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МАТЕРИАЛЫ XIII МЕЖДУНАРОДНОЙ КОНФЕРЕНЦИИ ПО ФИЗИКЕ ВЫСОКИХ ЭНЕРГИЙ

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I. INTRODUCTION

Since 1955 [1] physicists have been interested in exploiting analyticity properties of the scattering matrix. It appeared that microcausality allowed to obtain a certain number of analytic properties, in particular dispersion relations, for certain scattering or reaction amplitudes. All the relatively simple results were collected in a rather short period, from 1955 to 1958 [2],[3] [though some proofs were completed only recently [4]] and, at this stage, it looked as if no further progress was possible. It also appeared that the results thus far obtained were very insufficient if one wants to combine analyticity and unitarity to get physical information on the S matrix, beyond the mere testing of dispersion relations. For this reason, a much stronger postulate of analyticity was proposed by Mandelstam. The Mandelstam representation was built in such a way that it was consistent at least with the existing analyticity properties obtained from field theory and also with lowest order perturbation theory. Since nobody could see any prospect of establishing Mandelstam representation from local field theory mathematical physicists turned their efforts toward perturbation theory with the hope (based on an analogy with potential scattering) that even if perturbation theory does not converge, it predicts correctly the singularities of the S matrix. We know now that these attempts were unsuccessful in the sense that for $\pi\pi$, πN, NN scattering, one could not prove or disprove at all orders Mandelstam representation. In the meantime most of the quantum field theoreticians became sooner or later discouraged by a search for the extension of the analyticity domain of scattering and got interested in other problems, such as reformulating field theory in a nicer way or studying questions of consistency of the axioms. In 1962, I heard a talk given by one of the best specialists on the aims and methods of local field theory. The possibility of deriving analytic properties of the scattering matrix was just not mentioned !

However, a small minority of axiomatists continued the investigation of analytic properties and obtained what seems to me very useful results which I will describe later.

More recently, these results were combined with the unitarity condition and a rather important enlargement of the analyticity domain was obtained, and in some respects this gave better results than perturbation theory. In especially favourable cases, such as $\pi\pi$ scattering, crossing symmetry allowed to extend the analyticity region so far that it begins to resemble strongly the Mandelstam domain.

II. DISPERSION RELATIONS AND LEHMANN ELLIPSE

We begin by a short review of old results. To my knowledge, forward dispersion relations have been established from field theory for the following processes :

$$\pi\pi \to \pi\pi$$
$$\pi N \to \pi N$$
$$\pi \Lambda \to \pi \Lambda$$
$$\pi \Sigma \to \pi \Sigma$$
$$\pi \Lambda \to \pi \Sigma$$
$$\pi K \to \pi K$$
$$K K \to K K$$

It is sad to see that in this list only one process ($\pi N \to \pi N$) is directly accessible to experiment. However, perturbation theory allows a derivation of forward dispersion relations for $K\bar{N}$ scattering and also $N\bar{N}$ scattering.

The validity of dispersion relations for negative transfer is connected with the existence of the Lehmann ellipse. I remind you that for the elastic case, Lehmann [3] has shown that the scattering amplitude is analytic inside an ellipse, for fixed physical energies, in the $\cos\theta$ variable where $\cos\theta$ is the centre-of-mass scattering angle. This ellipse has foci at $\cos\theta = \pm 1$ and a semi-major axis :

$$x_0 = \left[1 + \frac{(m_1^2 - m_A^2)(m_2^2 - m_B^2)}{k^2[s - (m_1 - m_2)^2]}\right]^{\frac{1}{2}}$$

where m_A and m_B are the masses of the particles. m_1 and m_2 are the lowest mass states such that

$$\langle 0 | j_A | m_1 \rangle \neq 0 \qquad \langle 0 | j_B | m_2 \rangle \neq 0$$

k is the c.m. momentum and s the square of the c.m. energy :

$$s = \left[\sqrt{m_A^2 + k^2} + \sqrt{m_B^2 + k^2}\right]^2$$

In addition, the imaginary part of the scattering amplitude in the physical region can be continued in the $\cos\theta$ plane into an analytic function called the absorptive part, which is analytic inside a bigger ellipse, with foci -1, $+1$, and semi-major axis given by $2x_0^2 - 1$.

