of classical versus quantum equivalence of T-dualised σ-models has been strongly studied in recent years, and extensive reviews covering abelian, non-abelian dualities and their applications to string theory and statistical physics are available [1, 2, 3]. More recent developements on the geometrical aspects of duality can be found in [4].

The interpretation of T-duality as a canonical transformation, for constant backgrounds, was first given by [5, 6]. Its more general formulation [7] was applied to the non-abelian case in [8, 9].

Restricting ourselves to the non-abelian dualisations of Lie groups, the one-loop equivalence was established for the group manifold of $SU(2)$ in [10, 11]. A further decisive progress was to generalize the one-loop equivalence to an arbitrary Lie group [12]. However, as pointed out in [13], Tyurin's analysis considers only models with explicit invariance under the left group action (whose existence is crucial for the dualisation process) leaving aside the right action and the possible symmetry breaking schemes for it. The one-loop equivalence problem in this more general setting has been examined recently [13],[14] for the group manifold $(SU(2)_L \times SU(2)_R)/SU(2)_D$, where $SU(2)_R$ is broken down to a $U(1)$. The renormalisability does survive despite the lowering of the right isometries.

One of the purposes of the present paper is to analyze the geometry of the dualised model for any $(G_L \times G_R)/G_D$, with arbitrary breaking of G_R. While in [12] supersymmetry considerations à la Busher [15],[16] were of great help to derive the dualised geometry, we will show that a direct computation in local coordinates is fairly efficient to extract the Ricci tensor in the presence of symmetry breaking. An interesting expression for the Ricci tensor of the dualised geometry (with torsion) exhibits its dependance with respect to the geometrical quantities of the original model.

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Then, we ask ourselves the problem of the two-loop renormalisability of dualised models. Some years ago, it was noted that in the minimal dimensional scheme, two-loop renormalisability does not hold for the SU(2) T-dualised model [17, 18]. The second purpose of the present work is a more precise analysis of this two-loop (in)equivalence, still on the simple example of the original SU(2) T-dualised model.

The main remark is that, part of the isometries being somehow lost, the T-dualised models are not - as they should be if one wants to give an all-order analysis - defined by a sufficent system of Ward identities. For example, in our simple case there is, *a priori* only a linear SU(2) [or O(3)] invariance, and any O(3) invariant action is allowed (let us remind the reader that in higher-loop corrections to a classical action, all the terms which are not prohibited by some reason such as power counting, isometries or conservation laws..., would appear). To our present knowledge, the extra constraints coming from the origin of the model (dualisation of an $(SU(2)_L \times SU(2)_R)/SU(2)_D$ chiral model) are not understood. As it is highly probable that they are linked with the space-time dimension, it is not surprising that a minimal dimensional renormalisation scheme fails : as is well known, when the regularization method does not respect all the properties that define the theory, extra finite counter-terms are needed [19].

The content of this article is the following : after setting the notations, in section 2 we characterize the geometry of the group manifold $(G_L \times G_R)/G_D$. This is most conveniently done using frames and, despite symmetry breaking, one obtains a simple form for the Ricci tensor. In section 3 the dualised theory is examined. In section 4 we use the previous results to discuss the one-loop equivalence. In section 5, we focuse on the $SU(2)$ σ-model and its *a priori* quantum bare action and obtain through $\hbar$ expansion the possible counter-terms that may be added to the classical action in order to reabsorb the divergences. Then in section 6 we give the 2-loop divergences and in section 7 we discuss how they match with the candidates in section 5. Our result is that coupling constant and field renormalisations (infinite and finite ones) are not sufficient to ensure the two-loop existence of the T-dualised theory but the metric itself has to be deformed (in a **finite way**). Some concluding remarks are offered in section 8.

A more complete description of these derivations was published in [20, 21].

2 Geometry of the initial theory

Let us consider a target space with coordinates g, with a left and right action of some Lie group G with structure constants $f_{ij}^{\ \ s}$. Its corresponding action can be written

$$S = \frac{1}{2}\int d^2x\, B_{ij}\, \eta^{\mu\nu} J^i_\mu\, J^j_\nu, \tag{1}$$

where the J^i_μ are defined by

$$g^{-1}\partial_\mu g = J^i_\mu T_i, \qquad (T_i)_j^{\ k} = -f_{ij}^{\ \ k}.$$

The matrices of the adjoint representation T_i verify the Jacobi identity

$$[T_i, T_j] = f_{ij}^{\ \ k} T_k, \qquad i, j, k = 1, \ldots \nu = \dim(G).$$

For the reader's convenience we present the derivation [1, 10] of the dualised model. The essence of the dualisation process is to switch from the coordinates g to new coordinates

ψ_i defined as the Lagrange multipliers of the Bianchi identities. Concretely this transformation is carried out starting from the action

$$S = \frac{1}{4}\int d^2x \left[B_{ij}\,\eta^{\mu\nu} J^i_\mu J^j_\nu - \epsilon^{\mu\nu}\psi_i(\partial_\mu J^i_\nu - \partial_\nu J^i_\mu + f_{st}{}^i J^s_\mu \wedge J^t_\nu)\right].$$

The field equations obtained from the variations with respect to the currents J^i_μ give, using minkowskian coordinates on the worldsheet

$$J^{\mu i} = B^{ij}\,\epsilon^{\mu\nu}\left(\partial_\nu\psi_j - (A\cdot\psi)_{jk}\,J^k_\nu\right),$$

where $(A^s)_{ij} = (T_i)_j{}^s = -f_{ij}{}^s$ and $(A\cdot\psi)_{ij} = (A^s)_{ij}\psi_s$. Using this relation, the action can be written, up to total derivatives and in light-cone coordinates $(x_\pm = \frac{x^0 \pm x^1}{\sqrt{2}})$:

$$S = \frac{1}{2}\int d^2x\,\partial_+\psi_i\,(B + A\cdot\psi)^{ij}\partial_-\psi_j. \tag{2}$$

Let us emphasize that the final action of the dualised theory is independent of the choice of the representation for the fields parametrizing G.

As we have seen the starting point is the theory defined by the action (1), where the constant matrix B_{ij} is symmetric and invertible.

Let us begin with a discussion of the isometries. The groups $G_L \times G_R$ and G_D act on g according to

$$g \longrightarrow g' = G_L\,g\,G_R^{-1}, \qquad g \longrightarrow g' = G_D\,g\,G_D^{-1},$$

so that

$$g^{-1}\partial_\mu g \quad\longrightarrow\quad G_R\,g^{-1}\partial_\mu g\,G_R^{-1}.$$

It follows that the action (1) is invariant under G_L, while the matrix B_{ij} will generally break G_R down to some subgroup H (possibly trivial). For a simple group G the most symmetric action is given by the bi-invariant metric

$$B_{ij} = \frac{1}{\rho}g_{ij}, \qquad g_{ij} = \mathrm{Tr}\,(T_iT_j) = \tilde\rho\,\mathrm{Tr}\,(t_it_j), \tag{3}$$

where g_{ij} is the Killing metric and the t_i the defining representation of the simple algebra under consideration. For instance, for a simple compact group, with the standard normalization $\mathrm{Tr}\,(t_it_j) = -2\delta_{ij}$, we see that taking $\rho = -2\tilde\rho$ gives $B_{ij} = \delta_{ij}$.

This choice enlarges the isometry group to $G_L \times G_R$ because $(T_s)_i{}^k g_{kl} = -f_{sil}$ is fully skew-symmetric.

In order to have a better insight of the geometry of this model it is convenient to use a vielbein formalism, through the identification of the currents J^i_μ with the vielbein e^i. We shall write the metric $B_{ij}e^ie^j$ and the Bianchi identities (in fact the Maurer-Cartan equations)

$$de^i + \frac{1}{2}f_{st}{}^i e^s \wedge e^t = 0.$$

We follow the notations of [22] and define the spin-connection $\omega^i{}_j$ by

$$de^i + \omega^i{}_s \wedge e^s = 0, \qquad \omega^i{}_j = \omega^i{}_{j,s}e^s.$$

The frame indices are lowered or raised using the metric B_{ij} and its inverse $B^{ij} = B^{-1}_{ij}$. A straightforward computation gives

$$2\omega_{ij,k} = (T_jB)_{ki} - (T_iB)_{kj} - (T_iB)_{jk}.$$

The curvature is defined by

$$R^i{}_j = d\omega^i{}_j + \omega^i{}_s \wedge \omega^s{}_j = \frac{1}{2}R^i{}_{j,st}\,e^s \wedge e^t,$$

and the Ricci tensor by $ric_{ij} = R^s{}_{i,sj}$. One gets easily

$$ric_{ij} = -\omega^s{}_{i,t}\,\omega^t{}_{j,s} + \omega^t{}_{s,t}\,\omega^s{}_{i,j}.$$

Therefore, as expected for an homogeneous space, the scalar curvature $R = (B^{-1})_{ij}\,ric_{ij}$ is a constant.

A drastic simplification takes place for the bi-invariant metric (3), for which we have

$$ric_{ij} = -\frac{\rho}{4}\,B_{ij}. \tag{4}$$

The metric is therefore Einstein, i.e. the Ricci is proportional to the metric, and such a simple structure will have a counterpart in the dualised theory.

3 Geometry of the dualised theory

In (2) we come back to standard notations and change the coordinates ψ_i to ψ^i. Let us write the dual action

$$S = \frac{1}{2}\int d^2x\,G_{ij}\,\partial_+\psi^i\,\partial_-\psi^j, \quad G_{ij} = (B + A\cdot\psi)^{-1}_{ij},$$

which, written in minkowskian coordinates, gives

$$S = \frac{1}{2}\int d^2x\left[g_{ij}\,\eta^{\mu\nu}\partial_\mu\psi^i\partial_\nu\psi^j + h_{ij}\,\epsilon^{\mu\nu}\partial_\mu\psi^i\partial_\nu\psi^j\right],$$

where $g_{ij} = G_{(ij)}$ is the target space metric and $h_{ij} = G_{[ij]}$ is the torsion potential. The torsion T_{ijk} is defined by $T_{ijk} = \frac{3}{2}\partial_{[i}h_{jk]}$. The connections without torsion γ^i_{jk} and with torsion $\Gamma^i{}_{jk}$ respectively write :

$$\begin{aligned}
\gamma^i_{jk} &= \frac{1}{2}g^{is}(\partial_jg_{sk} + \partial_kg_{sj} - \partial_sg_{jk}),\\
\Gamma^i{}_{jk} &= \frac{1}{2}g^{is}(\partial_jG_{ks} + \partial_kG_{sj} - \partial_sG_{kj})\\
&= \gamma^i_{jk} + T^i_{jk},
\end{aligned}$$

and the coresponding covariant derivatives are :

$$\begin{aligned}
D_ik_j &= \partial_ik_j - \Gamma^s{}_{ij}k_s = \nabla_ik_j - T^s_{ij}v_s,\\
D_ik^j &= \partial_ik^j + \Gamma^j{}_{is}k^s = \nabla_ik^j + T^j_{is}v^s.
\end{aligned}$$

The Riemann tensor without torsion will be noted $R_{ij,kl}$ whereas we will denote the one with torsion as $\bar{R}_{ij,kl}$.

The Ricci tensor, defined by

$$Ric_{ij} = \bar{R}^s{}_{i,sj} = \partial_s\Gamma^s_{ji} - \Gamma^s_{jt}\Gamma^t_{si} - D_jD_i(\ln\sqrt{\det g}),$$

can be expressed as (see [20] for explicit details) :

$$Ric_{ij} = -G_{is}\, ric_{st}\, G_{tj} + D_j v_i, \tag{5}$$

where

$$v_i = -2\, G_{it}\, f_{st}{}^s - \partial_i \ln(\sqrt{\det g}).$$

One can notice that relation (5) relates Ricci tensors living on different manifolds.

Defining

$$w_i = g_{is}\psi^s, \qquad W_i = \psi^s G_{si}\ ,$$

it is also possible to show the useful relation :

$$D_j w_i + \partial_{[i} W_{j]} = \frac{1}{2} G_{ij} + \frac{1}{2}(GBG)_{ij}. \tag{6}$$

When we dualise the bi-invariant metric on G, the initial Ricci tensor is given by (4), and we get for the dual theory

$$Ric_{ij} = \frac{\rho}{4}\,(GBG)_{ij} + D_j v_i.$$

Relation (6) then becomes :

$$D_j w_i = \frac{1}{2}\, G_{ij} + \frac{1}{2}(GBG)_{ij}, \;\; w_i = W_i = (B^{-1})_{is}\psi^s,$$

which makes us deduce

$$Ric_{ij} = -\frac{\rho}{4}\, G_{ij} + D_j\, \mathcal{V}_i, \qquad \mathcal{V}_i = v_i + \frac{\rho}{2} w_i\ .$$

This last relation establishes the quasi-Einstein character of the bi-invariant torsionful dual metric.

4 One-loop equivalence

We are now in position to discuss the quantum properties of the dualised models at the one-loop level.

Let us first consider the broken principal models with classical action (1). Its one-loop divergence is

$$-\frac{1}{2\pi\epsilon}\,\frac{1}{2}\int d^2x\, ric_{ij}\,\eta^{\mu\nu} J^i_\mu\, J^j_\nu, \qquad d = 2-\epsilon,$$

where the Ricci components are computed in the vielbein basis.

Renormalisability in the strict field theoretic sense requires that these divergences have to be absorbed by (field independent) deformations of the coupling constants $\hat{\rho}_s$ hidden in the matrix B and possibly a non-linear field renormalisation. This last ingredient is not

required here because the scalar curvature is a constant. Therefore the renormalisability of the theory (1) is ensured by

$$ric_{ij} = \hat{\chi}_t(\hat{\rho}_s)\frac{\partial}{\partial\hat{\rho}_t}B_{ij}. \tag{7}$$

The one-loop renormalisability is clear for two extreme choices of metrics :

1. The bi-invariant metric, for which relation (4) shows that it is Einstein.

2. The maximally broken metric, for which the matrix B contains $\nu(\nu+1)/2$ independent coupling constants $\hat{\rho}_s$. Since the Ricci is also a symmetric matrix, it can always be absorbed by a deformation of the coupling constants.

For partial breakings of the group G_R, relation (7) may fail to hold and is indeed a constraint which mixes conditions involving the breaking matrix B and the algebra through its structure constants.

We will now assume that the initial theory is renormalisable at one-loop order so that (7) always holds.

In order to define cleanly the coupling constants and their renormalisation we switch from the couplings $\{\hat{\rho}_i,\ i = 1,\ldots,c\}$ to (λ, ρ_i) by the relations

$$\hat{\rho}_1 = \frac{1}{\lambda}, \qquad \hat{\rho}_{i+1} = \frac{\rho_i}{\lambda}, \qquad i = 1,\ldots c-1. \tag{8}$$

We scale similarly the breaking matrix

$$B_{ij} = \frac{1}{\lambda} S_{ij}(\rho_i),$$

where, for simplicity, the matrix S can be taken linear in the couplings ρ_i. Writing (7) with these new coupling constants, the full one-loop action for the initial theory becomes, with $\epsilon = 2-d$,

$$\int \frac{d^2x}{2\,\lambda}\left(1 + \frac{\lambda\chi_\lambda}{2\pi\epsilon} + \frac{\lambda}{2\pi\epsilon}\sum_s \chi_s \frac{\partial}{\partial\rho_s}\right) S_{ij}(\rho) J^i_\mu J^j_\nu. \tag{9}$$

We see that the divergences can be absorbed through coupling constant renormalisations.

In order to compare to the renormalisation properties of the dualised theory, let us recall that the most general conditions giving one-loop renormalisability (we work with worldsheet components) are

$$\begin{cases} Ric_{(ij)} = \hat{\chi}_s\,\dfrac{\partial}{\partial\hat{\rho}_s}\, g_{ij} + D_{(i}\, u_{j)}, \\ Ric_{[ij]} = \hat{\chi}_s\,\dfrac{\partial}{\partial\hat{\rho}_s}\, h_{ij} + u_s\, T^s_{ij} + \partial_{[i}\, U_{j]}, \end{cases}$$

where the $\hat{\rho}_s$ are the coupling constants in the principal model we started from, appearing now in a non trivial way in the dualised model. The only constraint on the functions $\hat{\chi}_s$, is that they should be field independent.

These relations can be gathered into the single one

$$Ric_{ij} = \hat{\chi}_s\,\frac{\partial}{\partial\hat{\rho}_s}\, G_{ij} + D_j u_i + \partial_{[i}(u+U)_{j]}.$$

In order to compute the divergences of the dualised theory in terms of the coupling constants defined in (8), we start from the dual classical action

$$\frac{1}{\lambda}\tilde{G}_{ij}(S,\tilde{\psi})\,\partial_+\tilde{\psi}\,\partial_-\tilde{\psi}^j\ ,$$

where $G(B,\psi) = \lambda\tilde{G}(S,\tilde{\psi})$ and $\tilde{\psi}^i = \lambda\psi^i$. The one-loop counterterms follow from the Ricci tensor taken from relation (5) and written

$$Ric_{ij} = \lambda^2\left\{-G_{is}(S,\tilde{\psi})ric_{st}G_{tj}(S,\tilde{\psi}) + D_j v_i\right\}.$$

Using (6) and (7) appropriately scaled out, we get finaly for the one-loop action of the dualised theory

$$\int\frac{d^2x}{\lambda}\left[\left(1+\frac{\lambda\chi_\lambda}{2\pi\epsilon}+\frac{\lambda}{2\pi\epsilon}\sum_s\chi_s\frac{\partial}{\partial\rho_s}\right)G_{ij}+\frac{\lambda}{2\pi\epsilon}\left(D_j\mathcal{V}_i-2\chi_\lambda\partial_{[i}W_{j]}\right)\right]\partial_+\psi^i\,\partial_-\psi^j,$$

where $\mathcal{V}_i = v_i - 2\chi_\lambda w_i$. Comparing this relation with (9) we conclude that, up to the non-linear field re-definition described by the vector $\mathcal{V}_i$ and the jauge transformation described by W_i, the coupling constants renormalise in exactly the same way than in the principal model we started from. We have thus established their equivalence, in the strict field theoretic sense, and proved that their beta functions do coincide. What is really new is that this equivalence survives even for the maximal breaking of G_R.

As a remark, let us point out that after dualisation, the isometries of G_L, essential for the dualisation process, are lost. Hence for the maximal breaking of G_R, no trace seems to remain of the original isometries in the dualised theory. This constitutes a nice example of non-homogeneous metrics with torsion, with no isometries to account for their one-loop renormalisability. Our computation which puts forward an experimental fact (the one-loop renormalisability) needs some basic theoretical explanation since we know that renormalisability is never accidental but the result of some underlying deeper symmetry.

5 The two-loop order bare action for the T-dualised $SU(2)$ σ-model

In the special case considered here, the original model is the $\frac{SU(2)\times SU(2)}{SU(2)}$ non-linear σ model, and its dualised metric writes :

$$G_{ij}[\vec{\phi}] = \frac{1}{1+\vec{\phi}^2}\left[\delta_{ij}+\phi^i\phi^j+\epsilon_{ijk}\phi^k\right]\ , \tag{10}$$

where $\vec{\phi}$ is a SU(2) (real) vector representation and the ϕ^i , $i = 1, 2, 3$, are the coordinates on the dualised manifold. Then $\vec{\phi}^2$ is a $SO(3)$ invariant and the symmetry is linearly realised. Torsion breaks parity, but the model is invariant under the simultaneous change $\phi\to-\phi$ and $\epsilon_{ijk}\to-\epsilon_{ijk}$. Let us emphasize that no other local symmetry exists for that model.

In order to analyse the two-loop renormalisability of the dualised $SU2$ σ-model, we first examine all the possible ways to reabsorb the divergences through local counter-terms. As usual, we should allow for finite and infinite renormalisations of both field and coupling. But, as it was already shown [17, 18, 21], this appears as insufficient to reabsorb the various divergences. Thus, we also allow for a finite deformation of the classical metric and torsion potential $g_{ij} + h_{ij} = G_{ij}$ to describe its quantum extension : of course, this *à la Friedan* [23] extension of the notion of renormalisability involves *a priori* an infinite number of new parameters. Let us emphasize that we shall consider **only finite deformations**. Furthermore, we would obviously introduce too many parameters by allowing finite counterterms of the field and the coupling and a finite deformation for the metric simultaneously. Indeed, the finite counterterms added in order to renormalise the coupling and the field could be seen as finite deformations of the original metric. So, without restricting the generality of our analysis, we will consider here only infinite counterterms for both coupling and field, while we will only take a finite deformation for the metric. This deformation must of course satisfy the same isometries as the original metric.

