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INTRODUCTION TO THEORETICAL INTERMEDIATE
ENERGY NUCLEAR PHYSICS

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1 Introduction

Intermediate Energy Physics deals mainly with the interaction of elementary particles (electrons, photon, muon, neutrino, pion, nucleon, etc.) with nuclei, but also problems of pure elementary particle physics are subject of investigations. One can give a gross subdivision of the field:

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- (a) Use of elementary particles as a probe to investigate nuclear structure. Scattering experiments using particles of a few hundred MeV kinetic energy up to a few GeV allow the investigation of some aspects of nuclear structure which are not amenable with the usual classic methods of nuclear spectroscopy (electromagnetic transitions, Coulomb excitation, etc.). In this case one has to assume that the interaction between the projectile and the individual nucleon in the nucleus is sufficiently well known and in addition that one has a reliable scattering theory to treat the scattering on a composite target (nucleus). In such cases experimental data can give information on nuclear structure (i.e. electron scattering on nuclei at higher energies can give information on certain form factors of nuclei, etc.)
- (b) Investigation of the properties of the projectile itself or of the basic interaction between projectile and nucleon.

In such cases the structure of the target nucleus has to be known sufficiently accurate.

Examples:

investigation of hydrogen like atoms constituted of μ^- , π^- , K^- , Σ^- , \bar{p} , etc. and a nucleus. Investigation of the transition energies gives information on the projectile masses, the fine structure allows the determination of the magnetic moment of projectiles with spin, etc.

Pion - nucleus scattering where a short - living resonance (Δ , with $m_\Delta \sim 1232$ MeV) can be studied.

In photon - nucleus scattering one can study the ρ - meson ($\pi\pi$ - resonance) production ($m_\rho \sim 765$ MeV).

μ^- - capture on light nuclei could probably be used to investigate the question of a finite neutrino mass (i.e., $\mu^- + {}^3\text{He} \rightarrow d + n + \nu_\mu$, $\mu^- + {}^6\text{Li} \rightarrow t + t + \nu_\mu$).

- (c) Investigation of multiply scattering theories.

One has to assume that nuclear structure as well as the elementary interaction between projectile and nucleons are sufficiently well known. This is particularly interesting for projectiles with strong interactions.

Examples:



pionic atoms (scattering of pions at very low energies), scattering of pions off nuclei at intermediate and high energies.

According to the large variety of phenomena to be investigated, theoretical intermediate energy physics needs many tools from various different fields: Nuclear Physics (nuclear structure, nuclear models, nuclear reactions), knowledge from elementary particle physics (strong, weak and electromagnetic interactions). The basis of all these investigations is of course quantum mechanics, in particular relativistic quantum mechanics. For many purposes at least some rudimentary knowledge of quantum field theory is very useful. These tools will be developed during the course as far as is necessary.

In the following the most important subfields of theoretical intermediate energy physics are listed.

1. Scattering of strong interacting projectiles on nuclei at intermediate and high energy :

Multiple scattering in nuclei,
Eikonal methods,
 Δ - resonance in nuclei,
dispersion relations in particle - nucleus scattering.

2. Exotic atoms with hadrons (π^- , K^- , \bar{p} , Σ^- , etc. - atoms) :

nuclear structure effects,
multiple scattering at low energies,
optical potentials at low energies.

3. Strong interactions and nuclei:

pion - absorption.

4. Leptons and nuclei:

μ^- - atoms and nuclear structure
electron scattering.

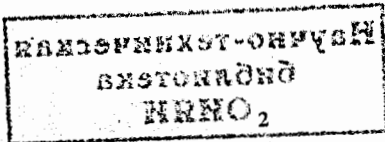
5. Weak interactions and nuclei:

μ^- - capture,

ν - scattering on nuclei.

6. Photons and nuclei:

low energy theorems applied to nuclei,
photoproduction of particles and resonances in nuclei.



Part I

1 Short introduction into scattering theory

1.1 Potential scattering

We restrict ourselves here to the scattering of two structureless particles with masses m_1 and m_2 interacting through a potential $V(\vec{x}_1, \vec{x}_2)$ with each other. Spins are not considered and we assume that non relativistic quantum mechanics is valid. In addition we assume that the potential depends only on the difference $\vec{x}_1 - \vec{x}_2$ and that it has a finite range. The stationary Schroedinger equation of this two - body problem is then

$$\left[-\frac{\hbar^2}{2m_1} \nabla_1^2 - \frac{\hbar^2}{2m_2} \nabla_2^2 + V(\vec{x}_1 - \vec{x}_2) \right] \Psi(\vec{x}_1, \vec{x}_2) = E_{tot} \Psi(\vec{x}_1, \vec{x}_2) \quad (1.1)$$

Introducing relative - and center of mass coordinates we are able to separate the center of mass motion from the rest.

$$\left. \begin{aligned} \vec{x} &= \vec{x}_1 - \vec{x}_2 & \vec{x}_1 &= \vec{R}_{cm} + \frac{m_2}{m_1+m_2} \vec{x} \\ \vec{R}_{cm} &= \frac{m_1 \vec{x}_1 + m_2 \vec{x}_2}{m_1+m_2} & \vec{x}_2 &= \vec{R}_{cm} - \frac{m_1}{m_1+m_2} \vec{x} \end{aligned} \right\} \quad (1.2)$$

Introducing these variables into (1.1) together with

$$\varphi(\vec{x}, \vec{R}_{cm}) \equiv \Psi(\vec{x}_1(\vec{x}, \vec{R}_{cm}), \vec{x}_2(\vec{x}, \vec{R}_{cm})) \quad \text{and} \quad m_{red} = \frac{m_1 m_2}{m_1 + m_2} \quad (\text{reduced mass}) \quad (1.3)$$

leads to

$$\left[-\frac{\hbar^2}{2(m_1 + m_2)} \nabla_{R_{cm}}^2 - \frac{\hbar^2}{2m_{red}} \nabla^2 + V(\vec{x}) \right] \varphi(\vec{x}, \vec{R}_{cm}) = E_{tot} \varphi(\vec{x}, \vec{R}_{cm}) \quad (1.4)$$

This allows one to make a product "Ansatz" for $\varphi(\vec{x}, \vec{R}_{cm})$.

$$\varphi(\vec{x}, \vec{R}_{cm}) = \phi_{cm}(\vec{R}_{cm}) \psi(\vec{x}) \quad (1.5)$$

where $\phi_{cm}(\vec{R}_{cm})$ is a solution of

$$-\frac{\hbar^2}{2(m_1 + m_2)} \nabla_{R_{cm}}^2 \phi_{cm} = E_{cm} \phi_{cm} \quad (1.6)$$

For $\Psi(\vec{x})$ we obtain the equation

$$\left[-\frac{\hbar^2}{2m_{red}} \nabla^2 + V(\vec{x}) \right] \Psi(\vec{x}) = E \Psi(\vec{x}) \quad (1.7)$$

together with

$$\left[-\frac{\hbar^2}{2(m_1 + m_2)} \nabla_{R_{cm}}^2 - E_{cm} \right] \phi_{cm}(\vec{R}_{cm}) = 0 \quad (1.8)$$

The center of mass motion is then according (1.6) described by a plane-wave

$$\phi_{cm}(\vec{R}_{cm}) = \frac{1}{\sqrt{V_{ol}}} e^{i\vec{k}_{cm} \cdot \vec{R}_{cm}} \quad \text{or} \quad \frac{1}{(2\pi)^3} e^{i\vec{k}_{cm} \cdot \vec{R}_{cm}} \quad (1.9)$$

with

$$\vec{k}_{cm}^2 = \frac{2(m_1 + m_2)}{\hbar^2} E_{cm} = \frac{2(m_1 + m_2)\hbar c}{\hbar^2} \epsilon_{cm} = 2(\mu_1 + \mu_2)\epsilon_{cm} \quad (1.10)$$

The quantities $\mu_i = \frac{m_i c}{\hbar}$, $\epsilon_i = \frac{E_i}{\hbar c}$ are the masses and energies measured in units $(length)^{-1}$, usually f_m^{-1} . In absence of the interaction potential also $\Psi(\vec{x})$ will be a plane wave

$$\Psi(\vec{x}) = \frac{1}{\sqrt{V_{ol}}} e^{i\vec{k}\vec{x}} \quad \text{or} \quad \frac{1}{(2\pi)^{\frac{3}{2}}} e^{i\vec{k}\vec{x}} \quad (1.11)$$

with

$$\vec{k}^2 = 2\mu_{red}\epsilon \quad (1.12)$$

A remark on the normalization:

- finite normalization volume V_{ol} , all physical quantities must be independent on V_{ol} .
- normalization to a δ -function.

In case the potential $V(\vec{x})$ has a finite range, one looks for scattering solutions of (1.7) with asymptotic form ($|\vec{x}| \rightarrow \infty$)

$$\Psi_{(+)}(\vec{x}) \stackrel{r=|\vec{x}| \rightarrow \infty}{=} \frac{1}{(2\pi)^{\frac{3}{2}}} \left[e^{i\vec{k}\vec{x}} + f(\Omega) \frac{e^{i\kappa r}}{r} \right] \quad (1.13)$$

which is a superposition of the incoming plane wave (describes the beam) and an outgoing spherical wave. In this asymptotic region, outside the range of the potential V , the influence of the interaction is given by a modulation $f(\Omega)$ of the outgoing wave.

$f(\Omega)$ is called the scattering amplitude. This picture has been borrowed from classical wave optics.

Starting from the Schrodinger equation one can derive an expression for the particle current density. We multiply (1.7) by Ψ^* and the complex conjugate equation with Ψ and

subtract both expressions from each other (for stationary solutions we can also substitute $E\Psi$ by $i\hbar \frac{\partial \Psi}{\partial t}$) and obtain:

$$-\frac{\hbar^2}{2m_{red}} \nabla [\Psi^* \nabla \Psi - \Psi \nabla \Psi^*] = i\hbar \frac{\partial}{\partial t} (\Psi^* \Psi) \quad (1.14)$$

This allows one to define the current density \vec{j} through

$$\vec{j} = \frac{i\hbar}{2m_{red}} [\Psi \nabla \Psi^* - \Psi^* \nabla \Psi] = Rc \left\{ \frac{\hbar}{im_{red}} \Psi^* \nabla \Psi \right\} \quad (1.15)$$

Together with the particle density (probability density) $\rho = \Psi^* \Psi$, \vec{j} satisfies a continuity equation

$$\nabla \vec{j} + \frac{\partial \rho}{\partial t} = 0 \quad (1.16)$$

Current density of the incoming particles

If we take the z -axis parallel to the direction of the incoming particle beam, we get for plane waves

$$\vec{j}_{in} = Rc \left\{ \frac{\hbar}{im_{red}} e^{-i\vec{k}\vec{x}} i\vec{k} e^{i\vec{k}\vec{x}} \right\} \frac{1}{V_{ol}} = \frac{\hbar \vec{k}}{m_{red}} \frac{1}{V_{ol}} = \frac{\hbar \kappa}{m_{red} V_{ol}} \vec{e}_z \quad (1.17)$$

Using

$$\vec{v}_{rel} = \vec{v}_1 - \vec{v}_2 = \frac{\vec{p}_1}{m_1} - \frac{\vec{p}_2}{m_2} = \hbar \frac{m_2 \vec{k}_1 - m_1 \vec{k}_2}{m_1 m_2} = \frac{\hbar}{m_{red}} \left(\frac{m_2 \vec{k}_1 - m_1 \vec{k}_2}{m_1 + m_2} \right) = \frac{\hbar}{m_{red}} \vec{k} \quad (1.18)$$

one obtains

$$\vec{j}_{in} = \frac{\vec{v}_{rel}}{V_{ol}} = \frac{\hbar \kappa}{m_{red} V_{ol}} \vec{e}_z \quad (1.19)$$

Current density of scattered particles

Starting from the asymptotic form of the wave function we get the scattering wave.

$$\Psi_{sc} = \frac{1}{\sqrt{V_{ol}}} f(\Omega) \frac{e^{i\kappa r}}{r} \quad (1.20)$$

and

$$\nabla \Psi_{sc} = \frac{1}{\sqrt{V_{ol}}} \left[f(\Omega) \frac{\partial}{\partial r} \left(\frac{e^{i\kappa r}}{r} \right) \cdot \vec{e}_r + \frac{e^{i\kappa r}}{r^2} \frac{\partial}{\partial \vartheta} f(\Omega) \cdot \vec{e}_\vartheta + \frac{e^{i\kappa r}}{r^2 \sin \vartheta} \frac{\partial}{\partial \varphi} f(\Omega) \cdot \vec{e}_\varphi \right] \quad (1.21)$$

and

$$\begin{aligned} \vec{j}_{sc} &= Re \left\{ \frac{\hbar}{im_{red}} \Psi_{sc}^* \nabla \Psi_{sc} \right\} = \\ &= \frac{1}{V_{ol}} \left\{ \frac{\hbar \kappa}{m_{red} r^2} |f(\Omega)|^2 \vec{e}_r + Re \left[\frac{-i\hbar f^* \frac{\partial f}{\partial \vartheta}}{m_{red} r^3} \vec{e}_\vartheta - \frac{i\hbar f^* \frac{\partial f}{\partial \varphi}}{m_{red} r^3 \sin \vartheta} \vec{e}_\varphi \right] \right\} \end{aligned} \quad (1.22)$$

For very large values of r the first term dominates and one gets

$$\vec{j}_{sc} \cong \frac{1}{V_{ol}} \frac{\hbar \kappa}{m_{red}} \frac{1}{r^2} |f(\Omega)|^2 \vec{e}_r = j_{in} \frac{1}{r^2} |f(\Omega)|^2 \vec{e}_r \quad (1.23)$$

Scattering cross section

One calculates now the number of particles dn going through the surface element $d\vec{S}$ of the detector placed at a distance r from the target.

$$dn = \vec{j}_{sc} d\vec{S} = \vec{j}_{sc} r^2 d\Omega \vec{e}_r = j_{in} |f(\Omega)|^2 d\Omega \quad (1.24)$$

$$\frac{dn}{d\Omega} = j_{in} |f(\Omega)|^2 \quad (1.25)$$

the differential elastic scattering cross section is then defined as

$$\frac{d\sigma}{d\Omega} = \frac{1}{j_{in}} \frac{dn}{d\Omega} = |f(\Omega)|^2 \quad (1.26)$$

and, being a physical measurable quantity, is independent of the normalization volume.

Relation between interaction potential and scattering amplitude

To obtain the relation between $V(\vec{x})$ and the scattering amplitude, we first put the Schroedinger equation (1.7) into the form of an integral equation. We rewrite (1.7) in the following form:

$$(\nabla^2 + \kappa^2) \Psi(\vec{x}) = U(\vec{x}) \Psi(\vec{x}) \quad (1.27)$$

with

$$U(\vec{x}) = \frac{2m_{red}}{\hbar^2} V(\vec{x}) \quad (1.28)$$

Introducing a δ -function on the right side of (1.27) leads to

$$(\nabla^2 + \kappa^2) \Psi(\vec{x}) = \int d^3 \vec{x}' \delta^{(3)}(\vec{x} - \vec{x}') U(\vec{x}') \Psi(\vec{x}')$$

A formal, particular solution to this equation is given by

$$\Psi_{part}(\vec{x}) = \int d^3 \vec{x}' (\nabla^2 + \kappa^2)^{-1} \delta(\vec{x} - \vec{x}') U(\vec{x}') \Psi(\vec{x}') \quad (1.29)$$

The Green function of this equation

$$G_0(\vec{x}, \vec{x}') \equiv (\nabla^2 + \kappa^2)^{-1} \delta(\vec{x} - \vec{x}') \quad (1.30)$$

is a solution of

$$(\nabla^2 + \kappa^2)G_0(\vec{x}, \vec{x}') = \delta(\vec{x} - \vec{x}') \quad (1.31)$$

Performing a Fourier transformation on both sides of equ.(1.31) leads to

$$G_0(\vec{x} - \vec{x}') = \frac{1}{(2\pi)^3} \int d^3\vec{k} \tilde{G}_0(\vec{k}) e^{i\vec{k}(\vec{x} - \vec{x}')} \quad (1.32)$$

$$\tilde{G}_0(\vec{k}) = -\frac{1}{\vec{k}^2 - \kappa^2}$$

In order to obtain particular solutions to (1.39) such as outgoing or incoming or standing waves, one has to choose particular paths in the complex k - plane. For the present purpose we choose however a quicker way to obtain the desired Green function. Consider the spherical symmetrical solution to (1.31) in a region $\vec{x} \neq \vec{x}'$:

$$(\nabla^2 + \kappa^2)G_0(R) = 0 \quad (1.33)$$

This may be solved with the ansatz ($R = |\vec{x} - \vec{x}'|$)

$$G_0^\pm(R) = C_0 R^\alpha e^{\pm i\kappa R}, \quad \alpha = -1$$

C_0 can be determined by comparison with (1.32') and we obtain the well known result

$$G_0^\pm(\vec{x} - \vec{x}') = -\frac{e^{\pm i\kappa|\vec{x} - \vec{x}'|}}{4\pi |\vec{x} - \vec{x}'|} \quad (1.34)$$

The positive sign in the exponent leads to outgoing waves, the negative sign to incoming waves. In order to obtain the full solution of (1.27) we have to add to Ψ_{part} (1.29) a solution of the homogeneous equation (a plane wave). The full scattering solution, describing outgoing spherical waves is then given by

$$\Psi_{\vec{\kappa}}^{(+)}(\vec{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \left[e^{i\vec{\kappa}\vec{x}} - \int d^3\vec{x}' \frac{e^{+i\kappa|\vec{x} - \vec{x}'|}}{4\pi |\vec{x} - \vec{x}'|} U(\vec{x}') \Psi_{\vec{\kappa}}^{(+)}(\vec{x}') \right] \quad (1.35)$$

Considering very large distances from the scattering center well outside the interaction region, i.e.outside the range of the potential $U(\vec{x})$ ($V(\vec{x})$), one may expand the exponent in the following way:

$$\begin{aligned} |\vec{x} - \vec{x}'| &= \sqrt{\vec{x}^2 + \vec{x}'^2 - 2\vec{x}\vec{x}'} = |\vec{x}| \sqrt{1 - \frac{2\vec{x}\vec{x}'}{|\vec{x}|^2} + \frac{\vec{x}'^2}{|\vec{x}|^2}} \sim |\vec{x}| \left(1 - \frac{\vec{x}\vec{x}'}{|\vec{x}|^2} \right) = \\ &= |\vec{x}| - \frac{\vec{x}\vec{x}'}{|\vec{x}|} \end{aligned} \quad (1.36)$$

Defining $\vec{\kappa}'$ through

$$\vec{\kappa}' = \kappa \frac{\vec{x}}{|\vec{x}|}; \quad \kappa'^2 = \kappa^2 \quad (1.37)$$

and replacing in the denominator $|\vec{x} - \vec{x}'|$ by $|\vec{x}|$ leads to the result

$$\Psi_{\vec{\kappa}}^{(+)}(\vec{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \left[e^{i\vec{\kappa}\vec{x}} - \frac{(2\pi)^2 \mu_{red}}{hc} \frac{c^{i\kappa r}}{r} \int d^3\vec{x}' \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-i\vec{\kappa}'\vec{x}'} V(\vec{x}') \Psi_{\vec{\kappa}}^{(+)}(\vec{x}') \right] \quad (1.38)$$

or introducing the normalized plane wave $\phi_{\vec{\kappa}}(\vec{x}) \equiv \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} e^{i\vec{\kappa}\vec{x}}$ to:

$$\Psi_{\vec{\kappa}}^{(+)}(\vec{x}) = \phi_{\vec{\kappa}}(\vec{x}) - \frac{(2\pi)^2 \mu_{red}}{hc} \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{c^{i\kappa r}}{r} \int d^3\vec{x}' \phi_{\vec{\kappa}'}(\vec{x}') V(\vec{x}') \Psi_{\vec{\kappa}}^{(+)}(\vec{x}') \quad (1.39)$$

Comparing this result with (1.13) gives us the scattering amplitude.

$$f_{\vec{k}'\vec{k}}(\Omega) = -\frac{(2\pi)^2 \mu_{red}}{\hbar c} \int d^3 \vec{x}' \phi_{\vec{k}'}^*(\vec{x}') V(\vec{x}') \Psi_{\vec{k}}^{(+)}(\vec{x}') \quad (1.40)$$

The integral in (1.40) is usually called T - matrix element. Therefore one has

$$t_{\vec{k}'\vec{k}}^{cm} = \int d^3 \vec{x}' \phi_{\vec{k}'}^*(\vec{x}') V(\vec{x}') \Psi_{\vec{k}}^{(+)}(\vec{x}') \quad (1.41)$$

and

$$f_{\vec{k}'\vec{k}}^{cm}(\Omega) = -\frac{(2\pi)^2 \mu_{red}}{\hbar c} t_{\vec{k}'\vec{k}}^{cm} \quad (1.42)$$

In general it is sufficient to know $\Psi_{\vec{k}}^{(+)}(\vec{x})$ in a region $|\vec{x}'| \leq R$ (R is here the range of the potential) to determine $t_{\vec{k}'\vec{k}}^{cm}$.

The optical theorem

We calculate now the total current density (1.15) using the asymptotic form of the wave function Ψ (1.13). With

$$\nabla \Psi = \frac{1}{V_{ol}} \left[i\vec{k} e^{i\vec{k}\vec{z}} + f \frac{\partial}{\partial r} \left(\frac{e^{i\kappa r}}{r} \right) \vec{e}_r + \frac{e^{i\kappa r}}{r^2} \frac{\partial f}{\partial \theta} \vec{e}_\theta + \frac{e^{i\kappa r}}{r^2 \sin \theta} \frac{\partial f}{\partial \phi} \vec{e}_\phi \right]$$

one gets

$$\vec{j} = \frac{\hbar}{m_{red}} \text{Re} \left\{ \frac{1}{i} \Psi^* \nabla \Psi \right\} = \frac{\hbar}{m_{red} V_{ol}} \text{Re} \left\{ \vec{k} - i e^{-i\vec{k}\vec{z}} f(\Omega) \left[-\frac{e^{i\kappa r}}{r^2} + i\kappa \frac{e^{i\kappa r}}{r} \right] \vec{e}_r - i \frac{e^{-i\vec{k}\vec{z} + i\kappa r}}{r^2} \frac{\partial f}{\partial \theta} \vec{e}_\theta - \right.$$

$$\left. i \frac{e^{-i\vec{k}\vec{z} + i\kappa r}}{r^2 \sin \theta} \frac{\partial f}{\partial \phi} \vec{e}_\phi + \vec{k} f^* \frac{e^{-i\kappa r + i\vec{k}\vec{z}}}{r} - i \frac{e^{-i\kappa r}}{r} |f|^2 \left[-\frac{-i\kappa r}{r^2} + \frac{i\kappa e^{i\kappa r}}{r} \right] \vec{e}_r - \frac{i}{r^3} f^* \frac{\partial f}{\partial \theta} \vec{e}_\theta - \frac{i}{r^3 \sin \theta} f^* \frac{\partial f}{\partial \phi} \vec{e}_\phi \right\}$$

One may decompose \vec{j} now into various parts

$$\vec{j} = \vec{j}_{in} + \vec{j}_{sc} + \vec{j}_{int}$$

Incoming current density

$$\vec{j}_{in} = \frac{\hbar \vec{k}}{m_{red} V_{ol}} \quad (1.43)$$

Current density of the scattered particles (only the dominant term for large r).

$$\vec{j}_{sc} = \frac{\hbar \vec{k}}{m_{red} r^2 V_{ol}} |f(\Omega)|^2 \vec{e}_r \quad (1.44)$$

Interference term

$$\vec{j}_{int} = \frac{\hbar}{m_{red} V_{ol}} \text{Re} \left\{ \vec{e}_r \frac{\kappa}{r} f(\Omega) e^{i\kappa r(1-\cos \theta)} + \frac{\vec{k}}{r} f^*(\Omega) e^{-i\kappa r(1-\cos \theta)} + \dots \right\} \quad (1.45)$$

If we integrate \vec{j} over the whole solid angle $\Omega = 4\pi$, there will be no contribution from \vec{j}_{in} to this integral. For asymptotically large values of r there will be also no contribution coming from \vec{j}_{int} except from a very small angular range $d\Omega$ around $\theta = 0$. This is so because for large r the integrand oscillates very rapidly. One can estimate this contribution by setting all the smoothly varying parts equal to their values at $\theta = 0$. Using

$$\int_{\theta=0}^{\theta=\delta\theta} \sin \theta d\theta e^{i\kappa r(1-\cos \theta)} = \int_{\cos \delta\theta}^1 d\mu e^{i\kappa r(1-\mu)} = \frac{e^{i\kappa r}}{i\kappa r} (e^{-i\kappa r \cos \delta\theta} - e^{-i\kappa r}) = \dots$$

$$= \frac{i}{\kappa r} + \text{oscillating terms}$$

and $\vec{k} \vec{e}_r = \cos \theta \kappa \sim \kappa$ we get

$$\int r^2 d\Omega \vec{j}_{int} \vec{e}_r = \frac{1}{V_{ol}} Re \left\{ \frac{\hbar}{m_{red}} \int d\Omega r^2 \left[\frac{\kappa}{r} f(0) e^{i\kappa r(1-\cos\theta)} + \vec{\kappa} \vec{e}_r f^*(0) \frac{1}{r} e^{-i\kappa r(1-\cos\theta)} \right] \right\}$$

$$= \frac{1}{V_{ol}} Re \left\{ \frac{2\pi \hbar \kappa r^2 i}{m_{red} \kappa r^2} (f(0) - f^*(0)) \right\} = -\frac{4\pi \hbar}{m_{red} V_{ol}} Im f(0)$$

If we integrate now the continuity equation (1.16) over the whole space, taking into account that for stationary states $\frac{\partial \rho}{\partial t} = 0$, we obtain

$$\int d^3 \vec{x} \nabla \vec{j} = \int d\vec{j} = \int r^2 d\Omega \vec{e}_r [j_{in} + j_{sc} + j_{int}] =$$

$$\frac{\hbar \kappa}{m_{red} V_{ol}} \left[\int d\Omega |f(\Omega)|^2 - \frac{4\pi}{\kappa} Im f(0) \right] = 0$$

The total scattering cross section is defined as

$$\sigma_{tot} = \int d\Omega |f(\Omega)|^2 = \int d\Omega \left(\frac{d\sigma}{d\Omega} \right) \quad (1.46)$$

and we get the optical theorem

$$\sigma_{tot} = \frac{4\pi}{\kappa} Im f(0) \quad (1.47)$$

Physically the optical theorem means the following:

The shadow cast by the target into the forward direction leads to a decrease of the intensity of the beam proportional to the total scattering cross section. This decrease arises because of destructive interferences between the incoming and the scattered waves and is a direct consequence of the conservation of probability.

Partial wave expansion of the scattering amplitude

One may try to solve the Schroedinger equation (1.7) (or (1.27)) directly. In order to do so we introduce spherical coordinates and get

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) - \frac{\vec{l}^2}{r^2} \quad (1.48)$$

$$\vec{l}^2 = - \left[\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right] \quad (1.49)$$

The eigenfunctions of \vec{l}^2 (operator of the square of the orbital angular momentum) are the usual spherical harmonics $Y_{lm}(\Omega)$ which satisfy

$$\vec{l}^2 Y_{lm}(\Omega) = l(l+1) Y_{lm}(\Omega) \quad l = 0, 1, 2, 3, \dots \quad -l \leq m \leq +l$$

$$l_z Y_{lm}(\Omega) = m Y_{lm} \quad \left(l_z = -i \frac{\partial}{\partial \phi} \right) \quad (1.50)$$

If the potential $V(\vec{x})$ (or $U(\vec{x})$) depends only on $r = |\vec{x}|$ one can use the following separation ansatz for $\Psi_{\vec{\kappa}}(\vec{x})$:

$$\Psi_{\vec{\kappa}}(\vec{x}) = \sum C_{lm}(\vec{\kappa}) R_l(r) Y_{lm}(\Omega) \quad (1.51)$$

This leads for each partial wave to a radial Schroedinger equation of the form

$$\left[\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + \kappa^2 - \frac{l(l+1)}{r^2} \right] R_l(r) = U(r) R_l(r) \quad (1.52)$$

or if the new variable $\rho = \kappa r$ is introduced.

$$\frac{d^2 R_l(\rho)}{d\rho^2} + \frac{2}{\rho} \frac{dR_l(\rho)}{d\rho} + \left(1 - \frac{l(l+1)}{\rho^2}\right) R_l(\rho) = \frac{1}{\kappa^2} U(r) R_l(\rho) \quad (1.53)$$

The homogeneous equation, i.e. the equation for free particles is then just the differential equation for the spherical Bessel functions ($j_l(\rho)$, $n_l(\rho)$, $h_l^{(\pm)}(\rho)$ or $h_l^{(1)}(\rho)$ and $h_l^{(2)}(\rho)$). We use here the definitions given in Messiah, Quantum Mechanics, vol.1, North. Holland (1961).