Using this information, it is possible to derive dispersion relations not only for $\cos\theta = 1$ but also for negative transfer. We define (for the elastic case) $t = -2k^2(1-\cos\theta)$. Then the fixed t amplitude satisfies dispersion relations, when t is not too large. The results are the following :

$\pi\pi \quad -28\mu^2 < t \leq 0$ [3] $\qquad \mu =$ pion mass

$\pi N \quad -12{,}4\mu^2 < t \leq 0$ [3] $\qquad$ (now $-18.8\,\mu^2 < t \leq 0$) [5),6)]

$\pi K \quad -27\mu^2 < t \leq 0$ [3]

$KK \quad -22\mu^2 < t \leq 0$ [3]

$\pi\Lambda \quad -12\mu^2 < t \leq 0$ [7]

$\pi\Sigma \quad -8\mu^2 < t \leq 0$ [7]

$\pi\Lambda \to \pi\Sigma \quad -8\mu^2 < t \leq 0$ [7]

One important thing I should say is that these results imply the property of crossing, i.e., the possibility of continuing (on the mass shell) from the amplitude $AB \to CD$ to the amplitude $A\bar{C} \to D\bar{B}$. This crossing property is not obvious in cases where dispersion relations have not been proved.

The major defects of these results are the following :

1) the analyticity domains one obtains are "flat" in one variable : only on a segment of t real one has analyticity in s. Only for real s has one analyticity in $\cos\theta$ (or equivalently t);

2) there are processes for which dispersion relations do not hold at all. In this case can one continue from the particle-particle amplitude to the particle-antiparticle amplitude without going off the mass shell ? What happens then to the Pomeranchuk theorem, which, originally was established under the assumption that forward dispersion relations are valid ?

3) the Lehmann ellipse is too small at high energies. This, I should qualify a little bit : we expect, if we decompose the scattering amplitude in partial waves that the number of partial waves contributing effectively will be of the order of $L \simeq kR(s)$ where $R(s)$ is the apparent range of the forces at the particular energy s. Since no sharp cut-off is expected, we believe that for large angular momentum, the behaviour of partial waves will be governed by a factor $\exp-L/kR(s)$. Now, if the Lehmann formula was the optimum, one could deduce the rate of decrease of partial waves; comparing the two expressions, one finds $R(s) \sim k \sim \sqrt{s}$. This, we believe, is difficult to accept.

4) we believe that there is a certain amount of truth in the Mandelstam representation, and if we just use the results we have mentioned up to now, we see that we are very far from it.

To all these subjects of dissatisfaction, I shall give in the following Sections an answer which should make us rather optimistic on the future of field theory. By optimistic, I mean that its consequences might become some day sufficiently accurate so that one can compare it with experiment and maybe decide that it was wrong from the beginning.

III. ANALYTICITY IN TWO VARIABLES

The first successful attempt to get a non-flat analyticity domain, i.e., a domain which contains points where both s and t are complex, was done by Mandelstam [8] in the particular case of $\pi\pi$ scattering (and the academic case of a neutral scalar theory). In this work extensive use of crossing symmetry in s,t,u was made ($s+t+u=4\mu^2$). The result was that the scattering amplitude was analytic in regions delimited by analytic hypersurfaces of the form

$$|s\,t| < \text{Const} \qquad |s\,t\,u| < \text{const}$$

For instance, for the $\pi\pi$ case, one has $|s\,t| < 256\,\mu^4$. Inside this region the scattering amplitude has only the normal cuts : $t_{real} > 4\mu^2$, $s_{real} > 4\mu^2$, $u_{real} > 4\mu^2$.

Then Lehmann [9] was able to prove that complete crossing symmetry (i.e., invariance of the scattering amplitude with respect to permutations of s,t and u) was not necessary to find a domain in two complex variables. He was able to prove that for any case for which dispersion relations are valid in $-t_o < t \leqslant 0$, such a two-dimensional domain exists. I shall not describe it in detail, as we shall show better results in a moment. All I shall say is that the dimensions of this domain can be computed explicitly from the dimensions of the Lehmann ellipse and, like the work of Mandelstam, does not need any continuation in the masses.