Let us first write the bare action :

$$S^o = \frac{1}{\lambda^o}\int G^o_{ij}\partial_+\phi^{oi}\partial_-\phi^{oj} \tag{11}$$

where :

$$\begin{cases} \dfrac{1}{\lambda^o} = \dfrac{1}{\lambda}\left[1+\dfrac{\hbar\lambda}{2\pi}\dfrac{\Lambda_1}{\varepsilon}+\left(\dfrac{\hbar\lambda}{2\pi}\right)^2\dfrac{\Lambda_2}{\varepsilon}+\cdots\right], \\ \vec{\phi}^o = \vec{\phi}+\dfrac{\hbar\lambda}{2\pi}\dfrac{\vec{v}_1(\vec{\phi})}{\varepsilon}+\left(\dfrac{\hbar\lambda}{2\pi}\right)^2\dfrac{\vec{w}_2(\vec{\phi})}{\varepsilon}+\cdots, \\ G^o_{ij} = G_{ij}+\dfrac{\hbar\lambda}{2\pi}\tilde{G}_{ij}+\left(\dfrac{\hbar\lambda}{2\pi}\right)^2\hat{G}_{ij}+\cdots \end{cases} \tag{12}$$

The double poles $\frac{\hbar^2}{\varepsilon^2}$ have been hidden in the dots : we only need the coefficient of $\frac{1}{\varepsilon}$ (the double poles are not new quantities as they are directly related to first order simple poles and it has already been proved that the dualised $SU2\,\sigma$-model is one-loop renormalisable [10]).

In order to express (11), we shall introduce the Lie derivative $\underset{\vec{k}}{\mathcal{L}}$. For any tensor S_{ij} defined on a manifold with coordinates ϕ^j , one has, in a change of coordinates :

$$S^0_{ij}(\vec{\phi^0})\partial_+\phi^{oi}\partial_-\phi^{oj} = S_{ij}(\vec{\phi})\partial_+\phi^i\partial_-\phi^j\ ,$$

and if $\vec{\phi^0} = \vec{\phi}+\eta\vec{k}$,

$$S_{ij}(\vec{\phi}) = S^0_{ij}(\vec{\phi}) - \eta\,\underset{\vec{k}}{\mathcal{L}}(S^0_{ij}(\vec{\phi})) + \mathcal{O}(\eta^2). \tag{13}$$

We remind the reader that

$$\underset{\vec{k}}{\mathcal{L}}(S_{ij}) = k^s\nabla_s S_{ij} + S_{sj}\nabla_i k^s + S_{is}\nabla_j k^s. \tag{14}$$

Expanding (11) with the help of (12), one gets the possible counter-terms at lowest orders :

- 0 order in $\dfrac{\hbar\lambda}{2\pi}$:

$$\frac{1}{\lambda}G_{ij}\partial_+\phi^i\partial_-\phi^j \tag{15}$$

- first order in $\frac{\hbar\lambda}{2\pi}$:

$$\frac{1}{\lambda}\left[\frac{1}{\varepsilon}\left(\Lambda_1 G_{ij} + \underset{\vec{v}_1}{\mathcal{L}}(G_{ij})\right) + \tilde{G}_{ij}\right]\partial_+\phi^i\partial_-\phi^j\,, \tag{16}$$

- second order in $\frac{\hbar\lambda}{2\pi}$:

$$\frac{1}{\lambda}\left[\frac{1}{\varepsilon^2}(\cdots) + \frac{1}{\varepsilon}\left(\Lambda_1\,\tilde{G}_{ij} + \Lambda_2\,G_{ij} + \underset{\vec{v}_1}{\mathcal{L}}(\tilde{G}_{ij}) + \underset{\vec{w}_2}{\mathcal{L}}(G_{ij})\right) + (\cdots)\right]\partial_+\phi^i\partial_-\phi^j\,. \tag{17}$$

As we don't consider the 3-loop order, only the term in $\frac{1}{\varepsilon}$ needs to be explicit in (17).

6 The two-loop order divergences

We use the expression of the covariant divergences given by Hull and Townsend [24], in the background field method and in the **minimal** dimensional scheme, up to the two-loop order :

$$\begin{cases} Div^1_{ij} = -\dfrac{\hbar}{2\pi\varepsilon}Ric_{ij} \\ Div^2_{ij} = -\dfrac{\hbar^2\lambda}{8\pi^2\varepsilon}\left(\bar{R}_i{}^{klm}(\bar{R}_{klmj} - \dfrac{1}{2}\bar{R}_{lmkj}) + 2T^k{}_{mn}T^{lmn}\bar{R}_{kijl}\right) \end{cases}$$

In order to ensure the renormalisability of the theory, these divergences should match with the candidate counter-terms given by (16) and (17) :

$$\begin{cases} CT^1_{ij} = \dfrac{\hbar}{2\pi\varepsilon}\left(\Lambda_1 G_{ij} + \underset{\vec{v}_1}{\mathcal{L}}(G_{ij})\right) \\ CT^2_{ij} = \dfrac{\hbar^2\lambda}{4\pi^2\varepsilon}\left(\Lambda_1\tilde{G}_{ij} + \underset{\vec{v}_1}{\mathcal{L}}(\tilde{G}_{ij}) + \Lambda_2 G_{ij} + \underset{\vec{w}_2}{\mathcal{L}}(G_{ij})\right) \end{cases} \tag{18}$$

It has been previously proven that the dualised metric is quasi-Einstein as soon as the original metric is Einstein. So, the first counterterm in (18) can match the one-loop divergences Div^1_{ij}, and in our special case, we get :

$$\Lambda_1 = \frac{1}{2} \quad , \quad \vec{v}_1 = \frac{1}{2}\left(\frac{1-\phi^2}{1+\phi^2}\right)\vec{\phi}$$

The addition to the effective action of a $\hbar$ finite deformation of the metric modifies the $\hbar^2$ divergences. The additional term is easily obtained as

$$-\frac{\hbar}{2\pi\varepsilon}\left\{Ric_{ij}\left(G_{kl} + \frac{\hbar\lambda}{2\pi}\tilde{G}_{kl}\right) - Ric_{ij}(G_{kl})\right\} \equiv -\frac{\hbar^2\lambda}{4\pi^2\varepsilon}\Delta_{ij} + \mathcal{O}(\hbar^3)\,.$$

Finally, the dualised $SU2$ σ-model will be renormalisable at two loops if and only if we can find $\left\{\tilde{G}_{ij}[\vec{\phi}]\,;\ \Lambda_2\,,\ \vec{w}_2[\vec{\phi}]\right\}$ such that :

$$\frac{\hbar^2\lambda}{4\pi^2\varepsilon}\left(\Lambda_1\tilde{G}_{ij} + \underset{\vec{v}_1}{\mathcal{L}}(\tilde{G}_{ij}) + \Lambda_2 G_{ij} + \underset{\vec{w}_2}{\mathcal{L}}(G_{ij})\right) = \frac{\hbar^2\lambda}{4\pi^2\varepsilon}\Delta_{ij} - Div^2_{ij}\,. \tag{19}$$

An important fact must be pointed out here as the last equation, once the one-loop divergences have been reabsorbed, is equivalent to the following one :

$$\begin{aligned} Ric\Big(G + \frac{\hbar\lambda}{2\pi}\tilde{G}\Big) &= \Big(\Lambda_1 + \frac{\hbar\lambda}{2\pi}\Lambda_2\Big)\Big(G + \frac{\hbar\lambda}{2\pi}\tilde{G}\Big) \\ &+ \underset{\vec{v}_1 + \frac{\hbar\lambda}{2\pi}\vec{w}_2}{\mathcal{L}}\Big(G + \frac{\hbar\lambda}{2\pi}\tilde{G}\Big) - \frac{2\pi\varepsilon}{\hbar}Div^2 + \mathcal{O}(\hbar^2)\,. \end{aligned} \tag{20}$$

This means that the finite deformation of the metric $\tilde{G}$ will be defined up to the ones that keep the quasi-Einstein property of the metric at the $\hbar$ order.

7 Results

According to the linearly realised symmetry of the T-dualised $SU2\,\sigma$-model, the finite deformation of the metric $\tilde{G}_{ij}$ writes :

$$\tilde{G}_{ij} = \alpha(\tau)\delta_{ij} + \beta(\tau)\phi^i\phi^j + \epsilon_{ijk}\gamma(\tau)\phi^k\,,$$

where $\tau = \vec{\phi}^2$ and the vector $\vec{w}_2$ writes $\vec{w}_2 = w_2(\tau)\vec{\phi}$. It is then possible to re-express (20) as a set of three linear differential equations (see [21] for more details). We didn't put any finite counterterms for the field and the coupling as it is always possible to see them as a deformation of the metric. Indeed, the possibility for a finite field renormalisation $\vec{W}$ and a finite coupling renomalisation b could be interprated as a deformation of the form $b\,G_{ij} + \underset{\vec{W}(\vec{\phi})}{\mathcal{L}}(G_{ij})$. In an opposite view, we can say now that there is always a part of the deformation $\tilde{G}_{ij}$ that can be transferred into such finite counterterms. Using this freedom, it is possible to reabsorb $\alpha(\tau)$ and redefine $\tilde{G}_{ij}$ such that :

$$\tilde{G}_{ij} = \bar{G}_{ij} + b\,G_{ij} + \underset{\vec{W}}{\mathcal{L}}\ G_{ij}\,, \tag{21}$$

with

$$\bar{G}_{ij} = \bar{\beta}(\tau)\phi^i\phi^j + \epsilon_{ijk}\bar{\gamma}(\tau)\phi^k\,.$$

Equation (20) depends now on the two functions $\bar{\beta}(\tau)$ and $\bar{\gamma}(\tau)$. It is possible to express $\bar{\beta}(\tau)$ as a function of $\bar{\gamma}(\tau)$ which itself satisfies a non-homogeneous linear fourth order differential equation :

$$\begin{aligned} \bar{\beta}(\tau) = &\frac{3(1-\tau)(13+6\tau+\tau^2)}{4(6+\tau)(1+\tau)^3} + \frac{2(17+3\tau)}{6+\tau}\bar{\gamma}(\tau) \\ &- \frac{4(5-6\tau-\tau^2)}{6+\tau}\bar{\gamma}'(\tau) - \frac{8\tau}{6+\tau}\bar{\gamma}''(\tau)\,, \\ &\bar{\gamma}^{(4)}(\tau) + \tfrac{(6-\tau)(7+\tau)}{\tau(6+\tau)}\bar{\gamma}^{(3)}(\tau) \\ &\quad + \tfrac{1260-276\tau-91\tau^2+3\tau^3+\tau^4}{4\tau^2(6+\tau)^2}\bar{\gamma}''(\tau) \\ &\quad + \tfrac{-120+254\tau+57\tau^2+3\tau^3}{8\tau^2(6+\tau)^2}\bar{\gamma}'(\tau) \\ &\quad - \tfrac{138+25\tau+\tau^2}{8\tau^2(6+\tau)^2}\bar{\gamma}(\tau) \\ &= -\tfrac{3(6402-8681\tau-5856\tau^2-22\tau^3+390\tau^4+39\tau^5)}{64\tau^2(1+\tau)^5(6+\tau)^2}\,. \end{aligned} \tag{22}$$

The two-loops quantities Λ_2 and $\vec{w}_2$ are then fixed as :

$$\Lambda_2 = -\frac{b}{2}\,,$$

$$\begin{aligned}\vec{w}_2 = &-\tfrac{342+2526\,\tau+1743\,\tau^2+244\,\tau^3-62\,\tau^4-30\,\tau^5-3\,\tau^6}{8\,(1+\tau)^4\,(6+\tau)^2}\\ &+\tfrac{1-2\tau-\tau^2}{(1+\tau)^2}\,W(\tau)+\tfrac{(1-\tau)\,\tau}{1+\tau}\,W'(\tau)\\ &+\tfrac{42-46\,\tau-21\,\tau^2-2\,\tau^3}{(6+\tau)^2}\,\bar{\gamma}(\tau)\\ &-\tfrac{120-110\,\tau+10\,\tau^2+11\,\tau^3+\tau^4}{(6+\tau)^2}\,\bar{\gamma}'(\tau)\\ &-\tfrac{6\,\tau\,(22-3\,\tau-\tau^2)}{(6+\tau)^2}\,\bar{\gamma}''(\tau)-\tfrac{4\,\tau^2}{6+\tau}\,\bar{\gamma}^{(3)}(\tau)\,.\end{aligned}$$

The model will be renormalisable up to two loop iff equation (22) has a solution which is analytic near $\tau = 0$. In order to reach such a conclusion, we use the method of Frobenius for linear differential equations [25]. Let us observe that $\tau = 0$ is a regular singularity (notice that we are only interested in $\tau \geq 0$). The indicial equation of the linear differential equation (22) around the singular point 0 has four different solutions : $\nu = -\frac{3}{2},\ -\frac{1}{2},\ 0,\ 1$. For each one, we can find convergent series $\tau^\nu \sum_{n=0}^{\infty} c_n \tau^n$ that are independent solutions of the homogeneous equation associated to (22). We give here the first terms of such series (it happens that for $\nu = -\frac{3}{2}$ we have an exact solution) :

$$\begin{aligned}\bar{\gamma}_{-\frac{3}{2}}(\tau) &= \frac{1}{\tau^{\frac{3}{2}}}+\frac{1}{20\sqrt{\tau}}-\frac{\sqrt{\tau}}{20}\,,\\ \bar{\gamma}_{-\frac{1}{2}}(\tau) &= \frac{1}{\sqrt{\tau}}\left(1-\frac{11}{6}\tau+\frac{35}{108}\tau^2+\cdots\right),\\ \bar{\gamma}_0(\tau) &= 1+\frac{23}{840}\tau^2+\cdots\,,\\ \bar{\gamma}_1(\tau) &= \tau\left(1+\frac{1}{42}\tau-\frac{1}{324}\tau^2+\cdots\right).\end{aligned}$$

Then, we use the method of variation of parameters to find $\lambda_{-\frac{3}{2}}(\tau)$, $\lambda_{-\frac{1}{2}}(\tau)$, $\lambda_0(\tau)$ and $\lambda_1(\tau)$ such that

$$\begin{aligned}\bar{\gamma}(\tau) &= \lambda_{-\frac{3}{2}}(\tau)\,\bar{\gamma}_{-\frac{3}{2}}(\tau)+\lambda_{-\frac{1}{2}}(\tau)\,\bar{\gamma}_{-\frac{1}{2}}(\tau)\\ &\quad+\lambda_0(\tau)\,\bar{\gamma}_0(\tau)+\lambda_1(\tau)\,\bar{\gamma}_1(\tau)\end{aligned}$$

is the general solution of the inhomogeneous equation (22).

The first terms in the expansion of these functions are :

$$\begin{aligned}\lambda_{-\frac{3}{2}}(\tau) &= \lambda^o_{-\frac{3}{2}}+\tau^{\frac{7}{2}}\left(\frac{1067}{1680}-\frac{13691}{3780}\tau+\cdots\right),\\ \lambda_{-\frac{1}{2}}(\tau) &= \lambda^o_{-\frac{1}{2}}+\tau^{\frac{5}{2}}\left(-\frac{1067}{240}+\frac{2543509}{100800}\tau+\cdots\right),\\ \lambda_0(\tau) &= \lambda^o_0+\frac{1067}{192}\tau^2-\frac{9805}{288}\tau^3+\cdots\,,\\ \lambda_1(\tau) &= \lambda^o_1-\frac{1067}{480}\tau+\frac{27887}{5760}\tau^2+\cdots\,.\end{aligned}$$

The analyticity requirement near $\tau = 0$ enforces the choice $\lambda^o_{-\frac{3}{2}} = \lambda^o_{-\frac{1}{2}} = 0$; $\bar{\gamma}(\tau)$ is then expressed as a convergent series in τ, and the same will be true for $\breve{\beta}(\tau)$.

The final expression (21) for the deformation $\widetilde{G}_{ij}$ depends on 3 constants (b, λ^o_0 and λ^o_1) and an arbitrary function ($W(\tau)$). The constant b is the finite coupling renormalisation while $\vec{W}$ is the finite field renormalisation. The dualised $\mathbf{SU(2)}$ σ-model is therefore renormalisable at the two-loop order if and only if we add a finite $\hbar$ deformation of the classical metric, depending on two parameters λ^o_0 and λ^o_1.

As these two parameters are in fact the factors of the analytic solutions of the homogenous equation given from (22), they can also be interprated as the factors of the analytic solutions of equation (20) where the divergences Div^2 would have been set as vanishing quantities. In other words, these two parameters that appear in the study of the two-loop renormalisability of the $SU(2)$ dualised σ-model are independant from the divergences at this order and reflect only the freedom of reabsorbing the divergences at the one-loop order without adding new divergences at the two-loop order.

8 Concluding remarks

We have been able to exhibit some set of counter-terms that ensures the two-loop renormalisability of the T-dualised chiral non-linear σ model. The one-loop effective metric is defined **up to two constants** (λ^o_0 and λ^o_1), and some finite arbitrary field and coupling renormalisations. As is well known (e.g. in [26]), the two-loop Callan-Symanzik β function (related to Λ_2) depends on these finite counterterms. Notice that the normalisation condition $b = 0$ (no $\hbar$ extra finite coupling constant renormalisation) enforces $\Lambda_2 = 0$.

We emphasize that, contrarily to D. Friedan's approach to σ models quantisation, where the classical metric receives **infinite** perturbative deformations, our candidate for the deformation of the classical metric is a **finite one**, depending on **only two parameters** (plus the usual infinite, and finite, renormalisations of the field and of the coupling constant) : our ansatz is that a proper understanding of the dualisation process will precisely offer the extra constraints that uniquely define the quantum extension of the classical theory, order by order in perturbation theory, in the same spirit as Ward identities determine what otherwise would appear as new parameters.

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New Representation for Lagrangians of Self-Dual Nonlinear Electrodynamics

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Abstract

We elaborate on a new representation of Lagrangians of $4D$ nonlinear electrodynamics including the Born-Infeld theory as a particular case. In this new formulation, in parallel with the standard Maxwell field strength $F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}$, an auxiliary bispinor field $V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}}$ is introduced. The gauge field strength appears only in bilinear terms of the full Lagrangian, while the interaction Lagrangian E depends on the auxiliary fields, $E = E(V^2, \bar{V}^2)$. The generic nonlinear Lagrangian depending on $F, \bar{F}$ emerges as a result of eliminating the auxiliary fields. Two types of self-duality inherent in the nonlinear electrodynamics models admit a simple characterization in terms of the function E. The continuous $SO(2)$ duality symmetry between nonlinear equations of motion and Bianchi identities amounts to requiring E to be a function of the $SO(2)$ invariant quartic combination $V^2\bar{V}^2$, which explicitly solves the well-known self-duality condition for nonlinear Lagrangians. The discrete self-duality (or self-duality under Legendre transformation) amounts to a weaker condition $E(V^2, \bar{V}^2) = E(-V^2, -\bar{V}^2)$. We show how to generalize this approach to a system of n Abelian gauge fields exhibiting $U(n)$ duality. The corresponding interaction Lagrangian should be $U(n)$ invariant function of n bispinor auxiliary fields.

1 Introduction

It is well known that the on-shell $SO(2)$ $(U(1))$ duality invariance of Maxwell equations can be generalized to the whole class of the nonlinear electrodynamics models, including the famous Born-Infeld theory. The condition of $SO(2)$ duality can be formulated as a nonlinear differential equation for the Lagrangian of these theories [1, 2, 3]. Up to now, the general solution of this equation was analysed only in the framework of proper power expansions. Here, based on our recent paper [6], we present details of new representation for the Lagrangians of these theories. Alongside with the electromagnetic field strength $F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}$, it involves an auxiliary tensor (bispinor) field $V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}}$. It will be shown that the $SO(2)$ duality condition can be explicitly solved in this formalism, without resorting to any perturbative expansion. The general Lagrangian solving this constraint is a sum of an interaction term $E(V^2, \bar{V}^2)$ depending only on the $U(1)$-invariant combination of the auxiliary fields, $E = \tilde{E}(V^2\bar{V}^2)$, and non-invariant terms which are bilinear in V and F. The Lagrangian involving only the Maxwell field strengths emerges as a result of eliminating the tensor auxiliary fields by their algebraic equations of motion. More general nonlinear electrodynamics Lagrangians respecting the so-called discrete self-duality (or duality under Legendre transformation) also admit a simple characterization in terms of the function E. In this case it should be even, $E(V^2, \bar{V}^2) = E(-V^2, -\bar{V}^2)$, and otherwise arbitrary.