<i>regular solution (for $r \rightarrow 0$)</i>	$j_l(\rho) = \sqrt{\frac{\pi}{2\rho}} J_{l+\frac{1}{2}}(\rho)$	}	(1.54)
<i>irregular solution (spherical Neumann function)</i>	$n_l(\rho) = (-1)^l \sqrt{\frac{\pi}{2\rho}} J_{-(l+\frac{1}{2})}(\rho)$		
<i>spherical Hankel function</i>	$h_l^{(\pm)}(\rho) = n_l(\rho) \pm i j_l(\rho)$		
	$h_l^{(-)}(\rho) = h_l^{(+)*}(\rho)$		
<i>spherical Hankel function of the 1. and 2. kind</i>	$h_l^{(1)}(\rho) = -i h_l^{(+)}(\rho)$ $h_l^{(2)}(\rho) = i h_l^{(-)}(\rho)$		

For large ρ , $\rho \gg 1$, these function have the following asymptotic behaviour:

$$j_l(\rho) =_{\rho \rightarrow \infty} \frac{\sin(\rho - \frac{l\pi}{2})}{\rho} \quad (1.55)$$

$$n_l(\rho) =_{\rho \rightarrow \infty} \frac{\cos(\rho - \frac{l\pi}{2})}{\rho} \quad (1.56)$$

$$h_l^{(\pm)}(\rho) =_{\rho \rightarrow \infty} (R_l \pm i S_l) \frac{e^{\pm i\rho}}{\rho} \quad (1.57)$$

$$R_l + i S_l = \sum_{s=0}^l \frac{i^{s-l} (l+s)!}{2^s s! (l-s)!} \left(\frac{1}{\rho}\right)^s \quad (1.58)$$

j_l and n_l , or h_l^{\pm} or $h_l^{(1)}$ and $h_l^{(2)}$ are pairs of linearly independent functions. We can now write down the solution of the homogeneous equation to (1.53):

$$R_l = C_l^1(\kappa) j_l(\rho) + C_l^2(\kappa) n_l(\rho) \quad (1.59)$$

or

$$R_l = D_l^{(1)}(\kappa) h_l^{(+)}(\rho) + D_l^{(2)}(\kappa) h_l^{(-)}(\rho) \quad (1.60)$$

To investigate the asymptotic behaviour of the solutions of the inhomogeneous equation (1.53) we start with the ansatz

$$R_l(\rho) = \Phi_l h_l^{\pm} \quad (1.61)$$

where we assume that $\Phi_l(\rho)$ is a function smoothly varying with ρ (this because we know that far outside the range of the potential the homogeneous equation should approximately be valid. Combining (1.53) and (1.61) one gets

$$\Phi_l'' h_l^{(\pm)} + 2\Phi_l' \left(h_l^{(\pm)'} + \frac{h_l^{(\pm)}}{\rho} \right) = \frac{1}{\kappa^2} U h_l^{\pm} \quad (1.62)$$

Since Φ_l was assumed to be a slowly varying function, one can neglect the term $\frac{\Phi_l''}{\Phi_l}$. One then has



$$\frac{\Phi_l'}{\Phi_l} \approx \frac{U}{2\kappa} \frac{h_l^{(\pm)'}}{h_l^{(\pm)} + \frac{h_l^{(\pm)'}}{\rho}} = \frac{U}{2\kappa^2} \frac{h_l^{(\pm)'}}{\frac{(l+1)h_l^{(\pm)'}}{\rho} - h_{l+1}^{(\pm)'}}$$

and

$$\frac{\Phi_l'}{\Phi_l} \approx_{\rho \rightarrow \infty} -\frac{U}{2\kappa^2} \frac{h_l^{(\pm)'}}{h_{l+1}^{(\pm)'}} \quad (1.63)$$

where use has been made of a recursion relation for h_l ($h_l' = \frac{l}{\rho} h_l - h_{l+1}$) If one uses now the asymptotic forms (1.58) for the Hankel functions, one gets for large ρ

$$\frac{h_l^{\pm}}{h_{l+1}^{\pm}} \approx_{\rho \rightarrow \infty} \pm i \left(1 + 0\left(\frac{1}{\rho}\right)\right) \quad (1.64)$$

and

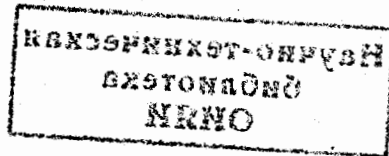
$$\frac{\Phi_l'}{\Phi_l} \approx_{\rho \rightarrow \infty} -\frac{U}{2\kappa^2} i \left(1 + 0\left(\frac{1}{\rho}\right)\right) (\pm 1) \quad (1.65)$$

Integration leads to

$$\lg \frac{\Phi_l}{\Phi_l^0} \approx_{\rho \rightarrow \infty} -\frac{1}{2\kappa^2} \int_{\rho_0}^{\rho} U(r) (\pm i) \left(1 + 0\left(\frac{1}{\rho}\right)\right) d\rho \quad (1.66)$$

For potentials decreasing stronger than $\frac{1}{r}$ for $r \rightarrow \infty$ (i.e. $|U(r)| < \frac{M}{r^{1+\epsilon}}$, $\epsilon > 0$) the integral on the right hand side of (1.66) becomes independent on ρ ($\rho \gg 1$) and approaches a constant value. We may set

$$\delta_l(\kappa) = -\frac{1}{2\kappa^2} \int_{\rho_0}^{\rho} U(r) \left(1 + 0\left(\frac{1}{\rho}\right)\right) d\rho \quad (1.67)$$



and get

$$\Phi_l = \Phi_l^0 e^{\pm i\delta_l(\kappa)} \quad (1.68)$$

and

$$R_l^{(\pm)}(r) \approx_{r \rightarrow \infty} h_l^{(\pm)}(\rho) e^{\pm i\delta_l(\kappa)} \approx \frac{e^{\pm i(\rho - \frac{l\pi}{2} + \delta_l(\kappa))}}{\rho} \quad (1.69)$$

Thus in the asymptotic range, the influence of the potential manifests itself only through a phaseshift. If U is a complex potential, then also $\delta_l(\kappa)$ will be complex.

The general solution (1.51) will be described in the asymptotic range by a linear combination of the R_l^{\pm} . This linear combination has to be such, that it is the asymptotic form of the exact solution, behaving regularly at the origin, i.e.

$$R_l(r) \approx_{r \rightarrow \infty} C_{lm} \frac{h_l^{(+)} e^{i\delta_l} - h_l^{(-)} e^{-i\delta_l}}{2i} = C_{lm} \frac{\sin(\rho - \frac{l\pi}{2} + \delta_l)}{\rho} \quad (1.70)$$

The plane wave $e^{i\vec{x}\vec{z}}$, which is also a solution of the homogeneous equation has the well known partial wave decomposition

$$e^{i\vec{x}\vec{z}} = e^{i\kappa z \cos\theta} = \sum_{l=0}^{\infty} (2l+1) i^l j_l(\kappa r) P_l(\cos\theta) = \sum_{lm} \delta_{m,0} i^l \sqrt{4\pi(2l+1)} j_l(\kappa r) Y_{lm}(\Omega) \quad (1.71)$$

Therefore we can use the following relation to determine the scattering amplitude:

$$\Psi_{\vec{x}}^{(+)}(\vec{x}) \approx_{r \rightarrow \infty} A \left[e^{i\vec{x}\vec{z}} + f(\Omega) \frac{e^{i\kappa r}}{r} \right] = A \left[\sum_{lm} \delta_{m,0} i^l \sqrt{4\pi(2l+1)} j_l(\kappa r) Y_{lm}(\Omega) + f(\Omega) \frac{e^{i\kappa r}}{r} \right] =$$

$$A \left[\sum_{lm} C_{lm} \frac{h_l^{(+)} e^{i\delta_l} - h_l^{(-)} e^{-i\delta_l}}{2i} Y_{lm} \right] = A \sum_{lm} C_{lm} \frac{h_l^{(+)} e^{2i\delta_l} - h_l^{(-)}}{2i} e^{-i\delta_l} Y_{lm}(\Omega) \quad (1.72)$$

Comparing the coefficients of the incoming waves ($h_l^{(-)} \sim \frac{e^{-i\kappa r}}{r}$) on both sides of (1.72) leads to

$$C_{lm} = \delta_{m,0} i^l e^{i\delta_l} \sqrt{4\pi(2l+1)} \quad (1.73)$$

Using this expression and comparing then the coefficients of the outgoing wave $\frac{e^{i\kappa r}}{r}$ leads to the following expression to determine $f(\Omega)$:

$$\sum_{lm} \delta_{m,0} i^l \sqrt{4\pi(2l+1)} (-i)^l \frac{Y_{lm}}{2i\kappa} + f(\Omega) = \sum_{lm} C_{lm} e^{-i\delta_l} \frac{(-i)^l e^{2i\delta_l}}{2i\kappa} Y_{lm}$$

or

$$f(\Omega) = \sum_{lm} \delta_{m,0} \sqrt{4\pi(2l+1)} \left(\frac{e^{2i\delta_l} - 1}{2i\kappa} \right) Y_{lm}(\Omega) = \sum_l (2l+1) f_l(\kappa) P_l(\cos\theta) \quad (1.74)$$

with

$$f_l(\kappa) \equiv \frac{e^{2i\delta_l(\kappa)} - 1}{2i\kappa} = \frac{S_l(\kappa) - 1}{2i\kappa} \quad (1.75)$$

Equation (1.74) is the partial wave expansion of the scattering amplitude $f(\Omega)$. f_l is the partial wave amplitude for the l 'th partial wave and S_l is the S -matrix element. From the optical theorem (1.47) we get

$$\sigma_{tot} = \frac{4\pi}{\kappa} \text{Im} f(0) = \frac{4\pi}{\kappa} \sum_{l=0}^{\infty} (2l+1) \text{Im} f_l(\kappa) \quad (P_l(1) = 1) \quad (1.76)$$

Next we calculate the total reaction cross section. We write the wave function $\Psi_{\vec{x}}^{(+)}(\vec{x})$ as follows:

$$\Psi_{\vec{x}}^{(+)}(\vec{x}) \stackrel{r \rightarrow \infty}{\cong} A \left[e^{i\vec{k}\vec{x}} + f(\Omega) \frac{e^{i\kappa r}}{r} \right] = A \left[\sum_l i^l (2l+1) j_l P_l + \frac{e^{i\kappa r}}{r} \sum_l (2l+1) \frac{S_l - 1}{2i\kappa} P_l \right] =$$

$$A \left[C_{in}(\Omega) \frac{e^{-i\kappa r}}{r} + C_{out} \frac{e^{i\kappa r}}{r} \right] \quad (1.77)$$

with

$$C_{in} = -\frac{1}{2i\kappa} \sum_{l=0}^{\infty} (-1)^l (2l+1) P_l(\cos\theta) \quad (1.78)$$

$$C_{out} = \frac{1}{2i\kappa} \sum_{l=0}^{\infty} (2l+1) S_l(\kappa) P_l(\cos\theta) \quad (1.79)$$

The current density \vec{j} is then given (up to terms of the order $O(\frac{1}{r^3})$) by

$$\vec{j} \approx \vec{e}_r j_{in} \frac{|C_{out}|^2 - |C_{in}|^2}{r^2} + \vec{O}\left(\frac{1}{r^3}\right) \quad (1.80)$$

$$dn_{nonel} \equiv -\vec{j} d\vec{S} = -\vec{j} \vec{e}_r r^2 d\Omega \cong j_{in} (|C_{in}|^2 - |C_{out}|^2) d\Omega$$

$$n_{nonel} = \int dn_{nonel} = j_{in} \int (|C_{in}|^2 - |C_{out}|^2) d\Omega \quad (1.81)$$

n_{nonel} is the number of all those particles that have been removed from the entrance channel, which therefore do not contribute to elastic scattering.

$$\sigma_{tot}^r \equiv \frac{n_{nonel}}{j_{in}} = \int (|C_{in}|^2 - |C_{out}|^2) d\Omega \quad (1.82)$$

σ_{tot}^r is called the total reaction cross section (sometimes it is also called non elastic cross section) Using (1.78) and (1.79) leads to

$$\sigma_{tot}^r = \frac{\pi}{\kappa^2} \sum_l (2l+1)(1 - |S_l|^2) = \frac{\pi}{\kappa^2} \sum_l (2l+1)(1 - \eta_l^2), \quad (1.83)$$

where we have introduced a new quantity, the inelasticity η_l . One may allow for complex phase shifts $\hat{\delta}_l = \delta_l + i\delta_l^{imag}$.

$$S_l = e^{2i\hat{\delta}_l} = e^{2i\delta_l - 2\delta_l^{imag}} = \eta_l e^{2i\delta_l} \quad (1.84)$$

Using this, we rewrite the expression for the elastic cross section:

$$\begin{aligned} \sigma_{el} &= \int d\Omega |f(\Omega)|^2 = \int \frac{d\Omega}{4\kappa^2} \left| \sum_{l=0}^{\infty} (2l+1) [\eta_l e^{2i\delta_l} - 1] P_l(\cos\theta) \right|^2 \\ &= 4\pi \sum_l (2l+1) |f_l(\kappa)|^2 = 4\pi \sum_l \frac{(2l+1)}{4\kappa^2} [1 + \eta_l^2 - 2\eta_l \cos 2\delta_l] \\ &= \frac{2\pi}{\kappa^2} \sum_l (2l+1)(1 - \eta_l \cos 2\delta_l) + \frac{\pi}{\kappa^2} \sum_l (2l+1)(\eta_l^2 - 1) \end{aligned} \quad (1.85)$$

or

$$\sigma_{el} = \frac{2\pi}{\kappa^2} \sum_l (2l+1)(1 - \eta_l \cos 2\delta_l) - \frac{\pi}{\kappa^2} \sum_l (2l+1)(1 - \eta_l^2) \quad (1.86)$$

From the optical theorem we obtain

$$\sigma_{tot} = \frac{2\pi}{\kappa^2} \sum_l (2l+1)(1 - \eta_l \cos 2\delta_l) \quad (1.87)$$

Therefore we have

$$\begin{aligned} \sigma_{tot} &= \sigma_{el} + \sigma_{tot}^r \\ \sigma_{tot} &\geq \sigma_{el} \\ \text{Im} f_l(\kappa) &\geq \kappa |f_l(\kappa)|^2 \end{aligned} \quad (1.88)$$

The value of the inelasticity may vary between 0 and 1.

$$\begin{aligned} \eta_l = 0 &: \text{maximal absorption,} & f_l &= \frac{1}{2i\kappa} \text{ (pure imaginary)} \\ \sigma_{el} &= \frac{\pi}{\kappa^2} \sum_l (2l+1) = \sigma_{tot}^r = \frac{1}{2} \sigma_{tot} \end{aligned}$$

$$\begin{aligned} \eta_l = 1 &: \text{pure elastic scattering} & \sigma_{tot}^r &= 0 \\ \sigma_{tot} &= \sigma_{el} = \frac{2\pi}{\kappa^2} \sum_l (2l+1)(1 - \cos 2\delta_l) = \\ &= \frac{4\pi}{\kappa^2} \sum_l (2l+1) \sin^2 \delta_l \end{aligned}$$

1.2 High - energy approximation in potential scattering - Glauber theory

We will derive a form of scattering theory valid for high energies ($\kappa R \gg 1$, R is the range of the interaction). One assumes that all partial waves behave in a similar way, namely that all $\delta_l(\kappa)$ are sufficiently smooth functions of l . In this case one can replace the sum over all partial amplitudes $f_l(\kappa)$ by an integral over l :

$$\begin{aligned} f(\Omega) &= \sum_l (2l+1) f_l(\kappa) P_l(\cos\theta) = \\ &= \sum_l (2l+1) \frac{e^{2i\delta_l} - 1}{2i\kappa} P_l(\cos\theta) \cong \int_0^{\infty} dl (2l+1) \frac{e^{2i\delta_l} - 1}{2i\kappa} P_l(\cos\theta) \end{aligned} \quad (1.89)$$

We substitute now $2l+1 = 2\kappa b$ (b ... collision parameter) in (1.89).

$$f(\Omega) \cong \frac{\kappa}{i} \int_{\frac{1}{2\kappa}}^{\infty} db b (e^{2ib} - 1) P_{\kappa b - \frac{1}{2}}(\cos\theta) = \frac{\kappa}{i} \int_{\frac{1}{2\kappa}}^{\infty} db b (e^{2ix(b)} - 1) P_{\kappa b - \frac{1}{2}}(\cos\theta) \quad (1.90)$$

with

$$\chi(b) \equiv \delta_l(\kappa) = \delta_{\kappa b - \frac{1}{2}}(\kappa) \quad (1.91)$$

If l approaches very large values, we may use an asymptotic formula (given for example in Abramowitz and Stegun, pg. 362):

$$\lim_{r \rightarrow \infty} \left[P_{\nu}^{-\mu} \left(\cos \frac{x}{\nu} \right) r^{\mu} \right] = J_{\mu}(x) \quad , \quad x \geq 0, \quad \mu \geq 0 \quad (1.92)$$

In our case here we have $\mu = 0$ and

$$\lim_{r \rightarrow \infty} P_{\nu} \left(\cos \frac{x}{\nu} \right) = J_0(x)$$

For large ν we may write

$$\cos \frac{x}{\nu} \sim 1 - \frac{x^2}{2\nu^2}$$

If we define x to be given by

$$x = 2\kappa b \sin \frac{\theta}{2} = (2\nu + 1) \sin \frac{\theta}{2}$$

then one has

$$\lim_{r \gg 1} \frac{x}{\nu} = 2 \sin \frac{\theta}{2} + \frac{1}{\nu} \sin \frac{\theta}{2} \approx 2 \sin \frac{\theta}{2}$$

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and

$$\cos \frac{x}{\nu} \approx 1 - 2 \sin^2 \frac{\theta}{2} = \cos \theta$$

Therefore we may replace $P_l(\cos\theta)$ by $J_0(x) = J_0(2\kappa b \sin \frac{\theta}{2})$ and obtain the following approximation for $f(\Omega)$:

$$f(\Omega) \approx -i\kappa \int_{\frac{1}{2\kappa}}^{\infty} db b (e^{2ix(b)} - 1) P_{\kappa b - \frac{1}{2}}(\cos\theta) \approx \frac{\kappa}{i} \int_0^{\infty} db b (e^{2ix(b)} - 1) J_0 \left(2\kappa b \sin \frac{\theta}{2} \right) \quad (1.93)$$

Below we will derive this approximation of the scattering amplitude once again for the case of potential scattering, using a different method. $\chi(b)$, the phase will be explicitly expressed through the potential V and we will see that $\Psi_{\kappa}^{(+)}$ has to be known only inside the range of V . We assume in the following again that we are dealing with high energies, therefore with very large momenta κ and small scattering angles. One can then approximate $\Psi_{\kappa}(\vec{x})$ by

$$\Psi_{\kappa}(\vec{x}) \sim A e^{i\vec{\kappa}\vec{x}} \phi(\vec{x}) \quad , \quad \kappa r \gg 1 \quad , \quad \theta \ll 1 \quad (1.94)$$

where $\phi(\vec{x})$ shall be a slowly varying function of \vec{x} ($|\phi''| \ll |\phi'|$, $|\phi|$). If we write (1.94) in the form

$$\Psi_{\kappa}(\vec{x}) \sim A e^{i\Phi(\vec{x})} = A e^{i(\vec{\kappa}\vec{x} + \hat{\Phi}(\vec{x}))} \quad (1.95)$$

then $\hat{\Phi}(\vec{x})$ is slowly varying with \vec{x} (A is a normalisation factor).

$$\nabla \Psi_{\kappa}(\vec{x}) = A i e^{i\Phi} \nabla \Phi = A i e^{i\Phi} (\vec{\kappa} + \nabla \hat{\Phi})$$

and

$$\nabla^2 \Psi_{\kappa}(\vec{x}) = A [i \nabla^2 \Phi - (\nabla \Phi)^2] e^{i\Phi} \approx -A (\nabla \Phi)^2 e^{i\Phi}$$

We have neglected $\nabla^2\Phi$. If we use this result now in the Schroedinger equation (1.27), we obtain the approximate result:

$$(\nabla\Phi)^2 = \kappa^2 - U(r) \equiv n^2(\vec{x})\kappa^2 \quad (1.96)$$

and

$$\nabla\Phi(\vec{x}) = \pm n(\vec{x})\vec{\kappa} = \vec{\kappa} + \nabla\hat{\Phi}(\vec{x}) \quad (1.97)$$

The solution of this equation is then given by

$$\int_{\vec{x}_0}^{\vec{x}} \nabla\hat{\Phi}(\vec{x}')d\vec{x}' = \hat{\Phi}(\vec{x}) - \hat{\Phi}(\vec{x}_0) = \int_{\vec{x}_0}^{\vec{x}} d\vec{x}'(n(\vec{x}') - 1)\vec{\kappa} \quad (1.98)$$

As initial condition we choose $\hat{\Phi}(\vec{x}_0) = 0$. \vec{x}_0 can be shifted to $-\infty$ without any restriction. If θ is small, one may replace $\vec{\kappa}\vec{x}$ by $\sim \kappa|\vec{x}|$.

$$\phi(\vec{x}) = \vec{\kappa}\vec{x} + \hat{\Phi}(\vec{x}) = \vec{\kappa}\vec{x} + \int_{\vec{x}_0}^{\vec{x}} d\vec{x}'\vec{\kappa}(n(\vec{x}') - 1) \approx \vec{\kappa}\vec{x} + \kappa \int_{z_0}^z dz'(n(\vec{x}') - 1) \quad (1.99)$$

On the other hand we have

$$\begin{aligned} \kappa(n(\vec{x}) - 1) &= \sqrt{\kappa^2 - 2\frac{\mu_{red}}{\hbar c}V(\vec{x})} - \kappa \stackrel{|\vec{x}|\rightarrow\infty}{\approx} \kappa \left[1 - \frac{\mu_{red}}{\hbar c\kappa^2}V(\vec{x}) \right] - \kappa \\ &= -\frac{\mu_{red}}{\hbar c} \frac{V(\vec{x})}{\kappa} = -\frac{c}{v_{rel}} \frac{V(\vec{x})}{\hbar c} \end{aligned}$$

If we use this in (1.99) we get

$$\Phi(\vec{x}) \approx \vec{\kappa}\vec{x} - \frac{c}{v_{rel}} \int_{z_0=-\infty}^z \frac{V(\vec{x}')}{\hbar c} dz' = \vec{\kappa}\vec{x} - \frac{\mu_{red}}{\kappa} \int_{z_0=-\infty}^z \frac{V(\vec{x}')}{\hbar c} dz' \quad (1.100)$$

The vector \vec{x}' can be decomposed into the components \vec{b} (in the xy plane) $z'\vec{e}_z$ ($\vec{x}' = (\vec{b}, z')$). Therefore one has:

$$\Psi_{\vec{\kappa}}(\vec{x}) \cong A e^{i \left[\vec{\kappa}\vec{x} - \frac{\mu_{red}}{\kappa} \int_{-\infty}^z \frac{V(\vec{b}, z')}{\hbar c} dz' \right]} = A e^{i \left[\vec{\kappa}\vec{x} - \frac{c}{v_{rel}} \int_{-\infty}^z \frac{V(\vec{b}, z')}{\hbar c} dz' \right]} \quad (1.101)$$

One sees immediately that this approximation will not be a good solution in the whole space, since it contains no outgoing spherical waves (analogous to classical ray optics). Inside the interaction range $|\vec{x}| \leq R$ the function $\Psi_{\vec{\kappa}}(\vec{x})$ given in (1.101) may be a reasonable approximation. The scattering amplitude can be obtained by using this approximate $\Psi_{\vec{\kappa}}(\vec{x})$ in equation (1.40). We will use here the δ -function normalisation ($A = (2\pi)^{-\frac{3}{2}}$) and obtain

$$f(\Omega) = -\frac{(2\pi)^2 \mu_{red}}{\hbar c} \int d^3\vec{x}' \left(\frac{1}{2\pi} \right)^{\frac{3}{2}} e^{-i\vec{\kappa}'\vec{x}'} V(\vec{x}') \left(\frac{1}{2\pi} \right)^{\frac{3}{2}} e^{i \left[\vec{\kappa}\vec{x} - \frac{c}{v_{rel}} \int_{-\infty}^{z'} \frac{V(\vec{b}, z'')}{\hbar c} dz'' \right]}$$

The integration is done using cylindrical coordinates ($d^3\vec{x}' = d^2b dz'$; $\vec{x}' = \vec{b} + z'\vec{e}_z$):

$$f(\Omega) = -\frac{\mu_{red}}{2\pi\hbar c} \int d^2b \int_{-\infty}^{+\infty} dz' e^{i\vec{q}\cdot(\vec{b}+z'\vec{e}_z)} V(\vec{b}+z'\vec{e}_z) e^{-i\frac{c}{v_{rel}} \int_{-\infty}^{z'} \frac{V(\vec{b}+z''\vec{e}_z)}{\hbar c} dz''}$$

where $\vec{q} = \vec{\kappa} - \vec{\kappa}'$. In the case of elastic scattering we have $|\vec{\kappa}'| = |\vec{\kappa}|$ and, $|\vec{q}| = 2\kappa \sin \frac{\theta}{2}$. For scattering into small values of θ one has approximately $\vec{q}\vec{e}_z \sim 0$ and therefore also $e^{i\vec{q}\vec{e}_z z'} \approx 1$. This leads to

$$\begin{aligned}
f(\Omega) &= -\frac{\mu_{red}}{2\pi\hbar c} \int d^2b \int_{-\infty}^{+\infty} dz' e^{iq\vec{b}} V(\vec{b} + z'\vec{e}_z) e^{-i\frac{c}{v_{rel}} \int_{-\infty}^{z'} \frac{V(\vec{b} + z''\vec{e}_z)}{\hbar c} dz''} \\
&= -\frac{i\mu_{red}v_{rel}}{2\pi c} \int d^2b \int_{-\infty}^{+\infty} dz' e^{iq\vec{b}} \frac{d}{dz'} \left(e^{-i\frac{c}{v_{rel}} \int_{-\infty}^{z'} \frac{V(\vec{b} + z''\vec{e}_z)}{\hbar c} dz''} \right) \\
&= -\frac{i\mu_{red}v_{rel}}{2\pi c} \int d^2b e^{iq\vec{b}} \left(e^{-i\frac{c}{v_{rel}} \int_{-\infty}^{+\infty} \frac{V(\vec{b} + z''\vec{e}_z)}{\hbar c} dz''} - 1 \right)
\end{aligned}$$

The so-called eikonal phaseshifts $\chi(\vec{b})$ are defined through

$$\chi(\vec{b}) \equiv -\frac{c}{2v_{rel}} \int_{-\infty}^{+\infty} \frac{V(\vec{b} + z\vec{e}_z)}{\hbar c} dz \quad (1.102)$$

Using this in $f(\Omega)$ we get

$$f(\Omega) \cong i\frac{\mu_{red}v_{rel}}{2\pi c} \int d^2b e^{iq\vec{b}} \left(1 - e^{2i\chi(\vec{b})} \right) \quad (1.103)$$

For spherical symmetric potentials $V(\vec{x}) = V(|\vec{x}|)$ one has also $\chi(\vec{b}) = \chi(|\vec{b}|) = \chi(b)$. With $d^2b = b db d\phi$ one gets

$$f(\Omega) \cong i\mu_{red} \frac{v_{rel}}{c} \int_0^{\infty} db b \left(1 - e^{2i\chi(b)} \right) J_0(qb) \quad (1.104)$$

where we have used

$$J_0(qb) = \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{iqb \cos\phi}$$

If we finally use $\frac{v_{rel}}{c} = \frac{\kappa}{\mu_{red}}$ and $qb = 2\kappa b \sin \frac{\theta}{2}$, we obtain

$$f(\theta) \cong i\kappa \int_0^{\infty} db b \left(1 - e^{2i\chi(b)} \right) J_0 \left(2\kappa b \sin \frac{\theta}{2} \right) \quad (1.105)$$

which is exactly the expression obtained earlier (1.93) from the partial wave expansion

1.3 The eikonal approximation in hadron - nucleus scattering

In the theory of potential scattering the interaction potential of a projectile with a nucleus is given as the sum over the potentials between projectile and all the individual nucleons.

$$V(\vec{x}) = \sum_{j=1}^A V_j(\vec{x} - \vec{x}_j) \quad (1.106)$$

From (1.102) follows then that the eikonal phaseshift $\chi(\vec{b})$ for the projectile - nucleus scattering is given by the sum of the phaseshifts of the scattering on the individual nucleons. Remember, however, that normally phaseshifts of potential scattering theory are not additive. This additivity of phases is a particular property of the eikonal method.

$$\chi(\vec{k}) = -\frac{C}{2v_{rel}} \int_{-\infty}^{+\infty} \sum_{j=1}^A \frac{V_j(\vec{b} - \vec{s}_j, z)}{\hbar c} dz = \sum_{j=1}^A \chi_j(\vec{b} - \vec{s}_j) \quad (1.107)$$

The following coordinates have been introduced

$$\vec{x} = \vec{b} + z\vec{e}_z$$

$$\vec{x}_j = (\vec{x}_j\vec{e}_z)\vec{e}_z + \vec{s}_j = z_j\vec{e}_z + \vec{s}_j$$

$$\vec{x} - \vec{x}_j = \vec{b} - \vec{s}_j + \vec{e}_z(z - z_j)$$

In the integrals we finally substituted $z' = z - z_j$.