Independently, Bros, Epstein and Glaser [10], starting from the initial domain of analyticity of the scattering amplitude, were able to prove the following theorem : in the neighbourhood of any physical point an arbitrary reaction amplitude $A+B \to C+D$ is analytic in both variables minus the energy cut. The only restriction is that particles A,B,C,D should be stable. This result is stronger than that of Mandelstam and Lehmann because not all physical points are contained in their domains, but it is also less accurate because through this complicated calculation B.E.G. were unable to keep track of the size of the neighbourhoods. A corollary of this is that the partial wave amplitudes are analytic in the neighbourhood of the physical cut. Another one is that if dispersion relations hold for $-t_0 < t \leqslant 0$ then, given $-t_0 < \bar{t} < 0$ and $\bar{s}$ outside the cuts for this particular $\bar{t}$, one has analyticity in $|s-\bar{s}| < \eta(\bar{s},\bar{t})$, $|t-\bar{t}| < \eta(\bar{s},\bar{t})$.

IV. THE CROSSING PROPERTY

As we already said, the existence of dispersion relations makes the crossing property automatic : more precisely, if the dispersion relations describe $A+B \to A+B$, then the physical scattering amplitude is $\lim_{\varepsilon \to 0} F(s+i\varepsilon,t)$ for $s > (M_A+M_B)^2$ and the limiting value on the left-hand cut from $s=-\infty$ to $s=(M_A-M_B)^2-t$ is connected with the crossed process $A+\bar{B} \to A+\bar{B}$, for a physical energy given by

$$u = 2M_A^2 + 2M_B^2 - s - t$$

More precisely, it is given by

$$\lim_{\varepsilon \to 0} F(s-i\varepsilon, t)$$

For t sufficiently close to 0 the two cuts do not overlap and one can continue from the $AB \to AB$ to the $A\bar{B} \to A\bar{B}$ amplitude. Among the consequences of this crossing property is the Pomeranchuk theorem, which can then be obtained with additional assumptions which will probably be discussed by R.J. Eden.

What Glaser, Epstein and Bros [11] have shown is that even if dispersion relations do not hold for the process $A+B \to C+D$, one still has a sufficiently large domain of analyticity for $t < 0$. t can now be made arbitrarily large negative. This domain is the intersection of a cut plane and a finite region (see Fig. 1). From the result of the previous Section, all the physical region points are accessible, and we see that one can continue from the amplitude $A+B \to C+D$ to the complex conjugate

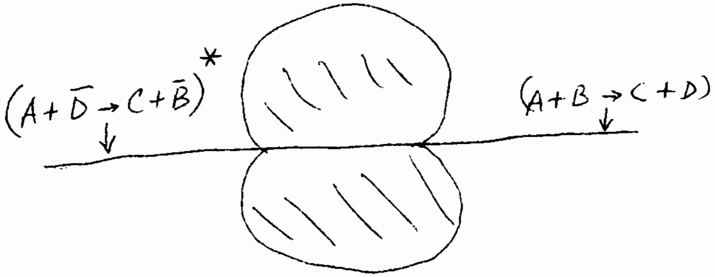

Fig. 1

of $A+\bar{D} \to C+\bar{B}$. This is enough to prove the Pomeranchuk theorem [12]. If one dislikes the fact that one goes from one amplitude to the complex conjugate of the other amplitude, one can follow another path, using fixed negative u first, and then fixed negative s. Anyway, in the NN forward case, the analyticity domain is like this

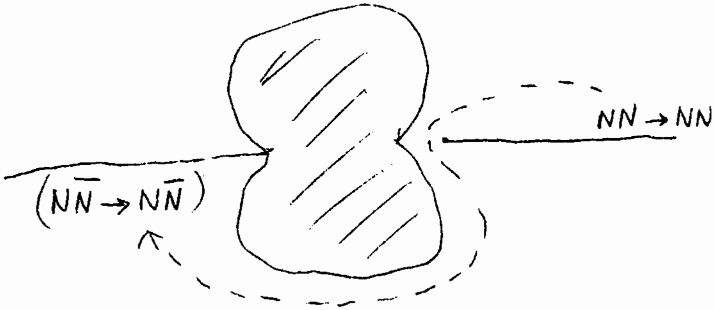

Fig. 2

so that the continuation from $NN \to NN$ to $N\bar{N} \to N\bar{N}$ can be made directly.

I should perhaps mention that the B.E.G. [11] results apply also to form factors, i.e., form factors are always analytic in a cut plane minus perhaps a finite region [13].