In Sect. 2 we give a brief account of the continuous and "discrete" dualities in nonlinear electrodynamics in the conventional approach. A novel representation of the appropriate

Lagrangians via bispinor auxiliary fields $V_{\alpha\beta}$ and $\bar{V}_{\dot{\alpha}\dot{\beta}}$ is discussed in Sect. 3. Two examples of duality-invariant models in the new setting, including the Born-Infeld theory, are presented in Sect. 4. An extension to $U(n)$ duality-invariant systems of n Abelian gauge fields is given in Sect. 5. The corresponding Lagrangian in the V,F representation is fully specified by the interaction term which is an $U(n)$ invariant function of n auxiliary fields $V^k_{\alpha\beta}$. The discrete self-duality also amounts to a simple restriction on the interaction function.

2 Self-dualities in nonlinear electrodynamics

We shall discuss nonlinear 4D electrodynamics models which reveal duality properties and include the free Maxwell theory and Born-Infeld theory as particular cases. Detailed motivations why such models are of interest to study can be found, e.g., in [3].

2.1 Continuous on-shell $SO(2)$ duality. In the spinor notation, the Maxwell field strengths are defined by

$$F_{\alpha\beta}(A) \equiv \frac{1}{2}(\partial^{\dot{\beta}}_{\alpha} A_{\beta\dot{\beta}} + \partial^{\dot{\beta}}_{\beta} A_{\alpha\dot{\beta}}) \,, \\ \bar{F}_{\dot{\alpha}\dot{\beta}}(A) \equiv \frac{1}{2}(\partial^{\beta}_{\dot{\alpha}} A_{\beta\dot{\beta}} + \partial^{\beta}_{\dot{\beta}} A_{\beta\dot{\alpha}}) \,, \tag{1}$$

where $\partial_{\alpha\dot{\beta}} = \frac{1}{2}(\sigma^m)_{\alpha\dot{\beta}}\partial_m$ and $A_{\alpha\dot{\beta}}$ is the corresponding vector gauge potential. Below we shall sometimes treat $F_{\alpha\beta}$ and $\bar{F}_{\dot{\alpha}\dot{\beta}}$ as independent variables, without assuming Eqs. (1).

Let us introduce the Lorentz-invariant complex variables

$$\varphi = F^{\alpha\beta}F_{\alpha\beta} \,, \quad \bar{\varphi} = \bar{F}^{\dot{\alpha}\dot{\beta}}\bar{F}_{\dot{\alpha}\dot{\beta}} \,. \tag{2}$$

In this representation, two independent invariants which one can construct out of the Maxwell field strength in the standard vector notation take the following form:

$$F^{mn}F_{mn} = 2(\varphi + \bar{\varphi}) \,, \\ F_{mn}\tilde{F}^{mn} = -2i(\varphi - \bar{\varphi}) \,. \tag{3}$$

Here

$$F_{mn} = \partial_m A_n - \partial_n A_m \,, \quad \tilde{F}^{mn} = \frac{1}{2}\varepsilon^{mnpq}F_{pq}.$$

It will be convenient to deal with dimensionless $F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}$ and $\varphi, \bar{\varphi}$, introducing a coupling constant f, $[f] = 2$. Then the generic nonlinear Lagrangian $\mathcal{L}(F, \bar{F}) = f^{-2}L(\varphi, \bar{\varphi})$, where

$$L(\varphi, \bar{\varphi}) = -\frac{1}{2}(\varphi + \bar{\varphi}) + L_{int}(\varphi, \bar{\varphi}) \tag{4}$$

and $L_{int}(\varphi, \bar{\varphi})$ collects all possible self-interaction terms of higher-order in $\varphi, \bar{\varphi}$.

We shall use the following notation for the derivatives of the Lagrangian $L(\varphi, \bar{\varphi})$ [1]

$$P_{\alpha\beta}(F) \equiv i\partial L/\partial F^{\alpha\beta} = 2iF_{\alpha\beta}L_{\varphi} \,, \\ L_{\varphi} = \partial L/\partial\varphi \,, \quad L_{\bar{\varphi}} = \partial L/\partial\bar{\varphi} \,, \\ L_{\varphi\bar{\varphi}} = \partial^2 L/\partial\varphi\partial\bar{\varphi} \ldots \,, \tag{5}$$

[1] In these and some subsequent relations it is assumed that the functional argument F stands for both $F_{\alpha\beta}$ and $\bar{F}_{\dot{\alpha}\dot{\beta}}$; we hope that this short-hand notation will not give rise to any confusion.

and for the bilinear combinations of them

$$\pi \equiv P^{\alpha\beta}P_{\alpha\beta} = -4\varphi(L_{\varphi})^2 \,, \\ \bar{\pi} \equiv \bar{P}^{\dot{\alpha}\dot{\beta}}\bar{P}_{\dot{\alpha}\dot{\beta}} = -4\bar{\varphi}(L_{\bar{\varphi}})^2 \,. \tag{6}$$

In the vector notation, the same quantities read

$$\tilde{P}^{mn} \equiv \frac{1}{2}\varepsilon^{mnpq}P_{pq} = 2\partial L/\partial F_{mn} \\ \frac{i}{2}P_{mn}\tilde{P}^{mn} = \pi - \bar{\pi} \,.$$

The nonlinear equations of motion have the following form in this representation:

$$\mathcal{E}_{\alpha\dot{\alpha}}(F) \equiv \partial^{\dot{\beta}}_{\alpha}\bar{P}_{\dot{\alpha}\dot{\beta}}(F) - \partial^{\beta}_{\dot{\alpha}}P_{\alpha\beta}(F) = 0 \,. \tag{7}$$

These equations, together with the Bianchi identities

$$\mathcal{B}_{\alpha\dot{\alpha}}(F) \equiv \partial^{\dot{\beta}}_{\alpha}\bar{F}_{\dot{\alpha}\dot{\beta}} - \partial^{\beta}_{\dot{\alpha}}F_{\alpha\beta} = 0 \,, \tag{8}$$

constitute a set of first-order equations in which one can treat $F_{\alpha\beta}$ and $\bar{F}_{\dot{\alpha}\dot{\beta}}$ as *unconstrained* conjugated variables.

This set is said to be duality-invariant if the Lagrangian $L(\varphi, \bar{\varphi})$ satisfies certain nonlinear condition [1, 2, 3]. The precise form of this self-duality condition in the spinor notation is

$$S[F, P(F)] \equiv F^{\alpha\beta}F_{\alpha\beta} + P^{\alpha\beta}P_{\alpha\beta} - \text{c.c.} \\ = \varphi + \pi - \bar{\varphi} - \bar{\pi} \\ = \varphi - \bar{\varphi} - 4\,[\varphi(L_{\varphi})^2 - \bar{\varphi}(L_{\bar{\varphi}})^2] = 0. \tag{9}$$

Let us define the nonlinear transformation

$$\delta_{\omega}F_{\alpha\beta} = \omega\, P_{\alpha\beta}(F) \equiv 2i\,\omega\, F_{\alpha\beta}L_{\varphi} \,, \tag{10}$$

where ω is a real parameter. This transformation is a nonlinear realization of the $SO(2)$ group, provided the condition (9) is satisfied. Indeed, in this case $F_{\alpha\beta}$ and $P_{\alpha\beta}(F)$ form an $SO(2)$ vector

$$\delta_{\omega}P_{\alpha\beta}(F) = -4\,\omega\, F_{\alpha\beta}\frac{\partial}{\partial\varphi}[\varphi(L_{\varphi})^2 - \bar{\varphi}(L_{\bar{\varphi}})^2] = -\omega F_{\alpha\beta} \,. \tag{11}$$

The set of equations (7), (8) is clearly invariant under these transformations

$$\delta_{\omega}\begin{pmatrix} \mathcal{B}_{\alpha\dot{\alpha}} \\ \mathcal{E}_{\alpha\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}\begin{pmatrix} \mathcal{B}_{\alpha\dot{\alpha}} \\ \mathcal{E}_{\alpha\dot{\alpha}} \end{pmatrix}. \tag{12}$$

Thus these transformations are an obvious generalization of the $SO(2)$ duality transformation in the Maxwell theory:

$$\delta_{\omega}F_{\alpha\beta} = -i\omega\, F_{\alpha\beta} \,, \quad \delta_{\omega}\bar{F}_{\dot{\alpha}\dot{\beta}} = i\omega\, \bar{F}_{\dot{\alpha}\dot{\beta}} \,, \tag{13}$$

which is a symmetry of the vacuum Maxwell equation $\partial^{\beta}_{\dot\alpha}F_{\alpha\beta}=0$.

The authors of [2] analyzed the self-duality condition (9) as an equation for the unknown L, using the power expansion in some variable. They have found that in each order of the perturbation theory the general solution for L is specified by some arbitrary function of the single variable $\varphi\bar\varphi$. In Sect. 3 we propose the explicit solution to this problem, without resorting to any power expansion.

It should be pointed out that the $SO(2)$ duality transformations in the standard setting described above cannot be realized on the vector potential A_m; they provide a symmetry between the equations of motion and Bianchi identity and as such define *on-shell* symmetry. The manifestly $SO(2)$ duality-invariant Lagrangians can be constructed in the formalism with additional vector and auxiliary fields [4]. Here we shall not discuss connections with this extended formalism.

Although the Lagrangian $L(\varphi,\bar\varphi)$ satisfying (9) is not invariant with respect to transformation (10), one can still construct, out of φ and $\bar\varphi$, the $SO(2)$ invariant function

$$\begin{aligned} I[F,P(F)] &\equiv L+\frac{i}{2}(FP-\bar F\bar P)\\ &= L-\varphi L_\varphi-\bar\varphi L_{\bar\varphi}\equiv I(\varphi,\bar\varphi)\ , \\ \delta_\omega I(\varphi,\bar\varphi) &= \frac{i}{2}\,\omega\,(\varphi+\pi-\bar\varphi-\bar\pi)=0\ , \end{aligned} \tag{14}$$

where $FP-\bar F\bar P=P_{\alpha\beta}F^{\alpha\beta}-\bar P_{\dot\alpha\dot\beta}\bar F^{\dot\alpha\dot\beta}$. This function cannot be taken as a Lagrangian since it contains no free part.

2.2 Self-duality under Legendre transformation. For what follows we shall need a first-order representation of the action corresponding to the Lagrangian (4), such that the Bianchi identities (8) are implemented in the action with the appropriate Lagrange multipliers and so $F_{\alpha\beta},\bar F_{\dot\alpha\dot\beta}$ are unconstrained complex variables. This form of the action is given by

$$\frac{1}{f^2}\int d^4x\tilde L[F,F^D(B)]=\frac{1}{f^2}\int d^4x\{L(\varphi,\bar\varphi)+i[FF^D(B)-\bar F\bar F^D(B)]\}\ , \tag{15}$$

where

$$F^D_{\alpha\beta}(B)\equiv\frac{1}{2}(\partial^{\dot\beta}_{\alpha}B_{\beta\dot\beta}+\partial^{\dot\beta}_{\beta}B_{\alpha\dot\beta})\ . \tag{16}$$

Varying with respect to the Lagrange multiplier $B_{\alpha\dot\beta}$, one obtains just the Bianchi identities for $F_{\alpha\beta},\bar F_{\dot\alpha\dot\beta}$ (8). Solving them in terms of the gauge potential $A_{\alpha\dot\beta}$ and substituting the result into (15), we come back to (4). On the other hand, the multiplier $B_{\alpha\dot\beta}$ is defined up to the standard Abelian gauge transformation, which suggests interpreting $B_{\alpha\dot\beta}$ and $F^D_{\alpha\beta}(B)$ as the *dual* gauge potential and gauge field strength, respectively. Using the algebraic equations of motion for the variables $F_{\alpha\beta},\bar F_{\dot\alpha\dot\beta}$, one can express the action (15) in terms of $F^D_{\alpha\beta}(B),\bar F^D_{\dot\alpha\dot\beta}(B)$. If the resulting action has the same form as the original one in terms of $F_{\alpha\beta}(A),\bar F_{\dot\alpha\dot\beta}(A)$, the corresponding electrodynamics model is said to enjoy the "discrete" self-duality. This sort of duality should not be confused with the on-shell continuous $SO(2)$ duality discussed earlier. The relevant $SO(2)$ symmetry is realized on the variables $F_{\alpha\beta}$ according to (10) and it is not defined on the dual vector potential $B_{\alpha\dot\beta}$. However, as we shall see soon, any $L(\varphi,\bar\varphi)$ solving the constraint (9) corresponds to a

system revealing the discrete self-duality. The inverse statement is not generally true, so the class of nonlinear electrodynamics actions admitting $SO(2)$ duality of equations of motion forms a subclass in the variety of actions which are self-dual in the "discrete" sense.

Let us elaborate on this in some detail. The dual picture is achieved by varying (15) with respect to the independent variables $F_{\alpha\beta},\bar F_{\dot\alpha\dot\beta}$, which yields

$$F^D_{\alpha\beta}(B)=i\partial L/\partial F^{\alpha\beta}\equiv P_{\alpha\beta}(F)=2iF_{\alpha\beta}L_\varphi\ , \tag{17}$$

where $P_{\alpha\beta}(F)$ is the same as in (5). Substituting the solution of this algebraic equation, $F_{\alpha\beta}=F_{\alpha\beta}(F^D)$, into (15) gives us the dual Lagrangian $\hat L(F^D)$

$$\hat L(F^D)=\hat L(\pi^D,\bar\pi^D)\equiv\tilde L[F(F^D),F^D]\ , \tag{18}$$

where $\pi^D\equiv F^{D\,\alpha\beta}F^D_{\alpha\beta}=\pi(F)$ and $\pi,\bar\pi$ were defined in (6).

Using (17) and its conjugate, as well as the definitions (15), (18), one can explicitly check the property

$$F_{\alpha\beta}=-i\partial\hat L/\partial F^{D\,\alpha\beta}\quad\text{and c.c.}\ . \tag{19}$$

Due to this relation, and keeping in mind the inverse one (17), one can treat the equation

$$\hat L(\pi,\bar\pi)=L(\varphi,\bar\varphi)+i(FP-\bar F\bar P) \tag{20}$$

as setting the direct and inverse Legendre transforms $L\leftrightarrow\tilde L$ between two functions of 6 complex variables

$$\begin{aligned} &F_{\alpha\beta}\ \Rightarrow\ P_{\alpha\beta}=i\partial L/\partial F^{\alpha\beta}\ ,\\ &dL=-iP_{\alpha\beta}dF^{\alpha\beta}+i\bar P_{\dot\alpha\dot\beta}d\bar F^{\dot\alpha\dot\beta}\ , \end{aligned} \tag{21}$$

$$\begin{aligned} &P_{\alpha\beta}\ \Rightarrow\ F_{\alpha\beta}=-i\partial\hat L/\partial P^{\alpha\beta}\ ,\\ &d\hat L=iF^{\alpha\beta}dP_{\alpha\beta}-i\bar F^{\dot\alpha\dot\beta}d\bar P_{\dot\alpha\dot\beta}\ . \end{aligned} \tag{22}$$

Within this interpretation, the "discrete" self-duality defined above and amounting to the condition

$$\hat L(\pi,\bar\pi)=L(\pi,\bar\pi)\ , \tag{23}$$

can be equivalently called "self-duality under Legendre transformation" [1, 3].

Let us show that the $SO(2)$ duality condition (9) indeed guarantees the self-duality under Legendre transformation (23). The simplest proof of this statement (see, e.g., [3]) makes use of the finite discrete $SO(2)$ transformation

$$F_{\alpha\beta}\to P_{\alpha\beta}\ ,\quad P_{\alpha\beta}\to -F_{\alpha\beta} \tag{24}$$

and the invariance of function (14) under the global version of the $SO(2)$ transformations (10). Due to the latter property, (14) is invariant with respect to (24) too

$$L(\varphi,\bar\varphi)+\frac{i}{2}PF-\frac{i}{2}\bar P\bar F=L(\pi,\bar\pi)-\frac{i}{2}FP+\frac{i}{2}\bar F\bar P.$$

Comparing this relation with (20), we arrive at the condition (23).

For the dual Lagrangian $\hat{L}(\pi,\bar{\pi})$ one can construct the 1st order action similar to (15)

$$f^{-2}\int d^4x\tilde{L}[P,F(A)] = f^{-2}\int d^4x[\hat{L}(\pi,\bar{\pi}) - iPF(A) + i\bar{P}\bar{F}(A)]. \tag{25}$$

where $F_{\alpha\beta}(A), \bar{F}_{\dot\alpha\dot\beta}(A)$ were defined in (1). The self-dual case (in the "discrete" sense) corresponds to identifying $\hat{L}(\pi,\bar{\pi}) = L(\pi,\bar{\pi})$. Varying (25) with respect to the gauge potential $A_{\alpha\dot\beta}$ produces Bianchi identities for the originally unconstrained variable $P_{\alpha\beta}, \bar{P}_{\dot\alpha\dot\beta}$ as the corresponding equations of motion and so implies

$$P_{\alpha\beta} = F^D_{\alpha\beta}(B)\ , \quad \bar{P}_{\dot\alpha\dot\beta} = \bar{F}^D_{\dot\alpha\dot\beta}(B)\ .$$

On the other hand, the equation of motion for $P^{\alpha\beta}$ in this representation yields

$$F_{\alpha\beta} = -i\partial L/\partial P^{\alpha\beta} = -2iP_{\alpha\beta}L_\pi\ . \tag{26}$$

Solving this equation for the unknown $P_{\alpha\beta}$ as a function of $F_{\alpha\beta}(A), \bar{F}_{\dot\alpha\dot\beta}(A)$, we come back to the action corresponding to the original Lagrangian (4).

The on-shell $SO(2)$ duality formulated earlier in terms of the variables $F_{\alpha\beta}, P_{\alpha\beta}(F)$ admits an equivalent formulation in terms of the dual variables $P_{\alpha\beta}, F_{\alpha\beta}(P)$:

$$\delta_\omega P_{\alpha\beta} = -\omega\, F_{\alpha\beta}(P) = 2i\omega\, P_{\alpha\beta}L_\pi\ . \tag{27}$$

The condition of $SO(2)$ self-duality has the following form in this representation:

$$S[F(P),P] = \pi[1-4(L_\pi)^2] - \text{c.c.} = 0\ . \tag{28}$$

The function $S[F(P),P]$ is obtained by substituting $F_{\alpha\beta}(P)$ from (26) into the universal bilinear form (9).

3 A new form of the actions of nonlinear electrodynamics and self-dualities

3.1 Nonlinear electrodynamics Lagrangians in a new setting. The recently constructed $N=3$ supersymmetric extension of the Born-Infeld theory [6] suggests a new representation for the actions of nonlinear electrodynamics discussed in the previous Section.

The infinite-dimensional off-shell $N=3$ vector multiplet contains gauge field strengths (1) and auxiliary fields $H_{\alpha\beta}$ and $\bar{H}_{\dot\alpha\dot\beta}$ [5]. The gauge field part of the off-shell super $N=3$ Maxwell component Lagrangian is [2]

$$\frac{1}{16}[h+\bar{h} - 6(\bar{H}\bar{F} + HF) + \varphi + \bar{\varphi}]\ , \tag{29}$$

where $h = H^{\alpha\beta}H_{\alpha\beta}$ and $HF = H^{\alpha\beta}F_{\alpha\beta}(A)$. Eliminating the auxiliary fields $H_{\alpha\beta}, \bar{H}_{\dot\alpha\dot\beta}$ by their algebraic equations of motion we arrive at the standard Maxwell action

$$L_2(F) = -\frac{1}{2}(\varphi+\bar{\varphi})\ . \tag{30}$$

[2]In the rest of the paper we put the overall coupling constant f equal to 1.