Another important property of the eikonal method is the fact, that the scattering amplitude is a simple Fourier transform of the so called profile function $\Gamma(\vec{b})$:

$$\Gamma(\vec{b}) \equiv 1 - e^{2i\chi(\vec{b})} \quad (1.108)$$

This allows one to determine the eikonal phase (or profile function) without the use of the potential $V(\vec{x})$, provided the elementary scattering amplitudes are known.

$$f_i \rightarrow \Gamma_i \rightarrow \chi_i \rightarrow \chi = \sum_j \chi_j \rightarrow \Gamma \rightarrow f(\Omega)$$

From this one concludes that the eikonal method to determine approximately the scattering amplitude may be valid even under less restrictive conditions. To illustrate this we consider the scattering of a spinless particle on a nucleus. One assumes the elementary scattering amplitudes for scattering on an individual nucleon to be known (f_i). Starting with f_i one calculates Γ_i by inverting the Fourier transform:

$$i\kappa\Gamma_j(\vec{b}') = \frac{1}{2\pi} \int e^{-i\vec{q}\vec{b}'} f_j(\vec{q}) d^2\vec{q} = i\kappa \left(1 - e^{2i\chi_j(\vec{b}')} \right)$$

$$e^{2i\chi_j(\vec{b}')} = 1 - \frac{1}{2\pi i\kappa} \int e^{-i\vec{q}\vec{b}'} f_j(\vec{q}) d^2\vec{q}$$

One integrates over those \vec{q} 's with constant energy (or constant κ). For very large energies this is approximately a plane. From $\chi = \sum_j \chi_j$ follows

$$e^{2i\chi(\vec{b})} = e^{2i\sum_j \chi_j(\vec{b}-\vec{s}_j)} = \prod_{j=1}^A e^{2i\chi_j(\vec{b}-\vec{s}_j)}$$

$$\Gamma(\vec{b}) = 1 - e^{2i\chi(\vec{b})} = 1 - \prod_{j=1}^A e^{2i\chi_j(\vec{b}-\vec{s}_j)}$$

and

$$F(\vec{q}) = \frac{i\kappa}{2\pi} \int d^2\vec{b} e^{i\vec{q}\vec{b}} \left(1 - \prod_{j=1}^A e^{2i\chi_j(\vec{b}-\vec{s}_j)} \right) \quad (1.109)$$

This is the projectile - nucleus scattering amplitude and it still depends on all nucleon coordinates. In the space of nuclear states $F(\vec{q})$ is still an operator. Considering a transition from initial state $|i\rangle$ to the final state $|f\rangle$ we obtain

$$F_{fi}(\vec{q}) = \langle f | F(\vec{q}) | i \rangle = \frac{i\kappa}{2\pi} \int d^2\vec{b} e^{i\vec{q}\vec{b}} \langle f | \left(1 - \prod_{j=1}^A e^{2i\chi_j(\vec{b}-\vec{s}_j)} \right) | i \rangle \quad (1.110)$$

Explicit calculation of the product in (1.110) using $e^{2i\chi_j} = 1 - \Gamma_j(\vec{b}-\vec{s}_j)$, leads to

$$\left. \begin{aligned} \prod_{j=1}^A (1 - \Gamma_j) &= 1 - \sum_{j=1}^A \Gamma_j + \sum_{i<j} \Gamma_i \Gamma_j - \sum_{i<j<k} \Gamma_i \Gamma_j \Gamma_k + \dots + (-1)^A \Gamma_1 \Gamma_2 \dots \Gamma_A \\ \Gamma_j(\vec{b} - \vec{s}_j) &= \frac{1}{2\pi i\kappa} \int d^2\vec{q}' e^{-i\vec{q}'(\vec{b}-\vec{s}_j)} f_j(\vec{q}') \end{aligned} \right\} \quad (1.111)$$

This is a polynomial in the amplitudes $f_j(\vec{q})$, where terms with 1 amplitude (single scattering), 2 amplitudes (double scattering), etc., and a term with maximal A amplitudes $f_1 f_2 \dots f_A$ (A - fold scattering) occur. Since in all these products $i \neq k$, etc.,

one sees that the projectile is scattered on each target nucleon only once. This is a consequence of the approximations made. We assumed small scattering angles. Multiple scattering on one and the same nucleon would require at least once scattering through a large angle. In deriving equ. (1.110) we made implicitly the assumption that all nucleons have fixed positions; we have "frozen in" the nucleons inside the nucleus during that time the projectile interacts with the nucleus. We have neglected the so called Fermi motion of the bound nucleons and evaluated all multiple scattering terms at one and the same momentum transfer.

2 Formal scattering theory

In this chapter the dynamics of a quantum system will be studied from a more general point of view, the time evolution operator playing the central role. The considerations will be valid in both the relativistic and the nonrelativistic case. We will develop the relation between the time evolution operator and the scattering matrix, which is the central object to be studied in scattering theory. We will also investigate Green's operators (propagators) and the Lippmann - Schwinger equation for the T - matrix and give some general formulae to calculate decay - probabilities and cross sections from the T - matrix.

2.1 The time evolution operator

We start with the derivation of the time evolution operator in the Schroedinger picture. The dynamical equation describing the time evolution of a quantum system is the Schroedinger equation.

$$H_s \Psi_s(t) = i\hbar \frac{\partial \Psi_s(t)}{\partial t} \quad (2.1)$$

The index s stands for Schroedinger picture. In this picture observables or dynamic variables are represented through time independent Hermitian operators (an explicit time dependence is of course possible). The state vector $\Psi_s(t)$ of the system is time dependent and evolves according the Schroedinger equation (2.1) and a given initial condition. We define the time evolution operator $U_s(t, t_0)$ through

$$\Psi_s(t) = U_s(t, t_0) \Psi_s(t_0) \quad (2.2)$$

with the initial condition

$$U_s(t_0, t_0) = 1 \quad (2.3)$$

Inserting this into the Schroedinger equation gives us

$$i\hbar \frac{\partial U_s(t, t_0)}{\partial t} = H_s U_s(t, t_0) \quad (2.4)$$

It is easy to show from (2.2) that U_s has multiplicative group properties.

$$U_s(t, t_0) = U_s(t, t') U_s(t', t_0) \quad (2.5)$$

$$U_s^{-1}(t, t_0) = U(t_0, t) \quad (2.6)$$

The conservation of probability requires $\langle \Psi_s(t) | \Psi_s(t) \rangle = \langle \Psi_s(t_0) | \Psi_s(t_0) \rangle$ from which follows the unitarity of U_s :

$$U_s^+(t, t_0) U_s(t, t_0) = U_s(t, t_0) U_s^+(t, t_0) = 1 \quad (2.7)$$

U_s is unitary since H_s is hermitian. H_s is the generator of an infinitesimal time translation $U_s(t + \delta t, t_0)$. A formal integration of (2.4) together with the initial condition (2.3) leads to

$$U_s(t, t_0) = 1 + \left(-\frac{i}{\hbar}\right) \int_{t_0}^t H_s U_s(t', t_0) dt' \quad (2.8)$$

If H_s does not explicitly depend on t , one can find a closed expression for the solution

$$U_s(t, t_0) = e^{-\frac{i}{\hbar} H_s (t-t_0)} \quad (2.9)$$

and

$$\Psi_s(t) = e^{-\frac{i}{\hbar} H_s (t-t_0)} \Psi_s(t_0)$$

In the Heisenberg picture the quantum system is described through time independent state vectors and time dependent operators. The transition from one picture to the next is given by the transformation

$$\Psi_H = U_s^+(t, t_0) \Psi_s(t) = U_s(t_0, t) \Psi_s(t) \quad (2.10)$$

$$O_{op,H}(t) = U_s^+(t, t_0) O_{op,s} U(t, t_0) \quad (2.11)$$

If the Hamiltonian H_s can be split into an "unperturbed" (or "free") part H_0 and a "perturbation" (or "interaction") V , it is convenient to use the interaction picture to describe the dynamics of the system. In this interaction picture the "free" motion corresponding to H_0 is separated from that of the whole system. This separation is performed in such a way that the time evolution of the observables is determined through H_0 , whereas the state vectors have a time evolution determined through the interaction V . In order to formulate the interaction picture, one performs a unitary transformation on the Schroedinger state vectors.

$$\Psi(t) = e^{\frac{i}{\hbar} H_0 (t-t_0)} \Psi_s(t) \quad (2.12)$$

$\Psi(t)$ is the state vector in the interaction picture. From the Schroedinger equation

$$H_s \Psi_s(t) = (H_0 + V) \Psi_s(t) = i\hbar \frac{\partial \Psi_s(t)}{\partial t} \quad (2.13)$$

we obtain

$$i\hbar \frac{\partial \Psi(t)}{\partial t} = V(t) \Psi(t) \quad \text{Tomonaga - Schwinger equation} \quad (2.14)$$

with

$$V(t) = e^{\frac{i}{\hbar}H_0(t-t_0)} V e^{-\frac{i}{\hbar}H_0(t-t_0)} \quad (2.15)$$

The Tomonaga-Schwinger equation describes the time evolution of the state vector $\Psi(t)$ in the interaction picture. One easily shows that the time evolution of all observables is determined by H_0 (one says also that the observables have only a kinematical time dependence, whereas the state vector $\Psi(t)$ has the dynamic time dependence determined through the interaction $V(t)$). This splitting of the dynamic time evolution from the " trivial " one is particularly useful in scattering theory. We define now a time evolution operator in the interaction picture through

$$\bar{\Psi}(t) = U(t, t')\Psi(t') \quad (2.16)$$

with

$$\left. \begin{aligned} U(t, t) &= 1 \\ U(t, t'')U(t'', t') &= U(t, t') \\ U^{-1}(t, t') &= U(t', t) \end{aligned} \right\} \quad (2.17)$$

The Tomonaga-Schwinger equation gives us

$$i\hbar \frac{\partial U(t, t')}{\partial t} = V(t)U(t, t') \quad (2.18)$$

and formal integration leads to

$$U(t, t') = 1 + \left(\frac{-i}{\hbar} \right) \int_{t'}^t V(t'')U(t, t'')dt'' \quad (2.19)$$

The conservation of probability leads as before to the unitarity of U:

$$U^+(t, t') = U^{-1}(t, t') = U(t', t) \quad (2.20)$$

From (2.16) and (2.12) one easily obtains the relation between $U(t, t')$ and $U_s(t, t')$:

$$U(t, t') = e^{\frac{i}{\hbar}H_0(t-t_0)} U_s(t, t') e^{-\frac{i}{\hbar}H_0(t'-t_0)} \quad (2.21)$$

and in particular for not time dependent H_s one gets

$$U(t, t') = e^{\frac{i}{\hbar}H_0(t-t_0)} e^{-\frac{i}{\hbar}H_s(t-t')} e^{-\frac{i}{\hbar}H_0(t'-t_0)} \quad (2.22)$$

Expectation values of operators are of course in all pictures the same.

2.2 The scattering matrix (S - matrix)

This chapter is devoted to the development of a rigorous formalism to describe a quantum collision process. The basic ideas are very simple. One assumes that in the past, at $t = -\infty$, a beam of free particles was prepared in the asymptotic region, by fixing the experimental conditions (the projectiles have a certain energy, spin, direction of motion). At time $t = 0$ the interaction between the projectiles and the target takes place. The scattered particles are detected at time $t = +\infty$ with detectors placed in the asymptotic region (i.e. far away from the scattering center). We therefore have to investigate the time evolution operator at 3 different times, $t = -\infty, 0, +\infty$. The following limiting procedure for operator functions given by Gell - Mann and Goldberger will turn out to be very useful for the further discussion.

$$\lim_{t \rightarrow \infty} F(t) = \lim_{\epsilon \rightarrow 0^+} \left\{ \epsilon \int_0^{\infty} e^{-\epsilon t'} F(t') dt' \right\} \quad (2.23)$$

$$\lim_{t \rightarrow -\infty} F(t) = \lim_{\epsilon \rightarrow 0^+} \left\{ \epsilon \int_{-\infty}^0 e^{\epsilon t'} F(t') dt' \right\} \quad (2.24)$$

One convinces oneself by partial integration from the correctness of this relation provided $F(t)$ has proper limits $F(\pm\infty)$ at $t = \pm\infty$. If $F(t)$ however oscillates at large $|t|$, then the exponential factor will damp these oscillations. It is important to note that first one has to perform the integration and then to take the limiting value for $\epsilon \rightarrow 0^+$. This limiting procedure may now be applied to the collision operator $U(t, t')$ and one may define the following 4 operators

$$\begin{aligned} U(t, -\infty) &= \lim_{\epsilon \rightarrow 0^+} \left\{ \epsilon \int_{-\infty}^0 e^{\epsilon t'} U(t, t') dt' \right\} \\ U(t, +\infty) &= \lim_{\epsilon \rightarrow 0^+} \left\{ \epsilon \int_0^{\infty} e^{-\epsilon t'} U(t, t') dt' \right\} \\ U(+\infty, t) &= \lim_{\epsilon \rightarrow 0^+} \left\{ \epsilon \int_0^{\infty} e^{-\epsilon t'} U(t', t) dt' \right\} \\ U(-\infty, t) &= \lim_{\epsilon \rightarrow 0^+} \left\{ \epsilon \int_{-\infty}^0 e^{\epsilon t'} U(t', t) dt' \right\} \end{aligned} \quad (2.25)$$

We note without proof that $U(t, \pm\infty)$ and $U(\pm\infty, t)$ satisfy the same differential equation as $U(t, t')$ and have the same multiplicative group properties. They are, however, not necessarily unitary since the inverse operators have not yet been defined. They satisfy however the relations $U^+(+\infty, t) = U(t, +\infty)$, etc.

In the following we will investigate the scattering problem (2.13) and assume that there exists only one way of splitting the hamiltonian H_s . The generalisation to many arrangement channels is not very difficult and will be given later. We start from

$$H_s = H_0 + V \quad (2.26)$$

and denote the eigenstates of H_0 by Φ_α . It is assumed that H_0 has no bound states. In the next step we define the so-called Moeller operators (or wave operators) by

$$\Omega^{(\pm)} = U(0, \mp\infty) \quad \text{and} \quad \Omega^{(\pm)\dagger} = U(\mp\infty, 0) \quad (2.27)$$

They transform the eigenstates $\{\phi_\alpha\}$ of H_0 into the eigenstates $\Psi_\alpha^{(\pm)}$ of the full hamiltonian H_s at time $t = 0$

$$|\Psi_\alpha^{(\pm)}\rangle = \Omega^{(\pm)} |\Phi_\alpha\rangle \quad \text{and} \quad \langle \Psi_\alpha^{(\pm)} | = \langle \Phi_\alpha | \Omega^{(\pm)\dagger} \quad (2.28)$$

From this we can calculate the matrix elements of the Moeller operators and obtain a useful representation of these operators.

$$\left. \begin{aligned} \langle \Phi_\beta | \Omega^{(\pm)} | \Phi_\alpha \rangle &= \langle \Phi_\beta | \Psi_\alpha^{(\pm)} \rangle \\ \langle \Phi_\beta | \Omega^{(\pm)\dagger} | \Phi_\alpha \rangle &= \langle \Psi_\beta^{(\pm)} | \Phi_\alpha \rangle \end{aligned} \right\} \quad (2.29)$$

The set of functions $|\Psi_\alpha\rangle$ may be assumed to be a complete set of orthogonal functions. As a consequence we can derive immediately a representation of the Moeller operators.

$$\left. \begin{aligned} \Omega^{(\pm)} &= \sum_\alpha |\Psi_\alpha^{(\pm)}\rangle \langle \Phi_\alpha | \\ \Omega^{(\pm)\dagger} &= \sum_\alpha |\Phi_\alpha\rangle \langle \Psi_\alpha^{(\pm)} | \end{aligned} \right\} \quad (2.30)$$

Using the operator expression (2.25) for $U(0, -\infty)$ allows one to obtain an explicit representation of Ω^\pm . We assume again that H_s has no explicit time dependence. With

$$U(t, t') = e^{\frac{i}{\hbar} H_0(t-t_0)} e^{-\frac{i}{\hbar} H_s(t-t')} e^{-\frac{i}{\hbar} H_0(t'-t_0)}$$

we get from (2.25):

$$U(0, -\infty) = \lim_{\epsilon \rightarrow 0^+} \left\{ \epsilon \int_{-\infty}^0 e^{i\epsilon t'} U(0, t') dt' \right\}$$

$$= \lim_{\epsilon \rightarrow 0^+} \left\{ \epsilon \int_{-\infty}^0 e^{i\epsilon t'} e^{-\frac{i}{\hbar} H_0 t_0} e^{\frac{i}{\hbar} H_s t'} e^{-\frac{i}{\hbar} H_0 (t' - t_0)} dt' \right\}$$

If one sets $t_0 = 0$ this reduces to

$$U(0, -\infty) = \lim_{\epsilon \rightarrow 0^+} \left\{ \epsilon \int_{-\infty}^0 e^{i\epsilon t'} e^{\frac{i}{\hbar} H_s t'} e^{-\frac{i}{\hbar} H_0 t'} dt' \right\}$$

Using the same steps in the calculation of $U(0, +\infty)$ one gets also

$$U(0, +\infty) = \lim_{\epsilon \rightarrow 0^+} \left\{ \epsilon \int_0^{\infty} e^{-i\epsilon t'} e^{\frac{i}{\hbar} H_s t'} e^{-\frac{i}{\hbar} H_0 t'} dt' \right\}$$

If we use now the completeness of the free states $|\Phi_\alpha\rangle$ we find

$$U(0, \mp\infty) = \lim_{\epsilon \rightarrow 0^+} \left\{ \mp\epsilon \int_0^{\mp\infty} e^{\pm i\epsilon t'} e^{\frac{i}{\hbar} H_s t'} \sum_{\alpha} |\Phi_\alpha\rangle \langle \Phi_\alpha| e^{-\frac{i}{\hbar} E_\alpha t'} dt' \right\}$$

$$= \lim_{\epsilon \rightarrow 0^+} \sum_{\alpha} \frac{\pm\epsilon}{\pm\epsilon + \frac{i}{\hbar}(H_s - E_\alpha)} |\Phi_\alpha\rangle \langle \Phi_\alpha|$$

or

$$\Omega^{(\pm)} = \lim_{\eta \rightarrow 0^+} \sum_{\alpha} \frac{\pm i\eta}{E_\alpha - H_s \pm i\eta} |\Phi_\alpha\rangle \langle \Phi_\alpha| \quad (2.31)$$

With this we obtain for the scattering solutions $|\Psi_\alpha^{(\pm)}\rangle$ the simple expression

$$|\Psi_\alpha^{(\pm)}\rangle = \Omega^{(\pm)} |\Phi_\alpha\rangle = \lim_{\eta \rightarrow 0^+} \frac{\pm i\eta}{E_\alpha - H_s \pm i\eta} |\Phi_\alpha\rangle \quad (2.32)$$

The functions $\{|\Psi_\alpha^{(\pm)}\rangle\}$ are eigenstates of the full Hamiltonian and therefore orthogonal

$$H_s |\Psi_\alpha^{(\pm)}\rangle = E_\alpha |\Psi_\alpha^{(\pm)}\rangle$$

$$\langle \Psi_\beta^{(\pm)} | \Psi_\alpha^{(\pm)} \rangle = \delta_{\beta\alpha} \quad (2.33)$$

This allows us to obtain another property of the Moeller operators:

$$\Omega^{(\pm)\dagger} \Omega^{(\pm)} = \sum_{\alpha\beta} |\Phi_\alpha\rangle \langle \Psi_\alpha^{(\pm)} | \Psi_\beta^{(\pm)} \rangle \langle \Phi_\beta| = \sum_{\alpha} |\Phi_\alpha\rangle \langle \Phi_\alpha| = 1 \quad (2.34)$$

$$\Omega^{(\pm)} \Omega^{(\pm)\dagger} = \sum_{\alpha\beta} |\Psi_\alpha^{(\pm)}\rangle \langle \Phi_\alpha | \Phi_\beta \rangle \langle \Psi_\beta^{(\pm)}| = \sum_{\alpha} |\Psi_\alpha^{(\pm)}\rangle \langle \Psi_\alpha^{(\pm)}| \quad (2.35)$$

The full hamiltonian H_s may have bound states $|\Psi_B\rangle$. The completeness of the eigenstates of H_s is therefore given by

$$1 = \sum_{\alpha} |\Psi_\alpha^{(\pm)}\rangle \langle \Psi_\alpha^{(\pm)}| + \sum_B |\Psi_B\rangle \langle \Psi_B| \equiv \sum_{\alpha} |\Psi_\alpha^{(\pm)}\rangle \langle \Psi_\alpha^{(\pm)}| + \Lambda_B \quad (2.36)$$

$$\Omega^{(\pm)} \Omega^{(\pm)\dagger} = 1 - \Lambda_B \quad (2.37)$$

The Moeller operators are therefore only unitary operators, if H_0 and H_s have no bound states. Next we show

$$E_\alpha | \Psi_\alpha^{(\pm)} \rangle = H_s | \Psi_\alpha^{(\pm)} \rangle = H_s \Omega^{(\pm)} | \Phi_\alpha \rangle$$

$$E_\alpha | \Psi_\alpha^{(\pm)} \rangle = \Omega^{(\pm)} E_\alpha | \Phi_\alpha \rangle = \Omega^{(\pm)} H_0 | \Phi_\alpha \rangle$$

and therefore

$$\left. \begin{aligned} H_s \Omega^{(\pm)} &= \Omega^{(\pm)} H_0 \\ \Omega^{(\pm)\dagger} H_s &= H_0 \Omega^{(\pm)\dagger} \\ \Omega^{(\pm)\dagger} | \Psi_B \rangle &= 0 \implies \Omega^{(\pm)\dagger} \Lambda_B = 0 \end{aligned} \right\} \quad (2.38)$$

After having discussed the properties of the Moeller operators we introduce the concept of the scattering operator S . This operator is constructed in such a way, that it connects the free states at time $-\infty$ with the free states at time $t = +\infty$, after the interaction has taken place. This operator S is defined through

$$S \equiv U(+\infty, 0) U(0, -\infty) = \Omega^{(-)\dagger} \Omega^{(+)} \quad (2.39)$$

and it acts by construction on the asymptotically free eigenstates $| \Phi_\alpha \rangle$ of H_0 . The S -matrix elements are then given by

$$\langle \Phi_\beta | S | \Phi_\alpha \rangle = \langle \Phi | \Omega^{(-)\dagger} \Omega^{(+)} | \Phi_\alpha \rangle = \langle \Psi_\beta^{(-)} | \Psi_\alpha^{(+)} \rangle \quad (2.40)$$

The S -matrix elements are time-independent and they represent the probability amplitude to find the system after a measurement in the state $| \Psi_\beta^{(-)} \rangle$ if it was originally in state $| \Psi_\alpha^{(+)} \rangle$. $| \Psi_\alpha^{(+)} \rangle$ is obtained by time evolution from the state $| \Phi_\alpha \rangle$, whereas $| \Psi_\beta^{(-)} \rangle$ goes by time evolution into the state $| \Phi_\beta \rangle$. We note that the definition of the S -matrix does not rely on a particular splitting of the hamiltonian H_s . It is a unitary operator and it commutes with H_0 :

$$\left. \begin{aligned} S^\dagger S &= S S^\dagger = 1 \\ [S, H_0] &= 0 \end{aligned} \right\} \quad (2.41)$$

In the next step we calculate the S -matrix explicitly.

$$\begin{aligned} \langle \Phi_\beta | S | \Phi_\alpha \rangle &= \langle \Psi_\beta^{(-)} | \Psi_\alpha^{(+)} \rangle = \left[\langle \Psi_\beta^{(+)} | - \langle \Psi_\beta^{(+)} | - \langle \Psi_\beta^{(-)} | \right] | \Psi_\alpha^{(+)} \rangle \\ &= \delta_{\beta\alpha} - \left[\langle \Psi_\beta^{(+)} | - \langle \Psi_\beta^{(-)} | \right] | \Psi_\alpha^{(+)} \rangle \end{aligned}$$

$$| \Psi_\beta^{(+)} \rangle - | \Psi_\beta^{(-)} \rangle = \lim_{\eta \rightarrow 0^+} i\eta \left[\frac{1}{E_\beta - H_s + i\eta} + \frac{1}{E_\beta - H_s - i\eta} \right] | \Phi_\beta \rangle$$

$$= \lim_{\eta \rightarrow 0^+} \frac{2i\eta(E_\beta - H_s)}{(E_\beta - H_s)^2 + \eta^2} | \Phi_\beta \rangle = \lim_{\eta \rightarrow 0^+} \frac{2i\eta(E_\beta - H_0 - V)}{(E_\beta - H_s)^2 + \eta^2} | \Phi_\beta \rangle$$

$$= \lim_{\eta \rightarrow 0^+} \frac{-2i\eta}{(E_\beta - H_s)^2 + \eta^2} V | \Phi_\beta \rangle = -2i\pi \delta(E_\beta - H_s) V | \Phi_\beta \rangle$$

$$\left[\langle \Psi_\beta^{(+)} | - \langle \Psi_\beta^{(-)} | \right] | \Psi_\alpha^{(+)} \rangle = 2i\pi \langle \Phi_\beta | V \delta(E_\beta - H_s) | \Psi_\alpha^{(+)} \rangle$$

$$= 2i\pi\delta(E_\beta - E_\alpha) \langle \Phi_\beta | V | \Psi_\alpha^{(+)} \rangle$$

With this we obtain

$$\langle \Phi_\beta | S | \Phi_\alpha \rangle = \delta_{\beta\alpha} - 2\pi i\delta(E_\beta - E_\alpha) \langle \Phi_\beta | V | \Psi_\alpha^{(+)} \rangle \quad (2.42)$$

In a similar way one obtains also

$$\langle \Phi_\beta | S | \Phi_\alpha \rangle = \delta_{\beta\alpha} - 2\pi i\delta(E_\beta - E_\alpha) \langle \Psi_\beta^{(-)} | V | \Phi_\alpha \rangle \quad (2.43)$$

If $E_\beta = E_\alpha$ (" on the energy shell " or simply " on - shell ") one has therefore

$$\langle \Phi_\beta | V | \Psi_\alpha^{(+)} \rangle = \langle \Psi_\beta^{(-)} | V | \Phi_\alpha \rangle \quad (2.44)$$

For $E_\beta \neq E_\alpha$ this is not the case. Comparing (2.44) with (1.40) and (1.41) shows the relation to the definition of the T - matrix and the scattering amplitude of potential scattering. However, in our present case we have not yet separated the center of mass motion and in particular we have not made any restriction to a two - body scattering problem. The equation (2.42) ... (2.44) play a central role in scattering theory.