V. EXTENSION OF DISPERSION RELATIONS TO COMPLEX t, ENLARGEMENT OF THE LEHMANN ELLIPSE

Here we want to answer point 4) of dissatisfaction. So far, in the derivation of dispersion relations, unitarity has never been explicitly used (though, perhaps implicitly, in the derivation of the large Lehmann ellipse). The new ingredient we shall use is precisely unitarity, or more precisely, positivity as expressed as follows : the absorptive part of the scattering amplitude and all its derivatives with respect to $\cos\theta$ are maximum in the forward direction :

$$\left.\left(\frac{d}{d\cos\theta}\right)^n A_s(s,\cos\theta)\right|_{\cos\theta=1} \geqslant \left|\left(\frac{d}{d\cos\theta}\right)^n A_s(s,\cos\theta)\right|_{-1<\cos\theta<+1}$$

This property, together with the existence of dispersion relation for $-t_o < t < 0$ with a finite number of subtractions, and the existence of analyticity neighbourhoods (B.E.G.) [10] allows to prove the following [14] : the scattering amplitude for any process for which dispersion relations hold for $t_o < t \leqslant 0$, (and for which no unphysical cut is present), is analytic in the product of a circle $|t| < R$ and a cut plane in s [with cuts $s_{real} > (M_A+M_B)^2$, $u_{real} > (M_A+M_B)^2$]. For any $|t| < R$, dispersion relations are valid. If the $t=0$ dispersion relations are valid without subtraction, this persists in $|t| < R$. If there are n subtractions for $t=0$, this persists in $|t| < R$ if n is even. If n is odd, it may increase by one unit. More extensive use of unitarity allows in fact to prove then that the Froissart bound $|F(s,t=0)| < \mathrm{const}\, s(\log s/s_o)^2$ holds at least for complex s, and hence $n=2$ [15].

From this result, it follows that the absorptive part is analytic in $|t| < R$ *) and then, by using again unitarity, that the absorptive part is analytic in the ellipse in the $\cos\theta$ variable, foci $\cos\theta = \pm 1$, and semi-major axis $x'_0 = 1+(R/2k^2)$.

This removes the difficulty we had mentioned previously and the range of the forces, as defined from the exponential decrease of partial waves for large angular momentum, is finite, independent of energy.

Now, what is R ? One of my collaborators, G. Sommer, has found a very simple lower limit for R [16].

Let $x_0(s)$ be the semi-major axis of the small Lehmann ellipse (using $\cos\theta$ as a variable). This ellipse contains the circle

$$|t| < 2k^2 (x_0(s) - 1) = R(s)$$

[Notice that $R(s) = 0$ for $k^2 = 0$ and $k^2 = \infty$). Then

$$R \geqslant \text{maximum} \quad R(s)$$

$$(\text{threshold} < s < \infty)$$

The results are the following :

$$\begin{array}{ll} \pi\pi & R = 4\mu_\pi^2 \\ K\pi & R = 4\mu_\pi^2 \\ KK & R = 4\mu_\pi^2 \\ \pi N & R = 1.83\mu_\pi^2 \end{array}$$

The anomaly for πN scattering is connected with the presence of the nucleon pole. We hope that it can be removed. Once R is known, one can still enlarge the domain of validity of dispersion relations by using the new ellipse for the absorptive part. For $\pi\pi$ scattering, one finds a region as indicated in Fig. 3.

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*) A similar result holds even if dispersion relations are not true, but only for sufficiently high energies.

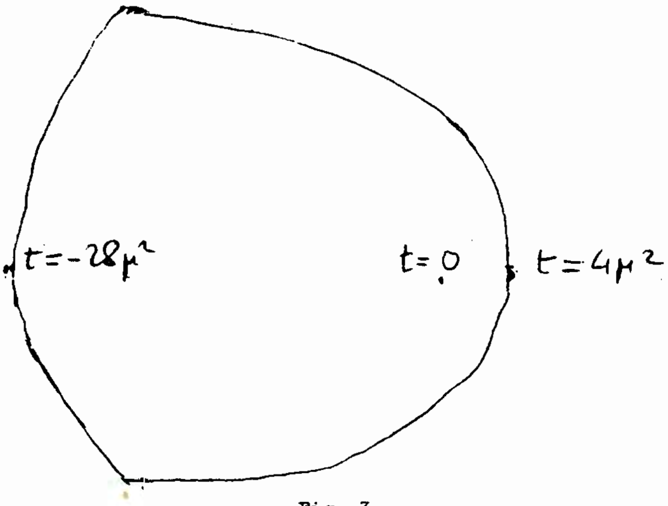

Fig. 3

For π K scattering, a similar thing can be done. For π N scattering, one gets also such a domain, which originally contains the real segment $-12.4\,\mu^2 < t < 1.83\,\mu^2$. This, however, can be then improved because the analyticity of the amplitude in a larger domain implies at least in the elastic region analyticity in a larger domain of the absorptive part. In this way one finds that the dispersion relations can be extended down to $t=-18.8\,\mu^2$, which constitutes a confirmation of the result of Jin [5]. It seems that the only limitations to the domain of validity of dispersion relations for π N are due to the singularities of the absorptive part for energies above the inelastic threshold. This is also the case for $\pi\pi$ scattering.