The $N=3$ off-shell superfield strengths contain the following combinations of fields [6]:

$$V_{\alpha\beta} = \frac{1}{4}(H_{\alpha\beta} + F_{\alpha\beta})\ ,$$
$$\bar{V}_{\dot\alpha\dot\beta} = \frac{1}{4}(\bar{H}_{\dot\alpha\dot\beta} + \bar{F}_{\dot\alpha\dot\beta})\ . \tag{31}$$

The free Maxwell Lagrangian (29), being rewritten through $V_{\alpha\beta}, \bar{V}_{\dot\alpha\dot\beta}$, reads

$$\mathcal{L}_2(V,F) = \nu + \bar{\nu} - 2\,(VF + \bar{V}\bar{F}) + \frac{1}{2}(\varphi+\bar{\varphi})\ , \tag{32}$$

where

$$\nu \equiv V^{\alpha\beta}V_{\alpha\beta}\ , \quad \bar{\nu} \equiv \bar{V}^{\dot\alpha\dot\beta}\bar{V}_{\dot\alpha\dot\beta}\ ,$$
$$VF \equiv V^{\alpha\beta}F_{\alpha\beta}\ , \quad \bar{V}\bar{F} \equiv \bar{V}^{\dot\alpha\dot\beta}\bar{F}_{\dot\alpha\dot\beta}. \tag{33}$$

Eliminating $V^{\alpha\beta}$ by its algebraic equation of motion,

$$V^{\alpha\beta} = F^{\alpha\beta}\ , \quad \bar{V}^{\dot\alpha\dot\beta} = \bar{F}^{\dot\alpha\dot\beta}\ , \tag{34}$$

we arrive at the free Lagrangian (30).

Our aim will be to find a nonlinear extension of the free Maxwell Lagrangian using the $N=3$ supersymmetry-inspired form $\mathcal{L}_2(V,F)$ of it, eq. (32), such that this extension becomes the generic nonlinear Lagrangian $L(F^2,\bar{F}^2)$, eq. (4), after eliminating the auxiliary fields $V_{\alpha\beta}, \bar{V}_{\dot\alpha\dot\beta}$ by their algebraic (*nonlinear*) equations of motion.

By Lorentz covariance, such a nonlinear Lagrangian has the following general form:

$$\mathcal{L}[V,F(A)] = \mathcal{L}_2[V,F(A)] + E(\nu,\bar{\nu})\ , \tag{35}$$

where E is a real function encoding self-interaction. Varying the action with respect to $V_{\alpha\beta}$, we derive the algebraic relation between V and $F(A)$ in this formalism

$$F_{\alpha\beta}(A) = V_{\alpha\beta}(1+E_\nu) \quad \text{and c.c.}\ , \tag{36}$$

where $E_\nu \equiv \partial E(\nu,\bar{\nu})/\partial\nu$. This relation is a generalization of the free equation (34) and it can be used to eliminate the auxiliary variable $V^{\alpha\beta}$ in terms of $F^{\alpha\beta}$ and $\bar{F}^{\dot\alpha\dot\beta}$, $V_{\alpha\beta} \Rightarrow V_{\alpha\beta}[F(A)]$ (see eq. (40) below). The natural restrictions on the interaction function $E(\nu,\bar{\nu})$ are

$$E(0,0) = 0\ , \quad E_\nu(0,0) = E_{\bar{\nu}}(0,0) = 0\ , \tag{37}$$

which mean that its $(\nu,\bar{\nu})$-expansion does not contain constant and linear terms. Clearly, given some non-singular interaction Lagrangian $L_{int}(\varphi,\bar{\varphi})$ in (4), one can pick up the appropriate function $E(\nu,\bar{\nu})$, such that the elimination of $V^{\alpha\beta}, \bar{V}^{\dot\alpha\dot\beta}$ by (36) yields just this self-interaction. Thus (35) with an arbitrary (non-singular) interaction function E is another form of generic nonlinear electrodynamics Lagrangian (4). The second equation of motion in this representation, obtained by varying (35) with respect to $A_{\alpha\dot\alpha}$, has the form

$$\partial^\beta_{\dot\alpha}[F_{\alpha\beta}(A) - 2V_{\alpha\beta}] + \text{c.c.} = 0\ . \tag{38}$$

After substituting $V_{\alpha\beta} = V_{\alpha\beta}[F(A)]$ from (36), eq. (38) becomes the dynamical equation for $F_{\alpha\beta}(A), \bar{F}_{\dot\alpha\dot\beta}(A)$ corresponding to the generic Lagrangian (4). Comparing (38) with (7) yields the relation

$$P_{\alpha\beta}(F) = i\,[F_{\alpha\beta} - 2V_{\alpha\beta}(F)]\ , \tag{39}$$

where $P_{\alpha\beta}(F)$ was defined in (5).

Let us elaborate in more detail on the relation of the V, F representation of the nonlinear electrodynamics Lagrangians to the original "minimal" one (4) which involves only $F_{\alpha\beta}$ and $\bar{F}_{\dot\alpha\dot\beta}$. The general solution of the algebraic equation (36) for $V_{\alpha\beta}$ can be written as

$$V_{\alpha\beta}(F) = F_{\alpha\beta}G(\varphi, \bar\varphi)\ . \tag{40}$$

The transition function $G(\varphi, \bar\varphi)$ can be found from the basic requirement that (35) coincides with the initial nonlinear action after eliminating $V_{\alpha\beta}, \bar{V}_{\dot\alpha\dot\beta}$

$$\mathcal{L}[V(F), F] = L(\varphi, \bar\varphi)\ . \tag{41}$$

Using eq.(40), one can obtain the relations

$$\nu = \varphi G^2\ , \quad \bar\nu = \bar\varphi \bar{G}^2\ , \tag{42}$$

$$V(F)F = \varphi G\ , \quad \bar{V}(F)\bar{F} = \bar\varphi\bar{G}\ . \tag{43}$$

After substituting these expressions in (41) with making use of the explicit expressions (32), (35), eq. (41) can be rewritten as

$$E(\nu, \bar\nu) = L(\varphi, \bar\varphi) - \frac{1}{2}(\varphi + \bar\varphi) + \varphi G(2 - G) + \bar\varphi\bar{G}(2 - \bar{G})\ . \tag{44}$$

One should also add the relations:

$$G^{-1} = 1 + E_\nu\ , \quad \bar{G}^{-1} = 1 + E_{\bar\nu}\ , \tag{45}$$

which follow from comparing (40) with (36).

Differentiating (44) with respect to φ and using the relations

$$\frac{\partial\nu}{\partial\varphi} = G^2 + 2\varphi G\frac{\partial G}{\partial\varphi}\ , \quad \frac{\partial\bar\nu}{\partial\varphi} = 2\bar\varphi\bar{G}\frac{\partial\bar{G}}{\partial\varphi}\ ,$$

one obtains the simple expression for transition functions

$$G(\varphi, \bar\varphi) = \frac{1}{2} - L_\varphi\ . \tag{46}$$

A useful corollary of this formula and eqs. (42), (45) is the relation

$$\nu E_\nu = \frac{1}{4}\varphi(1 - 4L_\varphi^2)\ . \tag{47}$$

Given a fixed $L(\varphi, \bar\varphi)$, one can express $\varphi, \bar\varphi$ in terms of $\nu, \bar\nu$ using eqs. (42), (46) and then restore the explicit form of $E(\nu, \bar\nu)$ by (44). Conversely, given $E(\nu, \bar\nu)$, one can restore

$L(\varphi, \bar\varphi)$. In practice, finding out such explicit relations is a rather complicated task, as we shall see on two examples in Sect. 4.

3.2 Self-dualities revisited. Until now we did not touch any issues related to self-dualities. A link with the consideration in the previous Section is established by Eq. (39) which relates the functions $P_{\alpha\beta}(F)$ and $V_{\alpha\beta}(F)$.

Substituting this into the $SO(2)$ duality condition (9) and making use of eq. (47) we find

$$S[F, P(F)] = [F^{\alpha\beta} - V^{\alpha\beta}(F)]V_{\alpha\beta}(F) - \text{c.c} = \nu E_\nu - \bar\nu E_{\bar\nu}\ . \tag{48}$$

Thus passing to the V, F representation allows one to reduce the nonlinear differential equation (9) to a *linear* differential equation for the function $E(\nu, \bar\nu)$:

$$\text{Eq.(9)} \quad \Leftrightarrow \quad \nu E_\nu - \bar\nu E_{\bar\nu} = 0\ . \tag{49}$$

The corresponding realization of the $SO(2)$ (or $U(1)$) transformations (10), (11) in terms of $F^{\alpha\beta}$ and $V^{\alpha\beta}(F)$ is given by

$$\delta_\omega V_{\alpha\beta}(F) = -i\omega V_{\alpha\beta}(F)\ , \tag{50}$$

$$\delta_\omega F_{\alpha\beta} = i\omega[F_{\alpha\beta} - 2V_{\alpha\beta}(F)]\ . \tag{51}$$

It is worth recalling that the on-shell $SO(2)$ duality transformation mixes the dynamical equations of motion for $F_{\alpha\beta}, \bar{F}_{\dot\alpha\dot\beta}$ with the Bianchi identities (8), and so in this extended set of equations one should treat $F_{\alpha\beta}, \bar{F}_{\dot\alpha\dot\beta}$ as *unconstrained* conjugated complex variables (without explicitly solving (8) in terms of gauge potential $A_{\alpha\dot\beta}$). Correspondingly, the algebraic relation (36) should be viewed to connect two independent sets of variables, $(F_{\alpha\beta}, \bar{F}_{\dot\alpha\dot\beta})$ and $(V_{\alpha\beta}, \bar{V}_{\dot\alpha\dot\beta})$:

$$F_{\alpha\beta} = V_{\alpha\beta}(1 + E_\nu)\ . \tag{52}$$

Due to this relation, one can equivalently formulate the dynamics and duality transformations in terms of either the set $(F_{\alpha\beta}, \bar{F}_{\dot\alpha\dot\beta})$ or the set $(V_{\alpha\beta}, \bar{V}_{\dot\alpha\dot\beta})$. Respectively, one can choose as the basic one either the transformation (51) or (50). It is important to be sure that (52) is covariant under the transformations (50),(51). It is straightforward to check that both (52) and the constraint (49) itself are indeed covariant under these transformations on the surface of eq. (49).

It is important to emphasize that the new form (49) of the self-duality constraint (9) admits a transparent interpretation as the condition of invariance of $E(\nu, \bar\nu)$ with respect to the $U(1)$ transformations (50)

$$\delta_\omega E = 2i\omega(\bar\nu E_{\bar\nu} - \nu E_\nu) = 0\ . \tag{53}$$

The general solution of (49) is a function $\tilde{E}(a)$ which depends on the single real $U(1)$ invariant variable $a = \nu\bar\nu$ quartic in the auxiliary fields $V_{\alpha\beta}$ and $\bar{V}_{\dot\alpha\dot\beta}$

$$E_{sd}(\nu, \bar\nu) = \tilde{E}(a) = \tilde{E}(\nu\bar\nu)\ , \ \tilde{E}(0) = 0\ . \tag{54}$$

Thus we come to the notable result that the *whole* class of nonlinear extensions of the Maxwell action admitting the on-shell $SO(2)$ duality is parametrized by an arbitrary $SO(2)$ invariant real function of one argument $E_{sd} = \tilde{E}(\nu\bar\nu)$ in the representation (35).

The remarkable property of E_{sd} is that only terms $\sim \nu^n\bar\nu^n$ can appear in its power expansion. Below we shall present this expansion for two examples, including the most interesting case of Born-Infeld theory.

In the V, F representation it is very easy to construct invariants of the $SO(2)$ duality rotations. Besides the function $E(\nu,\bar\nu)$ itself, one more real invariant combination of $V_{\alpha\beta}$ and $F_{\alpha\beta}(V)$ is as follows

$$I_0(V,F) = \nu + \bar\nu - VF - \bar V\bar F = -\nu E_\nu - \bar\nu E_{\bar\nu} \,. \tag{55}$$

Finally, let us examine which restrictions on the interaction Lagrangian $E(\nu,\bar\nu)$ are imposed by the requirement of the "discrete" self-duality with respect to the exchange $F(A) \leftrightarrow F^D(B)$. For this we shall need a first-order representation of the Lagrangian (35) similar to (15).

Let us treat $\mathcal{L}(V,F)$ in eq.(35) as a function of two independent variables and implement the Bianchi identities for $F_{\alpha\beta}, \bar F_{\dot\alpha\dot\beta}$ (amounting to the expressions (1)) in the Lagrangian via the dual field-strength $F^D_{\alpha\beta}(B)$ (16):

$$\tilde{\mathcal{L}}[V,F,F^D(B)] \equiv \mathcal{L}(V,F) + i[F^D(B)F - \bar F^D(B)\bar F] \,. \tag{56}$$

The algebraic $V^{\alpha\beta}$ equation of motion $\partial\tilde L/\partial V^{\alpha\beta} = 0$ is just the relation (52). On the other hand, varying (56) with respect to $F_{\alpha\beta}$, one obtains the linear relation

$$F_{\alpha\beta} - 2V_{\alpha\beta} = -iF^D_{\alpha\beta}(B) \quad \text{and c.c.} \tag{57}$$

as the corresponding equation of motion. The Bianchi identities for $F^D_{\alpha\beta}(B)$ following from the definition (16) imply for $F_{\alpha\beta} = F_{\alpha\beta}(A)$ just the dynamical equation (38) (with $V_{\alpha\beta}$ expressed in terms of $F_{\alpha\beta}, \bar F_{\dot\alpha\dot\beta}$ from (52)). This proves the equivalence of the dynamics described by (56) with that associated with (35) or (4).

The function $\tilde{\mathcal{L}}[V,F,F^D(B)]$ in (56) contains only quadratic and linear terms in F and $\bar F$, so one can explicitly find the dual form of (56) in terms of $F^D_{\alpha\beta}(B), \bar F^D_{\dot\alpha\dot\beta}(B)$ and $V_{\alpha\beta}, \bar V_{\dot\alpha\dot\beta}$, expressing $F_{\alpha\beta}$ and $\bar F_{\dot\alpha\dot\beta}$ from Eq. (57):

$$\tilde{\mathcal{L}}[V,F(V,F^D),F^D] \equiv \tilde{\mathcal{L}}(U,F^D) = \mathcal{L}_2(U,F^D) + E(-u,-\bar u) \,, \tag{58}$$

where

$$U_{\alpha\beta} \equiv -iV_{\alpha\beta} \,, \quad u = U^{\alpha\beta}U_{\alpha\beta} \,.$$

Comparing the dual Lagrangian (58) with the original one (35), we observe that the necessary and sufficient condition of the discrete self-duality is the following simple restriction on the function E [6]

$$E(\nu,\bar\nu) = E(-\nu,-\bar\nu) \,. \tag{59}$$

Obviously, an arbitrary $SO(2)$-invariant function $E_{sd} = \tilde E(\nu\bar\nu)$ corresponding to a $SO(2)$ self-dual system automatically satisfies the discrete self-duality condition (59). This elementary consideration provides us with a simple proof of the fact (mentioned in Sect. 2) that the $SO(2)$ self-dual systems constitute a subclass in the set of those enjoying the discrete self-duality.

4 Examples of self-dual systems

4.1 Born-Infeld theory. The Lagrangian of the Born-Infeld theory has the following form in terms of complex invariants (2)

$$L_{BI}(\varphi,\bar\varphi) = [1 - Q(\varphi,\bar\varphi)] \,, \tag{60}$$

where

$$Q(\varphi,\bar\varphi) = \sqrt{1+X} \,, \qquad X(\varphi,\bar\varphi) \equiv (\varphi+\bar\varphi) + (1/4)(\varphi-\bar\varphi)^2 \,. \tag{61}$$

The power expansion of the BI-lagrangian is

$$L_{BI} = -\frac{1}{2}(\varphi+\bar\varphi) + \frac{1}{2}\varphi\bar\varphi - \frac{1}{2^2}\varphi\bar\varphi(\varphi+\bar\varphi) + \frac{1}{2^3}\varphi\bar\varphi(3\varphi\bar\varphi + \varphi^2 + \bar\varphi^2) + O(\varphi^5) \,. \tag{62}$$

In the BI theory the function (5) has the following explicit form

$$P_{\alpha\beta}(F) = i\frac{\partial L_{BI}}{\partial F^{\alpha\beta}} = -iF_{\alpha\beta}Q^{-1}(\varphi,\bar\varphi)[1 + \frac{1}{2}(\varphi-\bar\varphi)] \,. \tag{63}$$

It is easy to check that this function satisfies the $SO(2)$ self-duality condition (9), so BI theory belongs to the class of self-dual models [1, 2].

Let us study the Lagrangian V, F function $\mathcal{L}_{BI}(V,F)$ (35) for this particular case. Our basic purpose will be to find the corresponding function $E_{BI}(\nu,\bar\nu)$.

The function $G(\varphi,\bar\varphi)$ relating the variables $V_{\alpha\beta}$ and $F_{\alpha\beta}$ and defined by eq. (46), is given by the expression

$$\begin{aligned} G \equiv g &= \frac{1}{2}\left\{1 + Q^{-1}\left[1 + \frac{1}{2}(\varphi-\bar\varphi)\right]\right\} \\ &= 1 - \frac{1}{2}\bar\varphi + \frac{1}{2}\varphi\bar\varphi + \frac{1}{4}(\bar\varphi)^2 + \dots \,. \end{aligned} \tag{64}$$

It is easy to find the inverse relation

$$\varphi = 2\,\bar g\,\frac{1-\bar g}{[1-(g+\bar g)]^2} \,. \tag{65}$$

Our aim is to find E_{BI} as a function of the variables $\nu = V^2, \bar\nu = \bar V^2$. As the first step, one expresses $\nu, \bar\nu$ in terms of g and $\bar g$, using (43) and (65)

$$\nu = \varphi\,g^2 = 2\,\bar g g^2\frac{1-\bar g}{[1-(g+\bar g)]^2} \,. \tag{66}$$

Introducing

$$t \equiv \frac{g\bar g}{1-(g+\bar g)} \,, \tag{67}$$

one finds that t, as a consequence of (66) and the fact that $g(\varphi=0) = 1$, satisfies the following quartic equation:

$$t^4 + t^3 - \frac{1}{4}\nu\bar\nu = 0 \,, \; t(\nu=\varphi=0) = -1 \,. \tag{68}$$

It allows one to express t in terms of $a \equiv \nu\bar{\nu}$

$$t(a) = -1 - \frac{a}{4} + \frac{3a^2}{16} - \frac{15a^3}{64} + \dots . \quad (69)$$

One can write a closed expression for $t(a)$ as the proper solution of (68), but we do not present it here in view of its complexity.

Now we are ready to find $E_{BI}(\nu,\bar{\nu})$. Taking into account the explicit expressions (65) and (66) and substituting all this into (44), one finally finds a simple expression for $E_{BI}(\nu,\bar{\nu})$ through the real variable $t(a)$ [6]

$$\begin{aligned} E_{BI}(a) &= 2[2t^2(a) + 3t(a) + 1] \\ &= \frac{a}{2} - \frac{a^2}{8} + \frac{3a^3}{32} + \dots . \end{aligned} \quad (70)$$

4.2 One more example. Let us now consider the self-dual system corresponding to the simplest choice of the function E in the action (35)

$$\check{E} = \frac{1}{2}\nu\bar{\nu} \quad (\nu\check{E}_\nu = \check{E}) \quad (71)$$

which is the lowest order self-dual approximation of E_{BI}. This model is distinguished in that the relation (52) and the corresponding representation of equations of motion via variables V are polynomial.

Using Eq.(45) one can obtain the relation between variables $\nu, \bar{\nu}$ and $\varphi, \bar{\varphi}$ in this case

$$\varphi = \nu(1 + \frac{1}{2}\bar{\nu})^2 \quad \text{and c.c.} . \quad (72)$$

Like in the previous example, we present here first terms in the power expansion of the solution

$$\nu = \varphi - \varphi\bar{\varphi} + \varphi^2\bar{\varphi} + \frac{3}{4}\varphi\bar{\varphi}^2 + \dots . \quad (73)$$

The corresponding expansion of the transition function is

$$\begin{aligned} \check{G} &= \frac{1}{2} - \check{L}_\varphi = (1 + \frac{1}{2}\bar{\nu})^{-1} \\ &= 1 - \frac{1}{2}\bar{\varphi} + \frac{1}{2}\varphi\bar{\varphi} + \frac{1}{4}(\bar{\varphi})^2 + \dots \end{aligned} \quad (74)$$

and one can directly find $\check{L}(\varphi,\bar{\varphi})$ for this case. The Lagrangian $\check{L}(\varphi,\bar{\varphi})$ turns out to be highly non-polynomial, despite the fact that the interaction in the V, F representation is specified by the simple monomial (71).