In the next steps we will generalize all the considerations developed so far to the case where the Hamiltonian H_s can be split in various ways into " unperturbed " and " perturbation " part. We call the various possible splittings of H_s , determined by the number and nature of the particles participating in a collision process, " arrangement channels " (sometimes they are also called fragmentation channels). In each of these arrangement channels c one has therefore

$$H_s = H_c + V_c \quad (2.45)$$

Each of these channels c contains many ordinary channels differing bei all the possible quantum numbers γ of fragments in that channel. We denote the asymptotic " free "

states in a given arrangement channel c by $\Phi_{c,\gamma}$. These functions are eigenfunctions of H_c with energy $E_{c,\gamma}$ and represent only a subset of the full complete set of eigenstates of H_c .

$$H_c \Phi_{c,\gamma} = E_{c,\gamma} \Phi_{c,\gamma} \quad (2.46)$$

To distinguish the $\Phi_{c,\gamma}$ from the full set of eigenstates of H_c we call the latter Φ_c^c . In the following we shall use the following short hand notation. A general state γ in arrangement channel c will be denoted by $n \equiv \{c, \gamma\}$. The initial state α in the initial arrangement channel $c = i$ will be denoted by $a \equiv \{i, \alpha\}$ and similar for the final state β in the final channel $c = f$ by $b \equiv \{f, \beta\}$. Next we define for each arrangement channel c a time evolution operator U_c ($t_0 = 0$).

$$U_c(t, t') = e^{\frac{i}{\hbar} H_c t} e^{-\frac{i}{\hbar} H_c (t-t')} e^{-\frac{i}{\hbar} H_c t'} \quad (2.47)$$

All the equations (2.25) remain valid when applied to $U_c(t, \mp\infty)$ and $U_c(\mp\infty, t)$. We introduce now the so-called " channel projection operator " Λ_c , projecting on the " channel space " R_c , spanned by the state vectors $\Phi_{c,\gamma}$ belonging to the channel c .

$$\Lambda_c = \sum_\gamma | \Phi_{c,\gamma} \rangle \langle \Phi_{c,\gamma} | = \Lambda_c^\dagger \quad (2.48)$$

The Moeller operators for channel c are then defined as

$$\Omega_c^{(\pm)} = U_c(0, \mp\infty) \Lambda_c \quad (2.49)$$

$$\Omega_c^{(\pm)\dagger} = \Lambda_c U_c(\mp\infty, 0) \quad (2.50)$$

The wave function $\Psi_n^{(\pm)} \equiv \Psi_{c,\gamma}^{(\pm)}$ of the whole system at time $t = 0$, originating from the " free " wave functions $\Phi_n \equiv \Phi_{c,\gamma}$ are then defined as previously

$$|\Psi_n^{(\pm)}\rangle = \Omega_c^{(\pm)} |\Phi_n\rangle \quad (2.51)$$

In particular, we have in the initial and final channels (with $a \equiv i, \alpha$ and $b \equiv f, \beta$)

$$\left. \begin{aligned} |\Psi_a^{(\pm)}\rangle &= \Omega_i^{(\pm)} |\Phi_a\rangle \\ |\Psi_b^{(\pm)}\rangle &= \Omega_f^{(\pm)} |\Phi_b\rangle \end{aligned} \right\} \quad (2.52)$$

The Moeller operators can be determined as before by

$$\begin{aligned} \Omega_c^{(\pm)} &= \lim_{t \rightarrow \mp\infty} \{U_c(0, t) \Lambda_c\} = \lim_{t \rightarrow \mp\infty} \left\{ e^{\pm i H_s t} e^{-\frac{i}{\hbar} H_c t} \Lambda_c \right\} \\ &= \lim_{\epsilon \rightarrow 0^+} \left\{ \mp \epsilon \int_0^{\mp\infty} e^{\pm i \epsilon t} e^{\pm i H_s t} e^{-\frac{i}{\hbar} H_c t} dt \Lambda_c \right\} \end{aligned} \quad (2.53)$$

Performing the integral leads to

$$\begin{aligned} \Omega_c^{(\pm)} &= \lim_{\eta \rightarrow 0^+} \sum_{\gamma} \frac{\pm i \eta}{E_{c\gamma} - H_s \pm i \eta} |\Phi_{c\gamma}\rangle \langle \Phi_{c\gamma}| \\ &= \Lambda_c + \lim_{\eta \rightarrow 0^+} \sum_{\gamma} \frac{1}{E_{c\gamma} - H_s \pm i \eta} V_c |\Phi_{c\gamma}\rangle \langle \Phi_{c\gamma}| \end{aligned} \quad (2.54)$$

and we obtain for the full scattering solution as before

$$\begin{aligned} |\Psi_{c\gamma}^{(\pm)}\rangle &= \Omega_c^{(\pm)} |\Phi_{c\gamma}\rangle = \lim_{\eta \rightarrow 0^+} \frac{\pm i \eta}{E_{c\gamma} - H_s \pm i \eta} |\Phi_{c\gamma}\rangle \\ &= |\Phi_{c\gamma}\rangle + \lim_{\eta \rightarrow 0^+} \frac{1}{E_{c\gamma} - H_s \pm i \eta} V_c |\Phi_{c\gamma}\rangle \end{aligned} \quad (2.55)$$

We remind again that the set of function $\Phi_{c\gamma}$ is not complete and that functions related to different arrangement channels are not necessarily orthogonal ($\langle \Phi_{c'\gamma'} | \Phi_{c\gamma} \rangle \neq 0$). Therefore in general we will have $\Lambda_{c'} \Lambda_c \neq 0$. The solution of the full problem belong however to a complete orthogonal set:

$$\langle \Psi_{c'\gamma'}^{(\pm)} | \Psi_{c\gamma}^{(\pm)} \rangle = \delta_{c'c} \delta_{\gamma'\gamma} \quad (2.56)$$

In the following we shall derive a few properties of the Moeller operators :

$$\begin{aligned} \Omega_{c'}^{(\pm)\dagger} \Omega_c^{(\pm)} &= \sum_{\gamma'\gamma} |\Phi_{c'\gamma'}\rangle \langle \Phi_{c\gamma}| \lim_{\eta' \rightarrow 0^+} \frac{\mp i \eta'}{E_{c'\gamma'} - H_s \mp i \eta'} \lim_{\eta \rightarrow 0^+} \frac{\pm i \eta}{E_{c\gamma} - H_s \pm i \eta} |\Phi_{c\gamma}\rangle \langle \Phi_{c\gamma}| \\ &= \sum_{\gamma'\gamma} |\Phi_{c'\gamma'}\rangle \langle \Psi_{c'\gamma'}^{(\pm)} | \Psi_{c\gamma}^{(\pm)} \rangle \langle \Phi_{c\gamma}| = \delta_{c'c} \sum_{\gamma} |\Phi_{c\gamma}\rangle \langle \Phi_{c\gamma}| = \Lambda_c \delta_{c'c} \end{aligned}$$

$$\Omega_{c'}^{(\pm)\dagger} \Omega_c^{(\pm)} = \Lambda_c \delta_{c'c} \quad (2.57)$$

$$\Omega_c^{(\pm)} \Omega_c^{(\pm)\dagger} = \sum_{\gamma'\gamma} |\Psi_{c\gamma'}^{(\pm)}\rangle \langle \Phi_{c\gamma'} | \Phi_{c\gamma}\rangle \langle \Psi_{c\gamma}^{(\pm)}| = \sum_{\gamma} |\Psi_{c\gamma}^{(\pm)}\rangle \langle \Psi_{c\gamma}^{(\pm)}| \equiv Q_c \quad (2.58)$$

The operator Q_c projects into the space spanned by the vectors $|\Psi_{c\gamma}^{(\pm)}\rangle$. This is a subspace of the space spanned by all the eigenstates $|\Psi_n^{(\pm)}\rangle$ of the full Hamiltonian. These subspaces are all mutual orthogonal to each other. Summing over all subspaces we get the full space of the scattering solutions of H_s . Adding to it the space spanned by the genuine bound states (all particles of the system bound) we get the full space of eigenstates of H_s . Therefore

$$\sum_c \Omega_c^{(\pm)} \Omega_c^{(\pm)\dagger} = \sum_c Q_c = 1 - \Lambda_B \quad (2.59)$$

with

$$\Lambda_B = \sum_B |\Psi_B\rangle\langle\Psi_B|$$

Equation (2.59) is the completeness relation for the eigenstates of H_s . We also can easily derive the relation

$$\left. \begin{aligned} H_s \Omega_c^{(\pm)} &= \Omega_c^{(\pm)} H_c \\ \Omega_c^{(\pm)\dagger} \Lambda_B &= 0 \end{aligned} \right\} \quad (2.60)$$

At this stage we can generalize the concept of the S - operator to the case where there is more than one arrangement channel. Instead of one S - operator we have now a set of S - operators defined by

$$S_{c'e} = \Omega_{c'}^{(-)\dagger} \Omega_e^{(+)} \quad (2.61)$$

With particular reference to the initial and final channels we are interested in calculating the matrix elements of

$$S_{fi} = \Omega_f^{(-)\dagger} \Omega_i^{(+)} \quad (2.62)$$

taken between the asymptotic eigenstates $\Phi_a = \Phi_{i,\alpha}$ and $\Phi_b = \Phi_{f,\beta}$ of the corresponding channel Hamiltonian H_i and H_f :

$$\langle \Phi_b | S_{fi} | \Phi_a \rangle = \langle \Phi_{f\beta} | S_{fi} | \Phi_{i\alpha} \rangle \quad (2.63)$$

Using (2.52) we obtain

$$\langle \Phi_b | S_{fi} | \Phi_a \rangle = \langle \Phi_{f\beta} | \Omega_f^{(-)\dagger} \Omega_i^{(+)} | \Phi_{i\alpha} \rangle = \langle \Psi_b^{(-)} | \Psi_a^{(+)} \rangle \quad (2.64)$$

Next we shall prove the unitarity of the multi - channel S - matrix. As a first step we calculate

$$\begin{aligned} \sum_{c''} S_{c'e''} S_{cc''}^+ &= \sum_{c''} \Omega_{c'}^{(-)\dagger} \Omega_{c''}^{(+)} \Omega_{c''}^{(+)\dagger} \Omega_c^{(-)} = \sum_{c''} \Omega_{c'}^{(-)\dagger} Q_{c''} \Omega_c^{(-)} \\ &= \Omega_{c'}^{(-)\dagger} (1 - \Lambda_B) \Omega_c^{(-)} = \Omega_{c'}^{(-)\dagger} \Omega_c^{(-)} = \Lambda_c \delta_{c'e} \end{aligned} \quad (2.65)$$

and similar

$$\sum_{c''} S_{c''e}^+ S_{c'e} = \Lambda_c \delta_{c'e} \quad (2.66)$$

Taking the matrix element of (2.66) or (2.67) between the states $|\Phi_{c',\gamma'}\rangle$ and $|\Phi_{c,\gamma}\rangle$ we obtain for example

$$\langle \Phi_{c',\gamma'} | \sum_{c''} S_{c''e}^+ S_{c'e} | \Phi_{c,\gamma} \rangle = \delta_{c'e} \delta_{\gamma'\gamma} \quad (2.67)$$

On the other hand we may introduce into the left hand side a complete set of functions $|\Phi_{c'',\gamma''}\rangle$. Since both S - operators contain the projection operator $\Lambda_{c''}$, only the functions $|\Phi_{c'',\gamma''}\rangle$ remain and we have

$$\left. \begin{aligned} \sum_{c'',\gamma''} \langle \Phi_{c',\gamma'} | S_{c''e}^+ | \Phi_{c'',\gamma''} \rangle \langle \Phi_{c'',\gamma''} | S_{c'e} | \Phi_{c,\gamma} \rangle &= \delta_{c'e} \delta_{\gamma'\gamma} \\ \sum_{c'',\gamma''} \langle \Phi_{c',\gamma'} | S_{c'e} | \Phi_{c'',\gamma''} \rangle \langle \Phi_{c'',\gamma''} | S_{c'e}^+ | \Phi_{c,\gamma} \rangle &= \delta_{c'e} \delta_{\gamma'\gamma} \end{aligned} \right\} \quad (2.68)$$

(1) These relations are the unitary relations for the multi-channel S-matrix. We proceed in giving a more explicit representation of the S-matrix. We start from (2.55).

$$\begin{aligned}
 |\Psi_b^{(\pm)}\rangle &= \Omega_f^{(\pm)} |\Phi_b\rangle = |\Phi_b\rangle + \lim_{\eta \rightarrow 0^+} \frac{1}{E_b - H_s \pm i\eta} V_f |\Phi_b\rangle \\
 |\Psi_a^{(\pm)}\rangle &= \Omega_i^{(\pm)} |\Phi_a\rangle = |\Phi_a\rangle + \lim_{\eta \rightarrow 0^+} \frac{1}{E_a - H_s \pm i\eta} V_i |\Phi_a\rangle
 \end{aligned} \quad (2.69)$$

and get

$$\begin{aligned}
 \langle \Phi_b | S_{fi} | \Phi_a \rangle &= \langle \Psi_b^{(-)} | \Psi_a^{(+)} \rangle = [\langle \Psi_b^{(+)} | - (\langle \Psi_b^{(+)} | - \langle \Psi_b^{(-)} |)] | \Psi_a^{(+)} \rangle \\
 &= \delta_{ba} - [\langle \Psi_b^{(+)} | - \langle \Psi_b^{(-)} |] | \Psi_a^{(+)} \rangle
 \end{aligned}$$

$$\begin{aligned}
 \langle \Psi_b^{(+)} | - \langle \Psi_b^{(-)} | &= \langle \Phi_b | V_f \lim_{\eta \rightarrow 0^+} \left[\frac{1}{E_b - H_s - i\eta} - \frac{1}{E_b - H_s + i\eta} \right] \\
 &= \langle \Phi_b | V_f \lim_{\eta \rightarrow 0^+} \frac{2i\eta}{(E_b - H_s)^2 + \eta^2} = 2\pi i \langle \Phi_b | V_f \delta(E_b - H_s)
 \end{aligned}$$

and

$$\langle \Phi_b | S_{fi} | \Phi_a \rangle = \delta_{ba} - 2\pi i \delta(E_b - E_a) \langle \Phi_b | V_f | \Psi_a^{(+)} \rangle \quad (2.70)$$

or

$$\langle \Phi_b | S_{fi} | \Phi_a \rangle = \delta_{ba} - 2\pi i \delta(E_b - E_a) \langle \Psi_b^{(-)} | V_i | \Phi_a \rangle \quad (2.71)$$

On the energy shell ($E_b = E_a$) we have therefore

$$\langle \Phi_b | V_f | \Psi_a^{(+)} \rangle = \langle \Psi_b^{(-)} | V_i | \Phi_a \rangle \quad (2.72)$$

In case we have only one arrangement channel ($V_f = V_i = V$) we obtain the result given in (2.44).

2.3 The Lippmann - Schwinger equation, T - matrix, propagators

Propagators (Green functions, Green operators):

Looking back at equation (2.69) we see that the full solution of a scattering problem was obtained using a Green operator (or Green function, or propagator):

$$G^{(\pm)}(E) = \lim_{\eta \rightarrow 0^+} \frac{1}{E - H_s \pm i\eta} \equiv \frac{1}{E - H_s \pm i\eta} \quad (2.73)$$

We suppress from now on the symbol of taking the limiting value of this expression, implicitly it is however always implied. Propagators for other Hamiltonians such as H_i , H_f , H_c , etc. can be defined in a similar way.

$$G_c^{(\pm)}(E) = \lim_{\eta \rightarrow 0^+} \frac{1}{E - H_c \pm i\eta} \equiv \frac{1}{E - H_c \pm i\eta} \quad (2.74)$$

In the special case of free particles we have

$$G_0^{(\pm)}(E) = \lim_{\eta \rightarrow 0^+} \frac{1}{E - H_0 \pm i\eta} \equiv \frac{1}{E - H_0 \pm i\eta} \quad (2.75)$$

where H_0 consists only of the sum of the kinetic energies of the particles or in the relativistic case of the sum of the kinetic energies and masses. Since the operators H_0 , H_c , H_s are all hermitian operators, we have

$$G^{(\pm)(+)}(E) = G^{(\mp)} \quad \text{etc.}$$

Some useful combinations between the various propagators can be constructed by using the following operator identities:

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1} \quad (2.76)$$

$$A^{-1} - B^{-1} = B^{-1}(B - A)A^{-1} \quad (2.77)$$

Choosing for example $A = E - H_s \pm i\eta$ and $B = E - H_c \pm i\eta$ ($H_s = H_c + V_c$) we derive

$$G^{(\pm)}(E) = G_c^{(\pm)} + G^{(\pm)}(E)V_c G_c^{(\pm)}(E) \quad (2.78)$$

$$G^{(\pm)}(E) = G_c^{(\pm)} + G_c^{(\pm)}(E)V_c G^{(\pm)}(E) \quad (2.79)$$

In case there is only 1 arrangement channel(we may call this a direct process, $H_s = H_d + V_d$) we have

$$G^{(\pm)}(E) = G_d^{(\pm)}(E) + G_d^{(\pm)}(E)V_d G^{(\pm)}(E) = G_d^{(\pm)}(E) + G^{(\pm)}(E)V_d G_d^{(\pm)}(E) \quad (2.80)$$

If we choose $A = E - H_c \pm i\eta$ and $B = E - H_0 \pm i\eta$ we obtain

$$G_c^{(\pm)}(E) = G_0^{(\pm)} + G_0^{(\pm)}(E)(V - V_c)G_c^{(\pm)}(E) = G_0^{(\pm)}(E) + G_c^{(\pm)}(E)(V - V_c)G_0^{(\pm)}(E) \quad (2.81)$$

Lippmann - Schwinger equation, T - matrix, S - matrix

Using (2.55) and (2.79) we obtain for the scattering solution in channel c:

$$|\Psi_{c\gamma}^{(\pm)}\rangle = |\Phi_{c\gamma}\rangle + G^{(\pm)}V_c |\Phi_{c\gamma}\rangle = (1 + G^{(\pm)}V_c) |\Phi_{c\gamma}\rangle$$

$$= (1 + (G_c^{(\pm)} + G_c^{(\pm)}V_c G)V_c) |\Phi_{c\gamma}\rangle = (1 + G_c^{(\pm)}V_c(1 + G^{(\pm)}V_c)) |\Phi_{c\gamma}\rangle$$

or

$$|\Psi_{c\gamma}^{(\pm)}\rangle = |\Phi_{c\gamma}\rangle + G_c^{(\pm)}(E)V_c |\Psi_{c\gamma}^{(\pm)}\rangle = |\Phi_{c\gamma}\rangle + \frac{1}{E - H_c \pm i\eta} V_c |\Psi_{c\gamma}^{(\pm)}\rangle. \quad (2.82)$$

This integral equation for the full scattering solution in channel c is called the Lippmann - Schwinger equation. It is completely analogous to (1.35), the solution of the simple problem of potential scattering. The main part of the S-matrix $\langle \Phi_b | S_{fi} | \Phi_a \rangle$ derived in eqs. (2.70) - (2.72), were the matrix elements $\langle \Phi_b | V_f | \Psi_a^{(+)} \rangle$ and $\langle \Psi_b^{(-)} | V_i | \Phi_a \rangle$. We may write

$$\langle \Phi_b | V_f | \Psi_a^{(+)} \rangle = \langle \Phi_b | V_f (|\Phi_a\rangle + G^{(+)}V_i |\Phi_a\rangle) = \langle \Phi_b | V_f + V_f G^{(+)}V_i | \Phi_a \rangle \quad (2.83)$$

$$\langle \Psi_b^{(-)} | V_i | \Phi_a \rangle = \langle \Phi_b | (1 + V_f G^{(-)+})V_i | \Phi_a \rangle = \langle \Phi_b | V_i + V_f G^{(+)}V_i | \Phi_a \rangle \quad (2.84)$$

This allows one to define transition operators T_{fi} and \bar{T}_{fi} (sometimes called Lovelace operators):

$$T_{fi}(E) = V_i + V_f G^{(\pm)}(E)V_i$$

$$\bar{T}_{fi}(E) = V_f + V_f G^{(\pm)}(E) V_i \quad (2.85)$$

Both T - operators give on - shell ($E = E_a = E_b$) the same value for the matrix element.

$$\begin{aligned} \langle \Phi_b | V_f | \Psi_a^{(+)} \rangle &= \langle \Phi_b | \bar{T}_{fi}(E) | \Phi_a \rangle = \langle \Psi_b^{(-)} | V_i | \Phi_a \rangle \\ &= \langle \Phi_b | T_{fi}(E) | \Phi_a \rangle \quad \text{for } E = E_b = E_a \end{aligned} \quad (2.86)$$

To prove this one has to show that on - shell $\langle \Phi_b | V_f | \Phi_a \rangle = \langle \Phi_b | V_i | \Phi_a \rangle$ is valid (using the hermiticity of the operators H_s and V_f). With the aid of the operator relation for the propagators one can derive integral equations satisfied by the Lovelace operators T_{fi} and \bar{T}_{fi} :

$$\begin{aligned} a) \quad T_{fi} &= V_i + V_f G^{(+)} V_i = V_i + V_f (G_f^{(+)} + G_f^{(+)} V_f G^{(+)} V_i) \\ &= V_i + V_f G_f^{(+)} V_i + V_f G_f^{(+)} V_f G^{(+)} V_i = V_i + V_f G_f^{(+)} (V_i + V_f G_f^{(+)} V_i) \\ &= V_i + V_f G_f^{(+)} T_{fi} \end{aligned}$$

$$\begin{aligned} b) \quad \bar{T}_{fi} &= V_f + V_f G^{(+)} V_i = V_f + V_f (G_i^{(+)} + G^{(+)} V_i G_i^{(+)} V_i) = V_f + V_f G_i^{(+)} V_i + V_f G^{(+)} V_i G_i^{(+)} V_i \\ &= V_f + V_f G_i^{(+)} V_i + (\bar{T}_{fi} - V_f) G_i^{(+)} V_i = V_f + \bar{T}_{fi} G_i^{(+)} V_i \end{aligned}$$

$$T_{fi}(E) = V_i + V_f G_f^{(+)}(E) T_{fi}(E) \quad (2.87)$$

$$\bar{T}_{fi}(E) = V_f + \bar{T}_{fi}(E) G_i^{(+)}(E) V_i \quad (2.88)$$

The S - matrix is then given by

$$\langle \Phi_b | S_{fi} | \Phi_a \rangle = \delta_{ba} - 2\pi i \delta(E_a - E_b) \langle \Phi_b | T_{fi}(E_b) | \Phi_a \rangle$$

$$= \delta_{ba} - 2\pi i \delta(E_a - E_b) \langle \Phi_b | \bar{T}_{fi}(E_a) | \Phi_a \rangle \quad (2.89)$$

2.4 Transition probabilities, decay probabilities, cross sections

According to our discussion after equ.(2.40) we can define the total transition probability for a genuine transition as

$$W_{ba} = |\langle \Phi_b | S_{fi} | \Phi_a \rangle - \delta_{ba}|^2 = (2\pi)^2 (\delta(E_b - E_a))^2 |\langle \Phi_b | T_{fi} | \Phi_a \rangle|^2 \quad (2.90)$$

We may use the following definition of the δ - function:

$$\delta(E_b - E_a) = \lim_{\tau \rightarrow \infty} \left\{ \frac{\tau}{2\pi\hbar} \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{+\frac{\tau}{2}} e^{i(E_b - E_a)t} dt \right\} = \lim_{\tau \rightarrow \infty} \left\{ \frac{\tau}{2\pi\hbar} \delta_{E_b, E_a} \right\}$$

in (2.90) we obtain

$$W_{ba} = (2\pi)^2 \delta(E_b - E_a) \lim_{\tau \rightarrow \infty} \left\{ \frac{\tau}{2\pi\hbar} \delta_{E_b, E_a} \right\} |\langle \Phi_b | T_{fi} | \Phi_a \rangle|^2 \quad (2.90')$$

The basic quantity to calculate decay probabilities and cross sections is the transition probability per unit time which can be defined as follows:

$$w_{ba} = \lim_{\tau \rightarrow \infty} \frac{W_{ba}}{\tau} = \frac{2\pi}{\hbar} \delta(E_b - E_a) |\langle \Phi_b | T_{fi} | \Phi_a \rangle|^2 \quad (2.91)$$

where we have made use of the fact $\delta(E_b - E_a) \delta_{E_b, E_a} = \delta(E_b - E_a)$. The dimension of w_{ba} is $[w_{ba}] = \text{sec}^{-1}$.