VI. HOW CLOSE ARE WE TO MANDELSTAM REPRESENTATION ?

It is quite clear that if we are in a situation where in all three channels fixed transfer dispersion relations are valid for complex transfer (complex t, or complex s, or complex u), the domain of analyticity of the scattering amplitude is not natural and has a holomorphy envelope which is definitely bigger than the union of the three domains. In this way one can extend the domain quite a lot but, as has been shown by Bros and Glaser [17], there is no hope of getting Mandelstam representation. However, we have many other information :

— analyticity in the neighbourhood of physical points.

— size of the ellipse of analyticity of the amplitude at high energy as deduced, by using the unitarity condition, from the enlarged ellipse of analyticity of the absorptive part;

— unitarity in the elastic region, which allows to extend the analyticity domain of the absorptive part in the elastic region and then, by using crossing symmetry in the inelastic region (this is a non-trivial step, as opposed to what happens when the Mandelstam representation is valid).

The most favourable case is undoubtedly the case of $\pi\pi$ scattering. I shall only present the most striking results [18].

1) Generalizing the Mandelstam method, one finds a family of domains

$$C_\lambda > |s-\lambda|\,|t-\lambda|$$

where λ lies inside the region of validity of fixed transfer dispersion relations. Inside these domains, the only singularities are real cuts. For $0 < \lambda < 4$ one finds, defining

$$\lambda = 16\,\frac{16-(s_0-4)^2}{64-(s_0-4)^2}$$

domains :

$$|s-\lambda|\,|t-\lambda| < \frac{3\times 64\, s_0^2}{64-(s_0-4)^2} + \lambda^2$$

It is easy to see that these domains envelope the double spectral function border

$$t = 16 + \frac{64}{s-4}$$

from $s=4$ to $s=8$, and

$$t = 4 + \frac{64}{s-16}$$

from $s = 32$ to $s = \infty$. So, except in the interval $8 < s < 32$, where one can continue till $t = 256/s$, one can continue the scattering amplitude till the border of the double spectral functions. One implication is that at low kinetic energy $(|s-4| \to 0)$ the scattering amplitude is analytic in a region $|t-4| < (64/|s-4)$, so that as the kinetic energy tends to zero, the domain approaches a cut plane, which, if a potential scattering description is possible, forces to use a Yukawa superposition.

2) For $-28 < \lambda < 0$, one finds other domains $|s-\lambda||t-\lambda| < c_\lambda$ which are tabulated

λ =	0	-2	-3	-4	-6	-8	-10	-12
c_λ =	256	324	361	362	338	319	303	292
λ =	-14	-16	-18	-20	-22	-24		
c_λ =	285	281	284	281.7	224.5	158		

From these, and from an analytic completion due to the validity of dispersion relations in all three channels, one gets a rather large domain of analyticity of partial waves, which contains the normal cuts, from $s = 4\mu^2$ to $s = 78\mu^2$ and from $s = -28\mu^2$ to $s = 0$ and extends rather far away in the imaginary direction $(|\mathrm{Im}\, s|\max \sim 80\,\mu^2)$ with a crudely roundish shape. In particular, we see that the ρ resonance is well inside the analyticity domain.

3) The result of Bros, Epstein and Glaser [11)] on crossing symmetry can be extended at least to the parabola with summit $t = 0$ and extremity $t = \mu^2$. For any t inside this parabola, the scattering amplitude is analytic in a cut plane with cuts $s_{real} > 4\mu^2$, $u_{real} > 4\mu^2$, minus a finite region.

This latter result, together with the analyticity in the neighbourhood of physical points is what makes it extremely difficult (hopefully impossible) to construct a counter-example to the Mandelstam representation consistent with all what we know. The question is : have we accumulated enough information to carry the final analytic completion and get Mandelstam representation ? This is a field in which I am rather a poor "amateur" knowing only a few tricks like the tube theorem, and I would like to urge experts, of whom many are here, to get interested in this problem.

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