Note that the first two orders of the solution (74) coincide with the corresponding terms in the Born-Infeld theory (64), while the terms starting from the 3-rd order are different. It can be also checked that the first three terms in the non-polynomial Lagrangian of this model $\check{L}(\varphi,\bar{\varphi})$ coincide with those in L_{BI} (62).

5 $U(n)$ self-duality

Let us consider n Abelian field-strengths

$$F^i_{\alpha\beta}\,, \quad \bar{F}^i_{\dot{\alpha}\dot{\beta}}\,, \quad (75)$$

where $i = 1, 2 \dots n$. As the first step, one can realize the group $SO(n)$ on these variables

$$\delta_\omega F^i_{\alpha\beta} = \xi^{ik} F^k_{\alpha\beta}\,, \quad \xi^{ki} = -\xi^{ik}\,. \quad (76)$$

This group is assumed to define an off-shell symmetry of the corresponding nonlinear Lagrangian $L(F^k,\bar{F}^k) = -\frac{1}{2}(F^iF^i) - \frac{1}{2}(\bar{F}^i\bar{F}^i) + L_{int}(F^k,\bar{F}^k)$.

The $U(n)$ self-duality conditions for the Lagrangian $L(F^k,\bar{F}^k)$ generalizing the $U(1)$ condition (9) have been analyzed in Refs.[7, 8, 3]. In the spinor notation, these conditions read

$$\mathcal{A}^{[kl]} = (F^kP^l) - (F^lP^k) - \text{c.c.} = 0\,, \quad (77)$$
$$\mathcal{S}^{(kl)} = (F^kF^l) + (P^kP^l) - \text{c.c.} = 0\,, \quad (78)$$

where

$$\begin{aligned} P^k_{\alpha\beta}(F) &\equiv i\frac{\partial L}{\partial F^{k\alpha\beta}} \\ (F^kP^l) &\equiv F^{k\alpha\beta}P^l_{\alpha\beta}\,, \quad \text{etc} . \end{aligned} \quad (79)$$

The condition (77) amounts to the $SO(n)$ invariance of the Lagrangian and holds off shell. The second condition is the true analog of (9). It guarantees the covariance of the equations of motion for $F^k_{\alpha\beta}, \bar{F}^k_{\dot{\alpha}\dot{\beta}}$ together with Bianchi identities,

$$\begin{aligned} \mathcal{E}^k_{\alpha\dot{\alpha}}(F) &\equiv \partial^{\dot{\beta}}_\alpha \bar{P}^k_{\dot{\alpha}\dot{\beta}}(F) - \partial^\beta_{\dot{\alpha}} P^k_{\alpha\beta}(F) = 0\,, \\ \mathcal{B}^k_{\alpha\dot{\alpha}}(F) &\equiv \partial^{\dot{\beta}}_\alpha \bar{F}^k_{\dot{\alpha}\dot{\beta}} - \partial^\beta_{\dot{\alpha}} F^k_{\alpha\beta} = 0\,, \end{aligned} \quad (80)$$

under the following nonlinear transformations:

$$\begin{aligned} \delta_\eta F^k_{\alpha\beta} &= -\eta^{kl} P^l_{\alpha\beta}(F)\,, \\ \delta_\eta P^k_{\alpha\beta}(F) &= \eta^{kl} F^l_{\alpha\beta} \end{aligned} \quad (81)$$

where $\eta^{kl} = \eta^{lk}$ are real parameters. On the surface of the condition (78) these transformations, together with (76), form the group $U(n)$. The $U(n)$ group structure becomes manifest after passing to the new variables:

$$\begin{aligned} V^k_{\alpha\beta}(F) &\equiv \frac{1}{2}[F^k_{\alpha\beta} + iP^k_{\alpha\beta}(F)]\,, \\ \delta V^k_{\alpha\beta}(F) &= \omega^{kl} V^l_{\alpha\beta}(F)\,, \\ \omega^{kl} &= \xi^{kl} + i\eta^{kl}\,, \quad \bar{\omega}^{lk} = -\omega^{kl}\,. \end{aligned} \quad (82)$$

The particular solution of the $U(n)$ self-duality conditions (77), (78) constructed so far [8] is formulated in terms of the algebraic equation for auxiliary scalar variables $\chi^{kl} = \chi^{lk}$

$$\begin{aligned} L &= -\frac{1}{2}(\chi^{kk} + \bar{\chi}^{kk})\,, \\ \chi^{kl} &+ \frac{1}{2}\chi^{kj}\bar{\chi}^{jl} = (F^kF^l)\,. \end{aligned} \quad (83)$$

It generalizes a similar representation for the BI Lagrangian (or its $N = 1$ extension) [9, 10]

$$L_{BI} = -\frac{1}{2}(\chi + \bar{\chi}) \,,$$
$$\chi + \frac{1}{2}\chi\bar{\chi} = \varphi = (FF) \,, \tag{84}$$

where χ is an auxiliary scalar. This $n = 1$ version of Eq.(83) can be readily solved

$$\chi = \frac{1}{2}(\varphi - \bar{\varphi}) - L_{BI} \,.$$

The solution for an arbitrary n has been constructed in [8] within a perturbative expansion.

Passing to an analog of the V, F representation in the $U(n)$ case will allow us to find the *general* solution to (77), (78).

Let us define a new representation for the $SO(n)$ invariant nonlinear electrodynamics Lagrangians in terms of the Abelian gauge field strengths $F^k_{\alpha\beta}(A^k)$ and auxiliary fields $V^k_{\alpha\beta}$

$$\mathcal{L}(V^k, F^k) = (V^kV^k) + (\bar{V}^k\bar{V}^k) - 2(V^kF^k) - 2(\bar{V}^k\bar{F}^k)$$
$$+\frac{1}{2}(F^kF^k) + \frac{1}{2}(\bar{F}^k\bar{F}^k) + E(V^k, \bar{V}^k) \,. \tag{85}$$

The real Lagrangian of interaction $E(V, \bar{V})$ is $SO(n)$ invariant by definition. In the case without interaction ($E = 0$), the bilinear part of (85) gives the standard free Lagrangian of n Abelian fields,

$$L_2(F^k, \bar{F}^k) = -\frac{1}{2}(F^kF^k) - \frac{1}{2}(\bar{F}^k\bar{F}^k) \,, \tag{86}$$

as the result of eliminating the auxiliary fields.

In the general case of $E \neq 0$ the algebraic equation for $V_{\alpha\beta}$ is

$$F^k_{\alpha\beta} = V^k_{\alpha\beta} + \frac{1}{2}\frac{\partial E}{\partial V^{\alpha\beta}_k} \,. \tag{87}$$

Using the relation

$$P^k_{\alpha\beta} = i[F^k_{\alpha\beta} - 2V^k_{\alpha\beta}] \,, \tag{88}$$

one can rewrite the $U(n)$ self-duality conditions (77) and (78) in this representation as follows

$$i(F^lV^k) - i(F^kV^l) - \text{c.c.} = 0 \,,$$
$$(F^lV^k) + (F^kV^l) - 2(V^kV^l) - \text{c.c.} = 0 \,. \tag{89}$$

One can readily show that, after making use of the relation (87), these conditions can be brought into the form quite similar to the $U(1)$ self-duality condition (49)

$$V^k_{\alpha\beta}\frac{\partial E}{\partial V^l_{\alpha\beta}} - \bar{V}^k_{\dot{\alpha}\dot{\beta}}\frac{\partial E}{\partial \bar{V}^l_{\dot{\alpha}\dot{\beta}}} = 0 \,. \tag{90}$$

This constraint is none other than the condition of invariance of $E(V, \bar{V})$ with respect to the $U(n)$ transformations (82).

Using the $U(n)$-invariant function E one can also construct the simple invariant

$$I_0 = -V^k_{\alpha\beta}\frac{\partial E}{\partial V^k_{\alpha\beta}} - \bar{V}^k_{\dot{\alpha}\dot{\beta}}\frac{\partial E}{\partial \bar{V}^k_{\dot{\alpha}\dot{\beta}}} \,. \tag{91}$$

It is easy to see that the whole Lagrangian (85) is not invariant with respect to the nonlinear part of the $U(n)$ transformations, being invariant only under the off-shell $SO(n)$ ones (corresponding to $\eta^{kl} = 0$ in (82)). This matches with the fact that these $U(n)/SO(n)$ transformations define an on-shell symmetry of the joint set of equations of motion and Bianchi identities. Just these transformations are true analogs of the standard $U(1)$ ($SO(2)$) duality rotations discussed in previous Sections.

The conclusion is that the V, F representation in the case of n Abelian gauge field strengths allows one to reduce the nonlinear $U(n)$ self-duality conditions to the $U(n)$ invariance condition (90) for the interaction function $E(V, \bar{V})$ and so to obtain a general description of the $U(n)$ self-dual models of nonlinear electrodynamics in terms of an $U(n)$ invariant function of n complex auxiliary variables $V^i_{\alpha\beta}, \bar{V}^i_{\dot{\alpha}\dot{\beta}}$.

The passing to the standard $F, \bar{F}$ representation of the corresponding Lagrangians requires solving the nonlinear algebraic equations (87) for $V^i_{\alpha\beta}, \bar{V}^i_{\dot{\alpha}\dot{\beta}}$. In general, this can be done only within power expansions. It would be interesting to find an example of self-dual system with a few gauge field strengths, where a solution to such equations can be found in a closed form, like in the BI example of Sect. 4, and to establish the precise connection with the Lagrangian (83).

Finally, let us notice that the condition of "discrete" self-duality in the general case is as follows

$$E(V^k_{\alpha\beta}, \bar{V}^k_{\dot{\alpha}\dot{\beta}}) = E(-iV^k_{\alpha\beta}, i\bar{V}^k_{\dot{\alpha}\dot{\beta}}) \,. \tag{92}$$

It is an obvious generalization of the $n = 1$ condition (59).

6 Conclusion

This talk is an extended version of Sect. 3 of our work [6] devoted to the construction of $N = 3$ supersymmetric Born-Infeld theory (see also [11]). We have introduced a new V, F representation for the Lagrangians of nonlinear electrodynamics and shown that it allows for a simple description of systems exhibiting the properties of on-shell $U(n)$ self-duality or/and off-shell discrete self-duality in terms of real function of auxiliary bispinor complex fields. In the V, F representation, the nonlinear self-duality conditions are reduced to the simple $U(1)$ or $U(n)$ invariance conditions (49), (90) for this function. The general condition of the discrete self-duality, or self-duality under Legendre transformation, is the invariance of this function under some discrete reflections of its arguments, Eqs. (59), (92).

It is an interesting and quite feasible task to extend this consideration to the case of $N = 1$ and $N = 2$ supersymmetric extensions of nonlinear electrodynamics [3, 8] in order to obtain a general characterization of the corresponding self-dual systems. Also, finding out a similar V, F representation for non-Abelian BI theory and its superextensions could shed more light on the structure of these theories.

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On Unconventional Integrations and Cross Ratio on Supermanifolds

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Abstract

The conventional integration theory on supermanifolds had been constructed so as to possess (an analog of) Stokes' formula. In it, the exterior differential d is vital and the integrand is a section of a fiber bundle of finite rank. Other, not so popular, but, nevertheless, known integrations are analogs of Berezin integral associated with infinite dimensional fibers. Here I offer other unconventional integrations that appear thanks to existence of several versions of traces and determinants and do not allow Stokes formula. Such unconventional integrations have no counterpart on manifolds except in characteristic p.
Another type of invariants considered are analogs of the cross ratio for "classical superspaces".
As a digression, homological fields corresponding to simple Lie algebras and superalgebras are described.

For the basics on Linear Algebra in Superspaces and Supermanifold theory see [1]; for notations and useful facts see [2], [3]. At the talk I also considered related issues partly collected in [4]. As compared with the talk, §§2, 3 are new; they are a part of the talk given 10 years earlier [5] but yet unpublished. Encouraged by Manin's selected examples [6] and recent results in classification of simple Lie superalgebras [7] I decided to draw attention to these issues.

1 Integration

1.1 Integration with Stokes' formula

In mid 1970's J. Bernstein and I discussed how to construct an analog of integration theory on supermanifolds. We had at our disposal (1) the differential forms, i.e., functions polynomial in differentials of the coordinates, the coefficients of these polynomials being usual functions and (2) volume forms, the latter constituted a rank one module Vol over the algebra of functions and under the change of coordinates the generator $\mathrm{vol}(x(y))$ of Vol accrued the Berezinian (superdeterminant) of the Jacobi matrix as the factor. Each of the above notions had to be carefully reconsidered in super setting because even the most innocent-looking notions and theorems (e.g., the Foubini theorem) displayed, in supersetting, funny signs at unexpected places, see [8], v. 31.

On manifolds, one *can* integrate differential forms; on supermanifolds, one can *not*: their transformation rule yields no analog of determinant, except in the absence of odd parameters. On the other hand, one, clearly, can integrate elements of Vol, provided they

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are with compact support, of course. But we wanted to have some analog of Stokes' formula, and, therefore, needed (1) elements of "degrees" lesser than that of volume forms, and (2) the notion of the supermanifold with boundary to overcome the puzzle demonstrated by "Rudakov's example"; for solution see [9].

To have integration theory, one needs not only what to integrate (the integrand), but over what (cycle), and orientation. The latter two notions turned out to be more involved than we originally thought; Shander clarified this in his development of integration theory, see [10] and the details in [8].

Actually, what we had had was sufficient to construct the integration theory desired: by setting $\Sigma_{-i} = \mathrm{Hom}_{\mathrm{F}}(\Omega^i, \mathrm{Vol})$, where F is the superspace of functions, we obtain a complex dual to the de Rham one with $\Sigma_0 = \mathrm{Vol}$ as forms of the highest degree. We called the elements of $\Sigma_\cdot$ *integrable* forms (the ones one can integrate) and described how to integrate such forms in [9].

1.2 Veblen's problem and Rudakov

We wondered for a while if there is another integration theory with Stokes' formula, and to investigate the options, considered the following problem: *describe all differential operators acting in the spaces of tensor fields and invariant with respect to any changes of variables.* Indeed, the exterior differential (instrumental in Stokes' formula) is, evidently, an invariant and, as is proven in [11], this is the only invariant unary differential operator between spaces of tensor fields whose fibers are irreducible $\mathfrak{gl}(n)$-modules with vacuum vector. So, in order to scan the options on supermanifolds, it was necessary to list all such operators.

This problem (to list all invariant differential operators) goes back to O. Veblen (see [4] for a review). For unary operators on manifolds it was solved by Rudakov [11] as a part of another problem (description of irreducible vacuum vector modules over simple Lie algebras of formal or polynomial vector fields).

1.3 Unconventional integrations

A. Shwarts and his students [12] attempted to integrate densities and objects depending on higher jets of the diffeomorphism but all their examples boil down to either pseudodifferential ([14]) or integral forms. Having obtained an analog of Rudakov's result for the general vectorial Lie *super*algebra [13], we can be sure that there is only one integration theory on supermanifolds *provided the integration involves tensors*

$$\textit{with irreducible finite dimensional fibers.} \tag{$*$}$$

The result of [13] do not preclude, however, unconventional integrations. For tensors other than $(*)$ constructions à la Shwarts may lead to an integration theory (perhaps, [15] leads to it). Such a theory does exist and in [16] the calculations from [13] are used to consider infinite dimensional fibers snubbed at in [13] for no reason except tradition. It turns out that in the spaces of such tensors there act invariant operators similar to Berezin integral. Next, observe that having stated that it is impossible to integrate differential forms on supermanifolds, we almost immediately published a paper [14] showing, nevertheless, how to do it if one is very eager to. More exactly, one has to consider *pseudodifferential*

forms, i.e., functions nonpolynomial in differentials. Of course, there are no such functions on manifolds. Certain types of pseudoforms lead to new invariants — semi-infinite cohomology of supermanifolds; quite criminally, no examples are calculated yet.

Here I consider still another type of "integrations".

1.4 Supertraces and superdeterminants

From the very beginning I wondered what if we stop insisting on having an analog of the Stokes' formula? What remains of the integration then? Only the Jacobian, one can say. Since it is easier to deal with Lie algebras than with groups, let me list analogs of trace for Lie superalgebras. Then, if the Lie superalgebra $\mathfrak{g}$ can be exponentiated to a Lie supergroup, we can consider the analog of the determinate defined via the formula

$$\det\exp X = e^{\mathrm{tr}(X)} \quad \text{for any} \quad X \in \mathfrak{g}. \tag{1}$$

In other words, I mean:

1) Let us consider the Lie superalgebras $\mathfrak{g}$ with a trace also denoted by tr (i.e., $\mathrm{tr}([x,y]) = 0$ for any $x, y \in \mathfrak{g}$); then for the role of Vol we can take tensor fields of type tr, its infinitesimal transformations being the Cartan prolongation (see [4]) of the pair $(\mathrm{id}, \mathfrak{g})$, where id is the "standard" or "identity" representation of $\mathfrak{g}$.

The prime example is provided by the Poisson Lie superalgebra $\mathfrak{g} = \mathfrak{po}(0|2n)$. Indeed, there is a parametric family (quantization) of Lie superalgebras $\mathfrak{g}_t$ which at $t = 0$ coincides with $\mathfrak{po}(0|2n)$ and $\mathfrak{g}_t \simeq \mathfrak{gl}(2^{n-1}|2^{n-1})$ for $t \neq 0$, see [17].

2) From various points of view it is clear that $\mathfrak{gl}(n)$ has at least two superanalogs: the "simple-minded" one, $\mathfrak{gl}(n|m)$, and the "queer" one, $\mathfrak{q}(n)$. On $\mathfrak{q}(n)$, the supertrace vanishes identically but there are specially designed for it its particular, queer, trace and determinant. Regrettably, the queer trace is odd and, therefore, to describe the corresponding representation, we need odd parameters, which causes extra difficulties.

So, still another versions of integration theory, if exist, are related with the queertrace and its "quasiclassical limit" as $t \longrightarrow 0$: the restriction of the above quantization is a parametric family $\mathfrak{g}_t$ which at $t = 0$ coinsides with $\mathfrak{po}(0|2n-1)$ and $\mathfrak{g}_t \simeq \mathfrak{q}(2^{n-1})$ for $t \neq 0$. In 1) and 2) the identity representation of $\mathfrak{po}(0|m)$ is the adjoint one and the Berezin integral serves as tr.

3) The analog of trace on the general vectorial algebra $\mathfrak{vect}(m|n)$ is the divergence. I do not know how to generalize formula (1) with divergence serves as tr, so let me mention two other, more obvious, analogs of tr: (a) in characteristic p such analog exists, e.g., for Lie algebras of contact vector fields (but not only), see [18]; another one is provided by (b) "superconformal" algebras of divergence-free series $\mathfrak{svect}(1|N)$and the exceptions related with $N = 4$ and $N = 5$ extended Neveu-Schwarz algebras, see [19]; these traces are of the same parity as N.

2 Jordan superalgebras

I consider here certain algebraic structures associated with certain selected "classical superdomains". So far, nobody knows yet (as far as I know) even a "right" definition of this basic notion, to start with: for finite dimensional manifolds all is clear, for supermanifolds we just consider the most easy to handle simple Lie supergroups and their Lie superalgebras whereas the elusive "right" definition requires, perhaps, semisimple or almost simple Lie superalgebras. So we take the road of least resistance:

Unless otherwise mentioned the ground field is $\mathbb{C}$, the classical superdomains are considered as quotients of *simple* or close to them "classical" Lie supergroups modulo *certain* maximal parabolic subsupergroups; for the list see [20].

This paper is an attempt to tackle the following questions: What are the criteria for selecting the above-mentioned subgroups among other maximal ones? Why cosets modulo other parabolic subsupergroups are seldom considered in Differential Geometry whereas only these "other" cosets are the main topic of study in analytical mechanics of nonholonomic dynamical systems and in supergravity ([21], [22])?