Whenever the interaction is translationaly invariant the transition matrix T_{fi} (T_{fi}) contains an overall momentum conserving δ - function which can be extracted from the

T - matrix element. This is best shown in the coordinate representation. We use as an example a system which in the initial channel consist of n_i composite particles interacting with each other. In the final channel we may have a different number n_f of composite particles. The asymptotic wave functions for the initial and final channel can be written in the following form:

$$\Phi_a = \prod_{j=1}^{n_i} \left(\frac{1}{\sqrt{V_{ol}}} e^{i\vec{k}_j^{(i)} \vec{R}_j^{(i)}} \phi_{\alpha_j}^{int} \right) \frac{1}{\sqrt{J_i}} \quad (2.92)$$

$$\Phi_b = \prod_{k=1}^{n_f} \left(\frac{1}{\sqrt{V_{ol}}} e^{i\vec{k}_k^{(f)} \vec{R}_k^{(f)}} \phi_{\beta_k}^{int} \right) \frac{1}{\sqrt{J_f}} \quad (2.92')$$

$\phi_{\alpha_j}^{int}$ and $\phi_{\beta_k}^{int}$ are the normalized internal wave functions of the composite particle j and k in the initial and final channel. $\vec{R}_j^{(i)}$ describes the center of mass of particle j and $\vec{k}_j^{(i)}$ is its center of mass momentum in the initial channel i and $\vec{R}_k^{(f)}$ and $\vec{k}_k^{(f)}$ are the corresponding quantities for particle k in the final channel f . The quantities J_i and J_f are the corresponding Jacobians when going from the usual Cartesian coordinates to internal and center of mass coordinates. The sets of the center of mass coordinates $\vec{R}_j^{(i)}$ and $\vec{R}_k^{(f)}$ may for example again be transformed into sets of Jacobi coordinates. Among this Jacobi coordinates is one, which is the center of mass coordinate of the whole system. Let $\vec{\rho}_j^{(i)}$ and $\vec{\rho}_k^{(f)}$ be those Jacobi coordinates. We have then

$$\sum_{j=1}^{n_i} \vec{k}_j^{(i)} \vec{R}_j^{(i)} = \sum_{j=1}^{n_i} \vec{k}_j^{(i)} \vec{R}_{CM} + \sum_{j=1}^{n_i-1} \vec{k}_j^{(i)'} \vec{\rho}_j^{(i)} = \vec{K}_{CM}^{(i)} \vec{R}_{CM} + \sum_{j=1}^{n_i-1} \vec{k}_j^{(i)'} \vec{\rho}_j^{(i)}$$

and a similar expression for the final states. The T-matrix element can now be written as follows:

$$\langle \Phi_b | T_{fi} | \Phi_a \rangle = \int d\tau \Phi_b^* T_{fi} \Phi_a$$

$$= \int d\tau \prod_{k=1}^{n_f} \left(\frac{e^{-i\vec{k}_k^{(f)} \vec{R}_k^{(f)}}}{\sqrt{V_{ol}}} \phi_{\beta_k}^{int*} \right) T_{fi} \prod_{j=1}^{n_i} \left(\frac{e^{i\vec{k}_j^{(i)} \vec{R}_j^{(i)}}}{\sqrt{V_{ol}}} \phi_{\alpha_j}^{int} \right) \frac{1}{\sqrt{J_f J_i}}$$

$$= \int d^3 \vec{R}_{cm} \frac{e^{i\vec{R}_{cm}(\vec{K}_{cm}^{(i)} - \vec{K}_{cm}^{(f)})}}{V_{ol}} \int d\tau' \frac{J}{\sqrt{J_f J_i}} \prod_{k=1}^{n_f-1} \left(\frac{e^{-i\vec{k}_k^{(f)'} \vec{\rho}_k^{(f)'}}}{\sqrt{V_{ol}}} \phi_{\beta_k}^{int*} \right) T_{fi} \prod_{j=1}^{n_i-1} \left(\frac{e^{i\vec{k}_j^{(i)'} \vec{\rho}_j^{(i)'}}}{\sqrt{V_{ol}}} \phi_{\alpha_j}^{int} \right)$$

J is the Jacobian of the transformation from the Cartesian coordinates to the total center of mass \vec{R}_{cm} and some other set of remaining coordinates which needs not to be specified for the present discussion. We may rewrite the expression for the T - matrix element once more to prepare it for a continuum normalisation of the wave functions.

$$\langle \Phi_b | T_{fi} | \Phi_a \rangle = \frac{1}{V_{ol}} \int d^3 \vec{R}_{cm} e^{i\vec{R}_{cm}(\vec{K}_{cm}^{(i)} - \vec{K}_{cm}^{(f)})} \prod_{j=1}^{n_i+n_f-2} \left(\frac{(2\pi)^{\frac{3}{2}}}{\sqrt{V_{ol}}} \right) \int d\tau' Int$$

$$Int = \frac{J}{\sqrt{J_f J_i}} \prod_{k=1}^{n_f-1} \left(\frac{e^{-i\vec{k}_k^{(f)'} \vec{\rho}_k^{(f)'}}}{(2\pi)^{\frac{3}{2}}} \phi_{\beta_k}^{int*} \right) T_{fi} \prod_{j=1}^{n_i-1} \left(\frac{e^{i\vec{k}_j^{(i)'} \vec{\rho}_j^{(i)'}}}{(2\pi)^{\frac{3}{2}}} \phi_{\alpha_j}^{int} \right)$$

If the interaction is translationaly invariant, then the inner integral $\int d\tau' \dots$ does no longer depend on \vec{R}_{cm} and the outer integral can be performed leading to a factor $\delta_{\vec{K}_{cm}^{(f)}, \vec{K}_{cm}^{(i)}}$. The final result is then

$$\langle \Phi_b | T_{fi} | \Phi_a \rangle = \delta_{\vec{K}_{cm}^{(f)}, \vec{K}_{cm}^{(i)}} T_{b,a} \left(\frac{(2\pi)^3}{V_{ol}} \right)^{\frac{n_i+n_f-2}{2}} \quad (2.93)$$

$T_{b,a}$ is sometimes called the reduced T - matrix element. According to its definition, $T_{b,a}$ has the dimension $\text{MeV}(f_m^{\frac{3}{2}})^{n_i+n_f-2}$. Inserting (2.93) into (2.91) one gets for the transition probability per unit time the expression

$$w_{ba} = \frac{2\pi}{\hbar} \delta(E_b - E_a) \delta_{\vec{K}_{cm}^f, \vec{K}_{cm}^i} |T_{b,a}|^2 \left(\frac{(2\pi)^3}{V_{ol}} \right)^{n_i + n_f - 2} \quad (2.94)$$

One may also consider the differential transition probability to measure the final reaction products with momenta lying in the interval $(\vec{k}_j, \vec{k}_j + d\vec{k}_j)$. In this case one has to multiply (2.94) with the corresponding probability, i.e. with the available phase space $\prod_{j=1}^{n_f} \left(\frac{V_{ol}}{(2\pi)^3} \right) d^3 \vec{k}_j^f$. The result is:

$$dw_{ba} = w_{ba} \prod_{j=1}^{n_f} \left(\frac{V_{ol}}{(2\pi)^3} \right) d^3 \vec{k}_j^f = \frac{2\pi}{\hbar} \delta(E_b - E_a) \frac{V_{ol} \delta_{\vec{K}_{cm}^f, \vec{K}_{cm}^i}}{(2\pi)^3} \left(\frac{(2\pi)^3}{V_{ol}} \right)^{n_i - 1} \prod_{j=1}^{n_f} d^3 \vec{k}_j^f |T_{b,a}|^2 \quad (2.94')$$

We consider next the two most important cases with $n_i = 1$ (decay probability) and $n_i = 2$ (reaction between to composite particles).

a) Decay probability.

We obtain for $n_i = 1$:

$$dw_{ba} = \frac{2\pi}{\hbar} \delta(E_b - E_a) \frac{V_{ol} \delta_{\vec{K}_{cm}^f, \vec{K}_{cm}^i}}{(2\pi)^3} |T_{b,a}|^2 \prod_{j=1}^{n_f} d^3 \vec{k}_j^f$$

If we go now to the continuum normalisation we have to take the limiting value for $V_{ol} \rightarrow \infty$. With

$$\lim_{V_{ol} \rightarrow \infty} \left[\frac{V_{ol} \delta_{\vec{K}_{cm}^f, \vec{K}_{cm}^i}}{(2\pi)^3} \right] = \delta^{(3)}(\vec{K}_{cm}^f - \vec{K}_{cm}^i)$$

we obtain

$$dw_{ba} = \frac{2\pi}{\hbar} \delta(E_b - E_a) \delta^{(3)}(\vec{K}_{cm}^f - \vec{K}_{cm}^i) |T_{b,a}|^2 \prod_{j=1}^{n_f} d^3 \vec{k}_j^f \quad (2.95)$$

With $E = \hbar c \epsilon = \hbar c k^0$ we can also write

$$dw_{ba} = \frac{2\pi c}{(\hbar c)^2} \delta^{(4)}(K_{cm}^f - K_{cm}^i) |T_{b,a}|^2 \prod_{j=1}^{n_f} d^3 \vec{k}_j^f \quad (2.95')$$

We see that this result is again independent from the normalisation volume. The transition rate for the decay $a \rightarrow b$ is obtained from (2.95') by integration over $\prod_j d^3 \vec{k}_j^f$ and the total decay - rate is obtained by summing over all the final states.

$$w_{total} = \sum_b w_{ba} = \sum_b \int dw_{ba}$$

The decay width's are given by

$$\Gamma_{tot} = \hbar w_{total} = \sum_b \hbar w_{ba} = \sum_b \Gamma_b$$

b) Cross section of reactions

We consider a reaction with $n_i = 2$ composite particles in the entrance channel. The current density of the incoming particles is $j_{in} = \frac{v_{rel}}{V_{ol}}$ (see equ.(1.17)) and the differential cross section is then defined by

$$d\sigma_{ba} = \frac{dw_{ba}}{j_{in}} = \frac{2\pi}{\hbar} \delta(E_b - E_a) \frac{V_{ol} \delta_{\vec{K}_{cm}^f, \vec{K}_{cm}^i}}{(2\pi)^3} \frac{V_{ol}}{v_{rel}} \frac{(2\pi)^3}{V_{ol}} |T_{b,a}|^2 \prod_{j=1}^{n_f} d^3 \vec{k}_j^f$$

$$= \frac{(2\pi)^4}{\hbar v_{rel}} \delta(E_b - E_a) \frac{V_{ol} \delta(\vec{K}_{cm}^f, \vec{K}_{cm}^i)}{(2\pi)^3} |T_{ba}|^2 \prod_{j=1}^{n_f} d^3 \vec{k}_j^f$$

where we have used (2.94'). Going over to the continuum normalisation one obtains

$$d\sigma_{ba} = \frac{(2\pi)^4}{\hbar v_{rel}} \delta(E_b - E_a) \delta^{(3)}(\vec{K}_{cm}^f - \vec{K}_{cm}^i) |T_{ba}|^2 \prod_{j=1}^{n_f} d^3 \vec{k}_j^f \quad (2.96)$$

or

$$d\sigma_{ba} = \frac{(2\pi)^4}{(\hbar c)^2} \delta^{(4)}(K_{cm}^f - K_{cm}^i) \frac{c}{v_{rel}} |T_{ba}|^2 \prod_{j=1}^{n_f} d^3 \vec{k}_j^f \quad (2.96')$$

We note that the wave functions used in T_{ba} have been normalized in a non covariant way. The volume elements $d^3 \vec{x}$ and $d^3 \vec{k}$ are not Lorentz - invariant and therefore also T_{ba} (or $|T_{ba}|^2$) is not invariant under Lorentz transformations. On the other hand one knows that $d\sigma_{ba}$ is necessarily an invariant quantity. It will therefore be very useful to write the expressions for the decay probability and the cross section in a manifest covariant form, i.e. to express all expressions through relativistic invariants. First we note that the four - dimensional δ - function is invariant under Lorentz transformations. Next we note that for any Lorentz transformation Λ^μ_ν we have

$$k'^\mu = \sum_\nu \Lambda^\mu_\nu k^\nu$$

and

$$\left| \frac{\partial(k'^0, k'^1, k'^2, k'^3)}{\partial(k^0, k^1, k^2, k^3)} \right| = |\det \Lambda| = 1 \quad (2.97)$$

Therefore also $d^4 k$ is an invariant quantity.

$$d^4 k' = \left| \frac{\partial(k'^0, k'^1, k'^2, k'^3)}{\partial(k^0, k^1, k^2, k^3)} \right| d^4 k = d^4 k \quad (2.98)$$

The transformation property of $d^3 \vec{k}$ we show best by using a special Lorentz transformation along the z - axis.

$$\Lambda^\mu_\nu = \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{-\beta}{\sqrt{1-\beta^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-\beta}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{1}{\sqrt{1-\beta^2}} \end{pmatrix} \quad \beta = \frac{v}{c} \quad (2.99)$$

This gives us

$$\left. \begin{aligned} k'^0 &= \frac{k^0 - \beta k^3}{\sqrt{1-\beta^2}}, & k'^1 &= k^1, \\ k'^2 &= k^2, & k'^3 &= \frac{-\beta k^0 + k^3}{\sqrt{1-\beta^2}} = \frac{-\beta \sqrt{k^2 + \mu^2} + k^3}{\sqrt{1-\beta^2}} \end{aligned} \right\} \quad (2.100)$$

From these equations we immediately get the Jacobian for the transformed momentum 3 - vectors:

$$\left| \frac{\partial(k'^1, k'^2, k'^3)}{\partial(k^1, k^2, k^3)} \right| = \frac{k^0 - \beta k^3}{k^0 \sqrt{1-\beta^2}} = \frac{k'^0}{k^0} \quad (2.101)$$

and consequently one has

$$d^3 \vec{k}' = \left| \frac{\partial(k'^1, k'^2, k'^3)}{\partial(k^1, k^2, k^3)} \right| d^3 \vec{k} = \frac{k'^0}{k^0} d^3 \vec{k} \quad (2.102)$$

Therefore

$$\frac{d^3 \vec{k}'}{k'^0} = \frac{d^3 \vec{k}}{k^0} \quad (2.103)$$

is an invariant quantity. If we integrate equ.(2.90') for the total transition probability W_{ba} over a small momentum range in channel a and b and use (2.103) we obtain

$$W_{ba} = \sum_{\alpha, \beta} \int (2\pi)^2 \delta(E_b - E_a) \cdot$$

$$\begin{aligned} & \lim_{T \rightarrow \infty} \left(\frac{T}{2\pi\hbar} \right) \delta_{\mathcal{R}_{im}^i, \mathcal{R}_{im}^f} |T_{ba}|^2 \left(\frac{(2\pi)^3}{V_{ol}} \right)^{n_i+n_f-2} \prod_{j=1}^{n_f} \left(\frac{V_{ol}}{(2\pi)^3} d^3 \vec{k}_j^f \right) \prod_{k=1}^{n_i} \left(\frac{V_{ol}}{(2\pi)^3} d^3 \vec{k}_k^i \right) \\ & = \lim_{V_{ol} T \rightarrow \infty} \left(\frac{V_{ol} c T}{(2\pi)^4 \hbar c} \right) \frac{(2\pi)^2}{\hbar c} \sum_{\alpha, \beta} \int \delta^{(4)}(K_{cm}^f - K_{cm}^i) \left| \sqrt{\prod_{j=1}^{n_f} \epsilon_j^f} T_{ba} \sqrt{\prod_{k=1}^{n_i} \epsilon_k^i} \right|^2 \prod_{j=1}^{n_f} \frac{d^3 \vec{k}_j^f}{\epsilon_j^f} \prod_{k=1}^{n_i} \frac{d^3 \vec{k}_k^i}{\epsilon_k^i} \end{aligned} \quad (2.104)$$

Since the total transition probability is by definition a Lorentz invariant quantity, also the only remaining quantity

$$M_{ba} \equiv \sqrt{\prod_{j=1}^{n_f} \epsilon_j^f} T_{ba} \sqrt{\prod_{k=1}^{n_i} \epsilon_k^i} \quad (2.105)$$

must be Lorentz invariant. An analogous investigation may be applied to the S - matrix element. M_{ba} is called the invariant amplitude. Using this form of the invariant amplitude and the invariant phase space factors in (2.96') we can rewrite the expression for cross section $d\sigma_{ba}$

$$d\sigma_{ba} = \frac{(2\pi)^4}{(\hbar c)^2} \delta^{(4)}(K_{cm}^f - K_{cm}^i) \frac{c}{v_{rel} \epsilon_1^i \epsilon_2^i} |M_{ba}|^2 \prod_{j=1}^{n_f} \frac{d^3 \vec{k}_j^f}{\epsilon_j^f} \quad (2.106)$$

The quantities $\epsilon_1^i, \epsilon_2^i$ and ϵ_j^f are on - shell quantities. They are the zero - components of the momentum four vectors of the corresponding particles. In order to bring (2.106) into a fully manifest covariant form we still have to show that the factor $\frac{c}{v_{rel} \epsilon_1^i \epsilon_2^i}$ can be represented by invariant quantities. Since $d\sigma_{ba}$ is invariant, we can be sure that at least in the laboratory - system (L - system) we will be able to express this factor through

invariants. We note that in the L - system one has $\vec{v}_2 = 0$, $\vec{v}_{rel} = \vec{v}_1$, $\epsilon_2^i = \mu_2^i$ and $\epsilon_1^i = \sqrt{\vec{k}_1^{i2} + \mu_1^{i2}} = \frac{\mu_1^i}{\sqrt{1-\beta_1^2}}$ and $\vec{k}_1^i = \frac{\vec{\beta}_1^i \mu_1^i}{\sqrt{1-\mu_1^{i2}}}$. Therefore we have in the L - system:

$$\frac{v_{rel}}{c} = \frac{|\vec{v}_1|}{c} = |\vec{\beta}_1^i| = \frac{|\vec{k}_1^i|}{\epsilon_1^i}$$

and

$$\frac{c}{v_{rel} \epsilon_1^i \epsilon_2^i} = \frac{\epsilon_1^i}{|\vec{k}_1^i| \epsilon_1^i \epsilon_2^i} = \frac{1}{|\vec{k}_1^i| \epsilon_2^i} = \frac{1}{|\vec{k}_1^i| \mu_2^{(i)}} \quad (Lab.s) \quad (2.107)$$

Consider next the relativistic invariant s (it is one of the so-called Mandelstam - variables):

$$s \equiv (k_1^i + k_2^i)^2 = k_1^{i2} + k_2^{i2} + 2k_1^i k_2^i = \mu_1^{i2} + \mu_2^{i2} + 2\epsilon_1^i \epsilon_2^i - 2\vec{k}_1^i \vec{k}_2^i = (\epsilon_1^{i*} + \epsilon_2^{i*})^2 \quad (2.108)$$

We characterize quantities referring to the center of mass system (CM - system) by a star. In the Lab. - system we have

$$s = \mu_1^{i2} + \mu_2^{i2} + 2\epsilon_1^i \mu_2^i$$

Using the three invariants k_1^{i2}, k_2^{i2} and s one can construct another invariant $\lambda(s, \mu_1^{i2}, \mu_2^{i2})$. The so-called triangular function λ is defined as $\lambda(x^2, y^2, z^2) \equiv (x^2 - y^2 - z^2)^2 - 4y^2 z^2$

$$\lambda(s, \mu_1^{i2}, \mu_2^{i2}) = (s - \mu_1^{i2} - \mu_2^{i2})^2 - 4\mu_1^{i2} \mu_2^{i2} = 4[(k_1^i k_2^i)^2 - \mu_1^{i2} \mu_2^{i2}] \quad (2.109)$$

In the Lab. - system one has

$$\lambda_{Lab.}(s, \mu_1^{i2}, \mu_2^{i2}) = 4 |\vec{k}_1^i|^2 \mu_2^{i2}$$

and therefore (2.107) can be rewritten as

$$\frac{c}{v_{rel} \epsilon_1^i \epsilon_2^i} = \frac{1}{|\vec{k}_1^i| \mu_2^i} = \frac{2}{\sqrt{\lambda(s, \mu_1^{i2}, \mu_2^{i2})}} = \frac{1}{\sqrt{(k_1^i k_2^i)^2 - \mu_1^{i2} \mu_2^{i2}}} \quad (2.110)$$

Next we shall make the same considerations for the CM - system. We start from the quantity $|\vec{v}_{rel}^*| \epsilon_1^{i*} \epsilon_2^{i*}$:

$$|\vec{v}_{rel}^*| \epsilon_1^{i*} \epsilon_2^{i*} = |\vec{v}_1^* - \vec{v}_2^*| \epsilon_1^{i*} \epsilon_2^{i*} = c \epsilon_1^{i*} \epsilon_2^{i*} \left| \frac{\vec{k}_1^{i*}}{\epsilon_1^{i*}} - \frac{\vec{k}_2^{i*}}{\epsilon_2^{i*}} \right| = c \epsilon_1^{i*} \epsilon_2^{i*} |\vec{\kappa}_i| \frac{\epsilon_1^{i*} + \epsilon_2^{i*}}{\epsilon_1^{i*} \epsilon_2^{i*}} = c |\vec{\kappa}_i| \sqrt{s}$$

In the above expression we have used $\vec{k}_1^{i*} = -\vec{k}_2^{i*} \equiv \vec{\kappa}_i$. We have therefore in the CM - system

$$\frac{c}{|\vec{v}_{rel}^*| \epsilon_1^{i*} \epsilon_2^{i*}} = \frac{1}{|\vec{\kappa}_i| \sqrt{s}} \quad (2.111)$$

If we evaluate $\lambda(s, \mu_1^{i2}, \mu_2^{i2})$ in the CM - system we obtain

$$\begin{aligned} \lambda_{cm}(c, \mu_1^{i2}, \mu_2^{i2}) &= 4[(k_1^{i*} k_2^{i*})^2 - \mu_1^{i2} \mu_2^{i2}] = 4[(\epsilon_1^{i*} \epsilon_2^{i*} - \vec{k}_1^{i*} \vec{k}_2^{i*})^2 - \mu_1^{i2} \mu_2^{i2}] \\ &= 4[(\vec{\kappa}_i^2 + \mu_1^{i2})(\vec{\kappa}_i^2 + \mu_2^{i2}) + 2\vec{\kappa}_i^2 \epsilon_1^{i*} \epsilon_2^{i*} + (\vec{\kappa}_i^2)^2 - \mu_1^{i2} \mu_2^{i2}] \\ &= 4\vec{\kappa}_i^2 [2\vec{\kappa}_i^2 + 2\epsilon_1^{i*} \epsilon_2^{i*} + \mu_1^{i2} + \mu_2^{i2}] = 4\vec{\kappa}_i^2 s \end{aligned}$$

Therefore we have also in the CM - system

$$\lambda_{cm}(c, \mu_1^{i2}, \mu_2^{i2}) = 4[(k_1^{i*} k_2^{i*})^2 - \mu_1^{i2} \mu_2^{i2}] = 4\vec{\kappa}_i^2 s \quad (2.112)$$

or

$$\frac{c}{|\vec{v}_{rel}^*| \epsilon_1^{i*} \epsilon_2^{i*}} = \frac{1}{\sqrt{(k_1^{i*} k_2^{i*})^2 - \mu_1^{i2} \mu_2^{i2}}} \quad (2.113)$$

exactly as for the Lab. - system. If we replace in (2.106) $\frac{c}{v_{rel} \epsilon_1^i \epsilon_2^i}$ by the so called relativistic flux factor $\frac{1}{\sqrt{(k_1^{i*} k_2^{i*})^2 - \mu_1^{i2} \mu_2^{i2}}}$, we have achieved our goal to write the expression for the cross section in manifest covariant way.

$$d\sigma_{ba} = \frac{(2\pi)^4}{(\hbar c)^2} \delta^{(4)}(K_{cm}^f - K_{cm}^i) \frac{|M_{ba}|^2}{\sqrt{(k_1^{i*} k_2^{i*})^2 - \mu_1^{i2} \mu_2^{i2}}} \prod_{j=1}^{n_f} \frac{d^3 \vec{k}_j^f}{\epsilon_j^f} \quad (2.114)$$

In this form we can calculate $d\sigma_{ba}$ in any arbitrary Lorentz frame. One should note that for arbitrary \vec{v}_1 and \vec{v}_2 in a general Lorentz frame the expression $\frac{c}{v_{rel} \epsilon_1^i \epsilon_2^i}$ is certainly not a relativistic invariant.

3 Multiple scattering theory and optical potentials

This chapter is devoted to the investigation of the scattering process of a projectile on a system of A bound nucleons, as for example scattering of hadrons on nuclei. We shall confine this investigation to the treatment of elastic scattering only. The target nucleons in the initial and final state will therefore be described by the fully antisymmetrised ground state wave function Φ_A^0 . All considerations will be based on the use of potential scattering theory, a fact that limits the validity to low and intermediate energies up to a few hundreds of MeV. The interaction of the projectile with the target nucleus is then given by

$$V(\vec{x}) = \sum_{j=1}^A V_j(\vec{x} - \vec{x}_j) \quad (3.1)$$

Let H_A be the Hamiltonian of the target nucleons and T_{pr}^{kin} be the kinetic energy operator of the projectile. We look for a scattering solution $\Psi_E^{(+)}$ of

$$H\Psi_E^{(+)} = E\Psi_E^{(+)} \quad (3.2)$$

where

$$H = H_A + T_{pr}^{kin} + V(\vec{x}) = H_0 + V(\vec{x}) \quad (3.3)$$

$$H_A = \sum_{j=1}^A T_j^{kin} + \frac{1}{2} \sum_{j \neq i} V(ij) \quad (3.4)$$

The formal solution of this problem is given by the Lippmann - Schwinger equation

$$\Psi_E^{(+)} = \Phi_E + G_0^{(+)}(E)V\Psi_E^{(+)} \quad (3.5)$$

where

$$G_0^{(+)}(E) = (E - H_0 + i\eta)^{-1} \quad (3.6)$$

and Φ_E is the solution of the homogeneous problem (without the term $V(\vec{x})$). Φ_E is a product of the plane wave describing the undisturbed motion of the projectile with the nuclear ground state wave function Φ_A^0 of the target nucleus. Using the definition of the T-matrix (equ.2.86) and (3.5) we get the Lippmann - Schwinger equation for the T-matrix

$$\langle \Phi_f | T | \Phi_E \rangle = \langle \Phi_f | V | \Psi_E^{(+)} \rangle$$

and

$$\langle \Phi_f | T | \Phi_E \rangle = \langle \Phi_f | V [| \Phi_E \rangle + G_0^{(+)}(E)V | \Psi_E^{(+)} \rangle] = \langle \Phi_f | [V + VG_0^{(+)}(E)T] | \Phi_E \rangle$$

or

$$T(E) = V + VG_0^{(+)}(E)T(E) \quad (3.7)$$

The operator $G_0^{(+)}(E)$ is a very complicated many body operator. The main goal of our investigation will be to understand the projectile - nucleus scattering problem as a sequence of collisions of the projectile with the individual nucleons of the target nucleus. To obey the Pauli principle the nuclear wave function must be fully antisymmetrised. Depending at which step of the development antisymmetry is introduced, one gets two different theories, namely the Foldy - Watson theory (FW) and the Kerman - Mc Manus - Thaler theory (KMT). The KMT - theory is formulated in such a way that it works exclusively in the restricted Hilbert space of antisymmetric target wave functions. In contrast to this the FW-theory is formulated in the entire Hilbert space and only initial and final target states are antisymmetric. In the following we shall develop both theories in a formal way, i.e. without use of wave functions. In principle one could try to solve (3.7) through iteration obtaining an expansion in terms of the potential V . This expansion is called the Born - series and it would be convergent only for weak potentials. It is however possible to obtain a much better convergence by reordering of the various terms of this expansion. One tries to combine all terms containing the interaction with one and the

same nucleon (i.e. terms of the form $V_j G_0^{(+)} V_j G_0^{(+)} \dots$). In this way we describe fully the scattering process on this nucleon and obtain then a series of consecutive scattering process on the various target nucleons.

3.1 Foldy - Watson theory

We start from equ.(3.7) and insert (3.1) to obtain

$$T(E) = \sum_{j=1}^A V_j + \sum_{j=1}^A V_j G_0^{(+)}(E) T(E) \quad (3.7')$$

Next we define an auxiliary operator T_j :

$$T_j = V_j + V_j G_0^{(+)}(E) T \quad (3.8)$$

which satisfies

$$T = \sum_{j=1}^A T_j \quad (3.9)$$

The right hand side of (3.8) may be reformulated in such a way as to isolate T_j . We obtain

$$T_j = V_j + V_j G_0^{(+)} T_j + V_j G_0^{(+)} \sum_{k \neq j} T_k$$

or

$$(1 - V_j G_0^{(+)}) T_j = V_j + V_j G_0^{(+)} \sum_{k \neq j} T_k$$

or

$$T_j = (1 - V_j G_0^{(+)})^{-1} V_j + (1 - V_j G_0^{(+)})^{-1} V_j G_0^{(+)} \sum_{k \neq j} T_k \quad (3.10)$$

Next a new quantity t_j is defined by

$$t_j \equiv (1 - V_j G_0^{(+)})^{-1} V_j \quad (3.11)$$

With this operator t_j we convert (3.10) into the form

$$T_j = t_j + t_j G_0^{(+)} \sum_{k \neq j} T_k \quad (3.12)$$

Multiplying (3.11) from the left by $(1 - V_j G_0^{(+)})$ we obtain

$$(1 - V_j G_0^{(+)}) t_j = V_j$$

or

$$t_j = V_j + V_j G_0^{(+)} t_j \quad (3.11')$$

This equation had exactly the form of a Lippmann - Schwinger equation for the scattering of the projectile on the single isolated nucleon j , if in place of $G_0^{(+)}$ the free propagator $g_0^{(+)}$ would occur. $G_0^{(+)}$ is however still a complicated many - body propagator. Usually one calls t_j the " in - medium " T - matrix for the scattering of the projectile on the nucleon j . It is the T - matrix for the scattering projectile on nucleon j which is bound in the target nucleus. The propagator $G_0^{(+)}$ describes the " medium effects ", i.e. it takes into account the interaction between the target nucleons. We may now insert (3.12) into (3.9) and obtain the Watson - series.

$$T = \sum_{j=1}^A t_j + \sum_{j=1}^A t_j G_0^{(+)} \sum_{k \neq j} t_k + \sum_{j=1}^A t_j G_0^{(+)} \sum_{k \neq j} t_k G_0^{(+)} \sum_{l \neq k} t_l + \dots \quad (3.13)$$

Each term in this series expansion describes a sequence of multiple collisions: single scattering, double scattering, triple scattering, etc. Usually one uses the Watson series as a starting point for various approximations (i.e. terminating the series after a certain order, replacing t_j by the free t - operators t_j^{free} (impulse approximation), etc.). In the next step we introduce the concept of the optical potential to describe elastic scattering.

It will allow to bring the Watson series into a different form. We remember once more that in the Foldy Watson theory the full Hilbert space is used. For our purpose it is sufficient to think that the ground state of the nucleus can be described by a Slater determinant. This is nothing else than a linear combination of $A!$ products of A single nucleon wave functions. All $A!$ such products are degenerate with respect to the ground state energy. We call P_W the projection operator on the space spanned by these $A!$ product wave functions. The projection operator Q_W projects on to the rest of Hilbert space, such that

$$P_W + Q_W = 1 \quad (3.14)$$

The basis functions from which Q_W is built up are not contained in the set of the $A!$ products of the ground state wave function. They are members of the set of "excited" states. With this we may now reorder equation (3.12) writing

$$T_j = t_j + t_j(P_W + Q_W)G_0^{(+)} \sum_{k \neq j} T_k = t_j + t_j G_0^{(+)} P_W T + t_j G_0^{(+)} Q_W \sum_{k \neq j} T_k - t_j G_0^{(+)} P_W T_j \quad (3.12')$$

or

$$(1 + t_j G_0^{(+)} P_W) T_j = t_j + t_j G_0^{(+)} P_W T + t_j G_0^{(+)} Q_W \sum_{k \neq j} T_k \quad (3.15)$$

We define a new t -operator t'_j by

$$t'_j \equiv (1 + t_j G_0^{(+)} P_W)^{-1} t_j \quad (3.16)$$

or

$$t'_j = t_j - t_j G_0^{(+)} P_W t'_j = V_j + V_j G_0^{(+)} Q_W t'_j \quad (3.17)$$

We use this now in equ.(3.15) and obtain

$$T_j = t'_j + t'_j G_0^{(+)} P_W T + t'_j G_0^{(+)} Q_W \sum_{k \neq j} T_k \quad (3.12'')$$

Inserting this into equ.(3.9) and iterating the 3rd term in (3.12'') leads to a new formulation of the scattering problem. We obtain

$$T(E) = U_{opt}(E) + U_{opt}(E) G_0^{(+)}(E) P_W T(E) \quad (3.18)$$

with

$$U_{opt}(E) = \sum_{j=1}^A t'_j + \sum_{j=1}^A t'_j G_0^{(+)}(E) Q_W \sum_{k \neq j} t'_k + \dots \quad (3.19)$$

$U_{opt}(E)$ as well as T are operators symmetric with respect of interchanging any two of the nucleon coordinates. Taking the matrix element between correctly antisymmetrized initial and final nuclear states leads therefore to correct physical results. Since $G_0^{(+)}$ is diagonal with respect to the target states, we may replace P_W by $P_0 | \Phi_A^0 \rangle \langle \Phi_A^0 |$ ($\mathcal{A} P_W \mathcal{A} = P_0$). We see that in Foldy - Watson theory we use a set of nonsymmetrized intermediate states in constructing a symmetrical operator U_{opt} , which can then be used in the Lippmann - Schwinger equation for the T matrix:

$$T(E) = U_{opt}(E) + U_{opt}(E) G_0^{(+)}(E) P_0 T(E) \quad (3.18')$$

3.2 Kerman - Mc. Manus - Thaler theory

In the KMT - theory we start from the following T - operator

$$T = V + V G_0^{(+)} \mathcal{A} T \quad (3.20)$$

where \mathcal{A} is a projection operator projecting on to the space of fully antisymmetrized target states. Initial and final state are also fully antisymmetrized. We proceed as in case of the FW theory by writing

$$T = \sum_{j=1}^A V_j + \sum_{j=1}^A V_j G_0^{(+)} \mathcal{A} T = \sum_{j=1}^A T_j$$

$$T_j = V_j + V_j G_0^{(+)} \mathcal{A} T_j + V_j G_0^{(+)} \mathcal{A} \sum_{k \neq j} T_k$$

With

$$\tau_i \equiv (1 - V_i G_0^{(+)} \mathcal{A})^{-1} V_i$$

or

$$\tau_i = V_i + V_i G_0^{(+)} \mathcal{A} \tau_i \quad (3.21)$$

we obtain

$$T_i = \tau_i + \tau_i G_0^{(+)} \mathcal{A} \sum_{j \neq i} T_j$$

and

$$T = \sum_{i=1}^A \tau_i + \sum_{i=1}^A \tau_i G_0^{(+)} \mathcal{A} \sum_{j \neq i} \tau_j + \sum_{i=1}^A \tau_i G_0^{(+)} \mathcal{A} \sum_{j \neq i} \tau_j G_0^{(+)} \mathcal{A} \sum_{k \neq j} \tau_k + \dots \quad (3.22)$$

This T - operator leads to the same physical answer as in the FW - theory provided one uses correctly antisymmetrized initial and final state wave functions. However, now we have also antisymmetric intermediate states, so that we can make further simplifications. First we note that the matrix elements of all τ_i 's are identically the same. We have for example

$$\langle \Phi_A^\alpha | \sum_{j \neq i} \tau_j | \Phi_A^\beta \rangle = (A-1) \langle \Phi_A^\alpha | \tau_1 | \Phi_A^\beta \rangle = \frac{A-1}{A} \langle \Phi_A^\alpha | \sum_{j=1}^A \tau_j | \Phi_A^\beta \rangle$$

We can use this fact in (3.22) and see that all the summations can be done without any restrictions $j \neq i$, etc. The error made by such a procedure is just compensated by the numerical factor $\frac{A-1}{A}$. In such a way we obtain from (3.22) the expression

$$T = \sum_{i=1}^A \tau_i + \frac{A-1}{A} \sum_{i=1}^A \tau_i G_0^{(+)} \mathcal{A} \sum_{j=1}^A \tau_j + \left(\frac{A-1}{A}\right)^2 \sum_{i=1}^A \tau_i G_0^{(+)} \mathcal{A} \sum_{j=1}^A \tau_j G_0^{(+)} \mathcal{A} \sum_{k=1}^A \tau_k + \dots \quad (3.22')$$

This series expansion is however nothing but the iterated equation

$$T = \sum_{i=1}^A \tau_i + \frac{A-1}{A} \sum_{i=1}^A \tau_i G_0^{(+)} \mathcal{A} T \quad (3.23)$$

If we define now

$$\tau' \equiv \frac{A-1}{A} \sum_{j=1}^A \tau_j \quad (3.24)$$

and

$$T' \equiv \frac{A-1}{A} T \quad (3.25)$$

we obtain from (3.23)

$$T = \frac{A}{A-1} \tau' + \tau' G_0^{(+)} \mathcal{A} T$$

or

$$T' = \tau' + \tau' G_0^{(+)} \mathcal{A} T' \quad (3.26)$$

This is a fully symmetric operator, commuting with \mathcal{A} . Knowing that we work entirely within the space of antisymmetrized states, \mathcal{A} could in principle be omitted. In full analogy to the FW-theory one can construct an optical potential \mathcal{U} . We introduce the two projection operators $P_0 = |\Phi_A^0\rangle\langle\Phi_A^0|$ and $Q = 1 - P_0$. Starting from (3.22')

$$T = V + V G_0^{(+)} \mathcal{A} (P_0 + Q_0) T = V + V G_0^{(+)} \mathcal{A} P_0 T + V G_0^{(+)} \mathcal{A} Q_0 T$$

we obtain by iterating the term $V G_0^{(+)} \mathcal{A} Q_0 T$ the optical potential \mathcal{U} .

$$T = \mathcal{U} + \mathcal{U} G_0^{(+)} \mathcal{A} P_0 T \quad (3.27)$$

$$U = V + VG_0^{(+)}AQ_0U \quad (3.28)$$

To determine U we use the same procedure as before with T :

$$U = \sum_{i=1}^A U_i = \sum_{i=1}^A V_i + \sum_{i=1}^A V_i G_0^{(+)} A Q_0 \sum_{j=1}^A U_j$$

$$U_i = V_i + V_i G_0^{(+)} A Q_0 U_i + V_i G_0^{(+)} A Q_0 \sum_{j \neq i} U_j \quad (3.29)$$

$$(1 - V_i G_0^{(+)} A Q_0) U_i = V_i + V_i G_0^{(+)} A Q_0 \sum_{j \neq i} U_j$$

Define now $\hat{\tau}_i$ by

$$\hat{\tau}_i \equiv (1 - V_i G_0^{(+)} A Q_0)^{-1} V_i = V_i + V_i G_0^{(+)} A Q_0 \hat{\tau}_i \quad (3.30)$$

We then have

$$U_j = \hat{\tau}_j + \hat{\tau}_j G_0^{(+)} A Q_0 \sum_{k \neq j} U_k \quad (3.29')$$

Iterating U_k and summing over all i we obtain

$$U = \sum_{j=1}^A U_j = \sum_{i=1}^A \hat{\tau}_i + \sum_{i=1}^A \hat{\tau}_i G_0^{(+)} A Q_0 \sum_{j \neq i} (\hat{\tau}_j + \hat{\tau}_j G_0^{(+)} A Q_0 \sum_{k \neq j} U_k)$$

or

$$U = \sum_{j=1}^A \hat{\tau}_j + \sum_{j=1}^A \hat{\tau}_j G_0^{(+)} A Q_0 \sum_{k \neq j} \hat{\tau}_k + \sum_{j=1}^A \hat{\tau}_j G_0^{(+)} A Q_0 \sum_{k \neq j} \hat{\tau}_k G_0^{(+)} A Q_0 \sum_{l \neq k} \hat{\tau}_l + \dots \quad (3.31)$$

Using the antisymmetry of the states one can again perform all sums without the restrictions and use the counting factors to correct for the errors:

$$U = \sum_{i=1}^A \hat{\tau}_i + \frac{A-1}{A} \sum_{j=1}^A \hat{\tau}_j G_0^{(+)} A Q_0 \sum_{j=1}^A \hat{\tau}_j + \left(\frac{A-1}{A}\right)^2 \sum_{i=1}^A \hat{\tau}_i G_0^{(+)} A Q_0 \sum_{j=1}^A \hat{\tau}_j G_0^{(+)} A Q_0 \sum_{k=1}^A \hat{\tau}_k + \dots \quad (3.31')$$

This expansion for U may be summed and gives

$$U = \sum_{i=1}^A \hat{\tau}_i + \frac{A-1}{A} \sum_{i=1}^A \hat{\tau}_i G_0^{(+)} A Q_0 U \quad (3.32)$$

We may introduce the following quantities

$$\hat{\tau}' \equiv \frac{A-1}{A} \sum_{i=1}^A \hat{\tau}_i \quad (3.33)$$

and

$$U' = \frac{A-1}{A} U, \quad T' = \frac{A-1}{A} T \quad (3.34)$$

to obtain from (3.32) the following result:

$$U' = \hat{\tau}' + \hat{\tau}' G_0^{(+)} A Q_0 U' \quad (3.35)$$

$$T' = U' + U' G_0^{(+)} P_0 T' \quad (3.36)$$

From the definitions of τ_j and $\hat{\tau}_j$ ((3.21),(3.30)) one obtains easily

$$\hat{\tau}_j = \tau_j - \tau_j G_0^{(+)} \mathcal{A} P_0 \hat{\tau}_j \quad (3.37)$$

In principle one can also establish a relation between the t_i 's of FW - theory and the τ_i 's of KMT - theory. To do so, we first eliminate V_i from the defining equation of the τ_i 's (3.21). From (3.11) we obtain

$$V_i = (1 - V_i G_0^{(+)} t_i) = t_i (1 - G_0^{(+)} V_i) \quad (3.38)$$

Then we write (3.21) in the form

$$\tau_i = V_i \mathcal{A} + V_i G_0^{(+)} \mathcal{A} \tau_i \quad (3.21')$$

where we have replaced V_i bei $V_i \mathcal{A}$. Before it was unnecessary to do this, since we were entirely working within the space of antisymmetric states. Inserting (3.38) into (3.21') gives

$$\begin{aligned} \tau_i &= t_i (1 - G_0^{(+)} V_i) \mathcal{A} + t_i (1 - G_0^{(+)} V_i) \mathcal{A} G_0^{(+)} \tau_i \\ &= t_i \mathcal{A} - t_i G_0^{(+)} V_i \mathcal{A} + t_i G_0^{(+)} \mathcal{A} \tau_i - t_i G_0^{(+)} V_i \mathcal{A} G_0^{(+)} \tau_i \\ &= t_i \mathcal{A} - t_i G_0^{(+)} (V_i \mathcal{A} + V_i \mathcal{A} G_0^{(+)} \tau_i) + t_i G_0^{(+)} \mathcal{A} \tau_i \\ &= t_i \mathcal{A} - t_i G_0^{(+)} \tau_i + t_i G_0^{(+)} \mathcal{A} \tau_i \end{aligned}$$

or

$$\tau_i = t_i \mathcal{A} + t_i G_0^{(+)} (\mathcal{A} - 1) \tau_i \quad (3.39)$$

The basic idea of multiple scattering theories is to reduce the calculation of the projectile nucleus amplitude to the knowledge of the "elementary" free projectile - nucleon amplitude. In the frame of potential scattering theory the free scattering matrix satisfies

$$t_i^{free}(E') = V_i + V_i g_0^{(i)+}(E') t_i^{free}(E') \quad (3.40)$$

where

$$g_0^{(i)+}(E') = \frac{1}{E' - T_{pr}^{kin} - T_i^{kin} + i\eta} \quad (3.41)$$

The energy E' can here be considered to be an arbitrary parameter. Since both, $t_i(E)$ and $t_i^{free}(E')$ satisfy a Lippmann Schwinger equation with the same interaction, one can establish a connection between t_i and t_i^{free} . We eliminate V_i from (3.11) and obtain:

$$V_i = t_i^{free} (1 - g_0^{(i)+} V_i)$$

and

$$\begin{aligned} t_i &= t_i^{free} (1 - g_0^{(i)+} V_i) + t_i^{free} (1 - g_0^{(i)+} V_i) G_0^{(+)} t_i \\ &= t_i^{free} - t_i^{free} g_0^{(i)+} V_i - t_i^{free} g_0^{(i)+} V_i G_0^{(+)} t_i + t_i^{free} G_0^{(+)} t_i \\ &= t_i^{free} - t_i^{free} g_0^{(i)+} (V_i + V_i G_0^{(+)} t_i) + t_i^{free} G_0^{(+)} t_i \\ &= t_i^{free} - t_i^{free} g_0^{(i)+} t_i + t_i^{free} G_0^{(+)} t_i \end{aligned}$$

One gets therefore the required relation in the form

$$t_i(E) = t_i^{free}(E') - t_i^{free}(E') [g_0^{(i)+}(E') - G_0^{(+)}(E)] t_i(E) \quad (3.42)$$

The energy E' is usually chosen such as to make $[g_0^{(i)+}(E') - G_0^{(+)}(E)]$ as small as possible. One has then the hope that the Watson - series will converge sufficiently rapid. We assume that $t_i^{free}(E')$ is known from experiment. How to chose in an optimal way the quantity E' has been discussed at length in the literature. (S.A. Gurvitz, J.P. Dedonder and P.D. Amado, Phys.Rev. C19(1979)142; H. Feshbach, Theoretical Nuclear Physics, Nuclear Reactions; J. Wiley and Sons Inc. (1992), pg.142).

Part II

4 Elementary introduction to relativistic quantum mechanics

4.1 Nonrelativistic classical mechanics versus nonrelativistic quantum mechanics

Classical nonrelativistic mechanics describes a system of N mass points by the Lagrange equations. In case the forces between the mass points are conservative, these equations are

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \quad i = 1, 2, \dots, 3N \quad (4.1)$$

$3N$ is the numbers of degrees of freedom of the system and the Lagrange function L is in this case given by

$$L(q, \dot{q}, t) = T - U \quad (4.2)$$

where $T(q, \dot{q}, t) = \sum_{k=1}^N \frac{m_k \dot{x}_k^2}{2}$ is the total kinetic energy of the system and $U(q, t) = U(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N, t)$ the total potential energy. The Lagrange equations are the Euler - Lagrange equations of a variational principle, namely of Hamilton's principle. In short, this principle says, if we can associate with a mechanical system a Lagrange function $L(q, \dot{q}, t)$, the latter being a continuous function and at least two times differentiable in q, \dot{q} , then the physical trajectories $q_i(t), \dot{q}_i(t)$ with the boundary conditions $q_i(t_1), q_i(t_2)$ make the integral

$$S = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t) dt \quad (4.3)$$

extremal.

$$\delta S = 0 \quad (4.4)$$

However, the Lagrange function is only determined up to the total differential of a function of the coordinates. The form $L = T - U$ is called the "natural form" of L . If L is two times continuously differentiable, one can locally resolve the equations

$$p_k \equiv \frac{\partial L}{\partial \dot{q}_k}(q, \dot{q}, t) \quad k = 1, 2, \dots, 3N \quad (4.5)$$

for \dot{q}_k . The quantities p_k are called the generalised momenta. Let $\dot{q}_k = \dot{q}_k(q, p, t)$ be these solutions. Then the Legendre transform of $L(q, \dot{q}, t)$ ($\mathcal{L} * L(q, \dot{q}, t)$) is a function of the coordinates $\{q, p, t\}$:

$$H(q, p, t) \equiv \mathcal{L} * L(q, \dot{q}, t) = \sum_{k=1}^{3N} \dot{q}_k p_k - L(q, \dot{q}, t) \quad (4.6)$$

$H(q, p, t)$ is called the Hamilton function of the system and in the present case of conservative forces it is identical with the total energy of the system $H = T + U$. The Legendre transformation can be uniquely inverted. (q_k, p_k) are called "canonical conjugate" variables. From the total differential dH and the Lagrange equations (4.1) we obtain easily the set of Hamiltonian's equations:

$$\frac{\partial H}{\partial q_k} = -\dot{p}_k, \quad \frac{\partial H}{\partial p_k} = \dot{q}_k, \quad \frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \quad (4.7)$$

These also called "canonical equations" are completely equivalent to the original Lagrange equations. From (4.7) we see that in case H does not explicitly depend on the time t ($\frac{dH}{dt} = 0$), H is a constant equal to the total energy $H = E_{tot}$. In Hamiltonian mechanics one uses quite frequently Poisson's brackets. Let $f(q, p, t)$ and $g(q, p, t)$ be two functions supposed to be at least once continuously differentiable with respect to the variables q and p . The Poisson bracket is then defined as

$$\{f, g\}_{Poisson} \equiv \sum_k \left(\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} \right) \quad (4.8)$$

It is easy to show the validity of the following relations:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}_{Poisson} \quad (4.9)$$

$$\{q_k, f\}_{Poisson} = \frac{\partial f}{\partial p_k} \quad (4.10)$$

$$\{p_k, f\}_{Poisson} = -\frac{\partial f}{\partial q_k} \quad (4.11)$$

$$\{q_k, q_{k'}\}_{Poisson} = \{p_k, p_{k'}\}_{Poisson} = 0 \quad (4.12)$$

$$\{q_k, p_{k'}\}_{Poisson} = \delta_{kk'} \quad (4.13)$$

$$\dot{q}_k = \{q_k, H\}_{Poisson} = \frac{\partial H}{\partial p_k} \quad (4.14)$$

$$\dot{p}_k = \{p_k, H\}_{Poisson} = -\frac{\partial H}{\partial q_k} \quad (4.15)$$

Usually canonical transformations are applied to the system to facilitate the solution of the equations of motion. A canonical transformation is defined as a transformation of the variables and the Hamilton function such that the new variables satisfy again Hamilton's equations. A necessary and sufficient condition to achieve this is, that Hamilton's principle is valid for the old and the new variables. This is the case if the Lagrange functions differ only by the total derivative with respect to time of a function of the old and / or new variables. We choose here the following transformation:

$$\left. \begin{aligned} q_k &= q_k(\bar{q}, \bar{p}, t) & \text{with } \bar{q}_k &= \bar{q}_k(q, \bar{p}, t) \\ p_k &= p_k(\bar{q}, \bar{p}, t) \\ M &= S(q, \bar{p}, t) - \sum_k \bar{q}_k(q, \bar{p}, t) \bar{p}_k \end{aligned} \right\} \quad (4.16)$$

and request:

$$\delta \int_{t_1}^{t_2} L(q, \dot{q}, t) dt = \delta \int_{t_1}^{t_2} \left[\bar{L}(\bar{q}, \dot{\bar{q}}, t) + \frac{dM}{dt} \right] dt = 0 \quad (4.17)$$

If we introduce H and \bar{H} instead of L and compare the integrands we obtain

$$\sum_k \dot{q}_k p_k - H = \sum_k \dot{\bar{q}}_k \bar{p}_k - \bar{H} + \frac{dM}{dt}$$

or

$$\sum_k dq_k p_k - H dt = \sum_k d\bar{q}_k \bar{p}_k - \bar{H} dt + \sum_k \left(\frac{\partial S}{\partial q_k} dq_k + \frac{\partial S}{\partial \bar{p}_k} d\bar{p}_k \right) + \frac{\partial S}{\partial t} dt - \sum_k d\bar{q}_k \bar{p}_k - \sum_k \bar{q}_k d\bar{p}_k$$

Comparing the coefficients of equal differentials leads to:

$$p_k = \frac{\partial S}{\partial q_k}, \quad \bar{H} = H + \frac{\partial S}{\partial t}, \quad \bar{q}_k = \frac{\partial S}{\partial \bar{p}_k} \quad (4.18)$$

Now we may request the new variables to be cyclic variables. Cyclic variables are defined such that the Lagrange function is independent of them (in our case \bar{L} should be independent of \bar{q}_k). In this case we have

$$\bar{p}_k = \frac{\partial \bar{L}}{\partial \dot{\bar{q}}_k}, \quad \dot{\bar{p}}_k = \frac{d}{dt} \left(\frac{\partial \bar{L}}{\partial \dot{\bar{q}}_k} \right) = \frac{\partial \bar{L}}{\partial \bar{q}_k} = 0$$

Therefore $\bar{p}_k = \alpha_k = \text{const.}$ The condition $\bar{H} = 0$ can be fulfilled if $S(q, \bar{p} = \alpha, t)$ satisfies the differential equation:

$$\bar{H} = H \left(q_k, \frac{\partial S}{\partial q_k}, t \right) + \frac{\partial S(q_k, \alpha_k, t)}{\partial t} = 0 \quad (4.19)$$

Equation (4.19) is called Hamilton - Jacobi equation. The function S has also an interesting physical meaning:

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum_k \frac{\partial S}{\partial q_k} \dot{q}_k = \frac{\partial S}{\partial t} + \sum_k p_k \dot{q}_k = -H + \sum_k p_k \dot{q}_k = L(q, \dot{q}, t)$$

$$S(t) - S(t_0) = \int_{t_0}^t L(q(t'), \dot{q}(t'), t') dt' \quad (4.20)$$

S is seen to be the action integral evaluated along the dynamic (physical) trajectories.

The transition from classical mechanics to quantum mechanics is formally performed by replacing the canonical conjugated variables q and p by operators. These operators shall satisfy the usual commutation relations. The Poisson brackets have to be replaced by the corresponding commutator expressions. In the following we list these analogies between classical Hamilton mechanics and quantum mechanics.

$$\begin{array}{ccc}
\text{Classical mechanics} & \Rightarrow & \text{Quantum mechanics} \\
q_k, p_k & & q_{op}^k, p_{op}^k = -i\hbar \frac{\partial}{\partial q_{op}^k} \\
E_{tot} & & E_{tot,op} = i\hbar \frac{\partial}{\partial t} \\
\sum_k H(q_k, p_k, t) & & H_{op}(q_{op}^k, p_{op}^k, t) \\
H(q_k, p_k = \frac{\partial S}{\partial q_k}, t) + \frac{\partial S}{\partial t} = 0 & & [H_{op}(q_{op}^k, p_{op}^k, t) - i\hbar \frac{\partial}{\partial t}] \Psi(q, p, t) = 0 \\
\{q_k, p_{k'}\}_{Poisson} = \delta_{kk'} & & \frac{1}{i\hbar} [q_{op}^k, p_{op}^{k'}] = \delta_{kk'} \\
\dot{q}_k = \{q_k, H\}_{Poisson} = \frac{\partial H}{\partial p_k} & & \dot{q}_{op}^k = \frac{1}{i\hbar} [q_{op}^k, H] = \frac{\partial H_{op}}{\partial p_{op}^k} \\
\dot{p}_k = \{p_k, H\}_{Poisson} = -\frac{\partial H}{\partial q_k} & & \dot{p}_{op}^k = \frac{1}{i\hbar} [p_{op}^k, H_{op}] = -\frac{\partial H_{op}}{\partial q_{op}^k} \\
\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\}_{Poisson} & & \frac{dF_{op}}{dt} = \frac{\partial F_{op}}{\partial t} + \frac{1}{i\hbar} [F_{op}, H_{op}]
\end{array}$$

The two formal conditions: $q_k, p_k \Rightarrow q_{op}^k, p_{op}^k = -i\hbar \frac{\partial}{\partial q_{op}^k}$ and $\{q_k, p_{k'}\}_{Poisson} = \delta_{kk'} \Rightarrow \frac{1}{i\hbar} [q_{op}^k, p_{op}^{k'}] = \delta_{kk'}$ are completely equivalent to each other. They are called the "canonical quantisation rules". This formal analogy between classical mechanics and quantum mechanics is kept as far as possible also in relativistic quantum mechanics and in quantum field theory.