In these questions "super" is beside the point, so we can very well begin with manifolds. The classical domains are distinguished among symmetric spaces by the fact that the Lie algebra of the symmetry group of any classical domain (Hermitian symmetric space) M is a simple complex Lie algebra of the form

$$\mathfrak{g} = \oplus_{|i|\leq 1}\, \mathfrak{g}_i; \qquad (d=1)$$

the tangent space to M at a fixed point can be identified with the $\mathfrak{g}_{-1}$ and, on it, one can always define a Jordan algebra structure, by fixing any element $p \in \mathfrak{g}_1$ and setting

$$x \circ y = [[p,x],y] \quad \text{for any } x, y \in \mathfrak{g}_{-1}. \tag{2}$$

Recall that a *Jordan algebra* is a commutative algebra J with product $\circ$ satisfying, instead of associativity, the identity

$$(x^2 \circ y) \circ x = x^2 \circ (y \circ x). \tag{JI}$$

In a very inspiring paper [23] McCrimmon gave an account of some applications of Jordan algebras from antiquity to nowadays, see also refs. in [24], [25]. The paper and books strengthen my prejudice that *general* Jordan algebras are, bluntly speaking, useless. Contrariwise, *simple* Jordan algebras give rise to several notions important in various problems. For simple Jordan algebras the so-called *general norm* [26] should be nondegenerate which imposes additional constraints on the parabolic subalgebra. This answers the above questions but tempts one to make use of the other coset spaces as well.

I wish to make similar use of simple Jordan *super*algebras, especially infinite dimensional ones, associated with infinite dimensional classical superdomains listed in [5]; for convenience I reproduce these tables.

In the '60s a remarkable correspondence between Jordan algebras and certain $\mathbb{Z}$-graded Lie algebras became explicit, cf. [27], [28]–[29] and [25]. Kantor used this correspondence to list simple Jordan algebras (over $\mathbb{C}$ and $\mathbb{R}$) by the, so far, simplest known method. He clarified the mysterious relation of Jordan algebras with classical domains and actively studied certain generalizations of Jordan algebras associated with $\mathbb{Z}$-graded simple Lie algebras of finite depth d (since all of them are of the form $\mathfrak{g} = \underset{|i|\leq d}{\oplus} \mathfrak{g}_i$, their *length* is equal to d). Supersymmetry, supertwistors etc. are related with gradings $d > 1$ almost without exceptions; so it is interesting to find generalizations of Jordan algebras (or, rather, useful related structures).

Following Freudental and Springer, Kantor generalized products (2) to several arguments which is natural for $d > 1$. I suggest, contrariwise, to stick to formula (2), even for $d > 1$, with the minimal modification: fix $p \in \mathfrak{g}_1$ and for $\mathfrak{g}_- = \underset{i\leq 0}{\oplus} \mathfrak{g}_i$ set

$$x \circ y = [[p,x],y] \quad \text{for any } x, y \in \mathfrak{g}_-. \tag{3}$$

In this way we obtain noncommutative generalizations of Jordan algebras (with unknown relations instead of (JI)) and it is interesting to investigate what type of integrable systems are associated with them under the Sokolov-Svinolupov's approach, cf. [30], [31].

Kac [32] has already applied this correspondence to list simple *finite* dimensional Jordan superalgebras.

Tables (borrowed from [5]) provide with a list of simple Jordan superalgebras associated with the known in 1991 simple $\mathbb{Z}$-graded Lie superalgebras of polynomial growth (SZGLSAPGs for short), cf. [32], [17], [33], namely, with $\mathbb{Z}$-gradings of depth 1 of SZGLSAPGs *including* finite dimensional ones. (This is the place where [32] contains an omission — cf. Kac' exceptional Jordan superalgebra K with our series $\mathfrak{sh}$ discovered by Serganova in 1983, see [8], and later rediscovered several times.)

2.0.1 Tits–Kantor–Köcher's functor $\mathfrak{kan}$

Let J be a Jordan superalgebra and p the tensor that determines the product in J, i.e., $p(x,y) = x \circ y$. To J, Kantor assigns (see [32]) a $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{kan}(J) = \underset{|i|\leq 1}{\oplus} \mathfrak{kan}(J)_i$, a $\mathbb{Z}$-graded Lie subalgebra in $\mathfrak{vect}(J)$ such that (here $L_a(x) = a \circ x$)

$$\begin{aligned}
\mathfrak{kan}(J)_{-1} &= \mathfrak{vect}(J)_{-1},\\
\mathfrak{kan}(J)_0 &= \mathrm{Span}(L_a, [L_a, L_b] \mid a, b \in J),\\
\mathfrak{kan}(J)_1 &= \mathrm{Span}(p, [L_a, p] \mid a \in J).
\end{aligned}$$

Conversely, for any $\mathbb{Z}$-graded Lie superalgebra of the form $\mathfrak{g} = \underset{i\geq -1}{\oplus} \mathfrak{g}_i$ we define a Jordan superalgebra structure on $\mathfrak{g}_{-1}$ if $(\mathfrak{g}_1)_{\bar 0} \neq 0$ in the following way. Take $p \in (\mathfrak{g}_1)_{\bar 0}$ and for $x, y \in \mathfrak{g}_{-1}$ set

$$x \circ y = [[p,x],y]. \tag{J}$$

2.0.2 Digressioin: on homological fields

I do not know what structure is related with an arbitrary odd p but if p is *homologic*, i.e., $[p,p] = 0$, then the formula

$$[x,y]' = [[p,x],y] \tag{L}$$

determines a Lie superalgebra structure on $\Pi(\mathfrak{g}_{-1})$. This structure had been first noticed, perhaps, by M. Gerstenhaber in '60s and rediscovered many times since then. It seemed interesting to describe in intrinsic terms the p's which determine *simple* Lie (super)algebras. Homological vector fields were first introduced, in connection with the problem of integration of differential equations on supermanifolds, by V. Shander [34], who gave a normal form for the nonsingular fields. However, Shander did not consider singularities of the fields in that work; this was recently done by Vaintrob in a series of articles (e.g.,[35]) in which he showed that the study of singularities of homological fields, and their classification, turns out to be rather similar to the case of singularities of smooth functions. Regrettably, the answer, as we have recently established with Grozman, is more trivial than expected:

Let, first, $\mathfrak{g} = \mathfrak{gl}(n)$; denote the matrix units by ∂_i^j, let $(\partial_i^j)^* = x_j^i$ be the dual basis. Then $p = \sum \xi_j^i \xi_k^j \delta_i^k$, where ξ_j^i is the odd copy of x_j^i and $\delta_i^k = \frac{\partial}{\partial \xi_k^i}$. Similarly, if the X_i form a basis of $\mathfrak{g}$ and $[X_i, X_j] = \sum c_{ij}^k X_k$, then the operator $p \in \mathfrak{vect}(0|\dim \mathfrak{g})$ is of the form $\frac{1}{2}\sum c_{ij}^k X_i^* X_j^* X_k$, where X_i^* is the dual of X_i.

Having observed that every simple finite dimensional Lie algebra (over $\mathbb{C}$) possesses a nondegenerate symmetric bilinear form, we see that for such algebras p is a hamiltonian vector field; to find the corresponding generating function is easy: it is the sum of all elements of degree 1 and weight 0 with respect to the Cartan subalgebra of $\mathfrak{po}(0|\dim \mathfrak{g})$. Generalization to Lie superalgebras is straightforward. Still, observe that some simple Lie superalgebras have no form at all, some (e.g., $\mathfrak{q}(n)$) possess an *odd* nondegenerate symmetric bilinear form in which case p belongs to the antibracket algebra, not to the Poisson one.

2.1 SZGLSAPGs of depth 1 and length 1

All possible $\mathbb{Z}$-gradings of SZGLSAPGs $\mathfrak{g}$ are listed in [32] for $\dim \mathfrak{g} < \infty$ and in [8] for most of the other cases. Our job is to pick those of them which are of depth 1, in particularly, of the form $\underset{|i|\leq 1}{\oplus} \mathfrak{g}_i$, see Tables.

Albert's notation for Jordan algebras were given in accordance with Cartan's notations for the corresponding Lie algebras. As follows from Serganova's classification of systems of simple roots of simple Lie superalgebras [36], Cartan's notations are highly inappropriate for Lie superalgebras.

2.1.1 Matrix Jordan superalgebras

Let $B_{m,2n} = \begin{pmatrix} 1_m & 0 \\ 0 & J_{2n} \end{pmatrix}$, where $J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$. Set

$$\begin{aligned}
&\mathtt{Mat}(m|n) = \{X \in \mathrm{Mat}(m|n)\},\\
&\mathtt{Q}(n|n) = \{X \mid [X, J_{2n}] = 0\},\\
&\mathtt{OSp}(m|2n) = \{X \mid X^{st} B_{m,2n} = B_{m,2n} X\},\\
&\mathtt{Pe}(n|n) = \{X \mid X^{st} J_{2n} = (-1)^{p(X)} J_{2n} X\}.
\end{aligned}$$

In the first two of these spaces the product is given by the formula

$$X \circ Y = XY + (-1)^{p(X)p(Y)} YX.$$

I leave it as an excersise to figure out the formula in the other two cases; for the answer see [32].

2.1.2 Jordan algebras from bilinear forms

Set

$$\mathtt{Q}_{m|2n} = \mathbb{C}^{m|2n}$$

with a nondegenerate even symmetric bilinear form $(\cdot,\cdot)$ and the product

$$x \circ y = (e, x)y + x(e, y) - (x, y)e \qquad (Q)$$

where $e \in (\mathtt{Q}_{m|2n})_{\bar 0}$ satisfies $(e, e) = 1$.

Set $\mathtt{HQ}_{m|2n} = \Pi(\mathbb{C}[p, q, \Theta])$, where $m > 0$, $p = (p_1, \dots, p_n)$, $q = (q_1, \dots, q_n)$, $\Theta = (\xi_1, \dots, \xi_r, \eta_1, \dots, \eta_r)$ for $m = 2r$, of $\Theta = (\xi, \eta, \theta)$ for $m = 2r + 1$ with the Jordan product defined with the help of the symplectic form ω on the supermanifold with coordinates p, q, Θ:

$$x \circ y = \omega(e, x)y + x\omega(e, y) - \omega(x, y)e, \qquad (H')$$

where $e \in (\mathtt{HQ}_{m|2n})_{\bar 0}$ satisfies $\omega(e, e) = 1$.

To explicitly give the product, consider the space $\mathbb{C}[p, q, \Theta, \alpha, \beta]$ with two extra odd indeterminates and the Poisson bracket such that p and q, ξ and η, and α, β are dual. Setting $\deg \alpha = -\deg \beta = -1$ the degrees of the other indeterminates being 0, we obtain the $\mathbb{Z}$-grading of the Poisson algebra, and its quotient modulo center, of the form ($d = 1$). On $\mathfrak{g}_{-1} = \mathbb{C}[p, q, \Theta]\alpha$, define the product

$$H_{f\alpha} \circ H_{g\alpha} = \{\{H_\beta, H_{f\alpha}\}, H_{g\alpha}\} = (-1)^{p(f)} H_{\{f,g\}\alpha}.$$

In other words, on the superspace of functions with shifted parity, we set

$$f \circ g = (-1)^{p(f)+1}\{f, g\}. \qquad (H, K)$$

2.1.3 Exceptional Jordan superalgebras

There are two of them associated with the gradings of $\mathfrak{osp}(4|2;\alpha)$ and $\mathfrak{ab}_3$ from Table 1 and the corresponding loops.

2.1.4 Stringy Jordan superalgebras

These are obtained from $\mathfrak{k}^L(1|n)$ by formula (J) with the grading from Table 1. They will be denoted, respectively, by

$$\mathtt{K^LQ}_{1|n} \cong \Pi(\mathbb{C}[t^{-1}, t, \theta_1, \dots, \theta_n])$$

$$\mathtt{K^MQ}_{1|n} \cong \Pi(\mathbb{C}[t^{-1}, t, \theta_1, \dots, \sqrt{t}\theta_n]).$$

The product is given by formula (H, K).

2.1.5 Loop Jordan superalgebras

For a finite-dimensional Jordan superalgebra J denote by $J^{(1)} = J \otimes \mathbb{C}[t^{-1}, t]$ the loops with values in J and point-wise product.

2.1.6 Twisted loop Jordan superalgebras

These are associated by formula (J) with the Lie superalgebras from Table 2.

3 Cross ratios

In [37] Kantor generalized the cross ratio of four points on $\mathbb{P}^1$ to most of the quotients G/P, where G is a simple Lie group and P is its parabolic subgroup. The Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ in these cases is of the form $\mathfrak{g} = \bigoplus\limits_{|i|\leq d} \mathfrak{g}_i$. I do not know any paper referring to [37], so Kantor's studies drew no attention at all. His constructions, however, naturally appear in supersetting [6]; this prompts me to try to decipher a part of [37]. I will consider here the simplest case, when $d = 1$ and the corresponding Jordan algebra is simple. In this case one can generalize the cross ratio from $\mathbb{P}^1 = \mathtt{Gr}_1^2$ to a collection of $\mathtt{Gl}(2m|2n)$-invariants of four points on $\mathtt{Gr}_{m|n}^{2m|2n}$. First, consider

$$(A, B, C, D) = (A - B)(C - B)^{-1}(C - D)(A - D)^{-1}. \qquad (CR)$$

Let $mn = 0$. Now, replace the rhs, X, of (CR) with $\det(X - \lambda E)$; the collection of all coefficients of the powers of λ is the analog of the cross ratio. These are all the invariants of four points for the general, orthogonal and Lagrangian grassmannians.

For their super counterparts we take the Berezinian (superdeterminant) and the amount of *polynomially independent* invariants is infinite, cf. [3]. If, however, we consider rational dependence, which is natural in super setting, the coefficients of the first $n + m$ powers of λ generate the algebra of invariants and is a natural candidate for the cross ratio.

On $\mathtt{Q}(n|n)$, we should take the queerdeterminant, qet, instead of det; the collection obtained is finite.

For loop Jordan superalgebras we consider matrix-valued functions and $\det(X - \lambda E)$ returns a collection of functions, rather than numbers.

I do not know the complete cross ratio for Jordan superalgebras related to quadrics and do not know at all what are they for twisted loops and in stringy cases, most interesting to me. One invariant is obvious (but there should be several if $\dim \mathtt{Q} > 1$): given the form $(\cdot,\cdot)$, or the symplectic form ω in the curved case, set (for the Λ-points (see [1]) of the Jordan algebra J, i.e., for $A, B, C, D \in (J \otimes \Lambda)_{\bar 0}$

$$(A, B, C, D) = \frac{(A - B, A - B)}{(C - B, C - B)} \frac{(C - D, C - D)}{(A - D, A - D)}. \qquad (CRQ)$$

For curved quadrics, take $\omega(H_{A-B}, H_{A-B})$ instead of $(A - B, A - B)$, etc.

4 Tables

Everywhere we assume the notational conventions of [3] and definitions adopted there.

In Table 1 we say that the homogeneous superspace G/P, where G is a simple Lie supergroup, P its parabolic subsupergroup corresponding to several omitted generators of a Borel subalgebra (description of these generators can be found in [38]), of *depth d* and *length l* if such are the depth and length of $\mathfrak{g} = \mathrm{Lie}(G)$ in the $\mathbb{Z}$-grading compatible with that of $\mathrm{Lie}(P)$. Note that all superspaces of Table 1 possess an hermitian structure (hence are of depth 1) except $PeGr$ (no hermitian structure), PeQ (no hermitian structure, length 2), $CGr_{0,k}^{0,n}$ and $SCGr_{0,k}^{0,n}$ (no hermitian structure, lengths $n - k$ and, resp., $n - k - 1$).

Let $\mathfrak{s}(\mathfrak{g})$ be the traceless part of $\mathfrak{g}$ and $\mathfrak{p}(\mathfrak{g}) = \mathfrak{g}/\text{center}$; let $\mathfrak{g}_\varphi^{(m)}$ be the stationary subalgebra of the loop algebra with values in $\mathfrak{g}$ singled out by the degree m automorphism φ of $\mathfrak{g}$;for $G = \mathfrak{g}_\varphi^{(m)}$ with the $\mathbb{Z}$-grading of type $(d = 1)$ the last column of Table 1 contains G_0; the map $-$st, "minus supertransposition", sends X to $-X^{st}$, the map Π

sends $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\begin{pmatrix} d & c \\ b & a \end{pmatrix}$, and δ_x sends $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\begin{pmatrix} a & xb \\ xc & d \end{pmatrix}$; the automorphism A of $\mathfrak{po}$ is defined on monomials $f(\theta)$ as id if $\frac{\partial f}{\partial \theta_1} = 0$ and otherwise; $irr(\ldots)$ is any of the two irrducible components; LGr and OGr stand for the Lagrangian and orthogonal Grassmannian, respectively; the dual domain is endowed with an asterisk as a left superscript.

Table 1: Gradings of twisted loop (super)algebras corresponding to hermitian superdomains.

$\mathfrak{g}_\varphi^{(m)}$	φ	grading elements from $\mathfrak{h}$	$(\mathfrak{g}_\varphi^{(m)})_0$
$\mathfrak{sl}(2m\|2n)^{(2)}$	$(-st) \circ \mathrm{Ad}\ \mathrm{diag}(\Pi_{2m}, J_{2n})$	$\mathrm{diag}(1_m, -1_m, 1_n, -1_n)$	$\mathfrak{sl}(m\|n)^{(1)}$
$\mathfrak{sl}(2m)^{(2)}$	$(-t) \circ \mathrm{Ad}(\Pi_{2m})$		$\mathfrak{sl}(m)^{(1)}$
$\mathfrak{sl}(2n)^{(2)}$	$(t) \circ \mathrm{Ad}(J_{2n})$		$\mathfrak{sl}(n)^{(1)}$
$\mathfrak{sl}(n\|n)^{(2)}$	Π	$\mathrm{diag}(1_p, 0_{n-p}, 1_p, 0_{n-p})$	$\mathfrak{s}(\mathfrak{gl}(p\|p)_\Pi^{(2)} \oplus \mathfrak{gl}(n-p\|n-p)_\Pi^{(2)})$
$\mathfrak{sl}(n\|n)^{(2)}$	$\Pi \circ (-st)$	$\mathrm{diag}(1_p, -1_{n-p}, -1_p, 1_{n-p})$	$\mathfrak{s}(\mathfrak{gl}(p\|p)_{\Pi\circ(-st)}^{(2)} \oplus \mathfrak{gl}(n-p\|n-p)_{\Pi\circ(-st)}^{(2)})$
$\mathfrak{osp}(2m\|2n)^{(2)}$	$\varphi_{m,n} \mathrm{Ad}\ \mathrm{diag}(1_{2m-1}, 1, 1_{2n})$	$\mathrm{diag}(J_2, O_{2(m+n-1)})$	$(\mathfrak{cosp}(2m-2\|2n))_{\varphi_{m-1,n}}^{(1)}$
$\mathfrak{o}(2m)^{(2)}$			$(\mathfrak{co}(2m-2))^{(1)}$
$\mathfrak{psq}(2n)^{(4)}$	$(-st) \circ \delta_{\sqrt{-1}}$	$\mathrm{diag}(J_{2n}, J_{2n})$	$\mathfrak{psq}(n)_{\delta_{-1}}^{(2)}$
$\mathfrak{sh}(2n)^{(2)}$	A		$(\mathfrak{sh}(2n-2) \oplus \Lambda(2n-2))_A^{(2)}$
$\mathfrak{psq}(n)^{(2)}$	δ_{-1}	$\mathrm{diag}(1_p, 0_{n-p}, 1_p, 0_{n-p})$	$\mathfrak{ps}(\mathfrak{q}(p)_{\delta_{-1}}^{(2)} \oplus \mathfrak{q}(n-p)_{\delta_{-1}}^{(2)})$

Table 2: for the lack of space I give an interpretation of the supergrassmannians here, linewise: the supergrassmannian of $p|q$-dimensional subsuperspaces in $\mathbb{C}^{m|n}$ and same for $n = m$, $p = q$; superquadric of $1|0$-dimensional isotropic (wrt a non-degenerate even form) lines in $\mathbb{C}^{m|n}$; ortolagrangian supergrassmannian; queergrassmannian; "odd" superquadric (wrt a non-degenerate even form) of $1|0$-lines in $\mathbb{C}^{n|n}$; odd-lagrangian supergrassmannian; curved supergrassmannian of $0|1$-dimensional subsupermanifolds in $\mathbb{C}^{0|n}$; curved superquadric; two exceptions.