4.2 Relativistic mechanics of a free particle

As is well known, the Hamilton function of a relativistic particle with rest mass m is given by

$$H_0 = \sqrt{c^2 \vec{p}^2 + (mc^2)^2} \quad (4.21)$$

However, unlike to classical mechanics, the Lagrange function L_0 is not equal to the kinetic energy T . This can easily be verified.

$$T = \sqrt{c^2 \vec{p}^2 + (mc^2)^2} - mc^2 = \frac{mc^2}{\sqrt{1-\beta^2}} - mc^2 \quad (\beta = \frac{v}{c})$$

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-\beta^2}} \neq \nabla_v T$$

One can use the Lagrange formalism to obtain a suitable Lagrange function. If L_0 is the Lagrange function of the free particle, we request:

$$\frac{\partial L_0}{\partial v^i} = p^i = \frac{mv^i}{\sqrt{1-\beta^2}}, \quad \frac{\partial L_0}{\partial x^i} = 0 \quad i = 1, 2, 3$$

Integration gives us:

$$L_0 = \int \frac{\partial L_0}{\partial v^i} dv^i = \int \frac{mv^i}{\sqrt{1-\beta^2}} dv^i = -mc^2 \int \frac{d}{dv^i} \sqrt{1-\beta^2} dv^i = -mc^2 \sqrt{1-\beta^2}$$

$$L_0 = -mc^2 \sqrt{1-\beta^2} \neq T \quad (4.22)$$

Using this L_0 we obtain of course the right Hamilton function H_0 .

$$H_0 = \sum_i v^i \frac{\partial L_0}{\partial v^i} - L_0 = \sum_i v^i \frac{mc^2 v^i}{\sqrt{1-\beta^2}} + mc^2 \sqrt{1-\beta^2} = mc^2 \left[\frac{\beta^2}{\sqrt{1-\beta^2}} + \sqrt{1-\beta^2} \right] = \frac{mc^2}{\sqrt{1-\beta^2}}$$

4.3 The relativistic particle in an electromagnetic field

A particle with mass m and charge q experiences in an electromagnetic field the Lorentz force.

$$\vec{K}_L = q\vec{E} + \frac{q}{c}\vec{v} \times \vec{B} = -q\nabla\Phi - \frac{q}{c}\frac{\partial\vec{A}}{\partial t} + \frac{q}{c}\vec{v} \times (\nabla \times \vec{A}) = \frac{d}{dt}\vec{p}_{mech} \quad (4.23)$$

Here we have used Maxwell's equations $\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$ and $\vec{B} = \nabla \times \vec{A}$. We observe that

$$\vec{v} \times (\nabla \times \vec{A}) = \nabla(\vec{v}\vec{A}) - (\vec{v}\nabla)\vec{A}$$

$$\frac{d\vec{A}}{dt} = \frac{\partial\vec{A}}{\partial t} + (\vec{v}\nabla)\vec{A}$$

and obtain from (4.23) the following expression:

$$\frac{d\vec{p}_{mech}}{dt} = -q\nabla\Phi - \frac{q}{c}\frac{\partial\vec{A}}{\partial t} + \frac{q}{c}[\nabla(\vec{v}\vec{A}) - (\vec{v}\nabla)\vec{A}] = -\frac{q}{c}\frac{d\vec{A}}{dt} - q\nabla\Phi + \frac{q}{c}\nabla(\vec{v}\vec{A})$$

or

$$\frac{d}{dt}(\vec{p}_{mech} + \frac{q}{c}\vec{A}) = -\nabla\left(q\Phi - \frac{q}{c}\vec{v}\vec{A}\right) \quad (4.24)$$

If we request (4.24) to be the Lagrange equations of the system, we have to set

$$p^i = \frac{\partial L}{\partial v^i} = p_{mech}^i + \frac{q}{c}A^i$$

$$\frac{\partial L}{\partial x^i} = -\frac{\partial}{\partial x^i}\left(q\Phi - \frac{q}{c}\vec{v}\vec{A}\right)$$

Integration of the second one of these equations gives us

$$L = -\left(q\Phi - \frac{q}{c}\vec{v}\vec{A}\right) + f(\vec{v})$$

where $f(\vec{v})$ is an arbitrary function. It can be determined from the equation defining p^i .

$$\frac{\partial L}{\partial v^i} = \frac{q}{c}A^i + \frac{df}{dv^i} = p_{mech}^i + \frac{q}{c}A^i$$

$$\frac{df}{dv^i} = p_{mech}^i = \frac{mv^i}{\sqrt{1-\beta^2}} = -mc^2 \frac{d}{dv^i} \sqrt{1-\beta^2}$$

From this we obtain by integration

$$f(\vec{v}) = -mc^2 \sqrt{1-\beta^2} = L_0 \quad (\text{Lagrange function of the free particle})$$

The Lagrange function of the interacting system is therefore given by

$$L = L_0 - \left(q\Phi - \frac{q}{c}\vec{v}\vec{A}\right) = -mc^2 \sqrt{1-\beta^2} - \left(q\Phi - \frac{q}{c}\vec{v}\vec{A}\right) \quad (4.25)$$

and the Hamilton function by

$$H = \sum_i v^i \frac{\partial L}{\partial v^i} - L = \vec{v}\vec{p}_{mech} + \frac{q}{c}\vec{v}\vec{A} + mc^2 \sqrt{1-\beta^2} + \left(q\Phi - \frac{q}{c}\vec{v}\vec{A}\right)$$

$$= \frac{m\vec{v}^2}{\sqrt{1-\beta^2}} + mc^2 \sqrt{1-\beta^2} + q\Phi = \frac{mc^2}{\sqrt{1-\beta^2}} + q\Phi$$

$$= c\sqrt{\vec{p}_{mech}^2 + (mc)^2} + q\Phi$$

or

$$H = H_0 + q\Phi = \sqrt{c^2 \vec{p}_{mech}^2 + (mc^2)^2} + q\Phi = c\sqrt{\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + (mc)^2} + q\Phi \quad (4.26)$$

Maxwell's equation are invariant under gauge transformations of the fields Φ , \vec{A} :

$$\Phi \rightarrow \Phi' = \Phi - \frac{1}{c} \frac{\partial \Lambda}{\partial t}$$

$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \nabla \Lambda$$

Also the equation of motion (4.24) is invariant under this gauge transformation, L and H are not invariant.

$$L' = -mc^2 \sqrt{1 - \beta^2} - \left(q\Phi' - \frac{q}{c} \vec{v} \cdot \vec{A}' \right) = -mc^2 \sqrt{1 - \beta^2} - \left(q\Phi - \frac{q}{c} \vec{v} \cdot \vec{A} \right) + \frac{q}{c} \left(\frac{\partial \Lambda}{\partial t} + \vec{v} \cdot \nabla \Lambda \right)$$

$$L' = L + \frac{q}{c} \frac{d\Lambda}{dt} \quad (4.27)$$

From

$$\vec{p}' = \nabla_{\vec{v}} L' = \nabla_{\vec{v}} L + \frac{q}{c} \nabla \Lambda = \vec{p} + \frac{q}{c} \nabla \Lambda$$

we obtain

$$H' = \vec{p}' \cdot \vec{v} - L' = \vec{p} \cdot \vec{v} + \frac{q}{c} \vec{v} \cdot \nabla \Lambda - L - \frac{q}{c} \left(\frac{\partial \Lambda}{\partial t} + \vec{v} \cdot \nabla \Lambda \right)$$

or

$$H' = H - \frac{q}{c} \frac{\partial \Lambda}{\partial t} \quad (4.28)$$

It is very easy to show that $H - q\Phi$ and $\vec{p} - \frac{q}{c}\vec{A}$ are gauge invariant quantities:

$$H' - q\Phi' = H - \frac{q}{c} \frac{\partial \Lambda}{\partial t} - q \left(\Phi - \frac{1}{c} \frac{\partial \Lambda}{\partial t} \right) = H - q\Phi \quad (4.29)$$

$$\vec{p}' - \frac{q}{c} \vec{A}' = \vec{p} + \frac{q}{c} \nabla \Lambda - \frac{q}{c} (\vec{A} + \nabla \Lambda) = \vec{p} - \frac{q}{c} \vec{A} \quad (4.30)$$

From these two gauge invariant quantities one can construct a four vector:

$$\left. \begin{aligned} \left(\frac{H - q\Phi}{c}, \vec{p} - \frac{q}{c} \vec{A} \right) &= (p^\mu - \frac{q}{c} A^\mu) \\ \text{with} & \\ (p^\mu - \frac{q}{c} A^\mu) (p_\mu - \frac{q}{c} A_\mu) &= (mc)^2 \end{aligned} \right\} \quad (4.31)$$

4.4 The electromagnetic field

We give here only a summary of the basic equations (we use the cgs - system and electrostatic units).

Maxwell's equations :

$$\left. \begin{aligned} \nabla \times \vec{B} &= \frac{4\pi}{c} \vec{j}_{tot} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}, & \nabla \vec{B} &= 0, & \vec{B} &= \nabla \times \vec{A} \\ \nabla \times \vec{E} &= -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, & \nabla \vec{E} &= 4\pi \rho_{tot}, & \vec{E} &= -\nabla \Phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \\ \nabla \cdot \vec{j}_{tot} + \frac{\partial \rho_{tot}}{\partial t} &= 0 \\ \vec{K}_L &= q\vec{E} + \vec{v} \times \vec{B} \end{aligned} \right\} \quad (4.32)$$

Completely equivalent to Maxwell's equations are the equations satisfied by the fields Φ and \vec{A} :

$$\left. \begin{aligned} \square \vec{A} &= \frac{4\pi}{c} \vec{j}_{tot} - \nabla \left(\frac{1}{c} \frac{\partial \Phi}{\partial t} + \nabla \vec{A} \right) \\ \square \Phi &= 4\pi \rho_{tot} + \frac{1}{c} \frac{\partial}{\partial t} \left(\frac{1}{c} \frac{\partial \Phi}{\partial t} + \nabla \vec{A} \right) \end{aligned} \right\} \quad (4.33)$$

In four - vector notation using $j_{tot}^\mu = (c\rho_{tot}, \vec{j}_{tot})$ and $A^\mu = (\Phi, \vec{A})$ we have

$$\left. \begin{aligned} \square A^\mu &= \frac{4\pi}{c} j_{tot}^\mu + \partial^\mu (\partial_\alpha A^\alpha) \\ \partial_\mu j_{tot}^\mu &= 0 \end{aligned} \right\} \quad (4.34)$$

As already mentioned, the Maxwell equations are gauge invariant. The same is true for the equation (4.33) or (4.34). ($A^\mu \rightarrow A'^\mu = A^\mu - \partial^\mu \Lambda$). The equations (4.34) can be written in different form, using the also gauge invariant, antisymmetric field tensor $F^{\mu\nu}$. The $F^{\mu\nu}$'s are defined by

$$\left. \begin{aligned} F^{\mu\nu} &= -F^{\nu\mu} = \partial^\mu A^\nu - \partial^\nu A^\mu \\ F_{\mu\nu} &= -F_{\nu\mu} = \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned} \right\} \quad (4.35)$$

and the equation (4.34) acquires the form:

$$\left. \begin{aligned} \partial_\mu F^{\mu\nu} &= \frac{4\pi}{c} j_{tot}^\nu \\ \partial_\nu \partial_\mu F^{\mu\nu} &= \frac{4\pi}{c} \partial_\nu j_{tot}^\nu = 0 \end{aligned} \right\} \quad (4.34')$$

Below we give the $F^{\mu\nu}$'s in explicit form.

$$F^{\mu\nu} = -F^{\nu\mu} = \begin{pmatrix} 0, & -E^1, & -E^2, & -E^3 \\ E^1, & 0, & -B^3, & B^2 \\ E^2, & B^3, & 0, & -B^1 \\ E^3, & -B^2, & B^1, & 0 \end{pmatrix}$$

$$F_{\mu\nu} = -F_{\nu\mu} = \begin{pmatrix} 0, & E^1, & E^2, & E^3 \\ -E^1, & 0, & -B^3, & B^2 \\ -E^2, & B^3, & 0, & -B^1 \\ -E^3, & -B^2, & B^1, & 0 \end{pmatrix}$$

$$F^\mu{}_\nu = -g_{\nu\beta} F^{\mu\beta} = \begin{pmatrix} 0, & E^1, & E^2, & E^3 \\ E^1, & 0, & B^3, & -B^2 \\ E^2, & -B^3, & 0, & B^1 \\ E^3, & B^2, & -B^1, & 0 \end{pmatrix}$$

$$F_\mu{}^\nu = -g_{\mu\alpha} F^{\alpha\nu} = \begin{pmatrix} 0, & -E^1, & -E^2, & -E^3 \\ -E^1, & 0, & B^3, & -B^2 \\ -E^2, & -B^3, & 0, & B^1 \\ -E^3, & B^2, & -B^1, & 0 \end{pmatrix}$$

Energy density of the field:

$$u = \frac{1}{8\pi} (\vec{E} \cdot \vec{D} + \vec{H} \cdot \vec{B}) = \frac{1}{8\pi} (\epsilon \vec{E}^2 + \mu \vec{H}^2) \quad \text{in case } \vec{D} = \epsilon \vec{E} \text{ and } \vec{B} = \mu \vec{H} \quad (4.36)$$

($\mu = \epsilon = 1$ in vacuum)

Poynting vector:

$$\vec{S} = \frac{c}{4\pi} (\vec{E} \times \vec{H}) \quad (4.37)$$

4.5 Transition to quantum mechanics

4.5.1 Relativistic particle in the electromagnetic field and the question of gauge invariance in quantum mechanics

As a first step we try to perform the same replacement of the canonical conjugate variables by the corresponding operators as in relativistic quantum mechanics.

$$E_{op} = i\hbar \frac{\partial}{\partial t} = cp_{op}^0 = \hbar ck_{op}^0$$

$$\vec{p}_{op} = -i\hbar \nabla = \hbar \vec{k}_{op}$$

or

$$p^\mu = i\hbar \partial^\mu = i\hbar (\partial^0, -\nabla)$$

$$p_\mu = i\hbar \partial_\mu = i\hbar (\partial_0, \nabla)$$

(4.38)

Before going into the details of the problems created by the relativistic form of the Hamilton function or Hamilton operator (square root form), we shall deal with the question of gauge invariance in quantum mechanics. In chapter 4.3 we have seen that the relativistic Lagrange function and the relativistic Hamilton function are not gauge invariant, but instead the four-vector $(\frac{H-q\Phi}{c}, \vec{p} - \frac{q}{c}\vec{A})$ was an gauge invariant quantity. In quantum theory, however, we require that

$$H'\Psi' = i\hbar \frac{\partial \Psi'}{\partial t} \quad \text{and} \quad H\Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

describe the same physics. Since H is not invariant, we look for a relation between Ψ' and Ψ which guarantees us that both equations describe the same physics. We start with the Ansatz

$$\Psi' = \phi \Psi \quad (4.39)$$

and investigate the action of the operator $H' - q\Phi'$ on Ψ' .

$$\begin{aligned} (H' - q\Phi') \Psi' &= \left(i\hbar \frac{\partial}{\partial t} - q\Phi' \right) \Psi' = \left(i\hbar \frac{\partial}{\partial t} - q\Phi + \frac{q}{c} \frac{\partial \Lambda}{\partial t} \right) \phi \Psi = \\ &= \phi \left(i\hbar \frac{\partial}{\partial t} - q\Phi \right) \Psi + \left(i\hbar \frac{\partial \phi}{\partial t} + \frac{q}{c} \frac{\partial \Lambda}{\partial t} \phi \right) \Psi \end{aligned}$$

or

$$(H' - q\Phi') \Psi' = \phi [H - q\Phi] \Psi + \left[i\hbar \frac{\partial \phi}{\partial t} + \frac{q}{c} \frac{\partial \Lambda}{\partial t} \phi \right] \Psi \quad (4.40)$$

The second term in (4.40) can be used as a condition to determine ϕ by requesting

$$i\hbar \frac{\partial \phi}{\partial t} + \frac{q}{c} \frac{\partial \Lambda}{\partial t} \phi = 0 \quad (4.41)$$

The solution of this equation is

$$\phi = e^{i \frac{q}{\hbar c} \Lambda(\vec{x}, t)} \quad (4.42)$$

where the integration constant has been set arbitrarily equal to 1. The difference between Ψ' and Ψ is therefore a pure phase factor.

$$\Psi' = e^{i\frac{q}{\hbar c}\Lambda(\vec{z},t)}\Psi \quad (4.39')$$

From (4.40) we have also

$$[H' - q\Phi']\Psi' = e^{i\frac{q}{\hbar c}\Lambda}[H - q\Phi]\Psi \quad (4.43)$$

Using (4.39') we are able to investigate also the transformation law of the momentum components. We get:

$$\left(\vec{p} - \frac{q}{c}\vec{A}'\right)\Psi' = \left(-i\hbar\nabla - \frac{q}{c}\vec{A} - \frac{q}{c}\nabla\Lambda\right)\Psi' = e^{i\frac{q}{\hbar c}\Lambda}\left(-i\hbar\nabla - \frac{q}{c}\vec{A}\right)\Psi$$

or

$$\left[\vec{p} - \frac{q}{c}\vec{A}'\right]^n \Psi' = e^{i\frac{q}{\hbar c}\Lambda}\left[\vec{p} - \frac{q}{c}\vec{A}\right]^n \Psi \quad (4.43') \quad (4.43')$$

for arbitrary powers n . If we consider $c\sqrt{\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + (mc)^2}$ to be defined by the corresponding power expansion, we have also:

$$c\sqrt{\left(-i\hbar\nabla - \frac{q}{c}\vec{A}'\right)^2 + (mc)^2}\Psi' = e^{i\frac{q}{\hbar c}\Lambda}c\sqrt{\left(-i\hbar\nabla - \frac{q}{c}\vec{A}\right)^2 + (mc)^2}\Psi \quad (4.44)$$

One may define the following operators:

$$\left. \begin{aligned} D_t &\equiv cD_0 = \left(\frac{\partial}{\partial t} + i\frac{q}{\hbar}\Phi\right) & D^\mu &\equiv \partial^\mu + i\frac{q}{\hbar c}A^\mu \\ \vec{D} &= \left(\nabla - i\frac{q}{\hbar c}\vec{A}\right) & D_\mu &\equiv \partial_\mu + i\frac{q}{\hbar c}A_\mu \end{aligned} \right\} \quad (4.45)$$

They are called the "covariant derivatives" and have the following transformation properties

$$D'_t\Psi' = e^{i\frac{q}{\hbar c}\Lambda}D_t\Psi \quad (4.46)$$

$$\vec{D}'\Psi' = e^{i\frac{q}{\hbar c}\Lambda}\vec{D}\Psi \quad (4.47)$$

Summary:

If we perform a gauge transformation on the fields A^μ , such that $A^\mu \rightarrow A'^\mu = A^\mu - \partial^\mu\Lambda$, corresponding to $\vec{A}' = \vec{A} + \nabla\Lambda$ and $\Phi' = \Phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}$, and if we perform on the wave function simultaneously the transformation $\Psi' = e^{i\frac{q}{\hbar c}\Lambda(\vec{z},t)}\Psi$, then

$$H'\Psi' = e^{i\frac{q}{\hbar c}\Lambda}H\Psi - \frac{q}{c}\frac{\partial\Lambda}{\partial t}e^{i\frac{q}{\hbar c}\Lambda}\Psi$$

$$i\hbar\frac{\partial\Psi'}{\partial t} = e^{i\frac{q}{\hbar c}\Lambda}i\hbar\frac{\partial\Psi}{\partial t} - \frac{q}{c}\frac{\partial\Lambda}{\partial t}e^{i\frac{q}{\hbar c}\Lambda}\Psi$$

or

$$\left(H' - i\hbar\frac{\partial}{\partial t}\right)\Psi' = e^{i\frac{q}{\hbar c}\Lambda}\left(H - i\hbar\frac{\partial}{\partial t}\right)\Psi = 0$$

and $H'\Psi' = i\hbar\frac{\partial\Psi'}{\partial t}$ and $H\Psi = i\hbar\frac{\partial\Psi}{\partial t}$ describe the same physics. In addition we have:

$$i\hbar D'^\mu\Psi' = e^{i\frac{q}{\hbar c}\Lambda}i\hbar D^\mu\Psi \quad (4.48)$$

and

$$|\Psi'|^2 = |\Psi|^2 \quad (4.49)$$

Operators containing the derivatives ∇ and $\frac{\partial}{\partial t}$ are not gauge invariant. An example is the operator of the current density of non relativistic quantum theory:

$\vec{j} \sim (\Psi^* \nabla \Psi - \nabla \Psi^* \Psi)$. One obtains gauge invariant expressions if one replaces the derivatives ∇ and $\frac{\partial}{\partial t}$ by the corresponding covariant derivatives (D, \bar{D}) . For example

$$\vec{j}' \sim \Psi'^* \bar{D}' \Psi' - (\bar{D}' \Psi')^* \Psi' = \Psi^* D \Psi - (D \Psi)^* \Psi \quad (4.50)$$

is a gauge invariant expression. The gauge invariance of Maxwell's equations and of $(H - q\Phi)$ remains valid in quantum mechanics, provided we perform the transformation on the fields and wave function:

$$\left. \begin{aligned} A^\mu &\rightarrow A'^\mu = A^\mu - \partial^\mu \Lambda \\ \Psi &\rightarrow \Psi' = e^{i\frac{q}{\hbar c} \Lambda(\vec{x}, t)} \Psi \end{aligned} \right\} \quad (4.51)$$

Any wave equation containing the operators $\partial^\mu (\partial_\mu)$ can be made gauge invariant by replacing the derivatives $\partial^\mu (\partial_\mu)$ by the corresponding covariant derivatives.

$$\left. \begin{aligned} \partial^\mu &\rightarrow D^\mu = \partial^\mu + i\frac{q}{\hbar c} A^\mu \\ \partial_\mu &\rightarrow D_\mu = \partial_\mu + i\frac{q}{\hbar c} A_\mu \end{aligned} \right\} \quad (4.52)$$

If this latter substitution is performed in the equation for the free particle, one gets automatically the corresponding equation for the particle moving in the field A^μ .

The gauge principle

One can reverse the foregoing arguments and request the theory to be invariant under local phase transformations (i.e. the phase depends on \vec{x} and t):

$$\Psi(\vec{x}, t) \rightarrow \Psi'(\vec{x}, t) = e^{i\alpha(\vec{x}, t)} \Psi(\vec{x}, t)$$

One notices then that this requirement cannot be fulfilled by a free theory. The theory must be a theory with interaction, where the interaction with the particle is described by

a four vector field. The form of the interaction is completely determined up to a gauge transformation:

$$A^\mu \rightarrow A'^\mu = A^\mu - \partial^\mu \Lambda \quad \text{if} \quad \Psi \rightarrow \Psi' = e^{i\frac{q}{\hbar c} \Lambda(\vec{x}, t)} \Psi$$

This is very easy to show since in our case we know already the answer. We first investigate the time dependence.

$$\Psi' = e^{i\alpha} \Psi$$

$$\frac{\partial \Psi'}{\partial t} = e^{i\alpha} \left[\frac{\partial}{\partial t} + i \frac{\partial \alpha}{\partial t} \right] \Psi$$

In full generality we can set

$$\frac{\partial \alpha}{\partial t} = \frac{q}{\hbar} (\Phi - \Phi')$$

where Φ and Φ' are two scalar functions and the factor $\frac{q}{\hbar}$ has been added, since we want to obtain $\alpha = \frac{q}{\hbar c} \Lambda$. We obtain in this way for arbitrary powers n :

$$\left(\frac{\partial}{\partial t} + i\frac{q}{\hbar} \Phi' \right)^n \Psi' = e^{i\alpha} \left(\frac{\partial}{\partial t} + i\frac{q}{\hbar} \Phi \right)^n \Psi$$

Next we consider the dependence on the space coordinates:

$$\vec{p}_{op} \Psi' = (-i\hbar \nabla) (e^{i\alpha} \Psi) = -i\hbar e^{i\alpha} [\nabla + i\nabla\alpha] \Psi$$

Again we can set in full generality.

$$\nabla\alpha = \frac{q}{\hbar c}(\vec{A}' - \vec{A}) = \frac{q}{\hbar c}\nabla\Lambda$$

and obtain

$$\left(-i\hbar\nabla - \frac{q}{c}\vec{A}'\right)^n \Psi' = e^{i\alpha} \left(-i\hbar\nabla - \frac{q}{c}\vec{A}\right)^n \Psi$$

This shows that $\frac{\partial\Psi'}{\partial t} \neq e^{i\alpha}\frac{\partial\Psi}{\partial t}$ and $c\sqrt{\vec{p}_{op}^2 + (m_0c)^2}\Psi' = i\hbar\frac{\partial\Psi'}{\partial t}$ is not invariant under local gauge transformations. The invariant quantity is

$$c\sqrt{\left(\vec{p}_{op} - \frac{q}{c}\vec{A}\right)^2 + (mc)^2}\Psi = i\hbar\left(\frac{\partial}{\partial t} + i\frac{q}{\hbar}\Phi\right)\Psi$$

where the vector field satisfies the transformation

$$\vec{A}' - \vec{A} = \frac{\hbar c}{q}\nabla\alpha = \nabla\Lambda$$

$$\Phi' - \Phi = -\frac{\hbar}{q}\frac{\partial\alpha}{\partial t} = -\frac{1}{c}\frac{\partial\Lambda}{\partial t}$$

The presence of the vector field A^μ which acts in a universal, prescribed manner on a particle with charge q is enforced by the requirement of local phase invariance of the wave function $\Psi \rightarrow \Psi' = e^{i\alpha}\Psi$. Such a vector field introduced to guarantee local phase invariance of Ψ is called "gauge field". The principle to enforce the presence of an interaction by requesting local phase (gauge) invariance is called "gauge principle".

One may decompose Ψ into its real - and imaginary part $\Psi = \Psi_R + i\Psi_I$. We have then

$$\Psi' = e^{i\alpha}\Psi = (\Psi'_R + i\Psi'_I) = \cos\alpha\Psi_R - \sin\alpha\Psi_I + i(\sin\alpha\Psi_R + \cos\alpha\Psi_I)$$

or

$$\Psi'_R = \cos\alpha\Psi_R - \sin\alpha\Psi_I$$

$$\Psi'_I = \sin\alpha\Psi_R + \cos\alpha\Psi_I$$

We see from this that $\Psi \rightarrow \Psi' = e^{i\alpha}\Psi$ corresponds to some kind of rotation in the (Ψ_R, Ψ_I) plane (This plane is a kind of internal space and the underlying symmetry is called an internal symmetry). The invariance under local gauge transformations corresponds to the invariance under rotations in the internal space. Here, in the case of the electromagnetic field, all such transformations form a commutative group (Abelian group), in this case U(1), the group of unitary one - dimensional matrices ($UU^+ = U^+U = 1$). We have dealt with the problem of gauge invariance in such an elaborate manner because the gauge principle is fundamental to all modern elementary particle physics although the underlying internal symmetry spaces in strong and weak interactions are much more complicated than the simple case of the electromagnetic field. The relevant symmetry groups (gauge groups) are then no longer commutative (they are non - Abelian).

4.5.2 Relativistic wave equations

4.5.2.1 The Klein - Gordon equation

a) Free particles

We start the discussion with the free particle Hamilton function H_0 (4.21), which provides us with the correct energy - momentum relation.

$$H_0 = \sqrt{c^2\vec{p}^2 + (mc^2)^2} = E \quad (4.21)$$

Since we want to follow the usual quantisation procedure (see (4.38)), we would have to deal with the square root of an operator function. To avoid this problem one takes the square of (4.21):

$$\left(\frac{E}{c}\right)^2 = \vec{p}^2 + (mc)^2 \quad (4.53)$$

or with $p^0 = p_0 = \frac{E}{c}$:

$$p_\mu p^\mu = (mc)^2 \quad (4.54)$$

If now the replacement $p^\mu \rightarrow i\hbar\partial^\mu$ ($E \Rightarrow i\hbar\frac{\partial}{\partial t} = i\hbar c\partial^0$) is performed, one arrives immediately at the Klein - Gordon equation for the free particle. The wave function Ψ in this case is a scalar function.

$$p_\mu p^\mu \Psi = (mc)^2 \Psi$$

If we set $\mu \equiv \frac{mc}{\hbar}$ and $\partial_\mu \partial^\mu \equiv \square$ (there should be no confusion in using the same greek character for mass (measured in fm^{-1} and the indices μ).

$$(\square + \mu^2)\Psi = 0 \quad (4.55)$$

In contrast to the Schroedinger equation of nonrelativistic quantum mechanics is the Klein - Gordon equation (4.55) Lorentz invariant. From the fact that we squared H_0 to derive (4.55) there arises immediately a problem. The Klein - Gordon equation must have solutions corresponding to positive and negative energies:

$$E = \pm c \sqrt{\vec{p}^2 + (mc)^2} \quad (4.56)$$

We note already at this point that negative energy solution will be associated with antiparticles. As in nonrelativistic quantum mechanics one can derive expressions for a "probability current density". The procedure is exactly the same: take the Klein - Gordon equation for Ψ and multiply by Ψ^* and subtract from it the complex conjugated equation for Ψ^* , multiplied by Ψ and obtain:

$$\frac{1}{c} \frac{\partial}{\partial t} \left(\frac{1}{c} \frac{\partial \Psi^*}{\partial t} \Psi - \frac{1}{c} \frac{\partial \Psi}{\partial t} \Psi^* \right) + \nabla(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) = 0$$

From this equation we can define a current density \vec{j} and a density ρ :

$$\left. \begin{aligned} \vec{j} &= \frac{c}{i} (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) \\ \rho &= \frac{1}{i} \left(\Psi \frac{1}{c} \frac{\partial \Psi^*}{\partial t} - \Psi^* \frac{1}{c} \frac{\partial \Psi}{\partial t} \right) \end{aligned} \right\} \quad (4.57)$$

where we have added arbitrarily the factor $\frac{c}{i}$. The solutions of the Klein - Gordon equation for the free particle will be plane waves.

$$\Psi = \mathcal{N} e^{-ik_\mu x^\mu} \quad (4.58)$$

The normalisation factor will be specified later. With this form for Ψ we obtain from (4.57) :

$$\left. \begin{aligned} \vec{j} &= 2\vec{k}c |\mathcal{N}|^2 \\ \rho &= 2k^0 |\mathcal{N}|^2 \end{aligned} \right\} \quad (4.57')$$

From the last of this two equations (4.57') we see that the "probability density" has the sign of the energy. ρ is therefore not a positive definite quantity. Therefore it can not be identified with a probability density. With $j^0 = c\rho$ we obtain from (4.57'):

$$j^\mu = \frac{c}{i} [\Psi \partial^\mu \Psi^* - \Psi^* \partial^\mu \Psi] \quad (4.57'')$$

and the continuity equation is as usual a conservation for the four current density j^μ

$$\partial_\mu j^\mu = 0 \quad (4.59)$$

To each \vec{k} there are two solutions k^0 :

$$k^0 = \pm \epsilon_{\vec{k}} = \pm \sqrt{\vec{k}^2 + \mu^2} \quad (\epsilon_{\vec{k}} > 0) \quad (4.60)$$

Our plane wave solutions will therefore be of the form

$$\tilde{\Psi}_{\pm} = \tilde{\mathcal{N}}_{\pm} e^{\mp i \epsilon_{\vec{k}} t + i \vec{k} \cdot \vec{x}} \quad (4.61)$$

To avoid the problems with the non-positive definite ρ we consider the electric charge density and electric current density. Integration of (4.59) leads then to charge conservation. We multiply j^{μ} with e ($e > 0$) and obtain for $\tilde{\Psi}_{(+)}$

$$c \tilde{\rho}_{(+)}^{el} = \frac{ec}{i} [\tilde{\Psi}_{(+)} \partial^0 \tilde{\Psi}_{(+)}^* - \tilde{\Psi}_{(+)}^* \partial^0 \tilde{\Psi}_{(+)}] = 2ec \epsilon_{\vec{k}} | \tilde{\mathcal{N}}_{(+)} |^2 \quad (4.62)$$

and for $\tilde{\Psi}_{(-)}$

$$c \tilde{\rho}_{(-)}^{el} = \frac{ec}{i} [\tilde{\Psi}_{(-)} \partial^0 \tilde{\Psi}_{(-)}^* - \tilde{\Psi}_{(-)}^* \partial^0 \tilde{\Psi}_{(-)}] = -2ec \epsilon_{\vec{k}} | \tilde{\mathcal{N}}_{(-)} |^2 \quad (4.63)$$

For the electric current densities we obtain:

$$\vec{j}_{(+)}^{el} = \frac{ec}{i} [\tilde{\Psi}_{(+)}^* \nabla \tilde{\Psi}_{(+)} - \tilde{\Psi}_{(+)} \nabla \tilde{\Psi}_{(+)}^*] = 2ec \vec{k} | \tilde{\mathcal{N}}_{(+)} |^2 \quad (4.64)$$

$$\vec{j}_{(-)}^{el} = \frac{ec}{i} [\tilde{\Psi}_{(-)}^* \nabla \tilde{\Psi}_{(-)} - \tilde{\Psi}_{(-)} \nabla \tilde{\Psi}_{(-)}^*] = 2ec \vec{k} | \tilde{\mathcal{N}}_{(-)} |^2 = -2ec(-\vec{k}) | \tilde{\mathcal{N}}_{(-)} |^2 \quad (4.65)$$

We see that $\tilde{\rho}_{(+)}^{el}$ and $\tilde{\rho}_{(-)}^{el}$ can be interpreted as charge densities for positive and negative charged particles, both, however with positive energy $k_0 = +\epsilon_{\vec{k}}$. A similar

interpretation is possible for the current densities. We have however to deal with a minor difficulty (as seen from (4.65)), - the solution for negative charge and positive energy occurs with momentum $(-\vec{k})$ instead of \vec{k} . If we write the expressions in four vector notation, we have

$$\tilde{j}_{(+)}^{el,\mu} = 2ec(\epsilon_{\vec{k}}, \vec{k}) | \tilde{\mathcal{N}}_{(+)} |^2 = 2eck^{\mu} | \tilde{\mathcal{N}}_{(+)} |^2$$

and

$$\tilde{j}_{(-)}^{el,\mu} = 2ec(-\epsilon_{\vec{k}}, \vec{k}) | \tilde{\mathcal{N}}_{(-)} |^2 = -2ec(\epsilon_{\vec{k}}, -\vec{k}) | \tilde{\mathcal{N}}_{(-)} |^2$$

For a particle with negative charge and positive energy we would however expect:

$$\tilde{j}_{(+)}^{el,\mu}(-e) = -2ec(\epsilon_{\vec{k}}, \vec{k}) | \tilde{\mathcal{N}}_{(+)} |^2 = 2ec(-\epsilon_{\vec{k}}, -\vec{k}) | \tilde{\mathcal{N}}_{(+)} |^2 = 2ec(-k^{\mu}) | \tilde{\mathcal{N}}_{(+)} |^2 = \tilde{j}_{(-)}^{el,\mu}(+e)$$

To obtain this result we have to choose for $\Psi_{(-)} = \mathcal{N}_{(-)} e^{ik_0 x^0 - i \vec{k} \cdot \vec{x}}$. With the definition

$$\Psi_{\pm} = \mathcal{N}_{\pm} e^{ik_{\mu} x^{\mu}} \quad k_0 = k^0 = +\epsilon_{\vec{k}} > 0 \quad (4.66)$$

we obtain the desired result. This choice of solution leads to the same results for the charge densities as before. The normalisation factor is obtained by integrating equation (4.57) (charge conservation). One obtains

$$| \mathcal{N}_{+} |^2 = | \mathcal{N}_{-} |^2 = \frac{1}{2\epsilon_{\vec{k}} V_{ol}}$$

and

$$j_{\pm}^{\mu} = \pm 2eck^{\mu} | \mathcal{N}_{\pm} |^2 = \pm cc \frac{k^{\mu}}{\epsilon_{\vec{k}} V_{ol}} = \pm \frac{c}{V_{ol}}(c, \vec{v}) \quad (4.67)$$

The two solutions Ψ_{\pm} correspond physically to particle and antiparticle. In principle it is completely irrelevant which of both is treated as particle or antiparticle. The same

interpretation of the current density j^μ can be applied to neutral particles. In this case one has to choose the wave function to be real. In this case $j^\mu = 0$ and there is no conservation law. Relativistic quantum theory leads us obviously to new degrees of freedom, the charge degree of freedom. In the present case of the motion of a particle with spin=0 and momentum \vec{k} , we see that there are 3 charge degrees of freedom ($\pm e$ and 0). In nature this is approximately realized for the π mesons (π^\pm, π^0) if we neglect the small mass difference between π^\pm 's and π^0 .

b) Spinless, charged particle in an electromagnetic field

The Klein - Gordon equation for the motion of the particle in an electromagnetic field A^μ is obtained from the free Klein - Gordon equation by replacing all derivatives ∂^μ by the corresponding covariant derivatives D^μ . From

$$(\partial_\mu \partial^\mu + \mu^2)\Psi = 0$$

one obtains

$$(D_\mu D^\mu + \mu^2)\Psi = 0 \quad (4.68)$$

or

$$\left(\partial_\mu + i\frac{q}{\hbar c}A_\mu\right)\left(\partial^\mu + i\frac{q}{\hbar c}A^\mu\right)\Psi + \mu^2\Psi = 0$$

or

$$(\square + \mu^2)\Psi = -V_{KG}^{op}\Psi \quad (4.69)$$

with

$$V_{KG}^{op} \equiv i\frac{q}{\hbar c}(\partial_\mu A^\mu + A_\mu \partial^\mu) - \left(\frac{q}{\hbar c}\right)^2 A_\mu A^\mu = 2A^\mu \partial_\mu + \partial_\mu A^\mu \quad (4.70)$$

If we go to the non relativistic approximation of (4.69), we obtain the usual Schroedinger equation for a spinless particle moving in the electromagnetic field A^μ .

4.5.2.2 The Dirac equation

a) The free Dirac - particle

In order to avoid the problem of the not positive definite probability density, Dirac chose a different way. Corresponding to the Schroedinger equation $H\Psi = i\hbar\frac{\partial\Psi}{\partial t}$, he postulated the existence of a wave equation linear in $\frac{\partial\Psi}{\partial t}$. Relativistic covariance then requires also linearity in ∇ . Dirac postulated the following equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = c[-i\hbar\vec{\alpha}\nabla + \beta mc]\Psi = c\left[-i\hbar\sum_{j=1}^3\alpha^j\frac{\partial}{\partial x^j} + \beta mc\right]\Psi \quad (4.71)$$

with yet to be determined quantities α^i, β . They can be determined if one imposes the following conditions:

1. The correct relation between energy and momentum must be preserved

$$E = c\sqrt{\vec{p}^2 + (mc)^2}$$

2. The equation should be covariant under Lorentz transformations.

In-order that condition 1) be satisfied, one can request that Ψ should also be a solution of the Klein - Gordon equation. Squaring (4.71) leads to

$$\begin{aligned} \left(i\hbar\frac{\partial}{\partial t}\right)^2\Psi &= c^2(-i\hbar\vec{\alpha}\nabla + \beta mc)(-i\hbar\vec{\alpha}\nabla + \beta mc)\Psi \\ &= c^2\left[(-i\hbar)^2\sum_{i=1}^3(\alpha^i)^2\partial_i\partial_i + (-i\hbar)^2\sum_{i>j}(\alpha^i\alpha^j + \alpha^j\alpha^i)\partial_i\partial_j - i\hbar cm\sum_{i=1}^3(\alpha^i\beta + \beta\alpha^i)\partial_i + (mc)^2\beta^2\right]\Psi \end{aligned} \quad (4.72)$$

If Ψ satisfies also the Klein - Gordon equation we have also

$$\left(-i\hbar\frac{\partial}{\partial t}\right)^2\Psi = -(\hbar c)^2\sum_{i=1}^3\partial_i\partial_i\Psi + (mc)^2\Psi \quad (4.73)$$

The quantities α^i and β can now be determined by comparing the coefficients of equal derivatives in (4.72) and (4.73). It is immediately clear that the α^i and β can not be ordinary numbers, but must be matrices. It turns out that the smallest rank of these matrices is 4. Therefore also Ψ has to be a four component object (it is called a four component spinor). If we do the comparison mentioned above, we get the following algebra for the matrices:

$$\alpha^i \beta + \beta \alpha^i = 0 \quad i = 1, 2, 3 \quad (4.74)$$

$$\alpha^i \alpha^j + \alpha^j \alpha^i = 0 \quad i, j = 1, 2, 3, i \neq j \quad (4.74')$$

$$(\alpha^i)^2 = \beta^2 = 1 \quad i = 1, 2, 3 \quad (4.74'')$$

In order to make the Hamiltonian H Hermitian, also the matrices α^i and β have to be Hermitian. These matrices are determined up to a unitary transformation. Below we give the so-called standard representation:

$$\alpha^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4.75)$$

where the σ_i are the usual 2×2 Pauli - spin matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4.76)$$

Usually a different set of matrices, γ^μ , γ^5 is used, which we present below also in their standard representation (see Bjorken and Drell, Relativistic Quantum Mechanics, New York, McGraw - Hill (1964)):

$$\vec{\gamma} = \beta \vec{\alpha}, \quad \gamma^0 = \beta, \quad \gamma^\mu = (\gamma^0, \vec{\gamma}), \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \quad (4.77)$$

With (4.75) one finds their explicit form:

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (4.77')$$

The γ matrices obey the commutation relations:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \quad (4.78)$$

A very often used combination of γ matrices is

$$\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] \quad (4.79)$$

In explicit form:

$$\left. \begin{aligned} \sigma^{ij} &= -\frac{i}{2} [\sigma_i, \sigma_j] \underline{1} = \sigma_k \underline{1} \quad i, j, k \text{ cyclic} \\ \sigma^{0i} &= i \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} = i\sigma_i \gamma^5 \end{aligned} \right\} \quad (4.79')$$

Writing the Dirac equation (4.71) in terms of the γ matrices leads to

$$H_{Dir} = hc\gamma^0(\vec{\gamma}\vec{k}_{op} + \mu\underline{1}) = c(\vec{\alpha}\vec{p}_{op} + \gamma^0 mc) \quad (4.80)$$

or

$$H_{Dir}\Psi = hc\gamma^0(-i\gamma^j\partial_j + \mu\underline{1})\Psi = ihc\partial^0\Psi = ihc\partial_0\Psi$$

or

$$\frac{\gamma^0}{\hbar c} H_{Dir} \Psi = (-i\gamma^j \partial_j + \mu \underline{1}) \Psi = i\gamma^0 \partial_0 \Psi$$

or finally

$$(i\gamma^\mu \partial_\mu - \mu \underline{1}) \Psi = \frac{\gamma^0}{\hbar c} \left[i\hbar \frac{\partial}{\partial t} - H_{Dir} \right] \Psi = 0 \quad (4.81)$$

Usually (4.81) is called Dirac equation.

In order to find the plane wave solution of the Dirac equation, corresponding to a free particle, we make the following Ansatz:

$$\Psi = \frac{1}{\sqrt{V_{ol}}} e^{-ik_\mu x^\mu} \omega \quad (4.82)$$

where ω is a four - component spinor still to be determined. One can write ω in the form

$$\omega = \begin{pmatrix} \omega_{up} \\ \omega_{low} \end{pmatrix} \quad (4.83)$$

where ω_{up} and ω_{low} are two 2 - component spinors. Using this Ansatz in the Dirac equations, one obtains

$$(\gamma^\mu k_\mu - \mu \underline{1}) \omega = 0 \quad (4.84)$$

In matrix notation one has

$$\gamma^\mu k_\mu = \begin{pmatrix} k_{0\underline{1}} & -\vec{\sigma} \vec{k} \\ \vec{\sigma} \vec{k} & -k_{0\underline{1}} \end{pmatrix}$$

and from (4.84):

$$\begin{pmatrix} (k_0 - \mu) \underline{1} & -\vec{\sigma} \vec{k} \\ \vec{\sigma} \vec{k} & -(k_0 + \mu) \underline{1} \end{pmatrix} \begin{pmatrix} \omega_{up} \\ \omega_{low} \end{pmatrix} = 0$$

or

$$\left. \begin{aligned} (k_0 - \mu) \omega_{up} - (\vec{\sigma} \vec{k}) \omega_{low} &= 0 \\ (\vec{\sigma} \vec{k}) \omega_{up} - (k_0 + \mu) \omega_{low} &= 0 \end{aligned} \right\} \quad (4.85)$$

Eliminating for example ω_{low} , we get from the second equation (4.85)

$$\omega_{low} = \frac{\vec{\sigma} \vec{k}}{k_0 + \mu} \omega_{up}$$

and

$$\omega = \left\{ \begin{array}{c} \omega_{up} \\ \frac{\vec{\sigma} \vec{k}}{k_0 + \mu} \omega_{up} \end{array} \right\} \quad (4.86)$$

Inserting ω_{low} into the first equation (4.85) we get

$$\left[(k_0 - \mu) \underline{1} - \frac{(\vec{\sigma} \vec{k})(\vec{\sigma} \vec{k})}{k_0 + \mu} \right] \omega_{up} = 0$$

which with $(\vec{\sigma} \vec{k})(\vec{\sigma} \vec{k}) = k^2 \underline{1}$ transforms into

$$\left(\frac{k_0^2 - \mu^2 - k^2}{k_0 + \mu} \right) \omega_{up} = 0 \quad (4.87)$$

From this we see that $k_0^2 = k^2 + \mu^2$ or

$$k_0 = \pm \epsilon_{\vec{k}} = \pm \sqrt{\vec{k}^2 + \mu^2} \quad (4.88)$$

As we could have expected, we are back to the problem of negative energies. Let us consider first the case of positive energies $k_0 = +\epsilon_{\vec{k}}$. In the rest frame of the particle ($\vec{k} = 0$) we obtain for ω :

$$\omega_{rest} = \begin{Bmatrix} \omega_{up} \\ \vec{0} \end{Bmatrix}$$

In the rest frame we can use for ω_{up} the usual two component spinor $\chi(s)$ known from nonrelativistic quantum mechanics:

$$\omega_{up} = \chi(s) = \begin{Bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad s = +1 \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad s = -1 \end{Bmatrix} \quad (4.89)$$

where $s = +1$ and -1 describe the two spin states of the particle. These form of ω_{up} can now be used in (4.86) and we have

$$\omega = \begin{Bmatrix} \chi(s) \\ \frac{\vec{\sigma}\vec{k}}{k_0 + \mu} \chi(s) \end{Bmatrix} \quad \text{for } k_0 = +\sqrt{\vec{k}^2 + \mu^2}$$

The spinor may be normalised:

$$\omega^\dagger \omega = \left(\chi^\dagger(s), \chi^\dagger(s) \frac{\vec{\sigma}\vec{k}}{k_0 + \mu} \right) \begin{Bmatrix} \chi(s) \\ \frac{\vec{\sigma}\vec{k}}{k_0 + \mu} \chi(s) \end{Bmatrix} = \chi^\dagger(s) \left[1 + \frac{(\vec{\sigma}\vec{k})(\vec{\sigma}\vec{k})}{(\epsilon_k + \mu)^2} \right] \chi(s) =$$

$$\chi^\dagger(s) \left[1 + \frac{\vec{k}^2}{(\epsilon_k + \mu)^2} \right] \chi(s) = \chi^\dagger(s) \chi(s) \frac{2\epsilon_k}{\epsilon_k + \mu} = \frac{2\epsilon_k}{\epsilon_k + \mu}$$

The normalized, positive energy spinor is therefore given by

$$u(\vec{k}, s) \equiv \omega_{(+)} = \sqrt{\frac{\epsilon_k + \mu}{2\epsilon_k}} \begin{Bmatrix} \chi(s) \\ \frac{\vec{\sigma}\vec{k}}{\epsilon_k + \mu} \chi(s) \end{Bmatrix} \quad s = \pm 1 \quad (4.90)$$

Next we consider the negative energy solutions ($k_0 = -\epsilon_{\vec{k}}$). In the rest frame we have $\omega_{up} = 0$. For ω_{low} we choose again a two - component Pauli spinor. To get consistent description of antiparticles we choose

$$\omega_{low} = \hat{\chi}(s) = \begin{Bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad s = +1 \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad s = -1 \end{Bmatrix} \quad (4.91)$$

This choice is based on the following idea. In Dirac's interpretation of antiparticles in the ground state all the negative energy states are filled. If a negative energy particle is excited above the ground state, a hole is left in the "filled sea".

It is the absence of a negative energy, negative charged, spin down state that corresponds to a antiparticle with positive energy, positive charge and spin up. This rather artificial reasoning is avoided in quantum field theory. If in (4.85) we eliminate ω_{up} we get for ω ($k_0 = -\epsilon_{\vec{k}} < 0$, $\omega_{low} = \hat{\chi}(s)$):

$$\omega = \begin{Bmatrix} -\frac{\vec{\sigma}\vec{k}}{\epsilon_k + \mu} \hat{\chi}(s) \\ \hat{\chi}(s) \end{Bmatrix}$$

This would correspond to a particle with four momentum $(-\epsilon_{\vec{k}}, \vec{k})$. We expect however

as we did in case of the Klein - Gordon equation, that the antiparticles should have a four momentum $(-\epsilon_{\vec{k}}, -\vec{k})$. Therefore we define the normalised spinor by the above given expression, however \vec{k} changed into $(-\vec{k})$:

$$v(\vec{k}, s) \equiv \omega_{(-)} = \sqrt{\frac{\epsilon_{\vec{k}} + \mu}{2\epsilon_{\vec{k}}}} \begin{Bmatrix} \frac{\vec{\sigma} \cdot \vec{k}}{\epsilon_{\vec{k}} + \mu} \chi(s) \\ \chi(s) \end{Bmatrix} \quad s = \pm 1 \quad (4.92)$$

The plane wave solutions are then given by

$$\left. \begin{aligned} \Psi_{(+)} &= \frac{1}{\sqrt{V_{ol}}} e^{-ik_{\mu}x^{\mu}} u(\vec{k}, s) \quad \text{particle} \\ \Psi_{(-)} &= \frac{1}{\sqrt{V_{ol}}} e^{+ik_{\mu}x^{\mu}} v(\vec{k}, s) \quad \text{antiparticle} \end{aligned} \right\} \quad (4.93)$$

Now the probability density and current density concept may be worked out. We start from the Dirac equation

$$i\hbar \frac{\partial \Psi}{\partial t} = c(-i\hbar \vec{\alpha} \nabla \Psi) + \gamma^0 mc^2 \Psi \quad (4.94)$$

and the Hermitian conjugate equation ($\gamma^{0+} = \gamma^0$, $\vec{\alpha}^+ = \vec{\alpha}$):

$$-i\hbar \frac{\partial \Psi^+}{\partial t} = c i \hbar \nabla \Psi^+ \vec{\alpha} + \Psi^+ \gamma^0 mc^2 \quad (4.95)$$

Multiplying (4.94) from the left with Ψ^+ and (4.95) from the right with Ψ and subsequent subtraction leads to

$$i\hbar \frac{\partial}{\partial t} (\Psi^+ \Psi) = -i\hbar c \nabla (\Psi^+ \vec{\alpha} \Psi)$$

Since $\Psi^+ \Psi$ is positive definite, it can be interpreted as probability density.

$$\rho = (\Psi^+ \Psi) > 0 \quad (4.96)$$

We can also define a current density \vec{j} by

$$\vec{j} = c \Psi^+ \vec{\alpha} \Psi \quad (4.97)$$

which can be interpreted as probability current density. ρ and \vec{j} satisfy the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$$

Usually one uses the adjoint spinors instead of Ψ^+ .

$$\bar{\Psi} \equiv \Psi^+ \gamma^0 \quad \text{adjoint spinor} \quad (4.98)$$

In four vector notation we have $j^{\mu} = (c\rho, \vec{j})$:

$$j^{\mu} = c \bar{\Psi} \gamma^{\mu} \Psi \quad (4.99)$$

and the conservation law

$$\partial_{\mu} j^{\mu} = 0 \quad (4.100)$$

Using (4.93) we are led to the explicit form of the current density:

$$j_{(+)}^{\mu} = \frac{c}{V_{ol}} \frac{k^{\mu}}{\epsilon_{\vec{k}}} = \frac{1}{V_{ol}} (c, \vec{v}) \quad (4.101)$$

$$j_{(-)}^{\mu} = \frac{c}{V_{ol}} \frac{k^{\mu}}{\epsilon_{\vec{k}}} = \frac{1}{V_{ol}} (c, \vec{v}) \quad (4.101')$$

In contrast to the Klein - Gordon equation, where we had

$$j_{(-),el}^{\mu}(+ | e |) = j_{(+),el}^{\mu}(- | e |)$$

we see from (4.101') that this is not true for the Dirac equation. Therefore, if we want to interpret the negative energy solutions of the Dirac equation as antiparticles of positive energy, one has to introduce by hand a minus sign in (4.101'). The presence of this minus sign is proved in field theory. It is there due to the fact that the current density operator j^{μ} is defined by a "normal product" $\mathcal{N}\{\bar{\Psi}(x)\gamma^{\mu}\Psi(x)\}$. There is an anticommutator between fermion operators involved which gives automatically the minus sign.

b) The spin $\frac{1}{2}$ particle in an electromagnetic field

As in case of the Klein - Gordon equation we will obtain the Dirac equation for the particle in the field A^{μ} by replacing all ∂^{μ} 's by the corresponding D^{μ} 's. For the free particle we have:

$$H_{Dir}^0 = \hbar c \gamma^0 [-i\gamma^i \partial_i + \mu] = i\hbar c \partial_0 \quad (4.102)$$

In the field A^{μ} we have

$$\begin{aligned} \hbar c \gamma^0 [-i\gamma^i D_i + \mu] &= i\hbar c D_0 \\ &= \hbar c \gamma^0 \left[-\gamma^i \left(\partial_i + i \frac{q}{\hbar c} A_i \right) + \mu \right] = i\hbar c \left(\partial_0 + i \frac{q}{\hbar c} A^0 \right) \end{aligned}$$

or

$$\hbar c \gamma^0 \left[-i\gamma^i \partial_i + \mu + \frac{q}{\hbar c} \gamma^i A_i \right] = i\hbar c \partial_0 - q A^0$$

or

$$\hbar c \gamma^0 [-i\gamma^i \partial_i + \mu] + q \gamma^0 \gamma^i A_i + q A^0 = i\hbar c \partial_0$$

With

$$H_{Dir}^0 = \hbar c \gamma^0 [-i\gamma^i \partial_i + \mu] \quad \text{and} \quad q \gamma^0 \gamma^i A_i + q A^0 = q \gamma^0 \gamma^{\mu} A_{\mu} = q(A^0 - \gamma^0 \vec{\gamma} \vec{A})$$

we get

$$H_{Dir} = H_{Dir}^0 + q(A^0 - \gamma^0 \vec{\gamma} \vec{A}) = i\hbar c \partial_0 \quad (4.103)$$

From (4.99) one sees that the four vector

$$j_{op}^{\mu} = c q \gamma^0 \gamma^{\mu} \quad (4.104)$$

can be interpreted as the current density operator. From (4.103) with (4.104) we obtain

$$H_{Dir} = H_{Dir}^0 + \frac{1}{c} j_{op}^{\mu} A_{\mu} = i\hbar c \partial_0 \quad (4.105)$$

4.5.2.3 The free electromagnetic field (massless spin=1 photons)

The free electromagnetic field outside of charge and current distributions is described according to (4.34) by

$$\square A^{\mu} = \partial^{\mu} (\partial_{\alpha} A^{\alpha}) \quad (4.106)$$

As a convenient gauge we can use the Lorentz gauge and obtain then

$$\square A^{\mu} = 0 \quad (4.107)$$

$$\partial_{\alpha} A^{\alpha} = 0 \quad (4.108)$$

The solutions to (4.107) are then plane waves of the form

$$A^{\mu} = N \epsilon^{\mu} e^{-ikx} \quad (4.109)$$

N is a normalisation constant and ϵ^μ the polarisation vector. The electromagnetic field is a massless field. Therefore we have

$$k^2 = k^\alpha k_\alpha = k^{02} - \vec{k}^2 = 0 \quad (4.110)$$

The Lorentz gauge gives us

$$\partial_\alpha A^\alpha = N \epsilon^\alpha \partial_\alpha e^{-ikx} = (-i)N e^{-ikx} \epsilon^\alpha k_\alpha = 0$$

or

$$\epsilon^\alpha k_\alpha = 0 \quad (4.111)$$

We can perform a further gauge transformation on the field, such that also the new field satisfy the Lorentz condition:

$$A'^\mu = A^\mu - \partial^\mu \Lambda \quad (4.112)$$

$$\partial_\mu A'^\mu = \partial_\mu A^\mu - \partial_\mu \partial^\mu \Lambda = -\square \Lambda$$

In order to lead to

$$\partial_\mu A'^\mu = 0$$

Λ must satisfy the equation

$$\square \Lambda = \partial_\mu \partial^\mu \Lambda = 0$$

In this case also Λ is a plane wave which can be written in the form

$$\Lambda = \Lambda_0 e^{-ikx} \quad (4.113)$$

Using this in (4.112) we obtain

$$N' \epsilon'^\mu e^{-ikx} = N \epsilon^\mu e^{-ikx} - i \Lambda_0 k^\mu e^{-ikx}$$

If we set $N' = N$ and define $\beta \equiv -i \Lambda_0 / N$, we get

$$\epsilon'^\mu = \epsilon^\mu + \beta k^\mu \quad (4.114)$$

Since A'^μ satisfies also the Lorentz condition, one has also

$$\epsilon'^\mu k_\mu = \epsilon^\mu k_\mu + \beta k^\mu k_\mu = 0 \quad (4.115)$$

One can choose β such that:

$$\epsilon'^0 = 0 \quad (4.116)$$

In this case the Lorentz condition is reduced to a condition amongst ordinary three dimensional vectors:

$$\vec{\epsilon}' \vec{k} = 0 \quad (4.117)$$

The meaning of this is, that there are only two linearly independent polarisation vectors, transversal to the momentum vector. If we choose

$$k^\mu = (k^0, 0, 0, k^0) \quad (4.118)$$

the two independent polarisation vectors are given by

$$\vec{\epsilon}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{\epsilon}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad (4.119)$$

corresponding to

$$\epsilon_1^\mu = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \epsilon_2^\mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad (4.120)$$

This situation corresponds to linear polarisation. If one choose the standard representation in the form

$$\vec{\epsilon}(\lambda = +1) = -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{\epsilon}(\lambda = -1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \quad (4.121)$$

then one has circular polarisation. In our envisaged case of a perturbation treatment of scattering processes the meaning of the plane wave solution is the following:

$$\left. \begin{aligned} A^\mu(x) &= N \epsilon^\mu(\lambda) e^{-ikx} && \text{incoming photon} \\ A^\mu(x) &= N \epsilon^{*\mu}(\lambda) e^{ikx} && \text{outgoing photon} \end{aligned} \right\} \quad (4.122)$$

The polarisation vectors satisfy the following orthogonality relation:

$$\vec{\epsilon}^*(\lambda') \vec{\epsilon}(\lambda) = \delta_{\lambda\lambda'} \quad (4.123)$$

or

$$\epsilon^{*\mu}(\lambda') \epsilon_\mu(\lambda) = -\delta_{\lambda'\lambda} \quad (4.124)$$

We note that the presence of only two independent polarisation vectors, despite the fact that photons are spin = 1 particles, is basically a consequence of the fact that they are massless particles.

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