Table 2: Classical superspaces of depth 1.

$\mathfrak{g}$	$\mathfrak{g}_0$	$\mathfrak{g}_{-1}$	Underlying domain	Name of the superdomain
$\mathfrak{sl}(m\vert n)$	$\mathfrak{s}(\mathfrak{gl}(p\vert q) \oplus \mathfrak{gl}(m-p\vert n-q))$	$id \otimes id^*$	$Gr_p^m \times Gr_q^n$	$Gr_{p,q}^{m,n}$
$\mathfrak{psl}(m\vert m)$	$\mathfrak{ps}(\mathfrak{gl}(p\vert p) \oplus \mathfrak{gl}(m-p\vert m-q))$	$id \otimes id^*$	$G_p^m \times Gr_p^m$	$Gr_{p,p}^{m,m}$
$\mathfrak{osp}(m\vert 2n)$	$\mathfrak{cosp}(m-2\vert 2n)$	id	Q_{m-2}	$Q_{m-2,n}$
$\mathfrak{osp}(2m\vert 2n)$	$\mathfrak{gl}(m\vert n)$	Λ^2	${}^*OGr_m \times LGr_n$	$OLGr_{m,n}$
$\mathfrak{sq}(n)$	$\mathfrak{s}(\mathfrak{q}(p) \oplus \mathfrak{q}(n-p))$	$irr(id \otimes id*)$	Gr_p^n	QGr_p^n
$\mathfrak{psq}(n)$	$\mathfrak{ps}(\mathfrak{q}(p) \oplus \mathfrak{q}(n-p))$			
$\mathfrak{pe}(n)$	$\mathfrak{cpe}(n-1)$	id	$\mathbb{C}P^{n-1}$	PeQ_{n-1}
$\mathfrak{spe}(n)$	$\mathfrak{cspe}(n-1)$			
$\mathfrak{pe}(n)$	$\mathfrak{gl}(p\vert n-p)$	$\Pi(S^2(id))$	Gr_p^n	$PeGr_p^n$
$(\mathfrak{spe}(n))$	$(\mathfrak{sl}(p\vert n-p))$	or $\Pi(\Lambda^2(id))$		
$\mathfrak{vect}(0\vert n)$	$\mathfrak{vect}(0\vert n-k) \oplus \mathfrak{gl}(k;\Lambda(n-k))$	$\Lambda(k) \otimes \Pi(id)$		$CGr_{0,k}^{0,n}$
$\mathfrak{svect}(0\vert n)$	$\mathfrak{vect}(0\vert n-k) \oplus \mathfrak{sl}(k;\Lambda(n-k))$	$\Pi(Vol)$ if $k=1$	–	$SCGr_{0,k}^{0,n}$
$\mathfrak{h}(0\vert m)$	$\mathfrak{h}(0\vert m-2) \oplus \Lambda(m-2) \cdot z$		–	$CQ_{m-2,0}$
$\mathfrak{sh}(m)$	$\mathfrak{sh}(m-2) \oplus \Lambda(m-2) \cdot z$	$\Pi(id)$		
$\mathfrak{osp}(4\vert 2;\alpha)$	$\mathfrak{cosp}(2\vert 2) \simeq (\mathfrak{gl}(2\vert 1))$	id	$\mathbb{CP}^1 \times \mathbb{CP}^1$	E_α
$\mathfrak{ab}(3)$	$\mathfrak{cosp}(2\vert 4)$	$L_{3\varepsilon_1}$	$\mathbb{CP}^1 \times Q_5$	$AB(3)$

The curved superquadric has infinite dimensional "stringy" counterpart with $\mathfrak{h}(0|m)$ replaced with the centerless N-extended Neveu-Schwarz algebra $\mathfrak{k}^L(1|N)$ or Ramond algebra $\mathfrak{k}^M(1|N)$.

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Supersymmetry Breaking from Coset Space Dimensional Reduction

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Abstract

The Coset Space Dimensional Reduction scheme is briefly reviewed. Then a ten-dimensional supersymmetric E_8 gauge theory is reduced over symmetric and non-symmetric six-dimensional coset spaces. In general a four-dimensional non-supersymmetric gauge theory is obtained in case the used coset space is symmetric, while a softly broken supersymmetric gauge theory is obtained if the used coset space is non-symmetric. In the process of exhibiting the above properties we also present two attractive models, worth exploiting further, which lead to interesting GUTs with three families in four dimensions.

1 Introduction

In the recent years the theoretical efforts to establish a deeper understanding of Nature have managed to achieve a number of successes in developing frameworks that aim to describe the fundamental theory at the Planck scale. On the other hand a real breakthrough in a deeper knowledge of Nature should include an understanding of the, at present, free parameters of the Standard Model (SM) in terms of a few fundamental ones. Clearly the apparent success of the SM is spoiled by the presence of a plethora of free parameters mostly related to the ad-hoc introduction of the Higgs and Yukawa sectors in the theory. It is worth recalling that the Coset Space Dimensional Reduction (CSDR) [1, 2, 3, 4] was suggesting from the beginning that a unification of the gauge and Higgs sectors can be achieved in higher dimensions. The four-dimensional gauge and Higgs fields are simply the surviving components of the gauge fields of a pure gauge theory defined in higher dimensions. In the next step of development of the CSDR scheme, fermions were introduced [5, 6] and then the four-dimensional Yukawa and gauge interactions of fermions found also a unified description in the gauge interactions of the higher-dimensional theory. The last step in this unified description in high dimensions is to relate the gauge and fermion fields that have been introduced. A simple way to achieve that is to demand that the higher-dimensional gauge theory is $\mathcal{N} = 1$ supersymmetric which requires that the gauge and fermion fields are members of the same supermultiplet.

In the spirit described above a very welcome additional input is that string theory suggests furthermore the dimension and the gauge group of the higher-dimensional supersymmetric theory [7]. Further support to this unified description comes from the fact that the reduction of the theory over coset [4] and CY spaces [7] provides the four-dimensional theory with scalars belonging in the fundamental representation of the gauge group as are introduced in the SM. In addition the fact that the SM is a chiral theory leads us

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to consider D-dimensional supersymmetric gauge theories with $D = 4n + 2$ [8, 4], which include the ten dimensions suggested by the heterotic string theory [7].

Concerning supersymmetry, the nature of the four-dimensional theory depends on the corresponding nature of the compact space used to reduce the higher-dimensional theory. Specifically the reduction over CY spaces leads to supersymmetric theories [7] in four dimensions, the reduction over symmetric coset spaces leads to non-supersymmetric theories, while a reduction over non-symmetric ones leads to softly broken supersymmetric theories [9, 10, 11]. Concerning the latter as candidate four-dimensional theories that describe the nature, in addition to the usual arguments related to the hierarchy problem, we should remind a further evidence established in their favor the last years. It was found that the search for renormalization group invariant (RGI) relations among parameters of softly broken supersymmetric GUTs considered as a unification scheme at the quantum level, could lead to successful predictions in low energies. More specifically the search for RGI relations was concerning the parameters of softly broken GUTs beyond the unification point and could lead even to all-loop finiteness [12, 13]. On the other hand in the low energies lead to successful predictions not only for the gauge couplings but also for the top quark mass, among others, and to interesting testable predictions for the Higgs mass [14].

The review is organized as follows. In section 2 we describe the CSDR. In section 3 we describe two representative examples of CSDR of a ten-dimensional supersymmetric E_8 gauge theory over symmetric coset spaces. In section 4 we present the CSDR of the same E_8 gauge theory over the three existing non-symmetric coset spaces. Finally section 5 contains our conclusions.

2 The Coset Space Dimensional Reduction (CSDR) scheme

Let us proceed by recalling the fundamental ideas of the dimensional reduction procedure. Dimensional reduction is the construction of a lower dimensional Lagrangian starting from higher dimensions. In our construction we will use the symmetries of the extra dimensions. The CSDR is a dimensional reduction scheme where the extra dimensions form a coset space S/R and the symmetry used is the group S of isometries of S/R.

Given this form of the extra coordinates one can construct a lower dimensional Lagrangian by demanding the fields to be symmetric. This means that one requires the fields to be form invariant under the action of the group S on extra coordinates,

$$\delta_\xi \Phi^i = L_\xi \Phi^i = 0, \tag{1}$$

where L_ξ is the Lie derivative with respect to the Killing vectors ξ of the extra dimensional metric. However this condition is too strong when the higher-dimensional Lagrangian possesses a symmetry. Then a generalized form of the symmetry condition (1) can be

$$\delta_\xi \Phi^i = L_\xi \Phi^i = U_\xi \Phi^i, \tag{2}$$

where U_ξ is the symmetry of the Lagrangian.

When we apply CSDR in a gauge theory the U_ξ is a gauge transformation. Then the original Lagrangian becomes independent of the extra coordinates because of gauge invariance.

To be specific we consider a Yang-Mills-Dirac Lagrangian with gauge group G in D dimensions

$$S = \int d^D x \sqrt{-g}\Big[-\frac{1}{4} Tr\,(F_{MN}F_{K\Lambda})\, g^{MK} g^{N\Lambda} + \frac{i}{2}\overline{\psi}\Gamma^M D_M \psi\Big], \tag{3}$$

where capital indices M, N run from $0 \dots D-1$. The field strength is

$$F_{MN} = \partial_M A_N - \partial_N A_M - [A_M, A_N], \tag{4}$$

with A_M the gauge field. Γ^M are the D-dimensional gamma matrices satisfying

$$\{\Gamma_M, \Gamma_N\} = 2g_{MN}, \tag{5}$$

ψ is an anticommuting spinor in D dimensions and $\overline{\psi}$ is the Dirac adjoint. D_M is the covariant derivative,

$$D_M = \partial_M - \theta_M - A_M, \tag{6}$$

with

$$\theta_M = \frac{1}{2}\theta_{MN\Lambda}\Sigma^{N\Lambda}, \tag{7}$$

the spin connection of M^D.

Then the original spacetime (M^D, g^{MN}), is assumed to be compactified to $M^4 \times S/R$, with S/R a coset space. The original coordinates z^M become coordinates of $M^4 \times S/R$, $z^M = (x^m, y^\alpha)$, where α is a curved index of the coset, and with a we will denote a flat tangent space index. The metric is

$$g^{MN} = \begin{bmatrix} \eta^{\mu\nu} & 0 \\ 0 & -g^{\alpha\beta} \end{bmatrix}, \tag{8}$$

where $\eta^{\mu\nu} = diag(1,-1,-1,-1)$ is the Minkowski spacetime metric and $g^{\alpha\beta}$ is the coset space metric.

To develop the geometry of coset spaces we divide the generators of S, Q_A in two sets, the generators of R, Q_i $(i = 1, \dots, dimR)$ and the generators of S/R, Q_a $(a = dimR + 1 \dots, dimS)$. Then the commutation relations for the S generators become

$$\begin{aligned} [Q_i, Q_j] &= f_{ij}^k Q_k, \\ [Q_i, Q_a] &= f_{ia}^b Q_b, \\ [Q_a, Q_b] &= f_{ab}^i Q_i + f_{ab}^c Q_c. \end{aligned} \tag{9}$$

The coset space S/R is a symmetric one when $f_{ab}^c = 0$.

The coordinates y define an element of S, $L(y)$, which is a coset representative. Then the Maurer-Cartan form with values in the Lie algebra of S is defined by

$$V(y) = L^{-1}(y)dL(y) = e_\alpha^A Q_A dy^\alpha \tag{10}$$

and obeys the Maurer-Cartan equation,

$$dV + V \wedge V = 0. \tag{11}$$

From the relation (11), and using standard techniques of differential geometry we can develop the geometry of coset spaces, and compute vielbeins, connections, curvature and torsion. For instance the vielbein and the R-connection can be computed at the origin $y = 0$, and the results are, $e^a_\alpha = \delta^a_\alpha$ and $e^i_\alpha = 0$.

Next we require that the fields are invariant up to a gauge transformation when S acts on S/R. Let ξ^α_A, $A = 1, \ldots, dimS$, be the Killing vectors which generate the isometries S of S/R and denote by W_A the compensating gauge transformation associated with ξ_A. Define, next, the infinitesimal coordinate transformation

$$\delta_A \equiv L_{\xi_A}. \tag{12}$$

Then the condition that a coordinate transformation is compensated by a gauge transformation takes the following specific form when it is applied to the higher-dimensional vector and spinor

$$\begin{aligned} \delta_A A_M &= \xi^\beta_A \partial_\beta A_M + \partial_M \xi^\beta_A A_\beta = \partial_M W_A - [W_A, A_M], & (13)\\ \delta_A \psi &= \xi^\alpha_A \partial_\alpha \psi - \frac{1}{2} G_{Abc} \Sigma^{bc} \psi = D(W_A)\psi. & (14) \end{aligned}$$

Inspecting eq. (13) we recognize the familiar form of the Lie derivative of a vector and the gauge transformation appropriate for the adjoint representation. The same is true for the spinor in eq. (14). There $D(W_A)$ represents a gauge transformation in the representation to which the spinor belongs. The quantities W_A may depend on internal coordinates y. The conditions (13), (14) should be covariant when $A_M(x, y)$ and $\psi(x, y)$ transform under a gauge transformation, therefore W's transform under a gauge transformation as

$$\widetilde{W}_A = g W_A g^{-1} + (\delta_A g) g^{-1}. \tag{15}$$

Eqs (13), (14) and (15) provide us with the gauge freedom to do all calculations at $y = 0$ and $W_a = 0$. Note also that the variations δ_A satisfy $[\delta_A, \delta_B] = f^C_{AB} \delta_C$ leading to a further consistency condition for the W's

$$\xi^\alpha_A \partial_\alpha W_B - \xi^\alpha_B \partial_\alpha W_A - [W_A, W_B] = f_{AB}{}^C W_C. \tag{16}$$

The above conditions imply certain constraints that the D-dimensional fields should obey.

The solution of the constraints provide

- The remaining gauge invariance in four dimensions,
- The four-dimensional spectrum,
- The scalar potential of the four-dimensional theory.

More specifically to find the four-dimensional gauge group we embed R in G

$$\begin{aligned} G &\supset R_G \times H, \\ H &= C_G(R_G). & (17) \end{aligned}$$

Then H, the centralizer of the image of R in G, is the four-dimensional gauge group.

The scalar fields that are obtained from the higher components of the vector field are obtained as follows. First we embed R in S and decompose the adjoint of S under R

$$\begin{aligned} S &\supset R \\ adjS &= adjR + \sum s_i. & (18) \end{aligned}$$

Then we embed R in G and decompose the adjoint of G under $R \times H$,

$$\begin{aligned} G &\supset R_G \times H \\ adjG &= (adjR, 1) + (1, adjH) + \sum (r_i, h_i). & (19) \end{aligned}$$

The rule is that when

$$s_i = r_i,$$

i.e. when we have two identical representation of R in the decompositions (18), (19) there is an h_i multiplet of scalar fields that survives in four dimensions.

To find the representation of H under which the four-dimensional fermions transform, we have to decompose the representation F of G to which the fermions belong, under $R_G \times H$, i.e.

$$F = \sum (t_i, h_i), \tag{20}$$

and the spinor of $SO(d)$ under R

$$\sigma_d = \sum \sigma_j. \tag{21}$$

Then for each pair t_i and σ_i, where t_i and σ_i are identical irreducible representations, there is an h_i multiplet of spinor fields in the four-dimensional theory.

In order to obtain chiral fermions in four-dimensions we have to impose the Weyl condition in D dimensions. In $D = 4n + 2$ dimensions, which is the case of interest, the decomposition of the left handed, say spinor under $SU(2) \times SU(2) \times SO(d)$ is

$$\sigma_D = (2, 1, \sigma_d) + (1, 2, \overline{\sigma}_d). \tag{22}$$

So we have in this case the decompositions

$$\sigma_d = \sum \sigma_k, \ \overline{\sigma}_d = \sum \overline{\sigma}_k. \tag{23}$$

Let us start from a vector-like representation F for the fermions. In this case each term (t_i, h_i) in eq.(20) will be either self-conjugate or it will have a partner $(\overline{t}_i, \overline{h}_i)$. According to the rule described in eqs. (20), (21) and considering σ_d we will have in four dimensions left-handed fermions transforming as $f_L = \sum h^L_k$. Since σ_d is non self-conjugate, f_L is non self-conjugate too. Similarly from $\overline{\sigma}_d$ we will obtain the right handed representation $f_R = \sum \overline{h}^R_k$ but as F is vector-like, $\overline{h}^R_k \sim h^L_k$. Therefore there will appear two sets of Weyl fermions with the same quantum numbers under H. This is already a chiral theory, but still one can go further and try to impose the Majorana condition in order to eliminate the doubling of the fermionic spectrum. If we had started with F complex, we should have again a chiral theory since in this case $\overline{h}^R_k$ is different from h^L_k (σ_d non self-conjugate). Starting with F vector-like along with the Majorana condition will be used in our discussions. The Majorana condition can be imposed in $D = 2, 3, 4 + 8n$ dimensions and is given by $\psi = C\overline{\psi}^T$, where C is the D-dimensional charge conjugation matrix. Majorana and Weyl conditions are compatible in $D = 4n + 2$ dimensions. Then in our case if we start with Weyl-Majorana spinors in $D = 4n + 2$ dimensions we force f_R to be the charge conjugate to f_L, thus arriving in a theory with fermions only in f_L. Furthermore if F is to be real, then we have to have $D = 2 + 8n$, while for F pseudoreal $D = 6 + 8n$.

The potential is obtained from the internal components of the field strength. In CSDR they are given by

$$F_{ab} = f^C_{ab} \phi_C - [\phi_a, \phi_b], \tag{24}$$

so the potential is

$$V(\phi) = -\frac{1}{4}g^{ac}g^{bd}Tr(f_{ab}^{C}\phi_C - [\phi_a, \phi_b])(f_{cd}^{D}\phi_D - [\phi_c, \phi_d]). \tag{25}$$

The minimization of the potential is in general a difficult problem. If however S has an isomorphic image S_G in G which contains R_G, the minimum is zero. Furthermore, the four-dimensional gauge group H breaks further by these non-zero vacuum expectation values of the Higgs fields to the centralizer K of the image of S in G, i.e. $K = C_G(S)$ [15]. More generally it can be proven [4] that dimensional reduction over a symmetric coset space always gives a potential of spontaneous breaking form.

The effective action in four dimensions can be written as

$$\begin{aligned} S &= C\int d^4x\Big(-\frac{1}{4}F^t_{\mu\nu}F^{t\mu\nu} - \frac{1}{2}(D_\mu\phi_a)^t(D^\mu\phi^a)^t + V(\phi) \\ &+\frac{i}{2}\overline{\psi}\Gamma^\mu D_\mu\psi - \frac{i}{2}\overline{\psi}\Gamma^a D_a\psi\Big), \end{aligned} \tag{26}$$

where

$$D_\mu = \partial_\mu - A_\mu \tag{27}$$

is the spacetime part of the gauge covariant derivative and

$$D_a = \partial_a - \theta_a - \phi_a, \tag{28}$$

is the internal part of the gauge covariant derivative, with

$$\theta_a = \frac{1}{2}\theta_{abc}\Sigma^{bc},$$

the spin connection of the coset space. C is the volume of the coset space.

Note that the second fermion term in eq. (26) can be written as

$$L_Y = \frac{i}{2}\overline{\psi}\Gamma^a\nabla_a\psi + \overline{\psi}V\psi, \tag{29}$$

where

$$\begin{aligned} \nabla_a &= -\partial_a + \frac{1}{2}f_{ibc}e^i_\gamma e^\gamma_a\Sigma^{bc} + \phi_a, \\ V &= \frac{i}{4}\Gamma^a G_{abc}\Sigma^{bc}. \end{aligned} \tag{30}$$

In eq. (30) we have used the full connection with torsion,

$$\theta^a_{cb} = -f^a_{ib}e^i_\alpha e^\alpha_c - (D^a_{cb} + \frac{1}{2}\Sigma^a_{cb}) = -f^a_{ib}e^i_\alpha e^\alpha_c - G^a_{cb} \tag{31}$$

with

$$D^a_{cb} = g^{ad}\frac{1}{2}[f^e_{db}g_{ec} + f^e_{cb}g_{de} - f^e_{cd}g_{be}], \tag{32}$$

and Σ^a_{cb} the contorsion. The general choice of the contorsion tensor is

$$\Sigma_{abc} = 2\tau(D_{abc} + D_{bca} - D_{cba}), \tag{33}$$

where τ is a free parameter. Furthermore the constraints imply, that $\nabla_a = \phi_a$, when ∇_a acts on a spinor field, and the term $\frac{i}{2}\overline{\psi}\Gamma^a\nabla_a\psi$ in eq. (29) is exactly the Yukawa term. The last term, $V = \frac{i}{4}\Gamma^a G_{abc}\Sigma^{bc}$ gives mass to the gaugino as will be explained in the next section.

3 Coset Space Dimensional Reduction over symmetric coset spaces

Recently we have examined the CSDR of supersymmetric gauge theories in ten dimensions over all six-dimensional coset spaces [11]. Our results are that when the coset space is symmetric, there is no remnant of the supersymmetric spectrum in the four-dimensional theory. Here we present two examples representing this case, one is instructive and the other is a promising candidate to become a realistic model.

3.1 CSDR over $SO(7)/SO(6)$

The ten-dimensional gauge group that we consider is $G = E_8$. First we examine the dimensional reduction over $S/R = SO(7)/SO(6)$, which is a symmetric coset space. To apply the rules we need the embedding of $R = SO(6)$ in $G = E_8$ and the decomposition of the adjoint of G,

$$\begin{aligned} E_8 &\supset SO(6)\times SO(10) \\ 248 &= (15,1)+(1,45)+(6,10)+(4,16)+(\overline{4},\overline{16}). \end{aligned} \tag{34}$$

Then the four-dimensional gauge group is $H = C_{E_8}(SO(6)) = SO(10)$.

To find the $R = SO(6)$ content of $SO(7)/SO(6)$ vector and spinor we need the decompositions

$$\begin{aligned} SO(7) &\supset SO(6) \\ 21 &= 15+6, \end{aligned} \tag{35}$$

and

$$\begin{aligned} SO(6) &\supset SO(6) \\ 4 &= 4. \end{aligned} \tag{36}$$

Therefore the $R = SO(6)$ content of the vector and spinor of $SO(7)/SO(6)$ are 6 and 4 respectively.

The rules stated in section 2 are telling us that from the ten-dimensional gauge field A^a_M we obtain in four dimensions the gauge field $A^{\alpha\beta}_\mu$ and the scalars χ^m, where a is a 248 E_8-index, $\alpha\beta$ a 45 and m a 10 $SO(10)$-index. From the ten-dimensional gaugino ψ^a, we obtain the four-dimensional left-handed matter fermions ψ^s, belonging to 16 of $SO(10)$. In summary

$$\begin{aligned} A^a_M &\longrightarrow A^{\alpha\beta}_\mu, \chi^m, \\ \psi^a &\longrightarrow \psi^s. \end{aligned} \tag{37}$$

Note that there is no supersymmetric partner of the four-dimensional gauge field and the scalar matter in four dimensions transforms as a 10-plet, while fermion matter as a left-handed 16-plet, therefore the spectrum of the four-dimensional theory is non-supersymmetric.

3.2 CSDR over $CP^2 \times S^2$

Choosing $G = E_8$ and $S/R = CP^2 \times S^2$ which is the coset $SU(3) \times SU(2)/SU(2) \times U(1) \times U(1)$ we have an interesting and promising example. The embedding of $R = SU(2) \times U(1) \times U(1)$ in E_8 is given by the decomposition

$$E_8 \supset SU(2) \times U(1) \times U(1) \times SO(10)$$

$$\begin{aligned} 248 \;=\; & (1,45)_{(0,0)} + (3,1)_{(0,0)} + (1,1)_{(0,0)} + (1,1)_{(0,0)} + (1,1)_{(2,0)} + (1,1)_{(-2,0)} \\ & +(2,1)_{(1,2)} + (2,1)_{(-1,2)} + (2,1)_{(-1,-2)} + (2,1)_{(1,-2)} + (1,10)_{(0,2)} + (1,10)_{(0,-2)} \\ & +(2,10)_{(1,0)} + (2,10)_{(-1,0)} + (2,16)_{(0,1)} + (1,16)_{(1,-1)} + (1,16)_{(-1,-1)} \\ & +(2,\overline{16})_{(0,-1)} + (1,\overline{16})_{(-1,1)} + (1,\overline{16})_{(1,1)}. \end{aligned} \quad (38)$$

The four-dimensional gauge group is $H = C_{E_8}(SU(2) \times U(1) \times U(1)) = SO(10) \times U(1) \times U(1)$. The vector and spinor content under R of the specific coset are, $1_{0,2a} + 1_{0,-2a} + 2_{b,0} + 2_{-b,0}$ and $1_{b,-a} + 1_{-b,-a} + 2_{0,a}$ respectively. Choosing $a = b = 1$ we find that the scalar fields of the four-dimensional theory transform as $10_{(0,2)}$, $10_{(0,-2)}$, $10_{(1,0)}$, $10_{(-1,0)}$ under H. Also, we find that the fermions of the four-dimensional theory are the following left-handed multiplets of H: $16_{(-1,-1)}$, $16_{(1,-1)}$, $16_{(0,1)}$. Therefore altogether we find

$$\begin{aligned} A_M^a &\longrightarrow A_\mu^{\alpha\beta},\ \chi_1^m,\ \chi_2^m,\ \chi_3^m,\ \chi_4^m \\ \psi^a &\longrightarrow \psi_1^s,\ \psi_2^s,\ \psi_3^s, \end{aligned} \quad (39)$$

where the index conventions are as in the previous subsection 3.1. Worth noting are the non-supersymmetric spectrum as in the previous example as well as the three fermion generations which is a clear improvement as compared to the previous case.

Analogous conclusions concerning supersymmetry can be drawn by examining the rest six-dimensional symmetric coset spaces [11].

4 Soft SUSY breaking by CSDR over non-symmetric coset spaces

In the present section we examine the reduction of a ten-dimensional E_8 gauge theory over the three non-symmetric coset spaces. We find that a softly broken supersymmetric four-dimensional gauge theory is obtained.

4.1 CSDR over $G_2/SU(3)$

As first example of CSDR over a non-symmetric coset space, we present the case of $G_2/SU(3)$ [9]. The rest two six-dimensional non-symmetric coset spaces i.e. $Sp(4)/(SU(2) \times U(1))_{non.max}$ and $SU(3)/U(1) \times U(1)$, are examined in the following subsections 4.2 and 4.3 according to refs. [10, 11].

We start again with $G = E_8$ in ten dimensions, $S/R = G_2/SU(3)$, and $R = SU(3)$. Now we need the decomposition

$$\begin{aligned} E_8 &\supset SU(3) \times E_6 \\ 248 &= (8,1) + (1,78) + (3,27) + (\overline{3},\overline{27}), \end{aligned} \quad (40)$$

from which we find that the four-dimensional gauge group is $H = C_{SU(3)}(E_8) = E_6$. Next, to find the $R = SU(3)$ content of $G_2/SU(3)$ vector and spinor, we examine the decompositions

$$\begin{aligned} G_2 &\supset SU(3) \\ 14 &= 8 + 3 + \overline{3} \end{aligned} \quad (41)$$

and

$$\begin{aligned} SO(6) &\supset SU(3) \\ 4 &= 1 + 3 \end{aligned} \quad (42)$$

from which we conclude that the $R = SU(3)$ content of $G_2/SU(3)$ vector and spinor are $3+\overline{3}$ and $1+3$ respectively. Then the CSDR rules determine that from the ten-dimensional gauge field A_M^a we obtain in four dimensions the gauge field A_μ^α and the scalars β^i, where a is a 248 E_8-index, α a 78 and i a 27-E_6 index. From the ten-dimensional gaugino ψ^a we obtain the four-dimensional gaugino λ^α and the left-handed matter fermions ψ_β^i. In summary we find

$$\begin{aligned} A_M^a &\longrightarrow A_\mu^\alpha, \beta^i, \\ \psi^a &\longrightarrow \lambda^\alpha, \psi_\beta^i. \end{aligned} \quad (43)$$

Note that the surviving fields are organized as four-dimensional $\mathcal{N} = 1$ vector and chiral multiplets V^α and B^i.

Moreover the scalar potential, using the metric of $G_2/SU(3)$,

$$g_{ab} = R^2 \delta_{ab}, \quad (44)$$

where R is the radius of $G_2/SU(3)$, was found [9] to be

$$\begin{aligned} V(\beta) \;=\; & \frac{8}{R^4} - \frac{40}{3R^2}\beta^2 - \left[\frac{4}{R} d_{ijk}\beta^i\beta^j\beta^k + h.c\right] + \beta^i\beta^j d_{ijk} d^{klm}\beta_l\beta_m \\ & + \frac{11}{4}\sum_\alpha \beta^i (G^\alpha)_i^j \beta_j \beta^k (G^\alpha)_k^l \beta_l. \end{aligned} \quad (45)$$

In turn it was found that the F-terms are obtained from the superpotential

$$\mathcal{W}(B) = \frac{1}{3} d_{ijk} B^i B^j B^k, \quad (46)$$

where d_{ijk} is the E_6-symmetric invariant tensor [16].

Similarly the D-terms were found to be

$$D^\alpha = \sqrt{\frac{11}{2}}\beta^i (G^\alpha)_i^j \beta_j, \quad (47)$$

where $(G^\alpha)_i^j$ are representation matrices for the 27 of E_6. The rest terms of the scalar potential that are not obtained from F- or D- terms belong to the scalar Soft Supersymmetry Breaking (SSB) part of the Lagrangian, given by

$$L_{SSB} = -\frac{40}{3R^2}\beta^2 - \left[\frac{4}{R} d_{ijk}\beta^i\beta^j\beta^k + h.c\right]. \quad (48)$$

The SSB sector is completed by the gaugino Mass M, which is obtained from the V operator of eqs.(29), (30), and was found to be in the present case

$$M = (1 + 3\tau)\frac{6}{\sqrt{3}R}. \quad (49)$$

4.2 CSDR over $Sp(4)/(SU(2)\times U(1))_{non.max}$

This time we have again $G=E_8$ in ten dimensions but $S/R=Sp(4)/(SU(2)\times U(1))_{non.max}$. The decomposition to be used is

$$\begin{aligned} E_8 &\supset SU(2)\times U(1)\times E_6 \\ 248 &= (3_0,1)+(1_0,1)+(1_0,78)+(2_3,1)+(2_{-3},1) \\ &\quad +(2_1,27)+(2_{-1},\overline{27})+(1_{-2},27)+(1_2,\overline{27}). \end{aligned} \tag{50}$$

Thus the four-dimensional gauge group is

$$H=C_{E_8}(SU(2)\times U(1))=E_6\times U(1). \tag{51}$$

Now the $R=SU(2)\times U(1)$ content of $Sp(4)/(SU(2)\times U(1))_{non-max.}$ vector and spinor are

$$2_1+2_{-1}+1_2+1_{-2}, \tag{52}$$

and

$$2_1+1_0+1_{-2}, \tag{53}$$

respectively. From the CSDR rules we read that from the higher-dimensional gauge field we obtain the four-dimensional gauge fields, A^α_μ, A_μ, with α a 78 E_6-index, and two complex scalar fields β^i, γ^i, with i a 27 E_6-index. From the ten-dimensional gaugino we obtain the four-dimensional gaugino λ^α, λ, and two spinors ψ^i_β, and ψ^i_γ belonging to the 27 of E_6, i.e.

$$\begin{aligned} A^a_M &\longrightarrow A^\alpha_\mu, A_\mu, \beta^i, \gamma^i, \\ \psi^a &\longrightarrow \lambda^\alpha, \lambda, \psi^i_\beta, \psi^i_\gamma. \end{aligned} \tag{54}$$

Note that the surviving fields can be organized as two vector multiplets V^α, V and two chiral multiplets B^i, and C^i.

As in the previous case, the Lagrangian is $\mathcal{N}=1$ supersymmetric supplemented by a soft SSB part. The F-terms of the supersymmetric part are obtained from superpotential

$$\mathcal{W}(B^i,C^j)=\sqrt{\frac{5}{7}}d_{ijk}B^iB^jC^k, \tag{55}$$

while the scalar SSB part is,

$$\begin{aligned} \mathcal{L}_{scalarSSB} &= -\frac{6}{R_1^2}\beta^i\beta_i-\frac{4}{R_2^2}\gamma^i\gamma_i \\ &+\Big[4\sqrt{\frac{10}{7}}R_2\Big(\frac{1}{R_2^2}+\frac{1}{2R_1^2}\Big)d_{ijk}\beta^i\beta^j\gamma^k+h.c\Big]. \end{aligned} \tag{56}$$

R_1 and R_2 are the coset space scales coming from the metric,

$$g_{ab}=diag(R_1^2,R_1^2,R_2^2,R_2^2,R_1^2,R_1^2). \tag{57}$$

Finally the gaugino Mass calculated from the V operator is,

$$M=(1+3\tau)\frac{R_2^2+2R_1^2}{8R_1^2R_2}. \tag{58}$$

4.3 CSDR over $SU(3)/U(1)\times U(1)$

The $G=E_8$, ten-dimensional theory is now reduced over the last non-symmetric coset space $S/R=SU(3)/U(1)\times U(1)$. From the decomposition

$$\begin{aligned} E_8 &\supset U(1)_1\times U(1)_2\times E_6 \\ 248 &= 1_{(0,0)}+1_{(0,0)}+1_{(3,\frac{1}{2})}+1_{(-3,\frac{1}{2})}+1_{(0,-1)}+1_{(0,1)} \\ &\quad +1_{(-3,-\frac{1}{2})}+1_{(-3,-\frac{1}{2})}+78_{(0,0)}+27_{(3,\frac{1}{2})} \\ &\quad +27_{(-3,\frac{1}{2})}+27_{(0,-1)}+\overline{27}_{(-3,-\frac{1}{2})}+\overline{27}_{(3,-\frac{1}{2})}+\overline{27}_{(0,1)}, \end{aligned} \tag{59}$$

we conclude that the four-dimensional gauge group is,

$$H=C_{E_8}(U(1)_1\times U(1)_2)=U(1)_1\times U(1)_2\times E_6. \tag{60}$$

The $R=U(1)\times U(1)$ content of $SU(3)/U(1)\times U(1)$ vector and spinor are

$$(3,\frac{1}{2})+(-3,\frac{1}{2})+(0,-1)+(-3,-\frac{1}{2})+(3,-\frac{1}{2})+(0,1), \tag{61}$$

and

$$(0,0)+(3,\frac{1}{2})+(-3,\frac{1}{2})+(0,-1), \tag{62}$$

respectively.

Applying the CSDR rules we find that from the ten-dimensional gauge fields we obtain the four-dimensional gauge fields A^α_μ, with α a 78 E_6-index, $A_{(1)\mu}$, $A_{(2)\mu}$, the two $U(1)$ gauge fields, three scalars α^i, β^i, γ^i, with i a 27 index, and three E_6 singlets but charged under the $U(1)$s, α, β, and γ. From the ten-dimensional gaugino we obtain the supersymmetric partners of the above scalars, i.e.

$$\begin{aligned} A^a_M &\longrightarrow A^\alpha_\mu, A_{(1)\mu}, A_{(2)\mu}, \alpha^i, \beta^i, \gamma^i, \alpha, \beta, \gamma, \\ \psi^a &\longrightarrow \lambda^\alpha, \lambda_{(1)}, \lambda_{(1)}, \psi^i_\alpha, \psi^i_\beta, \psi^i_\gamma, \psi_\alpha, \psi_\beta, \psi_\gamma. \end{aligned} \tag{63}$$

Therefore the surviving fields are organized as three vector multiplets, V^α, $V_{(1)}$, $V_{(2)}$, and six chiral multiplets, A^i, B^i, C^i, A, B, C.

As in the previous two cases the Lagrangian is softly broken supersymmetric. The F-terms come from the superpotential

$$\mathcal{W}(A^i,B^j,C^k,A,B,C)=\sqrt{40}d_{ijk}A^iB^jC^k+\sqrt{40}ABC. \tag{64}$$

The scalar SSB part of the Lagrangian is,

$$\begin{aligned} \mathcal{L}_{scalarSSB} &= \Big(\frac{4R_1^2}{R_2^2R_3^2}-\frac{8}{R_1^2}\Big)\alpha^i\alpha_i+\Big(\frac{4R_1^2}{R_2^2R_3^2}-\frac{8}{R_1^2}\Big)\overline{\alpha}\alpha \\ &+\Big(\frac{4R_2^2}{R_1^2R_3^2}-\frac{8}{R_2^2}\Big)\beta^i\beta_i+\Big(\frac{4R_2^2}{R_1^2R_3^2}-\frac{8}{R_2^2}\Big)\overline{\beta}\beta \\ &+\Big(\frac{4R_3^2}{R_1^2R_2^2}-\frac{8}{R_3^2}\Big)\gamma^i\gamma_i+\Big(\frac{4R_3^2}{R_1^2R_2^2}-\frac{8}{R_3^2}\Big)\overline{\gamma}\gamma \\ &+\Big[\sqrt{280}\Big(\frac{R_1}{R_2R_3}+\frac{R_2}{R_1R_3}+\frac{R_3}{R_2R_1}\Big)\,d_{ijk}\alpha^i\beta^j\gamma^k \\ &+\sqrt{280}\Big(\frac{R_1}{R_2R_3}+\frac{R_2}{R_1R_3}+\frac{R_3}{R_2R_1}\Big)\,\alpha\beta\gamma+h.c\Big]. \end{aligned} \tag{65}$$

The coset space radii, R_1, R_2, R_3 are entering the above formula (65) through the metric, which takes the following form for the present coset $SU(3)/U(1) \times U(1)$

$$g_{ab} = diag(R_1^2, R_1^2, R_2^2, R_2^2, R_3^2, R_3^2). \tag{66}$$

The SSB sector is completed by the gaugino mass,

$$M = (1 + 3\tau)\frac{(R_1^2 + R_2^2 + R_3^2)}{8\sqrt{R_1^2 R_2^2 R_3^2}}. \tag{67}$$

Thus we have established that besides the supersymmetric spectrum, we obtain a softly broken supersymmetric Lagrangian by CSDR over a non-symmetric coset space.

5 Conclusions

The CSDR was originally introduced as a scheme which, making use of higher dimensions, incorporates in a unified manner the gauge and the ad-hoc Higgs sector of the spontaneously broken gauge theories in four dimensions [1, 2]. Next fermions were introduced in the scheme and the ad-hoc Yukawa interactions have also been included in the unified description [5, 8].

Of particular interest for the construction of realistic theories in the framework of CSDR are the following virtues that complemented the original suggestion: (i) The possibility to obtain chiral fermions in four dimensions resulting from vector-like representations of the higher-dimensional gauge theory [8, 4]. This possibility can be realized due the presence of non-trivial background gauge configurations which are introduced by the CSDR constructions [17], (ii) The possibility to deform the metric of certain non-symmetric coset spaces and thereby obtain more than one scales [4, 18], (iii) The possibility to use coset spaces, which are multiply connected. This can be achieved by exploiting the discrete symmetries of the S/R [19, 4]. Then one might introduce topologically non-trivial gauge field [20] configurations with vanishing field strength and induce additional breaking of the gauge symmetry. It is the Hosotani mechanism [21] applied in the CSDR.

In the above list recently has been added the interesting possibility that the popular softly broken supersymmetric four-dimensional chiral gauge theories might have their origin in a higher-dimensional supersymmetric theory with only vector supermultiplet [9, 10, 11], which is dimensionally reduced over non-symmetric coset spaces.

Let us also note that the current discussion on the higher-dimensional theories with large extra dimensions [22], provides a new framework to examine further the CSDR since the classical treatment used in CSDR is justified in the case of large radii which are far away from the scales that the quantum effects of gravity are important.

On the other hand we should also note that the effective field theories resulting from compactification of higher-dimensional theories contain also towers of massive higher harmonic (Kaluza-Klein) excitations, whose contributions at the quantum level alter the behaviour of the running couplings from logarithmic to power [23]. As a result the traditional picture of unification may change drastically [24, 25]. Combining the quantum behaviour of the higher-dimensional theories [24, 25], with the unification at the classical level of the gauge-Higgs and the gauge-Yukawa sectors that can be achieved in the CSDR scheme, one might hope to achieve a *reduction of couplings* of the SM by reducing the dimensions of a higher-dimensional theory.

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